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ADHM DATA FOR FRAMED SHEAVES ON HIRZEBRUCH SURFACES

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Contents

1. Introduction

Moduli spaces of framed sheaves over projective surfaces have been the object of some interest over the last few years. When the projective surface is the complex projective plane, these moduli spaces are resolutions of singularities of the moduli space of ideal instantons on the four-sphere S^4, and as such, they have been used to compute of the partition function of $N = 2$ topological super Yang-Mills theory [27, 4]. More generally, they are at the basis of the so-called instanton counting [26]. Fine moduli spaces of framed sheaves were constructed by Huybrechts and Lehn [18, 19] by introducing a stability condition. Bruzzo and Markushevich [5] showed that framed sheaves on projective surfaces, even without a stability condition, give rise to fine moduli spaces.

ADHM data were first introduced to give a description of the moduli spaces of instantons on $\mathbb{R}^4$, or the four-sphere [1]. They were used by Donaldson to show that these moduli spaces are isomorphic to moduli spaces of framed bundles on the projective plane [9] (see also [10]). King then considered the case of the projective plane blown up at one point [21], and Buchdahl extended this to multiple blowups of the projective plane [8]. ADHM data for moduli spaces of torsion-free sheaves were considered by Nakajima for the projective plane [25] and by Henni for multiple blowups of the projective plane [15].

In this thesis we consider torsion-free sheaves on a Hirzebruch surface Σ_n, for $n > 0$, that are framed to the trivial bundle on a generic rational curve ℓ_∞ in Σ_n, with $\ell_\infty^2 = n$. We provide an ADHM description of the moduli space of such sheaves. This will allow us to construct a fine moduli space which is a smooth quasi-projective variety. The fineness of this moduli space implies that it is isomorphic to the moduli space constructed in [5] (and therefore embeds as an open subset in Huybrechts-Lehn's moduli space, in the sense of [5]).

We obtain ADHM data by generalizing to the torsion-free case a monad which was introduced by Buchdahl [7] for the locally free case (Buchdahl was actually interested in μ-stable vector bundles on Hirzebruch surfaces with $c_1 = 0$, $c_2 = 2$). Indeed we show in Corollary 4.6 that for any framed torsion-free sheaf $(\mathcal{E}, \theta)$ on Σ_n, the underlying sheaf $\mathcal{E}$ is isomorphic to the cohomology of a monad

$$\mathcal{U}_{\vec{k}} \xrightarrow{\alpha} \mathcal{V}_{\vec{k}} \xrightarrow{\beta} \mathcal{W}_{\vec{k}} \tag{1.1}$$

whose terms $\mathcal{U}_{\vec{k}}$, $\mathcal{V}_{\vec{k}}$ and $\mathcal{W}_{\vec{k}}$ depends only on the Chern character $\mathrm{ch}(\mathcal{E}) = (r, aE, -c-\frac{1}{2}na^2)$ (here E is the "exceptional curve" in Σ_n, i.e., the unique irreducible curve in Σ_n squaring to $-n$, and we have put $\vec{k} = (n, r, a, c)$). This provides a map

$$(\mathcal{E}, \theta) \longmapsto (\alpha, \beta) \in \mathrm{Hom}(\mathcal{U}_{\vec{k}}, \mathcal{V}_{\vec{k}}) \oplus \mathrm{Hom}(\mathcal{V}_{\vec{k}}, \mathcal{W}_{\vec{k}}). \tag{1.2}$$

We call $L_{\vec{k}}$ the image of (1.2) and we prove in Proposition 4.9 and in Lemma 4.10 that $L_{\vec{k}}$ is a smooth variety. We construct a principal $\mathrm{GL}(r, \mathbb{C})$-bundle $P_{\vec{k}}$ over $L_{\vec{k}}$ whose fibre over a point (α, β) is naturally identified with the space of framings for the cohomology of (1.1). This implies that the map (1.2) lifts to a map

$$(\mathcal{E}, \theta) \longmapsto \theta \in P_{\vec{k}}.$$

The group $G_{\vec{k}}$ of isomorphisms of monads of the form (1.1) acts on $P_{\vec{k}}$, and the moduli space $\mathcal{M}^n(r, a, c)$ of framed sheaves on Σ_n with the given Chern character is set-theoretically defined as the quotient $P_{\vec{k}}/G_{\vec{k}}$. Theorem 5.1 proves that $\mathcal{M}^n(r, a, c)$ inherits from $P_{\vec{k}}$ a structure of smooth algebraic variety; moreover Lemma 4.8 states that two monads of the form (1.1) are isomorphic if and only it their cohomologies are isomorphic, and this ensures that there is a bijection between $\mathcal{M}^n(r, a, c)$ and set of isomorphism classes of framed sheaves on Σ_n (isomorphisms of framed sheaves are introduced by Definition 3.3). This enables us to show that the set of these classes is nonempty if and only if $2c \geq na(1 - a)$.

We prove further the fineness of the moduli space by constructing a universal family $\left(\mathfrak{E}_{\vec{k}}, \Theta_{\vec{k}}\right)$ of framed sheaves on Σ_n parametrized by $\mathcal{M}^n(r, a, c)$ (for a precise notion of family see Definition 4.2). In other words, we show that any family $(\mathfrak{F}, \Theta)_{\vec{k}}$ of framed sheaves on Σ_n parametrized by a scheme S can be got by suitably pulling-back $\left(\mathfrak{E}_{\vec{k}}, \Theta_{\vec{k}}\right)$ from $\Sigma_n \times \mathcal{M}^n(r, a, c)$ to $\Sigma_n \times S$.

Further developments. It would be of some interest to investigate in some detail some particular cases. First of all, one should deal with the "minimal case" $2c = na(1-a)$. It is easy to deduce from the injection $\mathcal{E} \rightarrowtail \mathcal{E}^{**}$ that all framed sheaves reaching this bound are locally free. Some preliminary results suggest that $\mathcal{M}^n(r, a, \frac{1}{2}na(1-a))$ is likely to be isomorphic to the total space of the direct sum of $n-1$ copies of the tangent bundle to the Grassmanian variety $\mathrm{Gr}(a, r)$ of a-planes inside $\mathbb{C}^r$.

Another issue deserving consideration is the case given by the condition $r = 1$. By normalizing the framed sheaves, one can assume $a = 0$, so that $\mathcal{M}^n(1, 0, c)$ can be straightforwardly identified with the Hilbert scheme of c points on the total space of $\mathcal{O}_{\mathbb{P}^1}(-n)$.

It is worth to analyze the partial compactification of the Uhlenbeck-Donaldson type introduced in [6] for the moduli spaces constructed in this thesis, and study how it can be expressed in terms of ADHM data.

These developments will be the object of a future work.

One could also investigate how to substitute $\mathbb{C}$ with more general ground fields, or one could try to generalize the contruction to varieties other than the Hirzebruch surfaces. For example, following Henni [15] one could try to determine the effect of a multiple blow-up on the moduli space.

Throughout this thesis, if not otherwise specified, by "scheme" we mean a noetherian reduced scheme of finite type over $\mathbb{C}$.

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2. Families of framed sheaves on Σ_n. Monads.

If X and S are schemes, we use the following notation: if $\mathcal{F}$ is a sheaf of $\mathcal{O}_{X\times S}$-modules, and F is a morphism between two such sheaves, we shall denote by $\mathcal{F}_s$ (resp. F_s) the restriction of $\mathcal{F}$ (resp. F) to the fibre of $X \times S \longrightarrow S$ over the point $s \in S$.

2.1. **Monads.** A monad M on a scheme T is a three-term complex of locally free $\mathcal{O}_T$-modules, having nontrivial cohomology only in the middle term:

$$M : \qquad 0 \longrightarrow \mathcal{U} \xrightarrow{a} \mathcal{V} \xrightarrow{b} \mathcal{W} \longrightarrow 0\,. \tag{2.1}$$

The cohomology of the monad will be denoted by $\mathcal{E}(M)$. It is a coherent $\mathcal{O}_T$-module. A *morphism* (*isomorphism*) *of monads* is a morphism (isomorphism) of complexes.

With each monad M one can canonically associate the following commutative diagram, called the *display of the monad*, with exact rows and columns:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{U} & \xrightarrow{a} & \mathcal{B} & \xrightarrow{q} & \mathcal{E} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle i} & & \\ 0 & \longrightarrow & \mathcal{U} & \xrightarrow{a} & \mathcal{V} & \xrightarrow{\pi} & \mathcal{A} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \nu} & & \\ & & & & \mathcal{W} & = & \mathcal{W} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{2.2}$$

where $\mathcal{A} := \operatorname{coker} a$, $\mathcal{B} := \ker b$, $\mathcal{E} = \mathcal{E}(M)$. The morphisms ι, q and π are canonical, while the other ones are naturally induced.

We describe some basic facts about the cohomology of a monad.

Lemma 2.1. *Given a monad M as in eq.* (2.1)*, its cohomology $\mathcal{E}$ is torsion-free. In particular, if* $\dim T = 2$*, the morphism a, as a morphism between the fibres of $\mathcal{U}$ and $\mathcal{V}$, can fail to be injective at most in a finite number of points.*

Proof. From the display (2.2) one gets an injection $\mathcal{A} \rightarrowtail \mathcal{A}^{**}$. Since $\mathcal{A}^{**}$ is reflexive, it is torsion-free, so that $\mathcal{A}$ is torsion free as well. As $\mathcal{E}$ is a subsheaf of $\mathcal{A}$, the thesis follows. □

In particular, if $\dim T = 2$, the dimension of the fibres of $\mathcal{A}$ is constant outside at most a finite number of points.

Suppose now that T is a product: $T = X \times S$, where X is a smooth connected projective variety, and let t_i, $i = 1, 2$ the canonical projections onto the first and second factor respectively.

The following result will be very useful to identify three-term complexes as monads.

Lemma 2.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be locally-free sheaves on T, and let $F : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism. Then*

(1) *If for all* closed *points $s \in S$ the restricted morphism F_s is injective, then F is injective.*
(2) *If for all* closed *points $s \in S$ the restricted morphism F_s is surjective, then F is surjective.*

In particular this implies that, if F_s is bijective for all closed *points $s \in S$, then F is an isomorphism.*

Proof. Surjectivity: for all closed $s \in S$, we have $\operatorname{coker} F_s = 0$. This implies $\operatorname{coker} F = 0$.

Injectivity: to simplify the notation, we set $\mathcal{K} := \ker F$. Let $\mathcal{O}_X(1)$ be a very ample sheaf on X over $\operatorname{Spec}\mathbb{C}$, so that $\mathcal{O}_T(1) = t_1^*\mathcal{O}_X(1)$ is a very ample sheaf on T over S. By Theorem 8.8 at p. 252 of [14] one has the following surjection:

$$t_1^* t_{1*}\left(\mathcal{K}(n)\right) \longrightarrow \mathcal{K}(n) \longrightarrow 0 \qquad \text{for } n \gg 0\,. \tag{2.3}$$

At the same time one has the following vanishing result (see the proof of Theorem 2.1.5 on pp. 33-35 of [20]):

$$H^i\left(\mathcal{F}_s(n)\right) = H^i\left(\mathcal{G}_s(n)\right) = 0 \qquad \begin{array}{l} \text{for } i > 0 \\ \text{and} \\ \text{for all } s \in S \end{array} \qquad n \gg 0\,.$$

Let us fix any such n. For all closed points $s \in S$ it is easy to construct the following diagram :

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{T}(s) & \longrightarrow & \iota_s^* t_{2*}(\mathcal{K}(n)) & \longrightarrow & \iota_s^* t_{2*}\left(\mathcal{F}(n)\right) & \xrightarrow{\ \iota_s^* t_{2*}(F(n))\ } & \iota_s^* t_{2*}\left(\mathcal{G}(n)\right) \\ & & & & & \downarrow & & \downarrow \\ & & & 0 & \longrightarrow & H^0\left(\mathcal{F}_s(n)\right) & \xrightarrow{\ H^0(F_s(n))\ } & H^0\left(\mathcal{G}_s(n)\right) \end{array}$$

where $\mathcal{T}(s) := L_1\iota_s^*\left[\operatorname{im}\left(t_{2*}F(n)\right)\right]$, and ι_s is the closed immersion of s into S. The Semicontinuity Theorem implies that the vertical arrows in this diagram are isomorphisms, so that

$$\mathcal{T}(s) \simeq \iota_s^* t_{2*}(\mathcal{K}(n))\,.$$

Since there exists an open subset $U \subseteq S$ such that $\operatorname{im}\left(t_{2*}F(n)\right)|_U$ is locally free, this implies that

$$\iota_s^* t_{2*}(\mathcal{K}(n)) = 0 \qquad \text{for all closed points } s \in U.$$

Hence, the sheaf $t_{2*}(\mathcal{K}(n))$ is either zero or a pure torsion sheaf. In view of the surjection in eq. (2.3), it follows $t_{2*}(\mathcal{K}(n)) = 0$.

□

Lemma 2.3. *Let M be a monad over the product scheme $T = X \times S$ (see eq. (2.1)). The cohomology $\mathcal{E}(M)$ is flat over S if and only if the restricted morphisms a_s are injective for all closed points $s \in S$.*

Proof. Consider the following short exact sequence, which can be extracted from the display of M:

$$0 \longrightarrow \mathcal{U} \xrightarrow{a} \mathcal{B} \longrightarrow \mathcal{E} \longrightarrow 0\,.$$

Since $\mathcal{B}$ is locally free, it is flat on S. Then by Lemma 2.1.4 on p. 33 of [20], we only need to show that the injectivity of the morphisms a_s over the closed points of S is equivalent to the injectivity over *any* point $s \in S$. We proceed as follows. Since flatness is a local notion, we can assume that $S = \operatorname{Spec} R$, so that every point $s_0 \overset{\iota_{s_0}}{\hookrightarrow} S$ corresponds to a prime ideal $\mathfrak{p}$ of R. The latter defines an irreducible closed subscheme $Z = \operatorname{Spec}(R/\mathfrak{p}) \overset{\iota_Z}{\hookrightarrow} S$, and in particular the immersion ι_{s_0} factors as

$$\begin{array}{ccc} \operatorname{Spec} k(s_0) & & \\ {\scriptstyle j_{s_0}}\big\downarrow & \searrow^{\iota_{s_0}} & \\ Z & \overset{\iota_Z}{\hookrightarrow} & S \end{array}$$

where j_{s_0} is the immersion of the generic point into Z. This enables us to write the following equation:

$$a_{s_0} = j_{s_0}^* \left(a|_Z \right). \tag{2.4}$$

Now we claim that the restricted morphism $a|_Z$ is injective. Indeed we know that the restriction $\left(a|_Y\right)_s$ is injective for all closed points $s \in Z$. Since both $\mathcal{U}$ and $\mathcal{B}$ are locally free, the claim is a consequence of Lemma 2.2. Since the morphism j_{s_0} is flat, the thesis follows from eq. (2.4). □

3. Framed sheaves on Hirzebruch surfaces: the main result

Let us denote by Σ_n the n-th Hirzebruch surface. It is a ruled surface over $\mathbb{P}^1$, and we we shall denote by F the class in $\operatorname{Pic}(\Sigma_n)$ of the fibre of the ruling. The surface Σ_n can be seen also as $\Sigma_n = T_n \cup \ell_\infty$ where T_n is the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-n)$, and $\ell_\infty \simeq \mathbb{P}^1$ is a rational curve. We shall denote by H the class in $\operatorname{Pic}(\Sigma_n)$ of the latter. Since one has $\operatorname{Pic}\Sigma_n = \mathbb{Z}H \oplus \mathbb{Z}F$, it is customary to adopt the following notation:

$$\mathcal{E}(p,q) := \mathcal{E} \otimes \mathcal{O}_{\Sigma_n}(pH + qF) \qquad p, q \in \mathbb{Z},$$

where $\mathcal{E}$ is any sheaf of $\mathcal{O}_{\Sigma_n}$-modules.

It is important to know the cohomology of the sheaves $\mathcal{O}_{\Sigma_n}(p,q)$.

Lemma 3.1.

$$H^0(\mathcal{O}_{\Sigma_n}(p,q)) \neq 0 \qquad \Leftrightarrow \qquad \begin{cases} p \geq 0 \\ np+q \geq 0\,; \end{cases}$$

$$H^1(\mathcal{O}_{\Sigma_n}(p,q)) \neq 0 \qquad \Leftrightarrow \qquad \begin{cases} p \geq 0 \\ q \leq -2 \end{cases} \qquad or \qquad \begin{cases} p \leq -2 \\ q \geq n\,; \end{cases}$$

$$H^2(\mathcal{O}_{\Sigma_n}(p,q)) \neq 0 \qquad \Leftrightarrow \qquad \begin{cases} p \leq -2 \\ np+q \leq -(n+2)\,. \end{cases}$$

Proof. Similar to King's proof for the case $n = 1$ [21, pp. 22-23]. □

For more information about Hirzebruch surfaces see for example [3] or [2].

Definition 3.2. *A* framed sheaf *is a pair* $(\mathcal{E}, \theta)$, *where*

(1) $\mathcal{E}$ *is a torsion-free sheaf on* Σ_n *of rank* $r > 0$ *such that*

$$\mathcal{E}|_{\ell_\infty} \simeq \mathcal{O}_{\ell_\infty}^{\oplus r}; \tag{3.1}$$

(2) θ *is a fixed isomorphism* $\theta : \mathcal{E}|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r}$.

Condition (3.1) is called *triviality at infinity*, and implies $c_1(\mathcal{E}) \propto E$, where $E = H - nF$. The isomorphism θ is called the *framing at infinity*. By "sheaf trivial at infinity" we shall understand a torsion-free sheaf satisfying the condition (3.1) (i.e., we do not fix a framing).

Definition 3.3. *An isomorphism* Λ *between two framed sheaves* $(\mathcal{E}, \theta)$ *and* $(\mathcal{E}', \theta')$ *is an isomorphism* $\Lambda : \mathcal{E} \xrightarrow{\sim} \mathcal{E}'$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{E}|_{\ell_\infty} & \xrightarrow{\theta} & \mathcal{O}_{\ell_\infty}^{\oplus r} \\ {\scriptstyle\Lambda_\infty}\big\downarrow & \nearrow{\scriptstyle\theta'} & \\ \mathcal{E}'|_{\ell_\infty} & & \end{array} \tag{3.2}$$

where $\Lambda_\infty := \Lambda|_{\ell_\infty}$.

From now on we consider only the case $n \geq 1$ and assume that framed sheaves are normalized, that is, $0 \leq a \leq r-1$. The moduli space we are interested in may be set-theoretically defined as follows: $\mathcal{M}^n(r,a,c)$ is the set of isomorphism classes of framed sheaves on Σ_n having rank r, first Chern class aE, and second Chern class c. The main result in this thesis is Theorem 3.4, which states that this set can be endowed with a structure of a smooth algebraic variety, with an explicit description in terms of linear algebraic data (the ADHM data), which makes it into the fine moduli space of framed sheaves on Σ_n. To simplify the statement of the Main Theorem we need to introduce some notation. In particular we need to introduce a quasi-affine variety that we shall denote by $L_{\vec{k}}$. Let us denote by $\vec{k}$ a quadruple (n, r, a, c), and define $k_i \in \mathbb{Z}$, $i = 1, \ldots, 4$ as follows:

$$k_1 = c + \frac{1}{2}na(a-1) \qquad \text{and} \qquad \begin{cases} k_2 = k_1 + na \\ k_3 = k_1 + (n-1)a \\ k_4 = k_1 + r - a\,. \end{cases} \tag{3.3}$$

Now suppose $k_1 \geq 0$.

- We introduce the locally free sheaves on Σ_n

$$\begin{cases} \mathcal{U}_{\vec{k}} := \mathcal{O}_{\Sigma_n}(0,-1)^{\oplus k_1} \\ \mathcal{V}_{\vec{k}} := \mathcal{O}_{\Sigma_n}(1,-1)^{\oplus k_2} \oplus \mathcal{O}_{\Sigma_n}^{\oplus k_4} \\ \mathcal{W}_{\vec{k}} := \mathcal{O}_{\Sigma_n}(1,0)^{\oplus k_3}. \end{cases} \tag{3.4}$$

  In what follows, we shall write $\mathcal{U}_{\vec{k},\infty}$ instead of $\mathcal{U}_{\vec{k}}\big|_{\ell_\infty}$, and analogously for $\mathcal{V}_{\vec{k}}$ and $\mathcal{W}_{\vec{k}}$.
- We introduce the vector spaces

$$\mathbb{V}_{\vec{k}} := \mathbb{V}_{\vec{k},A} \oplus \mathbb{V}_{\vec{k},B} := \operatorname{Hom}\left(\mathcal{U}_{\vec{k}}, \mathcal{V}_{\vec{k}}\right) \oplus \operatorname{Hom}\left(\mathcal{V}_{\vec{k}}, \mathcal{W}_{\vec{k}}\right).$$

  We shall call $f_i' : \mathbb{V}_{\vec{k}} \longrightarrow \mathbb{V}_{\vec{k},i}$, $i = A, B$ the canonical projections. We shall adopt the letter α (resp. β) to denote an element of $\mathbb{V}_{\vec{k},A}$ (resp. $\mathbb{V}_{\vec{k},B}$).
- $\overline{L}_{\vec{k}}$ is the affine subvariety of $\mathbb{V}_{\vec{k}}$ given by the equation

$$\beta \circ \alpha = 0.$$

  So, one can define the complex

$$M(\alpha,\beta): \qquad \mathcal{U}_{\vec{k}} \xrightarrow{\alpha} \mathcal{V}_{\vec{k}} \xrightarrow{\beta} \mathcal{W}_{\vec{k}}\,.$$

- Finally, we introduce the quasi-affine variety $L_{\vec{k}}$ as the open subset of $\overline{L}_{\vec{k}}$ defined by the following four conditions:
  (c1) The sheaf morphism α is a monomorphism.
  (c2) The sheaf morphism β is an epimorphism. In particular, $M(\alpha,\beta)$ is a monad on Σ_n.
  (c3) The vector space morphisms $\alpha \otimes k(y)$ have maximal rank for all closed points $y \in \ell_\infty$.

(c4) Consider the display associated with $M(\alpha,\beta)$ as in eq. (2.2). By restricting it to ℓ_∞, by twisting by $\mathcal{O}_{\ell_\infty}(-1)$ and by taking cohomology, we get a vector space morphism $\Phi := H^0\left(\nu|_{\ell_\infty}(-1)\right) : H^0\left(\mathcal{A}|_{\ell_\infty}(-1)\right) \longrightarrow H^0\left(\mathcal{W}_{\vec{k},\infty}(-1)\right)$. We require that

$$\det\Phi \neq 0\,.$$

We shall see in Lemma 4.10 that $L_{\vec{k}}$ is a smooth variety. Now we come to the Main Theorem.

Theorem 3.4. *The set $\mathcal{M}^n(r,a,c)$ is nonempty if and only if $c+\frac{1}{2}na(a-1)\geq 0$ and, in this case, it can be given a structure of smooth algebraic variety of dimension $2rc+(r-1)na^2$. This is obtained by representing it as a quotient $\mathcal{M}^n(r,a,c) = P_{\vec{k}}/G_{\vec{k}}$, where $P_{\vec{k}}$ is a $\mathrm{GL}(r)$-principal bundle over $L_{\vec{k}}$, while $G_{\vec{k}}$ is an algebraic group, acting freely on $P_{\vec{k}}$, that can be described as*

$$G_{\vec{k}} = \mathrm{Aut}\left(\mathcal{U}_{\vec{k}}\right)\times\mathrm{Aut}\left(\mathcal{V}_{\vec{k}}\right)\times\mathrm{Aut}\left(\mathcal{W}_{\vec{k}}\right).$$

With this scheme structure $\mathcal{M}^n(r,a,c)$ is a fine moduli space of framed sheaves on Σ_n.

Note that

$$\dim\mathcal{M}^n(r,a,c) = 2rc+(r-1)na^2 = 2r\Delta,$$

where $\Delta = c_2 - \frac{r-1}{2r}c_1^2$ is the discriminant of the sheaves parametrized by $\mathcal{M}^n(r,a,c)$.

4. The varieties $L_{\vec{k}}$ and $P_{\vec{k}}$.

In this section we explain how the varieties $L_{\vec{k}}$ and $P_{\vec{k}}$ arise and construct a canonical family $\left(\widetilde{\mathfrak{E}}_{\vec{k}},\widetilde{\Theta}_{\vec{k}}\right)$ on the product $\Sigma_n\times P_{\vec{k}}$.

4.1. **Families of framed sheaves.** Let S be a scheme; let $T=\Sigma_n\times S$ and t_i, $i=1,2$ be the projections onto the first and the second factor, respectively. Analogously, we introduce the product scheme $T_\infty = \ell_\infty\times S$, with the projections u_i, $i=1,2$.

Definition 4.1. *Let $\vec{k}=(n,r,a,c)\in\mathbb{Z}^4$ with $r\geq 1$, $n\geq 1$ and $0\leq a\leq r-1$ as above. We say that a coherent sheaf $\mathfrak{F}$ on T satisfies condition $\vec{k}$ if and only if it is flat on S and for all closed points $s\in S$*

- *the restricted sheaf $\mathfrak{F}_s$ is torsion-free and trivial at infinity on $T_s\simeq\Sigma_n$;*
- *the Chern character of $\mathfrak{F}_s$ is $(r, aE, -c-\frac{r-1}{2r}na^2)$.*

We shall prove in Corollary 4.4 that a sheaf $\mathfrak{F}$ satisfing condition $\vec{k}$ is torsion-free.

Definition 4.2. *Given a vector $\vec{k}$ and a product scheme T as above, a family of framed sheaves on Σ_n is a pair $(\mathfrak{F}, \Theta)$, where:*

(1) *$\mathfrak{F}$ is a sheaf on T satisfing condition $\vec{k}$;*
(2) *Θ is an isomorphism $\mathfrak{F}|_{T_\infty} \to \mathcal{O}_{T_\infty}^{\oplus r}$.*

Now, we come to the main result of this section. To state it, we introduce the following notation. For any sheaf F of $\mathcal{O}_T$-modules we let

$$\mathcal{F}(p,q) = \mathcal{F} \otimes t_1^* \mathcal{O}_{\Sigma_n}(p,q) \qquad \text{for all } (p,q) \in \mathbb{Z}.$$

Proposition 4.3. *A sheaf $\mathfrak{F}$ on T that satisfies condition $\vec{k}$ is isomorphic to the cohomology of a canonically defined monad $M(\mathfrak{F})$ on T. This monad is of the form*

$$M(\mathfrak{F}): \qquad 0 \longrightarrow \mathfrak{U} \xrightarrow{A} \mathfrak{V} \xrightarrow{B} \mathfrak{W} \longrightarrow 0 \tag{4.1}$$

where the locally free sheaves $\mathfrak{U}$ and $\mathfrak{W}$ are

$$\begin{cases} \mathfrak{U} = \mathcal{O}_T(0,-1) \otimes t_2^* R^1 t_{2*}\left[\mathfrak{F}(-2, n-1)\right] \\ \mathfrak{W} = \mathcal{O}_T(1,0) \otimes t_2^* R^1 t_{2*}\left[\mathfrak{F}(-1,0)\right]. \end{cases}$$

The sheaf $\mathfrak{V}$ is defined as an extension

$$0 \longrightarrow \mathfrak{V}_- \xrightarrow{\mathfrak{c}} \mathfrak{V} \xrightarrow{\mathfrak{d}} \mathfrak{V}_+ \longrightarrow 0\,, \tag{4.2}$$

$$\text{where} \qquad \begin{cases} \mathfrak{V}_+ := t_2^* R^1 t_{2*}\left[\mathfrak{F}(-2, n)\right] \\ \mathfrak{V}_- := \mathcal{O}_T(1,-1) \otimes t_2^* R^1 t_{2*}\left[\mathfrak{F}(-1,-1)\right] \end{cases}$$

and the morphisms $\mathfrak{c}$ and $\mathfrak{d}$ are canonically determined by $\mathfrak{F}$.

Moreover the association $\mathfrak{F} \mapsto M(\mathfrak{F})$ is covariantly functorial.

Proof. See Appendix A. □

This Proposition and Lemma 2.1 have the following easy Corollary.

Corollary 4.4. *A sheaf $\mathfrak{F}$ on T which satisfies condition $\vec{k}$ is torsion-free.*

4.2. **Absolute monads and the variety $L_{\vec{k}}$.** As a straightforward consequence of Proposition 4.3 we get the following result.

Corollary 4.5. *Let the sheaf $\mathfrak{F}$ and the monad $M(\mathfrak{F})$ be as in the Proposition 4.3. Suppose that S is affine, and that the sheaves $R^1 t_{2*}(\mathfrak{F}(p,q))$ are trivial for $(p,q) \in \{(-2, n-1), (-1,0), (-2,n), (-1,-1)\} =: \mathscr{I}$ (by Proposition 4.3, these hypotheses are satisfied in* *an open neighborood of any point of S). One has the following isomorphisms*

$$\begin{cases} \mathfrak{U} \simeq t_1^* \mathcal{U}_{\vec{k}} \\ \mathfrak{V} \simeq t_1^* \mathcal{V}_{\vec{k}} \\ \mathfrak{W} \simeq t_1^* \mathcal{W}_{\vec{k}} \end{cases}$$

where the sheaves $\mathcal{U}_{\vec{k}}$, $\mathcal{V}_{\vec{k}}$ and $\mathcal{W}_{\vec{k}}$ were defined in eq. (3.4).

Proof. Let $\mathcal{F}$ be any of the sheaves $\mathfrak{F}(p,q)$ for $(p,q) \in \mathscr{I}$. A trivialization for $R^1 t_{2*}(\mathcal{F})$ corresponds to choosing a closed point $s_0 \in S$ and an isomorphism

$$R^1 t_{2*}(\mathcal{F}) \xrightarrow{\sim} \mathcal{O}_S \otimes \left[R^1 t_{2*}(\mathcal{F}) \otimes k(s_0) \right] .$$

Since $\mathfrak{F}_{s_0}$ is trivial at infinity, from Lemma A.1 and from the Semicontinuity Theorem one can obtain the isomorphism

$$R^1 t_{2*}(\mathcal{F}) \otimes k(s_0) \simeq H^1(\mathcal{F}_{s_0}) .$$

The dimensions of the vector spaces $H^1(\mathfrak{F}_{s_0}(p,q))$ can be computed by means of Riemann-Roch Theorem and Lemma A.1. Choosing bases for these four vector spaces, one gets trivializations

$$\begin{cases} R^1 t_{2*}(\mathfrak{F}(-2,n-1)) \simeq \mathbb{C}^{k_1} \otimes \mathcal{O}_S \\ R^1 t_{2*}(\mathfrak{F}(-1,0)) \simeq \mathbb{C}^{k_3} \otimes \mathcal{O}_S \\ R^1 t_{2*}(\mathfrak{F}(-2,n)) \simeq \mathbb{C}^{k_4} \otimes \mathcal{O}_S \\ R^1 t_{2*}(\mathfrak{F}(-1,-1)) \simeq \mathbb{C}^{k_2} \otimes \mathcal{O}_S \end{cases} \quad \text{so that} \quad \begin{cases} \mathfrak{U} \simeq \mathcal{O}_T(0,-1)^{\oplus k_1} \\ \mathfrak{W} \simeq \mathcal{O}_T(1,0)^{\oplus k_3} \\ \mathfrak{V}_- \simeq \mathcal{O}_T^{\oplus k_4} \\ \mathfrak{V}_+ \simeq \mathcal{O}_T(1,-1)^{\oplus k_2} \end{cases}$$

where k_i, $i = 1, \dots, 4$ are defined in eq. (3.3). The thesis follows for $\mathfrak{U}$ and $\mathfrak{W}$. By plugging these sheaves into the sequence eq. (4.2), the latter splits, since

$$\begin{aligned} \operatorname{Ext}^1(\mathcal{O}_T(1,-1), \mathcal{O}_T) &\simeq H^1(T, \mathcal{O}_T(-1,1)) \simeq \\ &\simeq H^0\left(S, R^1 t_{2*} \mathcal{O}_T(-1,1)\right) \simeq \qquad (*) \\ &\simeq H^0\left(S, R^1 t_{2*}\left[t_1^* \mathcal{O}_{\Sigma_n}(-1,1)\right]\right) \simeq \\ &\simeq H^0(S, \mathcal{O}_S) \otimes H^1(\Sigma_n, \mathcal{O}_{\Sigma_n}(-1,1)) = 0 \end{aligned}$$

(the isomorphism $(*)$ holds true since S is affine and the last step is a consequence of Lemma A.1). This ends the proof. □

The following result is the absolute case of Proposition 4.3, obtained by letting $S = \operatorname{Spec} \mathbb{C}$, and follows easily from the Corollary 4.5.

Corollary 4.6. *All sheaves $\mathcal{E}$ on Σ_n that are trivial at infinity are isomorphic to the cohomology of a monad $M_{\vec{k}}(\mathcal{E})$ of the form is $M(\alpha,\beta)$, for a suitable $(\alpha,\beta)\in\overline{L}_{\vec{k}}$.*

An immediate consequence of this Corollary is the following result.

Corollary 4.7. *If $k_1 = c + \frac{1}{2}na(a-1) < 0$, the set $\mathcal{M}^n(r,a,c)$ is empty.*

In the sequel we fix a vector $\vec{k}$ such that $k_1 \geq 0$.

The functoriality of $M_{\vec{k}}(-)$ implies that $M_{\vec{k}}(\mathcal{E}) \simeq M_{\vec{k}}(\mathcal{E}')$ whenever $\mathcal{E}\simeq\mathcal{E}'$. In particular, $M_{\vec{k}}(-)$ provides a set-theoretical map between the set of isomorphism classes of sheaves on Σ_n that are trivial at infinity and the set of isomorphism classes of monads of the form $M(\alpha,\beta)$. The following two results establish the injectivity of this map, and enable us to characterize its image.

Lemma 4.8. *Let $(\alpha,\beta),(\alpha',\beta')$ be any two points in $\overline{L}_{\vec{k}}$ satisfying the conditions (c1) and (c2) introduced at p. 10 (so that $M=M(\alpha,\beta)$ and $M'=M(\alpha',\beta')$ are monads). Then*

$$M\simeq M' \qquad \textit{if and only if} \qquad \mathcal{E}(M)\simeq\mathcal{E}(M').$$

Proof. Lemma 4.1.3 of [28], p. 276, which is proved for the locally free case, applies here as well. ☐

Proposition 4.9. *Given a point $(\alpha,\beta)\in\overline{L}_{\vec{k}}$ satisfying conditions (c1) and (c2) at p. 10, the cohomology $\mathcal{E}$ of the monad $M(\alpha,\beta)$ is trivial at infinity if and only if the morphisms (α,β) satisfy conditions (c3) and (c4) at p. 11, that is, if and only if the point (α,β) lies inside the variety $L_{\vec{k}}$.*

Proof. Condition (c3) is equivalent to the local freeness of $\mathcal{E}|_{\ell_\infty}$. As for condition (c4), from the display of $M(\alpha,\beta)$ one gets the exact sequence

$$H^0\left(\mathcal{E}|_{\ell_\infty}(-1)\right) \rightarrowtail H^0\left(\mathcal{A}|_{\ell_\infty}(-1)\right) \xrightarrow{\Phi} H^0\left(\mathcal{W}_{\vec{k},\infty}(-1)\right) \twoheadrightarrow H^1\left(\mathcal{E}|_{\ell_\infty}(-1)\right).$$

Condition (c4) is equivalent to the vanishing of $H^i\left(\mathcal{E}|_{\ell_\infty}(-1)\right)$, $i=0,1$. The thesis follows easily. ☐

In what follows, for any pair of morphisms $(\alpha,\beta)\in L_{\vec{k}}$, we call $\mathcal{E}_{\alpha,\beta}$ the cohomology of the monad $M(\alpha,\beta)$, and let $\mathcal{E}_{\alpha,\beta,\infty}=\mathcal{E}_{\alpha,\beta}|_{\ell_\infty}$.

Lemma 4.10. *The variety $L_{\vec{k}}$ is smooth of dimension* $\dim L_{\vec{k}} = \dim\mathbb{V}_{\vec{k}} - \dim\mathbb{W}_{\vec{k}}$, *where* $\mathbb{W}_{\vec{k}} = \mathrm{Hom}\left(\mathcal{U}_{\vec{k}},\mathcal{W}_{\vec{k}}\right)$.

Proof. The proof is essentially the same as in Lemma 4.1.7 at p. 293 of [28]. We define the map

$$\begin{array}{rccc} \zeta : & \mathbb{V}_{\vec{k}} & \longrightarrow & \mathbb{W}_{\vec{k}} \\ & (\alpha,\beta) & \longmapsto & \beta\alpha . \end{array}$$

$\overline{L}_{\vec{k}}$ coincides with the set $\{\zeta = 0\}$. For any point $(\alpha_0,\beta_0) \in L_{\vec{k}}$ the differential $\mathrm{d}\zeta$ is the linear map

$$\begin{array}{rccc} (\mathrm{d}\zeta)|_{(\alpha_0,\beta_0)} : & \mathbb{V}_{\vec{k}} & \longrightarrow & \mathbb{W}_{\vec{k}} \\ & (\alpha,\beta) & \longmapsto & \beta_0\alpha + \beta\alpha_0 . \end{array}$$

The rank of this map is bounded above by $\dim \mathbb{V}_{\vec{k}} - \dim L_{\vec{k}}$, and it coincides with this value on the nonempty open subset of nonsingular points of $L_{\vec{k}}$ (see [14], pp.31-33).

Given any point $(\alpha,\beta) \in L_{\vec{k}}$, one has the isomorphism

$$\operatorname{coker}(\mathrm{d}\zeta)|_{(\alpha,\beta)} \simeq H^2\left(\mathcal{E}^*_{\alpha,\beta} \otimes \mathcal{E}_{\alpha,\beta}\right) . \tag{4.3}$$

Indeed, put $\mathcal{E} = \mathcal{E}_{\alpha,\beta}$ and $M = M(\alpha,\beta)$, and consider the dual complex M^*. It is not a monad, since has nontrivial cohomology both in the middle term (isomorphic to $\mathcal{E}^*$) and in the right term (isomorphic to $\mathcal{E}^1 := \mathcal{E}xt^1(\mathcal{E},\mathcal{O}_{\Sigma_n})$). Let $T_\bullet$ be the total complex of the double complex $M \otimes M^*$. Since $\mathcal{E}^1$ is supported on points one has

$$H^i\left(\mathcal{E} \otimes \mathcal{E}^1\right) = H^i\left(\mathcal{T}or_1\left(\mathcal{E},\mathcal{E}^1\right)\right) = 0 \qquad \text{for } i > 0 .$$

As the sheaves $\mathcal{U}_{\vec{k}}$, $\mathcal{V}_{\vec{k}}$ and $\mathcal{W}_{\vec{k}}$ satisfy the hypotheses of Lemma 4.1.7 on p. 293 of [28], by applying Künneth's Theorem to the complex $T_\bullet$ ([32, Thm 3.6.3 p. 88]), condition (4.3) follows.

Being $\mathcal{E}^* \otimes \mathcal{E}$ trivial at infinity on Σ_n, Lemma A.1 implies that $\mathrm{d}\zeta$ has maximal rank everywhere on $L_{\vec{k}}$. □

4.3. **The variety $P_{\vec{k}}$.** Let us introduce the varieties $\breve{\mathfrak{T}} = \Sigma_n \times L_{\vec{k}}$ and $\breve{\mathfrak{T}}_\infty = \ell_\infty \times L_{\vec{k}}$, together with the canonical projections shown in the following diagram:

$$\begin{array}{ccc} \ell_\infty & \hookrightarrow & \Sigma_n \\ \uparrow{\scriptstyle \breve{\mathfrak{u}}_1} & & \uparrow{\scriptstyle \breve{\mathfrak{t}}_1} \\ \breve{\mathfrak{T}}_\infty & \hookrightarrow & \breve{\mathfrak{T}} \\ & {\scriptstyle \breve{\mathfrak{u}}_2}\searrow & \downarrow{\scriptstyle \breve{\mathfrak{t}}_2} \\ & & L_{\vec{k}}\,. \end{array} \tag{4.4}$$

Definition 4.11. *The complex $\breve{\mathbb{M}}_{\vec{k}}$ on $\breve{\mathfrak{T}}$ is defined as*

$$\breve{\mathbb{M}}_{\vec{k}}: \qquad \breve{\mathfrak{t}}_1^* \mathcal{U}_{\vec{k}} \xrightarrow{\breve{\mathfrak{t}}_2^* f_B} \breve{\mathfrak{t}}_1^* \mathcal{V}_{\vec{k}} \xrightarrow{\breve{\mathfrak{t}}_2^* f_B} \breve{\mathfrak{t}}_1^* \mathcal{W}_{\vec{k}}$$

where f_A and f_B are the restrictions to $L_{\vec{k}}$ of the projections f'_A and f'_B introduced in Section 3.

This complex is a monad by Lemma 2.2: we call its cohomology $\breve{\mathfrak{E}}_{\vec{k}}$. It satisfies condition $\vec{k}$, and more precisely one has the following isomorphism for all points $(\alpha,\beta) \in L_{\vec{k}}$:

$$\left(\breve{\mathfrak{E}}_{\vec{k}}\right)_{(\alpha,\beta)} \simeq \mathcal{E}_{\alpha,\beta}\,.$$

We call $\breve{\mathbb{M}}_{\vec{k},\infty}$ the restriction of $\breve{\mathbb{M}}_{\vec{k}}$ to $\breve{\mathfrak{T}}_\infty$. This is isomorphic to the monad

$$\breve{\mathbb{M}}_{\vec{k},\infty} \simeq \qquad 0 \longrightarrow \breve{\mathfrak{u}}_1^* \mathcal{U}_{\vec{k},\infty} \xrightarrow{\breve{\mathfrak{u}}_2^* g_B} \breve{\mathfrak{u}}_1^* \mathcal{V}_{\vec{k},\infty} \xrightarrow{\breve{\mathfrak{u}}_2^* g_B} \breve{\mathfrak{u}}_1^* \mathcal{W}_{\vec{k},\infty} \longrightarrow 0\,. \tag{4.5}$$

Here the morphisms (g_A, g_B) are the restrictions to $L_{\vec{k}}$ of (g'_A, g'_B), which are defined in their turn as the following compositions:

$$(g'_A, g'_B): \mathbb{V}_{\vec{k}} \xrightarrow{(f_A, f_B)} \mathbb{V}_{\vec{k},A} \oplus \mathbb{V}_{\vec{k},B} \xrightarrow{\cdot|_{\ell_\infty}} \begin{array}{c} \operatorname{Hom}\left(\mathcal{U}_{\vec{k},\infty}, \mathcal{V}_{\vec{k},\infty}\right) \\ \oplus \\ \operatorname{Hom}\left(\mathcal{V}_{\vec{k},\infty}, \mathcal{W}_{\vec{k},\infty}\right)\,. \end{array} \tag{4.6}$$

The cohomology of $\breve{\mathbb{M}}_{\vec{k},\infty}$ will be denoted by $\breve{\mathfrak{E}}_{\vec{k},\infty}$. This has the property that

$$\left(\breve{\mathfrak{E}}_{\vec{k},\infty}\right)_{(\alpha,\beta)} \simeq \mathcal{E}_{\alpha,\beta,\infty} \tag{4.7}$$

for all points $(\alpha,\beta) \in L_{\vec{k}}$.

Let $\mathfrak{N}_{\vec{k}}$ denote the direct image $\breve{\mathfrak{u}}_{2*} \breve{\mathfrak{E}}_{\vec{k},\infty}$. Since $\breve{\mathfrak{E}}_{\vec{k},\infty}$ is a trivial vector bundle on each fibre of $\breve{\mathfrak{u}}_{2*}$, the sheaf $\mathfrak{N}_{\vec{k}}$ is locally free of rank r. Let $P_{\vec{k}}$ be its bundle of linear frames, which is a principal $\mathrm{GL}(r)$ bundle on $L_{\vec{k}}$, whose fibre at $(\alpha,\beta)\in L_{\vec{k},\infty}$ may be identified with the bundle of linear frames of the vector bundle $\mathcal{E}_{\alpha,\beta,\infty}$. Moreover, if $\tau\colon P_{\vec{k}}\to L_{\vec{k}}$ is the projection, the vector bundle $\widetilde{\mathfrak{N}}_{\vec{k}}=\tau^*\mathfrak{N}_{\vec{k}}$ is trivial. Note that a framed sheaf $(\mathcal{E},\theta)$ determines at point of $P_{\vec{k}}$. Indeed the monad whose cohomology if $\mathcal{E}$ gives a point $(\alpha,\beta)\in L_{\vec{k}}$, and the framing θ gives a point of $P_{\vec{k}}$ in the fibre over (α,β).

4.4. **The $G_{\vec{k}}$-action on $P_{\vec{k}}$.** The group $G_{\vec{k}}$ acts naturally on $L_{\vec{k}}$ according to the following formulas:

$$\begin{cases}\alpha & \mapsto \quad \alpha'=\psi\alpha\phi^{-1}\\ \beta & \mapsto \quad \beta'=\chi\beta\psi^{-1}\end{cases} \qquad \bar{\psi}=(\phi,\psi,\chi)\in G_{\vec{k}}\,. \tag{4.8}$$

This action, that will be called $\rho_0: L_{\vec{k}}\times G_{\vec{k}}\to L_{\vec{k}}$, lifts to an action $\rho\colon P_{\vec{k}}\times G_{\vec{k}}\to P_{\vec{k}}$, by recalling that a point $\theta\in P_{\vec{k}}$ is a framing for the sheaf $\mathcal{E}$ corresponding to $(\alpha,\beta)=\tau(\theta)$, i.e., it is an isomorphism $\theta\colon \mathcal{E}_{\alpha,\beta,\infty}\to\mathcal{O}_{\ell_\infty}^{\oplus r}$. If $\bar{\psi}=(\phi,\psi,\chi)\in G_{\vec{k}}$, we let

$$\rho(\theta,\bar{\psi})=\theta\circ\Lambda_\infty\left(\alpha,\beta;\bar{\psi}\right)^{-1}\,,$$

where, after letting $(\alpha',\beta')=\bar{\psi}\cdot(\alpha,\beta)$, the isomorphism $\Lambda_\infty\left(\alpha,\beta;\bar{\psi}\right):\mathcal{E}_{\alpha,\beta,\infty}\longrightarrow\mathcal{E}_{\alpha',\beta',\infty}$ is induced by $\bar{\psi}: M(\alpha,\beta)\longrightarrow M\left(\alpha',\beta'\right)$. The fact that this is an action follows from the identity

$$\Lambda_\infty(\alpha,\beta,\bar{\psi}'\cdot\bar{\psi})=\Lambda_\infty(\alpha',\beta',\bar{\psi}')\circ\Lambda_\infty(\alpha,\beta,\bar{\psi})\,,$$

which is easily checked. With this action the projection $\tau\colon P_{\vec{k}}\to L_{\vec{k}}$ is a $G_{\vec{k}}$-equivariant morphism.

4.5. **The family $\left(\widetilde{\mathfrak{E}}_{\vec{k}},\widetilde{\Theta}_{\vec{k}}\right)$.** The geometrical environment of this subsection is given by the varieties $\widetilde{\mathfrak{T}}=\Sigma_n\times P_{\vec{k}}$ and $\widetilde{\mathfrak{T}}_\infty=\ell_\infty\times P_{\vec{k}}$, together with the canonical projections shown in the following diagram:

$$\begin{array}{ccc}\ell_\infty & \hookrightarrow & \Sigma_n\\ \uparrow{\scriptstyle \tilde{\mathfrak{u}}_1} & & \uparrow{\scriptstyle \tilde{\mathfrak{t}}_1}\\ \widetilde{\mathfrak{T}}_\infty & \hookrightarrow & \widetilde{\mathfrak{T}}\\ & {\scriptstyle \tilde{\mathfrak{u}}_2}\searrow & \downarrow{\scriptstyle \tilde{\mathfrak{t}}_2}\\ & & P_{\vec{k}}\,.\end{array}$$

Proposition 4.12. *Let $\widetilde{\mathfrak{E}}_{\vec{k}}$ be*

$$\widetilde{\mathfrak{E}}_{\vec{k}}:=\left(\mathrm{id}_{\Sigma_n}\times\tau\right)^*\breve{\mathfrak{E}}_{\vec{k}}\,.$$

This sheaf satisfies condition $\vec{k}$, and in particular for any point $\theta \in P_{\vec{k}}$ one has the natural isomorphism

$$\left(\widetilde{\mathfrak{E}}_{\vec{k}}\right)_{\theta} \simeq \mathcal{E}_{\tau(\theta)}\,. \tag{4.9}$$

We shall call $\widetilde{\mathfrak{E}}_{\vec{k},\infty}$ the restriction of $\widetilde{\mathfrak{E}}_{\vec{k}}$ to $\widetilde{\mathfrak{T}}_{\infty}$, so that $\widetilde{\mathfrak{E}}_{\vec{k},\infty} \simeq (\mathrm{id}_{\mathbb{P}^1} \times \tau)^* \,\breve{\mathfrak{E}}_{\vec{k},\infty} \simeq \tilde{\mathfrak{u}}_2^* \widetilde{\mathfrak{N}}_{\vec{k}}$.

Proof. The flatness of $\widetilde{\mathfrak{E}}_{\vec{k}}$ on $P_{\vec{k}}$ follows from $\widetilde{\mathfrak{T}} \simeq P_{\vec{k}} \times_{L_{\vec{k}}} \breve{\mathfrak{T}}$, while the isomorphism in eq. (4.9) comes from Proposition 9.3 at p. 255 of [14], since $\widetilde{\mathfrak{T}}_{\infty} \simeq P_{\vec{k}} \times_{L_{\vec{k}}} \breve{\mathfrak{T}}_{\infty}$. The last statement is trivial. □

One can extend the action ρ of $G_{\vec{k}}$ on $P_{\vec{k}}$ to actions $\tilde{\rho}$ on $\widetilde{\mathfrak{T}}$ and ρ_{∞} on $\widetilde{\mathfrak{T}}_{\infty}$ where

$$\begin{cases} \tilde{\rho} := \mathrm{id}_{\Sigma_n} \times \rho \\ \rho_{\infty} := \mathrm{id}_{\ell_{\infty}} \times \rho\,. \end{cases}$$

Lemma 4.13. *The sheaf $\widetilde{\mathfrak{E}}_{\vec{k}}$ has a $G_{\vec{k}}$-linearization Ψ that for any point $\left(\theta, \bar{\psi}\right) \in P_{\vec{k}} \times G_{\vec{k}}$, satisfy the isomorphism*

$$\left(\Psi|_{\widetilde{\mathfrak{T}}_{\infty} \times G_{\vec{k}}}\right)^{-1}_{\left(\theta,\bar{\psi}\right)} \simeq \Lambda_{\infty}\left(\alpha, \beta; \bar{\psi}\right)\,.$$

Proof. One has the isomorphism

$$m_{12}^* \breve{\mathbb{M}}_{\vec{k}} \xrightarrow{m_3^*\left(\mathrm{id}_{G_{\vec{k}}}\right)} (\mathrm{id}_{\Sigma_n} \times \rho_0)^* \,\breve{\mathbb{M}}_{\vec{k}}\,,$$

where

$$G_{\vec{k}} \xleftarrow{m_3} \breve{\mathfrak{T}} \times G_{\vec{k}} \xrightarrow{m_{12}} \breve{\mathfrak{T}}$$

are the canonical projections. □

As $P_{\vec{k}}$ is the bundle of linear frames of $\mathfrak{N}_{\vec{k}}$, there exists a canonical isomorphism $\widetilde{\mathfrak{N}}_{\vec{k}} \longrightarrow \mathcal{O}_{P_{\vec{k}}}^{\oplus r}$, which can be regarded as a framing $\widetilde{\Theta}_{\vec{k}}$ for the sheaf $\widetilde{\mathfrak{E}}_{\vec{k}}$. As a consequence of eq. (4.4), the morphism $\widetilde{\Theta}_{\vec{k}}$ is $G_{\vec{k}}$-equivariant, namely, the following diagram is commutative

$$\begin{array}{ccc} l_{12}^* \widetilde{\mathfrak{E}}_{\vec{k},\infty} & \xrightarrow{l_{12}^* \widetilde{\Theta}_{\vec{k}}} & \mathcal{O}_{\widetilde{\mathfrak{T}}_{\infty} \times G_{\vec{k}}} \\ {\scriptstyle \Psi_{\infty}} \downarrow & \nearrow {\scriptstyle \rho_{\infty}^* \widetilde{\Theta}_{\vec{k}}} & \\ \rho_{\infty}^* \widetilde{\mathfrak{E}}_{\vec{k},\infty} & & \end{array} \tag{4.10}$$

where $\Psi_{\infty} = \Psi|_{\widetilde{\mathfrak{T}}_{\infty} \times G_{\vec{k}}}$ and $l_{12} : \widetilde{\mathfrak{T}}_{\infty} \times G_{\vec{k}} \longrightarrow \widetilde{\mathfrak{T}}_{\infty}$ is the projection.

Proposition 4.14. *The sheaves* $\widetilde{\mathfrak{E}}_{\vec{k}}$ *and* $\widetilde{\mathfrak{E}}_{\vec{k},\infty}$ *are respectively isomorphic to the cohomologies of the monads*

$$\widetilde{\mathbb{M}}_{\vec{k}} := (\mathrm{id}_{\Sigma_n} \times \tau)^* \, \breve{\mathbb{M}}_{\vec{k}};$$
$$\widetilde{\mathbb{M}}_{\vec{k},\infty} := (\mathrm{id}_{\ell_\infty} \times \tau)^* \, \breve{\mathbb{M}}_{\vec{k},\infty}.$$

Both monads $\widetilde{\mathbb{M}}_{\vec{k}}$ *and* $\widetilde{\mathbb{M}}_{\vec{k},\infty}$ *are* $G_{\vec{k}}$*-equivariant.*

5. The moduli space $\mathcal{M}^n(r,a,c)$

In this section we shall give the moduli space $\mathcal{M}^n(r,a,c)$ its scheme structure, and prove the first part of the Main Theorem. The space $\mathcal{M}^n(r,a,c)$ can be set-theoretically identified with the quotient $P_{\vec{k}} \big/ G_{\vec{k}}$. We denote by $\pi : P_{\vec{k}} \longrightarrow \mathcal{M}^n(r,a,c)$ the natural projection.

Theorem 5.1. *The orbit space* $\mathcal{M}^n(r,a,c)$ *is a smooth algebraic variety, and* $P_{\vec{k}}$ *is a principal* $G_{\vec{k}}$*-bundle over it.*

Before proving this Theorem, we need to investigate some properties of the $G_{\vec{k}}$-action on $P_{\vec{k}}$.

Lemma 5.2. *If* $\mathcal{E}$ *and* $\mathcal{E}'$ *are sheaves on* Σ_n *trivial at infinity, there is an injection*

$$0 \longrightarrow \mathrm{Hom}(\mathcal{E},\mathcal{E}') \xrightarrow{R} \mathrm{Hom}(\mathcal{E}|_{\ell_\infty},\mathcal{E}'|_{\ell_\infty}) \simeq \mathrm{End}(\mathbb{C}^r)$$

where R *is the restriction morphism.*

Proof. If $\mathcal{E}$ and $\mathcal{E}'$ are locally free, one has $\mathrm{Hom}\,(\mathcal{E},\mathcal{E}') \simeq H^0(\mathcal{E}^* \otimes \mathcal{E}')$. The sheaf $\mathcal{E}^* \otimes \mathcal{E}'$ is locally free and trivial at infinity, so that the result follows by twisting by it the structure sequence of ℓ_∞, and then by taking cohomology, by virtue of Lemma A.1.

In the general case one has an injection

$$0 \longrightarrow \mathrm{Hom}(\mathcal{E},\mathcal{E}') \longrightarrow \mathrm{Hom}(\mathcal{E}^{**},\mathcal{E}'^{**})\,.$$

Since $\mathcal{E}^{**}$ and $\mathcal{E}'^{**}$ are locally free, the thesis follows from the first part of the proof. □

This generalizes to the relative situation. Let S be a scheme and let $\mathfrak{F}$ and $\mathfrak{F}'$ be two sheaves on $T = \Sigma_n \times S$ that satisfy condition $\vec{k}$.

Corollary 5.3. *The restriction morphism*

$$\mathrm{Hom}\,(\mathfrak{F},\mathfrak{F}') \xrightarrow{R} \mathrm{Hom}\,(\mathfrak{F}|_{T_\infty},\mathfrak{F}'|_{T_\infty})$$

is injective.

Proof. Since both $\mathfrak{F}$ and $\mathfrak{F}'$ are locally free along T_∞ one gets $\mathcal{T}or_i\left(\mathscr{H}, \mathcal{O}_{T_\infty}\right) = 0$ for $i > 0$ (see for example [11], p. 700). Thus if we twist the structure sequence of the divisor T_∞ on T by $\mathscr{H} = \mathcal{H}om\left(\mathfrak{F}, \mathfrak{F}'\right)$ we get

$$0 \longrightarrow \mathscr{H}(-T_\infty) \longrightarrow \mathscr{H} \longrightarrow \mathscr{H}|_{T_\infty} \longrightarrow 0\,,$$

$$\text{where} \qquad \mathscr{H}|_{T_\infty} \simeq \mathcal{H}om\left(\mathfrak{F}|_{T_\infty}, \mathfrak{F}'|_{T_\infty}\right).$$

It follows that

$$\ker R = H^0\left(\mathscr{H}(-T_\infty)\right) = H^0\left(t_{2*}\left(\mathscr{H}(-T_\infty)\right)\right).$$

By Propositions 4.3 and B.3, the sheaf $\mathscr{H}(-T_\infty)$ is flat on S. At the same time by Proposition B.1 and by Lemma 5.2 one has the following vanishing result for all closed points $s \in S$:

$$H^0\left(\mathscr{H}(-T_\infty)\right) = H^0\left(\mathrm{Hom}\left(\mathfrak{F}_s, \mathfrak{F}'_s\right)(-\ell_\infty)\right) = 0\,.$$

The Semicontinuity Theorem entails the vanishing of the sheaf $t_{2*}\left(\mathscr{H}(-T_\infty)\right)$. This ends the proof. □

Corollary 5.4. *The action of $G_{\vec{k}}$ on $P_{\vec{k}}$ is free.*

Proof. Let $\left(\alpha, \beta; \bar{\psi}\right) \in L_{\vec{k}} \times G_{\vec{k}}$, and put $(\alpha', \beta') = \bar{\psi} \cdot (\alpha, \beta)$. It follows from Lemma 5.2 that a morphism $\Lambda \in \mathrm{Hom}\left(\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'}\right)$ is fully determined by its restriction to ℓ_∞, which we called Λ_∞. Lemma 4.1.3 at p. 276 of [28] implies that Λ is induced by a unique isomorphism $\bar{\psi} : M(\alpha, \beta) \longrightarrow M(\alpha', \beta')$ between the corresponding monads.

Suppose that $\bar{\psi}$ lies in the stabilizer of a point $\theta \in P_{\vec{k}}$. It follows that $\Lambda_\infty\left(\alpha, \beta; \bar{\psi}\right) = \mathrm{id}_{\mathcal{E}_{\alpha,\beta,\infty}}$, where $(\alpha, \beta) = \tau(\theta)$. Since $\bar{\psi}$ is uniquely determined, this implies $\bar{\psi} = \mathrm{id}_{G_{\vec{k}}}$. □

Proposition 5.5. *The graph Γ of the action ρ is closed in $P_{\vec{k}} \times P_{\vec{k}}$.*

Proof. Let $x = \left(\theta_{\alpha,\beta}, \theta'_{\alpha',\beta'}\right)$ be a point in Γ, where the notation $\theta_{\alpha,\beta}$ means that θ belongs to the fibre over (α, β). One has

$$\Lambda_\infty := \left(\theta'_{\alpha',\beta'}\right)^{-1} \circ \theta_{\alpha,\beta} \in \mathrm{Iso}\left(\mathcal{E}_{\alpha,\beta,\infty}, \mathcal{E}_{\alpha',\beta',\infty}\right).$$

We define the vector space $\operatorname{Hom}_\Lambda\left(\mathcal{E}_{\alpha,\beta},\mathcal{E}_{\alpha',\beta'}\right)$ as the fibre product

$$\begin{array}{ccc}
\operatorname{Hom}_\Lambda\left(\mathcal{E}_{\alpha,\beta},\mathcal{E}_{\alpha',\beta'}\right) & \xrightarrow{\quad i \quad} & \mathbb{C} \\
\Big\downarrow{\scriptstyle j} & & \Big\downarrow{\scriptstyle \cdot\Lambda_\infty} \\
\operatorname{Hom}\left(\mathcal{E}_{\alpha,\beta},\mathcal{E}_{\alpha',\beta'}\right) & \xrightarrow{R_x} & \operatorname{Hom}\left(\mathcal{E}_{\alpha,\beta,\infty},\mathcal{E}_{\alpha',\beta',\infty}\right)
\end{array}$$

where the morphism R_x is given by restriction to ℓ_∞, while $\cdot\Lambda_\infty$ is the multiplication by Λ_∞. Both morphisms i and j are injective, since R_x is injective by Lemma 5.2, and $\cdot\Lambda_\infty$ is injective by the invertibility of Λ_∞ (cf. Lemma 1.2 at p. 86 of [16]).

Thus $\operatorname{Hom}_\Lambda\left(\mathcal{E}_{\alpha,\beta},\mathcal{E}_{\alpha',\beta'}\right)$ is the subspace of homomorphisms between $\mathcal{E}_{\alpha,\beta}$ and $\mathcal{E}_{\alpha',\beta'}$ that at infinity reduce to a multiple of Λ_∞. Thus,

$$\Gamma = \left\{ \left(\theta_{\alpha,\beta},\theta'_{\alpha',\beta'}\right) \in P_{\vec{k}} \times P_{\vec{k}} \,\middle|\, \dim \operatorname{Hom}_\Lambda\left(\mathcal{E}_{\alpha,\beta},\mathcal{E}_{\alpha',\beta'}\right) = 1 \right\} . \tag{5.1}$$

Let us consider the following product varieties, along with the canonical projections:

$$\mathfrak{X} := \Sigma_n \times P_{\vec{k}} \times P_{\vec{k}} \underset{q_{13}}{\overset{q_{12}}{\rightrightarrows}} \widetilde{\mathfrak{T}}$$

$$\mathfrak{Y} := \ell_\infty \times P_{\vec{k}} \times P_{\vec{k}} \underset{p_{13}}{\overset{p_{12}}{\rightrightarrows}} \widetilde{\mathfrak{T}}_\infty .$$

One can pull-back the family $\left(\widetilde{\mathfrak{E}}_{\vec{k}},\widetilde{\Theta}_{\vec{k}}\right)$ to $\mathfrak{X}$ in two different ways, getting $\left(q_{1i}^*\widetilde{\mathfrak{E}}_{\vec{k}},p_{1i}^*\widetilde{\Theta}_{\vec{k}}\right)$ for $i=2,3$. Out of these two pairs one defines

$$\begin{aligned}
\mathscr{H} &= \mathcal{H}om\left(q_{12}^*\widetilde{\mathfrak{E}}_{\vec{k}},q_{13}^*\widetilde{\mathfrak{E}}_{\vec{k}}\right) \\
\boldsymbol{\Lambda} &= \left(p_{13}^*\widetilde{\Theta}_{\vec{k}}\right)^{-1} \circ \left(p_{12}^*\widetilde{\Theta}_{\vec{k}}\right) \in \operatorname{Iso}\left(p_{12}^*\widetilde{\mathfrak{E}}_{\vec{k},\infty},p_{13}^*\widetilde{\mathfrak{E}}_{\vec{k},\infty}\right) .
\end{aligned}$$

Moreover, since $q_{1i}^*\widetilde{\mathfrak{E}}_{\vec{k}}$ for $i=2,3$ are locally free along $\mathfrak{Y}$, one has the isomorphism

$$\mathscr{H}_\infty := \mathcal{H}om\left(p_{12}^*\widetilde{\mathfrak{E}}_{\vec{k},\infty},p_{13}^*\widetilde{\mathfrak{E}}_{\vec{k},\infty}\right) \simeq \mathscr{H}|_{\mathfrak{Y}} .$$

We introduce the sheaf $\mathscr{H}_{\boldsymbol{\Lambda}}$ by means of the exact sequence

$$0 \longrightarrow \mathscr{H}_{\boldsymbol{\Lambda}} \longrightarrow \mathcal{O}_{\mathfrak{Y}} \oplus \mathscr{H} \xrightarrow{\left(\cdot\boldsymbol{\Lambda},-()|_{\mathfrak{Y}}\right)} \mathscr{H}_\infty \longrightarrow 0 . \tag{5.2}$$

For any point $x = \left(\theta_{\alpha,\beta}, \theta'_{\alpha',\beta'}\right) \in P_{\vec{k}} \times P_{\vec{k}}$ one has the commutative diagram

$$\begin{array}{ccc} H^0\left(\mathcal{O}_{\mathfrak{X}_x}\right) \oplus H^0\left(\mathscr{H}_x\right) & \longrightarrow & H^0\left(\mathscr{H}_{\infty,x}\right) \\ \downarrow & & \downarrow \\ \mathbb{C} \oplus \operatorname{Hom}\left(\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'}\right) & \longrightarrow & \operatorname{Hom}\left(\mathcal{E}_{\alpha,\beta,\infty}, \mathcal{E}_{\alpha',\beta',\infty}\right) \ ; \end{array}$$

where the vertical arrows are the natural morphisms, which are invertible by Proposition B.1. By taking into account the exact sequence (5.2), we get the isomorphism

$$H^0\left(\mathscr{H}_{\mathbf{\Lambda},x}\right) \simeq \operatorname{Hom}_{\Lambda}\left(\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'}\right) . \tag{5.3}$$

Since, by Proposition B.3, the sheaf $\mathscr{H}$ is flat on $P_{\vec{k}} \times P_{\vec{k}}$, the sheaf $\mathscr{H}_{\mathbf{\Lambda}}$ is flat on $P_{\vec{k}} \times P_{\vec{k}}$ in turn (cf. Proposition 9.1A.(e) at p. 254 of [14]). Eq. (5.3) and the Semicontinuity Theorem ensure that Γ, as characterized in (5.1), is closed. □

The smooth algebraic varieties $P_{\vec{k}}$ and $G_{\vec{k}}$ have unique compatible structures of complex manifolds $P^{an}_{\vec{k}}$ and $G^{an}_{\vec{k}}$. Note that Γ is closed in $P^{an}_{\vec{k}} \times P^{an}_{\vec{k}}$ as well.

Corollary 5.6. *The action of $G^{an}_{\vec{k}}$ on $P^{an}_{\vec{k}}$ is locally proper.*

Proof. Let K_{θ_0} be a compact neighbourhood of a point $\theta_0 \in P^{an}_{\vec{k}}$. We consider the morphism

$$\begin{array}{rccc} \gamma_{\theta_0}: & K_{\theta_0} \times G^{an}_{\vec{k}} & \longrightarrow & P^{an}_{\vec{k}} \times P^{an}_{\vec{k}} \\ & \left(\theta_{\alpha,\beta}; \bar{\psi}\right) & \longmapsto & \left(\theta_{\alpha,\beta}, \bar{\psi} \cdot \theta_{\alpha,\beta}\right) . \end{array}$$

Since the action of $G^{an}_{\vec{k}}$ is free, γ_{θ_0} is injective, so that its image is

$$\operatorname{im} \gamma_{\theta_0} = \Gamma \cap \left(K_{\theta_0} \times P^{an}_{\vec{k}}\right) . \tag{5.4}$$

We have to prove that, for any compact subset $K \subset P_{\vec{k}}$, the counterimage $\left(\rho|_{K_{\theta_0} \times G^{an}_{\vec{k}}}\right)^{-1}(K)$ is compact. But it is easy to see that

$$\left(\rho|_{K_{\theta_0} \times G^{an}_{\vec{k}}}\right)^{-1}(K) = \gamma_{\theta_0}^{-1}\left(\Gamma \cap \left(K_{\theta_0} \times K\right)\right) .$$

As Γ is closed by Proposition 5.5, the thesis follows. □

We recall that an algebraic group G is said to be *special* if every locally isotrivial G-principal bundle is locally trivial [29] (a fibration is said to be isotrivial if it is trivial in the étale topology).

Lemma 5.7. *The group $G_{\vec{k}}$ is special.*

Proof. For any two positive integers p, q, let $H_{p,q}$ the subgroup of $\mathrm{GL}(p+q)$ whose elements are the matrices

$$\begin{pmatrix} \mathbf{1}_q & A \\ 0 & \mathbf{1}_p \end{pmatrix}, \quad \text{where } A \in \mathrm{Hom}\left(\mathbb{C}^p, \mathbb{C}^q\right).$$

This group is isomorphic to the direct product of copies of the additive group $\mathbb{C}$, and therefore it is special (cf. Proposition 1 at p. 1 of [12]). We have

$$G_{\vec{k}} \simeq \mathrm{GL}(k_1) \times \mathrm{Aut}\left(\mathcal{V}_{\vec{k}}\right) \times \mathrm{GL}(k_3),$$

where $\mathrm{Aut}\left(\mathcal{V}_{\vec{k}}\right)$ can be embedded as a closed subgroup in $\mathrm{GL}(nk_2 + k_4)$. Morever H_{k_4,nk_2} is a normal subgroup of $\mathrm{Aut}\left(\mathcal{V}_{\vec{k}}\right)$, and we get the short exact sequence of groups:

$$0 \longrightarrow H_{k_4,nk_2} \longrightarrow G_{\vec{k}} \longrightarrow \mathrm{GL}(k_1) \times \mathrm{GL}(k_2)^{\times n} \times \mathrm{GL}(k_4) \times \mathrm{GL}(k_3) \longrightarrow 0.$$

Since the group $\mathrm{GL}(p)$ is special for any p (cf. Theorem 2 at p. 24 of [29]), it results that $G_{\vec{k}}$ is special as well (cf. Lemma 6 at p. 25, *ibid.*). □

We have now all ingredients to prove Theorem 5.1.

Proof of Theorem 5.1. Since $\mathcal{M}^n(r,a,c)$ is defined as a quotient set, the canonical projection π induces both the quotient topology, which makes $\mathcal{M}^n(r,a,c)$ into a noetherian topological space, and a canonical structure of locally ringed space (see for example [30]).

Let $\mathcal{M}^n(r,a,c)^{an} := P_{\vec{k}}^{an}/G_{\vec{k}}^{an}$ and let $\pi^{an} : P_{\vec{k}}^{an} \longrightarrow \mathcal{M}^n(r,a,c)^{an}$ be the projection. Since the action of $G_{\vec{k}}^{an}$ on $P_{\vec{k}}^{an}$ is free and locally proper, Satz 24 of [17] implies that $\mathcal{M}^n(r,a,c)^{an}$ with its natural structure of locally ringed space is a complex manifold.

We have a commutative diagram of locally ringed spaces:

$$\begin{array}{ccc} P_{\vec{k}}^{an} & \longrightarrow & P_{\vec{k}} \\ \downarrow{\scriptstyle \pi^{an}} & & \downarrow{\scriptstyle \pi} \\ \mathcal{M}^n(r,a,c)^{an} & \longrightarrow & \mathcal{M}^n(r,a,c). \end{array}$$

It follows plainly that $\mathcal{M}^n(r,a,c)$ is an algebraic variety, and is smooth since $\mathcal{M}^n(r,a,c)^{an}$ is (see [31], p. 109).

Moreover, the morphism $P_{\vec{k}} \times G_{\vec{k}} \longrightarrow P_{\vec{k}} \times P_{\vec{k}}$ defined by the action ρ is a closed immersion, and $\mathcal{M}^n(r,a,c)$ is a geometric quotient of $P_{\vec{k}}$ modulo $G_{\vec{k}}$. It follows from Proposition 0.9 at p. 16 of [24] that $P_{\vec{k}}$ is a principal $G_{\vec{k}}$-bundle over $\mathcal{M}^n(r,a,c)$, in particular it is locally isotrivial. Lemma 5.7 says that $P_{\vec{k}}$ is actually locally trivial, and this completes the proof. □

We come now to the first part of the proof of the Main Theorem.

Proof of the Main Theorem, first part. From Corollary 4.7, if $k_1 < 0$ the set $\mathcal{M}^n(r,a,c)$ is empty. *Vice versa*, let $\vec{k} = (n,r,a,c)$ be such that $k_1 \geq 0$, and define the sheaf $\mathcal{E}_{\vec{k}}$ as follows:

$$\mathcal{E}_{\vec{k}} = \begin{cases} \mathcal{I}_{c\cdot x} & \text{if } r = 1 \quad (\Rightarrow a = 0) \\ \mathcal{I}_{c\cdot x} \oplus \mathcal{O}_{\Sigma_n}(aE) & \text{if } r = 2 \\ \mathcal{I}_{c\cdot x} \oplus \mathcal{O}_{\Sigma_n}(aE) \oplus \mathcal{O}_{\Sigma_n}^{\oplus(r-2)} & \text{if } r > 2 \end{cases}$$

where $\mathcal{I}_{c\cdot x}$ is the ideal sheaf of a 0-dimensional scheme of length c concentrated at a point $x \notin \ell_\infty$. The Chern character of $\mathcal{E}_{\vec{k}}$ is $(r, aE, -c - \frac{1}{2}na^2)$. It follows that $\mathcal{M}^n(r,a,c)$ is empty if and only if $k_1 < 0$.

By Theorem 5.1 $\mathcal{M}^n(r,a,c)$ is a smooth algebraic variety, and its dimension can be computed from the dimensions of $L_{\vec{k}}$, $\mathrm{GL}(r)$ and $G_{\vec{k}}$. □

6. Fineness of the moduli space

In this section we prove that the moduli space $\mathcal{M}^n(r,a,c)$ is fine by constructing a universal family of framed sheaves on Σ_n.

6.1. **The universal family** $\left(\mathfrak{E}_{\vec{k}}, \Theta_{\vec{k}}\right)$**.** We aim at introducing the universal family of framed sheaves on Σ_n. Let us define the varieties $\mathfrak{T} = \Sigma_n \times \mathcal{M}^n(r,a,c)$ and $\mathfrak{T}_\infty = \ell_\infty \times \mathcal{M}^n(r,a,c)$, together with the canonical projections shown in the following diagram:

$$\begin{array}{ccc} \ell_\infty & \hookrightarrow & \Sigma_n \\ \uparrow{\scriptstyle \mathfrak{u}_1} & & \uparrow{\scriptstyle \mathfrak{t}_1} \\ \mathfrak{T}_\infty & \hookrightarrow & \mathfrak{T} \\ & {\scriptstyle \mathfrak{u}_2}\searrow & \downarrow{\scriptstyle \mathfrak{t}_2} \\ & & \mathcal{M}^n(r,a,c)\,. \end{array}$$

Out of the quotient morphism $\pi : P_{\vec{k}} \longrightarrow \mathcal{M}^n(r,a,c)$ we define the projections

$$\begin{aligned} \boldsymbol{q} = \mathrm{id}_{\Sigma_n} \times \pi : \quad & \widetilde{\mathfrak{T}} \longrightarrow \mathfrak{T}\,; \\ \boldsymbol{p} = \mathrm{id}_{\ell_\infty} \times \pi : \quad & \widetilde{\mathfrak{T}}_\infty \longrightarrow \mathfrak{T}_\infty\,. \end{aligned}$$

We define the sheaf

$$\mathfrak{E}_{\vec{k}} = \left(\boldsymbol{q}_* \widetilde{\mathfrak{E}}_{\vec{k}}\right)^G$$

where $()^G$ denotes taking invariants with respect to the action of $G_{\vec{k}}$ on $P_{\vec{k}}$.

Proposition 6.1. *$\mathfrak{E}_{\vec{k}}$ is rank r coherent sheaf, satisfying condition $\vec{k}$. Actually, for any point $[\theta] \in \mathcal{M}^n(r,a,c)$ with $\tau(\theta) = (\alpha,\beta)$ one has the isomorphism*

$$\left(\mathfrak{E}_{\vec{k}}\right)_{[\theta]} \simeq \mathcal{E}_{\alpha,\beta}\,. \tag{6.1}$$

Furthermore, by considering the restriction at infinity $\mathfrak{E}_{\vec{k},\infty} := \mathfrak{E}_{\vec{k}}\big|_{\mathfrak{T}_\infty}$ we get

$$\mathfrak{E}_{\vec{k},\infty} \simeq \left(\boldsymbol{p}_*\widetilde{\mathfrak{E}}_{\vec{k},\infty}\right)^G\,. \tag{6.2}$$

We need to prove a few preliminary results. First, we take the monad $\widetilde{\mathbb{M}}_{\vec{k}} = (\mathrm{id}_{\Sigma_n}\times\tau)^*\,\breve{\mathbb{M}}_{\vec{k}}$ as in Proposition 4.14, and we let

$$\widetilde{\mathbb{M}}_{\vec{k}} = \qquad 0 \longrightarrow \widetilde{\mathfrak{U}}_{\vec{k}} \xrightarrow{\widetilde{A}_{\vec{k}}} \widetilde{\mathfrak{V}}_{\vec{k}} \xrightarrow{\widetilde{B}_{\vec{k}}} \widetilde{\mathfrak{W}}_{\vec{k}} \longrightarrow 0\,.$$

Analogously, we let

$$\widetilde{\mathbb{M}}_{\vec{k},\infty} = \qquad 0 \longrightarrow \widetilde{\mathfrak{U}}_{\vec{k},\infty} \xrightarrow{\widetilde{A}_{\vec{k},\infty}} \widetilde{\mathfrak{V}}_{\vec{k}} \xrightarrow{\widetilde{B}_{\vec{k},\infty}} \widetilde{\mathfrak{W}}_{\vec{k},\infty} \longrightarrow 0\,.$$

We introduce the subsheaves

$$\mathfrak{U}_{\vec{k}} = \left(\boldsymbol{q}_*\widetilde{\mathfrak{U}}_{\vec{k}}\right)^G; \qquad \mathfrak{V}_{\vec{k}} = \left(\boldsymbol{q}_*\widetilde{\mathfrak{V}}_{\vec{k}}\right)^G; \qquad \mathfrak{W}_{\vec{k}} = \left(\boldsymbol{q}_*\widetilde{\mathfrak{W}}_{\vec{k}}\right)^G;$$

$$\mathfrak{U}_{\vec{k},\infty} = \left(\boldsymbol{p}_*\widetilde{\mathfrak{U}}_{\vec{k},\infty}\right)^G; \qquad \mathfrak{V}_{\vec{k},\infty} = \left(\boldsymbol{p}_*\widetilde{\mathfrak{V}}_{\vec{k},\infty}\right)^G; \qquad \mathfrak{W}_{\vec{k},\infty} = \left(\boldsymbol{p}_*\widetilde{\mathfrak{W}}_{\vec{k},\infty}\right)^G,$$

Lemma 6.2. *One has isomorphisms*

$$\mathfrak{U}_{\vec{k}} \simeq \mathfrak{t}_1^*\mathcal{U}_{\vec{k}}\,; \qquad \mathfrak{V}_{\vec{k}} \simeq \mathfrak{t}_1^*\mathcal{V}_{\vec{k}}\,; \qquad \mathfrak{W}_{\vec{k}} \simeq \mathfrak{t}_1^*\mathcal{W}_{\vec{k}}\,;$$

$$\mathfrak{U}_{\vec{k},\infty} \simeq \mathfrak{u}_1^*\mathcal{U}_{\vec{k},\infty}\,; \qquad \mathfrak{V}_{\vec{k},\infty} \simeq \mathfrak{u}_1^*\mathcal{V}_{\vec{k},\infty}\,; \qquad \mathfrak{W}_{\vec{k},\infty} \simeq \mathfrak{u}_1^*\mathcal{W}_{\vec{k},\infty}.$$

Thus the sheaves $\mathfrak{U}_{\vec{k}}$, $\mathfrak{V}_{\vec{k}}$ and $\mathfrak{W}_{\vec{k}}$ are locally free of rank k_1, k_2+k_4 and k_3, respectively. Furthermore, there is an isomorphism $\boldsymbol{q}^*\mathfrak{U}_{\vec{k}} \simeq \widetilde{\mathfrak{U}}_{\vec{k}}$, and similarly for the other sheaves.

Proof. It is a straigthforward consequence of the projection formula. □

Remark 6.3. *Since the restriction morphism $\cdot|_{\widetilde{\mathfrak{T}}_\infty}$ is $G_{\vec{k}}$-equivariant, the triples $\left(\mathfrak{U}_{\vec{k}},\mathfrak{V}_{\vec{k}},\mathfrak{W}_{\vec{k}}\right)$ on $\mathfrak{T}$ and $\left(\mathfrak{U}_{\vec{k},\infty},\mathfrak{V}_{\vec{k},\infty},\mathfrak{W}_{\vec{k},\infty}\right)$ on $\mathfrak{T}_\infty$ are related by restriction at infinity, that is,*

$$\begin{cases} \mathfrak{U}_{\vec{k},\infty} \simeq \mathfrak{U}_{\vec{k}}\big|_{\mathfrak{T}_\infty} \\ \mathfrak{V}_{\vec{k},\infty} \simeq \mathfrak{V}_{\vec{k}}\big|_{\mathfrak{T}_\infty} \\ \mathfrak{W}_{\vec{k},\infty} \simeq \mathfrak{W}_{\vec{k}}\big|_{\mathfrak{T}_\infty}\,. \end{cases}$$

We introduce now the "universal" monad.

Proposition 6.4. *One can introduce the following commutative diagram of monads on $\mathfrak{T}$:*

$$\begin{array}{llcccccccc} \mathbb{M}_{\vec{k}}: & 0 & \longrightarrow & \mathfrak{U}_{\vec{k}} & \xrightarrow{A_{\vec{k}}} & \mathfrak{V}_{\vec{k}} & \xrightarrow{B_{\vec{k}}} & \mathfrak{W}_{\vec{k}} & \longrightarrow & 0 \\ & & & \downarrow \cdot|_{\mathfrak{T}_\infty} & & \downarrow \cdot|_{\mathfrak{T}_\infty} & & \downarrow \cdot|_{\mathfrak{T}_\infty} & & \\ \mathbb{M}_{\vec{k},\infty}: & 0 & \longrightarrow & \mathfrak{U}_{\vec{k},\infty} & \xrightarrow{A_{\vec{k},\infty}} & \mathfrak{V}_{\vec{k},\infty} & \xrightarrow{B_{\vec{k},\infty}} & \mathfrak{W}_{\vec{k},\infty} & \longrightarrow & 0 \end{array} \tag{6.3}$$

where

$$\begin{cases} A_{\vec{k}} := \left(\boldsymbol{q}_*\widetilde{A}_{\vec{k}}\right)\Big|_{\mathfrak{U}_{\vec{k}}} \\ B_{\vec{k}} := \left(\boldsymbol{q}_*\widetilde{B}_{\vec{k}}\right)\Big|_{\mathfrak{V}_{\vec{k}}} \end{cases} \quad \text{and} \quad \begin{cases} A_{\vec{k},\infty} := \left(\boldsymbol{p}_*\widetilde{A}_{\vec{k},\infty}\right)\Big|_{\mathfrak{U}_{\vec{k},\infty}} \\ B_{\vec{k},\infty} := \left(\boldsymbol{p}_*\widetilde{B}_{\vec{k},\infty}\right)\Big|_{\mathfrak{V}_{\vec{k},\infty}} \end{cases}.$$

The sheaf $\mathfrak{E}_{\vec{k}}$ is isomorphic to the cohomology of the first monad, and analogously $\mathfrak{E}_{\vec{k},\infty}$ is isomorphic to the cohomology of $\mathbb{M}_{\vec{k},\infty}$.

Proof. The morphisms of $\mathbb{M}_{\vec{k}}$ are well defined due to by the $G_{\vec{k}}$-equivariance of $\widetilde{A}_{\vec{k}}$ and $\widetilde{B}_{\vec{k}}$. The condition $B_{\vec{k}} \circ A_{\vec{k}} = 0$ follows from $\left(\boldsymbol{q}_*\widetilde{A}_{\vec{k}}\right) \circ \left(\boldsymbol{q}_*\widetilde{B}_{\vec{k}}\right) = \boldsymbol{q}_*\left(\widetilde{A}_{\vec{k}} \circ \widetilde{B}_{\vec{k}}\right) = 0$. The injectivity of $A_{\vec{k}}$ is apparent since $A_{\vec{k}}$ is the restriction of the injective morphism $\boldsymbol{q}_*\widetilde{A}_{\vec{k}}$. We prove that $\operatorname{coker} B_{\vec{k}} = 0$. Lemma 6.2 implies $\boldsymbol{q}^*B_{\vec{k}} \simeq \widetilde{B}_{\vec{k}}$ and $\boldsymbol{q}^*\left(\operatorname{coker} B_{\vec{k}}\right) \simeq \operatorname{coker} \widetilde{B}_{\vec{k}} = 0$. The vanishing of $\operatorname{coker} B_{\vec{k}}$ follows from the faithful flatness of $\boldsymbol{q}$.

The proof for $\mathbb{M}_{\vec{k},\infty}$ is analogous, and the commutativity of the diagram is easy after Lemma 6.3.

The last statement follows from Proposition 4.14. □

Proof of Proposition 6.1. The coherence of $\mathfrak{E}_{\vec{k}}$ follows from $\mathfrak{E}_{\vec{k}} \simeq \mathcal{E}\left(\mathbb{M}_{\vec{k}}\right)$, and its rank can be computed easily from the ranks of $\mathfrak{U}_{\vec{k}}$, $\mathfrak{V}_{\vec{k}}$ and $\mathfrak{W}_{\vec{k}}$ given by Lemma 6.2.

From Lemma 6.2 and Proposition 6.4 we get the isomorphisms $\boldsymbol{q}^*\mathbb{M}_{\vec{k}} \simeq \widetilde{\mathbb{M}}_{\vec{k}}$, $\boldsymbol{q}^*\mathfrak{E}_{\vec{k}} \simeq \widetilde{\mathfrak{E}}_{\vec{k}}$ and $\boldsymbol{q}^*A_{\vec{k}} \simeq \widetilde{A}_{\vec{k}}$. Note that $\left(A_{\vec{k}}\right)_{[\theta]} = \alpha$ for all points $[\theta] \in \mathcal{M}^n(r,a,c)$, with $\tau(\theta) = (\alpha,\beta)$. Now 2.3 implies the flatness of $\mathfrak{E}_{\vec{k}}$, and eq. (6.1) follows from eq. (4.9). This is enough to show that $\mathfrak{E}_{\vec{k}}$ satisfies condition $\vec{k}$.

Finally, eq. (6.2) is a consequence of the commutativity of the diagram (6.3). □

The morphism $\widetilde{\Theta}_{\vec{k}}$ (cf. subsection 4.5) provides a framing for the sheaf $\mathfrak{E}_{\vec{k}}$. Note that this morphism is $G_{\vec{k}}$-equivariant.

Definition 6.5. *We define the isomorphism* $\Theta_{\vec{k}}$ *as the restriction of* $\boldsymbol{p}_*\widetilde{\Theta}_{\vec{k}}$ *to the* $G_{\vec{k}}$*-invariant subsheaf* $\mathfrak{E}_{\vec{k},\infty}$*:*

$$\Theta_{\vec{k}} := \left(\boldsymbol{p}_*\widetilde{\Theta}_{\vec{k}}\right)\Big|_{\mathfrak{E}_{\vec{k},\infty}} : \mathfrak{E}_{\vec{k},\infty} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{T}_\infty}^{\oplus r} .$$

We shall call $\Theta_{\vec{k}}$ *the universal framing.*

We shall see that the framed sheaf $\left(\mathfrak{E}_{\vec{k}}, \Theta_{\vec{k}}\right)$ is a universal family of framed sheaves on Σ_n.

6.2. **The morphism** $f_{\left[(\mathfrak{F},\Theta)_{\vec{k}}\right]}$**.** We show how to canonically associate a scheme morphism $f_{\left[(\mathfrak{F},\Theta)_{\vec{k}}\right]} : S \longrightarrow \mathcal{M}^n(r,a,c)$ with an isomorphism class $\left[(\mathfrak{F},\Theta)_{\vec{k}}\right]$ of families of framed sheaves on $T = \Sigma_n \times S$.

We begin by describing some properties of the monad $M(\mathfrak{F})$.

Lemma 6.6. *Let* $\mathfrak{F}$ *be a sheaf on* T *which satisfies condition* $\vec{k}$*, and let* $M(\mathfrak{F})$ *be the canonically associated monad.*

- *For any closed point* $s \in S$ *there is an isomorphism of complexes*
$$M(\mathfrak{F})_s \simeq M(\alpha(s), \beta(s)) , \tag{6.4}$$
where $(\alpha(s), \beta(s)) = (A_s, B_s) \in L_{\vec{k}}$*.*
- *The restriction* $M(\mathfrak{F})_\infty$ *of the monad* $M(\mathfrak{F})$ *to* T_∞ *is a monad, whose cohomology is isomorphic to* $\mathfrak{F}|_{T_\infty}$*. For any closed point* $s \in S$ *there is an isomorphism*
$$M(\mathfrak{F})_{\infty,s} \simeq \qquad 0 \longrightarrow \mathcal{U}_{\vec{k},\infty} \xrightarrow{\alpha(s)|_{\ell_\infty}} \mathcal{V}_{\vec{k},\infty} \xrightarrow{\beta(s)|_{\ell_\infty}} \mathcal{W}_{\vec{k},\infty} \longrightarrow 0 . \tag{6.5}$$

Proof. The proof splits naturally into two parts.

- Let $s \in S$ be any closed point. By Corollary 4.5, there is an isomorphism
$$M(\mathfrak{F})_s \simeq \mathcal{U}_{\vec{k}} \xrightarrow{A_s} \mathcal{V}_{\vec{k}} \xrightarrow{B_s} \mathcal{W}_{\vec{k}} .$$
It is enough to show that A_s is injective. This is a consequence of Lemma 2.1.4 at p. 33 of [20] to the short exact sequence.
$$0 \longrightarrow \mathfrak{U} \xrightarrow{A} \ker B \longrightarrow \mathfrak{F} \longrightarrow 0 .$$
Hence $M(\mathfrak{F})_s$ is a monad, whose cohomology is isomorphic to $\mathfrak{F}_s$, which is trivial at infinity. Proposition 4.9 implies $(A_s, B_s) \in L_{\vec{k}}$.

- Eq. (6.5) follows trivially from eq. (6.4), since the condition $(\alpha(s),\beta(s)) \in L_{\vec{k}}$ implies that $\alpha(s)|_{\ell_\infty}$ is injective. By Lemma 2.2 this condition ensures that A_∞ is injective.

□

Remark 6.7. *Suppose that $(\mathfrak{F},\Theta)_{\vec{k}}$ is a family of framed sheaves on T. For any closed point $s \in S$, let $\theta(s) = \Theta_s$ be the restricted framing. One has*

$$\theta(s) \in \operatorname{Iso}\left(\mathcal{E}_{\alpha(s),\beta(s),\infty}, \mathcal{O}_{\mathbb{P}^1}^{\oplus r}\right) .$$

We proceed with the construction of the morphism $f_{[(\mathfrak{F},\Theta)_{\vec{k}}]} : S \longrightarrow \mathcal{M}^n(r,a,c)$ first defining it on closed points. We choose an open affine cover $S = \bigcup_{a\in\mathscr{A}} S_a = \bigcup_{a\in\mathscr{A}} \operatorname{Spec}\mathscr{S}_a$ where the S_a's satisfy the conditions of Corollary 4.5. Moreover, if (A,B) is the pair of morphisms in the monad $M(\mathfrak{F})$, we introduce the following notation:

$$(A_a, B_a, \Theta_a) := (A|_{\Sigma_n\times S_a}, B|_{\Sigma_n\times S_a}, \Theta|_{\ell_\infty\times S_a}) .$$

Recall that $t_2 : T \longrightarrow S$ is the projection. Note that by applying the functor t_{2*} to the monad $M(\mathfrak{F})$ restricted to $\Sigma_n \times S_a$ we obtain a complex of trivial sheaves on S_a, so that

$$(t_{2*}A_a, t_{2*}B_a) \in \mathbb{V}_{\vec{k}} \otimes \mathscr{S}_a \simeq \mathscr{S}_a^{\oplus d} .$$

If we define $(\alpha_a(s),\beta_a(s)) = (t_{2*}A_a, t_{2*}B_a)\otimes_{\mathscr{S}_a} k(s)$ we obtain the same morphisms as in Lemma 6.6. We define a morphism $\bar{f}_a : S_a \longrightarrow L_{\vec{k}}$ by letting $\bar{f}_a(s) = (\alpha_a(s),\beta_a(s))$. We complete this to a scheme morphism by defining the rings homomorphism

$$\bar{f}_a^\sharp : \mathbb{C}[z_1,\dots,z_d] \longrightarrow \mathscr{S}_a$$

which maps the polynomial g to $\tilde{g}\,(A_a,B_a)$, where $\tilde{g}$ is the natural extension of g to the ring $\mathscr{S}_a[z_1,\dots,z_d]$.

In the same way, we define $\theta_a(s) = u_{2*}\Theta\otimes_{\mathscr{S}_a} k(s)$. This allows us to lift the morphisms $\bar{f}_a$ to morphisms $\tilde{f}_a : S_a \longrightarrow P_{\vec{k}}$ which we define on closed points as $\bar{f}_a(s) = \theta_a(s)$. Again, these extend to scheme morphisms. By composing these morphisms with the projection $\pi : P_{\vec{k}} \longrightarrow \mathcal{M}^n(r,a,c)$ we obtain morphisms $f_a : S_a \longrightarrow \mathcal{M}^n(r,a,c)$, which glue to a morphism $f : S \longrightarrow \mathcal{M}^n(r,a,c)$, since on overlaps the different $\tilde{f}_a$'s differ by the action of $G_{\vec{k}}$.

6.3. **The natural isomorphism $\boldsymbol{\mu}$.** In this section we prove that $\mathcal{M}^n(r,a,c)$ represents a moduli functor, i.e., it is a fine moduli space. Let $\mathfrak{Sch}$ the category of noetherian reduced schemes of finite type over $\mathbb{C}$ and $\mathfrak{Sets}$ the category of sets.

Definition 6.8. *For any vector $\vec{k}$ such that $k_1 \geq 0$ we introduce the contravariant functor $\mathcal{M}od_{\vec{k}} : \mathfrak{Sch} \longrightarrow \mathfrak{Sets}$ by the following prescriptions:*

- *for any object $S \in \mathrm{Ob}\,(\mathfrak{Sch})$ we define the set $\mathcal{M}od_{\vec{k}}(S)$ as*
$$\mathcal{M}od_{\vec{k}}(S) := \frac{\left\{ \begin{array}{l} \text{families } (\mathfrak{F}, \Theta)_{\vec{k}} \\ \text{of framed sheaves} \\ \text{on } \Sigma_n \times S \end{array} \right\}}{\{\text{framing-preserving isomorphisms}\}} ;$$
- *for any morphism $\varphi : S' \longrightarrow S$ we define the set-theoretic map*
$$\begin{array}{rccc} \mathcal{M}od_{\vec{k}}(\varphi) : & \mathcal{M}^n(r,a,c)\,(S) & \longrightarrow & \mathcal{M}od_{\vec{k}}(S') \\ & \left[(\mathfrak{F}, \Theta)_{\vec{k}}\right] & \longmapsto & \left[(\varphi_\Sigma^* \mathfrak{F}, \varphi_\infty^* \Theta)_{\vec{k}}\right] , \end{array}$$
where $\varphi_\Sigma = \mathrm{id}_{\Sigma_n} \times \varphi$ and $\varphi_\infty = \mathrm{id}_{\ell_\infty} \times \varphi$.

We call this functor the moduli functor.

Observe that $\mathcal{M}od_{\vec{k}}\,(\mathrm{Spec}\,\mathbb{C})$ is the set underlying $\mathcal{M}^n(r,a,c)$. The key property of this functor is the following.

Proposition 6.9. *The functor $\mathcal{M}od_{\vec{k}}(-)$ is represented by the scheme $\mathcal{M}^n(r,a,c)$, that is there is a natural isomorphism of functors*
$$\mathcal{M}od_{\vec{k}}(-) \simeq \mathrm{Hom}\,(-, \mathcal{M}^n(r,a,c)) \ .$$
This implies that $\mathcal{M}^n(r,a,c)$ is a fine moduli space of framed sheaves on Σ_n. The pair $\left(\mathfrak{E}_{\vec{k}}, \Theta_{\vec{k}}\right)$ on $\Sigma_n \times \mathcal{M}^n(r,a,c)$, is the universal family of framed sheaves on Σ_n.

We divide the proof of this Proposition into several steps.

Lemma 6.10. *Let S be any scheme.*

- *We define the map $\boldsymbol{\mu}_S$ as*
$$\begin{array}{rccc} \boldsymbol{\mu}_S : & \mathcal{M}od_{\vec{k}}(S) & \longrightarrow & \mathrm{Hom}\,(S, \mathcal{M}^n(r,a,c)) \\ & \left[(\mathfrak{F}, \Theta)_{\vec{k}}\right] & \longmapsto & f_{\left[(\mathfrak{F},\Theta)_{\vec{k}}\right]} , \end{array}$$
where the morphism $f_{\left[(\mathfrak{F},\Theta)_{\vec{k}}\right]}$ was introduced in the preceding subsection.
- *We define the map $\boldsymbol{\nu}_S$ as*
$$\begin{array}{rccc} \boldsymbol{\nu}_S : & \mathrm{Hom}\,(S, \mathcal{M}^n(r,a,c)) & \longrightarrow & \mathcal{M}od_{\vec{k}}(S) \\ & f & \longmapsto & \left[\left(f_\Sigma^* \mathfrak{E}_{\vec{k}}, f_\infty^* \Theta_{\vec{k}}\right)\right] . \end{array}$$

With these definitions, $\boldsymbol{\mu}$ *is a natural tranformation from the functor* $\mathcal{M}od_{\vec{k}}(-)$ *to the functor* $\operatorname{Hom}(-,\mathcal{M}^n(r,a,c))$*, while* $\boldsymbol{\nu}$ *is a natural tranformation from* $\operatorname{Hom}(-,\mathcal{M}^n(r,a,c))$ *to* $\mathcal{M}od_{\vec{k}}(-)$*.*

Proof. The naturality of $\boldsymbol{\nu}$ is straightforward since

$$\left[\left(\varphi_\Sigma^* f_\Sigma^* \mathfrak{E}_{\vec{k}}, \varphi_\infty^* f_\infty^* \Theta_{\vec{k}}\right)\right] = \left[\left((f\circ\varphi)_\Sigma^* \,\mathfrak{E}_{\vec{k}}, (f\circ\varphi)_\infty^*\, \Theta_{\vec{k}}\right)\right] ,$$

whenever a composition of morphisms $S' \xrightarrow{\varphi} S \xrightarrow{f} \mathcal{M}^n(r,a,c)$ is given.

To prove the naturality of $\boldsymbol{\mu}$ we need to show that, given any morphism $S' \xrightarrow{\varphi} S$ and any family $(\mathfrak{F},\Theta)_{\vec{k}}$ on T, the following equality holds true:

$$f_{\left[(\mathfrak{F},\Theta)_{\vec{k}}\right]} \circ \varphi = f_{\left[\left(\varphi_\Sigma^*\mathfrak{F},\varphi_\infty^*\Theta\right)_{\vec{k}}\right]} . \tag{6.6}$$

To simplify the notation; we let $f = f_{\left[(\mathfrak{F},\Theta)_{\vec{k}}\right]}$ and $f' = f_{\left[\left(\varphi_\Sigma^*\mathfrak{F},\varphi_\infty^*\Theta\right)_{\vec{k}}\right]}$. We can assume that $S = \operatorname{Spec}\mathscr{S}$ and $S' = \operatorname{Spec}\mathscr{S}'$, so that φ is induced by a ring homomorphism $\mathscr{S} \xrightarrow{\varphi^\sharp} \mathscr{S}'$. If we let (A,B) and (A',B') be the morphisms in the monads, respectively, $M(\mathfrak{F})$ and $M(\varphi^*\mathfrak{F})$ respectively,

$$(A',B') = \left(\varphi^\sharp\right)^{\oplus d}(A,B) .$$

In particular, for any polynomial $g \in \mathbb{C}[z_1,\ldots,z_d]$ one has

$$\operatorname{ev}_{(A',B')}\, g = \left(\varphi^\sharp \circ \operatorname{ev}_{(A,B)}\right) g ,$$

so that $\bar{f}\circ\varphi = \bar{f}' : S' \longrightarrow L_{\vec{k}}$. We can assume that $\operatorname{im}\bar{f}'$ is contained in a trivializing open affine subset $V = \operatorname{Spec}\mathscr{L}$ of $L_{\vec{k}}$, and we put $N_{\vec{k}} = \Gamma\left(V,\mathfrak{N}_{\vec{k}}\right)$. Moreover, we introduce $N = N_{\vec{k}} \otimes_{\mathscr{L}} \mathscr{S} \simeq \Gamma\left(T_\infty, \mathfrak{F}|_{T_\infty}\right)$ and similarly $N' \simeq \Gamma\left(T'_\infty, \varphi_\infty^*\left(\mathfrak{F}|_{T_\infty}\right)\right)$; it turns out that so that N is a free $\mathscr{S}$-module and N' is a free $\mathscr{S}'$-module, both of rank r. One has $\varphi_\infty^*\Theta \simeq \Theta \otimes_{\mathscr{S}} 1_{\mathscr{S}'}$.

For any polynomial $h \in \operatorname{Sym}_{\mathscr{L}}(H^*)$, where $H \simeq \operatorname{Hom}_{\mathscr{L}}\left(N_{\vec{k}}, \mathscr{L}^{\oplus r}\right)$, we get

$$\begin{aligned}\operatorname{ev}_{\varphi_\infty^*\Theta}\left(h\otimes_{\mathscr{L}} 1_{\mathscr{S}'}\right) &= \operatorname{ev}_{(\Theta\otimes_{\mathscr{S}}1_{\mathscr{S}'})}\left[\left(h\otimes_{\mathscr{L}}1_{\mathscr{S}}\right)\otimes_{\mathscr{S}}1_{\mathscr{S}'}\right] = \\ &= \left[\operatorname{ev}_\Theta\left(h\otimes_{\mathscr{L}}1_{\mathscr{S}}\right)\right]\otimes_{\mathscr{S}}1_{\mathscr{S}'} ,\end{aligned}$$

Hence

$$\tilde{f}\circ\varphi = \tilde{f}' : S' \longrightarrow P_{\vec{k}} .$$

By applying the projection π to both sides of this equation, we obtain eq. (6.6).

□

Lemma 6.11. *For any scheme S one has*

$$\boldsymbol{\nu}_S \circ \boldsymbol{\mu}_S = \mathrm{id}_{\mathcal{M}od_{\vec{k}}(S)} \ .$$

Proof. We need to prove that

$$\left(f_\Sigma^* \mathfrak{E}_{\vec{k}}, f_\infty^* \Theta_{\vec{k}}\right) \simeq (\mathfrak{F}, \Theta)_{\vec{k}} \ ,$$

for any family $(\mathfrak{F}, \Theta)_{\vec{k}}$ of framed sheaves on Σ_n parametrized by S, where $f = f_{\left[(\mathfrak{F},\Theta)_{\vec{k}}\right]}$. It is enough to show that there is an isomorphism

$$f_\Sigma^* \mathbb{M}_{\vec{k}} \simeq M(\mathfrak{F}) \tag{6.7}$$

and that this isomorphism is compatible with the framings. By Lemma 6.2 there are isomorphisms

$$\begin{cases} f_\Sigma^* \mathfrak{U}_{\vec{k}} \simeq t_1^* \mathcal{U}_{\vec{k}} \\ f_\Sigma^* \mathfrak{V}_{\vec{k}} \simeq t_1^* \mathcal{V}_{\vec{k}} \\ f_\Sigma^* \mathfrak{W}_{\vec{k}} \simeq t_1^* \mathcal{W}_{\vec{k}} \end{cases}$$

and $\left(\boldsymbol{q}^* A_{\vec{k}}, \boldsymbol{q}^* B_{\vec{k}}, \boldsymbol{p}^* \Theta_{\vec{k}}\right) \simeq \left(\widetilde{A}_{\vec{k}}, \widetilde{B}_{\vec{k}}, \widetilde{\Theta}_{\vec{k}}\right)$.

When $S = \operatorname{Spec} \mathscr{S}$ and satisfies the conditions of Corollary 4.5, we have in addition

$$\begin{cases} \mathfrak{U} \simeq t_1^* \mathcal{U}_{\vec{k}} \\ \mathfrak{V} \simeq t_1^* \mathcal{V}_{\vec{k}} \\ \mathfrak{W} \simeq t_1^* \mathcal{W}_{\vec{k}} \end{cases}$$

$$\begin{aligned} \text{and} \qquad f_\Sigma^* \left(A_{\vec{k}}, B_{\vec{k}}\right) &= f_\Sigma^* \boldsymbol{q}^* \left(\widetilde{A}_{\vec{k}}, \widetilde{B}_{\vec{k}}\right) = \\ &= f_\Sigma^* \left(\widetilde{A}_{\vec{k}}, \widetilde{B}_{\vec{k}}\right) = \\ &= f_\Sigma^* \tilde{\mathfrak{t}}_2^* \tau^* \left(\mathrm{id}_{L_{\vec{k}}}\right) = \\ &= t_2^* \tilde{f}^* \tau^* \left(\mathrm{id}_{L_{\vec{k}}}\right) = \\ &= t_2^* \bar{f}^* \left(\mathrm{id}_{L_{\vec{k}}}\right) = \\ &= (A, B) \, . \end{aligned}$$

This proves eq. (6.7) locally, and similarly the compatibility of $\Theta_{\vec{k}}$ can be shown. By Corollary 5.3 we get the thesis. □

Lemma 6.12. *For any vector $\vec{k}$ such that $k_1 \geq 0$ and for any scheme S one has*

$$\boldsymbol{\mu}_S \circ \boldsymbol{\nu}_S = \mathrm{id}_{\mathrm{Hom}(S, \mathcal{M}^n(r,a,c))} \ .$$

Proof. Let $g : S \longrightarrow \mathcal{M}^n(r,a,c)$ be any morphism. We need to show that:

$$g = f_{\left[\left(g_\Sigma^*\mathfrak{E}_{\vec{k}},\, g_\infty^*\Theta_{\vec{k}}\right)\right]}\,;$$

for simplicity's sake we set $f = f_{\left[\left(g_\Sigma^*\mathfrak{E}_{\vec{k}},\, g_\infty^*\Theta_{\vec{k}}\right)\right]}$. Let $M\left(g_\Sigma^*\mathfrak{E}_{\vec{k}}\right)$ be the monad

$$0 \longrightarrow \mathfrak{U} \xrightarrow{A} \mathfrak{V} \xrightarrow{B} \mathfrak{W} \longrightarrow 0$$

canonically associated with $g_\Sigma^*\mathfrak{E}_{\vec{k}}$. We can work locally by assuming that $S = \operatorname{Spec}\mathscr{S}$ satisfies the hypotheses of Corollary 4.5 for the sheaves $\mathfrak{U}$, $\mathfrak{V}$, $\mathfrak{W}$ and that $\operatorname{im} g \subseteq W$, where $W \subseteq \mathcal{M}^n(r,a,c)$ is a trivializing open subset for the $G_{\vec{k}}$-principal bundle $P_{\vec{k}}$. Thus, there exists a local section $\sigma : W \longrightarrow P_{\vec{k}}$ lifting g to $P_{\vec{k}}$:

$$\begin{array}{ccc} & & P_{\vec{k}} \\ & \overset{\sigma\circ g =: \tilde{g}}{\nearrow} & \downarrow \pi \\ S & \xrightarrow[g]{} & \mathcal{M}^n(r,a,c)\,. \end{array} \tag{6.8}$$

Under our assumptions, the complex

$$g_\Sigma^*\mathbb{M}_{\vec{k}} : \qquad g_\Sigma^*\mathfrak{U}_{\vec{k}} \xrightarrow{g_\Sigma^* A_{\vec{k}}} g_\Sigma^*\mathfrak{V}_{\vec{k}} \xrightarrow{g_\Sigma^* B_{\vec{k}}} g_\Sigma^*\mathfrak{W}_{\vec{k}} \longrightarrow 0$$

is a monad. Indeed the morphism $g_\Sigma^* A_{\vec{k}}$ is injective, as it follows from diagram (6.8) and Lemma 2.2. This monad is isomorphic to $M\left(g_\Sigma^*\mathfrak{E}_{\vec{k}}\right)$: as a matter of fact, their cohomologies are isomorphic and Lemma 4.1.3 at p. 276 of [28] applies. Because of this isomorphism, in view of Proposition 6.1 one has

$$\left(g_\Sigma^*\mathfrak{E}_{\vec{k}}\right)_s \simeq \mathcal{E}_{\alpha(s),\beta(s)},$$

where $g(s) = [\theta(s)]$ and $(\alpha(s),\beta(s)) = \tau(\theta(s))$ for any closed point $s \in S$. This is enough to end the proof. ∎

Appendix A. Proof of Proposition 4.3

Lemma A.1. *Let $\mathcal{E}$ be a sheaf trivial at infinity on Σ_n. One has*

$$\begin{aligned} H^0(\mathcal{E}(p,q)) &= 0 \qquad \text{for} \qquad np+q \leq -1\,, \\ H^2(\mathcal{E}(p,q)) &= 0 \qquad \text{for} \qquad np+q \geq -(n+1)\,. \end{aligned}$$

Proof. When $\mathcal{E}$ is locally free, the proof is essentially the same as in [21], p. 24.

If $\mathcal{E}$ is not locally free, we get the thesis by using the exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}^{**} \longrightarrow \mathcal{Q} \longrightarrow 0\,. \tag{A.1}$$

□

Let us now consider the product scheme $\mathfrak{X} := \Sigma_n \times \Sigma_n \times S$, together with the canonical projections p_{12}, p_{13} and p_{23}. We shall write

$$\mathcal{O}_{\mathfrak{X}}(p,q)(p',q') := p_{13}^*\mathcal{O}_T(p,q) \otimes p_{23}^*\mathcal{O}_T(p',q')\,.$$

Buchdahl [7] proves the existence of a vector bundle $\mathcal{R}$ on $\Sigma_n \times \Sigma_n$, which enables us to construct a locally free resolution of the structure sheaf $\mathcal{O}_\Delta$ of the diagonal $\Delta \subseteq \Sigma_n \times \Sigma_n$. By pulling-back $\mathcal{R}$ on $\mathfrak{X}$, we get the resolution

$$0 \longrightarrow \mathcal{O}_{\mathfrak{X}}(-1,-1)(-1,n-1) \longrightarrow \mathfrak{R}^* \longrightarrow \mathcal{O}_{\mathfrak{X}} \longrightarrow \mathcal{O}_{\mathfrak{D}} \longrightarrow 0\,, \tag{A.2}$$

where $\mathfrak{D} := \Delta \times S \subset \mathfrak{X}$ is the relative diagonal, and $\mathfrak{R} = p_{12}^*\mathcal{R}$ is an extension of two line bundles:

$$0 \longrightarrow \mathcal{O}_{\mathfrak{X}}(1,0)(1,-n) \xrightarrow{p_{12}^*c} \mathfrak{R} \xrightarrow{p_{12}^*d} \mathcal{O}_{\mathfrak{X}}(0,1)(0,1) \longrightarrow 0 \tag{A.3}$$

(c and d are the morphisms defining $\mathcal{R}$ on $\Sigma_n \times \Sigma_n$).

Let $\mathfrak{F}$ be a sheaf on T satisfing condition $\vec{k}$, and let $\mathfrak{F}' = \mathfrak{F}(-1,0)$. By twisting (A.2) by $p_{23}^*\mathfrak{F}'$, and deleting the last term, we get the complex

$$\mathcal{C}^\bullet \quad : \quad 0 \longrightarrow \mathcal{C}^{-2} \longrightarrow \mathcal{C}^{-1} \longrightarrow \mathcal{C}^0 \longrightarrow 0\,.$$

One has

$$\begin{aligned}
\mathcal{H}^0(\mathcal{C}^\bullet) &= p_{23}^*\mathfrak{F}'|_{\mathfrak{D}} \\
\mathcal{H}^{-1}(\mathcal{C}^\bullet) &= \mathcal{T}or_1\,(\mathcal{O}_{\mathfrak{D}}, p_{23}^*\mathfrak{F}') \\
\mathcal{H}^{-2}(\mathcal{C}^\bullet) &= \mathcal{T}or_1\,(\mathcal{I}_{\mathfrak{D}}, p_{23}^*\mathfrak{F}') \simeq \mathcal{T}or_2(\mathcal{O}_{\mathfrak{D}}, p_{23}^*\mathfrak{F}')\,.
\end{aligned}$$

We claim that

$$\mathcal{T}or_i\,(\mathcal{O}_{\mathfrak{D}}, p_{23}^*\mathfrak{F}') = 0 \qquad \text{for any } i > 0\,. \tag{A.4}$$

Indeed, if we apply the functor p_{23}^* to any flat resolution $\mathcal{P}^\bullet \xrightarrow{h^\bullet} \mathfrak{F}' \longrightarrow 0$ of $\mathfrak{F}'$ on T, we get

$$p_{23}^*\,(\mathcal{P}^\bullet) \xrightarrow{p_{23}^*h^\bullet} p_{23}^*\mathfrak{F}' \longrightarrow 0\,.$$

Since p_{23} is a flat morphism, this is a resolution of $p_{23}^*\mathfrak{F}'$ by flat $\mathcal{O}_{\mathfrak{X}}$-modules. By twisting it by $\mathcal{O}_{\mathfrak{D}}$ one has the isomorphism

$$\mathcal{T}or_i\left(\mathcal{O}_{\mathfrak{D}}, p_2^*\mathcal{F}\right) \simeq \mathcal{H}^i\left(p_{23}^*\left(\mathcal{P}^\bullet\right) \otimes \mathcal{O}_{\mathfrak{D}}\right)$$

(see [13]). The claim is then a consequence of the natural isomorphism $(\otimes_{\mathcal{O}_T}\mathcal{O}_{\mathfrak{X}})\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}_T \simeq \mathrm{id}_{\mathfrak{Mod}(T)}$.

Let us consider a Cartan-Eilenberg resolution of the complex $\mathcal{C}^\bullet$:

$$0 \longrightarrow \mathcal{C}^\bullet \longrightarrow \mathfrak{I}^{\bullet,\bullet}\,.$$

By applying the functor p_{13*} to this double complex we get a new double complex, with which two exact sequences ${}'E_r^{p,q}$ and ${}''E_r^{p,q}$ are associated. They both abut to the hyper-cohomology $\mathbb{R}^{p+q}p_{13*}(\mathcal{C}^\bullet)$ of the complex $\mathcal{C}^\bullet$. We compute the second sequence, by taking first cohomology of the rows. Due to the fact that the sheaves $\mathfrak{I}^{p,q}$ are p_{13*}-acycle, we get $\mathcal{H}^p\left(p_{13*}\mathfrak{I}^{\bullet,q}\right) = p_{13*}\mathcal{H}^p\left(\mathfrak{I}^{\bullet,q}\right)$. Since the complex $\mathcal{H}^p\left(\mathfrak{I}^{\bullet,\bullet}\right)$ is an injective resolution of $\mathcal{H}^p\left(p_{13*}\mathcal{C}^\bullet\right)$ for any p, because of the properties of Cartan-Eilenberg resolutions, one has

$${}''E_2^{p,q} = R^q p_{13*}\mathcal{H}^p(\mathcal{C}^\bullet)\,.$$

Eq. (A.4), and the fact that $H^0(\mathcal{C}^\bullet)$ is supported on $\mathfrak{D}$, imply that

$$R^q p_{13*}\mathcal{H}^p\left(\mathcal{C}^\bullet\right) = \begin{cases} \mathfrak{F}' & \text{if} \quad p = 0\,,\ q = 0 \\ 0 & \text{otherwise}\,. \end{cases}$$

Since the differential d_2 has degree $(2, -1)$, one has that ${}''E_\infty^{p,q} = {}''E_2^{p,q}$, so that

$$\mathbb{R}^i p_{13*}(\mathcal{C}^\bullet) = \begin{cases} \mathfrak{F}' & \text{if } i = 0 \\ 0 & \text{otherwise}\,. \end{cases} \tag{A.5}$$

As for the first spectral sequence, we have

$$
\begin{aligned}
{}'E_1^{p,q} &= R^q p_{13*}\mathcal{C}^p = \\
&= \begin{cases} R^q p_{13*}\left(p_{23}^*\mathfrak{F}'\right) & p=0 \\ R^q p_{13*}\left(\mathfrak{R}^*\otimes p_{23}^*\mathfrak{F}'\right) & p=-1 \\ R^q p_{13*}\left[p_{13}^*\mathcal{O}_T(-1,-1)\otimes p_{23}^*(\mathfrak{F}(-2,n-1))\right] & p=-2 \\ 0 & p\neq 0,-1,-2 \end{cases} \simeq \qquad \text{(A.6)} \\
&\simeq \begin{cases} t_2^* R^q\left[t_{2*}(\mathfrak{F}(-1,0)\right] & p=0 \\ R^q p_{13*}\left(\mathfrak{R}^*\otimes p_2^*\mathfrak{F}'\right) & p=-1 \\ \mathcal{O}_T(-1,-1)\otimes t_2^* R^q\left[t_{2*}(\mathfrak{F}(-2,n-1))\right] & p=-2 \\ 0 & p\neq 0,-1,-2\,, \end{cases}
\end{aligned}
$$

where the last isomorphism is a consequence, for $p=0,-2$, of Proposition 9.3 at p. 255 of [14], when referred to the diagram

$$
\begin{array}{ccc}
\mathfrak{X} & \xrightarrow{p_{23}} & T \\
\downarrow{\scriptstyle p_{13}} & & \downarrow{\scriptstyle t_2} \\
T & \xrightarrow{t_2} & S\,.
\end{array}
$$

We can get more explicit information on the case $p=-1$ by dualizing the exact sequence in eq. (A.3), by twisting the resulting sequence by $p_2^*\mathfrak{F}'$, and by applying the functor p_{13*}. Again by Proposition 9.3 at p. 255 of [14], we get the long exact sequence

$$
\cdots \xrightarrow{\partial_q} \mathcal{O}_T(-1,0)\otimes t_2^* R^q t_{2*}[\mathfrak{F}(-2,n)] \xrightarrow{\mathfrak{c}_q} {}'E_1^{-1,q} \xrightarrow{\mathfrak{d}_q}
$$
$$
\xrightarrow{\mathfrak{d}_q} \mathcal{O}_T(0,-1)\otimes t_2^* R^q t_{2*}[\mathfrak{F}(-1,-1)] \xrightarrow{\partial_{q+1}} \cdots\,. \qquad \text{(A.7)}
$$

We claim that

$$
R^i t_{2*}[\mathfrak{F}(p,q)] = \begin{cases} 0 & \text{if } i\neq 1 \\ \text{loc. free} & \text{if } i=1 \end{cases} \qquad \text{when} \qquad \begin{aligned} (p,q) =& (-2,n-1),(-1,-1), \\ & (-1,0),(-2,n)\,. \end{aligned}
$$

Indeed, for any closed point $s\in S$, the sheaf $\mathfrak{F}_s$ satisfies the vanishing conditions in Lemma A.1. So the claim follows from the Semicontinuity Theorem and from the density of closed points inside S. As a consequence, the sequence (A.7) yields a short exact sequence for the case $q = 1$; therefore one has

$$ {}'E_1^{p,q} = \begin{cases} \text{locally free} & \text{if } \begin{cases} p = -2, -1, 0 \\ q = 1 \end{cases} \\ 0 & \text{otherwise} \end{cases} $$

and the complex

$$ 0 \longrightarrow {}'E_1^{-2,1} \longrightarrow {}'E_1^{-1,1} \longrightarrow {}'E_1^{0,1} \longrightarrow 0 \,. \tag{A.8} $$

Hence the spectral sequence stabilizes at the first step. By eq. (A.5) the complex (A.8) has nontrivial cohomology only in the middle term, and this is isomorphic to ${}'E_2^{-1,1} \simeq \mathbb{R}^0 p_{13*}(\mathcal{C}^\bullet) \simeq \mathfrak{F}(-1,0)$. By twisting the complex A.8 by $\mathcal{O}_T(1,0)$ one gets the monad (4.1), while by twisting the exact sequence coming from eq. (A.7) by $\mathcal{O}_T(1,0)$ one gets the short exact sequence (4.2) with

$$ \begin{cases} \mathfrak{c} = \mathfrak{c}_1 = R^1 p_{13*} \left[\mathrm{id}_{p_{23}^* \mathfrak{F}'} \otimes (p_{12}^* d)^* \right] \\ \mathfrak{d} = \mathfrak{d}_1 = R^1 p_{13*} \left[\mathrm{id}_{p_{23}^* \mathfrak{F}'} \otimes (p_{12}^* c)^* \right] \,. \end{cases} $$

The functoriality of $M(-)$ is a direct result of the construction.

Appendix B. Two technical results

B.1. **Base change in** $\mathrm{Hom}(M, \bullet)$ **for** M **flat.** Let A be a commutative ring, and let B and R be *not necessarily flat* A-algebras, with R noetherian. We shall call $R' := R \otimes_A B$, and we shall denote by $\cdot_{(B)}$ the functor $\otimes_A B$. Note that there is a natural isomorphism of functors $\cdot_{(B)} \simeq \cdot_{(R')}$, where $\cdot_{(R')} = \otimes_R R'$.

Proposition B.1. *Let M, N be R-modules, and suppose that M is finitely generated as a R-module, and flat as an A-module. One has the following isomorphism for $i \geq 0$:*

$$ \mathrm{Ext}^i_R(M,N)_{(R')} \simeq \mathrm{Ext}^i_{R'}\left(M_{(R')}, N_{(R')}\right) \,. $$

The proof of this Proposition relies on a Lemma.

Lemma B.2. *The module M admits a resolution by free finitely generated R-modules:*

$$ F_\bullet \longrightarrow M \longrightarrow 0 \quad = $$

$$ = \quad \cdots \longrightarrow F_2 \xrightarrow{f_1} F_1 \xrightarrow{f_0} F_0 \longrightarrow M \longrightarrow 0 \,. $$

Proof. It is a consequence of Proposition 1.4 at p. 415 of [22]. □

Proof of Proposition B.1. This is a modification of the proof of Proposition (3.E) at p. 20 of [23]. We choose a resolution $F_\bullet$ for M as in Lemma B.2. Since M is flat as an A-module, by applying Proposition 9.1A.(e) at p. 254 of [14] one can prove inductively that the modules $\ker f_i$'s are flat as A-modules. If we twist $F_\bullet$ by B we get a resolution for $M_{(B)}$ by free finitely generated B-modules:

$$F_{\bullet,(B)} \longrightarrow M_{(B)} \longrightarrow 0 \quad =$$

$$= \quad \cdots \longrightarrow F_{2,(B)} \xrightarrow{f_{1,(B)}} F_{1,(B)} \xrightarrow{f_{0,(B)}} F_{0,(B)} \longrightarrow M_{(B)} \longrightarrow 0\,.$$

Since $\cdot_{(B)} \simeq \cdot_{(R')}$ one has the isomorphism

$$F_{\bullet,(B)} \longrightarrow M_{(B)} \longrightarrow 0 \quad \simeq \quad F_{\bullet,(R')} \longrightarrow M_{(R')} \longrightarrow 0\,.$$

In particular $F_{i,(R')}$ is a free finitely generated R-module for $i \geq 0$. If we apply the functor $\mathrm{Hom}_{R'}\left(-, N_{(R')}\right)$ to this sequence, we get

$$0 \longrightarrow \mathrm{Hom}_{R'}\left(M_{(R')}, N_{(R')}\right) \longrightarrow \mathrm{Hom}_{R'}\left(F_{\bullet,(R')}, N_{(R')}\right) \quad =$$

$$= \quad 0 \longrightarrow \mathrm{Hom}_{R'}\left(M_{(R')}, N_{(R')}\right) \longrightarrow \mathrm{Hom}_{R}\left(F_\bullet, N\right)_{(R')}\,,$$

where the second step holds true since the F_i's are free and finitely generated. From the definition of Ext^i, the thesis follows. □

B.2. **The flatness of a $\mathcal{H}om$ sheaf.** Let X be a smooth connected projective variety over $\mathbb{C}$ of dimension $\dim X = 2$, and let S be a smooth algebraic variety over $\mathbb{C}$. Consider the product $T = X \times S$ with the canonical projections π_i, $i = 1, 2$. Suppose one has two monads on T

$$M: \quad 0 \longrightarrow \mathcal{U} \xrightarrow{A} \mathcal{V} \xrightarrow{B} \mathcal{W} \longrightarrow 0$$

$$M': \quad 0 \longrightarrow \mathcal{U}' \xrightarrow{A'} \mathcal{V}' \xrightarrow{B'} \mathcal{W}' \longrightarrow 0\,.$$

We shall call $\mathcal{E}$ and $\mathcal{E}'$ the cohomologies of M and of M', respectively.

Proposition B.3. *If the sheaves $\mathcal{E}$ and $\mathcal{E}'$ are flat on S, then also the sheaf $\mathcal{H}om(\mathcal{E}, \mathcal{E}')$ is flat on S.*

In order to prove this Proposition, we shall state several intermediate results for the sheaf $\mathcal{E}$. The case for $\mathcal{E}'$ is completely analogous. The first Lemma is an easy consequence of Lemmas 2.1 and 2.3.

Lemma B.4. • *The sheaf $\mathcal{E}$ is torsion-free.*

- *Suppose that $\mathcal{E}$ is flat on S. For every point $s \in S$ there is an isomorphism*

$$\mathcal{E}_s \simeq \mathcal{E}\left(M_s\right)$$

In particular $\mathcal{E}_s$ is torsion-free for all $s \in S$.

Lemma B.5. *Let $s \in S$ be any point. One has the isomorphisms*

$$\left[\mathcal{E}xt^i_{\mathcal{O}_T}\left(\mathcal{E}, \mathcal{O}_T\right)\right]_s \simeq \mathcal{E}xt^i_{\mathcal{O}_{T_s}}\left(\mathcal{E}_s, \mathcal{O}_{T_s}\right) \qquad i \geq 0\,. \tag{B.1}$$

In particular, for $i = 0$ one has

$$\left(\mathcal{E}^*\right)_s \simeq \left(\mathcal{E}_s\right)^* \,. \tag{B.2}$$

Since $\dim X = 2$ *it follows that $\left(\mathcal{E}^*\right)_s$ is locally free on T_s.*

Proof. The first statement follows by applying Proposition B.1 to the sheaf $\mathcal{H}om_{\mathcal{O}_T}\left(\mathcal{E}, \mathcal{O}_T\right)$.

The second statement is a consequence of the fact that in dimension 2 a reflexive sheaf is locally free. □

Proposition B.6. *The sheaf $\mathcal{E}^*$ is flat on S.*

Proof. Consider the display of the monad M: the sheaf $\mathcal{A} = \operatorname{coker} A$ is an extension of the sheaf $\mathcal{W}$ by $\mathcal{E}$, and both sheaves are flat on S. By Proposition 9.1A.(e) at p. 254 of [14] that $\mathcal{A}$ is flat on S, so that the isomorphisms in eq. (B.1) hold for this sheaf as well. Now put $\mathcal{B} = \ker B$ and let $s \in S$ be any point. If we take the dual of the display of M and then we restrict the result to T_s we get the commutative diagram

$$\begin{array}{ccccccc} \mathcal{W}_s^* & \xrightarrow{B_s^*} & \left(\mathcal{A}^*\right)_s & \longrightarrow & \left(\mathcal{E}^*\right)_s & \longrightarrow & 0 \\ \| & & \downarrow{\scriptstyle F_s} & & \downarrow & & \\ \mathcal{W}_s^* & \xrightarrow{B_s^*} & \mathcal{V}_s^* & \longrightarrow & \mathcal{B}_s^* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle A_s^*} & & \downarrow & & \\ & & \mathcal{U}_s^* & = & \mathcal{U}_s^* & & \end{array} \quad . \tag{B.3}$$

By dualizing the display of the restricted monad M_s one gets:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{W}_s^* & \xrightarrow{B_s^*} & (\mathcal{A}_s)^* & \longrightarrow & (\mathcal{E}_s)^* & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{W}_s^* & \xrightarrow{B_s^*} & \mathcal{V}_s^* & \longrightarrow & \mathcal{B}_s^* & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle A_s^*} & & \downarrow & & \\
 & & & & \mathcal{U}_s^* & = & \mathcal{U}_s^* & &
\end{array} \qquad (B.4)$$

By the local freeness of $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ and by the isomorphisms in eq. (B.1) the diagrams (B.3) and (B.4) are isomorphic, so that

$$\begin{cases} \ker F_s = (\mathcal{A}^*)_s \\ \ker B_s^* = 0\,. \end{cases}$$

The thesis follows from Proposition 9.1A.(e) at p. 254 of [14] and from Lemma 2.1.4 at p. 33 of [19]. □

Corollary B.7. *The sheaf $\mathcal{E}^*$ is locally free.*

Proof. Since $\mathcal{E}^*$ is flat on S, and since its restrictions $(\mathcal{E}^*)_s$ are locally free for all $s \in S$, the thesis follows from Lemma 2.1.7 at p. 35 of [19]. □

Proof of Proposition B.3. Observe that one has the following natural morphisms of sheaves on T:

$$\mathcal{H}om\,(\mathcal{E}, \mathcal{E}') \xrightarrow{f} \mathcal{H}om\,(\mathcal{E}^{**}, (\mathcal{E}')^{**})\ . \qquad (B.5)$$

Since $\mathcal{E}$ and $\mathcal{E}'$ are torsion-free, this morphism is injective. Moreover, since the sheaves $\mathcal{E}^*$ and $(\mathcal{E}')^*$ are locally free, the sheaf $\mathcal{H}om\,(\mathcal{E}^{**}, (\mathcal{E}')^{**})$ is locally free too and in particular it is flat on S. By restricting eq. (B.5) to the fibre T_s, for some point $s \in S$, one gets

$$\mathcal{H}om\,(\mathcal{E}, \mathcal{E}')_s \xrightarrow{f_s} \mathcal{H}om\,(\mathcal{E}^{**}, (\mathcal{E}')^{**})_s\ .$$

By the local freeness of $\mathcal{E}^*$ and $(\mathcal{E}')^*$, by the isomorphism in eq. (B.2) and by Proposition B.1 the restricted morphism f_s becomes

$$\mathcal{H}om\,(\mathcal{E}_s, \mathcal{E}'_s) \xrightarrow{f_s} \mathcal{H}om\,((\mathcal{E}_s)^{**}, (\mathcal{E}'_s)^{**})$$

up to isomorphism. Since $\mathcal{E}_s$ and $\mathcal{E}'_s$ are torsion-free, this morphism is injective. By Lemma 2.1.4 at p. 33 of [19] the sheaf $\operatorname{im} f$ is flat on S, and the thesis follows from Proposition 9.1A.(e) at p. 254 of [14]. □

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