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Global properties of higher dimensional models and their cosmological implications

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Contents

In this thesis, I will concentrate the discussion on two papers, [28, 43]. During my PhD activity, I have also written, in collaboration with Marco Fabbrichesi and Maurizio Piai, the papers [72, 73], and, with Lotfi Boubekeur, [74].

Introduction

In our world, we experience only three spatial dimensions, and the hypothetical presence of additional ones seems in direct conflict with observation. From the point of view of particle physics, an argument against the existence of extra dimensions comes from scattering processes: no missing additional transverse momenta indicates that particles do not escape into extra dimensions. Nevertheless, the possibility that our space-time contains more than the observed four dimensions has been considered for a long time, at least since the observation by Kaluza [1] and Klein [2] that, in a five dimensional world, gravitation and electromagnetism can be unified in a unique interaction.

In fact, this lack of evidence does not exclude the possible presence of extra dimensions compactified on manifolds with small size. For instance, consider the possibility that one extra dimension is compactified on a circle of radius R. Moving a particle with small mass m ($mc \ll \hbar/R$) into the extra dimension requires an energy

$$E = \frac{\hbar c}{R}. \tag{1}$$

The accessible energies at current experiments are of the order of 1 TeV. This means that the extra dimension can not be detected by these experiments if it is compactified with a radius $R \leq \hbar c/1\text{TeV} \simeq 10^{-13}$ μm. This simple example shows that, with a suitable procedure of compactification, it is possible to construct models based on the existence of extra dimensions, and at the same time compatible with experimental data.

In the original approach of Kaluza and Klein, all the fields that constitute our world live in the higher dimensional space. More recently, the possibility that different fields can live in different submanifolds of the entire space-time, thanks to some phenomena of field localization, has been increasingly studied. One of the early examples is the model of Rubakov and Shaposhnikov [3], which realizes a dynamical binding mechanism of fermions on a three dimensional space by means of a higher dimensional domain wall. In this model, fermions are localized on a hypersurface of the higher dimensional space-time, while bosons can propagate through the entire space. Motivated by this solitonic realization, many other models of field localization appeared in the literature. Visser [4] has pointed out that matter might be gravitationally bound on submanifolds: specifically, he proposed a model where a U(1) gauge field in $4+1$ dimensions induces a background metric that binds particles on a three dimensional brane orthogonal to the $U(1)$ gauge field. Later, Squires [5] pointed out that this effect can also be produced by a non-zero cosmological constant in the higher dimensional background.

The possibility that different fields probe different regions of space is quite natural in string theory. In this framework, consistency conditions require the presence of extended objects, called Dirichlet branes (D-branes), that, like membranes, occupy a finite region of space time [6]. They have the property that open strings end on their surface; for this reason, they have been used to construct models in which the fields of the standard model (described by open string modes) are localized on the brane surface, while gravity (described by closed string modes) can probe the entire space-time.

In the past few years, starting from the seminal paper by Arkani–Hamed, Dimopoulus, and Dvali [7] on models with large extra dimensions, the interest on the phenomenological consequences of field localization has increased. These models offer the possibility to rephrase the hierarchy problem in an interesting way. Consider in fact a model with d flat extra dimensions, compactified on torii of radius R. Let all the fields of the standard model (SM) be localized on a three dimensional brane, while gravity propagates through the entire space-time. For distances much larger than R, gravity will behave in the usual manner, and the effective four dimensional action for gravitation is obtained after integrating out the degrees of freedom relative to the extra dimensions. The resulting effective scale of gravity M_4 is corrected by volume factors, and its relationship with the fundamental scale M_d in $d+4$ dimensions becomes

$$M_4^2 = M_d^{(d+2)} (R/\hbar c)^d \,. \tag{2}$$

With two extra dimensions, choosing R close to the experimental limits coming from gravimetric experiments on deviations from Newtonian law (around 0.1 mm), one can take the fundamental scale M_d of the same order as the electroweak scale. In this way, the hierarchy problem is turned into the geometrical problem of explaining why R takes a certain value and not others. The interest on this class of models is mainly phenomenological, since their predictions (existence of Kaluza-Klein modes for graviton that influence particle physics experiments, and deviations of gravitational potential from the Newtonian form at distances not much smaller than the millimeter) can be tested in the next generation of experiments.

However, flat backgrounds are not particularly suitable to construct cosmological models. For example, a system constituted by a brane with non-zero energy density does not satisfy Einstein equations when embedded in a flat space containing only gravity: this means that a cosmological model in which the (non-zero) energy density of our universe is localized on a brane cannot be embedded in these space-times.

For the purposes of this thesis, a more interesting approach is the one of Randall and Sundrum [8]. These authors show that even gravity can be localized on a lower dimensional submanifold of a d-dimensional space, thanks to the strong curvature of the background produced by the presence of a non-zero cosmological constant. This means that it is possible to construct models with *infinitely large* extra dimensions and still compatible with the observed physics in the gravitational sector. In particular, they consider a five dimensional model in which SM fields are localized on a three brane, corresponding to a fixed point of a $\mathcal{Z}_2$ symmetry acting along the extra dimension. By choosing carefully the cosmological constant on the brane and on the bulk, it is possible to obtain a Poincaré invariant four dimensional subspace that corresponds to the observed four dimensions. Moreover, the gravitational potential measured by an observer on the brane takes the correct Newtonian form.

The Randall-Sundrum (RS) model is a simple example of fully consistent five dimensional model, from the point of view of Einstein theory of gravity, that contains a special submanifold represented by a brane with non-zero energy density, and that admits a consistent four dimensional effective description of gravity. The background of the RS model is given by an Anti-de Sitter space, a space characterized by a constant, negative curvature.

Is it possible to extend the approach of RS to general spaces, with more complex global geometries, containing some boundaries that we interpret as branes? Do these generalizations admit a consistent four dimensional form for the gravitational potential? And, most importantly, have these models interesting phenomenological consequences? In the first part of the thesis we try to answer these questions by means of specific examples.

From the point of view of general relativity, the simplest way to look at the brane is to consider it as a boundary, containing localized matter, that is described by an energy momentum tensor T_{ab}. The question is therefore how to define in a consistent way a space that admits boundaries. In general relativity, as it is normally formulated, the notion of *physical boundary* to space-time (that is, a boundary reachable at finite distance) is carefully avoided. The reason is that boundaries are artificial special places where appropriated boundary conditions have to be imposed on the physics. Without such postulated boundary conditions all predictability is lost, and the theory is not physically acceptable. Since without some deeper underlying theory there is no physically justifiable reason for choosing any particular type of boundary condition, the attitude in standard general relativity has been to simply exclude boundaries.

In string theory, the situation can be different. When a D-brane is used as a boundary there is a specific and well-defined boundary condition for the physics: D-branes are defined as the loci on which the fundamental open strings end, and Dirichlet-type boundary conditions are imposed. D-branes are therefore capable of providing both a physical boundary for the space-time *and* a plausible boundary condition for the physics in the whole space-time. However, in general, the path from string theory to low-energy effective field theory is rather indirect, and elucidation of the proper boundary conditions may be obscure.

In any case, whatever the underlying theory may be, we work only at the level of an *effective description*: the systems that we study are constituted by higher dimensional Einstein gravity coupled to a boundary, and we must face the problem to obtain a low-energy effective theory compatible with general relativity [1].

We will use the following technique to deal in a consistent way with a manifold with boundary. Take the manifold, and make a second copy of it (including a second boundary); then, sew the two manifolds together along their respective boundaries, creating a single manifold without boundary, which contains the doubled brane and exhibits a $\mathcal{Z}_2$ symmetry on reflection around the brane. In other words, the boundary is transformed in a (thin shell) brane, and the entire system can be analyzed using a generalization of the Israel–Lanczos thin-shell formalism of general relativity [9, 10]. The boundary conditions for the space-time become particularly simple: on one side of the brane, the fields behave as in the case without the brane. The other side is an exact replication of the first one [2]. Considering a 3+1 shell propagating in a 4+1 space, one finds that the metric is continuous, the connection (containing derivatives of the metric) exhibits a step-function discontinuity, and the Riemann tensor a delta-like function at the position of the brane. The discontinuity of the Riemann tensor on the brane requires some source in order to be produced: the energy density on the brane [3]. The junction conditions relate energy on the brane with discontinuities in the Riemann tensor. The dynamics of the brane is determined by these Israel-Lanczos conditions, that, analogously to the Einstein equations, rule the cosmological behavior of the projected four dimensional geometry on the brane once the localized energy density has been specified.

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[1] This fact does not means that we are not interested in the fundamental theory behind a specific model: the field content of the backgrounds we will consider is the typical one of low energy string theory.

[2] Actually, the condition of $\mathcal{Z}_2$ symmetry is not strictly necessary, and one can also consider different space-times patched at the boundary that in this way becomes an edge. However, we will limit our discussion to the simplest case of gluing $\mathcal{Z}_2$-replicated spaces.

[3] This is analogous to what happens in electromagnetism, where discontinuities in the electric field are produced by charge distributions.

Models constructed following this procedure are called *brane world* models [12]. It is interesting to notice that the energy density on the brane is chosen in such a way to adapt the boundary to the space-time: to construct a brane-world model simply means to join two space-times with an edge, where the energy density that constitutes our universe lives. Chapter 1 is dedicated to present the general formalism on which brane world models are based. Once one has constructed in a consistent way such a model, there remains the problem of understanding if it admits a correct four dimensional behavior for gravity. This is not a simple task, and in our discussion we will consider various models in which the correct behavior of gravity is obtained in different ways.

The definition of brane world shows that the brane (which corresponds to an edge) is a very important part of a space-time, and cannot be considered as a *test object* that moves without modifying the background. At the same time, it is very reasonable that the properties of the branes, and consequently the induced four dimensional quantities, are very sensitive to the global properties of the space-time in which they are embedded. The procedure of dimensional reduction via junction conditions shows that this intuition is correct: the presence of horizons affects the cosmology of the brane universe since they modify the Friedmann equation for the four dimensional model. Moreover, the presence of coordinate singularities, in the metric that describes the higher dimensional background, has in general another interesting consequence: the model does not admit a low energy effective limit in which the gravitational action assumes the usual Einstein-Hilbert form in four dimensions.

This observation allows us to relate these properties of brane world models embedded in non-trivial geometries with the problem of the cosmological constant. This problem arises because of the huge difference between the expected contributions of quantum effects to the energy of the vacuum, and its actual value. In general, quantum contributions to the vacuum energy can be eliminated by some symmetry: a typical example is a theory with exact supersymmetry, in which the cosmological constant turns out to be zero. However, phenomenology requires that supersymmetry must be broken at a scale larger than the electroweak one, predicting again a too large value for the vacuum energy. This fact turns out to be general: any symmetry that cancels the contributions to the vacuum energy must, for phenomenological reasons, be broken at some scale larger than the experimental bound on cosmological constant. A different approach to the problem consists on trying to compensate any contribution to the vacuum energy with the vacuum expectation value of some other field. However, Weinberg [13] has shown that, starting from the four dimensional covariant Einstein-Hilbert action for gravity, and admitting minimal coupling of fields to gravity, it is not possible to realize this mechanism without a great deal of fine tuning on the parameters.

Now, brane worlds embedded in non trivial backgrounds do not admit an effective four dimensional Einstein-Hilbert form for the gravitational action, and in principle can avoid Weinberg's arguments. In particular, they provide the possibility of constructing a model in which vacuum energy density is compensated by some fields living in the extra dimensions, without the necessity of fine tuning. We will discuss in Chapter 2 how to construct such a model, considering some interesting phenomenological consequences due to Lorentz violating effects.

The previous examples show in specific cases how global properties of space-time influence four dimensional physics. In Chapter 3 we address a more general question regarding the *nature* of higher dimensional singularities of space-time. In four dimensions, the study of global properties of space-time, and in particular of the behavior of singularities, represents an important part of general relativity. In particular, applications of the results by Hawking and Penrose [14] to the entire universe are generally used to state that the universe begun from an initial, *space-like* singularity. These theorems require some general condition on the nature of space-time, and on the structure of the stress-energy tensor of matter embedded in the universe. In particular, the energy density must satisfy certain *energy conditions*, that are a general requirement for the stability of the system under gravitational fluctuations.

Taking into account the results of four dimensional general relativity, we discuss whether considering higher dimensional spaces modifies the situation. In particular, one can ask if it is possible to find some geometry for the higher dimensional space-time that describes a universe with the characteristics of the observed one, but that avoids an initial singularity.

Let us clarify what we intend for a geometry describing the observed universe. It has often been noted that the space-time inside a Schwarzschild black hole is analogous to an (anisotropic) cosmology with a final big crunch. In particular, when the black hole is infinitely large, and there is no space outside the horizon, this analogy becomes exact, and the cosmological space-time reproduces exactly the features of this kind of universe. It is possible to show that, considering a higher dimensional version of this solution, one can also manage the model to obtain *isotropic* versions of universes with these properties, with respect to three dimensional submanifolds corresponding to the observed spatial dimensions. Another alternative model is the white hole solution obtained by an analytic continuation *á la* Kruskal of the usual Schwarzschild solution. In this case, the model describes an expanding, anisotropic universe starting from an initial singularity. These examples, as fascinating as they are, do not help to solve the problem of an initial singularity in a cosmological background. Indeed, also in the case of the white hole solution, in the limit of the horizon going to infinity, we obtain a flat Friedmann-Robertson-Walker universe, with an initial, unavoidable space-like singularity.

Intuitively, a possible structure for a space-time suitable for our purpose of avoiding a singularity is obtained by simply turning over a Schwarzschild black hole. The (previously internal) time dependent region, where future directed geodesics point toward the singularity, now becomes an external, expanding time dependent region, with future directed geodesics pointing to infinity. The infinite, external static region now becomes an internal static region, separated by a Cauchy horizon from the time dependent one, and contains a (potentially avoidable) *time-like singularity.* We will show how to obtain geometries with these features from a proper analytic continuation of known solutions to Einstein equations for supergravity backgrounds. In particular, the same analytic continuation applied to the Schwarzschild geometry corresponds topologically to the operation of turning the inside out, that we have just described. Another nice feature of these solutions is that their maximal analytical extension contains a time dependent contracting region that can be interpreted as a contracting phase for the universe, separated by regular horizons from the expanding phase: in other words, these space-times represent examples of higher dimensional *bouncing universes.*

Can these examples avoid the celebrated singularity theorems? A careful study of these solutions suggest that they describe a pair of infinitely distant, singular, negative tension branes. Negative tension objects do not satisfy the energy conditions necessary for the validity of these theorems. Issues regarding the stability of these negative tension objects under perturbations can be addressed considering them as lying on fixed points of some higher dimensional orbifold symmetry: this implies that they are non-dynamical objects and consequently not affected by fluctuations. For these reasons, these higher dimensional space-times become interesting *per se* since they present an example of negative-tension objects corresponding to solutions of Einstein equations, providing a step toward the identification of gravity analogues of negative tension objects, like *orientifold planes*, that naturally arise in string theory.

Chapter 1

Brane worlds and cosmology

As discussed in the Introduction, various mechanisms can be used to localize fields on lower dimensional spaces that we will call in general branes. From the point of view of gravity, these models can be embedded in a consistent way in Einstein general relativity, by considering the brane as a thin shell object containing a localized energy density. Its energy density must satisfy suitable junction conditions that adapt the shell to the global geometry. In particular, regardless of the fundamental theory that produces it, any boundary can be treated with this method at the level of an effective theory of gravity. Indeed, the boundary can be seen as a thin brane obtained by imposing a suitable $\mathcal{Z}_2$ symmetry to the space-time: in this approach, one eliminates the region of space-time that resides on one side of the brane, and substitutes this region with a duplicate of the other side (see Fig(1.1)). The resulting space-time is constituted by two replica of the same region, with a brane on the edge between them. While the metric tensor results continuous at the edge, the corresponding Riemann tensor gets contributions of the form of delta-like functions.

Once one considers the Einstein equations, matter localized on the brane, which gives a delta-like contribution to the stress energy tensor, must compensate the singular terms coming from the Riemann tensor. These compensation conditions are called junction conditions. The first part of this chapter is dedicated to present a general formalism, developed by Israel and Lanczos [9, 10] starting from Einstein theory of gravity, that allows to express these junction conditions.

Einstein equations in four dimensions give interesting predictions on the evolution of our universe, once the properties of matter that constitutes it are specified. In the same way junction conditions characterize the evolution of a universe localized on the brane: this will be the subject of the second part of this chapter. After a brief presentation of the tools of the standard cosmological model that we need in the following discussion, we will present the cosmological evolution of a general class of brane world models, specifying the conditions one must impose to ensure a

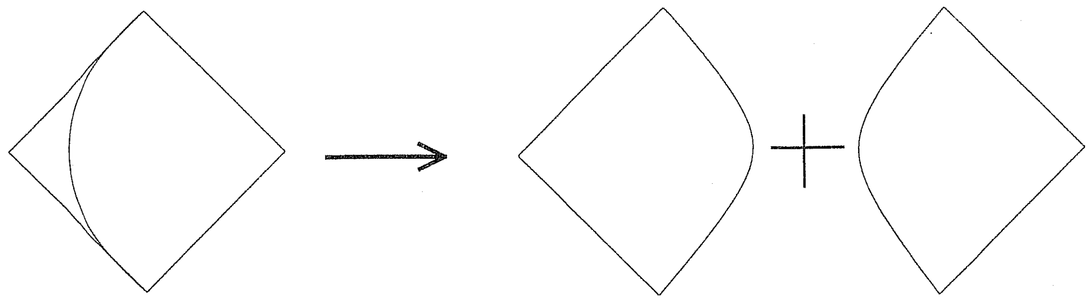

Figure 1.1: The simplest example of brane surgery: a brane world in Minkowski space. The region on the left of the brane is eliminated, and substituted with a copy of the region on the right.

cosmological behavior compatible with the observed universe. In the last section, we will work out a simple example, also addressing the question to determine how to obtain an acceptable behavior for gravity at large distances.

1.1 Israel-Lanczos junction conditions

In Einstein theory of relativity, gravity, in a d-dimensional space-time, is described by the following set of equations:

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G_d T_{ab}\,, \tag{1.1}$$

where G_d is the Newtonian constant in d dimensions (calling M_d the d-dimensional Planck mass, we define $8\pi G_d = M_d^{2-d}$), and we drop a possible cosmological constant term [1]. We use these equations to deduce the junction conditions on the brane.

Let us work in a five dimensional space-time, that we call bulk, that contains a four dimensional hypersurface, that we call brane. As we have anticipated in the introduction, in a brane world model where the brane separates two folds of the space-time identified by a $\mathcal{Z}_2$ symmetry, the Riemann tensor generally presents terms proportional to delta-like functions.

To express in a closed way the complete Riemann tensor, it is necessary to define the induced quantities of the higher dimensional space on the brane. Calling g_{ab} the bulk metric, we need the definition of the *unit normal vector* n to the brane. This is a space-like vector orthogonal to the brane surface: this means that calling V^a the

[1] A contribution coming from a cosmological constant Λ can always be re-absorbed in the definition of the energy momentum tensor: $T^a_b \to T^a_b + \frac{\Lambda}{8\pi G_d} diag(-1, 1, ...)$.

velocity of the three spatial coordinates of the brane, it satisfies $V^a \cdot n_a = 0$. It also obeys the following normalization condition

$$g_{ab} n^a n^b = 1 \,. \tag{1.2}$$

Starting from the normal vector, the *induced metric* on the brane is defined as

$$h_{ab} = g_{ab} - n_a n_b \,. \tag{1.3}$$

It is easy to see, calculating its trace, that h_{ab} has dimension 4, and consequently represents the metric of a four dimensional space.

Now, we need the definition of the *extrinsic curvature* (also called second fundamental form) of the brane, given by

$$K_{ab} = h_a^l \nabla_l n_b \,, \tag{1.4}$$

This quantity is very important for our arguments, because it allows to express the delta-like part of the Riemann tensor. Indeed, starting from the bulk metric, using standard definitions, one realizes that the Riemann tensor results [15]:

$$R_{abcd} = -\delta(\eta)\ [K_{ac}\, n_b\, n_d + K_{bd}\, n_a\, n_c - K_{ad}\, n_b\, n_c - K_{bc}\, n_a\, n_d] + R^{\rm bulk}_{abcd}. \tag{1.5}$$

In the previous formula, there is singular contribution proportional to the delta function $\delta(\eta)$, where η indicates the direction normal to the brane, and a regular contribution due to bulk quantities.

At this point, we have to specify the right hand side of Eq. (1.1), namely the stress energy tensor of the space-time. It gets a contributions from the fields that live in the bulk, and from matter localized on the brane:

$$T_{ab} = \delta(\eta)\, T^{\rm surface}_{ab} + T^{\rm bulk}_{ab} \,. \tag{1.6}$$

Now, substitute (1.5) and (1.6) in the Einstein equations (1.1). Isolating the delta-like part of the equations, and using the $\mathcal{Z}_2$ symmetry, one ends with the following *Israel-Lanczos junction conditions*:

$$K_{ab} = -4\pi G_5 \left(T_{ab} - \frac{1}{3}\, g_{ab} T \right) , \tag{1.7}$$

A variational derivation of the Israel-Lanczos junction conditions

Let us derive explicitly the equations (1.7) following a variational procedure: the methods used will result useful in the following discussion.

We know that the metric is continuous everywhere and its derivatives as well, except on the brane, that we call Σ, where the derivatives have a discontinuity. The action for the brane-bulk system is given by the sum of the Einstein-Hilbert action in five dimensions, plus an action describing the energy density on the brane, plus an important boundary term that constitutes the point of the present discussion:

$$S = S_{EH} + S_{br} + S_{GH} \, ; \tag{1.8}$$

where

$$S_{EH} = \frac{1}{8\pi G_5} \int d^{4+1}x \sqrt{-g_5}\, R \, , \tag{1.9}$$

$$S_{br} = - \int_{\Sigma_\pm} d^{3+1}x \sqrt{-h_4}\, \mathcal{L}_{br} \, . \tag{1.10}$$

Here, h_{ab} is the induced metric on the hypersurfaces Σ_+ and Σ_-, since for generality we consider as different the two sides of the brane. With $\Sigma_\pm$ we indicate the sum of the contributions of the two sides. Moreover, $\mathcal{L}_{br}$ is the Lagrangian for the matter living on the brane. The last term in (1.8) is the boundary Gibbons-Hawking term and will be introduced in order to render the variational procedure fully consistent: indeed, recall that in a variational procedure all the final quantities must be well defined and continuous. Let us perform the variation of the action (1.8). It receives the following contribution due to the presence of discontinuities on the brane:

$$\delta S_{EH} = - \frac{1}{8\pi G_5} \int_{\Sigma\pm} d^{3+1} \sqrt{-h} g^{ab} n^c (\nabla_a \delta g_{bc} - \nabla_c \delta g_{ab}) \, , \tag{1.11}$$

Replacing $g^{ab} = h^{ab} + n^a n^b$ in (1.11) one gets

$$\delta S_{EH} = - \frac{1}{8\pi G_5} \int_{\Sigma\pm} d^{3+1} \sqrt{-h}\, h^{ab} n^c (\nabla_a \delta g_{bc} - \nabla_c \delta g_{ab}) \, . \tag{1.12}$$

It is possible to identify the term $(\nabla_a \delta g_{bc} - \nabla_c \delta g_{ab})$ as the discontinuity of the derivative of the metric across the surface. To compensate this discontinuity let us introduce in the action the following Gibbons-Hawking boundary term

$$S_{GH} = - \frac{1}{8\pi G_5} \int_{\Sigma\pm} d^{3+1} \sqrt{-h}\, K \, , \tag{1.13}$$

where $K = h^{ab} K_{ab}$ and K_{ab} is the extrinsic curvature defined in (1.4). Varying the Gibbons-Hawking term one gets

$$\delta S_{GH} = - \frac{1}{8\pi G_5} \int_{\Sigma\pm} d^{3+1} \sqrt{-h} \left(\delta K - \frac{1}{2} K h^{ab} \delta g_{ab} \right) \, . \tag{1.14}$$

Now, we need the variation of δK. Using the variation

$$\delta n_a = \frac{1}{2} n_a n^b n^c \delta g_{bc} \,, \tag{1.15}$$

after some algebra one gets

$$\delta K = -K^{ab}\delta g - h^{ab}n^c(\nabla_a \delta g_{bc} - \nabla_c \delta g_{ab}) + \frac{1}{2} K\, n^c n^d \delta g_{cd} \,. \tag{1.16}$$

Using these quantities in Eq. (1.14), let us consider the sum

$$\begin{aligned} \delta S_{EH} + \delta S_{GH} \quad &\propto \int d^{3+1}x\sqrt{-h}\Big(\frac{1}{2}h^{ab}n^c\nabla_a g_{bc} + K^{ab}\delta g_{ab} \\ &- \frac{1}{2}K\, n^a n^b \delta g_{ab} - \frac{1}{2} K\, h^{ab}\delta g_{ab}\Big) \,. \end{aligned} \tag{1.17}$$

To simplify this formula, consider the following argument. Take a vector X^a tangential to $\Sigma\pm$, so that

$$\nabla_a X^a = h^{ab}\nabla_a X_b + n^a n^b \nabla_a X_b \,. \tag{1.18}$$

Define the derivative operator $\tilde{\nabla}$ on Σ by

$$\tilde{\nabla}_c X^a = h^a_{\ d} \nabla_c X^d \,; \tag{1.19}$$

then, using (1.18) and (1.19) one has

$$\nabla_a X^a = \tilde{\nabla}_a X^a - X^a n^b \nabla_b X_a \,, \tag{1.20}$$

and

$$h^{ab}n^c\nabla_a \delta g_{bc} = \nabla_a(h^{ab}n^c\delta g_{bc}) - \delta g_{bc}\nabla_a(h^{ab}n^c) \,. \tag{1.21}$$

But using the definition of extrinsic curvature K_{ab} given in (1.4) one finds

$$h^{ab}n^c\nabla_a \delta g_{bc} = \nabla_a(h^{ab}n^c\delta g_{bc}) + K\, n^a n^b \delta g_{ab} - K^{ab}\delta g_{ab} \,. \tag{1.22}$$

Then, the substitution of (1.22) in (1.17) and the integration of the total derivative term gives the final result

$$\delta S_{EH} + \delta S_{GH} = \frac{1}{8\pi G_5}\int_{\Sigma\pm} d^{3+1}x\sqrt{-h}(K^{ab} - K h^{ab})\delta g_{ab} \,. \tag{1.23}$$

Therefore, the Gibbons-Hawking term renders this derivation fully consistent, canceling the discontinuities in the variational procedure.

Let us include now the variation of matter on the surface Σ:

$$\delta S_{br} = \int_{\Sigma\pm} d^{3+1}x\sqrt{-h}\, T^{ab}\delta g_{ab} \,, \tag{1.24}$$

where $\sqrt{-h}T^{ab}/2 \equiv \delta S_{br}/\delta h_{ab}$ is the energy momentum tensor of the brane. Now, requiring the variation of the whole action

$$\delta S = \delta S_{EH} + \delta S_{GH} + \delta S_{br} \tag{1.25}$$

to be zero, using (1.23) and (1.24), we get the Israel-Lanczos matching conditions

$$K_{ab}|_+^- - g_{ab}K_c{}^c|_+^- = -8\pi G_5\, T_{ab}\,, \tag{1.26}$$

where $K_{ab}|_+^- = K_{ab}^+ - K_{ab}^-$ and the plus and minus signs indicate on which side of the brane the extrinsic curvature is evaluated. This relation is equivalent to

$$K_{ab}|_+^- = -8\pi G_5 \left(T_{ab} - \frac{1}{3}\, g_{ab}T\right), \tag{1.27}$$

where $T \equiv T_{ab}g^{ab}$ is the trace of T_{ab}. Imposing the $\mathcal{Z}_2$ symmetry means to take

$$K_{ab}^+ = -K_{ab}^- = K_{ab} \qquad \Rightarrow\; K_{ab}|_+^- = 2K_{ab}\,. \tag{1.28}$$

Substituting (1.28) in (1.27), one recovers Eq. (1.7).

1.2 An example of brane surgery

We have presented in the previous Section the junction conditions satisfied by matter on the brane. This Section is devoted to determine the geometrical properties of a brane embedded into a bulk described by a static, maximally symmetric metric of the form

$$\mathrm{d}s^2 = -h(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{h(r)} + r^2\,\mathrm{d}x_{3,k}^2. \tag{1.29}$$

Here, $\mathrm{d}x_{3,k}$ indicates the metric of a three dimensional spatial maximally symmetric submanifold of constant curvature k (with $k = 0, \pm 1$). It can be expressed as

$$\mathrm{d}x_{3,k}^2 \equiv \mathrm{d}\chi^2 + \frac{\sin^2(\sqrt{k}\chi)}{\sqrt{k}}\,(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{1.30}$$

The form (1.29) is not the most general form for a metric with the required properties, but it is fully representative for the illustrative purposes of this chapter.

As we know, the brane corresponds to an hypersurface on this space, defined in terms of a parameter τ that will subsequently define the proper time of the induced four dimensional metric:

$$X^a(\tau, \chi, \theta, \phi) = (t(\tau), a(\tau), \chi, \theta, \phi)\,. \tag{1.31}$$

The velocity of the (χ, θ, ϕ) element of the brane is defined as

$$V^a = \left(\frac{dt}{d\tau}, \frac{da}{d\tau}, 0, 0, 0\right). \tag{1.32}$$

By definition, the normal to the brane is a vector such that $V^a n_a = 0$. It is immediate to check that a suitable normal vector is given by

$$n_a = \pm\left(-\frac{da}{d\tau}, \frac{dt}{d\tau}, 0, 0, 0\right), \tag{1.33}$$

where the choice of sign is connected to the direction of the normal: a normal pointing outwards correspond to a positive sign, and viceversa.

The normalization condition $n^2 = 1$ translates in the condition

$$h(a)\dot{t}^2 - \frac{\dot{a}^2}{h(a)} = 1, \tag{1.34}$$

using Eq. (1.34) and the assumed form of the metric, and defining $\dot{a} = da/d\tau$. The induced four dimensional metric takes the well known form

$$ds^2_{3+1} = -d\tau^2 + a(\tau)^2\, dx^2_{3,k}. \tag{1.35}$$

This means that despite the rather unusual form for the five dimensional metric, this procedure furnishes a Friedmann-Robertson-Walker form of the four dimensional projected metric. In this approach, the scale factor of the four dimensional universe corresponds to the position of the brane in the five dimensional background: we will widely use this fact in the next Section.

At this point, let us present the results of the calculation of the extrinsic curvature for this space:

$$K_{ab} = \frac{1}{2}\, n^c\, \frac{\partial g_{ab}}{\partial x^c}. \tag{1.36}$$

If we go to an orthonormal basis, the $\hat{\chi}\hat{\chi}$ component is easily evaluated

$$K_{\hat{\chi}\hat{\chi}} = K_{\hat{\theta}\hat{\theta}} = K_{\hat{\phi}\hat{\phi}} == -\frac{\sqrt{h(a) + \dot{a}^2}}{a}, \tag{1.37}$$

while for the $\tau\tau$ component one gets

$$K_{\hat{\tau}\hat{\tau}} = +\frac{d}{da}\left(\sqrt{h(a) + \dot{a}^2}\right). \tag{1.38}$$

Furnishing the extrinsic curvature, we have provided the geometrical information about the left hand side of Eq. (1.7). The second part of the chapter is devoted to specify the right hand side of this equation and to study the cosmological consequences of this choice.

1.3 Cosmology

1.3.1 Standard cosmology

As a first step, let us briefly present the main results of standard cosmology that we will use in the rest of the thesis [2]: the great successes reached by this model in describing the observed universe makes it a point of reference when one tries to construct alternative models.

The standard cosmological model (SCM) uses both general relativity and quantum field theory. It is based on some simplifying assumptions concerning the nature and large scale properties of the energy density that constitutes the universe. In this Section, in particular, we will show how the theory of gravitation (general relativity, but also Newtonian theory) yields equations that describe the geometrical evolution of our universe.

The Einstein equations in four dimensions are given in Eq. (1.1) for $d = 4$. Consider matter on the brane as a perfect fluid, with stress energy tensor of the diagonal form

$$T^a_b = \mathrm{diag}(-\rho, p, p, p)\,; \tag{1.39}$$

here, ρ is the energy density, while p is the pressure of the fluid. They are related by an *equation of state*

$$p = \omega\rho\,. \tag{1.40}$$

We will require some ansatz on the form of the metric that simplifies the Einstein equations. Our observed universe is expanding, and it appears homogeneous and isotropic along three dimensional spatial sections. The ansatz for the metric we will consider is the FRW metric that we already met at the end of the previous subsection:

$$ds^2 \equiv g_{ab}dx^a dx^b = -d\tau^2 + a^2(\tau)dx^2_{3,k}\,, \tag{1.41}$$

where $dx^2_{3,k}$ is the metric of the three dimensional maximally symmetric space with constant curvature parameter $k = \pm 1, 0$, see Eq. (1.30).

The energy momentum tensor obeys an equation of conservation of the form $T^{ab}{}_{;b} = 0$. The $\mu = 0$ component of this equation, together with a FRW form for the metric, gives the equation of conservation for the energy:

$$\dot\rho + 3H(\rho + p) = 0\,, \tag{1.42}$$

where $H = \frac{\dot a}{a}$ is the *Hubble parameter* and a dot means derivative with respect to the cosmic time τ.

[2] For a complete treatment of these subjects, see for example [16].

This equation allows us to determine the cosmological evolution of the energy density as a function of the scale factor. For the three forms of energy density normally considered in cosmology one has

- Matter: $\omega = 0 \quad \Rightarrow \rho \propto a(\tau)^{-3}$,
- Radiation: $\omega = \frac{1}{3} \quad \Rightarrow \rho \propto a(\tau)^{-4}$,
- Vacuum energy $\omega = -1 \quad \Rightarrow \rho =$constant.

Returning to Einstein equations, let us write the non zero components of the Ricci tensor and the Ricci scalar for a FRW metric:

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a}\,, \\ R_{ij} &= \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2}\right) g_{ij}\,, \\ R &= -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)\,. \end{aligned} \tag{1.43}$$

It is a simple task to show that a suitable combination of Einstein equations give us the so called Friedmann equation:

$$H^2 = \frac{8\pi G_4}{3}\rho - \frac{k}{a^2}\,, \tag{1.44}$$

while a linear combination of the 0-0 and i-j components give the Raychaudhuri equation, that describe the acceleration of our universe:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_4}{3}(\rho + 3p)\,. \tag{1.45}$$

Equations (1.42), (1.44) and (1.45) are the basic equations of the standard cosmological model. Since they describe observational results with a great deal of accuracy, any alternative cosmological model should give, for the evolution of the scale factor of the present universe, the same predictions of these equations.

While further consequences and developments of the previous equations are beyond our scope, we want to stress that also within the Newtonian theory of gravitation one find the same evolution equations for the scale factors: see [17] for a detailed discussion. This means that, in general, a model that predicts the correct behavior for gravity at large distances normally also gives the correct cosmological equations for the evolution of the universe. Since the large scale constituents of our universe (galaxies, etc) actually are in slow motion and feel weak fields, it is not a surprise that they are well described by Newtonian theory.

1.3.2 Brane world cosmology

After this brief overview of the standard cosmological equations, let us proceed to consider what happens in our brane world model. We know that in this case the equations that rule the system are the junction conditions (1.7). Let us consider the general form of the five dimensional metric considered in Section (1.2), and let us again consider a perfect fluid form for matter on the brane: the stress energy tensor is given by (1.39).

Under these assumptions, by means of (1.7), and using the expression for the extrinsic curvature in Section (1.2), we get the following junction conditions:

$$8\pi\, G_5\, \rho = \pm 3\, \frac{\sqrt{h(a)+\dot{a}^2}}{a}. \tag{1.46}$$

Choosing the plus sign the energy density results positive definite: a plus sign corresponds to take a normal that points inward, and the side of the $\mathcal{Z}_2$ replicated space that corresponds with this choice [3].

For the pressure we obtain

$$8\pi\, G_5\, p = -\frac{1}{a^2}\,\frac{\mathrm{d}}{\mathrm{d}a}\left(a^2\sqrt{h(a)+\dot{a}^2}\right). \tag{1.47}$$

These equations are compatible with the conservation of the stress energy localized on the brane. That means that the model satisfy an equation of conservation of energy of the standard form:

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{1.48}$$

While the conservation equation is identical to that for standard cosmology, we find a surprise calculating the equation that should correspond to the Friedmann equation. One gets

$$H^2 = -\frac{h(a)}{a^2} + \left(\frac{8\pi\, G_5\, \rho}{3}\right)^2, \tag{1.49}$$

in contrast, the *standard* Friedmann equation is

$$H^2 = -\frac{k}{a^2} + \frac{\Lambda}{3} + \frac{8\pi\, G_4\, \rho}{3}, \tag{1.50}$$

where we have explicitly written a term due to the cosmological constant (see footnote 1).

Consider Eq. (1.49): as we have already anticipated, the scale factor of the four dimensional universe corresponds to the position of the brane in the higher

[3] As an example, notice that in Fig. (1.1) the brane contains negative energy density.

dimensional background. It is possible to choose the parameters of the model in such a way to obtain a static, Poincaré invariant four dimensional universe: indeed, it is enough to choose the position of the brane and its energy density in such a way that the right hand side of equation (1.49) vanishes. This situation corresponds to a *static* configuration, with brane that does not move in the space. However, considering a perturbation on the energy density, one obtains a model in which the brane moves, producing a variation of the observed scale factor [18, 19].

Let us render quantitative these statements. Starting from (1.49), to get a brane cosmology that is not in conflict with observation, one split the (3+1)-dimensional energy into a constant λ, determined by the brane tension, plus ordinary matter ρ_{br}, with $\rho_{br} \ll \lambda$ to suppress the quadratic term in comparison to the linear one [12]. Then, being $\rho = \lambda + \rho_{br}$, one gets

$$H^2 = -\frac{h(a)}{a^2} + \left(\frac{8\pi\, G_5\, \lambda}{3}\right)^2 + \left(\frac{16\pi\, G_5\, \lambda}{3}\right)\left(\frac{8\pi\, G_5}{3}\right)\left[\rho_{br} + \frac{1}{2}\frac{\rho_{br}^2}{\lambda}\right]. \qquad (1.51)$$

Singling out the term linear in ρ_{br}, this allows us to identify

$$G_4 = G_5\left(\frac{16\pi\, G_5\, \lambda}{3}\right); \qquad \text{that is} \qquad G_5 = \sqrt{\frac{3\, G_4}{16\pi\, \lambda}}. \qquad (1.52)$$

Therefore

$$H^2 = -\frac{h(a)}{a^2} + \left(\frac{8\pi\, G_4}{3}\right)\left[\frac{1}{2}\lambda + \rho_{br} + \frac{1}{2}\frac{\rho_{br}^2}{\lambda}\right]. \qquad (1.53)$$

Since we want $\lambda \gg \rho_{br}$ to suppress the quadratic term, this leaves us with a large (3+1)-dimensional cosmological constant that we will need to eliminate by canceling it (either fully or partially) with some term in $h(a)$ [12].

This result is in its own way quite remarkable: up to this point no assumptions had been made about the size of the brane tension, or even whether or not the brane tension was zero. Nor had any assumption been made up to this point about the existence or otherwise of any cosmological constant in the five dimensional bulk. It is observational cosmology that first forces us to take λ large (electro-weak scale or higher to avoid major problems with nucleosynthesis), and then forces us to deduce the presence of an almost perfectly countervailing cosmological constant in the bulk.

Naturally, another important constraint that the model must satisfy regards the behavior of gravitational interactions. For distances larger than the millimeter, indeed, it is necessary to recover a four dimensional behavior for the Newtonian potential. The fact that one recovers a correct form for the Friedmann equation is a significant clue that gravity becomes four dimensional at large distances. But how large? In the next section, we will try to answer, in a simple example, this question.

1.3.3 The Randall-Sundrum surgery

Let us discuss a simple choice for the function $h(r)$ in Eq. (1.29), namely, the case of pure AdS space, with a flat three dimensional submanifold $k = 0$: this is the Randall-Sundrum model [8]. We choose

$$h(r) = \frac{\Lambda}{6} r^2 \,. \tag{1.54}$$

Let us consider initially a brane-world containing only vacuum energy: with the notation of the previous subsection, $\rho = \lambda$, and $\rho_{br} = 0$. The condition that ensures a Poincaré invariant brane results

$$\Lambda = 8\pi G_4 \lambda \,. \tag{1.55}$$

This is called condition of *criticality*: notice that the vacuum energy on the brane and the one in the bulk must be chosen with great accuracy to ensure this condition. In this situation, the brane does not move and the universe on the brane is static and Poincaré invariant. A small perturbation from this static situation, with the addition of matter with $\rho_{br} \ll \lambda$, forces the brane to move, and this fact produces an interesting cosmology on the brane. The Friedmann equation turns out to be

$$H^2 = \left(\frac{8\pi \, G_4}{3}\right) \left[\rho_{br} + \frac{1}{2}\frac{\rho_{br}^2}{\lambda}\right] \,, \tag{1.56}$$

and it describes, neglecting sub-dominant quadratic corrections, a flat FRW universe with vanishing cosmological constant.

What about gravity in this model? Let us change coordinates to recast the metric in a more familiar form. Define

$$\eta \equiv \sqrt{\frac{6}{\Lambda}} \, \ln r \,. \tag{1.57}$$

This implies

$$\mathrm{d}s^2 = +\mathrm{d}\eta^2 + \exp\left(-2\sqrt{\frac{\Lambda}{6}}\,\eta\right)\left[-\frac{\Lambda}{6}\,\mathrm{d}t^2 + a^2\,\mathrm{d}x_{3,0}^2\right] \,. \tag{1.58}$$

Re-label the time parameter in terms of proper time of a cosmologically comoving observer

$$\tau \equiv \sqrt{\frac{\Lambda}{6}}\, t \,. \tag{1.59}$$

One ends with a metric of the form

$$ds^2 = +d\eta^2 + \exp\left(-2\sqrt{\frac{\Lambda}{6}}\,\eta\right)\left[-d\tau^2 + dx^2 + dy^2 + dz^2\right] , \tag{1.60}$$

that is, exactly the metric of Randall and Sundrum as presented in [8]. These authors have shown that, when suitable junction conditions hold (that correspond exactly to the conditions (1.55)), gravity at large distances behaves as four dimensional. This is due to the fact that, due to the negative cosmological constant, the five dimensional space is curved in such a way to localize gravity. More precisely, there is a graviton bound state attached to the brane with an exponential falloff controlled by the distance scale parameter

$$L_{\text{graviton}} = \sqrt{\frac{6}{|\Lambda_{4+1}|}}. \tag{1.61}$$

Now the experimental fact that we do not see short distance deviations from the inverse square law of gravity at least down to millimeter scales implies that L_{graviton} must be less than one millimeter. For distances larger than this, this model predicts a four dimensional behavior for gravity.

To conclude, let us return to the cosmological side of the discussion. With the condition of criticality, one obtains that

$$\lambda = \frac{3}{4\pi\, G_{3+1}\, L^2_{\text{graviton}}} = \frac{3}{4\pi}\frac{L^2_{\text{Planck}}}{L^2_{\text{graviton}}}\, M_4\,. \tag{1.62}$$

This relation implies that if L_{graviton} is as large as allowed by experimental constraints, then the quadratic terms in the density become important once temperatures reach the electroweak scale (about 100 GeV). In particular, this model offers the possibility of seeing deviations from the standard cosmology as we go back with time: while the present universe can be described with great accuracy, the early universe should have been different from the one described by SCM.

Chapter 2

Cosmological constant and self tuning mechanisms

In the previous chapter we have considered the Randall-Sundrum space-time as a background to embed a brane world model. In this background, we have shown that the five dimensional metric can be re-casted in a form that admits four dimensional Poincaré invariant slices. Moreover, due to a negative cosmological constant in the bulk, the coefficient in front of the four dimensional slice in (1.60), called warp factor, succeeds on localize gravity.

Four dimensional Poincaré invariance, although simplifies the tractation, is not strictly necessary to describe phenomenology. Indeed, what we observe is an universe homogeneous and isotropic in the three spatial dimensions, not in time. Starting from this observation, one can ask if it is possible to construct brane world models in backgrounds that do not admit four dimensional Poincaré invariant reparametrizations: we will call these backgrounds asymmetrically warped [20, 21]. These backgrounds exist, and in general the associated space-times have non trivial global properties, with horizons that cover singularities. A higher dimensional version of the Schwartzschild black hole is an example of these spaces. A part from the intrinsic interest to analyze the behavior of brane world models in more general backgrounds, these attempts are interesting for at least two reasons.

The first is the fact that the low energy four dimensional limit is not Lorentz invariant, predicting a different behavior for gravitational and electromagnetic waves. The effects of violation of the Lorentz invariance are subtle and very interesting from the cosmological point of view.

The second is represented by the new insights that these models offer toward the understanding of the cosmological constant problem. In general, the most interesting attempts to solve the cosmological constant problem are based on models that try to absorb the cosmological constant on the vacuum expectation value of some field, like a scalar field. However, various models based on this idea have failed to solve the problem. Weinberg [13] presented a general argument, essentially based on the assumption of general covariance of the four dimensional gravitational action, which shows why models of this type cannot work.

Now, brane world models, which by definition are models in which gravity is described as a five dimensional theory, can help in facing the problem. In this chapter we will discuss how to look at the cosmological constant problem in this extra dimensional setting. In the first part of the chapter we will review the cosmological constant problem. The second part is instead dedicated to a brane world approach to the problem: we will show that, for this purpose, the most interesting models are those based on asymmetrically warped backgrounds, discussing a specific example in which we will also study the effects of Lorentz violation.

The cosmological constant problem

The cosmological constant problem is one of the difficulties that arise in trying to join general relativity with quantum field theory.

Let us briefly recall the problem, starting from the Einstein equations with a cosmological constant term:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_4 T_{\mu\nu} + \Lambda g_{\mu\nu} \, . \tag{2.1}$$

Here, Λ is an integration constant, that arises in the variational procedure that yields the equations: it has units of the inverse of a squared length. Taking as an ansatz a FRW form for the metric, one can parametrise the present bounds on the value of Λ in terms of the present value of the Hubble parameter, H_0:

$$\Lambda \lesssim H_0^2 = h_0^2 \cdot 10^{-52} m^{-2} \tag{2.2}$$

where h_0 is a constant of order one that reflects our error in the determination of the Hubble parameter.

Consider the introduction of quantum fields in the description. First of all, notice that the fundamental scale of gravity is M_{Pl}, with $M_{Pl} \simeq 2 \cdot 10^{18}$ GeV. One can expect the integration constant Λ to be of the same order of M_{Pl}, since this is the typical scale of the theory. However, this expectation is far to be correct, since the experimental limit (2.2), in terms of M_{Pl}, reads

$$\Lambda \lesssim 10^{-120} M_{Pl}^2 \, . \tag{2.3}$$

Moreover, even if we set $\Lambda = 0$, we know that in any quantum field theory there is a non zero vacuum energy (the energy of the ground state) due to contributions of vacuum fluctuations. This can be written, in terms of the energy momentum tensor, as

$$\langle T_{\mu\nu}\rangle = -\langle\rho\rangle g_{\mu\nu}. \tag{2.4}$$

This implies that the vacuum energy contributes to the cosmological constant via the following quantity

$$\lambda_{eff} = 8\pi G_4\langle\rho\rangle = \frac{\lambda^4}{M_{Pl}^2}, \tag{2.5}$$

where λ represents some characteristic scale. The inequality (2.3) becomes

$$\lambda \lesssim 10^{-3} eV. \tag{2.6}$$

The smallness of the parameter λ contrasts with our knowledge of standard model of particle physics. Indeed, in this framework one can expect that the vacuum fluctuations of the quantum fields produce an energy density μ^4, with μ at least of the order of $\mathcal{O}$(TeV), the natural cut-off of the theory.

Actually, one can consider the possibility that some high-energy symmetry exists, that cancels exactly λ_{eff} at a certain high scale. However, any low energy phase transition (think for example to the QCD phase transition) contributes to the effective vacuum energy, in a way that is not canceled by the previous symmetry: in other words, the cosmological constant problem is actually an *infrared* problem.

An alternative approach to the problem is the idea of dynamical adjustment: we would like to find a mechanism that compensates new contributions to the vacuum energy density with an equal (but opposite in sign) contribution. For example, a scalar field could adsorb new contributions to the vacuum energy on its expectation value.

Unfortunately, Weinberg [13] has shown that this idea is not realizable without reintroducing an unadmissibly high level of fine tuning in the model. For a simplified version of his proof, consider the following argument [23]. Take the standard minimal form for the action that describes gravity minimally coupled to matter and one adjusting scalar:

$$S = \frac{1}{16\pi G_4}\int d^4x\,\sqrt{-g}\,\left(R + \mathcal{L}_{SM} + \mathcal{L}(\phi)\right), \tag{2.7}$$

Now, integrate out the standard model degrees of freedom, and choose the extremizing value for the scalar field ϕ_0. One finds

$$S = \frac{1}{16\pi G_4}\int d^4x\,\sqrt{-g}\,\left(R - \Lambda_{SM} - V(\phi_0)\right). \tag{2.8}$$

To cancel the cosmological constant means to render zero the sum of the potential for the scalar and of Λ_{SM}: $\Lambda_{SM} + V(\phi_0) = 0$.

It is clear that it is not possible to choose once for ever the potential $V(\phi)$ in such a way that its minimum satisfies the previous equation for *any value* of the parameter Λ_{SM}. This means that for a given value of $V(\phi_0)$, one must impose a great fine tuning to the quantity Λ_{SM}.

However, notice that this example, and Weinberg's general proof, are based on the hypothesis of four dimensional general covariance. There is the possibility that models based on the brane world picture can give new insights to the problem, since gravity is actually described by an higher dimensional theory.

For example, we have already shown in Section (1.3.3) that, in the RS model, it is possible to compensate a four dimensional vacuum energy with bulk quantities, in such a way to obtain a Poincaré invariant brane, with $H = 0$, also in the presence of a non-zero cosmological constant in the bulk. However this model requires a fine tuning between cosmological constant on the brane and the one in the bulk. In the past years, the possibility to compensate vacuum energy density with bulk quantities has been analyzed in detail, looking for brane world models that provide this compensation without fine tuning. Such mechanisms of adjustment of the cosmological constant are called *self tuning mechanisms* [26].

The next sections are devoted to present some examples of these attempts for a brane world model embedded into a five dimensional bulk. In Section (2.1) we will consider a bulk containing gravity and a scalar field: in this case an apparent self tuning solution has results to be erroneous, for reasons that is very interesting to point out. Section (2.2) shows an example of how to cure the problems of the previous model.

Self tuning mechanisms

2.1 A bulk containing a scalar field

In [24] and [25], an adjustment mechanism of the cosmological constant has been presented, for a brane world embedded into a bulk containing gravity and a scalar field. In this approach, any new contribution to the vacuum energy on the brane is transmitted to the expectation value of the scalar and of the cosmological constant in the bulk.

Let us present briefly the model. Consider the action

$$S = \int d^5x\sqrt{-g_5}\Big[R - \frac{4}{3}(\nabla\phi)^2 - \Lambda e^{a\phi}\Big] - \int d^4x\sqrt{-g_4}e^{b\phi}\lambda \qquad (2.9)$$

Here, g_5 and g_4 correspond to the determinants of the five dimensional and four dimensional metric, respectively, and we have set the five dimensional Newton constant equal to one. This action describes a bulk containing gravity and a scalar field, and a brane containing vacuum energy density λ conformally coupled to the bulk scalar field.

To analyze the model, the first step is to find exact solutions to the Einstein and scalar equations in the bulk, using some ansatz. First, we look for solutions of Einstein equations that are four dimensional Poincaré invariant, with the five dimensional metric that takes the form

$$ds^2 = e^{2A}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + dr^2 \,. \tag{2.10}$$

We will require that the metric and the fields depend only on one variable, let us say r, in such a way that $A = A(r)$, and $\phi(r)$. Strictly speaking, this condition is true only when Birkhoff theorem holds: however, the presence of a scalar field generically violates the theorem, and one needs additional conditions to ensure the dependence only on one variable. In this case, a condition that guarantees this fact is

$$A' = \alpha\phi' \tag{2.11}$$

for some constant α, and a prime indicates derivative along r [1]. In any case, we will take this as an ansatz, that moreover simplifies the calculations.

With these conditions, the solutions to the Einstein equations and to the equation of motion for the scalar field are easily found. For the scalar field, one gets

$$\phi = \pm\frac{3}{4}\ln|\frac{4}{3}r - c| + d \tag{2.12}$$

where c and d are arbitrary integration constants. One obtains the expression for the scale factor $A(r)$ starting from (2.11). Actually, this solution presents a singularity of the Ricci scalar at the point $r = \frac{3}{4}c$.

The brane will be located on a point of the extra dimension, and the physical region of the space-time will be represented by the interior region between the singularity and the brane. The exterior region between the brane and infinity must be excluded because, for any choice of the parameters, the warp factor is not sufficiently steep to localize gravity.

On the contrary, taking the interior region, the singularity represents a natural boundary for the space-time, that will be cut at this point. The fact that the extra dimension is finite indicates that the four dimensional gravity behaves in the correct way for distances larger than the experimental bound on deviation from the Newtonian form for the gravitational potential, when one chooses the parameter c sufficiently small.

[1] See Appendix A for a proof of this statement in a more general background.

Using the methods developed in the previous chapter, one can find the junction conditions that the vacuum energy density λ must satisfy to render consistent the model. We will not present here the explicit results of the calculation since we will obtain them in the next Section with a particular choice of the parameters. We limit ourselves to the simplest case of a bulk without cosmological constant: $\Lambda = 0$. The authors of [24] propose a simple way to understand how the adjustment mechanism work. Looking at the action (2.9), one sees that any correction to the parameter λ, coming from some new contribution to the vacuum energy, can be absorbed into the expectation value of the extra dimensional scalar field ϕ_0, without changing the form of the action and without requiring a fine tuned choice of λ.

However, a simple variation of this argument shows the problems behind this model [23]. The field configuration corresponding to the solution of the equation of motion of the scalar field should extremize the action (2.9). This condition is not true in this case. This is immediately clear since the only non-derivative coupling of ϕ appears in the factor multiplying λ: thus, for nonzero λ, changing ϕ by an infinitesimal constant leads to a change in S proportional to that constant.

The point is that the space time presents a boundary, being cut at the singularity. When varying the action on a finite interval (as appropriate if a singularity is present), one gets, in addition to the equations of motion, a boundary term, as we have already seen in Section (1.1), which has not been considered. Notice indeed that the action (2.9) does not present a Gibbons-Hawking term, necessary to compensate discontinuities when boundaries are present.

This additional boundary term will reintroduce a fine tuning on the model. To see this fact, consider the method used in [27] to solve the problems of this model: add a second brane that covers the singularity, containing a dark energy σ. The resulting system is a brane world model with two branes, each of them containing a specific energy density. We have seen in the previous chapter that the Gibbons-Hawking terms in this system has the effect to impose precise junction conditions for the two dark energies λ and σ. One finds, solving the equation, that a new fine tuning between the two dark energies is required, exactly like in the RS model, and in this way the model does not solve the cosmological constant problem.

Let us also present another clue of the fact that some version of Weinberg no-go theorem should hold also in this case. The point is that a five dimensional solution like (2.10) is still Poincaré invariant in four dimensions. This means that integrating out the additional spatial dimension, one finds, for a scale lower than the five dimensional Planck scale, a covariant four dimensional effective action for gravity: the no-go theorem applies to this effective theory.

To conclude, we must look for a model constructed in such a way that

- It does not present dangerous boundaries, a part from the brane, that would need a separate discussion. This means, in particular, that the physical region of the space-time does not contain naked singularities.

- It does not admit, *at any scale*, an effective description for gravity in terms of a four dimensional Einstein-Hilbert action. This fact leads to possible interesting consequences, since a model constructed in these terms should present effects of Lorentz violation.

The following section is devoted to present an example of such a model.

2.2 A bulk containing a scalar and a gauge field

Let us consider the possibility to add a gauge field to the background considered up to now, constituting a charged dilatonic background [28]. In Section (2.2.1) we will present the total bulk action, and the general solutions to the associated Einstein and field equations. The interesting fact is that the resulting space-time does *not* share the bad properties of the case analyzed in the previous section.

Indeed, this geometry presents horizons that cover the naked singularities. These are points where the metric component g^{tt} vanish. They describe null-like hypersurfaces in which the space-time can safely be cut off without the necessity to introduce additional boundaries: in particular, a gravitational action does not need additional Gibbons-Hawking terms due to their presence. They can be seen as physical boundaries since, from the bulk point of view, an object (in particular a graviton) takes an infinite time to reach them. The resulting five dimensional metric, moreover, is not four dimensional Poincaré invariant, offering a background in which Weinberg no-go theorem definitively does not hold. One expects to recover the correct behavior for the gravitational potential since, considering as physical the region between the brane and the horizon, one ends with a finite extra dimension.

Section (2.2.2) is devoted to construct a brane world model in a space-time with this field content. We will present the junction conditions, that show how to compensate bulk quantities and energy on the brane in such a way that is possible to find a Poincaré invariant brane without fine tuning. Actually, junction conditions impose also severe constraints that render matter on the brane exotic. Using however the properties of a scalar-tensor theory of gravity, we will show a method to render energy density on the brane physically acceptable.

In Section (2.3) we will discuss some phenomenological consequences of this approach, at the level of Lorentz violating effects, that are naturally present since gravity is not Poincaré invariant in four dimensions. In Section (2.4) we will conclude with a discussion on the progresses that this class of models provide towards the comprehension of the cosmological constant problem.

2.2.1 Space-time geometry

Let us consider the following action, in the Einstein frame [2]:

$$\begin{aligned} S &= \int d^5x\,\sqrt{-g_5}\,\left[R-\frac{1}{2}(\nabla\phi)^2-e^{-\sigma\phi}F_{\mu\nu}F^{\mu\nu}\right] \\ &- \int d^4x\,\sqrt{-g_4}\,e^{(2\beta+\gamma)\phi}\mathcal{L}_m(\psi,\nabla\psi,e^{\beta\phi}g_{ab})+S_{GH}\,, \end{aligned} \tag{2.13}$$

where ϕ is the scalar field, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the field strength tensor of an abelian gauge field, A_μ, and we use units in which $16\pi G_5=1$. This field interacts in the standard minimal way with gravity, and it conformally couples with the scalar field. The Lagrangian $\mathcal{L}_m$ describes matter fields, ψ, living on the brane that we model as a perfect fluid with equation of state $p=\omega\rho$; g_4 is the four dimensional induced metric. The conformal coupling of the scalar to projected metric is specified by β and we have introduced an additional conformal coupling, parametrized by γ, of the scalar to matter on the brane. The Gibbons-Hawking term S_{GH} is added for consistency.

In this section, we will concentrate on the analysis of the five dimensional background corresponding to the first line of formula (2.13). This background is very interesting *per se* since the action (2.13) is a typical action for a model based on low energy string theory. Indeed, the gauge field in five dimensions, corresponds to the dual of the string theoretical antisymmetric two form field, $B_{\mu\nu}$. Alternatively, it can be obtained from the Ramond-Ramond field present in the spectrum of a ten dimensional type II string model. For these reasons the scalar will be called from now on dilaton field.

Varying the action (2.13), one obtains the following Einstein equations:

$$\begin{aligned} R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R &= -\frac{1}{4}(\nabla\phi)^2\,g_{\mu\nu}+\frac{1}{2}\,\nabla_\mu\phi\,\nabla_\nu\phi \\ &+ e^{-\sigma\phi}\left(2F_\mu^\lambda F_{\nu\lambda}-\frac{1}{2}g_{\mu\nu}F^2\right)\,. \end{aligned} \tag{2.14}$$

The equation of motion for the dilaton is

$$\nabla^2\phi=-\sigma e^{-\sigma\phi}F^2\,. \tag{2.15}$$

while for the gauge field, one gets

$$\nabla_\mu\left(e^{-\sigma\phi}F^{\mu\nu}\right)=0\,. \tag{2.16}$$

[2] The Einstein frame is related to the string frame by the Weyl transformation $g^E_{\mu\nu}=\mathrm{e}^{-\beta\phi}g^s_{\mu\nu}$.

To solve the Einstein and field equations we need some ansatz for the form of the fields. We will require that the all the quantities depend only on *one* variable, in particular r. For a general background, a metric can always be re-casted in a form depending only on one variable only if Birkhoff theorem holds: however, as we have said, for a background containing a scalar field this theorem does not hold. In this case, one must impose some additional condition to ensure the fact that the metric depends only on one variable, as discussed in Appendix A.

Let us take the theorem as granted, and consider the following ansatz for the form of the metric and the fields:

$$\begin{cases} ds^2 = -A(r)^2 dt^2 + B(r)^2 dr^2 + r^2 C(r)^2 dx_{k,3}^2 \\ \phi = \phi(r) \\ F_{tr} = F_{tr}(r) \end{cases} \tag{2.17}$$

Here, $dx_{k,3}^2$ represents the metric for a maximally symmetric three dimensional subspace, see Eq.(1.30). This form for the metric describes a space isotropic and homogeneous in the three spatial dimensions. However, notice that the coefficients $A(r)$ and $C(r)$ are a priori different: in this case, such a metric is *not* four dimensional Poincaré invariant. This fact in particular induces violations of Lorentz invariance, as we will discuss later.

Starting from this ansatz, one finds the solutions

$$\begin{cases} ds^2 = -h_+ h_-^{1-2b} dt^2 + h_+^{-1} h_-^{-1+b} dr^2 + r^2 h_-^b dx_{k,3}^2 \\ \phi = \frac{2b}{\sigma} \ln h_- = \pm\sqrt{3b(1-b)} \ln h_- \\ F_{tr} = Q/r^3 \end{cases} \tag{2.18}$$

where the functions h_+ and h_- are defined as

$$h_+(r) = s(r)\left[1 - \left(\frac{r_+}{r}\right)^2\right], \qquad h_-(r) = \left|k - \left(\frac{r_-}{r}\right)^2\right|. \tag{2.19}$$

The sign function is

$$s(r) = \operatorname{sgn}\left[k - \left(\frac{r_-}{r}\right)^2\right], \tag{2.20}$$

and finally the constant b is given by

$$b = \frac{\sigma^2}{\sigma^2 + 4/3}. \tag{2.21}$$

It is important to notice that the form of the five dimensional metric depends explicitly on the curvature k of the three dimensional submanifold. This metric corresponds to a a black hole only in the case $k = 1$. The charge of these objects, for every k, is given by $Q^2 = \frac{4(r_+ r_-)^2 b}{\sigma^2}$. In the case $k = 1$, the mass of the black hole is proportional to $\mu = r_+^2 + r_-^2(1-2b)$; for the case of $k = 0, -1$, the calculation of this quantity is more delicate, and will be discussed only in the next chapter in a more general context. In general, r_+, r_- are arbitrary constants of integration, but in order to have a singularity hidden by a horizon, for $k = 1$, one requires $r_- < r_+$. To have a real electric charge, we must also impose r_-^2, $r_+^2 > 0$.

Let us describe in more details the global properties of the solutions (2.18):

- ***k=1:*** This case has been study for the first time in [29]. By computing the scalar curvature, it is possible to realize that $r = r_-$ is a scalar singularity for an arbitrary value of b. The background is asymptotically flat and corresponds, in the case of dilaton field and written in the Einstein frame, to the well-known Reissner-Nördstrom solution as it can be seen from the Penrose diagram shown in Fig. (2.1).

- ***k=0* and *k=−1:*** It is clear from the expressions for h_- above, that $r = r_-$ is just a regular point, whereas $r = r_+$ remains a horizon. Furthermore, the coordinate r becomes time-like in the region $r > r_+$ but remains space-like for $r < r_+$: exactly the opposite behavior of Schwarzschild black hole. The region $r > r_+$ is interesting for cosmology: in this region r becomes the time coordinate and the horizon r_+ is a past Cauchy horizon for this cosmological solution. Unlike the standard cosmological singularity $r = 0$ becomes a time-like singularity behind the horizon, resembling the 'white hole' region of the Reissner-Nördstrom solution, but having a single horizon instead of two. Moreover, unlike the Schwarzschild solution, since the singularity is time-like it may be avoided by a future directed time-like curve in the region beyond the horizon, as an analysis of the Penrose diagram in Fig. (2.1) shows. These and other properties render the background very interesting also outside a pure brane world context, and we will discuss these issues in the next chapter.

Let us consider the case $k = 0$. Switching off the gauge field means to take $r_+ = 0$: in this case, the metric does not present any horizon, but a naked singularity in $r = 0$. For the special case $b = \frac{1}{3}$, moreover, the metric becomes four dimensional Poincaré invariant, and corresponds to the space-time examined in the previous section. Notice that, contrarily to that case, the presence of horizons with $r_+ \neq 0$ does not allow, for any choice of the parameters of the model, to recast the metric in a form presenting a four dimensional Poincaré invariant slice.

2.2.2 Embedding the brane

In the previous section, we have studied the properties of the background metric. This subsection is devoted to the problem of embedding the brane in this back-

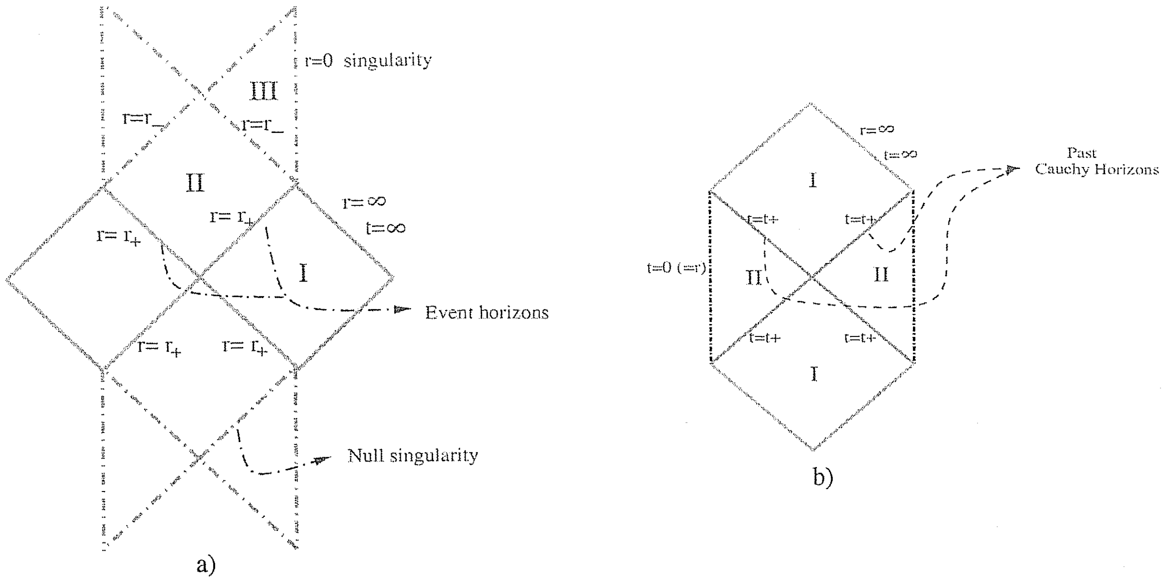

Figure 2.1: Penrose diagrams for the $k = 1$ solution (on the left) and for the $k = 0, -1$ ones (on the right).

ground, and to obtain the properties that characterize the brane world, i.e. the induced metric and the behavior of matter on the brane. Let us not fix the brane curvature k for the moment, but, for simplicity, let us locate the brane in a region of the metric that is static. We will use the receipt developed in the previous chapter to construct a brane world model.

The brane corresponds to a hypersurface on the five dimensional space-time parametrized by the cosmic time τ:

$$X^\mu = (t(\tau), \mathcal{R}(\tau), x_1, x_2, x_3) \tag{2.22}$$

Taking the brane velocity

$$V^\mu = (\dot{t}(\tau), \dot{\mathcal{R}}(\tau), 0, 0, 0)\,, \tag{2.23}$$

where the dot indicates derivative along τ. It is easy to see that the normal to the brane must satisfy the following normalization condition:

$$-h_+ h_-^{1-2b}\dot{t}^2 + h_+^{-1} h_-^{b-1}\dot{\mathcal{R}}^2 = -1\,. \tag{2.24}$$

It is immediate to show that the induced metric on the boundary becomes

$$ds^2 = -d\tau^2 + a^2(\tau) dx_{3,k}^2\,, \tag{2.25}$$

with $a(\tau)$ is the scale factor on the brane, i.e.

$$a(\tau) = r\, h_-^{b/2}(r)_{|r=\mathcal{R}(\tau)} \,. \tag{2.26}$$

We know that the components of the extrinsic curvature must satisfy the junction conditions

$$[K_{ab}]_-^+ = -\frac{1}{2}\left(T_{ab} - \tfrac{1}{3} g_{ab} T^c{}_c\right) \,, \tag{2.27}$$

while the junction condition for the dilaton reads (n is the unit normal to the brane):

$$[n \cdot \partial\phi]_-^+ = \left(-\frac{\beta}{2}(3p - \rho) - \gamma\omega_{\mathcal{L}}\,\rho\right) \,. \tag{2.28}$$

Junction conditions, and self tuning

Assuming the $\mathcal{Z}_2$ symmetry, the junction conditions for the metric become

$$\rho = \mp 12 \left(\frac{1}{\mathcal{R}} + \frac{b\, h_-'}{2\, h_-}\right) h_+^{1/2} h_-^{(1-b)/2} \sqrt{1 + h_+^{-1} h_-^{-1+b} \dot{\mathcal{R}}^2} \,, \tag{2.29}$$

$$3p + 2\rho = \pm 12 \frac{h_+^{-1/2} h_-^{(b-1)/2}}{\sqrt{1 + h_+^{-1} h_-^{-1+b}\dot{\mathcal{R}}^2}} \left(\ddot{\mathcal{R}} - \frac{b\, h_-'}{2\, h_-}\dot{\mathcal{R}}^2 \right.$$

$$\left. + \tfrac{1}{2} h_-^{1-b} h_+' + \tfrac{(1-2b)}{2} h_+ h_-^{-b} h_-' \right) \,, \tag{2.30}$$

$$\frac{\beta}{2}(3p - [1 - 2\omega_{\mathcal{L}}\gamma/\beta]\rho) = \mp 4\,\frac{b}{\sigma}\, h_-' h_+^{1/2} h_-^{-(1+b)/2} \sqrt{1 + h_+^{-1} h_-^{-1+b}\dot{\mathcal{R}}^2} \,. \tag{2.31}$$

where a dot means derivative with respect to τ and a prime means derivative with respect to the variable on which the function depend. The choice of the plus or the minus, as we have seen in the previous chapter, depends to the side of the $\mathcal{Z}_2$ symmetric space that one consider as physical region of the space-time.

Let us concentrate on the case $k = 0$: the other cases can be obtained with similar methods, without adding additional information. As we have seen, in this case the static region of the space-time is located *inside* the horizon, and it lies between the time-like singularity and the horizon.

To have a brane with positive energy density, located in the static region, one must take as physical the interior region of the space, between the singularity and the position of the brane $\mathcal{R}(\tau)$. We will however consider the opposite case, that avoids the presence of naked singularities, taking as physical the exterior region between the position of the brane and the horizon. This region of the space time is perfectly well defined because it does not present any boundary besides the brane: however, in this case one must deal with a negative energy density on the brane.

Various combinations of the junction conditions give us the physically interesting relations. We will let the equation of conservation of the energy for the next Section, while let us consider the Friedmann equation, that as we know gives important information about the eventual realization of the condition of Poincaré invariance.

The Friedmann equation on the brane results

$$H^2 = \frac{1}{144}\rho^2 - h_+ h_- \left(\frac{1}{a} + \frac{b}{2}\frac{h'_-}{h_-^{b/2+1}}\right)^2 ; \tag{2.32}$$

where the Hubble parameter is defined by

$$H(\tau) = \frac{\dot{a}(\tau)}{a(\tau)} = \dot{\mathcal{R}}(\tau)\left(\frac{1}{\mathcal{R}(\tau)} + \frac{b}{2}\frac{h'_-}{h_-}\right) . \tag{2.33}$$

In particular, for any choice of ρ and $\mathcal{R}$, one considers, the couplings β and γ as fixed, and tunes the parameters r_+, $r_-$and σ, connected with the properties of the dilaton and the gauge field, in such a way to compensate the contribution of the energy density on the brane, and to obtain a vanishing right hand side in Eq. (2.32). This procedure is perfectly analogous to the choice of the bulk cosmological constant that leads to the condition of criticality in the case of RS. Contrarily to RS, in this case there exist regions on the parameter space that ensure complete compensation without fine tuning.

Behavior of energy density on the brane

Although one can construct a self tuning mechanism in this model, it is also necessary to check if matter on the brane has the necessary characteristics to be compatible with the observed universe. To understand this fact, let us continue the analysis of the junction conditions for this model.

A combination of (2.29, 2.30, 2.31) gives the equation of (non)-conservation of the energy on the brane:

$$\dot{\rho} + 3H(\rho + p) = \frac{\beta}{2}\dot{\phi}\left([1 - 2\omega_{\mathcal{L}}\gamma/\beta]\rho - 3p\right) . \tag{2.34}$$

Substituting the known solution for the bulk scalar field, one finds the non standard equation

$$\dot{\rho} + 3H\left(1 + \omega + (3\omega - 1)\frac{\beta\sigma}{2} + \sigma\rho\right)\rho = 0 \tag{2.35}$$

This means that, in the Einstein frame, one does not obtain the usual equation of conservation of energy, and this fact is due to the presence of the dilaton in the bulk. This field couple with matter on the brane, that transmits to it part of its energy. Choosing the parameter $\sigma = 0$ corresponds to eliminate the dilaton, and indeed Eq. (2.34) takes the usual form.

Another suitable combination of the junction conditions imposes the following relation between ω, the coefficient of proportionality that appears in the equation of state, and various parameters of the model

$$\omega = \frac{1}{3} - \frac{4b}{9\beta\sigma(1-b)} + \frac{2\gamma}{3\beta}. \tag{2.36}$$

The previous formula shows that, considering the couplings β and γ as fixed, there is a precise relation between the parameter ω and σ. This fact actually reduces, but does not completely eliminate, the freedom in the choice of the parameters to compensate the energy of the brane without fine tuning.

However, consider a static universe with a Poincaré invariant brane, and call R_0 the position of the brane, that is located behind the horizon: $R_0 \leq r_+$. Another combination of the junction conditions imposes the relation

$$r_+ = R_0\sqrt{1 + \frac{1}{3(1-b)\omega}} \tag{2.37}$$

the previous requirement imposes, since $b \leq 1$ (see the definition in Eq.(2.21)), the condition

$$\omega \geq 0 \tag{2.38}$$

Since we have a negative energy density on the brane, Eq. (2.38) means that matter on the brane violates the condition $\rho + p \geq 0$, called *weak energy condition*. These problems with energy density are actually common to any brane world model. Indeed, the authors of [30] have shown that *any* brane world model based on Einstein theory of gravity, that provides a self tuning mechanism, requires exotic energy density on the brane.

In the recent literature some group has tried to overcome these difficulties considering corrections to Einstein gravity given by Gauss-Bonnet terms [31]. In the next section, we will consider the possibility, more natural in this context, to consider a Brans-Dicke theory [32].

The model as a Brans-Dicke theory

The results of the previous section seem to throw doubts on the validity of this approach: it seems that it is possible to compensate various contributions without fine-tuning in such a way to obtain a Poincaré invariant brane, but only at the price to work with an exotic form of energy density on the brane.

Luckily, it is possible to see that the last problem can be overcome in the context of a higher dimensional Brans-Dicke theory of gravity. Indeed, let us perform a Weyl transformation on the action (2.13). We know that in a dilaton gravity theory, the particular conformal frame is not fixed a priori by any physical reason: in particular, one can change the coupling of the scalar curvature with the scalar field via a conformal transformation, and different choices of the conformal factor gives different models. In this case, it is possible to perform a Weyl transformation of the metric such that one recover energy conservation on the brane: consider the transformation

$$\tilde{g}_{\mu\nu} = e^{-\alpha\phi} g_{\mu\nu} \,. \tag{2.39}$$

Let us show that choosing carefully the parameter α, one recovers an exact conservation of the energy density on the brane. After the conformal transformation, the action (2.13) takes the form of a generalized Brans-Dicke theory in five dimensions

$$\begin{aligned} S_5 + S_4 &= \int d^5x \sqrt{-\tilde{g}_5}\left[\Phi\tilde{R} - \frac{\mathrm{w_{BD}}}{\Phi}(\nabla\Phi)^2 - \Phi^{\eta} F_{\mu\nu}F^{\mu\nu}\right] \\ &- \int d^4x \sqrt{-\tilde{g}_4}\,\tilde{\mathcal{L}}_{br}. \end{aligned} \tag{2.40}$$

In the previous expression, $\tilde{\mathcal{L}}_{br} = e^{(2\alpha+2\beta+\gamma)\phi}\mathcal{L}_m(...e^{(\alpha+\beta)\phi}\tilde{g}_{ab},...)$, $\eta = 1/3 - 2\sigma/3\alpha$. We have also defined a Brans-Dicke-type field, Φ, by:

$$\Phi = e^{3\alpha\phi/2} \,, \tag{2.41}$$

and the Brans-Dicke parameter $\mathrm{w_{BD}}$ is given by,

$$2\mathrm{w_{BD}} + 3 = \frac{4}{9\alpha^2} + \frac{1}{3} \,. \tag{2.42}$$

In the new frame, eq. (2.34) becomes

$$\dot{\tilde{\rho}} + 3\tilde{H}(\tilde{\rho} + \tilde{p}) = \dot{\phi}\frac{\delta\tilde{\mathcal{L}}_{br}}{\delta\phi} = \frac{\dot{\phi}}{2}[(\alpha+\beta+2\gamma)\tilde{\rho} - 3\tilde{p}(\alpha+\beta)] \,. \tag{2.43}$$

where $\tilde{\rho} = e^{2\alpha\phi}\rho$. Thus energy will be conserved on the brane, and the right hand side of (2.43) will vanish as soon as the following relation for α holds:

$$\alpha = \frac{-2\gamma - \beta(1-3\omega)}{1-3\omega} \,. \tag{2.44}$$

where ω is the parameter in the equation of state of the perfect fluid: $p = \omega\rho$. For the case $\omega = \frac{1}{3}$ (radiation dominated universe), the only way to eliminate the right hand side of (2.43) is to take $\gamma = 0$, or $\dot{\phi} = const.$ and formula (2.44) does not applies in that case. Notice that for the pure conformal coupling ($\gamma = 0$) one recovers energy conservation in the Jordan frame as usual.

This analysis shows that in this situation there is a preferred conformal frame, in which energy is conserved on the brane, and the higher dimensional gravitational theory becomes a dilaton-gravity theory. In this frame, the dilaton field result decoupled from the energy on the brane. It is also possible to see that relations like (2.36) does not hold in this case, and the parameter ω can take the preferred value (although we must always take $\rho \leq 0$).

Actually, what we obtain is a five dimensional version of the Brans-Dicke theory. Notice however that the usual limits, coming from gravitational experiments, on the parameter w_{BD} of Brans-Dicke theories do not apply in our higher dimensional case. Those limits come essentially from the evaluation of the differences between general relativity and scalar tensor theories predictions in four dimensions, and do not directly apply to an higher dimensional case. The only constraint we must impose is that the BD parameter given by (2.42) result positive, in such a way to have the correct sign for the scalar kinetic term.

2.3 Violation of four dimensional Lorentz invariance

Let us return in this section to the action written in the Einstein frame, that is, describing the usual Einstein theory of gravity. It has been emphasized [20, 21, 33, 34] that brane world models can present interesting effects regarding the different behavior for the speed of propagation of light and gravitational signals. Indeed, the general solution (2.18) breaks the four dimensional Poincaré invariance in the bulk. As a consequence, a gravitational wave produced on the brane, that is not constrained to move on it, can reach a second point of the brane passing through the bulk. This means that is then possible to foresee gravitational signals that travel faster through the bulk than on the brane.

Now, it has been argued since long time that faster than light propagation and/or variation of the speed of light can solve cosmological puzzles like the horizon problem [34]. The elaboration of brane world models has raised the interested for these possibilities [33]. Indeed, since a gravitational signal can bring information faster than light, it carries information from one region of the universe to another that is outside the causal horizon of the first (determined on the hypothesis that nothing can carry information faster than light). Disconnected regions of the universe, that

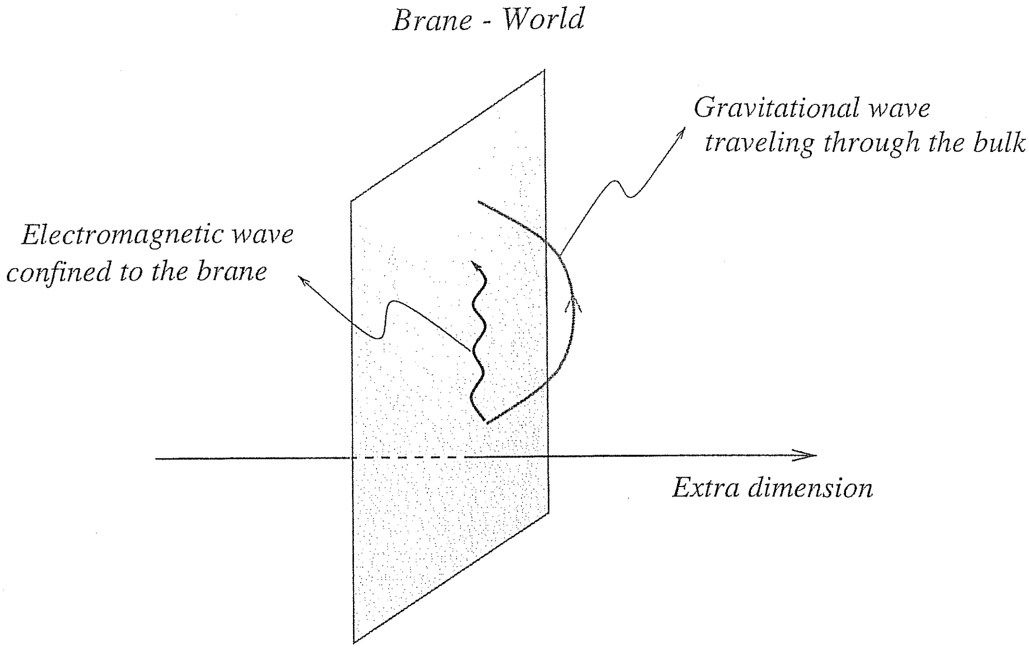

Figure 2.2: An artistic representation of the different behaviors of photons and gravitons

are inexplicably similar one to the other, becomes connected by gravitational signals.

However, these effects, although interesting, are also dangerous for phenomenology, and must be carefully controlled. The violation of four dimensional Lorentz invariance was extensively discussed in [20, 21], where the authors shown the correctness of the intuitive idea for which if one has a decreasing speed of gravitational waves moving away from the brane, then the brane Lorentz invariance can be recovered, in the sense that the gravitational waves prefer to move on the brane, due to the Fermat's principle. As an example of the method employed, let us estimate, the conditions for which one has a negative derivative for the speed of gravitational waves running away of the brane. This means, according to Fermat's principle, that the gravitational waves prefer to propagate on the brane rather than through the bulk. Choosing appropiately the parameters of the model, it is possible to control the effects of Lorentz violation to render them compatible with phenomenological constraints.

Let us first examine the case of a static brane with a a vanishing curvature $k = 0$. From the expression (2.18) of the metric, we deduce the local speed of propagation of gravitational waves in a direction parallel to the brane is:

$$c^2_{\text{grav}}(r) = \left(\frac{r_+^2}{r^2} - 1\right)\left(\frac{1}{r^2}\right)^{(2-3b)} r_-^{2(1-3b)} . \tag{2.45}$$

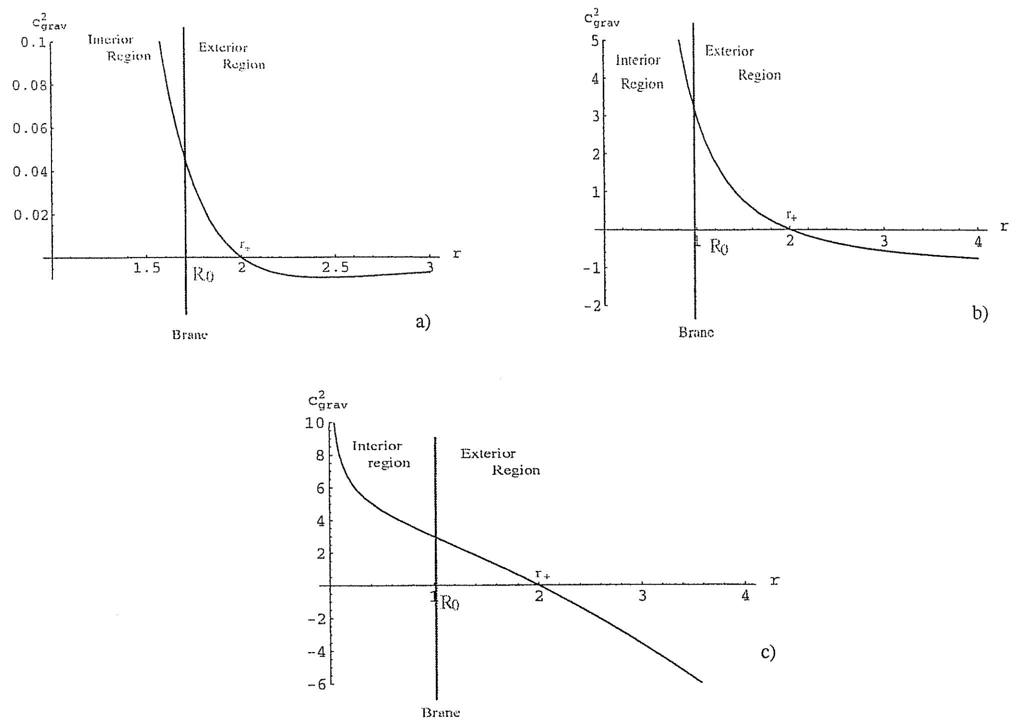

Figure 2.3: Speed of gravitational waves as a function of the extradimension r for $k = 0$ and a) $b = 0.001$, b) $b = 2/3$, c) $b = 0.95$. The brane is always located inside the horizon, sited at $r = r_+$. We took $r_- = 1$, $r_+ = 2$.

This expression is of course valid only in the region where r is a space-like coordinate. It is easy to see that, since $0 < b < 1$, the local speed of propagation $c^2_{\text{grav}}(r)$ is always a decreasing function of r in the region $0 < r < r_+$. This means that this local speed of propagation will be either decreasing or increasing away from the brane in R_0 depending on the sign of the brane energy density (see figure 2.3):

- Positive energy density. We keep the interior region ($r < R_0$) and thus the speed of propagation is increasing away from the brane and the gravitational waves will prefer to propagate through the bulk. Note that in this case the naked singularity at $r = 0$ is not shielded by a horizon.
- Negative energy density. We keep now the exterior region ($R_0 < r < r_+$), therefore in this case the gravitational waves will prefer to travel through the brane instead of the bulk and there will be no evidence of Lorentz violation.

We can then conclude that Lorentz violation would be manifest only in the case with positive energy density on the brane, and in the presence of naked singularity in the physical space.

2.4 Is the cosmological constant problem solved?

In this chapter, we have presented an explicit example of higher dimensional model that tries to face the cosmological constant problem via a compensation mechanism. Can this be considered a solution of this problem?

We have already pointed out some problems connected with the nature of the energy density localized on the brane. However, considering models that are not based on the Einsteinian theory of gravity, but, for example, on Brans-Dicke theories, these problems can be overcome.

Maybe, the real problem of the previous mechanism does not lie in the behavior of the energy density, but on the same realization of the self-tuning mechanism. Indeed, start from a static brane, and consider a small perturbation from this position. As expected, one obtains the correct, linear behavior on the energy density. However, as the brane moves further away from the static point, the corrections to the Friedmann equation will start to become sizeable. One has to overcome this problem requiring a mild time dependence for the parameters defining the charge and the mass of the higher dimensional background: $\dot{r}_+/r_+ \sim \dot{r}_-/r_- \sim H$.

One can expect that a modification of charge and mass can be due to some phase transition on the brane: similar phenomena has been for example considered in the holographic context of the AdS-CFT correspondence [35]. However, in this case the time dependence of the parameters seems not associated to any phase transition, so it is less clear the source of this behavior. Moreover, a consistent holographic interpretation of the present space-time, that is asymptotically flat and not AdS, up to now has not been worked out.

A part from this problem, there is no doubt that this approach to the cosmological constant via asymmetrically warped space-times is interesting and corresponds to a first step toward further developments.

First, it presents a simple model that shows in an explicit, calculable case how to renounce in a consistent way to a generally covariant four dimensional action for gravity with the aim to approach the cosmological constant problem. This approach, by the way, evidentiate how unavoidable effects of Lorentz violations can be controlled and used also in other cosmological contexts.

Second, it points out in an interesting example the importance of the global properties of the bulk space, showing how the presence of singularities and horizons really affects the physics on the brane.

Chapter 3

Negative tension objects and their cosmological consequences

In the common view, our universe begun with an initial singularity, in which the scale factor vanishes and the energy density diverges, and that we cannot describe in terms of known physics. After this initial point, a phase of expansion started (maybe with a primordial phase of inflation), and this phase continues today.

A simple model that presents this behavior is the flat FRW universe. Let us start from the Raychaudhuri equation (3.1):

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)\,. \tag{3.1}$$

Today, we observe a universe in expansion, with $\dot{a} \geq 0$. If the following inequality holds

$$\rho + 3p \geq 0\,, \tag{3.2}$$

then $\ddot{a}$ was always negative, and thus at some finite time in the past a must be have been equal to zero. This argument is based on the so called strong energy condition (3.2).

Notice that a universe dominated by a positive cosmological constant violates this condition. This exception is important because it is believed that during the inflationary period a positive cosmological constant provided the necessary force to exponentially expand the universe. Interestingly, recent observations seem to favor the possibility that our universe is presently accelerating [22]: the most natural explanation of this fact, in terms of the SCM, is to admit the possibility that a fraction of the present energy density is constituted by positive cosmological constant.

However, the previous argument concerning the initial singularity can be generalized to the following statement: given a flat universe, filled with energy density satisfying the null energy condition, a possible phase of contraction is connected to the expanding one by a singularity [36]. The null energy condition prescribes that $(\rho + p) \geq 0$: a positive cosmological constant satisfies this condition, and, although it describes an accelerating universe, it does not help, alone, to avoid an initial singularity in a flat universe. On the other hand, it is interesting to observe that examples of systems that do not satisfy the weak energy condition exist: for example, the negative energy density of the flat dilatonic charged brane world model of the previous chapter violates this condition. But working with an energy density that does not satisfy the weak energy conditions is usually a delicate issue, since this fact generally induces instabilities.

In the present chapter, we start from the problem of the initial singularity posing the following question: are there higher dimensional backgrounds suitable to describe the general cosmological properties of the present universe, namely expansion and acceleration, and at the same time help to consistently face the problem of the initial singularity?

Acceleration is due to a force. The same cosmological constant can be seen as an external force that causes the acceleration of the universe. Is it possible to find, thinking always from a geometrical point of view, some different sources of acceleration? A simple answer to this question is the following: negative mass objects. These can be the responsible of this acceleration: exactly like a positive mass object attracts another positive mass object, a negative mass one repels, accelerating, another object.

In this chapter, we will present a general receipt that, starting from known solutions to Einstein equations, furnishes examples of backgrounds that describes an accelerating, expanding universe, and that contain *negative tension objects*. These backgrounds have also the characteristic to be asymptotically flat, a notable difference in respect to the asymptotically dS space-time corresponding to a universe with positive energy density.

As we will discuss, space-times containing negative tension objects can help to avoid an initial singularity in an interesting way: intuitively, their repelling action does not allow a contracting universe to contract up to a zero size. More precisely, we will show that the structure of the solution is such that it can be reinterpreted as a time-like wormhole, that smoothly connects two cosmological regions, one that can be interpreted as a contracting phase, the other as the expanding one that corresponds to the observed universe. We will also discuss how to avoid the instability generally associated with negative tension objects, giving some indications of the fact that they lie on *fixed point* of some orbifold symmetry.

The solutions of Einstein equations we will present are interesting *per se*, since provide an explicit example of how to obtain, in Einstein theory of gravitation, objects with negative tension, and can give some hints of how to include, in a theory of gravity, negative mass objects, like orientifolds, that naturally arise in string theory.

It is interesting to observe that a particular case of the time-dependent regions of the solutions that we will describe have been recently re-discovered in a totally different context, in the attempts to find time dependent string backgrounds: they are called *S-branes*.

The first part of the chapter is dedicated to a brief overview of recent attempts to describe what happened at the beginning of our universe. They are based on a geometrical approach, and are constructed using the concept of *time-like* orbifolds, an orbifold defined not only along a spatial coordinate, but also along a time-like one. These approaches, although very interesting, should be used with care for the still unclear status of the concept of time-like orbifolds. Indeed, it is not clear if they are consistent or not in a theory containing gravity. The attempts to obtain the same results of these models, but at the level of the well based Einstein theory of gravity, lead naturally to a class of space-times corresponding to a generalization of S-branes.

In the second part of the chapter, we will concentrate on a careful analysis of these new solutions, showing how to obtain them from a simple analytical continuation of known solutions of Einstein equations in supergravity backgrounds. We will present two ways of interpret these solutions, interesting for possible cosmological applications: in terms of space-times containing negative mass objects, and in terms of wormholes connecting two cosmological regions of the complete space-time. We will also discuss the delicate issues of the thermodinamical properties of these objects, and their stability under classical and quantum fluctuations.

3.1 New directions towards the initial singularity

It is a challenge for modern physics to try to understand what really happened at the singularity: does it correspond to the beginning of time? Or the universe has also a past, interesting history, potentially important to describe at least the first stages of the phase of expansion? One can imagine, for example, that the universe underwent a period of contraction that preceded the present period of expansion. In the past decade many efforts have been done to construct models that describe this type of universe, in the context of fundamental theories like string theory. A model constructed in these lines is the *pre-Big Bang scenario*, developed by Veneziano, Gasperini et al. (see [37] for a recent review).

In this model, a contracting phase is still connected to an expanding one via a singularity. However, the singularity (or more precisely the run-away instability of

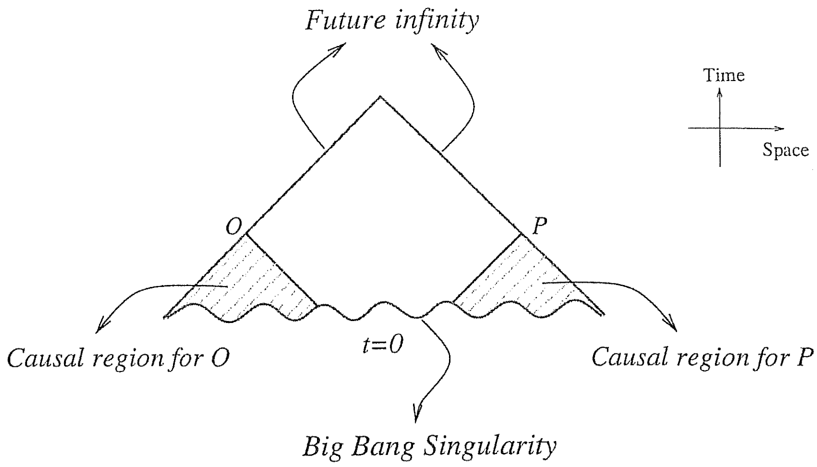

Figure 3.1: Penrose diagram for a FRW universe with the initial space-like singularity.

the fields at the singularity), is tamed by string effects and transformed in conventional cosmology, possibly by means of a T-duality that maps an higher curvature regime to a low curvature one.

Recently, other approaches to the initial singularity have been presented, based on different methods always coming from string theory. In particular, new developments of string theory models constructed in time-dependent backgrounds [47] have received a great attention for their possible relevance in cosmology: this will be the subject of the next two sections.

Models based on time-like orbifolds

Past year, a new cosmological scenario has been presented, based on a higher dimensional brane world approach: the ekpyrotic model [38]. In this scenario, the background is constituted by a five dimensional space, with a spatial extra dimension compactified on a circle. Matter that forms our universe is localized on a three dimensional brane, localized on a fixed point of a $\mathcal{Z}_2$ symmetry acting along the extra dimension. The model contains, besides our fixed brane, other branes in a nearly BPS state, that move along the extra dimension. In particular, the authors propose a scenario to interpret the big bang in terms of the collision between our brane with another one. Quantum fluctuations produced by the process of approaching of the two branes result in the observed nearly scale invariant spectrum of the CMB.

In this thesis, we are interested on the methods that the authors use to construct the space-time where the model will be embedded. To explain these methods, let us consider the illustrative example of a flat space–time in $2+1$ dimensional gravity (our three dimensions are reduced to a singular one, called X). The line element is

$$ds^2 = -dT^2 + dX^2 + dY^2 \,. \tag{3.3}$$

We define in this space time an orbifold projection in the following way. Let us identify points on this space along orbits of a subgroup of its isometry group, i.e. a subgroup of the three–dimensional Poincaré group. Define k to be the Killing vector

$$k = 2\pi i(\Delta J), \tag{3.4}$$

where

$$iJ = T\frac{\partial}{\partial X} + X\frac{\partial}{\partial T} \,, \tag{3.5}$$

is the generator of Lorentz boosts along the X direction. The Killing vector k defines a one parameter subgroup of isometries. Let me identify points Q along the orbits of this subgroup according to

$$Q \sim \exp{(k)}\, Q \,. \tag{3.6}$$

Now, perform a change of frame from the usual Minkowski frame to the Milne form:

$$\begin{aligned} T &= t\cosh(\phi)\,, \\ X &= t\sinh(\phi)\,. \end{aligned} \tag{3.7}$$

This change of frame allows to write the metric of the orbifold space in the following way

$$ds^2 = -dt^2 + t^2 d\phi^2 + dY^2 \tag{3.8}$$

where ϕ is a periodic coordinate, corresponding to an extra spatial dimension.

The passage from a period of contraction to a period of expansion, in this model for the Universe, is represented by the behavior of the scale factor for the compactified extra-dimension ϕ. For t that goes from $-\infty$ to $+\infty$, the scale factor decreases, crosses a space-like singularity in $t = 0$, and after starts an expansion phase. The size of the extra dimension, in this way, shrinks to a point, and after begins to increase.

The flat metric (3.3), with the additional orbifold conditions, represents an exact solution of a string theory system, that utilizes a time-dependent orbifold, an orbifold that involves also a time-like coordinate. In this sense this approach is new, because it utilizes a novel ingredient in string theory, and produces an interesting time dependent background in which it is possible to construct a higher dimensional cosmological model.

However, let us notice some potential problems of this construction. The metric in the frame (3.8) actually excises two important regions of the space time,

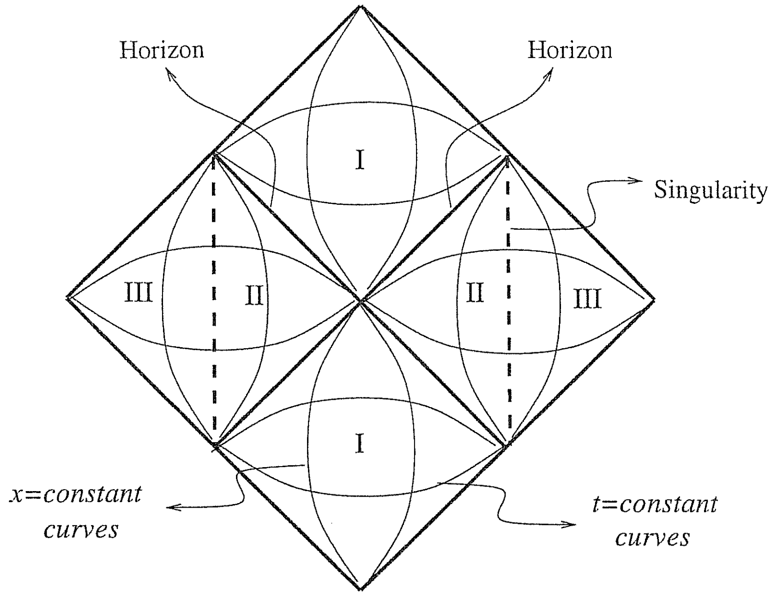

Figure 3.2: Penrose diagram for a space that avoids the singularity problem

just because the choice of the Milne coordinate system eliminates two folds of the Minkowski space. Indeed, the system of metric (3.8) contains also additional regions, and there is not any compelling reason to eliminate them (the fact that they lies behind a singularity is not enough, because, we repeat, this is due only to the bad choice of coordinates in the dimensional reduction). The point is that, in these excites regions, it is possible to see that the identification (3.6) implies the presence of *closed time-like curves.* In general, their presence corresponds to a violation of causality, and signals an instability for the system: we will reconsider this problem at the end of this subsection.

Is it possible to extend the methods introduced in [38] in some interesting way, at least from the point of view of the initial singularity problem? Recently, another group [46] has shown that a simple extension of the receipt of [38] can provide the possibility to avoid completely an initial singularity, obtaining a model for a universe passes in a smooth way from a phase of contraction to a phase of expansion. Consider again, as illustrative example, the same $2+1$ dimensional system of metric (3.3). This time, let k be the Killing vector

$$k = 2\pi i \left(\Delta J + R\, P\right), \tag{3.9}$$

where J is just as before, while P is given by

$$iP = \frac{\partial}{\partial Y}\,, \tag{3.10}$$

and corresponds to the translation along the Y direction. Also in this case we shall identify points Q along the orbits of this subgroup according to

$$Q \sim \exp{(k)}\, Q\ . \tag{3.11}$$

Also in this case, it is possible to find a particular coordinate system that allows to write the metric in a space with a compactified extra dimension. The corresponding Penrose diagram is given in Figure (3.2). In this diagram, the interesting cosmological regions I are not connected by a singularity, but by a horizon that separates them from the two static regions II. The universe can pass from a period of contraction (region I in the lower part of the diagram) to a region of expansion (region I in the upper part) without meeting any singularity. However, this space time has the problem that the Penrose diagram comprehends again regions containing closed time-like curves (the regions indicted by III).

These space-times, defined in terms of time dependent orbifolds, have stirred up the attention of string theorists, that have tried to understand if it is really possible to define a consistent string theory in this background [47]. Very recently, Horowitz and Polchinski [39] have pointed out that, for the unavoidable presence of regions containing closed time-like curves, these systems are unstable. It is sufficient to embed in these space-times a massive particle, to produce a giant black hole that overcloses the universe producing a big crunch.

Any way, we believe that this approach to the initial singularity is extremely interesting, and we pose the following question: is it possible to find an exact solution for a gravitational system with the field content of low energy string theory, that reproduces the interesting behavior of [46], with the same Penrose diagram, but without violating the energy conditions? The answer to this question builds an interesting bridge with other interesting recently considered time-dependent solutions in string theory, the so called *S-branes*, and constitutes the argument of the following subsection.

Space-like branes

Very recently, a different approach has been considered to study time-dependence in string backgrounds. Remember that, in string theory, D-branes are constructed defining a Dirichlet boundary condition in one space-like coordinate. Gutperle and Strominger, in [40], ask the following question: is it possible to define Dirichlet boundary conditions in the time direction, in such a way to construct a space-like brane? The space-like brane (S-brane) would be an object with a purely space-like world volume, that exists only for one instant in time: it corresponds to a sort of time-like kink.

This question is interesting, because one of the concepts one hopes to generalize from a spatial to a temporal context is holography. In the AdS/CFT correspondence, the D-brane field theory holographically reconstructs a spatial dimension. By analogy, S-branes should reconstruct a time dimension: such temporal reconstruction was argued to be a key ingredient of a proposed dS/CFT correspondence. In analogy with AdS/CFT, one can indeed imagine that the world volume field theory on a S-brane is holographically dual to the supergravity solution sourced by the S-brane. This holographical interpretation suggests that a Sp-brane (with $p+1$ dimensional Euclidean worldvolume) in d dimension should have $ISO(p+1) \times SO(d-p-2,1)$ symmetry. The non compact $SO(d-p-2,1)$ can be interpreted as the R-symmetry of an Euclidean field theory living on on the S-brane.

In [40], few examples have been provided of supergravity solutions corresponding to S-branes. Subsequently, more general analysis were performed in [41, 42] and [43]. Let us consider here the simplest example discussed in [40], to present some features that will subsequently developed: this corresponds to a charged S0-brane in $d = 4$ Einstein-Maxwell gravity. The Einstein-Maxwell action is

$$\int d^4x\sqrt{-g}(R - F^2)\,. \tag{3.12}$$

Consider the ansatz

$$\begin{aligned} ds^2 &= -\frac{dt^2}{\lambda^2} + \lambda^2 dz^2 + R^2 dH_2^2, \\ F &= Q\epsilon_2 \end{aligned} \tag{3.13}$$

where R and λ are functions of t only, and dH_2^2 (ϵ_2) is the line (volume) element on the unit H_2 space with negative curvature. Solving the corresponding equations, one finds as a solution

$$ds^2 = -\frac{Q^2}{\tau_0^2}\frac{\tau^2}{\tau^2 - \tau_0^2}d\tau^2 + \frac{\tau_0^2}{Q^2}\frac{\tau^2 - \tau_0^2}{\tau^2}dz^2 + \frac{Q^2\tau^2}{\tau_0^2}dH_2^2 \tag{3.14}$$

where τ_0^2 is an integration constant.

Interestingly, this geometry corresponds, in four dimensions, to a special case of the solutions previously presented in [28], that we have considered in the previous chapter for the five dimensional case. The geometry given by (3.14), in particular, is obtained switching off the dilaton field, and has the same Penrose diagram of Fig (2.1).

Remarkably, this corresponds exactly to the Penrose diagram of Fig (3.2), but without the regions denominated by III in that figure. It presents two cosmological regions (denominated by I), one corresponding to an expanding phase, the other to a contracting one. They are divided by horizons from static regions, that present time-like singularities for $r = 0$. This means that supergravity solutions corresponding to S-branes, constructed to analyze time-solutions of string theory, can constitute suitable spaces to construct models describing a bouncing universe.

Regarding this purpose, we ask the following question: are the time-dependent regions of the solutions the only important ones, or the analysis of the static part of the solution, containing time-like singularities, provide additional interesting information about the global behavior of the space time?

The remaining part of this chapter is devoted to show that the answer to the previous question is affirmative. We will show that time-dependent solutions of the form (3.14) are, in a certain sense, typical in supergravity and low-energy string theory, and a natural interpretation of the corresponding space-time is a space that contains a pair of negative mass objects lying in the static part of the geometry. We will interpret the time dependence of the space time (in particular, the characteristic behavior of contraction followed by an expansion) as due to a sort of destabilization of the space time due to the negative mass objects. The presence of negative mass objects avoids the various theorems that predict an initial singularity: indeed, we will see that the solutions can be written in a particular frame that resembles a time-like kink, or wormhole [1].

3.2 Cosmological space-times from negative tension brane backgrounds

Recently, there has been considerable interest in the dynamics of brane interactions. The interest was motivated partly by the insights that static brane configurations have already given to long-standing low-energy issues like the hierarchy problem, and partly by the potential application of brane collision/annihilation processes to the cosmology of the very early universe [44, 38, 45, 28, 40, 41, 42, 46, 47, 39, 29].

A remarkable feature, which has emerged from studies of brane physics, is the existence of physically sensible objects with negative tension — a prime example being orientifolds [48]. These objects are expected to bear important implications for cosmology. For example, some negative-tension objects do not satisfy the standard positive-energy conditions which underlie the singularity theorems. As such, they may open up qualitatively new kinds of behavior for the very early Universe.

It then behooves us to construct a cosmology built out of objects having negative-tension. In doing so, it is imperative to understand first the large-scale gravitational fields produced by these objects. We will take a step towards improving our under-

[1] The following part of the chapter is based on [43].

standing of these objects by providing a class of simple space-times which describe gravitating negative-tension objects, based on the solutions of [28] that we have studied, in a particular case, in the previous chapter. These solutions describe cosmological space-times with a horizon, but with singularities only in the static region of the full space-time. We will see that these singularities correspond to negative-tension branes of opposite charge. We believe these space-times are useful for developing intuition concerning such objects, as they are no more complicated to analyze than the well-known Schwarzschild black-hole. The space-times we discuss in this chapter enjoy the following properties:

- They are classical solutions to the combined field equations involving dilaton, metric, $(q+1)$-form ($q \geq 0$) antisymmetric tensor fields. For particular choices of coupling parameters, they are classical solutions to bosonic field equations of supergravity and string effective field theory at low-energy.
- They describe field configurations of a pair of q-branes carrying mutually opposite q-form charge and equal but *negative* tension. These q-branes constitute *time-like* singularities of the space-time metric which are separated from one another by an infinite proper distance.

The space-time is time-independent in the immediate vicinity of each brane. The static nature of the space-time metric may be understood as a consequence of Birkhoff's and Israel's theorems for negative- and zero-tension objects. By contrast, part of the space-time which lies to the future of both branes is time-dependent. The boundary between the two regions – time-independent versus time-dependent regions – is a horizon of the space-time.

Outline

The remaining part of the chapter is structured in the following way. After a general discussion about stability problems related to negative mass objects, we review and generalize the solutions studied in the previous chapter, starting from a particularly simple Schwarzschild-like limit, for which a generalization of the Birkhoff and the Israel theorems [54] to negative-tension objects applies. After, in Section (3.5) we will support the interpretation in terms of negative-tension sources in two ways. First, the conserved charges which are carried by the source branes are computed using the curved-space generalizations of Noether's theorem. Second, the response of a test particle to the gravitational field is examined through the study of time-like and null geodesics. Section (3.6) describes how the throat between the two cosmological regions can be interpreted as a time-like bounce. Section (3.7) investigates small fluctuations about the solutions, with evidence presented for the instability of some of their remote-past features. We believe the late-time metric to be stable, and we regard the calculations of this section as a first step towards a more comprehensive stability analysis. We also discuss in that section the relevance of the time-like singularities, and why these can make sense of space-time's overall causal structure. In that section, we show that a Hawking temperature can be defined, and we present preliminary arguments that this reflects the spectrum of particles seen by static observers. Finally, in Section (3.8), we comment on some future directions for research that our calculations suggest, above all on the construction of the cosmological models.

Negative tension versus stability

As many of the unusual features of these space-times are traceable to the fact that the source carries negative tension, it is worth recalling why such branes are believed to make sense [48] – and potentially to be virtues [52] – in the low-energy world. Traditionally, negative-tension (and negative-mass) objects were considered pathological on the following grounds. Consider the world-volume action of a single q-brane, which has the form

$$S_b = -T \int d^{q+1}y \, \sqrt{-\det \gamma} + \cdots , \tag{3.15}$$

where T denotes the brane's tension [2], $\gamma_{ab} = g_{MN}\, \partial_a x^M \, \partial_b x^N$ refers to the metric induced on brane's world-volume by the space-time metric $g_{MN}(x)$, and the ellipses represent contributions of other low-energy modes of the brane dynamics. If the embedding of the brane were free to fluctuate about some fixed value, x_0^M, in the ambient space-time, then $x^M = x_0^M + \xi^M$ and ξ^M is a *dynamical* variable. A negative-definite value of the tension, $T < 0$, poses a problem since it implies a negative-definite kinetic energy – and hence an instability – for the fluctuation ξ^M [53]. This being so, one always assumes that the tension T of a dynamical object is positive-definite.

The explicit construction of sensible negative-tension objects such as the orientifolds within string theory hints how the aforementioned instability and no-go argument are avoidable. Specifically, the argument does not apply in the instances of the space-time studied in this chapter, simply because these objects are not free to move in the ambient space-time. Rather, negative-tension branes are arranged to be localized at special points, such as orbifold fixed points or space-time boundaries, and hence do not carry dynamical variables such as ξ^M, causing an instability

[2]Here, we tacitly assume that the object moves relativistically so that the energy density ρ (as measured per unit q-dimensional volume) equals to the tension T.

as the tension T becomes negative-valued. The immobility is consistent with the equations of motion because it is the equation of motion for the missing dynamical variables ξ^M which would have required the brane's centre-of-mass to follow a geodesic trajectory (if the brane were neutral).

We believe that the immobility of the negative-tension branes helps explaining several otherwise puzzling features of the space-time we describe in this chapter. For instance, as will be shown later, the source branes do not follow geodesics in the space-time, even when the branes are arranged not to carry any electric charge. On the other hand, despite not following the geodesics, the space-time contains no nodal singularity. The situation is unlike what arises with the C-metric solution, where the nodal singularities are interpreted as consequences of the external stress-energy which is required to force the sources to move along their non-geodesic trajectories. This kind of stress-energy is not required for negative-tension objects since, by construction, they are not required to move along geodesics in any case. The immobility of these objects might also help explaining why the late-time regions of the metric are time-dependent [3].

3.3 Simple solutions: Schwarzschild revisited

Before presenting the solutions in their most general form, we pause here first to build intuition by describing their simplest variant: vacuum solution to Einstein's field equations, $R_{\mu\nu} = 0$, in four dimensions.

Let us start with the well-known Schwarzschild black-hole, whose space-time geometry is given – in the asymptotically flat region $r \geq 2M$ – by:

$$ds_I^2 = -\left[1 - \frac{2M}{r}\right] dt^2 + \left[1 - \frac{2M}{r}\right]^{-1} dr^2 + r^2 \left(\sin^2\theta d\phi^2 + d\theta^2\right), \qquad (3.16)$$

whose constant r and t surface is the two-sphere S_2. *Birkhoff's theorem* states that Eq.(3.16) is the unique solution for representing *spherically symmetric* non-rotating black holes [4]. *Israel's theorem* [54] states further that eq.(3.16) is also the unique solution for representing *static* non-rotating black holes [5].

As is well known, the apparent singularity of the metric Eq.(3.16) on the surface $r = 2M$ is a coordinate artifact. For $r < 2M$, the metric goes over to that of

[3]We are largely concerned with *classical* aspects of the negative-tension objects. The stability issue creeps out again once quantum effects such as pair-creation/annihilation of these objects are taken into account. We discuss this issue further in later sections.

[4]This theorem assumes nothing regarding time-(in)dependence of the solution.

[5]Although we describe in detail in this section the four-dimensional case, the discussion trivially generalizes to $2 + n$ dimensions – with $n \geq 2$ – through the replacement $1/r \to 1/r^{n-1}$.

the interior region, for which the role of r and t gets interchanged, leading to a time-dependent metric of the form:[6]

$$ds^2_{II} = -\left[\frac{2M}{t}-1\right]^{-1} dt^2 + \left[\frac{2M}{t}-1\right] dr^2 + t^2\left(\sin^2\theta d\phi^2 + d\theta^2\right). \quad (3.17)$$

Note that the surface of constant r and t remains the same two-sphere S_2. A real, *space-like* singularity occurs for $t \to 0$, which is to the future of any observer falling into the Schwarzschild black-hole.

A particularly simple form of the solutions which are of interest may be obtained from Eq.(3.17) by an *analytic continuation*, $\theta \to i\theta$ followed by an overall sign change of the metric. [7] This leads to the following time-dependent vacuum solution:

$$ds^2_{I} = -\left[1-\frac{2P}{t}\right]^{-1} dt^2 + \left[1-\frac{2P}{t}\right] dr^2 + t^2\left(\sinh^2\theta d\phi^2 + d\theta^2\right). \quad (3.18)$$

Note that, after the analytic continuation, the surface of constant r and t has turned from the two-sphere, S_2, to the hyperbolic surface, $\mathcal{H}_2$, viz. sign of the curvature scalar is flipped from positive to negative. The metric is explicitly time-dependent but homogeneous otherwise – it has a space-like Killing vector $\xi = \partial_r$ in addition to the symmetries of the hyperbolic surface $\mathcal{H}_2$ at fixed values of r and t.

Eq.(3.18) is well-defined for $t > 2P$, but as is clear from its connection with the Schwarzschild black-hole, the degeneracy at $t = 2P$ is merely a coordinate artifact. An extension of the metric to $t < 2P$ is given by performing the same continuation as the one leading to Eq.(3.18):

$$ds^2_{II} = -\left[\frac{2P}{r}-1\right] dt^2 + \left[\frac{2P}{r}-1\right]^{-1} dr^2 + r^2\left(\sinh^2\theta d\phi^2 + d\theta^2\right). \quad (3.19)$$

The metric in this region is static and retains the hyperbolic space $\mathcal{H}_2$ at constant r and t. A real, *time-like* singularity occurs for $r \to 0$, and this is the structure we are primarily interested.

Just as $r = 2M$ does for the Schwarzschild black hole, the surface $t = 2P$ defines a *non-compact* horizon of the space-time described by Eqs.(3.18, 3.19). This is most transparently seen from the Penrose diagram of the space-time, given in Fig.2.1. It is simply a $\pi/2$-rotation of the Penrose diagram for the Schwarzschild space-time.

An observer in region I experiences a time-dependent, expanding region of the space-time, which becomes *flat* as $t \to \infty$. The observer sees no singularity in null or time-like future, but will experience two time-like singularities in the past.

[6] We adopt here the convention of always labelling the time coordinate as t, both inside and outside the horizon.

[7] Equivalently one can take $\theta \to i\theta, \phi \to i\phi, t \to ir, r \to it, M \to iP$.

By contrast, an observer in regions II and IV experiences a static space-time, and sees only a single time-like singularity in the past. Observers in region III see no singularities to their past, but have both time-like singularities in their future light cones. On the other hand, observers at fixed values of r, θ and ϕ in the static regions – including the singularities themselves as a limiting case – do not follow geodesics and so experience a proper acceleration.

The above description suggests a viable interpretation for this solution, as well as for many of the other more general ones which we present in subsequent sections. Regions II and IV describe the space-time external to two objects which we argue to be negative-tension branes. These branes may also carry other conserved charges. Region I gives the time-varying transient gravitational fields which these branes produce at late times. Region III similarly describes the time-reversal of this last time-dependent process.

In this interpretation, the horizons, which are reminiscent of S-branes (in a precise sense explained below) [40], describe the locus of instants when observers make the transition from having only one of the branes in their past light cone to having them both in their past.

3.4 General solutions

We now turn to the description of a wider class of solutions which extend the simple considerations of Section (3.3) to various space-time and brane's world-volume dimensions, and to a system involving metric, dilaton, and $(q+1)$-form tensor fields — a system encompassing bosonic fields of diverse supergravity or superstring theories and their compactifications. This wider class of solutions was already obtained in [28], in which the primary interest was generalization of the well-known black branes of string theory to all possible signs of the curvature parameter, k, of the maximally-symmetric transverse space.

3.4.1 Dilaton-generalized Maxwell-Einstein solutions

The system we will consider is defined by the following Einstein-frame action in $d = (n + q + 2)$-dimensional space-time:

$$S = \int_{\mathcal{M}_d} d^d x \sqrt{g} \left[\alpha R - \lambda (\partial\phi)^2 - \eta e^{-\sigma\phi} F_{q+2}^2 \right] , \qquad (3.20)$$

where $g_{\mu\nu}, \phi, F$ denotes metric, dilaton field, and $(q+2)$-form tensor field strength, respectively. Stability requires the constants α, λ and η to be positive, and, if so, they are removable by absorbing them into redefinition of the fields [8]. It is nevertheless useful to keep them arbitrary since this in principle allows to examine various reduced systems, where each constant is taken zero (to decouple the relevant fields) or negative (*e.g.* to reproduce E-brane solutions in the hypothetical type II* string theories, related to the type II string theories via time-like T-duality [56]). Eq.(3.20) includes supergravity, and so also low-energy string theory, for specific choices of d, σ and q (for instance $d = 10$, $q = 1$ and $\sigma = 1$).

The field equations obtained from Eq.(3.20) are given by:

$$\alpha G_{\mu\nu} = \lambda T_{\mu\nu}[\phi] + \eta e^{-\sigma\phi} T_{\mu\nu}[F] \tag{3.21}$$

$$2\,\lambda\,\nabla^2\phi = -\sigma\,\eta\,e^{-\sigma\phi}F^2\,, \tag{3.22}$$

$$\nabla_\mu\left(e^{-\sigma\phi}F^{\mu\cdots}\right) = 0\,, \tag{3.23}$$

where

$$T_{\mu\nu}[\phi] = \nabla_\mu\phi\nabla_\nu\phi - \tfrac{1}{2}g_{\mu\nu}(\nabla\phi)^2 \qquad \text{and} \qquad T_{\mu\nu}[F] = (q+2)F_\mu{}^{\cdots}F_{\nu\ldots} - \tfrac{1}{2}g_{\mu\nu}F^2_{\ldots}.$$

We are interested in classical solutions whose space-time geometry take the form of an asymmetrically warped product between q-dimensional flat space-time and n-dimensional maximally-symmetric space, parametrized by a constant curvature $k = 0, \pm1$. For this ansatz, the solutions depend only on one warping variable – either t or r – and ought to exhibit isometry $SO(1,1) \times O_k(n) \times ISO(q)$, where $O_k(n)$ refers to $SO(n-1,1)$, $ISO(n)$ or $SO(n)$ for $k = -1$, 0 and 1, respectively. The ansatz is motivated for describing a flat q-brane propagating in $d = (n+q+2)$-dimensional ambient space-time, where n-dimensional transverse hyper-surface is a space of maximal symmetry, and constitute an extension of Birkhoff's and Israel's theorems.

A solution satisfying these requirement is readily obtained as [28]

$$ds^2 = h_-^A\left(-h_+h_-^{1-(n-1)b}dt^2 + h_+^{-1}h_-^{-1+b}dr^2 + r^2h_-^b\,dx_{n,k}^2\right) + h_-^B dy_q^2\,, \tag{3.24}$$

$$\phi = \tfrac{(n-1)\sigma b}{\Sigma^2}\ln h_-\,, \tag{3.25}$$

$$F_{try_1\ldots y_q} = Q\epsilon_{try_1\ldots y_q}r^{-n}, \qquad \epsilon_{try_1\ldots y_q} = \pm1\,. \tag{3.26}$$

The notations are as follows. The metric of an n-dimensional maximally symmetric space, whose Ricci scalar equals to $n(n-1)k$ for $k = 0, \pm1$, is denoted as

[8] The canonical choices are $\lambda = \alpha = 1/2$, and $\eta = 1/2(q+2)!$ in units where $8\pi G = 1$.

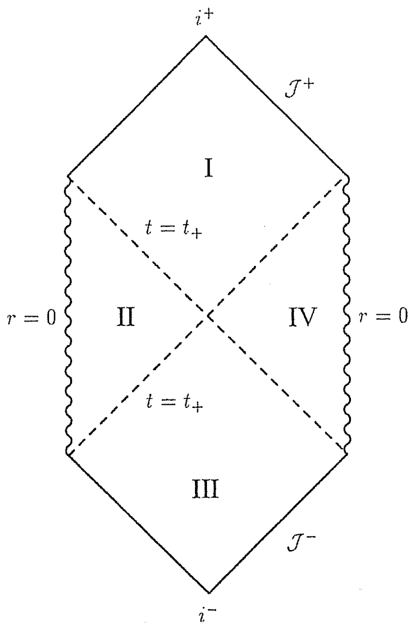

Figure 3.3: Penrose diagram for the $k = 0, -1$ case. This diagram is very similar to the Schwarzschild black hole (rotated by $\pi/2$), but now region I (III) is not static, but time-dependent with a Cauchy horizon and region II (IV) is static.

$dx^2_{n,k}$. The harmonic functions $h_\pm(r)$ depend on two first-integral constants, $r_\pm$, and are given by:

$$h_+(r) = s(r)\,\left(1 - (r_+/r)^{n-1}\right)\,, \qquad h_-(r) = \left|k - (r_-/r)^{n-1}\right|\,, \tag{3.27}$$

where

$$s(r) = \text{sgn}\left(k - (r_-/r)^{n-1}\right)\,. \tag{3.28}$$

The constant Q is given by

$$Q = \left(\frac{4n(n-1)^2\alpha\,\lambda(r_+r_-)^{n-1}}{(q+2)!\,\eta\,(\alpha n\Sigma^2 + 4(n-1)\,\lambda)}\right)^{1/2}\,, \tag{3.29}$$

where Σ and b are constants defined in terms of parameters of the action as

$$\Sigma^2 = \sigma^2 + \frac{4\lambda}{\alpha}\frac{q(n-1)^2}{n(n+q)}\,, \qquad b = \frac{2\alpha n\Sigma^2}{(n-1)(\alpha n\Sigma^2 + 4(n-1)\lambda)}\,.$$

Likewise, the exponents A, B in Eq.(3.24) are defined in terms of the same parameters as

$$A = -\frac{4\lambda q(n-1)^2 b}{\alpha n(n+q)\Sigma^2} \qquad \text{and} \qquad B = \frac{4\lambda(n-1)^2 b}{\alpha(n+q)\Sigma^2} = -\frac{n}{q}A.$$

The solution defined by Eqs.(3.24 - 3.26) is unique modulo trivial field redefinitions: $\phi \to \phi(r) + 2\phi_0$ and $F \to F e^{\sigma\phi_0}$, which in turn can be compensated by rescaling of the space-time coordinates and the first-integral constants, $r_\pm$.

The two first-integral constants, $r_\pm$, are related intimately to two conserved charges associated with the solution. One of these is the q-form electric charge Q – see Eq.(3.26) – acting as the source of the $(q+2)$-form tensor field strength. The electric charge is measurable from the flux integral $\oint {}^*F_{q+2}$ over the n-dimensional symmetric space. The explicit integral yields the electric charge given precisely by Eq.(3.29), so is a function of the first-integral constants. For the q-brane to be physically sensible, the electric charge Q ought to be real-valued. From Eq.(3.29) and from the stability condition $\eta > 0$ [9], the condition renders an inequality $(r_- r_+)^{n-1} \geq 0$. Note that $r_- = 0$ if and only if $Q = 0$, and the point $r = r_-$ is potentially singular (or a horizon) only if $k = 1$. The second conserved charge is associated with the Killing vector of the metric, and so can be understood as a mass, in a sense which will be made explicit later.

A dual, magnetically charged solution is obtainable from the electrically charged solution by making the duality transformation: $F_{q+2} \to \tilde{F}_n = {}^*F_{q+2}$, $\sigma \to -\sigma$ and $q \to (d-4-q)$ in Eqs.(3.24), (3.25) and (3.26), where F_{q+2} is related to $\tilde{F}_n$ through the dilaton-dependent expression:

$$F_{q+2} = \mathrm{e}^{\sigma\phi} \epsilon_{q+2,n} \tilde{F}_n. \tag{3.30}$$

The solution presented above is expressed as a function of the coordinate r. One readily finds that r denotes a spatial coordinate for $k = -1, 0$ in so far as $r < r_+$. For $r > r_+$, the harmonic function h_+ flips the overall sign, so the r coordinate becomes temporal. As such, we will relabel the coordinates as $r \leftrightarrow t$ for $r > r_+$ so that t labels always the time coordinate.

Drawing lessons from the simple solution presented in Section 2, we are primarily interested in $k = -1, 0$ cases. Note that the $k = -1$ solution is obtainable from the $k = 1$ solution in much the same way via the following *analytic continuation*:

$$t \to ir, \quad r \to it, \Omega_n \to i\Omega_n \quad \text{and} \quad r_+ \to ir_+, \quad r_-^{n-1} \to -(ir_-)^{n-1}.$$

Note that this is precisely the same as that defined the simpler, Schwarzschild-type solution in the previous section. The above procedure also suggests that one can obtain yet another solution with $h_+ = |k - (r_+/r)^{n-1}|$ and $h_- = 1 - (r_-/r)^{n-1}$ via an alternative analytic continuation $r_- \to ir_-$ and $r_+^{n-1} \to -(ir_+)^{n-1}$ [10]. It turns out these new solutions are singular at $r = r_-$ for generic values of the parameters.

[9] This last conclusion does not follow for E-branes, for which η may be chosen negative (see later sections).

[10] The additional minus sign is required to ensure the real-valuedness of the electric charge Q.

As such, they would correspond to more standard cosmology evolving from a past singularity. Further new (and generically singular) solutions are also obtainable by T-dualizing the above solutions with respect to the coordinate r in the time-dependent region and t in the static region. In this case the corresponding element of the metric – g_{rr} or g_{tt} – in the string frame gets inverted and the dilaton field is shifted accordingly (see for instance, ref. [55]).

3.4.2 Asymptotic and near-horizon geometries

For foregoing discussions, we pause here to examine both the asymptotic and the near-horizon geometries of our solution, Eqs.(3.24 - 3.26). As the $k = 1$ case parallels to the standard black-brane studies, we focus primarily on the $k = 0, -1$ cases. For the ease of the analysis, we adopt the *isotropic coordinates*, defined by

$$\tau^{n-1} = \left(t^{n-1} - r_+^{n-1}\right). \tag{3.31}$$

The near-horizon and the asymptotic limits then correspond to $\tau \to 0$ and $\tau \to \infty$, respectively.

The metric Eq.(3.24) takes, in the isotropic coordinates, the form:

$$ds^2 = \left(\frac{H_-}{H_+}\right)^{A+b} \left[-\frac{H_+^{2/(n-1)}}{H_-} d\tau^2 + \frac{H_-^{1-nb}}{H_+^{2-nb}} dr^2 + \tau^2 H_+^{2/(n-1)} dx_{n,k}^2\right] + \left(\frac{H_-}{H_+}\right)^B dy_q^2 . \tag{3.32}$$

The harmonic functions $H_\pm(\tau)$ are given, for the $k = -1$ case, by

$$H_+ = 1 + \left(\frac{r_+}{\tau}\right)^{(n-1)}, \qquad H_- = H_+ + \left(\frac{r_-}{\tau}\right)^{(n-1)}, \tag{3.33}$$

and, for $k = 0$ case, by

$$H_+ = 1 + \left(\frac{r_+}{\tau}\right)^{(n-1)}, \qquad H_- = \left(\frac{r_-}{\tau}\right)^{(n-1)}. \tag{3.34}$$

Likewise, the dilaton field and the $(q+2)$-form tensor field strength are given in the isotropic coordinates by

$$\phi(\tau) = \tfrac{(n-1)\sigma b}{\Sigma^2} \ln\left(H_+^{-1} H_-\right) \quad \text{and} \quad F_{try_1 \ldots y_q}(\tau) = Q \epsilon_{try_1 \ldots y_q} \tau^{-n} H_+^{-n/(n-1)} . \tag{3.35}$$

From Eqs.(3.32) and (3.35), we now analyze the limiting geometries for the two cases $k = -1, 0$ separately.

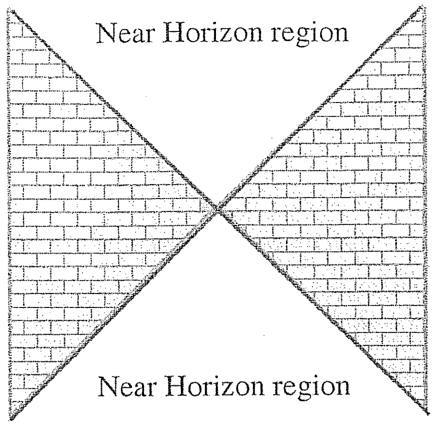

Figure 3.4: Penrose diagram for the near horizon limit. It is simply the Milne Universe, where the shaded zone is excluded and an apparent singularity sits at $\tilde{t} = 0$.

The $k = -1$ brane:

In the asymptotic region, $\tau \to \infty$, and both H_+ and H_- approach the unity. That is, the asymptotic geometry is flat:

$$ds^2|_{\text{asymptotic}} = -d\tau^2 + dr^2 + \tau^2 d\mathcal{H}_n^2 + dy_q^2\,, \tag{3.36}$$

where $d\mathcal{H}_n^2 = dx_{n,-1}^2$. Moreover, both the dilaton field and the $(q+2)$-form field strength become zero: ϕ, $F \to 0$. This is very interesting as the time-dependent regions I and III tend asymptotically to a vacuum state corresponding to (a patch of) flat space-time, both in the asymptotic past and future infinity.

In case the system under consideration is the bosonic part of a supersymmetric theory, the asymptotic region could constitute a supersymmetric vacuum. For instance, as the asymptotic geometries are flat, in- and out-states might be defined naturally having anywhere up to the maximal number of unbroken supersymmetries. Clearly, cosmologies with asymptotic supersymmetry could have many interesting features.

In the near-horizon region, $\tau \to 0$, and the harmonic functions are reduced to

$$H_+ \to \left(\frac{r_+}{\tau}\right)^{n-1} \qquad \text{and} \qquad H_- \to \left(\frac{\overline{r}}{\tau}\right)^{n-1},$$

where $\overline{r}^{n-1} := \left(r_-^{n-1} + r_+^{n-1}\right)$. For simplicity, consider a particular choice of the parameters so that $r_- = r_+$ – the result does not change if they are different. The metric then behaves as

$$ds^2|_{\text{near-horizon}} = -d\tilde{t}^2 + \left(\frac{\tilde{t}}{r_+}\right)^2 dr^2 + r_+^2 d\mathcal{H}_n^2 + dy_q^2\,, \tag{3.37}$$

where unimportant numerical factors are absorbed by rescaling coordinate variables, and the time coordinate is newly defined as $\tilde{t} := \tau^{(n-1)/2} r_+^{(n-3)/2} 2^{(1-A-b)/2}$. Note that the near-horizon geometry does not depend explicitly on the dimension n of the transverse space. Moreover, the dilaton field and the $(q+2)$-form field strength tend to constants in this limit.

We thus find that the near-horizon geometry of the time-dependent regions I and III ($t > r_+$) is described by the direct product of a two-dimensional Milne Universe with coordinates $\tilde{t}$ and r, an n-dimensional hyperbolic space with coordinates x_n, and a q-dimensional flat space with coordinates y_q. In the near-horizon geometry, the Penrose diagram of Fig.3.3 goes over to that of the Milne Universe, illustrated in Fig.3.4. The apparent singularity at $\tilde{t} = 0$ is harmless, as it corresponds to a regular point at the horizon.

Alternatively, the near-horizon limit can be taken from the static interior regions – regions II and IV of Fig.3.3. In this case, we find that the two-dimensional space-time with coordinates r, t is reduced to Rindler space-time — the shaded region in Fig.3.4.

The $k = 0$ brane:

The $k = 0$ branes exhibit several marked differences from the $k = \pm 1$ ones. The main difference is in the asymptotic geometry, which in this case does not become flat as $\tau \to \infty$. In particular, in this limit, the coefficient of $d\mathcal{E}_n^2 = dx_{n,0}^2$ goes to zero and the dilaton field runs logarithmically to $\phi = -\infty$.

The result for the metric in the near-horizon limit –again taking $r_- = r_+$ for simplicity – is:

$$ds^2|_{\text{near-horizon}} = -d\tilde{t}^2 + \left(\frac{\tilde{t}}{r_+}\right)^2 dr^2 + r_+^2 d\mathcal{E}_n^2 + dy_q^2 \,. \tag{3.38}$$

As might be expected starting from the original causal structure, the geometry is again a direct product of a two-dimensional Milne Universe, an n-dimensional flat space and a q-dimensional flat space. The Milne Universe geometry of the near-horizon region seems to be quite generic for all these solutions. Note, however, that, in this case, the near horizon geometry is an exactly flat space-time, in contrast to the $k = -1$ brane.

3.5 Interpretation I: negative tension brane

In this section, we shall be drawing a viable interpretation of our solution Eqs.(3.24 - 3.26). We will first investigate in further detail the conserved charges for these systems. alluded we will show that, while the definition of the electric charge of the source object do not pose problems, the definition of the gravitational mass requires careful treatment. We will then explore the space-time geometries and causal structures by studying geodesic motion of a test particle.

3.5.1 Conserved quantities

We start by identifying two conserved quantities as Noether charges carried by the source branes, whose metric, dilaton field, and $(q+2)$-form field strength are given as in Eqs.(3.24 - 3.26).

Electric charge

We have argued earlier that the constant Q, Eq.(3.29), defined roughly by a flux integral of the Poincaré dual n-form field strength ${}^*F_{q+2} = \tilde{F}_n$ over the n-dimensional maximally symmetric space, is interpretable as a conserved electric charge. We now elaborate the argument, and associate the electric charge with q-branes located at each of the two time-like singularities.

From the field equation Eq.(3.26) of the $(q+2)$-form tensor field strength, a conserved charge density can be defined through $d^*F_{q+2} = {}^*J$. This leads to the following expression for the electric charge:

$$Q = \int_{\Sigma} d\Sigma_{\mu i \ldots} \nabla_{\nu} \left(e^{-\sigma\phi} F^{\mu\nu i \ldots} \right) = \int_{\partial\Sigma} d\Sigma_{\mu\nu i \ldots} e^{-\sigma\phi} F^{\mu\nu i \ldots}, \tag{3.39}$$

where Σ refers to any $(n+2)$-dimensional space-like hyper-surface transverse to the q-brane. Advantage of the above expression of the electric charge lies in the observation that the integrand vanishes almost everywhere by virtue of the field equation Eq.(3.23). It does not vanish literally everywhere, however, because the integrand behaves like a delta function at each of the two time-like singularities. Conservation of Q is also clear in this formulation, as the second equality of Eq.(3.39) exhibits that Q is independent of Σ so long as the boundary conditions on $\partial\Sigma$ are not changed.

Evaluating the flux integral Eq.(3.39) over a space-like hyper-surface $t =$ constant within either of the two static regions (regions II and IV of Fig.3.3), we retrieve the result Eq.(3.29), up to an overall normalization, for the electric charge at each of the two time-like singularities. The electric charge turns out *equal* but *opposite* for each of the q-branes located at the two time-like singularities in the fully extended space-time: $Q_{\text{II}} = -Q_{\text{IV}}$. One can draw this conclusion by directly applying Eq.(3.39) to a choice of the space-like hyper-surface Σ, which extends from the immediate right of the singularity located in region II to the immediate left of the singularity located in region IV, and passes through the 'throat' where these regions touch (see Fig.3.3). As this choice of the hyper-surface does not enclose the singularities, the flux integral in Eq.(3.39) necessarily vanishes. This implies that the (outward-directed) electric fluxes through the two components of the boundary, $\partial\Sigma = \Sigma_{II} + \Sigma_{IV}$, are equal and opposite to one another, and so the same is true for the electric charges which source the dilaton and the tensor fields on the two boundaries.

We are led in this way to identify the conserved quantities, $\pm Q$, with electric charges carried by each of the two q-branes located at the time-like singularities. Which brane carries which sign of the electric charge may be determined as follows. As Eq.(3.26) defines the constant Q relative to a coordinate patch labelled by r and t, the key observation is that the coordinate t can increase into the future only for one of the two regions, II or IV. Then, the charge $+Q$ applies to the brane whose static region t increases into the future, and $-Q$ applies to the brane whose t increases into the past.

Gravitational mass

Recall that the metric Eq.(3.24) is static only in the regions II and IV, but not in the regions I and III. This means that only in the static regions is it possible to define a conserved gravitational mass (or tension) in the usual sense for the branes located at time-like singularities.

A procedure for evaluating the gravitational mass in the present situation is to adopt the *Komar integral* formalism [63], which cleanly associates a conserved quantity with any Killing vector field, ξ^μ, by defining a flux integral:

$$K[\xi] := c\alpha \oint_{\partial\Sigma} dS_{\mu\nu} D^\mu \xi^\nu \,. \tag{3.40}$$

Here, c denotes a normalization constant, and Σ is again an $(n+2)$-dimensional space-like hyper-surface transverse to the q-brane, and $\partial\Sigma$ refers to the boundary of Σ. The Komar charge K is manifestly conserved since it is invariant under arbitrary deformations of the space-like hyper-surface Σ for a fixed value of the fields on the boundary $\partial\Sigma$.

The connection between the flux integral Eq.(3.40) and the more traditional representation of $K[\xi]$ as an integral over Σ of a current density is obtained by using the identity $D^2\xi^\mu = -R^\mu{}_\nu \xi^\nu$ and Gauss' law:

$$K[\xi] = 2c\alpha \int_\Sigma dS_\mu D_\nu D^\mu \xi^\nu = \int_\Sigma dS_\mu J^\mu(\xi) \,; \tag{3.41}$$

where the current density

$$J^\mu(\xi) = c \left(T^\mu_\nu \xi^\nu - \frac{1}{d-2} T^\lambda_\lambda \, \xi^\mu \right) , \tag{3.42}$$

is conserved in the sense that $D_\mu J^\mu = 0$. This last expression utilizes the properties of Killing vector fields, as well as Einstein's equations for relating $R_{\mu\nu}$ to the total stress-tensor, $T_{\mu\nu}$. As we see explicitly later, if $T_{\mu\nu}$ is nonzero, then the value taken by K can depend on the location of the boundary $\partial\Sigma$ in Eq.(3.40).

We now argue that, if one adopts the Komar integral for the definition of the q-brane tension $\mathcal{T}$, the *sign* of the tension ought to be the *same* for both static regions, II and IV. This is most transparently seen for the Schwarzschild-like solution for which $T_{\mu\nu} = 0$, by applying the definition of Eq.(3.40) to the two-component boundary of a surface, Σ_t, of constant t. The boundary extends from near the singularity in region II over to near the singularity in region IV. Then, the vanishing of $T_{\mu\nu}$ leads to the conclusion that the contribution from each boundary component is equal and opposite: $K_{II}(\partial_t) = -K_{IV}(\partial_t)$. However, since the globally-defined time-like Killing vector is only future-directed in one of the two regions, II or IV, local observers will identify $\mathcal{T} = -K[\partial_t]$ in the region where ∂_t is past-directed, leading to the conclusion $\mathcal{T}_{II} = \mathcal{T}_{IV}$.

To evaluate the tension $\mathcal{T} = K[\partial_t]$ in the patch for which ∂_t is future-directed, we will choose for the hyper-surface Σ a constant-t spatial slice and for the boundary $\partial\Sigma$ a $r = r_0$ (viz. a constant radius) slice in the regions II and IV, respectively. It turns out that, if $Q \neq 0$, the expression for the tension depends on the value r at which the boundary $\partial\Sigma$ is defined. This is also true for the radius-dependent mass of the Reissner-Nordström black-hole. Likewise, we would expect that the gravitational mass of the q-brane depends on the stress-energy of the $(q+2)$-form tensor field for which the brane is a source if $Q \neq 0$. Explicitly, we find the tension is given by:

$$\begin{aligned} \frac{\mathcal{T}(r)}{V} &= -\,2\alpha(n-1)\left[r_-^{n-1} - k r_+^{n-1} + r_-^{n-1}(2 + A - (n-1)\,b)\left(\left(\frac{r_+}{r}\right)^{n-1} - 1\right)\right] \\ &= -\,\frac{(n-1)}{8\pi G}\left[r_-^{n-1} - k r_+^{n-1} + \frac{2Q^2}{(n+q)(n-1)}\left(\frac{1}{r^{n-1}} - \frac{1}{r_+^{n-1}}\right)\right], \qquad (3.43) \end{aligned}$$

where the normalization constant c has been chosen to ensure that $\mathcal{T}/V$ takes the conventional (positive) value for the black-brane solutions ($c = 4$), and V is the volume of the $(n+q)$-dimensional hyper-surface over which the integration is performed. The standard normalization choices $16\pi G\eta = 1/(q+2)!$ and $16\pi G\alpha = 1$ are made in the second equality above.

Recall that for $k = -1, 0$ in the static part of the space-time r must satisfy $r \leq r_+$, and this shows that the tension as defined above is negative throughout the static region. For the special case of the simple Schwarzschild-type solution discussed in Section 3.3, the tension becomes simply $\mathcal{T}/V = M/V = -P/V_n$, where V_n denotes the finite volume of the n-sphere (so $V_2 = 4\pi$). As the Schwarzschild case is a vacuum space-time ($T_{\mu\nu} = 0$), this result is independent of the choice of

r, and is only nonzero due to the δ-function singularity in $T_{\mu\nu}$ which the solution displays as the time-like singularities are approached. This again shows how the tension may be identified with q-branes sitting at these singularities. For $k = 1$ one recovers the standard charge dependence of the tension, in this case the calculation is done in the region inside the second horizon.

Negative tension, $\mathcal{T} < 0$, for both branes is in accord with the form of the Penrose diagram of Fig.3.3, which, in the static regions, II and IV, is similar to the Penrose diagram for a negative-mass Schwarzschild black-hole [64], or the overcharged region of the Reissner-Nordström black-hole. As we shall show next, negative-valued gravitational mass or tension is also borne out by the behavior of the geodesics of a test particle in these regions.

Note that the Komar integral technique used above can also be used to compute a 'conserved' charge in the time-dependent regions, which may be relevant for confirming the S-brane interpretation of the horizons of these regions. The quantity obtained in this way involves the ti-components of the stress-tensor, and defines a generalized momentum corresponding to the symmetry under shifting r in this region. We regard however the existence of the static regions, where conserved quantities like tension can be clearly defined, as being very helpful in providing a physical interpretation of geometries like S-branes.

3.5.2 Repulsive geodesics

To substantiate why the negative-tension interpretation is a viable one, we will study geodesic motion of a test particle in the background of the solution Eqs.(3.24 - 3.26). Specifically, we will be primarily interested in the $k = -1$ case, and study geodesic motion of a massless or a massive test particle, which couples only to the metric but not to the dilaton or the $(q+1)$-form tensor fields. To understand the nature of the solution beyond the static regions, we will follow the geodesic motion of these particles starting from the past time-dependent region III, passing through the static regions II and IV, and eventually ending in the future time-dependent region I.

Null geodesics

In the static regions II and IV, the radial coordinate ranges over $r < r_+$, so we consider the *radial* null geodesics defined by $ds^2 = d\mathcal{H}_n^2 = dy_q^2 = 0$. This implies that:

$$-h_-^{A+1-(n-1)b} h_+ \, \dot{t}^2 + h_-^{A-1+b} h_+^{-1} \, \dot{r}^2 = 0 \tag{3.44}$$

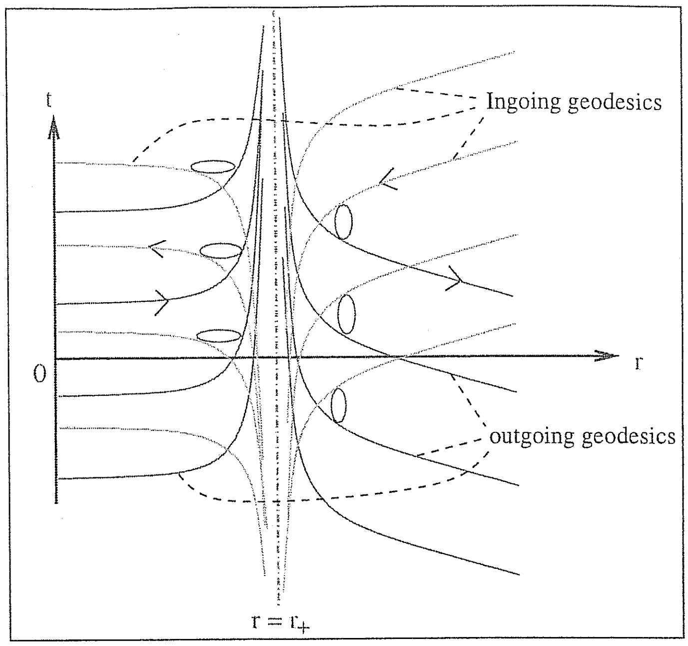

Figure 3.5: Null-like geodesics form in the simple case, $b = 0$, $n = 3$.

where the dots refer to differentiation with respect to the affine parameter along the world-line. Thus,

$$\frac{dt}{dr} = \frac{\dot{t}}{\dot{r}} = \pm h_+^{-1} h_-^{-1+nb/2} , \tag{3.45}$$

where the $\pm$ sign is for outgoing/ingoing geodesics.

As the regions II and IV are time-independent, a first-integral of the geodesic equation renders energy conservation:

$$-\xi_m \dot{x}^m = h_-^{A+1-(n-1)b} h_+ \dot{t} = E , \tag{3.46}$$

where $\xi = \partial_t$ denotes the time-like Killing vector. Eqs.(3.46) and (3.44) together furnish

$$\dot{r} = \pm E h_-^{-A+(n-1)b/2} . \tag{3.47}$$

Though we have derived them in the static region, Eqs.(3.47) and (3.45) are applicable equally well in other regions too. One can integrate them numerically for a generic initial condition, and we illustrate the result in Fig.3.5. The outgoing geodesics are those which travel outside the horizon and pass into the region I of Fig.3.3. Similarly, ingoing geodesics are those which come from the past time-dependent region III in Fig.3.3.

Since $h_\pm \to \infty$ as $r \to 0$, there is no difficulty for integrating either Eq.(3.47) or (3.45) right down to $r = 0$, indicating that null geodesics reach the singularities in a finite interval of both affine parameter and coordinate time.

Time-like geodesics

Radially directed time-like geodesic motion is characterized by $\dot{s}^2 = -1$, $d\mathcal{H}_n^2 = dy_q^2 = 0$, and so:

$$-h_-^{A+1-(n-1)b} h_+ \dot{t}^2 + h_-^{A-1+b} h_+^{-1} \dot{r}^2 = -1 \,. \tag{3.48}$$

Combining this with the first-integral of energy conservation, Eq.(3.46), we find the following condition for time-like geodesics in all regions:

$$\dot{r} = \pm \left(E^2 \, h_-^{-2A+(n-2)b} - h_-^{-A-b+1} h_+ \right)^{1/2} \,, \tag{3.49}$$

where again the sign is + for outgoing and − for incoming radial time-like geodesics.

As before, the geodesic equations can be integrated numerically in the general case, but it is clear that the observer takes a finite proper-time to reach the horizon ($h_+ \to 0$), across which the observer can pass freely. In terms of the coordinate time, we have:

$$\frac{dt}{dr} = \pm \frac{E \, h_-^{(n-1)b-A}}{h_+ h_- \sqrt{E^2 \, h_-^{-2A+(n-2)b} - h_-^{1-A-b} h_+}} \,. \tag{3.50}$$

The integral of Eq.(3.50) diverges as $h_+ \to 0$, so we see that it takes an infinite time for a particle to reach the horizon as seen by a static observer inside the horizon ($r < r_+$).

On the other hand, as $r \to 0$, $h_\pm \sim 1/r^{n-1} \to \infty$. In this limit, the first term inside the square root of Eq.(3.49) grows slower than the second term, which renders the square-root to become complex-valued if r becomes sufficiently small. One sees from this that an infalling time-like geodesic never hits the singularity. Instead the infalling observer reaches a point of closest approach, $r_m > 0$, at which the square root in Eq.(3.49) becomes zero, and reflected outward. The turning point for a time-like geodesic is given by the value r_c of the coordinate r, for which the following expression holds:

$$r_c = \frac{r_+}{(1 + E^2[h_-(r_c)]^{(n-1)b+A-1})^{1/(n-1)}} \,. \tag{3.51}$$

Note that r_c is always between $0 \le r_c \le r_+$. For example, for $r_- = 0$, and $k = -1$, we have that

$$r_c = \frac{r_+}{(1 + E^2)^{1/(n-1)}} \,. \tag{3.52}$$

Gravitational repulsion

We see from the above considerations concerning geodesic motion of a test particle that the two time-like singularities act gravitationally as *repulsive* centers, as no

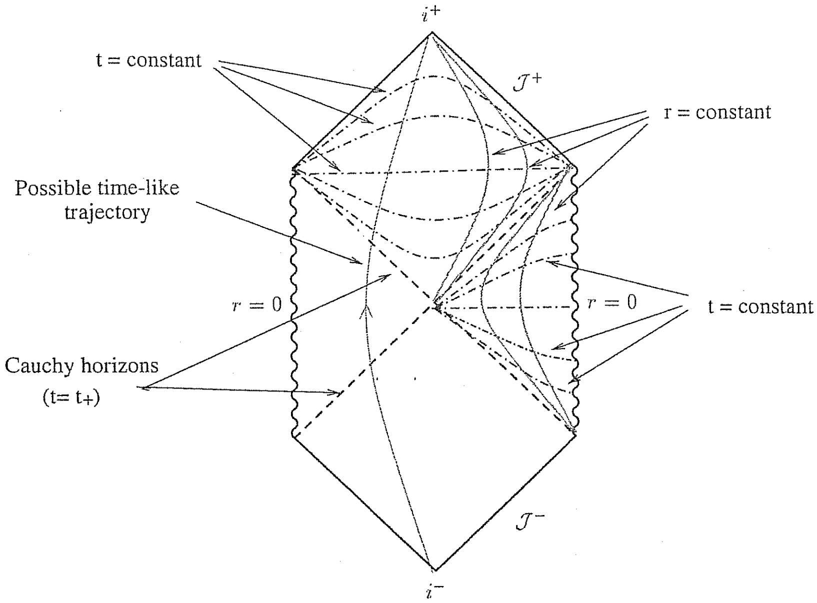

Figure 3.6: Typical time-like trajectory. Shown also are constant-r, constant-t surfaces and the Cauchy horizon of the solution.

infalling time-like geodesic can hit them [11]. In this sense, the space is time-like (although not null) geodesically complete. Observers who originate in the remote past – region III – enter one of the static regions by passing through the past horizon, and then leave this through the future horizon of the future time-dependent region, I. This resembles what happens in other geometries, such as the Reissner-Nordström black-hole.

Putting together the above results, we are led to draw the following conclusion. Static observers in regions II and IV are those whose world trajectories follow lines of constant r, and these observers have proper accelerations which are directed towards the nearest singularity. I find the following expression for the proper acceleration, for $k = -1, 0$ and $q = 0$. We have, in the coordinates adopted,

$$a^r = -\frac{(n-1)h_+ h_-^{1-b}}{2r^n}\left[\frac{r_+^{n-1}}{h_+} + \frac{r_-^{n-1}}{h_-}\right], \tag{3.53}$$

so the value of the acceleration is always negative.

The singularities themselves are special instances of these observers for whom $r \to 0$, in which limit the proper acceleration becomes infinitely large. As discussed in the Introduction, this behavior does not contradict with the equations of motion

[11]However, infalling null geodesics can hit the singularity.

for the branes at the singularities since for negative-tension branes these do *not* imply motion along a geodesic (or otherwise) within the space-time.

This is in contrast to what is found for accelerating positive-mass particles, as described by the C-metric. For this metric, the particle world-lines are also not geodesics, so the particles follow trajectories which are not self-consistently determined by the fields which the particles source. For positive-mass particles this inconsistency shows up through the appearance of nodal defects, which are conical singularities along the line connecting the two particles. These singularities are interpreted as being the gravitational influence of whatever additional stress-energy is responsible for the particle motion [51, 50].

3.6 Interpretation II: time-like wormhole

Comparison with the Schwarzschild black-hole permits another interpretation of our solution. After re-expressing our solutions Eqs.(3.24 - 3.26) in conformal frame, the geometry of the n-dimensional slices turns out that of a *time-like bounce.* On the other hand, the scale factor for the r coordinate resembles an object localized in time, and so is a kind of *time-like kink.* Such bounce/kink behavior would help explain what precisely the S-brane configuration is.

We shall be interested in foliating the geometry with respect to the time in the time-dependent regions, I and III. We will be finding that the geometry exhibits bounce and kink behavior for the symmetric space and the radial direction, respectively.

3.6.1 Einstein-Rosen wormhole: a review

We begin by recapitulating the interior dynamics of the Schwarzschild black-hole relevant for our foregoing discussions.

Consider the maximally extended space-time of the Schwarzschild black-hole. We am interested in describing time-evolution of the space-time geometry. One may foliate the space-time as a stack of constant t surfaces. Then, the space-time at sufficiently early epoch consists of two disconnected asymptotically flat components, each containing a space-like singularity surrounded by a past horizon. The two components evolve and, at some early epoch, the two singularities join together and smooth out by forming a 'wormhole' connecting the two components. The wormhole neck widens, reaching a maximal proper size $r = 2M$ at the time-symmetric point $t = 0$. This is the instance when the wormhole neck is instantaneously static and the event horizon of the two components join instantaneously. Evolving further, the

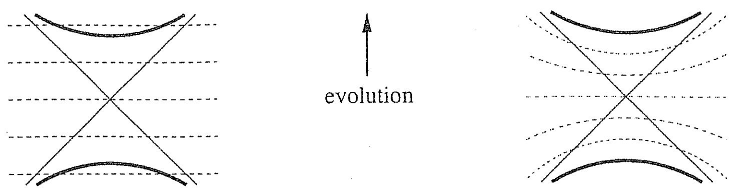

Figure 3.7: Two possible foliation of the maximally extended space-time of the Schwarzschild black-hole. Both cases lead to the Einstein-Rosen bridge connecting the two static regions.

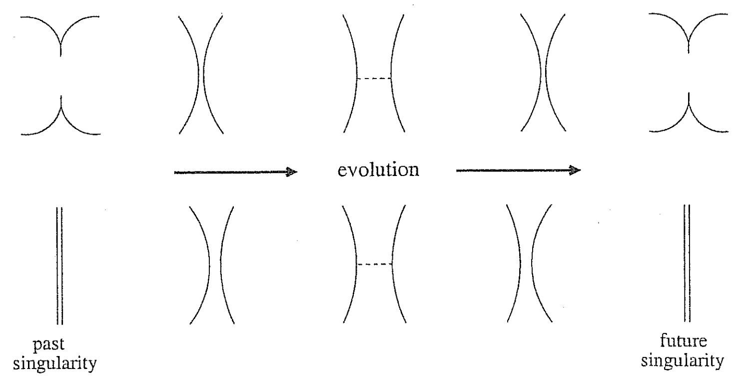

Figure 3.8: Cartoon view of time evolution of the Einstein-Rosen bridges. The upper/lower sequence corresponds to the evolution for the left/right choice of the foliation in Fig.3.7.

wormhole neck recontracts, eventually pinching off as the two singularities reappear and the space-time disconnects.

Two remarks are in order. First, as it is evident from the Kruskal coordinates, the process of wormhole formation and recollapse occurs so rapidly that it is impossible to traverse the wormhole and communicate between the two asymptotic regions without encountering the singularity. Second, the picture of the time-evolution depends on the foliation. Consider, for instance, an alternative foliation illustrated in the right of Fig.3.7. In this case, the geometry starts as a space-like singularity in the asymptotic past, grows out as a hyperboloid, reaches a *maximal* neck size of the hyperboloid, and recollapses to a space-like singularity in the asymptotic future. See Fig.3.8 for the comparison.

A natural question is whether a foliation similar to the Schwarzschild black-hole is possible for our solution as well. We find that it is, although a marked difference would be that the time-evolution is with respect to the regions outside the horizon (inside out compared to the Schwarzschild black-hole case) and details

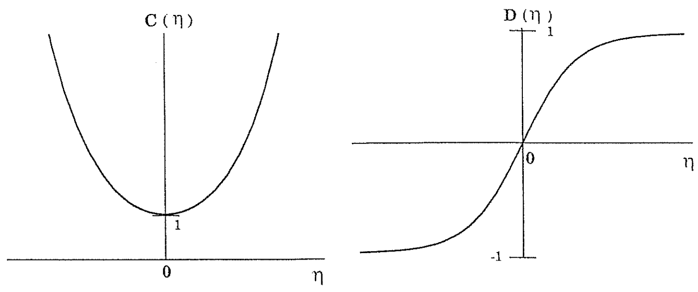

Figure 3.9: A bounce and kink for the $k = -1$ brane.

of the evolution are somewhat different for $k = -1$ and $k = 0$ branes, although the properties we end up finding turn out similar. As such, we will again separate the discussion for the $k = 0$ and $k = -1$ cases.

3.6.2 The $k = -1$ brane

Consider, for simplicity, the case $r_- = 0$ and $q = 0$, for which the singularities are point-like [12]. Recall that the metric in the original coordinates is

$$ds^2 = -\frac{1}{h_+}dt^2 + h_+ dr^2 + t^2 dx_{n,-1}^2, \tag{3.54}$$

where $h_+ = 1 - \left(\frac{r_+}{t}\right)^{n-1}$. We now rewrite this metric in terms of the conformal time η (not to be confused with the normalization constant used in earlier sections) as

$$ds^2 = C^2(\eta)\left[-d\eta^2 + dx_{n,-1}^2\right] + D^2(\eta)dr^2\,, \tag{3.55}$$

where the conformal time is defined by

$$C(\eta) = t(\eta) = r_+ \cosh^{2/(n-1)}\left[\frac{(n-1)}{2}\eta\right] \geq r_+\,, \tag{3.56}$$

and so η ranges over $-\infty < \eta < \infty$. Then, the scale factor for the r-direction becomes

$$D(\eta) = \tanh\left[\frac{(n-1)}{2}\eta\right] \tag{3.57}$$

and has the same functional dependence for all values of n. These expressions exhibit the bouncing structure of the $(n+1)$-dimensional space and the (time-like) kink structure of the radial dimension. We illustrate the behavior of the scale factor in Fig. 3.9 for the example of $n = 3$ and $r_+ = 1$.

[12] The case $r_- \neq 0$ for $b = 0$ can also be integrated analytically, but gives rise to a more complicated result.

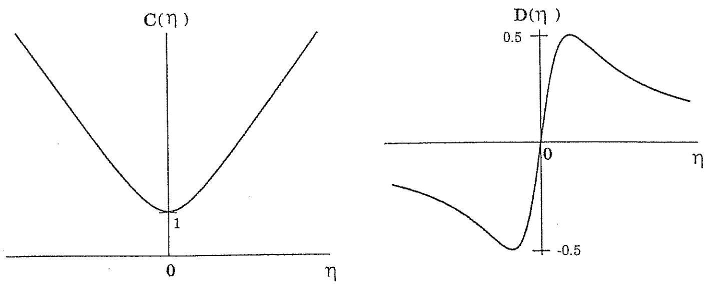

Figure 3.10: A bounce and kink for the $k = -0$ brane.

3.6.3 The $k = 0$ brane

In this case, one cannot take a vanishing charge ($r_- = 0$), as then h_- would vanish too. We instead concentrate on the limit $b = 0$ and $q = 0$ but $r_- \neq 0$. The starting metric has the same form as Eq.(3.54) and h_+ has the same form also, but now $h_- = (r_-/t)^{n-1}$. In terms of the conformal time η, the metric becomes, as before,

$$ds^2 = C^2(\eta)\left[-d\eta^2 + dx_{n,0}^2\right] + D^2(\eta)dr^2\,, \tag{3.58}$$

but now with the conformal time defined by

$$C(\eta) = t(\eta) = r_+\left[1 + \frac{(n-1)^2 r_-^{n-1}}{4r_+^{n-1}}\eta^2\right]^{\frac{1}{n-1}} \geq r_+\,, \tag{3.59}$$

and ranging again over $-\infty < \eta < \infty$. The scale factor for r is similarly obtained for general n, and is

$$D(\eta) = \frac{2(n-1)r_-^{(n-1)}\eta}{4r_+^{n-1} + (n-1)^2 r_-^{(n-1)}\eta^2}. \tag{3.60}$$

One sees once again the bounce behavior of the $(n+1)$-dimensional symmetric space and the kink behavior of the scale factor for the r-direction. We can see this clearly in Fig.3.10, where as before we plot an example for $n = 3$. Note that, at $\eta = 0$ viz. at $t = r_+$, nothing special happens to the $(n+1)$-dimensional subspace, but the scale-factor for the "extra dimension", r, degenerates to zero!

3.6.4 Cosmological bounce/kink and time-like wormhole

As anticipated, the cosmological bounce behavior of this solution offers yet another physical interpretation: the solution is reminiscent of a *time-like* version of the Schwarzschild wormhole or Einstein-Rosen bridge, which connects the two asymptotically flat regions in the maximally extended Kruskal coordinate space-time. The solution corresponds to a $\pi/2$-rotation of the foliation illustrated in Fig.3.7 in the sense that the two time-dependent regions — instead of the static regions — are connected by a *time-like wormhole*. Note that, according to Figs.3.9 and 3.10, the geometry of each fixed r slice start out contracting, reaching the minimum volume, and subsequently expanding.

The bounce/kink interpretation of our solution fits also nicely with the interpretation that, in particular cases, our solution reduces to the S-brane (as alluded earlier), and with the proposal that the S-branes are time-like kinks. Our solution clarifies the proposal further in that the S-brane is in fact located at the horizon r_+.

3.6.5 Comparison with Reissner-Nordström black-hole

There is a resemblance between the solutions presented here and (part of) the space-time of the Reissner-Nordstrom black-hole. More specifically, if we let the outer horizon of the Reissner-Nordstrom black-hole go to infinity, then the geometry and the Penrose diagram of the two space-times are the same.

We believe that the previous interpretation of the Kruskal diagram for our $k = 0, -1$ solutions in terms of interactions due to negative-tension objects remains valid also for the non-extremal Reissner-Norsdröm black-hole in four dimensions, whose horizon is given by S_2 and $k = 1$ (switching off the dilaton, see [57], pg. 158). In this analysis, the time-dependent region, the region between the inner and the outer horizons, is interpretable as a destabilization of the space-time due to the combined gravitational field of two negative-mass objects. Inspecting the Penrose diagram of the non-extremal Reissner-Norsdröm black-hole, one notes that the same considerations are applicable. First of all, the two singularities in the region inside the inner horizon, where the space-time is static, still exhibit *opposite charges* and *equal* but *negative masses*. The negative value of the mass obtained from the Komar integral calculation is essentially due to contributions coming from the electromagnetic field.

The past light-cone of an observer in the 'time-dependent' region — the region between the inner and the outer horizons – is aware of both the negative-mass objects located inside the inner horizons: the simultaneous repulsion of the two objects propel the observer toward increasing values of the coordinate r. Once the observer crosses the horizon corresponding to $r = r_+$, entering into the region outside the outer horizon, the observer's past light-cone does not see any longer two negative-mass objects, but only one. The interaction with only one object is not sufficient to destabilize the space-time. In the asymptotically flat static region outside the outer horizon, the Komar integral calculation gives a positive mass object: indeed, the effect of the electromagnetic field is suppressed in comparison with the gravitational one.

Passing to a conformal frame, constructing the wormhole solution connecting the time-dependent regions of the metric, one finds a "bounce structure" with a periodic cosine dependence, instead of the hyperbolic-cosine one obtained for $k = -1, 0$, describing in this way a cyclic universe (for related ideas see for instance [58]).

3.7 Stability, singularity and thermodynamics

An immediate question is whether our solution Eqs.(3.24 - 3.26) is stable. In this section, for definiteness, we shall be taking again the particular solution: $k = -1$ brane with $q = 0$, and make a first step toward the complete stability analysis, both at classical and quantum levels. At the same time, based on these results, we draw definitive statements concerning the physical nature of the time-like singularities inherent to our solution.

3.7.1 The Cauchy horizon

An analysis of the stability of – or the particle production by – a given space-time starts with initially-small fluctuations of the fields involved, and propagates them forward in time throughout the space-time. The set-up therefore presupposes that the initial-value problem is well-posed. In the space-time of Eq.(3.24), this is not clear as there exists a *Cauchy horizon*, which separates the past time-dependent region III from the static regions II and IV. The Cauchy horizon exists because initial conditions specified in region III do not uniquely determine the future evolution of the fluctuation fields. They do not do so because all points after the Cauchy horizon have at least one singularity in their past light cone, and so can potentially receive signals from these singularities. This implies that a unique time evolution of a field fluctuation from the past time-dependent region III into the future time-dependent region I must also involve a specification of some sort of boundary condition at the location of the two time-like singularities.

From the perspective of brane physics, the existence of such Cauchy horizons is physically reasonable. Imagine that the time-like singularities are the positions of real branes. There then exists a possibility that these branes might emit radiation into the future time-dependent region I, and the possible choices for boundary conditions at the singularities simply encode the possible emission processes which can occur on branes' world-volume. A well-posed time-evolution problem in the embedding space-time thus requires specification as to whether or not the branes are emitting or absorbing radiation.

When necessary, we shall choose the simplest possible brane boundary condition: we assume the brane neither emits nor absorbs any radiation.

3.7.2 The Klein-Gordon equation

I first consider the Klein-Gordon equation for a scalar field propagating in the background Eqs.(3.24 - 3.26), with particular attention paid to these equations' limiting behavior at asymptotic infinity, and near the horizons. We then explore some relevant properties of the solutions in these regions.

Consider the Klein-Gordon equation of a massive scalar field:

$$-\frac{1}{\sqrt{g}}\partial_M\left[\sqrt{g}g^{MN}\partial_N\right]\psi+M^2\psi=0$$

in the time-dependent regions I and III. Adopting the isotropic coordinates, the equation is given by

$$-\frac{1}{\sqrt{g}}\partial_\tau\left[\sqrt{g}g^{\tau\tau}\partial_\tau\right]\psi-\;g^{rr}\partial_r^2\psi-\frac{1}{\omega^2\sqrt{h}}\partial_i\left[\sqrt{h}h^{ij}\partial_j\right]\psi+M^2\psi=0. \tag{3.61}$$

Here, for clarity, we denote $h_{ij}(x)$ for the metric on the n-dimensional maximally-symmetric hyperbolic space $\mathcal{H}_n$, whose coordinates are x^i, and write $g_{ij}(\tau,x)=\omega^2(\tau)h_{ij}(x)$. The relevant metric components are:

$$\begin{aligned} g_{\tau\tau} &= -\left(\frac{H_-}{H_+}\right)^b\frac{H_+^{2/(n-1)}}{H_-}, \\ g_{rr} &= \left(\frac{H_-}{H_+}\right)^{1-(n-1)b}\frac{1}{H_+}, \\ \omega^2 &= \tau^2\left(\frac{H_-}{H_+}\right)^b H_+^{2/(n-1)}. \end{aligned} \tag{3.62}$$

The functional form of the metric involved permits separation of variables, so we take $\psi(r,t,x)=\mathrm{e}^{iPr}\,f(t)\,L_K(x)$, where P and K are separation constants determined by the eigenvalue equations:

$$-\partial_r^2e^{iPr}=P^2e^{iPr}\qquad\text{and}\qquad-\frac{1}{\sqrt{h}}\partial_i\left[\sqrt{h}h^{ij}\partial_j\right]L_K=K^2L_K.$$

Both eigenvalue equations can be solved explicitly, and delta-function or $\mathcal{L}_2$ normalizability of the solutions require both $P^2\geq 0$ and $K^2\geq 0$. The temporal eigenvalue equation then becomes:

$$-\frac{1}{\sqrt{g}}\frac{d}{d\tau}\left[\sqrt{g}g^{\tau\tau}\frac{df}{d\tau}\right]+\left[g^{rr}P^2+\frac{K^2}{\omega^2}+M^2\right]f=0. \tag{3.63}$$

Asymptotic past/future

In the asymptotic future and past regions I and III, $\tau \to \infty$, so the metric becomes flat with $H_\pm \to 1$, and the mode functions go over to standard forms. In this limit, Eq.(3.63) is reduced to

$$\ddot{f} + \frac{n}{\tau}\dot{f} + \left(P^2 + M^2 + \frac{K^2}{\tau^2}\right) f = 0\,, \tag{3.64}$$

where the dots represent derivatives with respect to τ. The solution is expressible in terms of the Bessel functions:

$$f(\tau) = \tau^{(1-n)/2}\left[\alpha_1\, J_y(\rho\,\tau) + \alpha_2\, Y_y(\rho\,\tau)\right]\,, \tag{3.65}$$

where $y = -\frac{1}{2}\sqrt{(n-1)^2 - 4\,K^2}$, the α_1, α_2 are constants of integration, and the parameter in the argument is $\rho = \sqrt{P^2 + M^2}$.

At future infinity in the time-dependent region I (or past infinity in region III), we find the asymptotic behavior of the solution is $f(\tau) \sim \tau^{-n/2} e^{\pm iP\tau}$, if $P \neq 0$. If $P = 0$ then $f(\tau) \sim \tau^{a_\infty}$, with

$$a_\infty = -\frac{1}{2}\left[(n-1) \pm \sqrt{(n-1)^2 - 4K^2}\right]. \tag{3.66}$$

These solutions are oscillatory for all $K^2 > \frac{1}{4}\,(n-1)^2$, and do not grow with τ for large τ so long as $K^2 \geq 0$.

Near-horizon limit

[13] Near the horizon, $\tau \to 0$ and the asymptotic form is governed by the limits $H_+ \to (r_+/\tau)^{n-1}$ and $H_- \to (\overline{r}/\tau)^{n-1}$, with $\overline{r}^{n-1} = \left(r_-^{n-1} - k r_+^{n-1}\right)$. The metric functions therefore reduce to $g_{\tau\tau} \to \alpha_\tau \tau^{n-3}$, $g_{rr} \to \alpha_r \tau^{n-1}$ and $\omega \to \alpha_\omega$. The precise values of the constants α_τ, α_r and α_ω are not required, apart from the following ratio:

$$\frac{\alpha_\tau}{\alpha_r} = r_+^2 \left(\frac{\overline{r}}{r_+}\right)^{(nb-2)(n-1)} = r_+^2 \left[\left(\frac{r_-}{r_+}\right)^{n-1} - k\right]^{nb-2}. \tag{3.67}$$

With these limits, the Klein-Gordon equation becomes, in the near-horizon limit:

$$\ddot{f} + \frac{1}{\tau}\dot{f} + \left[\frac{\alpha_\tau P^2}{\alpha_r}\frac{1}{\tau^2} + \alpha_\tau \tau^{n-3}\left(M^2 + \frac{K^2}{\alpha_\omega^2}\right)\right] f = 0\,, \tag{3.68}$$

[13] In this subsection, we relax the restriction to $k = -1$, and treat all possible cases on equal footing.

If $P \neq 0$, then the solutions are oscillatory, having the form $f \sim \tau^{a_0}$, with $a_0 = \pm iP\sqrt{\alpha_\tau/\alpha_r}$. If $P = 0$, then a similar argument shows that the solutions are nonsingular as $\tau \to 0$.

The logarithmic singularity which is implied by the form τ^{a_0} found above has a familiar source, which is most easily seen by transforming to 'tortoise' coordinates: $t_* = t + r_+ \log[(t/r_+) - 1]$, whose range is $-\infty < t_* < \infty$, with $t_* \to -\infty$ corresponding to the horizon due to the logarithmic singularity as $t \to r_+$. In terms of the tortoise coordinate, the dominant part of the Klein-Gordon equation governing the r and t_* dependence of ψ becomes

$$(-\partial_{t_*}^2 + \partial_r^2)\psi = 0 .$$

This simple wave equation describes waves propagating in *both* directions across the horizon. Note that the mass term drops out of these asymptotic expressions, and so, near the horizon, a massive field behaves like a massless one, approximately propagating along the light-cone. Just as for our discussion of the geodesics, these ingoing and outgoing modes describe motion into and out of the static regions, II and IV, evolving from the past time-dependent region III and to the future time-dependent region I.

3.7.3 Classical stability

One may now ask whether our solutions Eqs.(3.24-3.26) are classically stable in the time-dependent regions, I and III. Classical instability is understood here to mean that initially-small fluctuations grow much more strongly with time than does the background metric. Although a complete stability analysis is beyond the scope of this chapter, we perform the first steps here for scalar fluctuations which are governed by the Klein-Gordon equation. For simplicity, we focus in this discussion on the massless case, $M = 0$.

There are two parts to be studied for the stability analysis. First, identify the modes which grow uncontrollably, and then determine whether well-behaved initial conditions can generate the uncontrollably growing modes, if these exist. In the present instance, we have just seen that the asymptotic forms for the Klein-Gordon solutions do not include any growing modes, due to the conditions $P^2 \geq 0$ and $K^2 \geq 0$, which follow from the normalizability of the spatial mode functions.

Potentially more dangerous are growing metric modes near the past horizons, which divide the past time-dependent region III from the static regions II and IV. These are more dangerous because of the infinite blue-shift which infalling modes from the region III would experience as they fall into the horizon. This blue-shift boosts their energy (as seen by infalling observers) to arbitrarily large values, and one suspects that such large energy densities drive runaway behavior in the gravitational modes, much as has been found to be so for the *inner horizon* of the Reissner-Nordström black-hole. Naively one might have expected that the horizon in our case could be better behaved than the Reissner-Nordström case [57, 59] due to the presence in that case of the asymptotically flat static region from where the signals sent to the horizon are infinitely blue-shifted. However in this case that region is absent. Nevertheless this does not guarantee the stability of the horizon and a careful stability analysis needs to be performed.

As a preliminary estimate of whether such an instability does exist, let us compute the energy, $E = -u^m \partial_m \psi$ of the Klein-Gordon modes considered above as seen by an observer whose velocity, $u = M\partial_t + N\partial_r$, is well-behaved as it crosses the horizon. The normalization condition $u^2 = -1$ in the vicinity of the horizon allows a determination of how M and N must behave as $\tau \to 0$ (in isotropic coordinates) in order to remain non-singular. One finds in this way $u^2 \sim -\alpha_\tau M^2 \tau^{n-3} + \alpha_r N^2 \tau^{n-1}$, which is regular near $\tau \to 0$ provided $M \sim \tau^{-(n-3)/2}$ and $N \sim \tau^{-(n-1)/2}$ near the horizon. With this choice, one then finds

$$-E = M\partial_\tau \psi + N\partial_r \psi \sim \psi\, \tau^{-(n-1)/2}\,. \tag{3.69}$$

Using the asymptotic solution below Eq.(3.68): $\psi \sim \tau^{a_0}$ with $a_0 = \pm i P\sqrt{\alpha_\tau/\alpha_r}$, we see that $E \to \infty$ as the horizon is approached. This suggests that the stress-energy density of the mode under consideration diverges as well in this limit. As such, this mode is likely to destabilize the metric modes near the past horizon, much like what is found for the Reissner-Nordström black-hole near $r = r_-$. Notice that if the horizon were stable, we would have a counter-example to the strong version of the cosmic censorship hypothesis, since observers coming from the past cosmological region III could examine the singularity without having to fall into it (see for instance [60]).

There is a second kind of instability of the Reissner-Nordström black-hole, which the present solutions do *not* share. This second stability problem for the Reissner-Nordström horizon is seen as soon as the Einstein-Maxwell system is extended to include also a scalar field, e.g. Einstein-dilaton-Maxwell system: in this case, the inner horizon turns into a genuine singularity. A similar problem does not arise for our solution, since our solution is already a solution to the combined Einstein-dilaton-$(q+2)$-form Maxwell system. One can see explicitly that turning the dilaton on or off does not change the structure of the horizon. Of course, a more detailed calculation of the metric modes is required to establish definitively whether this instability does really arise .

One sees in this way that the horizons to the past of the static regions are likely to be unstable to becoming singularities in response to small perturbations. On the other hand, we do not expect a similar instability for the horizons to the future of the static regions. Certainly, a more detailed stability analysis of these space-times is desirable.

3.7.4 Issue of quantum stability

Before proceeding describing some aspects of particle production on these space-times, we first pause to remind the reader of some general stability issues.

The Hawking-Ellis vacuum stability theorem

Hawking and Ellis [57] have proposed a generalization to curved space of the familiar flat-space stability condition that a system's energy must be bounded from below. They propose that the energy of a physically sensible theory should be required to satisfy the following positivity conditions, at least on classical macroscopically averaged scales. Specifically, for an arbitrary future-directed, time-like unit-vector, t^μ, the corresponding energy flux vector $E^\mu = -T^{\mu\nu}t_\nu$ ought to be null- or time-like and future-directed:

$$|E^\mu|^2 \leq 0, \qquad \text{and} \qquad -E^\mu t_\mu = T^{\mu\nu}t_\mu t_\nu \geq 0. \tag{3.70}$$

This last inequality implies that the energy density seen by all observers is non-negative.

Physically, this condition ensures the vacuum is stable against the spontaneous pair creation of positive- and negative-mass objects. Given that the present solution is interpreted here in terms of objects — more precisely, a pair of equal-tension q-branes — whose tensions clearly violate the weak energy condition, one might be concerned about instability due to runaway particle production.

It is important in this kind of discussion to distinguish carefully between the energy density defined by the stress tensor of the fields of the problem, and the tension of the q-branes which are their sources. For field fluctuations it is the local field stress energy which is important, and although the q-brane tension is negative, the field stress energy is everywhere positive or zero. For instance, the simple, four-dimensional Schwarzschild-type solution studied in section 2 has a vanishing energy-momentum tensor except at the location $r = 0$. The geodesically complete space-time of the solution, however, does not include this point, implying that the energy condition is satisfied *globally*.

Further insight is provided by the consideration of the non-extremal Reissner-Nordström black-hole in four dimensions, the situation elucidated in section 4.3. There, we have shown that the region inside the outer horizon exhibits precisely the same physical characteristics as our solutions: the region between the outer and the inner horizon is cosmological, while the region inside the inner horizon corresponds to the static region, and the black-hole singularity inside the static region is *time-like*. We have argued that the Komar mass is negative if measured inside the static region, i.e. inside the inner horizon. The negative-mass, however, does not imply violation of the energy condition. This is because, as is well-known, the stress-tensor of the electromagnetic field is well-behaved everywhere, and can be related to the the local mass $M(r)$ via, for example, $dM(r)/dr = 4\pi r^2 T_{tt}$. Thus, though T_{tt} is positive everywhere, the local mass $M(r)$ can become negative inside the inner horizon because the large electromagnetic field digs up a deep gravitational potential well. The latter is precisely what renders the Komar mass negative when measured inside the inner horizon. By the same line of reasoning, one can understand why the Komar mass turns out positive *if* measured outside the outer horizon.

Indeed, we have a situation similar to the above cases: the stress-tensor of matter fields in the right-hand side of Eq.(3.21) are well-defined, and are positive-definite. Despite being so, the Komar mass, defining a local mass, can become negative inside the horizon, as the positivity of the matter stress-tensor imposes the positivity of radial variation of the tension but not that of the tension itself.

3.7.5 How singular is the time-like singularity?

Let us now examine the behavior of waves near the time-like singularity at $r = 0$, and ask whether the singularity is ameliorated when it is probed by waves rather than by particles[14].

This sort of the problem has been studied previously [62, 61] in the context of static space-times having time-like singularities. In some cases, it can happen that space-times which appear singular when probed by classical particles are not singular when these test particles are treated quantum mechanically as waves. Qualitatively, this occurs when an effective repulsive barrier is produced that does not permit the particles to enter into the singularity, and instead scatters them. More precisely, the singular region is not singular to waves if these waves propagate through the singularity in a definite and unique way. As explained in [62], mathematically, this condition is equivalent to the condition that the time-translation operator for the waves must be *self-adjoint*. A sufficient condition to ensure this property is if only one of the two linearly-independent solutions to the equation

$$D^\mu D_\mu \psi \pm i\psi = 0\,, \tag{3.71}$$

is square-integrable.

[14]We limit our discussion to the massless case: for the massive one, the singularity is already well behaved.

In the present case, let us examine the solutions to the massless Klein-Gordon equation near the singularity $r = 0$, where the equation becomes equivalent to Eq.(3.71). The condition of non-integrability of a solution translates into the following condition on the wave function's radial part, $f(r)$:

$$\| f \|^2 \propto \int_0 dr r^n h_+ h_- \left(\frac{df}{dr}\right)^2 \to \infty, \tag{3.72}$$

as $r \to 0$.

Since the Klein-Gordon equation reduces, for r near 0, to:

$$f'' - \frac{(n-2)}{r} f' = 0\,, \tag{3.73}$$

the two independent solutions to this equation behave as

$$f(r) \sim c_0 + c_1 r^{n-1}\,, \tag{3.74}$$

for any dimension n, with arbitrary constants c_0, c_1. It is clear that both of these solutions are normalizable, implying the singularity is wave-singular.

3.7.6 Temperature and entropy

Given the explicit time dependence of the space-time in the time-dependent regions I and III, one would expect particle production takes place in these regions. This radiation would indicate a quantum instability for the future region. A calculation of this radiation is beyond the scope of the present work, but we will make a preliminary analysis which shows that a Hawking temperature can be associated with the static regions II and IV of the space-time.

Hawking temperature

An indication that some observers may see excitations with a thermal spectrum is offered by adopting the Hartle-Hawking computation of the Hawking temperature for a black-hole [64]. These steps also lead to the definition of a Hawking temperature for the space-time under consideration, when applied to the static regions II and IV.

The estimate proceeds by performing a Euclidean continuation of the metric in this region by sending $t \to i\tau$, and then demanding no conical singularity at the horizon in this Euclidean space-time. This condition requires the Euclidean time coordinate to be periodic $\tau \sim \tau + 2\pi/\kappa$, and so implicitly defines a temperature: $T = \kappa/(2\pi)$.

The r- and τ-dependent parts of the Euclidean metric in the static region are:

$$\begin{aligned} ds_E^2 &= |h_+|^{-1} h_-^{A+b-1} dr^2 + |h_+| h_-^{A+1-(n-1)b} d\tau^2 \,, \\ &\approx h_-^{A+b-1} \left(\frac{r_+}{(n-1)\rho} \right) d\rho^2 + h_-^{A+1-(n-1)b} \left(\frac{(n-1)\rho}{r_+} \right) d\tau^2 \\ &\equiv dR^2 + \kappa^2 R^2 d\tau^2 \,, \end{aligned} \tag{3.75}$$

where $\rho = r_+ - r \ll r_+$ gives the coordinate distance from the horizon. The last equality of Eqs.(3.75) defines the rescaled radial coordinate R and the parameter

$$\kappa = \frac{(n-1)}{2r_+} h_-^{1-nb/2} \,, \tag{3.76}$$

which determines the temperature. Here, $h_- = h_-(r_+) = |k - (r_-/r_+)^{n-1}|$ denotes the value of this quantity at the horizon.

We find in this way the temperature:

$$T = \frac{\kappa}{2\pi} = \frac{n-1}{4\pi r_+} \left| k - \left(\frac{r_-}{r_+} \right)^{n-1} \right|^{1-nb/2} . \tag{3.77}$$

This reduces to previously obtained expressions for the special cases where these metrics agree with those considered elsewhere. In particular, it vanishes for extremal black-branes, for which $k = 1$ and $r_- = r_+$. For the four-dimensional Schwarzschild-type solution presented in Section3.3, we have $r_- = 0$ and so $T = |k|(n-1)/(4\pi r_+)$.

Entropy

The possibility of associating a temperature with a space-time involving horizons immediately suggests that it may also be possible to associate to it an entropy, using the thermodynamic relation

$$\frac{\partial S}{\partial(-M)} = \frac{1}{T} \,. \tag{3.78}$$

The unusual negative sign in this expression arises because of a technical complication in defining an entropy in the present instance. The complication arises because the entropy is associated with degrees of freedom behind the horizon, where the globally-defined time-like Killing vector changes direction. This situation is very much like what happens for the de Sitter space, for which the above expression is used to define the entropy [65].

For simplicity, we shall be restricting ourselves to the simplest Schwarzschild solution with $r_- = 0$, $k = -1$ and $n = 2$, for which we have $T = 1/(4\pi r_+)$ and

$\mathcal{T}/V = M/V = -P/V_n = -r_+/2GV_n$, with V_n denoting the volume of the n-sphere. In this case, the entropy becomes:

$$\frac{\partial S}{\partial(-M)} = -8\pi G V_n \,\frac{M}{V}\,, \tag{3.79}$$

from which we integrate to find

$$\frac{S}{V} = 4\pi G V_n \left(\frac{M}{V}\right)^2\,, \tag{3.80}$$

where the integration constant is chosen to ensure $S(M=0)=0$.

Note that, although both S and M both diverge due to infinite volume of the planar or the hyperbolic directions, the entropy and tension per unit volume are finite, and are related in the same way as are these quantities for a black-brane. Notice also that we retrieve the usual expression, $S = 4\pi G M^2$ when we specialize to the $k=1$ case of a black brane.

In the general case, the expression for the entropy will depend on the electric charge as well. In order to extract the general form of the entropy we follow the standard prescription in terms of the Euclidean action. Consider first the definition of the Gibbs free energy:

$$W = -T\log Z = TS + Q\Phi(r) - \mathcal{T}(r)\,, \tag{3.81}$$

where $\mathcal{T}(r)$ is given in Eq.(3.43), while $\Phi(r)$ is the potential associated with the $q+2$ form. Note that we have here a sign change in the right-hand side of Eq.(3.81) with respect to the usual definition of the free energy as explained above.

In the semiclassical approximation, one can identify the partition function Z with e^{-I_E}, where I_E corresponds to the Euclidean action for the system. From this fact we obtain immediately

$$TS + Q\Phi(r) - \mathcal{T}(r) = T I_E\,. \tag{3.82}$$

At this point, we need an expression for the Euclidean action for our system. This takes the form

$$I_E = -\int d^dx\sqrt{g}\left(\alpha R - \lambda(\partial\phi)^2 - \eta e^{-\sigma\phi}F^2\right) - 2\alpha\int d^{d-1}x\sqrt{h}\,K\,. \tag{3.83}$$

where we have included the Gibbons-Hawking boundary action.

The contribution from a boundary at a surface of a fixed r is

$$\begin{aligned}\int d^{d-1}x\sqrt{h}\,K &= \frac{V(1-n)}{2T}\left[r_-^{n-1} - k\,r_+^{n-1} + (2+A-(n-1)b)r_-^{n-1}\left(\left(\frac{r_+}{r}\right)^{n-1} - 1\right)\right] \\ &= \frac{\mathcal{T}}{4\alpha T}\,,\end{aligned} \tag{3.84}$$

where in the second equality we have used Eq.(3.43). Consider now the solutions that we have found for our system. Following [65] we take the magnetic rather than the electric solution, using the duality transformations given in section 3.1. Substituting the solutions in Eq. (3.83), it is straightforward to obtain a general expression for the Euclidean action in terms of the parameters of the model. We then find the following expression that relates Eq.(3.83) to other global quantities, and that allows interesting manipulations of Eq. (3.82):

$$I_E = \frac{1}{2T}\left(\mathcal{T}(r) - Q\Phi(r)\right) . \tag{3.85}$$

Substituting Eq.(3.85) into Eq.(3.82), a simple calculation yields the general relation

$$S = -I_E . \tag{3.86}$$

At this point, one can write the general expression for the entropy density s, using the known value of the temperature T, for any curvature k. Indeed, for $k = 1$, in which the entropy is calculated outside the outer horizon, it is enough to change sign on the last expression in Eq.(3.81). We obtain the following compact form

$$s = \frac{S}{V} = \alpha 4\pi\, r_+^n \left| k - \left(\frac{r_-}{r_+}\right)^{n-1} \right|^{nb/2} = \frac{1}{4G}\sqrt{g_{nn}}|_{r_+} , \tag{3.87}$$

where in the last equation we have used $\alpha = (16\pi G)^{-1}$. It is remarkable that, for any k, the expression for the entropy does not depend on the coordinate r: g_{nn} corresponds to the determinant of the induced metric on the n spatial dimensions, and it is calculated at the horizon r_+. In case $k = 1$, we obtain the well-known relation

$$S = \frac{\mathcal{A}}{4G} \tag{3.88}$$

where $\mathcal{A}$ is the area of the black-hole horizon. Again for $k = -1$ and 0, the area of the horizon is infinite but we can still consider the entropy per unit volume. It is worth noting that these quantities can be made finite by modding out the planar or the hyperbolic subspace by discrete subgroups of $ISO(n)$ and $SO(n-1, 1)$, respectively, as the operation would leave the volume of the horizon finite.

3.8 Future developments

There are two important issues which we were able to resolve only partially here. One is the question of the stability of these space-times (for a recent discussion see [68]). We have shown that the past horizons of these space-times are likely to be unstable in precisely the same manner as the inner horizon of the Reissner-Nordström black-hole. A more complete investigation of stability is obviously of considerable interest. The second is the question of quantum instability, and whether the time-dependent fields in regions I and III give rise to particle production. We have argued that there is a natural definition for the Hawking temperature for the static space-times near the q-branes, and this strongly suggests that this is associated with thermal radiation as seen by the static observers. A more detailed calculation of particle production is certainly desirable.

The time-dependent regions, I and III, of the space-times are also of considerable interest, because they may open up a new avenue for the cosmology of the early Universe. In the future time-dependent region I, the space-time exhibits expansion of the hyperbolic directions, and a past horizon without a past space-like singularity. Since region III corresponds to the time-reverse of region I, taken together the two regions I and III offer an interesting realization of a singularity-free cosmology, which bounces from a contracting to an expanding Universe. One is eventually interested in a realistic bouncing cosmology free from instabilities at the past horizon. A possible prescription ensuring the stability would be a periodic identification of the Killing coordinate, r, in the time-dependent regions.

An obvious obstacle to constructing a realistic cosmology out of the solutions Eqs.(3.18) and (3.19) is that co-moving observers do not see a homogeneous and isotropic space. This objection needs not be fatal, as it may describe space-time during the very early universe – perhaps during inflation – before the Universe is really required to be isotropic and homogeneous. Indeed, an attractive brane cosmology for these early epoch has been proposed by utilizing brane-antibrane interactions [44]. Alternatively, it may be the higher-dimensional solutions which are of cosmological relevance. After all, these space-times do have three-dimensional hyper-surfaces which are homogeneous and isotropic. In the specific metrics presented in [28], this usually requires that the radial coordinate, r, describes a compact direction. This would be problematic if the space-time also includes the static regions because, there, the r-coordinate would correspond to a compact time direction. Having closed time-like curves, it may also lead to orbifold instabilities [47]. In a brane-world picture, this instability however does not appear: a direction for future work can be to contruct a cosmological model in this framework.

Appendix A

General cosmological solutions and Birkhoff's theorem

Birkhoff's theorem states that, for gravitational systems with maximal symmetry, each solution of Einstein equations can be re-conducted to a form depending only on one variable, via a coordinate transformation.

Extensions of this theorem for systems in more than four dimensions that contain also a gauge field and a cosmological constant have been studied [69, 20]. However, if a scalar field is present in the system, the theorem does not hold [28, 70, 71].

In the present appendix, we will consider a five dimensional background constituted by gravity, a scalar field, and a gauge field. We will show that Birkhoff theorem does not hold for this background, if one only requires maximal symmetry for a three dimensional spatial submanifold. However, we will find an additional condition that ensures, with this field content, that the metric can always be re-casted on a form depending only on one variable.

Let us consider the following action, in the Einstein frame:

$$S \;=\; \int d^5x\,\sqrt{-g_5}\,\left[R-\frac{1}{2}(\nabla\phi)^2-e^{-\sigma\phi}F_{\mu\nu}F^{\mu\nu}\right]; \tag{A.1}$$

here ϕ is a scalar field (called dilaton), while $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ corresponds the field strength tensor of an abelian gauge field, A_μ. The gravitational coupling $16\pi G_5$ is set to one.

Varying the action (2.13), we obtain the following Einstein equations:

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=-\frac{1}{4}(\nabla\phi)^2\,g_{\mu\nu}+\frac{1}{2}\,\nabla_\mu\phi\,\nabla_\nu\phi+e^{-\sigma\phi}\left(2F_\mu^{\lambda}F_{\nu\lambda}-\frac{1}{2}g_{\mu\nu}F^2\right)\,. \tag{A.2}$$

The equation of motion for the dilaton is

$$\nabla^2\phi = -\sigma e^{-\sigma\phi}F^2 \,, \tag{A.3}$$

while for the gauge field we get

$$\nabla_\mu \left(e^{-\sigma\phi}F^{\mu\nu}\right) = 0 \,. \tag{A.4}$$

We look for solutions that are isotropic, homogeneous and maximally symmetric in a three spatial submanifold. The most general ansatz consistent with these symmetries is given by

$$ds^2 = e^{2B(t,y)}(-dt^2 + dy^2) + e^{2A(t,y)}dx^2_{k,3} \tag{A.5}$$

where y is the coordinate that labels the extra spatial dimension and $dx^2_{k,3}$ represents a three dimensional maximally symmetric metric with constant curvature parametrized by k. We also assume that the scalar and gauge fields depend only on t and y; that is, $\phi = \phi(t,y)$, $F_{\mu\nu} = F_{\mu\nu}(t,y)$.

A.1 Bulk equations

With the ansatz (A.5), the equations of motion take a simple form written in the light cone coordinates defined as $u = t + y$, $v = t - y$. Moreover, the equation for the gauge field is easily solved, giving

$$F_{tr} = de^{\sigma\phi+2B-3A} \,. \tag{A.6}$$

Here, d is a constant that corresponds to the conserved charge. After some algebraic manipulations, and by means of Eq. (A.6), the Einstein and dilaton equations of motion (A.2, A.3) become

$$3A_{,u}A_{,v} + A_{,uv} = -2ke^{2(B-A)} + d^2e^{\sigma\phi+2B-6A} \,, \tag{A.7}$$

$$B_{,uv} + A_{,uv} + \frac{1}{4}\phi_{,u}\phi_{,v} = ke^{2(B-A)} - \frac{3}{2}d^2e^{\sigma\phi+2B-6A} \,, \tag{A.8}$$

$$3\phi_{,u}A_{,v} + 3\phi_{,v}A_{,u} + 2\phi_{,uv} = -2d^2\sigma e^{\sigma\phi+2B-6A} \,, \tag{A.9}$$

$$2A_{,u}B_{,u} = A_{,uu} + A^2_{,u} + \frac{1}{6}\phi^2_{,u} \,, \tag{A.10}$$

$$2A_{,v}B_{,v} = A_{,vv} + A^2_{,v} + \frac{1}{6}\phi^2_{,v} \,. \tag{A.11}$$

Now, let us show that, already in the simplest case when $k = 0$ and $d = 0$ (null curvature of the three dimensional submanifold and no gauge field), the effect of the dilaton is such that Birkhoff's theorem does not hold, and in general solutions depend strictly on two variables.

A.2 Non validity of Birkhoff's theorem

The case $k = 0$ with vanishing gauge field corresponds to a system containing a dilaton field without any potential. Equation (A.7) simplifies and the general solution can be found exactly: it is given by

$$A(u,v) = \frac{1}{3}\ln\left[\xi(u) + \chi(v)\right], \tag{A.12}$$

where $\xi(u)$ and $\chi(v)$ are arbitrary functions of u and v respectively. (A possible additive integration constant can always be absorbed by an overall scaling of ξ and χ). Using Eq. (A.12), it is possible to find the most general form for the fields ϕ and B. These read

$$\phi(u,v) = \int dk e^{ik(\xi-\chi)}(a(k)Y_0[k(\xi+\chi)] + b(k)J_0[k(\xi+\chi)]) \tag{A.13}$$

and

$$\begin{aligned} B(u,v) = & \ \frac{1}{2}\ln\left(\xi_{,u}\chi_{,v}\right) - \frac{1}{3}\ln\left(\xi+\chi\right) + \frac{1}{4}\int d\xi(\xi+\chi)(\phi_{,\xi})^2 \\ + & \ \frac{1}{4}\int d\chi(\xi+\chi)(\phi_{,\chi})^2 + \frac{1}{4}\int d\xi d\chi \phi_{,\xi}\phi_{,\chi} + C \end{aligned} \tag{A.14}$$

We can explicitly see that in the absence of a dilaton field both A and B depend only on the combination $\xi(u) + \chi(v)$, establishing Birkhoff's theorem in that case. In the presence of the dilaton field there is a dependence on $\xi(u) - \chi(v)$ which ruins this result indicating that Birkhoff's theorem will not be valid.

However, we may in principle foresee another change of variables that could make all the functions still depend on one single parameter: we will now prove that this is not the case. In the form above, the general solution does not offer much information. For the purpose of proving the non validity of Birkhoff's theorem we only need a counterexample, and we will restrict to the following particular solution for the dilaton field:

$$\phi(u,v) = \kappa\ln\left(\xi+\chi\right) + \lambda(\xi-\chi), \tag{A.15}$$

where κ and λ are two arbitrary real constants.

Inserting the solution (A.15) into (A.14), we get

$$B(u,v) = \frac{1}{2}\ln\left(\xi_{,u}\chi_{,v}\right) + \frac{(l-1)}{3}\ln\left(\xi+\chi\right) + \frac{\lambda^2}{8}(\xi+\chi)^2 + \frac{\kappa\lambda}{2}(\xi-\chi), \tag{A.16}$$

where $l = 3\kappa^2/4$. Notice that, while A is exactly determined, the fields ϕ and B depend on the free parameters κ and λ. At this point, one can perform the following transformation of coordinates

$$r = \frac{3}{2}(\xi+\chi)^{1/3}, \qquad t = \frac{1}{2}(\xi-\chi), \tag{A.17}$$

and in these coordinates, the metric (A.5) takes the form

$$ds^2 = r^{2(l-1)} e^{\frac{\lambda^2}{4} r^6 - \kappa\lambda t^3} dt^2 - r^{2(l+1)} e^{\frac{\lambda^2}{4} r^6 - \kappa\lambda t^3} dr^2 + r^2 dx_{0,3}^2 \,. \tag{A.18}$$

The metric coefficients depend both on t and r (moreover, there is an exchange between time-like and space-like coordinates). Now, we show that, for a general choice of the parameters, the metric depends strictly on the two variables r and t.

Our argument will be slightly more general, and does not apply only to the metric (A.18), but to all metrics of the form

$$ds^2 = -g(r) dt^2 + f(t) dr^2 + h(r) dx_{0,3}^2 \,. \tag{A.19}$$

It is evident that the metric (A.18) can be recasted to this form with a simple rescaling of the coordinates r and t. The explicit structure of the functions g, f and h is not important for our argument. With the exception of the trivial case in which $f(t) = constant$, a metric of the form (A.19) can not be rewritten on a form depending only on one variable.

The point is that when we perform a change of coordinates, we limit ourselves only on the two dimensional plane (r, t), while the three dimensional spatial coordinates will not be involved in the transformation, not to spoil the symmetry of the three dimensional submanifold. In general, we send $(t, r) \to (x_0, x_1)$, with x_0 the time-like variable, via a coordinate transformation

$$\begin{aligned} t &= p(x_0, x_1), \\ r &= k(x_0, x_1). \end{aligned} \tag{A.20}$$

Let us search a coordinate transformation of the previous form such that the transformed metric depends only on one of the new coordinates, let us say x_1. Notice that the only way to ensure that the coefficent of the transformed metric $h(k(x_0, x_1))$ depends only on x_1 is to force $k(x_0, x_1)$ to depend only on x_1: we must have $k = k(x_1)$.

With this information, the metric (A.19) becomes

$$\begin{aligned} ds^2 &= -\left[\dot{p}^2 g(k(x_1)\right] dx_0^2 - \left[2\dot{p}p' g(k(x_1))\right] dx_0 dx_1 \\ &+ \left[(k')^2 f(p(x_0, x_1)) - (p')^2 g(k(x_1))\right] dx_1^2 + \left[h(k(x_1))\right] dx_{0,3}^2 \,, \end{aligned} \tag{A.21}$$

where in the previous expression the dot corresponds to a derivative along x_0, and a prime to a derivative along x_1. Let us analyse each term of the transformed metric (A.21). The only way to have a non zero coefficient for the first term of the metric that does not depend on x_0 is to impose $\ddot{p} = 0$, with $\dot{p} \neq 0$. With this condition, the second term of the metric does not depend on x_0 only when $\dot{p}' = 0$. These two requirements are satisfied only if p can be expressed as $p(x_0, x_1) = c \cdot x_0 + m(x_1)$, where c is a nonvanishing constant and and m an arbitrary function. With a function p of this form, it is finally easy to see that the third term of the metric will be independent of x_0 if and only if the function f is a constant.

The previous argument shows that in general, with a function f different from a constant, it is not possible, starting from a metric of the form (A.19), to arrive to a metric that depends only on one variable. Starting from our specific metric (A.18), the condition $f = constant$ is reached when $\lambda = 0$. In this case, it is clear that the metric depends only on the variable r.

A.3 A condition for the validity of Birkhoff theorem

Additional assumptions must be made to ensure that the solutions to Einstein's equations depend only on one variable: in this section, we consider an example of these requirements for the charged dilatonic background. We suppose that the dilaton field can be expressed as

$$\phi = G(A), \tag{A.22}$$

where G is a smooth function of the unique variable $A(t, y)$[1]. With this ansatz, Eq. (A.10) and (A.11) become [2]:

$$2B_{,u} = \left(1 + \frac{G'^2}{6}\right) A_{,u} + \frac{A_{,uu}}{A_{,u}} , \tag{A.23}$$

$$2B_{,v} = \left(1 + \frac{G'^2}{6}\right) A_{,v} + \frac{A_{,vv}}{A_{,v}} . \tag{A.24}$$

These equations imply that $A(u, v)$ is necessarily of the form

$$A(u, v) = f(U(u) + V(v)), \tag{A.25}$$

where f, U and V are arbitrary functions of a single variable. The integration of the two above differential equations (A.23) and (A.24) then gives

$$\begin{aligned} B(u, v) &= \frac{1}{2} f(U + V) + \frac{1}{2}\mathcal{H}(f(U + V)) + \frac{1}{2} \ln|f'(U + V)| \\ &+ \frac{1}{2} \ln|U'| + \frac{1}{2} \ln|V'| + \zeta , \end{aligned} \tag{A.26}$$

[1]This is a generalization of the requirement of proportionality of [71], where $G(A) = kA$ for some constant k. Actually, all the following discussion is a generalization, for every brane curvature and with a gauge field, of the argument of that paper in the case of proportional solutions.

[2]Imposing the condition that $A_{,u}$ and $A_{,v}$ are different from zero, otherwise it is simple to show that the theorem is satisfied.

where ζ is a constant of integration, which can be arbitrarily chosen by an appropriate rescaling (we will take $\zeta = 0$). The function $\mathcal{H}$ depends on G in the following way:

$$\mathcal{H}(f) \equiv \int G'^2(f) df . \tag{A.27}$$

Plugging A and B in Eq. (A.7), it is straightforward to see that f' can be expressed as a (complicated) function of f, that we call $f' \equiv \mathcal{F}(f)$.

Finally, let us put our results in the metric (A.5) (written in the light-cone coordinates), obtaining

$$ds^2 = -4\,|U'\,V'\,f'|\; e^{f+\mathcal{H}(f)}\,du\,dv \,+\, e^{2f}\,dx^2_{k,3}\,. \tag{A.28}$$

Let us introduce new coordinates defined by

$$R = e^f \tag{A.29}$$
$$T = U - V. \tag{A.30}$$

The previous metric can be rewritten as

$$ds^2 = -P(R)dT^2 + Q(R)dR^2 + R^2 dx^2_{k,3}\,, \tag{A.31}$$

with

$$P(R) \equiv Re^{\mathcal{H}(\ln R)}$$
$$Q(R) \equiv \frac{1}{R}\frac{e^{\mathcal{H}(\ln R)}}{\mathcal{L}(\ln R)}. \tag{A.32}$$

Although the explicit form of the functions P and Q is complicated, we have reached the promised result, showing that the initial metric (A.5), with the additional requirement (A.22), can be re-casted on a form depending only on one variable. Notice that, in the simple example of the previous section, the condition (A.22) corresponds exactly to chose the parameter $\lambda = 0$, that turned out to be the condition to obtain a metric depending on one variable.

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