ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

CONFORMAL FIELD THEORIES IN HIGHER GENUS RIEMANN SURFACES.

Thesis submitted for the degree

of

Doctor Philosophiæ

Candidate

Francesco Toppan

Supervisor

Prof. Loriano Bonora

Academic Year 1988/89

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INTRODUCTION.

Two dimensional conformal field theories have received a lot of attention due to their relevance in string theory, as well as the fact that, in statistical mechanics, they describe systems at second order phase transitions.

The motivation to study conformal field theories in higher genus Riemann surfaces is both physical (in connection with the Polyakov perturbative scheme for string theory, or with models in statistical mechanics having particular boundary conditions) and mathematical: the higher genus shares the same features as the genus zero case but the complications arising from the non-trivial topology are introduced.

This thesis is devoted to the study of the conformal field theories in higher genus with a particular concern on two aspects, namely in establishing a prescription to compute the correlation functions of these theories, and in elaborating an operatorial framework for them. For what concerns the former aspect we recall that one of the key properties of the CFT is the factorizability of the correlation functions in chiral and antichiral blocks. Our aim is to give a prescription to compute such chiral blocks. We have to remember that in CFT such as the minimal models the correlation functions are fully constrained by the conformal invariance. To solve the partial differential equation so implied is however, from a practical point of view, extremely hard. At genus zero the Coulomb gas method elaborated in [1] gives an integral representation of the PDE. The basic idea underlying this method is the existence of a certain free field theory whose correlation functions, suitably constrained, have to be interpreted as correlation functions for the minimal models. The free field theory considered is that of a free bosonic field Φ with a background charge. However such a theory is not easily treatable in higher genus.

Motivated by the duality property between bosons and fermions in two dimension, we will introduce a new system, a *real weight* chiral anticommuting $b-c$ system, which gives rise to a fermionized version of the Coulomb gas and can be formulated in a straightforward way on any genus Riemann surface. This system allows to compute chiral blocks for any Conformal Field Theory which admit a representation in terms of free fields (not only

minimal models then, but also WZW models for instance). The $b-c$ system allows to compute, in a very easy way, also quantities which are relevant for string theories, like spin-field correlation functions and their generalizations. Regarding this, one should remark that the $b-c$ system theory is a well-defined theory interesting in itself, not only as a recipe to compute correlation functions for given CFT.

Besides knowing the correlation functions for the Conformal Field Theories, one should also be able to identify their operatorial content. In ch. 4 we will generalize the construction of the chiral vertex operator to higher genus Riemann surfaces. We will work in the Krichever-Novikov framework [2] which has the striking advantage that it mimics to higher genus the procedure already carried out on the sphere. Morever, the global data concerning the Riemann surfaces are present from the very beginning. An interesting question concerns the possibility that in higher genus the Krichever-Novikov algebra, which should replace in higher genus the Virasoro algebra as a classifying tool, could be more refined than the Virasoro one.

The material is so organized: in ch. 1 a discussion on conformal field theories is carried out, emphasizing particularly the aspects which will be relevant later. In ch. 2 the real weight $b-c$ system is introduced on the sphere and a detailed presentation about how the information about the vertex operators of the theory under consideration, as well as the procedure to get primary field correlation functions, is given. In ch. 3 the generalization of such $b-c$ systems to higher genus is performed. In ch. 4, besides the construction of the vertex operator, a detailed analysis of the higher genus oscillators is presented.

1 THE CONFORMAL FIELD THEORIES.

1.1 INTRODUCTION TO CONFORMAL FIELD THEORIES.

In this section we will briefly review the features of the Conformal Field Theories on the complex plane, emphasizing particularly those aspects which will be relevant later on.

The standard reference as an introduction to the minimal models is of course the work by Belavin, Polyakov and Zamolodchikov [3]. A huge amount of review articles is now available.[4-10]

The subject of two dimensional conformal field theories has received considerable attention mainly because 2-D CFT are the building blocks [11-13] of classical string compactifications. A classification of 2-D CFT would provide useful information about the classical solution space of string theory.

On the other hand, in statistical mechanics, conformally invariant quantum field theories describe the critical behaviour of systems at second order phase transition. [14,15] This happens because the fluctuations of the dynamical variables are scale-invariant at the critical point. In three or more dimensions the conformal group is finite-dimensional and it does not provide too many informations about the theory at hand . In two dimensions however the conformal algebra becomes infinite- dimensional [16] and places significant restrictions on two-dimensional conformally invariant theories. This could lead to a classification of possible critical phenomena in two dimensions. In certain cases we will see that conformal invariance is sufficient to determine all the correlation functions of the theory.

A 2-dimensional Minkowsky space spanned by the coordinates σ, τ $(0 < \sigma \leq \pi)$ is rotated to the euclidean and a radial quantization [17] is performed. The cylinder can be conformally mapped to the complex plane $\mathbb{C} - \{0\}$ through

$$z = e^{w} \quad , \quad \bar{z} = e^{\bar{w}}, \tag{1.1}$$

where $w = \tau + i\sigma$, $\bar{w} = \tau - i\sigma$.

$\tau = -\infty$ is mapped to the origin and $\tau = +\infty$ to the point at infinity.

The conformal group G in two dimensions consists of all the transformations

$$z \to \xi(z) \quad , \quad \bar{z} \to \bar{\xi}(\bar{z}), \tag{1.2}$$

with ξ and $\bar{\xi}$ holomorphic.

G is given by the direct product

$$G = \Gamma \otimes \bar{\Gamma}, \tag{1.3}$$

where Γ ($\bar{\Gamma}$) is the group of analytical substitutions of the variable z ($\bar{z}$). This means that one can concentrate on studying just the chiral part of the conformal theories.

The Lie algebra L of the group Γ is generated by the basis of the vectors in $\mathbb{C} - \{0\}$: [18]

$$l_n = z^{n+1} \frac{d}{dz}, \tag{1.4}$$

where n is an integer.

This algebra coincides with the algebra of the vector fields on the circle S^1.

L admits a unique 1-dimensional $\mathbb{C}c$ central extension, [19] $L_c = L \oplus \mathbb{C}c$ which is the Virasoro algebra:

$$\begin{aligned} [l_n, c] &= 0 \\ [l_m, l_n] &= (m-n) l_{m+n} + \delta_{m,-n} \frac{(m^3 - m)}{12} c, \end{aligned} \tag{1.5}$$

The generators l_{-1},l_0,l_1 form the subalgebra $sl(2,\mathbb{R}) \subset L_c$ whose corresponding subgroup consists of the projective transformations. Their commutation relations are not anomalous even in the presence of the center c.

A Conformal Field Theory is characterized by the existence of a stress-energy tensor $T(z)$ which can be Laurent-expanded through

$$\begin{aligned} T(z) &= \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \\ L_n &= \frac{1}{2\pi i} \oint_C dz z^{n+1} T(z), \end{aligned} \tag{1.6}$$

where C encircles the origin.

The stress energy tensor $T(z)$ is the generator of the infinitesimal transformations of the group Γ, i.e., for any combination X of local fields and their derivatives up to a finite number, the conformal transformation $\delta_\epsilon X$ satisfies

$$< \delta_\epsilon X >= \frac{1}{2\pi i} \oint_C d\xi \epsilon(\xi) < T(\xi) X >, \tag{1.7}$$

where ϵ is an infinitesimal holomorphic function and C encloses all the singular points z_k where the local fields in X are located. (*)

The correlation function

$$< T(z) X >$$

is analytic in z, single-valued and regular everywhere but at the points z_k.

Inside a correlation function $T(z)$ satisfies the following O.P.E.:

$$T(z)T(w) = \frac{1}{(z-w)^4}\frac{c}{2} + \frac{2T(w)}{(z-w)^2} + \frac{1}{(z-w)}\partial_w T(w) + ... \tag{1.8}$$

which expresses the conformal properties of the field $T(z)$ itself.

From (1.8) it turns out that the Laurent coefficients L_n introduced in (1.6) satisfy the Virasoro algebra (1.5) where, by definition:

$$[L_n, L_m] = \frac{1}{(2\pi i)^2}\Big[\oint dz \oint_{|z|>|w|} dw - \oint dw \oint_{|w|>|z|} dz\Big] z^{n+1} T(z) w^{m+1} T(w). \tag{1.9}$$

(*) The local fields $A_i(z_i)$ are assumed to be operators and the correlation functions are defined as follows:

$$< A_i(z_1)...A_n(z_n) >=< 0|\mathcal{R}\{A_1(z_1)...A_n(z_n)\}|0 >,$$

where $\mathcal{R}$ is the radial ordering, i.e.:

$$\mathcal{R}A(z)B(w) = \begin{cases} A(z)B(w) & \text{if } |z| > |w| \\ B(w)A(z) & \text{if } |w| > |z| \end{cases}$$

or with a minus sign for fermionic operators.

As a basic assumption, in the BPZ approach to the conformal theories an infinite set of local fields A_j is postulated to exist. The set of operators $A_j(z)$ is assumed to be complete, which means that any state in the theory can be generated by a linear action of these operators. The identity, as well as any coordinate derivative of the fields involved, belongs to the set $\{A_j\}$.

The completeness assumption means that an operator algebra

$$A_i(\xi)A_j(0) = \sum_k C_{ij}^k(\xi)A_k(0) \tag{1.10}$$

holds in a weak sense (inside a correlation function).
The structure constants C_{ij}^k are single-valued c-number functions.

The operator algebra is furthermore assumed to be associative:

$$\sum_k C_{ij}^k C_{kl}^m = \sum_k C_{ik}^m C_{jl}^k. \tag{1.11}$$

Among the fields $A_j(\xi)$ there are some (let us denote them as Φ_j),called primary, which under conformal transformation behaves like

$$\Phi_j(z) \to \left(\frac{\partial \xi}{\partial z}\right)^{\Delta_j} \Phi_j(\xi), \tag{1.12}$$

where Δ_j is a real, non-negative parameter (the conformal dimension).

The conformal property (1.12) can also be expressed as a commutation relation:

$$[L_m, \Phi_n] = z^{m+1}\frac{\partial}{\partial z}\Phi_n(z) + \Delta_n(m+1)z^m\Phi_n(z). \tag{1.13}$$

If all the fields $A_j(\xi)$ entering a correlation function are primary, the general relation (1.7) can be reduced to the Ward identity

$$< T(z)\Phi_1(z_1)...\Phi_n(z_n) >= \sum_{i=1}^{n}[\frac{\Delta_i}{(z-z_i)^2} + \frac{1}{(z-z_i)}\frac{\partial}{\partial z_i}] < \Phi_1(z_1)...\Phi_n(z_n) > \tag{1.14}$$

which relates a correlation function of T and Φ_i with that of Φ_i only, through the application of a differential operator. The same procedure can be iterated to express a correlation function with an arbitrary number of stress-energy tensor T insertions in terms of primary fields correlation functions.

The O.P.E. of the stress-energy tensor T with a given primary field Φ_i generates a whole conformal family (denoted by $[\Phi_i]$) of fields associated to the primary field Φ_i. The conformal family $[\Phi_i]$ forms a representation of the conformal algebra and it is spanned by the fields $L_{n_k}...L_{n_1}\Phi_i$, with L_n given by eq. (1.6). In particular $L_n\Phi_i = 0$ if $n > 0$, $L_0\Phi_i = \Delta_i\Phi_i(z)$, $L_1\Phi_i(z) = \partial_z\Phi_i(z)$.

The new fields $L_{-n_k}...L_{-n_1}\Phi_i$, with $n_i > 0$, are called descendents.

It follows from eq. (1.14) that the descendent fields correlation functions are completely determined by the primary field correlation functions; therefore it is only needed to specify the dynamics for the primary fields. The whole set of fields $\{A_j\}$ introduced above consists of some number (which can be infinite) of conformal families:

$$\{A_j\} = \oplus_i[\Phi_i]. \tag{1.15}$$

In a conformal theory the ket vacuum $|0>$ must satisfy the relation

$$L_n|0>= 0 \qquad if \quad n \geq -1 \tag{1.16}$$

in order to have a non-singular stress-energy tensor at $z = 0$. This equation reflects the conformal invariance of the vacuum.

In the same way the bra vacuum must satisfy

$$<0|L_{-n} = 0 \qquad if \quad n \geq -1, \tag{1.17}$$

so that $T(z)$ be non-singular at $z = \infty$.

The reality condition for $T(z)$ in the Minkowsky (imaginary values of τ) space is implied by the relation

$$L_n^\dagger = L_{-n}. \tag{1.18}$$

The primary fields Φ_j are put in correspondence with the highest weight vector which generates a highest weight representation of the Virasoro algebra. [20]

A h.w. vector w is characterized by the following properties:

$$\begin{aligned} L_0 w &= hw \\ L_n w &= 0 \qquad if \quad n > 0. \end{aligned} \tag{1.19}$$

The h.w.r. is a representation of the Virasoro algebra spanned by the vectors

$$L_{i_k} ... L_{i_1} w.$$

A h.w. vector $|m>$ which satisfies (1.19) can be introduced simply by setting

$$\begin{aligned} |m> &= \Phi_m(0)|0> \\ L_0|m> &= \Delta_m|m> . \end{aligned} \tag{1.20}$$

In our theory the system of primary fields can be chosen to be orthonormal:

$$< \Phi_n(z_1)\Phi_m(z_2) >= \delta_{nm}(z_1 - z_2)^{-2\Delta_n}. \tag{1.21}$$

We can introduce the "out" primary states by defining

$$< n| = \lim_{z \to \infty} < 0|\Phi_n(z) z^{2L_0} \tag{1.22}$$

and the orthonormality condition (1.21) can be reexpressed as

$$< n|m >= \delta_{nm}. \tag{1.23}$$

1.2 THE MINIMAL MODELS.

In this section we will exploit the properties of the highest weight representations of the Virasoro algebra in order to completely solve those particular conformal theories called minimal models.

The detailed analysis of the properties of the h.w.r. is originally due to Feigin-Fuchs [21].

Let us denote as $V_{c,h}$ a h.w.r. with highest weight h and central charge c . All the vectors

$$L_{-i_1}...L_{-i_n}|h>$$

with a fixed value $i = i_1 + ... + i_n$ span the eigenspace V_{h+i} of L_0 with eigenvalue $h + i$, so that

$$V_{c,h} = \oplus_{i \in \mathbf{Z}_+} V_{h+i}.$$

The formal power series

$$chV_{c,h} = \sum_{i \in \mathbf{Z}_+} (dimV_{h+i}) q^{h+i} \tag{1.24}$$

is called the character of the representation $V_{c,h}$.

A h.w.r. is called a Verma representation if all the vectors

$$L_{-i_1}...L_{-i_n}|h>$$

are linearly independent.

It turns out that a representation $V_{c,h}$ is reducible provided there exists some vector $|\chi>$ belonging to V_{h+i} , $i > 0$, (null vector) which satisfies the relation

$$L_n|\chi>= 0 \qquad if \quad n > 0. \tag{1.25}$$

For instance at the level $i = 2$ the vector

$$|\chi>= \left(L_{-2} + \frac{3}{2(h+1)} {L_{-1}}^2\right)|h> \tag{1.26}$$

is easily seen to be a null vector if h satisfies the relation:

$$h = \frac{1}{16}(5 - c \pm \sqrt{(c-1)(c-25)}). \tag{1.27}$$

A null vector is orthogonal to any state $|\psi> \in V_{c,h}$ and in particular it has zero norm:

$$<\psi|\chi> = 0. \tag{1.28}$$

A h.w. vector which admits null vectors is called degenerate.

A h.w.r. uniquely defines an irreducible representation, the Verma representation $M_{c,h}$, obtained from $V_{c,h}$ by taking the quotient of the subspaces generated by the null vectors.

The analysis about which couple of values (c,h) characterizing the h.w.r. makes $V_{c,h}$ reducible has been originally performed by Kac [22]; he considered the determinants $detM_i(h,c)$ of the inner products of the states belonging to V_{h+i}. It turns out that

$$detM_i(h,c) = const \prod_{pq \leq i} (h - h_{p,q}(c))^{P(i-pq)} \tag{1.29}$$

where $p, q \in \mathbb{Z}_+$. Here

$$h_{p,q} = \frac{1}{48}[(13-c)(p^2+q^2) + \sqrt{(c-1)(c-25)}(p^2-q^2) - 24pq - 2 + 2c]. \tag{1.30}$$

Let us parametrize c as follows:

$$c = 1 - \frac{6}{m(m+1)}. \tag{1.31}$$

$P(k)$ is defined by the relation

$$\sum_{k=0}^{\infty} P(k)t^k = \prod_{i=1}^{\infty} \frac{1}{(1-t^n)}. \tag{1.32}$$

The determinant (1.29) vanishes for $h = h_{p,q}$, which means that for this value the representation $V_{c,h}$ is reducible. The corresponding null vector appears at the lowest order

at the level pq and has dimension $h = pq$. The h.w. vector corresponding to the degenerate representation labelled by p, q will be denoted by $|\Phi_{p,q}>$. A detailed analysis conducted by Friedan, Qiu and Shenker [23] shows that if $c < 1$, only a discrete set of values of c gives rise to unitary theories. The allowed values of c turn out to be those of eq. (1.31) with m restricted to be $m = 3, 4, 5, ...$ and $p \leq m$, $q \leq m - 1$.

An important property satisfied by the correlation functions which involve a null field χ corresponding to a null vector is the following: they are all vanishing. This property implies that the correlation functions involving degenerate primary fields must satisfy linear partial differential equations. If the null field appears at the level pq the maximal order of derivatives in the P.D.E. will be pq. For instance the correlation function

$$< \chi(z)\Phi(z_1)...\Phi(z_n) >$$

involving the null vector $|\chi>$ introduced in (1.26) (here $|h> \equiv |\Phi_{1,2}>$) will satisfy the equation

$$\left(\frac{3}{2(h+1)} \frac{\partial^2}{\partial z^2} - \sum_{i=1}^{n} \frac{h_i}{(z-z_i)^2} - \sum_{i=1}^{n} \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \right) < \Phi_{12}(z)\Phi_1(z_1)\Phi_n(z_n) >= 0. \quad (1.33)$$

In the case of a 4-point correlation function, the $SL(2, \mathbb{R})$ invariance, which is an exact symmetry, allows us to reexpress the P.D.E. as an ordinary differential equation, whose solutions are given in terms of hypergeometric functions.

A P.D.E. like (1.33) puts a constraint on the O.P.E. satisfied by a degenerate field with any given primary field $\Phi_i(z)$. In order to understand such a constraint one has simply to insert the O.P.E. (1.10) inside the P.D.E.. The conformal dimension of the primary fields appearing in the r.h.s. of (1.10) will be restricted.

The O.P.A. of the degenerate field Φ_{p_1,q_1} with another degenerate field Φ_{p_2,q_2} will look as follows ("fusion rule"):

$$[\Phi_{p_1,q_1}][\Phi_{p_2,q_2}] = \sum_{l_1=|p_1-q_1|+1}^{p_1+q_1-1} \sum_{l_2=|p_2-q_2|+1}^{p_2+q_2-1} [\Phi_{l_1,l_2}] \quad (1.34)$$

The r.h.s. of (1.34) means that in the O.P.E. only fields belonging to those particular conformal families will appear. The "fusion algebra" involving only degenerate conformal families is a closed algebra.

For discrete values of $c < 1$ there are conformal theories, the minimal models, which involve only a finite number of degenerate conformal families. In particular this is true when c belongs to the unitary series. The conformal dimension of the primary fields in a a minimal theory is given by the "conformal grid" introduced in ref.[3].

The simplest example of a minimal model is the Ising model [24,25], characterized by having central charge $c = \frac{1}{2}$ ($m = 3$ in (1.31)).
The Ising model involves 3 different conformal families whose primary fields have conformal dimension

$$h_{(1,1)} = 0, \quad h_{(2,1)} = \frac{1}{2}, \quad h_{(1,2)} = \frac{1}{16}.$$

So far we have analyzed the operatorial content of the minimal models. In order to fully specify this kind of theories one should give a prescription about how to compute the whole set of correlation functions. The determination of the coefficients C_{nml} appearing in the 3- point function

$$< n|\Phi_m(z,\bar{z})|l >= C_{nml} z^{\Delta_n - \Delta_m - \Delta - l} \bar{z}^{\bar{\Delta}_n - \bar{\Delta}_m - \bar{\Delta}_l} \tag{1.35}$$

(for the time being we use the full correlation functions, analytic times antianalytic parts) is sufficient to fully solve the problem.

In (1.34) we have used the $SL(2,\mathbb{C})$ invariance to put the fields Φ_l and Φ_m respectively to 0 and ∞ .

The coefficients C_{nml} are however not fixed by the conformal invariance and to determine them one has to make use of some dynamical principle. In the bootstrap approach of BPZ the dynamical principle which plays the role is the associativity property (1.11) of the O.P.A .. Let us consider a primary fields 4-point correlation function

$$< \Phi_k(\xi_1,\bar{\xi}_1)\Phi_l(\xi_2,\bar{\xi}_2)\Phi_n(\xi_3,\bar{\xi}_3)\Phi_m(\xi_4,\bar{\xi}_4) > .$$

The projective invariance allows us to fix $\xi_1 = \infty$, $\xi_2 = 1$, $\xi_4 = 0$ and

$$\xi_3 = \eta = \frac{(\xi_1 - \xi_2)(\xi_3 - \xi_4)}{(\xi_1 - \xi_3)(\xi_2 - \xi_4)} \tag{1.36}$$

(η is a conformally invariant anharmonic ratio).

The associativity condition is equivalent to the crossing symmetry which is expressed by the following relation: let

$$G^{lk}_{nm}(\eta, \bar{\eta}) = < k|\Phi_l(1)\Phi_n(\eta, \bar{\eta})|m >,$$

then

$$G^{lk}_{nm}(\eta, \bar{\eta}) = G^{mk}_{nl}(1 - \eta, 1 - \bar{\eta}) \tag{1.37}$$

After substitution of the product $\Phi_n(\eta)\Phi_m(0)$ with the O.P.E., eq. (1.37) leads to a system of equations which allows to determine the constants C_{nml} .

1.3 THE COULOMB GAS METHOD.

In the previous sections we have reviewed the BPZ approach to CFT. We have seen in particular that the conformal invariance, together with the associativity of the operator algebra, fixes in a univoque way the correlation functions of the minimal model theories. To solve the partial differential equations derived in the bootstrap approach is however, from a practical point of view, not an easy task.

In this section we will review the Coulomb gas method elaborated by Dotsenko and Fateev [1,26], which gives an integral representation of the PDE. In the simplest cases, it has been explicitely shown that the results obtained by using the Coulomb gas method are in agreement with the requirements placed by the conformal invariance.

The basic idea underlying the Coulomb gas method is the existence of a certain free field theory whose correlation functions, suitably constrained, have to be interpreted as correlation functions for the minimal model under consideration. The free field theory considered is that of a free bosonic field $\Phi(z,\bar{z})$, whose equation of motion in the (uncompactified) complex plane is given by

$$\partial_z \partial_{\bar{z}} \Phi = 0, \tag{1.38}$$

so that the chiral and antichiral parts decouple

$$\Phi(z,\bar{z}) = \Phi(z) + \Phi(\bar{z}). \tag{1.39}$$

If we take the action on the sphere S^2 to be

$$S = \frac{1}{2}\int d^2x\sqrt{g}[g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi + i\alpha_0\Phi R] \tag{1.40}$$

(here $g_{\alpha\beta}$ is the metric on the sphere, R is the 2-dim. curvature and $-2\alpha_0$ represents the insertion of a background charge) and we assume, which can be done without loss of generality, that the curvature is all concentrated in the point at ∞, then we are led to the equation of motion (1.38) on the complex plane.

The correlation functions are defined by means of a path integral:

$$< \Phi(x_1)...\Phi(x_n) >= \frac{1}{N}\int [D\Phi]\Phi(x_1)...\Phi(x_n)e^{-S}. \tag{1.41}$$

The propagator for the chiral part reads as follows

$$< \Phi(z)\Phi(w) >= -\log(z-w). \tag{1.42}$$

The stress-energy tensor T for the theory under consideration is obtained by varying the action w.r.t. the metric $g_{\alpha\beta}$; we get

$$T(z) = -\frac{1}{4} : \partial_z\Phi\partial_z\Phi : +i\alpha_0\partial_z^2\Phi. \tag{1.43}$$

The normal ordering is defined as usual by requiring the subtraction of the singular terms in the operator expansion.

It is straightforward to apply the Wick theorem to compute the O.P.E. $T(z)T(w)$. In particular one obtains for the central charge c the value

$$c = 1 - 24\alpha_0^2. \tag{1.44}$$

The presence of the background charge is therefore necessary if we want the free bosonic theory to describe a minimal model whose charge $c < 1$ is given by eq. (1.31).

The primary fields $V_\alpha(z)$ of the Virasoro algebra introduced in the previous section are expressed in this framework as follows, by taking the chiral part of their r.h.s.:

$$V_\alpha(z) =: e^{i\alpha\Phi(z)} : \quad . \tag{1.45}$$

From the general O.P.E. between a stress-energy tensor $T(z)$ and any given weight h primary field $V(P)$:

$$T(z)V(P) = \frac{hV(P)}{(z-P)^2} + \frac{1}{(z-P)}\partial_P V(P), \tag{1.46}$$

we can derive in our case the conformal dimension h_α of the vertex $V_\alpha(z)$:

$$h_\alpha = \alpha^2 - 2\alpha\alpha_0. \tag{1.47}$$

The conformal dimension satisfies a duality relation

$$h_\alpha = h_{2\alpha_0-\alpha}. \tag{1.48}$$

This means that the vertex $V_{2\alpha_0-\alpha}(z)$ describes the same primary field as $V_\alpha(z)$.

The 2-point correlation function $< V_\alpha(z)V_{2\alpha_0-\alpha}(w) >$ is given by

$$< V_\alpha(z)V_{2\alpha_0-\alpha}(w) >= (z-w)^{2\alpha(2\alpha_0-\alpha)}. \tag{1.49}$$

A generic correlation function is given by:

$$< V_1(P_1)...V_n(P_n) >= const \prod_{i<j}(z_{P_i} - z_{P_j})^{2\alpha_i\alpha_j}. \tag{1.50}$$

In (1.50) the constraint

$$\sum_i \alpha_i = 2\alpha_0 \tag{1.51}$$

is assumed to be satisfied. Any other correlation function which does not satisfy (1.51) is automatically zero. The origin of the constraint (1.51) can be understood by noting that, for $c = 1$ and $\alpha_0 = 0$, there is a conformal weight 1 conserved current $\bar{J}(z)$:

$$\begin{aligned} \bar{J}(z) &= i\partial_z\Phi \\ \partial_{\bar{z}}\bar{J}(z) &= 0 \end{aligned} \tag{1.52}$$

The last equation follows from the equation of motion (1.38). A charge neutrality condition imposes in this case [27]

$$\sum_i \alpha_i = 0.$$

When $c < 1$, even if $\bar{J}(z)$ is no longer a primary field, it still measures the charge inserted by the vertex V_α. The charge neutrality is now given by eq. (1.51) because the linear term in Φ in the action (1.40) inserts a charge $-2\alpha_0$ at ∞.

At this point we have to specify which kind of prescription is needed in order to compute the correlation functions of the conformal minimal models in terms of the bosonic theory. Firstable one has to take into account the fact that both the fields V_α and $V_{2\alpha_0-\alpha}$ represent a primary field with conformal weight given by (1.47). Then, one has to notice that non-local screening charge operators having conformal dimension $h = 0$ can be inserted in a correlator without changing its conformal properties.

A screening charge operator Q is constructed simply by taking the contour integral $\oint J$ of a screening operator $J =: e^{i\alpha\Phi} :$. In order to make sense of the contour integral J must have conformal dimension $h_\alpha = 1$. This condition can be satisfied by 2 values of α, namely

$$\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 1}. \tag{1.53}$$

We get therefore 2 screening operators $J_\pm =: e^{i\alpha_\pm \phi} :$, and the relative charge operators $Q_\pm = \oint J_\pm$. The screening operator $J_\pm$ carries a charge $\alpha_\pm$. Inside a correlation function the insertion of a suitable number of charge operators $Q_\pm$ can be done in order to balance the charge neutrality condition (1.51). In this way the correlation function for the minimal model under consideration is obtained from the correlation function of the vertex fields V_α "dressed" by the presence of the charge operators .

Let us consider the 4-point correlation function of the primary field Φ_i, represented by the vertex V_α. Many choices are in principle available but the one which works is the following:

$$< \Phi_i(z_1)...\Phi_i(z_4) >=< V_\alpha V_\alpha V_\alpha V_{2\alpha_0-\alpha} \oint J_+ ... \oint J_- ... > \tag{1.54}$$

(in the r.h.s. the number of screening operators has been inserted in order to match the condition (1.51)). The screening contours must be suitably chosen to enclose the vertices V in order to get a non-vanishing result.

With the choice (1.54) the condition on the neutrality of the charge can be satisfied only if α is quantized and assumes the following values:

$$\alpha = \alpha_{n,m} = \frac{1}{2}[(1-n)\alpha_- + (1-m)\alpha_+] \tag{1.55}$$

(n, m non-negative integers).

It is quite a remarkable property that the conformal dimensions $h_{\alpha_{n,m}}$ corresponding to these values of α precisely coincide with the conformal dimensions given by the Kac formula.

Any correlation function which involves primary fields of minimal models can be represented as in (1.54) with the insertion of screening charges.

In the above formulation the method proposed by Dotsenko and Fateev is somewhat heuristic. A deeper understanding of why it works has been given by Felder [28], who studied the properties of a cohomology underlying the representation of the minimal models. The reason why one has to perform the contour integration over the screening charges is due to the fact that in this way one projects out from the highest weight representation the subspace spanned by the null vectors.

As an example of the above procedure let us consider the 4-point correlation function

$$< \Phi_{n,m}(\infty)\Phi_{1,2}(z)\Phi_{1,2}(1)\Phi_{n,m}(0) >$$

which is expressed through

$$\oint dv < V_{2\alpha_0-\alpha_{n,m}}(\infty)V_{\alpha_{1,2}}(z)V_{\alpha_{1,2}}(1)V_{\alpha_{n,m}}(0)J_+(v) > .$$

There are two independet contour integration (either from 0 to ∞ or from 1 to ∞) which correspond to the two independent hypergeometric functions I_1,I_2 which solve the second order differential equation satisfied by our correlator. The integrals I_1,I_2 are given by:

$$\begin{aligned} I_1 &= \int_1^\infty dv v^a(v-1)^b(v-z)^c = \\ &= \frac{\Gamma(b+1)\Gamma(-a-b-c-1)}{\Gamma(-a-c)}F(-c,-a-b-c-1,-a-c;z) \\ I_2 &= \int_0^z dv v^a(1-v)^b(z-v)^c = \\ &= z^{a+c+1}\frac{\Gamma(a+1)\Gamma(c+1)}{\Gamma(a+c+2)}F(-b,a+1,a+c+2;z) \end{aligned} \tag{1.56}$$

$F(a,b,c,d)$ is the hypergeometric function [29].

So far we have limited ourselves to consider only the chiral part of the conformal theories. However the complete correlator has both analytic and antianalytic pieces. Assuming that the physical vertex operators have spin 0 ($h = \bar{h}$), the full correlator is of the form

$$G(z,\bar{z}) = \sum_{i,j} X_{i,j} I_i(z) I_j(\bar{z}). \tag{1.57}$$

The coefficients $X_{i,j}$ are found by requiring $G(z,\bar{z})$ to be monodromy invariant (the chiral pieces $I_i(z)$ transform linearly into themselves: $I_i(z) \to g_{ik}I_k(z)$ when z winds around the points 0,1,∞).

For the case at hand we get

$$G(z,\bar{z}) = \sin\pi(a+b+c)\sin\pi b|I_1(z)|^2 + \sin\pi a\sin\pi c|I_2(z)|^2 \tag{1.58}$$

This expression can be shown to be crossing symmetric.

1.4 OSCILLATORIAL REPRESENTATION OF THE VIRASORO ALGEBRA AND CHIRAL VERTEX OPERATORS.

In the previous section we have seen how the Coulomb gas can be formulated in a field theoretical language. It is possible however to rephrase the Coulomb gas emphasizing another point of view, namely its algebraic aspect.

It turns out that there are two equivalent algebraic versions of the Coulomb gas; one is bosonic, the other one is fermionic.

It is important to know all the different versions of the Coulomb gas because, though they are all equivalent, one version can be preferred to solve specific problems. In particular the algebraic formulations are more suited to study the representation theory, while the field theoretic versions are more suited to explicitely compute the correlation functions.

As for the two different algebraic formulations, the fermionic one goes back to the original work of Feigin-Fuchs [21] and it is realized in terms of semiinfinite forms constructed with tensor fields on the circle S^1; the space of the semiinfinite forms carries a representation of the Virasoro algebra. The bosonic version [30-32] gives a representation of the Virasoro algebra realized in terms of oscillators. The equivalence of these two representations goes under the name of boson-fermion correspondence [33].

Here we will present the oscillatorial version following in particular the reference [30].

The oscillator (Heisenberg) algebra is a complex Lie algebra defined by the basis $\{a_n, n \in \mathbb{Z}\}$ with the commutation relations

$$[a_m, a_n] = m\delta_{m,n}. \tag{1.59}$$

a_0 is a central element since it commutes with any given a_n .

An irreducible representation R of the Heisenberg algebra is spanned by the monomials

$$a_{-1}^{k_1} ... a_{-n}^{k_n} |0, \mu >,$$

where $k_i \in \mathbb{Z}_+$; $|0, \mu >$ is called the vacuum vector of R and satisfies the properties:

$$\begin{aligned} a_n|0, \mu > &= 0 \qquad for \quad n > 0 \\ a_0|0, \mu > &= \mu|0, \mu > . \end{aligned} \tag{1.60}$$

We limit ourselves to consider μ a real parameter.

A positive-definite controvariant hermitian form $< *|* >$ is introduced on R assuming that the monomials

$$\frac{1}{\sqrt{k_1!...k_n!}} a_{-1}^{k_1}...a_{-n}^{k_n}|0,\mu>$$

form an orthonormal basis (in particular the normalization condition $< 0,\mu|0,\mu >$ holds).

The hermiticity is introduced by the relation

$$a_n{}^\dagger = a_{-n} \tag{1.61}$$

In terms of the oscillator a_n we can write down an explicit representation of the Virasoro operators L_m which satisfy the Virasoro algebra (1.5).

They are defined as follows:

$$\begin{aligned} L_n(\mu) &= (a_0 + \mu n)a_n + \sum_{j=1}^{\infty} a_{-j}a_{n+j} + \frac{1}{2}\sum_{j=1}^{n-1} a_j a_{n-j} \\ L_{-n}(\mu) &= (a_0 - \mu n)a_n + \sum_{j=1}^{\infty} a_{-n-j}a_j + \frac{1}{2}\sum_{j=1}^{n-1} a_{-n+j}a_{-j} \\ L_0(\mu) &= \frac{1}{2}({a_0}^2 - \mu^2) + \sum_{j=1}^{\infty} a_{-j}a_j \end{aligned} \tag{1.62}$$

(n is a positive integer).

The central charge c is related to the c-number parameter μ by:

$$c = 1 - 12\mu^2. \tag{1.63}$$

It is also possible to introduce a hermitian coordinate q_0 as conjugate to the momentum a_0:

$$[q_0, a_n] = i\delta_{n,0}. \tag{1.64}$$

In terms of a complex variable z we can write a free scalar field $\phi(z)$ as:

$$\phi(z) = q_0 - i(a_0 - \mu)\log z - i\sum_{n=1}^{\infty}[\left(\frac{a_{-n}}{n}\right)z^n - \left(\frac{a_n}{n}\right)z^{-n}]. \tag{1.65}$$

A vertex operator of momentum t is defined by:

$$V_t(z) =: \exp it\phi(z) := \exp\,(t\sum_{n=1}^{\infty}(\frac{a_{-n}}{n})z^n)\exp\,(-t\sum_{n=1}^{\infty}(\frac{a_n}{n})z^{-n})\exp\,(itq_0)z^{t(a_0-\mu)} \quad (1.66)$$

(in a normal ordered expression $a_n, n > 0$, is moved on the right of a_{-n} and a_0 on the right of q_0).

$V_t(z)$ is a primary field since it satisfies the commutation relation (1.13) with $L_n(\mu)$. Its conformal dimension is

$$h(t) = \frac{t^2}{2} + \mu t = h(-2\mu - t).$$

As we have seen in the previous section we can introduce two screening operators $J_\pm \equiv V_{t_\pm}(z)$, having conformal dimension 1 ($t_\pm = -\mu \pm \sqrt{\mu^2+2}$). The screening charge operators $Q_\pm = \oint J_\pm$ commute with all $L_n(\mu)$ but satisfy

$$[q_0, Q_\pm] = t_\pm Q_\pm. \quad (1.67)$$

The ket vacuum $|0,\mu>$ satisfies the right property of conformal invariance (1.16) ($L_n(\mu)|0,\mu>= 0$ for $n \geq -1$). However since from (1.61) it is

$$L_n(\mu)^\dagger = L_{-n}(-\mu) \quad (1.68)$$

it turns out that the bra vacuum which satisfies the conformal property (1.17) is $< 0,-\mu|$:

$$< 0,-\mu|L_{-n}(\mu) = 0 \qquad for \quad n \geq -1. \quad (1.69)$$

The bra vacuum $< 0,-\mu|$ is however the hermitian conjugate of the ket vector

$$|0,-\mu>= \lim_{z\to 0} V_{-2\mu}(z)|0,\mu>= \exp -(2i\mu q_0)|0,\mu> . \quad (1.70)$$

This has the consequence that the correlation functions must be computed using the positive-definite inner product on R, but the bra vacuum must be reinterpreted according

to (1.70). Moreover, unless $\mu = 0$, the oscillator representation of the Virasoro operators is not unitary.

The primary fields correlation functions $< V_{t_1}(z_1)...V_{t_n}(z_n) >$ are defined through the radial ordered product

$$< V_{t_1}(z_1)...V_{t_n}(z_n) >=< 0, -\mu|R[V_{t_1}(z_1)...V_{t_n}(z_n)]|0, \mu > . \tag{1.71}$$

As a consequence the charge neutrality condition (1.51) follows naturally:

$$\sum_i t_i + 2\mu = 0 \tag{1.72}$$

Of course the correlation functions computed using the prescription (1 .71) give the same result of eq. (1.50).

In this framework the singular vertex operators $\Phi_r{}^{\pm}(t, z)$ which correspond to a null vector $|\chi >$ of eq. (1.25) is introduced by the relation:

$$\Phi_r{}^{\pm}(t, z) = \oint_{C_r} dz_r \int_z^{z_r} dz_{r-1} ... \int_z^{z_1} dz_1 V_{t_\pm}(z_r)...V_{t_\pm}(z_1)V_t(z) \tag{1.73}$$

The contour integration C_r encloses all the other points.

The equivalence between this procedure and the other one presented in the previous section is quite obvious.

A final remark concerns the fact that this procedure can be easily generalized to the superVirasoro algebras, which can be realized in terms of boson and fermion oscillators.[30]

1.5 THE WZW MODELS.

So far we have discussed conformal theories characterized by having central charge $c < 1$. There is a theorem due to Cardy [34]

which states that theories having $c \geq 1$ have necessarily an infinite number of primary fields. It is however possible to characterize a class of $c \geq 1$ theories, the so called rational conformal field theories, in the same way as done before for the minimal models. [35,36]

The theories of this class are invariant under a chiral algebra $\mathcal{A}$ which admits the Virasoro algebra as subalgebra. In these models the states are organized into a finite number of representations of $\mathcal{A}$ and it is possible to extend the notion of highest weight vector and primary fields introduced for Virasoro to the full chiral algebra $\mathcal{A}$ (in general primary fields for the Virasoro subalgebra are descendents fields w.r.t. the full chiral algebra $\mathcal{A}$).

Among the RCFT, there exist the so called Wess-Zumino-Witten models [37−40] which admit a lagrangian formulation.

The fundamental fields in these theories are a set of scalars $g(\xi)$, taking values in a compact Lie group G.

The action $S_{\lambda,k}$ for these models is given by

$$\begin{aligned} S_{\lambda,k} &= \frac{1}{4\lambda^2}\int d^2\xi tr(\partial_\alpha g^{-1}\partial^\alpha g) + k\Gamma(g) \\ \Gamma(g) &= \frac{1}{24}\int d^3x\epsilon^{\alpha\beta\gamma}tr(g^{-1}\partial_\alpha g g^{-1}\partial_\beta g g^{-1}\partial_\gamma g); \end{aligned} \tag{1.74}$$

λ and k are dimensionless coupling constants (k necessarily integral). The Wess-Zumino term $\Gamma(g)$ is an integral over a 3-dim. manifold whose boundary is the 2-dim. space.

The action is invariant under the transformation

$$g \to \Omega(z)g\bar{\Omega}(\bar{z})$$

(where $\Omega, \bar{\Omega} \in G$) which gives rise to the Kač-Moody algebra.[41] As a consequence of this symmetry the chiral current

$$J(z) = J^a(z)t^a = -\frac{1}{2}k\partial_z g g^{-1} \tag{1.75}$$

(here t^a are the antihermitian matrices representing the Lie algebra $[t^a, t^b] = f^{abc}t^c$ of the group G) is conserved:

$$\partial_{\bar{z}} J(z) = 0. \tag{1.76}$$

The current $J^a(z)$ is on the same footing as the stress-energy tensor T; it generates a current algebra G, while the full chiral algebra $\mathcal{A}$ is given by the semidirect product of G with the Virasoro algebra. Besides (1.8) the currents $T(z)$, $J^a(z)$ satisfy the following O.P. E.:

$$\begin{aligned} T(z)J^a(w) &= \frac{1}{(z-w)^2}J^a(w) + \frac{1}{(z-w)}\partial_w J^a(w) + ... \\ J^a(z)J^b(w) &= \frac{k\delta^{ab}}{(z-w)^2} + \frac{f^{abc}}{z-w}J^c(w). \end{aligned} \tag{1.77}$$

Here k is the level of the Kač-Moody algebra and it is related to c by the following formula:

$$c = \frac{kD}{c_V + k}, \tag{1.78}$$

where D is the dimension of the Lie algebra and c_V is defined through

$$c_V \delta^{ab} = f^{acd} f^{bcd}.$$

The current algebra (1.8,1.77) continues to be satisfied if one expresses the stress-energy tensor $T(z)$ through the Sugawara construction:

$$T(z) = \frac{1}{k + c_V} : J^a(z)J^a(z) := \frac{1}{k + c_V} \lim_{z \to w} [J^a(z)J^a(z) - \frac{k dim G}{(z-w)^2}]. \tag{1.79}$$

The primary fields for the chiral algebra $\mathcal{A}$ are defined to be the operators which satisfy the O.P.E. (1.46) together with

$$J^a(z)V_l(P) = \frac{t^a{}_l}{z - P}V_l(P) + ... \tag{1.80}$$

(1.80) implies the following Ward identity:

$$< J^a(z)V_{i_1}(z_1)...V_{i_n}(z_n) >= \sum_{k=1}^{n} \frac{t^a{}_{i_k}}{(z - z_{i_k})} < V_{i_1}(z_1)...V_{i_n}(z_n) > . \qquad (1.81)$$

Using the projective invariance the eq.(1.81) can be rewritten as:

$$\sum_{k=1}^{n} t^a{}_{i_k} < V_{i_1}(z_1)...V_{i_n}(z_n) >= 0. \qquad (1.82)$$

The modes of the $J^a(z)$ current become the Kač-Moody generators:

$$J^a{}_n = \oint dw w^n J^a(w) \qquad (1.83)$$

(n is an integer).

The operators L_n,$J^a{}_m$ obey the following commutation relations:

$$\begin{aligned} [L_n, J^a{}_m] &= -m j^a{}_{m+n} \\ [J^a{}_n, J^a{}_m] &= \frac{k}{2} n \delta^{ab} \delta_{m+n,0} + f^{abc} J^c{}_{n+m} \end{aligned} \qquad (1.84)$$

(the commutator $[L_n, L_m]$ is given by eq. (1.5)).

An operatorial interpretation for the WZW theories exists as well as for the minimal models. The h.w. vector $|V_j>$ is defined to satisfy eq. (1.19) and

$$\begin{aligned} J^a{}_0|V_j> &= t^a{}_j|V_j> \\ J^a{}_n|V_j> &= 0 \qquad for \quad n > 0. \end{aligned} \qquad (1.85)$$

The h.w. representation of the chiral algebra $\mathcal{A}$ is spanned by the vectors

$$L_{-n_1}...L_{-n_s} J^a{}_{-m_1}...J^a{}_{-m_t}|V_j> . \qquad (1.86)$$

In terms of the modes-operators the Sugawara relation (1.79) can be rexpressed as:

$$-(k + c_V)L_n = \sum_{-\infty}^{\infty} : J^a{}_m J^a{}_{n-m} : \qquad (1.87)$$

This relation means that the chiral algebra $\mathcal{A}$ is contained in the Kač-Moody enveloping algebra. The Virasoro descendent L_{-n} in (1.86) can therefore be written in terms of the Kač-Moody descendents $J^a{}_{-n}$.

A consequence of eq. (1.87) is that any h.w.r. of $\mathcal{A}$ admits a null vector: applying (1.87) to $|V_j>$ we find

$$(J^a{}_{-1}t^a{}_j + \frac{1}{2}(k + c_V))|V_j >= 0. \tag{1.88}$$

Equation (1.88) fixes the conformal dimension of $|V_j>$ to be:

$$\Delta_j = \frac{c_j}{k + c_V} \tag{1.89}$$

where c_i is the Casimir $c_i = t^a{}_i t^a{}_i$ in the i representation.

The null state (1.88) implies that the differential equations which follow from the Ward identities for the primary fields correlation functions can be completely solved, as with the minimal models.

We now follow the ref. [42–44] to show that the WZW theories admit a Coulomb gas representation realized in terms of free bosonic fields. In order to be definite we limit ourselves to consider the $s\hat{u}(2)_k$ Kač-Moody algebra. The extension of this procedure to generic WZW theories is quite straightforward, the main complication involves group theorethical factors.

In our particular case the Lie algebra structure constants f^{abc} are precisely $f^{abc} = \epsilon^{abc}$; the parameter $c_V = 2$ and the dimension $D = 3$.

The Kač-Moody currents $J^{\pm}(z)$, $J^3(z)$ can be constructed from a free boson ϕ_0 compactified on a circle of radius $\sqrt{k}$ and the Z_k parafermions $\psi, \psi^\dagger$ [45] *:

* We remember that the parafermions $\psi, \psi^\dagger$ describe a CFT with central charge $c = \frac{2(k-1)}{k+2}$; they have fractional conformal dimension $(1 - \frac{1}{k})$ and are called in this way since they do not obey neither commutation nor anticommutation relations.

$$
\begin{aligned}
J^+(z) &= \sqrt{k}\psi(z)\exp\,[i\frac{\phi(z)}{\sqrt{k}}] \\
J^-(z) &= \sqrt{k}\psi^\dagger(z)\exp\,[-i\frac{\phi(z)}{\sqrt{k}}] \\
J^3(z) &= \sqrt{k}\partial_z\phi.
\end{aligned}
\tag{1.90}
$$

The parafermions $\psi, \psi^\dagger$, in turn, can be constructed in terms of 2 bosonsϕ_1, ϕ_2 on a lorentzian lattice of signature $(+,-)$. The former is compactified on a circle with radius $\sqrt{k+2}$ and has background charge

$$\alpha_{0_1} = \frac{1}{2}\sqrt{k+2};$$

the latter has negative signature and is compactified on a circle of radius $\sqrt{k}$:

$$
\begin{aligned}
\psi(z) &= \frac{1}{2}\left(\partial_z\phi_2 - \sqrt{\frac{k+2}{k}}\partial_z\phi_1\right)\exp\,[\frac{i}{\sqrt{k}}\phi_2] \\
\psi^\dagger(z) &= \frac{1}{2}\left(\partial_z\phi_2 + \sqrt{\frac{k+2}{k}}\partial_z\phi_1\right)\exp\,[\frac{-i}{\sqrt{k}}\phi_2].
\end{aligned}
\tag{1.91}
$$

Using this representation for the parafermions it is possible to represent the $s\hat{u}(2)_k$ algebra in terms of the 3 bosons ϕ_1, ϕ_2, ϕ_0.

The stress-energy tensor $T(z)$ for this WZW model is the stress- energy tensor of a free-fields system:

$$
\begin{aligned}
T(z) = &-\frac{1}{4}\partial_z\phi_1\partial_z\phi_1 + \frac{1}{4}\partial_z\phi_2\partial_z\phi_2 - \frac{1}{4}\partial_z\phi_0\partial_z\phi_0 \\
&+ \frac{i}{2\sqrt{k+2}}\partial_z\partial_z\phi_1
\end{aligned}
\tag{1.92}
$$

In this theory there are several conformal dimension 1 screening operators J; only one of them however, $\tilde{J}$, commutes with the Kač-Moody currents:

$$\tilde{J} = [\partial_z\phi_2 + \sqrt{(k+2)k}\partial_z\phi_1]\exp\,[\frac{i}{\sqrt{k+2}}\phi_1] \tag{1.93}$$

For what concerns the full chiral algebra primary fields V, they are expressed as

$$V_{\lambda,\mu,\nu} = \exp\,[\lambda\phi_0 + \mu\phi_1 + \nu\phi_2].$$

The coefficients λ, μ, ν have to be determined in such a way to satisfy the O.P.E. (1.46,1.80).

The primary fields can also be labelled as $V_{j,m}$ since

$$\begin{aligned} T(z)V_{j,m}(0) &= \frac{\Delta_j}{z^2}V_{j,m}(0) + \frac{1}{z}\partial_z V_{j,m}(0) \\ J^3(z)V_{j,m}(0) &= \frac{m}{z}V_{j,m}(0). \end{aligned} \tag{1.94}$$

with

$$\Delta_j = \frac{j(j+1)}{k+2} \tag{1.95}$$

and j, m are integers or half-integers ($-j \leq m \leq j$).

The correlation functions can now be computed in the usual way. For instance the 4-point correlation function of the fields in the fundamental representation $j = \frac{1}{2}$ is given by:

$$< \tilde{V}_-(0)V_+(z)V_+(1)V_-(\infty) \oint \tilde{J} > . \tag{1.96}$$

The tilde means that in the operator $\tilde{V}$ is inserted the "vacuum charge"; we have put $\pm \equiv (\frac{1}{2}, \pm\frac{1}{2})$. Choosing two independent integration contours we get for the result the following:

$$F\left(\frac{1}{k+2}, -\frac{1}{k+2}, \frac{k}{k+2}, z\right)$$

and

$$F\left(\frac{1}{k+2}, \frac{3}{k+2}, \frac{k+4}{k+2}, z\right), \tag{1.97}$$

where $F(a,b,c,d)$ is the hypergeometric function. The above results correspond to the two different solutions of the Knizhnik-Zamolodchikov equations of ref. [39].

We conclude this section pointing out that WZW theories provide a scheme to analyze rational conformal field theories. The so-called coset construction [46] is in fact a powerful method to produce RCFT. The basic idea goes as follows: given a compact Lie group G and a subgroup $H \subseteq G$, the Virasoro generators ${L_n}^G$ and ${L_n}^H$ with central charge c^G and c^H respectively can be constructed in terms of the Sugawara relation (1.87).

The generators ${L_n}^{\frac{G}{H}} = {L_n}^G - {L_n}^H$ can be associated to the coset $\frac{G}{H}$. The ${L_n}^{\frac{G}{H}}$ satisfy a Virasoro algebra with central charge

$$c^{\frac{G}{H}} = c^G - c^H \tag{1.98}.$$

In particular the minimal models of the unitary series can be obtained from the coset

$$\frac{SU(2)_k \times SU(2)_1}{SU(2)_{k+1}}$$

with $m = k + 2 = 3, 4, 5, ...$

2 THE FERMIONIZATION OF THE COULOMB GAS

2.1 INTRODUCTION.

In the previous chapter we have discussed in detail the Coulomb gas prescription which enables us to compute the correlation functions for the conformal minimal models and WZW theories.

As we have already stated, so far the Dotsenko- Fateev realization has been the only field theoretical formulation of the Coulomb gas.

The aim of this chapter is to provide a fermionic realization of the Coulomb gas, exploiting the possibilities offered by *real weight* chiral $b-c$ systems.

Firstable we should say that it is reasonable to expect that there exists a fermionic field theory formulation of the Coulomb gas. The duality between bosons and fermions in two dimensions is in fact a well-established property (free fermions can be bosonized on any Riemann surface [47−50]). Moreover, the semiinfinite forms which appear in the Feigin-Fuchs [21] formulation and are constructed starting from bases of real (or complex) tensor fields, can be envisaged as bases of $b-c$ systems.

The fact that a free chiral anticommuting $b-c$ system of suitable (in general irrational) weight can be used to describe any given minimal model has been indeed proven in ref. [51,52]. It is worth to mention that the interest in considering such kind of fermionic theories is not merely an academic one. The main advantage of the $b-c$ system approach w.r.t. the usual one lies on the fact that the $b-c$ systems can be formulated from the very beginning as chiral theories. This in turn has the consequence that the generalization of the Coulomb gas prescrition in the $b-c$ system framework to CFT formulated on higher genus Riemann surfaces can be done in a straightforward way. That is not true when the Coulomb gas is described as a free boson. Moreover in higher genus Riemann surfaces the $b-c$ framework takes automatically into account geometrical factors such as curvature and holonomy.

Even the "charge at infinity" acquires a topological meaning and the " charge neutral-

ity condition" (1.51) is seen to be a consequence of a "generalized Riemann-Roch index".

This chapter is structured as follows: at first we introduce the real weight $b-c$ systems. We remark that these are new objects, since in the standard literature [53-56], only integral or half-integral weight $b-c$ systems are considered. The quantization of the $b-c$ system is performed in an operatorial framework. Then we are able to prove that such a $b-c$ system is equivalent to the chiral part of a free boson with a charge at infinity. This equivalence is a reflection of the above mentioned duality betwen bosons and fermions. It is likely to emphasize that such a duality appears at this very fundamental level and the equivalence between the bosonic and the fermionic Coulomb gas prescriptions is just a consequence of it. As a corollary any CFT which can be realized in terms of free bosons (like the WZW models and the parafermionic theories discussed in the previous chapter), can be expressed in terms of the corresponding $b-c$ systems too. [57]

The key point of our strategy to use the $b-c$ system as a recipe to compute correlation functions for CFT is based on the fact that it is possible to encode in the bases themselves over which the b and c tensor fields are expanded, the information about the primary fields insertions under consideration.

Along the same line real weight commuting $\beta-\gamma$ systems can be quantized too. It is not clear if these systems play any role in the CFT. It is likely however that they are someway connected with the Coulomb gas representations of the superconformal theories.

In this chapter we limit ourselves to treat the genus zero case only. The higher genus case will be dealt in the next chapter.

2.2 REAL WEIGHT $b-c$ SYSTEMS.

At first we have to introduce a real weight chiral $b-c$ system and the corresponding bases. This system is described in terms of two (anti)-commuting fields b and c of weight λ and (1-λ) respectively. Unless otherwise specified it will be understood that λ is a real number. Let us start with the anticommuting case. The energy momentum tensor of the system is

$$T = (1-\lambda)\partial bc - \lambda b\partial c \tag{2.1}$$

A system of bases for the b fields on the sphere is given by: [1]

$$f_j^{(\lambda,l)}(z) = (P_+ - P_-)^{j+\lambda}(z-P_+)^{j-\lambda}(z-P_-)^{-j-\lambda}(dz)^{\lambda} \tag{2.2}$$

where $j \in \mathbb{Z} + \lambda l$ and $l \in \mathbb{Z}^+$ is a sector index, namely when going once around P_+

$$f_j^{(\lambda,l)} \to e^{2\pi i\lambda l} f_j^{(\lambda,l)}. \tag{2.3}$$

When λ is rational, i.e. $\lambda = \frac{m}{n}$, with m,n relatively prime integers, then there are only n distinct sectors $l = 1,2,...,n$. When λ is half-integer we recover the Neveu-Schwarz ($l=1$) and the Ramond ($l=2$) sectors. For the c fields we have similarly the bases

$$f^j_{(1-\lambda,l)}(z) = (P_+ - P_-)^{-j-\lambda-1}(z-P_+)^{-j+\lambda-1}(z-P_-)^{j+\lambda-1}(dz)^{1-\lambda} \tag{2.4}$$

with the same conventions as above, so that the following duality relation holds:

$$\frac{1}{2\pi i}\oint_C f^i_{(1-\lambda,l)}(z) f_j^{(\lambda,l)}(z) = \delta^i_j \tag{2.5}$$

where $i,j \in \mathbb{Z} + \lambda l$. The contour integral winds once around P_+. Notice that for integer λ eq.(2.4) gives the $2\lambda - 1$ zero-modes of the c fields.

We expand now, in each sector l, the fields b and c :

$$b^l(z) = \sum_j b^{j,l} f_j^{(\lambda,l)}(z) \qquad c^l(z) = \sum_j c^l_j f^j_{(1-\lambda,l)}(z) \tag{2.6}$$

---

[1] P_- and P_+ are two points singled out on our sphere. We use this non-standard notation as an introduction to the higher genus case. The normalization is chosen in such a way that the usual notation is recovered by sending P_+ to 0 and P_- to infinity .

and assume the following anticommutation relations

$$\{b^{i,l}, c^l_j\} = \delta^i_j \qquad \{b^{i,l}, b^{j,l}\} = \{c_{i,l} c_{j,l}\} = 0 \tag{2.7}$$

For each sector the ket and bra vacua are defined as follows [3]:

$$\begin{aligned} b^{j,l}|0>_l = {}_l<0|c_j{}^l = 0, \qquad & for\ j \leq \lambda + \overline{\lambda(l-1)} - 1; \\ c^l_j|0>_l = {}_l<0|b^{j,l} = 0, \qquad & for\ j \geq \lambda + \overline{\lambda(l-1)} \end{aligned} \tag{2.8}$$

The bar denotes the non-integer part of $\lambda(l-1)$. When λ is integer or half-integer the bra vacuum so defined does not coincide with the one usually introduced in the literature because it automatically takes into account the zero-modes insertions [58] and moreover ${}_l<0|0>_l = 1$. This situation reflects in this language the discussion after eq.(1.70).

Our choice of the vacuum comes from the requirement that $b^l(z)|0>_l$ be non singular in $z = P_+$; on the other hand the first value of j corresponding to non-negative frequency in the expansion of $b(z)$ is just

$$j = \lambda + \overline{\lambda(l-1)}.$$

The propagator is defined as follows:

$$S^{(\lambda,l)}(z,w) \equiv {}_l<0|R\left(b^l(z)c^l(w)\right)|0>_l = \begin{cases} {}_l<0|b^l(z)c^l(w)|0>_l, & \text{if } |\zeta(z)| > |\zeta(w)| \ ; \\ -{}_l<0|c^l(w)b^l(z)|0>_l, & \text{if } |\zeta(w)| > |\zeta(z)| \ . \end{cases} \tag{2.9}$$

where R denotes the radial ordered product and

$$\zeta(x) = \frac{(x - P_+)}{(x - P_-)}.$$

Inserting the expansion (2.6) in eq. (2.9) we get

$$S^{(\lambda,l)}(z,w) = \begin{cases} \sum_{j \leq \lambda + \overline{\lambda(l-1)} - 1} f^{(\lambda,l)}_j(z) f^j_{(1-\lambda,l)}(w), & \text{if } |\zeta(z)| > |\zeta(w)|; \\ -\sum_{j \geq \lambda + \overline{\lambda(l-1)}} f^{(\lambda,l)}_j(z) f^j_{(1-\lambda,l)}(w), & \text{if } |\zeta(w)| > |\zeta(z)| \ . \end{cases} \tag{2.10}$$

Then the expression for the propagator $S^{(\lambda,l)}(z,w)$ is:

$$S^{(\lambda,l)}(z,w) = \frac{1}{(z-w)} \left(\frac{z - P_+}{w - P_+}\right)^{\overline{\lambda(l-1)}} \left(\frac{z - P_-}{w - P_-}\right)^{-2\lambda + 1 - \overline{\lambda(l-1)}} (dz)^\lambda (dw)^{1-\lambda}. \tag{2.11}$$

The sum $1-2\lambda$ of the exponents plays the role of a total charge as it will be seen later on. It is easy to see that there is a direct connection of $S^{(\lambda,l)}(z,w)$ with the covariant delta function for λ differentials in the sector l

$$\Delta^l(z,w) = \sum_j f_j^{(\lambda,l)}(z) f^j_{(1-\lambda,l)}(w) \tag{2.12}$$

i.e.

$$g(z) = \oint_C \Delta^l(z,w) g(w) \tag{2.13}$$

where $g(z)$ is any smooth λ-differential with the multivaluedness given in (2.3).

2.3 EQUIVALENCE BETWEEN FERMIONIC AND BOSONIC COULOMB GAS.

As explained in the introduction our aim now is to generalize the bases (2.6) in such a way as to record in the bases themselves the insertions of suitable fields which will be assimilated to the vertex operators of the bosonized formulation. Specifically our purpose is to arrive at eq.(2.23). To this end we notice that the propagator $S^{(\lambda,l)}(z,w)$ can be seen as the propagator in the sector $l=1$ with the insertion of a "λ-spin fields" at the points $P_\pm$:

$$S^{(\lambda,l)}(z,w) = \frac{{}_1<0|R\left(S^{-,l}(P_-)b^1(z)c^1(w)S^{+,l}(P_+)\right)|0>_1}{{}_1<0|R\left(S^{-,l}(P_-)S^{+,l}(P_+)\right)|0>_1} \tag{2.14}$$

where $S^{\pm,1}$ are the identity operators. The factors $\left(\frac{z-P_+}{w-P_+}\right)^{\overline{\lambda(l-1)}}$ and $\left(\frac{z-P_-}{w-P_-}\right)^{-\overline{\lambda(l-1)}}$ in eq.(2.11) are the effects of the λ-spin field insertions $S^{+,l}(P_+)$ and $S^{-,l}(P_-)$. The weights of the λ-spin fields can be calculated as shown below and are equal to

$$\pm\frac{1}{2}\overline{\lambda(l-1)}\left[\pm\overline{\lambda(l-1)}+2\lambda-1\right].$$

The factor $\left(\frac{z-P_+}{w-P_+}\right)^{1-2\lambda}$ is due to the total charge (see below). In the standard formalism, when λ is half-integral the spin fields can be inserted at any points on the sphere. In our framework, in order to reproduce (2.14) with the insertion of the λ-spin fields and of the charge in arbitrary points we have to modify the bases in the following multiplicative way:

$$\begin{aligned}\tilde{f}_j^{(\lambda,l)}(z) =&(P_+-P_-)^{j-\lambda-\lambda(\bar{l-1})}(z-P_+)^{j-\lambda-\overline{\lambda(l-1)}}(z-P_-)^{-j+\lambda-1+\overline{\lambda(l-1)}}\cdot\\ &\cdot(z-P_1)^{\overline{\lambda(l-1)}}(z-P_2)^{-\overline{\lambda(l-1)}}(z-P_3)^{1-2\lambda}(dz)^\lambda\end{aligned} \tag{2.15}$$

$$\begin{aligned}\tilde{f}^j_{(1-\lambda,l)}(z) =&(P_+-P_-)^{-j+\lambda+\lambda(\bar{l-1})}(z-P_+)^{-j-1+\lambda+\overline{\lambda(l-1)}}(z-P_-)^{j-\lambda-\overline{\lambda(l-1)}}\cdot\\ &\cdot(z-P_1)^{-\overline{\lambda(l-1)}}(z-P_2)^{\overline{\lambda(l-1)}}(z-P_3)^{2\lambda-1}(dz)^{1-\lambda}\end{aligned}$$

The exponents of $(z-P_\pm)$ in eq.(2.15) is an integer and l-independent. In particular

$$\begin{aligned}(z-P_+)^{j-\lambda-\overline{\lambda(l-1)}}(z-P_-)^{-j+\lambda-1+\overline{\lambda(l-1)}} =&(z-P_+)^k(z-P_-)^{-k-1}\\ k=(j-\lambda-\overline{\lambda}(l-1))\in&\mathbb{Z}\end{aligned}$$

Therefore in this formalism we have only the vacuum $|0>_1$[2].

Using these modified bases we obtain the propagator

$$\tilde{S}^{(\lambda,l)}(z,w) = \frac{1}{(z-w)}\left(\frac{z-P_1}{w-P_1}\right)^{\overline{\lambda(l-1)}}\left(\frac{z-P_2}{w-P_2}\right)^{-\overline{\lambda(l-1)}}\left(\frac{z-P_3}{w-P_3}\right)^{1-2\lambda}(dz)^{\lambda}(dw)^{1-\lambda} \tag{2.16}$$

Notice that with the use of the bases in eq.(2.15) the propagator is independent of the points $P_{\pm}$ where the bra and ket vacua are defined. Just like (2.11), the sum $(1-2\lambda)$ of the exponents in eq.(2.16) plays the role of the total charge. When λ is an integer there are no λ-spin fields and the factor $\left(\frac{z-P_3}{w-P_3}\right)^{-2\lambda+1}$ is just the effect of the zero-mode insertions, the number $1-2\lambda$ being the Riemann-Roch index on the sphere. Obviously the c zero-modes can be inserted at arbitrary points on the sphere. In our formalism this is achieved by a further modification of the bases:

$$\hat{f}_j^{(\lambda,1)} = (z-P_+)^{j-\lambda}(z-P_-)^{-j+\lambda-1}\prod_{i=1}^{2\lambda-1}(z-P_i)^{-1}(dz)^{\lambda}$$

and

$$\hat{f}^j_{(1-\lambda,1)}(z) = (z-P_+)^{-j+\lambda-1}(z-P_-)^{j-\lambda}\prod_{i=1}^{2\lambda-1}(z-P_i)(dz)^{1-\lambda} \tag{2.17}$$

The propagator becomes

$$\begin{aligned}\hat{S}^{(\lambda,1)}(z,w) &= \frac{1}{(z-w)}\prod_{i=1}^{2\lambda-1}\left(\frac{w-P_i}{z-P_i}\right)(dz)^{\lambda}(dw)^{1-\lambda} \\ &=_1 <0|R\left(b^1(z)c^1(w)\right)|0>_1 = \frac{<\prod_i c(P_i)b(z)c(w)>}{<\prod_i c(P_i)>}\end{aligned} \tag{2.18}$$

The RHS refers to the standard formalism. From eq.(2.16), for $\lambda \in \mathbb{R}$, the charge $1-2\lambda$ can be seen as a sort of generalized Riemann-Roch index. As we will show later this is true in any genus. This result plays an essential role in our approach; in particular the background charge for the bosonized version of the minimal models in higher genus is just equal to the generalized Riemann-Roch index $(2\lambda-1)(g-1)$, $\lambda \in \mathbb{R}$.

---

[2] With this basis the first non-negative frequency in P_+ in the expansion of $b(z)|0>$ is just $(z-P_+)^0$. This reflects the fact that we have inserted the λ-spin fields away from the points $P_{\pm}$ where the bra and ket vacua are defined

As a natural extension of the above procedure we introduce a " fat " $b-c$ system (we denote it as $B-C$ system). To do this we reshuffle the location of the λ-spin fields and of the zero modes and represent them as insertions of V-fields at points P_i. As we will show these fields $V^i(P_i)$ have to be assimilated to (the chiral part of) insertions of vertex operators : $e^{i\alpha_k\phi(P_k)}$:; namely the basic relation which allows to identify this formalism with the standard one is the following:

$$S(z,w) = {}_1 < 0|R(B(z)C(w))\,|0>_1 = \frac{<\prod_k V^k(P_k)b(z)c(w)>}{<\prod_k V^k(P_k)>} \tag{2.19}$$

These insertions are obtained by modifying in a suitable way the $b-c$ system bases. The principle which drives us is the following: the bases are assumed no more to be holomorphic outside P_- and P_+, but a well defined behaviour is assumed around the points P_i where the primary fields are thought to be inserted.

For the B field we have the expansion:

$$\begin{aligned} B(z) &= \sum_j B^j g_j^{(\lambda)}(z) \\ g_j^{(\lambda)}(z) &= \frac{(P_+ - P_-)^{j-\lambda+1}(z-P_+)^{j-\lambda}}{(z-P_-)^{j-\lambda+1}} \prod_{i=1}^{n}(z-P_i)^{\tilde{\alpha}_i}(dz)^{\lambda}. \end{aligned} \tag{2.20}$$

The corresponding expansion for the dual C fields is

$$\begin{aligned} C(z) &= \sum_j C_j g^j_{(1-\lambda)}(z) \\ g^j_{(1-\lambda)}(z) &= (P_+ - P_-)^{-j+\lambda}\frac{(z-P_+)^{-j+\lambda-1}}{(z-P_-)^{-j+\lambda}} \prod_{i=1}^{n}(z-P_i)^{-\tilde{\alpha}_i}(dz)^{1-\lambda}, \end{aligned} \tag{2.21}$$

where $j \in \mathbb{Z}+\lambda$. Also in this case we get a total charge conservation expressed by the constraint

$$\sum_i \tilde{\alpha}_i = 1-2\lambda. \tag{2.22}$$

This constraint is required as a geometrical consistency condition for the B and C bases, specifically by the fact that they must satisfy the right tensorial properties. The "fat" propagator $S(z,w)$ is given by:

$$S(z,w) = \frac{1}{(z-w)}\prod_i \left(\frac{z-P_i}{w-P_i}\right)^{\tilde{\alpha}_i}(dz)^{\lambda}(dw)^{1-\lambda} \tag{2.23}$$

According to eq. (2.19), the correlation functions $< \prod_i V^i(P_i) >$ can now be computed with a procedure similar to the one introduced by Dixon *et al.* in [59]. The stress-energy tensor satisfies the following OPE with any weight h primary field $V(P)$:

$$T(z)V(P) = \frac{hV(P)}{(z-P)^2} + \frac{1}{(z-P)}\partial_P V(P) \tag{2.24}$$

Therefore

$$< T(z)V(P_1)...V(P_n) >= \sum_i [\frac{h_i}{(z-P_i)^2} + \frac{\partial_{P_i}}{(z-P_i)}] < V(P_1)...V(P_n) > \tag{2.25}$$

(2.25) gives us a set of first order differential equations which can be solved to obtain the correlator $< V(P_1)...V(P_n) >$. In our case we have

$$\frac{< T(z)V(P_1)...V(P_n) >}{< V(P_1)...V(P_n) >} = \lim_{z\to w} [(1-\lambda)\partial_z - \lambda\partial_w][S(z,w) - \frac{1}{z-w}] \tag{2.26}$$

By solving the equations we get

$$< \prod_i V^i(P_i) >= const\delta\left(\sum_i \tilde{\alpha}_i + 2\lambda - 1\right) \prod_{i<j} (z_{P_i} - z_{P_j})^{\tilde{\alpha}_i \tilde{\alpha}_j} \tag{2.27}$$

The conformal weight of the vertex operator in P_i is given by: [3]

$$h_{\tilde{\alpha}_i} = \frac{1}{2}\tilde{\alpha}_i(\tilde{\alpha}_i + 2\lambda - 1) \tag{2.28}$$

The vertex operators satisfy a duality relation since the same conformal weight is obtained both from $\tilde{\alpha}_i$ and from $2\tilde{\alpha}_0 - \tilde{\alpha}_i$ (here $2\tilde{\alpha}_0 \equiv 1 - 2\lambda$). The central charge of the fermionic system is:

$$c(\lambda) = -12\lambda^2 + 12\lambda - 2 = 1 - 12\tilde{\alpha}_0^2 \tag{2.29}$$

It is clear at this point that the Ansatz we have made, namely that the modified bases represent primary fields vertex operators is indeed correct.

It is also evident that the $b-c$ system of real weight λ share the same features of a bosonic field Φ with a background charge at infinity.

[3] For $\lambda \in \frac{\mathbb{Z}}{2}$ a similar formula can be found in [60].

We recall a few basic facts about it. The stress energy tensor is

$$T_{zz} = -\frac{1}{4}\partial_z\phi\partial_z\phi + i\alpha_0\partial^2{}_z\phi$$

where $-2\alpha_0$ is the charge placed at ∞. The central charge is

$$c = 1 - 24\alpha^2{}_0$$

The primary fields are represented by vertex operators $V_\alpha =: e^{i\alpha\phi} :$ of conformal weight

$$h_{\alpha_i} = \alpha_i{}^2 - 2\alpha_i\alpha_0 = h_{2\alpha_0-\alpha_i}$$

Therefore the relation between the two formalisms is established by setting:

$$\tilde{\alpha}_i = \sqrt{2}\alpha_i \qquad \tilde{\alpha}_0 = \sqrt{2}\alpha_0 \tag{2.30}$$

In particular

$$< \prod_i V_i(P_i) > = < \prod_i e^{i\alpha_i\phi(P_i)} >$$

as for the chiral part. We notice however that the condition

$$\sum{}_i \alpha_i = 2\alpha_0 \tag{2.31}$$

for the non-vanishing of the correlator in the Coulomb gas case, is obtained in the $b-c$ formalism as a topological prescription (generalized Riemann-Roch index).

The electromagnetic duality [61] in our framework is expressed by the transformation $\lambda \to (1-\lambda)$.

Therefore a free boson with charge α_0 at ∞ is equivalent to a $b-c$ system of weight

$$\lambda = \frac{1}{2} \pm \sqrt{2}\alpha_0.$$

As far as concerns the Coulomb gas, the two formalisms can now proceed in a parallel way. In order to fulfill the condition (2.22) we have in general to introduce screening charges in the correlators. A screening charge is determined by the condition $h_{\tilde{\alpha}} = 1$, i.e.

$$\tilde{\alpha} = \tilde{\alpha}_\pm = \tilde{\alpha}_0 \pm \sqrt{(\tilde{\alpha}_0)^2 + 2} \tag{2.32}$$

As a consequence the allowed charges $\tilde{\alpha}$ are quantized:

$$\tilde{\alpha}_{r,s} = \frac{1}{2}(1-r)\tilde{\alpha}_+ + \frac{1}{2}(1-s)\tilde{\alpha}_- \tag{2.33}$$

If we specialize ourselves to the models of the unitary series we get in particular

$$c = 1 - \frac{6}{m(m+1)} \quad \tilde{\alpha}_0 = \pm\frac{1}{\sqrt{2m(m+1)}} \tag{2.34}$$

Therefore the relevant $b-c$ system one has to consider will have

$$\lambda = \frac{1}{2} \pm \frac{1}{\sqrt{2m(m+1)}} \tag{2.35}$$

So, in general λ will be irrational, apart from an infinite series of special values ($m = 8, 49, 288, ...$).

We conclude this chapter with a couple of remarks. The former concerns the fact that the procedure we have outlined allows us to compute not only primary fields correlation functions but also correlation functions involving composite operators, built up with primary field operators. To be explicit let us consider the case of the current $\bar{J}(z) = i\partial_z\Phi$ introduced in (1.52).

In the fermionized version $\bar{J}(z)$ is represented by $\bar{J}(z) =: bc : (z)$. Since the fields $b(z)$ and $c(z)$ turn out to have charge $+1$ and -1 respectively, the correlation functions involving $\bar{J}(z)$ can be computed by taking the finite part of the insertion

$$\lim_{\epsilon\to 0} V_1(z+\epsilon)V_{-1}(z) :$$

$$< \bar{J}(z)... >=<: V_1V_{-1} : (z)... > . \tag{2.36}$$

The other remark concerns the application of this formalism to the conformal theories which are expressed in terms of more bosonic fields. In this case the total correlation function is given by the product of the correlation functions of any constituent $b-c$ systems. For instance the correlation function $< V_{\alpha_1}...V_{\alpha_n} >$ for the $\hat{su}(2)_k$ WZW model which involves 3 bosonic fields, can be expressed as follows

$$< V_{\alpha_1}...V_{\alpha_n} >=< V_{1,1}...V_{1,n} >_1 < V_{2,1}...V_{2,n} >_2 < V_{3,1}...V_{3,n} >_3, \tag{2.37}$$

where $< ... >_i \ , i = 1,2,3$ refers to the correlation functions realized in terms of the b_i,c_i system. The corresponding charges for the primary fields operators are associated with the values λ, μ, ν which enter in the primary field operator after eq. (1.93).

3 REAL WEIGHT $b-c$ SYSTEMS IN HIGHER GENUS RIEMANN SURFACES.

3.1 INTRODUCTION.

Conformal field theories formulated on higher genus Riemann surfaces have received attention mainly in string theory, in the context of multiloop amplitudes within the Polyakov perturbative expansion scheme .[62]

However higher genus CFT are not relevant only in that case. The modular invariance requirement for a CFT on a torus, for instance, [63,64] specifies the operatorial content of the theory. Besides that the problem of defining CFT in higher genus Riemann surfaces is an interesting mathematical problem for itself, since it deals with the complications introduced by the non-trivial topology.

Here we will face the problem of extending the results of the previous chapter to real weight chiral $b-c$ systems in higher genus R.S. The main motivation for that lies of course in the application of the results we get to analyze higher genus CFT along the same lines as for genus 0. This means that we want to mimic the construction carried on the complex plane to describe a CFT on a generic Riemann surface. It makes sense to do so because the factorization property between chiral and antichiral pieces discussed in ch.1 holds in any genus. We are therefore motivated [2] to single out on the Riemann surface two points P_+ and P_- which play the same role as the North Pole and the South Pole on the Riemann sphere. A suitably normalized abelian differential of the third kind allows us to introduce an euclidean time τ on the Riemann surface. $P_\pm$ can be set to correspond to $\tau = \pm\infty$ and can therefore be pictorially visualized as the point where a strings comes from the very past and goes into the very future.

The material of this chapter is organized as follows: We deal first of all (section 2) with the problem of generalizing bases of real weight λ differentials to arbitrary genus. Then, inspired by the calculations of the spin field correlation functions, we apply these results to calculate correlation functions of field insertions whose interpretation depends on the particular $b-c$ system we are considering. To this end we introduce suitable $b-c$

systems, both commuting and anticommuting, essentially characterized by the bases over which we expand them. The bases contain the information concerning the field insertions we want to describe.

In section 4 we test our formalism by rederiving known results of spin field correlation functions and generalize them, by calculating correlation functions of spin fields for rational λ. In this case field insertions represent exactly spin fields. Finally in section 5 we apply our method to bases of $b-c$ systems ("fat" bases) such that the corresponding field insertions represent chiral vertex operators in minimal models. Finally, Appendix A and Appendix B contain details relevant to Section 2 and Section 5, respectively. Appendix C is devoted to our notation and conventions concerning theta function theory.

A few distinctive features of our approach deserve to be pointed out. Our bases allow us to define bra and ket vacua both depending on the Riemann surface. This is why we can generalize in a natural way our previous approach on the sphere. Furthermore we obtain the conservation of the total charge in the correlation functions, eq.(3.30) below, as a consequence of geometrical consistency on the basis (the total charge corresponds to the higher genus "charge at infinity" of the Coulomb gas approach). This is an example of the connection between conformal field theory and geometry exposed by the $b-c$ system approach.

3.2 REAL WEIGHT DIFFERENTIALS.

In this section we introduce and give the explicit form of bases for real weight differentials. This generalizes the work of refs.[2,58].

Real weight differentials in general are well defined only on a covering (in general with infinite sheets) of a Riemann surface Σ. However for brevity we will refer to them as λ - differentials on Σ. Let $P_+, P_- \in \Sigma$ be two distinguished points in general position which for $g = 0$ can be identified with $0, \infty$. In the following, $z_\pm$ will denote local coordinates vanishing at $P_\pm$. For $\lambda \in \mathbb{R}$ we define a λ - differential holomorphic outside $P_\pm$ with the following behaviour in a neighborhood of these points:

$$f_j^{(\lambda,l)}(z_\pm) = a_j^{(\lambda)\pm}(l) z_\pm^{\pm j - s(\lambda)}(1 + O(z_\pm))(dz_\pm)^\lambda = h_j^{(\lambda,l)}(z_\pm)(dz_\pm)^\lambda, \tag{3.1}$$

$$s(\lambda) \equiv \frac{g}{2} - \lambda(g-1),$$

where $j \in \mathbb{Z} + s(\lambda) + \lambda(l-1)$, $l \in \mathbb{Z}^+$ and $a_j^{(\lambda)\pm}(l)$ are constants. When $\lambda = \frac{m}{n}$, with m and n relatively prime numbers, there are only n distinct sectors, i.e. l =1,...,n.

When we go once around P_+ along a trivial homological cycle separating P_+ and P_-, $h_j^{(\lambda,l)}$ picks up a phase:

$$h_j^{(\lambda,l)} \to e^{2\pi i \lambda(l \mp 1 - g \pm g)} h_j^{(\lambda,l)}. \tag{3.2}$$

For instance when $\lambda \in \mathbb{Z} + \frac{1}{2}$ the Neveu - Schwarz (l =1) and the Ramond (l =2) sectors are recovered [51].

We remark that $f_j^{(\lambda,l)}(z)$ is a well defined λ - differential on Σ only if $l = 1$ (single-valuedness of $h_j^{(\lambda,1)}$ in P_+), and

$$Q(\lambda) \equiv -2s(\lambda) + 1 \equiv (2\lambda - 1)(g-1)$$

is an integer (single-valuedness in P_-). Notice that the same information can be gotten from the Riemann-Roch theorem. Indeed when $\lambda > 1$, $g > 1$, the λ - form $f_j^{(\lambda,1)}(z)$ is holomorphic for $s(\lambda) \leq j \leq -s(\lambda)$, therefore eq.(3.1) gives all the zero modes of the Cauchy - Riemann operator $\bar{\partial}$ coupled to λ - differentials and therefore $Q(\lambda)$ must be an integer.

Due to the Riemann - Roch theorem eq.(3.1) must be modified in a few specific cases which are listed and treated in detail in Appendix A.

For $\lambda \in \frac{\mathbb{Z}}{2}$ the Riemann - Roch theorem guarantees the existence and uniqueness of $f_j^{(\lambda,l)}(z)$ up to the multiplicative constants $a_j^{(\lambda)+}(l)$ ($a_j^{(\lambda)-}(l)$ is completely determined once the $a_j^{(\lambda)+}(l)$ is chosen). For arbitrary λ we will show the existence of $f_j^{(\lambda,l)}(z)$ by explicit construction. The uniqueness follows from the fact that given two λ - differentials satisfying eq.(3.1), their quotient is a meromorphic function with g poles in general position; therefore, by the Weierstrass (Noether) gap theorem [65], this function is a constant.

On Σ the euclidean time can be defined to be the harmonic function

$$\tau = Re \int_{Q_0}^{Q} f_{-\frac{g}{2}}^{(1)}(z), \tag{3.3}$$

where $f_{-\frac{g}{2}}^{(1)}(z)$ is the third kind differential with simple poles in $P_\pm$ with residues ± 1, normalized in such a way that the periods along the homology cycles be purely imaginary; its explicit expression is given in Appendix A. The level lines

$$C_\tau = \{Q \in \Sigma | Re \int_{Q_0}^{Q} f_{-\frac{g}{2}}^{(1)}(z) = \tau\}, \tag{3.4}$$

for $\tau \to \pm\infty$ become small circles around $P_\mp$.

The dual $f_{(1-\lambda,l)}^{j}(z)$ of $f_j^{(\lambda,l)}(z)$ is defined by means of

$$\frac{1}{2\pi i} \oint_{C_\tau} f_i^{(\lambda,l)}(z) f_{(1-\lambda,l)}^{j}(z) = \delta_i^j. \tag{3.5}$$

Let us start now with the construction of the λ - differential $f_j^{(\lambda,l)}(z)$ in terms of theta functions and prime forms. The expression

$$\frac{E(z,P_+)^{j+s(\lambda)-2\lambda}}{E(z,P_-)^{j+s(\lambda)}},$$

is a (multivalued) λ - differential with degree in P_+ exceeding that of $f_j^{(\lambda,l)}(z)$ by $2s(\lambda) - 2\lambda = g(1-2\lambda)$. Since by the Riemann vanishing theorem [66,67] the divisor of $\theta(P - gP_+ + \Delta)$ is gP_+, we put

$$f_j^{(\lambda,l)}(z) = \frac{E(z,P_+)^{j+s(\lambda)-2\lambda}}{E(z,P_-)^{j+s(\lambda)}} \frac{\theta(z + (j-s(\lambda))P_+ - (j+s(\lambda))P_- + (1-2\lambda)\Delta)}{\theta(z - gP_+ + \Delta)^{1-2\lambda}}. \tag{3.6}$$

where Δ is the Riemann class. The θ - function in the denominator insures the correct singularity in P_+, while the θ - function in the numerator guarantees the single-valuedness of $f_j^{(\lambda,l)}(z)$. Since the latter has g zeroes, it follows that the degree of $f_j^{(\lambda,l)}(z)$ is precisely $2\lambda(g-1)$.

Using the definition of the σ - differential (see Appendix C) and inserting the theta characteristics, eq.(3.6) is generalized to the λ - differential with $[{}^{\delta}_{\epsilon}]$ - structure, i.e.

$$f_j^{(\lambda,l)}(z) = \frac{\theta[{}^{\delta}_{\epsilon}](z+(j-s(\lambda))P_+ - (j+s(\lambda))P_- + (1-2\lambda)\Delta)}{E(z,P_+)^{-j+s(\lambda)}E(z,P_-)^{j+s(\lambda)}}\sigma(z)^{1-2\lambda}, \tag{3.7}$$

where $j \in \mathbb{Z} + s(\lambda) + \lambda(l-1)$, $l \in \mathbb{Z}^+$ (*). For $\delta, \epsilon = 0$ eq.(3.7) differs from eq.(3.6) by the constant term $s(P_+,...,P_+)$ defined in eq.(C.8). When $\lambda = \frac{m}{n}$ then $\delta_i, \epsilon_i \in \{0, \frac{1}{n},..., \frac{n-1}{n}\}$, so that there are n^{2g} "$[{}^{\delta}_{\epsilon}]$ - *structures*", n for each one of the a_i and b_i homology cycles. However, for the time being, we will leave the "$[{}^{\delta}_{\epsilon}]$ - *structures*" undetermined. The dual of $f_j^{(\lambda,l)}(z)$ is

$$f^j_{(1-\lambda,l)}(z) = \frac{N(\lambda,l,j)\theta[{}^{-\delta}_{-\epsilon}](z-(j-s(\lambda)+1)P_+ + (j+s(\lambda)-1)P_- + (2\lambda-1)\Delta)}{E(z,P_+)^{j-s(\lambda)+1}E(z,P_-)^{-j-s(\lambda)+1}\sigma(z)^{2\lambda-1}}, \tag{3.8}$$

where $N(\lambda,l,j)$ is a suitable normalization constant fixed by the condition (3.5), which,for the sake of brevity, we do not write down explicitly.

The multivaluedness of $f^j_{(1-\lambda,l)}(z)$ is given by (3.2) with λ replaced by $1-\lambda$, so that the integrand in eq.(3.5) is a well defined 1-differential.

To conclude this section we give the expression of the covariant delta function for λ - differentials in the sector l:

$$\Delta^{(l)}(z,w) = \sum_j f_j^{(\lambda,l)}(z) f^j_{(1-\lambda,l)}(w), \tag{3.9}$$

that is, if $g(z)$ is a smooth λ - differential with the multivaluedness given in eq.(3.2), we have

$$g(z) = \oint_{C_\tau} \Delta^{(l)}(z,w)g(w). \tag{3.10}$$

(*) Of course when $\lambda = \frac{m}{n}$, with m and n relatively prime, $l = 1,...,n$ represent the only distinct possibilities.

3.3 THE "FAT" b – c SYSTEM IN THE OPERATOR FORMALISM.

We start with the chiral anticommuting b - c system. It is a first order system whose action is

$$S = \int_\Sigma b\bar{\partial}c, \tag{3.11}$$

the fields b and c being λ and $1-\lambda$ differentials, respectively. The anticommutation relations are

$$\{b^{i,l}, c^l_j\} = \delta^i_j, \qquad \{b^{i,l}, b^{j,l}\} = 0 = \{c^l_i, c^l_j\}, \tag{3.12}$$

where $j \in \mathbb{Z} + s(\lambda) + \lambda(l-1)$, $l \in \mathbb{Z}^+$, $\lambda \in \mathbb{R}$. In the operator formalism these fields are expanded in a basis of λ - differentials in such a way that the equations of motion $\bar{\partial}b = 0$ and $\bar{\partial}c = 0$ are satisfied everywhere except possibly at $P_\pm$. We use the differentials defined in section 2 as bases to expand the b and c fields

$$b^{(l)}(z) = \sum_j b^{j,l} f_j^{(\lambda,l)}(z), \qquad c^{(l)}(z) = \sum_j c^l_j f^j_{(1-\lambda,l)}(z). \tag{3.13}$$

For any $\lambda \in \mathbb{R}$ we define the vacuum $|0>_l$ associated to Σ, following ref.[68] (for $\lambda = 0, 1$ or $g = 1$ there are some modifications, see Appendix A), by requiring that

$$\begin{aligned} b^{j,l}|0>_l &= {}_l<0|c^l_j = 0, \qquad for \;\; j \le s(\lambda) + \mu(l) - 1; \\ c^l_j|0>_l &= {}_l<0|b^{j,l} = 0, \qquad for \;\; j \ge s(\lambda) + \mu(l), \end{aligned} \tag{3.14}$$

$$ {}_l<0|0>_l = 1,$$

$$\mu(l) = \overline{\lambda(l-1)},$$

where the bar denotes the non integral part of $\lambda(l-1)$. Notice that the requirement ${}_l<0|0>_l = 1$ is consistent with the algebra (3.12).

Since they are going to play a very important role in the following, let us digress a bit on the properties of the vacua we have just defined. They are different from the vacua one usually meets in the literature [47–50]. First of all both $|0>_l$ and ${}_l<0|$ depend on the moduli of Σ. The second remark is that the vacuum ${}_1<0|$ accounts for the insertion of the total charge $Q(\lambda)$, which is entirely concentrated in P_-. The subscript l means a

further insertion of "λ-spin fields" $S^{-,l}(P_-)$ and $S^{+,l}(P_+)$ in $_1<0|$ and $|0>_1$, respectively. In particular as we will show below we have

$$|0>_l = S^{+,l}(P_+)|0>_1, \qquad {}_l<0| = \frac{{}_1<0|S^{-,l}(P_-)}{{}_1<0|S^{-,l}(P_-)S^{+,l}(P_+)|0>_1},$$

where $S^{-,1}(P_-)$ and $S^{+,1}(P_+)$ are the identity operators. We recall that for $Q(\lambda) \in \mathbb{Z}^+$ the higher genus generalization of the standard (i.e. without zero mode insertion) vacuum are defined by the requirement that $b^{(1)}(z)|0>_\Sigma$, $c^{(1)}(z)|0>_\Sigma$ and ${}_\Sigma<0|b^{(1)}(z)$, ${}_\Sigma<0|c^{(1)}(z)$ be holomorphic in P_+ and P_- respectively. This condition gives $|0>_\Sigma = |0>_1$ and

$$\begin{aligned} {}_\Sigma<0|c_j^1 &= 0, \qquad for \;\; j \leq -s(\lambda); \\ {}_\Sigma<0|b^{j,1} &= 0, \qquad for \;\; j \geq 1-s(\lambda). \end{aligned}$$

If $Q(\lambda) \notin \mathbb{Z}^+$, it is not possible to define a bra vacuum such that the exponents of z_- in the expansion of ${}_\Sigma<0|b^{(1)}(z)$, ${}_\Sigma<0|c^{(1)}(z)$ be integral. The solution of this problem is to use a modified basis, where an "amount $Q(\lambda)$ of singularity" in P_- of the differential $f_j^{(\lambda,1)}(z_\pm)$ ($f^j_{(1-\lambda,1)}(z_\pm)$) is shifted to other points. As we will show, this corresponds to a shift in the location of the charge $Q(\lambda)$. Moreover, using this modified basis in the expansion of the fields $b^{(1)}(z)$ and $c^{(1)}(w)$, the vacuum ${}_1<0|$ is defined by the holomorphicity condition on ${}_1<0|b^{(1)}(z)$, ${}_1<0|c^{(1)}(z)$ in P_-. An analogous argument holds for the vacua in an arbitrary sector l. The bases representing λ-spin field insertions at arbitrary points will be discussed later.

After this digression concerning the vacua (3.13), let us compute the following propagator

$$S^{(l)}(z,w) \equiv {}_l<0|\mathcal{R}(b^{(l)}(z)c^{(l)}(w))|0>_l = \begin{cases} {}_l<0|b^{(l)}(z)c^{(l)}(w)|0>_l, & \text{if } \tau_z > \tau_w; \\ -{}_l<0|c^{(l)}(w)b^{(l)}(z)|0>_l, & \text{if } \tau_w > \tau_z \; . \end{cases} \tag{3.15}$$

Inserting the expansions (3.12) into eq.(3.15) we obtain

$$S^{(l)}(z,w) = \begin{cases} \sum_{j\leq s(\lambda)+\mu(l)-1} f_j^{(\lambda,l)}(z) f^j_{(1-\lambda,l)}(w), & \text{if } \tau_z > \tau_w; \\ -\sum_{j\geq s(\lambda)+\mu(l)} f_j^{(\lambda,l)}(z) f^j_{(1-\lambda,l)}(w), & \text{if } \tau_w > \tau_z. \end{cases} \tag{3.16}$$

To evaluate $S^{(l)}(z,w)$ one looks at the behaviour of the right hand side of eq.(3.16) in a neighborhood of $P_\pm$ and in the limit $z \to w$. A careful analysis similar to the one carried out in [6] shows that

$$S^{(l)}(z,w) = \frac{1}{E(z,w)} \left(\frac{E(z,P_-)}{E(w,P_-)}\right)^{Q(\lambda)-\mu(l)} \left(\frac{E(z,P_+)}{E(w,P_+)}\right)^{\mu(l)} \cdot$$

$$\cdot \left(\frac{\sigma(z)}{\sigma(w)}\right)^{2\lambda-1} \frac{\theta[^{\delta}_{\epsilon}](z-w+(Q(\lambda)-\mu(l))P_- + \mu(l)P_+ - (2\lambda-1)\Delta)}{\theta[^{\delta}_{\epsilon}]((Q(\lambda)-\mu(l))P_- + \mu(l)P_+ - (2\lambda-1)\Delta)}. \tag{3.17}$$

That equations (3.16) and (3.17) coincide, can be seen also in another way: one considers the propagator $S^{(l)}(z,w)$ in (3.17) as λ - differential in z and expands it in the basis $f_j^{(\lambda)}(z)$

$$S^{(l)}(z,w) = \sum_j a^j(w) f_j^{(\lambda)}(z), \tag{3.18}$$

where

$$a^j(w) = \frac{1}{2\pi i} \oint_{C_\tau} S^{(l)}(z,w) f^j_{(1-\lambda)}(z), \tag{3.19}$$

It is easy to verify that

$$a^j(w) = \begin{cases} f^j_{(1-\lambda)}(w), & \text{if } j \le s(\lambda)+\mu(l), \\ 0, & \text{if } j \ge s(\lambda)+\mu(l)+1, \end{cases} \qquad \tau_z > \tau_w;$$

$$a^j(w) = \begin{cases} 0, & \text{if } j \le s(\lambda)+\mu(l), \\ -f^j_{(1-\lambda)}(w), & \text{if } j \ge s(\lambda)+\mu(l)+1, \end{cases} \qquad \tau_w > \tau_z. \tag{3.20}$$

Notice also that $S^{(l)}(z,w)$ can be seen as the generalization of the Szegö kernel to λ - and $(1-\lambda)$ - differentials in the w and z variables respectively.

Let us come now to the main point of this section (and a crucial point of our construction), i.e. generalizing the bases (3.6,3.7) in such a way as to record in the bases themselves the insertion of suitable fields (V-fields) which, as in ref.[51], will be eventually assimilated to vertex operators of the bosonized formulation. The goal will be achieved with the introduction of the bases (3.28) and (3.29). In the following we would like to

motivate the introduction of these bases by discussing several intermediate steps that lead to these equations, bearing in mind that the basic idea is simply to shift (inside the bases) the singularities corresponding to the charge $Q(\lambda)$ and the spin-fields, from P_+ and P_- to generic points on Σ. The first step in this direction consists, following the procedure already seen for the genus zero case, in locating the zero modes (whose total number is $Q(\lambda)$) outside the point P_- (zero modes exist only when $Q(\lambda) \in \mathbb{Z}^+$ in the sector $l = 1$; however their presence is reflected in any sector l). This corresponds to expanding the b field in terms of [68]

$$\tilde{f}_j^{(\lambda,l)}(z) = \frac{\prod_{i=1}^{Q(\lambda)} E(z,P_i)\theta[^{\delta}_{\epsilon}](z+u)}{E(z,P_-)^{j-s(\lambda)+1}E(z,P_+)^{-j+s(\lambda)}\sigma(z)^{1-2\lambda}}, \tag{3.21}$$

$$u = (j-s(\lambda))P_+ - (j-s(\lambda)+1)P_- + \Sigma_{i=1}^{Q(\lambda)}P_i + (1-2\lambda)\Delta,$$

where $j \in \mathbb{Z} + s(\lambda) + \lambda(l-1)$, $l \in \mathbb{Z}^{+(*)}$ and $Q(\lambda) \in \mathbb{Z}^+$. The dual basis (up to a normalization) is

$$\tilde{f}^j_{(1-\lambda,l)}(z) = \frac{\theta[^{-\delta}_{-\epsilon}](z-(j-s(\lambda)+1)P_+ + (j-s(\lambda))P_- - \Sigma_{i=1}^{Q(\lambda)}P_i + (2\lambda-1)\Delta)}{E(z,P_+)^{j-s(\lambda)+1}E(z,P_-)^{-j+s(\lambda)}\sigma(z)^{2\lambda-1}\prod_{i=1}^{Q(\lambda)}E(z,P_i)}. \tag{3.22}$$

The propagator with the insertion of zero-modes at the points $P_1, ..., P_{Q(\lambda)}$ is

$$S^{(l)}(z,w) = \frac{1}{E(z,w)}\left(\frac{E(z,P_+)E(w,P_-)}{E(w,P_+)E(z,P_-)}\right)^{\mu(l)}\left(\prod_{i=1}^{Q(\lambda)}\frac{E(z,P_i)}{E(w,P_i)}\right)\cdot$$

$$\cdot\left(\frac{\sigma(z)}{\sigma(w)}\right)^{2\lambda-1}\frac{\theta[^{\delta}_{\epsilon}](z-w-\mu(l)P_- + \mu(l)P_+ + \Sigma_{i=1}^{Q(\lambda)}P_i - (2\lambda-1)\Delta)}{\theta[^{\delta}_{\epsilon}](-\mu(l)P_- + \mu(l)P_+ + \Sigma_{i=1}^{Q(\lambda)}P_i - (2\lambda-1)\Delta)}. \tag{3.23}$$

When $\lambda \in \mathbb{R}$ is generic we cannot do the same as above, i.e. insert vertex fields with charge ± 1. However we can, for example, insert a vertex field at the point $P \in \Sigma$ that absorbs the entire charge $Q(\lambda)$. This can be done substituting in eqs.(3.21-22) the terms $\prod_{i=1}^{Q(\lambda)}E(z,P_i)$ and $\Sigma_{i=1}^{Q(\lambda)}P_i$ with $E(z,P)^{Q(\lambda)}$ and $Q(\lambda)P$, respectively. The propagator becomes

(*) The range of distinct values of l is determined by the value of λ (see comments after eq.(3.1)).

$$S^{(l)}(z,w) = \frac{1}{E(z,w)} \left(\frac{E(z,P_+)E(w,P_-)}{E(w,P_+)E(z,P_-)}\right)^{\mu(l)} \left(\frac{E(z,P)}{E(w,P)}\right)^{Q(\lambda)} \cdot$$
$$\cdot \left(\frac{\sigma(z)}{\sigma(w)}\right)^{2\lambda-1} \frac{\theta[^{\delta}_{\epsilon}](z-w-\mu(l)P_- + \mu(l)P_+ + Q(\lambda)P - (2\lambda-1)\Delta)}{\theta[^{\delta}_{\epsilon}](-\mu(l)P_- + \mu(l)P_+ + Q(\lambda)P - (2\lambda-1)\Delta)}. \tag{3.24}$$

Notice that $S^{(l)}(z,w)$ can be seen as the propagator in the sector $l=1$ with the insertion of λ-spin fields at the points $P_\pm$:

$$S^{(l)}(z,w) = \frac{{}_1<0|R\left(S^{-,l}(P_-)b^1(z)c^1(w)S^{+,l}(P_+)\right)|0>_1}{{}_1<0|R(S^{-,l}(P_-)S^{+,l}(P_+))|0>_1} \tag{3.25}$$

where $S^{\pm,1}$ are identity operators. The factors $\left(\frac{E(z,P_+)}{E(w,P_+)}\right)^{\mu(l)}$ and $\left(\frac{E(z,P_-)}{E(w,P_-)}\right)^{-\mu(l)}$ in eqs.(3.13-14) are the effects of the λ-spin field insertions $S^{+,l}(P_+)$ and $S^{-,l}(P_-)$.

The remarks just made, concerning eq. (3.25), suggests that also the spin-fields can be moved away from P_+ and P_-. Let us consider a simple example. We want to find the analog of eqs.(3.21-22) with the insertion of the λ- spin fields at arbitrary points Q_1 and Q_2. The bases fit for that are for $Q(\lambda) \in \mathbb{Z}^+$ (for $\lambda \in \mathbb{R}$ the modification is analogous)

$$\hat{f}^{(\lambda,l)}_j(z) = \left(\frac{E(z,Q_1)}{E(z,Q_2)}\right)^{\mu(l)} \frac{\prod_{i=1}^{Q(\lambda)} E(z,P_i)\sigma(z)^{2\lambda-1}\theta[^{\delta}_{\epsilon}](z+u)}{E(z,P_+)^{-j+s(\lambda)+\mu(l)}E(z,P_-)^{j-s(\lambda)+1-\mu(l)}}, \tag{3.26}$$

where $u = (j-s(\lambda)-\mu(l))(P_+-P_-)-P_-+\mu(l)(Q_1-Q_2)+(1-2\lambda)\Delta, \quad j \in \mathbb{Z}+s(\lambda)+\lambda(l-1)$. For the sake of conciseness we do not write down explicitly the dual basis. We remark that the numbers $j-s(\lambda)-\mu(l)$ are l-independent and integral. So the conditions (3.14) define a unique (l-independent) vacuum.

Using such modified bases we find a propagator $\hat{S}^{(\lambda,l)}$, which is equal to (3.23) with $P_+(P_-)$ replaced by $Q_1(Q_2)$. As a consequence this propagator is independent of the points $P_\pm$ where the vacua are defined.

As a final natural extension of the procedure outlined above, we introduce now "fat" $b-c$ systems, referred to as $B-C$ systems. The idea is to reshuffle the location of the λ-spin fields and of the total charge, and to represent them as insertions of V-fields at the points P_i. As we have shown in the $g=0$ case the fields $V^i(P_i)$ can be regarded as the fermionized counterparts of (chiral) insertions of vertex operators $: e^{i\alpha_i\phi(P_i)} :$ (for suitable α_i). In other words

$$S(z,w) \;=<0|\mathcal{R}((B(z)C(w))|0>=\;\frac{<\prod_k V^k(P_k)b(z)c(w)>}{<\prod_k V^k(P_k)>}. \tag{3.27}$$

The last expression refers to the standard formalism.

In order to implement the idea just outlined, we have to use modified bases

$$g_j^{(\lambda)}(z)=\frac{\prod_{i=1}^n E(z,P_i)^{\tilde{\alpha}_i}\theta[^{\delta}_{\epsilon}](z+u)}{E(z,P_-)^{j-s(\lambda)+1}E(z,P_+)^{-j+s(\lambda)}\sigma(z)^{1-2\lambda}}, \tag{3.28}$$

where $u=(j-s(\lambda))P_+-(j-s(\lambda)+1)P_-+\sum_{i=1}^n\tilde{\alpha}_iP_i+(1-2\lambda)\Delta,\quad j\in\mathbb{Z}+s(\lambda)$. We expand

$$B(z)\;=\;\sum_j B^j g_j^{(\lambda)}(z).$$

The dual bases are

$$g^j_{(1-\lambda)}(z)=\frac{\theta[^{-\delta}_{-\epsilon}](z+(-j+s(\lambda)-1)P_+-(-j+s(\lambda))P_--\sum_{i=1}^n\tilde{\alpha}_iP_i+(2\lambda-1)\Delta)}{E(z,P_+)^{j-s(\lambda)+1}E(z,P_-)^{-j+s(\lambda)}\prod_{i=1}^n E(z,P_i)^{\tilde{\alpha}_i}\sigma(z)^{2\lambda-1}}, \tag{3.29}$$

and

$$C(z)\;=\;\sum_j C_j g^j_{(1-\lambda)}(z).$$

The vacuum $|0>$ in eq.(3.27) is defined by the analog of eq.(3.24) and since in eqs.(3.28-29) $j\in\mathbb{Z}+s(\lambda)$ it corresponds to $|0>_1$. The requirement that $g_j^{(\lambda)}$ be of weight λ in z gives the constraint

$$\sum_{i=1}^n\tilde{\alpha}_i=(2\lambda-1)(g-1). \tag{3.30}$$

From this equation we can see the topological origin of the constraint over the total charge of the V-fields.

The propagator of the $B-C$ system is

$$S(z,w)\equiv<B(z)C(w)>=$$

$$\frac{1}{E(z,w)}\prod_{i=1}^n\left(\frac{E(z,P_i)}{E(w,P_i)}\right)^{\tilde{\alpha}_i}\left(\frac{\sigma(z)}{\sigma(w)}\right)^{2\lambda-1}\frac{\theta[^{\delta}_{\epsilon}](z-w+\sum_{i=1}^n\tilde{\alpha}_iP_i-(2\lambda-1)\Delta)}{\theta[^{\delta}_{\epsilon}](\sum_{i=1}^n\tilde{\alpha}_iP_i-(2\lambda-1)\Delta)}. \tag{3.31}$$

3.4 SPIN FIELD CORRELATION FUNCTIONS.

As an introduction to the calculation of V-field correlation functions in the next section, here we show how one can compute correlation functions of spin fields in a straightforward way. In particular we test the formalism introduced in the previous section by recovering the 2-point spin-field insertions in the $\lambda = \frac{1}{2}$ and in the commuting $\lambda = \frac{3}{2}$ cases [69,70]. We then generalize these calculations to rational λ. We recall that λ-spin fields are a particular case of V-fields. We start with the $\lambda = \frac{1}{2}$ case. Let $S^{\pm}(P_{\pm})$ be the spin-field connecting the Neveu-Schwarz with the Ramond vacuum(*):

$$S^{(2)}(z,w) = {}_2\!<0|\mathcal{R}(b(z)c(w))|0>_2 = \frac{< S^-(P_-)b(z)c(w)S^+(P_+) >}{< S^-(P_-)S^+(P_+) >}. \tag{3.32}$$

Once the propagator $S^{(2)}(z,w)$ is known, the correlation function $< S^-(P_-)S^+(P_+) >$ can be computed with a procedure similar to the one introduced by Dixon et al. in [59].

The normal ordered stress energy tensor $T(z)$ of a $b-c$ system of weight λ is

$$T(z) = (1-\lambda)\partial bc - \lambda b\partial c. \tag{3.33}$$

By means of the operator product expansion

$$< T(z)V^1(P_1)...V^n(P_n) >= \sum_i \left(\frac{h_i}{(z-P)^2} + \frac{\partial_{P_i}}{(z-P_i)} \right) < V^1(P_1)...V^n(P_n) > \tag{3.34}$$

we obtain a set of first order differential equations which can be solved to get the correlator $< V^1(P_1)...V^n(P_n) >$.

For $\lambda = \frac{1}{2}$ we have the following correlation function for the stress-energy tensor in presence of spin-field insertions:

$$\frac{< S^-(P_-)T(z)S^+(P_+) >}{< S^-(P_-)S^+(P_+) >} = \lim_{z\to w} \frac{1}{2}(\partial_z - \partial_w)\left(S^{(2)}(z,w) - \frac{1}{(z-w)} \right) \tag{3.35}$$

with

$$S^{(2)}(z,w) = \frac{1}{E(z,w)} \left(\frac{E(z,P_+)E(w,P_-)}{E(w,P_+)E(z,P_-)} \right)^{\frac{1}{2}} \frac{\theta[^{\delta}_{\epsilon}](z-w+\frac{1}{2}(P_+-P_-))}{\theta[^{\delta}_{\epsilon}](\frac{1}{2}(P_+-P_-))} \tag{3.36}$$

(*) For brevity here we do not consider odd spin structures.

We obtain

$$< S^-(P_-)S^+(P_+) >= K_{\delta,\epsilon}E(P_-,P_+)^{-\frac{1}{4}}\theta[^{\delta}_{\epsilon}](\frac{1}{2}(P_+ - P_-)) \tag{3.37}$$

where $K_{\delta,\epsilon}$ is an integration constant which carries a dependence on the spin structure. The conformal weight of the spin-fields $S^{\pm}(P_{\pm})$ is $\frac{1}{8}$ as can be seen in two independent ways: either by looking at the residue in the leading singularity in (4.4) or directly from the geometrical weight in eq. (3.37).

Before addressing the commuting $\lambda = \frac{3}{2}$ case, we consider the anticommuting $\lambda = \frac{3}{2}$ system. In order for the propagator to be non vanishing we must put $2g-2$ insertions of b zero-modes at the points P_i. Therefore the propagator for the $\lambda = \frac{3}{2}$ case is

$$\tilde{S}^{(2)}(z,w) =_2< 0|\mathcal{R}(b(z)c(w))|0 >_2= \frac{< S^-(P_-)b(z)c(w)\prod_{i=1}^{2g-2} b(z_i)S^+(P_+) >}{< S^-(P_-)\prod_{i=1}^{2g-2} b(z_i)S^+(P_+) >} \tag{3.38}$$

where

$$\tilde{S}^{(2)}(z,w) = \frac{1}{E(z,w)}\left(\frac{E(z,P_+)E(w,P_-)}{E(w,P_+)E(z,P_-)}\right)^{\frac{1}{2}}\prod_{i=1}^{2g-2}\frac{E(z,P_i)}{E(w,P_i)}\left(\frac{\sigma(z)}{\sigma(w)}\right)^2\frac{\theta[^{\delta}_{\epsilon}](z-w+u)}{\theta[^{\delta}_{\epsilon}](u)} \tag{3.39}$$

with $u = \frac{1}{2}(P_+ - P_-) + \sum_{i=1}^{2g-2} P_i - 2\Delta$.

The same procedure outlined above allows us to compute the correlation function with zero-modes and spin-field insertions:

$$\begin{aligned} &< S^-(P_-)b(P_1)...b(P_{2g-2})S^+(P_+) >= K_{\delta,\epsilon}\frac{\sigma(P_+)}{\sigma(P_-)}\prod_i \sigma(P_i)^2 E(P_-,P_+)^{-\frac{1}{4}}\cdot \\ &\cdot\prod_{j,k} E(P_-,P_j)^{-\frac{1}{2}}E(P_k,P_+)^{\frac{1}{2}}\prod_{l<m} E(P_l,P_m)\theta[^{\delta}_{\epsilon}](\frac{1}{2}(P_+ - P_-) + \sum_i P_i - 2\Delta). \end{aligned} \tag{3.40}$$

In P_-,P_+ the conformal weight is respectively $-\frac{3}{8}$ and $\frac{5}{8}$. The $2g-2$ points P_i represent the zero-mode insertions, as can be seen by noting that the conformal weight at P_i is precisely $\frac{3}{2}$.

Let us now discuss the commuting case. The quantization of a generic commuting $\beta-\gamma$ system of weight λ and $1-\lambda$ respectively is achieved by imposing the commutation relations

$$[\beta^{i,l},\gamma^l_j]=\delta^i_j, \qquad [\beta^{i,l},\beta^{j,l}]=0=[\gamma^l_i,\gamma^l_j]. \tag{3.41}$$

The bra and ket vacua of the bosonic system are assumed to satisfy the relation (3.14) as for the anticommuting case.

Once a specific choice of the zero-mode insertions is made, the bosonic propagator

$$\bar{S}^{(l)}(z,w)\equiv_l <0|\mathcal{R}(\beta^{(l)}(z)\gamma^{(l)}(w))|0>_l,$$

coincides with the fermionic one

$$S^{(l)}(z,w)\equiv_l <0|\mathcal{R}(b^{(l)}(z)c^{(l)}(w))|0>_l .$$

The regularized bosonic stress energy tensor is

$$T(z)=-\lim_{z\to w}((1-\lambda)\partial_z-\lambda\partial_w)\left(\beta(z)\gamma(w)-\frac{1}{z-w}\right). \tag{3.42}$$

For $\lambda=\frac{3}{2}$ a possible expression for the bosonic propagator is the one given in eq. (3.39). The points P_i represent the insertion of vertex operators $V_1(P_i)$ which carry a charge 1. These vertex operators can no longer (as in the fermionic case) be considered as zero-mode insertions because their conformal weight, as shown below, is $-\frac{3}{2}$ (*). Starting from the correlator of the stress-energy tensor with vertex operators V_1 and spin-field insertions

$$\frac{<S^-(P_-)\left(\prod_{i=1}^{2g-2}V_1(P_i)\right)T(z)S^+(P_+)>}{<S^-(P_-)\left(\prod_{i=1}^{2g-2}V_1(P_i)\right)S^+(P_+)>},$$

we get the correlation function:

$$\begin{aligned}<S^-(P_-)V_1(P_1)...V_1(P_{2g-2})S^+(P_+)>&=K_{\delta,\epsilon}\frac{\sigma(P_-)}{\sigma(P_+)}\prod_i\sigma(P_i)^{-2}E(P_-,P_+)^{\frac{1}{4}}\cdot\\ \cdot\prod_{j,k}E(P_-,P_j)^{\frac{1}{2}}E(P_k,P_+)^{-\frac{1}{2}}&\prod_{l<m}E(P_l,P_m)^{-1}\frac{1}{\theta[^{\delta}_{\epsilon}](\frac{1}{2}(P_+-P_-)+\sum_i P_i-2\Delta)}.\end{aligned} \tag{3.43}$$

In P_-,P_+ the conformal weight is respectively $\frac{3}{8}$ and $-\frac{5}{8}$. In order to compare our result with Atick-Sen's [69,70] we notice that they absorbed the extra charge by inserting, instead

(*) In the path integral approach these vertex operators are represented by a delta function $\delta(\beta)$.

of $2g-2$ vertex operators with charge 1, $g-1$ vertex operators V_2 with charge 2 (which turn out to have conformal weight -4)$^{(*)}$. In our formalism their result can be recovered starting from the propagator

$$\tilde{S}^{(2)}(z,w) = \frac{1}{E(z,w)} \left(\frac{E(z,P_+)E(w,P_-)}{E(w,P_+)E(z,P_-)} \right)^{\frac{1}{2}} \prod_{i=1}^{g-1} \left(\frac{E(z,P_i)}{E(w,P_i)} \right)^2 \left(\frac{\sigma(z)}{\sigma(w)} \right)^2 \frac{\theta[^{\delta}_{\epsilon}](z-w+u)}{\theta[^{\delta}_{\epsilon}](u)} \tag{3.44}$$

with $u = \frac{1}{2}(P_+ - P_-) + 2\sum_{i=1}^{g-1} P_i - 2\Delta$. With this choice the correlator is

$$\begin{aligned} &< S^-(P_-)V_2(P_1)...V_2(P_{g-1})S^+(P_+) >= K_{\delta,\epsilon} \frac{\sigma(P_-)}{\sigma(P_+)} \prod_i \sigma(P_i)^{-4} E(P_-,P_+)^{\frac{1}{4}} \cdot \\ &\cdot \prod_{j,k} E(P_-,P_j)E(P_k,P_+)^{-1} \prod_{l<m} E(P_l,P_m)^{-4} \frac{1}{\theta[^{\delta}_{\epsilon}](\frac{1}{2}(P_+-P_-)+2\sum_i P_i - 2\Delta)}. \end{aligned} \tag{3.45}$$

This result completely agrees with Atick and Sen's. From the discussion carried out in section 3, it is clear that our approach allows us to compute the correlation function for an arbitrary number of spin-field insertions and, moreover, that the vertex operators, which must be inserted in order to fulfill the condition (3.30) on the total charge, can be placed in general position and assumed to have the most general charge. The corresponding formulas can be obtained by specializing the expressions given in the next section for real λ, eq.(3.56).

In this section however we limit ourselves to give some more examples with rational λ. The general expression for the spin-field correlation functions in the case of anticommuting $b-c$ systems of half-integer weight $\frac{N}{2}$, N odd, is:

$^{(*)}$ The relation between the conformal weight and the charge in the case of a fermionic $b-c$ system of weight λ is shown in the next section and it is given by eq. (3.53). We remark here that since the stress-energy tensor for the bosonic system differs by a minus sign from the fermionic one, in the bosonic case the relation between conformal weight and charge is just the opposite: $h_{\tilde{\alpha}_i} = -\frac{1}{2}\tilde{\alpha}_i^2 + \frac{1}{2}(1-2\lambda)\tilde{\alpha}_i$. An immediate consequence of this fact is that only in a certain range of values for λ is it possible to introduce real charge operators which reproduce the $(\lambda, 1-\lambda)$ $\beta-\gamma$ system.

$$
\begin{aligned}
&< S^+(P_1)...S^+(P_q)b(Q_1)...b(Q_r)c(R_1)...c(R_s)S^-(T_1)...S^-(T_t) >= \\
&= K_{\delta,\epsilon} \prod_i \sigma(P_i)^{\frac{1}{2}(N-1)} \prod_j \sigma(Q_j)^{N-1} \prod_k \sigma(R_k)^{1-N} \prod_l \sigma(T_l)^{\frac{1}{2}(1-N)}. \\
&\cdot \prod_{i<j} E(P_i,P_j)^{\frac{1}{4}} \prod_{k<l} E(Q_k,Q_l) \prod_{m<n} E(R_m,R_n) \prod_{p<q} E(T_p,T_q)^{\frac{1}{4}}. \\
&\cdot \prod_{i,j,k,l} E(P_i,Q_j)^{\frac{1}{2}} E(P_i,R_k)^{-\frac{1}{2}} E(P_i,T_l)^{-\frac{1}{4}} E(Q_j,R_k)^{-1} E(Q_j,T_l)^{-\frac{1}{2}} E(R_k,T_l)^{\frac{1}{2}}. \\
&\cdot \theta[{}^{\delta}_{\epsilon}](\frac{1}{2}(\sum_i P_i - \sum_l T_l) + \sum_j Q_j - \sum_k R_k - (N-1)\Delta).
\end{aligned}
\tag{3.46}
$$

The charge conservation condition requires the constraint

$$\frac{1}{2}(q-t)+r-s=(N-1)(g-1)$$

to be satisfied, otherwise the correlation function vanishes. The spin-fields $S^{\pm}$ have charge $\pm\frac{1}{2}$ and their conformal weight is $\frac{2N-1}{8}$ for S^+ and $\frac{3-2N}{8}$ for S^-.

Likewise the general spin-field correlation function for a commuting $\beta-\gamma$ system with $\lambda=\frac{N}{2}$ is:

$$
\begin{aligned}
&< S^+(P_1)...S^+(P_q)V(Q)S^-(T_1)...S^-(T_s) >= \\
&= K_{\delta,\epsilon} \prod_i \sigma(P_i)^{-\frac{1}{2}(N-1)} \sigma(Q)^{\gamma(1-N)} \prod_j \sigma(T_j)^{\frac{1}{2}(N-1)}. \\
&\cdot \prod_{i<j} E(P_i,P_j)^{-\frac{1}{4}} \prod_{k<l} E(T_k,T_l)^{-\frac{1}{4}} \prod_m E(P_m,Q)^{-\frac{1}{2}\gamma} \prod_n E(T_n,Q)^{\frac{1}{2}\gamma} \prod_{t,u} E(P_t,T_u)^{\frac{1}{4}}. \\
&\cdot \left[\theta[{}^{\delta}_{\epsilon}](\frac{1}{2}(\sum_i P_i - \sum_l T_l) + \gamma Q - (N-1)\Delta)\right]^{-1}.
\end{aligned}
\tag{3.47}
$$

The spin-fields $S^{\pm}$ have charge $\pm\frac{1}{2}$ and conformal weight $\frac{1-2N}{8}$ for S^+ and $\frac{2n-3}{8}$ for S^-. The extra charge has been absorbed by inserting at the point Q a single operator V, whose charge γ is given by

$$\gamma=\frac{2(N-1)(g-1)+s-q}{2}. \tag{3.48}$$

Finally we consider the case of a fermionic $b-c$ system of rational weight $\lambda=\frac{N}{M}$, with N,M relatively prime integers. The λ-spin fields with charge $\pm k, k=\frac{1}{M},...,\frac{M-1}{M}$ are denoted by $S_k^{\pm}$. These fields, of course, live on a suitable covering of the Riemann surface

Σ. The correlation function with the insertion of $N_{\pm k}$ λ-spin fields $S_k^{\pm}$ at the points $P_{\pm k,i}$, where $i_k = 1, ..., N_{\pm k}$, is given by:

$$
\begin{aligned}
&< S_1^+(P_{1,1})..S_1^+(P_{1,N_1})S_2^+(P_{2,1})..S_{M-1}^+(P_{M-1,N_{M-1}})b(Q_1)..b(Q_r)\cdot \\
&\cdot c(R_1)..c(R_s)S_{M-1}^-(P_{M-1,1})..S_1^-(P_{-1,N_{-1}}) >= \\
&= K_{\delta,\epsilon}[\prod_k \prod_{l=1,...,N_{\pm k}} \sigma(P_{\pm k,l})^{\pm(2N-M)\frac{k}{M^2}}] \prod_i \sigma(Q_i)^{\frac{2N-M}{M}} \prod_j \sigma(R_j)^{\frac{M-2N}{M}}\cdot \\
&\cdot [\prod_{k,i}\prod_{k',j} E(P_{\pm k,i}, P_{\mp k',j})^{-\frac{kk'}{2M^2}}] \prod_{k,l,m} E(P_{\pm k,l}, Q_m)^{\pm\frac{k}{M}} \prod_{k,n,q} E(P_{\pm k,n}, R_q)^{\mp\frac{k}{M}}\cdot \\
&\cdot \prod_{(k,i)\neq(k',j)} E(P_{\pm k,i}, P_{\pm k',j})^{\frac{kk'}{2M^2}} \prod_{l<m} E(Q_l, Q_m) \prod_{n<p} E(R_n, R_p)\cdot \\
&\cdot \theta[{}^{\delta}_{\epsilon}]\left(\pm\sum_{k,i}\frac{k}{M}P_{\pm k,i} + \sum_j Q_j - \sum_l R_l - \frac{(2N-M)}{M}\Delta\right).
\end{aligned}
\tag{3.49}
$$

The correlator is non-vanishing only if the constraint

$$
\frac{2N-M}{M}(g-1) = r - s + \sum_{k=1}^{M-1}(\frac{k}{M}N_{+k} - \frac{k}{M}N_{-k}) \tag{3.50}
$$

is satisfied. The conformal weight of $S_k^{\pm k}$ is given by

$$
\frac{k^2 \mp (M-2N)k}{2M^2}. \tag{3.51}
$$

The correlation functions calculated so far are in general non-single-valued as the points, where the insertions occur, are shifted by a homology cycle. So the above formulas are to be considered as starting points where single-valuedness and modular invariance are still to be implemented.

3.5 VERTEX INSERTION CORRELATION FUNCTIONS FOR MINIMAL MODELS.

In this section we return to the general case. We consider the "fat" $b-c$ systems defined in section 3 (eqs. (3.27)-(3.31)) and apply to them the method applied in the last section to a few simple cases. We will use

$$\frac{< T(z)V^1(P_1)\dots V^n(P_n) >}{< V^1(P_1)\dots V^n(P_n) >} = \lim_{z\to w}((1-\lambda)\partial_z - \lambda\partial_w)\left(S(z,w) - \frac{1}{z-w}\right) \tag{3.52}$$

where $S(z,w)$ is given by eq. (3.31).

Analyzing the residues of the leading and subleading singularities at P_i, and comparing them with the OPE of $T(z)V^i(P_i)$ we are able to identify the weights h_i of the fields V^i and to extract differential equations which allow us to determine the form of $< \prod_i V^i(P_i) >$. Specifically we obtain the relation which determines the conformal weight in terms of the charge $\tilde{\alpha}_i$:

$$h_i \equiv h_{\tilde{\alpha}_i} = \frac{1}{2}\tilde{\alpha}_i^2 - \frac{1}{2}(1-2\lambda)\tilde{\alpha}_i, \tag{3.53}$$

and

$$\partial_{P_i} ln < \prod_k V^k(P_k) > = (2\lambda-1)\tilde{\alpha}_i\frac{\sigma'(P_i)}{\sigma(P_i)} + \tilde{\alpha}_i\frac{\theta'[^{\delta}_{\epsilon}](v)}{\theta[^{\delta}_{\epsilon}](v)} + \sum_{i\neq j}\tilde{\alpha}_i\tilde{\alpha}_j\frac{E'(P_i,P_j)}{E(P_i,P_j)}, \tag{3.54}$$

where

$$v = \sum_i \tilde{\alpha}_i P_i - (2\lambda-1)\Delta. \tag{3.55}$$

Integrating (3.54) we obtain

$$< \prod_k V^k(P_k) > = K_{\delta,\epsilon}\prod_k \sigma(P_k)^{\tilde{\alpha}_k(2\lambda-1)}\prod_{i<j}E(P_i,P_j)^{\tilde{\alpha}_i\tilde{\alpha}_j}\theta[^{\delta}_{\epsilon}](v) \tag{3.56}$$

where $K_{\delta,\epsilon}$ is an integration constant.

As for eq.(3.53) we remark that the weight of V^i can be obtained directly from eq.(3.56) by calculating the conformal weight of the RHS at the points P_i and applying the constraint (3.30).

In general the correlation functions given by eq.(3.56) are not single-valued on Σ. They have cuts with endpoints P_i as expected. But they, in general, pick up a phase and the theta characteristics are shifted when P_i winds around a homology cycle. Precisely, when $P_i \to P_i + na + mb$, eq.(3.56) goes over to

$$K_{\delta,\epsilon} \prod_k \sigma(P_k)^{\tilde{\alpha}_k(2\lambda-1)} \prod_{i<j} E(P_i,P_j)^{\tilde{\alpha}_i\tilde{\alpha}_j} e^{-2\pi i\tilde{\alpha}_i^2 mn - 2\pi i\tilde{\alpha}_i m\epsilon} \theta[{}^{\delta+\tilde{\alpha}_i m}_{\epsilon+\tilde{\alpha}_i n}](v). \tag{3.57}$$

The phase and the shifts can be rational or irrational depending on the values taken by $\tilde{\alpha}_i, \epsilon$ and δ. In order to carry on the discussion of this very important point we need to be more specific, so let us refer as in ch. 2 to the minimal models.

Let us add the information that $B-C$ systems represent minimal models. It has already been shown indeed that in genus 0 "fat" $b-c$ systems constitute a dual (fermionized) version of the Coulomb gas approach, so that by the use of $B-C$ systems we can represent primary fields and reconstruct their correlation functions. The correspondence between the $b-c$ system formalism and Dotsenko-Fateev's Coulomb gas approach is established by means of the identifications

$$\tilde{\alpha}_i = \sqrt{2}\alpha_i, \qquad \tilde{\alpha}_0 \equiv \frac{1}{2}(1-2\lambda) = \sqrt{2}\alpha_0, \tag{3.58}$$

and the value of λ suitable for a given minimal model is obtained by equating the central charges

$$c(\lambda) = -12\lambda^2 + 12\lambda - 2 = 1 - 24\alpha_0^2 \equiv 1 - \frac{6(p-q)^2}{pq}.$$

Here α_0 and α_i are the same symbols appearing in [1]. So for the minimal model identified by the integers p and q we have

$$\lambda = \frac{1}{2} + \frac{p-q}{\sqrt{2pq}}. \tag{3.59}$$

As we have already seen, in general λ is irrational. Incidentally we remark that exchanging p and q turns λ into $1-\lambda$. This is the way the electromagnetic duality of the bosonic formulation is recovered in our formalism. We remember that the constraint (3.30) on the bases implies that the V-fields correlation functions are non-vanishing only if the total charge $\sum_i \tilde{\alpha}_i$ satisfies the relation:

$$\sum_i \tilde{\alpha}_i = -2\tilde{\alpha}_0(g-1). \tag{3.60}$$

We can now introduce screening charges, i.e. weight one V-fields that by eq.(3.53) have charge

$$\tilde{\alpha}_{\pm} = \tilde{\alpha}_0 \pm \sqrt{\tilde{\alpha}_0^2 + 2}. \tag{3.61}$$

Eqs.(3.60) and (3.61) imply charge quantization, the same as in the case $g = 0$:

$$\tilde{\alpha}_{r,s} = \frac{1}{2}(1-r)\tilde{\alpha}_+ + \frac{1}{2}(1-s)\tilde{\alpha}_- = (1-r)\sqrt{\frac{q}{2p}} - (1-s)\sqrt{\frac{p}{2q}}, \tag{3.62}$$

where $1 \leq r \leq p-1, 1 \leq s \leq q-1$. We remark that in higher genus, w.r.t. the genus 0 case, an extra charge $2\tilde{\alpha}_0 g$ must be reabsorbed (we can think of $2\tilde{\alpha}_0(g-1)$ as a "bra vacuum charge"). In general this can easily be done by inserting in the correlation function an appropriate number of screening charges and relative contour integrations; this is possible thanks to the fact that for the $\tilde{\alpha}_i's$ given by eqs.(3.58-60) the following relation holds

$$(p-1)\tilde{\alpha}_+ + (q-1)\tilde{\alpha}_- = -2\tilde{\alpha}_0 = \sqrt{2}\frac{p-q}{\sqrt{pq}}, \tag{3.63}$$

with the parametrization of $\tilde{\alpha}_i$ and λ given by eqs. (3.57-59).

For the sake of simplicity, from now on, we will be dealing only with unitary minimal models. The relevant formulas are obtained from the previous ones by the substitutions $q \to p$ and $p \to p+1$.

Let us now return to eq. (3.56) with the parametrization of $\tilde{\alpha}_i$ and λ given by eqs. (3.62) and (3.59), respectively, and the above substitutions. Looking at eq.(3.57) we see that the shifts of the theta-characteristics are in general irrational. This fact is not surprising since it is inherited from the bases (3.28) we started from: they are well-behaved as far as the z-dependence is concerned, but not with respect to the P_i's (well-behaved meaning that the relevant shifts along homology cycles involve rational phases and shifts). Now, if phases and shifts in (3.57) were rational (as they are in some particular cases), we could take the attitude of the previous section and consider eq.(3.56) as our basic result, from which single-valued quantities can be calculated by taking linear combinations of the RHS of (3.56) with suitable coefficients $K_{\delta,\epsilon}$. Since this is not the case, we take another attitude (see however the remark at the end of this section): we change the bases (3.28). In order to construct the new bases we argue as follows.

When a P_i, i.e. a point where a vertex is inserted, winds around a homology cycle, in general the theta-characteristics change and a phase appears. The phase and the theta-characteristic shifts must correspond to the behaviour of the vertex insertions at the points P_i. We describe the latter in the following way: we say that there exists a finite covering Σ' of the Riemann surface Σ, such that on Σ' the correlation functions $< \prod_k V^k(P_k) >$ are single-valued (up to a possible phase depending only on the θ-characteristics). We will see that this simple statement will allow us not only to characterize the covering, but also to determine the θ-characteristics δ and ϵ.

To see this let us modify the bases (3.28) as follows

$$g_j^{(\lambda)}[{}^{\delta}_{\epsilon}](z,P_1,...,P_n) = \frac{E(z,P_+)^{j-s(\lambda)}}{E(z,P_-)^{j-s(\lambda)+1}} \prod_{i=1}^{n} \left(\frac{E(P_i,P_+)^{j-s(\lambda)}}{E(P_i,P_-)^{j-s(\lambda)+1}} \right)^{\tilde{\alpha}_i} \prod_{i=1}^{n} E(z,P_i)^{\tilde{\alpha}_i} \cdot$$

$$\cdot \prod_{i\leq j}^{n} E(P_i,P_j)^{\tilde{\alpha}_i\tilde{\alpha}_j} \sigma(z)^{2\lambda-1} \prod_{i=1}^{n} \sigma(P_i)^{\tilde{\alpha}_i(2\lambda-1)} \theta[{}^{\delta}_{\epsilon}](cz+cv|d\Omega), \qquad c,d \in \mathbb{R}, \qquad (3.64)$$

where $u = (j-s(\lambda))P_+ - (j-s(\lambda)+1)P_- + \sum_{i=1}^{n} \tilde{\alpha}_i P_i + (1-2\lambda)\Delta)$, $j \in \mathbb{Z} + s(\lambda)$, and c, d are numerical constants to be determined. We recover the bases (3.28) by setting $c = d = 1$, up to a normalization. This normalization is actually very important since it allows us to put the conformal weights $h_{\tilde{\alpha}_i} = \frac{1}{2}\tilde{\alpha}_i^2 - \frac{1}{2}(1-2\lambda)\tilde{\alpha}_i$ at the points P_i. We point out that this normalization is not *ad hoc*: it is possible only if condition (3.30) is satisfied, being thus another indication of the geometrical consistency of our method.

Our aim now is to determine c and d in such a way that the bases be single-valued (up to theta-characteristic dependent phases) when one of the P_i winds around a homology cycle of the covering Σ'. This will allow us to determine also δ and ϵ. The detailed derivation is contained in Appendix B. Here we write down the result.

Using the freedom in the choice of theta-function basis, pointed out in Appendix B, we can write the new bases (3.64) in the form:

$$g_j^{(\lambda)}[{}^{\delta}_{\epsilon}](z,P_1,...,P_n) = \frac{E(z,P_+)^{j-s(\lambda)}}{E(z,P_-)^{j-s(\lambda)+1}} \prod_{i=1}^{n} \left(\frac{E(P_i,P_+)^{j-s(\lambda)}}{E(P_i,P_-)^{j-s(\lambda)+1}} \right)^{\tilde{\alpha}_i} \prod_{i=1}^{n} E(z,P_i)^{\tilde{\alpha}_i} \cdot$$

$$\cdot \prod_{i\leq j}^{n} E(P_i,P_j)^{\tilde{\alpha}_i\tilde{\alpha}_j} \sigma(z)^{2\lambda-1} \prod_{i=1}^{n} \sigma(P_i)^{\tilde{\alpha}_i(2\lambda-1)} \theta[{}^{u}_{0}{}^{\frac{u}{2p(p+1)}}](\sqrt{2p(p+1)}(z+v)|2p(p+1)\Omega)$$

$$(3.65).$$

where $u = 0, 1, ..., 2p(p+1) - 1$. These bases are characterized by the fact that when any P_i winds $2p(p+1)$ times around any homology cycle of Σ, they remain unchanged. This is not true for the z-dependence: the effect of a z-shift along the homology cycles implies an irrational shift both for the δ and ϵ characteristics. Evidently the z-dependence can be understood only in terms of an infinite covering of Σ. The propagator of the new $b - c$ system is given by

$$\mathcal{S}(z,w) = \frac{1}{E(z,w)} \prod_{i=1}^{n} \left(\frac{E(z,P_i)}{E(w,P_i)} \right)^{\tilde{\alpha}_i} \left(\frac{\sigma(z)}{\sigma(w)} \right)^{2\lambda-1} \cdot$$
$$\cdot \frac{\theta[\overset{u}{\underset{0}{\overset{}{\underset{}{}}}}_{2p(p+1)}](\sqrt{2p(p+1)}(z - w + \sum_{i=1}^{n} \tilde{\alpha}_i P_i - (2\lambda - 1)\Delta)|2p(p+1)\Omega)}{\theta[\overset{u}{\underset{0}{\overset{}{\underset{}{}}}}_{2p(p+1)}](\sqrt{2p(p+1)}(\sum_{i=1}^{n} \tilde{\alpha}_i P_i - (2\lambda - 1))\Delta)|2p(p+1)\Omega)} \cdot \tag{3.66}$$

Redoing now the calculations from (3.52) to (3.56) with these new bases we find

$$< \prod_k V^k(P_k) >= K_\delta \prod_k \sigma(P_k)^{\tilde{\alpha}_k(2\lambda-1)} \prod_{i<j} E(P_i,P_j)^{\tilde{\alpha}_i\tilde{\alpha}_j} \theta[\overset{\delta}{\underset{0}{\overset{}{\underset{}{}}}}](y|2p(p+1)\Omega), \tag{3.67}$$

where v is given by:

$$v = \sqrt{2p(p+1)}(\Sigma_j \tilde{\alpha}_j P_j + (1 - 2\lambda)\Delta),$$

and the characteristic δ is of the form $\frac{u}{2p(p+1)}$ where $u = 0, ..., 2p(p+1) - 1$.

A few comments are in order. The result (3.67) is now single-valued when any P_i winds $n = 2p(p+1)$ times around any homology cycle. We have already noticed that this in general cannot be true for any integer n for the formula (3.56). However it is likely that one can recover eq.(3.67) starting from the formula (3.56), by means of an averaging procedure over the theta characteristics similar to the one outlined in the third article of ref. [21].

An interesting question concerns the possible relation between our bases and the bases for the Krichever-Novikov algebra with more punctures described in [71-73].

Eq.(3.67) is our final result in this section. It is the higher genus counterpart of eq. (1.50).

A similar result was obtained via the bosonic approach in ref. [74] (see also [24, 75-83]). It can be taken as a starting point for computing conformal blocks of minimal models in a generic Riemann surface.

Of course, when we want to compute the full correlation functions, we have to put together the chiral and the antichiral pieces, as discussed in ch. 1. Such a combination must satisfy some properties, like monodromy invariance, which give rise to constraint on the constant factor K_δ. Besides that contour integral are still to be carried on when one deals with minimal models.

We point out that our formalism allows us to compute not only the correlation functions, but also the partition functions. Take for instance the case $g = 1$, which is a favourite one since in this case one does not need to insert any screening charge. The chiral part of the partition function can be obtained by inserting the identity operator 1 as follows:

$$< 1 >= \lim_{\tilde{\alpha} \to 0} < V_{\tilde{\alpha}} V_{-\tilde{\alpha}} > .$$

The full partition function (chiral and antichiral pieces together) has to be determined satisfying the modular invariance.

A final remark concerns the Ising model. In our framework the Ising model is exceptional because it can be described in two different ways. In one procedure one treats it on the same foot as any other conformal minimal model. This means that one has simply to specialize the formulae given above to the particular value $p = 3$ which gives $c = \frac{1}{2}$ as central charge.

There is anyway another procedure [84] which reflects the well-known fact that the Ising model is equivalent to a theory of Majorana fermions. Taking into account this fact, the Ising model can be described by a $\lambda = \frac{1}{2}$ $b - c$ system, where now the fields are interpreted as Majorana.

Since a Dirac stress-energy tensor T_D is twice a Majorana stress- energy tensor T_M:

$$T_D(z) = 2T_M(z),$$

we can compute the Majorana correlation functions $< V...V >_{(\lambda=\frac{1}{2},M)}$ which describe our Ising model, by means of the Dirac $\lambda = \frac{1}{2}$ correlation functions $< V...V >_{(\lambda=\frac{1}{2},D)}$ computed above.

The relation between the two is the following

$$< V...V >_{(\lambda=\frac{1}{2},M)} = \sqrt{< V...V >_{(\lambda=\frac{1}{2},D)}} \tag{3.68}$$

For instance the Ising partition function Z on the torus can be expressed as

$$Z = \frac{1}{2|\eta|}\left(|\theta[^{\frac{1}{2}}_{0}](0|\tau) + |\theta[^{0}_{\frac{1}{2}}](0|\tau) + |\theta[^{\frac{1}{2}}_{\frac{1}{2}}](0|\tau)\right) \tag{3.69}$$

where

$$\eta(\tau) = \tau^{\frac{1}{24}} \prod_{n=0}^{\infty}(1-\tau^n)$$

is the Dedekind η function.

4 THE HIGHER GENUS VERTEX OPERATOR

4.1 INTRODUCTION.

In the previous chapter we have elaborated a system which allows to compute the chiral blocks of primary fields correlation functions in higher genus Riemann surfaces. A prescription was also outlined for computing, starting with these blocks, the minimal models correlation functions.

Just like the CFT on the complex plane, a full understanding of the minimal models on higher genus Riemann surfaces can not however be obtained unless we are able to specify their operatorial content.

The oldest operatorial formulations of conformal field theories [85–89] over generic Riemann surfaces privileged the local description of CFT over a disk cut out from the Riemann surface. The globalization was essentially obtained (gluing or sewing procedure) via Bogoljubov transformations relating states over the disk to states over the Riemann surface without disk.

Here we tackle the problem of constructing the operatorial formalism from a different point of view, namely privileging from the very beginning the global data. To do so we make use of the Krichever-Novikov formalism [2,58,90–92] (already encountered in ch. 3) in which the meromorphic tensor fields on genus g Riemann surface holomorphic outside the points P_+ and P_- are considered.*.

The Krichever-Novikov bases over which such meromorphic tensor fields are expanded play the same role as the monomials z^n over the sphere ; therefore they provide a mean to perform a Laurent-like expansion of any tensor field on the genus g Riemann surface; such an expansion turns out to be globally defined. As we have already pointed out in ch.3 we can mimic for the case at hand the conformal field theory over a sphere.

---

* Maybe that a natural framework for studying the correlation functions with insertions of vertices at the points P_i makes use of the system of bases holomorphic outside P_i. See the remark at the end of ch. 3.

The conformal field theories on the sphere are characterized by their representations of the Virasoro algebra (1.5). In higher genus Riemann surface they should be classified in terms of the Krichever-Novikov algebra. Loosely speaking the KN algebra is the analogue of the Virasoro algebra, but now the circle S^1 is replaced by the equal-time curves C_τ of eq. (3.4). A Bogoljubov transformation connects locally the two algebras.

While the representation theory of the Virasoro algebra is a well established matter, the representation theory of the KN algebra is still an open problem. The results presented in this section are part of a long research project [93] whose aim is to set up a chiral vertex operator representation for the KN algebra. In this context the chiral vertex operators will be realized in terms of higher genus oscillators, along the same lines as shown in section 1.4 for the genus 0 case. Even if the work is still in progress we have some new interesting results which sheds light on these vertex operator representations.

The material of this chapter is organized in the following way: in sect. 2 we will review standard facts about the KN algebra and we will establish our notations. In sect. 3 the higher genus analogue of the Heisenberg (oscillator) algebra is introduced and its relevance in representing the KN algebra is discussed. Moreover a system of bases, different from the usual KN one, is introduced; such system allows a striking simplification of the oscillator algebra. In the next section the theory of a free chiral boson is considered; such a bosonic field allows us to arrive at the main result, namely the construction of a vertex operator which has a geometrical meaning, since its commutation relations with the Krichever-Novikov generators give rise to a Lie derivative. The transformation properties around the homology cycles for the vertex operator are discussed. The last section is devoted to draw the conclusion and to discuss the future perspectives.

4.2 THE KRICHEVER-NOVIKOV ALGEBRA.

In this chapter we will consider only a particular set of meromorphic tensor fields, namely the functions, the meromorphic differentials, the vector fields and the weight 2 differentials. The Krichever-Novikov bases over which these tensor fields are expanded are labelled respectively by A_J, ω^J, e_J and Ω^J.

The local behaviour of these bases around $P_\pm$, is obtained by specializing the equations (3.1) and (A.1,A.2). Since for later convenience we have to modify slightly the conventions used in ch.3 we report here their local behaviour:

$$\begin{aligned} A_j(z_\pm) &= a^\pm{}_{0,j}(z_\pm)^{\pm j-\frac{g}{2}}(1+O(z_\pm)) \qquad for \quad |j|>\frac{g}{2} \\ A_j(z_\pm) &= a^\pm{}_{0,j}(z_\pm)^{\pm j-\frac{g}{2}-(\frac{1\mp 1}{2})}(1+O(z_\pm)) \qquad for \quad -\frac{g}{2}\le j<\frac{g}{2} \end{aligned} \tag{4.1}$$

$A_{\frac{g}{2}}$ is taken to be =1.

$$\begin{aligned} \omega^j(z_\pm) &= a^\pm{}_{1,j}(z_\pm)^{\mp j+\frac{g}{2}-1}(1+O(z_\pm))(dz_\pm) \qquad for \quad |j|>\frac{g}{2} \\ \omega^j(z_\pm) &= a^\pm{}_{1,j}(z_\pm)^{\mp j+\frac{g}{2}-\frac{1\pm 1}{2}}(1+O(z_\pm))(dz_\pm) \qquad for \quad -\frac{g}{2}\le j<\frac{g}{2}. \end{aligned} \tag{4.2}$$

the ω^j with $-\frac{g}{2}\le j<\frac{g}{2}$ are holomorphic abelian differentials.

$\omega^{\frac{g}{2}}$ is assumed to be the abelian differential of the third kind with simple poles in $P_\pm$ and residues ± 1, normalized in such a way that the periods around the homology cycles be purely imaginary.

For the vectors we get

$$e_j(z_\pm) = a^\pm{}_{-1,j}(z_\pm)^{j-\frac{3}{2}g+1}(1+O(z_\pm))\frac{\partial}{\partial z_\pm} \tag{4.3}$$

Finally,

$$\Omega^j(z_\pm) = a^\pm{}_{2,j}(z_\pm)^{\mp j+\frac{3}{2}g-2}(1+O(z_\pm))(dz_\pm)^2 \tag{4.4}$$

The coefficients $a^+{}_{i,J}$ can be assumed to be normalized $=1$.

Let us call i_τ the map which restricts the tensor fields f^λ of weight λ, defined on the Riemann surface Σ, to the equal-time curves C_τ. The image of i_τ is dense in the space of the holomorphic tensor fields $f^\lambda{}_{C_\tau}$ defined on C_τ. Therefore any differentiable tensor field over C_τ can be developped over the relevant Krichever-Novikov bases.

The bases (4.1)-(4.4) satisfy a duality relation which reads as follows:

$$\begin{aligned} \delta_I{}^J &= \frac{1}{2\pi i}\oint_{C_\tau} e_I \Omega^J \\ \delta_I{}^J &= \frac{1}{2\pi i}\oint_{C_\tau} A_I \omega^J. \end{aligned} \tag{4.5}$$

The completeness of the bases guarantees that we can write a Dirac delta function $\Delta(Q,\bar{Q})$:

$$\Delta(Q,\bar{Q}) = \frac{1}{2\pi i}\sum_K A_K(Q)\omega^K(\bar{Q}) \tag{4.6}$$

The delta function must be understood as follows:

$$\begin{aligned} A_K(Q) &= \oint_{C_\tau} \Delta(Q,\bar{Q}) A_K(\bar{Q}) \\ \omega^K(\bar{Q}) &= \oint_{C_\tau} \omega^K(Q)\Delta(Q,\bar{Q}). \end{aligned} \tag{4.7}$$

A similar delta function can be constructed for vectors and 2- differentials replacing in the above formulas $A_K \to e_K$, $\omega^K \to \Omega^K$.

Let us now recall the notion of the Lie derivative L_e (e is a vector field); in local coordinates, when acting on a tensor field of weight λ, is defined through

$$L_e f^\lambda = e\frac{\partial f^\lambda}{\partial z} + \lambda f^\lambda \frac{\partial e}{\partial z} \tag{4.8}$$

In the r.h.s. the ordinary derivative can be replaced by the covariant derivative.

The meromorphic vector fields e_I satisfy an almost- graded Lie algebra L:*

$$[e_I, e_J] = L_{e_I} e_J = \sum_{-g_0}^{g_0} C^S{}_{IJ} e_{I+J-S} \tag{4.9}$$

where $g_0 = \frac{3}{2}g$ and the structure constants $C^S{}_{IJ}$ are expressed by

* An almost-graded Lie algebra L is an algebra which can be expanded in subspaces L_i: $L = \sum_i L_i$ and the L_i are such that $L_i L_j \subset \sum_{|k|\leq N} L_{i+j-k}$.

$$C^S{}_{IJ} = \frac{1}{2\pi i} \oint_{C_\tau} [e_I, e_J] \Omega^{I+J-S}. \tag{4.10}$$

Using the leading behaviour in the expansion (4.1-4.4) it is trivial to compute the value of $C^{g_0}_{IJ}$:

$$C^{g_0}_{IJ} = J - I.$$

The g_0 gradation structure is obtained because the structure constants are different from zero only in the range

$$I + J - g_0 \geq S \geq I + J + g_0,$$

as it can be easily understood from the polar structure of the tensor fields.

The algebra L admits subalgebras identified by the vector spaces

$$L_\pm{}^s = \{e_I : \pm I \geq g_0 - s\}.$$

The subset L_0 of the holomorphic vector fields with poles both in P_+ and P_- has dimension $3g-3$; it generates the non- trivial transformations of the Riemann surface and can be identified with the space tangent to the moduli.

The algebra L is the higher genus extension of the algebra of the vector fields over S^1; just like the vector fields algebra on S^1 L admits a unique (up to trivial cocycles) central extension, the KN algebra:

$$\begin{aligned} [L_I, L_J] &= \sum_S C^S{}_{IJ} e_{I+J-S} + \frac{c}{12} t \chi_{IJ} \\ [L_I, t] &= 0. \end{aligned} \tag{4.11}$$

The cocycle χ_{IJ} is given by

$$\chi_{IJ} = \frac{1}{2\pi i} \oint_{C_\tau} \frac{1}{2} \partial^3 \bar{e_I} \bar{e_J} - (I \leftrightarrow J) - R[(\partial \bar{e_I} \bar{e_J}) - (I \leftrightarrow J)] \tag{4.12}$$

$(e_I \equiv \bar{e_I} \frac{\partial}{\partial z})$.

R is the schwarzian connection, holomorphic outside $P_\pm$, which under diffeomorphism $z \to w(z)$ transforms as follows:

$$R'(w)d^2w = R(z)d^2z + \{w,z\}d^2z \tag{4.13}$$

where

$$\{w,z\} = \frac{\partial^3{}_z w}{\partial_z w} - \frac{3}{2}\left(\frac{\partial^2{}_z w}{\partial_z w}\right)^2$$

Two algebras specified by different schwartzian connections are isomorphic since they differ only by trivial cocycles. The cocycle χ_{IJ} is obviously antisymmetric; moreover it satisfies a locality condition

$$\chi_{IJ} = 0 \qquad if \quad |I+J| > 3g.$$

The algebra (4.11) replaces in higher genus the Virasoro algebra.

4.3 THE OSCILLATORS IN HIGHER GENUS.

Let us introduce now the generalization of the Heisenberg algebra (1.59) to higher genus Riemann surfaces. The Heisenberg algebra can be seen as a trivial Lie algebra centrally extended by the introduction of a cocycle. The commutator of the pointwise product of $\mathbb{C}$-valued functions of the basis (4.1) gives rise to the trivial Lie algebra structure. As for the cocycle, it can be introduced as follows: if A_I,A_J are meromorphic functions of the basis (4.1), then we can define the cocycle to be:

$$\gamma_{IJ} = \frac{1}{2\pi i} \oint_{C_\tau} A_I dA_J \tag{4.14}$$

(4.14) has the right properties to be considered a cocycle: it is of course antisymmetric and satisfies in a trivial way the cocycle condition (consequence of the Jacobi relation of the algebra):

$$\gamma(A,[B,C]) + \gamma(B,[C,A]) + \gamma(C,[A,B]) = 0 \tag{4.15}$$

From the expansion (4.1), due to the polar structure in $P_\pm$, it is clear that, if $|I|, |J| > \frac{g}{2}$ the relation

$$\gamma_{IJ} = 0 \qquad for \quad |I+J| > g \tag{4.16}$$

holds. For any I,J, the slightly weak condition is held: $\gamma_{IJ} = 0$ if $|I+J| > 2g$. This has the consequence that the cocycle γ_{IJ} is "local" and univocally defined (trivial cocycles are necessarily zero).

Since $A_{\frac{g}{2}}$ is a constant, it is clear that $\gamma_{\frac{g}{2}I} = 0$.

Then the higher genus centrally extended Heisenberg algebra is introduced by the relation

$$[\alpha_I, \alpha_J] = -\gamma_{IJ} \tag{4.17}$$

The operators α_J are the higher genus version of the oscillators (1.59) (specializing 4.17 with $g = 0$ we reproduce precisely 1.59).

Using the oscillator (4.17) one can give an explicit realization of the centrally extended KN algebra (4.11) with $c \leq 1$. This is in complete analogy with the construction carried on in sect. (1.4).

We introduce the operators

$$\bar{L}_n = L_n + \hat{L}_n \tag{4.18}$$

(n is an integer)

where

$$L_n = \frac{1}{2} l_n{}^{IJ} \alpha_I \alpha_J = \frac{1}{2} \frac{1}{2\pi i} \oint_{C_\tau} e_n(Q) \omega^I(Q) \omega^J(Q) \alpha_I \alpha_J \tag{4.19}$$

and

$$\hat{L}_n = -\frac{\mu}{2\pi i} \oint_{C_\tau} e_n \alpha_I \nabla_z \omega^I. \tag{4.20}$$

Here ∇_z is the covariant derivative (acting on a λ-tensor we have $\nabla_z \equiv \partial_z - \lambda \Gamma_z$, with Γ_z a given, specified, connection: different choices for the connection give rise to different projective connections in the cocycle).

At a fixed value of n, $l_n{}^{IJ}$ differs from zero only on a strip in the plane of the pairs (I, J).

It is easily proven, since the relation (4.17) and (4.6) that the commutator $[\bar{L}_n, \bar{L}_m]$ gives the centrally extended Krichever-Novikov algebra (4.11) with central charge $c = 1 - 12\mu^2$.

We remark that this is true no matter which possible normal ordering is introduced for the operators L_n.

It turns out that the bases defined in eq. (4.1,4.2) are not the most suited to analyse the properties of the generalized oscillator algebra.

Let us introduce the following conventions about the index I: we denote as i the index running over the set $\hat{I} = \{-\frac{g}{2} \leq i < \frac{g}{2}\}$; $i_{1'}$ and $i_{2'}$ define the set $\bar{I} = \{i_{1'} < -\frac{g}{2}, i_{2'} > \frac{g}{2}\}$. Then

$$I \equiv \{i_{1'}, i, \frac{g}{2}, i_{2'}\}. \tag{4.21}$$

In this notation the 1-forms holomorphic everywhere are denoted as ω^i $(i \in \hat{I})$.

The standard choice of the basis η^i for the holomorphic differentials is introduced through the normalization

$$\begin{aligned} \frac{1}{2\pi i}\oint_{a_j} \eta^i &= \delta^i{}_j \\ \frac{1}{2\pi i}\oint_{b_j} \eta^i &= \Omega^i{}_j. \end{aligned} \tag{4.22}$$

The contour integrals go around the homology cycles a and b. $\Omega^i{}_j$ is the period matrix of our Riemann surface.

We introduce the following notation concerning the ω^I of the basis (4.2):

$$\begin{aligned} \frac{1}{2\pi i}\oint_{a_j} \omega^I &= N^I{}_j \\ \frac{1}{2\pi i}\oint_{b_j} \omega^I &= M^I{}_j. \end{aligned} \tag{4.23}$$

In particular if we connect the different system of bases for the holomorphic differentials:

$$\begin{aligned} \eta^i &= C^i{}_j \omega^j \\ \omega^i &= (C^{-1})^i{}_j \eta^j, \end{aligned} \tag{4.24}$$

we get the set of relations

$$\begin{aligned} C^i{}_k N^k{}_j &= \delta^i{}_j \\ C^i{}_k M^k{}_j &= \Omega^i{}_j \end{aligned} \tag{4.25}$$

(which imply that $(C^{-1})^i{}_j = N^i{}_j$ and $M^i{}_k = N^i{}_l \Omega^l{}_j$ are satisfied).

Since the matrix $M^i{}_k$ is non-singular we can also introduce a matrix $D^i{}_j$ defined by the relation

$$D^i{}_j M^j{}_k = \delta^i{}_k. \tag{4.26}$$

Integrating along the homology cycles a_i,b_i the expression

$$dA_I = \gamma_{JI}\omega^J \tag{4.27}$$

we get the relations

$$
\begin{aligned}
0 &= N^J{}_i \gamma_{JI} \\
0 &= M^J{}_i \gamma_{JI}.
\end{aligned}
\tag{4.28}
$$

We point out that the sum in the r.h.s. is finite since, at a fixed value I, γ_{JI} is different from zero only for a limited range of values of J.

Eq. (4.28) tells us that the matrix γ_{JI} admits $2g+1$ null vectors (eigenvector with zero eigenvalue): they are $N_i \equiv \{N^J{}_i\}$; $M_i \equiv \{M^J{}_i\}$ and $K_{\frac{g}{2}} \equiv \{\delta^J{}_{\frac{g}{2}}\}$.

We introduce now the "momentum" field

$$
P(Q) = \sum_I \alpha_I \omega^I(Q). \tag{4.29}
$$

From eq. (4.17) it follows that $P(Q)$ satisfies the commutation relation

$$
[P(Q), P(\bar{Q})] = -2\pi i d_Q \Delta(Q, \bar{Q}). \tag{4.30}
$$

At this point we are motivated to introduce the operators $\tilde{\alpha}_i$ and $\tilde{\tilde{\alpha}}_i$ through the infinite sums:

$$
\begin{aligned}
\tilde{\alpha}_i &= N^J{}_i \alpha_J \\
\tilde{\tilde{\alpha}}_i &= M^J{}_i \alpha_J.
\end{aligned}
\tag{4.31}
$$

Since

$$
\begin{aligned}
&[\tilde{\alpha}_i, \alpha_L] = N^J{}_i [\alpha_J, \alpha_L] = N^J{}_i \gamma_{JL} = 0 \\
&[\tilde{\tilde{\alpha}}_i, \alpha_L] = 0.
\end{aligned}
\tag{4.32}
$$

then $\tilde{\alpha}_i$ and $\tilde{\tilde{\alpha}}_i$, together with $\alpha_{\frac{g}{2}}$ are $2g+1$ central elements of the oscillator algebra (4.17) (we remark that even if $\tilde{\alpha}_i$ and $\tilde{\tilde{\alpha}}_i$ are introduced via an infinite sum, the commutators (4.32) are well-defined since they involve only a finite sum).

Therefore the algebra (4.17) admits $2g+1$ conserved moments

$$
p \equiv \alpha_{\frac{g}{2}} \quad , \quad \tilde{p}_i \equiv \tilde{\alpha}_i \quad , \quad \tilde{\tilde{p}}_i \equiv \tilde{\tilde{\alpha}}_i.
$$

Two new systems of bases for functions and 1-forms can be introduced: $\tilde{A}_I$,$\tilde{\omega}^I$ and $\tilde{\tilde{A}}_I$, $\tilde{\tilde{\omega}}^I$.

They are introduced through the position

$$\begin{aligned}
\tilde{\omega}^I &\equiv \{\tilde{\omega}^i, \tilde{\omega}^{\frac{g}{2}}, \tilde{\omega}^{i'}\} \equiv \\
&\equiv C^i{}_j\omega^j, \omega^{\frac{g}{2}}, \omega^{i'} - N^{i'}{}_i C^i{}_j \omega^j \\
\tilde{A}_I &\equiv \{\tilde{A}_i, \tilde{A}_{\frac{g}{2}}, \tilde{A}_{i'}\} \equiv \\
&\{N^I{}_i A_i, A_{\frac{g}{2}}, A_{i'}\}
\end{aligned} \tag{4.33}$$

Similar relations hold for $\tilde{\tilde{A}}_I$ and $\tilde{\tilde{\omega}}^I$ with $N^I{}_i \to M^I{}_i$, $C^i{}_j \to D^i{}_j$.

These new systems of bases satisfy the duality property and the completeness relation (4.6):

$$\begin{aligned}
\delta_I{}^J &= \frac{1}{2\pi i}\oint_{C_\tau} \tilde{A}_I\tilde{\omega}^J = \frac{1}{2\pi i}\oint_{C_\tau} \tilde{\tilde{A}}_I\tilde{\tilde{\omega}}^J \\
\Delta(Q,\bar{Q}) &= \frac{1}{2\pi i}\sum_K \tilde{A}_K(Q)\tilde{\omega}^K(\bar{Q}) = \frac{1}{2\pi i}\sum_K \tilde{\tilde{A}}_K(Q)\tilde{\tilde{\omega}}^K(\bar{Q}).
\end{aligned} \tag{4.34}$$

Since $\tilde{A}_i$ is introduced in terms of an infinite sum, the relation (4.33) must be understood in the distribution theory context. In later discussions however we do not need to consider them.

Outside $P_\pm$ we have

$$d\tilde{A}_i \equiv d\tilde{\tilde{A}}_i = 0. \tag{4.38}$$

If we define

$$\begin{aligned}
\tilde{\alpha}_{i'} &= \tilde{\tilde{\alpha}}_{i'} = \alpha_{i'} \\
\tilde{\alpha}_{\frac{g}{2}} &= \tilde{\tilde{\alpha}}_{\frac{g}{2}} = \alpha_{\frac{g}{2}}.
\end{aligned} \tag{4.35}$$

then we obtain that

$$P(Q) = \sum_K \alpha_K\omega^K(Q) = \tilde{\alpha}_K\tilde{\omega}^K(Q) = \tilde{\tilde{\alpha}}_K\tilde{\tilde{\omega}}^K(Q). \tag{4.36}$$

We can derive from eq. (4.17) the full algebra for $\tilde{\alpha}_I$, $\tilde{\tilde{\alpha}}_I$:

$$
\begin{aligned}
[\tilde{\alpha}_{i'}, \tilde{\alpha}_{j'}] &= -\gamma_{i'j'} \\
[\tilde{\alpha}_I, \tilde{\alpha}_j] &= [\tilde{\alpha}_I, \tilde{\alpha}_{\frac{g}{2}}] = 0.
\end{aligned}
\tag{4.37}
$$

The same relation holds when we replace $\tilde{} \rightarrow \tilde{\tilde{}}$.

The geometrical meaning of the $\tilde{\omega}$ is expressed by the property:

$$
\frac{1}{2\pi i} \oint_{a_i} \tilde{\omega^j} = \delta^j{}_i \qquad \frac{1}{2\pi i} \oint_{a_i} \tilde{\omega^{j'}} = 0 \tag{4.39}
$$

(the same relation holds when we replace $\tilde{} \rightarrow \tilde{\tilde{}}$ and the homology cycles a_i with b_i).

The systems $(A_I, \omega^I, \alpha_I)$, $(\tilde{A}_I, \tilde{\omega^I}, \tilde{\alpha_I})$, $(\tilde{\tilde{A}}_I, \tilde{\tilde{\omega^I}}, \tilde{\tilde{\alpha}}_I)$ are all on the same foot: they are all interchangeable, since it is possible to express any two of them in terms of the third one. E.g. the relation which expresses α_I in terms of $\tilde{\alpha}_I$ is given by:

$$
\begin{aligned}
\alpha_i &= \tilde{\alpha}_j C^j{}_i - \tilde{\alpha}_{j'} N^{j'}{}_j C^j{}_i \\
\alpha_{i'} &= \tilde{\alpha}_{i'} \\
\alpha_{\frac{g}{2}} &= \tilde{\alpha}_{\frac{g}{2}}.
\end{aligned}
\tag{4.40}
$$

(4.40) inverts the relations (4.31) and (4.35). Since, as usual we can get a similar relation by replacing $\tilde{} \rightarrow \tilde{\tilde{}}$, $C^i{}_j \rightarrow D^i{}_j$, $N^{j'}{}_j \rightarrow M^{j'}{}_j$, we are able to express $\tilde{\tilde{\alpha}}$ in terms of $\tilde{\alpha}$. In particular we obtain the following relation

$$
\tilde{\tilde{\alpha}}_i = \tilde{\alpha}_k C^k{}_j M^j{}_i + \tilde{\alpha}_{\frac{g}{2}} (M^{\frac{g}{2}}{}_i - C^k{}_j M^j{}_i N^{\frac{g}{2}}{}_k) + \tilde{\alpha}_{k'} (M^{k'}{}_i - C^k{}_j M^j{}_i N^{k'}_k) \tag{4.41}
$$

Here $k' = \{k'_1, k'_2\}$.

The importance of (4.41) lies on the fact that it is a consistency condition which constraints the central elements $\tilde{\tilde{\alpha}}_i$, $\tilde{\alpha}_k$, $\tilde{\alpha}_{\frac{g}{2}} = \alpha_{\frac{g}{2}}$, which label the representations of the oscillator algebra. The last term in the r.h.s. $\tilde{\alpha_{k'}} (M^{k'}{}_i - C^k{}_j M^j{}_i N^{k'}_k)$ is a c-number since it commutes with any α_I as it can be easily verified.

4.4 THE CONSTRUCTION OF THE VERTEX OPERATOR

The next step in our construction involves the introduction of a bosonic Fubini-Veneziano field $\Phi(Q)$ which satisfies the relation

$$d_Q \Phi(Q) = P(Q) \tag{4.42}$$

This is obtained by setting *:

$$\Phi(Q) = q + \alpha_I \int_{Q_0}^{Q} \omega^I(\bar{Q}) \tag{4.43}$$

Q_0 is a reference point in a generic position on the Riemann surface but different from $P_\pm$ so that the integral in the r.h.s. is well-defined. The physical results we get at the end should be independent from the point Q_0.

The constant $q \equiv \Phi(Q_0)$ is assumed to satisfy the following commutation relations with the operators α_I:

$$[q, \alpha_I] = -A_I(Q_0) \tag{4.44}$$

(notice that in particular $[q, \alpha_{\frac{g}{2}}] = -1$ for any Q_0, which is precisely what we want from our knowledge of the genus zero case).

(4.44) guarantees the right commutation property between $\Phi(Q)$ and $P(Q')$:

$$[\Phi(Q), P(Q')] = -2\pi i \Delta(Q, Q'). \tag{4.45}$$

The field $\Phi(Q)$ is multivalued, its multivaluedness being given by

$$\Phi(Q + n^i a_i + m^i b_i) = \Phi(Q) + 2\pi i(\tilde{\alpha}_i n^i + \tilde{\tilde{\alpha}}_i m^i) \tag{4.46}$$

$n^i, m^i \in \mathbb{Z}$ are the winding numbers around the homology cycles a_i, b_i respectively.

* Naively one could expect that the most general solution to the eq. (4.42) contains terms proportional to $\tilde{A}_i$, $\tilde{\tilde{A}}_i$; but these terms can be reabsorbed in the constant q

$\Phi(Q)$ represents the chiral part in the decomposition of the harmonic bosonic field Φ in ref. [94-96].

We can now introduce the higher genus analogue of the vertex (1.66).

It must be a well-defined tensorial object and therefore its commutator with the Krichever-Novikov generators $\bar{L}_I$ must be equal to the Lie derivative (4.8), in analogy with eq. (1.13) for the genus zero case. From our previous knowledge about the genus zero case we already know that the vertex should be normal ordered. Before going ahead let us introduce some useful notations: we define

$$\begin{aligned} \Phi(Q)_< &= \sum_{i_1} \alpha_{i_1} \int_{Q_0}^{Q} \omega^{i_1} \qquad i_i = \{i_{1'}, i\} \\ \Phi_>(Q) &= \sum_{i_2} \alpha_{i_2} \int_{Q_0}^{Q} \omega^{i_2} \qquad i_2 = \{\frac{g}{2}, i_{2'}\} \end{aligned} \tag{4.47}$$

(all the operators appearing in $\Phi_>(Q)$ commute one another since , for any i'_2, j'_2, we have $\gamma_{i'_2 j'_2} = 0$).

The following commutation relations will be useful:

$$\begin{aligned} [\alpha_I, e^{\beta q}] &= \beta A_I(Q_0) e^{\beta q} \\ [\alpha_I, e^{\beta \Phi_<(Q)}] &= -\beta \gamma_{I j_1} \int_{Q_0}^{Q} \omega^{j_1} e^{\beta \Phi_<} \end{aligned} \tag{4.48}$$

(same relation replacing $<\rightarrow>$ and $1 \rightarrow 2$).

Moreover

$$\begin{aligned} [L_I, \alpha_J] &= \frac{-1}{2\pi i} \oint_{C_\tau} e_I \omega^K dA_J \alpha_K \\ [L_I, q] &= e_I(Q_0) \omega^K(Q_0) \alpha_K \end{aligned} \tag{4.49}$$

and

$$\begin{aligned} [L_I, e^{\beta q}] &= [\frac{-\beta}{2\pi i} e_I(Q_0) \omega^K(Q_0) \alpha_K - \frac{\beta^2}{2} e_I(Q_0) \omega^K(Q_0) A_K(Q_0)] e^{\beta q} \\ [L_I, e^{\beta \Phi_<}] &= [\frac{-\beta}{2\pi i} \oint_{C_\tau} e_I \omega^K dA_{j_1} \alpha_K \int_{Q_0}^{Q} \omega^{j_1}] e^{\beta \Phi_<} - \\ &\quad - [\frac{\beta^2}{2} \frac{1}{2\pi i} \oint_{C_\tau} (e_I dA_{j_1} dA_{i_1} \int_{Q_0}^{Q} \omega^{j_1} \int_{Q_0}^{Q} \omega^{i_1}] e^{\beta \Phi_<} \end{aligned} \tag{4.50}$$

As a preliminary step towards the construction of the chiral vertex operator of charge β, we introduce the following operator which is motivated by the analogy with the genus zero case:

$$V_\beta(Q) = e^{\beta\Phi_<} e^{\beta q} e^{\beta\Phi_>}. \tag{4.51}$$

The commutation property of $V_\beta(Q)$ with the operators $\hat{L}_I$ introduced in (4.20) are easily seen to be the right one:

$$[\hat{L}_I, V_\beta(Q)] = \beta\mu\nabla e_I(Q) V_\beta(Q). \tag{4.52}$$

Notice that the r.h.s. is actually not dependent on the reference point Q_0.

More involved is the computation of the commutation relations with the L_I. After lenghty computations, and making use of the relation

$$d_Q e^{A(Q)} = (B + \frac{C}{2}) e^A$$

which holds for generic operators $A(Q)$, $B(Q)$, $C(Q)$ such that $dA = B$ and $[A, B] = C$ is a c-number, we get at the end:

$$\begin{aligned}[L_I, V_\beta(Q)] =& e_I \partial V_\beta(Q) + \frac{\beta^2}{2}[\left(\sum_{K,L\geq\frac{g}{2}} - \sum_{K,L<\frac{g}{2}}\right) {l_I}^{KL} A_K(Q) A_L(Q) V_\beta(Q) \\ & - e_I \gamma_{i_1 j_1} \omega^{i_1}(Q) \int_{Q_0}^{Q} (Q)] V_\beta(Q)\end{aligned} \tag{4.53}$$

where ${L_I}^{KL}$ is introduced in eq. (4.19).

If we define

$$\bar{V}_\beta(Q) = e^{\frac{\beta^2}{2}\gamma_{i_1 j_2} \int_{Q_0}^{Q} \omega^{i_1}(\bar{Q}) \int_{Q_0}^{\bar{Q}} \omega^{j_2}} V_\beta(Q) \tag{4.54}$$

then we have that $\bar{V}_\beta(Q)$ satisfies the relation

$$[L_I, \bar{V}_\beta(Q)] = e_I \partial \bar{V}_\beta(Q) + \frac{\beta^2}{2} \left(\sum_{K,L\geq\frac{g}{2}} - \sum_{K,L<\frac{g}{2}}\right) {l_I}^{KL} A_K(Q) A_L(Q) \bar{V}_\beta(Q). \tag{4.55}$$

The next step requires the identification of the term

$$\left(\sum_{K,L\geq\frac{g}{2}} - \sum_{K,L<\frac{g}{2}}\right) {l_I}^{KL} A_K(Q)A_L(Q). \tag{4.56}$$

Firstable we must say that this object is well-defined since the sum which appears is a finite one (see the remark after (4.20)). We will see now that there exists a connection Γ which ensures the following identification:

$$\left(\sum_{K,L\geq\frac{g}{2}} - \sum_{K,L<\frac{g}{2}}\right) {l_I}^{KL} A_K(Q)A_L(Q) = \nabla e_I(Q) = de_I + \Gamma e_I. \tag{4.57}$$

Such a connection is the same for any I. We notice at first that both sides are linearly dependent on the vector e_I. (4.57) can be proved analysing the leading behaviour of both sides around $P_\pm$. The reasoning goes as follows: the leading behaviour of the term $e_I(Q)\omega^K(Q)\omega^L(Q)$ in the limit $Q \to P_\pm$ can be computed from (4.2) and (4.4) and it is given by

$$(z_+)^{I-K-L-\frac{g}{2}-1}(dz_+)(1+O(z_+)). \tag{4.58}$$

(4.58) makes clear that ${l_I}^{KL}$ is different from zero on the range $I-K-L-\frac{g}{2}-1 \leq -1$, namely for

$$K+L \geq I-\frac{g}{2}. \tag{4.59}$$

A similar analysis can be performed around the point P_- and gives that, for values of I very large ($I >> \frac{g}{2}$), then a further condition for ${l_I}^{KL}$ to be different from zero is obtained:

$$K+L \leq I+\frac{g}{2}. \tag{4.60}$$

A constraint derived from P_- which leads to a condition like (4.60) actually holds for any I, but there are slight modifications according to the range of values taken by I. The leading term in (4.56) is obtained precisely when $K+L = I-\frac{g}{2}$. In fact in this case $A_K(Q)A_L(Q)$ goes as

$$A_K A_L \sim (z_+)^{I-\frac{3}{2}g}(1+O(z_+)) \tag{4.61}$$

It is a simple exercise to count the number of terms which give contributions to (4.56). With respect to the normalization after (4.4) we get

$$\left(\sum_{K,L\geq\frac{g}{2}} - \sum_{K,L<\frac{g}{2}}\right) l_I{}^{KL} A_K(Q)A_L(Q) \sim (I+\frac{g}{2}+1)(z_+)^{I-\frac{3}{2}g}(1+...). \tag{4.62}$$

Meanwhile we get for de_I:

$$de_I \sim (I-\frac{3}{2}g+1)(z_+)^{I-\frac{3}{2}g}(1+...) \tag{4.63}$$

The same analysis, performed for P_- gives us:

$$\begin{aligned}\left(\sum_{K,L\geq\frac{g}{2}} - \sum_{K,L<\frac{g}{2}}\right) l_I{}^{KL} A_K(Q)A_L(Q) &= \alpha(-I+\frac{g}{2}-1)(z_-)^{-I-\frac{3}{2}g}(1+...) \\ de_I &= \alpha(-I-\frac{3}{2}g+1)(z_-)^{-I-\frac{3}{2}g}(1+...).\end{aligned} \tag{4.64}$$

It turns out that the connection Γ which plays the game has no pole in P_+ and a simple pole in P_- with residue $2g-2$. Γ can be explicitely constructed in the following way: given a specific abelian differential of the third kind $\bar{\omega}$ with simple poles in P_+ and in P_- and residues $+1$ and -1 respectively, together with $2g$ simple zeros at the points P_i, we can set:

$$\Gamma = \frac{\partial\bar{\omega}}{\bar{\omega}} + \chi \tag{4.65}$$

Here χ is a 1-form with the following characteristics: it is the sum of $2g+1$ abelian differentials of the third kind ($\chi = \sum_i \omega_{P_iP_-} - \omega_{P_+P_-}$), having poles in P_i,P_+ and P_- with residue $+1$ in P_-. The poles at P_i in χ precisely cancel the poles in $\frac{\partial\bar{\omega}}{\bar{\omega}}$ deriving from the zeroes of $\bar{\omega}$.

Γ has the right polar behaviour (we remember that since (4.56) is holomorphic outside $P_\pm$, so Γ is). The 1-form χ is determined up to the addition of an unknown holomorphic abelian differential.

Therefore the commutation relation of the vertex $\bar{V}_\beta$ does not give rise to the Lie derivative. Moreover $\bar{V}_\beta$ even at the classical level is not a tensor field. The vertex operator with the right commutation property is

$$\hat{V}_\beta = \bar{\omega}^{\frac{\beta^2}{2}} e^{\frac{\beta^2}{2}\int_{Q_0}^{Q}\chi}\bar{V}_\beta(Q) \tag{4.66}$$

which satisfies:

$$[L_I, \hat{V}_\beta(Q)] = e_I\partial\hat{V}_\beta(Q) + \frac{\beta^2}{2}\partial e_I\hat{V}_\beta(Q) \tag{4.67}.$$

If we are interested in the commutation property with the operator $\hat{L}_I$ (see eq. (4.10)), then we must take

$$\begin{aligned}\hat{V}'_\beta &= \bar{\omega}^{(\frac{\beta^2}{2}+\beta\mu)} e^{(\frac{\beta^2}{2}+\beta\mu)\int_{Q_0}^{Q}\chi}\bar{V}'_\beta(Q)\\ [L_I, \hat{V}'_\beta(Q)] &= e_I\partial\hat{V}'_\beta(Q) + (\frac{\beta^2}{2}+\beta\mu)\partial e_I\hat{V}'_\beta(Q).\end{aligned} \tag{4.68}$$

Doing some more exercise one can compute the multivaluedness of the operator $\hat{V}_\beta(Q)$ around the homology cycles a, b. The important thing is that its multivaluedness is expressed by a numerical factor which does not depend on the point Q. For instance we have

$$\hat{V}_\beta(Q + a_i) = e^{\beta c_i}\hat{V}_\beta(Q) \tag{4.69}$$

with

$$e^{\beta c_i} = e^{\beta^2(\frac{1}{2}N^I{}_i A_I(Q_0) + \frac{1}{2}\oint_{a_i} A_{j_1}\omega^{j_1} - \frac{1}{2}\oint_{a_i} A_{j_2}\omega^{j_2})} e^{\beta\tilde{\alpha}_i} e^{x_i}$$

(here x_i derives from the multivaluedness of the term containing χ).

It is possible to set to 0 the multivaluedness around the homology cycles a (or, since they are on the same foot, b) by reabsorbing these terms with insertions of holomorphic abelian differentials, namely defining

$$V'_\beta(Q) = e^{-\beta c_i\int_{Q_0}^{Q}\eta^i}\hat{V}_\beta(Q). \tag{4.70}$$

It turns out that $V'_\beta(Q)$ is single-valued around the homology cycles a_i and satisfies the right commutation relations w. r.t. the operator L'_I obtained from L_I by substituting the operators α_i with $\alpha'_i = \alpha_i + c_i$:

$$[L'_I, V'_\beta(Q)] = e_I \partial V'_\beta(Q) + \frac{\beta^2}{2} e_I(Q) \partial V'_\beta(Q). \tag{4.71}$$

The operator L'_I gives trivially a representation of the Krichever-Novikov algebra which differs from the one obtained from L_I only up to a trivial cocycle (see the remark after (4.13).

To get rid completely of the multivaluedness of $V'_\beta(Q)$ one has to choose in a very specific way the set of values $\tilde{\alpha}_i$, $\tilde{\tilde{\alpha}}_i$ and $\alpha_{\frac{g}{2}}$ which label the particulasr representation. The constraint (4.41) inmplies that this can not be implemented in general: only on some Riemann surfaces characterized by particular values of the period matrix this can be achieved. Anyway, if we allow the previous redefinition to be done in general, namely if we let for any I to put $V"_\beta(Q) = e^{-\beta c_I \int_{Q_0}^{Q} \omega^I} V"_\beta(Q)$ and $\alpha'_I = \alpha_I + c_I$, then $V"_\beta(Q)$ is single-valued and has the right commutation property. This position has the further consequence that the parameters $\tilde{\alpha}_i$, $\tilde{\tilde{\alpha}}_i$ and $\alpha_{\frac{g}{2}}$ have no real physical meaning and can be all set equal to zero without loss of generality. The latter position is however delicate since it involves an "infinite" redefinition. A more accurate evaluation of its feasibility is required.

4.5 Conclusion and perspectives.

In this chapter we have discussed how to set up a global operatorial formalism to study CFT on higher genus Riemann surfaces. The very basic problem, namely the construction of the vertex operator, has been carried out (in a different context see [97]). This, in a certain sense, is the most difficult part of the job since, before knowing it, one had to guess the explicit form of the vertex having no hints about what was going on higher genus. Most of the work which has still to be done concerns the application of this framework. Apart from the obvious generalization to the superconformal case, let us discuss the most important open problems.

At first we have to mention that the single-valued operator of charge β should provide, together with the operators α_I, a representation for the higher genus generalization of the Kač- Moody affine simply laced $\beta^2 = 2$ algebra (along the same lines as shown in [41] for the genus zero case). The work in this direction is in progress: everything seems to fit in in our framework, the only trouble being due to the technical complexity of evaluating the commutator of two vertices, which requires to perform a resummation of an infinite sum. Once this goal is achieved, the coset construction can provide a method to realize an explicit vertex representation of the Krichever-Novikov algebra.

Another very interesting problem concerns the connection between the framework of this chapter and the real weight $b - c$ system approach: it would be very nice to reobtain the correlation functions of ch. 3 within the language of this chapter. In this case one has already some hints that the central elements $\tilde{\alpha}_i$, $\tilde{\tilde{\alpha}}_i$ of the oscillator algebra are somehow related to the theta functions characteristics a_i and b_i.

The computation of the correlation functions would require the introduction of a ket and of a bra vacuum over which the operators α_I act, together with an inner product. By the way we notice that the operators $\tilde{\alpha}_I$ (or $\tilde{\tilde{\alpha}}_I$) allows to solve a problem concerning the normal ordering of the operators α_i mentioned in [90]: the ket and bra vacua are introduced in such a way that $< 0|\tilde{\alpha_{i'}}_1 =< 0|\alpha_{i'_1} = 0$ and $\tilde{\alpha_{i'}}_2|0 >= \alpha_{i'_2}|0 >= 0$; our $\tilde{\alpha}_i$ are c-numbers and their normal ordering causes no trouble.

One of the long-standing problem for the higher genus oscillators concerns the intro-

duction of their hermitian conjugate, which provides a Hilbert space structure to their representation space (see e.g. [98]). It may be possible that the "diagonalization" of the oscillators we have performed could help to solve this problem. The introduction of such a hermitian conjugation could provide a definite parallelism between the higher genus and the genus 0 case.

APPENDIX A

In this Appendix we discuss some particular cases in which bases defined in Section 3.2 need to be modified. Due to the Riemann - Roch theorem, eq.(3.1) does not work in the following cases (this follows also from the application of the Riemann vanishing theorem to the explicit form of eq.(3.1) given in eq.(3.7))

$\forall g \; for \; |j| \leq \frac{g}{2} \; and \; \lambda = 0, 1;$

$\forall g \; for \; |j| = \frac{1}{2} \; and \; \lambda = \frac{1}{2}, \; l = 1 \; with \; odd \; spin \; structure;$

$g = 1 \; for \; |j| = \frac{1}{2} \; and \; \lambda \in \mathbb{Z};$

$g = 1 \; for \; |j| = \frac{1}{2} \; and \; \lambda \in \mathbb{Z} + \frac{1}{2}, \; l = 1.$

Then we define

$$f_j^{(0)}(z_\pm) = a_j^{(0)\pm} z_\pm^{\pm j - \frac{g}{2} - \frac{1 \mp 1}{2}} (1 + O(z_\pm)), \qquad -\frac{g}{2} \leq j \leq \frac{g}{2} - 1, \tag{A.1}$$

and $f_{\frac{g}{2}}^{(0)}(z) = 1$. For $\lambda = 1$ we put

$$f_j^{(1)}(z_\pm) = a_j^{(1)\pm} z_\pm^{\pm j + \frac{g}{2} - \frac{1 \pm 1}{2}} (1 + O(z_\pm)) dz_\pm, \qquad -\frac{g}{2} + 1 \leq j \leq \frac{g}{2}, \tag{A.2}$$

and choose $f_{-\frac{g}{2}}^{(1)}(z)$ to be the third kind abelian differential with simple poles at $P_\pm$ with residue ± 1, normalized in such a way that its periods be purely imaginary, i.e. $Re \oint_{a_i} f_{-\frac{g}{2}}^{(1)}(z) = Re \oint_{b_i} f_{-\frac{g}{2}}^{(1)}(z) = 0$. For $g = 1$ we define

$$f_j^{(\lambda)}(z) = f_j^{(0)}(z) (f_{\frac{1}{2}}^{(1)}(z))^\lambda, \quad \lambda \in Z, \tag{A.3}$$

$$f_j^{(\lambda,1)}(z) = f_j^{(0)}(z) (f_{\frac{1}{2}}^{(1)}(z))^\lambda, \quad \lambda \in Z + \frac{1}{2}, \tag{A.4}$$

where the spin structure of $(f_{\frac{1}{2}}^{(1)}(z))^{\frac{1}{2}}$ is chosen to be odd. Finally, for $\lambda = \frac{1}{2}$, l =1 and odd spin structure:

$$f_{-\frac{1}{2}}^{(\frac{1}{2},1)}(z_\pm) = a_{-\frac{1}{2}}^{(\frac{1}{2})\pm}(1) z_\pm^{-1} (1 + O(z_\pm)) (dz_\pm)^{\frac{1}{2}}, \tag{A.5}$$

$$f_{\frac{1}{2}}^{(\frac{1}{2},1)}(z_\pm) = a_{\frac{1}{2}}^{(\frac{1}{2})\pm}(1) (1 + O(z_\pm)) (dz_\pm)^{\frac{1}{2}}. \tag{A.6}$$

The multivaluedness of $f_{(1-\lambda,1)}^j(z)$ is given by (3.2) with λ replaced by $1 - \lambda$, so that the integrand in eq.(3.5) is a well defined 1 - differential.

The explicit form of the differentials in eqs.(A.1 - 2) is

$$f_j^{(0)}(z) = \frac{\theta(z + (j - \frac{g}{2})P_+ + R - (j + \frac{g}{2} + 1)P_+ + \Delta)}{E(z,R)^{-1}E(z,P_-)^{j+\frac{g}{2}+1}E(z,P_+)^{-j+\frac{g}{2}}\sigma(z)}, \qquad -\frac{g}{2} \leq j \leq \frac{g}{2} - 1, \qquad (A.7)$$

$$f_j^{(1)}(z) = \frac{\theta(z + (j + \frac{g}{2} - 1)P_+ - R - (j - \frac{g}{2})P_- - \Delta)}{E(z,R)E(z,P_-)^{j-\frac{g}{2}}E(z,P_+)^{-j-\frac{g}{2}+1}\sigma(z)^{-1}}, \qquad -\frac{g}{2} + 1 \leq j \leq \frac{g}{2}, \qquad (A.8)$$

where the point $R \in \Sigma$ is arbitrary; its presence in (A.7) reflects the fact that the meromorphic function defined in (A.7) is unique up to addition of a constant. In eq.(A.8) the pole in R is cancelled by the zero of the θ - function; indeed, being $\theta(-z) = \theta(z)$, from the Riemann vanishing theorem we have for $-\frac{g}{2} + 1 \leq j \leq \frac{g}{2}$

$$\theta((j + \frac{g}{2} - 1)P_+ - (j - \frac{g}{2})P_- - \Delta) = 0.$$

Note that this result allows us to identify the differential $f_{\frac{1}{2}}^{(1)}(z)$ in eq.(A.3) with $\sigma(z)^2$.

For $j = -\frac{g}{2}$ the 1 - differential is

$$f_{-\frac{g}{2}}^{(1)}(z) = d\,ln\frac{E(z,P_+)}{E(z,P_-)} - 2\pi i \sum_{j,k=1}^{g} Im\left(\int_{P_-}^{P_+} \omega_j\right)(Im\Omega)_{jk}^{-1}\omega_k(z). \qquad (A.9)$$

Finally the conditions (A.5 - 6) give

$$f_{-\frac{1}{2}}^{(\frac{1}{2},1)}(z) = \frac{E(z,S)}{E(z,P_+)E(z,P_-)}\theta[{}^{\delta}_{\epsilon}](z + S - P_+ - P_-), \qquad (A.10)$$

$$f_{\frac{1}{2}}^{(\frac{1}{2},1)}(z) = \frac{\theta[{}^{\delta}_{\epsilon}](z - P_-)}{E(z,P_-)}, \qquad (A.11)$$

where the θ - characteristics are odd. The presence in eq.(A.10) of the arbitrary point $S \in \Sigma$ is due to the fact that eq.(A.5) does not fix uniquely $f_{-\frac{1}{2}}^{(\frac{1}{2},1)}(z)$. Note that $f_{\frac{1}{2}}^{(\frac{1}{2},1)}(z)$, as defined in (A.11), is equal to the expression (3.7) with $j = \frac{1}{2}$, $\lambda = \frac{1}{2}$, $l = 1$. This means that in the framework of θ - functions theory the modification (A.6) to eq.(3.1) is automatically taken into account. Moreover since $P_- \in div\theta[{}^{\delta}_{\epsilon}](z - P_-)$, $f_{\frac{1}{2}}^{(\frac{1}{2},1)}(z)$ does

not have poles, it is the zero - mode associated to the odd spin structure $\begin{bmatrix}\delta\\ \epsilon\end{bmatrix}$. In particular using the definition of the prime form, we have

$$f_{\frac{1}{2}}^{(\frac{1}{2},1)}(z) = h(P_-)h(z), \tag{A.12}$$

so that we can identify $f_{\frac{1}{2}}^{(\frac{1}{2},1)}(z)$ with the $\frac{1}{2}$ - differential which appears in eq.(A.4).

APPENDIX B

Our aim in this Appendix is to determine the values of c, d, δ and ϵ in eq.(3.64). To this end we will use the formula

$$\theta[{}^{\delta}_{\epsilon}](cz+cn+cm\Omega|d\Omega)=e^{-(1/d)(\pi icm\Omega cm+2\pi icm(cz+\epsilon+cn))}\theta[{}^{\delta+cm/d}_{\epsilon+cn}](cz|d\Omega), \qquad (B.1)$$

which holds for any b, c, δ and ϵ. As a consequence of this the change of the basis under a shift along the homology cycles is

$$g_j^{(\lambda)}[{}^{\delta}_{\epsilon}](z,P_1,.,P_k+na+mb,.,P_n)=e^{-2\pi im\left(\tilde{\alpha}_k^2 n+\tilde{\alpha}_k\epsilon/c\right)}g_j^{(\lambda)}[{}^{\delta+\tilde{\alpha}_k m/c}_{\epsilon+c\tilde{\alpha}_k n}](z,P_1,.,P_k,.,P_n), \qquad (B.2)$$

where we have put $d=c^2$ in order to eliminate the dependence on z, P_+, P_-, P_i, Ω and Δ. We notice that, had we started from an even more general expression of the bases, in which z, P_+, P_-, P_i, Ω and Δ all had a different coefficient in front of them in the argument of the θ function, the requirement of the phase being independent on z, P_+, P_-, P_i, Ω and Δ in eq.(3.64), would have implied the equality of the coefficients in front of z, P_+, P_-, P_i and Δ, and therefore would have entailed the same conclusion.

Now we impose the new bases (3.64) to live on the covering Σ', as far as the points P_i are concerned (while we ignore the z-dependence). In other words we assume that there exists an integer N such that when any P_i winds N times around an a or b cycle, the bases return to the initial form (up to a phase). From eq.(B.2) we have

$$g_j^{(\lambda)}[{}^{\delta}_{\epsilon}](z,P_1,.,P_k+Na,.,P_n)=g_j^{(\lambda)}[{}^{\delta}_{\epsilon+c\tilde{\alpha}_k N}](z,P_1,.,P_k,.,P_n), \qquad (B.3)$$

and

$$g_j^{(\lambda)}[{}^{\delta}_{\epsilon}](z,P_1,.,P_k+Nb,.,P_n)=e^{-2\pi iN(\tilde{\alpha}_k\epsilon/c)}g_j^{(\lambda)}[{}^{\delta+\tilde{\alpha}_k N/c}_{\epsilon}](z,P_1,.,P_k,.,P_n), \qquad (B.4)$$

Therefore, first of all, $c\tilde{\alpha}_k N$ and $\tilde{\alpha}_k N/c$ must be integers. Now due to eq.(3.62) with $q\to p$ and $p\to p+1$, we can write

$$\tilde{\alpha}_k=\frac{n_k}{\sqrt{2p(p+1)}},$$

where n_k is an integer. Therefore we have

$$\frac{cNn_k}{\sqrt{2p(p+1)}} = m_k \in \mathbb{Z}, \qquad \frac{Nn_k}{c\sqrt{2p(p+1)}} = l_k \in \mathbb{Z}. \tag{B.5}$$

It follows that

$$m_k = an_k, \qquad l_k = bn_k, \tag{B.6}$$

and since in any given minimal unitary model n_k always takes on the value 1, a and b must be integers. Therefore we can rewrite eq.(B.7) as follows

$$cN = a\sqrt{2p(p+1)} \qquad \frac{N}{c} = b\sqrt{2p(p+1)}, \tag{B.8}$$

from which we have

$$N^2 = ab\ 2p(p+1) \qquad c^2 = \frac{a}{b}. \tag{B.9}$$

Now we choose $N = 2p(p+1)$ (we will discuss later all the other possible choices). Then

$$ab = 2p(p+1), \qquad c = \frac{\sqrt{2p(p+1)}}{b}. \tag{B.10}$$

The theta-characteristics δ and ϵ can be deduced from the same formulas (B.3-4). Under the shift $P_k \to P_k + a$ the characteristic ϵ becomes $\epsilon + c\tilde{\alpha}_k$. So ϵ must be an integer times $c\tilde{\alpha}_k = n_k/b$. Similarly, as a consequence of a shift of a b-cycle, we deduce that δ must be an integer times $\tilde{\alpha}_k/c = \frac{bn_k}{2p(p+1)}$. Finally we arrive at the bases (3.64) where the theta function is

$$\theta[^{bu/2p(p+1)}_{t/b}]\left(\frac{\sqrt{2p(p+1)}}{b}(z+v)|\frac{2p(p+1)}{b^2}\Omega\right) \tag{B.11}$$

Where $t, u \in \mathbb{Z}$. Recalling now that b is an integer we remark that an equivalent basis of b-th order theta-functions [66] is given by eq. (B.11) with $b = 1$. We will use this freedom in the choice of a theta-function basis to write our new bases (3.64) in the form (3.65).

Let us recall now the choice we did after eq.(B.9) for N. We could have chosen also $N = 2p(p+1)s$ where s is an integer. In this case $ab = 2p(p+1)s$ and $c = \sqrt{2p(p+1)s}/b$; but this would have entailed a simple redefinition of the basis of theta functions of the type

considered above. Similarly we could have chosen $N = p(p+1)$, so that $ab = p(p+1)/2$ etc.. But again this corresponds to a redefinition of the basis of the theta functions, as above. Other choices for N are possible for some particular values of p, but these cases too can be treated in the same way. So, in conclusion, there is no loss of generality in choosing the bases (3.65).

APPENDIX C

Here we recall some facts about theta functions theory [65−67,99]. The θ - function with characteristic $[^{\delta}_{\epsilon}]$ is defined by

$$\theta[^{\delta}_{\epsilon}](z) = \sum_{n\in\mathbb{Z}^g} e^{\pi i(n+\delta)\Omega(n+\delta)+2\pi i(n+\delta)(z+\epsilon)} = e^{\pi i\delta\Omega\delta+2\pi i\delta(z+\epsilon)}\theta(z+\epsilon+\Omega\delta), \qquad (C.1)$$

$$\theta(z) = \theta[^{0}_{0}](z), \qquad z\in\mathbb{C}^g, \quad \delta,\epsilon\in\mathbb{R}^g,$$

where $\Omega_{i,j} \equiv \oint_{b_i}\omega_j$, $\Omega_{i,j} = \Omega_{j,i}$, $Im(\Omega) > 0$. The holomorphic differentials ω_i, $i = 1,...,g$ are normalized in such a way that $\oint_{a_i}\omega_j = \delta^i_j$, a_i, b_i being the homology cycle basis.

When $\delta_i, \epsilon_i \in \{0, \frac{1}{2}\}$, the θ - function is even or odd depending on the parity of $4\delta\epsilon$. The θ - function is multivalued under a lattice shift in the z - variable:

$$\theta[^{\delta}_{\epsilon}](z+n+\Omega m) = e^{-\pi im\Omega m-2\pi imz+2\pi i(\delta n-\epsilon m)}\theta[^{\delta}_{\epsilon}](z). \qquad (C.2)$$

The prime form is defined by ($I(z)$ is the Jacobi map)

$$E(z,w) = \frac{\theta[^{\delta}_{\epsilon}](I(z)-I(w))}{h(z)h(w)} = -E(w,z), \qquad z,w\in\Sigma, \qquad (C.3)$$

it is a holomorphic (multivalued) $(-\frac{1}{2},-\frac{1}{2})$ - differential with a simple zero in $z = w$:

$$E(z,w) \sim z-w, \ as\, z\to w. \qquad (C.4)$$

$h(z)$ is the square root of $\Sigma^g_{i=1}\omega_i(z)\partial_{u_i}\theta[^{\delta}_{\epsilon}](u)|_{u_i=0}$, it is the holomorphic $\frac{1}{2}$ - differential with non singular (i.e. $\partial_{u_i}\theta[^{\delta}_{\epsilon}](u)|_{u_i=0} \neq 0$) odd spin structure $[^{\delta}_{\epsilon}]$. Notice that $E(z,w)$ does not depend on the particular choice of $[^{\delta}_{\epsilon}]$. The prime form has the following multivaluedness around the b's homology cycles:

$$E(z+na+mb,w) = e^{-\pi im\Omega m-2\pi im(I(z)-I(w))}E(z,w). \qquad (C.5)$$

The σ - differential is defined by

$$\sigma(z) = e^{-\Sigma^g_{j=1}\oint_{a_j}\omega_j(w)lnE(w,z)} \qquad (C.6)$$

It is a $\frac{g}{2}$-differential without zeroes and poles defined on a covering of Σ. It has the following multivaluedness

$$\sigma(z+na+mb) = e^{\pi i(g-1)m\Omega m - 2\pi i m(I(\Delta)-(g-1)I(z))}\sigma(z). \qquad (C.7)$$

A useful relation involving $\sigma(z)$ is

$$\theta(I(z) - I(P_1 + ... + P_g) + I(\Delta)) = s(P_1, ..., P_g)\sigma(z)E(z, P_1)...E(z, P_g), \qquad (C.8)$$

where $s(P_1, ..., P_g)$ is a holomorphic section of a line bundle of degree $g-1$ in each variable.

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