ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

Superstring Perturbation Theory up to Two Loops and Beyond

Thesis Submitted for the Degree of

Doctor Philosophiae

Candidate:
Chuan-Jie Zhu

Supervisor:
Prof. Roberto Iengo

Academic Year 1988/89

TRIESTE

SISSA - SCUOLA
INTERNAZIONALE
SUPERIORE
STUDI AVANZATI
TRIESTE
Strada Costiera 11

Superstring Perturbation Theory up to Two Loops and Beyond

Thesis Submitted for the Degree of

Doctor Philosophiae

Candidate: Chuan-Jie Zhu

Supervisor: Prof. Roberto Iengo

Academic Year 1988/89

Acknowledgement

At first and foremost I would like to thank my supervisor Prof. Roberto Iengo. I thank him for his teaching about quantum theories and superstring perturbation theory (from him I learnt the most important and recent developments in string perturbation theory). I also thank him for his guidance which led to the works described in this thesis (some of them are done together). I admire him a lot for his constant pursuing for clear, simple and correct physical interpretation of the results obtained after long and complex computations. To this point I also want to make a comment: eventhough most of the works described here are just long and complex calculations, the results nevertheless have clear and simple physical interpretations, for which most of them are owed to Prof. Roberto Iemgo. Finally I also thank him for careful reading of the thesis, for important comments and suggestions.

Second I want to think Profs. D. Amati, L. Bonora, E. Gava, G. Sotkov and Dr. F. Ferrari for discussions and their interests in my works. I also thank Profs. E. Gava and G. Sotkov for collaboration.

Part of this thesis was typesetted by my wife – Mrs. Man-Lian Zhang. To her I would like to give my special thanks.

Contents

1. Introduction

Superstring theories [1,2] are serious candidates for a unified theory including gravity. This became manifest by the important proof of anomaly cancellation for the type I superstring theory [3], with gauge group $SO(32)$ and the relationship then proven between this and one-loop finiteness [4]. Almost all the previous approaches to unified theory [5] suffer from the defects that either they don't care about gravity – like Grand Unified Theories (GUTs) [6] – or they are at most some low energy effective theories – like theories including N=1 supergravity [7], because of the nonrenormalizibility of quantum gravity. In superstring theories the general belief is that superstring theories are finite theories (in (quantum) loop perturbation expansion order by order, at least). If one believes that the (ultraviolet) infinity in quantum field theory comes from the assumption of the point likeness of the fundamental particles, there is no puzzlings that string theories are finite because strings are one-dimensional extended objects. The inverse square of the string tension provides a natural cut off. Bosonic string theories contain some divergences only because they contain tachyons or there is no symmetry to ensure the vanishing of the massless tadpole. Explicit computations up to one-loop level show that superstring theories are finite [1,2], having neither ultraviolet nor infrared divergences (in ten dimensions). Due to some heuristic arguments, one believes that this finiteness should be persist to all orders in string perturbation expansion [9]. It is certainly an important problem to give a rigorous proof of this "fact". Despite the efforts of many people the proofs has always been incomplete [10,11]. However our efforts in this direction is rather plain: we will explore as much as we can at two loops [12-17]. We will see that we can give a full treatment of superstring perturbation theory at two loops. Even more some sensible physical results can also be drawn, like high energy behaviour of the scattering of gravitons [18-23]. Before running into the details of two-loop calculations, let us briefly review the various approaches to string perturbation theory.

There are various approaches to string pertubation theory. Roughly speaking one can divide all these approaches into two types. The first type is by using the light-cone gauge [24]. In this gauge all the ghosts decouple completely and the non-physical modes like x^- may be expressed directly in terms solely of the physical ones. Nevertheless this approach is non-covariant and Lorentz invariance is not manifest. This drawback sometimes turns out to be fatal because string theory is a theory plagued with anomalies. Anomaly is very difficult to analyse in a non-covariant gauge. In fact the critical dimension of string theory was determined by Lorentz invariance* [25]. There is a vast amount of literatures on light-cone gauge string theory [26-30] and there is also a review article [31] written by A.Restuccia and J.G.Taylor on light-cone gauge analysis of superstrings which gives a comparatively complete analysis of light-cone superstring perturbation theories. So we will not discuss thie approach here. The second type is the Lorentz covariant approach. Let us recall some of them.

The old operator formalism [32] is very suitable to do calculations up to one loop level [1,2]. Now this operator formalism was extended to high loops by including the contributions of ghosts [33-36]. One virtue of this approach is its explicitness by using the Schottky parametrization (from the process of "sewing" reggeons) of the higher genus Riemann surface†. Being quite explicit, this approach may turn out to have potential to give an explicit proof of the finiteness of superstring theories. The approach taken by string field theory [37-41] is a more ambitious approach. Hopefully string field theory should provide tools to do something nonperturbatively. The complaints often come from the fact that we don't have a good string field theory which possesses (in a manifest way) all the miracles of string theory. So it is at least premature to talk about this approach, eventhough we can do a lot with (bad) string field theory. Another approach [42] which is a natural path integral extention from random (particle trajectory) line to random surface (string trajectory), see fig.1, seems more promising. This approach is usually called Polyakov String Theory, because it was Polyakov who firstly introduced it as a functional integration over both the metric and string coordinates. This seemingly trivial extension is actually quite non-trivial [43,44]. In ordinary field theory it is necessary to introduce interactions for particles to interact. In Polyakov string theory it seems trivial to introduce interactions for strings to interact. The trajectory of string is a two-dimensional surface (called Riemann surface in mathematics). All Riemann surfaces can be classified by their topology – the handles and boundaries attached to them, see fig.2. Because we are obliged to do a summation over all surfaces (for fixed boundaries), this summation necessaryly

* That is to say that Lorentz invariance is violated off critical dimension.

† The connection between string perturbation theory and Riemann surface will be explained below.

includes a summation over topology. It is one of the miracles in string theory that the first two terms in this topological summation (according to the well known topological characterization of the (closed) Riemann surface – the genus g (the number of handles of the Riemann surface)) are precisely the tree and one-loop level terms in string theory, as required by the general standards of quantum field theory – like factorization and unitarity, etc. So it is natural to suppose that this topological summation is actually the whole story of string perturbation theory, see fig.3. Nevertheless this approach is not manifestly unitary. Unitarity of Polyakov string (perturbation) theory has been proved for bosonic string theory [45] by relating the Polyakov amplitudes with those in light-cone gauge approach which is manifestly unitary. Various tests for superstring theories are also in agreement with unitarity and factorization. Also the rules for calculations should be derived from a more fundamental theory – hopefuly a very good string theory (which nobody knows at present) which explains all the miracles of string theory. All these problems are left for further research works in the next century partially because string theory belongs to 21th century physics [46], which accidently discovered in this century. For detail discussions of Polyakov string theory and extensive references, see for instance [47].

So what is the present art of string perturbation theory, specifically the Polyakov string theory? Despite many efforts we are still a long way from a thorough understanding of string perturbation theory. It is generally believed that the S-matrix of closed superstirng theory is finite, at least in loop perturbation expansion order by order. Part of this conjecture is a stringy non-renormalization theorem [48] which states that the zero- up to three-particle (massless) amplitudes do not get renormalized. In particular it includes the the (perturbative) vanishing of the cosmological constant. Surprisingly, this question has not been solved completely. In the last couple of years, we have seen a number of elegant but erroneous arguments to proof the above assertions on a formal level. Part of the cruxes of the matter lie in our ignorance of supermanifolds and, in particular, of supermoduli space. Bosonic strings are easier to master due to more than a hundred years of work in Riemann surface. Recently the generalization of the notion of a Riemann surface to a super Riemann surface parametrized by coordinates (z, θ) where θ is a Grassmann number has attracted a lots of interests [49-52]. Here the moduli and the supermoduli characterize the superconformal structures of the super Riemann surfaces. Nevertheless the study of supermoduli space is just at the beginning. How the theory of supermoduli space will give a full resolution of the subtleties [53, 54, 55] of the present formulation (of superstring perturbation theory) and possibly show manifestly the supersymmetry and finiteness is not clear to us.

At present, there are basically two mutually orthogonal strategies of pursuing superstring perturbation theory. First, given the fact that the partition function can be written as a total derivative in ordinary moduli space* [56], it seems natural to focus on the modular boundary, i.e., on degenerate Riemann surfaces [57, 58]. The boundary integrals for the vacuum amplitude have been investigated [54, 55], with the result that under certain assumptions their one-loop vanishing extends to all orders in perturbation theory. If one understands their arguments, there still remains much work to be done, in particular for understanding factorization and modular invariance in the general genus case.

On the other hand, amplitudes with n external legs still require (at least for $n > 3$ which is presumbaly nonvanishing) the computation of the full integrand in the moduli space of (punctured) Riemann surfaces. At the one-loop level, this can be done in a straightforward manner [1, 2]. The important observation here is that the summation over spin structures [59] is crucial for obtaining a genuine result: it makes the zero- to three-particle (massless of course) amplitudes vanish pointwise in moduli space and enormously simplifies the four-particle amplitude [60]. In this thesis we will show that this same pattern occurs also at two loops [12-16]. By using a purely algebraic parametrization of genus two Riemann surfaces by hyperelliptic coordinates, the summation over spin structures can be done in a straightforward way. Here the modular invariance can be implemented explicitly†. In fact the requirement of modular invariance ensures the vanishing of the cosmological constant [13]. The pointwise vanishing of n-particle amplitudes ($n < 4$) (the nonrenormalization theorem) can be proven by a straightforward algebraic calculation [13, 14]. Then the calculation of the four-particle amplitude simplifies enormously and can be explicitly carried out [15]. All of these details will be explained in this thesis. It is organised as follows:

* That is, we have done the trival integration over supermoduli space.

† This fixes completely the phase in the summation over spin structures.

In section 2, we review briefly the general strategy of multi-loop calculation following [61, 56, 62] in Polyakov string perturbation theory. Following [61], we explain how moduli space comes out as a result of integration over world-sheet metric – the geometry – and derive the correct measure for the partition function and scattering amplitudes for closed bosonic string theories [63]. This derivation was also extend to superstring theories [64-66] in the Nevea-Schwarz-Ramond covariant formalism [67, 68]. In order to do explicit calculation in superstring theories, it is convenient first to do integration over supermoduli – leaving two insertions of supercurrent at two loops, for example – and then over moduli. The necessity of a summation over spin structures [59] – GSO projection [69] – is also explained, ensuring modular invariance and finiteness of the amplitudes.

In section 3, we give some relevant mathematical backgrounds about Riemann surface in general and hyperelliptic Riemann surface in particular [70, 71]. Some usefull relations among Θ-constants, period matrix and branch point (see section 3), i.e. Thomae formula and variational formula [72] etc. are also given. We also gave the proof of a formula (which will be used in section 6) given by V.G.Knizhnik in [12] and some new results [16].

In section 4, we use the results of section 2 to do some sample calculations in string theories up to one-loop level. I will derive the Virsoro-Shapiro dual amplitiude [73, 74] in closed bosonic string theory. For type II superstring theories I will do the tree amplitude calculations in some details to see how complex the calculation is because I don't want to cheat those people who haven't done this kind of calculations, by saying that it is also easy to get what and what. The calculation at one-loop level was done in hyperelliptic formalism to demonstrate that one can also do explicit calculation in hyperelliptic formalism. The results obtained are shown to be identical with those obtained in Θ-function formalism [1, 2].

In section 5, we first calculate the n-particle amplitudes up to $n = 3$ and verify the nonrenormalization theorem, i.e. the vanishing of the n-prticle (massless) ($n < 4$) amplitudes, explicitly [12, 13, 14]. This calculation is based on a set of identities called Lianzi identities in [14, 15]. We give also in this section the proof of Lianzi identities and also the derivation of several summation formulas used in [15] and later.

In section 6, we present the full details about the computation of the four-particle amplitude [15, 16]. By using the nonrenormalization proven in the last section, the calculation of the four-particle (all bosons) amplitude simplifies enormously. The standard kinematic factor comes out as a result of summation over spin structures. The contribution from ghost and superghost is also calculated. As we shall see in section 7, this contribution is necessary to ensure the right properties of the four-particle amplitude.

In section 7, we verify the main properties of the four-particle amplitude. The indenpendence of the amplitudes on the insertion points of supercurrent was checked explicitly by carefully studying the boundary terms (or total derivatives) [56, 15, 16]. We show that the amplitude obtained is finite, having neither ultraviolet nor infrared divergences and has the right factorization properties [75]. All our discussions go through both heterotic string (HST) [76] and type II superstring (SST II) theories [1,2] although sometimes we discuss HST only.

In section 8, starting from the exact integral representation of the superstring scattering amplitude at two loops for four bosonic massless external particles obtained in section 6, we evaluated its asymptotic behaviour in the limit of high energy and small momentum tansfer [22]. By using a simple compactification scheme (see section 9) we study also the case of four dimensional space-time. We find a rescattering eikonal form due to multi-graviton exchange in the t channel. A subleading term like a Schwarzschild correction due to the interaction among the exchanged gravitons can also be obtained [22, 23]. These results are similar, although not quite identical, to what obtained in [19].

In section 9, we review briefly a simple compactification scheme called fermionic construction [77, 78]. Then we show that how the previous prescription for two-loop calculations can be applied to these four-dimensional models by explicitly doing some calculations. The cosmological constants of all the supersymmetric string models (in fermionic construction of course) at two loops are shown to be zero [79, 80]. The nonvanishing amplitudes (and also the nonrenomalization theorem) can be readily calculated although we will not present all the details and formulas [81].

In section 10, we present the calculation of two-loop fermionic amplitude [17]. This calculaion is very technical. Nevertheless we feel that it is worth presenting such complex calculation here not only because one can do it by the techniques at hand but also it points out many questions unanswered. In particular we discuss in detail the modular invariance and show in full details how we obtained a modular invariant result for the fermionic amplitude.

In sction 11 we will survey briefly some topics of multi-loop calculations [82-85]. We will mainly discuss the cosmological constant. By explicitly calculating the cosmolgical constant for high genus hyperelliptic Riemann surfaces (which unfortunately consist of only a measure zero part of the moduli space for genus higher than 2), we can see how modular invariance can be implemented and ensures the vanishing of the cosmological constant [85]. Nevertheless an explicit proof of the vanishing of the cosmological constant is still lacking. At present only partial results have been obtained.

In the last section (section 12) wee would like to summarize what we have achieved. As to the future of string theory we are no more optimistic than some people but we are also no more pessimistic than other people. It is certainly true that even if we do have a complete proof and answers to all the questions raised in this thesis we are still a long way from the goals that string theories would be reached: a truely unified theories of everything.

There are three appendixes which explain some technical points. In appendix A I present some details for the calculation of the chiral determinants following [86-89], see also [90, 91]. In appendix B the derivation of eq.(5.14) is done in some details [13]. Finally I reproduce appendix A of ref.[22] as appendix C here for completeness.

2. String Perturbation Theory

In this section, we review briefly the general strategy of multiloop calculation following [61, 56]. We will discuss first what Polyakov's approach to string theory means. We discuss how to fix the gauge and reduce the functional integration to the integration over moduli space. Then we extend all these discussions to superstring theories and derive the loop measure for superstring theories [56] (for both HST and SST II).

In the Polyakov approach to string theory, quantization is performed by summing the functional integration over all geometry and string coordinates. The vacuum amplitude (partition function) is then

$$Z = \sum_{\text{topologies}} \int \frac{D(\text{geometry})D(\text{string coordinates})}{\text{Vol.(symmetry group)}} e^{-S}. \tag{2.1}$$

Please note that we always assume Wick rotation both for two-dimensional world-sheet and target space-time where string moves. S is the action. The n-particle amplitude is computed by inserting vertices on Riemann surface in the partition function, i.e. we have

$$\begin{aligned} A_n(k_i, \epsilon_i) = & \sum_{\text{topologies}} \int \frac{D(\text{geometry})D(\text{string coordinates})}{\text{Vol.(symmetry group)}} \\ & \times \int_{\Sigma_g} \prod_{i=1}^{n} \mathrm{d}^2 z_i V(k_i, \epsilon_i, z_i) e^{-S}, \end{aligned} \tag{2.2}$$

where $V(k_i, \epsilon_i, z_i)$ is the vertex for the emission of i-th particle with momentum k_i and polarization tensor ϵ_i.

For closed bosonic string, we have

$$S = \int \mathrm{d}^2\sigma \sqrt{g} g^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X, \tag{2.3}$$

where X are string coordinates describing the embedding of string in space-time and $g_{\alpha\beta}(g^{\alpha\beta} = (g^{-1})_{\alpha\beta})$ is the world-sheet metric ($\sigma^\alpha(\alpha = 1,2)$ are local coordinates on the world-sheet). It is not difficult to see that this action (2.3) is invariant under the following reparametrization of the local coordinates:

$$\begin{aligned} \sigma^\alpha & \longrightarrow f^\alpha(\sigma), \\ g_{\alpha\beta} & \longrightarrow \frac{\partial f^\gamma(\sigma)}{\partial \sigma^\alpha} \frac{\partial f^\delta(\sigma)}{\partial \sigma^\beta} g_{\gamma\delta}(f(\sigma)), \end{aligned} \tag{2.4}$$

and under the following rescaling of the metric:

$$g_{\alpha\beta} \longrightarrow e^{\varphi(\sigma)} g_{\alpha\beta}. \tag{2.5}$$

To quantize this theory properly, one should factorize out the volume of this symmetry group and get the correct measure for the path integral. We will follow Faddeev-Popov procedure to factorize out this (infinite) volume.

To choose a gauge condition, we would like to choose the conformal gauge in which the metric takes the form

$$g_{\alpha\beta} = e^{\varphi} \delta_{\alpha\beta}. \tag{2.5}$$

But this extremely convenient gauge has some topological limitations. Let us discuss now both the derivation of (2.6) and these limitations.

The first naive argument which shows that (2.6) is possible is the following. The possibility of the choice (2.6) means that any metric $g_{\alpha\beta}$ can be given in the form

$$g_{\alpha\beta} = (e^{\varphi(\sigma)} \delta_{\alpha\beta})^{f} = e^{\varphi(f(\sigma))} \frac{\partial f^{\gamma}}{\partial \sigma^{\alpha}} \frac{\partial f^{\gamma}}{\partial \sigma^{\beta}}, \tag{2.7}$$

where $\{f^{\gamma}(\sigma)\}$ defines the necessary coordinate transfor Hence, the right hand siude of (2.7) depends on three arbitrary functions $f^{1}(\sigma)$, $\varphi(\sigma)$. But $g_{\alpha\beta}(\sigma)$ also has three independent number of independent functions matches. However, this is not enough. We must show that the transformation (2.7) is nonsingular, i.e. the jacobian for passing to the variables (φ, f^{α}) is non zero. To show this we shall consider a small variation of (2.7):

$$\delta g_{\alpha\beta} = \delta\varphi g_{\alpha\beta} + \nabla_{\alpha} V_{\beta} + \nabla_{\beta} V_{\alpha}, \tag{2.8}$$

where $V^{\alpha} = \delta f^{\alpha}$. The nonsigular nature of the transformation will be proved if for any $\delta g_{\alpha\beta}$ we can find $\delta\varphi$ and V such that (2.8) will hold. In other words, we must be able to solve the equation:

$$\delta\varphi \delta_{\alpha\beta} + \nabla_{\alpha} V_{\beta} + \nabla_{\beta} V_{\alpha} = \delta g_{\alpha\beta} \equiv \gamma_{\alpha\beta}, \tag{2.9}$$

or

$$(PV)_{\alpha\beta} = \nabla_{\alpha} V_{\beta} + \nabla_{\beta} V_{\alpha} - g_{\alpha\beta} \nabla^{\delta} V_{\delta} = \gamma_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \gamma^{\delta}_{\delta}, \tag{2.10}$$

which is obtained from (2.9) by subtracting the trace. The question, whether the conformal gauge is always accessible, is reduced now to the possibility of solving (2.10) which we shall rewrite symbolically:

$$PV = \gamma. \tag{2.11}$$

Here we have denoted by P the differential operator, defined by (2.10) which takes vector fields into traceless tensors (notice that the number of independent components is the same). There exists a conjugate operator P^+ which acts in the opposite direction—transforming tensors into vectors. It is easy to realize that equation (2.11) will be solvable if and only if the conjugate operator P^+ doesn't have zero modes. On the other hand, the solution of eq. (2.11) is not unique if P has zero modes.

So, our conclusion is that the existence of zero modes of the operator P^+ means that the conformal gauge is not accessible, and zero modes of P that it is not unique (and one should further fix the remaining gauge freedom).

The number of zero modes is regulated by index theorem. We will not go into the details of these mathematics and only recall the relevant results. We have

$$N_0(P) - N_0(P^+) = 3\chi = -(6g-6), \tag{2.12}$$

where N_0 denotes the number of zero modes, χ is the Euler character of the Riemann surface Σ_g and g is the genus. In particular, we have the following list:

$$\begin{aligned} &N_0(P) = 6, \qquad N_0(P^+) = 0, \qquad &&\text{for} \quad g = 0 \ \text{(sphere)};\\ &N_0(P) = 2, \qquad N_0(P^+) = 2, \qquad &&\text{for} \quad g = 1 \ \text{(torus)};\\ &N_0(P) = 0, \qquad N_0(P^+) = 6g-6, \qquad &&\text{for} \quad g \geq 2. \end{aligned} \tag{2.13}$$

So we found that on a sphere we can always introduce a conformal gauge, which is defined modulo $SL(2,C)$ transformations (with six ($= N_0(P)$) real parameters) which requires extra gauge fixing, e.g. the fixing of three out of four complex $z_i, i = 1,2,3,4$ (the locations of the inserted vertices) in the case of four-particle amplitude at tree level. In the case of Riemann surface with higher genus we have topological obstructions for the conformal gauge. The best thing which can be done is the following choice of gauge

$$g_{\alpha\beta}(\sigma) = e^{\varphi(\sigma)} g^{(0)}_{\alpha\beta}(\sigma; \tau_1, \tau_2, \cdots, \tau_{6g-6}), \tag{2.14}$$

where $g^{(0)}_{\alpha\beta}$ is a metric which depends on $6g-6$ extra parameters and, e.g. which can be chosen to have constant negative curvature. Integration over all metrics (i.e. geometry) must include not only functional integration over $\varphi(\sigma)$ but also $6g-6$ dimensional integration over $\{\tau_i, i = 1, 2, \cdots, 6g-6\}$—the moduli space. Let us now derive the explicit measure for such integration. Before doing that, let us mention an important mathematical result. Roughly speaking, this moduli space is a complex space. So we will use complex coordinates for this moduli space and also for the Riemann surface. In complex coordinates, the metric tensor on the Riemann surface are given by the components $g_{z\bar{z}}, g_{zz}$ and $g_{\bar{z}\bar{z}}$. Then eq.(2.8) can be written in the following form:

$$\begin{aligned}
\delta g_{zz} &= \nabla_z V_z, \\
\delta g_{\bar{z}\bar{z}} &= \nabla_{\bar{z}} V_{\bar{z}}, \\
\delta g_{z\bar{z}} &= \delta\varphi g_{z\bar{z}} + g_{z\bar{z}}(\nabla^z + \nabla_z V^z).
\end{aligned} \tag{2.15}$$

Here ∇_z and ∇^z are covariant derivatives:

$$\begin{aligned}
\nabla^z_{(1)} V_z &= g^{z\bar{z}} \partial_{\bar{z}} V_z, \\
\nabla^{(1)}_z V_z &= g_{z\bar{z}} \partial_z (g^{z\bar{z}} V_z) \quad \text{etc.},
\end{aligned} \tag{2.16}$$

and

$$(\nabla^{(1)}_z)^+ = -g^{z\bar{z}} \partial_{\bar{z}}, \tag{2.17}$$

where we used index (n) to distinguish the covariant derivatives acting on different tensor fields (see, e.g. [43, 44] for more details).

From the previous discussions and (2.17), we see that an arbitrary variation of δg_{zz} can be written in the following form:

$$\delta g_{zz} = \nabla_z V_z + \delta\tau_i \phi^i_{zz}, \tag{2.18}$$

where $\tau_i, i = 1, 2, \cdots, 3g-3$ are the complex coordinates for moduli space and $\{\phi^i_{zz}\}$ are a basis of the zero modes of $(\nabla^{(1)}_z)^+$—the 2-differentials. Similarly, we have

$$\delta g_{\bar{z}\bar{z}} = \nabla_{\bar{z}} V_{\bar{z}} + \delta\bar{\tau}_i \bar{\phi}^i_{\bar{z}\bar{z}}. \tag{2.19}$$

In order to find the integration measure, one defines a metric in the space of all metrics:

$$\begin{aligned}
\| \delta g_{z\bar{z}} \|^2 &= \int d^2 z g_{z\bar{z}} \delta g_{z\bar{z}} \delta g^{z\bar{z}}, \\
\| \delta g_{zz} \|^2 &= \int d^2 z g_{z\bar{z}} \delta g_{zz} \delta g^{zz}, \\
&= \int d^2 z g_{z\bar{z}} \, \nabla^{(1)}_z \, V_z \, \nabla^z_{(-1)} \, V^z + \delta\tau_i \delta\bar{\tau}_j \langle \phi^j, \phi^i \rangle \quad \text{etc.}
\end{aligned} \tag{2.20}$$

where $\langle \phi^i, \phi^j \rangle = \int d^2z (g_{z\bar{z}})^{-1} \bar{\phi}^i_{\bar{z}\bar{z}} \phi^j_{zz}$. Then we have

$$Dg = D[\varphi V_z V_{\bar{z}}] \prod_{i=1}^{3g-3} d^2\tau_i \det{}'(\nabla_z \nabla^z_{(-1)}) \det\langle \phi^i, \phi^j \rangle. \tag{2.21}$$

Notice that the functional integration over conformal factor $e^{\varphi(\sigma)}$ can be trivially factorize out in the critical dimens $d = 26$ for closed bosonic string as shown by Polyakov in [3], the partition function can be written as

$$Z = \sum_g \int_{M_g} \prod_i d^2\tau_i \det{}'(\nabla_z \nabla^z_{(-1)}) \det\langle \phi^i, \phi^j \rangle \int DX e^{-S}, \tag{2.22}$$

where M_g is the moduli space. In this expression we decomposed the variation of δg_{zz} as in (2.18). In other words, this is a choice of gauge slice. We can also choose other gauge slice, e.g.

$$\begin{aligned} \delta g_{zz} &= \nabla_z V'_z + \delta y_i \mu^i_{zz}, \\ \delta g_{\bar{z}\bar{z}} &= \nabla_{\bar{z}} V'_{\bar{z}} + \delta \bar{y}_i \bar{\mu}^i_{\bar{z}\bar{z}}, \end{aligned} \tag{2.23}$$

as shown schematically in Fig.4, and where $\bar{\mu}^i_{\bar{z}\bar{z}} = g_{z\bar{z}} \mu^{iz}_{\bar{z}}$, $\mu^{iz}_{\bar{z}}$ are called Beltrami differentials.

From (2.18) and (2.23), we have

$$\nabla_z V_z + \delta\tau_i \phi^i_{zz} = \nabla_z V' + \delta y_i \mu^i_{zz}, \tag{2.24}$$

or

$$\delta\tau_i \langle \phi^j, \phi^i \rangle = \delta y_i \langle \phi^j, \mu^i \rangle. \tag{2.25}$$

Doing wedge product over $j = 1, 2, ..., 3g-3$ with eq.(2.25), we have

$$\prod_{i=1}^{3g-3} dy_i \cdot \det\langle \phi^i, \mu^j \rangle = \prod_{i=1}^{3g-3} d\tau_i \cdot \det\langle \phi^i, \phi^j \rangle. \tag{2.26}$$

Substituting this expression into (2.22), we get

$$Z = \sum_g \int_{M_g} \prod_i d^2 y_i \frac{|\det\langle \phi^i, \mu^j \rangle|^2}{\det\langle \phi^i, \phi^j \rangle} \det{}'(\nabla_z \nabla^z_{(-1)}) \int DX e^{-S}. \tag{2.27}$$

Following the standard Faddeev-Popov procedure, the gauge parameter V^z for reparametrization invariance can be replaced by an anticommuting ghost field c^z. Introducing its conjugate antighost field b_{zz}, we have the following reparametrization invariant ghost action

$$S_{\text{gh}} = \int \mathrm{d}^2 z g_{z\bar{z}} b_{zz} \nabla^z c^z + \text{C.C.} \quad . \tag{2.28}$$

Then we can represent $\det'(\nabla_z \nabla^z_{(-1)})$ by a path integral over ghost fields. We have

$$\int D[bc\bar{b}\bar{c}] \prod_i b(z_i)\bar{b}(\bar{z}_i) e^{-S_{gh}} = \det'(\nabla_z \nabla^z_{(-1)}) \frac{|\det\phi^i(z_k)|^2}{\det\langle\phi^i,\phi^j\rangle}. \tag{2.29}$$

Substituting $\det'(\nabla_z \nabla^z_{(-1)})/\det\langle\phi^i,\phi^j\rangle$ by the above expression, we found that the partition function (2.27) can be expressed as

$$\begin{aligned} Z &= \sum_g \int_{M_g} \prod_i d^2 y_i \left| \frac{\det\langle\phi^i,\mu^j\rangle}{\det\phi^i(z_k)} \right|^2 \int D[Xbc\bar{b}\bar{c}] \prod_i b(z_i)\bar{b}(\bar{z}_i) e^{-(S+S_{gh})} \\ &= \sum_g \int_{M_g} \prod_i d^2 y_i \int D[Xbc\bar{b}\bar{c}] \prod_i |\langle\mu^i, b\rangle|^2 \; e^{-(S+S_{gh})}, \end{aligned} \tag{2.30}$$

where $\langle\mu^i, b\rangle$ is the standard notation for the pairing between b field and the Beltrami differentials:

$$\langle\mu^i, b\rangle = \int d^2 z \mu^{iz}_{\bar{z}} b_{zz}. \tag{2.31}$$

All the above discussinos can be extended to supersymmetric string theories. Here the complication comes mainly from the fermion fields on Riemann surface. First, we have supersymmetric (2-dimensional) partners for all the bosonic fields in closed bosonic string theory and have to integrate over all these fields. The functional integration can be carried out straightforwardly. In the end, because of topological obstruction one should also integrate over a $2g-2$ dimensional space $\{\rho^a, a = 1, 2, \cdots, 2g-2\}$—the supermoduli space, in addition to the $6g-6$ dimensional moduli space. However, the integration over supermoduli space is a Grassmannian integration and can be explicitly carried out and we have the following expression for the partition function derived in [56]:

$$\begin{aligned} Z = &\sum_g \int_{M_g} \prod_i d^2 m_i \int D[X\psi bc\beta\gamma] e^{-(S[X,\psi]+S_{gh}[b,c,\beta,\gamma])} \\ &\times \prod_a \delta(\langle\chi_a,\beta\rangle)(\langle\chi_a, J\rangle + \frac{\partial}{\partial\rho_a}) \prod_i \langle\mu_i, b\rangle \times (\text{left sector}), \end{aligned} \tag{2.32}$$

where J is the total super current (see eq.(2.56)), β and γ are ghost fields for the super reparametrization transformation. Here $\frac{\partial}{\partial\rho_a}$ acts on $\prod_i\langle\mu^i, b\rangle$ as $\frac{\partial}{\partial\rho_a}\mu^i = \frac{\partial}{\partial m_i}\chi_a$. Let us now try to derive the above expression for the loop measure of superstring theory.

Our starting point is the supersymmetric generalization of the bosonic string action. It is [92, 93]

$$S = \int d^2\sigma \cdot e\{g^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X + \psi \cdot \gamma^\alpha \partial_\alpha \psi + \chi_\alpha \gamma^\beta \gamma^\alpha \psi \cdot (\partial_\beta X - \frac{1}{4}\chi_\beta \psi)\}, \tag{2.33}$$

where* $e = \det(e^a_\alpha)$, ψ^μ ($\mu = 1, 2, \cdots, D$, the space-time index) is the supersymmetric partner of X^μ and χ_α (α=1,2) is the supersymmetric partner of the two-dimensional "vierbein" field e^a_α ($g_{\alpha\beta} = e^a_\alpha e^b_\beta \delta_{ab}$). Apart from the usual reparameterization invariance of the local coordinates (see eq.(2.4)), the action (2.33) is also invariant under the following local supersymmetric transformations:

$$\begin{aligned} \delta X_\mu &= \varepsilon \psi_\mu, \\ \delta \psi_\mu &= \sigma^\alpha(\partial_\alpha X_\mu - \frac{1}{2}\chi_\alpha \psi_\mu)\varepsilon, \\ \delta g_{\alpha\beta} &= \varepsilon(\sigma_\alpha \chi_\beta + \sigma_\beta \chi_\alpha, \\ \delta \chi_\alpha &= 2\hat{\nabla}_\alpha \varepsilon, \end{aligned} \tag{2.34}$$

with ε being a Majorana spinor. The Weyl rescaling of the metric (see eq.(2.5)) is generalized to

$$\begin{aligned} g_{\alpha\beta} &\longrightarrow e^{2\varphi(\sigma)} g_{\alpha\beta}, \\ \chi_\alpha &\longrightarrow e^{\frac{1}{2}\varphi(\sigma)} \chi_\alpha, \\ X_\mu &\longrightarrow X_\mu, \\ \psi_\mu &\longrightarrow e^{-\frac{1}{2}\varphi(\sigma)} \psi_\mu. \end{aligned} \tag{2.35}$$

As in bosonic string theory, in order to get the correct measure for the path integral, we should factorize out the volume of the symmetry group. We will follow the Faddeev-Popov procedure to factorize out this (infinite) volume.

To choose a gauge condition, we would like to choose the supercoformal gauge in which the metric and the gravitino fields take the form

$$\begin{aligned} g_{\alpha\beta} &= e^\varphi \delta_{\alpha\beta}, \\ \chi_\alpha &= \gamma_\alpha \chi. \end{aligned} \tag{2.36}$$

Then the action reduced to a quadratic form and the path integral can be easily computed. But this extremely convenient gauge has some topological limitations. Let us discuss now both the derivation of (2.36) and these limitations.

* $\gamma^\alpha = e^\alpha_a \sigma^a$, $\sigma^a (a = 1, 2)$ are the Pauli matrices.

The first naive argument which shows that (2.36) is possible is the following. The possibility of the choice (2.36) means that any metric and gravitino configuration can be transformed to the form of (2.36) by repeating use of either general coordinate transformation or supersymmetric transformation (2.34). Let us first of all count the number of independent functions in the gauge (2.36) which gives us a rough orientation of the situation. We have 3 components of $g_{\alpha\beta}$ and two Majorana spinors χ_a, or in other words 3 bosonic functions and 4 fermionic ones. We replace them by one bosonic fuctions and two fermionic functions χ. This is reasonable, since we have two extra bosonic functions, describing general coordinate transformation from the gauge (2.36) to an arbitrary one, and two fermionic ones which enter in (2.34). All in all, the number of independent function matches. However, this is not enough. We must show that the transformations are nonsingular, i.e. the Jacobian for passing to the variables $(\varphi,\ V_\alpha,\ \chi,\ \varepsilon)^*$ is non zero. To show this we shall consider an arbitrary variation of χ_α and examine whether we can write it as some variation of χ plus a supersymmetric transformation. We have

$$\begin{aligned} \delta\chi_\alpha &= \sigma_\alpha\delta\chi + 2\hat{\nabla}_\alpha\varepsilon \\ &= \sigma_\alpha(\delta\chi + \sigma^\beta\ \nabla_\beta\, \varepsilon) + 2(\hat{\nabla}_\alpha\varepsilon - \frac{1}{2}\sigma_\alpha\sigma^\beta\ \nabla_\beta\, \varepsilon). \end{aligned} \tag{2.37}$$

The nonsingular nature of the supersymmetric transformation will be proved if for any $\delta\chi_\alpha$ we can find $\delta\chi$ and ε such that (2.37) will hold. In other words, we must be able to solve the equation:

$$\sigma_\alpha\delta\chi + 2\hat{\nabla}_\alpha\varepsilon = \delta\chi_\alpha, \tag{2.38}$$

or

$$(L\varepsilon)_\alpha \equiv 2(\hat{\nabla}_\alpha\varepsilon - \frac{1}{2}\sigma_\alpha\sigma^\beta\ \nabla_\beta\, \varepsilon) = \delta\chi_\alpha - \frac{1}{2}\sigma_\alpha\sigma^\beta\delta\chi_\beta, \tag{2.39}$$

which is obtained from (2.38) by subtracting the trace. All this is quite analogous to the bosonic case (2.10).

The question, whether the conformal gauge is accessible, is reduced now to the possibility of solving (2.39) which we shall write symbolically:

$$L\varepsilon = \delta\chi. \tag{2.40}$$

---

* where V_α and ε describe the general coordinate and supersymmetric transformation.

Here we have denoted by L the differential operator, defined by (2.39) which takes Majorana spinor fields into traceless vector spinor fields (note that the number of independent components is the same). There exists a conjugate operator L^+ which acts in the opposite direction—transforming (traceless) vector spinors into spinors. It is easy to realize that equation (2.40) will be solvable if and only if the conjugate operator L^+ doesn't have zero modes. On the other hand, the solution of (2.40) is not unique if L has zero modes.

So, our conclusion is that zero modes of the operator L^+ means that the superconformal gauge is not accessible, and zero modes of L that it is not unique (and one should further fix the remaining gauge freedom).

The number of zero modes of L ans L^+ are not readily found from the classical theory of Riemann surface because here we are dealing with anticommuting (or Grassmanian) fields. The theory of super Riemann surface is still in its fancy form. A straightforward extension of ordinary Riemann surface may not be enough. Here we will restrict our discussions to the particular background where the gravitino field χ_α vanishes: $\chi_\alpha = 0$. In the final results we shall be able to restore χ_α dependence by the use of (world-sheet) supersymmetry. As in the bosonic case we also use the (super) complex coordinates on the world-sheet. By taking the world-sheet metric as $ds^2 = \rho(d\sigma^2 + d\tau^2) = \rho dz d\bar{z}$, $\frac{1}{2}\rho = g_{z\bar{z}}$, $z(\bar{z}) = \sigma \pm i\tau$, the covariant derivative acting on spinor field can be found to be:

$$\begin{aligned}(L\varepsilon)_\alpha &\equiv \nabla_\alpha \varepsilon - \frac{1}{2}\sigma_\alpha \sigma^\beta \nabla_\beta \varepsilon \\ &= \sqrt{g_{z\bar{z}}} \left\{ \partial_\alpha(\sqrt{g^{z\bar{z}}}\varepsilon) - \frac{1}{2}\sigma_\alpha\sigma^\beta \partial_\beta(\sqrt{g^{z\bar{z}}}\varepsilon) \right\}.\end{aligned} \tag{2.41}$$

Changing to complex coordinates, this is exactly the covariant derivatives acting on right spinor (or $(\frac{1}{2},\ 0)$ tensor) field:

$$\nabla_z^{(\frac{1}{2})} \varepsilon_\theta = \sqrt{g_{z\bar{z}}}\partial_z(\sqrt{g^{z\bar{z}}}\varepsilon_\theta), \tag{2.42}$$

and we have

$$(\nabla_z^{(\frac{1}{2})})^+ = -g^{z\bar{z}}\partial_{\bar{z}}, \tag{2.43}$$

acting on $(\frac{3}{2},\ 0)$ tensor (or right vector spinor) field. The existence of zero modes of the operator $(\nabla_z^{(\frac{1}{2})})^+$—the holomorphic $\frac{3}{2}$-differentials—is the topological obstruction to the accessibility of the superconformal gauge. Of course, the bosonic part of (2.41) should also be analyzed.

The number of zero modes of the operators $\nabla_z^{(n)}$ and $(\nabla_z^{(n)})^+$ are regulated by index theorem. We will not go into the details of these mathematics and only recall the results here. We have

$$\begin{aligned} N_0(\nabla_z^{(n-1)}) - N_0((\nabla_z^{(n-1)})^+) &= +\frac{1}{2}(2n-1)\chi \\ &= -(2n-1)(g-1), \end{aligned} \tag{2.44}$$

where N_0 denotes the number of zero modes, χ is the Euler character of the Riemann surface Σ_g and g is the genus. In particular, we have the following list for $n = \frac{3}{2}$*:

$$\begin{aligned} &N_0(\nabla_z^{(\frac{1}{2})}) = 2, \qquad N_0((\nabla_z^{(\frac{1}{2})})^+) = 0, \qquad &&\text{for } \; g = 0 \text{ (sphere)}; \\ &N_0(\nabla_z^{(\frac{1}{2})}) = 0, \qquad N_0((\nabla_z^{(\frac{1}{2})})^+) = 2g-2, \qquad &&\text{for } \; g \geq 2. \end{aligned} \tag{2.45}$$

For $g = 1$ (the torus), the number of zero modes also depends on the spin structures (see below). For even spin structures we have $N_0(\nabla_z^{(\frac{1}{2})}) = N_0((\nabla_z^{(\frac{1}{2})})^+) = 0$ and $N_0(\nabla_z^{(\frac{1}{2})}) = N_0((\nabla_z^{(\frac{1}{2})})^+) = 1$ for the odd spin structure.

Combining the above discussions withe the ones for bosonic string, we found that on a sphere we can always introduce a superconformal gauge, which is defined modulo $SL(2|1, C)$ transformations (with six $(= N_0(P))$ even real parameters and four $(= N_0(L))$ odd parameters) which requires extra gauge fixing, e.g. the fixing of three out four complex z_i, $i = 1, 2, 3, 4$ (the even coordinates of the inserted vertices) and two out of four complex θ_i, $i = 1, 2, 3, 4$ (the odd coordinates) in the case of four-particle amplitude at tree level in type II superstring theory. In the case of Riemann surface with high genus we have topological obstructions for the superconformal gauge. The best thing which can be done is the following choice of gauge

$$\begin{aligned} g_{\alpha\beta}(\sigma) &= e^{\varphi(\sigma)} g^0_{\alpha\beta}(\sigma;\ m_i,\ \bar{m}_i), \\ \chi_\alpha(\sigma) &= \sum_{a=1}^{4g-2} \rho^a \chi^a_\alpha(\sigma;\ m_i,\ \bar{m}_i) + \sigma_\alpha \chi, \end{aligned} \tag{2.46}$$

where $(m_i,\ \bar{m}_i)$ are even moduli parameters and $\rho^a (a = 1, \cdots, 4g-4)$ are odd moduli or supermoduli parameters. $\chi^a_\alpha(\sigma;\ m_i,\ \bar{m}_i)$ are super Beltrami differentials which consist of a basis of zero modes of L^+. As in the bosonic case, we also use the complex coordinates for the (even and odd) moduli space and for the Riemann surface. Then the metric $g_{\alpha\beta}$

* For $n = 2$, please see (2.13).

are given by the components $g_{z\bar{z}}$, g_{zz} and $g_{\bar{z}\bar{z}}$. The gravitino field χ_α are decomposed into a σ-trace part (a $(\frac{1}{2},\ 0)$ tensor χ_z^θ) and a traceless part (a $(1,\ -\frac{1}{2})$ tensor $\chi_z^{\bar{\theta}}$) and their complex conjugate fields ($\chi_{\bar{z}}^{\bar{\theta}}$ and $\chi_{\bar{z}}^{\theta}$).

To compute the measure, we would like to go a two-step way [62]. First we write the measure on the split surface Σ_0 which is obtained from the super-Riemann surface Σ_ρ by setting all the super moduli to zero. Second we will restore the super moduli dependence of the string integrand and do the integration over super moduli. By completely analogous to the bosonic string, we write the g-loop measure of the type II superstring theory as follows

$$Z_g = \int_{sM_g} \prod_i \mathrm{d}^2 m_i \prod_a \mathrm{d}^2 \rho^a \ Z_{\mathbf{X}} |Z_{\mathbf{BC}}|^2, \tag{2.47}$$

where

$$\begin{aligned} Z_{\mathbf{X}} &= \int [D\mathbf{X}^\mu] \exp(-S[bfX^\mu]), \\ Z_{\mathbf{BC}} &= \int [D\mathbf{B}D\mathbf{C}] \exp(-S[\mathbf{B},\mathbf{C}]) \prod_a \delta(<\hat{\mu}_a, \mathbf{B}>) \prod_i <\mu^i, \mathbf{B}> . \end{aligned} \tag{2.48}$$

Here $S[\mathbf{X}^\mu]$ and $S[\mathbf{B},\ \mathbf{C}]$ are the actions for the matter superfield $\mathbf{X}^\mu(z,\ \bar{z},\ \theta,\ \bar{\theta}) = \mathbf{X}^\mu(z,\ \theta) + \bar{\mathbf{X}}^\mu(\bar{z},\ \bar{\theta})$, $\mathbf{X}^\mu(z,\ \theta) = X^\mu(z) + \theta\psi^\mu(z)$ and the superconformal ghost superfields $\mathbf{B}(z,\ \theta) = \beta(z) + \theta b(z)$, $\mathbf{C}(z,\ \theta) = c(z) + \theta\gamma(z)$:

$$\begin{aligned} S[\mathbf{X}^\mu] &= \frac{1}{2} \int D\mathbf{X}^\mu \bar{D}\mathbf{X}_\mu, \\ S[\mathbf{B},\ \mathbf{C}] &= \frac{1}{2\pi} \int \mathbf{B}\bar{D}_{-1}\mathbf{C}. \end{aligned} \tag{2.49}$$

The innerproducts $< \mu_i,\ \mathbf{B} >$ and $< \hat{\mu}_a,\ \mathbf{B} >$ are defined by integration over the super-Riemann surface. The above functional integrals are evaluated by expanding the fields in an arthonormal basis of eigen modes of the corresponding Laplacians. The $\mathbf{B}$-field has $3g-3$ anti-commuting and $2g-2$ commuting zero modes proportional to the holomorphic $(\frac{3}{2},\ 0)$-differentials. The corresponding zero mode integrals are projected out by the operators $\delta(\langle\hat{\mu}_a,\ \mathbf{B}\rangle)$ and $\langle\mu_i,\ \mathbf{B}\rangle$, making the ghost partition function well-defined. Furthermore, since under analytic coordinate transformations on supermoduli space $(m,\ \rho) \longrightarrow (\tilde{m},\ \rho)$, the basis of super-Beltrami differentials transform as tangent vectors we readily verify that the fermionic string partition function $Z_{\mathbf{X}}|Z_{\mathbf{BC}}|^2$ transforms as coordinate invariant density on sM_g.

At this point we would like to point out the second complication coming from the fermion fields on Riemann surface and motivate the integration over the supermoduli. Because fermion fields are half integer differentials, they can change a sign when travel around a non-contractible cycle (path) on Riemann surface. In order to define a fermion field, one should specify its properties when traveling around all the non-contractible cycles on Riemann surface [59]. A specification of this properties is called a spin structures, and there can be 2^{2g} different spin structures on a genus g Riemann surface. Because large reparametrization (those which cann't be continuously deformed to identity and are related to modular transformation) mixes spin structures (changing one spin structures to another one), one has to discuss all the spin structures and do the appropriate summation over spin structures in the partition function in order to get a sensible (e.g. supersymmetric, tachyon free, etc.) theory [59]. As claimed by many people superstring loop amplitudes are space-time supersymmetric. In the covariant NSR-formulation discussed here (following Polyakov), space-time supersymmetry arises only after performing the GSO-projection on even word-sheet fermion parity [69], which in the path-integral is implemented by summing over all the spin structures [59]. However, because spin structure is an intrinsic part of the geometry of a super Riemann surface, this sum can only be performed after integrating over the supermoduli. In other words, whereas on sM_g space-time supersymmetry is a symmetry relating the contribution of different super Riemann surface, and therefore difficult to analyze, after reducing the integrand to M_g, it is realized as a symmetry on each individual Riemann surface. So in order to discuss the space-time properties of the loop amplitudes, we should first do the integration over supermoduli and then do the summation over spin structures. The integration over supermoduli is a Grassmanian integration and can easily be done. Up to now, the problem of summation over spin structures was not completely solved. However, at two-loops (not mention the complete solution at one loop) this was solved in [13] by using modular invariance and the cosmological constant was shown to be zero. We will discuss this solution in section 5 and also the nonrenormalization theorem there.

The first step in the calculation of integral over the supermoduli ρ^a is to isolate all ρ^a-dependence of the string integrand. This means we have to know the difference of the integrand on a super-Riemann surface Σ_ρ with that on the split surface Σ_0 obtained from

Σ_ρ by setting all odd moduli to zero. A direct procedure for calculating this difference is to perform in the functional integral on Σ_0 a redefinition of the fields, such that the new fields satisfy the boundary conditions of Σ_ρ. This redefinition is obtained as follows.

Let $(z,\ \theta)$ denote the complex super coordinates on Σ_0 and $(\tilde{z},\ \tilde{\theta})$ those on Σ_ρ. These two coordinate systems are related to each other by a so-called quasi-superconformal transformation, $(z,\ \theta) \to (\tilde{z},\ \tilde{\theta})$, which can always be chosen to have the following form [62]

$$\begin{aligned} \tilde{z} &= z + \theta\varepsilon(z,\bar{z}), \\ \tilde{\theta} &= \theta + \varepsilon(z,\bar{z}) + \frac{1}{2}\theta\varepsilon\partial\varepsilon(z,\bar{z}), \end{aligned} \tag{2.50}$$

where $\varepsilon(z,\bar{z})$ is a multi-valued, anti-commuting $(-\frac{1}{2},0)$-differential on Σ_0. This coordinate transformation can be read as the operation switching on the odd supermoduli. The redefinition relating the fields on Σ_0 to those on Σ_ρ can now be expressed by the help of the above differential ε as follows

$$\begin{aligned} X &\to X + \varepsilon\psi + \bar{\varepsilon}\bar{\psi}, & \psi &\to \psi + \varepsilon\partial X, \\ b &\to b + 3\partial\varepsilon\beta + \varepsilon\partial\beta, & \beta &\to \beta + \varepsilon b, \\ c &\to c + \varepsilon\gamma, & \gamma &\to \gamma - 2\partial\varepsilon c + \varepsilon\partial c, \end{aligned} \tag{2.51}$$

From this one finds that the action functionals on Σ_ρ are expressed in component fields on Σ_0 as

$$\begin{aligned} S[\mathbf{X}]_\rho &= \frac{1}{2\pi}\int \left[\partial X\bar{\partial}X + \psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi} + \hat{\chi}\psi\partial X + \bar{\hat{\chi}}\bar{\psi}\bar{\partial}X + \bar{\hat{\chi}}\hat{\chi}\bar{\psi}\psi\right], \\ S[\mathbf{B},\mathbf{C}]_\rho &= \frac{1}{\pi}\int \left[b\bar{\partial}c + \beta\bar{\partial}\gamma + \hat{\chi}(\frac{1}{2}b\gamma + \frac{3}{2}\beta\partial c + \frac{1}{2}(\partial\beta)c)\right] + \text{C.C} \end{aligned} \tag{2.52}$$

Here the matter action $S[\mathbf{X}]$ is the well-known Brink-Di Vecchia-Howe action of a two-dimensional massless scalar coupled to N=1 supergracity in the Wess-Zumino gauge [92, 93]. The anti-commuting supermoduli are contained in the two-dimensional gravitino field $\hat{\chi}$, which is an odd $(-\frac{1}{2},1)$-differentiual on Σ_0. It is related to the differential ε by

$$\hat{\chi}(z,\bar{z}) = 2\bar{\partial}\varepsilon(z,\bar{z}). \tag{2.53}$$

Before we procced, let us make a short comment on the matter action $S[\mathbf{X}]^\rho$ in (2.52). At first sight the presence of the last term in this action is a bit surprising, since classically it destroys the decoupling of the left- and right-moving modes. At the quantum level, however, this additional term is in fact necessary in order to have chiral factorization, because it precisely cancels the correlation between $\hat{\chi}\psi\partial X$ and $\bar{\hat{\chi}}\bar{\psi}\bar{\partial}X$ caused by the contraction: $\partial X(z)\bar{\partial}X(w) = -\pi\delta(z-w)$. Unless stated otherwise, we will in the following consider only the holomorphic part (or right part) of the partition function.

It is clear from the above discussion, that all dependence of the action on Σ_ρ on the anti-commuting supermoduli ρ^a resides in the gravitino field $\hat{\chi}$. To simplify the following analysis, we will now choose the coordinates on sM_g such that the gravitino $\hat{\chi}$ can be expanded as

$$\hat{\chi}(z,\bar{z}) = \sum_{a=1}^{2g-2} \rho^a \chi_a(z,\bar{z}), \tag{2.54}$$

where the $2g-2$ differentials $\chi_a(z,\bar{z})$ are all independent of the odd supermoduli. Having made this choice, we can expand the action on Σ_ρ as

$$S[\mathbf{X},\mathbf{B},\mathbf{C}]_\rho = S[\mathbf{X},\mathbf{B},\mathbf{C}]_0 + \sum_{a=1}^{2g-2} \rho^a \langle \chi_a, J\rangle, \tag{2.55}$$

where J is the analytic two-dimensional supercurrent of the matter-ghost system

$$J = \psi\cdot\partial X + 2c\partial\beta - \gamma b + 3\partial c\beta, \tag{2.56}$$

and the pairing $\langle\chi_a, J\rangle$ is defined by integration over Σ_0. If we choose the even differentials μ^i to be independent of the supermoduli, and allow the odd differentials χ_a to be dependent of the even moduli, we have

$$\langle \mu_i, \mathbf{B}\rangle_\rho = \langle \mu_i, b\rangle_0 + \langle \frac{\partial \chi}{\partial m^i}, \beta\rangle_0, \tag{2.57}$$

and

$$\langle \hat{\mu}_a, \mathbf{B}\rangle^\rho = \langle \chi_a, \beta\rangle_0. \tag{2.58}$$

Combining all these we find the following expression for the string partition function on Σ_ρ, in which all dependence on the odd supermoduli ρ_a has been made manifest:

$$Z(m,\rho) = \int e^{-S_0}\left[\prod_a[(1+\rho^a\langle\chi_a,J\rangle)\delta(\langle\chi_a,\beta\rangle)]\prod_i(\langle\mu_i,b\rangle + \rho^a\langle\frac{\partial\chi_a}{\partial m_i},\beta\rangle)\right] \times (\text{left part}), \tag{2.59}$$

where we introduced the short-hand notation:

$$\int e^{-S_0} = \int D[X\psi bc\beta\gamma]\exp\left\{-\frac{1}{\pi}\int\left[\frac{1}{2}\partial X\bar{\partial}X + \frac{1}{2}\psi\bar{\partial}\psi + b\bar{\partial}c + \beta\bar{\partial}\gamma\right]\right\} \times (\text{left part}). \tag{2.60}$$

Here all fields and differentials are defined on the split surface Σ_0. We can now explicity perform the Grassmann integration over the odd supermoduli ρ_a, leading to:

$$\begin{aligned} Z_g = &\int_{M_g}\prod_i d^2m_i \int D[X\psi bc\beta\gamma]e^{-(S[X,\psi]+S_{gh}[b,c,\beta,\gamma])} \\ &\times\prod_a \delta(\langle\chi_a,\beta\rangle)(\langle\chi_a,J\rangle + \frac{\partial}{\partial\xi_a})\prod_i\langle\mu_i,b\rangle \times (\text{left sector}). \end{aligned} \tag{2.61}$$

This is exactly (2.32) as quoted before.

3. Hyperelliptic Riemann Surface and Theta Function

As it is well-known that the evolution of a closed string sweeps out a world sheet, which is a two dimensional surface or Riemann surface embedded in a target space-time, the theory of Riemann surface as well as the theory of the moduli space of the Riemann surface plays a prominent role in string theory. In this section we will give some relevant mathematical background materials about Riemann surface in general and hyperelliptic Riemann surface in particular. Let us start with the definition of a Riemann surface.

A Riemann surface is a real two dimensional manifold with coordinates σ^α and euclidean metric $g_{\alpha\beta}, \alpha, \beta = 1, 2$. One can always choose a set of local coordinates (σ^1, σ^2) such that the Riemann metric takes the form

$$\begin{aligned} \mathrm{d}S^2 &= e^{\varphi(\sigma^1,\sigma^2)} \mathrm{d}\sigma^\alpha \mathrm{d}\sigma^\beta \delta_{\alpha\beta} \\ &= e^{\varphi(z,\bar{z})} \mathrm{d}z \mathrm{d}\bar{z}, \end{aligned} \tag{3.1}$$

by setting $z = \sigma^1 + i\sigma^2$, $\bar{z} = \sigma^1 - i\sigma^2$. From (3.1) one sees that the transition function from one local chart to another one is either a holomorphic or an anti-holomorphic function. So every Riemann surface is an one dimensional complex manifold. Of particular interesting is closed Riemann surface which has no boundaries. The topological classification of the closed Riemann surfaces is by their genus—the handles on the surface. See fig.2.

Because of the handles on the Riemann surface, one can have non-contratible path(s) or cycles on the surface. In fact on a genus g Riemann surface we have and only have $2g$ homological different cycles. Furthermore a canonical homology basis of cycles can be chosen such that the intersection paring of cycles (counting orientation) satisfies

$$\begin{aligned} (\alpha_A, \alpha_B) &= (\beta_A, \beta_B) = 0, \\ (\alpha_A, \beta_B) &= -(\beta_A, \alpha_B) = \delta_{AB}. \end{aligned} \tag{3.2}$$

A basis of cycles satisfying these conditions is showing in fig.5.

From cohomology theory a choice of homology basis corresponds to a choice of g holomorphic and g anti-holomorphic closed one forms (on complex manifold, of course). They are known as the (first kind) Abelian differentials: ω_A, $\bar{\omega}_A$. A standard way of normalizing the ω_A's is to require

$$\oint_{\alpha_A} \omega_B = \delta_{AB}. \tag{3.3}$$

Then the periods over the β cycles are completely determined

$$\oint_{\beta_A} \omega_B = \tau_{AB}, \tag{3.4}$$

which is known as the period matrix of the Riemann surface. It is a symmetric matrix with a positive definite immaginary part.

The space of all the matrices $\mathcal{H}_g = \{\tau; \tau_{ij} = \tau_{ji},\ \mathrm{Im}\tau > 0\}$ is a complex $\frac{1}{2}g(g+1)$-dimensional space known as Siegel's upper half plane. We have seen that every Riemann surface with a chosen homology basis gives rise to a τ. In fact it can be shown that no two inequivalent Riemann surfaces have the same period matrice τ (Torelli's theorem). So we can use $\mathcal{H}_g$ to parametrize the moduli space of the Riemann surfaces. This is a highly redudant description, however, since the same surface with two different basis will in general have two fifferent matrices τ. Suppose the two canonical basis are related by

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{3.5}$$

where A, B, C, D are $g \times g$ integer matrices. In order to preserve the conditions (3.2), the matrix in (3.5) must be a symplectic modular matrix with integer coefficients, i.e. an element of $Sp(2g,,\mathbf{Z})$:

$$\begin{aligned} DC^T - CD^T &= BA^T - AB^T = 0, \\ DA^T - CB^T &= AD^T - BC^T = 1. \end{aligned} \tag{3.6}$$

The change of the abelian differentials and the period matrix under the change of basis (3.5) can easily be computed. We have

$$\begin{aligned} \omega' &= (C^T\tau + D^T)^{-1}\omega, \\ \tau' &= (A\tau + B)(C\tau + D)^{-1}. \end{aligned} \tag{3.7}$$

So in the parametrization of the moduli space of Riemann surface by $\mathcal{H}_g$, we can identify two period matrices τ and τ' when they are related as in (3.7). But this parametrization can not be a good paremetrization because the dimensions do not equal: $\frac{1}{2}g(g+1) > 3g-3$ for $g > 3$. Only for $g = 2,\ 3$ we have $\frac{1}{2}g(g+1) = 3g-3$ and $g = 1$, $\frac{1}{2}g(g+1) = 1$ which also equals to the dimension of the moduli space of the torus (the genus 1 Riemann surface). For $g > 3$, we must impose some constraints on the entries of the matrix τ for τ being a period matrix of a Riemann surface. To find these constraints and a good parametrization of the moaduli space of the Riemann surface by period matrix is known as Schottky problem. This problem was solved recently [94, 95]. It may have potential applications in string theory and physics [96, 97].

The function analyses on the Riemann surface are based on the Riemann theta function [70, 71] which is defined as

$$\Theta(z|\tau) = \sum_{n \in \mathbf{Z}^g} e^{i\pi n\cdot\tau\cdot n + 2\pi i n\cdot z}. \tag{3.8}$$

For describing spin structures it is useful to generalize (3.8) to include characteristic $a,\ b \in R^g$. The Riemann theta function with characteristic is defined by

$$\begin{aligned} \Theta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) &= \sum_{n \in \mathbf{Z}^g} e^{i\pi(n+a)\cdot\tau\cdot(n+a) + 2\pi i(n+a)\cdot(z+b)} \\ &= e^{i\pi a\cdot\tau\cdot a + 2\pi i a\cdot(z+b)}\Theta(z|\tau). \end{aligned} \tag{3.9}$$

Then we have

$$\begin{aligned} \Theta\begin{bmatrix} a+n \\ b+m \end{bmatrix}(z+\tau\cdot n+m) &= e^{-i\pi n\cdot\tau\cdot n - 2\pi i n\cdot(z+b)}\Theta\begin{bmatrix} a \\ b \end{bmatrix}(z) \\ &= e^{-i\pi n\cdot\tau\cdot n - 2\pi i n\cdot(z+b) + 2\pi i m\cdot a}\Theta\begin{bmatrix} a \\ b \end{bmatrix}(z), \end{aligned} \tag{3.10}$$

and

$$\Theta\begin{bmatrix} a+n \\ b+m \end{bmatrix}(z) = e^{2\pi i m\cdot a}\Theta\begin{bmatrix} a \\ b \end{bmatrix}(z), \tag{3.11}$$

describing the (quasi) periodicity properties in z and in the characteristics, with $m,\ n \in \mathbf{Z}^g$.

The theta functions with half integer (and integer) characteristics can be divided into even and odd depending on whether they are symmetric under $z \to -z$. From (3.9) we easily get

$$\Theta\begin{bmatrix} a \\ b \end{bmatrix}(-z) = e^{4\pi i a\cdot b}\Theta\begin{bmatrix} a \\ b \end{bmatrix}(z). \tag{3.12}$$

Then Θ is even if $4a\cdot b$ is even and odd if $4a\cdot b$ is odd. From what follows we will restrict $a,\ b$ to integer and half integer values in R^g. Then every Θ-function has a definite parity.

For these Θ-functions we have the following Riemann Θ-identity [70, 71]

$$\sum_{a,b\in\{0,\frac{1}{2}\}^g} e^{4\pi i(a\cdot b_0 - b\cdot a_0)}\Theta\begin{bmatrix} a+a_0 \\ b+b_0 \end{bmatrix}(x)\Theta\begin{bmatrix} a+a_0 \\ b+b_0 \end{bmatrix}(y)\Theta\begin{bmatrix} a+a_0 \\ b+b_0 \end{bmatrix}(u)\Theta\begin{bmatrix} a+a_0 \\ b+b_0 \end{bmatrix}(v)$$

$$= 2^g\Theta\begin{bmatrix} a_0 \\ b_0 \end{bmatrix}\left(\frac{1}{2}(x+y+u+v)\right)\Theta\begin{bmatrix} a_0 \\ b_0 \end{bmatrix}\left(\frac{1}{2}(x+y-u-v)\right) \tag{3.13}$$

$$\times\,\Theta\begin{bmatrix} a_0 \\ b_0 \end{bmatrix}\left(\frac{1}{2}(x-y+u-v)\right)\Theta\begin{bmatrix} a_0 \\ b_0 \end{bmatrix}\left(\frac{1}{2}(x-y-u+v)\right),$$

The proof of this identity is very easy. One simply calculate the left hand side of (3.13) by using the definition of Θ-function and make a change of summation. The result will be seen to be the right hand side of (3.13).

For fixed value of z one defines the multivalued function on the Riemann surface

$$f(P) = \Theta(z + \int_{P_0}^{P} \omega). \tag{3.14}$$

The Riemann vanishing theorem [70, 71] says that $f(P)$ either vanishes identically or $f(P)$ has g-zeroes $P_1, \cdots, P_g$ satisfying the relation

$$z + \sum_{i=1}^{g}\int_{P_0}^{P} \omega = \Delta, \tag{3.15}$$

where Δ is some vector $\in\ J(\Sigma)^*$ depending on P_0 and the canonical homology basis. Conversely, for all $P_1, \cdots, P_g \in \Sigma_g$, if we define z according to (3.15), then $f(P_i) = 0$. Δ is known as the vector of Riemann constant. The set of points $z \in J(\Sigma)$ for which the theta functions vanishes is a subset of complex codimension one in the Jacobian varity known as the theta divisor. A simple consequence of the Riemann vanishing theorem is that $\Theta(e|\Omega) = 0$ if and only if there exist $g-1$ points in Σ_g such that

$$e = \Delta - \sum_{i=1}^{g-1}\int_{P_0}^{P_i} \omega. \tag{3.16}$$

An important consequence of this theorem is that it allows us to characterize the spin structure of a Riemann surface [98].

---

* $J(\Sigma) = \mathbf{C}^g/L_\tau$, is the Jacobian varity of the Riemann surface. Here $L_\tau = \mathbf{Z}^g + \tau\cdot\mathbf{Z}^g$ is the Jacobian lattice.

There is a one to one correspondence between characteristic a, b where $a_i = 0,\ \frac{1}{2}$, $b_i = 0,\ \frac{1}{2}$ and spin structures.

Let a_0, b_0 be an odd characteristic corresponding to a spin bundle s. We can construct explicitly its holomorphic section. Let us consider the function

$$\Theta[^{a_0}_{b_0}](\int_w^z \omega), \tag{3.17}$$

where z and w are two arbitrary points on the Riemann surface. Keeping w fixed, eq.(3.17) will vanish as a function of z in $g-1$ points $P_1, \cdots, P_{g-1}$. Similarly, as a function of w, keeping z fixed, it will also vanish at the same points. Since the spin structure is odd it also vanishes to first order when z and w coincide. If we now take z and w very close to each other and to one of the P_i's, eq.(3.17) behaves like

$$(z-w)(z-P_i)(w-P_i). \tag{3.18}$$

Thus if we differentiate eq.(3.17) with respect to z and then set $z = w$, we obtain a holomorphic one form

$$h^2(z) = \sum_{A=1}^{g} \omega_A(z)\partial_{u_A}\Theta[^{a_0}_{b_0}](u)|_{u=0}. \tag{3.19}$$

From eq.(3.18) we know that $h^2(z)$ has only double zeroes at the P_i's and therefore we can take its square root.

Then the prime form is define as [71, 70]

$$E(z,w) = \frac{\Theta[^{a_0}_{b_0}](u)}{h(z)h(w)}, \tag{3.20}$$

where

$$u = \int_w^z \omega. \tag{2.21}$$

$E(z,w)$ is a $-\frac{1}{2}$ differential in z and w with only one zero at $z = w$ and $E(z,w) = -E(w,z)$. It is independent of the choice of the odd spin structure $\begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$. From the transformation property of the Riemann theta function (3.10), $E(z,w)$ is single valued when z is moved around the α_i cycles, but when z is moved around the β_i cycles n_i times it transforms as

$$E(z,w) \;\rightarrow\; e^{-i\pi n\cdot\tau\cdot n - 2\pi n\cdot u}\; E(z,w). \tag{3.22}$$

The theta-function and the prime form depend on the basis ω_A of the Abelian differentials and on the period matrix τ, which are fixed once we have chosen a canonical homology basis α_A, β_A. The action of a nontrival diffeomorphism on the homology basis is given by eq.(3.5) and the corresponding transformations of the Abilian differentials and the period matrix are given by eq.(3.7), then

$$\tilde{u} = u \cdot (C\tau + D)^{-1}. \tag{3.23}$$

The transformation rule for the theta-function is [70, 71, 98, 99]

$$\Theta[^a_b](u|\tau) \;\rightarrow\; \Theta[^{a'}_{b'}](\tilde{u}|\tilde{\tau}) = e^{-i\pi\phi(a,b)}\det{}^{-1/2}(C\tau + D)e^{i\pi u\cdot(C\tau+D)^{-1}C\cdot u}\Theta[^a_b](u|\tau), \tag{3.24}$$

where $\phi(a,b)$ is some phase indenpendent of u and τ and the new characteristic $\begin{bmatrix} a' \\ b' \end{bmatrix}$ is related to $\begin{bmatrix} a \\ b \end{bmatrix}$ by

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} + \frac{1}{2}\,\mathrm{diag}\begin{pmatrix} CD^T \\ AB^T \end{pmatrix}. \tag{3.25}$$

Notice that if $\begin{bmatrix} a \\ b \end{bmatrix}$ is an even (odd) characteristic $\begin{bmatrix} a' \\ b' \end{bmatrix}$ is also even (odd) since if $\Theta[^a_b](u|\tau)$ is an even (odd) function of u also $\Theta[^{a'}_{b'}](\tilde{u}|\tilde{\tau})$ is an even (odd) function of u.

The prime form depends on the choice of the homology basis. In fact from the definitions (3.19) and (3.20) we have

$$h^2(z) \;\rightarrow\; \tilde{h}^2(z) = e^{i\pi\phi(a,b)}\det{}^{1/2}(C\tau + D)\omega_i(z)\partial_{u_i}\Theta[^{a'_0}_{b'_0}](u|\tau), \tag{3.26}$$

and the spin structure $\begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$ is related to $\begin{bmatrix} a'_0 \\ b'_0 \end{bmatrix}$ by eq.(3.25). Then

$$\begin{aligned} E^2(z,w) \;\rightarrow\; \tilde{E}^2(z,w) &= \frac{\Theta^2[^{a_0}_{b_0}](\tilde{u}|\tilde{\tau})}{\tilde{h}^2(z)\tilde{h}^2(w)} \\ &= \frac{e^{2\pi i u\cdot(C\tau+D)^{-1}C\cdot u}\Theta^2[^{a'_0}_{b'_0}](u|\tau)}{\omega_i(z)\partial_{u_i}\Theta^2[^{a'_0}_{b'_0}](u|\tau)|_{u=0}\,\omega_j(w)\partial_{u_j}\Theta^2[^{a'_0}_{b'_0}](u|\tau)|_{u=0}}. \end{aligned} \tag{3.27}$$

Since $E(z,w)$ is independence of the particular odd spin structure chosen, the modular transformatiom of E is given by:

$$E(z,w) \;\rightarrow\; \tilde{E}(z,w) = e^{i\pi u\cdot\,(C\tau+D)^{-1}C\cdot\,u}E(z,w). \tag{3.28}$$

To complete the discussion about Riemann surface, we would like to mention a very important identity for the prime form—the Fay's triscecant identity (and its genernization). The generalized identity is [71, 70, 100]

$$\frac{\Theta_s(\sum_{i=1}^n u_i - \sum_{i=1}^n v_i)}{\Theta_s(0)}\frac{\prod_{i<j}^n E(u_i,u_j)E(v_i,v_j)}{\prod_{i,j}^n E(u_i,v_j)} = (-1)^{n(n-1)/2}\det\langle\psi(u_i)\psi(v_j)\rangle_s, \tag{3.29}$$

For $n=2$, we get Fay's triscecant identity.

So far so much for the general theory of Riemann surface. In what follows we will view Riemann surface as an algebraic curve. We will try to derive some explicit formulas for a special class of Riemann surface: the hyperelliptic Riemann surface. For $g=1,\ 2$ all the Riemann surfaces are hyperelliptic. For $g\geq 3$, hyperelliptic surfaces only consist of a measure zero part of the whole moduli space. We will restrict our discussions to $g=2$. The generalization (reduction) to higher (lower) genus hyperelliptic surface is straighforward.

It is a well-known fact that every genus two Riemann surface can be realized as a hyperelliptic surface in CP^2

$$y^2 = \prod_{i=1}^{6}(z-a_i) \tag{3.30}$$

where $a_i (i=1,2,\cdots,6)$ are the six branch points. From (3.30) one readily solves y in terms of z:

$$y = y(z) = \pm\sqrt{\prod_{i=1}^{6}(z-a_i)} \tag{3.31}$$

Then every genus two Riemann surface can be thought of as a double covering of S^2 (the Riemann sphere) with cutting and gluing appropraitly. We will see this in connection with canonical homology basis soon (Fig. 6).

There are two independent holomorphic abelian differentials on a genus two Riemann surface:

$$\Omega_1 = \frac{\mathrm{d}z}{y(z)}, \qquad \Omega_2(z) = \frac{z\mathrm{d}z}{y(z)} \tag{3.32}$$

To see that $\Omega_1(z)$ and $\Omega_2(z)$ are holomorphic differentials, one recalls that the uniformizer coordinate near branch point is u:

$$z - a_i = u^2 \tag{3.33}$$

and the coordinate near infinite point is v:

$$z = \frac{1}{v} \tag{3.34}$$

Set $\Omega(z) = \frac{z-x}{y(z)}dz$, one sees that $\Omega(z)$ has two zeros: one $z = x$ on the upper sheet of S^2, one $z = x$ on the lower sheet. We denote it simply as $z = x\pm$.

On hyperelliptic Riemann surface, spin structures are in one-to-one correspondence with the splitting of the branch point $\{a_i\}$ into two non-intersecting sets $\{A_k\}$ and $\{B_l\}$. In particular, the ten even spin structure (at genus two) are corresponded with the case when both set $\{A_k\}$ and $\{B_l\}$ has exact three elements. If we use the canonical homology basis as shown in Fig.2, the ten even spin structures are calculated to be [13]

$$s_1 \sim \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \{A_k\} = \{a_1 a_2 a_3\},\ \{B_l\} = \{a_4 a_5 a_6\}, \tag{3.35}$$

which is abbreviated as (123 | 456)

$$\begin{aligned} &s_2 \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \sim (124 \mid 356), \quad s_3 \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \sim (125 \mid 346), \quad s_4 \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim (126 \mid 345), \\ &s_5 \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \sim (134 \mid 256), \quad s_6 \sim \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \sim (135 \mid 246), \quad s_7 \sim \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \sim (136 \mid 245), \\ &s_8 \sim \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \sim (145 \mid 236), \quad s_9 \sim \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \sim (146 \mid 235), \quad s_{10} \sim \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sim (156 \mid 234). \end{aligned} \tag{3.36}$$

where the symbol $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ is the standard symbol to denote spin structure in connection with Θ-function with characteritics (see, for example [70, 71, 98]). The ordering of the ten even spin structures is arranged following the convention: $A_1 = a_1, A_2 = a_i, A_3 = a_j$ with $i < j$ and $s_1 < s_2$ if $i_1 \leq i_2$ or $j_1 < j_2$, which has been used in [12-16].

It is easy to see that the holomorphic abelian differentials $\Omega_i(z)$ (eq. (3.32)) are not normalized in the standard way:

$$\oint_{\alpha_i} \omega_j = \delta_{ij}, \qquad \oint_{\beta_i} \omega_j = \tau_{ij} = \tau_{ji} \tag{3.37}$$

where τ is the 2×2 period matrix. In fact, these differentials are related as follows:

$$\Omega_i = \sum_j \omega_j \oint_{\alpha_j} \Omega_i. \tag{3.38}$$

Set $(K)_{ij} = \oint_{\alpha_i} \Omega_j$, we have

$$\Omega_i = \omega_j K_{ji}. \tag{3.39}$$

It is not difficult to solve ω_i in terms of Ω_i:

$$\begin{aligned} \omega_1 &= \frac{K_{22}\Omega_1 - K_{21}\Omega_2}{\det K}, \\ \omega_2 &= \frac{-K_{12}\Omega_1 + K_{11}\Omega_2}{\det K}. \end{aligned} \tag{3.40}$$

In what follows, we give some useful formulae which will be used later. The first one is the Thomae formula [71]

$$\Theta_s^4(0) = \pm \det{}^2 K \prod_{i<j}^{3} A_{ij} B_{ij}, \tag{3.41}$$

where $A_{ij} = A_i - A_j, B_{ij} = B_i - B_j$. Because of the sign ambiguity of the above expression, we will use another quantity Q_s intead of $\Theta_s^4(0)$ [13]:

$$Q_s = \prod_{i<j}^{3} A_{ij} B_{ij}. \tag{3.42}$$

The second formula is

$$\begin{aligned} Q(z,\bar{w}) &= \omega(z) \cdot (\mathrm{Im}\tau)^{-1} \cdot \bar{\omega}(\bar{w}) \\ &= \frac{2}{T} \frac{1}{y(z)\bar{y}(\bar{w})} \int \frac{(z-u)(\bar{w}-\bar{u})}{\mid y(u) \mid^2} \mathrm{d}^2 u, \end{aligned} \tag{3.43}$$

where

$$T = \int \left| \frac{z_1 - z_2}{y(z_1)y(z_2)} \right|^2 d^2 z_1 d^2 z_2 = 2 \mid \det K \mid^2 \det(\mathrm{Im}\tau). \tag{3.44}$$

Finally we have a variational formula [72]

$$\frac{\partial \tau_{ij}}{\partial a_n} = \frac{i\pi}{2} \hat{\omega}_i(a_n) \hat{\omega}_j(a_n), \tag{3.45}$$

where $\hat{\omega}(a_i)$ is defined as follows:

$$\begin{aligned}\omega(z) &= (\hat{\omega}(z_0) + \hat{\omega}'(z_0)(z - z_0) + \cdots)dz \\ &= 2u\omega(u^2 + a_i)du, \\ \hat{\omega}(a_i) &= \lim_{u \to 0} 2u\omega(u^2 + a_i),\end{aligned} \tag{3.46}$$

where one should use the uniformizer coordinate u instead of z near the branch point.

All these formulas can be proved quite easily by explicit computation. The only trick is using the standard formula:

$$\int \omega_i \wedge \bar{\omega}_j = \sum_k \left\{ \oint_{\alpha_k} \omega \oint_{\beta_k} \bar{\omega}_j - (\alpha \longleftrightarrow \beta) \right\} = -2i(\mathrm{Im}\tau)_{ij}. \tag{3.47}$$

and the explicit formulas for τ in terms of K and G: $G = \oint_\beta \Omega$. For example, we have

$$\tau_{12} = \frac{-G_{11}K_{12} + G_{12}K_{11}}{\det K}. \tag{3.48}$$

To conclude this section, we recall another formula which is given by V. G. Knizhnik in [12]:

$$\begin{aligned}\langle \partial X^\mu(x_1) \partial X^\nu(x_2) \rangle = &- g^{\mu\nu} \left\{ \frac{1}{4(x_1 - x_2)^2} + \frac{1}{4T} \cdot \frac{\partial}{\partial x_2} \left\{ \frac{y(x_2)}{y(x_1)} \cdot \frac{1}{x_1 - x_2} \right.\right. \\ &\left.\left. \times \int \frac{(x_1 - z_1)(x_1 - z_2)}{(x_2 - z_1)(x_2 - z_2)} \cdot \left| \frac{z_1 - z_2}{y(z_1)y(z_2)} \right|^2 d^2 z_1 d^2 z_2 \right\} + (x_1 \leftrightarrow x_2) \right\}.\end{aligned} \tag{3.49}$$

We now give a derivation of this quite important formula (see section 5). Because $\langle \partial X^\mu(x_1)$ $\partial X^\nu(x_2) \rangle$ has a double pole when $x_1 = x_2$, and no simple pole, one can postulate the general form of it as:

$$\begin{aligned}\langle \partial X^\mu(x_1) \partial X^\nu(x_2) \rangle &= -g^{\mu\nu} \left\{ \frac{\partial}{\partial x_2} \left[\frac{1}{2(x_1 - x_2)} \left(1 + \frac{y(x_2)}{y(x_1)} \right) \right] + \omega_i(x_1) C_{ij} \omega_j(x_2) \right\} \\ &= -g^{\mu\nu} \left\{ \frac{\partial}{\partial x_2} \left[\frac{1}{2(x_1 - x_2)} \left(1 + \frac{y(x_2)}{y(x_1)} \right) \right] + \Omega_i(x_1) C'_{ij} \Omega_j(x_2) \right\}.\end{aligned} \tag{3.50}$$

Here we include the factor $\left(1 + \frac{y(x_2)}{y(x_1)}\right)$ (instead of the factor 2) to cancell the pole when x_1 and x_2 are on different sheet of the Riemann sphere.

Introducing the following anti-holomorphic $(0,1)$ differantials:

$$
\begin{aligned}
\bar{P}^1(\bar{x}) &= \frac{1}{\bar{y}(\bar{x})}\int\left|\frac{z}{y(z)}\right|^2 d^2z - \frac{\bar{x}}{\bar{y}(\bar{x})}\int\frac{z}{|\,y(z)\,|^2}d^2z, \\
\bar{P}^2(\bar{x}) &= -\frac{1}{\bar{y}(\bar{x})}\int\frac{\bar{z}}{|\,y(z)\,|^2}d^2z + \frac{\bar{x}}{\bar{y}(\bar{x})}\int\frac{1}{|\,y(z)\,|^2}d^2z,
\end{aligned}
\tag{3.51}
$$

we have

$$
\int \bar{P}^i(\bar{x})\Omega_j(x)\mathrm{d}^2x = \frac{1}{2}T\delta^i_j. \tag{3.52}
$$

Then it is easy to find $C'_{ij}\Omega_j(x_2)$ from the holomorphicity of $\langle\partial X(x_1)\partial X(x_2)\rangle$:

$$
\int \bar{P}^i(\bar{x}_1)\langle\partial X(x_1)\partial X(x_2)\rangle\mathrm{d}^2x_1 = 0, \tag{3.53}
$$

i.e. we have

$$
\int \bar{P}^i(\bar{x}_1)\left\{\frac{\partial}{\partial x_2}\left[\frac{1}{2(x_1-x_2)}\left(1+\frac{y(x_2)}{y(x_1)}\right)\right] + \Omega_i(x_1)C'_{ij}\Omega_j(x_2)\right\}d^2x_1 = 0. \tag{3.54}
$$

From the above expression, we get:

$$
C'_{ij}\Omega_j(x_2) = -\frac{1}{T}\int \bar{P}^i(\bar{x}_1)\frac{\partial}{\partial x_2}\left\{\frac{1}{x_1-x_2}\cdot\frac{y(x_2)}{y(x_1)}\right\}d^2x_1. \tag{3.55}
$$

Then

$$
\begin{aligned}
\langle\partial X^\mu(x_1)\partial X^\nu(x_2)\rangle =& -g^{\mu\nu}\left\{\frac{1}{2(x_1-x_2)^2} + \frac{1}{2}\frac{\partial}{\partial x_2}\left[\frac{1}{x_1-x_2}\cdot\frac{y(x_2)}{y(x_1)}\right]\right. \\
&\left. -\frac{1}{T}\Omega_i(x_1)\int \bar{P}^i(\bar{z}_1)\frac{\partial}{\partial x_2}\left[\frac{1}{z_1-x_2}\cdot\frac{y(x_2)}{y(z_1)}\right]d^2z_1\right\} \\
=& -g^{\mu\nu}\left\{\frac{1}{2(x_1-x_2)^2} + \frac{1}{2T}\frac{\partial}{\partial x_2}\int\frac{y(x_2)}{y(x_1)}\cdot\frac{1}{x_1-x_2}\right. \\
&\left.\times\frac{(x_1-z_1)(x_1-z_2)}{(x_2-z_1)(x_2-z_2)}\cdot\left|\frac{z_1-z_2}{y(z_1)y(z_2)}\right|d^2z_1d^2z_2\right\}.
\end{aligned}
\tag{3.56}
$$

Symmetrizing in x_1 and x_2, we get (3.49) as given by V. G. Knizhnik in [12] with minor modification.

From (3.56), one can also write the expression for $\sum_i\alpha_i\langle\partial X(z)X(w_i)\rangle$ with $\sum_i\alpha_i = 0$. It is

$$
\begin{aligned}
\sum_i\alpha_i\langle\partial X^\mu(z)X^\nu(w_i)\rangle =& -g^{\mu\nu}\sum_i\alpha_i\left\{\frac{1}{2(z-w_i)} + \frac{1}{2T}\int\frac{y(w_i)}{y(z)}\cdot\frac{1}{z-w_i}\right. \\
&\left.\times\frac{(z-z_1)(z-z_2)}{(w_i-z_1)(w_i-z_2)}\left|\frac{z_1-z_2}{y(z_1)y(z_2)}\right|d^2z_1d^2z_2\right\},
\end{aligned}
\tag{3.57}
$$

which satisfies

$$\int \bar{P}^i(\bar{z}) \sum_i \alpha_i \langle \partial X^\mu(z) X^\nu(w_i) \rangle d^2 z = 0. \tag{3.58}$$

By differentiating (3.57) with respect to $\bar{w}_i$, we get

$$\begin{aligned}\langle \partial X^\mu(z) \bar{\partial} X^\nu(w) \rangle &= -\pi \delta^{(2)} + \frac{\pi}{T} \cdot \frac{1}{y(z)\bar{y}(\bar{w})} \int \frac{(z-u)(\bar{w}-\bar{u})}{\mid y(u) \mid^2} d^2 u \\ &= -\pi \delta^{(2)}(z-w) + \frac{\pi}{2} \omega(z) \cdot (\mathrm{Im}\tau)^{-1} \cdot \bar{\omega}(\bar{w}).\end{aligned} \tag{3.59}$$

Finally, we would like to give the expression of Sezgö kernel—the propagator of 1/2-differential field ψ. It is [71]

$$\langle \psi(x) \psi(y) \rangle_s = \frac{1}{x-y} \cdot \frac{u(x)+u(y)}{2\sqrt{u(x)u(y)}}, \tag{3.60}$$

where

$$u(x) = \prod_{i=1}^{3} \sqrt{\frac{x-A_i}{x-B_i}}. \tag{3.61}$$

All these formulas will be used later in two-loop computation in superstring theories. That completes our review of the mathematics about genus 2 hyperelliptic Riemann surface. Before plunging into the very details of the two-loop calculations we would like first to do some easy and simple calculations at lower genus. This will be done in the next section.

4. Some Sample Calculations at Low Genus

In this section we present some sample calculations at low genus. First we are going to derive the Virasoro-Shapiro amplitude [73, 74] for the closed bosonic string theory. Second we will derive the tree-level amplitude for type II superstring theory. And finally we will do the one-loop calculations for superstring theories in the hyperelliptic language. The result obtained was proved to be the same as the result obtained in the Θ-function language.

For $g = 0$, there are three holomorphic conformal Killing vectors (zero modes of the operator P, see eq.(2.10)). In the calculation of the amplitude, there is an extra gauge freedom which should be fixed. We can fix three points to some arbitrarily chosen points. The measure for this gauge fixing is

$$\mathrm{d}\mu = \frac{d^2 z_i d^2 z_j d^2 z_k}{|(z_i - z_j)(z_j - z_k)(z_k - z_i)|^2}, \tag{4.1}$$

which is $PSL(2, C) \equiv SL(2, C)/\{1, -1\}$ invariant. Here i, j, k denote any three distinct points among the inserted points $1, \cdots, n$. The n-particle amplitude is then

$$A_n(k_1, \cdots, k_n) = \int \frac{\prod_{i=1}^n d^2 z_i}{\mathrm{d}\mu} \langle \prod_{i=1}^n V(k_i, \epsilon_i, z_i) \rangle, \tag{4.2}$$

where $V(k_i, \epsilon_i, z_i)$ is the vertex operator for the emission of the i-th particle (with polarization tensor ϵ_i and momentum k_i). For the tachyon we have

$$V(k_i, \epsilon_i, z_i) =: e^{ik_i \cdot X(z_i)} : \ , \tag{4.3}$$

with mass shell condition: $k^2 = 2$. For $n = 4$ we get from (4.3) and (4.2) the four-tachyon amplitude

$$\begin{aligned} A_4(k_1, \cdots, k_4) &= \int \frac{\prod_{i=1}^n d^2 z_i}{\mathrm{d}\mu} \exp \left\{ \sum_{i<j} k_i \cdot k_j \langle X(z_i) X(z_j) \rangle \right\} \\ &= \int \frac{\prod_{i=1}^n d^2 z_i}{\mathrm{d}\mu} \prod_{i<j} |z_i - z_j|^{2k_i \cdot k_j}, \end{aligned} \tag{4.4}$$

by using the contraction $\langle X(z_i) X(z_j) \rangle = \ln |z_i - z_j|^2$. This amplitude is independent of the arbitrary points z_i, z_j and z_k chosen. We can choose what we want to simplify the calculation. By choosing $z_2 = 0,\ z_3 = 1$ and $z_4 = \infty$, we get*

$$
\begin{aligned}
A_4(k_1, \cdots, k_4) &= \int d^2 |z_1|^{2k_1\cdot k_2} |1 - z_1|^{2k_1\cdot k_3} \\
&= \pi \frac{\Gamma(-1 - k_1 \cdot k_2 - k_1 \cdot k_3)\Gamma(1 + k_1 \cdot k_2)\Gamma(1 + k_1 \cdot k_3)}{\Gamma(-k_1 \cdot k_2)\Gamma(-k_1 \cdot k_3)\Gamma(2 + k.k - 2 + k_1 \cdot k_3)} \\
&= \pi \frac{\Gamma\left(-1 - \frac{s}{2}\right)\Gamma\left(-1 - \frac{t}{2}\right)\Gamma\left(-1 - \frac{u}{2}\right)}{\Gamma\left(2 + \frac{s}{2}\right)\Gamma\left(2 + \frac{t}{2}\right)\Gamma\left(2 + \frac{u}{2}\right)},
\end{aligned}
\tag{4.6}
$$

which is the manifestly dual amplitude of Shapiro [74] and Virasoro [73]. Here we have used the definition of the Mandelstam variables for the four-particle amplitude: $s = -(k_1+k_2)^2$, $t = -(k_3 + k_2)^2$ and $u = -(k_1 + k_3)^2$.

The four-particle amplitude in type II superstring theory can be calculated similarly. By using the super coordinate notation $\mathbf{z} = (z, \theta)$ and $\mathrm{d}\mathbf{z} = \mathrm{d}z + \theta \mathrm{d}\theta$, the invariant measure is now as follows [47]

$$
\mathrm{d}\mu = \frac{\mathrm{d}z_i \mathrm{d}z_j \mathrm{d}z_k \mathrm{d}\theta_i \mathrm{d}\theta_j \mathrm{d}\theta_k}{(z_{ij} z_{jk} z_{ki})^{1/2}} \delta,
\tag{4.7}
$$

where $\delta = \frac{z_{ij}\theta_k + z_{jk}\theta_i + z_{ki}\theta_j + \theta_i\theta_j\theta_k}{(z_{ij}z_{jk}z_{ki})^{1/2}}$. The measure $\mathrm{d}\mu$ is invariant under the superconformal group which is isomorphic to the complexified $OSp(1,1)$—the superconformal extension of $PSL(2,C)$. The four-particle amplitude is then

$$
A_4(k_1, \cdots, k_4) = \int \frac{\prod_{i=1}^4 d^2 z_i d^2 \theta_i}{\mathrm{d}\mu \mathrm{d}\bar{\mu}} \langle \prod_{i=1}^4 V(k_i, \epsilon_i, \mathbf{z}_i) \rangle.
\tag{4.8}
$$

For graviton the emission vertex is given by

$$
V(k_i, \epsilon_i, \mathbf{z}_i) = \epsilon_i^{\mu\nu} D\mathbf{X}_\mu(\mathbf{z}_i) \bar{D}\mathbf{X}_\nu(\mathbf{z}_i) e^{ik\cdot \mathbf{X}(\mathbf{z}_i)}.
\tag{4.9}
$$

The contractions needed in the calculation of the amplitude are†:

$$
\begin{aligned}
&\langle \mathbf{X}(\mathbf{z}_i)\mathbf{X}(\mathbf{z}_j) \rangle = \ln |z_{ij}|^2 - \frac{\theta_{ij}}{z_{ij}}, \\
&D_j \langle \mathbf{X}(\mathbf{z}_i)\mathbf{X}(\mathbf{z}_j) \rangle = \frac{\theta_{ij}}{z_{ij}}, \qquad \bar{D}_j \langle \mathbf{X}(\mathbf{z}_i)\mathbf{X}(\mathbf{z}_j) \rangle = \frac{\bar{\theta}_{ij}}{\bar{z}_{ij}}.
\end{aligned}
\tag{4.10}
$$

---

* Rememner the integration formula

$$
\int d^2 z z^a \bar{z}^{a'} (1 - z)^b (1 - \bar{z})^{b'} = \pi \frac{\Gamma(1 + a)\Gamma(1 + b)\Gamma(-1 - a' - b')}{\Gamma(2 + a + b)\Gamma(-a')\Gamma(-b')}.
\tag{4.5}
$$

† $D_j = \frac{\partial}{\partial \theta_j} + \theta_j \frac{\partial}{\partial z_j}$.

It seems an easy task to carry out the calculation to lead to the four-particle amplitude:

$$A_4(k_1,\cdots,k_4) = \epsilon_1^{\mu\mu'}\epsilon_2^{\nu\nu'}\epsilon_3^{\rho\rho'}\epsilon_4^{\sigma\sigma'}K_{\mu\nu\rho\sigma}K_{\mu'\nu'\rho'\sigma'}\frac{\Gamma(-\frac{s}{2})\Gamma(-\frac{t}{2})\Gamma(-\frac{u}{2})}{\Gamma(1+\frac{s}{2})\Gamma(1+\frac{t}{2})\Gamma(1+\frac{u}{2})}, \tag{4.11}$$

first obtained by Green and Schwarz in 1982 [1, 2]. Let us try.

First we would like to set some z_i and θ_i to some specifically chosen points in order to simplify the algebra of calculation. We set $z_2 = 0$, $z_3 = 1$, $z_4 = \infty$ and $\theta_3 = \theta_4 = 0$. We have then*

$$\begin{aligned}
\delta &= \theta_2, \\
\mathrm{d}\mu &= \frac{\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_4\mathrm{d}\theta_2\mathrm{d}\theta_3\mathrm{d}\theta_4}{|z_{23}z_{34}z_{42}|^{1/2}}\theta_2 \\
&= \frac{\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_4\mathrm{d}\theta_3\mathrm{d}\theta_4}{|z_{23}z_{34}z_{42}|^{1/2}}.
\end{aligned} \tag{4.12}$$

The computations of the correlation functions in $\langle\prod_{i=1}^4 V(k_i,\epsilon_i,\mathbf{z}_i)\rangle$ simplify a lot by using this choice. We have

$$\begin{aligned}
\langle\prod_{i=1}^4 V(k_i,\epsilon_i,\mathbf{z}_i)\rangle =& \epsilon_1^{\mu_1\nu_1}\epsilon_2^{\mu_2\nu_2}\epsilon_3^{\mu_3\nu_3}\epsilon_4^{\mu_4\nu_4}\langle\{(\psi_{\mu_1}+\theta_1\partial X_{\mu_1})(\psi_{\mu_2}+\theta_2\partial X_{\mu_2}) \\
&\psi_{\mu_3}\psi_{\mu_4}\times(\mathrm{C.\ C},\ \mu\to\nu)\}\prod_{i=1}^4 e^{ik_i\cdot\mathbf{X}(\mathbf{z}_i)}\rangle.
\end{aligned} \tag{4.13}$$

After integration over $d^2\theta_1 d^2\theta_2$ we get

$$\begin{aligned}
\int\langle\prod_{i=1}^4 V(k_i,\epsilon_i,\mathbf{z}_i)\rangle d^2\theta_1 d^2\theta_2 =& \epsilon_1^{\mu_1\nu_1}\epsilon_2^{\mu_2\nu_2}\epsilon_3^{\mu_3\nu_3}\epsilon_4^{\mu_4\nu_4}\langle\{(\partial X_{\mu_1}+ik_1\cdot\psi\psi_{\mu_1}) \\
&(\partial X_{\mu_2}+ik_2\cdot\psi\psi_{\mu_2})\psi_{\mu_3}\psi_{\mu_4}\times(\mathrm{C.\ C},\ \mu\to\nu)\}\prod_{i=1}^4 e^{ik_i\cdot X(z_i)}\rangle.
\end{aligned} \tag{4.14}$$

To simplify the writing we write $\epsilon_i^{\mu_i\nu_i} = \epsilon_i^{\mu_i}\bar{\epsilon}_i^{\nu_i}$. By using the Wick theorem the contractions

* We don't care the over all multiplicative factor.

in (4.14) can be easily carried out. We get

$$
\begin{aligned}
&\int\langle\prod_{i=1}^{4}V(k_i,\epsilon_i,\mathbf{z}_i)\rangle d^2\theta_1 d^2\theta_2\\
&=\Big\{\langle\partial(\epsilon_1\cdot X(z_1))\sum_i ik_i\cdot X(z_i)\rangle\langle ik_2\cdot\psi\epsilon_2\cdot\psi(z_2)\epsilon_3\cdot\psi(z_3)\epsilon_4\cdot\psi(z_4)\rangle\\
&\quad+\langle\partial(\epsilon_2\cdot X(z_2))\sum_i ik_i\cdot X(z_i)\rangle\langle ik_1\cdot\psi\epsilon_1\cdot\psi(z_1)\epsilon_3\cdot\psi(z_3)\epsilon_4\cdot\psi(z_4)\rangle\\
&\quad+\langle\partial(\epsilon_1\cdot X(z_1))\partial(\epsilon_2\cdot X(z_2))\rangle\langle\epsilon_3\cdot\psi(z_3)\epsilon_4\cdot\psi(z_4)\rangle\\
&\quad+\langle\partial(\epsilon_1\cdot X(z_1))\sum_i ik_i\cdot X(z_i)\rangle\langle\partial(\epsilon_2\cdot X(z_2))\sum_i ik_i\cdot X(z_i)\rangle\langle\epsilon_3\cdot\psi(z_3)\epsilon_4\cdot\psi(z_4)\rangle\\
&\quad+\langle\partial(\epsilon_1\cdot X(z_1))\sum_i ik_i\cdot X(z_i)\rangle\langle ik_1\cdot\psi\epsilon_1\cdot\psi(z_1)ik_2\cdot\psi\epsilon_2\cdot\psi(z_2)\epsilon_3\cdot\psi(z_3)\epsilon_4\cdot\psi(z_4)\rangle\Big\}\\
&\quad\times(\text{C.C.},\ \epsilon\to\tilde{\epsilon})\exp\Big[\sum_{i<j}k_i\cdot k_j\langle X(z_i)X(z_j)\rangle\Big].
\end{aligned}
\tag{4.15}
$$

The contractions $\langle X(z_i)X(z_j)\rangle$ etc. can be read from eq.(4.10):

$$
\begin{aligned}
&\langle X(z_i)X(z_j)\rangle=\ln|z_i-z_j|^2,\\
&\langle\partial X^\mu(z_i)X^\nu(z_j)\rangle=-g^{\mu\nu}\frac{1}{z_i-z_j},\\
&\langle\partial X^\mu(z_i)\partial X^\nu(z_j)\rangle=-g^{\mu\nu}\frac{1}{(z_i-z_j)^2},\\
&\langle\psi^\mu(z_i)\psi^\nu(z_j)\rangle=-g^{\mu\nu}\frac{1}{z_i-z_j}.
\end{aligned}
\tag{4.16}
$$

By using the above formulas and keeping only the leading terms in $\frac{1}{z_4}$ ($z_4\to\infty$), the quantiaty in the big curly parenthesis in eq.(4.15) is

$$
\begin{aligned}
RA=&-\frac{1}{z_4}\Big\{\frac{1}{z_1}\{\epsilon_1\cdot k_2(\epsilon_3\cdot k_2\epsilon_2\cdot\epsilon_4-\epsilon_4\cdot k_2\epsilon_2\cdot\epsilon_3)-\epsilon_4\cdot k_1(\epsilon_1\cdot k_2\epsilon_2\cdot\epsilon_3-\epsilon_1\cdot\epsilon_2\epsilon_3\cdot k_2)\\
&+k_1\cdot k_2\epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3-\epsilon_2\cdot k_1\epsilon_1\cdot\epsilon_4\epsilon_3\cdot k_2-\epsilon_3\cdot\epsilon_4\epsilon_1\cdot k_2\epsilon_2\cdot k_3\}\\
&-\frac{1}{1-z_1}\{\epsilon_1\cdot k_3(\epsilon_3\cdot k_2\epsilon_2\cdot\epsilon_4-\epsilon_4\cdot k_2\epsilon_2\cdot\epsilon_3)+\epsilon_2\cdot k_3(\epsilon_3\cdot k_1\epsilon_1\cdot\epsilon_4-\epsilon_1\cdot\epsilon_3\epsilon_4\cdot k_1)\\
&-\epsilon_1\cdot k_3\epsilon_2\cdot k_3\epsilon_3\cdot\epsilon_4\}+\frac{1}{z_1^2}(1-k_1\cdot k_2)\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4\\
&-\frac{1}{z_1(1-z_1)}\{\epsilon_2\cdot k_1(\epsilon_3\cdot k_1\epsilon_1\cdot\epsilon_4-\epsilon_4\cdot k_1\epsilon_1\cdot\epsilon_3)-\epsilon_2\cdot k_1\epsilon_1\cdot k_3\epsilon_3\cdot\epsilon_4\\
&+k_1\cdot k_2\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4-\epsilon_2\cdot k_1\epsilon_1\cdot\epsilon_3\epsilon_4\cdot k_2-\epsilon_3\cdot k_1(\epsilon_1\cdot k_2\epsilon_2\cdot\epsilon_4-\epsilon_1\cdot\epsilon_2\epsilon_4\cdot k_2)\}\Big\}.
\end{aligned}
\tag{4.17}
$$

The next problem is to do the integration voer z_1. This can be easily done with the help of the integration formula (4.5). We have

$$\begin{aligned}
&\int d^2 z \frac{1}{(1-z)\bar{z}} |z|^{2k_1\cdot k_2} |1-z|^{2k_1\cdot k_3} = k_1\cdot k_2 \frac{1}{k_1\cdot k_3} \int d^2 z |z|^{2k_1\cdot k_2 - 2} |1-z|^{2k_1\cdot k_3}, \\
&\int d^2 z \frac{1}{z} |z|^{2k_1\cdot k_2 - 2} |1-z|^{2k_1\cdot k_3} = \frac{k_1\cdot k_2 k_1\cdot k_4}{1-k_1\cdot k_2} \frac{1}{k_1\cdot k_3} \int d^2 z |z|^{2k_1\cdot k_2 - 2} |1-z|^{2k_1\cdot k_3}, \\
&\int d^2 z \frac{1}{(1-z)} |z|^{2k_1\cdot k_2 - 2} |1-z|^{2k_1\cdot k_3} = -k_1\cdot k_4 \frac{1}{k_1\cdot k_3} \int d^2 z |z|^{2k_1\cdot k_2 - 2} |1-z|^{2k_1\cdot k_3}, \\
&\int d^2 z |z|^{2k_1\cdot k_2 - 2} |1-z|^{2k_1\cdot k_3} = -\pi (k_1\cdot k_3)^2 \frac{\Gamma(-\frac{s}{2})\Gamma(-\frac{t}{2})\Gamma(-\frac{u}{2})}{\Gamma(1+\frac{s}{2})\Gamma(1+\frac{t}{2})\Gamma(1+\frac{u}{2})}.
\end{aligned} \tag{4.18}$$

Of course we have used the mass shell condition: $k^2 = 0$ and momentum conservation law: $\sum_{i=1}^{4} k_i = 0$. The Mandelstam variables s, t and u are the same as defind before. By using eq.(4.18) we get finally the four-particle amplitude in type II superstring theories

$$A_4(k_1, \cdots, k_4) = K \cdot \tilde{K} \frac{\Gamma(-\frac{s}{2})\Gamma(-\frac{t}{2})\Gamma(-\frac{u}{2})}{\Gamma(1+\frac{s}{2})\Gamma(1+\frac{t}{2})\Gamma(1+\frac{u}{2})}, \tag{4.19}$$

where the kinematic factors K and $\tilde{K}$ are calculated to be:

$$\begin{aligned}
K = & -\frac{1}{4}(st\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4 + su\epsilon_2\cdot\epsilon_3\epsilon_1\cdot\epsilon_4 + tu\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4) \\
& + \frac{1}{2}s(\epsilon_1\cdot k_4\epsilon_3\cdot k_2\epsilon_2\cdot\epsilon_4 + \epsilon_2\cdot k_3\epsilon_4\cdot k_1\epsilon_1\cdot\epsilon_3 + \epsilon_1\cdot k_3\epsilon_4\cdot k_2\epsilon_2\cdot\epsilon_3 + \epsilon_2\cdot k_4\epsilon_3\cdot k_1\epsilon_1\cdot\epsilon_4) \\
& + \frac{1}{2}t(\epsilon_2\cdot k_1\epsilon_4\cdot k_3\epsilon_1\cdot\epsilon_3 + \epsilon_3\cdot k_4\epsilon_1\cdot k_2\epsilon_2\cdot\epsilon_4 + \epsilon_2\cdot k_4\epsilon_1\cdot k_3\epsilon_3\cdot\epsilon_4 + \epsilon_3\cdot k_1\epsilon_4\cdot k_2\epsilon_1\cdot\epsilon_2) \\
& + \frac{1}{2}u(\epsilon_1\cdot k_2\epsilon_4\cdot k_3\epsilon_2\cdot\epsilon_3 + \epsilon_3\cdot k_4\epsilon_2\cdot k_1\epsilon_1\cdot\epsilon_4 + \epsilon_1\cdot k_4\epsilon_2\cdot k_3\epsilon_1\cdot\epsilon_4 + \epsilon_3\cdot k_2\epsilon_4\cdot k_1\epsilon_1\cdot\epsilon_2), \\
\tilde{K} = & K(\epsilon\to\bar{\epsilon}),
\end{aligned} \tag{4.20}$$

which are the same as obtained before [1, 2] by different method. That completes our calculation of the four-particle tree amplitude for type II superstring theory.

The last calculation which I would like to show in this section is the one-loop amplitudes in superstring theories. Here the complications come mainly from the summation over spin structures. There is really a great deal of literature on one loop calculations. Here we will do the calculations in hyperelliptic formalism and prove that the results obtained are identical with those of Θ-function formalism [1, 2]. In hyperelliptic language the genus 1

Riemann surface are realized as as algebraic curve in CP^2:

$$y^2 = \prod_{i=1}^{4}(z - a_i), \tag{4.21}$$

where a_i $(i = 1, \cdots, 4)$ are the four branch points. By choosing a canonical homology basis of cycles as shown in fig.7, we have the following list of three even spin structures (at one loop):

$$s_2 \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sim (12 \mid 34), \quad s_3 \sim \begin{bmatrix} 0 \\ 0 \end{bmatrix} \sim (13 \mid 24), \quad s_4 \sim \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sim (14 \mid 23). \tag{4.22}$$

By using the Thomae formula we obtain the modular invariant Riemann identity [59]

$$\Theta_2^4 - \Theta_3^4 + \Theta_4^4 = 0, \tag{4.23}$$

or

$$a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0, \tag{4.24}$$

where $a_{ij} = a_i - a_j$. This fixes the phases in the summation over spin structures.

The n-particle amplitudes in superstring theories are computed as follows

$$A_n(k_1, \cdots, k_n) = \int \mathrm{d}\mu \prod_{i=1}^{n} d^2 z_i \sum_s \eta_s Q_s \langle \prod_{i=1}^{n} V(k_i, \epsilon_i, z_i) \rangle_s, \tag{4.25}$$

where the measure $\mathrm{d}\mu$ is (in terms of the branch points):

$$\mathrm{d}\mu = \frac{1}{T^6} \frac{\prod_{i=1}^{n} d^2 a_i / \mathrm{d}V_{pr}}{|\prod_{i<j} a_{ij}|^4}, \tag{4.26}$$

for type II superstring theory or

$$\mathrm{d}\mu = \frac{1}{T^6} \frac{\prod_{i=1}^{n} d^2 a_i / \mathrm{d}V_{pr}}{\prod_{i<j} a_{ij}^2 \bar{a}_{ij}^3}, \tag{4.27}$$

for heterotic string theory. Here $T = \int \frac{d^2 z}{|y(z)|^2}$ and $\mathrm{d}V_{pr} = \frac{d^2 a_i d^2 a_j d^2 a_k}{|a_{ij} a_{jk} a_{ki}|^2}$. In (4.25) the summation over spin structures was written only for the right part. Appropriate factor of $Q_{s'}$ and the summation over s' should also be included for the left part. For graviton amplitude we have

$$V(k_i, \epsilon_i, z_i) = \{(\partial(\epsilon_i \cdot X) + i k_i \cdot \psi \epsilon_i \cdot \psi(z_i)) \times (\text{left part})\} e^{i k_i \cdot X(z_i)}. \tag{4.28}$$

By substituting the above expression into (4.25) we get

$$RA_n = \sum_s \eta_s Q_s \langle \prod_{i=1}^{n} \{(\partial(\epsilon_i \cdot X) + ik_i \cdot \psi \epsilon_i \cdot \psi(z_i)) \times (\text{left part})\} e^{ik_i \cdot X(z_i)} \rangle \\ = \text{various terms.} \tag{4.29}$$

Let us compute the various terms in RA_n.

For $n < 4$ the maximal number of $\langle \psi\psi \rangle$ contraction is three. For example we have a term (only the spin structure dependent part)

$$\begin{aligned} RA1 &= \sum_s \eta_s Q_s \langle \psi(z_1)\psi(z_2) \rangle_s^2 \\ &= \sum_s \eta_s Q_s \left(\frac{1}{z_1 - z_2} \frac{u(z_1) + u(z_2)}{2\sqrt{u(z_1)u(z_2)}} \right)^2 \\ &= \frac{1}{4(z_1 - z_2)^2} \sum_s \eta_s Q_s \left(2 + \frac{u(z_1)}{u(z_2)} + \frac{u(z_2)}{u(z_1)} \right) = 0, \end{aligned} \tag{2.30}$$

by explicit calculation. Let us see the term involing the maximal number of $\langle \psi\psi \rangle$ contraction:

$$\begin{aligned} RA2 = &\sum_s \eta_s Q_s \langle \psi(z_1)\psi(z_2) \rangle_s \langle \psi(z_2)\psi(z_3) \rangle_s \langle \psi(z_3)\psi(z_1) \rangle_s \\ = &\frac{1}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)} \\ &\times \sum_s \eta_s Q_s \frac{(u(z_1) + u(z_2))(u(z_2) + u(z_3))(u(z_3) + u(z_1))}{u(z_1)u(z_2)u(z_3)} = 0. \end{aligned} \tag{4.31}$$

So all the terms are zero after summation over spin structures. This is what we expected from the nonrenormalization theorem. For $n = 4$ the only nonvanishing terms are those with four $\langle \psi\psi \rangle$ contraction. We have

$$\begin{aligned} RA_4 = &(\text{kinematic factor}) \times \sum_s \eta_s Q_s \langle \psi(z_1)\psi(z_2) \rangle_s^2 \langle \psi(z_3)\psi(z_4) \rangle_s^2 \\ &\times (\text{left part}) \times \langle \prod_{i=1}^{4} e^{ik_i \cdot X(z_i)} \rangle \\ = &(\text{kinematic factor}) \times \frac{\prod_{i<j}^{4} a_{ij}}{\prod_{i=1}^{4} y(z_i)} \times (\text{left part}) \times \exp\big[\sum_{i<j} k_i \cdot k_j \langle X(z_i)X(z_j) \rangle\big]. \end{aligned} \tag{4.32}$$

The full amplitude (for type II superstring theory) is then

$$A_4(k_1,\cdots,k_4)=K\int\frac{1}{T^6}\frac{\prod_{i=1}^{n}d^2a_i/\mathrm{d}V_{pr}}{|\prod_{i<j}a_{ij}|^2}\prod_{i=1}^{4}\frac{d^2z_i}{|y(z_i)|^2}\exp\big[\sum_{i<j}k_i\cdot k_j\langle X(z_i)X(z_j)\rangle\big], \quad (4.33)$$

where K is the kinematic factor. This result is identical with the one obtained in the Θ-function formalism [1, 2]

$$A_4(k_1,\cdots,k_4)=K\int_F\frac{d^2\tau}{(\mathrm{Im}\tau)^6}\prod_{i=1}^{4}d^2z_i'\exp\big[\sum_{i<j}k_i\cdot k_j\langle X(z_i')X(z_j')\rangle\big], \quad (4.34)$$

This can be proved by using the following transformations:

$$\begin{aligned}\frac{\partial\tau}{\partial a_i}&=\frac{i\pi}{2}\omega(a_i)\omega(a_i),\\ \frac{\mathrm{d}z_i}{Ky(z_i)}&=\mathrm{d}z_i'\\ T=|K|^2\mathrm{Im}\tau,&\quad\omega(z)=\frac{1}{Ky(z)},\end{aligned} \quad (4.35)$$

which are relations proved for $g=2$, but can be similarly proved for $g=1$. That completes our calculations in lower genus cases. Now we start to do computations at two loops.

5. Two-Loop Calculations – Nonrenormalization Theorem

Since odd spin structures give trivially no contributions to the n-particle amplitudes up to $n = 5$, we shall consider only even ones. From (2.2), (2.30) and (2.32), we know the expression for two-loop n-particle amplitude in HST for a given choice of the spin structures:

$$A^n_{ss'} = \int \mathrm{d}\mu(m_i, \bar{m}_i) \prod_{i=1}^{N} d^2 z_i (\mathrm{det}\mathrm{Im}\tau)^{-5} \bar{L}_{s'} R_s. \tag{5.1}$$

To get the right amplitude, we have to perform the sum over all spin structures:

$$\begin{aligned} A^n &= \int \mathrm{d}\mu(m_i, \bar{m}_i) \prod_{i=1}^{N} d^2 z_i (\mathrm{det}\mathrm{Im}\tau)^{-5} \sum_{ss'} \eta_s \varphi_{s'} \bar{L}_{s'} R_s \\ &= \int \mathrm{d}\mu(m_i, \bar{m}_i) \prod_{i=1}^{N} d^2 z_i (\mathrm{det}\mathrm{Im}\tau)^{-5} \sum_{s'} \varphi_{s'} \bar{L}_{s'} \sum_{s} \eta_s R_s, \end{aligned} \tag{5.2}$$

where η_s and φ_s are phases.

To be specific, we consider the gauge boson vertex in HST. That is, we take the following form of $V(k, \epsilon, z)$:

$$\begin{aligned} V &= V_R \cdot V_L, \\ V_R &= \{\partial(\epsilon \cdot X) + ik \cdot \psi \epsilon \cdot \psi\} e^{ik \cdot X}, \\ V_L &= \lambda^I \lambda^J, \end{aligned} \tag{5.3}$$

where λ^I are the left moving (i.e. antiholomorphic) two dimenional spinors. Then we have

$$\begin{aligned} R^n_s &= \int D[X\psi bc\beta\gamma] e^{-(S[x,\psi] + S_{gh}[b,c,\beta,\gamma])} \prod_{i=1}^{n} V_R(k_i, \epsilon_i, z_i) \\ &\times \Big\{ \prod_{a=1}^{2} \delta(\langle \chi_a, \beta \rangle) \langle \chi_a, J \rangle \prod_{j=1}^{3} \langle \mu_i, b \rangle + \sum_{j=1}^{3} \prod_{i \neq j} \langle \mu^i, b \rangle \\ &\times \big[\langle \frac{\partial \chi_1}{\partial m_j}, \beta \rangle \delta(\langle \chi_1, \beta \rangle) \langle \chi_2, J \rangle \delta(\langle \chi_2, \beta \rangle) + (1 \leftrightarrow 2) \big] \Big\}, \\ \bar{L}^n_{s'} &= \int D[\lambda \bar{b} \bar{c}] e^{-(S[\lambda] + S_{gh}[\bar{b}, \bar{c}])} \prod_{j=1}^{3} \langle \bar{\mu}^j, \bar{b} \rangle \prod_{i=1}^{n} V_L(\bar{z}_i), \end{aligned} \tag{5.4}$$

where s refers to the spin structure of ψ, β and γ*; s' refers to the spin structure of λ^I; $\eta_i, i = 1,2,3$ and $\chi_a, a = 1,2$ are Beltrami and super Beltrami differentials respectively. All the scalar products are defined as

$$\langle \chi_a, \beta \rangle = \int d^2 z \chi_a \beta, \quad \text{etc.} \tag{5.5}$$

and

$$J(z) = \psi \cdot \partial X + 2c\partial\beta - \gamma b + 3\partial c \beta, \tag{5.6}$$

is the total supercurrent. In eq.(5.4) we have assumed that the metric is independent of supermoduli but allowed the super Beltrami differentials χ_a to depend on moduli. Due to the local world-sheet supersymmtry, there is a freedom in choosing χ_a and different choices are related by total derivatives in moduli space. In the following we shall make the choice that χ_a are δ-functions located in moduli independent points $x_a (a = 1,2)$ on the Riemann surface and μ^i are also δ-functions located in b_i. In particular we make the convienient choice [13] of taking $x_{1,2}$ to be the zeroes of a holomorphic abelian differential $\Omega(z) = \frac{z-x}{y(z)} dz$ to simplify computations. Then $x_{1,2} = x\pm$, i.e. the two corresponding points in the upper and lower Riemann sheet. Then eq.(5.4) simplifies to the form [13]:

$$R_s = (\det{}'\bar\partial_1)^{-5} (\det{}'\bar\partial_2)(\det\bar\partial_{1/2})^5_s (\det{}'\bar\partial_{3/2})^{-1}_s \frac{\langle J(x_1)J(x_2)\prod_{i=1}^N V_R(k_i,\epsilon_i,z_i)\rangle_s}{\det\varphi^a(x_b)}, \tag{5.7}$$

where $\varphi^a(z)$ are the holomorphic $\frac{3}{2}$-differentials and $\langle J(x_1)J(x_2)\cdots\rangle_s$ denotes the normalized correlator (the spin structure dependent part in R_s):

$$\langle J(x_1)J(x_2)\prod_{i=1}^n V_R(k_i,\epsilon_i,z_i)\rangle_s \equiv \frac{\ll \prod_{a=1}^2 J(x_a \delta(\beta(x_a)) \prod_{i=1}^3 b(b_i) \prod_{i=1}^N V_R(k_i,\epsilon_i,z_i) \gg_s}{\ll \prod_{a=1}^2 \delta(\beta(x_a)) \prod_{i=1}^3 b(b_i) \gg_s}, \tag{5.8}$$

where the double bracket $\ll \cdots \gg$ indicates the functional integration over all the right fields (including X also). The chiral determinats appearing in (5.7) are coming from the path integration over all the matter fields X, ψ and ghost fields b, c, β and γ. These determinats (for hyperelliptic surface) can be calculated by the method of conformal field theory [101]. This calculation was presented in appendix A. The results are (for genus 2

* They should have the same spin structure.

hyperelliptic surface and in an obvious notation):

$$
\begin{aligned}
\det{}_s\bar{\partial}_{1/2} &= \prod_{i<j}^{6}(a_{ij})^{-\frac{1}{8}}\left(\prod_{i<j}^{6}A_{ij}B_{ij}\right)^{\frac{1}{4}}, \\
\det{}_s\bar{\partial}_{3/2} &= \prod_{i<j}^{6}(a_{ij})^{\frac{3}{8}}\left(\prod_{i<j}^{6}A_{ij}B_{ij}\right)^{\frac{1}{4}}, \\
\det\bar{\partial}_2 &= \prod_{i<j}^{6}(a_{ij})^{\frac{5}{4}}, \quad \det\bar{\partial}_0 = \det K\prod_{i<j}^{6}(a_{ij})^{\frac{1}{4}},
\end{aligned}
\tag{5.9}
$$

where K is the periods of the differentials Ω_A (see eq.(3.32)) along the α cycles (see eq.(3.39)). By using these results we get

$$
(\det{}'\bar{\partial}_1)^{-5}(\det{}'\bar{\partial}_2)(\det\bar{\partial}_{1/2})^5_s(\det{}'\bar{\partial}_{3/2})^{-1}_s = \frac{\prod_{i<j}^6 A_{ij}B_{ij}}{(\det K)^5\prod_{i<j}^6 a_{ij}} = \frac{Q_s}{(\det K)^5\prod_{i<j}^6 a_{ij}}. \tag{5.10}
$$

So the spin structure dependent factor is just Q_s which is defined before (see eq.(3.42)). What left are the computations of correlation functions in $\langle J(x_1)J(x_2)\cdots\rangle$ and the summation over spin structures

$$
\begin{aligned}
R^n_s &= \frac{1}{(\det K)^5\prod_{i<j}^6 a_{ij}}\Lambda^n_s, \\
\Lambda^n_s &\equiv \langle J(x_1)J(x_2)\prod_{i=1}^{n}V_R(k_i,\epsilon_i,z_i)\rangle_s Q_s.
\end{aligned}
\tag{5.11}
$$

To begin with, let us first consider the case $n=0$, i.e. the vacuum amplitude. Using the explicit form of the supercurrent, we can represent Λ^0 as a sum of a matter part

$$
\Lambda^0_m = \langle\psi_\mu(x_1)\psi_\nu(x_2)\rangle_s\langle\partial X^\mu(x_1)\partial X^\nu(x_2)\rangle_s Q_s, \tag{5.12}
$$

and a ghost part

$$
\Lambda^0_{gh} = \langle J_{gh}(x_1)J_{gh}(x_2)\rangle_s Q_s, \tag{5.13}
$$

where $J_{gh} = J - \psi\cdot\partial X$. Here we will not do the calculations which have been done in [13] and only recall the relevant results which will be used in the present and the next section. The details can be found in appendix B. We have (following the notation of [13]):

$$
\begin{aligned}
\Lambda^0_{gh} = \Big\{&-2\partial_2 P(x_1x_2)R(x_1x_2) - 2P(x_2x_1)R(x_2x_1)\frac{\partial\varphi_1(x_1)}{\varphi_1(x_1)} \\
&-(\partial_2 R(x_2x_1) + 2\Lambda(x_2)R(x_2x_1))\frac{\partial\varphi_2(x_1)}{\varphi_2(x_2)} - (1\longleftrightarrow 2)\Big\}\,Q_s.
\end{aligned}
\tag{5.14}
$$

From [13] we know that all the terms in Λ^0_{gh} can be reduced, up to spin structure independent factors, to $\langle\psi(x_1)\psi(x_2)\rangle_s Q_s$ and $\langle\partial\psi(x_1)\psi(x_2)\rangle_s Q_s$. Then the summation over spin structures was reduced to the calculation of the following two expressions:

$$
\begin{aligned}
Ex_1 &= \sum_s \eta_s Q_s \langle\psi(x_1)\psi(x_2)\rangle_s, \\
Ex_2 &= \sum_s \eta_s Q_s \langle\partial\psi(x_1)\psi(x_2)\rangle_s.
\end{aligned}
\tag{5.15}
$$

Roughly speaking, these two expressions should be modular invariant (in the sense explained below) if the amplitude is modular invariant. To see how the requirement of modular invariance completely determined the phases η_s, let us recall first how modular transformation is realized in the hyperelliptic language.

A modular transformation is just a change of canonical homology basis chosen. In hyperelliptic formalism a change of canonical homology basis can be induced by a permutation of the branch points. So a modular transformation simply corresponds to a permutation of the six branch points $a_i, i = 1, 2, \cdots, 6$ in hyperelliptic language. What modular invariance means is that* $\sum_s \eta_s R_s$ should be invariant invariant under all the permutations of a_i's. From (5.11) we see that $\sum_s \eta_s \Lambda_s$ [13] should be antisymmetric for every interchange $a_i \longleftrightarrow a_j, i \neq j$. Reasoning along this line we get finally that Ex_1 and Ex_2 should be also antisymmetric for every interchange of the branch points. This is a sufficient but not a necessary condition ensuring the modular invariance of the (vacuum) amplitude at two loops.

To proceed further let us take $x_{1,2} = \infty\pm$. Then we have

$$
\begin{aligned}
\langle\psi(x_1)\psi(x_2)\rangle &= \frac{1}{4}\sum_{i=1}^{3}(A_i - B_i), \\
\langle\partial\psi(x_1)\psi(x_2)\rangle &= \frac{1}{8}\sum_{i=1}^{3}(A_i^2 - B_i^2).
\end{aligned}
\tag{5.16}
$$

---

* Strictly speaking this is not quite true because of the prefactor $(\det \mathrm{Im}\tau)^{-5}$ in eqs.(5.1) and (5.2) which is not modular invariant. There is a factor $(\det K)^{-5}$ in R_s which is also not modular invariant but spin structure independent. These two factors will combined (with also another factor of $(\det\bar{K})^{-5}$ coming from the left part $\bar{L}_{s'}$) to give a modular invariant expression: $(\det \mathrm{Im}\tau)^{-5}(\det K)^{-5}(\det\bar{K})^{-5} = \left(\frac{1}{2}T\right)^{-5}$. See eq.(3.44).

Then the two expressions Ex_1 and Ex_2 are polynominal functions of the six branch points $a_i, i = 1, \cdots, 6$. Good! Starting from one term, for example $\eta_1 Q_1(a_1+a_2+a_3-a_4-a_5-a_6)$ in Ex_1 or $\eta_1 Q_1(a_1^2+a_2^2+a_3^2-a_4^2-a_5^2-a_6^2)$ in Ex_2, all the rest terms can be generated by permutations of the six branch points by imposing of the modular invariance (in the sense of that whenever we interchange two branch points we get a minus sign explained before). By taking $\eta = 1$ we get the following unique determination of all the phases

$$\eta_1 = -\eta_2 = \eta_3 = -\eta_4 = \eta_5 = -\eta_6 = \eta_7 = \eta_8 = -\eta_9 = \eta_{10} = 1. \tag{5.17}$$

Then we can easily prove that

$$\begin{aligned} Ex_1 &= \frac{1}{4}\sum_s \eta_s Q_s \sum_{i=1}^{3}(A_i - B_i) = 0, \\ Ex_2 &= \frac{1}{8}\sum_s \eta_s Q_s \sum_{i=1}^{3}(A_i^2 - B_i^2) = 0. \end{aligned} \tag{5.18}$$

The argument goes as follows (which is quite useful in what follows in the proof of non-renormalization theorem and the calculation of the four-particle amplitude). Because these expressions are homogeneous polynominal (of degree 7 and 8 respectively) in a_i, they should be proportional to $P(a) = \prod_{i<j}(a_i - a_j) \equiv \prod_{i<j} a_{ij}$ (a homogeneous polynominal of degree 15 in a_i). One sees immediately that the powers of a_i cann't be matched. So Ex_1 and Ex_2 must vanish. That completes our proof of the vanishing of the vacuum amplitude at two loop for superstring theory*.

To prove nonrenormalization theorem at two-loops, we have to study the following quantities

$$\Lambda^n = \sum_s \eta_s Q_s \langle \prod_{i=1}^{n} V_R(k_i, \epsilon_i, z_i) J(x_1) J(x_2) \rangle_s, \tag{5.19}$$

obtained from (5.11) by summing over spin structires. Substituting (5.3) into (5.19), we can calculate Λ^n by using the Wick theorem. We consider first the contractions of ψ which are relevant for the summation over spin structures. There are two types of contractions:

* At thi point I would like to mention some early works about two-loop calculations for superstrings [102-106, 90]. Nevertheless all of these works pay little attentions to modular invariance which is very important to detemine uniquely the phases and to ensure the vanishing of the cosmological constant as we see here.

Type A: contractions $\langle J(x\pm)\psi(z_i)\rangle_s$ appear;

Type B: only the contraction $\langle J(x+)J(x-)\rangle_s$ appears,

$$\text{i.e. } \Big\langle \prod_{i=1}^{n} V_R(k_i,\epsilon_i,z_i)\Big\rangle_s \langle J(x_1)J(x_2)\rangle_s, \tag{5.20}$$

Let us compute Λ_s^3:

$$\Lambda_s^3 = \langle \prod_{i=1}^{3}\{\epsilon_i\cdot\partial X(z_i) + ik_i\cdot\psi(z_i)\epsilon_i\cdot\psi(z_i)\}e^{ik_i\cdot X(z_i)}\cdot J(x_1)J(x_2)\rangle_s Q_s. \tag{5.21}$$

We have

$$\Lambda_s^3 = \{A_s + B_s + C_s\}Q_s, \tag{5.22}$$

where

$$\begin{aligned}
A_s =& \Big\langle \epsilon_1\cdot\partial X(z_1)\epsilon_2\cdot\partial X(z_2)\epsilon_3\cdot\partial X(z_3)\prod_{i=1}^{3} e^{ik_i\cdot X(z_i)}J(x_1)J(x_2)\Big\rangle_s,\\
B_s =& -i\Big\langle\big\{\epsilon_1\cdot\partial X(z_1)\epsilon_2\cdot\partial X(z_2)\epsilon_3\cdot\psi(z_3)k_3\cdot\psi(z_3) + (1\leftrightarrow 3) + (2\leftrightarrow 3)\big\}\\
&\times\prod_{i=1}^{3} e^{ik_i\cdot X(z_i)}J(x_1)J(x_2)\Big\rangle_s,\\
C_s =& -\Big\langle\big\{\epsilon_1\cdot\partial X(z_1)\epsilon_2\cdot\psi(z_2)k_2\cdot\psi(z_2)\epsilon_3\cdot\psi(z_3)k_3\cdot\psi(z_3) + (1\leftrightarrow 2) + (1\leftrightarrow 3)\big\}\\
&\times\prod_{i=1}^{3} e^{ik_i\cdot X(z_i)}J(x_1)J(x_2)\Big\rangle_s,\\
D_s =& i\Big\langle\epsilon_1\cdot\psi(z_1)k_1\cdot\psi(z_1)\epsilon_2\cdot\psi(z_2)k_2\cdot\psi(z_2)\epsilon_3\cdot\psi(z_3)k_3\cdot\psi(z_3)\prod_{i=1}^{3} e^{ik_i\cdot X(z_i)}J(x_1)J(x_2)\Big\rangle_s.
\end{aligned} \tag{5.23}$$

For $\sum_s \eta_s Q_s A_s$, one sees that it is similar to R_s up to spin structure independent factor because $X(z,\bar{z})$ is independent of spin structure s. Then we have

$$\sum_s \eta_s Q_s A_s = 0, \tag{5.24}$$

by using eq.(5.18).

As to B_s, C_s and D_s, they lead to the following various spin structure dependent factors:

$$
\begin{aligned}
&E_s = \langle\psi(x_1)\psi(z_1)\rangle_s\langle\psi(z_1)\psi(x_2)\rangle_s, \\
&E_s = \langle\psi(x_1)\psi(z_1)\rangle_s\langle\psi(z_1)\psi(z_2)\rangle_s\langle\psi(z_2)\psi(x_2)\rangle_s, \\
&E_s = \langle\psi(z_1)\psi(z_2)\rangle_s^2 \times \begin{cases} \langle\psi(x_1)\psi(x_2)\rangle_s, \\ \langle\partial\psi(x_1)\psi(x_2)\rangle_s, \end{cases} \\
&E_s = \langle\psi(x_1)\psi(z_1)\rangle_s\langle\psi(z_1)\psi(z_2)\rangle_s\langle\psi(z_2)\psi(z_3)\rangle_s\langle\psi(z_3)\psi(x_2)\rangle_s, \\
&E_s = \langle\psi(z_1)\psi(z_2)\rangle_s\langle\psi(z_2)\psi(z_3)\rangle_s\langle\psi(z_3)\psi(z_1)\rangle_s \times \begin{cases} \langle\psi(x_1)\psi(x_2)\rangle_s, \\ \langle\partial\psi(x_1)\psi(x_2)\rangle_s, \end{cases} \\
&E_s = \langle\psi(z_1)\psi(z_2)\rangle_s^2\langle\psi(x_1)\psi(z_3)\rangle_s\langle\psi(z_3)\psi(x_2)\rangle_s.
\end{aligned}
\tag{5.25}
$$

Setting $x_{1,2} = \infty\pm$ and recalling (3.60) and (3.61):

$$
\langle\psi(z_1)\psi(z_2)\rangle_s = \frac{1}{z_1 - z_2}\cdot\frac{u(z_1)+u(z_2)}{2\sqrt{u(z_1)u(z_2)}}, \qquad u(z) = \prod_{i=1}^{3}\sqrt{\frac{z-A_i}{z-B_i}}\ , \tag{5.26}
$$

we get

$$
\begin{aligned}
&\langle\psi(z_1)\psi(x_1)\rangle_s = \frac{u(z_1)-1}{2\sqrt{u(z_1)}}, \qquad &&\langle\psi(z_2)\psi(x_2)\rangle_s = \frac{u(z_2)+1}{2\sqrt{u(z_2)}}, \\
&\langle\psi(x_1)\psi(x_2)\rangle = \frac{1}{4}\sum_{i=1}^{3}(A_i - B_i), \qquad &&\langle\partial\psi(x_1)\psi(x_2)\rangle = \frac{1}{8}\sum_{i=1}^{3}(A_i^2 - B_i^2).
\end{aligned}
\tag{5.27}
$$

By using these relations we see that

1). $\sum_s \eta_s E1_s Q_s = 0$ leads to the following identity

$$
\sum_s \eta_s \left\{ u(z) - \frac{1}{u(z)} \right\} Q_s = 0. \tag{5.28}
$$

2). $\sum_s \eta_s E2_s Q_s = 0$ leads to the identity

$$
\sum_s \eta_s \left\{ \frac{u(z_1)}{u(z_2)} - \frac{u(z_2)}{u(z_1)} \right\} Q_s = 0, \tag{5.29}
$$

and if (IV.23) is true.

3). $\sum_s \eta_s E3_s Q_s = 0$ leads to the following identities

$$
\sum_s \eta_s Q_s \left\{ \frac{u(z_1)}{u(z_2)} + \frac{u(z_2)}{u(z_1)} \right\} \sum_{i=1}^{3}(A_i^n - B_i^n) = 0, \quad n = 1, 2, \tag{5.30}
$$

where we have used (5.18).

4). $\sum_s \eta_s E4_s Q_s = 0$ if (5.28) is true.

5). $\sum_s \eta_s E5_s Q_s = 0$ if (5.30) is true.

6). $\sum_s \eta_s E6_s Q_s = 0$ leads to

$$\sum_s \eta_s \left\{ \frac{u(z_1)}{u(z_2)u(z_3)} - \frac{u(z_2)u(z_3)}{u(z_1)} \right\} Q_s = 0, \tag{5.31}$$

and if (5.28) is true.

So if we want to show that $\sum_s \eta_s \Lambda_s^3 = 0$ is true, it is sufficient (but not necessary) to show that (5.28)—(5.31) are true. What we will show below is that this is really the case. All these identities (called Lianzi identities in [14,15]) are true. In fact, (5.31) implies (5.28) and (5.29) as it can be easily seen by setting $z_1 = z_3$ and $z_3 = \infty$ respectively. Moreover, we have one more general identity:

$$\sum_s \eta_s \left\{ \frac{u(z_1)u(z_2)}{u(z_3)u(z_4)} - \frac{u(z_3)u(z_4)}{u(z_1)u(z_2)} \right\} Q_s = 0. \tag{5.32}$$

From this identity, one can easily derive (5.31) by settin $z_4 = \infty$.

All these identities (5.28) — (5.32) can be proved quite easily. Let us see, for example, (5.29). Substituting $u(z_1)$ and $u(z_2)$ by (5.26) into (5.29), we have:

$$\begin{aligned}
\text{LHS of } (5.29) &= \sum_s \eta_s \left\{ \prod_{i=1}^{3} \sqrt{\frac{(z_1 - A_i)(z_2 - B_i)}{(z_2 - A_i)(z_1 - B_i)}} - (z_1 \leftrightarrow z_2) \right\} Q_s \\
&= \sum_s \eta_s \frac{\prod_{i=1}^{3}(z_1 - A_i)(z_2 - B_i) - (z_1 \leftrightarrow z_2)}{\sqrt{\prod_{i=1}^{6}(z_1 - a_i)(z_2 - a_i)}} Q_s \\
&= \frac{1}{y(z_1)y(z_2)} \sum_s \eta_s \left\{ \prod_{i=1}^{3}(z_1 - A_i)(z_2 - B_i) - (A \leftrightarrow B) \right\} Q_s.
\end{aligned} \tag{5.33}$$

An important point is that this expression is modular invariant in the sense of that whenever we interchange a_i and a_j $(i \neq j)$ we get a minus sign for this expression. So this expression should be proportional to $P(a)$. By simple power counting, one sees that $\sum_s \eta_s (\prod_{i=1}^{3}(z_1 - A_i)(z_2 - B_i) - (A \leftrightarrow B)) \cdot Q_s$ is a homogeneous polynominal of degree $6 + 6 = 12$ in a_i and z_j. But the degree of $P(a)$ is 15. So we must have

$$\sum_s \eta_s \left\{ \prod_{i=1}^{3}(z_1 - A_i)(z_2 - B_i) - (A \leftrightarrow B) \right\} Q_s = 0. \tag{5.34}$$

That completes the proof of (5.29) (and also of (5.28) by setting $z_2 = \infty$). This same argument can also be used to prove (5.30). We have

$$\text{LHS of (5.30)} = \frac{1}{y(z_1)y(z_2)} \sum_s \eta_s Q_s \left\{ \prod_{i=1}^{3} (z_1 - A_i)(z_2 - B_i) + (A \leftrightarrow B) \right\} \sum_{i=1}^{3} (A_i^n - B_i^n). \tag{5.35}$$

One easily sees that this expression is also modular invariant and should be proportional to $P(a)$. But the degree of $\sum_s \eta_s Q_s \Big\{ \prod_{i=1}^3 (z_1 - A_i)(z_2 - B_i) + (A \leftrightarrow B) \Big\} \sum_{i=1}^3 (A_i^n - B_i^n)$ is at most 14 ($n = 1, 2$). So we must have

$$\sum_s \eta_s Q_s \Big\{ \prod_{i=1}^{3} (z_1 - A_i)(z_2 - B_i) + (A \leftrightarrow B) \Big\} \sum_{i=1}^{3} (A_i^n - B_i^n) = 0, \quad n = 1, 2, \tag{5.36}$$

To prove (5.32), one follows the same strategy as above. We have

$$\begin{aligned} \text{LHS of (5.32)} = \frac{1}{\prod_{i=1}^4 y(z_i)} \sum_s \eta_s \Bigg\{ & \prod_{i=1}^{3} (z_1 - A_i)(z_2 - A_i) \\ & \times (z_3 - B_i)(z_4 - B_i) - (A \leftrightarrow B) \Bigg\} Q_s. \end{aligned} \tag{5.37}$$

Notice that when $z_1 = z_3$ or z_4, or $z_2 = z_3$ or z_4, (5.32) is true because of (5.29) (which has been proved). Then the last factor $\sum_s \eta_s (\cdots) Q_s$ in (5.37) should be proportional to $P(a)(z_1 - z_3)(z_1 - z_4)(z_2 - z_3)(z_2 - z_4)$: a homogeneous polynominal of degree 19 in a_i and z_j. But the degree of $\sum_s \eta_s (\cdots) Q_s$ is $3 \times 4 + 6 = 18$. We have then

$$\sum_s \eta_s \Big\{ \prod_{i=1}^{3} (z_1 - A_i)(z_2 - A_i)(z_3 - B_i)(z_4 - B_i) - (A \leftrightarrow B) \Big\} Q_s = 0. \tag{5.38}$$

In summary, we have proved the following Lianzi identities:

$$\begin{aligned} & \sum_s \eta_s \left\{ u(z) - \frac{1}{u(z)} \right\} Q_s = 0, \\ & \sum_s \eta_s \left\{ \frac{u(z_1)}{u(z_2)} - \frac{u(z_2)}{u(z_1)} \right\} Q_s = 0, \\ & \sum_s \eta_s \left\{ \frac{u(z_1)}{u(z_2)u(z_3)} - \frac{u(z_2)u(z_3)}{u(z_1)} \right\} Q_s = 0, \\ & \sum_s \eta_s \left\{ \frac{u(z_1)u(z_2)}{u(z_3)u(z_4)} - \frac{u(z_3)u(z_4)}{u(z_1)u(z_2)} \right\} Q_s = 0, \\ & \sum_s \eta_s Q_s \left\{ \frac{u(z_1)}{u(z_2)} + \frac{u(z_2)}{u(z_1)} \right\} \sum_{i=1}^{3} (A_i^n - B_i^n) = 0, \quad n = 1, 2. \end{aligned} \tag{5.39}$$

Using this set of identities, one readily proves the nonrenormalization theorem, i.e. the vanishing of all the n-particle amplitude up to $n = 3$. We have

$$\sum_s \eta_s \Lambda_s^n \propto \sum_s \eta_s R_s^n = 0, \quad n = 1, 2, 3. \tag{5.40}$$

For example, for $n = 1$ we have

$$\begin{aligned}\sum_s \eta_s \Lambda_s^2 =& \sum_s \eta_s \langle J(x_1) J(x_2) V_R(k, \epsilon, z)\rangle_s Q_s \\ =& F_1' \sum_s \eta_s \langle \psi(x_1)\psi(x_2)\rangle_s Q_s + F_s' \sum_s \langle \partial\psi(x_1)\psi(x_2)\rangle_s Q_s \\ &+ F'' \sum_s \eta_s \langle \psi(x_1)\psi(z)\rangle_s \langle \psi(z)\psi(x_2)\rangle_s Q_s \\ =& -\frac{F''}{4} \sum_s \eta_s \left\{ u(z) - \frac{1}{u(z)} \right\} Q_s = 0.\end{aligned} \tag{5.41}$$

To conclude this section, we would like to prove the following summation formula:

$$\begin{aligned}&\sum_s \eta_s \left\{ \frac{u(z_1)u(z_2)}{u(z_3)u(z_4)} + \frac{u(z_3)u(z_4)}{u(z_1)u(z_2)} \right\} \sum_{i=1}^{3} (A_i^n - B_i^n) Q_s \\ =& \frac{2P(a)(z_1 - z_3)(z_1 - z_4)(z_2 - z_3)(z_2 - z_4)}{\prod_{i=1}^4 y(z_i)} \times \begin{cases} 1 & n = 1, \\ \sum_{i=1}^{6} a_i - \sum_{k=1}^{4} z_k & n = 2. \end{cases}\end{aligned} \tag{5.42}$$

which has been used in [15] in the calculation of the four-particle amplitude.

For $n = 1$, we have

$$\begin{aligned}\text{LHS of (5.42)} =& \frac{1}{\prod_{i=1}^4 y(z_i)} \sum_s \eta_s \left\{ \prod_{i=1}^{3} (z_1 - A_i)(z_2 - A_i)(z_3 - B_i)(z_4 - B_i) + \right. \\ & \left. + (A \leftrightarrow B) \right\} \sum_{i=1}^{3} (A_i - B_i) Q_s \\ =& \frac{1}{\prod_{i=1}^4 y(z_i)} \times (\text{a homogeneous polynominal of degree 19 in } a_i \text{ and } z_j).\end{aligned} \tag{5.43}$$

From Lianzi identities (5.30), this expression vanishes when $z_1 = z_3$ or z_4, or $z_2 = z_3$ or z_4. It should be proportional to $(z_1 - z_3)(z_1 - z_4)(z_2 - z_3)(z_2 - z_4)$. It is also modular invariant and should be proportional to $P(a)$. Then we have

$$\sum_s \eta_s (\cdots) \sum_{i=1}^{3} (A_i - B_i) Q_s = \frac{cP(a)(z_1 - z_3)(z_1 - z_4)(z_2 - z_3)(z_2 - z_4)}{\prod_{i=1}^4 y(z_i)}, \tag{5.44}$$

where c is a constant and can be calculated to be: $c = 2$.

For $n = 2$, we should have

$$
\begin{aligned}
&\sum_s \eta_s(\cdots) \sum_{i=1}^{3} (A_i^2 - B_i^2) Q_s \\
&\quad = \frac{cP(a)(z_1 - z_3)(z_1 - z_4)(z_2 - z_3)(z_2 - z_4)}{\prod_{i=1}^{4} y(z_i)} \left\{ \sum_{i=1}^{6} a_i + F(z_i) \right\},
\end{aligned}
\tag{5.45}
$$

from the above experience (and power counting). Here $F(z_i)$ is a linear function of z_i (without constant term). From the symmetry of the original expression, we have $F(z_i) = a \sum_i z_i$ and $a = -1$ (by explicit computation). That completes the proof of (5.42). Let us now turn to the computation of four-particle amplitude which is presumably non-vanishing.

6. Two-Loop Calculations – Four-Particle Bosonic Amplitude

In this section we are going to compute the following expression

$$\begin{aligned}\Lambda^4 &= \sum_s \eta_s \langle \prod_{i=1}^4 V_R(k_i, \epsilon_i, z_i) \cdot J(x_1)J(x_2)\rangle_s Q_s \\ &= \sum_s \eta_s \langle \prod_{i=1}^4 k_i \cdot \psi(z_i)\epsilon_i \cdot \psi(z_i) e^{ik\cdot X(z_i} \cdot J(x_1)J(x_2)\rangle_s Q_s,\end{aligned} \tag{6.1}$$

because of nonrenormalization theorem which was proved in the last section.

First, we want to show that Type A contractions give zero contributions, i.e. we have

$$\sum_s \eta_s(\cdots)\langle \psi^\mu(x_1)\psi^\nu(z_i)\rangle_s \langle \psi^\rho(x_2)\psi^\sigma(z_j)\rangle_s Q_s = 0. \tag{6.2}$$

In fact, all the type A contractions are the following kinds:

$$\begin{aligned} A4_S &= \langle\psi(x_1)\psi(z_1)\rangle_s\langle\psi(z_1)\psi(z_2)\rangle_s\langle\psi(z_2)\psi(z_3)\rangle_s\langle\psi(z_3)\psi(z_4)\rangle_s\langle\psi(z_4)\psi(x_2)\rangle_s, \\ B4_s &= \langle\psi(z_1)\psi(z_2)\rangle_s\langle\psi(z_2)\psi(z_3)\rangle_s\langle\psi(z_3)\psi(z_1)\rangle_s\langle\psi(x_1)\psi(z_4)\rangle_s\langle\psi(z_4)\psi(x_2)\rangle_s, \\ C4_s &= \langle\psi(z_1)\psi(z_2)\rangle_s^2\langle\psi(x_1)\psi(z_3)\rangle_s\langle\psi(z_3)\psi(z_4)\rangle_s\langle\psi(z_4)\psi(x_2)\rangle_s, \end{aligned} \tag{6.3}$$

or sometimes the expressions permutated among z_1, z_2, z_3 and z_4. By using the explicit formulas of $\langle\psi(z_1)\psi(z_2)\rangle_s$ etc., one readily shows that

$$\sum_s \eta_s A4_s Q_s = \sum_s \eta_s B4_s Q_s = \sum_s \eta_s C4_s Q_s = 0, \tag{6.4}$$

by using the Lianzi identities (5.39). Here one should use the more general identity (5.32) which is not needed in the verification of nonrenormalization theorem.

Using (6.2), we have

$$\Lambda^4 = \sum_s \eta_s \langle \prod_{i=1}^4 k_i \cdot \psi(z_i)\epsilon_i \cdot \psi(z_i)\rangle_s \langle J(x_1)J(x_2) \prod_{i=1}^4 e^{ik\cdot X(z_i)}\rangle_s Q_s. \tag{6.5}$$

By using the summation formula (5.42) derived in the last section, we have

$$\begin{aligned} &\sum_s \eta_s \langle\psi(z_1)\psi(z_2)\rangle_s\langle\psi(z_2\psi(z_3)\rangle_s\langle\psi(z_3)\psi(z_4)\rangle_s\langle\psi(z_4)\psi(z_1)\rangle_s \sum_{i=1}^3 (A_i^n - B_i^n)Q_s \\ &= \sum_s \eta_s \langle\psi(z_1)\psi(z_2)\rangle_s^2 \langle\psi(z_3)\psi(z_4)\rangle_s^2 \sum_{i=1}^3 (A_i^n - B_i^n)Q_s \\ &= \frac{P(a)}{8\prod_{i=1}^4 y(z_i)} \times \begin{cases} 1 & n=1, \\ \sum_{i=1}^6 a_i - \sum_{k=1}^4 z_k & n=2. \end{cases} \end{aligned} \tag{6.6}$$

Then one can do the summation over spin structures in (6.5). We have

$$\begin{aligned}
&\sum_s \eta_s \langle \prod_{i=1}^{4} k_i \cdot \psi(z_i) \epsilon_i \cdot \psi(z_i) \rangle_s \sum_{i=1}^{3} (A_i^n - B_i^n) Q_s \\
&= \frac{P(a)}{4 \prod_{i=1}^{4} y(z_i)} \cdot K(k,\epsilon) \times \begin{cases} 1 & n=1, \\ \sum_{i=1}^{6} a_i - \sum_{k=1}^{4} z_k & n=2. \end{cases}
\end{aligned} \tag{6.7}$$

where the kinematic factor $K(k,\epsilon)$ is computed to be:

$$\begin{aligned}
K = &-\frac{1}{4}(st\epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 + su\epsilon_2 \cdot \epsilon_3 \epsilon_1 \cdot \epsilon_4 + tu\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4) \\
&+ \frac{1}{2} s(\epsilon_1 \cdot k_4 \epsilon_3 \cdot k_2 \epsilon_2 \cdot \epsilon_4 + \epsilon_2 \cdot k_3 \epsilon_4 \cdot k_1 \epsilon_1 \cdot \epsilon_3 + \epsilon_1 \cdot k_3 \epsilon_4 \cdot k_2 \epsilon_2 \cdot \epsilon_3 + \epsilon_2 \cdot k_4 \epsilon_3 \cdot k_1 \epsilon_1 \cdot \epsilon_4) \\
&+ \frac{1}{2} t(\epsilon_2 \cdot k_1 \epsilon_4 \cdot k_3 \epsilon_1 \cdot \epsilon_3 + \epsilon_3 \cdot k_4 \epsilon_1 \cdot k_2 \epsilon_2 \cdot \epsilon_4 + \epsilon_2 \cdot k_4 \epsilon_1 \cdot k_3 \epsilon_3 \cdot \epsilon_4 + \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_2 \epsilon_1 \cdot \epsilon_2) \\
&+ \frac{1}{2} u(\epsilon_1 \cdot k_2 \epsilon_4 \cdot k_3 \epsilon_2 \cdot \epsilon_3 + \epsilon_3 \cdot k_4 \epsilon_2 \cdot k_1 \epsilon_1 \cdot \epsilon_4 + \epsilon_1 \cdot k_4 \epsilon_2 \cdot k_3 \epsilon_1 \cdot \epsilon_4 + \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_1 \epsilon_1 \cdot \epsilon_2),
\end{aligned} \tag{6.8}$$

which coincides with the standard kinematic factor at tree and one-loop level [1, 2] (see eq.(4.20)).

Recalling the relevant computations done in [13] and appendix B, we have

$$\begin{aligned}
I(x) &\equiv \langle J(x+) J(x-) \prod e^{ik\cdot X} \rangle \\
&= -\frac{1}{4} g^{\mu\nu} \langle \partial X_\mu(x+) \partial X_\nu(x-) \prod e^{ik\cdot X} \rangle + \langle \prod e^{ik\cdot X} \rangle I_{gh}(x),
\end{aligned} \tag{6.9}$$

where $I_{gh}(x) = \langle J_{gh}(x+) J_{gh}(x-) \rangle$ is the contribution from the ghost part. See (5.14) for explicit expression.

The various factors appearing in (6.9) can be calculated following [13] by Taylor expansion. The calculations are tedious and sometimes very complicated but straightforward. We only give the results of all the calculation for completeness.

1): x_1, x_2 are the zeroes of $\Omega_1(z) = \frac{dz}{y(z)}$, $x_{1,2} = \infty\pm$;

2):

$$\begin{aligned}
&P(x,y) \equiv \frac{1}{\Omega_2(x)} \langle \psi(x) \psi(y) \rangle \Omega_2(y), \\
&P(x_1, x_2) = -\langle \psi(x_1) \psi(x_2) \rangle = -P(x_2, x_1), \\
&\partial_1 P(x_2, x_1) = -\partial_2 P(x_1, x_2) = \langle \partial\psi(x_1) \psi(x_2) \rangle - \Lambda(x_2) \langle \psi(x_1) \psi(x_2) \rangle,
\end{aligned} \tag{6.10}$$

where $\Lambda(x)$ is the finite part of $P(x,y)$ when $y \to x$ and $\Lambda(x_1) = \Lambda(x_2) = -\frac{1}{2}\sum_{i=1}^{6} a_i$;

3):

$$\begin{aligned}
&\varphi_i(x) = \pm\Omega_1(x)\langle\psi(x)\psi(x_i)\rangle, \qquad \varphi_i(x_j) = \delta_{ij}, \quad i = 1,2, \\
&\partial\varphi_2(x_1) = -\partial\varphi_1(x_2) = -\langle\psi(x_1)\psi(x_2)\rangle, \\
&\partial\varphi_1(x_1) = \partial\varphi_2(x_2) = \frac{1}{2}\sum_{i=1}^{6} a_i;
\end{aligned} \tag{6.11}$$

4):

$$\begin{aligned}
&R(xy) = -\langle c(y)b(x)\prod_{j=1}^{3} b(b_j)\rangle \quad R(x_2x_1) = -\frac{1}{4}(\sum_{i=1}^{6} a_i - 2\sum_{i=1}^{3} b_i) + \frac{1}{2}\Sigma(b), \\
&\partial_2 R(x_2x_1) = -\frac{1}{16}(5\sum_{i=1}^{6} a_i^2 + 6\sum_{i<j}^{6} a_ia_j) + \frac{1}{2}\sum_{i=1}^{6} a_i \sum_{i=1}^{3} b_i \\
&\qquad - \frac{1}{2}\sum_{i<j}^{3} b_ib_j + \frac{1}{4}\left\{\sum_{i=1}^{6} a_i\Sigma(b) - 2\Sigma'(b)\right\};
\end{aligned} \tag{6.12}$$

where

$$\begin{aligned}
\Sigma(b) &= \frac{y(b_1)}{(b_1 - b_2)(b_1 - b_3)} + (123 \to 231) + (123 \to 312), \\
\Sigma'(b) &= \frac{(b_2 + b_3)y(b_1)}{(b_1 - b_2)(b_1 - b_3)} + (123 \to 231) + (123 \to 312);
\end{aligned} \tag{6.13}$$

5):

$$\langle\psi(x_1)\psi(x_2)\rangle = \frac{1}{4}\sum_{i=1}^{3}(A_i - B_i), \qquad \langle\partial\psi(x_1)\psi(x_2)\rangle = \frac{1}{8}\sum_{i=1}^{3}(A_i^2 - B_i^2), \tag{6.14}$$

where b_i are the locations of the Beltrami differentials.

Putting the above together and doing some computation, we get

$$\begin{aligned}
I_{gh} = &-\frac{1}{8}(\sum_{i=1}^{6} a_i - 2\sum_{i=1}^{3} b_i)\sum_{i=1}^{3}(A_i^2 - B_i^2) \\
&-\frac{1}{32}(\sum_{i=1}^{6} a_i^2 - 2\sum_{i<j}^{6} a_ia_j + 8\sum_{i<j}^{3} b_ib_j)\sum_{i=1}^{3}(A_i - B_i).
\end{aligned} \tag{6.15}$$

The factors $\sum_i(A_i^2 - B_i^2)$ and $\sum_i(A_i - B_i)$ appearing in (6.15) will have to be substituted by $\sum_i a_i - \sum_k z_k$ and 1 respectively in (6.9) due to the summation formula (5.42).

By making use of the formula (3.49), we have

$$\langle \partial X^\mu(x_1)\partial X^\nu(x_2)\rangle = \frac{g^{\mu\nu}}{2T}\int\Bigg\{-\frac{1}{8}\sum_{i=1}^{6}a_i^2+\frac{1}{4}\sum_{i<j}^{6}a_ia_j+u_1^2+u_1u_2+u_2^2- \\ -\frac{1}{2}(u_1+u_2)\sum_{i=1}^{6}a_i\Bigg\}\left|\frac{u_1-u_2}{y(u_1)y(u_2)}\right|^2 d^2u_1d^2u_2 \\ =\frac{g^{\mu\nu}}{2}\left\{-\frac{1}{8}\sum_{i=1}^{6}a_i^2+\frac{1}{4}\sum_{i<j}^{6}a_ia_j+\sum_{i=1}^{6}a_i^3\frac{\partial}{\partial a_i}\ln(T\prod_{j=1}^{6}a_j)\right\}. \tag{6.16}$$

Putting all these results together and doing some algebraic calculation, we have*

$$I(x=\infty)=-\frac{1}{2}\sum_{i<j}^{6}a_ia_j-\frac{1}{4}\sum_{i<j}^{3}b_ib_j+\frac{1}{4}\sum_{i=1}^{6}a_i\sum_{i=1}^{3}b_i \\ +\frac{1}{8}(\sum_{i=1}^{6}a_ia_j-2\sum_{i=1}^{3}b_i)\sum_{k=1}^{4}z_k-\frac{5}{4}\sum_{i=1}^{6}a_i^3\frac{\partial}{\partial a_i}\ln(T\prod_{j=1}^{6}a_j), \tag{6.17}$$

where we write only the leading terms when $k\to 0$, i.e. we put $\prod e^{ik\cdot X}\to 1$ in (6.9). We see that, for generic b_i, $I(x)$ is symmetric for the permutations of a_i, i.e. it is modular invariant. In the following we will take $a_{1,2,3}$ to be the moduli, i.e. our integration variables over moduli, and therefore we fix $b_i=a_i$ for $i=1,2,3$.

Notice that we have presented the expression for I(x) when $x_{1,2}=\infty\pm$, i.e. the zeros of Ω_1. To get the generic case $x_{1,2}=x\pm$, one simply performs a Möbius transformation: $z,a\longrightarrow -\frac{1}{(z-x)},-\frac{1}{(a-x)}$. Then

$$\prod_{i=1}^{4}dz_i\Omega(z_i)I(x)=\prod_{i=1}^{4}dz_k\frac{z_k-x}{y(z_k)}\cdot\Bigg\{-\frac{1}{2}\sum_{i<j}^{6}\frac{1}{a_i-x}\frac{1}{a_j-x}+\frac{1}{4}\sum_{i=1}^{6}\frac{1}{a_i-x}\sum_{i=1}^{3}\frac{1}{b_i-x} \\ -\frac{1}{4}\sum_{i<j}^{3}\frac{1}{b_i-x}\frac{1}{b_j-x}+\frac{1}{8}(\sum_{i=1}^{6}\frac{1}{a_i-x}-2\sum_{i=1}^{3}\frac{1}{b_i-x})\sum_{i=1}^{4}\frac{1}{z_i-x}+\frac{5}{4}\sum_{i=1}^{6}\frac{1}{a_i-x}\frac{\partial}{\partial a_i}\ln T\Bigg\}. \tag{6.18}$$

* For generic k, there is one more term coming from the contraction $\langle\sum_i ik_i\cdot X(z_i)\partial X^\mu(x_1)\rangle$ $\langle\sum_i ik_i\cdot X(z_i)\partial X_\mu(x_2)\rangle$ and there is an overall factor $\langle\prod e^{ik\cdot X}\rangle$ which is not equal to 1. For most of our discussions we will restrict to the case $k\to 0$ to simplify the presentation. But the discussions can be easily modified for generic k. A exact formula for the four-particle amplitude in type II superstring theory will be given at the end of this section and will be used to discuss the factorization in the next section and to study the ultra-high energy scattering or quantum corrections in section 8.

Putting everything together (see [13, 15, 16]), we get finally the following expression for the four-particle two-loop amplitude for HST (choosing $SO(32)$ and when $k \to 0$):

$$\begin{aligned} A(K) =& cK(k,\epsilon) \int d^2a_1 d^2a_2 d^2a_3 \mid a_{45}a_{56}a_{64} \mid^2 \\ & \times \frac{1}{T^5 \prod_{i<j}^6 a_{ij}\bar{a}_{ij}^3} \times \int \prod_{k=1}^4 d^2 z_k \Omega(z_k) I(x) \sum_s \langle \prod V_L \rangle \bar{Q}_s^4, \end{aligned} \tag{6.19}$$

where the integration runs over the complex plane, c is an undetermined constant and s denotes the spin structures of the left sector*. We will discuss the various properties of this amplitude (6.19) in the next section.

Similar calculation can be done for SST II. Here the relevant supercurrent insertion is

$$\langle J(r+)\bar{J}(\bar{s}+)J(r-)\bar{J}(\bar{s}-)\cdots\rangle_s, \tag{6.20}$$

where we have taken $x_{1,2} = r\pm$ for right sector and $\bar{x}_{1,2} = \bar{s}\pm$ for the left sector. We remark that it is necessary to take $r \neq s$ to get rid of some sigularities arising by simply taking $r = s$.

By using eq.(3.59), it is an easy matter to arrive at the following expression for the four-particle† amplitude at two-loops ($k \to 0$):

$$\begin{aligned} AII(k \to 0) =& c'\tilde{K} \int \frac{d^2a_1 d^2a_2 d^2a_3 \mid a_{45}a_{56}a_{64} \mid^2}{T^5 \prod_{i<j}^6 \mid a_{ij} \mid^2} \prod_{i=1}^4 d^2 z_i \frac{(z_i - r)(\bar{z}_i - \bar{s})}{\mid y(z_i) \mid^2} \\ & \times \left\{ I(r)\bar{I}(\bar{s}) + \frac{5}{4} \left(\frac{\pi}{T} \frac{1}{y(r)\bar{y}(\bar{s})} \int \frac{d^2v (v-r)(\bar{v}-\bar{s})}{\mid y(v) \mid^2} \right)^2 \right\}, \end{aligned} \tag{6.21}$$

where $\tilde{K}$ is the same kinematic factor as in tree and one-loop level, see ref.[1, 2] or $\tilde{K} = K(k,\epsilon) \cdot K(k,\bar{\epsilon})$ for $\epsilon_i^{\mu\nu} = \epsilon_i^\mu \bar{\epsilon}_i^\nu$, see eq.(4.20). For generic k an exact formula for this

---

* Needless to say, $\langle \prod V_L \rangle$ can also be explicity expressed in terms of $\langle \lambda\lambda \rangle = \langle \bar{\psi}\psi \rangle$.

† The vertex function is $V(k,\epsilon,z) = (\partial X^\mu + ik \cdot \psi\psi^\mu)\epsilon_{\mu\nu}(\bar{\partial} X^\nu + ik \cdot \bar{\psi}\bar{\psi}^\nu)e^{ik \cdot X}$.

amplitude can also be obtained. We have

$$\begin{aligned}AII(k,\epsilon) =& c'\tilde{K}\int \frac{d^2a_1d^2a_2d^2a_3|a_{45}a_{56}a_{64}|^2}{T^5\prod_{i<j}^6|a_{ij}|^2}\prod_{i=1}^4\frac{d^2z_i(z_i-r)(\bar{z}_i-\bar{s})}{|y(z_i)|^2}\\ &\times\left\{\left(I(r)\bar{I}(\bar{s})+\frac{5}{4}\left(\frac{\pi}{T}\frac{1}{y(r)\bar{y}(\bar{s})}\int\frac{d^2v(v-r)(\bar{v}-\bar{s})}{|y(v)|^2}\right)^2\right)\langle\prod e^{ik\cdot X}\rangle\right.\\ &+\frac{1}{16}\Big(\langle\partial X(r+)\cdot\partial X(r-)\bar{\partial}X(\bar{s}+)\cdot\bar{\partial}X(\bar{s}-)\prod e^{ik\cdot X}\rangle\\ &\left.-\langle\partial X(r+)\cdot\partial X(r-)\bar{\partial}X(\bar{s}+)\cdot\bar{\partial}X(\bar{s}-)\rangle\cdot\langle\prod e^{ik\cdot X}\rangle\Big)\right\}.\end{aligned} \tag{6.22}$$

By further evaluating the $\langle XX\rangle$ correlator, we can rewrite

$$\begin{aligned}AII(k,\epsilon) =& c'\tilde{K}\int \frac{d^2a_1d^2a_2d^2a_3|a_{45}a_{56}a_{64}|^2}{T^5\prod_{i<j}^6|a_{ij}|^2}\prod_{i=1}^4\frac{d^2z_i(z_i-r)(\bar{z}_i-\bar{s})}{|y(z_i)|^2}\\ &\times\left\{I(r)\bar{I}(\bar{s})+\frac{5}{4}\left(\frac{\pi}{T}\frac{1}{y(r)\bar{y}(\bar{s})}\int\frac{d^2v(v-r)(\bar{v}-\bar{s})}{|y(v)|^2}\right)^2\right.\\ &+\frac{\pi}{16T}\frac{1}{y^2(r)\bar{y}^2(\bar{s})}\int\frac{d^2v(v-r)(\bar{v}-\bar{s})}{|y(v)|^2}R(r)\cdot\bar{R}(\bar{s})\\ &-\frac{1}{64}\langle\partial X(r+)\cdot\partial X(r-)\rangle\bar{S}(\bar{s})-\frac{1}{64}\langle\bar{\partial}X(\bar{s}+)\cdot\bar{\partial}X(\bar{s}-)\rangle S(r)\\ &\left.+\frac{1}{256}S(r)\bar{S}(\bar{s})\right\}\left\langle\prod e^{ik\cdot X}\right\rangle,\end{aligned} \tag{6.23}$$

where

$$\begin{aligned}R(x) &= \frac{1}{T}\sum_i\frac{k_iy(z_i)}{x-z_i}\int\frac{(x-u_1)(x-u_2)}{(z_i-u_1)(z_i-u_2)}\left|\frac{(u_1-u_2)du_1du_2}{y(u_1)y(u_2)}\right|^2,\\ S(x) &= 4\sum_{i,j}k_i\cdot k_j\langle\partial X(x+)X(z_i)\rangle\langle\partial X(x-)X(z_j)\rangle\\ &= \left[\left(\sum_i\frac{k_i}{x-z_i}\right)^2-\frac{1}{y^2(x)}R(x)\cdot R(x)\right],\end{aligned} \tag{6.24}$$

and

$$\begin{aligned}\langle\partial X(x+)\cdot\partial X(x-)\rangle = \frac{5}{8}&\left\{-\sum_{i=1}^6\frac{1}{(a_i-x)^2}+2\sum_{i<j}^6\frac{1}{(a_i-x)(a_j-x)}\right.\\ &\left.-8\sum_{i=1}^6\frac{1}{a_i-x}\frac{\partial}{\partial a_i}\ln T\right\}.\end{aligned} \tag{6.25}$$

In the next section we will discuss the various properties and in particular the finiteness of the four-particle amplitudes (6.19) and (6.21). The factorization of the amplitude (6.23) will be also studied.

7. Finiteness and Factorization of the Four-Particle Amplitude

Now let us study the various properties, and in particular the finiteness and factorization, of the four-particle amplitudes calculated in the last section.

First we would like to show that the amplitudes (6.19) and (6.21) does not depend on arbitrary parameters, i.e. the choice for x and for the values of $a_{4,5,6}$. It is seen from the starting expression (it is actually simpler to do it before summing over spin structures) that $A(k)$ (eq.(6.19)) is invariant if we simultaneously make the same Möbius transformation for x and $a_{4,5,6}$. Since a generic Möbius transformation depends on three parameters, it follows that if we show that $A(k)$ is independent of x, then it will also be independent of $a_{4,5,6}$. The x independence is expected by a general argument [56] and indeed we can explicitly verify that it is true [15, 16]. In fact, the integrand on the right hand side of (6.19) is a meromorphic function of x and the poles, i.e. $x \to a_i$, can be expressed as total derivatives in $a_{1,2,3}$ and $z_{1,2,3,4}$. The strategy to show the independenceof the amplitudes on x is to study these total derivatives and show that they are vanishing. Then the amplitude as a meromorphic function of x has no poles. They are constants and independent of x.

Now let us study the pole structure of the amplitude (6.19). The sigularities of the integrand for $x \to a_i$, $i = 1, 2, 3$, can be isolated in the form:

$$\begin{aligned} cK \sum_{i=1}^{3} \prod_{k=1}^{4} \frac{z_k - a_i}{y(z_k)} \cdot \Bigg\{ \frac{1}{4} \frac{1}{(a_i - x)^2} - \frac{1}{4} \frac{1}{a_i - x} \sum_{j \neq i}^{6} \frac{1}{a_j - a_i} + \frac{1}{8} \frac{1}{a_i - x} \sum_{k=1}^{4} \frac{1}{z_k - a_i} \\ + \frac{5}{4} \frac{1}{a_i - x} \frac{\partial}{\partial a_i} \ln T \Bigg\} \prod_{k<l}^{6} \frac{1}{a_{kl} \bar{a}_{kl}^3} \frac{1}{T^5} \sum_s \langle \prod V_L \rangle \bar{Q}_s^4 \\ = - \frac{cK}{4} \sum_{i=1}^{3} \frac{\partial}{\partial a_i} \left\{ \frac{1}{a_i - x} \frac{1}{T^5 \prod_{k<l}^{6} a_{kl}} \prod_{k=1}^{4} \frac{z_k - a_i}{y(z_k)} \right\} \prod_{k<l}^{6} \bar{a}_{kl}^{-3} \sum_s \langle \prod V_L \rangle \bar{Q}_s^4, \end{aligned} \tag{7.1}$$

which are total derivatives in $a_{1,2,3}$. Similarly, the sigularities of the integrand of (6.19) for $x \to a_i$, $i = 4, 5, 6$, can be isolated in the form (choosing $x \to a_4$ for specifics):

$$\begin{aligned} cK \frac{1}{a_4 - x} \prod_{k=1}^{4} \frac{z_k - a_4}{y(z_k)} \cdot \Bigg\{ \frac{1}{2} \cdot \frac{2a_4 - a_5 - a_6}{a_{45} a_{46}} - \frac{1}{4} \sum_{i=1}^{3} \frac{1}{a_i - a_4} \\ + \frac{1}{8} \sum_{k=1}^{4} \frac{1}{z_k - a_4} + \frac{5}{4} \frac{\partial}{\partial a_4} \ln T \Bigg\} \prod_{k<l}^{6} a_{kl}^{-1} \bar{a}_{kl}^{-3} \cdot T^{-5} \sum_s \langle \prod V_L \rangle \bar{Q}_s^4. \end{aligned} \tag{7.2}$$

By making use of the formula

$$\frac{\partial}{\partial a_4}\ln T=\frac{1}{a_{45}a_{46}}\left\{2(a_5+a_6)-\sum_{i=1}^{4}a_i-\sum_{i=1}^{3}a_{i5}a_{i6}\frac{\partial}{\partial a_i}\ln T\right\}, \tag{7.3}$$

which can be derived from the projectve invariance of T^*, we can write (7.2) as:

$$cK\left\{\frac{1}{4}\sum_{i=1}^{3}\frac{\partial}{\partial a_i}\left\{\frac{1}{a_4-x}\frac{a_{i5}a_{i6}}{a_{45}a_{46}}\prod_{i<j}^{6}a_{ij}^{-1}\cdot T^{-5}\prod_{k=1}^{4}\frac{z_k-a_4}{y(z_k)}\right\}\right.$$

$$\left.+\frac{1}{2}\sum_{k=1}^{4}\frac{\partial}{\partial z_k}\left\{\frac{(z_k-a_5)(z_k-a_6)}{a_{45}a_{46}}\prod_{k=1}^{4}\frac{z_k-a_4}{y(z_k)}\right\}\prod_{i<j}^{6}a_{ij}^{-1}\cdot T^{-5}\right\}\prod_{i<j}^{6}\bar{a}_{ij}^{-3}\cdot\sum_s\langle\prod V_L\rangle\bar{Q}_s^4, \tag{7.4}$$

which are also total derivatives in $a_{1,2,3}$ and $z_{1,2,3,4}$. When $x\to z_k$, one can easily see that there is no sigular terms because of the prefactor $\prod_k(z_k-x)$. There is also no sigular terms when $x\to\infty$†. When we set $x=\infty$ in (6.18), we get precisely (6.17).

Actually, the above expressions for the sigularities in x can be generalized to include the $\prod e^{ik\cdot X}$ part of the vertices, by making use of the formula

$$\begin{aligned}&\langle\partial X^\mu(x+)\partial X_\mu(x-)\prod e^{ik\cdot X}\rangle\\&\quad=\langle T(x+)\prod e^{ik\cdot X}\rangle+\langle T(x-)\prod e^{ik\cdot X}\rangle-\sum_{i<j}k_i\cdot k_j\frac{1}{x-z_i}\cdot\frac{1}{x-z_j}\langle\prod e^{ik\cdot X}\rangle,\end{aligned} \tag{7.5}$$

where $T(x)$ is the (normal ordered) energy momentum tensor $T(x)=-\frac{1}{2}:\partial X(x)\cdot\partial X(x):$. This formula can be proved by calculating both side of (7.5) explicitly by making use of

---

* From the definition of T (eq.(3.44) we get $T(a')=|\prod_{i=1}^{6}(Ca_i+D)|^2T(a)$ for $a_i'=\frac{Aa_i+B}{Ca_i+D}$, $AD-BC=1$. Then we have

$$\sum_{i=1}^{6}a_i^n\frac{\partial}{\partial a_i}\ln T=\begin{cases}0, & n=0;\\ -3, & n=1;\\ -\sum_{i=1}^{6}a_i, & n=2\end{cases}$$

Solving $\frac{\partial}{\partial a_i}\ln T$ $(i=4,5,6)$ in terms of $\frac{\partial}{\partial a_i}\ln T$ $(i=1,2,3)$, we get exactly (7.3).

† In fact all the sigular terms can be shown to vanish by making use of the summation formulas for $\sum_{i=1}^{6}a_i^n\frac{\partial}{\partial a_i}\ln T$ obtained in the previous footnote.

the expressions of $\langle\partial X\partial X\rangle$ (eq.(3.49)) and $\langle X\partial X\rangle$ (eq.(3.57)). For $k\to 0$, we have

$$\lim_{x\to a_i}\langle\partial X(x+)\cdot\partial X(x-)\rangle$$
$$=5\left\{-\frac{1}{8}\frac{1}{(a_i-x)^2}+\frac{1}{4}\frac{1}{a_i-x}\sum_{j\neq i}^{6}\frac{1}{a_j-a_i}-\frac{1}{a_i-x}\frac{\partial}{\partial a_i}\ln T\right\}+(\text{regular terms}), \tag{7.6}$$

which has, implicitly, been used previously, see eq.(6.16) for $x=\infty$. By using eq.(7.5), we have

$$\lim_{x\to a_i}\langle\partial X(x+)\cdot\partial X(x-)\prod e^{ik\cdot X}\rangle=5\left\{-\frac{1}{8}\frac{1}{(a_i-x)^2}\langle\prod e^{ik\cdot X}\rangle\right.$$
$$\left.-\frac{1}{a_i-x}\frac{\partial}{\partial a_i}\ln\left(T\prod_{k<l}^{6}\mid a_k-a_l\mid^{1/2}\langle\prod e^{ik\cdot X}\rangle\right)\cdot\langle\prod e^{ik\cdot X}\rangle\right\}+(\text{regular terms}). \tag{7.7}$$

Then one easily sees that the above discussions go through by including also the factor $\prod e^{ik\cdot X}$. Therefore the present discussion holds for the general case.

The proof of independence of x is completed by checking that there are no boundary terms, i.e. that the total derivatives give vanishing contributions. This is true and can be checked by studying in detail the potentially dangerous degenerate configurations in the moduli space. For instance, consider the integration of the expression (7.1) or (7.4) in the region $u\to 0$ where $u=a_2-a_1$. The boundary term will be proportional to

$$\lim_{u\to 0}\oint\frac{\mathrm{d}\bar{u}}{\bar{u}^3}\cdot\mid u\mid^3\cdot F(u,\bar{u}), \tag{7.8}$$

where we have taken into account that $T\to\ln\mid u\mid$. The integration over d^2z_i is included in F: in the degeneration limit $\mathrm{d}z/y(z)\sim\mathrm{d}t/t$ in the uniformizer coordinate $t^2=z-a_1$. Since the left part is regular for $\bar{z}\to\bar{a}_1$, there is no sigularity coming from the integration over $\mathrm{d}\bar{z}$. Therefore F is regular and the above expression (7.8) vanishes.

Of course, when many z_k collide together, in particular in the point a_1, possible singularities have to be interpreted as physical singularities in the external momenta and one has to take into account the factor $\prod e^{ik\cdot X}$. As always in string theory, the integration by parts, like the ones we are discussing here, are meant to be done in the region of the momenta k where the integrand is regular, and then analytically continued everywhere [107].

As an extra exmple, in the "dividing" degeneration case $a_2 - a_1 = u$, $a_3 - a_1 = vu$, taking into account $T \to \frac{1}{|u|}$, we get the following boundary term for (7.1) and (7.4):

$$\lim_{u\to 0} \oint \frac{d\bar{u}}{\bar{u}^3} \cdot \mid u \mid^3 \cdot F(u,\bar{u}). \tag{7.9}$$

In this degeneration limit $dz/y(z) \sim dt/t^2$ but the left part is regular (for all the spin structures but one, where however $\bar{Q}_s^4$ gives a further factor $(\bar{u})^8$) and F is regular so that (7.9) vanishes. The conclusion is that $A(k)$ is independent of x, and therefore also of $a_{4,5,6}$.

The independence of the four-particle amplitude for SST II, eq.(6.21) on r (and $\bar{s}$) and $a_{4,5,6}$ can be discussed similarly. Let us compute the sigular terms when $r \to a_i$, $i = 1,2,3$ and $\bar{s}$ kept arbitrary. They are

$$c\bar{K} \prod_{k<l}^{6} \frac{1}{\bar{a}_{kl}} \cdot \prod_{k=1}^{4} \frac{\bar{z}_k - \bar{s}}{\bar{y}(\bar{z}_k)} \cdot \left\{ -\frac{1}{4}\frac{\partial}{\partial a_i} \left(\frac{1}{a_i - r} \prod_{k<l}^{6} \frac{1}{a_{kl}} \frac{1}{T^5} \prod_{k=1}^{4} \frac{z_k - a_i}{y(z_k)} \right) \bar{I}(\bar{s}) \right.$$
$$\left. + \frac{5}{4}\frac{1}{a_i - r} \frac{1}{\prod_{j\neq i}^{6}(a_j - a_i) \prod_{k<l}^{6} a_{kl} \cdot T^5} \prod_{k=1}^{4} \frac{z_k - a_i}{y(z_k)} \left(\frac{\pi}{T}\frac{1}{\bar{y}(\bar{s})} \int \frac{(v - a_i)(\bar{v} - \bar{s})}{\mid y(v) \mid^2} d^2 v \right)^2 \right\}. \tag{7.10}$$

where we have used the results of the previous analysis. One notices that the only obstruction for the first term to be a total derivative in a_i, $i = 1,2,3$, is coming from $\bar{I}(\bar{s}) = \cdots + \frac{5}{4} \sum_j \frac{1}{\bar{a}_j - \bar{s}} \frac{\partial}{\partial \bar{a}_j} \ln T$, the last term. What we will show below is that this gives a contribution which cancels exactly the second term in (7.10), i.e. we have

$$\frac{1}{4} \sum_{j=1}^{6} \frac{1}{\bar{a}_j - \bar{s}} \cdot \frac{\partial^2}{\partial \bar{a}_j \partial a_i} \ln T + \frac{1}{\prod_{j\neq i}^{6}(a_j - a_i)} \left(\frac{\pi}{T}\frac{1}{\bar{y}(\bar{s})} \int \frac{(v - a_i)(\bar{v} - \bar{s})}{\mid y(v) \mid^2} d^2 v \right)^2 = 0. \tag{7.11}$$

This identity can be proved as follows. From (3.45), we have

$$\frac{\partial \tau_{ij}}{\partial a_n} = \frac{i\pi}{2} \hat{\omega}_i(a_n) \hat{\omega}_j(a_n). \tag{7.12}$$

Notice that $T = 2 \mid \det K \mid^2 \det \mathrm{Im}\tau$, we have

$$\begin{aligned} \frac{\partial^2}{\partial a_i \partial \bar{a}_j} \ln T &= \frac{\partial^2}{\partial a_i \partial \bar{a}_j} \ln \det \mathrm{Im}\tau = \mathrm{Tr} \left(\frac{\partial^2}{\partial a_i \partial \bar{a}_j} \ln \mathrm{Im}\tau \right) \\ &= -4 \left(\frac{\pi}{T} \frac{1}{\hat{y}(a_i)\hat{\bar{y}}(\bar{a}_j)} \int \frac{(v - a_i)(\bar{v} - \bar{a}_j)}{\mid y(v) \mid^2} d^2 v \right)^2, \end{aligned} \tag{7.13}$$

where $\hat{y}^2(a_i) = \prod_{j\neq i}(a_i - a_j)$. So we have

$$\begin{aligned}\frac{1}{4}\sum_{j=1}^{6}\frac{1}{\bar{a}_j-\bar{s}}\frac{\partial^2}{\partial a_i \partial \bar{a}_j}\ln T &= -\sum_{j=1}^{6}\frac{1}{\bar{a}_j-\bar{s}}\left(\frac{\pi}{T}\frac{1}{\hat{y}(a_i)\hat{\bar{y}}(\bar{a}_j)}\int\frac{(v-a_i)(\bar{v}-\bar{a}_j)}{\mid y(v)\mid^2}d^2v\right)^2 \\ &= \frac{1}{\hat{y}^2(a_i)}\left(\frac{\pi}{T}\frac{1}{\bar{y}(\bar{s})}\int\frac{(v-a_i)(\bar{v}-\bar{s})}{\mid y(v)\mid^2}d^2v\right)^2.\end{aligned} \tag{7.14}$$

This is precisely (7.11). Then the sigular terms when $r \to a_i$, $i = 1, 2, 3$, are

$$-\frac{1}{4}c\tilde{K}\sum_{i=1}^{3}\frac{\partial}{\partial a_i}\left[\frac{1}{a_i - r}\frac{1}{T^5\mid\prod_{i<j}^{6}a_{ij}\mid^2}\prod_{k=1}^{4}\frac{(z_k-a_i)(\bar{z}_k-\bar{s})}{\mid y(z_k)\mid^2}\bar{I}(\bar{s})\right]. \tag{7.15}$$

Similarly, the singular terms when $r \to a_4$ are

$$\begin{aligned}+\frac{1}{4}c\tilde{K}\sum_{i=1}^{3}\frac{\partial}{\partial a_i}&\left[\frac{1}{a_4 - r}\frac{a_{i5}a_{i6}}{a_{45}a_{46}}\frac{1}{T^5\mid\prod_{i<j}^{6}a_{ij}\mid^2}\prod_{k=1}^{4}\frac{(z_k-a_4)(\bar{z}_k-\bar{s})}{\mid y(z_k)\mid^2}\bar{I}(\bar{s})\right] \\ &+ (\text{total derivatives in } z_k),\end{aligned} \tag{7.16}$$

by making use of the formula

$$\sum_{i=1}^{3}\frac{a_{i5}a_{i6}}{a_{45}a_{46}}\frac{\partial^2}{\partial a_i\partial\bar{a}_j}\ln T = -\frac{\partial^2}{\partial a_4\partial\bar{a}_j}\ln T. \tag{7.17}$$

which can be proved by making use of (7.13).

It is an easy matter to show that all these total derivatives eqs.(7.15) and (7.16) give zero contributions. That completes our verification of the independence of $AII(k)$ on r (and likewise on $\bar{s}$) and therefore also on $a_{4,5,6}$.

We metion that a form of the two-loop four-particle amplitude for SST II was conjectured in ref.[108]. While in some aspects it resembles our above formula eq.(6.20), in some others, in particular in the important ones related to the supercurrent contribution, it does not seem to agree with the result of our explicit computation.

Next, we would like to discuss the finiteness of $A(k)$, considering for definiteness the HST case. In the "handle" case we consider the corner $u \to 0$ where $u = a_2 - a_1$, and in the "dividing" case the corner $u \to 0$ and v keep fixed, where $u = a_2 - a_1$, $vu = a_3 - a_1$. We can read the corresponding expression for HST from eq.(6.19). By taking the appropriate variable $y = u^2$ and doing some computations, we can put the "handle" degeneration expression in the canonical form [109, 110]

$$\frac{d^2y}{\bar{y}^2y}\cdot(\ln\mid y\mid)^{-5}\cdot F, \tag{7.18}$$

where one recognizes a tachyon $1/\bar{y}^2$ in the left sector and a massless state $1/y$ in the right sector, as is expected in HST. For the "dividing" degeneration case the appropriate variable is $u = y^2$ and from (6.19) we get the canonical form

$$\frac{d^2y}{\bar{y}^2} \cdot y^2 \cdot F, \tag{7.19}$$

where we recognize again a tachyon in the left sector and a level 3 massive excitation (y^2, compare with the zero level $1/y$) in the right sector as is to be expected from the norenormalization theorems, implying that the one-loop tadpole vanishes if it is attached to a vertex $(\psi\psi)^n$ with $n < 4$. Of course, the integration over $\arg(y)$ will select the same contribution from the left sector as it does from the right sector, and therefore finally the degeneration expression will be

$$\begin{aligned} &\frac{d^2y}{\mid y \mid^2}(ln \mid y \mid)^{-5} \cdot F', && \text{for the "handle" case;} \\ &d^2y \mid y \mid^4 \cdot F', && \text{for the "dividing" case.} \end{aligned} \tag{7.20}$$

The amplitude is thus finite, for generic values of the external momenta k_i, taking into account the part $\prod e^{ik\cdot X}$.

A more subtle question is whether the leading term which we have obtained for $A(k)$, i.e. the coefficient mutiplying $K(k,\epsilon)$ in eq.(6.19), where we dropped $\prod e^{ik\cdot X}$, is also finite. The question arises because in the "handle" case, taking $z - a_1 = t^2$ we get $y(z) \sim t\cdot(t^2+u)^{1/2}$ and the integration over $\mathrm{d}z_i$ looks like $\prod_1^4 \mathrm{d}t_i/(t_i^2+u)^{1/2}$ which combined with an appropriate left sector contribution could give $\sim (\ln \mid y \mid)^4$, making the first expression in (7.20) divergent [108] (notice that the divergence comes from the integration region $z_i \sim z_j$ and therefore disappears for generic k_i). We will not done the rather involved explicit computation for the left sector, but we can nevertheless argue that even this divergence in $k_i \to 0$ is removed. In fact we can make use of the arbitrariness in x to choose* $x \to a_1$: we have then to take the finite part of (6.18) in this limit, and we can see that it contains a factor $(z_i - a_1) = t_i^2$ at least for two values of i. The resulting divergences from the integration over $\mathrm{d}z_i$ will then actually be

$$\prod_{i=1}^{2} \frac{\mathrm{d}t_i}{\sqrt{t_i^2+u}} \times (\text{left sector}) \sim (\ln \mid y \mid)^2, \tag{7.21}$$

---

* This means choosing the supercurrent insertion on a branch point, as considered in refs.[12, 111, 102].

making the first expression in (7.20) finite.

Similarly, in the "dividing" case: $z - a_1 = t^2$, $z - a_2 = t^2 + u$, $z - a_3 = t^2 = uv$, putting $x = a_1$, we would get, from the corner $z \to a_1$, a divergence like

$$\prod_{i=1}^{2} \frac{dt_i}{t_i^2 + y^2} \times (\text{left} \quad \text{sector}) \sim \frac{1}{\mid y \mid^4}$$

and the second expression in (7.20) will remain finite.

The last point which we want to discuss is factorization. Factorization is a very important and basic requirement for a sensible theory. In the early days of string theory (called dual resonance models) factorization plays an important rule to discover the physical spectrum of the Veniziano amplitude. We will show in what follows the four-particle amplitude (6.22) also has the right factorization property [75].

The factorized diagram fig.8 corresponds in moduli space the following limit:

$$\begin{aligned} &a_2 - a_1 = u, && a_3 - a_1 = vu, \\ &z_1 - a_1 = x_1 u, && z_2 - a_1 = x_2 u, \\ &u \to 0, && v, \quad x_1, \quad x_2 \qquad \text{fixed.} \end{aligned} \tag{7.23}$$

This can be easily understood. We knew that the limit $u \to 0$, v fixed corresponds to the dividing of a genus 2 Riemann surface into two genus one Riemann surfaces (two tori). One torus seems to be a point on the other torus, but it is actually a genus one Riemann surface with modular parameter v. This is due to conformal invariance. If we want to get the factorization configuration shown in fig.8, we should insert two particles one one torus and the other two on the other torus. Then we get the limit (7.23). The colliding of $a_{1,2,3}$, z_1 and z_2 to one point, for example a_1, should be simutaneous. Otherwise we will get other factorization limit, like fig.9, which is obtained by first taking $z_2 \to z_1$ and then z_1, $a_{2,3} \to a_1$.

Under the limit of (7.23) we have

$$T = \frac{2}{|u|} T_1 T_2 \frac{1}{|a_{14} a_{15} a_{16}|} + O(\ln |u|), \tag{7.24}$$

where

$$\begin{aligned} T_1 &= \int \frac{d^2 z}{|y_1(z)|^2}, \qquad y_1^2(z) = z(z-1)(z-v); \\ T_2 &= \int \frac{d^2 z}{|y_2(z)|^2}, \qquad y_2^2(z) = (z-a_1)(z-a_4)(z-a_5)(z-a_6). \end{aligned} \tag{7.25}$$

Then the measure transforms as follows:

$$\frac{d^2a_1d^2a_2d^2a_3|a_{45}a_{56}a_{64}|^2}{T^5\prod_{i<j}^6|a_{ij}|^2}=\frac{1}{2^5}\frac{|u|d^2ud^2vd^2a_1}{T_1^5T_2^5|v(1-v)|^2|a_{14}a_{15}a_{16}|^1}. \tag{7.26}$$

Apart from the kinematic factor and $\prod e^{ik\cdot X}$ the rest terms in (6.23) can be divided into two parts: one part which doesn't depend on the momentum and the other part which does. For the first part we have

$$\begin{aligned}\prod_{i=1}^4\frac{d^2z_i(z_i-r)(\bar z_i-\bar s)}{|y(z_i)|^2}&\left\{I(r)\bar I(\bar s+\frac{5}{4}\left(\frac{\pi}{T}\frac{1}{y(r)\bar y(\bar s)}\int\frac{d^2v(v-r)(\bar v-\bar s)}{|y(v)|^2}\right)^2\right\}\\&=\frac{1}{2^6}\frac{1}{|u|^2}\frac{1}{|a_{14}a_{15}a_{16}|^2}\frac{d^2x_1d^2x_2}{|y_1(x_1)y_1(x_2)|^2}\frac{d^2z_3d^2z_4}{|y_2(z_3)y_2(z_4)|^2},\end{aligned} \tag{7.27}$$

by subtracting all the divergent terms (in $\frac{1}{(a_i-r)^2}$ and $\frac{1}{a_i-r}$, $i=1,2,3$), setting $r=s=a_1$ and then taking the limit $u\to 0$.

The second part can be computed similarly. Let us write $R(x)=\sum_i R(x,z_i)$ and

$$R(x,z)=ky(z)\cdot\left(\frac{1}{x-z}+f_1(z)+(x-z)f_2(z)\right), \tag{7.28}$$

where

$$f_1(z)=-\frac{2}{T}\int\frac{1}{u_1-z}\mathrm{d}\mu(u_1,u_2),\qquad f_2(z)=\frac{1}{T}\int\frac{1}{(u_1-z)(u_2-z)}\mathrm{d}\mu(u_1,u_2), \tag{7.29}$$

and

$$\mathrm{d}\mu=\left|\frac{(u_1-u_2)\mathrm{d}u_1\mathrm{d}u_2}{y(u_1)y(u_2)}\right|^2. \tag{7.30}$$

The important observation here is that $f_{1,2}(z)$ are finite in the limit of eq.(7.23), irrespective whether $z\to a_1$ or $z\not\to a_1$. The point is that, taking for instance $z\sim a_1$, the potentially singular region $u_1\sim a_1$ contributing to $f_{1,2}$ can be written as a total deivative and then:

$$\begin{aligned}\frac{2}{3}\int\frac{1}{u_1-a_1}\frac{F(u_1,\bar u_1)d^2u_1}{|u_1-a_1|^3}&=-\int\left(\frac{\partial}{\partial u_1}\frac{1}{|u_1-a_1|^3}\right)F(u_1,\bar u_1)d^2u_1\\&=-\frac{1}{2i}\oint\frac{d\bar u_1}{\bar u_1-\bar a_1}\frac{1}{|u_1-a_1|}\cdot\frac{1}{u_1-a_1}\cdot F(u_1,\bar u_1)\\&\quad+\int\partial_{u_1}F(u_1,\bar u_1)\frac{d^2u_1}{|u_1-a_1|^3},\end{aligned} \tag{7.31}$$

where the contour integral is around a_1, and due to the angular integration it is only linear divergent. Noting that also T in the denominator of $f_{1,2}$ is linear divergent, we see that $f_{1,2}$ is finite. When z is away from a_1, we can also derive an explicit expression for $f_{1,2}(z)$. We have

$$\begin{aligned} f_1(z) &= \frac{1}{z-a_1} + \frac{1}{T_2}\int \frac{1}{z-w}\left|\frac{dw}{y_2(w)}\right|^2, \\ f_2(z) &= \frac{1}{z-a_1}\cdot\frac{1}{T_2}\int \frac{1}{z-w}\left|\frac{dw}{y_2(w)}\right|^2. \end{aligned} \tag{7.32}$$

First let us consider the term $\frac{1}{y^2(x)}R(x)\cdot R(x)$ in $S(x)$. By using eq.(7.28) and factorizing the singular terms in $\frac{1}{a_i - x}$, we get

$$\begin{aligned} \frac{1}{2}\prod_{l=1}^{4}(z_l - x)\frac{1}{y^2(x)}R(x)\cdot R(x) =& \sum_{i<j} k_i\cdot k_j y(z_i)y(z_j)\frac{\prod_{l=1}^4 (z_l - x)}{\prod_{n=1}^6 (x-a_n)} \\ &\times\left(\frac{1}{x-z_i} + f_1(z_i) + (x-z_i)f_2(z_i)\right)(z_i\to z_j) \\ =& \sum_{i<j} k_i\cdot k_j y(z_i)y(z_j)\left(\sum_{n=1}^{6}\frac{Q_n(i,j)}{a_n - x} + P(i,j)\right), \end{aligned} \tag{7.33}$$

where

$$\begin{aligned} Q_n(i,j) =& -\prod_{l=1}^{4}(z_l - a_n)\cdot\frac{1}{\prod_{l\neq n}(a_n - a_l)} \\ &\times\left(\frac{1}{a_n - z_i} + f_1(z_i) + (a_n - z_i)f_2(z_i)\right)(z_i\to z_j) \\ P(i,j) =& f_2(z_i)f_2(z_j). \end{aligned} \tag{7.34}$$

We will show that the expression (7.33) is zero after substracting all the divergent terms, setting $x = a_1$ and then taking the limit $u\to 0$. For z_i or z_j near a_1 (this is the case for $i = 1,2$), this is true because of the factor $y(z_i)y(z_j)$ which is zero in the limit $u\to 0$. For

z_i and z_j away from a_1, we get*

$$
\begin{aligned}
\sum_{n=4,5,6} \frac{Q_n(i,j)}{a_n - a_1} + P(i,j) =& f_2(z_i)f_2(z_j) - \sum_{n=4,5,6} \frac{1}{a_n - a_1} \frac{\prod_{l=1}^{4}(z_l - a_n)}{(a_n - a_1)^3 \prod_{l\neq n}^{4,5,6}(a_n - a_l)} \\
& \times \left(\frac{a_1 - a_n}{(a_1 - z_i)(a_n - z_i)} - \frac{a_1 - a_n}{a_1 - z_i} \frac{1}{w - z_i} \left|\frac{\mathrm{d}w}{y_2(w)}\right|^2 \right) \cdot (z_i \to z_j) \\
=& f_2(z_i)f_2(z_j) - \sum_{4,5,6} \frac{1}{\prod_{l\neq n}^{4,5,6}(a_n - a_l)} \frac{(z_i - a_n)(z_j - a_n)}{(z_i - a_i)(z_j - a_1)} \\
& \times \left(\frac{1}{a_n - z_i} - \frac{1}{T_2} \int \frac{1}{w - z_i} \left|\frac{\mathrm{d}w}{y_2(w)}\right|^2 \right) \cdot (z_i \to z_j) \\
=& 0
\end{aligned}
\tag{7.35}
$$

by using the expressions for $f_{1,2}(z)$ in eq.(7.32) and doing some "trivial" algebraic calculation like

$$
\sum_{n=4,5,6} \frac{(z_i - a_n)(z_j - a_n)}{\prod_{l\neq n}^{4,5,6}(a_n - a_l)} = 1. \tag{7.36}
$$

We conclude that

$$
\prod_{l=1}^{4}(z_l - x)\frac{1}{y^2(x)}R(x)\cdot R(x) - \left(\text{divergent terms in } \frac{1}{a_i - x}(i = 1,2,3)\right) = 0. \tag{7.37}
$$

Second we discuss the other disturbing term in (6.23):

$$
\prod_{l=1}^{4}(z_l - r)(\bar{z}_l - \bar{s})\frac{1}{y^2(r)\bar{y}^2(\bar{s})} \int \frac{\mathrm{d}^2w(w - r)(\bar{w} - \bar{s})}{|y(w)|^2} R(r)\cdot \bar{R}(\bar{s}). \tag{7.38}
$$

Let us compute the right (holomorphic) part of (7.38):

$$
\begin{aligned}
R_p &= \prod_{l=1}^{4}(z_l - r)\frac{r - w}{y^2(r)}R(r) \\
&= \prod_{l=1}^{4}(z_l - r)\frac{r - w}{\prod_{n=1}^{6}(r - a_n)} \sum_i k_i y(z_i) \left(\frac{1}{r - z_i} + f_1(z_i) + (r - z_i)f_2(z_i) \right) \\
&= \sum_i k_i y(z_i) \left(\sum_{n=1}^{6} \frac{\tilde{Q}_n(i)}{a_n - r} + f_2(z_i) \right),
\end{aligned}
\tag{7.39}
$$

---

* Notice that $\prod_{l=1}^{4}(z_l - a_n) \approx (a_1 - a_i)^2(z_i - a_n)(z_j - a_n)$ in this case.

where

$$\tilde{Q}_n(i) = -\prod_{l=1}^{4}(z_l - a_n) \cdot \frac{a_n - w}{\prod_{l\neq n}^{6}(a_n - a_l)} \cdot \left(\frac{1}{a_n - z_i} + f_1(z_i) + (a_n - z_i) f_2(z_i)\right). \quad (7.40)$$

For $z_i \sim a_1$ $(i = 1,2)$, these gives no contribution to R_p after subtracting the divergent terms because of the factor $y(z_i)$. For z_i away from a_1 we have also $R_p = 0$ which can be proved following the same reasoning in (7.35). The conclusion is

$$\prod_{l=1}^{4}(z_l - r)\frac{(r-w)}{y^2(r)}R(r) - \left(\text{divergent terms in } \frac{1}{a_i - r}(i = 1,2,3)\right) = 0. \quad (7.41)$$

By using eqs.(7.37) and (7.41) we easily compute the second part of eq.(6.23). We get

$$\begin{aligned} E2\text{nd} =& \prod_{i=1}^{4} \frac{d^2 z_i (z_i - r)(\bar{z}_i - \bar{s})}{|y(z_i)|^2} \left\{ \frac{\pi}{16T} \frac{1}{y^2(r)\bar{y}^2(\bar{s})} \int \frac{d^2 w (w-r)(\bar{w} - \bar{s})}{|y(w)|^2} R(r) \cdot \bar{R}(\bar{s}) \right. \\ & \left. - \frac{1}{64}\langle \partial X(r+) \cdot \partial X(r-)\rangle \bar{S}(\bar{s}) - \frac{1}{64}\langle \bar{\partial} X(\bar{s}+) \cdot \bar{\partial} X(\bar{s}-)\rangle S(r) + \frac{1}{256} S(r)\bar{S}(\bar{s}) \right\} \\ =& \frac{1}{|u|^2 |a_{14}a_{15}a_{16}|^2} \frac{d^2 x_1 d^2 x_2}{|y_1(x_1)y_1(x_2)|^2} \frac{d^2 z_3 d^2 z_4}{|(z_3 - a_1)(z_4 - a_1)|^2 |y_2(z_3) y_2(z_4)|^2} \\ & \times \prod_{i=1}^{4}(z_i - r)(\bar{z}_i - \bar{s}) \left(-\frac{1}{64}\langle \partial X(r-) \cdot \partial X(r+)\rangle \left(\sum_i \frac{k_i}{\bar{s} - \bar{z}_i}\right)^2 \right. \\ & \left. - \frac{1}{64}\langle \bar{\partial} X(\bar{s}-) \cdot \bar{\partial} X(\bar{s}+)\rangle \left(\sum_i \frac{k_i}{r - z_i}\right)^2 + \frac{1}{256}\left(\sum_i \frac{k_i}{r - z_i}\right)^2 \left(\sum_i \frac{k_i}{\bar{s} - \bar{z}_i}\right)^2 \right) \\ =& \frac{1}{2^6} k_1 \cdot k_2 \left(\frac{5}{2} + k_1 \cdot k_2\right) \frac{1}{|u|^2 ||a_{14}a_{15}a_{16}|^2} \frac{d^2 x_1 d^2 x_2}{|y_1(x_1) y_1(x_2)|^2} \frac{d^2 z_3 d^2 z_4}{|y_2(z_3) y_2(z_4)|^2}. \end{aligned} \quad (7.42)$$

Combining (7.27) and (7.42) we get ($2k_1 \cdot k_2 = -2$, see below)

$$\begin{aligned} AII(k,\epsilon) =& -\frac{1}{2^{12}} c' \bar{K} \int \frac{d^2 u d^2 v d^2 a_1}{T_1^5 T_2^5 |u||v(1-v)|^2 |a_{14}a_{15}a_{16}|^3} \\ & \times \frac{d^2 x_1 d^2 x_2}{|y_1(x_1)y_1(x_2)|^2} \frac{d^2 z_3 d^2 z_4}{|y_2(z_3)y_2(z_4)|^2} \langle \prod e^{ik\cdot X}\rangle. \end{aligned} \quad (7.43)$$

To see the factorization we should also compute $\langle \prod e^{ik\cdot X}\rangle$ in the factorization limit.

We have

$$
\begin{aligned}
\langle\prod e^{ik\cdot X}\rangle =&\exp\left\{-\frac{s}{2}\langle(X(z_1)-X(z_2))(X(z_3)-X(z_4))\rangle\right\}\\
&\times\exp\left\{-\frac{t}{2}\langle(X(z_1)-X(z_3))(X(z_2)-X(z_4))\rangle\right\}\\
=&(|u|)^{-t/2}\exp\left\{-\frac{t}{2}\Big[\langle X(x_1)X(x_2)\rangle_1\Big]\right\}\\
&\times\exp\left\{\frac{t}{2}\Big[+\langle X(z_3)X(z_4)\rangle_2-\langle X(a_1)X(z_3)\rangle_2-\langle X(a_1)X(z_4)\rangle_2\Big]\right\},
\end{aligned}
\tag{7.44}
$$

where the index 1 and 2 mean that the propagators $\langle X(z_i)X(z_j)\rangle$ are defined on the (factorized) first and second torus respectively.

Remember that the appropriate variable for the "dividing" degeneration case is y: $u=y^2$, we have

$$
\begin{aligned}
AII(k,\epsilon)=&-\frac{1}{2^{10}}c'\tilde{K}\int d^2y|y^2|^{-\frac{t}{2}}\\
&\times\frac{1}{T_1^5}\frac{d^2v}{|v(1-v)|^2}\frac{d^2x_1d^2x_2}{|y_1(x_1)y_1(x_2)|^2}\exp\left\{-\frac{t}{2}\langle X(x_1)X(x_2)\rangle_1\right\}\\
&\times\frac{d^2a_1}{T_2^5|a_{14}a_{15}a_{16}|^3}\frac{d^2z_3d^2z_4}{|y_2(z_3)y_2(z_4)|^2}\\
&\times\exp\left\{-\frac{t}{2}\Big[\langle X(z_3)X(z_4)\rangle_2-\langle X(a_1)X(z_3)\rangle_2-\langle X(a_1)X(z_4)\rangle_2\Big]\right\}\\
=&\frac{2\pi}{2^{10}}c'\tilde{K}\frac{1}{t-2}\int\frac{1}{T_1^5}\frac{d^2v}{|v(1-v)|^2}\frac{d^2x_1d^2x_2}{|y_1(x_1)y_1(x_2)|^2}\exp\left\{-\frac{t}{2}\langle X(x_1)X(x_2)\rangle_1\right\}\\
&\times\int\frac{d^2a_1}{T_2^5|a_{14}a_{15}a_{16}|^3}\frac{d^2z_3d^2z_4}{|y_2(z_3)y_2(z_4)|^2}\\
&\times\exp\left\{-\frac{t}{2}\Big[\langle X(z_3)X(z_4)\rangle_2-\langle X(a_1)X(z_3)\rangle_2-\langle X(a_1)X(z_4)\rangle_2\Big]\right\},\\
&\qquad t=-2k_1\cdot k_2\sim 2,
\end{aligned}
\tag{7.45}
$$

which shows apparently the right factorization property of the two-loop four-particle amplitude.

To complete the study of two-loop factorization, we'd like to show that the kinematic factor $\tilde{K}$ in eq.(6.23) can also be decomposed into a product of two one-loop kinematic factor. This has been done in [112]. Nevertheless the author of [112] was not able to obtain the standard kinematic factor. The point is that at one-loop the amplitude with the external line of the massive rank-two symmetric tensor also contributes to the two-loop factorization as can be easily seen from the presciption given in [113]. If this contribution is included we get the standard kinematic factor*. Let us now show some details of this calculation.

From the prescription given in ref.[113] we know the general vertex operator (right part) for the emission of a level one ($k^2 = -2$) massive state:

$$\begin{aligned} V_R &\sim \int d\theta \left\{ a\epsilon_\mu \partial D\mathbf{X}^\mu + b\epsilon_{\mu\nu}\partial \mathbf{X}^\mu D\mathbf{X}^\nu + c\epsilon_{\mu\nu\rho} D\mathbf{X}^\mu D\mathbf{X}^\nu D\mathbf{X}^\rho \right\} e^{ik\cdot\mathbf{X}} \\ &\sim \left\{ a\epsilon_\mu (r^2 X^\mu + ik\cdot\psi\partial\psi) + b\epsilon_{\mu\nu}(\partial X^\mu ik\cdot\psi\psi^\nu + \partial\psi^\mu\psi^\nu + \partial X_\mu \partial X^\nu) \right. \\ &\quad \left. + c\epsilon_{\mu\nu\rho}(ik\cdot\psi\psi^\mu\psi^\nu\psi^\rho + \partial X^\mu\psi^\nu\psi^\rho - \partial X^\nu\psi^\mu\psi^\rho + \partial X^\rho\psi^\mu\psi^\rho + \partial X^\rho\psi^\mu\psi\nu \right\} e^{ik\cdot X}, \end{aligned} \tag{7.46}$$

where $\epsilon_{\mu\nu\rho}$ is an antisymmetric tensor and $\epsilon_{\mu\nu}$ is a symmetric tensor. At one loop level the three point function is computed as follows (see section 4)†:

$$\begin{aligned} &\langle V_1(k_1,\epsilon_1,z_1)V_2(k,\epsilon_2,z_2)V_R(k,\epsilon,z)\rangle_s \\ &\qquad = \sum_s \eta_s Q_s \langle V_1(k_1,\epsilon_1,z_1)V_2(k_2,\epsilon_2,z_2)V_2(k,\epsilon,z_1)\rangle \\ &\qquad = -ic\sum_s \eta_s Q_s \langle k_1\psi\epsilon_1\cdot\psi(z_1)k_2\cdot\varphi\epsilon_2\cdot\varphi(z_2)k\cdot\psi\psi^\mu\psi^\nu\psi^\rho(z)\rangle_s \epsilon_{\mu\nu\rho} \\ &\qquad\quad + b\sum_s \eta_s Q_s \langle k_1\cdot\psi\epsilon_1,\cdot\psi(z_1)k_2\cdot\epsilon_2\cdot\psi(z_2)\partial\psi^\mu\psi^\nu(z)\rangle_s \epsilon^{\mu\nu} \\ &\qquad\quad - ia\sum_s \eta_s Q_s \langle k_1\cdot\psi\epsilon_1\cdot\psi(z_1)k_2\cdot\psi\epsilon_2\psi(z_2)\rangle_s \langle\partial\psi\psi(z)\rangle_s, \end{aligned} \tag{7.47}$$

by using the non renormalization theorem. From the above expression for the three-point function we found the following kinematic factor for the three contributions:

$$\begin{aligned} S_1(k_1,\epsilon_1,k_2,\epsilon_2,k,\epsilon) &= (k_1\cdot k\epsilon_1^\mu k_2^\nu\epsilon_2^\rho - \epsilon_1\cdot kk_1^\mu k_2^\nu\epsilon_2^\rho + k_2\cdot kk_1^\mu\epsilon_1^\nu\epsilon_2^\rho - \epsilon_2\cdot kk_1^\mu\epsilon_1^\nu k_2^\rho)\,\epsilon_{\mu\nu\rho}, \\ S_2(k_1,\epsilon_1,k_2,\epsilon_2,k,\epsilon) &= (k_1\cdot k_2\epsilon_1^\mu\epsilon_2^\nu - k_1\cdot\epsilon_2\epsilon_1^\mu k_2^\nu - \epsilon_1\cdot k_2 k_1^\mu\epsilon_2^\nu + \epsilon_1\cdot\epsilon_2 k_1^\mu k_2^\nu)\,\epsilon_{\mu\nu}, \\ S_3(k_1,\epsilon_1,k_2,\epsilon_2,k,\epsilon) &= (k_1\cdot k_2\epsilon_1\cdot\epsilon_2 - k_1\cdot\epsilon_2\epsilon_1\cdot k_2). \end{aligned} \tag{7.48}$$

We will use the following nonmalization for the summation over intermediate states:

$$\begin{aligned} \sum_\epsilon \epsilon_{\alpha\beta\gamma}\epsilon^{\mu\nu\rho} &= \delta^\mu_\alpha\delta^\nu_\beta\delta^\rho_\gamma + \delta^\nu_\alpha\delta^\rho_\beta\delta^\mu_\alpha + \delta^\rho_\alpha\delta^\mu_\beta\delta^\nu_\gamma - \delta^\nu_\alpha\delta^\mu_\beta\delta^\rho_\gamma - \delta^\rho_\alpha\delta^\nu_\beta\delta^\mu_\gamma - \delta^\mu_\alpha\delta^\rho_\beta\delta^\nu_\gamma, \\ \sum_\epsilon \epsilon_{\mu\nu}\epsilon^{\alpha\beta} &= \delta^\alpha_\mu\delta^\beta_\nu + \delta^\beta_\mu\delta^\alpha_\nu. \end{aligned} \tag{7.49}$$

---

* This has also been noted in an erratum to [112].

† The other two are massless particles.

Then we found

$$
\begin{aligned}
\sum_{\epsilon} & S_1(k_1,\epsilon_1,k_2,\epsilon_2,k,\epsilon)S_1(k_3,\epsilon_3,k_4,\epsilon_4,-k,\epsilon)/2k_1\cdot k_2 \\
&= -\epsilon_2\cdot\epsilon_3(k_1\cdot k_3(\epsilon_1\cdot k_4\epsilon_4\cdot k_2+\epsilon_1\cdot k_3\epsilon_4\cdot k_1)-k_2\cdot k_3(\epsilon_1\cdot k_3\epsilon_4\cdot k_2+\epsilon_1\cdot k_4\epsilon_4\cdot k_1)) \\
&\quad -\epsilon_1\cdot\epsilon_4(k_2\cdot k_4(\epsilon_2\cdot k_3\epsilon_3\cdot k_1+\epsilon_2\cdot k_4\epsilon_3\cdot k_2)-k_1\cdot k_4(\epsilon_2\cdot k_4\epsilon_3\cdot k_1+\epsilon_2\cdot k_3\epsilon_3\cdot k_2)) \\
&\quad -\epsilon_1\cdot\epsilon_3(k_2\cdot k_3(\epsilon_2\cdot k_4\epsilon_4\cdot k_1+\epsilon_2\cdot k_3\epsilon_4\cdot k_2)-k_1\cdot k_3(\epsilon_2\cdot k_3\epsilon_4\cdot k_1+\epsilon_2\cdot k_4\epsilon_4\cdot k_2)) \\
&\quad -\epsilon_2\cdot\epsilon_4(k_1\cdot k_4(\epsilon_1\cdot k_3\epsilon_3\cdot k_2+\epsilon_1\cdot k_4\epsilon_3\cdot k_1)-k_2\cdot k_4(\epsilon_1\cdot k_4\epsilon_3\cdot k_2+\epsilon_1\cdot k_3\epsilon_3\cdot k_1)) \\
&\quad -k_1\cdot k_2(k_2\cdot k_3-k_2\cdot k_4)(\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4-\epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3) \\
&\quad -(\epsilon_1\cdot k_3\epsilon_2\cdot k_4-\epsilon_1\cdot k_4\epsilon_2\cdot k_3)(\epsilon_3\cdot k_1\epsilon_4\cdot k_2-\epsilon_3\cdot k_2\epsilon_4\cdot k_1),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{\epsilon} & S_2(k_1,\epsilon_1,k_2,\epsilon_2,k,\epsilon)S_2(k_3,\epsilon_3,k_4,\epsilon_4,-k,\epsilon) \\
&=(k_1\cdot k_2)^2(\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4+\epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3)+\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4(k_1\cdot k_3k_2\cdot k_4+k_2\cdot k_3k_1\cdot k_4) \\
&\quad +(\epsilon_1\cdot k_3\epsilon_2\cdot k_4-\epsilon_1\cdot k_4\epsilon_2\cdot k_3)(\epsilon_3\cdot k_1\epsilon_4\cdot k_2-\epsilon_3\cdot k_2\epsilon_4\cdot k_1) \\
&\quad +\epsilon_1\cdot k_2\epsilon_2\cdot k_1\epsilon_3\cdot k_4\epsilon_4\cdot k_3 \\
&\quad +\epsilon_2\cdot\epsilon_3(k_1\cdot k_4\epsilon_1\cdot k_2\epsilon_4\cdot k_3-k_1\cdot k_2\epsilon_4\cdot k_3\epsilon_1\cdot k_4+\epsilon_1\cdot k_2\epsilon_4\cdot k_1) \\
&\quad +\epsilon_2\cdot\epsilon_4(k_1\cdot k_3\epsilon_1\cdot k_2\epsilon_3\cdot k_4-k_1\cdot k_2\epsilon_3\cdot k_4\epsilon_1\cdot k_3+\epsilon_1\cdot k_2\epsilon_3\cdot k_1) \\
&\quad +\epsilon_1\cdot\epsilon_3(k_2\cdot k_4\epsilon_2\cdot k_1\epsilon_4\cdot k_3-k_2\cdot k_1\epsilon_4\cdot k_3\epsilon_2\cdot k_4+\epsilon_2\cdot k_1\epsilon_4\cdot k_2) \\
&\quad +\epsilon_1\cdot\epsilon_4(k_2\cdot k_3\epsilon_2\cdot k_1\epsilon_3\cdot k_4-k_2\cdot k_1\epsilon_3\cdot k_4\epsilon_2\cdot k_3+\epsilon_2\cdot k_1\epsilon_3\cdot k_2) \\
&\quad +\epsilon_1\cdot\epsilon_2(-2k_1\cdot k_3\epsilon_3\cdot k_2\epsilon_4\cdot k_1-2k_2\cdot k_3\epsilon_3\cdot k_1\epsilon_4\cdot k_2-k_1\cdot k_2\epsilon_3\cdot k_4\epsilon_4\cdot k_3) \\
&\quad +\epsilon_3\cdot\epsilon_4(-2k_2\cdot k_3\epsilon_2\cdot k_4\epsilon_1\cdot k_3-2k_1\cdot k_3\epsilon_1\cdot k_4\epsilon_2\cdot k_3-k_3\cdot k_4\epsilon_3\cdot k_1\epsilon_1\cdot k_2),
\end{aligned}
\tag{7.50}
$$

and

$$
\begin{aligned}
2K(k,\epsilon) &= \sum_{\epsilon} S_1(k_1,\epsilon_1,k_2,\epsilon_2,k,\epsilon)S_1(k_3,\epsilon_3,k_4,\epsilon_4,-k,\epsilon)/2k_1\cdot k_2 \\
&\quad +\sum_{\epsilon} S_2(\cdots)S_2(\cdots)-S_3(\cdots)S_3(\cdots),
\end{aligned}
\tag{7.51}
$$

where $K(k,\epsilon)$ is the standard kinematic factor, see eq.(4.20) or eq.(6.8). Thus we recovered the standard kinematic factor from the product of two 3-particle kinematic factors by summing over all the intermediate states.

As a last comment to the factorization, we point out that before our explicit computation was finished, A Morozov [108] suggested a formula for the two-loop contribution to the four-particle amplitude which can easily be seen not satisfying the factorization condition. We also remark that the factorization of the two-loop four-particle amplitude has been discussed by Yasuda [112] by using the rather implicit representation of period matrix and Θ-constants, and claimed that only matter supercurrent part gives contribution to the lowest order in the factorization limit. But what we found is that the ghost supercurrent also gives a non-vanishing contribution at the same order.

8. Ultra-High Energy Scattering or Quantum Gravity Corrections from Superstring Theory

In this section we will study the high energy behaviour of the superstring scattering amplitudes. There are many motivations for doing that, perhaps the most important being the fact that superstring theory is supposed to be a scheme in which quantum gravitational effects are in principle computable, making possible the important goal of understanding the quantum corrections to general relativity. More precisely two regions of interest have been explored for the four massless bosonic particle scattering amplitude: the large s, small t limit [19] and the large s, fixed angle limit [21]. We shall be concerned in this section with the first region which could provide the computation of string corrections to the large distance gravitational interactions. This has been studied in [19, 20] up to one-loop level within the framework of standard superstring theories [1, 2] and to arbitrary loops by Regge-Gribov techniques [19] as far as the leading term is concerned. The two-loop stringy analysis was carried out in [22] by using the two-loop amplitudes obtained in [15, 16]. This is important because, as also noticed in [19], it is at two loops that one starts probing the full string interaction and therefore expecting genuine new effects appearing. It is also important to check that the two-loop amplitudes we have obtained can be used in practice to get physical results like this asymptotic behaviour.

To begin with let us study the high energy behaviour of the tree amplitude in type II superstring, eq.(4.19):

$$A_4(1b \to 2c) = K \cdot \tilde{K} g^2 \frac{\Gamma(-\frac{s}{2})\Gamma(-\frac{t}{2})\Gamma(-\frac{u}{2})}{\Gamma(1+\frac{s}{2})\Gamma(1+\frac{t}{2})\Gamma(1+\frac{u}{2})}, \tag{8.1}$$

where we have appended the coupling constant g into the amplitude. Here we are interested in the following high energy limit:

$$\begin{aligned} s &= -2k_1 \cdot k_b = -2k_2 \cdot k_c \to \infty, \\ t &= -2k_1 \cdot k_2 = -2k_b \cdot k_c \to 0. \end{aligned} \tag{8.2}$$

See fig.10 for the hinematics of the scattering. By using the Stirling formula

$$\Gamma(x+1) = \sqrt{2\pi} \frac{x^{x+\frac{1}{12}}}{e^x}, \tag{8.3}$$

we get

$$
\begin{aligned}
\frac{\Gamma(-\frac{s}{2})\Gamma(-\frac{t}{2})\Gamma(-\frac{u}{2})}{\Gamma(1+\frac{s}{2})\Gamma(1+\frac{t}{2})\Gamma(1+\frac{u}{2})} &= \frac{\Gamma(-\frac{s}{2})\Gamma(-\frac{t}{2})\Gamma(\frac{s}{2}+\frac{t}{2})}{\Gamma(1-\frac{s}{2})-\frac{t}{2})\Gamma(1+\frac{t}{2})\Gamma(1-\frac{s}{2})} \\
&= -\frac{\sin\pi(\frac{s}{2}+\frac{t}{2})}{\sin\pi\frac{s}{2}} \frac{\Gamma(-\frac{t}{2})}{\Gamma(1+\frac{t}{2})} \left(\frac{\Gamma(\frac{s}{2}+\frac{t}{2})}{\Gamma(1+\frac{s}{2})}\right)^2 \\
&= -\frac{\Gamma(-\frac{t}{2})}{\Gamma(1+\frac{t}{2})} \left(\frac{s}{2}\right)^{2+t} \exp\left(-i\frac{1}{2}\pi t\right),
\end{aligned}
\tag{8.4}
$$

where we have replaced $\exp\left(-i\frac{1}{2}\pi t\right)$ for $\frac{\sin\pi(\frac{s}{2}+\frac{t}{2})}{\sin\pi\frac{s}{2}}$ [18]. This is obtained by the following limiting procedure: setting $s = \tilde{s}(1+i\epsilon)$ ($\tilde{s}$ positive real) and taking $\tilde{s}$ to be large and then $\epsilon \to 0^+$. The asymptotic behaviour of the kinematic factor $K \cdot \tilde{K}$ is [19]

$$
K \cdot \tilde{K} \sim \left(\frac{s}{2}\right)^4 \epsilon_1 \cdot \epsilon_2 \epsilon_b \cdot \epsilon_c. \tag{8.5}
$$

By using eqs.(8.4) and (8.5) we get the following asymptotic behaviour of the graviton-graviton scattering

$$
\begin{aligned}
&A_{\text{tree}}(1b \to 2c) = a_{\text{tree}}(s,t), \\
&a_{\text{tree}}(s,t) \to \beta(t) s^{\alpha(t)} = -g^2 \frac{\Gamma(-\frac{t}{2})}{\Gamma(1+\frac{t}{2})} \left(\frac{s}{2}\right)^{2+t} \exp\left(-i\frac{1}{2}\pi t\right),
\end{aligned}
\tag{8.6}
$$

which is Regge-behaved. The corresponding Regge amplitude is a simple pole in the t-channel angular momentum at $J = \alpha(t) = 2 + t$, i.e.,

$$
a_J(t) \sim \frac{g^2}{J - \alpha(t)}, \tag{8.7}
$$

corresponding to the graviton trajectory exchange. It is a simple matter to show that the impact parameter transform of (8.6) violates partial wave unitarity at high energy [18], thus indicating the loop corrections are important.

The one-loop correction to the asympototic behaviour of the tree amplitude was calculated in [19, 20] by using the exact integral representation of the 4-garviton scattering amplitude. The following result was obtained

$$
a_{\text{one loop}}(s,t) \longleftarrow iF(D) g^4 \frac{\left(\frac{s}{2}\right)^3}{|t|^{(6-D)/2}}. \tag{8.8}
$$

Here $F(D)$ is a function depending on the dimension D of the compactified superstring theory. We will not repeat their derivation here. Nevertheless I would like to make a few comments about (8.8). First the asymptotic contribution at one loop was obtained from the integration over a corner of the moduli space: the Im$\tau \to \infty$ region. This is a general feature. When we make some asumptions we simplify the problem and we can get exlicit result. We will show later this is also what happens at two loops. Second the result is in accordance with the general Regge-behaviour at g-loops: $(\frac{s}{2})^{g+2+\frac{t}{g+1}} \times$ (t dependent function). As to the leading behaviour this result can be derived by using the Regge-Gribov method. It is also generalized to multi-loops [19]. Evenmore one can do a resummation over all loops and get a result which has an operator eikonal form. The s-channel unitarity was recovered. Due to the lack of space we will not discuss all these aspects here. See [19] for detail discussions. In what follows we will present some details how we obtain the two-loop Regge-behaviour from the explicit two-loop four-particle amplitude, eq.(6.23).

In hyperelliptic language the rescattering term depicted in fig.11 corresponds to the following corner in moduli space:

$$\begin{aligned} &a_2 \to a_1, \qquad && u = a_2 - a_1 \to 0, \\ &a_4 \to a_3, && \lambda = a_4 - a_3 \to 0, \\ &a_6 \to a_5, && v = a_6 - a_5 \to 0. \end{aligned} \tag{8.9}$$

This degeneration limit of the moduli space is expected from the diagrammatic analysis as shown in fig.12 and fig.13. Fig.12 shows the realization of the genus two Riemann surface as a double covering of the (cut) Riemann sphere. We indicated there also the canonical homology cycles chosen. From general arguments the rescattering term corresponds to the pinching of the three cycles α_1, α_2 and α_3 by taking $z_{1,2}$ in the upper and $z_{b,c}$ in the lower Riemann sheet (see fig.13). All these cycles can be identified in the hyperelliptic language as shown in fig.12, and one immediately sees that the rescattering term correspdonds to the corner in moduli space as given by eq.(8.9).

Under the limit (8.9) we have

$$T = 32\pi^4 \frac{\Im}{|a_{13}a_{35}a_{51}|^2}, \tag{8.10}$$

where we have defined $\Im$ as

$$\begin{aligned} \Im &\equiv \det \mathrm{Im}\tau \\ &= \frac{1}{\pi^2}(\ln|u|\ln|v| + \ln|v|\ln|\lambda| + \ln|\lambda|\ln|u|). \end{aligned} \tag{8.11}$$

τ is the standard period matrix.

By using the explicit formulas of $\langle \partial X(x+) \cdot \partial X(x-) \rangle$ and $I(x)$ and eq.(8.11), one can easily prove the following (for generic x):

$$\begin{aligned} &\langle \partial X(x+) \cdot \partial X(x-) \rangle = 0, \\ &I(x) = 0. \end{aligned} \tag{8.12}$$

As to $R(x)$ appearing in (6.23) we have, in this limit,

$$R(x) = y(x) \left(-\sum_i \frac{\delta_i k_i}{z_i - x} + \sum_i \delta_i k_i \sum_{i=1,3,5} \frac{c_i}{a_i - x} \right), \tag{8.13}$$

where we have defined c_i as follows:

$$\begin{aligned} c_1 &= \frac{\ln|v| \ln|\lambda|}{\ln|u| \ln|v| + \ln|v| \ln|\lambda| + \ln|\lambda| \ln|u|}, \\ c_3 &= \frac{\ln|u| \ln|v|}{\ln|u| \ln|v| + \ln|v| \ln|\lambda| + \ln|\lambda| \ln|u|}, \\ c_5 &= \frac{\ln|\lambda| \ln|u|}{\ln|u| \ln|v| + \ln|v| \ln|\lambda| + \ln|\lambda| \ln|u|}. \end{aligned} \tag{8.14}$$

Here the δ_i ($i = 1,2,\cdots,4$) are equal to 1 if z_i lies on the upper sheet and -1 if z_i lies on the lower sheet.

By using the fact that z_1 and z_2 are on the same sheet, z_b and z_c on the other one, and substituting the above results ((8.10)–(8.13)) into (6.23), we see that in the limit (8.9):

$$S(x) \to -2s \frac{(z_1 - z_2)(z_b - z_c)}{\prod_i (x - z_i)}, \tag{8.15}$$

and that the right hand side of eq.(6.23) is dominated by the $S(r)\bar{S}(\bar{s})$ term*, so we get:

$$\begin{aligned} AII(k,\epsilon) \sim \tilde{K}(k,\epsilon) \int \frac{d^2u d^2v d^2\lambda}{|uv\lambda|^2} \cdot \frac{1}{\Im^5} \frac{|a_{13}a_{35}a_{51}|^4 d^2z_1 d^2z_2 d^2z_b d^2z_c}{|y(z_1)y(z_2)y(z_b)y(z_c)|^2} \\ \times s^2 |(z_1 - z_2)(z_b - z_c)|^2 \langle \prod e^{ik\cdot X} \rangle. \end{aligned} \tag{8.16}$$

where (and in what follows) we drop the overall constants.

---

* The second and third terms in the right hand side of eq.(6.23) are easily seen to be: $O(\frac{1}{\ln|u|}, \frac{1}{\ln|v|}, \frac{1}{\ln|\lambda|})^2 \to 0$ and $s \cdot O(\frac{1}{\ln|u|}, \frac{1}{\ln|v|}, \frac{1}{\ln|\lambda|}) \to 0$ respectively.

From (8.16) one sees that the dependence on r and s disappears and the integration over z_i is manifestly Möbius invariant. This should be the case because we started from an amplitude which is Möbius invariant (modulo total derivatives in moduli space which comes from the arbitrariness of the supercurrent insertion points on the Riemann surface). Since the dependence on r and s disappears, we should get a manifestly Möbius invariant expression.

The integration over z_i can be simplified by a change of variables. This is motivated by the exponential factor $\langle\prod e^{ik\cdot X}\rangle$ as we shall see later. Choosing the homology cycles as shown in fig.3, we have

$$\begin{aligned}\omega_1 &= \frac{1}{2\pi i}\left(\frac{1}{z-a_1}-\frac{1}{z-a_3}\right)dz,\\ \omega_2 &= \frac{1}{2\pi i}\left(\frac{1}{z-a_5}-\frac{1}{z-a_3}\right)dz,\end{aligned} \tag{8.17}$$

in the limit (8.9) and when z is away from the branch points. Then if we define ρ and σ as

$$\begin{aligned}\rho &= 2\pi i\int_{z_1}^{z_2}\omega,\\ \sigma &= 2\pi i\int_{z_b}^{z_c}\omega,\end{aligned} \tag{8.18}$$

it follows that

$$\begin{aligned}d^2\rho_1 d^2\rho_2 &= \left|\frac{a_{13}a_{35}a_{51}(z_1-z_2)}{y(z_1)y(z_2)}\right|^2 d^2z_1 d^2z_2,\\ d^2\sigma_1 d^2\sigma_2 &= \left|\frac{a_{13}a_{35}a_{51}(z_b-z_c)}{y(z_b)y(z_c)}\right|^2 d^2z_b d^2z_c.\end{aligned} \tag{8.19}$$

Here we have used

$$y(z_i) = \pm(z_i-a_1)(z_i-a_3)(z_i-a_5). \tag{8.20}$$

By substituting (8.19) into (8.16) we get

$$AII(k,\epsilon) \sim \tilde{K}(k,\epsilon)s^2\int\frac{d^2u d^2v d^2\lambda}{|uv\lambda|^2}\cdot\frac{1}{\Im^5}d^2\rho_1 d^2\rho_2 d^2\sigma_1 d^2\sigma_2\langle\prod e^{ik\cdot X}\rangle. \tag{8.21}$$

The last piece of computations needed is the factor $\langle \prod e^{ik\cdot X} \rangle$. We have*

$$\langle \prod e^{ik\cdot X} \rangle = \exp \left\{ -\frac{s}{2} \langle (X(z_1) - X(z_2))(X(z_b) - X(z_c)) \rangle \right. \\ \left. -\frac{t}{2} \langle (X(z_1) - X(z_b))(X(z_2) - X(z_c)) \rangle \right\} . \tag{8.22}$$

General arguments give the expression for the propagator $\langle X(z)X(w) \rangle$ as follows [115, 116]:

$$\langle X(z)X(w) \rangle = \ln |E(z,w)|^2 + \frac{\pi}{2} \left(\int_z^w \omega - \text{c.c.} \right) \cdot (\text{Im}\tau)^{-1} \cdot \left(\int_z^w \omega - \text{c.c.} \right), \tag{8.23}$$

where $E(z,w)$ is the prime form defined on the Riemann surface [71, 70]. By using this expression, we have

$$\langle (X(z_1) - X(z_2))(X(z_b) - X(z_c)) \rangle = \ln \left| \frac{E(z_1,z_b)E(z_2,z_c)}{E(z_2,z_b)E(z_1,z_c)} \right|^2 \\ - \pi \left(\int_{z_1}^{z_2} \omega - \text{c.c.} \right) \cdot (\text{Im}\tau)^{-1} \cdot \left(\int_{z_b}^{z_c} \omega - \text{c.c.} \right) . \tag{8.24}$$

In appendix C it is shown that the first term is zero under the limit (8.9), then by taking into account the definitions of ρ and σ, eq.(8.18), we have

$$\langle (X(z_1) - X(z_2))(X(z_b) - X(z_c)) \rangle = \frac{1}{\pi} \text{Re}\rho \cdot (\text{Im}\tau)^{-1} \cdot \text{Re}\sigma. \tag{8.25}$$

The calculation of the other term appearing in (8.22) is more conveniently done by making use of the following integral representation:

$$\langle (X(z_1) - X(z_b))(X(z_2) - X(z_c)) \rangle = \\ - \frac{1}{\pi} \int d^2z \langle \partial X(z)(X(z_1) - X(z_b)) \rangle \langle \bar{\partial} X(\bar{z})(X(z_2) - X(z_c)) \rangle. \tag{8.26}$$

From the explicit expression of $\langle \partial X(z)(X(z_1) - X(z_b)) \rangle$ derived in section 3, and evaluating the integral in z by looking at the dominant regions, we get

$$\langle (X(z_1) - X(z_b))(X(z_2) - X(z_c)) \rangle = \frac{4 \ln|u| \ln|v| \ln|\lambda|}{\ln|u|\ln|v| + \ln|v|\ln|\lambda| + \ln|\lambda|\ln|u|}. \tag{8.27}$$

---

* The convention is

$$\langle X^\mu(x) X^\nu(y) \rangle = -g^{\mu\nu} \langle X(x)X(y) \rangle \to -g^{\mu\nu} \ln |x-y|^2,$$

for $x \to y$.

Substituting (8.25) and (8.27) into (8.21) we have

$$AII(k,\epsilon) \sim \tilde{K}(k,\epsilon)s^2 \int \frac{d^2u d^2v d^2\lambda}{|uv\lambda|^2} \cdot \frac{1}{\Im^5} d^2\rho_1 d^2\rho_2 d^2\sigma_1 d^2\sigma_2$$
$$\times \exp\left\{-\frac{s}{2\pi}\mathrm{Re}\rho \cdot (\mathrm{Im}\tau)^{-1} \cdot \mathrm{Re}\sigma - 2t\frac{\ln|u|\ln|v|\ln|\lambda|}{\ln|u|\ln|v| + \ln|v|\ln|\lambda| + \ln|\lambda|\ln|u|}\right\}. \tag{8.28}$$

The integration over $\mathrm{Re}\rho$ and $\mathrm{Re}\sigma$ can be easily done by using the following formula:

$$\int_{-\infty}^{+\infty} e^{-su_1u_2} du_1 du_2 = \frac{2\pi i}{s} \qquad \text{for } s \to i\infty \tag{8.29}$$

giving finally:

$$AII(k,\epsilon) \sim \tilde{K}(k,\epsilon) \int d\ln|u| d\ln|v| d\ln|\lambda|$$
$$\frac{1}{\Im^4} \exp\left\{-2t\frac{\ln|u|\ln|\lambda|\ln|v|}{\ln|u|\ln|v| + \ln|v|\ln|\lambda| + \ln|\lambda|\ln|u|}\right\}. \tag{8.30}$$

By remembering the definition of $\Im$, eq.(8.11), and comparing with the standard Feynman parametric representation of fig.1, we recognize that this result can be rewritten as ($q^2 = t$):

$$AII(k,\epsilon) \sim \tilde{K}(k,\epsilon) \int d^8q_1 d^8q_2 \frac{1}{q_1^2 q_2^2 (q - q_1 - q_2)^2}. \tag{8.31}$$

This gives exactly the eikonal approximation for high energy scattering of gravitons at two loops by taking into account the asymptotic behaviour of $\tilde{K}(k,\epsilon)$:

$$\tilde{K}(k,\epsilon) \sim s^4 \epsilon_1 \cdot \epsilon_2 \epsilon_b \cdot \epsilon_c, \qquad \text{for} \qquad s \to \infty. \tag{8.32}$$

The above analysis can be extended to lower dimensional superstring theories. Explicit calculations can be done for a class of interesting compactified models called D-dimensional fermionic strings [77, 78]. From eq.(9.34) we know that for the maximally space-time supersymmetric case the four-particle amplitude can be written as (with a suitable choice of the gauge group):

$$AII_D(k,\epsilon) \sim \tilde{K}(k,\epsilon) \int \frac{d^2a_2 d^2a_4 d^2a_6 |a_{13}a_{35}a_{51}|^2}{T^{D/2}\prod_{i<j}^6 |a_{ij}|^2}$$
$$\prod_i \left|\frac{dz_i(z_i - x)}{y(z_i)}\right|^2 \sum_s |Q_s|^{5-D/2} J_D(x,\bar{x}) \left\langle \prod e^{ik\cdot X} \right\rangle. \tag{8.33}$$

From the explicit expression of $J_D(x,\bar{x})$ given in eq.(9.35) we know that for very large s the relevant terms contributing to the rescattering term are the same as for $D=10$. Under the limit of (8.9) it is easily seen that the relevant summation in (8.33) is coming from $s=6,7,8$ and 9 (following the convention used in [12-16]), and the rest are all suppressed by powers of u, v or λ. We have

$$\begin{aligned}\frac{|Q_s|^{5-D/2}}{T^{D/2}} &\sim |a_{13}a_{35}a_{51}|^{10-D+D}\frac{1}{\Im^{D/2}}\\ &= |a_{13}a_{35}a_{51}|^{10}\frac{1}{\Im^{D/2}},\end{aligned} \tag{8.34}$$

which is dependent on D (the dimension of the space-time) through the power of $\Im$ only. So we get finally:

$$\begin{aligned}AII_D(k,\epsilon) &\sim \tilde{K}(k,\epsilon)\int d\ln|u|d\ln|v|d\ln|\lambda|\\ &\quad \frac{1}{\Im^{D/2-1}}\exp\left\{-2t\frac{\ln|u|\ln|\lambda|\ln|v|}{\ln|u|\ln|v|+\ln|v|\ln|\lambda|+\ln|\lambda|\ln|u|}\right\},\\ &\sim \tilde{K}(k,\epsilon)\int d^{D-2}q_1 d^{D-2}q_2\frac{1}{q_1^2q_2^2(q-q_1-q_2)^2}.\end{aligned} \tag{8.35}$$

This is the correct result of the eikonal approximation for the high energy scattering of gravitons in D dimension. From (8.35) one sees that $AII_D(k,\epsilon)$ is divergent for $D\to 4$ as

$$AII_D(k,\epsilon)\sim s^4\cdot\frac{1}{t}\cdot\left(\frac{1}{D-4}\right)^2, \tag{8.36}$$

In the above we have implicitly taken $t\ln s\to 0$, so we cannot see the full Regge behaviour $s^{4+t/3}$ at two loops [19]. The divergence in (8.36) for $D\to 4$ is an infrared divergence. The detail analysis of this infrared divergence and also the comparision with Feymann diagrams can be found in [22, 23], which we will not repeat here.

Another asymptotic configuration, which is sub-leading by a power of s, corresponds to a 2-graviton exchange in te t-channel which interact in the "middle" by exchange string excitations. This configuration was called H-term in [19, 22], for reasons evident from fig.14. For the calculation of this configuration and its physical implication, please see [22, 23] for details.

9. Four-Dimensional Superstrings and Their Two-Loop Computations

Up to now all of our calculations were done for ten-dimensional superstring theories – the heterotic string and type II superstring theories. But these theories are not realistic in that our space-time is four-dimensional, not ten-dimensional. So we should compactify six dimensions to escape observations (up to now), or construct some realistic string theories directly in four dimensions – four-dimensional string theories. After the resurgence of string theory in the fall of 1984, there are a lot of proposals of compactifying superstring theory, like compactification on Calabi-Yau manifold [117], orbifold compactification [118], covariant lattice costruction [119, 120] and (free) fermionic construction [77, 78]. The simplest scheme is the fermionic construction. In this section we will review briefly this construction and present some two-loop calculations for these models. Before doing that let us make some general coments about low dimensional string theory.

Everyone who once studied string theory knows that bosonic string theory (described by bosonic coordinates $X^\mu(\sigma,\tau)$, $\mu = 1, \cdots, D$) is consistent in $D = 26$ dimensions (called critical dimension) only. By introducing fermionic coordinates $\psi^\mu(\sigma,\tau)$, $\mu = 1, \cdots, D$ which combined with X^μ form a supermultiplet (in two dimensions of course) and supersymmetry, the critical dimension changes to $D_c = 10$. But what is the essence of this construction—by introducing 2-dimnsional fields, like X^μ and ψ^μ ? We are not obliged to interpret that all these bosonic coordinates are related to space-time coordinates. Probably only part of X^μ are related to space-time coordinates. Also peculiar to two-dimensional models is boson-fermion equivalence. A fermionic field can be constructed from bosonic field(s). We can bosonize some bosonic fields into fermionic fields or vice versa. These fields will not have any space-time interpretation—they are just hidden or internal coordinates. Then what is (are) the consistent condition(s) for the construction of string theoy? They turn out to be conformal invariance and modular invariance. Conformal invariance restricts the number of internal coordinates we should choose. For heterotic string theory in D dimensions, we have the following list of two-dimensional fields appearing in the construction of the models:

$X^\mu, \psi^\mu(\ \mu = 1, \cdots, D)$ which have a space-time interpretation;

internal fermions:

$\chi^I, y^I, \omega^I, I = 1, \cdots, 10 - D$ which are 3(10-D) free (right) Majorana fermions;

$\phi^A, A = 1, \cdots, 2(26 - D)$ which are 2(26-D) free (left) Majorana fermions.

The matter supercurrent is realized nonlinearly by these fields as [121]

$$J_m = \psi \cdot \partial X + \sum_{I=1}^{10-D} \chi^I y^I \omega^I. \tag{9.1}$$

As we discused in section 2, fermion fields have non-trivial boundary conditions. For different boundary conditions these fermion fileds give different contributions to the partition function. The partition fuction should be modular invariant, guarantee the cancellation of gauge and gravitational anomalies [122] (in the low energy or field limit of string theory). How modular invariance restricts the possible choice of boundary conditions was discused in [77, 78]. By studying the one-loop partition function and its multi-loop counterparts, they found a set of rules of constructing modular invariant string theories. By using their rules one can construct a lot of consistent models. Some of them are supersymmetric, having $N = 4$, $N = 2$ or $N = 1$ space-time supersymmetry. The one-loop vanishing of the partition function of all these supersymmmetric models can easily be proved [77]. The two-loop vanishing was proved in [80] by making use of th method of ref.[13], based on hyperelliptic description of genus $g = 2$ Riemann surface. Nevertheless one can also show that the prescription for two-loop computations (for non-vanishing amplitudes in particular) developed in the previous sections can also be extended to these low dimensional models. Befor doing that let us review briefly their construction.

To illustrate their construction let us first set some notations. For the torus it can be represented by a flat parallelogram in the complex plane with side 1 and τ corresponding to its two non-contractible loops. We denote the one-loop spin structure as $\begin{bmatrix}\alpha \\ \beta\end{bmatrix}$ where α and β are subsets of F^* containing those fermions that are periodic around 1 and τ respectively. For nay set α of fermions we define its characteristic function as

$$\alpha(f) = 1, \qquad \text{if} \quad f \in \alpha; \qquad\qquad \alpha(f) = 0, \quad \text{otherwise}. \tag{9.2}$$

* F denotes the set of all fermions.

We also define addition and multiplication of fermion sets as ordinary addition and multiplication modulo 2 of their characteristic functions. Thus addition is the symmetric difference and multiplication the intersection:

$$\begin{aligned} \alpha + \alpha' &= \alpha \dot{\cup} \alpha' - \alpha \cap \alpha', \\ \alpha\alpha' &= \alpha \cap \alpha'. \end{aligned} \tag{9.3}$$

The one-loop vacuum amplitude can be written as

$$Z_{one-loop} = \int \frac{d^2\tau}{(\mathrm{Im}\tau)^{\frac{D}{2}+1}\eta^{12}\bar{\eta}^{24}} \sum_{\alpha,\beta} C\begin{bmatrix}\alpha\\ \beta\end{bmatrix} X_{long,3/2} \prod_{f\in F'} \Theta\begin{bmatrix}\alpha(f)\\ \beta(f)\end{bmatrix}^{1/2}(\tau). \tag{9.4}$$

Here $\eta(\tau)$ is the Didekin eta function; $\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}$ is the Jacobi theta function with characteristic $\begin{bmatrix}\epsilon/2\\ \epsilon'/2\end{bmatrix}$. By slight abuse of notation we suppress the fact that this theta function should be complex conjugated when f is a left (anti-holomorphic) fermion. F' is the set all "transverse" fermions, i.e. F minus two ψ^{μ}'s. The contribution of two "longitudinal" fermions and superghost is denoted by $X_{long,3/2}$. It is different from 1 beyond one loop. Finally the coefficients $C\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ take values equal to either $+1$ or -1 which should be determined from modular invariance.

For torus the modular group (all the modular transformations from a group) is generated by

$$\tau \to \tau + 1, \tag{9.5}$$

and

$$\tau \to -\frac{1}{\tau}. \tag{9.6}$$

Under these transformations the eta and theta functions are transformed as [77]

$$\eta \to e^{i\pi/12}\eta,\ \Theta_1 \to e^{i\pi/4}\Theta_1,\ \Theta_2 \to e^{i\pi/4}\Theta_2,\ \Theta_3 \leftrightarrow \Theta_4; \tag{9.7}$$

and

$$\eta \to (-i\tau)^{1/2}\eta,\ \frac{\Theta_1}{\eta} \to e^{i\pi/2}\frac{\Theta_1}{\eta},\ \frac{\Theta_2}{\eta} \leftrightarrow \frac{\Theta_4}{\eta},\ \frac{\Theta_3}{\eta} \to \frac{\Theta_3}{\eta}, \tag{9.8}$$

respectively. To ensure the modular invariance of eq.(9.4) we should impose the following conditions on the coefficients C:

$$\begin{aligned} C\begin{bmatrix}\alpha\\ \beta\end{bmatrix} &= -\exp\left(\frac{1}{8}i\pi\sum_f \alpha(f)\right)\ C\begin{bmatrix}\alpha\\ \alpha+\beta+F\end{bmatrix}, \\ C\begin{bmatrix}\alpha\\ \beta\end{bmatrix} &= \exp\left(\frac{1}{4}i\pi\sum_f \alpha(f)\beta(f)\right)\ C\begin{bmatrix}\beta\\ \alpha\end{bmatrix}, \end{aligned} \tag{9.9}$$

where $\sum_f$ stands for $\Sigma_{right\ fermions} - \Sigma_{lef\ fermions}$. This is what one loop modular invariance tells us. At multi-loops, modular invariance requires the coefficient $C\begin{bmatrix}\alpha_1\alpha_2\cdots\alpha_g\\ \beta_1\beta_2\cdots\beta_g\end{bmatrix}$ factorized into a product of g one loop coefficients:

$$C\begin{bmatrix}\alpha_1\cdots\alpha_g\\ \beta_1\cdots\beta_g\end{bmatrix} = C\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}\cdots C\begin{bmatrix}\beta_1\\ \beta_g\end{bmatrix} \tag{9.10}$$

But there is additional constraint among one-loop coefficients that reads [77]

$$(-1)^{\alpha(\psi)+\alpha'(\psi)}\exp\left(\frac{1}{4}i\pi\sum_f \alpha(f)\alpha'(f)\right)\ C\begin{bmatrix}\alpha\\ \beta\end{bmatrix}C\begin{bmatrix}\alpha'\\ \beta'\end{bmatrix} = C\begin{bmatrix}\alpha\\ \beta+\alpha'\end{bmatrix}C\begin{bmatrix}\alpha'\\ \beta'+\alpha\end{bmatrix} \tag{9.11}$$

Eqs.(9.9)—(9.11) are all the equations required for modular invariance.

The analysis of these equations is rather lengthy [77], but the results can be presented with reasonable dispatch. The sets of fermions which enter the summation in eq.(9.4) form an additive group Ξ of subsets of F, containing in particular F. The precise choice of Ξ and $C\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ defines the theory. This choice is restricted by a series of consistency conditions, which are derived from eqs.(9.9)—(9.11). Firstly, admissible spin structure assignments to the fermions must be such that the supercurrent J_m has also a well-defined spin structure. This leads to the requirement that for all elements $\alpha \in \Xi$:

$$(-1)^\alpha J_m (-1)^\alpha = \delta_\alpha J_m, \tag{9.12}$$

where $(-1)^\alpha$ is the ferimon number operator that counts the ferimons in α modulo 2 and δ_α is a sign that takes the value -1 if $\psi^\mu \in \alpha$ and $+1$ otherwise. Note in particular that for any assigment $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ the supercurrent J_m, the gravitino and superghosts, as well as the D

ferimon field ψ^μ all have the same spin structure $\begin{bmatrix} \alpha(\psi^\mu) \\ \beta(\psi^\mu) \end{bmatrix}$. The second condition is that for all $\alpha \in \Xi$

$$\begin{aligned} n(\alpha) &= 0 \bmod 8, \\ n(\alpha\alpha') &= 0 \bmod 4, \\ n(\alpha\alpha'\alpha''\alpha''') &= 0 \bmod 2, \end{aligned} \tag{9.13}$$

where $n(\alpha)$ is the number of transverse right ferimons minus the number of left ferimons in set α. Finally the coefficients $C\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ must obey

$$\begin{aligned} C\begin{bmatrix} \alpha \\ \beta \end{bmatrix} &= \varepsilon^2 C\begin{bmatrix} \beta \\ \alpha \end{bmatrix}, \\ C\begin{bmatrix} \alpha \\ \alpha \end{bmatrix} &= -\varepsilon_\alpha C\begin{bmatrix} \alpha \\ F \end{bmatrix}, \\ C\begin{bmatrix} \alpha \\ \beta \end{bmatrix} C\begin{bmatrix} \alpha \\ \gamma \end{bmatrix} &= \delta_\alpha C\begin{bmatrix} \alpha \\ \beta+\gamma \end{bmatrix}, \end{aligned} \tag{9.14}$$

where $\varepsilon_\alpha = \exp(\frac{1}{8} i\pi n(\alpha))$. Note that condition (9.13) ensures the consistency of eq.(9.14).

By choosing a basis $\{b_0 = F,\ b_1, \cdots,\ b_N\}$ which generates Ξ, we can solve (9.14) explicitly. We have

$$\begin{aligned} \alpha &= \sum_{i=1}^{A} b_{I(i)}, \qquad \beta = \sum_{j=1}^{B} b_{J(j)}, \\ C\begin{bmatrix} \alpha \\ \beta \end{bmatrix} &= \prod_{i=1}^{A} \delta_{b_{J(j)}}^{B-1} \prod_{j=1}^{B} \delta_{b_{J(j)}}^{A-1} \varepsilon^2_{b_{J(j)}\alpha} \prod_{i=1}^{A}\prod_{j=1}^{B} C_{J(j)I(i)}, \end{aligned} \tag{9.15}$$

where $C_{J(j)I(i)} \equiv C\begin{bmatrix} b_{J(j)} \\ b_{I(i)} \end{bmatrix}$ are not all independent. C_{ij} with $i \le j$ $(j \ne 0)$ can be determined from (9.14) by C_{00} and C_{kl} for $k > l = 0, \cdots, N$.

The application of these rules is straightforward. By carefully analysing the spectrum of non-interacting string states, we found that a string model is supersymmetric if and only if there exists a set $s \in \Xi$ such that s is a set of precisely ten right ferimons including the ψ^μ. Without loss of generality we may assume that

$$s = \{\psi^\mu,\ \chi^I\}. \tag{9.16}$$

The one-loop vacuum amplitudes of these supersymmetric models can easily be proved to be vanishing. In what follows we prove that the two-loop vacuum amplitudes of these models also vanish. Evenmore explicit calculation can also be done for non-vanishing amplitude, for example the four-particle amplitude in $N=4$ space-time SUSY model.

To be specific, we shall consider the following choice of Ξ, indicating the corresponding four-dimensional string models they define [77, 79, 80]:

(1) N=4 space-time SUSY model, $\Xi(4)=\{F,\emptyset,s,F+s\}$, where $\emptyset$ is the empty set;

(2) N=2 space-time SUSY model, $\Xi(2)=\Xi(4)\cup\{s_1,s_1+\gamma \text{ for every } \gamma\in\Xi(4)\}$, where $s_1=\{\psi^\mu,(\chi_1,\chi_2),(y_3,\cdots,y_6),(\phi_1,\cdots,\phi_{16})\}$;

(3) N=1 space-time SUSY model, $\Xi(1)=\Xi(2)\cup\{s_2,s_2+\gamma \text{ for every } \gamma\in\Xi(2)\}$, where $s_2=\{\psi^\mu,(\chi_3,\chi_4),(y_5,y_6),(\omega_1,\omega_2),(\phi^1,\cdots,\phi^{16})\}$.

The two-loop vacuum amplitude can be written generally as:

$$Z_{\text{two loops}}=\int \mathrm{d}\mu\sum_{\alpha,\beta}C\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\langle J(x_1)J(x_2)\rangle_{\begin{bmatrix}\alpha\\ \beta\end{bmatrix}}\prod_{f\in F'}\left(Q\begin{bmatrix}\alpha(f)\\ \beta(f)\end{bmatrix}\right)^{1/8}, \tag{9.17}$$

where $J(x)$ is the supercurrent:

$$\begin{aligned}J(x)&=J_m(x)+J_{gh}(x)\\ &=\psi\cdot\partial X+\sum_I\chi^I y^I\omega^I+2c\partial\beta-\gamma b+3\partial c\beta.\end{aligned} \tag{9.18}$$

The calculation of the correlator $\langle J(x_1)J(x_2)\rangle$ appearing in (9.17) proceeds the same way as before. We have

$$\begin{aligned}\langle J(x_1)J(x_2)\rangle&=K_1\langle\psi(x_1)\psi(x_2)\rangle_{s(\psi)}+K_2\langle\partial\psi(x_1)\psi(x_2)\rangle_{s(\psi)}\\ &+K_3\sum_I\langle\psi(x_1)\psi(x_2)\rangle_{s(\chi^I)}\langle\psi(x_1)\psi(x_2)\rangle_{s(Y^I)}\langle\psi(x_1)\psi(x_2)\rangle_{s(\omega^I)},\end{aligned}$$

where $s(\psi)$ denotes the structure of ψ^μ, K_1, K_2 and K_3 are spin structure independent factors. By choosing $x_{1,2}=\infty\pm$, we have

$$\begin{aligned}Y&\equiv\langle\psi(x_1)\psi(x_2)\rangle_{s(\psi)}=\frac{1}{4}\sum_{i=1}^{3}(A_i-B_i),\\ \tilde{Y}&\equiv\langle\partial\psi(x_1)\psi(x_2)\rangle_{s(\psi)}=\frac{1}{8}\sum_{i=1}^{3}(A_i^2-B_i^2),\end{aligned} \tag{9.19}$$

as we used before*.

To proceed further, we shall do the summation in eq.(9.17) replacing the contribution of each spin structure $\begin{bmatrix}\alpha(f)\\ \beta(f)\end{bmatrix}$ by an average $\frac{1}{16}\sum_{n,m}\begin{bmatrix}(\alpha+ns)(f)\\ (\beta+ms)(f)\end{bmatrix}$, over all its translation by s (here and in the following s means the transverse part and $n=(n_1,\ n_2)$ etc.). We get

$$\begin{aligned} Z_{two\ loops} = \int d\mu \sum \prod_{f\notin s} Q^{1/8}_{s(f)} \Bigg(&\mathrm{SUM}_1\begin{bmatrix}\alpha\\ \beta\end{bmatrix} + \mathrm{SUM}_2\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \\ &+ \sum_I Y\begin{bmatrix}\alpha(y^I)\\ \beta(y^I)\end{bmatrix} Y\begin{bmatrix}\alpha(\omega^I)\\ \beta(\omega^I)\end{bmatrix} \mathrm{SUM}^I_3\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\Bigg), \end{aligned} \tag{9.20}$$

where the sum is over $\alpha_{1,2},\ \beta_{1,2}\in\Xi(N)$, for the N-SUSY model, and SUM_i are, apart from unimportant spin structure independent factors, given by

$$\mathrm{SUM}_1\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \sum_{n,m} C\begin{bmatrix}\alpha+ns\\ \beta+ms\end{bmatrix} Y\begin{bmatrix}\alpha(\psi)+n\\ \beta(\psi)+m\end{bmatrix} \prod_{f\in s} Q^{1/8}\begin{bmatrix}\alpha(f)+n\\ \beta(f)+m\end{bmatrix}. \tag{9.21a}$$

SUM_2 is the same with Y replacing $\tilde{Y}$, and

$$\mathrm{SUM}^I_3 = \sum_{n,m} C\begin{bmatrix}\alpha+ns\\ \beta+ms\end{bmatrix} Y\begin{bmatrix}\alpha(\chi^I)+n\\ \beta(\chi^I)+m\end{bmatrix} \prod_{f\in s} Q^{1/8}\begin{bmatrix}\alpha(f)+n\\ \beta(f)+m\end{bmatrix}. \tag{9.21b}$$

For the case N=4 it is always possible, by redefining if need $n_i\to n_i+1$, to put $\alpha(f)=0$ and also $\beta(f)=0$. Then by using

$$C\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \prod_i C\begin{bmatrix}\alpha_i\\ \beta_i\end{bmatrix}, \qquad C\begin{bmatrix}\alpha_i+n_is\\ \beta_i+m_is\end{bmatrix} = (-)^{n_i+m_i} C\begin{bmatrix}\alpha_i\\ \beta_i\end{bmatrix}, \tag{9.22}$$

each of SUM_i is proportional to

$$\sum_{n,m}(-)^{(n_1+n_2+m_1+m_2)} Y\begin{bmatrix}n_1 & n_2\\ m_1 & m_2\end{bmatrix} Q\begin{bmatrix}n_1 & n_2\\ m_1 & m_2\end{bmatrix} = 0, \tag{9.23}$$

(or $\tilde{Y}$ replacing Y) which vanishes as was shown before and in ref.[13].

By the explicit exprexssions of eq.(9.19) one can see that

$$Y\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} - Y\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} - Y\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + Y\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} = 0. \tag{9.24}$$

---

* For s=1, 8, 9, and 10, we interchange A and B which is more convenient for the discussions here.

The same also holding for $\tilde{Y}$, together with four additional identities obtained from eq.(9.24) by modular transformations. This is precisely what was conjectured to hold in ref.[79], where the result of eq.(9.23) is indicated to follow from eq.(9.24) and some Θ-function identity. For the model with N=2 an analysis like the one of ref.[79] shows that, in addition to expressions of the form of eq.(9.23), one has additional terms which are modular transformations of

$$G_1' = \sum_{n,m} C\begin{bmatrix}\alpha+ns\\ \beta+ms\end{bmatrix} Y\begin{bmatrix}1+n_1 & n_2\\ m_1 & m_2\end{bmatrix} Q^{1/2}\begin{bmatrix}1+n_1 & n_2\\ m_1 & m_2\end{bmatrix} Q^{1/2}\begin{bmatrix}n_1 & n_2\\ m_1 & m_2\end{bmatrix}, \tag{9.25}$$

or else of G_2' where $Y \to \tilde{Y}$, or else G_3' where (depending on the value of I) one has $Y\begin{bmatrix}n_1 & n_2\\ m_1 & m_2\end{bmatrix}$ instead of $Y\begin{bmatrix}1+n_1 & n_2\\ m_1 & m_2\end{bmatrix}$. The only non-zero contribution can come from the spin structures giving $m_1 = 0$ and $n_2 m_2 = 0$. Taking into account eq.(9.22) and eq.(9.19) we obtain ($q = 1$ for Y and $q = 2$ for $\tilde{Y}$)

$$\begin{aligned}(a_3^q - a_2^q)\cdot \Big\{ Q^{1/2}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} Q^{1/2}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} - Q^{1/2}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} Q^{1/2}\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}& \\ - Q^{1/2}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} Q^{1/2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\Big\} &= 0.\end{aligned} \tag{9.26}$$

This result can be checked from the explicit expressions of Q_s, and it is quoted as a Theta identity in ref.[79].

Finally the model with N=1 contains terms like eq.(9.23), terms like eq.(9.26) and additional terms which are modular transformations of

$$\begin{aligned}G_1'' = \sum_{n,m} C\begin{bmatrix}\alpha+ns\\ \beta+ms\end{bmatrix} Y\begin{bmatrix}1+n_1 & 1+n_2\\ m_1 & m_2\end{bmatrix} Q^{1/4}\begin{bmatrix}1+n_1 & 1+n_2\\ m_1 & m_2\end{bmatrix} Q^{1/4}\begin{bmatrix}1+n_1 & n_2\\ m_1 & m_2\end{bmatrix}\\ \times Q^{1/4}\begin{bmatrix}n_1 & 1+n_2\\ m_1 & m_2\end{bmatrix} Q^{1/4}\begin{bmatrix}n_1 & n_2\\ m_1 & m_2\end{bmatrix},\end{aligned} \tag{9.27}$$

or G_2'' with $Y \to \tilde{Y}$ or G_3'', where (depending on I) the characteristic of $Y[\,]$ can be the one of some other factor $Q^{1/4}[\,]$ in the right hand side of eq.(9.27). Using the property of C, this reduces to

$$\begin{aligned}\left(Y\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} + Y\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} - Y\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} - Y\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right)&\\ \times Q^{1/4}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} Q^{1/4}\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} Q^{1/4}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} Q^{1/4}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} &= 0,\end{aligned} \tag{9.28}$$

or else $Y \to \tilde{Y}$, which vanishes by virtue of eq.(9.24). Again as before the SUMs corresponding to all other choices of α_i and β_i are modular transformations of the generating one or they are identically zero because of the appearce of $Q^{1/4}$ with odd spin structure. That completes our proof that the vacuum amplitudes of all these supersymmetric models are vanishing at two loops.

The above discussions can be easily extended to check the nonrenormalization theorem. The nonrenormalization theorem at one-loop level has been studied in [123]. They found that the three-particle amplitude is no longer vanishing for $N \leq 2$ aupersymmetric string models*. This is also true at two loops [81]. We mention that the nonrenormalization theorem at two loops for $N = 4$ supersymmetric string theory was also discussed in [124], although this is just our results [14] extending to low dimensional string theory. Here I will not present the explicit formulas for the nonvanishing amplitudes for $N \leq 2$ SUSY string models at two loops. They would occupy all the rest of this thesis and I would never come to an end. Nevertheless I encourage those people who have computer accounts and like to practise with symbolic manipulation (with MACSYMA or REDUCE, for example) to prove the assertion made in [77] (the first one) in page 102, lines 23 to 27 directly at two loops†.

To conclude this section let us compute the four-particle amplitude of the D-dimensional string theoies with maximal SUSY $N = 4$. Here the relevant correlator is

$$\langle J(x+)J(x-)\prod_{i=1}^{4} k_i \cdot \psi(z_i)\epsilon_i \cdot \psi(z_i)e^{ik_i \cdot X(z_i)}\rangle_s. \tag{9.29}$$

The summation over spin structures can be easily done to yield ($k \to 0$)

$$\begin{aligned}&\sum_s \eta_s Q_s \langle J(x+)J(x-)\prod_{i=1}^{4} k_i \cdot \psi(z_i)\epsilon_i \cdot \psi(z_i)e^{ik_i \cdot X(z_i)}\rangle_s \\ &= c' K(k,\epsilon)P(a)\prod_{i=1}^{4}\frac{z_i - x}{y(z_i)} \times (I_m + I_{gh} + I(\chi y \omega)),\end{aligned} \tag{9.30}$$

---

* Nevertheless I don't agree with their expilcit expression for the kinematic factors in three-particle amplitude.

† To be honest I'd like to tell you that some of the computations in this thesis were carried out by using MACSYMA and REDUCE.

where

$$
\begin{aligned}
I_m =& -\frac{D}{8}\left\{-\frac{1}{8}\sum_{i=1}^{6}\frac{1}{(z_i-x)^2}+\frac{1}{4}\sum_{i<j}^{6}\frac{1}{a_i-x}\frac{1}{a_j-x}-\sum_{i=1}^{6}\frac{1}{a_i-x}\frac{\partial}{\partial a_i}\ln T\right\},\\
I_{gh} =& -\frac{1}{8}\left\{\sum_{i=1}^{6}\frac{1}{a_i-x}-2\sum_{i=1}^{3}\frac{1}{b_i-x}\right\}\left\{\sum_{i=1}^{6}\frac{1}{a_i-x}-\sum_{i=1}^{4}\frac{1}{z_i-x}\right\}\\
&-\frac{1}{32}\left\{\sum_{i=1}^{6}\frac{1}{(a_i-x)^2}-2\sum_{i<j}^{6}\frac{1}{a_i-x}\frac{1}{a_j-x}+8\sum_{i<j}^{3}\frac{1}{b_i-x}\frac{1}{b_j-x}\right\},\\
I(\chi y\omega) =& \frac{10-D}{64}\left\{\sum_{i=1}^{3}\frac{1}{x-A_i}-\frac{1}{x-B_i}\right\}^2.
\end{aligned}
\tag{9.31}
$$

The amplitude is then

$$
\begin{aligned}
A_D(k,\epsilon) =& cK(k,\epsilon)\int\frac{d^2a_1d^2a_2d^2a_3\mid a_{45}a_{56}a_{64}\mid^2}{T^{D/2}\prod_{i<j}^{6}a_{ij}\bar{a}_{ij}^3}\int\prod_{k=1}^{4}d^2z_k\frac{z_k-x}{y(z_k)}\\
&\sum_s\bar{Q}_s^4|Q_s|^{5-D/2}I_D(x)_s\langle\prod V_L\rangle,
\end{aligned}
\tag{9.32}
$$

where

$$
\begin{aligned}
I_D(x)_s =& -\frac{10-D}{64}\sum_{i=1}^{6}\frac{1}{(a_i-x)^2}-\frac{D+6}{32}\sum_{i<j}^{6}\frac{1}{a_i-x}\frac{1}{a_j-x}-\frac{1}{4}\sum_{i<j}^{3}\frac{1}{b_i-x}\frac{1}{b_j-x}\\
&+\frac{1}{4}\sum_{i=1}^{6}\frac{1}{a_i-x}\sum_{i=1}^{3}\frac{1}{b_i-x}+\frac{1}{8}(\sum_{i=1}^{6}\frac{1}{a_i-x}-2\sum_{i=1}^{3}\frac{1}{b_i-x})\sum_{i=1}^{4}\frac{1}{z_i-x}\\
&+\frac{D}{8}\sum_{i=1}^{6}\frac{1}{a_i-x}\frac{\partial}{\partial a_i}\ln T+\frac{10-D}{64}\left\{\sum_{i=1}^{3}\frac{1}{x-A_i}-\frac{1}{x-B_i}\right\}^2.
\end{aligned}
\tag{9.33}
$$

It is an easy matter to check that the above amplitude does not depend on the arbitrary chosen points x and $a_{4,5,6}$ and is finite.

All the above discussions can be extended to low dimensional type II string theories. Here we will only present an explicit formula for the four-particle amplitude in D-dimensional type II string theory with maximal supersymmetry. The model is

$$\Xi=\{F,\ \emptyset,\ s,\ F+s,\ \bar{s},\ F+\bar{s}.\cdots\}$$

The four-particle amplitude can be calculated to be

$$
\begin{aligned}
AII_D(k,\epsilon) =& c'\tilde{K}\int \frac{d^2a_1 d^2a_2 d^2a_3 |a_{45}a_{56}a_{64}|^2}{T^{D/2}\prod_{i<j}^{6}|a_{ij}|^2}\prod_{i=1}^{4}\frac{d^2z_i(z_i-r)(\bar{z}_i-\bar{s})}{|y(z_i)|^2} \\
&\times\sum_{s'}|Q_{s'}|^{5-D/2}J_D(r,\bar{s})\left\langle\prod e^{ik\cdot X}\right\rangle.
\end{aligned}
\tag{9.34}
$$

where

$$
\begin{aligned}
J_D(r,\bar{s}) =& \Bigg\{ I_D(r)_{s'}\bar{I}_D(\bar{s})_{s'} + \frac{D}{8}\left(\frac{\pi}{T}\frac{1}{y(r)\bar{y}(\bar{s})}\int\frac{d^2v(v-r)(\bar{v}-\bar{s})}{|y(v)|^2}\right)^2 \\
&-\frac{1}{64}\langle\partial X(r+)\cdot\partial X(r-)\rangle\bar{S}(\bar{s}) - \frac{1}{64}\langle\bar{\partial}X(\bar{s}+)\cdot\bar{\partial}X(\bar{s}-)\rangle S(r) \\
&+\frac{\pi}{16T}\frac{1}{y^2(r)\bar{y}^2(\bar{s})}\int\frac{d^2v(v-r)(\bar{v}-\bar{s})}{|y(v)|^2}R(r)\cdot\bar{R}(\bar{s}) + \frac{1}{256}S(r)\bar{S}(\bar{s})\Bigg\}.
\end{aligned}
\tag{9.35}
$$

10. Two-Loop Calculations – Four-Particle Fermionic Amplitude

In this setion we will calculate the 2 fermion—2 boson scattering amplitude at two loops [17] by using the method developed in the previous sections and the covariant fermion emission vertex [125, 126]. This computation is very technical. Nevertheless I think it is worth presenting it here not only because one can do it with the help of Riemann identity but also it points out many questions unanswered. For example, we don't know how to extend this calculation to 4F scattering amplitude. A more profound question is how explicit modular invariance is implemented in the general case (at two loops). For an arbitrary number of bosons, the13scattering amplitude can be easily shown to be modular invariant* [13]. But we don't know how this can be extended to include an arbitrary number of fermions. Because of spin fields in the covariant fermion emission vertex, we can not separate the contribution of even spin structures from the contribution of odd spin structures. Even for 2F–2B scattering amplitude, both even and odd spin structures contribute, which is contrary to 4B scattering case where only even spin structures give a contribution. So the foregoing proof [13] cann't be applied to show the modular invariance of the fermionic amplitudes. As we have seen in the previous sections and shall see in the next section, modular invariance is a very useful argument to show the vanishing of the vacuum amplitude and also the nonrenormalization theorem. Nevertheless in the calculation of the fermionic amplitude modular invariance seems violated, probably in a modular covariant way because of the choice of a spin bundle over the Riemann surface†. In my opinion it is dangerous to argue in this way. One should prove that the final reult does not depend on the particular spin structure chosen and is modular invariant. In this section we will show that this is what happens at two loops. Previous computations at one-loop level have been done by Atick and Sen in [127]. We will see that some of their calculations can be adapted to two loops, and we will use their results without further constantly referring to [127].

Before going into the details we'd like first to recapitulate what the strategy we have used. In the covariant formalism [125] the vertex describing fermion emission involves

---

* At least for summation over even spin structures.

† Remember that we are dealing with sipn fields here.

spin fields of matter and superconformal (bosonic) ghost, and the calculation of fermionic scattering amplitudes would necessarily require correlation functions of these fields (in particular, the spin fields). A priori, these are not so straightforward to compute. Nevertheless we know these correlation functions because of the efforts of many people [111, 56, 128]. These correlation functions are written in terms of Θ-functions and prime form. They can be derived by using conformal field theory method [101, 111], or by bosonization [56, 116] or other methods [128]. We will briefly review these results in section 2 and 3. In section 2 we also present some two-loop computations by using the bosonization techniques. The results are shown to agree with those of [12-16]. By using these correlation functions one can readily compute the (nonvanishing) fermion scattering amplitude. The summation over spin structures can be easily done by using the Riemann identity. Nonrenormalization theorem [14] involving fermions can be easily proved. The result obtained for 2F–2B scattering amplitude can be written in a modular invariant form. This is done in section 4. We will see also there that the insertions of supercurrent is very important to ensure the modular invariance. The calculation of the 4F amplitude is a more difficult problem and we have not successed doing that. It is certainly true that one must use some generalized Riemann identities as those used in [127].

To start with let us first review some of the basic facts about the bosonic ghost system and its spin fields. We start from the ansatz of ref.[129], formula (43), where the result of the integration over supermoduli is proposed to be the correlation function of the appropriate number of picture-changing operators $Q\xi$ introduced in [125]. The partition function is (right part only and for a specific choice of even spin structure s) (see eq.(10.6) for bosonization prescription)

$$\Lambda_s^0 = \langle\langle \xi(x_0) Q\xi(x_1) Q\xi(x_2) \rangle\rangle_s, \tag{10.1}$$

where $\langle\langle\cdots\rangle\rangle_s$ denotes functional integration over all (holomorphic) fields ψ, β, γ, b and c and also X. The action of the BRST charge Q on ξ is defined as follows [125]:

$$Q\xi(x) = c(x)\partial\xi(x) + \oint_x \frac{dz}{2\pi i}\left(\frac{1}{2}\gamma\psi\cdot\partial X + \frac{1}{4}\gamma^2 b\right)(z)\xi(x). \tag{10.2}$$

By substituting (10.2) into (10.1), we have

$$
\begin{aligned}
-4\Lambda^0_s =& \langle\langle \xi(x_0)\xi(x_1)\xi(x_2) \oint_{x_1} \gamma\psi\cdot\partial X \oint_{x_2} \gamma\psi\cdot\partial X \rangle\rangle_s \\
&+ \langle\langle \xi(x_0)c(x_1)\partial_1\xi(x_1)\xi(x_2) \oint_{x_2} \gamma^2 b \rangle\rangle_s \\
&+ \langle\langle \xi(x_0)\xi(x_1)c(x_2)\partial_2\xi(x_2) \oint_{x_1} \gamma^2 b \rangle\rangle_s .
\end{aligned}
\tag{10.3}
$$

In this paper we will compute only the first term–the contribution from the matter part of the supercurrent. The remaining terms–the contributions from the ghosts part of the supercurrent–could be fixed by BRST invariance, as we have checked in [15, 16]. Then the correlation function for the bosonic ghost that we want is

$$
F(y_1,y_2;x_0,x_1,x_2) = \langle \xi(x_0)\xi(x_1)\xi(x_2)\gamma(y_1)\gamma(y_2)\rangle_s, \tag{10.4}
$$

and the correlation function involving two spin fields is

$$
\tilde{F}(y_1,y_2;x_0,x_1,x_2;z_3,z_4) = \langle \xi(x_0)\xi(x_1)\xi(x_2)\gamma(y_1)\gamma(y_2)S_g^-(z_3)S_g^+(z_4)\rangle_s, \tag{10.5}
$$

where we have used the standard bosonization prescription:

$$
\begin{aligned}
\beta &\sim \partial\xi e^{-\phi}, & \gamma &\sim \eta e^{+\phi}, \\
S_g^- &\sim e^{-\frac{1}{2}\phi}, & S_g^+ &\sim e^{+\frac{1}{2}\phi}.
\end{aligned}
\tag{10.6}
$$

From [5,7,17,18], we have

$$
\begin{aligned}
F(y_1,y_2;x_0,x_1,x_2) =& -\frac{1}{4}\frac{y(y_1)y(y_2)}{\sqrt{u(y_1)u(y_2)}}\left\{\frac{1}{u(x_1)-u(x_2)}\right. \\
&\left(\frac{u(x_1)+u(y_1)}{x_1-y_1} - \frac{u(x_2)+u(y_1)}{x_2-y_1}\right)\left(\frac{u(x_1)+u(y_2)}{x_1-y_2} - \frac{u(x_2)+u(y_2)}{x_2-y_2}\right) \\
&\left. + (x_1\to x_0,\ x_2\to x_1) + (x_2\to x_0,\ x_1\to x_2)\right\},
\end{aligned}
\tag{10.7}
$$

and

$$
\begin{aligned}
\tilde{F}(y_1,y_2;x_0,x_1,x_2;z_3,z_4) =& F(y_1,y_2;x_0,x_1,x_2)\,(E(z_3,z_4))^{1/4}\,\frac{\sigma(z_3)}{\sigma(z_4)}\left\{\frac{E(y_1,z_3)E(y_2,z_3)}{E(y_1,z_4)E(y_2,z_4)}\right\}^{1/2} \\
&\frac{\Theta_s(x_0+x_1-2\Delta)\Theta_s(x_1+x_2-2\Delta)\Theta_s(x_2+x_0-2\Delta)}{\Theta_s(-y_1+x_0+x_1+x_2-2\Delta)\Theta_s(-y_2+x_0+x_1+x_2-2\Delta)} \\
&\frac{\Theta_s(-y_1+x_0+x_1+x_2+\frac{1}{2}(z_4-z_3)-2\Delta)\Theta_s(-y_2+x_0+x_1+x_2+\frac{1}{2}(z_4-z_3)-2\Delta)}{\Theta_s(x_0+x_1+\frac{1}{2}(z_4-z_3)-2\Delta)\Theta_s(x_1+x_2+\frac{1}{2}(z_4-z_3)-2\Delta)\Theta_s(x_2+x_0+\frac{1}{2}(z_4-z_3)-2\Delta)}.
\end{aligned}
\tag{10.8}
$$

To proceed further, we'd like to point out that the ansatz (10.1) works quite elegantly for the calculation of the bosonic amplitudes as one can easily checked.

By using the explicit formula of $F(y_1, y_2; x_0, x_1, x_2)$, we have*

$$
\begin{aligned}
\frac{-\Lambda_s^0}{\langle\langle 1\rangle\rangle_s} =& \frac{1}{4}\frac{y(x_1)y(x_2)}{u(x_1)-u(x_2)}\sqrt{u(x_1)u(x_2)}\langle\psi_\mu(x_1)\psi_\nu(x_2)\rangle\langle\partial X^\mu(x_1)\partial X^\nu(x_2)\rangle \\
&-\frac{1}{4}\left\{R(x_2x_1)y^2(x)\partial_1\left(\frac{u(x_1)+u(x_2)}{(x_1-x_2)(u(x_1)-u(x_2))}\right)\right. \\
&\left.+\partial_2(R(x_2x_1)y^2(x_2))\partial_1\left(\frac{u(x_2)}{u(x_1)-u(x_2)}\right)+(x_1\leftrightarrow x_2)\right\},
\end{aligned}
\tag{10.10}
$$

where we have defined $R(x_1x_2)$ as follows [24]:

$$
R(x,y) = \langle b(x)c(y)\prod_{i=1}^{3}(\eta_i * b)\rangle,
\tag{10.11}
$$

where η_i are Beltrami differentials which can be taken as δ-functions.

To compare with the computations done in previous sections, we also make the convenient choice of taking $x_{1,2}$ to be the zeros of a holomorphic abelian differential $\Omega(z) = \frac{z-x}{y(z)}$, i.e. $x_{1,2} = x\pm$, the two corresponding points in the upper and lower Riemann sheet. Then we have

$$
\begin{aligned}
\partial_1\left(\frac{u(x_2)}{u(x_1)-u(x_2)}\right) &= \frac{1}{4}\partial_2\ln u(x_2) \\
&= \frac{1}{8}\sum_{i=1}^{3}\left(\frac{1}{x-A_i}-\frac{1}{x-B_i}\right) \\
&= \frac{1}{2}\langle\psi(x+)\psi(x-)\rangle_s, \\
\partial_1\left(\frac{u(x_1)+u(x_2)}{(x_1-x_2)(u(x_1)-u(x_2))}\right) &= \frac{1}{4}\left(\frac{u''(x_1)}{u(x_1)}-\left(\frac{u'(x_1)}{u(x_1)}\right)^2\right) \\
&= -\frac{1}{8}\sum_{i=1}^{3}\left(\frac{1}{(x-A_i)^2}-\frac{1}{(x-B_i)^2}\right) \\
&= \langle\partial\psi(x+)\psi(x-)\rangle_s,
\end{aligned}
\tag{10.12}
$$

* Notice that the explicit dependence on x_0 drops here due to theintegration over y_1 and y_2 around x_1 and x_2 respectively.

and it follows:

$$
\begin{aligned}
-\Lambda_s^0/\langle\langle 1\rangle\rangle_s =& \frac{1}{8}y^2(x)\langle\psi(x+)\cdot\psi(x-)\rangle_s\langle\partial X(x+)\cdot X(x-)\rangle \\
&-\frac{1}{8}\left\{\partial(R(x_2x_1)y^2(x_2))+\partial_1(R(x_1x_2)y^2(x_1))\right\}\langle\psi(x+)\psi(x-)\rangle_s \\
&-\frac{1}{4}y^2(x)(R(x_2x_1)+R(x_1x_2)\langle\partial\psi(x+)\psi(x-)\rangle_s,
\end{aligned} \tag{10.13}
$$

which is identical with the expression (6.9) after substituting the various factors. The inclusion of the bosonic vertex operators is straightforward. So we see complete argeement between this ansatz and the one we used previously. However this ansatz is more suitable for the calculation of the fermion emission amplitude.

If we also make the choice of $x_{1,2} = x\pm$ in (10.8) we have then

$$
\begin{aligned}
\Lambda_s =& 2\langle\langle\xi(x_0)\xi(x_1)\xi(x_2)\oint_{x_1}\gamma\psi\cdot\partial X\oint_{x_2}\gamma\psi\cdot XS_g^-(z_3)S_g^+(z_4)\cdots\rangle\rangle_s \\
=& \frac{y^2(x)}{\Theta_s(\frac{1}{2}(z_3-z_4))}\frac{\sigma(z_3)}{\sigma(z_4)}\left\{\frac{E(x+,z_3)E(x-,z_3)}{E(x+,z_4)E(x-,z_4)}\right\}^{1/2} \\
&\langle\langle\psi(x+)\cdot\partial X(x+)\psi(x-)\cdot\partial X(x-)\cdots\rangle\rangle_s,
\end{aligned} \tag{10.14}
$$

where $\cdots$ denotes fields other than bosonic ghost fields or its spin fields. This formula will be used later in the calculation of the 2F–2B scattering amplitude in closed superstring theories.

We will not repeat the procedure to arrive at the following expression for the correlation function in $SO(2)$ spin model† [111, 127]

$$
\begin{aligned}
F_s(y_i,z_i,u_i,v_i) =& \langle\langle\prod_{i=1}^{N_1}S^+(y_i)\prod_{i=1}^{N_2}S^-(z_i)\prod_{i=1}^{N_3}\psi(u_i)\prod_{i=1}^{N_4}\bar\psi(v_i)\rangle\rangle_s \\
=& K_s\cdot\frac{\prod_{i<j}E(y_i,y_j)^{1/4}\prod_{i<j}E(z_i,z_j)^{1/4}\prod_{i,j}E(y_i,u_j)^{1/2}\prod_{i,j}E(z_i,v_j)^{1/2}}{\prod_{i,j}E(y_i,z_j)^{1/4}\prod_{i,j}E(y_i,v_j)^{1/2}\prod_{i,j}E(z_i,u_j)^{1/2}} \\
&\times\frac{\prod_{i<j}E(u_i,u_j)\prod_{i<j}E(v_i,v_j)}{\prod_{i,j}E(u_i,v_j)}\Theta_s\left(\frac{1}{2}(\sum_i y_i-\sum_i z_i)+\sum_i u_i-\sum_i v_i\right),
\end{aligned} \tag{10.15}
$$

where $S^+(y)$ and $S^-(z)$ are $SO(2)$ spin fields with fermionic charge $+1/2$ and $-1/2$ respectively. They have conformal dimension $1/8$. $\psi(u)$ and $\bar\psi(v)$ are $SO(2)$ complex fermion fields and have conformal dimension $1/2$. The conservation of fermionic charge requires that $\frac{1}{2}(N_1-N_2)+N_3-N_4=0$. Notice that since the argument of Θ_s contains* $\frac{1}{2}y_i$ and

† We will write $\langle\cdots\rangle$ instead of $\langle\langle\cdots\rangle\rangle$ in the sequel to simplify TEX typing.

* It is understood that we use the symbol y_i to denote its value under the Jacobi mapping: $\int^{y_i}\omega$.

$\frac{1}{2}z_i$, it changes to a Θ-function with a different spin structure as we translate y_i or z_i by n and $\tau \cdot m$ on the Jacobi variety where τ is the period matrix, so F_s is not well defined on the Riemann surface. As have been seen in [111] correlation functions of the physical vertex operators in string theories involve appropriate powers of the correlation functions of the spin fields given above and of the bosonic ghost, which make them well defined after summing over spin structures. The relative phases and normalization of the contributions from different spin structures are fixed as in [15,16], which we will not repeat. Here one should choose a reference spin structure. Nevertheless we will show later that the dependence on a reference spin structure will drop and we do get a modular invariant result.

The spin operators appearing in the covariant fermion emission vertex is $SO(10)$ spin operators. This can be written as a product of five $SO(2)$ spin operators $(S_i^+,\ S_i^-)$, $i = 1, 2, \cdots, 5$:

$$S_1^\pm S_2^\pm S_3^\pm S_4^\pm S_5^\pm \tag{10.16}$$

by combining the right-moving Majorana-Weyl fermions $\psi^\mu(z)$ in the NSR formulation of the superstrings into five complex fermions. There are 32 such operators which go into two sets according to their chiralities. We adopt the same convention as Atick and Sen did in [15,16]. Then the correlation functions of the $SO(10)$ spin operators will simply be products of the correlation functions of the $SO(2)$ spin fields which we have quoted the right formula, eq.(10.15). Now we begin to calculate the 2F–2B scattering amplitude.

We will use the covariant fermion emission vertices constructed by Friedan, Martinec and Shenker [125], and by Knizhnik [126]. Let us briefly review their construction. The basic fermion emission vertex is given by

$$V_{-1/2}(u, k, z) = u^\alpha(k) S_\alpha(z) S_g^-(z) e^{ik\cdot X}, \tag{10.17}$$

which carries a ghost charge of $-1/2$. Here u^α is a Majorana spinor characterizing the polarization of the external state. $S_\alpha(z)$ and $S_g^-(z)$ are the $SO(10)$ and the ghost spin fields which have conformal weight 5/8 and 3/8 respectively. So $V_{-1/2}$ has conformal weight 1 ($k^2 = 0$). Because $V_{-1/2}$ carries a ghost charge of $-1/2$ the correlation function involving several $V_{-1/2}$'s on the Riemann surface vanishes identically due to the ghost charge conservation. The solution to this problem given in [125] is to introduce another fermion emission vertex

$$V_{+1/2}(u,k,z) = u^{\alpha}(k)S_g^{+}(z)\lim_{w\to z}(w-z)^{1/2}\left(\psi(w)\cdot\partial X(w)S_{\alpha}(z)e^{ik\cdot X(z)}\right), \tag{10.18}$$

which carries a ghost charge of $+1/2$. Here S_g^+ is the ghost spin fields carrying ghost charge $+1/2$ and having conformal weight $-5/8$. By using the operator product expansions of ψ, S_α and X, and the on-shell condition $\gamma\cdot ku = 0$, we have

$$\begin{aligned} V_{+1/2}(u,k,z) =& u^{\alpha}(k)S_g^{+}(z)e^{ik\cdot X(z)} \\ & \left((\gamma_\mu)_{\alpha\beta}S^{\beta}(z)\partial X^{\mu}(z) - ik^{\mu}(\lim_{w\to z}(w-z)^{-1/2}\psi_\mu(w)S_\alpha(z)\right). \end{aligned} \tag{10.19}$$

The prescription for calculating an amplitude with $(2n)$ external fermions, as given in [125] is to use the vertex $V_{-1/2}$ for n of the fermions, and the $V_{+1/2}$ for the other n. Then the total charge adds up to zero and we expect a nonzero answer. In this paper, we will compute the 2F–2B scattering amplitude in closed superstrings. The boson emission vertex (carrying zero ghost charge) is given by

$$V_0(\epsilon,k,z) = (\partial(\epsilon\cdot X) + ik\cdot\psi\epsilon\cdot\psi)\, e^{ik\cdot X}. \tag{10.20}$$

Certainly all the above various vertex operators serve only half (right part) of a complete vertex operator. For different theories and different particles, one should supplement the other part in order to get the complete amplitude. However this is quite easy because the left part and the right part decouple completely from each other for the spin structure dependent factors. So we will only consider the right part in the sequel.

The relevant correlation function for the calculation of 2-fermion and 2-boson scattering amplitude is

$$\Lambda_s^4 = \langle\xi(x_0)Q\xi(x_1)Q\xi(x_2)V_0(\epsilon_1,k_1,z_1)V_0(\epsilon_2,k_2,z_2)V_{-1/2}(u_3,k_3,z_3)V_{+1/2}(u_4,k_4,z_4)\rangle_s. \tag{10.21}$$

By taking $x_{1,2} = x\pm$ and using (10.17)–(10.20), (10.22) and (10.14), we have†

$$\begin{aligned}\Lambda_s^4 =& E(z_3,z_4)^{1/4}\frac{y^2(x)}{\Theta_s(\frac{1}{2}(z_3-z_4))}\frac{\sigma(z_3)}{\sigma(z_4)}\left(\frac{E(x+,z_3)E(x-,z_3)}{E(x+,z_4)E(x-,z_4)}\right)^{1/2}\\ &\langle(\partial(\epsilon_1\cdot X(z_1))+ik_1\cdot\psi(z_1)\epsilon_1\cdot\psi(z_1))\,(\partial(\epsilon_2\cdot X(z_2))+ik_2\cdot\psi(z_2)\epsilon_2\cdot\psi(z_2))\\ &u_3^\delta S_\delta(z_3)u_4^\alpha\left((\gamma_\mu)_{\alpha\beta}S^\beta(z_4)\partial X^\mu(z_4)-ik_4^\mu(\lim_{w\to z_4}(w-z_4)^{-1/2}\psi_\mu(w)S_\alpha(z_4)\right)\\ &\psi(x+)\cdot\partial X(x+)\psi(x-)\cdot\partial X(x-)\prod_{i=1}^4 e^{ik_i\cdot X(z_i)}\rangle_s+(\text{other terms})\\ \equiv&\Lambda_s^m+(\text{other terms}).\end{aligned}\tag{10.22}$$

where (other terms) denotes the contributions from the ghost part of the supercurrent which we will not calculate explicitly. They could be fixed by BRST invariance as has been checked in [15, 16]. It is not difficult to see that Λ_s^m can be written as a sum of eight different terms. For each of them, we can use eq.(10.15) to calculate the correlation functions involving the spin operators S_α and the fermion fields ψ, and then sum over the spin structures. Because of the nonrenormalization theorem [48, 14]*, most of the terms give a zero contribution to the amplitude after summing over spin structures. In fact the only term contribute to the amplitude is the term involving maximal number (7 for 2F–2B scattering) of ψ fields. All the rest terms give zero contributions. Let us see the term involving six ψ fields (for spin structure dependent factors only):

$$\begin{aligned}\tilde{\Lambda}_s =& \frac{1}{\Theta_s(\frac{1}{2}(z_3-z_4))}\langle\psi^\mu(x_1)\psi^\nu(x_2):\psi^{\mu_1}(z_1)\psi^{\nu_1}(z_1):\\ &:\psi^{\mu_2}(z_2)\psi^{\nu_2}(z_2)S_\alpha(z_3)S^\beta(z_4):\rangle_s.\end{aligned}\tag{10.23}$$

The strategy to show that $\tilde{\Lambda}_s$ also gives zero contribution to the amplitude after summing over spin structures is as follows. First we write down the general expression for $\tilde{\Lambda}_s$ according to the tensor structures of $SO(10)$. It is easy to see that $\tilde{\Lambda}_s$ has 76 independent tensor structures. Then one can calculate the coefficients by a specific choice of the configuration of $(\mu\nu\mu_1\nu_1\mu_2\nu_2\alpha\beta)$. And finally all the coefficients can be easily shown to vanish

† Some trivial overall constants may have been omitted.

* We mention that the nonrenormalization theorem has also been discussed in ref.[130].

by making use of the Riemann identity (eq.(3.13) for $g = 2$):

$$
\begin{aligned}
&\sum_{a,b} e^{4\pi i(a\cdot b_0 - b\cdot a_0)}\Theta\begin{bmatrix} a+a_0 \\ b+b_0 \end{bmatrix}(x)\Theta\begin{bmatrix} a+a_0 \\ b+b_0 \end{bmatrix}(y)\Theta\begin{bmatrix} a+a_0 \\ b+b_0 \end{bmatrix}(u)\Theta\begin{bmatrix} a+a_0 \\ b+b_0 \end{bmatrix}(v) \\
&\quad = 4\Theta_T\left(\frac{1}{2}(x+y+u+v)\right)\Theta_T\left(\frac{1}{2}(x+y-u-v)\right) \\
&\qquad \Theta_T\left(\frac{1}{2}(x-y+u-v)\right)\Theta_T\left(\frac{1}{2}(x-y-u+v)\right),
\end{aligned} \tag{10.24}
$$

and

$$
\Theta_T\left(\frac{1}{2}(x_1 - x_2)\right) = \Theta_T\left(\int_{P_0}^{x_1}\omega\right) = 0, \tag{10.25}
$$

where $T = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$ is an odd spin structure and P_0 is the corresponding Weierstrass point or branch point† (or the zero point (other than x_1) of the function $f(z) = \Theta_T(\int_{x_1}^{z}\omega)$ defined on the Riemann surface). Here we have chosen a reference spin structure T (P_0 is the corresponding Weierstrass point) and the integration contours from x_1 to x_2 such that $x_1 - P_0 = -(x_2 - P_0)$. Now let us show some details of this calculation. The 76 independent tensor structures can be written as follows

$$
\begin{aligned}
\tilde{\Lambda}_s = &+ a_0(\gamma^{\mu\nu\mu_1\nu_1\mu_2\nu_2})_\alpha^{\ \beta} \\
&+ a_1\delta^{\mu\nu}(\gamma^{\mu_1\nu_1\mu_2\nu_2})_\alpha^{\ \beta} + (14 \text{ more terms}) \\
&+ a_{16}\delta^{\mu\nu}\delta^{\mu_1\nu_1}(\gamma^{\mu_2\nu_2})_\alpha^{\ \beta} + (44 \text{ more terms}) \\
&+ a_{61}\delta^{\mu\nu}\delta^{\mu_1\nu_1}\delta^{\mu_2\nu_2}\delta_\alpha^{\ \beta} + (14 \text{ more terms}),
\end{aligned} \tag{10.26}
$$

where $\gamma^{\mu\nu\cdots} = \gamma^\mu\gamma^\nu\cdots$. From (10.26) we see that some of the coeffivcients, like a_{16} and a_{61}, can be consistently set to zero because of the physical conditions. Now we prove that all these coefficients are zero.

By choosing $\mu = 1$, $\nu = \bar{1}$, $\mu_1 = 2$, $\nu_1 = 3$, $\mu_2 = \bar{2}$, $\nu_2 = \bar{3}$ and $\alpha = \beta = (-----)$, one sees that there is only one term in $\tilde{\Lambda}_s$:

$$
\begin{aligned}
\tilde{\Lambda}_s = &a_i\delta^{\mu\nu}\delta^{\mu_1\nu_1}\delta^{\mu_2\nu_2}\delta_\alpha^{\ \beta} \qquad (61 \le i \le 75) \\
= &\frac{1}{\Theta_s(\frac{1}{2}(z_3 - z_4))}\langle\psi^1(x_1)\bar{\psi}^1(x_2)\psi^2(z_1)\psi^3(z_1)\bar{\psi}^2(z_2)\bar{\psi}^3(z_2) \\
&S_1^+(z_3)S_2^+(z_3)S_3^+(z_3)S_4^+(z_3)S_5^+(z_3)S_1^-(z_4)S_2^-(z_4)S_3^-(z_4)S_4^-(z_4)S_5^-(z_4)\rangle_s \\
\sim &\Theta_s(\frac{1}{2}(z_3 - z_4))\Theta_s(\frac{1}{2}(z_3 - z_4) + x_- x_2)\Theta_s^2(\frac{1}{2}(z_3 - z_4) + z_1 - z_2),
\end{aligned} \tag{10.27}
$$

† Remenber that an odd spin structure is corresponded to the partition of the six branch points into two sets: $\{A_1\}$ and $\{A_2A_3A_4A_5A_6\}$ in the hyperelliptic language, see [71].

which is zero after summation over spin structures*. Thus we have $a_i = 0$ for $i = 61, \cdots, 75$ (the rest can be proved similarly by appropriately choosing the indexes μ, ν, etc.). Similarly by choosing $\mu = 1$, $\nu = \bar{1}$, $\mu_1 = 2$, $\nu_1 = 3$, $\mu_2 = \bar{2}$, $\nu_2 = \bar{4}$, $\alpha = (--+--)$ and $\beta = (---+-)$, we have

$$
\begin{aligned}
\tilde{\Lambda}_s =& a_i \delta^{\mu\nu} \delta^{\mu_1\nu_1} (\gamma^{\mu_2\nu_2})_\alpha^\beta \quad (16 \leq i \leq 60) \\
=& \frac{1}{\Theta_s(\frac{1}{2}(z_3 - z_4))} \langle \psi^1(x_1) \bar{\psi}^1(x_2) \psi^2(z_1) \psi^3(z_1) \bar{\psi}^2(z_2) \bar{\psi}^4(z_2) \\
& S_1^+(z_3) S_2^+(z_3) S_3^-(z_3) S_4^+(z_3) S_5^+(z_3) S_1^-(z_4) S_2^-(z_4) S_3^-(z_4) S_4^+(z_4) S_5^-(z_4) \rangle_s \qquad (10.28) \\
\sim& \Theta_s(\frac{1}{2}(z_3 - z_4) + x_1 - x_2) \Theta_s(\frac{1}{2}(z_3 - z_4) + z_1 - z_2) \\
& \times \Theta_s(-\frac{1}{2}(z_3 - z_4) + z_1) \Theta_s(\frac{1}{2}(z_3 - z_4) - z_2),
\end{aligned}
$$

which is also zero after summation over spin structures. We have then $a_i = 0$ for $i = 16, \cdots, 60$. Setting $\mu = 1$, $\nu = \bar{1}$, $\mu_1 = 2$, $\nu_1 = 3$, $\mu_2 = \bar{2}$, $\nu_2 = \bar{3}$ and $\alpha = \beta = (-++--)$ we get

$$
\begin{aligned}
\tilde{\Lambda}_s =& a_1 \delta^{\mu\nu} (\gamma^{\mu_1\nu_1\mu_2\nu_2})_\alpha^\beta \\
=& \frac{1}{\Theta_s(\frac{1}{2}(z_3 - z_4))} \langle \psi^1(x_1) \bar{\psi}^1(x_2) \psi^2(z_1) \psi^3(z_1) \bar{\psi}^2(z_2) \bar{\psi}^3(z_2) \\
& S_1^+(z_3) S_2^-(z_3) S_3^-(z_3) S_4^+(z_3) S_5^+(z_3) S_1^-(z_4) S_2^+(z_4) S_3^+(z_4) S_4^-(z_4) S_5^-(z_4) \rangle_s \qquad (10.29) \\
\sim& \Theta_s(\frac{1}{2}(z_3 - z_4) + x_1 - x_2) \Theta_s(\frac{1}{2}(z_3 - z_4) + z_1 - z_2) \\
& \times \Theta_s(-\frac{1}{2}(z_3 - z_4) + z_1 - z_2) \Theta_s(\frac{1}{2}(z_3 - z_4)),
\end{aligned}
$$

which is again zero after summation over spin structures. We get $a_1 = 0$ and like wise $a_i = 0$, $i = 2, \cdots, 15$. The proof of $a_0 = 0$ is achieved by choosing $\mu = 1$, $\nu = \bar{1}$, $\mu_1 = 2$,

---

* The strategy for the choice of a specific choice of the configuration of $(\mu\nu\mu_1\nu_1\mu_2\nu_2\alpha\beta)$ is that we should have at least a factor $\Theta_s(\frac{1}{2}(z_3 - z_4))$ coming from the remaining correlators (so we can use Riemann identity) and there is only one term contributing to $\tilde{\lambda}_s$. By using the Riemann identity one has always a multiplicative factor $\Theta_s(\frac{1}{2}(x_1 - x_2))$ which is zero by eq.(10.25). Similar remark holds for what follows.

$\nu_1 = 3$, $\mu_2 = \bar{2}$, $\nu_2 = \bar{3}$ and $\alpha = \beta = (+++--)$:

$$\begin{aligned}
\tilde{\Lambda}_s =& a_0 (\gamma^{\mu\nu\mu_1\nu_1\mu_2\nu_2})_\alpha^{\ \beta} \\
=& \frac{1}{\Theta_s(\frac{1}{2}(z_3 - z_4))} \langle \psi^1(x_1)\bar{\psi}^1(x_2)\psi^2(z_1)\psi^3(z_1)\bar{\psi}^2(z_2)\bar{\psi}^3(z_2) \\
& S_1^-(z_3)S_2^-(z_3)S_3^-(z_3)S_4^+(z_3)S_5^+(z_3)S_1^+(z_4)S_2^+(z_4)S_3^+(z_4)S_4^-(z_4)S_5^-(z_4)\rangle_s \\
\sim& \Theta_s(\frac{1}{2}(z_3 - z_4) + x_1 - x_2)\Theta_s^2(\frac{1}{2}(z_3 - z_4) + z_1 - z_2)\Theta_s(\frac{1}{2}(z_3 - z_4)),
\end{aligned} \tag{10.30}$$

which is zero after summation over spin structures and so $a_0 = 0$. That completes the proof of the nonrenormalization theorem for fermionic amplitudes at two loops. It is amusing to note that what we have proved is (K_s are some phases)

$$\sum_s K_s \tilde{\Lambda}_s = 0, \tag{10.31}$$

for generic configuration of $(\mu\nu\mu_1\nu_1\mu_2\nu_2\alpha\beta)$. By taking $(\mu\nu\mu_1\nu_1\mu_2\nu_2) = (\bar{1}12345)$ and appropriately adjusting $(\alpha\beta)$, we could get

$$\sum_s K_s \frac{\Theta_s(\frac{1}{2}(z_3 + z_4) - z_1)^2 \Theta_s(\frac{1}{2}(z_3 + z_4) - z_2)^2 \Theta_s(\frac{1}{2}(z_3 - z_4) + x_1 - x_2)}{\Theta_s(\frac{1}{2}(z_3 - z_4))} = 0, \tag{10.32}$$

which is an identity beyond the Riemann identity, eq.(10.24). It is also possible that one can get some generalized Riemann identities as those used in [15], but their role in the calculation of the 4F amplitude has not been sorted out.

In summary, the only contribution to the amplitude is

$$\begin{aligned}
\Lambda_s^m =& E(z_3, z_4)^{1/4} \frac{y^2(x)}{\Theta_s(\frac{1}{2}(z_3 - z_4))} \frac{\sigma(z_3)}{\sigma(z_4)} \left(\frac{E(x+, z_3)E(x-, z_3)}{E(x+, z_4)E(x-, z_4)} \right)^{1/2} \\
& k_1^{\mu_1} k_2^{\mu_2} \epsilon_1^{\nu_1} \epsilon_2^{\nu_2} k_4^{\mu} u_3^{\alpha_3}(k_3) u_4^{\alpha_4}(k_4) \lim_{w \to z_4} \left\{ (w - z_4)^{-1/2} \langle \partial X(x+) \cdot \partial X(x-) \prod e^{ik\cdot X} \rangle \right. \\
& \left. \langle \psi(x+) \cdot \psi(x-) \psi_{\mu_1}(z_1)\psi_{\nu_1}(z_1)\psi_{\mu_2}(z_2)\psi_{\nu_2}(z_2)\psi_\mu(w) S_{\alpha_3}(z_3) S_{\alpha_4}(z_4)\rangle_s \right\}.
\end{aligned} \tag{10.33}$$

Following Atick and Sen [127], a simple group theoretic analysis shows that Λ_s^m has 26 independent tensor structures. (This is the number of independent singlets in the $10 \otimes 10 \otimes 10 \otimes 10 \otimes 10 \otimes 16 \otimes 16$ representation of $SO(10)$). We write down the general expansion for the correlator as

$$\begin{aligned}
\Lambda =&\langle\psi(x+)\cdot\psi(x-)\psi^{\mu_1}(z_1)\psi^{\nu_1}(z_1)\psi^{\mu_2}(z_2)\psi^{\nu_2}(z_2)\psi^{\mu}(w)S_{\alpha_3}(z_3)S_{\alpha_4}(z_4)\rangle_s\\
&\times E(z_3,z_4)^{1/4}\frac{y^2(x)}{\Theta(\frac{1}{2}(z_3-z_4))}\frac{\sigma(z_3)}{\sigma(z_4)}\left(\frac{E(x+,z_3)E(x-,z_3)}{E(x+,z_4)E(x-,z_4)}\right)^{1/2}\\
=&A\delta^{\mu_1\nu_1}\delta^{\mu_2\nu_2}(\gamma^{\mu})_{\alpha_3\alpha_4}+B\delta^{\mu_1\nu_1}\delta^{\mu_2\mu}(\gamma^{\nu_2})_{\alpha_3\alpha_4}+C\delta^{\mu_1\nu_1}\delta^{\mu\nu_2}(\gamma^{\mu_2})_{\alpha_3\alpha_4}\\
&+D\delta^{\mu_1\mu}\delta^{\mu_2\nu_2}(\gamma^{\nu_1})_{\alpha_3\alpha_4}+E\delta^{\mu\nu_1}\delta^{\mu_2\nu_2}(\gamma^{\mu_1})_{\alpha_3\alpha_4}+F\delta^{\mu_1\mu_2}\delta^{\nu_1\nu_2}(\gamma^{\mu})_{\alpha_3\alpha_4}\\
&+G\delta^{\mu_1\nu_2}\delta^{\mu_2\nu_1}(\gamma^{\mu})_{\alpha_3\alpha_4}+H(\gamma^{\mu}\gamma^{\mu_1}\gamma^{\nu_1})_{\alpha_4\alpha_3}\delta^{\mu_2\nu_2}+I(\gamma^{\mu}\gamma^{\mu_1}\gamma^{\mu_2})_{\alpha_4\alpha_3}\delta^{\nu_1\nu_2}\\
&+J(\gamma^{\mu}\gamma^{\mu_1}\gamma^{\nu_2})_{\alpha_4\alpha_3}\delta^{\mu_2\nu_1}+K(\gamma^{\mu}\gamma^{\mu_2}\gamma^{\nu_1})_{\alpha_4\alpha_3}\delta^{\mu_1\nu_2}+L(\gamma^{\mu}\gamma^{\mu_2}\gamma^{\nu_2})_{\alpha_4\alpha_3}\delta^{\mu_1\nu_1}\\
&+M(\gamma^{\mu}\gamma^{\nu_1}\gamma^{\nu_2})_{\alpha_4\alpha_3}\delta^{\mu_2\mu_1}+O(\gamma^{\mu}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\nu_1}\gamma^{\nu_2})_{\alpha_4\alpha_3}\\
&+N\delta^{\mu_2\nu_1}\delta^{\mu\nu_2}(\gamma^{\mu_1})_{\alpha_3\alpha_4}+P\delta^{\nu_2\nu_1}\delta^{\mu_2\mu}(\gamma^{\mu_1})_{\alpha_3\alpha_4}+Q\delta^{\mu_1\mu}\delta^{\nu_1\nu_2}(\gamma^{\mu_2})_{\alpha_3\alpha_4}\\
&+R\delta^{\mu_1\nu_2}\delta^{\mu\nu_1}(\gamma^{\mu_2})_{\alpha_3\alpha_4}+S\delta^{\mu_1\nu_2}\delta^{\mu_2\mu}(\gamma^{\nu_1})_{\alpha_3\alpha_4}+T\delta^{\mu_1\mu_2}\delta^{\mu\nu_2}(\gamma^{\nu_1})_{\alpha_3\alpha_4}\\
&+U\delta^{\mu_1\mu_2}\delta^{\mu\nu_1}(\gamma^{\nu_2})_{\alpha_3\alpha_4}+V\delta^{\mu_1\mu}\delta^{\mu_2\nu_1}(\gamma^{\nu_2})_{\alpha_3\alpha_4}+W(\gamma^{\mu_1}\gamma^{\nu_2}\gamma^{\nu_1})_{\alpha_4\alpha_3}\delta^{\mu\mu_2}\\
&+X(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\nu_1})_{\alpha_4\alpha_3}\delta^{\mu\nu_2}+Y(\gamma^{\mu_2}\gamma^{\nu_1}\gamma^{\nu_2})_{\alpha_4\alpha_3}\delta^{\mu\mu_1}+Z(\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\nu_2})_{\alpha_4\alpha_3}\delta^{\mu\nu_1}.
\end{aligned}\tag{10.34}$$

The tensor structure have been chosen in such a way that the contribution from the tensors multiplying A to O vanish by on-shell constraints when substituted in eq.(10.33). Hence we only need to calculate the coefficients from N to Z. The calculation simplifies by noting that the result must be symmetric under the simultaneous exchange $\mu_1 \leftrightarrow \mu_2$, $\nu_1 \leftrightarrow \nu_2$, $z_1 \leftrightarrow z_2$. Symmetry under this exchange gives

$$\begin{aligned}
W(z_1,z_2)=Y(z_2,z_1),\qquad R(z_1,z_2)=N(z_2,z_1),\qquad P(z_1,z_2)=Q(z_2,z_1)\\
X(z_1,z_2)=Z(z_2,z_1),\qquad S(z_1,z_2)=V(z_2,z_1),\qquad T(z_1,z_2)=U(z_2,z_1)
\end{aligned}\tag{10.35}$$

This has the advantage of cutting down the number of terms that we need to calculate by half.

Let us now set $\mu_1=3$, $\nu_1=\bar{1}$, $\mu_2=1$, $\nu_2=\bar{2}$, $\mu=2$, $\alpha_3=(----+)$ and $\alpha_4=(++-+-)$. The only term in (10.34) that contributes for this configuration is N. It is

$$\begin{aligned}
N =&E(z_3,z_4)^{1/4}\frac{y^2(x)}{\Theta_s(\frac{1}{2}(z_3-z_4))}\frac{\sigma(z_3)}{\sigma(z_4)}\left(\frac{E(x+,z_3)E(x-,z_3)}{E(x+,z_4)E(x-,z_4)}\right)^{1/2}\\
&\langle\psi^3(z_1)\bar{\psi}^1(z_1)\psi^1(z_2)\bar{\psi}^2(z_2)\psi^2(w)\sum_{i=1}^{5}(\psi^i(x+)\bar{\psi}^i(x-)-\psi^i(x-)\bar{\psi}^i(x+))\\
&S_1^-(z_3)S_2^-(z_3)S_3^-(z_3)S_4^-(z_3)S_5^+(z_3)S_1^+(z_4)S_2^+(z_4)S_3^-(z_4)S_4^+(z_4)S_5^-(z_4)\rangle_s.
\end{aligned}\tag{10.36}$$

Next consider the configuration $\mu_1 = 3$, $\nu_1 = \bar{1}$, $\mu_2 = 1$, $\nu_2 = \bar{2}$, $\mu = 2$, $\alpha_3 = (+----)$ and $\alpha_4 = (-+-++)$. From (10.34) we see that this contribution is given by $-N - X$. On the other hand the application of the Riemann Θ-identity shows that this correlator vanishes (apart from a prefactor $E(z_4, w)^{1/2}$ which is needed to cancel the divergence factor $(w - z_4)^{-1/2}$ in (10.34)) in the limit $w \to z_4$. Thus we conclude that

$$N \simeq -X, \tag{10.37}$$

where $\simeq$ denotes equality up to the desired accuracy. Evaluation of the correlator with $\mu_1 = 1$, $\nu_1 = 3$, $\mu_2 = \bar{2}$, $\nu_2 = \bar{1}$, $\mu = 2$, $\alpha_3 = (---+-)$ and $\alpha_4 = (++--+)$ gives

$$S \simeq 0, \tag{10.38}$$

and with $\mu_1 = \bar{2}$, $\nu_1 = 3$, $\mu_2 = 2$, $\nu_2 = \bar{1}$, $\mu = 1$, $\alpha_3 = (++-+-)$ and $\alpha_4 = (----+)$ gives

$$T \simeq 0. \tag{10.39}$$

Finally setting $\mu_1 = 3$, $\nu_1 = \bar{1}$, $\mu_2 = \bar{2}$, $\nu_2 = 1$, $\mu = 2$, $\alpha_3 = (----+)$ and $\alpha_4 = (++-+-)$ gives

$$\begin{aligned} P =& E(z_3, z_4)^{1/4} \frac{y^2(x)}{\Theta_s(\frac{1}{2}(z_3 - z_4))} \frac{\sigma(z_3)}{\sigma(z_4)} \left(\frac{E(x+, z_3)E(x-, z_3)}{E(x+, z_4)E(x-, z_4)} \right)^{1/2} \\ & \langle \psi^3(z_1)\bar{\psi}^1(z_1)\bar{\psi}^2(z_2)\psi^1(z_2)\psi^2(w) \sum_{i=1}^{5} (\psi^i(x+)\bar{\psi}^i(x-) - \psi^i(x-)\bar{\psi}^i(x+)) \\ & S_1^-(z_3)S_2^-(z_3)S_3^-(z_3)S_4^-(z_3)S_5^-(z_3)S_1^+(z_4)S_2^+(z_4)S_3^-(z_4)S_4^+(z_4)S_5^-(z_4)\rangle_s \\ =& -N, \end{aligned} \tag{10.40}$$

and $\mu_1 = 3$, $\nu_1 = \bar{1}$, $\mu_2 = \bar{2}$, $\nu_2 = 1$, $\mu = 2$, $\alpha_3 = (+----)$ and $\alpha_4 = (-+-++)$ gives

$$P = -W. \tag{10.41}$$

Here we have determined the phase η introduced in [9]. So what we need to calculate is N only and all the others can be obtained by the above relations and by using eq.(10.35).

By using eq.(10.15), we have

$$
\begin{aligned}
N =& E(w,z_4)^{1/2}\frac{y^2(x)}{E(z_1,z_2)E(z_3,z_4)E(z_2,z_4)E(z_1,z_4)}\frac{\sigma(z_3)}{\sigma(z_4)}\\
&\frac{1}{E(x_1,x_2)}\left(\frac{E(x_1,z_3)E(x_2,z_4)}{E(x_1,z_4)E(x_2,z_3)}\right)^{1/2}\left(\frac{E(x+,z_3)E(x-,z_3)}{E(x+,z_4)E(x-,z_4)}\right)^{1/2}\\
&K_s\Theta_s(\frac{1}{2}(z_3-z_4)+z_1-z_2)\Theta_s(\frac{1}{2}(z_3-z_4)+z_2-z_4)\\
&\Theta_s(\frac{1}{2}(z_3+z_4)-z_1)\Theta_s(\frac{1}{2}(z_3-z_4)+x_1-x_2),
\end{aligned}
\tag{10.42}
$$

where we write only one term. All the remaining nine terms give identical results as can be easily seen from what follows. With the help of Riemann Θ-identity (10.24), one can do the summation over spin structures and gets

$$
\begin{aligned}
N =& E(w,z_4)^{1/2}\frac{y^2(x)}{E(z_1,z_2)E(z_3,z_4)E(z_2,z_4)E(z_1,z_4)}\frac{\sigma(z_3)}{\sigma(z_4)}\\
&\frac{1}{E(x_1,x_2)}\left(\frac{E(x_1,z_3)E(x_2,z_4)}{E(x_1,z_4)E(x_2,z_3)}\right)^{1/2}\left(\frac{E(x+,z_3)E(x-,z_3)}{E(x+,z_4)E(x-,z_4)}\right)^{1/2}\\
&\times 4\Theta_T(z_1-z_2+\frac{1}{2}(x_1-x_2))\Theta_T(z_1-z_4-\frac{1}{2}(x_1-x_2))\\
&\Theta_T(z_4-z_2-\frac{1}{2}(x_1-x_2))\Theta_T(z_3-z_4+\frac{1}{2}(x_1-x_2)).
\end{aligned}
\tag{10.43}
$$

The next problem is to write $\Theta_T(z_1-z_2+\frac{1}{2}(x_1-x_2))$ in hyperelliptic language. This can be done also with the help of Riemann Θ-identity. We have

$$
\begin{aligned}
\Theta_T^4(z_1-z_2+\frac{1}{2}(x_1-x_2)) =& \frac{1}{4}\sum_s \eta'_s\Theta_s^3(0)\Theta_s(2(z_1-z_2)+x_1-x_2)\\
=& -c\sum_s \eta_s Q_s\frac{\Theta_s(2(z_1-z_2)+x_1-x_2)}{\Theta_s(0)},
\end{aligned}
\tag{10.44}
$$

where we have used the Thomae formula [20] (see also eq.(3.41)):

$$
\Theta_s^4(0) = \pm\det\ {}^2K\prod_{i<j}^{3}(A_i-A_j)(B_i-B_j) \equiv \pm 4c\cdot Q_s,
\tag{10.45}
$$

where Q_s is defined as in [1] for even spin structures. Notice that the constant c depends on the homology basis chosen. Therefore it is not invariant under modular transformation eventhough it is spin structure independent. Notice that in eq.(10.44) the dependence on the reference spin structure drops (see eq.(10.48) below). This seem a little bit puzzling. The point is that on the left hand side of eq.(10.44) the choice for the integration contour from x_1 to x_2 also depends on the spin structure T. The dependence on T then drops due to these two factors.

By using the generalized Fay triscecant identity, eq.(3.29)

$$\frac{\Theta_s(\sum_{i=1}^n u_i - \sum_{i=1}^n v_i)}{\Theta_s(0)} \frac{\prod_{i<j}^n E(u_i,u_j)E(v_i,v_j)}{\prod_{i,j}^n E(u_i,v_j)} = (-1)^{n(n-1)/2} \det\langle\psi(u_i)\psi(v_j)\rangle_s, \tag{10.46}$$

we have

$$\begin{aligned} &\frac{\Theta_s(2(z_1-z_2)+x_1-x_2)}{\Theta_s(0)} \frac{(E(z_1,x_1)E(z_2,x_2))^2}{(E(z_1,x_2)E(z_2,x_1))^2 E(z_1,z_2)^4 E(x_1,x_2)} \\ &= -\begin{vmatrix} \langle\psi(x_1)\psi(x_2)\rangle_s & \langle\psi(x_1)\psi(z_2)\rangle_s & \langle\psi(x_1)\partial\psi(z_2)\rangle_s \\ \langle\psi(z_1)\psi(x_2)\rangle_s & \langle\psi(z_1)\psi(z_2)\rangle_s & \langle\psi(z_1)\partial\psi(z_2)\rangle_s \\ \langle\partial\psi(z_1)\psi(x_2)\rangle_s & \langle\partial\psi(z_1)\psi(z_2)\rangle_s & \langle\partial\psi(z_1)\partial\psi(z_2)\rangle_s \end{vmatrix}. \end{aligned} \tag{10.47}$$

By substituting (10.47) into (10.44) we get

$$\begin{aligned} &\Theta_T(z_1-z_2+\frac{1}{2}(x_1-x_2)) \frac{(E(z_1,x_1)E(z_2,x_2))^{1/2}}{(E(z_1,x_2)E(z_2,x_1))^{1/2} E(z_1,z_2) E(x_1,x_2)^{1/4}} \frac{1}{\sqrt[4]{c}} \\ &= \left(\sum_s \eta_s Q_s \begin{vmatrix} \langle\psi(x_1)\psi(x_2)\rangle_s & \langle\psi(x_1)\psi(z_2)\rangle_s & \langle\psi(x_1)\partial\psi(z_2)\rangle_s \\ \langle\psi(z_1)\psi(x_2)\rangle_s & \langle\psi(z_1)\psi(z_2)\rangle_s & \langle\psi(z_1)\partial\psi(z_2)\rangle_s \\ \langle\partial\psi(z_1)\psi(x_2)\rangle_s & \langle\partial\psi(z_1)\psi(z_2)\rangle_s & \langle\partial\psi(z_1)\partial\psi(z_2)\rangle_s \end{vmatrix} \right)^{1/4}, \end{aligned} \tag{10.48}$$

which can be a modular invariant expression if we choose the phases η_s as those used in [12-16] and in sections 6 and 7. At this point it is quite clear that the insertions of the supercurrent is very crucial to obtain a modular invariant expression. It is not clear how this can be recognized from the very begining of the calculation. The modular invariance of the fermionic amplitude (at two loops) is still an open problem.

The calculation of the modular invariant expression, eq.(10.48), is a little bit difficult, but nevertheless it can be carried out. We will sketch the main steps of the calculation here.

On genus 2 hyperelliptic Riemann surface we have (eq.(3.60))

$$\langle\psi(z)\psi(w)\rangle = \frac{1}{z-w} \frac{u(z)+u(w)}{2\sqrt{u(z)u(w)}}, \tag{10.49}$$

where the function $u(z)$ is defined by eq.(2.9). Then we have

$$\begin{aligned}
\langle\partial\psi(z)\psi(w)\rangle &= -\frac{1}{(z-w)^2}\frac{u(z)+u(w)}{2\sqrt{u(z)u(w)}} + \frac{1}{z-w}\frac{u(z)-u(w)}{2\sqrt{u(z)u(w)}}\cdot\frac{1}{2}\partial\ln u(z), \\
\langle\partial\psi(z)\partial\psi(w)\rangle &= -\frac{2}{(z-w)^3}\frac{u(z)+u(w)}{2\sqrt{u(z)u(w)}} \\
&\quad + \frac{1}{(z-w)^2}\frac{u(z)-u(w)}{2\sqrt{u(z)u(w)}}\cdot\left(\frac{1}{2}\partial\ln u(z)+\frac{1}{2}\partial\ln u(w)\right) \\
&\quad + \frac{1}{z-w}\frac{u(z)+u(w)}{2\sqrt{u(z)u(w)}}\cdot\frac{1}{2}\partial\ln u(z)\cdot\frac{1}{2}\partial\ln u(w).
\end{aligned} \tag{10.50}$$

In order to do the summation over spin structures, we write eq.(10.60) as follows:

$$\begin{aligned}
\Theta T &\equiv \sum_s \eta_s Q_s \begin{vmatrix} \langle\psi(x_1)\psi(x_2)\rangle_s & \langle\psi(x_1)\psi(z_2)\rangle_s & \langle\psi(x_1)\partial\psi(z_2)\rangle_s \\ \langle\psi(z_1)\psi(x_2)\rangle_s & \langle\psi(z_1)\psi(z_2)\rangle_s & \langle\psi(z_1)\partial\psi(z_2)\rangle_s \\ \langle\partial\psi(z_1)\psi(x_2)\rangle_s & \langle\partial\psi(z_1)\psi(z_2)\rangle_s & \langle\partial\psi(z_1)\partial\psi(z_2)\rangle_s \end{vmatrix} \\
&\equiv \Theta T_1 + \Theta T_2 + \Theta T_3,
\end{aligned} \tag{10.51}$$

by expanding the determinant with the first column. It is not quite easy to see that

$$\Theta T_2 \equiv -\sum_s \eta_s Q_s \langle\psi(z_1)\psi(x_2)\rangle_s \begin{vmatrix} \langle\psi(x_1)\psi(z_2)\rangle_s & \langle\psi(x_1)\partial\psi(z_2)\rangle_s \\ \langle\partial\psi(z_1)\psi(z_2)\rangle_s & \langle\partial\psi(z_1)\partial\psi(z_2)\rangle_s \end{vmatrix} = 0, \tag{10.52}$$

and

$$\Theta T_3 \equiv \sum_s \eta_s Q_s \langle\partial\psi(z_1)\psi(x_2)\rangle_s \begin{vmatrix} \langle\psi(x_1)\psi(z_2)\rangle_s & \langle\psi(x_1)\partial\psi(z_2)\rangle_s \\ \langle\psi(z_1)\psi(z_2)\rangle_s & \langle\psi(z_1)\partial\psi(z_2)\rangle_s \end{vmatrix} = 0, \tag{10.53}$$

but they are true. The reason why these two expressions are vanishing is very simple: they do not have the right pole structure as $\frac{1}{y^2(x)}$ as a function of x. One can also prove $\Theta T_2 = \Theta T_3 = 0$ by explicit computation. Let us see, for example ΘT_3. We have (omitting some spin structure independent factors)

$$
\begin{aligned}
\Theta T_3 \sim & \sum_s \eta_s Q_s \left(\frac{u(z_1)-u(x)}{\sqrt{u(z_1)u(x)}} + \frac{u(z_1)+u(x)}{\sqrt{u(z_1)u(x)}} \partial \ln u(z_1) \right) \\
& \times \begin{vmatrix} \frac{u(z_2)+u(x)}{\sqrt{u(z_2)u(x)}} & \frac{u(z_2)+u(x)}{\sqrt{u(z_2)u(x)}} + \frac{u(z_2)-u(x)}{\sqrt{u(z_2)u(x)}} \partial \ln u(z_2) \\ \frac{u(z_1)+u(z_2)}{\sqrt{u(z_1)u(z_2)}} & \frac{u(z_1)+u(z_2)}{\sqrt{u(z_1)u(z_2)}} + \frac{u(z_1)-u(z_2)}{\sqrt{u(z_1)u(z_2)}} \partial \ln u(z_2) \end{vmatrix} \\
\sim & + \sum_s \eta_s Q_s \frac{u(z_1)-u(x)}{\sqrt{u(z_1)u(x)}} \begin{vmatrix} \frac{u(z_2)+u(x)}{\sqrt{u(z_2)u(x)}} & \frac{u(z_2)+u(x)}{\sqrt{u(z_2)u(x)}} \\ \frac{u(z_1)+u(z_2)}{\sqrt{u(z_1)u(z_2)}} & \frac{u(z_1)+u(z_2)}{\sqrt{u(z_1)u(z_2)}} \end{vmatrix} \\
& + \sum_s \eta_s Q_s \frac{u(z_1)-u(x)}{\sqrt{u(z_1)u(x)}} \begin{vmatrix} \frac{u(z_2)+u(x)}{\sqrt{u(z_2)u(x)}} & \frac{u(z_2)-u(x)}{\sqrt{u(z_2)u(x)}} \\ \frac{u(z_1)+u(z_2)}{\sqrt{u(z_1)u(z_2)}} & \frac{u(z_1)-u(z_2)}{\sqrt{u(z_1)u(z_2)}} \end{vmatrix} \partial \ln u(z_2) \\
& + \sum_s \eta_s Q_s \frac{u(z_1)+u(x)}{\sqrt{u(z_1)u(x)}} \begin{vmatrix} \frac{u(z_2)+u(x)}{\sqrt{u(z_2)u(x)}} & \frac{u(z_2)+u(x)}{\sqrt{u(z_2)u(x)}} \\ \frac{u(z_1)+u(z_2)}{\sqrt{u(z_1)u(z_2)}} & \frac{u(z_1)+u(z_2)}{\sqrt{u(z_1)u(z_2)}} \end{vmatrix} \partial \ln u(z_1) \\
& + \sum_s \eta_s Q_s \frac{u(z_1)+u(x)}{\sqrt{u(z_1)u(x)}} \begin{vmatrix} \frac{u(z_2)+u(x)}{\sqrt{u(z_2)u(x)}} & \frac{u(z_2)-u(x)}{\sqrt{u(z_2)u(x)}} \\ \frac{u(z_1)+u(z_2)}{\sqrt{u(z_1)u(z_2)}} & \frac{u(z_1)-u(z_2)}{\sqrt{u(z_1)u(z_2)}} \end{vmatrix} \partial \ln u(z_1) \partial \ln u(z_2).
\end{aligned}
\tag{10.54}
$$

The first term in(10.54) can be easily seen to vanish by using of the Lianzi identities (eq.(5.39)). The vanishing of the second and the third terms leads to

$$
\begin{aligned}
& \sum_s \eta_s Q_s \partial \ln u(z) = 0, \\
& \sum_s \eta_s Q_s \left(\frac{u(z_1)}{u(z_2)} - \frac{u(z_2)}{u(z_1)} \right) \partial \ln u(z_3) = 0,
\end{aligned}
\tag{10.55}
$$

which are in fact Lianzi identities. They are obtained from (5.30) just by a Möbius transformation. The vanishing of the last term in (10.54) requires that

$$
\sum_s \eta_s Q_s \left(\frac{u(z_1)}{u(x)} - \frac{u(x)}{u(z_1)} \right) \partial \ln u(z_1) \partial \ln u(z_2) = 0. \tag{10.56}
$$

This identity can be proved as follows. By Möbius transformation it is enough to prove taht (10.56) is true for $z_1 = \infty$. We have ($z_1 = \infty$)

$$\begin{aligned}
\text{LHS of (10.56)} &\sim \sum_s \eta_s Q_s \left(\prod_{i=1}^{3}(x - A_i) - \prod_{i=1}^{3}(x - B_i) \right) \\
&\quad \times \sum_{i=1}^{3} \left(\frac{1}{z_2 - A_i} - \frac{1}{z_2 - B_i} \right) \sum_{i=1}^{3}(A_i - B_i) \\
&\sim \sum_s \eta_s Q_s \left(\prod_{i=1}^{3}(x - A_i) - \prod_{i=1}^{3}(x - B_i) \right) \sum_{i=1}^{3}(A_i - B_i) \\
&\quad \times \sum_{i=1}^{3} \left(\prod_{j=1}^{3}(z_2 - B_j) \prod_{j \neq i}(z_2 - A_j) - (A \leftrightarrow B) \right) \equiv \Theta T_3'.
\end{aligned} \tag{10.57}$$

By power counting the last expression ($\Theta T_3'$) is a homogeneous polynominal of degree 15 and therefore it is just a multiple of $P(a) = \prod_{i<j}^{6} a_{ij}$. We can set x and z_2 to zero to calculate this expression. We have then

$$\Theta T_3' \sim \sum_s \eta_s Q_s (\prod_{i=1}^{3} A_i - \prod_{i=1}^{3} B_i \sum_{i=1}^{3}(A_i - B_i) \sum_{i=1}^{3} \left(\prod_{j=1}^{3} B_j \prod_{j \neq i} A_j - (A \leftrightarrow B) \right) . \tag{10.58}$$

By explicitly calculating one term, $a_1^5 a_2^4 a_3^3 a_4^2 a_5$ for example, one can show that it is zero*. That completes our proof of eq.(10.56) and therefore eq.(10.53). Similarly one can prove eq.(10.52).

---

* Keeping only the term $a_1^5 a_2^4 a_3^3 a_4^2 a_5$ we have (first keeping the terms containing a_1^5 and no a_6)

$$\begin{aligned}
\Theta T_3' \sim &+ a_1^2(a_2 - a_3)P(a_4 \underline{a_5 a_6})a_1 a_2 a_3(-a_1 a_2 a_3(a_4 \underline{a_5}))a_1 \\
&- a_1^2(a_2 - a_4)P(a_3 \underline{a_5 a_6})a_1 a_2 a_4(-a_1 a_2 a_4(a_3 \underline{a_5}))a_1 \\
&+ a_1^2(a_2 - a_5)P(a_3 a_4 a_6)a_1 a_2 \underline{a_5}(-a_1 a_2 \underline{a_5}(a_3 a_4))a_1 \\
&+ a_1^2(a_3 - a_4)P(a_2 \underline{a_5 a_6})a_1 a_3 a_4(-a_1 a_3 a_4(a_2 \underline{a_5}))a_1 \\
&- a_1^2(a_3 - a_5)P(a_2 a_4 a_6)a_1 a_3 \underline{a_5}(-a_1 a_3 \underline{a_5}(a_2 a_4))a_1 \\
&+ a_1^2(a_4 - a_5)P(a_2 a_3 a_6)a_1 a_4 \underline{a_5}(-a_1 a_4 \underline{a_5}(a_2 a_3))a_1,
\end{aligned}$$

whee $P(a_i a_j a_k) = (a_i - a_j)(a_i - a_k)(a_j - a_k)$. The various terms vanish because they either give a_5^2 or a_6 as indicated above.

By using of eqs.(10.52) and (10.53) we get

$$\begin{aligned}
\Theta T = \Theta T_1 &\equiv \sum_s \eta_s Q_s \langle\psi(x_1)\psi(x_2)\rangle_s \begin{vmatrix} \langle\psi(z_1)\psi(z_2)\rangle_s & \langle\psi(z_1)\partial\psi(z_2)\rangle_s \\ \langle\partial\psi(z_1)\psi(z_2)\rangle_s & \langle\partial\psi(z_1)\partial\psi(z_2)\rangle_s \end{vmatrix} \\
&= \frac{1}{16}\frac{1}{(z_1-z_2)^2}\sum_s \eta_s Q_s \sum_{i=1}^{3}\left(\frac{1}{x-A_i}-\frac{1}{x-B_i}\right) \\
&\times \left\{\frac{1}{(z_1-z_2)^2}\cdot\frac{(u(z_1)+u(z_2))^2}{u(z_1)u(z_2)} + \partial\ln u(z_1)\cdot\partial\ln u(z_2)\right\} \\
&= \frac{1}{64}\frac{1}{(z_1-z_2)^2}\sum_s \eta_s Q_s \sum_{i=1}^{3}\left(\frac{1}{x-A_i}-\frac{1}{x-B_i}\right)\prod_{j=1}^{2}\sum_{i=1}^{3}\left(\frac{1}{z_j-A_i}-\frac{1}{z_j-B_i}\right).
\end{aligned} \tag{10.59}$$

The calculation of the last factor is simplified by modular invariance and projective invariance considerations. First it is modular invariant and therefore should be proportional to $P(a)=\prod_{i<j}^{6}(a_i-a_j)\equiv\prod_{i<j}^{6}a_{ij}$. Second, it should have a factor $y^2(x)y^2(z_1)y^2(z_2)=\prod_{i=1}^{6}(x-a_i)(z_1-a_i)(z_2-a_i)$ in the denominator. And finally by taking into account of the projective transformation properties of ΘT and power counting, the rest factor can only be proportional to $(x-z_1)^2(x-z_2)^2(z_1-z_2)^2$. By assembling all these together we have

$$\Theta T = c'\frac{(x-z_1)^2(x-z_2)^2P(a)}{y^2(x)y^2(z_1)y^2(z_2)}, \tag{10.60}$$

where c' is a constant: $c'=-\frac{1}{16}$.

By substituting (10.60) and (10.48) into (10.53), we get

$$\begin{aligned}
N =& C'P(a)E(z_4,w)^{1/2}\frac{(z_1-x)(z_2-x)\sqrt{z_3-x}\sqrt{(z_4-x)^3}}{y(z_1)y(z_2)\sqrt{y(z_3)}\sqrt{y^3(z_4)}} \\
&\times\frac{\sigma(z_3)}{\sigma(z_4)}\left(\frac{E(x+,z_3)E(x-,z_3)}{E(x+,z_4)E(x-,z_4)}\right)^{1/2} \\
=& C'P(a)E(z_4,w)^{\frac{1}{2}}\prod_{i=i}^{4}\frac{z_i-x}{y(z_i)},
\end{aligned} \tag{10.61}$$

by using of the following expression for $\sigma(z)$:

$$\sigma(z)=\frac{1}{\sqrt{y(z)}}\left(\frac{x-z}{E(x+,z)E(x-,z)}\right)^{1/2}\times(\text{factors independent of } z). \tag{10.62}$$

One sees that the above expression for $\sigma(z)$ has the main properties expected: one differential in z, no poles and zeros, but we don't have a proof of this relation at the moment.

By using of (10.36)–(10.41) and (10.35), one can calculate all the coefficients from N to Z appearing in (10.34). We get

$$\begin{aligned}
&N = +R = -X = -Z = -P \\
&\quad = -Q = +W = +Y = C'P(a)\sqrt{z_4 - w}\prod_{i=1}^{4}\frac{z_i - x}{y(z_i)}, \\
&T = S = U = V = 0,
\end{aligned} \tag{10.63}$$

with this (10.35) reduces to

$$\begin{aligned}
&C'\sqrt{z_4 - w}\,[(\gamma^{\mu_1}\gamma^{\nu_1}\gamma^{\mu_2})_{\alpha_4\alpha_3}\delta^{\mu\nu_2} + (\gamma^{\mu_2}\gamma^{\nu_2}\gamma^{\mu_1})_{\alpha_4\alpha_3}\delta^{\mu\nu_1} \\
&\quad -(\gamma^{\mu_1}\gamma^{\nu_1}\gamma^{\nu_2})_{\alpha_4\alpha_3}\delta^{\mu\mu_2} - (\gamma^{\mu_2}\gamma^{\nu_2}\gamma^{\nu_1})_{\alpha_4\alpha_3}\delta^{\mu\mu_1}]\prod_{i=1}^{4}\frac{z_i - x}{y(z_i)},
\end{aligned} \tag{10.64}$$

and the final amplitude takes the form (see [13, 15, 16]):

$$\begin{aligned}
A_{2F-2B} =& C\cdot K_{2F-2B}\int\frac{\prod_{i=1}^{6} d^2a_i/dV_{pr}}{T^5\prod_{i<j}a_{ij}}\prod_{i=1}^{4}\frac{d^2z_i(z_i - x)}{y(z_i)}\times\bar{F}(\bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4) \\
&\left\{\langle\partial X(x+)\cdot\partial X(x-)\prod e^{ik\cdot X}\rangle + (\text{ghost contributions})\right\},
\end{aligned} \tag{10.65}$$

where

$$\begin{aligned}
T &= \int\left|\frac{(z_1 - z_2)dz_1dz_2}{y(z_1)y(z_2)}\right|^2, \\
dV_{pr} &= \frac{d^2a_id^2a_jd^2a_k}{|a_{ij}a_{jk}a_{ki}|^2},
\end{aligned} \tag{10.66}$$

and

$$\begin{aligned}
K_{2F-2B} =& 2k_2\cdot k_4u_4\epsilon_1\cdot\gamma(k_1 + k_4)\cdot\gamma\epsilon_2\cdot\gamma u_3 \\
&+ 2k_1\cdot k_4u_4\epsilon_2\cdot\gamma(k_2 + k_4)\cdot\gamma\epsilon_1\cdot\gamma u_3,
\end{aligned} \tag{10.67}$$

is the standard kinematic factor for 2F–2B scattering [1, 2]. Here $\bar{F}$ is the contribution to the amplitude from the left (anti-analytic) part. It is seen from the above result that the only difference between 2F–2B scattering and 4B scattering (for matter supercurent part) is the kinematic factor. So we have

$$\langle\partial X(x+)\cdot\partial X(x-)\prod e^{ik\cdot X}\rangle + (\text{ghost contributions}) = -4I(x), \qquad \text{when } k\to 0, \tag{10.68}$$

where $I(x)$ is the same quantity which is defined in section 6, eq.(6.18) appearing in two-loop computations. This could also be derived by imposing of BRST invariance. This concludes our calculation of the 2F-2B scattering amplitude in closed superstrings.

11. Multi-Loop Calculation

So much about two loops. How about multi-loops? Does the vacuum amplitues vanish? Everybody answers: Yes, it does. Some people even offer one or two proofs for arbitrary loops. Nevertheless all of them are erroneous, partially because of our ignorance of the supermoduli. By choosing the super-Beltrami differentials located at moduli independent points x_a: $\chi_a = \delta(z - x_a)$, the vacuum amplitude can be written as

$$\begin{aligned} Z_g = \int_{M_g} \frac{\prod_i d^2 m_i}{(\det \mathrm{Im}\tau)^5} \sum_s \eta_s (\det \bar{\partial}_0)^{-5} \det \bar{\partial}_2 (\det{}_s \bar{\partial}_{1/2})^5 (\det{}_s \bar{\partial}_{3/2})^{-1} \\ \times \frac{\langle J(x_1) \cdots J(x_{2g-2}) \rangle_s}{\det \varphi^a(x_b)} \times (\text{left sector}), \end{aligned} \tag{11.1}$$

where $\langle J(x_1) \cdots \rangle_s$ is defined as before in (5.8) (with more insertions of $\delta(\beta(x_a))$ and $b(b_i)$). The chiral determinats appearing in (11.1) can be explicitly computed by using the Θ-constants. Roughly speaking the spin structure dependent determinants are [131, 98, 116]

$$\det{}_s \bar{\partial}_{3/2} \sim \det{}_s \bar{\partial}_{1/2} = (\det \bar{\partial}_0)^{-1/2} \Theta_s(0|\tau). \tag{11.2}$$

By using these results we get

$$(\det \bar{\partial}_0)^{-5} \det \bar{\partial}_2 (\det{}_s \bar{\partial}_{1/2})^5 (\det{}_s \bar{\partial}_{3/2})^{-1} \sim \Theta_s^4(0|\tau). \tag{11.3}$$

So the spin structure dependent factor is just $\Theta_s^4(0)$. If there were no insertions of super-curent, we would get

$$Z_g \sim \frac{\prod_i d^2 m_i}{(\det \mathrm{Im}\tau)^5} \sum_s \eta_s \Theta_s^4(0) \times (\text{left sector}). \tag{11.4}$$

With appropriate choice of the phases η_s, we have

$$\sum_s \eta_s \Theta_s^4(0) = 0, \tag{11.5}$$

by using the Riemann Θ-identity, eq.(3.13). Then the vacuum amplitude vanishes. However there are a number of questions arising by using this ansatz. First modular invariance plays no roles here. In fact the Riemann Θ-identity (11.5) is not modular invariant. It depends on a reference odd spin structure. One way to overcome this difficult is to argue that modular invariance was broken and could be restored by a resummation over all the odd spin structures [106]. In my opinion this is too naive to be true. Second the Riemannn Θ-identity does not have the right factorization property. This was pointed out by Moore and Morozov in [106] at two loops. As shown in [13] if the contribution of the supercurrent is included, the vacuum amplitude does have the right factorization limit. That shows supercurrent insertions in the vacuum amplitude (which is induced by integration over supermoduli) must play a crucial role to ensure modular invariance and factorization.

Another approach [82, 83, 84, 106] to prove the vanishing of the vacuum amplitude at low genus ($g \leq 5$) is to explicitly calculate the supercurent correlator $\langle J(x_1)\cdots J(x_{2g-2})\rangle$ by an appropriate choice of the insertion points x_a ($a = 1, \cdots, 2g-2$). A natural choice of these points seems to be the zeroes of an abelian differential, which leads to a pointwise vanishing of the vacuum amplitude and n-particle amplitudes ($n \leq 3$) in the moduli space, see previous sections and [14]. At two loops, by using hyperelliptic representation of genus $g = 2$ Riemann surface one sees that the requirement of modular invariance completely determines the phases in the summation over spin structures. The proper interpretation of these results in the Θ-function language leads to the concept of modular covariance. That is, one introduces an arbitrary reference spin structure at the intermediate steps. The final result does not depend on this spin structure and is modular invariant. They [84] further extend these results to multi-loops, claiming the vanishing of the vacuum amplitudes up to $g \leq 5$ and also part of the nonrenomalization theorem. Apart from some technical points which I don't understand fully, I agree with their computations but I would like to make some comments about these calculations [84]. First from our experience at one and two loops modular invariance was used as a principle to determine the phases in the summation over spin structures. Modular invarianc is also important in the proof of the vanishing of the vacuum amplitude. In the calculation of the non-vanishing amplitudes modular invariance also plays an important role as one can see from sections 6 and 10 in the calculation of the four-particle (bosonic and fermionic) amplitudes. The problem is that how the modular invariance of the final result is guaranteed. Probably this is not important for the vanishing amplitudes, but may be fatal for the nonvanishing amplitudes. Second eventhough one can express the various chiral determinants and propagators appearing in loop amplitudes in terms of Θ-functions, these expressions are not explicit enough (in my opinion). Some algebraic extension of the hyperelliptic representation of Riemann surface may be useful [132]. At high loops hyperelliptic Riemann surfaces consist of only a measure zero part of the whole moduli space. Nevertheless the study of multi-loop calculations on hyperelliptic Riemann surface may be helpful [85]. To conclude this section I would like to show that all the problems raising before can be solved easily for these hyperelliptic Riemann surfaces. Here we have an algebraic (i.e. hyperelliptic) reprensentation of high genus Riemann surface. The requirment of modular invariance leads to a unique determination of the phases in the summation over spin structures. The vacuum amplitude can easily be proved to be vanishing pointwise in moduli space.

A genus g hyperelliptic Riemann surface is an algebraic curve in CP^2:

$$y^2 = \prod_{i=1}^{2g+2} (z - a_i), \tag{11.6}$$

where a_i $(i = 1, \cdots, 2g+2)$ are $2g+2$ branch points. By solving y in terms of z, one can view the genus g hyperelliptic Riemann surface as a double covering of the sphere with cutting and gluing appropriately. See fig.15 for illustration and also a choice of the homology basis of cycles.

At genus g there are $\begin{pmatrix} 2g+1 \\ g \end{pmatrix}$ even spin structures which correspond to the splitting of the branch points into two sets, each has $g+1$ elements. We will not consider other even spin structures which give no contribution to the vacuum amplitude because of the presence of zero modes. We choose the insertion points of supercurrent to be zero points of the following abelian differential

$$\Omega(z) = \frac{\prod_{a=1}^{g-1}(z - x_a)}{y(z)} \mathrm{d}z, \tag{11.7}$$

i.e. the $2g-2$ points $x = x_a \pm$ $(a = 1, 2, \cdots, g-1)$. Then we have $\det\varphi^a(x_b) = 1$. The various chiral determinants appearing in (11.1) can be computed as in two loops. From appendix A we have

$$
\begin{aligned}
\det{}_s \bar\partial_{\frac{1}{2}} &= \prod_{i<j}(a_i - a_j)^{-1/8}\left(\prod_{k<l} A_{kl}B_{kl}\right)^{1/4}, \\
\det{}_s \bar\partial_{\frac{3}{2}} &= \prod_{i<j}(a_{ij})^{3/8}\left(\prod_{k<l} A_{kl}B_{kl}\right)^{1/4}, \qquad (11.8)\\
\det{}_s \bar\partial_2 &= \prod_{i<j}(a_{ij})^{5/4}, \qquad \det \bar\partial_0 = \det K \prod_{i<j}(a_{ij})^{1/4}.
\end{aligned}
$$

By using these formulas we get

$$
(\det \bar\partial_0)^{-5} \det \bar\partial_2 (\det{}_s \bar\partial_{1/2})^5 \det{}_s \bar\partial_{3/2}^{-1} = \frac{Q_s}{(\det K)^5 \prod_{i<j} a_{ij}}, \qquad (11.9)
$$

where $Q_s = \prod_{i<j} A_{ij}B_{ij}$. The spin structure dependent factor is again Q_s as at two loops.

The correlator for supercurrent insertions can also be computed as at two loops. At multi-loops I don't think that the ghost part will give new spin structure dependent factors other than $\langle\psi(x_i\pm)\psi(x_j\pm)\rangle$, $\langle\partial\psi(x_i\pm)\psi(x_j\pm)\rangle$ and $\langle\partial\psi(x_i\pm)\partial\psi(x_j\pm)\rangle$. I will use a diagramatic representation for these spin structure dependent factors. A closed loop starting from and ending in x_a (see fig.16) represents

$$
\langle\psi(x_a+)\psi(x_a-)\rangle = \frac{1}{4}\sum_{i=1}^{g+1}\left(\frac{1}{x_a - A_i} - \frac{1}{x_a - B_i}\right), \qquad (11.10)
$$

A closed loop starting from x_1, passing through x_2, $\cdots$, x_a and ending in x_1 represents $\langle\psi(x_1\pm)\psi(x_2\pm)\rangle\langle\psi(x_2\pm)\psi(x_3\pm)\rangle\cdots\langle\psi(x_{a-1}\pm)\psi(x_a\pm)\rangle$, see fig.17. For example we have a digram (see fig.18) which represents either

$$
\begin{aligned}
\langle\psi(x_1+)\psi(x_2-)\rangle\langle\psi(x_2+)\psi(x_1-)\rangle &= \frac{1}{x_1 - x_2}\frac{u(x_1) - u(x_2)}{2\sqrt{u(x_1)u(x_2)}} \times \frac{1}{x_2 - x_1}\frac{u(x_2) - u(x_1)}{\sqrt{u(x_2)u(x_1)}} \\
&= \frac{1}{4(x_1 - x_2)^2}\left(\frac{u(x_1)}{u(x_2)} + \frac{u(x_2)}{u(x_1)} - 2\right).
\end{aligned} \qquad (11.11)
$$

or

$$
\begin{aligned}
\langle\psi(x_1+)\psi(x_2+)\rangle\langle\psi(x_2-)\psi(x_1-)\rangle &= \frac{1}{x_1 - x_2}\frac{u(x_1) + u(x_2)}{\sqrt{u(x_1)u(x_2)}} \times \frac{1}{x_2 - x_1}\frac{-u(x_2) - u(x_1)}{2\sqrt{u(x_1)u(x_2)}} \\
&= \frac{1}{4(x_1 - x_2)^2}\left(\frac{u(x_1)}{u(x_2)} + \frac{u(x_2)}{u(x_1)} + 2\right).
\end{aligned} \qquad (11.12)
$$

Doing some more exercises with $\langle\psi(x_1+)\psi(x_2-)\rangle\langle\psi(x_2+)\psi(x_3-)\rangle\langle\psi(x_3+)\psi(x_1-)\rangle$, for example:

$$\begin{aligned}
&\langle\psi(x_1+)\psi(x_2-)\rangle\langle\psi(x_2+)\psi(x_3-)\rangle\langle\psi(x_3+)\psi(x_1-)\rangle \\
&= \frac{1}{x_1-x_2}\frac{u(x_1)-u(x_2)}{2\sqrt{u(x_1)u(x_2)}} \times \frac{1}{x_2-x_3}\frac{u(x_2)-u(x_3)}{2\sqrt{u(x_2)u(x_3)}} \times \frac{1}{(x_3-x_1)}\frac{u(x_3)-u(x_1)}{2\sqrt{u(x_3)u(x_1)}} \\
&= \frac{1}{8(x_1-x_2)(x_2-x_3)(x_3-x_1)}\left\{\left(\frac{u(x_1)}{u(x_2)}-\frac{u(x_2)}{u(x_1)}\right)+(12\to 23)+(12\to 31)\right\},
\end{aligned} \tag{11.13}$$

it is easy to prove that the spin structure dependent factor appearing in fig.17 will be

$$\begin{aligned}
&\langle\psi(x_1\pm)\psi(x_2\pm)\rangle\langle\psi(x_2\pm)\psi(x_3\pm)\rangle\cdots\langle\psi(x_a\pm)\psi(x_1\pm)\rangle \\
&\sim \frac{u(x_1)u(x_2)\cdots u(x_{[\frac{a}{2}]})}{u(x_{[\frac{a}{2}]+1})u(x_{[\frac{a}{2}]+2})\cdots u(x_a)} + (-1)^a\frac{u(x_{[\frac{a}{2}]+1})u(x_{[\frac{a}{2}]+2})\cdots u(x_a)}{u(x_1)u(x_2)\cdots u(x_{[\frac{a}{2}]})} + (\text{other terms}).
\end{aligned} \tag{11.14}$$

Here "other terms " means that they are either obtained by interchanging some x_i and x_j from the first term or terms with less (always even number) $u(x_i)$ appearing in (11.14), like a term

$$\frac{u(x_2)u(x_3)\cdots u(x_{[\frac{a}{2}]})}{u(x_{[\frac{a}{2}]+2})u(x_{[\frac{a}{2}]+3})\cdots u(x_a)} + (-1)^a\frac{u(x_{[\frac{a}{2}]+2})u(x_{[\frac{a}{2}]+3})\cdots u(x_a)}{u(x_2)u(x_3)\cdots u(x_{[\frac{a}{2}]})}. \tag{11.15}$$

For a = odd number, there is no constant term in (11.14). The other spin structure dependent factors, with partial derivatives, will be represented almost the same way as before, but with an arrow indicating the partial derivative*. For example fig.19 represents

$$\langle\partial\psi(x_1+)\psi(x_1-)\rangle = +\frac{1}{8}\sum_{i=1}^{g+1}\left(\frac{1}{(x_1-A_i)^2}-\frac{1}{(x_1-B_i)^2}\right), \tag{11.16}$$

and fig.20 represents either

$$\begin{aligned}
&\langle\partial\psi(x_1+)\psi(x_2-)\rangle\langle\psi(x_2+)\psi(x_1-)\rangle \\
&= -\frac{1}{x_1-x_2}\langle\partial\psi(x_1+)\psi(x_2-)\rangle\langle\psi(x_2+)\psi(x_1-)\rangle \\
&\quad + \frac{1}{2}\partial\ln u(x_1)\times\frac{1}{x_1-x_2}\frac{u(x_1)+u(x_2)}{2\sqrt{u(x_1)u(x_2)}}\times\frac{1}{x_2-x_1}\frac{u(x_2)-u(x_1)}{2\sqrt{u(x_1)u(x_2)}} \\
&= -\frac{1}{x_1-x_2}\langle\partial\psi(x_1+)\psi(x_2-)\rangle\langle\psi(x_2+)\psi(x_1-)\rangle \\
&\quad + \frac{1}{8(x_1-x_2)^2}\left(\frac{u(x_1)}{u(x_2)}-\frac{u(x_2)}{u(x_1)}\right)\times\partial\ln u(x_1),
\end{aligned} \tag{11.17}$$

---

* An arrow emitted from a point x_a will represent either $\langle\partial\psi(x_a+)\cdots\rangle$ or $\langle\partial\psi(x_a-)\cdots\rangle$. We will not have a diagram with two arrows emitted from the same point.

or $\langle\partial\psi(x_1+)\psi(x_2+)\rangle\langle\psi(x_2-)\psi(x_1-)\rangle$, $\langle\partial\psi(x_1-)\psi(x_2+)\rangle$ $\langle\psi(x_2-)\psi(x_1+)\rangle$ and $\langle\partial\psi(x_1-)$ $\psi(x_2-)\rangle$ $\langle\psi(x_2+)$ $\psi(x_1-)\rangle$ which gives the same spin structure dependent factors as in (11.17). We note here a very important observation: the partial derivative in $\langle\partial\psi(x_1+)$ $\psi(x_2-)\rangle$ $\langle\psi(x_2+)\psi(x_1-)\rangle$ can be interchanged, i.e. we can first evaluate $\langle\psi(x_1+)\psi(x_2-)\rangle$ $\langle\psi(x_2+)$ $\psi(x_1-)\rangle$ and then do the partial derivation with respect to x (not just to one of $x+$ or $x-$). The difference is only a term with the same spin structure dependent factor as the original expression and other terms with some spin structure dependent factors which also appear in the expansion (see eq.(11.14)) of the original expression. I encourage the readers to calculate some expressions, like $\langle\partial\psi(x_1+)\psi(x_2-)\rangle\langle\psi(x_2+)\psi(x_3-)\rangle\langle\psi(x_3+)$ $\psi(x_1-)\rangle$ and $\langle\partial\psi(x_1+)\psi(x_2-)\rangle\langle\psi(x_2-)\psi(x_1-)\rangle$ for example, to justify this important observation. Then in order to prove the vanishing of the vacuum amplitude we need only to prove that the diagrams without arrows give zero contribution to the vacuum amplitude. The diagrams with arrows will automatically give zero contribution because they are just partial derivation of zero, which is also zero.

Now let us begin with the first diagram, fig.21 in the expansion of $\langle J(x_1+)J(x_1-)\cdots$ $J(x_{g-1}+)J(x_{g-1}-)\rangle$:

$$\langle\psi(x_1+)\psi(x_1-)\rangle\langle\psi(x_2+)\psi(x_2-)\rangle\cdots\langle\psi(x_{g-1}+)\psi(x_{g-1}-)\rangle \qquad (11.18),$$

which leads to following spin structure summation:

$$\mathrm{SUM}=\sum_s \eta_s Q_s \prod_{a=1}^{g-1}\sum_{i=1}^{g+1}\left(\frac{1}{x_a-A_i}-\frac{1}{x_a-B_i}\right) \qquad (11.19)$$

where $Q_s=\prod_{i<j}^{g+1}(A_i-A_j)(B_i-B_j)$, $s=(A_i|B_j)$ denotes a spin structure. The phase η_s in the summation over spin structure in (11.19) can be determined by the requirement of modular invariance*. We fix first $\eta_1=1$, $s_1=(a_1a_2\cdots a_{g+1}|a_{g+2}\cdots a_{2g+2})$. Then the rest phases can be calculated by interchanging a_i and a_j. We get, for example,

$$\begin{aligned}\eta_2&=-1, &\quad s_2&=(a_1a_2\cdots a_g a_{g+2}|a_{g+1}a_{g+3}\cdots a_{2g+2});\\ \eta_{g+3}&=1, &\quad s_{g+3}&=(a_1a_2\cdots a_{g-1}a_{g+2}a_{g+3}|a_g a_{g+1}a_{g+4}\cdots a_{2g+2});\quad \text{etc.}\end{aligned} \qquad (11.20)$$

* We remind you to remember that modular invariance simply means SUM should be antisymmetric under every interchange $a_i\leftrightarrow a_j$ $(i\neq j)$.

This unique determination of phase readily leads to a modular invariant summation as we can check. The supercurrent insertion part $\prod_{a=1}^{g-1}\sum_{i=1}^{g+1}\left(\frac{1}{x_a-A_i}-\frac{1}{x_a-B_i}\right)$ is very important to ensure modular invariance. By interchanging all $\{A_i\}$ and $\{B_j\}$, the supercurrent part will give a sign $(-1)^{g-1}$ and Q_s will not change. Modular invariance requires the sign to be† $(-)^{g+1}=(-)^{g-1}$ which is exactly the sign coming from the supercurrnt part.

By making use of modular invariance one can easily prove the vanishing of the modular invariant expression SUM, eq.(11.19). As a function of a_1, SUM has $2g+1$ zeroes, $a_1=a_i$, $i=1,2,\cdots,2g+1$ because of modular invariance (SUM is antisymmetric for interchanges $a_1\leftrightarrow a_i$, $i=2,\cdots,2g+2$). It has also $g-1$ simple poles: $a_1=x_a$, $a=1,2,\cdots,g-1$ and a multi-pole at $a_1=\infty$ with multiplicities g (it comes from the factors Q_s). As a complex function, SUM(a_1) should satisfy the well-known relation ($n(x)$ denotes the multiplicities of the zero or poles)

$$\sum_{zeroes}n(x)-\sum_{poles}n(x)=0. \tag{11.21}$$

But this can not be satisfied for the above counting:

$$2g+1\ >\ g-1+g. \tag{11.22}$$

Therefore we must have

$$\mathrm{SUM}(a_1)=\mathrm{const.}=0. \tag{11.23}$$

Bu using eq.(11.14), other diagrams can also be easily proved to vanish. For example fig.22 gives

$$\begin{aligned}\mathrm{SUM}_1&=\sum_s\eta_sQ_s\langle\psi(x_1+)\psi(x_2-)\rangle\langle\psi(x_2+)\psi(x_1-)\rangle\prod_{a=3}^{g-1}\langle\psi(x_a+)\psi(x_a-)\rangle\\&\sim\sum_s\eta_sQ_s\left(\frac{u(x_1)}{u(x_2)}+\frac{u(x_2)}{u(x_1)}-2\right)\prod_{a=3}^{g-1}\sum_{i=1}^{g+1}\left(\frac{1}{x_a-A_i}-\frac{1}{x_a-B_i}\right)\\&\sim\sum_s\eta_sQ_s\left(\prod_{i=1}^{g+1}(x_1-A_i)(x_2-B_i)+(A\leftrightarrow B)\right)\prod_{a=3}^{g-1}\sum_{i=1}^{g+1}\left(\frac{1}{x_a-A_i}-\frac{1}{x_a-B_i}\right).\end{aligned} \tag{11.24}$$

† Remember that $(A_i|B_j)\to(B_j|A_i)$ can be obtained by $g+1$ interchanges $A_i\leftrightarrow B_i$, $i=1,2,\cdots,g+1$.

The counting (of the zeroes and poles) is as follows:

$$\begin{aligned}
&2g+1 \text{ zeroes, } a_1 = a_i,\ i = 2,3,\cdots,2g+1,\\
&g-3 \text{ simple ploes, } a_1 = x_a,\ a_3,\cdots,g-1,\\
&1 \text{ multi-pole } a_1 = \infty \text{ with multiplicities } \ g+1.
\end{aligned} \tag{11.25}$$

We have then

$$2g+1 \ > \ g-3+g+1 \implies \text{SUM}_1 = 0. \tag{11.26}$$

Other terms, like

$$\begin{aligned}
&\sum_s \eta_s Q_s \left(\frac{u(x_1)\cdots u(x_{[\frac{a}{2}]})}{u(x_{[\frac{a}{2}+1]})\cdots u(x_a)} + (-1)^a \frac{u(x_{[\frac{a}{2}+1]})\cdots u(x_a)}{u(x_1)\cdots u(x_{[\frac{a}{2}]})} \right)\\
&\qquad \times \prod_{a'=a+1}^{g-1} \sum_{i=1}^{g+1} \left(\frac{1}{x_{a'}-A_i} - \frac{1}{x_{a'}-B_i} \right)\\
&\sim \sum_s \eta_s Q_s \left(\prod_{i=1}^{g+1} (x_1 - A_i)\cdots(x_{[\frac{a}{2}]} - A_i)(x_{[\frac{a}{2}+1]} - B_i)\cdots(x_a - B_i) + (-)^a (A \leftrightarrow B) \right)\\
&\qquad \times \prod_{a'=a+1}^{g-1} \sum_{i=1}^{g+1} \left(\frac{1}{x_{a'}-A_i} - \frac{1}{x_{a'}-B_i} \right)
\end{aligned} \tag{11.27}$$

which is appearing in the diagram fig.23, can also be shown to vanish by the same reasoning as used above. Thus we have proved the vanishing of the vacuum amplitude.

Certainly the above computations were done only for a class of high genus Riemann surface—the hyperelliptic Riemann surface. One would like to go beyond the hyperelliptic Riemann surface. At $g = 3$, hyperelliptic surfaces consist of a codimension one submanifold of the $g = 3$ moduli space. In hyperelliptic language the deformation away from hyperelliptic surface was induced by an odd (with respect the $\mathbf{Z}_2$ symmetry of the hyperelliptic surface) 2-differential*: $\frac{b}{y(z)} d^2 z$. One can then use the method of conformal field to calculate the partial derivatives of the vacuum amplitude $Z_{3\ loops}$ with respect to the moduli parameter b. The vanishing of $Z_{3\ loops}$ will be proved if we have

$$\frac{\partial^n}{\partial b^n} Z_{3\ loops}\Big|_{b=0} = 0,\ n > 0. \tag{11.28}$$

---

* The even 2-differentials are given by $\frac{\sum_{i=0}^{4} c_i z^i}{y^2(z)} d^2 z$.

We have tried to do that, but it seems hopeless.

In summary I do not think that the vanishing of the vacuum amplitude has been proved rigorously beyond two loops, not mentioning the calculation of the nonvanishing amplitudes.

12. Conclusions and Perspectives

In previous sections we have done a lot of calculations at two loops for superstrings. Starting from the Polyakov prescription for the calculation of string scattering amplitudes, we derived the correct measure for the integration over moduli space for both bosonic and fermionic strings. Some mathematics about Riemann surface and Θ-functions are reviewed in section 3. In order to do calculation at two loops we use the hyperelliptic representation for genus 2 Riemann surface. Various formulas with detail proofs about hyperelliptic Riemann surface are presented also in section 3. In section 4 we show some sample calculations at tree and one-loop level by making use of the techniques developed before. Sections 5—7 and 10 constitute the main body of this thesis. We present in these sections the full details about the calculation of the two-loop bosonic and fermionic amplitudes. We also checked that the main properties and in particular the finiteness and factorization of the amplitudes obtained. In section 8 we showed that the amplitudes obtained can also be used to drawn some interesting physical results, like high energy behaviour of the scattering of gravitons. All these calculations can also be done for a special class of compactified string models (called fermionic construction) as we showed in section 9. In this section we also gave a brief review about this compactification scheme. As to high loops (beyond two loops) what we can say and we are sure is only for (high genus) hyperelliptic Riemann surfaces which unfortunately consist of only a measure zero part of the whole moduli space. Here modular invariance plays an important role and ensures the vanishing of the vacuum amplitude. All these are discussed in this thesis. From the above discussions we see that quite explicit calculations can be carried out at two loops. By using these results we have arrived at a very good understanding of superstrings at two loops. The two loop four-particle amplitudes are finite, in accordance with what we expected for superstrings. Here the arbitrarineess of the locations of supercurrents is very important to ensure the finiteness of the amplitudes. This arbitrariness also plays an important role to ensure the right factorization properties (and unitarity). All these are good at two loops, mostly due to the explicit parametrization of the genus 2 moduli space by branch points. How about multi-loops? I have almost nothing to say. But I believe that our explicit computations may shed some light on the general theory of high loop computations in superstring theories.

Superstring theory is believed to be a **T**heory **O**f **E**verything (a **TOE**). At the moment there seems less enthusiasm in this theory as before. But I must say that superstring theory is a very beautiful theory as before, in every aspect. It is the most natural and nontrivial extension of the point quantum field theory. Conformal symmetry plays an important role in string theory. The number of consistent string models increases without limit only indicating that we have not understood the undelying beauty of string theory. Many beautiful mathematics, like Riemann surface and their moduli space, Θ-functions, Calabi-Yau manifold, find their applications in string theory. And vice versa, the study of strings also contributes a lots to mathematics, like Virasoro algebar, Kac-Moody algebra and their vertex representation, infinite dimensional algebra and super Riemann surface. Recently the study of conformal field theory has merged with the study of exactly solvable models in statistical mechanics to lead such idea as quantum group. These are also related with some beautiful mathematics, like link and knot theories. All these are related to string theory. Probably some people will forget string, but I will not, at least.

Appendix A: Calculation of the Chiral Determinants

In this appendix I will show some details for the calculation of the chiral determinants appearing in the measure of string amplitudes. This calculation was based on conformal field theory on hyperelliptic Riemann surface developed in [86-89]. See also [90-91]. As shown in these papers conformal field theory techniques applied quite nicely to a class of complex curves described by the polynominal equation

$$y^N = \prod_{i=1}^{L}(z - a_i), \tag{A.1}$$

where we assume $a_i \neq a_j$ and $L = 0 \pmod N$. For $N = 2$ we have hyperelliptic Riemann surface. The genus of the Riemann surface is $g = (N-1)(L-2)/2$. By solving y in terms of z, one can view the Riemann surface as an N-sheeted branching coverings of the sphere. Clearly they possess the symmmetry given by permutation of the sheets, and permutation of the branch points which is just modular transformation. One can also fix three branch points by using the projective invariance $SL(2, \mathbf{C})$.

We shall be concerned with the computations of the chiral determinants of b, c system [125,49], where b and c are weight λ, $1-\lambda$ analytic Fermi fields on genus g Riemann surface. λ can be either in $\mathbf{Z}^+$or in $\mathbf{Z}^+ + \frac{1}{2}$, $\mathbf{Z}^+$ being the set of positive integers. The action is $= \int b\bar{\partial}c \; \mathrm{d}^2 z$ and the usual short distance expansions hold for b, c.

The energy momentum tensor is given by:

$$T_z = -\lambda : b\partial c : + (1-\lambda) : \partial b c : \tag{A.2}$$

where : : means subtraction of the double ploes: this regularization procedure is Weyl invariant but obviously breaks reparametrization invariance, turning T_z into a projective connection. In any case the following operator product expansion holds for any conformal field Φ of dimension h:

$$T_z(z)\Phi(w) \sim \frac{h}{(z-w)^2}\Phi(w) + \frac{1}{z-w}\partial_w \Phi(w) + \cdots \tag{A.3}$$

This formula will be crucial in the following.

Given N zero modes for $c(z)$, $\nu_i(z)$ $i = 1, \cdots, N$, and M zero modes for $b(z)$, $\mu_i(z)$ $i = 1, \cdots, M$, related by the Riemann-Roch theorems, the chiral determinant to be computed is formally represented by:

$$\det \bar{\partial}_\lambda \equiv \frac{\int DbDc \prod_{i=1}^{M} b(z_i) \prod_{j=1}^{N} c(y_j) \exp(-S)}{\det \mu_1(z_j) \det \nu_i(y_j)} \tag{A.4}$$

The computation of the path integral in (A.4) is done by the important observation: we can treat the fields b and c as free fields defined on the spheres with appropriate monodormy transformation from one sheet to another, and with some insertion of conformal fields at the branch points. That is, the determinats is computed as

$$\det \bar{\partial}_\lambda \sim F_\lambda \equiv \langle \prod_i V(a_i) \prod_j b(z_j) \prod_k c(w_k) \rangle, \tag{A.5}$$

where $V(a_i)$ are conformal fields inserted at the branch points, $b(z_j)$ and $c(w_k)$ are insertions of b and c fields needed to absorb the zero modes.

The computation of the correlatoor follows the standard method of conformal field theory. First one computes the normalized correlator (c_k and b_{N-k} are fields defined on the N sheets)

$$G_{k\ N-k}(z, w) = \frac{\langle c_k(z) b_{N-k}(w) \prod_i V(a_i) \prod_j b(z_j) \prod_k c(w_k) \rangle}{\langle \prod_i V(a_i) \prod_j b(z_j) \prod_k c(w_k) \rangle}. \tag{A.6}$$

Then we can compute

$$\ll T_z \gg = \frac{\langle T_z \prod_i V(a_i) \prod_j b(z_j) \prod_k c(w_k) \rangle}{\langle \prod_i V(a_i) \prod_j b(z_j) \prod_k c(w_k) \rangle}, \tag{A.7}$$

where $T_z = \sum_{k=0}^{N-1} [-\lambda : b_k(z) \partial c_{N-k}(z) : + (1-\lambda) : \partial b_k(z) c_{N-k}(z) :]$. By using the operator product expansion of T_z with $V(a_i)$ [86-89],

$$T(z) V(a_i) \sim \frac{h}{(z - a_i)^2} V(a_i) + \frac{1}{z - a_i} \partial V(a_i), \tag{A.8}$$

one can derive some partial differential equations for F_λ:

$$\begin{aligned} \frac{\partial}{\partial a_i} \ln F_\lambda &= \text{Residue of } \ll T_z \gg \text{ for } z \to a_i, \\ \frac{\partial}{\partial z_j} \ln F_\lambda &= \text{Residue of } \ll T_z \gg \text{ for } z \to z_j, \\ \frac{\partial}{\partial w_k} \ln F_\lambda &= \text{Residue of } \ll T_z \gg \text{ for } z \to w_k. \end{aligned} \tag{A.9}$$

By solving these differential equations we can get explicit expressions for F_λ (in term of a_i). Let us show some examples.

First for $N=2,\ L=2g+2,\ \lambda=2$ (the bosonic ghost system defined on the genus g hyperelliptic Riemannsurface), we have

$$
\begin{aligned}
G_{00}(z,w) &= \frac{1}{z-w}\prod_{i=1}^{2g+2}\frac{z-a_i}{w-a_i}\cdot\prod_{j=1}^{3g-3}\frac{w-z_j}{z-z_j} \equiv G_0(z,w),\\
G_{11}(z,w) &= \frac{1}{z-w}\prod_{i=1}^{2g+2}\frac{(z-a_i)^{1/2}}{(w-a_i)^{1/2}} \equiv G_1(z,w).
\end{aligned}
\tag{A.10}
$$

Seting $z-w=\Delta$, we compute $G_0(z,w)$ and $G_1(z,w)$ in Laurant expression in Δ:

$$
\begin{aligned}
G_0(z,w) =& \frac{1}{\Delta}+\sum_i\frac{1}{w-a_i}-\sum_j\frac{1}{w-z_j}+\Delta\left(\sum_{i<j}\frac{1}{(w-a_i)(w-a_j)}\right.\\
&\left.+\sum_{j<k}\frac{1}{(w-z_j)(w-z_k)}-\sum_{i,j}\frac{1}{(w-a_i)(w-z_j)}+\sum_j\frac{1}{(w-z_j)^2}\right)+O(\Delta^2),\\
G_1(z,w) =& \frac{1}{\Delta}+\frac{1}{2}\sum_i\frac{1}{w-a_i}\\
&+\frac{\Delta}{8}\left(2\sum_{i<j}\frac{1}{(w-a_i)(w-a_j)}-\sum_i\frac{1}{(w-a_i)^2}\right)+O(\Delta^2)
\end{aligned}
\tag{A.11}
$$

Then we have

$$
\begin{aligned}
\ll T_z \gg =& [2\partial_z(G_0(z,w)+G_1(z,w))+\partial_z(G_0(w,z)+G_1(w,z))]\,|_{z=w},\ \text{finite part}\\
=& \frac{5}{4}\sum_{i<j}\frac{1}{(z-a_i)(z-a_j)}-\frac{13}{8}\sum_i\frac{1}{(z-a_i)^2}\\
&+2\sum_j\frac{1}{(z-z_j)^2}-\sum_{i,j}\frac{1}{(z-a_i)(z-z_j)}+\sum_{j<k}\frac{1}{(z-z_j)(z-z_k)}.
\end{aligned}
\tag{A.12}
$$

The partial differential equations obtained are

$$
\begin{aligned}
\frac{\partial}{\partial a_i}\ln F_2 &= \frac{5}{4}\sum_{j\neq i}\frac{1}{a_i-a_j}-\sum_j\frac{1}{a_i-z_j},\\
\frac{\partial}{\partial z_j}\ln F_2 &= \sum_{k\neq j}\frac{1}{z_j-z_k}-\sum_i\frac{1}{z_j-a_i},
\end{aligned}
\tag{A.13}
$$

which can be integrated to yield:

$$F_2 = \left\langle \prod_i V(a_i) \prod_j b(z_j) \right\rangle = \text{const} \times \prod_{k<l} (a_k - a_l)^{\frac{5}{4}} \frac{\prod_{k<l}(z_k - z_l)}{\prod_k y^2(z_k)} \tag{A.14}$$

Notice that $\frac{\prod_{k<l}(z_k - z_l)}{\prod y_i^2(z_k)}$ is exactly $\det \Omega_k(z_l)$, where $\{\Omega_k(z)\}$ is a basis of holomorphic 2-differentials. So we have

$$\det \bar{\partial}_2 = \prod_{i<j}^{2g+2} (a_i - a_j)^{\frac{5}{4}}. \tag{A.15}$$

Similarly we can compute $\det \bar{\partial}_\lambda$ for $\lambda = 1$ and we have

$$\det \bar{\partial}_1 = \det K \prod_{i<j}^{2g+2} (a_i - a_j)^{\frac{1}{4}}. \tag{A.16}$$

The other determinants $\det_s \bar{\partial}_{3/2}$ and $\det_s \bar{\partial}_{1/2}$ can also be computed. Here we have a complication from the spin structure dependence. But no trouble at all. For example for $\lambda = 3/2$ ($N = 2,\ L = 2g = 2$ as before) the relevant formulas are

$$\begin{aligned} G_{00} &= \frac{1}{z-w} \prod_i \frac{(w - A_i)^{-1/4}(w - B_i)^{-3/4}}{(z - A_i)^{-1/4}(z - B_i)^{-3/4}} \prod_j \frac{(w - z_j)}{(z - z_j)}, \\ G_{11} &= \frac{1}{z-w} \prod_i \frac{(w - A_i)^{-3/4}(w - B_i)^{-1/4}}{(z - A_i)^{-3/4}(z - B_i)^{-1/4}} \prod_j \frac{(w - z_j)}{(z - z_j)}. \end{aligned} \tag{A.17}$$

By using these correlators and doing some algebraic calculation as before one arrives the following expression for $\det_s \bar{\partial}_{3/2}$

$$\det{}_s \bar{\partial}_{3/2} = \prod_{i<j} (a_{ij})^{3/8} \left(\prod_{k<l} A_{kl} B_{kl} \right)^{1/4}. \tag{A.18}$$

For $\det_s \bar{\partial}_{1/2}$ we have

$$\det{}_s \bar{\partial}_{1/2} = \prod_{i<j} (a_{ij})^{-1/8} \left(\prod_{k<l} A_{kl} B_{kl} \right)^{1/4}. \tag{A.19}$$

Appendix B: Calculation of the Ghost Correlators

In this appendix I would like to show in some details for the calculation of ghost correlators, eq.(5.14). By definition we have

$$I_{gh} = \langle J_{gh}(x_1)J_{gh}(x_2)\rangle = \frac{\langle J_{gh}(x_1)J_{gh}(x_2)\delta(\beta(x_1))\delta(\beta(x_2))\prod_i(\eta_i * b)\rangle}{\langle \delta(\beta(x_1))\delta(\beta(x_2))\prod_i(\eta_i * b)\rangle}. \tag{B.1}$$

Here we have taken a generic basis of the Beltrami differentials η_i and we use $(\eta_i * b)$ to denote the integration of $\eta_i(z)b(z)$ over the Riemann surface. Because of the appearance of the correlators at the same point in (B.1), one should use some regularizations. We choose a gaussion representation for the δ-functions χ_a:

$$\chi_a(y) \sim \lim_{\alpha\to 0}\frac{1}{\alpha}\exp\left(-\frac{|y-x_a|^2}{\alpha}\right), \tag{B.2}$$

to represent $J_{gh}(x_1)$ and $J_{gh}(x_2)$ as

$$J_{gh}(x_a) = (\chi_a * J_{gh}) = \int d^2z\chi_a(z)J_{gh}(z). \tag{B.3}$$

Then we have

$$\begin{aligned} I_{gh} =& \langle J_{gh}(x_1)J_{gh}(x_2)\rangle \\ =& \int d^2y_1d^2y_2\ \chi_1(y_1)\chi_2(y_2)\langle J_{gh}(y_1)J_{gh}(x_2)\rangle \\ =& \int d^2y_1d^2y_2\big[-\chi_1(y_1)\chi_2(y_2)+\chi_2(y_1)\chi_1(y_2)\big] \\ & \times \big[2R(y_1y_2)\partial_2S(y_1y_2)+2\partial_2R(y-1y_2)S(y_1y_2)\big], \end{aligned} \tag{B.4}$$

by using the explict expression of ghost supercurrent, eq.(5.6). Here we have defined $R(y_1y_2)$ and $S(y_1y_2)$ as [115]

$$\begin{aligned} R(y_1y_2) =& \langle b(y_1)c(y_2)\prod_i(\eta_i * b)\rangle \\ =& -\det{}^{-1}(\eta_i * f_j)\int\prod_{i=1}^{3}d^2\chi_k\ \eta_k(z_k)B(y_2y_1z_1z_2z_3), \\ B(yx_0x_1x_2x_3) =& \varepsilon^{\mu\nu\rho\sigma}H(yx_\mu)f_1(x_\nu)f_2(x_\rho)f_3(x_\sigma), \\ H(zx) =& \frac{1}{2(z-x)}\left(1+\frac{y(x)}{y(x)}\right)\frac{y(z)}{y(x)}, \qquad f^i(z) = \frac{z^{i-1}}{y^2(z)}, \end{aligned} \tag{B.5}$$

and

$$
\begin{aligned}
S(y_1y_2) &= \langle\gamma(y_1)\beta(y_2)\delta(\chi_1 * \beta_1)\delta(\chi_2 * \beta_2)\rangle \\
&= \det{}^{-1}(\chi_a * \varphi_b)\int \mathrm{d}^2w_1\mathrm{d}^2w_2\chi_1(w_1)\chi_2(w_2)G(y_1y_2w_1w_2), \\
G(yx_0x_1x_2) &= \varepsilon^{\mu\nu\rho}P(yx_\mu)\varphi_1(x_\nu)\varphi_2(x_\rho), \\
P(zy) &= \frac{1}{\Omega_2(z)}\langle\psi(z)\psi(y)\rangle\Omega_2(y), \\
\varphi_i(x) &= \pm\Omega_1(x)\langle\psi(x)\psi(x_i)\rangle.
\end{aligned} \tag{B.6}
$$

We will set from the beginning $(\chi_i * \varphi_j) = 2\delta_{ij}C_i,\ C_i = \varphi_i(x_i)$ which is consistent with our regulanization. We have then

$$
S(y_1y_2) = P(y_1y_2) - \frac{P(y_1w_1)\cdot\chi_1(w_1)\varphi_1(y_2)}{C_1} - \frac{P(y_1w_2)\cdot\chi_2(w_2)\varphi_2(y_2)}{C_2}. \tag{B.7}
$$

where intergration over w_1 and w_2 is understood. Notice that all the possible singularitis are coming from $S(y_1y_2)$ because integration over y_1 and y_2 can be either around x_1 or x_2, and sometimes $y_1 \simeq y_2 \simeq x_1$ or x_2. Let us compute first, for example a term

$$
\begin{aligned}
R(y_1y_2)\frac{\partial S(y_1y_2)}{\partial y_2}\chi_1(y_1)\chi_2(y_2) &= R(y_1y_2)\chi_1(y_1)\chi_2(y_2) \\
&\times\left\{\partial_2P(y_1y_2) - \frac{P(y_1w_1)\chi_1(w_1)\partial\varphi_1(y_2)}{C_1} - \frac{P(y_1w_2)\chi_2(w_2)\chi_2(w_2)\partial\varphi_2(y_2)}{C_2}\right\}.
\end{aligned} \tag{B.8}
$$

The only singular term is $P(y_1w_1)\chi_1(w_1)$ which appears in the following combination

$$
R(y_1y_2)\chi_1(y_1)P(y_1w_1)\chi_1(w_1)\partial\varphi_2(y_2)\chi_2(y_2) = R(y_1x_1)\chi_1(y_1)P(y_1w_1)\chi_1(w_1)\partial\varphi_2(x_2). \tag{B.9}
$$

The integration over y_1 and y_2 are computed as follows:

$$
\begin{aligned}
R(y_1x_2)\chi_1(y_1)P(y_1w_1)\chi_1(w_1) &= \frac{\int R(y_1x_2)\chi_1(y_1)P(y_1w_1)\chi_1(w_1)\mathrm{d}^2y_1\mathrm{d}^2w_1}{\int \chi_1(y_1)\chi_1(w_1)\mathrm{d}^2y_1\mathrm{d}^2w_1} \\
&= \frac{\int R(y_1x_2)e^{-\frac{|y_1-x_1|^2}{\alpha}}e^{-\frac{|w_1-x_1|^2}{\alpha}}P(y_1w_1)\mathrm{d}^2y_1\mathrm{d}^2w_1}{\int e^{-\frac{|y_1-x_1|^2}{\alpha}}e^{-\frac{|w_1-x_1|^2}{\alpha}}\mathrm{d}^2y_1\mathrm{d}^2w_1.}
\end{aligned} \tag{B.10}
$$

In (B.10) only $y_1,\ w_1 \to x_1$ contributes, so we can substitute $P(y_1w_1)$ by

$$
P(y_1w_1) \simeq \frac{1}{y_1 - w_1} + \Lambda(x_1) + O(y_1 - x_1, w_1 - x_1). \tag{B.11}
$$

Then by a changing of variables from $y_1 - w_1 = z_1$ and $y_1 + w_1 = 2x_1 + z_2$ to $y_1 = x_1 + \frac{1}{2}(z_1 + z_2)$ and $w_1 = x_1 - \frac{1}{2}(z_1 - z_2)$ we have

$$
\begin{aligned}
R(y_1x_2)\chi_1(y_1)P(y_1w_1)\chi_1(w_1) =& \Lambda(x_1)R(x_1x_2) \\
&+ \frac{\int R(x_1 + \frac{1}{2}(z_1+z_2), x_2)\frac{1}{z_1}e^{-\frac{|z_1|^2+|z_2|^2}{2\alpha}} \mathrm{d}^2 z_1 \mathrm{d}^2 z_2}{\int e^{-\frac{|z_1|^2+|z_2|^2}{2\alpha}} \mathrm{d}^2 z_1 \mathrm{d}^2 z_2} \\
=& \Lambda(x_1)R(x_1x_2) + \frac{1}{2}\partial_{x_1} R(x_1x_2).
\end{aligned}
\tag{B.12}
$$

Second we compute the other term in (B.4)

$$
\begin{aligned}
&\frac{\partial R(y_1y_2)}{\partial y_2} S(y_1y_2)\chi_1(y_1)\chi_2(y_2) \\
&\quad = \frac{\partial R(y_1y_2)}{\partial y_2}\chi_1(y_1)\chi_2(y_2)\left(P(y_1y_2) - \frac{P(y_1w_1)\chi_1(w_1)\varphi_1(y_2)}{\varphi_1(x_1)}\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. - \frac{P(y_1w_2)\chi_2(w_2)\varphi_2(y_2)}{\varphi_2(x_2)}\right) \\
&\quad = -\frac{\partial R(y_1y_2)}{\partial y_2}\chi_1(y_1)\chi_2(y_2)P(y_1w_1)\chi_1(w_1)\frac{\varphi_1(y_2)}{\varphi_1(x_1)}.
\end{aligned}
\tag{B.13}
$$

Here the singular factor is $\chi_1(y_1)P(y_1w_1)\chi_1(w_1)$ comimg from integration around x_1. However the integration over y_2 is regular (at least around x_2) which gives a factor $\varphi_1(x_2) = 0$. So we have

$$
\frac{\partial R(y_1y_2)}{\partial y_2} S(y_1y_2)\chi_1(y_1)\chi_2(y_2) = 0, \tag{B.14}
$$

where integration over y_1 and y_2 is understood.

Substituting (B.8), (B.12) and (B.14) into (B.4) we get

$$
\begin{aligned}
I_{gh} =& -2R(x_1x_2)\left\{\partial_2 P(x_1x_2) - \Lambda(x_1)\frac{\partial\varphi_1(x_2)}{\varphi_1(x_1)} - P(x_1x_2)\frac{\partial\varphi_2(x_2)}{\varphi_2(x_2)}\right\} \\
&+ \partial_2 R(x_1x_2)\frac{\partial\varphi_1(x_2)}{\varphi_1(x_1)} - (1 \leftrightarrow 2) \\
=& -2\partial_2 P(x_1x_2)R(x_1x_2) - (\partial_2 R(x_2x_1) + 2\Lambda(x_2)R(x_2x_1))\frac{\partial\varphi_2(x_1)}{\varphi_2(x_2)} \\
&- 2P(x_2x_1)R(x_2x_1)\frac{\partial\varphi_1(x_1)}{\varphi_1(x_1)} - (1 \leftrightarrow 2),
\end{aligned}
\tag{B.15}
$$

as given in(5.14).

Appendix C: Calculation of the Prime Form Part

In this appendix we show, in the rescattering cases, the contributions of the prime form part of the X-correlators appearing in (8.24) is suppressed with respect to the period matrix part, the one which we have taken into account in our estimates.

The part in question can be compactly expressed in terms of the unique abelian differential of the 3rd kind $\omega_{bc}(z)$, having simple poles with residues $+1$, -1 at z_b and z_c respectively, with zero periods along α-cycles. Indeed one has [71]:

$$\left|\frac{E(z_1,z_c)E(z_2,z_b)}{E(z_2,z_c)E(z_1,z_b)}\right|^s = \left|\exp\left\{-\frac{s}{2}\int_{z_1}^{z_2}\omega_{bc}(z)\right\}\right|^2. \qquad (C.1)$$

The quantity $\omega_{bc}(z)$ can be easily written, to leading order in the degeneration limit u, v, $\lambda \to 0$, in terms of the normalized abelian differentials $\omega_i(z)$ introduced in (8.17):

$$\begin{aligned} 2\omega_{bc} =& \frac{1}{z-z_b} + \frac{1}{2}\frac{\omega_1(z)}{\omega_1(z_b)}\oint_{\alpha_1}\omega_1(z')\left(\frac{1}{z-z_b}-\frac{1}{z'-z_b}\right) \\ &+ \frac{1}{2}\frac{\omega_2(z)}{\omega_2(z_b)}\oint_{\alpha_2}\omega_2(z')\left(\frac{1}{z-z_b}-\frac{1}{z'-z_b}\right) - (z_b \to z_c). \end{aligned} \qquad (C.2)$$

One can then do explicitly the contour integration to get:

$$\begin{aligned} 2\omega_{bc}(z) =& \frac{1}{z-z_b}\left(1+\frac{1}{2}\frac{\omega_1(z)}{\omega_1(z_b)}+\frac{1}{2}\frac{\omega_2(z)}{\omega_2(z_b)}\right) \\ &-\frac{1}{2}\frac{\omega_1(z)}{\omega_1(z_b)}\cdot\frac{1}{a_1-z_b}-\frac{1}{2}\frac{\omega_2(z)}{\omega_2(z_b)}\cdot\frac{1}{a_5-z_b} - (z_b \to z_c). \end{aligned} \qquad (C.3)$$

Consider now the first part of (C.3): as a function of z_b it is holomorphic everywhere; indeed it is holomorphic at ∞ and also at $z_b = z$: for the latter case, notice that $z_{1,2}$ and $z_{b,c}$ lie on opposite sheets, so that in (C.1) z and z_b are also on opposite sheets, implying: $\frac{1}{2}\frac{\omega_1(z)}{\omega_1(z_b)} + \frac{1}{2}\frac{\omega_2(z)}{\omega_2(z_b)} \to -1$ for $z_b \to z$. We conclude that in (C.3) the pole at $z_b = z$ is cancelled. As a result this expression is independent of z_b so that when we substract the term $(z_b \to z_c)$ we get zero. In fact one can see that $\int_{z_1}^{z_2}\omega_{bc} = O(u,\ v,\ \lambda)$.

Figure Captions

Fig.1 The trajectory of a point particle is a line. The trajecctory of a string is a (two-dimensional) surface – Riemann surface.

Fig.2 The topological classification of Riemann surfaces is by their genus – the handles attached to the sphere.

Fig.3 String perturbation theory represented as a topological summation.

Fig.4 Decomposition of ∇g_{zz}: $\delta g_{zz} = \nabla_z V_z + \delta\tau_i \phi^i_{zz} = \Delta_z V'_z + \delta y_i \mu^i_{zz}$.

Fig.5 A Riemann surface with a chosen canonical homology basis of cycles.

Fig.6 A genus two Riemann surface as a double covering of the (cut) Riemann sphere. We also indicated the canonical homology basis chosen. "..." meana that the path is on the lower sheet.

Fig.7 A genus one Riemann surface as a double covering of the sphere. We also indicated the canonical homology basis chosen.

Fig.8 Factorized diagramm of a four-particle amplitude at two loops.

Fig.9 Another factorization limit.

Fig.10 Kinematics for two-body scattering: $k_1 + k_b \to k_2 + b_c$.

Fig.11 Rescattering diagrams.

Fig.12 Genus two Riemann surface as a double covering of the (cut) Riemann sphere. Notice the identification of the cycle $\alpha_3 = \alpha_1 + \alpha_2$.

Fig.13 "Rescattering" term as the pinching limit of the cycles α_1, α_2 and α_3 on the Riemann surface.

Fig.14 H-diagrams.

Fig.15 A genus g (hyperelliptic) Riemann surface as a double covering of the (cut) Riemann sphere. We also indicated a canonical homology basis.

Fig.16 Diagramatic representation of $\langle\psi(x_a+)\psi(x_a-)\rangle$.

Fig.17 Diagramatic representation of $\langle\psi(x_1\pm)\psi(x_2\pm)\rangle\langle\psi(x_2\pm)\psi(x_3\pm)\rangle\cdots\langle\psi(x_a\pm)\psi(x_1\pm)\rangle$.

Fig.18 Diagramatic representation of $\langle\psi(x_1\pm)\psi(x_2\pm)\rangle\langle\psi(x_2\pm)\psi(x_1\pm)\rangle$.

Fig.19 Diagramatic representation of $\langle\partial\psi(x_1+)\psi(x_1-)\rangle$.

Fig.20 Diagramatic representation of $\langle\partial\psi(x_1\pm)\psi(x_2\pm)\rangle\langle\psi(x_2\pm)\psi(x_1\pm)\rangle$.

Fig.21 A term $\langle\psi(x_1+)\psi(x_1-)\rangle\cdots\langle\psi(x_{g-1}+)\,\psi(x_{g-1}-)\rangle$ in the expansion of $\langle J(x_1+)J(x_1-)\rangle$ $\cdots\langle J(x_{g-1}+)\,J(x_{g-1}-)\rangle$.

Fig.22 Another term in the expansion of $\langle J(x_1+)J(x_1-)\rangle\cdots\langle J(x_{g-1}+)J(x_{g-1}-)\rangle$.

Fig.23 A somewhat generic term in the expansion of $\langle J(x_1+)J(x_1-)\rangle\cdots\langle J(x_{g-1}+)J(x_{g-1}-)\rangle$.

References

[1] J. H. Schwarz, "Superstring Theory," *Phys. Reports* **89** (1982) 223-322.

[2] M.B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, vols. I and II (Cambridge University Press, 1986) .

[3] M.B. Green and J. H. Schwarz, "Anomaly Cancellations in Supersymmetric D=10 Gauge Theory and Superstring Theory," *Phys. Lett.* **149B** (1984) 117-122.

[4] M.B. Green and J. H. Schwarz, "Infinity Cancellations in $SO(32)$ Superstring Theory," *Phys. Lett.* **151B** (1985) 21-25.

[5] S. L. Glashow, "Partial-Symmetries of Weak Interactions," *Nucl. Phys.* **22** (1961) 579-588;
S. Weinberg, "A Model of Leptons," *Phys. Rev. Lett.* **19** (1967) 1264-1266;
A. Salam, "Weak and Electromagnetic Interactions," in *Elementary Particle Theory*, ed. N. Svartholm (Almqvist and Wiksell, Stockholm, 1968) 367-377.

[6] H. Georgi and S. L. Glashow, "Unity of All Elementary-Particle Forces," *Phys. Rev. Lett.* **32** (1974) 438-441.

[7] See for example P. van Nieuwenhuizen, "Supergravity," *Phys. Reports* **68** (1981) 189-398.

[8] M.B. Green and J. H. Schwarz, "Supersymmetric Dual String Theory (III). Loops and Renormalization," *Nucl. Phys.* **B198** (1982) 441-460.

[9] M. B. Green, "Lectures on Superstrings," in *Unified String Theory*, eds. M. Green and D. Gross (World Scientific, Singapore, 1986) 294-356.

[10] Anyone who think his (or her) work(s) should be cited here can send a mail to me. I will then cite his (or her) work(s) here.

[11] Some works I will not cite here. Probably I don't fully understand their meanings.

[12] V. G. Knizhnik, "Explicit Expression for the Two-Loop Measure in the Heterotic String Theory," *Phys. Lett.* **196B** (1987) 473-476.

[13] E. Gava, R. Iengo and G. Sotkov, "Modular Invariance and the Two-Loop Vanishing of the Cosmological Constant," *Phys. Lett.* **207B** (1988) 283-291.

[14] R. Iengo and C.-J. Zhu, "Notes on Non-Renormalization Theorem in Superstring Theories," *Phys. Lett.* **212B** (1988) 309-312.

[15] R. Iengo and C.-J. Zhu, "Two-Loop Computation of the Four-Particle Amplitude in Heterotic String Theory," *Phys. Lett.* **212B** (1988) 313-319.

[16] C.-J. Zhu, "Two-Loop Computations in Superstring Theoy," *Int. J. Mod. Phys.* **A4** (1989) 3877-3906.

[17] C.-J. Zhu, "Covariant Two-Loop Fermion Amplitude in Closed Superstrig Theoories," preprint SISSA-EP-5/89 (Jan., 1989) 25 pages, to appear in *Nucl. Phys. B.*

[18] M. Soldate, "Partial-Wave Unitarity and Closed-String Amplitudes," *Phys. Lett.* **186B** (1987) 321-327.

[19] D. Amati, M.Ciafaloni and G. Veneziano, "Superstring Collisions at Planckian Energies," *Phys. Lett.* **197B** (1987) 81-88; "Classical and Quantum Gravity Effects from Planckian Energy Superstring Collisions," *Int. J. Mod. Phys.* **A3** (1988) 1615-1661.

[20] B. Sundborg, "High-Energy Asymptotics: the One-Loop String Amplitudes and REsummation," *Nucl. Phys.* **B306** (1988) 545-566.

[21] D. J. Gross and P. F. Mende, "The High-Energy Behaviour of String Scattering Amplitudes," *Phys. Lett.* **197B** (1987) 129-134; "String Theory Beyond the Planck Scale," *Nucl. Phys.* **B303** (1988) 407-454.

[22] E. Gava, R. Iengo and C.-J. Zhu, "Quantum Gravity Corrections from Superstring Theory," *Nucl. Phys.* **B323** (1989) 585-613.

[23] R. Iengo, "Quantum Gravity Corrections from Multiloop Superstring Scattering Amplitudes," preprint IC/89/282 (Sept., 1989)

[24] M. B. Green and J. H. Schwarz, "Supersymmetric String Theories," *Phys. Lett.* **109B** (1982) 444-448.

[25] P. Goddard, J. Goldstone, C. Rebbi and C. B. Thorn, "Quantum Dynamics of a Massless Relativistic String," *Nucl. Phys.* **B56** (1973) 109-135.

[26] S. Mandelstam, "Interacting-String Picture of Dual-Resonnace Models," *Nucl. Phys.* **B64** (1973) 205-235; "Interacting-String Picture of Neveu-Schwarz-Ramond Model," *Nucl. Phys.* **B69** (1974) 77-106; "The Interacting-String Picture and Functional Integration," in *Unified String Theory*, eds. M. Green and D. Gross (World Scientific, Singapore, 1986) 46-102; "The n-Loop String Amplitude," *Physica Scripta*, **T15** (1987) 109-113.

[27] E. Cremmer and J. L. Gervais, "Combining and Splitting Relativistic Strings," *Nucl.*

Phys. **B76** (1974) 209-230; "Infinite Component Field Theory of Interacting Relativistic Strings and Dual theory," *Nucl. Phys.* **B90** (1975) 410-460.

[28] M. Kaku and K. Kikkawa, "Field Theory of Relativstic Strings. I. Trees," *Phys. Rev.* **D10** (1974) 1110-1133; "Field Theory of Relativstic Strings. II. Loops and Pomerons," *Phys. Rev.* **D10** (1974) 1823-1843.

[29] M. B. Green and J. H. Schwarz, "Superstring Interactions," *Nucl. Phys.* **B218** (1983) 43-88; "Superstring Field Theory," *Nucl. Phys.* **B243** (1984) 475-536.

[30] M. B. Green, J. H. Schwarz and L. Brink, "Superfield Theory of Type (II) Superstrings," *Nucl. Phys.* **B219** (1983) 437-478.

[31] A. Restuccia and J. G. Taylor, "Light-Cone Gauge Analysis of Superstrings," *Phys. Reports* **174** (1989) 283-407.

[32] V. Alessandrini, D. Amati, M. Le Bellac and D. Olive, "The Operator Approach to Dual Multiparticle Theory," *Phys. Reports* **1C** (1971) 269-346.

[33] P. Di Vecchia, R. Nakayama, J. L. Petersen, J. Sidenius and S. Sciuto, "BRST Invariant N-Reggeon Vertex," *Phys. Lett.* **182B** (1986) 164;
P. Di Vecchia, R. Nakayama, J. L. Petersen and S. Sciuto, "Properties of the Three-Reggeon Vertex in String Theories," *Nucl. Phys.* **B282** (1987) 103-124.

[34] P. Di Vecchia, R. Nakayama, J. L. Petersen, S. Sciuto and J. R. Sidenius, "Covariant N-String Amplitude," *Nucl. Phys.* **B287** (1987) 621-657.

[35] J. L. Petersen, and J. R. Sidenius, "Covariant Loop Calculus for the Closed Bosonic String," *Nucl. Phys.* **B301** (1987) 247-266; J. L. Petersen, J. R. Sidenius and A. K. Tollsten, "Covariant Loop Calculus for Neveu-Schwarz String," *Phys. Lett.* **213B** (1988) 30-36; and many other contributions.

[36] For reviews see for example J. L. Petersen, "On the Explicit Construction of the Superstring Loop Measure in thr Super-Schottky-Reggeon Formalism," preprint CERN-TH.5309/89 (Feb., 1989) 32 pages.

[37] W. Siegel, "Covariantly Second-Quantized Strings II, III," *Phys. Lett.* **151B** (1985) 391-395; 396-400.

[38] W. Siegel and B. Zwibach, "Gauge String Fields," *Nucl. Phys.* **B263** (1986) 105-128. T. Banks and M. Peskin, "Gauge Invariance of String Fields," *Nucl. Phys.* **B264** (1986) 513-546.

K. Itoh, T. Kugo, H. Kunitomo and H. Ooguri, "Gauge Invariant Local Action of String Field from BRS Formalism," *Prog. Theor. Phys.* **75** (1986) 162-174.

[39] A. Neveu, H. Nicolai and P. West, "New Symmetries and Ghost Structure of Covariant String Theories," *Phys. Lett.* **167B** (1986) 307-314.

H. Hata, K. Itoh, T. Kugo, H. Kunitomo and K. Ogawa, "Manifestly Covariant Field Theory of Interacting String I, II," *Phys. Lett.* **172B** (1986) 186-194; 195-203.

[40] E. Witten, "Non-Commutative Geometry and String Field Theory," *Nucl. Phys.* **B268** (1986) 253-294; "Interacting Field Theory for Open Superstrings," *Nucl. Phys.* **B276** (1986) 291-324.

[41] For reviews see for example C. B. Thorn, "String Field Theory," *Phys. Reports* **175** (1989) 1-101.

[42] A. M. Polyakov, "Quantum Geometry of Bosonic Strings," *Phys. Lett.* **103B** (1981) 207-210; "Quantum Geometry of Fermionic Strings," *Phys. Lett.* **103** (1981) 211-213.

[43] D. Friedan, "Introduction to Polyakov's String Theory," in *Recent Advances in Field Theory and Statistical Mechanics,* eds. J. B. Zuber and R. Stora (Elsevier, Amsterdam, 1984) 839-867.

[44] O. Alvarez, "Theory of Strings with Boundaries: Fluctuations, Topology and Quantum Geometry," *Nucl. Phys.* **B216** (1983) 125-184.

[45] E. D'Hoker and S. B. Giddings, "Unitarity of the Closed Bosonic Polyakov String," *Nucl. Phys.* **B291** (1987) 90-112.

[46] However I don't know who should have the credit.

[47] E. D'Hoker and D. H. Phong, "The Geometry of String Perturbation Theory," *Rev. Mod. Phys.* **60** (1988) 917-1065.

[48] E. Martinec, "Nonrenormalization Theorems and Fermionic String Finiteness," *Phys. Lett.* **171B** (1986) 189-194.

[49] D. Friedan, "Notes on String Theory and Two-Dimensional Conformal Field Theory," in *Unified String Theory*, eds. M. Green and D. Gross (World Scientific, Singapore, 1986) 162-213.

[50] L. Crane and J. M. Rabin, "Super Riemann Surfaces: Uniformization and Teichmüller Theory," *Comm. Math. Phys.* **113** (1988) 601-623.

[51] See for example M. A. Baranov, I. V. Frolov and A. S. Shvarts, "Geometry of Two-

Dimensional Superconformal Field Theories," *Theor. Math. Phys.* **70** (1987) 64-72.

[52] P. Nelson, "Lectures on Supermanifolds and Strings," preprint UPR-0363T (June, 1988) 82 pages.

[53] J. Atick, J. Rabin and A. Sen, "An Ambiguity in Fermionic String Perturbation Theory," *Nucl. Phys.* **B299** (1988) 279-294.

[54] J. Atick, G. Moore and A. Sen, "Some Global Issues in String Perturbation Theory," *Nucl. Phys.* **B308** (1988) 221-273.

[55] J. Atick, G. Moore and A. Sen, "Catoptric Tadpoles," *Nucl. Phys.* **B307** (1988) 221-273.

[56] E. Verlinde and H. Verlinde, "Multiloop Calculations in Covariant Superstring Theory," *Phys. Lett.* **192B** (1987) 95-102.

[57] D. Friedan and S. Shenker, "The Integrable Analytic Geometry of Quantum String," *Phys. Lett.* **175B** (1986) 287-294.

[58] D. Friedan and S. Shenker, "The Analytic Geometry of Two-Dimensional Conformal Field Theory," *Nucl. Phys.* **B281** (1987) 509-545.

[59] N. Seiberg and E. Witten, "Spin Structures in String Theory," *Nucl. Phys.* **B276** (1986) 272-292.

[60] See for example A. Namazie, K. Narain and M. Sarmadi, "Fermionic String Loop Amplitudes with External Bosons," *Phys. Lett.* **177B** (1986) 329-334;
Tanii, "Superstring One-Loop Nonrenonormalization Theorem in Covariant Path Integral Formalism," *Phys. Lett.* **179B** (1986) 251-256.

[61] A. M. Polyakov, *Gauge Fields and Strings*, (Chur, Harwood Academic Publs., 1988).

[62] E. Verlinde and H. Verlinde, "Lectures on String Pertubation Theory," in *Superstrings '88, Proceedings of the Trieste Spring School*, eds. M. Green et al. April 1988 (World Scientific, Singapore, 1989) 189-250.

[63] E. D'Hoker and P. H. Phong, "Multiloop Amplitudes for the Bosonic Polyakov String," *Nucl. Phys.* **B269** (1986) 205-234.

[64] E. D'Hoker and P. H. Phong, "Loop Amplitudes for the Fermionic String," *Nucl. Phys.* **B278** (1986) 225-241.

[65] G. Moore, P. Nelson and J. Polchinski, "Strings and Supermoduli," *Phys. Lett.* **169B** (1986) 47-53.

[66] M. Bonini and R. Iengo, "The Measure for the Supermoduli," *Phys. Lett.* **191B** (1987) 56-62.

[67] A. Neveu and J. H. Schwarz, "Factorizable Dual Model of Pions," *Nucl. Phys.* **B31** (1971) 86-112.

[68] P. Ramond, "Dual Theory for Free Fermions," *Phys. Rev.* **D3** (1971) 2415-2418.

[69] F. Gliozzi, J. Scherk and D. Olive, "Supersymmetry, Supergravity Theories and the Dual Spinor Model," *Nucl. Phys.* **B122** (1976) 253-290.

[70] D. Mumford, *Tata Lecture on Theta,* 2 vols. (Birkhäuser, Cambridge, Massachusetts, 1983).

[71] J. Fay, *Theta Functions on Riemann Surfaces*, Lectures Notes in Mathematics vol. 352 (Springer-Verlag, Berlin, 1973).

[72] D. Lebdev and A, Morozov, "Statistical Sums of Strings on Hyperelliptic Srfaces," *Nucl. Phys.* **B302** (1988) 163-188.

[73] M. Virasoro, "Alternative Constructions of Crossing Symmetric Amplitudes," *Phys. Rev.* **177** (1969) 2309-2311.

[74] J. A. Shapiro, "Electrostatic Analogue fir the Virasoro Model," *Phys. Lett.* **33B** (1970) 361-362.

[75] C.-J. Zhu, "Factorization of a Two-Loop Four-Particle Amplitude in Superstring Theory," *Phys. Lett.* **221B** (1989) 27-30.

[76] D. J. Gross, J. A. Harvey, E. Martinec and R. Rohm, "Heterotic String," *Phys. Rev. Lett.* **54** (1985) 502-505; "Heterotic String Theory (I). The Free Hetrotic String," *Nucl. Phys.* **B256** (1985) 253-284; "Heterotic String Theory (II). The Interacting Hetrotic String," *Nucl. Phys.* **B267** (1986) 75-124.

[77] I. Antoniadis, C. P. Bachas and C. Kounnas, "Fou-Dimensional Superstrings," *Nucl. Phys.* **B289** (1987) 87-108; I. Antoniadis and C. P. Bachas, "4d Fermionic Superstrings with Arbitrary Twists," *Nucl. Phys.* **B298** (1988) 586-612.

[78] K. Kawai, D. C. Lewellen, J. A. Schwartz and S.-H. H. Tye, "The Spin Structure Construction of String Models and Multi-Loop Modular Invariance," *Nucl. Phys.* **B299** (1988) 431-470; and many (> 10) other their papers.

[79] D. Arnaudon, C. P. Bachas, V. Rivasseau and P. Végreville, "On the Vanishing of the Cosmological Constant in Four-Dimensional Superstring Models," *Phys. Lett.* **195B**

(1987) 167-176.

[80] R. Iengo, G. Sotkov and C.-J. Zhu, "Two-Loop Vacuum Amplitude in Four-Dimensional Heterotic String Models," *Phys. Lett.* **211B** (1988) 425-430.

[81] C.-J. Zhu, unpublished works.

[82] A. Morozov and A. Perelomov, "A Note on Many-Loop Calculations for Superstrings in the NSR Formalism," *Int. J. Mod. Phys.* **A4** (1989) 1773-1780.

[83] O. Lechtenfeld, "The Three Loop Vanishing of the Superstring Cosmogical Constants," *Nucl. Phys.* **B309** (1988) 361-378.

[84] O. Lechtenfeld and A. Parkes, "On Covariant Multi-Loop Amplitudes," preprint CCNY-HEP-89/04 and UCD-89-8 (April, 1989) 60 pages.

[85] R. Iengo and C.-J. Zhu, unpublished works.

[86] L. Dixon, D. Friedan, E. Martinec and S. Shenker, "The Conformal Field Theory of Orbifolds," *Nucl. Phys.* **B282** (1987) 13-73.

[87] Al. B. Zamolodchikov, "Conformal Scalar Field on the Hyperelliptic Curves and Critical Ashkin-Teller Multipoints Correlation Functions," *Nucl. Phys.* **B285** **[FS19]** (1987) 481-503.

[88] M. A. Bershadsky and A. O. Radul, "Conformal Field Theories with Additional $\mathbf{Z}_N$ Symmetry," *Int. J. Mod. Phys.* **A2** (1987) 165-178; "g-Loop Amplitudes in Bosonic String Theory in Terms of Branch Points," *Phys. Lett.* **193B** (1987) 213-218.

[89] V. G. Knizhnik, "Analytic Fields on Riemann Surfaces. II," *Comm. Math. Phys.* **112** (1987) 567-590.

[90] D. Montano, "Superstrings on Hyperelliptic Surfaces and the Two-Loop Vanishing of the Cosmological Constant," *Nucl. Phys.* **B297** (1988) 125-140.

[91] E. Gava, "Conformal Fields on Complex Curves," preprint LEPTENS 88/1 (Jan., 1988) 15 pages.

[92] L. Brink, P. Di Vecchia and P. Howe, "A Locally Supersymmetric and Reparametrization Invariant Action for the Spinning String," *Phys. Lett.* **65B** (1976) 471-474.

[93] S. Deser and B. Zumino, "A Complete Action for the Spinning String," *Phys. Lett.* **65B** (1976) 369-373.

[94] B. A. Dubriovin, "The Kadomtsev-Petviashvili Equation and the RElations Netween the Periods of Holomorphic Differentials on Riemann Surfaces," *Math. USSR Izv.* **19**

(1982) 285-296.

[95] M. Mulase, "Cohomological Structure in Soliton Equations and Jacobian Varieties," *J. Diff. Geom.* **19** (1984) 403-430;

T. Shiota, "Characterization of Jacobian Varieties in Terms of Soliton Equations," *Inv. Math.* **83** (1986) 333-382.

[96] Y. Kitazawa, "Explicit Paramerization of Degenerate Surfaces from the KP Equation and Closed Bosonic String Amplitudes," *Nucl. Phys.* **B297** (1988) 338-348.

[97] G. Gilbert, "The Kadomtsev-Petviashvili Equations and Fundamental String Theory," *Comm. Math. Phys.* **117** (1988) 331-348.

[98] L. Alvarez-Gaume, G. Moore and C. Vafa, "Theta Functions, Modular Invariance, and Strings," *Comm. Math. Phys.* **106** (1986) 1-40.

[99] J. Igusa, *Theta Functions* (Springer-Verlag, Berlin, 1972).

[100] T. Eguchi and H. Ooguri, "Chiral Bosonization on a Riemann Surface," *Phys. Lett.* **187B** (1987) 127-134.

[101] A. Belavin, A. M. Polyakov and A. Zamolodchikov, "Infinite Conformal Symmetry in Two-Dimensional Quantun Field Theory," *Nucl. Phys.* **B241** (1984) 333-380.

[102] A. Morozov and A. Perelomov, "Partition Functions in Superstring Theory. The Case of Genus Two," *Phys. Lett.* **197B** (1987) 115-118.

[103] A. Morozov, "Two-Loop Statsum of Superstrings," *Nucl. Phys.* **B303** (1988) 343-372.

[104] O. Lechtenfeld and A. Parkes, "On the Vanishing of the Genus Two Superstring Vacuum Amplitude," *Phys. Lett.* **202B** (1988) 75-80.

A. Parkes, "The Two Loop Superstring Vacuum Amplitude and Canonical Divisors," *Phys. Lett.* **217B** (1988) 458-462.

[105] M. A. Bershadsky, "The Superbranch Point and Two-Loop Correlation in Heterotic String Theory," *Mod. Phys. Lett.* **A3** (1988) 91-108.

[106] G. Moore and A. Morozov, "Some Remarkes on Two-Loop Superstring Calculations," *Nucl. Phys.* **B306** (1988) 387-404.

[107] M. B. Green and N. Seiberg, "Contact Interactions in Superstring Theory," *Nucl. Phys.* **B299** (1988) 559-586.

[108] A. Morozov, "On the Two-Loop Contribution to the Superstring Four-Point Function," *Phys. Lett.* **209B** (1988) 473-476.

[109] A. A. Belavin and V. G. Knizhnik, "Algebraic Geometry and the Geometry of Quantum Strings," *Phys. Lett.* **B168** (1986) 201-206; "Complex Geometry and the Theory of Quantum Strings," *Sov. Phys. JETP* **64** (1986) 214-228.

[110] E. Gava, R. Iengo, T. Jayarama and R. Ramachandran, "Multilooop Divergences in the Closed Bosonic String Theory," *Phys. Lett.* **168B** (1986) 207-211.

[111] J. Atick and A. Sen, "Spin Field Correlators on an Arbitrary Genus Riemann Surface and Non-Renormalization Theorems in String Theories," *Phys. Lett.* **186B** (1987) 339-346.

[112] O. Yasuda, "Factorization of a Two-Loop Four-Particle Amplitude in Superstring Theory," *Phys. Rev. Lett.* **60** (1988) 1688-1691.

[113] G. Aldazabal, M. Bonini, R. Iengo and C. Nunez, "Superstring Vertex Operators and Scattering Amplitudes on Arbitrary Riemann Surfaces," *Nucl. Phys.* **B307** (1988) 291-318.

[114] See, e.g., V. N. Gribov, "A Reggeon Diagram Technique," *Sov. Phys. JETP* **26** (1968) 414-423. V. A. Abramovskii, V. N. Gribov and O. V. Kancheli, "Character of Inclusive Spectra and Fluctation Produced in Inelastic Processes by Multipomeron Exchange," *Sov. J. Nucl. Phys.* **18** (1974) 308-317.

[115] M. Bonini and R. Iengo, "Correlation Functions and Zero Modes on Higher Genus Riemann Surfaces," *Int. J. Mod. Phys.* **A3** (1988) 841-860.

[116] E. Verlinde and H. Verlinde, "Chiral Bosonization, Determinants and String Partition Functions," *Nucl. Phys.* **B288** (1987) 357-396.

[117] P. Candelas, G. Horowitz, A. Strominger and E. Witten, "Vacuum Configurations for Superstrings," *Nucl. Phys.* **B258** (1985) 46-74.

[118] L. Dixon, J. H. Harvey, C. Vafa and E. Witten, "Strings on Orbifolds," *Nucl. Phys.* **B261** (1985) 678-686; "Strings on Orbifolds (II)," *Nucl. Phys.* **B274** (1986) 285-314.

[119] K. Narain, "New Heterotic String Theories in Uncompactified Dimensions < 10 ," *Phys. Lett.* **169B** (1986) 41-46.

[120] For reviews see for example W. Lerche and A. N. Schellekens, "The Covariant Lattice Construction of Four-Dimensional Strings," preprint CERN–TH.4925 (Dec., 1987) 75 pages.

[121] I. Antoniadis, C. Bachas, C. Kounnas and P. Windey, "Supersymmetry Among Free

Fermions and Superstrings," *Phys. Lett.* **171B** (1986) 51-56.

[122] A. N. Shellekens and N. P. Warner, "Anomalies, Characters and Strings," *Nucl. Phys.* **B287** (1987) 317-361.

[123] J. Soda, N. Nakazawa, K. Sakai and S. Ojima, "Comments on the Non-Renormalization Theorem in the Four-Dimensional Superstrings," *Phys. Lett.* **201B** (1988) 73-77.

[124] J. Ellis and L. Mizrachi, "Multi-Loop No-Renormalization Theorems for a Class of Four-Dimensional String Models," preprint CERN-TH.5264/88 (Dec., 1988) 11 pages; "Multiloop No-Renormalization Theorems for the Ten-Dimensional Heterotic String," preprint CERN-TH.5302/89 (Feb., 1989) 29 pages.

[125] D. Friedan, E. Martinec and S. H. Shenker, "Conformal Invariance, Supersymmetry and String Theory," *Nucl. Phys.* **B271** (1986) 93-165.

[126] V. G. Knizhnik, "Covariant Fermionic Vertex in Superstrings," *Phys. Lett.* **160B** (1985) 403-407.

[127] J. J. Atick an A. Sen, "Covariant One-Loop Fermion Amplitudes in Closed String Theories," *Nucl. Phys.* **B293** (1987) 317-347.

[128] M. Bonini, R. Iengo and C. Nenuez, "Superconformal Ghost Correlators and Picture Changing," *Phys. Lett.* **219B** (1989) 427-434.

[129] E. Martinec, "Conformal Field Theory on a (Super-) Riemann Surface," *Nucl. Phys.* **B281** (1986) 157-210.

[130] O. Lechtenfeld, "On the Finiteness of Superstring," *Nucl. Phys.* **B322** (1989) 82-106.

[131] V. G. Knizhnik, "Analytic Fields on Riemann Surfaces," *Phys. Lett.* **180B** (1986) 247-254.

[132] J. Sobczyk and W. Urbanik, "On Application of Algebraic Curves in String Theory," preprint ITP UWr 88/7112 (Dec., 1988) 45 pages.

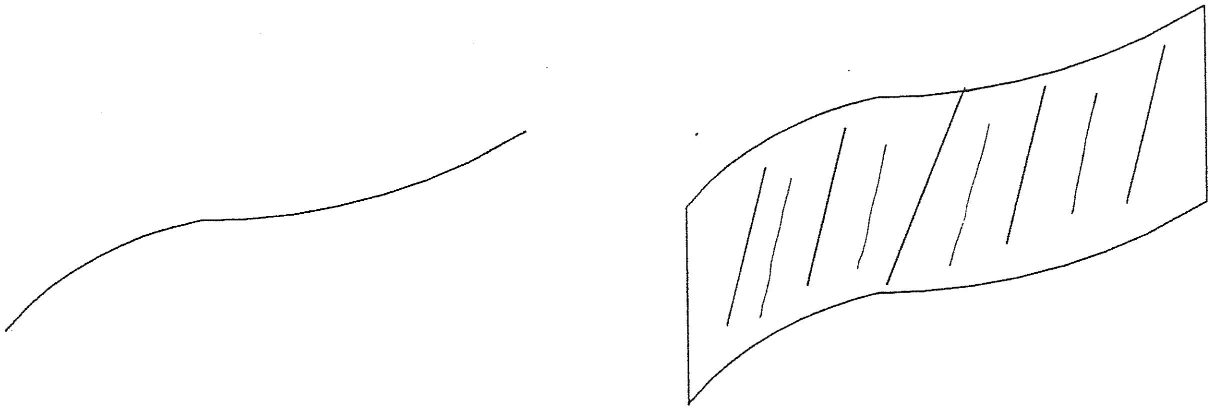

Fig.1

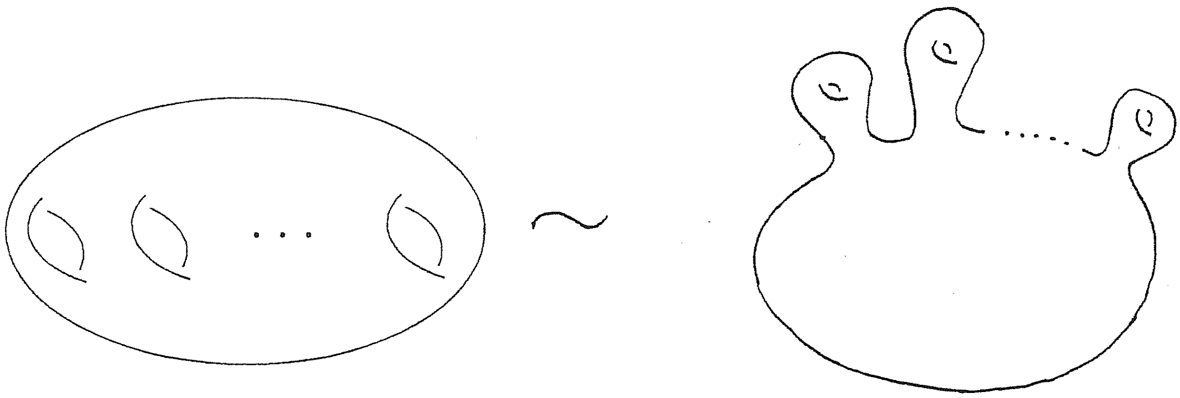

Fig.2

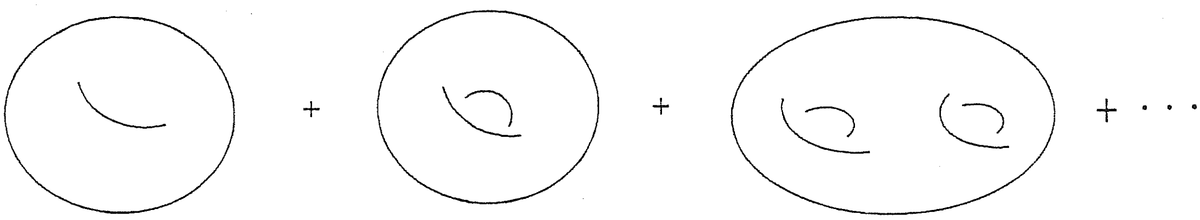

Fig.3

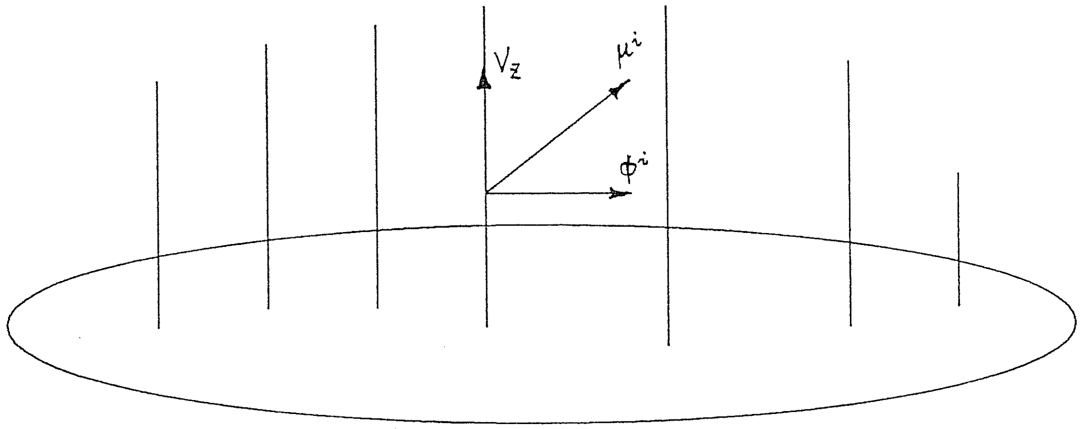

Fig.4

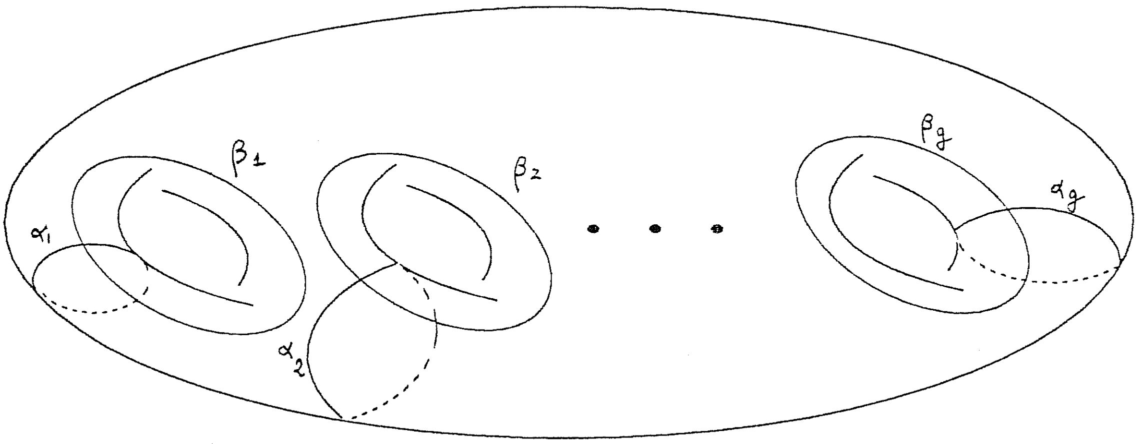

Fig.5

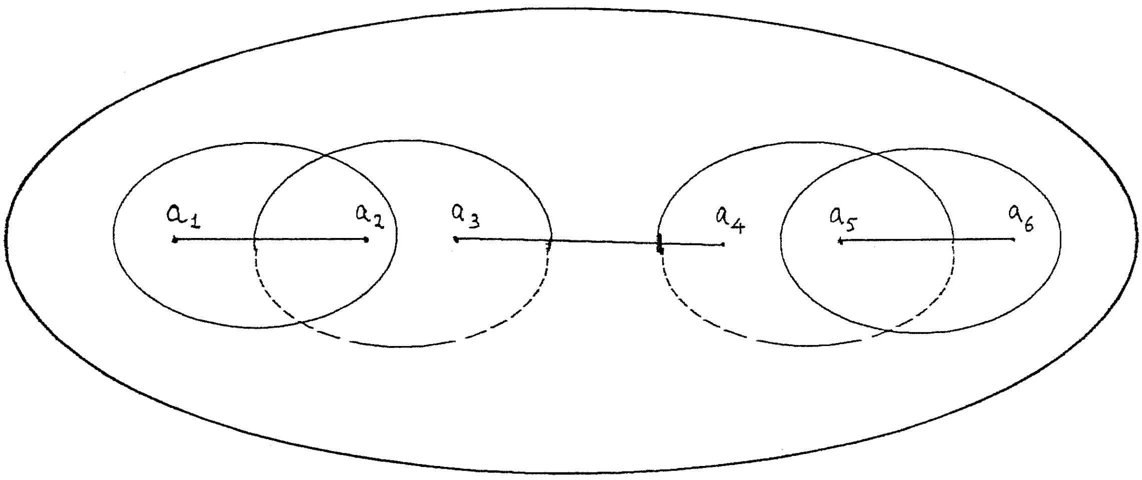

Fig.6

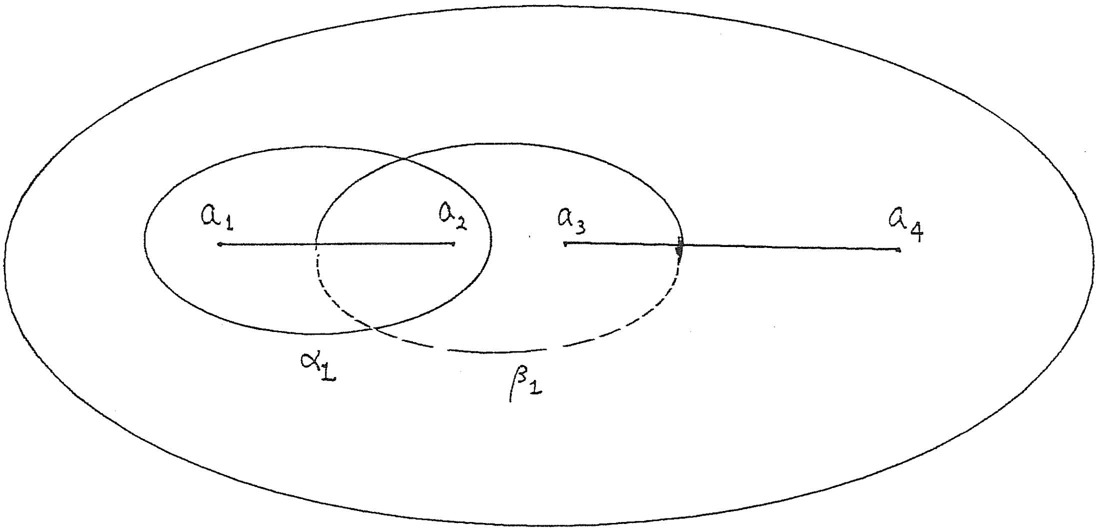

Fig.7

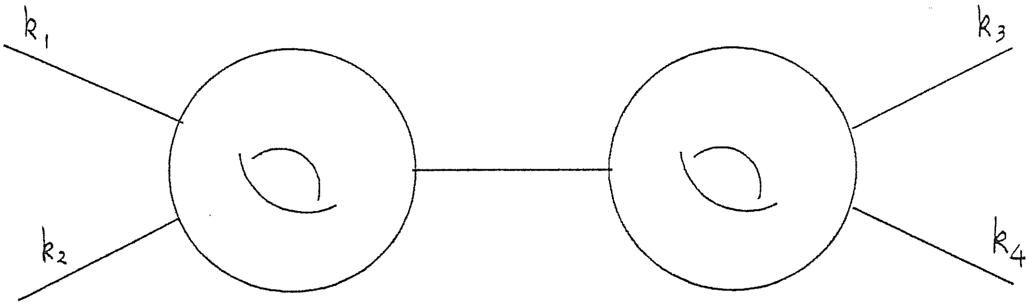

Fig.8

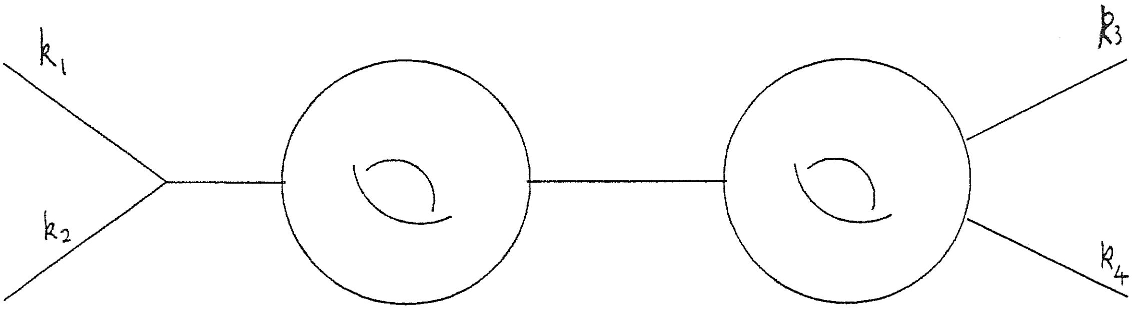

Fig.9

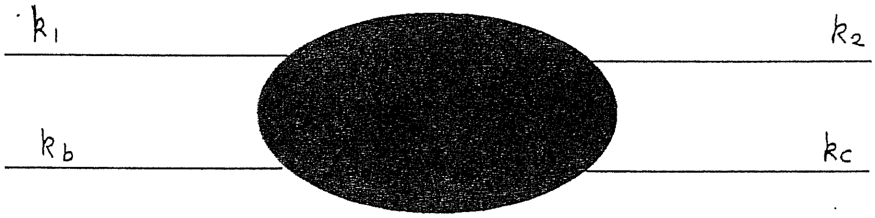

Fig.10

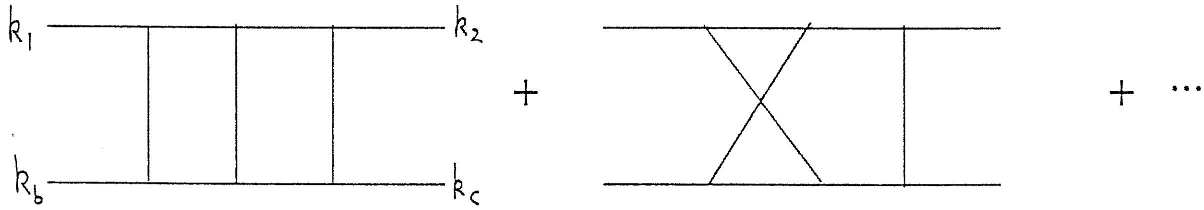

Fig.11

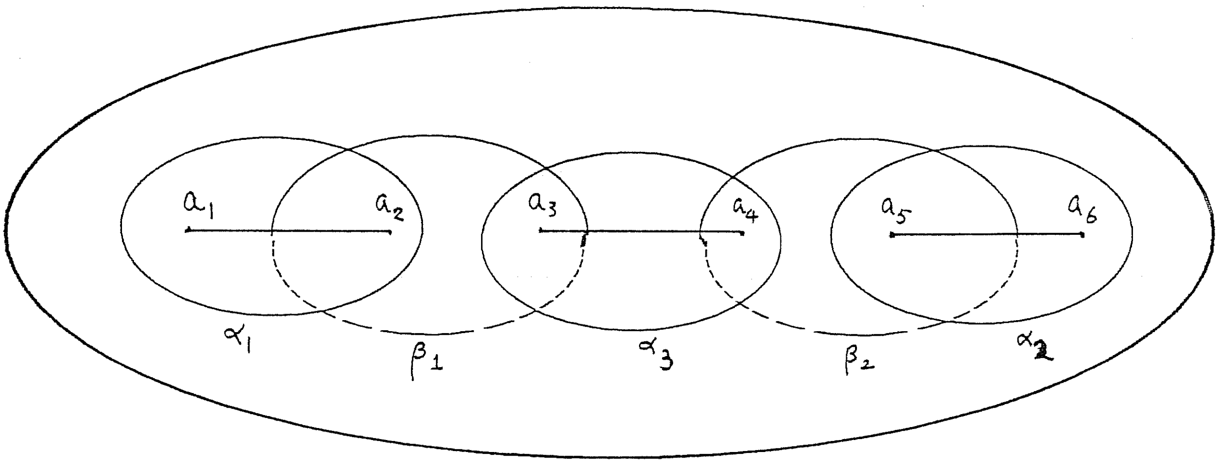

Fig.12

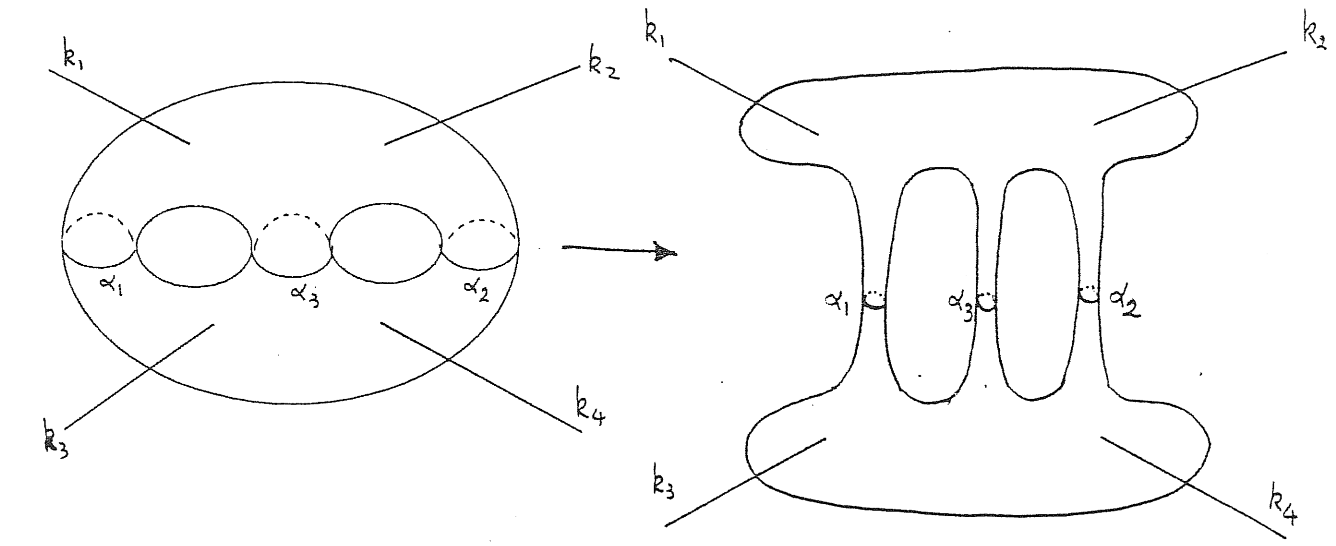

Fig.13

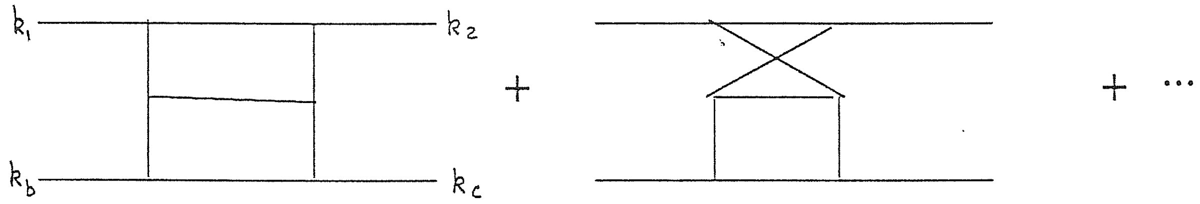

Fig.14

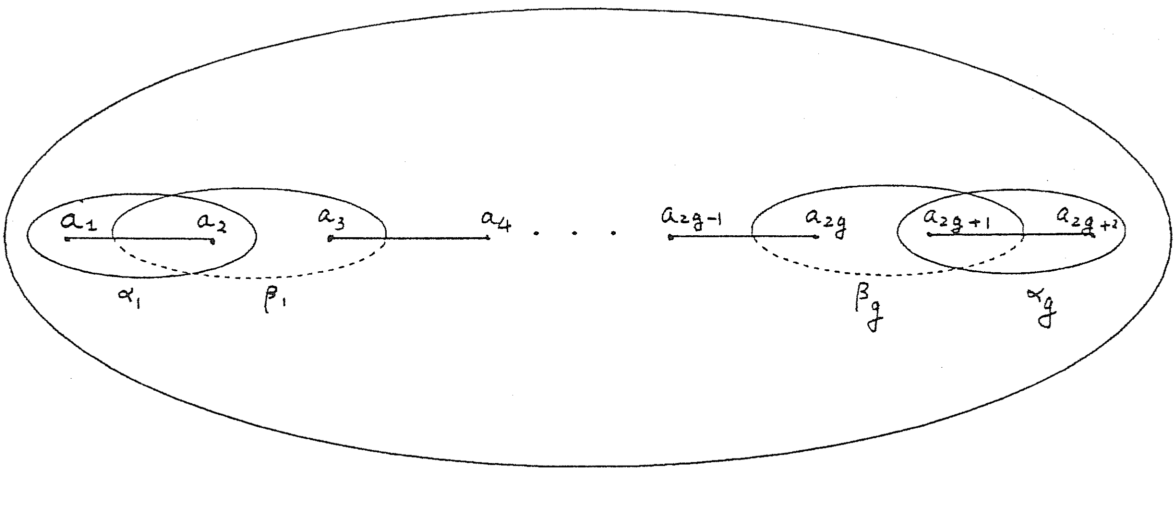

Fig.15

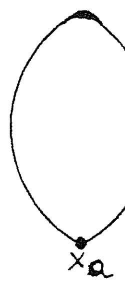

Fig. 16

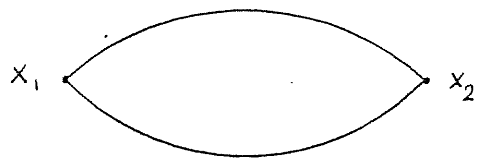

Fig.18

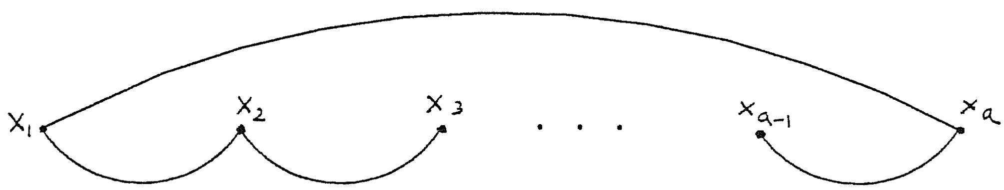

Fig.17

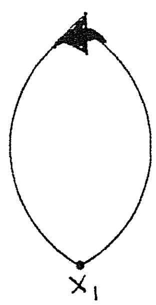

Fig. 19

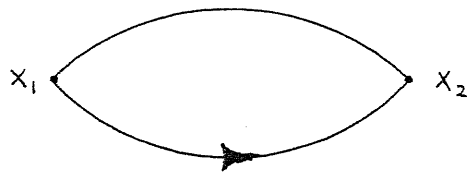

Fig.20

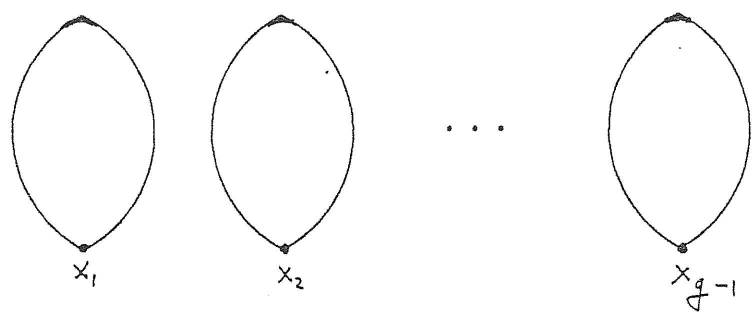

Fig.21

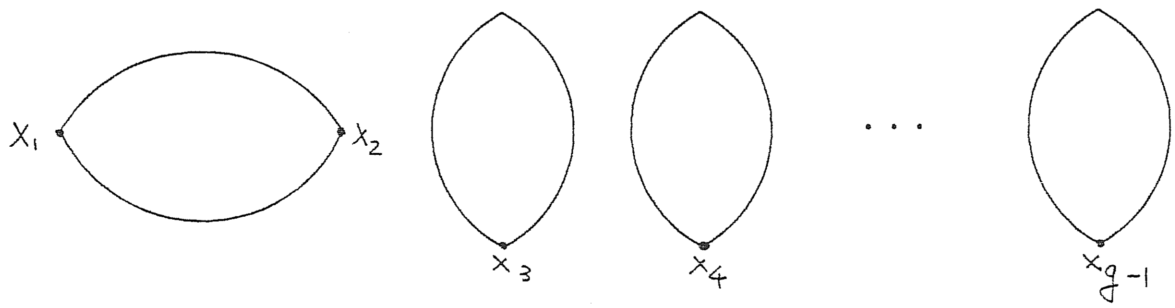

Fig.22

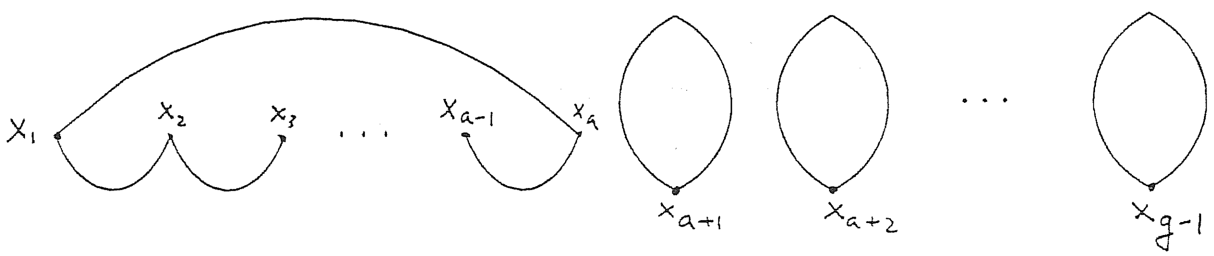

Fig.23