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Virtual Intersections

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To my family

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Introduction

The main ingredient in Fulton-MacPherson's *Intersection Theory* is the definition, for every local complete intersection morphism of schemes $f : X \to Y$, of a pull-back morphism on Chow groups $f^! : A_*(Y) \to A_*(X)$ (see [10], Chapter 6). This pull-back morphism enjoys a number of good properties, which can be formalized by saying that it defines a bivariant class $f^! \in A^*(X \to Y)$ (see [10], Chapter 17).
This theory has been extended to Deligne-Mumford stacks by Vistoli ([32]) and later to Artin stacks by Kresch ([23]).
Fulton describes another case in which a bivariant class is defined. Let $i : X \to Y$ be a closed embedding of schemes and $C_{X/Y}$ the normal cone of X in Y. If $C_{X/Y} \subset E$ is a given closed embedding of the normal cone of i into a rank-r vector bundle, then these data determine a bivariant class $i^!_E \in A^r(X \to Y)$ (see Example 17.6.4 in [10]). The main result of this thesis is a wide generalization of this example as follows. Let $f : F \to G$ be a Deligne-Mumford type morphism of algebraic stacks (see Definition 1.1.1). Kresch defines a normal cone C_f (or $C_{F/G}$) to this morphism (see Section 1.1.3). Assuming we have a closed embedding of C_f into a vector bundle stack $\mathfrak{E}$ (see Definition 1.1.10), or equivalently a perfect obstruction theory for the morphism f in the sense of [3], then we define a *virtual pull-back* morphism $f^!_{\mathfrak{E}} A_*(G) \to A_*(F)$. We show that $f^!_{\mathfrak{E}}$ defines a bivariant class and it enjoys the same properties as the Fulton-MacPherson counterpart. In particular it is functorial, that is, given morphisms $f : F \to G$ and $g : G \to H$ which have compatible obstruction theories (see Definition 1.3.3), then one has $(g \circ f)^! = f^! \circ g^!$.
These generalizations allow us to view Behrend-Fantechi virtual classes (see [3]) as certain virtual pull-backs. This approach will allow us to deduce certain relations between Gromov-Witten invariants, which were the initial reasons for doing this work.

It should be said that the idea is not entirely new, although we did not find this approach in the literature. The main inspiration point was the "functoriality property of the Behrend-Fantechi class" of Kim, Kresch and Pantev in [19]. Also, a similar situation appears in Jun Li's $[M, N]^{\mathrm{virt}}$-construction (see [26]).

When the core of this thesis was in an advanced state, we have been informed of Hsin-Hong Lai's paper on "Gromov-Witten invariants of blow-ups along manifolds with convex normal bundle" ([24]). The key ideas in Lai's work are closely related to ours, with the main difference that he treats one specific map of Deligne-Momford stacks which possesses a perfect relative obstruction theory.

The thesis is divided into two chapters. In the in the first chapter we develop the general theory of Virtual pull-backs. In the second Chapter we give a number of applications. In the following we give a detailed description of the contents.

Chapter 1

In the first section we recall the notions of normal cones of Behrend-Fantechi and Kresch and prove that these two notions are canonically isomorphic. This allows us to show that that normal cones in this generalized sense have similar properties to the normal cones defined in Fulton's book *Intersection theory* ([10]). In particular, Kresch's "deformation to the normal cone" is the correct analogue of the classical "deformation to the normal cone" explained in [10].

The main idea of the second section is to replace the normal sheaf $N_{F/G}$ with a "virtual normal bundle". The appropriate context for this is given by obstruction theories. Precisely, if f is a DM-type morphism of Artin stacks (see Definition 1.1.1) that admits a perfect relative obstruction theory $E^{\bullet}_{F/G}$ (see [3]), then we take the virtual normal bundle to be $h^1/h^0((E^{\vee}_{F/G})^{\bullet})$. Using this, we obtain a well-defined morphism $f^! : A_*(G) \to A_*(F)$, that we call a *virtual pull-back*. As a byproduct of our construction we obtain a generalized notion of virtual fundamental class that applies to some examples of Artin stacks.

In Section 3 we show that the virtual pull-back satisfies the basic properties enjoyed by Gysin maps. The only point where we need to be careful is the functoriality property, where we need a compatibility condition between the vector bundle stacks that replace the normal bundles. The statement of the functoriality property may be seen as a generalization of the functoriality property in [3] and [19]. In particular, when we deal with stacks possessing virtual classes we prove that, subject to a very natural compatibility relation between obstructions (see Definition 1.3.3), the virtual pull-back sends the virtual class of G to the virtual class of F.

The last section of the first chapter concerns push-forwards along a surjective morphism of stacks $f : F \to G$, with F and G stacks which possess perfect obstruction theories. We denote the virtual dimension of F by k_1 and the virtual dimension of G by k_2. The main result in this section (see Theorem 1.4.6) states that if the induced relative obstruction theory for f is perfect and $k_1 \geq k_2$, then the push-forward of the virtual class of F along f is equal to a scalar multiple of the virtual class of G. This result is a generalization of the straight-forward fact that given a surjective morphism of schemes $f : F \to G$, with G irreducible, then $f_*[F]$ is a scalar multiple of the fundamental class of G (possibly zero).

Chapter 2

The second chapter treats several applications of the theory developed in the first chapter to Gromov-Witten theory.

In the first section we investigate the relation between the four moduli spaces of stable maps associated to a cartesian diagram of smooth projective varieties. This situation will appear in most of the following sections.

In the second section we show that given an embedding of smooth projective varieties $i : X \to \mathbb{P}$, with $\mathbb{P}$ convex, the pullback along the natural map induced by i on the corresponding moduli spaces of stable maps sends boundary divisors of $\bar{M}_{0,n}(\mathbb{P}, i_*\beta)$ to boundary divisors of $\bar{M}_{0,n}(X, \beta)$.

In the third section we provide the answer to a very natural question. Given a smooth projective variety X and its blow-up $p : \tilde{X} \to X$ along some smooth projective subvariety, we would like to know when do certain Gromov-Witten invariants of X and $\tilde{X}$ agree. More precisely, if we start with a given homology class $\beta \in A_1(X)$ and a collection of cohomology classes $\gamma_i \in A^*(X)$, then we can associate a "lifted" homology class in $A_1(\tilde{X})$ (see Definition 2.3.1 for a precise statement) and cohomology classes $p^*\gamma_i \in A^*(\tilde{X})$. One could expect that the Gromov-Witten invariants associated to these data should be equal. This was first analyzed by Gathmann ([12]) where X was a convex space and Y a point and by Hu and collaborators ([15], [16] [17]) where it was treated the blow-up along points, curves and surfaces. Recently, it was shown by Lai ([24]) that (subject to a minor condition) the expectation is true for genus zero Gromov-Witten invariants of blow-ups along subvarieties with convex normal bundles. Our idea is to show the equality of rational Gromov-Witten invariants for X convex and then "pull the relation back" to an arbitrary X (see Proposition 2.3.6). The statement we get should be compared with Theorem 1.6 in [24]. We also added a short proof of this result for points and curves. The method we use is different from the one of Hu and it relies more on the degeneration method than on the virtual analysis.

The fourth section concerns rational Gromov-Witten invariants of projective bundles $p_X : \mathbb{P}_X(V) \to X$. These were studied by Qin and Ruan ([31]) where X was taken to be a projective space and by Elezi ([7], [8]) when V is a splitted bundle and X is a toric variety. Here, we analyze the map induced by p_X between the corresponding moduli spaces of stable maps to $\mathbb{P}_X(V)$ and X. Assuming we know the genus zero Gromov-Witten theory of X, we can compute certain genus zero Gromov-Witten invariants of arbitrary projective bundles using Elezi's result.

In Section 2.5 we give a proof of the conservation of number principle for virtually smooth morphisms. As a consequence we obtain that the virtual Euler characteristic is constant in virtually smooth families (see Definition 1.4.4). This statement is a generalization of Proposition 4.14 in [11] of Fantechi and Göttsche.

In Section 2.6 we consider a situation similar to the Quantum hyperplane section principle. The starting point data of the Quantum hyperplane section principle is a smooth complete intersection X in a smooth projective variety $\mathbb{P}$ which is obtained by cutting out $\mathbb{P}$ by r hyperplanes $H_1, ..., H_r$. Let us denote by $V := \mathcal{O}(H_1) \oplus ... \oplus \mathcal{O}(H_r)$ the splitted vector bundle on $\mathbb{P}$ associated to these hyperplanes. Then, the Quantum hyperplane section principle states that one can find a formula relating the Gromov-Witten invariants of X (twisted by V) in terms of Gromov-Witten invariants of $\mathbb{P}$. This situation was studied by many people, but we mainly have in mind the work of Kim, Kresch and Pantev [19] and the Quantum Lefschetz formula of Coates and Givental ([5]). In this thesis we are intersted in a smooth morphism $p : \mathbb{P} \to X$ and we relate certain Gromov-Witten invariants of $\mathbb{P}$ (twisted by the relative cotangent bundle $\Omega_{\mathbb{P}/X}$) with the Gromov-Witten invariants of X.

In the last section we show that Costello's push-forward formula follows as an easy consequence of our formalism.

Notation and conventions. Unless otherwise stated we denote inclusions by i and projections by p.
We work over a fixed ground field.
An Artin stack is an algebraic stack in the sense of [25] of finite type over the ground field.
Unless otherwise specified we will try to respect the following convention: we will usually denote schemes by X, Y, Z, etc, Artin stacks by F, G, H, etc. and Artin stacks for which we know that they are not Deligne-Mumford stacks (such as the moduli space of genus-g curves or vector bundle stacks) by gothic letters $\mathfrak{M}_g$, $\mathfrak{E}$, $\mathfrak{F}$, etc.
By a commutative diagram of stacks we mean a 2-commutative diagram of stacks and by a cartesian diagram of stacks we mean a 2-cartesian diagram of stacks.
Chow groups for schemes are defined in the sense of [10]; this definition has been extended to DM stacks (with $\mathbb{Q}$-coefficients) by Vistoli ([32]) and to algebraic stacks (with $\mathbb{Z}$-coefficients) by Kresch ([23]). We will consider Chow groups (of schemes/stacks) with $\mathbb{Q}$-coefficients.
For a fixed stack F we denote by $\mathcal{D}_F$ the derived category of coherent $\mathcal{O}_F$ modules.
For a fixed stack F we denote by L_F its cotangent complex defined in [30].

Contents

Chapter 1

Virtual pull-backs

In this chapter we develop the general theory of Virtual pull-backs which are generalizations of Fulton-MacPherson's Gysin pull-backs. More precisely, given a DM-type morphism of algebraic stacks $f : F \to G$ (see Definition 1.1.1) and a perfect obstruction theory for f (see Definition 1.1.4) we will define a morphism $f : A_*(G) \to A_*(F)$ and we will show that it satisfies the same properties as the classical pull-back, namely: compatibility with pull-backs and push-forwards, functoriality, commutativity and excess. In the last section we will also treat the problem of push-forwards along morphisms which posses a perfect obstruction theory.

1.1 Preliminaries

Given a morphism of algebraic stacks $f : F \to G$, we will give two equivalent definitions of the normal cone of f and we will prove some basic properties of normal cones. This will be the key geometric object which will allow us to develop the theory of virtual pull-backs.

1.1.1 Background

We shortly review the basic notions we will use in this thesis. Most of the definitions in this section appeared for the first time in "The intrinsic normal cone" of Behrend and Fantechi ([3]).

DM-type morphisms

Definition 1.1.1. A morphism $f : F \to G$ of Artin stacks is called of Deligne-Mumford type (or shortly of DM-type) if for any morphism $V \to G$, with V a scheme, $F \times_G V$ is a Deligne-Mumford stack.

Remark 1.1.2. Let us consider the following Cartesian diagram

$$\begin{array}{ccc} F' & \xrightarrow{f'} & G' \\ \downarrow & & \downarrow \\ F & \xrightarrow{f} & G. \end{array} \tag{1.1}$$

If f is a DM-type morphism, then f' is a DM-type morphism.

Remark 1.1.3. Let $f : F \to G$ be morphism of stacks and let L_f be the relative cotangent complex. Then f is of DM-type if and only if $L_f \in \mathcal{D}_F^{\leq 0}$

Obstruction Theories

Definition 1.1.4. Let $E^\bullet \in \mathcal{D}_X$. $E^\bullet$ is said to be of perfect amplitude if there exists $n \geq 0$ such that $E^\bullet$ is locally isomorphic to $[E^{-n} \to ... \to E^0]$, where $\forall i \in \{-n, ..., 0\}$, E^i is a locally free sheaf.

Definition 1.1.5. Let $E^\bullet \in \mathcal{D}_X^{\leq 0}$. Then a homomorphism $\Phi : E^\bullet \to L^\bullet$ in $\mathcal{D}_F$ is called an obstruction theory if $h^0(\Phi)$ is an isomorphism and $h^{-1}(\Phi)$ is surjective. If moreover, $E^\bullet$ is of perfect amplitude, then $E^\bullet$ is called a perfect obstruction theory.

Convention 1.1.6. Unless otherwise stated by a perfect obstruction theory we will always mean of perfect amplitude contained in $[-1, 0]$.

Definition 1.1.7. Let $f : F \to G$ be morphism of stacks such that $L_{F/G}$ is perfect. In this case we call $L_{F/G}$ the tautological obstruction.
Let $E^\bullet := [E^\vee \to 0]$ be a perfect complex with $E \neq 0$. Then, under the assumptions above we call $L_{F/G} \oplus E^\bullet$ a superfluous obstruction.

Cone stacks

Definition 1.1.8. Let X be a scheme and $\mathcal{F}$ be a coherent sheaf on X. We call $C(\mathcal{F}) := SpecSym(\mathcal{F})$ an abelian cone over X.

As described in [3], Section 1, every abelian cone $C(\mathcal{F})$ has a section $0 : X \to C(\mathcal{F})$ and an $\mathbb{A}^1$-action.

Definition 1.1.9. An $\mathbb{A}^1$-invariant subscheme of $C(\mathcal{F})$ that contains the zero section is called a cone over X.

Similarly, Behrend and Fantechi define in [3] Section 1, abelian cone stacks and cone stacks. Let us recall the definition.

Definition 1.1.10. Let F be a stack and let $E^\bullet$ be an element in $(\mathcal{D}_F)_{\geq 0}$. We call the stack quotient $h^1/h^0(E^\bullet)$ (in the sense of [3] Section 2) an abelian cone stack over stack F.
A cone stack is a closed substack of an abelian cone stack invariant under the action of $\mathbb{A}^1$ and containing the zero section.
If a cone is flat over F, then it is called a vector bundle stack.

Convention 1.1.11. From now on, unless otherwise stated, by cones we will mean cone-stacks.

Example 1.1.12. (i)Let $i : X \to Y$ be a closed embedding of schemes. If $\mathcal{I}$ denotes the ideal sheaf of X in Y, then $N_{X/Y} = SpecSym\, \mathcal{I}/\mathcal{I}^2$ is called the normal sheaf of X in Y and $C_{X/Y} := Spec \oplus_{k\geq 0} \mathcal{I}^k/\mathcal{I}^{k+1} \hookrightarrow N_{X/Y}$ is called the normal cone of X in Y.
(ii) If $f : F \to G$ is a local immersion of DM-stacks, then Vistoli defines (see [32], Definition 1.20) the normal cone to f as described below. Let us consider a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\tilde{f}} & V \\ \downarrow & & \downarrow \\ F & \xrightarrow{f} & G \end{array} \tag{1.2}$$

with U, V schemes, the upper horizontal arrow a closed immersion and the vertical arrows étale. Then $C_{F/G}$ is the cone obtained by descent from $C_{U/V}$. Note that $C_{F/G} \hookrightarrow N_{F/G} = SpecSym\ h^{-1}(L_{F/G})$.

1.1.2 Intrinsic normal cones to DM-type morphisms

In this section we recall the definition of (relative) intrinsic normal cones of Behrend and Fantechi ([3]) and we generalize it to DM-type morphisms of Artin stacks. We also prove that intrinsic normal cones behave nicely in Cartesian diagrams.

Definition 1.1.13. Let $f : F \to G$ be a DM-type morphism and let $L_{F/G} \in$ ob $\mathcal{D}(\mathcal{O}_F)$ be the cotangent complex. Then we denote the stack $h^1/h^0(L_{F/G})^\vee$ by $\mathfrak{N}_{F/G}$ and we call it the *intrinsic normal sheaf.*

Proposition 1.1.14. *(Behrend-Fantechi) Let us consider diagram (1.2) with the upper horizontal arrow a closed immersion, $U \to F$ an étale morphism and $V \to G$ a smooth morphism. Then for any U and V as above, there exists a unique cone-stack $\mathfrak{C}_{F/G} \subset \mathfrak{N}_{F/G}$ such that $\mathfrak{C}_{F/G} \times_F U = [C_{U/V}/\tilde{f}^* T_{V/G}]$.*

Definition 1.1.15. We call $\mathfrak{C}_{F/G}$ the *intrinsic normal cone to f.*

Lemma 1.1.16. *Let*

$$\begin{array}{ccc} F' & \xrightarrow{f'} & G' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ F & \xrightarrow{f} & G \end{array}$$

be a commutative diagram of Artin stacks with f and f' of DM-type. Then, there is an induced morphism of abelian cone stacks $\beta : \mathfrak{N}_{F'/G'} \to p^*\mathfrak{N}_{F/G}$.

Proof. The claim follows easily from the morphisms of cotangent complexes

$$p^* L_{F/G} \to L_{F'/G} \to L_{F'/G'}.$$

□

Proposition 1.1.17. *Let*

$$\begin{array}{ccc} F' & \xrightarrow{f'} & G' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ F & \xrightarrow{f} & G \end{array}$$

be a commutative diagram of DM-stacks stacks with f and f' of DM-type. Then, the morphism of Lemma 1.1.16 induces a morphism of cone stacks $\alpha : \mathfrak{C}_{F'/G'} \to p^*\mathfrak{C}_{F/G}$. If the diagram is cartesian, then α is a closed immersion. If moreover, q is flat, then α is an isomorphism.

Proof. This is a generalization of Proposition 7.1 in [3] where the authors treat the case G and G' are smooth. We will do the proof in several steps.
Step 0. If f and f' are locally closed embeddings of DM-stacks the claim follows from [32] Section 1.

Step 1. Given $F' \xrightarrow{p} F \xrightarrow{f} G$ morphisms of DM-stacks we show that the natural morphism $\mathfrak{N}_{F'/G} \to p^*\mathfrak{N}_{F/G}$ induces a morphism $\mathfrak{C}_{F'/G} \to p^*\mathfrak{C}_{F/G}$. For this, let M be a scheme such that $j : F \hookrightarrow M$ is a locally closed embedding. Moreover, we can choose M smooth over G such that the following diagram

$$\begin{array}{ccc} & & M \\ & {\scriptstyle j}\nearrow & \downarrow \\ F & \longrightarrow & G \end{array}$$

commutes. In the same way we choose N smooth over M such that $F' \hookrightarrow N$ is a locally closed embedding. Putting all together we have a commutative diagram

$$\begin{array}{ccccc} & & & & N \\ & & & {\scriptstyle i}\nearrow & \downarrow{\scriptstyle \pi} \\ & & & & M \\ & & & {\scriptstyle j}\nearrow & \downarrow \\ F' & \xrightarrow{p} & F & \longrightarrow & G \end{array}$$

with i and j locally closed embeddings. Using Step 0 for these maps we obtain a morphism $C_{F'/N} \to p^*C_{F/M}$. On the other hand we have a morphism $T_{N/G} \to \pi^*T_{M/G}$. From the commutative diagram

$$\begin{array}{ccc} i^*T_{N/G} & \longrightarrow & C_{F'/N} \\ \downarrow & & \downarrow \\ p^*j^*T_{M/G} & \longrightarrow & p^*C_{F/M} \end{array}$$

we obtain a morphism $[C_{F'/N}/i^*T_{N/G}] \to [p^*C_{F/M}/p^*j^*T_{M/G}]$ and therefore the conclusion.

Step 2. Let us first treat the case in which the given diagram is cartesian. As before, let V a scheme such that $F \hookrightarrow V$ is a locally closed embedding and V smooth over G. Let us now consider $V' = V \times_G G'$. Then we have the following diagram

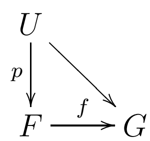
$$\begin{array}{ccccc} F' & \longrightarrow & V' & \longrightarrow & G' \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle q} \\ F & \longrightarrow & V & \longrightarrow & G. \end{array}$$

Let us moreover consider $F'' := F \times_V V'$. By the universal property of cartesian diagrams we obtain a map $\phi : F'' \to F'$. As the diagram on the left and big rectangle are both cartesian we obtain that ϕ is an isomorphism. This implies that $F' \to V'$ is a closed embedding. Using Step 0, we obtain a map $\alpha : [C_{F'/V'}/T_{V'/G'}] \to [p^* C_{F/V}/T_{V'/G'}]$.
If moreover, f is flat, the proof follows from the corresponding statement in Step 0.
Step 3. Let us consider $F'' := F \times_G G'$ with maps $p' : F' \to F''$ and $p'' : F'' \to F'$. By Step 2, we have a morphism $\mathfrak{C}_{F''/G'} \to p''^* \mathfrak{C}_{F/G}$ and by Step 1 we have a natural morphism $\mathfrak{C}_{F'/G'} \to p'^* \mathfrak{C}_{F''/G'}$. Composing the two morphisms we obtain a morphism $\mathfrak{C}_{F'/G'} \to p^* \mathfrak{C}_{F/G}$. □

Remark 1.1.18. It can be easily seen that the canonical morphism $\mathfrak{C}_{F'/G'} \to p^* \mathfrak{C}_{F/G}$ is injective if and only if $\mathfrak{N}_{F'/G'} \to p^* \mathfrak{N}_{F/G}$ is injective.

In the following we generalize the notion of intrinsic normal cone to a DM-type morphism $f : F \to G$ to the case F is an Artin stack (not necessarily a DM-stack).

Construction 1.1.19. Let us consider the following commutative diagram



(1.3)

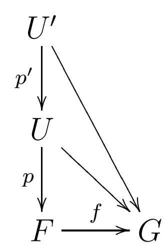
with U a scheme and p a smooth morphism. By Lemma 1.1.16 we have a morphism

$$\gamma : \mathfrak{N}_{U/G} \to p^* \mathfrak{N}_{F/G}. \tag{1.4}$$

Let us consider the restriction $\tilde{\gamma} : \mathfrak{C}_{U/G} \to p^* \mathfrak{N}_{F/G}$. We denote the image of $\tilde{\gamma}$ by $\mathfrak{C}_{F/G}|_U$ and we call it the local normal cone on U of F to G .

Let us now show that local normal cones glue. For this, we need the following easy lemma.

Lemma 1.1.20. *Let us consider the following commutative diagram*



with p and p' smooth morphisms. Then $\mathfrak{C}_{F/G}|_U \times_U U'$ is naturally isomorphic to $\mathfrak{C}_{F/G}|_{U'}$.

Proof. The claim easily reduces to showing that the following diagram is commutative

$$\begin{array}{ccc} \mathfrak{C}_{U'/G} & \longrightarrow & (p \circ p')^* \mathfrak{N}_{F/G} \\ \downarrow & & \downarrow \\ p'^* \mathfrak{C}_{U/G} & \longrightarrow & (p \circ p')^* \mathfrak{N}_{F/G} \end{array}$$

and this is obvious from the corresponding diagram between normal sheaves. □

This lemma shows that there exists a unique closed subcone $\mathfrak{C}_{F/G} \hookrightarrow \mathfrak{N}_{F/G}$ such that for every diagram (1.3) we have $\mathfrak{C}_{F/G} \times_F U = \mathfrak{C}_{F/G}|_U$.

Definition 1.1.21. The cone $C_{F/G}$ is called the intrinsic normal cone of f, or when there is no risk of confusion the intrinsic normal cone of F to G.

Remark 1.1.22. Let us consider diagram (1.3) with F a DM-stack. Let us show that restricting the natural map $\mathfrak{N}_{U/G} \to p^* \mathfrak{N}_{F/G} \to 0$ to $\mathfrak{C}_{U/G}$ we obtain a morphism of cones $\mathfrak{C}_{U/G} \to p^* \mathfrak{C}_{F/G} \to 0$. For this, let us consider $V \to G$ a smooth morphism and let us construct the following diagram

$$\begin{array}{ccc} F \times_G V & \xrightarrow{f'} & V \\ {\scriptstyle p}\downarrow & & \downarrow \\ F & \xrightarrow{f} & G \end{array}.$$

By Proposition 1.1.2 we have that $C_{f'}$ is canonically isomorphic to C_f. Let now $q : U \to F$ an étale morphism and $U \hookrightarrow M$, with M smooth over V. Then $q^* C_f = [C_{U/M}/T_{M/V}]$. This shows that restricting the natural map $\mathfrak{N}_{U/G} \to p^* \mathfrak{N}_{F/G} \to 0$ to $\mathfrak{C}_{U/G}$ we obtain a morphism of cones $[C_{U/M}/T_{M/G}] \to [C_{U/M}/T_{M/V}]$.
Therefore our definition is consistent with Definition 1.1.15.

Remark 1.1.23. Let us consider diagram (1.3), with p a smooth morphism. By Proposition 1.1.2 and Remark 1.1.18 we obtain a closed embedding $\mathfrak{C}_{U/F} \to \mathfrak{C}_{U/G}$. This shows that the sequence $0 \to \mathfrak{N}_{U/F} \to \mathfrak{N}_{U/G} \to \mathfrak{N}_{F/G}|_U \to 0$ induces the exact sequence of cones (in the sense of Definition 1.12 in [3])

$$0 \to \mathfrak{C}_{U/F} \to \mathfrak{C}_{U/G} \to \mathfrak{C}_{F/G}|_U \to 0. \tag{1.5}$$

Proposition 1.1.24. *Let*

$$\begin{array}{ccc} F' & \xrightarrow{f'} & G' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ F & \xrightarrow{f} & G \end{array}$$

be a commutative diagram of Artin stacks with f of DM-type. Then, there is an induced morphism of cone stacks $\alpha : \mathfrak{C}_{F'/G'} \to p^*\mathfrak{C}_{F/G}$. If moreover, the diagram is cartesian, then α is a closed immersion. If q is flat, then α is an isomorphism.

Proof. We argue as in the proof of Proposition 1.1.2.
Step 1. Let us consider the diagram is cartesian. Let $U \to F$ a flat scheme over F and $U' := U \times_F F'$. Then from the cartesian diagram

$$\begin{array}{ccccc} U' & \longrightarrow & F' & \longrightarrow & G' \\ {\scriptstyle q}\downarrow & & \downarrow & & \downarrow{\scriptstyle q} \\ U & \longrightarrow & F & \longrightarrow & G \end{array}$$

and Proposition 1.1.2 we obtain that

$$\mathfrak{C}_{U'/G'} \to q^*\mathfrak{C}_{U/G} \tag{1.6}$$

is a closed embedding. Using sequence (1.5) for $\mathfrak{C}_{F/G}|_U$ and $\mathfrak{C}_{F'/G'}|_{U'}$ we see that the embedding (1.6) induces an embedding $\mathfrak{C}_{F/G}|_U \hookrightarrow q^*(\mathfrak{C}_{U'/G'}|_{U'})$ which glues to a closed embedding $\mathfrak{C}_{F/G} \hookrightarrow q^*\mathfrak{C}_{U'/G'}$ from the corresponding morphism between normal sheaves.
If moreover, f is flat, the proof follows similarly.
Step 2. The general case follows from Step 1 and Proposition 1.1.2, in the same way as in the proof of Proposition 1.1.2 Step 3 follows from Step 2 and Step 1. □

1.1.3 Normal cones to DM-type morphisms

Here we recall Kresch's fundamental notion of normal cone of a DM-type morphism and we compare it to the intrinsic normal cones from the previous subsection.
Let $f : F \to G$ be a morphism of DM-type. The normal cone of f, denoted C_f or $C_{F/G}$ was defined by Kresch in [23], section 5.1 under the assumption

f representable and locally separated; it is a cone stack over F. In [22], Section 5.1 and in the proof of Proposition 1 in [19], Kresch mentions that the definition of C_f and its abelian hull N_f extends to DM-type morphisms. We spell out the definition.

Lemma 1.1.25. *Let $f : F \to G$ be a DM-type morphism of Artin stacks. Then one can construct a commutative diagram (not unique)*

$$\begin{array}{ccc} U & \xrightarrow{\tilde{f}} & V \\ \downarrow & & \downarrow \\ F & \xrightarrow{f} & G \end{array} \tag{1.7}$$

where U and V are schemes, the vertical arrows are smooth surjective and the top arrow $U \to V$ is a closed immersion. Moreover, U and V in diagram (1.8) can be taken such that the natural map $R \to S$ is a locally closed immersion.

Proof. Let W be a smooth atlas of G. As f is a DM-type morphism $F \times_G W$ is a DM-stack. Let U be an affine étale atlas of $F \times_G W$. Then, there exists a smooth scheme M such that $U \hookrightarrow M$ is a closed embedding of schemes. Taking V to be $M \times W$, we obtain the following commutative diagram with the vertical arrows smooth morphisms and the natural map $\tilde{f} : U \to V$ a closed immersion

$$\begin{array}{ccc} U & \xrightarrow{\tilde{f}} & V := M \times W \\ {\scriptstyle \acute{e}t}\downarrow & & \downarrow \\ F \times_G W & \longrightarrow & W \\ \downarrow & & \downarrow \\ F & \xrightarrow{f} & G. \end{array} \tag{1.8}$$

□

Lemma 1.1.26. *Let $R := U \times_F U$ and $S := V \times_G V$. Then the natural map $R \to S$ is a locally closed immersion.*

Proof. We can factor the morphism $R \to S$ as $R \to U \times_G U \to S = V \times_G V$. The last map is a closed immersion. Let us now show that the first map is a locally closed immersion. But this follows easily from the fact that the following diagram is Cartesian

$$\begin{array}{ccc} U \times_F U & \longrightarrow & U \times_G U \\ \downarrow & & \downarrow \\ F & \xrightarrow{\Delta} & F \times_G F. \end{array} \tag{1.9}$$

□

Proposition 1.1.27. *(Kresch) Let us consider the cone $C_{R/S}$. There are natural morphisms making $C_{R/S} \rightrightarrows C_{U/V}$ into a smooth groupoid in the category of schemes.*

Proof. (Sketch) Let $q_1,\ q_2 : S \to V$ be the obvious projections. Then we have natural maps

$$R = U \times_F U \to U \times_G U \to U \times_G V \simeq U \times_V S,$$

the last isomorphism depending on q_i. These maps induce natural maps $s_1,\ s_2 : C_{R/S} \to C_{U/V}$

$$C_{R/S} \rightrightarrows (C_{U \times_V S/S}) \times_{U \times_V S} R \simeq C_{U/V} \times_U R \to C_{U/V}. \tag{1.10}$$

In the same manner as in [22] Section 5.1 the maps s_i are smooth and determine a groupoid. □

In a completely analogous manner one can define a groupoid $[N_{R/S} \rightrightarrows N_{U/V}]$, where $N_{R/S}$, $N_{U/V}$ are the normal sheaves (where the normal sheaf $N_{R/S}$ is the abelian hull of the normal cone $C_{R/S}$ of [32], Definition 1.20). This groupoid defines a stack that we denote $N_{F/G}$.

Definition 1.1.28. Let $C_{F/G}$ be the stack associated to the groupoid $[C_{R/S} \rightrightarrows C_{U/V}]$ and $N_{F/G}$ the stack associated to the groupoid $[N_{R/S} \rightrightarrows N_{U/V}]$. We call $C_{F/G}$ the normal cone of f and $N_{F/G}$ the normal sheaf of f.

Theorem 1.1.29. *(Kresch) Let $f : F \to G$ be a DM-type morphism of Artin stacks. One can define a deformation space, i.e. a flat morphism $M^{\circ}_F G \to \mathbb{P}^1$ with general fibre G and special fibre the normal cone $C_{F/G}$. Moreover, for any cartesian diagram*

$$\begin{array}{ccc} F' & \xrightarrow{f'} & G' \\ \downarrow & & \downarrow \\ F & \xrightarrow{f} & G \end{array}$$

there exists an induced morphism $M^\circ_{F'}G' \to M^\circ_F G$ that fits into a cartesian diagram

$$\begin{array}{ccc} C_{F'/G'} & \longrightarrow & M^\circ_{F'}G' \\ \downarrow & & \downarrow \\ C_{F/G} & \longrightarrow & M^\circ_F G \end{array}$$

Proof. A detailed proof can be found in [23], proposition 13.52 for locally closed immersions. Let us sketch the construction in the general case. As in the case of cones there are natural morphisms making $M^\circ_{R/S} \rightrightarrows M^\circ_{U/V}$ into a smooth groupoid. Let us denote by $M^\circ_{F/G}$ the stack (in general it is not algebraic) associated to the groupoid $[M^\circ_{R/S} \rightrightarrows M^\circ_{U/V}]$.
Let us consider the diagram in Lemma 1.1.25. Taking $V' := V \times_G G'$ and $U' := U \times_V V'$ we obtain a similar diagram for f'. This gives a morphism of groupoids $M^\circ_{F/G} \to M^\circ_{F'/G'}$ which induces a morphism of cones $C_{F/G} \to C_{F'/G'}$. The diagram we obtain it can be easily seen to be cartesian. □

Remark 1.1.30. From Theorem 1.1.29 it follows that whenever G is of pure dimension r, then $C_{F/G}$ is again of pure dimension r.

Let us now compare the normal cone defined by Kresch with the intrinsic normal cone.
The following Lemma in probably well-known to experts, but as we did not find it in the literature, we give a detailed proof for completeness.

Proposition 1.1.31. *If $f : F \to G$ is a DM-type morphism, then the cone stack $C_{F/G}$ of Definition 1.1.28 is canonically isomorphic to the intrinsic normal cone $\mathfrak{C}_{F/G}$ of Definition 1.1.15.*

Proof. We divide the proof in several cases. In what follows we use the notation "=" for canonical isomorphisms.

Case 1. If f is a closed embedding of schemes the statement is trivial.

Case 2. If f is a locally closed embedding of stacks, then $N_{F/G}$ and $C_{F/G}$ are obtained by descent on F (see [32]) and hence it suffices to check the statement locally. This shows that the statement follows by the first case.

Case 3. If f factors as

$$\begin{array}{ccc} & & W := G \times M \\ & \overset{i}{\nearrow} & \downarrow \\ F & \xrightarrow{f} & G \end{array}$$

with i a locally closed embedding and M a smooth scheme, then $\mathfrak{N}_{F/G} = N_{F/W}/i^*T_M$. Let us take U, V étale covers of F and $G \times M$ such that $U \to V$ is a closed embedding of schemes. Then, it suffices to show we have an isomorphism

$$N_{U/V} \times_{N_{F/W}/T_M} N_{U/V} \simeq N_{U \times_F U/V \times_G V}$$

compatible with the groupoid structure. For this, we see the first term is isomorphic to $p^*i^*T_M \times N_{U/V} \times_{N_{F/W}} N_{U/V}$ and using $V \to W$ is étale we obtain the first term is isomorphic to $p^*i^*T_M \times N_{U \times_F U/V}$. On the other hand, we know by the previous case that $N_{U \times_F U/V \times_G V}$ is canonically isomorphic to $\mathfrak{N}_{U \times_F U/V \times_G V}$ for which we know it is isomorphic to $\mathfrak{N}_{U \times_F U/V} \times p^*i^*T_{V/G}$. This shows $\mathfrak{N}_{F/G} = N_{F/G}$.

Case 4. In general, we show $\mathfrak{N}_{F/G} = N_{F/G}$. The proof is very similar to Case 3, above. Let us consider diagram (1.8) with the diagonal map $g : U \to G$. By Case 3 we have

$$N_{U/G} = \mathfrak{N}_{U/G} = N_{U/V}/T_{V/G}. \tag{1.11}$$

In order to analyze the lower triangle of diagram (1.8), we consider the distinguished triangle of relative cotangent complexes

$$p^*L_{F/G} \to L_{U/G} \to L_{U/F} \to p^*L_{F/G}[1].$$

As $p : U \to F$ is smooth it is easy to see that we are in the conditions of Proposition 2.7 in [3] and thus we get a short exact sequence of intrinsic normal sheaves

$$0 \to \mathfrak{N}_{U/F} \to \mathfrak{N}_{U/G} \to p^*\mathfrak{N}_{F/G} \to 0. \tag{1.12}$$

By (1.11) and (1.12), in a similar way as before we get *local* isomorphisms. Moreover, the same equations (1.11) and (1.12) give a smooth morphism of abelian cone stacks $N_{U/V} \to \mathfrak{N}_{F/G}$ and in a completely analogous fashion we get morphisms of abelian cone stacks $N_{U \times_F U/V \times_G V} \to N_{U/V}$. This shows we obtain a morphism of abelian cone stacks $N_{U \times_F U/V \times_G V} \to N_{U/V} \times_{\mathfrak{N}_{F/G}} N_{U/V}$.

Since as remarked above, this morphism is a local isomorphism, we obtain an isomorphism $N_{U/V} \times_{\mathfrak{N}_{F/G}} N_{U/V} \simeq N_{U \times_F U/V \times_G V}$. Checking the diagram below is commutative

$$\begin{array}{ccc} N_{U \times_F U/V \times_G V} & \rightrightarrows & N_{U/V} \\ \downarrow & & \downarrow \\ N_{U/V} \times_{\mathfrak{N}_{F/G}} N_{U/V} & \rightrightarrows & N_{U/V} \end{array}$$

we obtain an isomorphism of groupoids and therefore the conclusion.

Case 5. By Case 4 above, it is enough to check that $C_{F/G}$ is canonically isomorphic to the relative intrinsic normal cone $\mathfrak{C}_{F/G}$ locally. For this, we look at the groupoid $[C_{U/V \times_G V} \rightrightarrows C_{U/V}]$ with the two maps obtained by replacing F with U. It is easy to see that $N_{U/V \times_G V}$ is isomorphic to $N_{U/V} \times \tilde{f}^* T_{V/G}$. Via this isomorphism, the two maps defining the groupoid are the projection and the natural action of $\tilde{f}^* T_{V/G}$ on $C_{U/V}$. This shows $C_{U/G}$ is locally isomorphic to $[C_{U/V}/\tilde{f}^* T_{V/G}]$ and therefore the claim follows. □

Remark 1.1.32. By the above Lemma we are allowed to identify the normal cone to a morphism with the intrinsic normal cone. In particular, the above Lemma shows that Definition 1.1.28 is independent of the choice of U and V in diagram 1.8. Although normal cones are cone stacks, we will use for simplicity the notation $C_{F/G}$ instead of $\mathfrak{C}_{F/G}$.

If X is a scheme, E is a vector bundle on X and $i : X \to E$ is the zero section, then $C_{X/E}$ is naturally isomorphic to E. We prove a series of successive generalizations of this result.

Example 1.1.33. Let G be a DM-stack, E a vector bundle on G and $G \to E$ the zero section. Then $C_{G/E}$ is canonically isomorphic to E.

Proof. Let V be an étale atlas of G and E_V the pull-back of E to V, then we can construct a commutative diagram as above and $C_{G/E}$ is obtained by descent from $C_{V/E_V} \simeq E_V$. This shows that $C_{G/E}$ is canonically isomorphic to E. □

Example 1.1.34. Let $F \xrightarrow{f} G$ be a DM-type morphism and $p : E \to G$ a vector bundle on G. Let $i : G \to E$ be the zero section, and let $g : F \to E$ be $g := i \circ f$. Then $C_{F/E}$ is canonically isomorphic to $C_{F/G} \times_F f^* E$.

Proof. Let us consider the distinguished triangles corresponding to g and f respectively. The morphism i induces a morphism $i_E^L \to L_G$ and therefore we obtain the following morphism of distinguished triangles

$$\begin{array}{ccccccc} f^*i^*L_E & \longrightarrow & L_F & \longrightarrow & L_{F/E} & \longrightarrow & f^*i^*L_E[1] \; . \\ \downarrow & & \uparrow\downarrow & & \downarrow & & \downarrow \\ f^*L_G & \longrightarrow & L_F & \longrightarrow & L_{F/G} & \longrightarrow & f^*L_G[1] \end{array}$$

Using p instead of i we obtain in the same way a morphism $L_{F/G} \to L_{F/E}$ and thus we get a morphism $f^*L_{G/E} \oplus L_{F/G} \to L_{F/E}$. To show it is an isomorphism it suffices to show the statement locally. As we may assume G is an affine scheme, it is easy to see that $i^*L_E = L_G \oplus E^\vee$. On the other hand, $L_{G/E} = [E^\vee \to 0]$, where $E^\vee$ stays in degree -1 and therefore we reduced the problem to showing the triangle

$$f^*L_G \oplus E^\vee \to L_F \to L_{F/G} \oplus [f^*E^\vee \to 0]$$

is distinguished. But this follows trivially from the definition of the mapping cone. This shows that $h^1/h^0(L_{F/E}^\vee)$ is isomorphic to $h^1/h^0(L_{F/G}^\vee) \times_F h^1/h^0(f^*L_{G/E}^\vee)$. We have thus obtained $C_{F/E}$ is isomorphic to $C_{F/G} \times_F f^*E$. □

Example 1.1.35. Let $F \xrightarrow{f} G$ be DM-type morphism, $\mathfrak{E} := E^1/E^0$ a vector bundle stack on G. Let $G \xrightarrow{i} \mathfrak{E}$ denote the zero section. If $g : F \to G$ is the composition $F \xrightarrow{f} G \xrightarrow{i} \mathfrak{E}$, then $C_{F/G}$ is naturally isomorphic to $C_{F/G} \times_F f^*\mathfrak{E}$

Proof. Using the above factorization of the morphism $F \to G$, we see that $C_{F/\mathfrak{E}} = [C_{F/E^1}/E^0]$. Using (ii) above for C_{F/E^1}, we obtain that the normal cone of F in $\mathfrak{E}$ is isomorphic to $C_{F/G} \times_F f^*\mathfrak{E}$. □

We include two examples in which the normal cone is a vector-bundle stack.

Example 1.1.36. Let $F \to G$ be a smooth morphism of DM-stacks. Then $C_{F/G}$ is isomorphic to $[F/T_{F/G}]$, hence is a vector bundle stack.

Example 1.1.37. Let $X \xrightarrow{f} Y$ be a morphism of smooth schemes. Then, U and V above can be taken to be X and $X \times Y$ as below

$$\begin{array}{ccc} X & \xrightarrow{id \times f} & X \times Y \\ \downarrow & & \downarrow \pi_2 \\ X & \longrightarrow & Y \end{array}$$

where π_2 is the projection on Y. It is then easy to see that the normal cone is $[N_{X/X\times Y}/T_X]$ that is a vector bundle stack.

1.1.4 Deformation of cones

Proposition 1.1.38. *(Behrend-Fantechi) Let us consider a distinguished triangle of elements of* $\mathcal{D}_F$

$$E'' \xrightarrow{\varphi} E \to E' \to E''[1]$$

with E' *a complex of perfect amplitude contained in* $[-1, 0]$*. Let us consider the associated cone-stacks* $\mathfrak{E} := h^1/h^0(E^\vee)$*,* $\mathfrak{E}' := h^1/h^0(E')^\vee$ *and* $\mathfrak{E}'' := h^1/h^0(E'')^\vee$*. Then*

$$0 \to \mathfrak{E}' \to \mathfrak{E} \to \mathfrak{E}'' \to 0$$

is an exact sequence of cone-stacks.

Proof. This is Proposition 2.7 in [3]. □

Proposition 1.1.39. *Let us consider a distinguished triangle of elements of* $\mathcal{D}_F$

$$E'' \xrightarrow{\varphi} E \to E' \to E''[1]$$

with E' *a complex of perfect amplitude contained in* $[-1, 0]$*. Then, there exists a cone* $\mathfrak{C}$ *over* $F \times \mathbb{P}^1$ *flat over* $\mathbb{P}^1$ *with* $\mathfrak{C}_0$ *isomorphic to* $\mathfrak{E}' \oplus \mathfrak{E}''$ *and* $\mathfrak{C}_1$ *isomorphic to* $\mathfrak{E}$*. Moreover, we have a commutative diagram*

$$\begin{array}{ccc} \mathfrak{E}' \times \mathbb{P}^1 & \longrightarrow & C \\ & \searrow & \downarrow \\ & & \mathbb{P}^1. \end{array}$$

Proof. Consider the morphism $v := (T \cdot id, U \cdot \varphi) : E_{\prime\prime} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \to E'' \oplus E$ in $\mathcal{D}(F \times \mathbb{P}^1)$ and the associated distinguished triangle

$$E'' \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{v} E'' \oplus E \to c(v) \to E'' \otimes \mathcal{O}_{\mathbb{P}^1}(-1)[1].$$

Then, by construction $c(v)$ has a morphism to $\mathbb{P}^1$. The restriction of $c(v)$ to $T \neq 0$ is isomorphic to E and the restriction to $T \neq 0$ isomorphic to $E'' \oplus E'$. Looking now to $\mathfrak{C} := h^1/h^0 c(v) \to \mathbb{P}^1$ we see that the general fiber is isomorphic to $\mathfrak{E}$ and the special fibre over $T = 0$ is isomorphic to $\mathfrak{E}'' \oplus \mathfrak{E}'$. By the proposition above $\mathfrak{E}$ and $\mathfrak{E}'' \oplus \mathfrak{E}'$ are locally isomorphic and therefore $\mathfrak{C}$ is flat. The morphism of distinguished triangles in Definition 1.3.3 gives a morphism of distinguished triangles

$$\begin{array}{ccccccc}
(fp)^*E''^\bullet(-1) & \xrightarrow{w^*v} & (fp)^*E''^\bullet \oplus p^*E^\bullet & \longrightarrow & w^*c(v) & \longrightarrow & (fq)^*E''^\bullet(-1)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(fp)^*L_{G/\mathfrak{M}}(-1) & \xrightarrow{w^*u} & (fp)^*L_{G/\mathfrak{M}} \oplus p^*L_{F/\mathfrak{M}} & \longrightarrow & w^*c(u) & \longrightarrow & (fq)^*L_{G/\mathfrak{M}}(-1)[1]
\end{array}$$

over $F' \times \mathbb{P}^1$. □

Proposition 1.1.40. *Let Y be a variety of pure dimension n which is flat over $\mathbb{P}^1$ and let $X \hookrightarrow Y$ be a closed subscheme of Y flat over $\mathbb{P}^1$. Then the push-forward of $s(C_{X_t/Y_t})$ to X does not depend on t. Here X_t, Y_t indicate the fiber of $X \to \mathbb{P}^1$, respectively of $Y \to \mathbb{P}^1$ over the point $\{t\}$.*

Proof. Step 1. Let us first show that the restriction of $C_{X/Y} \to \mathbb{P}^1_1$ to any point t in $\mathbb{P}^1$ is canonically isomorphic to C_{X_t/Y_t}.
We have the following diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{O}_Y & \longrightarrow & \mathcal{O}_{X_t} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle i_Y} & & \downarrow{\scriptstyle i_X} & & \\
0 & \longrightarrow & \mathcal{I}_t & \longrightarrow & \mathcal{O}_{Y_t} & \longrightarrow & \mathcal{O}_{X_t} & \longrightarrow & 0.
\end{array}$$

As X and Y are flat over $\mathbb{P}^1_1$ we have that the restriction of $\mathcal{I}$ to $\{t\}$ is isomorphic to $\mathcal{I}_t$. This shows that $(C_{X/Y})_t$ is canonically isomorphic to C_{X_t/Y_t}. In particular the restriction of $\mathcal{O}_{C_{X/Y}}$ to t is canonically isomorphic to $\mathcal{O}_{C_{X_t/Y_t}}$. From now on, without loss of generality we may assume that $t_0 = 0$ and $t_1 = 1$.
Step 2. Introduce deformation space $M := M^\circ_{X/Y} \to \mathbb{P}^1_1 \times \mathbb{P}^1_2$ with general

fiber over $\mathbb{P}^1_2$ isomorphic to Y and $i : C \to M$ the special fibre over $\mathbb{P}^1_1 \times \{0\}$ isomorphic to $C_{X/Y}$. As Y is a variety of pure dimension M is flat over $\mathbb{P}^1_i$, $i = 1, 2$. Let us denote the fibers of $p_2 : M \to \mathbb{P}^1_1$ over the points 0 and 1 by D_0, respectively D_1. As p_1 is flat we have that D_0 is a Cartier divisor isomorphic to $M^\circ_{X_0/Y_0}$ and D_1 is a Cartier divisor isomorphic to $M^\circ_{X_1/Y_1}$. Similarly, the fibers of $p_2 : M \to \mathbb{P}^1_1$ are Cartier divisors isomorphic to Y respectively C. Let us show that $[C_{X_0/Y_0}] = [C_{X_1/Y_1}]$ in $A_*(C)$. Using the fact that $M^\circ_{X_i/Y_i}$ is flat over $\mathbb{P}^1_2$ for $i = 1, 2$ we obtain that

$$[C] \cdot [D_i] = [C_{X_i/Y_i}]$$

in $A_*(D_i)$. By the definition and the commutativity of intersections with divisors this shows that $[i^* D_i] = [C_{X_i/Y_i}]$ in $A_*(C)$. As $[D_0] = [D_1]$ in $A_*(M)$, this implies that

$$[C_{X_0/Y_0}] = [C_{X_1/Y_1}] \tag{1.13}$$

in $A_*(C)$. Passing to the closure we also obtain that

$$[\mathbb{P}(C_{X_0/Y_0} \oplus \mathcal{O})] = [\mathbb{P}(C_{X_1/Y_1} \oplus \mathcal{O})] \tag{1.14}$$

in $A_*(\mathbb{P}(C_{X/Y} \oplus \mathcal{O}_X))$.

Step3. We are now ready to conclude the proof. Let us denote the first Chern class of $\mathcal{O}_{\mathbb{P}(C_{X_j/Y_j} \oplus \mathcal{O})}(1)$ by ξ_j. Intersecting $[\mathbb{P}(C_{X_j/Y_j} \oplus \mathcal{O})]$ with ξ^i, pushing forward to $X \times \mathbb{P}^1$ and summing up, by Step 1 we have that

$$\pi_* \sum_{i \geq 0} \xi^i [\mathbb{P}(C_{X_j/Y_j} \oplus \mathcal{O})] = (\pi|_j)_* \sum_{i \geq 0} \xi^i_j [\mathbb{P}(C_{X_j/Y_j} \oplus \mathcal{O})] \tag{1.15}$$

$$= s(C_{X_j/Y_j}). \tag{1.16}$$

Using relation (1.14) we note that the first term in (1.15) does not on j and therefore $s(C_{X_0/Y_0}) = s(C_{X_1/Y_1})$. □

1.2 Construction

In the following we will use a result of Kresch.

Proposition 1.2.1. *([22], Proposition 5.3.2) Let F admit a stratification by global quotients (see [22], Definition 3.5.3). Then, for any* vector bundle stack $\mathfrak{E}$*, we have a canonical isomorphism* $s^* : A_*(F) \to A_*(\mathfrak{E})$.

Remark 1.2.2. Every DM-stack admits a stratification by global quotients. If G admits a stratification by global quotients and $F \to G$ is a DM-type morphism then F admits a stratification by global quotients.

1.2.1 Definition of virtual pull-backs

Condition 1.2.3. We say that a morphism $F \to G$ of algebraic stacks and a vector bundle stack $\mathfrak{E} \to F$ satisfy condition $(\star)$ if

1. f is of DM-type,
2. we have fixed a closed embedding $C_f \hookrightarrow \mathfrak{E}$.

Convention 1.2.4. Will shortly say that the pair $(f, \mathfrak{E})$ satisfies condition $(\star)$.

Remark 1.2.5. Let us consider a Cartesian diagram

$$\begin{array}{ccc} F' & \longrightarrow & G' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ F & \xrightarrow{f} & G. \end{array}$$

If $\mathfrak{E}$ is a vector bundle on F such that $C_{F/G} \hookrightarrow \mathfrak{E}$ is a closed embedding, then $C_{F'/G'} \hookrightarrow p^*\mathfrak{E}$ is a closed embedding.

Construction 1.2.6. Let F be an Artin stack that admits a stratification by global quotient stacks and $\mathfrak{E}$ a vector bundle stack of (virtual) rank n on F such that $(f, \mathfrak{E})$ that satisfies condition $(\star)$ for f, we construct a pull-back map $f^!_{\mathfrak{E}} : A_*(G) \to A_{*-n}(F)$ as the composition

$$A_*(G) \xrightarrow{\sigma} A_*(C_{F/G}) \xrightarrow{i_*} A_*(\mathfrak{E}) \xrightarrow{s^*} A_{*-n}(F).$$

where

1. σ is defined on the level of cycles by $\sigma(\sum n_i[V_i]) = \sum n_i[C_{V_i \times_G F/V_i}]$
2. i_* is the push-forward via the closed immersion i
3. s^* is the morphism of Proposition 1.2.1.

The fact that σ is well defined is a consequence of Lemma 1.1.29 (see [22]). Going further, for any cartesian diagram

$$\begin{array}{ccc} F' & \xrightarrow{f'} & G' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ F & \xrightarrow{f} & G \end{array}$$

such that F' admits a stratification by global quotient stacks and $\mathfrak{E} \to F$ satisfies condition $(\star)$ for f, let $f^!_{\mathfrak{E}} : A_*(G') \to A_{*-n}(F')$ be the composition

$$A_*(G') \xrightarrow{\sigma} A_*(C_{F'/G'}) \xrightarrow{i_*} A_*(C_{F/G} \times_F F') \xrightarrow{i_*} A_*(p^*\mathfrak{E}) \xrightarrow{s^*} A_{*-n}(F').$$

Definition 1.2.7. In the notation above, we call $f^!_{\mathfrak{E}} : A_*(G) \to A_*(F)$ a *virtual pull-back*. When there is no risk of confusion we will omit the index.

Remark 1.2.8. In this remark we do not respect Convention 1.1.11. If E is a *vector bundle* such that (f, E) satisfies $(\star)$, then the above construction can be applied to any Artin stack F. It is clear that in order to have E a vector bundle N_f must necessarily be a cone and not a cone stack.
If f is a locally closed embedding, then N_f is a cone and not a cone stack. Under this assumption if E is a *vector bundle* such that (f, E) satisfies $(\star)$, then all the properties in the following section hold without any assumptions of F.

Remark 1.2.9. Note that in case X, Y are schemes such that X is regularly embedded in Y, then the normal bundle of X in Y satisfies condition $\star$ and $i^!_{N_{X/Y}}$ is precisely the refined Gysin pull-back of [10], Chapter 6 (Section 6.2). We remark that the pull-back depends on the chosen bundle. For example, if (i, E) satisfies condition $(\star)$ we can construct $i^!_{E\oplus E'}$, where E' is any other vector bundle. These morphisms will be obviously different from each other.

Remark 1.2.10. If $f : X \to Y$ is a smooth morphism of schemes, then by Example 1.1.36 $C_{X/Y}$ is a vector bundle stack and hence we can construct the associated virtual pull-back $f^!_{C_{X/Y}} : A_*(Y) \to A_*(X)$. We will show later that our definition agrees with the usual flat pull-back.

Proposition 1.2.11. *If $F \xrightarrow{f} G$ is a DM-type morphism and there exists a perfect relative obstruction theory $E^\bullet_{F/G}$, then condition $(\star)$ is fulfilled.*

Conversely, if $F \xrightarrow{f} G$ is a morphism that satisfies condition $(\star)$, then there exists a unique perfect obstruction theory $E^\bullet_{F/G} \to L_{F/G}$ such that $\mathfrak{E} = h^1/h^0(E^\bullet_{F/G})$

Proof. By Proposition 1.1.31, the normal sheaf $N_{F/G}$ is nothing but $\mathfrak{N}_{F/G}$. On the other hand we know that $\mathfrak{N}_{F/G}$ embeds in $\mathfrak{E}_{F/G} := h^1/h^0((E^\bullet_{F/G})^\vee)$ if and only if $(E^\bullet_{F/G})^\vee$ is an obstruction theory ([3], Proposition 2.6.). Our condition on the relative obstruction theory is equivalent to $\mathfrak{E}_{F/G}$ being a vector bundle stack ([ibid.]). □

Corollary 1.2.12. *If $F \xrightarrow{f} G$ is a DM-type morphism such that there exists a perfect relative obstruction theory $E^\bullet_{F/G}$ and G is a stack of pure dimension, then F has a virtual class in the sense of [3].*

Proof. We define $[F]^{\text{virt}}$ to be $f^!_{E^\bullet_{F/G}}([G])$. □

1.2.2 Two fundamental examples of Obstruction Theories

The purpose of this section is to explain two examples of obstruction theory which will play a fundamental role in the last chapter.

Example 1

Construction 1.2.13. Let $f : F \to G$ be a DM-type morphism and let F and G be DM-stacks having relative obstruction theories with respect to some smooth Artin stack $\mathfrak{M}$. Let us denote them by $E^\bullet_{F/\mathfrak{M}}$ and $E^\bullet_{G/\mathfrak{M}}$ respectively. Given a morphism $\varphi : f^*E^\bullet_{G/\mathfrak{M}} \to E^\bullet_{F/\mathfrak{M}}$ commuting with $f^*L_{G/\mathfrak{M}} \to L_{F/\mathfrak{M}}$, we construct a relative obstruction theory $E^\bullet_{F/G}$.
The morphism $f : F \to G$ induces a distinguished triangle of cotangent complexes

$$f^*L_{G/\mathfrak{M}} \to L_{F/\mathfrak{M}} \to L_{F/G} \to f^*L_{G/\mathfrak{M}}[1].$$

Similarly, φ gives rise to a distinguished triangle

$$f^*E^\bullet_{G/\mathfrak{M}} \xrightarrow{\varphi} E^\bullet_{F/\mathfrak{M}} \to E^\bullet_{F/G} \to f^*E_{G/\mathfrak{M}}[1] \tag{1.17}$$

hence we have a morphism of distinguished triangles that induces the following morphism in cohomology

$$\begin{array}{ccccccccccc} h^{-1}(f^*E^\bullet_{G/\mathfrak{M}}) & \to & h^{-1}(E^\bullet_{F/\mathfrak{M}}) & \to & h^{-1}(E^\bullet_{F/G}) & \to & h^0(f^*E^\bullet_{G/\mathfrak{M}}) & \to & h^0(E^\bullet_{F/\mathfrak{M}}) & \to & h^0(E^\bullet_{F/G}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ h^{-1}(f^*L^\bullet_{G/\mathfrak{M}}) & \to & h^{-1}(L^\bullet_{F/\mathfrak{M}}) & \to & h^{-1}(L^\bullet_{F/G}) & \to & h^0(f^*L^\bullet_{G/\mathfrak{M}}) & \to & h^0(L^\bullet_{F/\mathfrak{M}}) & \to & h^0(L^\bullet_{F/G}) \end{array}$$

We know that the first vertical arrows are surjective and by the definition of obstruction theories we get by a simple diagram chase that $E^\bullet_{F/G}$ is also an obstruction theory.

Remark 1.2.14. If G is smooth over $\mathfrak{M}$ then the above diagram shows that $E_{F/G}$ is perfect in $[-1,0]$.

Example 1.2.15. A special case of this construction is when $F \to G$ is a locally closed immersion and G is taken to be smooth over $\mathfrak{M}$. Then, $h^{-1}(f^*E^\bullet_{G/\mathfrak{M}}) = 0$ and this shows that $h^{-2}(E^\bullet_{F/G}) = 0$. This makes $E^\bullet_{F/G}$ into a perfect obstruction theory concentrated in degree -1 and consequently $\mathfrak{E}$ into a vector bundle.

Let us now motivate Definition 1.2.7. For this, let us assume $E^\bullet_{F/\mathfrak{M}}$ and $E^\bullet_{G/\mathfrak{M}}$ are perfect in $[-1,0]$. Then on F and G we have well defined virtual classes $[F]^{\mathrm{virt}}$ and $[G]^{\mathrm{virt}}$ respectively and we will show in the following that $f^!_{\mathfrak{E}^\vee_{F/G}}$ sends the virtual class of G to the virtual class of F. As remarked in the previous example, the situation is particularly nice when G is taken to be smooth over $\mathfrak{M}$.

Example 1.2.16. The basic case.

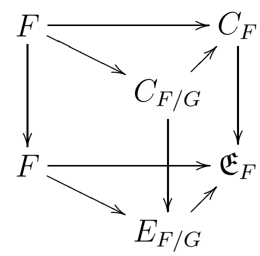
In the notation above, let us suppose G is smooth and $F \overset{i}{\hookrightarrow} G$ is a closed substack and there exists a morphism $f^*L_G \xrightarrow{\varphi} E^\bullet_F$. For simplicity we take $\mathfrak{M}$ to be Spec k. Then we have

(i) $(C_{F/G}, E^\bullet_{F/G})$ induces the same virtual class on F as $(C_F, E^\bullet_F)$.

(ii) The pull back defined by $E^\bullet_{F/G}$ respects the relation

$$i^![G] = [F]^{\mathrm{virt}}.$$

Proof. As G is smooth, the intrinsic normal cone $\mathfrak{C}_F$ defined in [3] is nothing but $[C_{F/G}/T_G]$. Moreover, $i^*L^\bullet_G$ can be represented by a complex concentrated in 0 and $E^\bullet_{F/G}$ by a complex concentrated in -1. By abuse of notation, we will indicate the corresponding sheaves by i^*L_G and $E_{F/G}$ respectively. Taking the long exact cohomology sequence of the exact triangle (1.18), we see that $E^\bullet_F$ is quasi isomorphic to $[E_{F/G} \to i^*L_G]$. Therefore the vector bundle stack $\mathfrak{E}_F := h^1/h^0((E^\bullet_F)^\vee)$ is equal to $[(E_{F/G})^\vee/T_G]$. Thus we have the diagram with cartesian faces



In other words, the morphism $A_*(\mathfrak{E}_F) \to A_*(F)$ factorizes through $A_*(E_{F/G})$ as follows:

$$\begin{array}{c} A_*(\mathfrak{E}_F) \to A_*(E_{F/G}) \to A_*(F) \\ [C_F] \mapsto [C_{F/G}] \ \mapsto [F]^{\mathrm{virt}}. \end{array}$$

For the second statement, we just have to note that by our definition $i^![G] = s^*([C_{F/G}])$, and by (i) is precisely $[F]^{\mathrm{virt}}$ as defined in [3]. □

Example 2

Construction 1.2.17. Let $f : F \to G$ be a DM-type morphism with an obstruction theory $E^\bullet_{F/G}$ and let G be a stack having a relative obstruction theory with respect to some smooth Artin stack $\mathfrak{M}$. Let us denote it $E^\bullet_{G/\mathfrak{M}}$. Given a morphism $\psi : E^\bullet_{F/G}[-1] \to f^*E^\bullet_{G/\mathfrak{M}}$ commuting with $L_{F/G} \to f^*L_{G/\mathfrak{M}}[1]$, we construct a relative obstruction theory $E^\bullet_{F/\mathfrak{M}}$.
The morphism $f : F \to G$ induces a distinguished triangle of cotangent complexes

$$L_{F/G}[-1] \to f^*L_{G/\mathfrak{M}} \to L_{F/\mathfrak{M}} \to L_{F/G}.$$

Similarly, ψ gives rise to a distinguished triangle

$$E^\bullet_{F/G}[-1] \xrightarrow{\psi} f^*E^\bullet_{G/\mathfrak{M}} \to E^\bullet_{F/\mathfrak{M}} \to E^\bullet_{F/G}. \tag{1.18}$$

Noting that $h^i(E^\bullet_{F/G}[-1]) = h^{i-1}(E^\bullet_{F/G})$ we obtain as in the previous example a morphism in cohomology

$$\begin{array}{ccccccccccc} h^{-1}(f^*E^\bullet_{G/\mathfrak{M}}) & \to & h^{-1}(E^\bullet_{F/\mathfrak{M}}) & \to & h^{-1}(E^\bullet_{F/G}) & \to & h^0(f^*E^\bullet_{G/\mathfrak{M}}) & \to & h^0(E^\bullet_{F/\mathfrak{M}}) & \to & h^0(E^\bullet_{F/G}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ h^{-1}(f^*L^\bullet_{G/\mathfrak{M}}) & \to & h^{-1}(L^\bullet_{F/\mathfrak{M}}) & \to & h^{-1}(L^\bullet_{F/G}) & \to & h^0(f^*L^\bullet_{G/\mathfrak{M}}) & \to & h^0(L^\bullet_{F/\mathfrak{M}}) & \to & h^0(L^\bullet_{F/G}) \end{array}$$

Using the definition of obstruction theories we get by a simple diagram chase that $E^\bullet_{F/\mathfrak{M}}$ is also an obstruction theory.
Moreover if $E^\bullet_{F/G}$ and $E^\bullet_{G/\mathfrak{M}}$ are perfect, then $E^\bullet_{F/\mathfrak{M}}$ is perfect.

1.3 Basic properties

Once we have defined a “pull-back”, we want to show it has good properties. Due to the geometric properties of the normal cone (1.1.29), the proofs follow essentially in the same way as the ones in [10]. The fact that our pull-back defines a bivariant class is analogous to Example 17.6.4 in [10]. The only point we need to be careful, is the functoriality property, where we need a compatibility condition between the vector bundle stacks that replace the normal bundles.

1.3.1 Pull-back and push-forward

Theorem 1.3.1. *Consider a fibre diagram of Artin stacks*

$$\begin{array}{ccc} F'' & \longrightarrow & G'' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ F' & \xrightarrow{f'} & G' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ F & \xrightarrow{f} & G \end{array}$$

and let us assume

1. *F', F'' admit stratifications by global quotient stacks,*
2. *$\mathfrak{E}$ is a vector bundle stack of rank d such that $(f, \mathfrak{E})$ satisfies condition $(\star)$ for f.*

(i) (Push-forward) If p is either a projective morphism of Artin stacks or a proper morphism of DM-stacks and $\alpha \in A_k(G'')$, then $f^!_{\mathfrak{E}} p_*(\alpha) = q_* f^!_{\mathfrak{E}} \alpha$ in $A_{k-d}(F')$.
(ii) (Pull-back) If p is flat of relative dimension n and $\alpha \in A_k(G')$, then $f^!_{\mathfrak{E}} p^*(\alpha) = q^* f^!_{\mathfrak{E}} \alpha$ in $A_{k+n-d}(F'')$
(iii)(Compatibility) If $\alpha \in A_k(G'')$, then $f^!_{\mathfrak{E}} \alpha = f'^!_{g^*\mathfrak{E}} \alpha$ in $A_{k-d}(F'')$.

Proof. (i) It is enough to show that the diagram of groups commutes

$$\begin{array}{ccc} A_*(G'') & \xrightarrow{\sigma''} & A_*(C_{F''/G''}) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle q_*} \\ A_*(G') & \xrightarrow{\sigma'} & A_*(C_{F'/G'}) \end{array} \tag{1.19}$$

where q is the map induced by the map between the deformation spaces $P : M^{\circ}_{G''}F'' \to M^{\circ}_{F'}G'$. This follows similarly to Prop 4.2 in [10]. More precisely, let us consider the following factorizations of σ' and σ''

$$\begin{array}{ccccc} A_*(G'') & \xrightarrow{pr^*} & A_*(G'' \times \mathbb{A}^1) & \longrightarrow & A_*(C_{F''/G''}) \; . \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle (p\times id)_*} & & \downarrow{\scriptstyle q_*} \\ A_*(G') & \xrightarrow{pr^*} & A_*(G' \times \mathbb{A}^1) & \longrightarrow & A_*(C_{F'/G'}) \end{array}$$

The diagram on the left commutes and we are left to show that the diagram on the right commutes. But the diagram on the right is induced by the commutative diagram below

$$\begin{array}{ccc} A_*(M^{\circ}_{F''}G'') & \longrightarrow & A_*(C_{F''/G''}) \\ {\scriptstyle P_*}\downarrow & & \downarrow{\scriptstyle q_*} \\ A_*(M^{\circ}_{F'}G') & \longrightarrow & A_*(C_{F'/G'}) \end{array}$$

where the horizontal maps are the ones induced by the natural inclusions of $C_{F''/G''}$ (and $C_{F'/G'}$) in $M^{\circ}_{G''}F''$ (and respectively $M^{\circ}_{F'}G'$). The commutativity of this diagram shows that diagram 1.19 commutes.
(ii) By (i) it is enough to show the statement for G' irreducible and $\alpha = G'$. Let $s_1 : F' \to g^*\mathfrak{E}$ and $s_2 : F'' \to g^*q^*\mathfrak{E}$ be the zero sections. Then using the definition of virtual pull-backs we have that

$$q^* f^!_{\mathfrak{E}}[G'] = q^* s_1^* C_{F'/G'}.$$

By the flatness of p we obtain that $f^!_{\mathfrak{E}}p^*(G') = f^!_{\mathfrak{E}}G''$ and using again the definition of virtual pull-backs we obtain $f^!_{\mathfrak{E}}G'' = s_2^* C_{F''/G''}$. Using now Proposition 1.1.24 we have that $C_{F''/G''} = q^* C_{F'/G'}$. We are thus left to show that $q^* s_1^* C_{F'/G'} = s_2^* r^* C_{F'/G'}$, where we denoted by r the obvious flat morphism $q^* C_{F'/G'} \to C_{F'/G'}$. Noting that $s_2^* = (s_1)^!_{\mathfrak{E}}$ the last statement is true by the corresponding statement for s_1.
(iii) Is obvious. ☐

Remark 1.3.2. As remarked before, the generalized Gysin pull-back is well-defined for smooth pull-backs. Let us show that the two definitions agree. By (i) above, it is enough to prove the claim for $\alpha = [G]$, for which it follows trivially by construction.

1.3.2 Functoriality

Definition 1.3.3. Let $F \xrightarrow{f} G \xrightarrow{g} \mathfrak{M}$ be DM-type morphisms of stacks. If we are given a distinguished triangle of relative obstruction theories which are perfect in $[-1, 0]$

$$g^* E^\bullet_{G/\mathfrak{M}} \xrightarrow{\varphi} E^\bullet_{F/\mathfrak{M}} \to E^\bullet_{F/G} \to g^* E^\bullet_{G/\mathfrak{M}}[1]$$

with a morphism to the distinguished triangle

$$g^* L_{G/\mathfrak{M}} \to L_{F/\mathfrak{M}} \to L_{F/G} \to g^* L_{G/\mathfrak{M}}[1],$$

then we call $(E^\bullet_{F/G}, E^\bullet_{G/\mathfrak{M}}, E^\bullet_{F/\mathfrak{M}})$ a compatible triple.

Remark 1.3.4. As in Construction 1.2.13, if there is a morphism $E^\bullet_{F/G} \xrightarrow{\psi} g^* E^\bullet_{G/\mathfrak{M}}[1]$ compatible with the corresponding morphism between the cotangent complexes, then ψ determines a complex $E^\bullet_{F/\mathfrak{M}}$ that fits in a distinguished triangle as above. Moreover, $E^\bullet_{F/\mathfrak{M}}$ defines a relative obstruction theory. If $E^\bullet_{F/G}$ and $E^\bullet_{G/\mathfrak{M}}$ are perfect, then $E^\bullet_{F/\mathfrak{M}}$ is perfect.

Lemma 1.3.5. *Consider a fibre diagram*

$$\begin{array}{ccccc} F' & \xrightarrow{f'} & G' & \xrightarrow{0} & \mathfrak{F}' \\ \downarrow p & & \downarrow q & & \downarrow r \\ F & \xrightarrow{f} & G & \xrightarrow{0} & \mathfrak{F}. \end{array}$$

with F' an Artin stack that admits a stratification by global quotients, $\mathfrak{F}$ a vector bundle stack on F and $\mathfrak{F}'$ its pullback to G'. Let us assume $\mathfrak{E}'$ is a vector bundle stack such that $(f, \mathfrak{E}')$ satisfies condition $(\star)$. Then

$$C_{F/\mathfrak{F}} \to \mathfrak{E} := \mathfrak{E}' \oplus f^*\mathfrak{F}$$

and for any $\alpha \in A_k(\mathfrak{F}')$

$$(0 \circ f)^!_{\mathfrak{E}}(\alpha) = f^!_{\mathfrak{E}'}(0^!_{\mathfrak{F}}(\alpha)).$$

Proof. For the first part it suffices to show that $C_{F'/\mathfrak{F}'}$ is canonically isomorphic to $C_{F'/G'} \times_{F'} (C_{G/\mathfrak{F}} \times_G G')$, that is example 1.1.35 (iii).
The equality follows in the same way as in ([10]) Let us notice that by theorem 1.3.1 (i) and the homotopy property for vector bundle stacks ([21]) we may assume α to be represented by $\mathfrak{F}'$ and G' can be taken to be irreducible. Now, the problem reduces to

$$(0 \circ f)^![\mathfrak{F}'] = f^![G']. \tag{1.20}$$

If $\pi_1 : E \to p^*\mathfrak{E}'$ and $\pi_2 : \mathfrak{E}' \to F'$ are the natural projections, then we have by the above

$$[C_{F'/\mathfrak{F}'}] = [\pi_1^* C_{F'/G'}] \in A_*(E).$$

From the construction of Gysin pull-backs

$$[C_{F'/G'}] = \pi_2^* f^![G'] \in A_*(p^*\mathfrak{E}')$$

and

$$[C_{F'/\mathfrak{F}'}] = (\pi_1\pi_2)^*(0 \circ f)^![\mathfrak{F}'] \in A_*(E).$$

Combining the three equalities we get equality (1.20) above, and therefore the conclusion. □

Theorem 1.3.6. *(Functoriality) Consider a fibre diagram*

$$\begin{array}{ccccc} F' & \xrightarrow{f'} & G' & \xrightarrow{g'} & \mathfrak{M}' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\ F & \xrightarrow{f} & G & \xrightarrow{g} & \mathfrak{M}. \end{array}$$

Let us assume f, g and $g \circ f$ are DM-type morphisms and have perfect relative obstruction theories $E'^{\bullet}$, $E''^{\bullet}$ and $E^{\bullet}$ respectively and let us denote the associated vector bundle stacks by $\mathfrak{E}'$, $\mathfrak{E}''$ and $\mathfrak{E}$ respectively. If F', F' admit stratifications by global quotients and $(E'^{\bullet}, E''^{\bullet}, E^{\bullet})$ is a compatible triple, then for any $\alpha \in A_k(\mathfrak{M}')$

$$(g \circ f)^!_{\mathfrak{E}}(\alpha) = f^!_{\mathfrak{E}'}(g^!_{\mathfrak{E}''}(\alpha)).$$

Proof. We argue as in the proof of Theorem 1 in [19] (or Theorem 6.5 of [10]).
In the same way as in the proof of the previous lemma $\mathfrak{M}'$ may be assumed

irreducible and reduced and $\alpha = [\mathfrak{M}']$.

Consider the vector bundle stacks: $\rho : \mathfrak{E} \to F$, $\pi : \mathfrak{E}'' \to G$ and $\sigma : \mathfrak{E}' \oplus f^*\mathfrak{E}'' \to F$.
By definition

$$\begin{aligned}(g \circ f)^! \mathfrak{M}' =& (\rho^*)^{-1}([C_{F'/\mathfrak{M}'}]) \\ g^! \mathfrak{M}' =& (\pi^*)^{-1}([C_{G'/\mathfrak{M}'}]).\end{aligned}$$

Let us now look at the cartesian diagram

$$\begin{array}{ccccc} F' & \xrightarrow{f'} & G' & \longrightarrow & C_{G'/\mathfrak{M}'} \\ \downarrow & & \downarrow & & \downarrow \\ F' & \xrightarrow{f'} & G' & \xrightarrow{0} & q^*\mathfrak{E}_{G/\mathfrak{M}}. \end{array}$$

From the definition of the pull-back we know that $f^!(g^!\mathfrak{M}')$ is equal to $f^!(0^![C_{G'/\mathfrak{M}'}])$ and by the previous lemma

$$f^!(0^![C_{G'/\mathfrak{M}'}]) = (0 \circ f)^![C_{G'/\mathfrak{M}'}].$$

If we denote $C_{G'/\mathfrak{M}'}$ by C_0, then the above shows that $f^!(g^!\mathfrak{M}')$ is represented in $\mathfrak{E}_{F/G} \oplus f^*\mathfrak{E}_{G/\mathfrak{M}}$ by the cycle $[C_{F'/C_0}]$. The construction respects equivalence in Chow groups and so we are reduced to showing

$$(\sigma^*)^{-1}([C_{F'/C_0}]) = (\rho^*)^{-1}([C_{F'/\mathfrak{M}'}]) \tag{1.21}$$

in A_*F'.

Introduce the double deformation space $M' := M^\circ_{F' \times \mathbb{P}^1 / M^\circ_{G'/\mathfrak{M}'}} \to \mathbb{P}^1 \times \mathbb{P}^1$ with general fiber $M^\circ_{G'/\mathfrak{M}'}$ and special fibre $C_{F' \times \mathbb{P}^1 / M^\circ_{G'/\mathfrak{M}'}}$ over $\{0\} \times \mathbb{P}^1$ (see [19], proof of Theorem 1). Restricting to this special fibre and considering the rational equivalence on the second $\mathbb{P}^1$ we see that

$$[C_{F'/C_0}] \sim [C_{F'/\mathfrak{M}'}] \tag{1.22}$$

in $A_*(C_{F' \times \mathbb{P}^1 / M^\circ_{G'/\mathfrak{M}'}})$.
In a completely analogous fashion there exists a double deformation space

$M := M^\circ_{F\times\mathbb{P}^1/M^\circ_{G/\mathfrak{M}}}$. If we consider the map $w : F' \times \mathbb{P}^1 \overset{p\times 1_{\mathbb{P}^1}}{\to} F \times \mathbb{P}^1$, then the general fibers of M and M' are related by the cartesian diagram

$$\begin{array}{ccc} F' \times \mathbb{P}^1 & \longrightarrow & M^\circ_{G'/\mathfrak{M}'} \\ {\scriptstyle w}\downarrow & & \downarrow \\ F \times \mathbb{P}^1 & \longrightarrow & M^\circ_{G/\mathfrak{M}}. \end{array}$$

This implies $C_{F'\times\mathbb{P}^1/M^\circ_{G'/\mathfrak{M}'}} \overset{i}{\to} (p\times 1_{\mathbb{P}^1})^* C_{F\times\mathbb{P}^1/M^\circ_{G/\mathfrak{M}}}$ is a closed immersion and consequently we can push forward relation (1.22) in $A_*(w^*C_{F\times\mathbb{P}^1/M^\circ_{G/\mathfrak{M}}})$. Now, by Proposition 1, in [19], we have a morphism

$$A_*(C_{F\times\mathbb{P}^1/M^\circ_{G/\mathfrak{M}}}) \overset{i_*}{\hookrightarrow} A_*(h^1/h^0(c(u)^\vee))$$

where $u := (T\cdot id, U\cdot can)$ is the map

$$f^*L_{G/\mathfrak{M}} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \overset{u}{\to} f^*L_{G/\mathfrak{M}} \oplus L_{F/\mathfrak{M}}$$

in $\mathcal{D}(F\times\mathbb{P}^1)$ and $c(u)$ its mapping cone. Here we denoted by T and U the homogeneous coordinates on $\mathbb{P}^1$. Let us consider the closed immersion $w^*i : A_*(w^*C_{F\times\mathbb{P}^1/M^\circ_{G/\mathfrak{M}}}) \hookrightarrow A_*(w^*h^1/h^0(c(u)^\vee))$. Then pushing forward via w^*i the equivalence relation we have in $A_*(w^*C_{F\times\mathbb{P}^1/M^\circ_{G/\mathfrak{M}}})$, we obtain the equivalence relation (1.22) in $A_*(w^*h^1/h^0(c(u)^\vee))$.
Let us now use the notation of Construction 1.2.13. Consider the morphism $v := (T\cdot id, U\cdot\varphi) : f^*E_{G/\mathfrak{M}} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \to f^*E_{G/\mathfrak{M}} \oplus E_{F/\mathfrak{M}}$ in $\mathcal{D}(F\times\mathbb{P}^1)$. The morphism of distinguished triangles in Definition 1.3.3 gives a morphism of distinguished triangles

$$\begin{array}{ccccccc} (fp)^*E''^\bullet(-1) & \xrightarrow{w^*v} & (fp)^*E''^\bullet \oplus p^*E^\bullet & \longrightarrow & w^*c(v) & \longrightarrow & (fq)^*E''^\bullet(-1)[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (fp)^*L_{G/\mathfrak{M}}(-1) & \xrightarrow{w^*u} & (fp)^*L_{G/\mathfrak{M}} \oplus p^*L_{F/\mathfrak{M}} & \longrightarrow & w^*c(u) & \longrightarrow & (fq)^*L_{G/\mathfrak{M}}(-1)[1] \end{array}$$

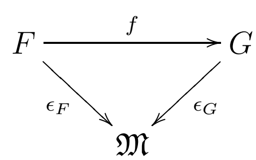
over $F'\times\mathbb{P}^1$. Dualizing and taking h^1/h^0 of the map $w^*c(v) \to w^*c(u)$, we obtain a morphism of Picard stacks $w^*h^1/h^0(c(u)^\vee) \to w^*h^1/h^0(c(v)^\vee)$ that is a closed immersion. Therefore, we can push forward the rational equivalence (1.22) on $w^*h^1/h^0(c(v)^\vee)$ that is a vector bundle stack on $F'\times\mathbb{P}^1$. The fact that the above map between cone stacks is a closed immersion follows from Prop 2.6 in [3] and the fact that the maps in cohomology induced by the vertical maps in the above diagram are isomorphisms in degree 0 and surjective in degree -1.

Let us now conclude the proof. We have obtained $[C_{F'/C_0}] \sim [C_{F'/\mathfrak{M}'}]$ in $A_*(w^*h^1/h^0(c(v)^\vee))$. Looking at $w^*h^1/h^0(c(v)^\vee) \to \mathbb{P}^1$, we see that $w^*h^1/h^0(c(v)^\vee)$ restricts to $F_0 := p^*\mathfrak{E}' \oplus p^*f^*\mathfrak{E}''$ and $F_1 := p^*\mathfrak{E}$ in $F' \times \{0\}$ respectively $F' \times \{1\}$. As the map

$$A_*(w^*h^1/h^0(c(v)^\vee)) \to A_*(F_i) \to F'$$

does not depend on i we deduce equality (1.21). □

Corollary 1.3.7. *Let us assume we have a commutative diagram*



with $\mathfrak{M}$ of pure dimension. If F and G admit stratifications by global quotients and $(E^\bullet_{F/G}, E^\bullet_{G/\mathfrak{M}}, E^\bullet_{F/\mathfrak{M}})$ a compatible triple, then

$$f^!_{\mathfrak{E}_{F/G}}[G]^{\mathrm{virt}} = [F]^{\mathrm{virt}}.$$

Proof. By the definition of virtual classes we have

$$[F]^{\mathrm{virt}} = (\epsilon_F)^!_{\mathfrak{E}_{F/\mathfrak{M}}}[\mathfrak{M}]$$
$$[G]^{\mathrm{virt}} = (\epsilon_G)^!_{\mathfrak{E}_{G/\mathfrak{M}}}[\mathfrak{M}].$$

Moreover, by the construction of $E_{F/G}$ we are in the hypotheses of Theorem 1.3.6 and therefore

$$(\epsilon_G \circ f)^!_{\mathfrak{E}_{F/\mathfrak{M}}}[\mathfrak{M}] = f^!_{\mathfrak{E}_{F/G}}(\epsilon_G)^!_{\mathfrak{E}_{G/\mathfrak{M}}}[\mathfrak{M}].$$

The two equations above show that $f^!_{\mathfrak{E}_{F/G}}[G]^{\mathrm{virt}} = [F]^{\mathrm{virt}}$.

□

Remark 1.3.8. Let us consider a cartesian diagram of DM stacks

$$\begin{array}{ccc} F' & \longrightarrow & G' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ F & \xrightarrow{i} & G \end{array}$$

with obstruction theories $E^\bullet_F$, $E^\bullet_G$, $E^\bullet_{F'}$, $E^\bullet_{G'}$ and let us assume $E^\bullet_{F/G}$ and $E^\bullet_{F'/G'}$ exist and are perfect. If $g^*\mathfrak{E}_{F/G} = \mathfrak{E}_{F'/G'}$, then $i^![G']^{\mathrm{virt}} = [F']^{\mathrm{virt}}$. This is a version of Proposition 5.10 in [3] and Theorem 1 in [19]. The advantage is that looking at obstruction theories is much easier than looking at cotangent complexes, which in general are difficult to compute.

1.3.3 Commutativity

Theorem 1.3.9. *(Commutativity) Consider a fiber diagram of Artin stacks*

$$\begin{array}{ccccc} F'' & \xrightarrow{v} & G'' & \xrightarrow{u} & H \\ {\scriptstyle q}\downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\ F' & \longrightarrow & G' & \longrightarrow & K \\ {\scriptstyle p}\downarrow & & \downarrow & & \\ F & \xrightarrow{f} & G & & \end{array}$$

such that F' and G'' admit stratifications by global quotients. Let us assume f and g are morphisms of DM-type and let $\mathfrak{E}$ and $\mathfrak{F}$ be vector bundle stacks of rank d, respectively e such that $(f, \mathfrak{E})$ and $(g, \mathfrak{F})$ satisfy condition $(\star)$. Then for all $\alpha \in A_k(G')$,

$$g^!_{\mathfrak{F}} f^!_{\mathfrak{E}}(\alpha) = f^!_{\mathfrak{E}} g^!_{\mathfrak{F}}(\alpha)$$

in $A_{k-d-e}(F'')$.

Proof. Using Theorem 1.3.1 we may assume $\alpha = [G']$. We see that the pull-back of $g^! f^! [G']$ to $p^*q^*\mathfrak{E} \oplus v^*u^*\mathfrak{F}$ is equal to $C_{C_{F'/G'} \times_{G'} G''/C_{F'/G'}}$ and the pull-back of $f^! g^! [G']$ to $p^*q^*\mathfrak{E} \oplus v^*u^*\mathfrak{F}$ is equal to $C_{C_{G''/G'} \times_{G'} F'/C_{G''/G'}}$. By Vistoli's rational equivalence ([32] Lemma 3.16, or equivalently [23])

$$[C_{C_{F'/G'} \times_{G'} G''/C_{F'/G'}}] = [C_{C_{G''/G'} \times_{G'} F'/C_{G''/G'}}]$$

in $A_*(C_{F'/G'} \times_{G'} C_{G''/G'})$. This equivalence pushes forward to $A_*(p^*q^*\mathfrak{E} \oplus v^*u^*\mathfrak{F})$ and therefore the equality in the theorem follows. $\square$

Remark 1.3.10. In Fulton's Intersection theory ([10]), the commutativity of Gysin maps is used in the proof of the functoriality. We did not use the commutativity of virtual pull-backs in our proof of property functoriality property.
In the same manner as in Example 6.5.3 in [10] the commutativity of virtual pull-backs is a consequence of the functoriality property.

1.3.4 Excess

Setting 1.3.11. Consider a fibre diagram of Artin stacks

$$\begin{array}{ccc} F'' & \longrightarrow & G'' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ F' & \xrightarrow{f'} & G' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ F & \xrightarrow{f} & G \end{array}$$

and $\mathfrak{F} \to F$ a vector bundle stack of rank d which satisfies condition $(\star)$ for f. If f' is an lci morphism of codimension d', then $\pi : \mathfrak{N} \to F'$ with $\mathfrak{N} := h^1/h^0(L_{f'})$ is a vector bundle stack of rank d' and $p^*\mathfrak{F}$ satisfies condition $(\star)$ for f'. Let us denote by E the cokernel of the inclusion $\mathfrak{N} \to p^*\mathfrak{F}$. Then E is a vector bundle of rank $e := d - d'$ which we call the excess bundle.

Theorem 1.3.12. *(Excess Intersection Formula) In notations as above*

$$f^!\alpha = c_e(E) \cdot (f')^!\alpha.$$

for any $\alpha \in A_*(G'')$.

Let us first prove a technical lemma.

Lemma 1.3.13. *Let us consider a distinguished triangle of elements of* $\mathcal{D}_F$

$$E'' \xrightarrow{\varphi} E \to E' \to E''[1]$$

with E, E', E'' *complexes of perfect amplitude contained in* $[-1, 0]$. *Consider* $\mathfrak{C}$ *the cone of Proposition 1.1.39 flat over* $\mathbb{P}^1$ *with* $\mathfrak{C}_0$ *isomorphic to* $\mathfrak{E}' \oplus \mathfrak{E}''$ *and* $\mathfrak{C}_1$ *isomorphic to* $\mathfrak{E}$. *Then, we have a commutative diagram*

$$\begin{array}{ccc} \mathfrak{E}' \times \mathbb{P}^1 & \longrightarrow & \mathfrak{C} \\ & \searrow & \downarrow \\ & & \mathbb{P}^1. \end{array}$$

Proof. Let us consider the morphism of distinguished triangles

$$\begin{array}{ccccccc} E''(-1) & \xrightarrow{v} & E''\oplus E & \longrightarrow & c(v) & \longrightarrow & E''^{\bullet}(-1)[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E''(-1) & \longrightarrow & E''\oplus E' & \longrightarrow & E'' & \longrightarrow & E''(-1)[1] \end{array}$$

over $F' \times \mathbb{P}^1$. By Example 1.2.2 $c(v) \to E'$ is an obstruction theory. This gives us a closed embedding $\mathfrak{E}' \hookrightarrow \mathfrak{C}$. $\square$

Proof of the Theorem. Let $F^\bullet \in \mathcal{D}_F$ be an obstruction theory for f such that $\mathfrak{F} := h^1/h^0(F)$. Let us consider the distinguished triangle

$$E^\bullet \to F^\bullet \to L_{f'} \to E^\bullet[1].$$

Then $E^\bullet$ is a complex quasi-isomorphic to $[E^\vee \to 0]$ and therefore we are under the hypothesis of the above lemma. This shows that we have a vector-bundle stack $\mathfrak{C}$ over $F'' \times \mathbb{P}^1$ flat over $\mathbb{P}^1$ with general fiber isomorphic to $\mathfrak{F}$ and special fiber isomorphic to $\mathfrak{N} \oplus E$ and a commutative diagram

$$\begin{array}{ccc} \mathfrak{N}\times\mathbb{P}^1 & \longrightarrow & \mathfrak{C} \\ & \searrow & \downarrow \\ & & \mathbb{P}^1. \end{array}$$

Let us denote by t the embedding $\{t\} \hookrightarrow \mathbb{P}^1$, by i the zero-section embedding $i : F'' \times \mathbb{P}^1 \to \mathfrak{C}$ and by i_t the zero-section embedding $i_t : F'' \times \{t\} \to \mathfrak{C}_t$. Then for any cycle $[\mathfrak{C}_\alpha] \in A_*(\mathfrak{N})$

$$i_t^![\mathfrak{C}_\alpha] = t^! i^! [\mathfrak{C}_\alpha \times \mathbb{P}^1]$$

and the second term of the above equality does not depend on the point $\{t\}$. We have thus obtained that

$$i_0^![\mathfrak{C}_\alpha] = i_1^![\mathfrak{C}_\alpha]. \tag{1.23}$$

Let us now fix $\alpha \in A_*(G'')$ and let us consider $\mathfrak{C}_\alpha$ the normal cone associated to α. Then by the definition of virtual pull-backs we have that

$$f^!\alpha = i_1^![\mathfrak{C}_\alpha] \tag{1.24}$$

Let us consider the following cartesian diagram

$$\begin{array}{ccccc} F'' & \longrightarrow & \mathfrak{C}_\alpha & \longrightarrow & \mathfrak{C}_\alpha \\ \downarrow & & \downarrow & & \downarrow \\ F'' & \xrightarrow{0} & \mathfrak{N} & \xrightarrow{1} & \mathfrak{N} \oplus E \end{array}$$

where 0 denotes the zero-section embedding and 1 indicates the inclusion on the first factor. Then by the excess formula for the square on the right and Lemma 1.3.5 we have that

$$\begin{aligned} i_0^! \mathfrak{C}_\alpha &= (1 \circ 0)^! [\mathfrak{C}_\alpha] && (1.25) \\ &= 0^! 1^! [\mathfrak{C}_\alpha] && (1.26) \\ &= 0^! (c_e(\pi^* E)[\mathfrak{C}_\alpha]) && (1.27) \\ &= c_e(E) 0^! ([\mathfrak{C}_\alpha]) && (1.28) \\ &= c(E) f'^! \alpha. && (1.29) \end{aligned}$$

Let us now conclude the proof. By equation (1.23), equation (1.24) and equation (1.25) we have that $f^! \alpha = c_e(E) f'^! \alpha$.

□

Remark 1.3.14. By the definition of virtual pull-backs we may write the conclusion of the Excess Theorem in the following form

$$(f'^!_{\mathfrak{F}})^! \alpha = c_e(E) f'^!_{\mathfrak{N}} \alpha.$$

Remark 1.3.15. The proof of the Excess Theorem is very similar to the proof of the Functoriality of virtual pull-backs. Let us show that we may interpret it as a functoriality property. Let us consider the following cartesian diagram

$$\begin{array}{ccccc} F'' & \xrightarrow{id} & F'' & \longrightarrow & G'' \\ \downarrow & & \downarrow & & \downarrow \\ F' & \xrightarrow{id} & F' & \xrightarrow{f'} & G' \end{array}$$

with f' an lci morphism and equipped with a (possibly nontautological) perfect obstruction theory $F^\bullet$ and a superfluous obstruction theory $E^\bullet := [E^\vee \to 0]$ for the identity morphism $id : F \to F$. The excess formula we proved shows that $f'^{!}_{\mathfrak{F}}\alpha = c_e(E) \cdot (f'^{!}_{\mathfrak{N}}\alpha)$ and the same excess formula implies that $id^{!}_{E}\beta = c_e(E) \cdot \beta$. Putting these together we obtain that

$$(f'_{\mathfrak{F}})^{!}\alpha = id^{!}_{E}(f'^{!}_{\mathfrak{N}}\alpha).$$

Let us emphasize that the triple $(E^{\bullet}, F^{\bullet}, L_{F'/G'})$ is *not* a compatible triple (instead, the rotated triple $(F^{\bullet}, L_{F'/G'}, E^{\bullet})$ is a compatible triple) and therefore the Excess Intersection Formula in *not* implied by the Functoriality Theorem.

Remark 1.3.16. More generally, if $F'^{\bullet}$ is an obstruction theory for f' with a morphism $\psi : F^{\bullet} \to F'^{\bullet}$, then we have a distinguished triangle

$$E^{\bullet} \to F^{\bullet} \to F'^{\bullet} \to E^{\bullet}[1].$$

with $E^{\bullet}$ quasi-isomorphic to the two-term perfect complex. Let $\mathfrak{E} := h^1/h^0(E^{\bullet})$ be the associated vector-bundle stack. With the arguments above we obtain that

$$(f'^{!}_{\mathfrak{F}})^{!}\alpha = 1^{!} f'^{!}_{\mathfrak{F}'}\alpha$$

where $1 : \mathfrak{F}' \to \mathfrak{F}' \oplus \mathfrak{E}$ is the identity on the first factor of $\mathfrak{F}' \oplus \mathfrak{E}$.

Remark 1.3.17. In the above remark, we do not have any way of "explicitly computing" $1^{!}$. Let us assume that $\mathfrak{E}$ is a global quotient E^1/E^0. By the basic properties of virtual pull-backs it is enough to determine $0^{!}$ in the following cartesian diagram

$$\begin{array}{ccc} W & \longrightarrow & F \\ \downarrow & & \downarrow \\ E^0 & \xrightarrow{d} & E^1 \\ \downarrow & & \downarrow \\ F & \xrightarrow{0} & \mathfrak{E} \end{array}$$

which means to determine $d^{!}$. This shows that there is no "easy" analogue excess formula in the general case as we need to take into account the morphism d. Note that under the assumptions of the theorem d is the zero-section embedding $d = 0 : F \to E^1$.

1.3.5 Conclusion

The properties in the previous sections (Theorem 1.3.1 and Theorem 1.3.9) show the following fundamental result which we anticipated several times.

Theorem 1.3.18. *Let F admit a stratification by global quotients, let $f : F \to G$ be a morphism and $\mathfrak{E} \to F$ be a rank-n vector bundle stack on F such that $(f, \mathfrak{E})$ satisfies Condition ($\star$). Then $f^!_{\mathfrak{E}}$ defines a bivariant class in $A^n(X \to Y)$ in the sense of [10], Definition 17.1.*

We will often use the following statement.

Corollary 1.3.19. *Let F admit a stratification by global quotients, let $f : F \to G$ be a morphism and $\mathfrak{E} \to F$ be a rank-n vector bundle stack on F such that $(f, \mathfrak{E})$ satisfies Condition ($\star$). Let us moreover consider the following cartesian diagram*

$$\begin{array}{ccc} F' & \xrightarrow{f'} & G' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ F & \xrightarrow{f} & G \end{array} .$$

and let E be a vector bundle on G'. Then for any $\alpha \in A_k(G')$ and any $m \geq 0$

$$f^!(c_m(E) \cdot \alpha) = c_m(f^*E) \cdot f^!\alpha$$

in $A_*(F')$

1.4 Virtual push-forward

In this section we consider a proper surjective morphism $f : F \to G$ of stacks which possess perfect obstruction theories and we analyze the push-forward of the virtual class of F along f. The strongest statement can only be obtained in *homology* and therefore in this section we will work with homology groups instead of Chow groups. The main result of this section states that if the virtual dimension of F is greater or equal to the virtual dimension of G and the induced relative obstruction theory is perfect, then the push-forward of the virtual class of F along f is equal to a scalar multiple of the virtual class of G. This result is a generalization of the straight-forward fact that given a surjective morphism of schemes $f : F \to G$, with G irreducible, then $f_*[F]$ is a scalar multiple of the fundamental class of G.

Let us first recall a definition of Gathmann ([13])

Definition 1.4.1. Let $p : F \to G$ be a proper morphism of stacks possessing virtual classes $[F]^{\text{virt}} \in A_{k_1}(F)$ and $[G]^{\text{virt}} \in A_{k_2}(G)$ with $k_1 \geq k_2$ and let

$\gamma \in A^{k_3}(F)$, with $k_3 \leq k_1 - k_2$ be a cohomology class. We say that p satisfies the virtual pushforward property for $[F]^{\mathrm{virt}}$ and $[G]^{\mathrm{virt}}$ if the following two conditions hold:
(i) If the dimension of the cycle $\gamma \cdot [F]^{\mathrm{virt}}$ is bigger than the virtual dimension of G then $p_*(\gamma \cdot [F]^{\mathrm{virt}}) = 0$.
(ii) If the dimension of the cycle $\gamma \cdot [F]^{\mathrm{virt}}$ is equal to the virtual dimension of G then $p_*(\gamma \cdot [F]^{\mathrm{virt}})$ is a scalar multiple of $[G]^{\mathrm{virt}}$.

Remark 1.4.2. Let $p : F \to G$ be a morphism as above. If G is smooth of the expected dimension, then p satisfies the virtual pushforward property.

Lemma 1.4.3. *Let $p : F \to G$ be a proper morphism of stacks possessing virtual classes of virtual dimensions k_1 respectively k_2 with $k_1 \geq k_2$ and let $\gamma \in A^{k_3}(F)$, with $k_3 \leq k_1 - k_2$ be a cohomology class. Let $[G]_1^{\mathrm{virt}}, ..., [G]_s^{\mathrm{virt}}$ be cycles corresponding to the irreducible components of G such that $[G]^{\mathrm{virt}} = [G]_1^{\mathrm{virt}} + ... + [G]_s^{\mathrm{virt}}$. If the relative obstruction theory $E^\bullet_{F/G}$ is perfect (in the sense of [3]), then*

$$p_*(\gamma \cdot [F]^{\mathrm{virt}}) = n_1[G]_1^{\mathrm{virt}} + ... + n_s[G]_s^{\mathrm{virt}} \tag{1.30}$$

for some $n_1, ..., n_s \in \mathbb{Q}$. Moreover, all n_i are zero if $k_3 < k_1 - k_2$.

Proof. Let $\mathfrak{C}_F := h^1/h^0(E^\bullet_F)$, $\mathfrak{C}_G := h^1/h^0(E^\bullet_G)$ and $\mathfrak{C}_{F/G} := h^1/h^0(E^\bullet_{F/G})$. Let $0_F : F \to \mathfrak{C}_F$, $0_G : F \to \mathfrak{C}_G$ and $0_{F/G} : F \to \mathfrak{C}_{F/G}$ be the zero-section embeddings. Then by the definition of the virtual class we have that $[G]^{\mathrm{virt}} = 0^!_G C_G$ and $[F]^{\mathrm{virt}} = 0^!_F C_F$. Let us denote by G^{v} any closed substack of G such that $[G]^{\mathrm{virt}} = [G^{\mathrm{v}}]$ in $A_*(G)$. With this notation we have that

$$[G]^{\mathrm{virt}} = 0^!_G[\mathfrak{C}_G|_{G^{\mathrm{v}}}]. \tag{1.31}$$

Let us now consider the following cartesian diagram

$$\begin{array}{ccc} F' & \xrightarrow{q} & G^{\mathrm{v}} \\ {\scriptstyle i}\downarrow & & \downarrow \\ F & \xrightarrow{p} & G. \end{array}$$

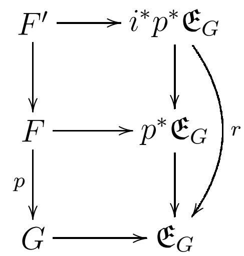
By the proof of Theorem 1.3.6 we have that

$$\begin{aligned} [F]^{\mathrm{virt}} &= 0^!_{F/G}[C_{F'/G^{\mathrm{v}}}] \\ &= 0^!_{F/G} 0^!_G[C_{F'/G^{\mathrm{v}}} \times_{F'} q^*\mathfrak{C}_G|_{G^{\mathrm{v}}}] \\ &= 0^!_G 0^!_{F/G}[C_{F'/G^{\mathrm{v}}} \times_{F'} q^*\mathfrak{C}_G|_{G^{\mathrm{v}}}] \end{aligned}$$

Let us denote $[C'] := 0^!_{F/G}[C_{F'/G^{\mathrm{v}}} \times_{F'} q^*\mathfrak{E}_G|_{G^{\mathrm{v}}}] \in A_*(q^*\mathfrak{E}_G|_{G^{\mathrm{v}}})$. Then the above computation shows that

$$\gamma \cdot [F^{\mathrm{virt}}] = 0^!_G \pi^* \gamma \cdot [C']$$

where $\pi : q^*\mathfrak{E}_G \to F$ denotes the canonical projection. By the commutativity of the pull-back with proper (projective) push-forward in the following cartesian diagram



we obtain that

$$p_*\gamma \cdot [F]^{\mathrm{virt}} = r_*\pi^*\gamma \cdot [C']. \tag{1.32}$$

By construction C' has a natural map to $\mathfrak{E}_G|_{G^{\mathrm{v}}}$ compatible with r and therefore $p_*\gamma \cdot [F]^{\mathrm{virt}} = \sum n_i [\mathfrak{E}_G|_{G^{\mathrm{v}}}]_i$, where the sum is taken over all the irreducible components of $\mathfrak{E}_G|_{G^{\mathrm{v}}}$. We can now conclude the proof using equation 1.31.
If $k_3 < k_1 - k_2$, then $r_*\pi^*\gamma \cdot [C'] = 0$ for dimensional reasons and therefore $p_*\gamma \cdot [F]^{\mathrm{virt}} = 0$. □

Definition 1.4.4. Let $p : F \to G$ be a proper morphism of stacks possessing virtual classes of virtual dimensions k_1 respectively k_2 with $k_1 \geq k_2$ such that f maps each connected component of F surjectively to some connected component of G. Then we call p a virtually surjective morphism.
If moreover, p has a perfect relative obstruction theory $E^\bullet_{F/G}$, then we call p a virtually smooth morphism.

Remark 1.4.5. This definition is very similar to Definition 3.14 in [11] of *a family of proper virtually smooth schemes.* The main difference is that we do not ask the base G to be smooth.

Theorem 1.4.6. *Let $p : F \to G$ be a virtually smooth morphism. If G is connected, then p satisfies the virtual push-forward property.*

Proof. By the above lemma all we need to show is that all coefficients n_i which appear in formula (1.30) are equal.

Let $G_1^{\mathrm{v}}, ..., G_s^{\mathrm{v}}$ be the irreducible components of G^{v} and let $m_1, ..., m_s$ be their geometric multiplicity. Then $[G]^{\mathrm{virt}} = m_1[G_1^{\mathrm{v}}] + ... + m_s[G_s^{\mathrm{v}}]$ and therefore $[C'] = \sum_{i=1}^{s} m_i 0^!_{F/G}[C'_i]$, where $C'_i := C_{F'_i/G_i^{\mathrm{v}}} \times_{F'_i} q^*\mathfrak{E}_G|_{G_i^{\mathrm{v}}}$. By equation (1.32) we have that $p_*\gamma \cdot [F]^{\mathrm{virt}} = r_*\pi^*\gamma \cdot (\sum_{i=1}^{s} m_i[C'_i])$. With this we have shown that it is enough to show the statement for G reduced.
Let us consider the cartesian diagram

$$\begin{array}{ccc} X_P & \xrightarrow{j} & F' \\ \downarrow{\scriptstyle q_P} & & \downarrow{\scriptstyle q} \\ P & \xrightarrow{i} & G^{\mathrm{v}} \end{array}$$

where P is a general point in G^{v} and X_P is the fiber of q over P. As G^{v} is reduced we may assume that P is a smooth point and therefore i is a regular embedding. By the commutativity of pull-backs with proper push-forwards we have that

$$(q_P)_* i^! \gamma \cdot [F]^{\mathrm{virt}} = i^* q_* \gamma \cdot [F]^{\mathrm{virt}}. \tag{1.33}$$

This shows that $(q_P)_* i^! \gamma \cdot [F]^{\mathrm{virt}} = i^* \sum_i n_i [G_i^{\mathrm{v}}]$. Without loss of generality we may assume that P is a point on G_1^{v} and with this we obtain that

$$(q_P)_* i^! \gamma \cdot [F]^{\mathrm{virt}} = n_1[P]. \tag{1.34}$$

On the other hand by the commutativity of pull-backs we have that

$$i^! q^! [G]^{\mathrm{virt}} = q_P^! i^* [G^{\mathrm{v}}] \tag{1.35}$$

$$= q_P^! [P]. \tag{1.36}$$

By the functoriality property of pull-backs we have that

$$i^! q^! [G^{\mathrm{v}}] = i^! [F]^{\mathrm{virt}}. \tag{1.37}$$

Equations (1.34), (1.35) and (1.37) imply that

$$n_1[P] = (q_P)_* \gamma \cdot q_P^! [P].$$

As G is connected the right-hand side of the above equation does not depend on P, hence p satisfies the push-forward property. □

Remark 1.4.7. The only point where we need to work with homology is the last part of the proof of the above theorem. For any connected G we have that $H_0(G) = \mathbb{Q}$, but this is usually no longer true for the corresponding Chow group.

Remark 1.4.8. Let us consider a cartesian diagram of stacks

$$\begin{array}{ccc} F' & \longrightarrow & F \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ G' & \xrightarrow{i} & G. \end{array}$$

If F, G, G' posses virtual classes and the relative obstruction theory $E^\bullet_{F/G}$ is perfect, then we have an induced virtual class on F', namely $[F']^{\mathrm{virt}} := q^![G']^{\mathrm{virt}}$.

Corollary 1.4.9. *Let us consider a cartesian diagram of stacks*

$$\begin{array}{ccc} F' & \longrightarrow & F \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ G' & \xrightarrow{i} & G \end{array}$$

such that p is proper and F, G, G' posses virtual classes. If the relative obstruction theory $E^\bullet_{F/G}$ is perfect, G is connected and no connected component of $G' \cap G_{red}$ is contained in the singular locus of G_{red}, then q satisfies the virtual push-forward property for $[F']^{\mathrm{virt}}$ and $[G']^{\mathrm{virt}}$, where $[F']^{\mathrm{virt}}$ is the one defined in Remark 1.4.8.

Proof. By Lemma 1.4.3 we have that $p_*(\gamma \cdot [F']^{\mathrm{virt}}) = n_1[G']_1^{\mathrm{virt}} + ... + n_s[G']_s^{\mathrm{virt}}$ for some $n_1, ..., n_s \in \mathbb{Q}$. We have to show that all non-zero n_i's are equal.
As is the proof of the theorem we may assume that G and G' are reduced. The hypothesis translates in no connected component of G' is contained in the singular locus of G.
Let us consider the following cartesian diagram

$$\begin{array}{ccccc} X & \longrightarrow & F' & \longrightarrow & F \\ {\scriptstyle q_P}\downarrow & & {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ P & \longrightarrow & G' & \xrightarrow{i} & G \end{array} .$$

By the hypothesis P can be taken to be a smooth point of both G and G'. Looking at the big diagram, the proof of Theorem 1.4.6 implies that

$$q_*(\gamma \cdot [F]^{\mathrm{virt}}) = (q_P)_*\gamma \cdot q_P^![P]$$

and in the same manner for the small diagram

$$q_*(\gamma \cdot [F']^{\mathrm{virt}}) = (q_P)_* \gamma \cdot q_P^! [P].$$

As G is connected $(q_P)_* \gamma \cdot q_P^! [P]$ does not depend on P and therefore we obtain that $q_*(\gamma \cdot [F']^{\mathrm{virt}})$ is a scalar multiple of $[G']^{\mathrm{virt}}$. □

Chapter 2

Applications to Gromov-Witten Theory

In this chapter we collect some applications of the virtual pull-back we defined. We take the ground field to be $\mathbb{C}$. By a homology class of a curve we will mean an element of $A_1^{alg}(X)$– the group of 1-cycles modulo algebraic equivalence (see [10], Chapter 10).

2.1 Preliminaries

Let us fix notations. Let X be a smooth projective variety and $\beta \in A_1(X)$ a homology class of a curve in X. We denote by $\overline{M}_{g,n}(X,\beta)$ the moduli space of stable genus-g, n-pointed maps to X of homology class β (see [9]). Let $\epsilon_X : \overline{M}_{g,n}(X,\beta) \to \mathfrak{M}_{g,n}$ be the morphism that forgets the map (and does not stabilize the pointed curve) and $\pi_X : \overline{M}_{g,n+1}(X,\beta) \to \overline{M}_{g,n}(X,\beta)$ the morphism that forgets the last marked point and stabilizes the result. Then it is a well-known fact that $E^{\bullet}_{\overline{M}_{g,n}(X,\beta)/\mathfrak{M}} := (\mathcal{R}^{\bullet}(\pi_X)_* ev_X^* T_X)^{\vee}$ defines an obstruction theory for the morphism p, where ev_X indicates the evaluation map $ev_X : \overline{M}_{g,n+1}(X,\beta) \to X$ (see [1]). We call $[\overline{M}_{g,n}(X,\beta)]^{\mathrm{virt}} := (\epsilon_X)^!_{\mathfrak{E}_{\overline{M}_{g,n}(X,\beta)/\mathfrak{M}}} \mathfrak{M}_{g,n}$ the virtual class of $\overline{M}_{g,n}(X,\beta)$. The dimension of $[\overline{M}_{g,n}(X,\beta)]^{\mathrm{virt}}$ is called the virtual dimension of $\overline{M}_{g,n}(X,\beta)$ and we denote it by $\mathrm{vdim}\overline{M}_{g,n}(X,\beta)$. To a collection of Chow (or cohomology) classes $\gamma_i \in A^{k_i}(X)$ such that $\sum_{i=1}^n k_i = \mathrm{vdim}\overline{M}_{g,n+1}(X,\beta)$, one can associate a Gromov-Witten (shortly GW) invariant defined to be $I^X_{g,n,\beta} := \prod_{i=1}^n ev_i^* \gamma_i \cdot [\overline{M}_{g,n}(X,\beta)]^{\mathrm{virt}}$.

Remark 2.1.1. Let $f : X \to Y$ be a morphism of smooth algebraic varieties. Let $\beta \in A_1(X)$ and $g,\ n$ be any natural numbers such that

- either $g \geq 2$
- either $g < 2$ and $f_*\beta \neq 0$
- either $g = 1$, $f_*\beta = 0$ and $n \geq 1$, either $g = 0$, $f_*\beta = 0$ and $n \geq 3$.

Then f induces a morphism of stacks $\bar{f} : \overline{M}_{g,n}(X, \beta) \to \overline{M}_{g,n}(Y, f_*\beta)$.
Convention: Given a morphism of smooth algebraic varieties $f : X \to Y$, we will indicate the induced morphism between moduli spaces of stable maps by the same letter with a bar.
Convention: In the following, everytime we write $\bar{f} : \overline{M}_{g,n}(X, \beta) \to \overline{M}_{g,n}(Y, f_*\beta)$ we will assume that $\overline{M}_{g,n}(Y, f_*\beta)$ is non-empty.

Let us now state the version of Cohomology and Base Change we will use in our applications. We refer to Proposition 1.15 in [29] which is the stack-version of Corollary 6.9.9 in EGA, III, Second part.

Theorem 2.1.2. *(Cohomology and Base Change) Let $G' \to^p G$ be a flat morphism of tame separated DM stacks and E a locally free sheaf on G'. If $\mathcal{R}^i p_* E$ are locally free sheaves for any i, then for any base change*

$$\begin{array}{ccc} F' & \xrightarrow{g} & G' \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ F & \xrightarrow{f} & G \end{array}$$

we have that

$$Lf^* Rp_* E = Rq_* Lg^* E.$$

Corollary 2.1.3. *Let $f : X \to \mathbb{P}$ be a morphism of smooth projective schemes and let $T^\bullet := [T^0 \to T^1]$ be a two term complex of bundles in $\mathcal{D}_{\mathbb{P}}$ concentrated in $[0, 1]$. Then we have the following canonical isomorphism in $\mathcal{D}_{\overline{M}_{0,n}(X,\beta)}$*

$$\bar{f}^* \mathcal{R}^\bullet(\pi_{\mathbb{P}})_* ev_{\mathbb{P}}^* T^\bullet \simeq \mathcal{R}^\bullet(\pi_X)_* ev_X^* f^* T^\bullet.$$

Proof. Step 1. By abuse of notation we denote by T^j the complex having T^j in degree 0 and zero in all other degrees. Let us show that $ev_{\mathbb{P}}^* T^j$ can be replaced by a quasi-isomorphic complex of vector bundles on $\overline{M}_{0,n+1}(\mathbb{P}, i_*\beta)$,

K^i such that $\mathcal{R}^i(\pi_{\mathbb{P}})_*K^j$ are all vector bundles. By [1], Prop. 5, we can construct for any $k \in \{0,1\}$ an exact sequence

$$0 \to E^{0,k} \to E^{1,k} \to T^k \to 0$$

such that $\mathcal{R}^i(\pi_{\mathbb{P}})_*E^{j,k}$ is a bundle $\forall i,j,k$. By Theorem 2.1.2 applied to K^j and the fact that $ev_X^* f^* = \bar{f}^* ev_{\mathbb{P}}^*$ we obtain that $\mathcal{R}^i(\pi_X)_* f^*K = \bar{f}^*\mathcal{R}^i(\pi_{\mathbb{P}})_*K^j$ and therefore $\mathcal{R}^i(\pi_X)_* f^*T^i = \bar{f}^*\mathcal{R}^i(\pi_{\mathbb{P}})_*T^j$.

Step 2. Let us consider the distinguished triangle

$$ev_{\mathbb{P}}^*T^0 \to ev_{\mathbb{P}}^*T^1 \to ev_{\mathbb{P}}^*T^\bullet \to ev_{\mathbb{P}}^*T^0[1].$$

Applying $\bar{f}^*\mathcal{R}^\bullet(\pi_{\mathbb{P}})_*$ we obtain a distinguished triangle

$$\bar{f}^*\mathcal{R}^\bullet(\pi_{\mathbb{P}})_*ev_{\mathbb{P}}^*T^0 \to \bar{f}^*\mathcal{R}^\bullet(\pi_{\mathbb{P}})_*ev_{\mathbb{P}}^*T^1 \to \bar{f}^*\mathcal{R}^\bullet(\pi_{\mathbb{P}})_*ev_{\mathbb{P}}^*T^\bullet \to \bar{f}^*\mathcal{R}^\bullet(\pi_{\mathbb{P}})_*ev_{\mathbb{P}}^*T^0[1]$$

and applying $\mathcal{R}^\bullet(\pi_X)_*\bar{f}^*$ to the same triangle we obtain

$$\mathcal{R}^\bullet(\pi_X)_*\bar{f}^*ev_{\mathbb{P}}^*T^0 \to \mathcal{R}^\bullet(\pi_X)_*\bar{f}^*ev_{\mathbb{P}}^*T^1 \to \mathcal{R}^\bullet(\pi_X)_*\bar{f}^*ev_{\mathbb{P}}^*T^\bullet \to \mathcal{R}^\bullet(\pi_X)_*\bar{f}^*ev_{\mathbb{P}}^*T^0[1].$$

Using Step 1 in order to compare the two triangles we deduce that $\bar{f}^*\mathcal{R}^\bullet(\pi_{\mathbb{P}})_*ev_{\mathbb{P}}^*T^\bullet \simeq \mathcal{R}^\bullet(\pi_X)_*ev_X^*f^*T^\bullet$. ☐

Proposition 2.1.4. *Let $f : X \to \mathbb{P}$ be a morphism of smooth projective varieties and let $T_{X/\mathbb{P}}$ be the dual of the cotangent complex of X to $\mathbb{P}$. Then, in notations as above*
(i)$\bar{f} : \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}(\mathbb{P}, f_*\beta)$ has a dual obstruction theory $E^\bullet_{\overline{M}_{g,n}(X,\beta)/\overline{M}_{g,n}(\mathbb{P},f_*\beta)}$ isomorphic to $\mathcal{R}^\bullet(\pi_X)_*ev_X^*T_{X/\mathbb{P}}$ in $\mathcal{D}_X$.
(ii) If $f = i$ is an embedding and $N_{X/\mathbb{P}}$ denotes the normal bundle of X in $\mathbb{P}$, then $\bar{i} : \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}(\mathbb{P}, f_*\beta)$ has a dual obstruction theory $E^\bullet_{\overline{M}_{g,n}(X,\beta)/\overline{M}_{g,n}(\mathbb{P},i*\beta)}$

$$[0 \to (\mathcal{R}^0(\pi_X)_*ev_X^*N_{X/\mathbb{P}})^\vee \to (\mathcal{R}^1(\pi_X)_*ev_X^*N_{X/\mathbb{P}})^\vee]$$

in $[0,2]$.
(iii) If $g = 0$ and $\mathbb{P}$ is convex, then $E^\bullet_{\overline{M}_{g,n}(X,\beta)/\overline{M}_{g,n}(\mathbb{P},f_*\beta)}$ is perfect.

Proof. (i) In notations as in the beginning of the section, the relative obstruction theories are $E^\bullet_{\overline{M}_{0,n}(\mathbb{P},f_*\beta)/\mathfrak{M}} := (\mathcal{R}^i(\pi_{\mathbb{P}})_*ev_{\mathbb{P}}^*T_{\mathbb{P}})^\vee$ and $E^\bullet_{\overline{M}_{0,n}(X,\beta)/\mathfrak{M}} := (\mathcal{R}^i(\pi_X)_*ev_X^*T_X)^\vee$. The distinguished triangle

$$f^*\Omega_{\mathbb{P}} \to \Omega_X \to L_{X/\mathbb{P}} \to f^*\Omega_{\mathbb{P}}[1] \qquad (2.1)$$

induces a distinguished triangle

$$\mathcal{R}^\bullet(\pi_X)_* ev_X^* f^*\Omega_{\mathbb{P}} \xrightarrow{\varphi} \mathcal{R}^\bullet(\pi_X)_* ev_X^*\Omega_X \to \mathcal{R}^\bullet(\pi_X)_* ev_X^* L_{X/\mathbb{P}} \to \mathcal{R}^\bullet(\pi_X)_* ev_X^* f^*\Omega_{\mathbb{P}}[1].$$

We now need to show that

$$\mathcal{R}^\bullet(\pi_X)_* ev_X^* f^*\Omega_{\mathbb{P}} = \bar{f}^*\mathcal{R}^\bullet(\pi_{\mathbb{P}})_* ev_{\mathbb{P}}^*\Omega_{\mathbb{P}}$$

in the derived category of $\overline{M}_{0,n}(X,\beta)$ and this follows by Corollary 2.1.3.
(ii) If $f = i$ is an embedding, then $L_{X/\mathbb{P}}$ is quasi-isomorphic to $[N^\vee_{X/Y} \to 0]$ in degrees $[-1,0]$ and the claim follows by (i).
(iii) If $\mathbb{P}$ is convex, then $\overline{M}_{0,n}(\mathbb{P}, i_*\beta)$ is smooth and the claim follows from Remark 1.2.14. □

Proposition 2.1.5. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{j} & \mathbb{P}' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{i} & \mathbb{P} \end{array}$$

be a cartesian diagram of smooth projective varieties and let $\beta \in H_2(X')$ be any homology class of a curve.
(i) Then the induced diagram of moduli spaces of stable maps

$$\begin{array}{ccc} \overline{M}_{g,n}(X',\beta) & \xrightarrow{\bar{j}} & \overline{M}_{g,n}(\mathbb{P}', j_*\beta) \\ \downarrow{\scriptstyle \bar{p}} & & \downarrow{\scriptstyle \bar{q}} \\ \overline{M}_{g,n}(X, p_*\beta) & \xrightarrow{\bar{i}} & \overline{M}_{g,n}(\mathbb{P}, (i\circ p)_*\beta) \end{array}$$

is commutative. If i is a closed embedding, then it induces an open and closed embedding of $\overline{M}_{g,n}(X',\beta)$ in the fiber product $\overline{M}_{g,n}(X, p_*\beta) \times_{\overline{M}_{g,n}(\mathbb{P},(i\circ p)_*\beta)} \overline{M}_{g,n}(\mathbb{P}', j_*\beta)$.
(ii)If the natural map $p^*L_{X/\mathbb{P}} \to L_{X'/\mathbb{P}'}$ is an isomorphism, then it induces an isomorphism

$$E^\bullet_{\overline{M}_{g,n}(X',\beta)/\overline{M}_{g,n}(\mathbb{P}',j_*\beta)} \simeq \bar{p}^* E^\bullet_{\overline{M}_{g,n}(X,p_*\beta)/\overline{M}_{g,n}(\mathbb{P},(i\circ p)_*\beta)}.$$

If moreover, $g = 0$ and $\mathbb{P}$ is convex then, $E^\bullet_{\overline{M}_{g,n}(X',\beta)/\overline{M}_{g,n}(\mathbb{P}',j_*\beta)}$ is perfect.

Proof. The first statement is clear, so let us treat the second one. The proof of part (ii) follows by Proposition 2.1.4 (i) applied to the morphism i, followed by Corollary 2.1.3 with $f = p$ and $T := L^{\vee}_{X/\mathbb{P}}$.
If $g = 0$ and $\mathbb{P}$ is convex then $\overline{M}_{g,n}(\mathbb{P},(i\circ p)_*\beta)$ is smooth and therefore $E^{\bullet}_{\overline{M}_{g,n}(X,p_*\beta)/\overline{M}_{g,n}(\mathbb{P},(i\circ p)_*\beta)}$ is perfect. The claim now follows from the first part of the proof. □

Example 2.1.6. Let us consider a cartesian diagram of smooth projective varieties as above. The induced commutative diagram of moduli spaces of stable maps will not be cartesian in general. One counterexample is the case of $\mathbb{P}$ is a point $P := \mathit{Spec}\ k$, $\mathbb{P}' := Y$ and $X' := X \times Y$. The diagram

$$\begin{array}{ccc} \overline{M}_{g,n}(X\times Y,(\beta_1,\beta_2)) & \xrightarrow{\bar{p_2}} & \overline{M}_{g,n}(Y,\beta_2) \\ \downarrow{\scriptstyle \bar{p_1}} & & \downarrow \\ \overline{M}_{g,n}(X,\beta_1) & \longrightarrow & \overline{M}_{g,n}(P,0) \end{array}$$

is *not* cartesian. Let M be the cartesian product $\overline{M}_{g,n}(Y,\beta_2)\times_{\overline{M}_{g,n}(P,0)} \overline{M}_{g,n}(X,\beta_1)$. Then, we still have a nice relation between the virtual class of M and the virtual class of $\overline{M}_{g,n}(X\times Y,(\beta_1,\beta_2))$. This was studied by Behrend (see [2], Theorem 1).

Remark 2.1.7. Let us consider the commutative diagram

$$\begin{array}{ccc} \overline{M}_{g,n}(X',\beta) & \xrightarrow{\bar{j}} & \overline{M}_{g,n}(\mathbb{P}',j_*\beta) \\ \downarrow{\scriptstyle \bar{p}} & & \downarrow{\scriptstyle \bar{q}} \\ \overline{M}_{g,n}(X,p_*\beta) & \xrightarrow{\bar{i}} & \overline{M}_{g,n}(\mathbb{P},(i\circ p)_*\beta) \end{array}$$

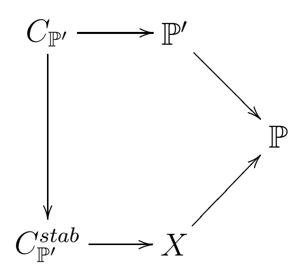
with i a closed embedding. If j induces an isomorphism $H_2(X')\to H_2(\mathbb{P}')$, then the above diagram is cartesian. In general, the cartesian product will be a disjoint union of components corresponding to all homology classes $\delta\in H_2(X')$ such that $j_*\delta = j_*\beta$.

Example 2.1.8. Let

$$\begin{array}{ccc} X' & \xrightarrow{j} & \mathbb{P}' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{i} & \mathbb{P} \end{array}$$

be a cartesian diagram of smooth projective varieties as in the above proposition. Let us moreover suppose that i and j induce injective morphisms of groups $i_* : H_2(X) \to H_2(\mathbb{P})$, respectively $j_* : H_2(X') \to H_2(\mathbb{P}')$. Then, the corresponding commutative diagram between moduli spaces of stable maps is cartesian.

Proof. Let us fix a scheme S and let us consider $f : C_{\mathbb{P}} \to \mathbb{P}$ an element in $\overline{M}_{g,n}(\mathbb{P}, (i \circ p)_*\beta)(S)$. Now let us consider $f : C_{\mathbb{P}'} \to \mathbb{P}'$ an object in $\overline{M}_{g,n}(\mathbb{P}', j_*\beta)(S)$ and $f : C_X \to X$ an object in $\overline{M}_{g,n}(X, p_*\beta)(S)$ which map to $f : C_{\mathbb{P}} \to \mathbb{P}$. We have that $f : C_{\mathbb{P}} \to \mathbb{P}$ is canonically isomorphic to $C_{\mathbb{P}'}^{stab} \to \mathbb{P}$ the stabilization of the composite map $C_{\mathbb{P}'} \to \mathbb{P}' \to \mathbb{P}$. Then, by our hypothesis on i we have that the curve C_X is also canonically isomorphic to $C_{\mathbb{P}'}^{stab}$. We have thus obtained a commutative diagram



and therefore by the universal property of cartesian products, we have obtained a canonical map $C_{\mathbb{P}'} \to X'$. By our hypothesis on j all maps $C_{X'} \to X'$ are obtained in this way. □

Example 2.1.9. Let us now look at an example where our construction of virtual pull-backs does not apply. If we consider $\overline{M}_{g,n}(X, \beta) \xrightarrow{\bar{i}} \overline{M}_{g,n}(\mathbb{P}^r, i_*\beta)$ with $\mathbb{P}$ convex, then the construction applies without further conditions only in genus zero. In general $h^{-2}(E_{\overline{M}_{g,n}(X,\beta)/\overline{M}_{g,n}(\mathbb{P}^r, i_*\beta)})$ might not vanish.
To see an example, let us consider $\mathbb{P}^r \hookrightarrow \mathbb{P}^r \times \mathbb{P}^s$, the inclusion into the first factor. Then we have an induced map $\overline{M}_{g,n}(\mathbb{P}^r, d_1) \hookrightarrow \overline{M}_{g,n}(\mathbb{P}^r \times \mathbb{P}^s, (d_1, 0))$. The above argument (see 2.3) shows that the dual relative obstruction theory we obtain is $\mathcal{R}^\bullet \pi_* ev^* N_{\mathbb{P}^r/\mathbb{P}^r \times \mathbb{P}^s}$. We have that the normal bundle $N_{\mathbb{P}^r/\mathbb{P}^r \times \mathbb{P}^s}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^r}^{\oplus s}$. Since the map $\mathcal{R}^0\pi_* f^*(\mathcal{O}_{\mathbb{P}^r}^{\oplus s}) \to \mathcal{R}^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^r}^{\oplus s})$ is obviously not surjective for $g \geq 1$, the (dual) relative obstruction theory will never be perfect.

2.2 Pulling back divisors

Let $\mathbb{P}$ be a convex variety and $d \in A_1(\mathbb{P})$ be the class of a curve. If $X \overset{i}{\hookrightarrow} \mathbb{P}$ is an embedding of smooth projective varieties, then i induces a morphism $\overline{M}_{0,n}(X,d) \overset{\bar{i}}{\hookrightarrow} \overline{M}_{0,n}(\mathbb{P},d)$ where we made the convention that $\overline{M}_{0,n}(X,d)$ is the union of all $\overline{M}_{0,n}(X,\beta)$ such that $i_*\beta = d$. Let $D_{\mathbb{P}} := D_{\mathbb{P}}(0,n_1,d_1 \mid 0,n_2,d_2)$ be a boundary divisor in $\overline{M}_{0,n}(\mathbb{P},d)$ that comes with a virtual class obtained by pull-back along the obvious forgetful morphism

$$D_{\mathbb{P}} \to \mathfrak{M}_{0,n_1+1} \times \mathfrak{M}_{0,n_2+1}$$

and analogously we have a boundary divisor $D_X := D_X(0,n_1,d_1 \mid 0,n_2,d_2)$ in $\overline{M}_{0,n}(X,d)$ equipped with a virtual fundamental class. Constructing the following cartesian diagram

$$\begin{array}{ccccc} D_X & \longrightarrow & D_{\mathbb{P}} & \longrightarrow & \mathfrak{M}_{0,n_1+1} \times \mathfrak{M}_{0,n_2+1} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{M}_{0,n}(X,d) & \xrightarrow{\bar{i}} & \overline{M}_{0,n}(\mathbb{P},d) & \longrightarrow & \mathfrak{M} \end{array}$$

we get

$$\bar{i}^![D_{\mathbb{P}}(0,n_1,d_1 \mid 0,n_2,d_2)]^{\text{virt}} = [D_X(0,n_1,d_1 \mid 0,n_2,d_2)]^{\text{virt}}.$$

Indeed, it is easy to check that the obstructions are compatible.

Remark 2.2.1. One could naïvely hope to obtain new relations between the rational GW invariants of X by pulling back the WDVV relations in $\mathbb{P}$. The above shows that for any $X \hookrightarrow \mathbb{P}$ pulling-back the WDVV equations in $\overline{M}_{0,n}(\mathbb{P},d)$ gives the WDVV equations in $\overline{M}_{0,n}(X,d)$.

2.3 Blow-ups

Let X be a smooth r-dimensional projective variety, $Y \subseteq X$ a smooth r'-codimensional subvariety and $p_X : \tilde{X} \to X$ the blow-up of X in Y, with exceptional divisor E. We are interested in comparing GW invariants of X with GW invariants of its blow-up $\tilde{X}$.

Definition 2.3.1. For every blow up $p_X : \tilde{X} \to X$ and every class $\beta \in A_1(X)$ we call the class $p^!\beta$ the lifting of β and we denote it by $\tilde{\beta}$, where $p^!$ is the refined intersection product of [10], Chapter 8.

Remark 2.3.2. The lifting of β satisfies two basic properties that follow trivially from the projection formula, namely $(p_X)_*\tilde{\beta} = \beta$ and $\tilde{\beta} \cdot E = 0$.

Lemma 2.3.3. *The moduli space of stable maps to $\tilde{X}$ of class $\tilde{\beta}$ and the moduli space of stable maps to X of class β have the same virtual dimension.*

Proof. By [10] we know that

$$K_{\tilde{X}} = p_X^* K_X + (r' - 1)E$$

and therefore the virtual dimension of $\overline{M}_{g,n}(\tilde{X}, \tilde{\beta})$ is

$$\begin{aligned}
\mathrm{vdim}(\overline{M}_{g,n}(\tilde{X}, \tilde{\beta})) &= (1-g)(r-3) - K_{\tilde{X}} \cdot \tilde{\beta} + n \\
&= (1-g)(r-3) - [p_X^* K_X + (r'-1)E] \cdot \tilde{\beta} + n \\
&= (1-g)(r-3) - p_X^* K_X \cdot \tilde{\beta} + n \\
&= (1-g)(r-3) - K_X \cdot \beta + n \\
&= \mathrm{vdim}\overline{M}_{g,n}(X, \beta).
\end{aligned}$$

□

Remark 2.3.4. Let $\bar{p}_X : \overline{M}_{g,n}(\tilde{X}, \tilde{\beta}) \to \overline{M}_{g,n}(X, \beta)$ be the morphism induced by the natural projection $p_X : \tilde{X} \to X$. After the above lemma it is natural to ask whether $(\bar{p}_{\mathbb{P}})_*[\overline{M}_{g,n}(\tilde{X}, \tilde{\beta})]^{\mathrm{virt}} = [\overline{M}_{g,n}(X, \beta)]^{\mathrm{virt}}$. We will show that under additional assumptions the answer to this question is positive. However, we cannot hope this result to hope in general, and not even in genus zero (for a counter example see [4]).

Lemma 2.3.5. *In notations as before, the natural projection $p_X : \tilde{X} \to X$ induces a morphism $\bar{p}_X : \overline{M}_{0,n}(\tilde{X}, \tilde{\beta}) \to \overline{M}_{0,n}(X, \beta)$. If $X = \mathbb{P}$ is convex and $\tilde{X} = \tilde{\mathbb{P}}$ the blow up of $\mathbb{P}$, then*

$$(\bar{p}_{\mathbb{P}})_*[\overline{M}_{0,n}(\tilde{\mathbb{P}}, \tilde{\beta})]^{\mathrm{virt}} = [\overline{M}_{0,n}(\mathbb{P}, \beta)]^{\mathrm{virt}}.$$

Proof. The proof is a straightforward generalization of [12], Proposition 2.2. Let us write the proof for completeness. Since $\mathbb{P}$ is convex the stack $\overline{M}_{0,n}(\mathbb{P}, \beta)$ is smooth of expected dimension d. Let $Z_1, ..., Z_k$ the connected components of $\overline{M}_{0,n}(\mathbb{P}, \beta)$. As $\overline{M}_{0,n}(\tilde{\mathbb{P}}, \beta)$ has expected dimension d we have

$$(\bar{p}_{\mathbb{P}})_*[\overline{M}_{0,n}(\tilde{\mathbb{P}}, \tilde{\beta})]^{\mathrm{virt}} = \alpha_1[Z_1] + ... + \alpha_k[Z_k]$$

for some $\alpha_i \in \mathbb{Q}$. If we show that p is a local isomorphism around a generic point $\mathcal{C} := (C, x_1, ..., x_n, f) \in Z_i$ for some $1 \leq i \leq k$ then by [12] we have

$$(\bar{p}_{\mathbb{P}})_*[\overline{M}_{0,n}(\tilde{\mathbb{P}}, \tilde{\beta})]^{\text{virt}} = [Z_1] + ... + [Z_k] = \overline{M}_{0,n}(\mathbb{P}, \beta).$$

Let us suppose that for some generic $\mathcal{C} \in Z_i$, $p^{-1}(\mathcal{C})$ is not a point. As $\mathcal{C}$ is generic, we may assume that C is irreducible. Then $f(C)$ must intersect the blown up locus and the subscheme M of $\overline{M}_{0,n}(\mathbb{P}, \beta)$ consisting of such maps must have dimension d. But $\mathbb{P}$ is convex and Y has codimension at least two, hence M has codimension at least 1. This leads to a contradiction. □

Proposition 2.3.6. *Let X, Y be smooth projective subvarieties of some smooth projective convex variety $\mathbb{P}$ and let us assume that there exists Z a smooth subvariety of $\mathbb{P}$, such that X and Z intersect transversely. Let $Y := X \cap Z$ and $\tilde{X}$ be the blow-up of X along Y. Then for any non-negative integer n and any $\beta \in A_1(X)$ with lifting $\tilde{\beta} \in A_*(\tilde{X})$*

$$(\bar{p}_X)_*[\overline{M}_{0,n}(\tilde{X}, \tilde{\beta})]^{\text{virt}} = [\overline{M}_{0,n}(X, \beta)]^{\text{virt}}.$$

Proof. If $Y = X \cap Z$, $\tilde{X}$ is the blow-up of X along Y and $\tilde{\mathbb{P}}$ is the blow-up of $\mathbb{P}$ along Z then the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{j} & \tilde{\mathbb{P}} \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_{\mathbb{P}}} \\ X & \xrightarrow{i} & \mathbb{P} \end{array}$$

is cartesian. By Proposition 2.1.5 this induces a cartesian diagram of DM-stacks

$$\begin{array}{ccc} \overline{M}_{0,n}(\tilde{X}, \tilde{\beta}) & \xrightarrow{\bar{j}} & \overline{M}_{0,n}(\tilde{\mathbb{P}}, \widetilde{i_*\beta}) \\ {\scriptstyle \bar{p}_X}\downarrow & & \downarrow{\scriptstyle \bar{p}_{\mathbb{P}}} \\ \overline{M}_{0,n}(X, \beta) & \xrightarrow{\bar{i}} & \overline{M}_{0,n}(\mathbb{P}, i_*\beta). \end{array}$$

In order to apply the virtual push-forward machinery to this diagram, we first need to analyze the obstruction theories involved. By Construction 1.2.17 and Corollary 1.3.7 applied to $\bar{i}$, we have

$$\bar{i}^![\overline{M}_{0,n}(\mathbb{P}, i_*\beta)]^{\text{virt}} = [\overline{M}_{0,n}(X, \beta)]^{\text{virt}}. \quad (2.2)$$

We know that $p_X^* N_{X/\mathbb{P}} = N_{\tilde{X}/\tilde{\mathbb{P}}}$. By Proposition 2.1.5 (ii) we obtain that

$$E^{\bullet}_{\overline{M}_{0,n}(\tilde{X},\tilde{\beta})/\overline{M}_{0,n}(\tilde{\mathbb{P}},\widetilde{i_*\beta})} = \bar{p}_X^* E^{\bullet}_{\overline{M}_{0,n}(X,\beta)/\overline{M}_{0,n}(\mathbb{P},i_*\beta)}.$$

This shows in particular that $E^{\bullet}_{\overline{M}_{0,n}(\tilde{X},\tilde{\beta})/\overline{M}_{0,n}(\tilde{\mathbb{P}},\widetilde{i_*\beta})}$ is perfect. Applying Corollary 1.3.7 to $\bar{j}$ we get

$$\bar{j}^![\overline{M}_{0,n}(\tilde{\mathbb{P}},\widetilde{i_*\beta})]^{\text{virt}} = [\overline{M}_{0,n}(\tilde{X},\tilde{\beta})]^{\text{virt}}$$

and by Proposition 1.3.1 (iii) we obtain that

$$\bar{i}^![\overline{M}_{0,n}(\tilde{\mathbb{P}},\widetilde{i_*\beta})]^{\text{virt}} = [\overline{M}_{0,n}(\tilde{X},\tilde{\beta})]^{\text{virt}}. \tag{2.3}$$

Proposition 1.3.1 (i) gives

$$\bar{i}^!(\bar{p}_{\mathbb{P}})_*[\overline{M}_{0,n}(\tilde{\mathbb{P}},\widetilde{i_*\beta})]^{\text{virt}} = (\bar{p}_X)_*\bar{i}^![\overline{M}_{0,n}(\tilde{\mathbb{P}},\widetilde{i_*\beta})]^{\text{virt}}. \tag{2.4}$$

By Proposition 2.3.5 we have

$$\bar{i}^!(\bar{p}_{\mathbb{P}})_*[\overline{M}_{0,n}(\tilde{\mathbb{P}},\widetilde{i_*\beta})]^{\text{virt}} = \bar{i}^![\overline{M}_{0,n}(\mathbb{P},i_*\beta)]^{\text{virt}}. \tag{2.5}$$

Gathering all together, equations 2.2, 2.3, 2.4, 2.5 translate in

$$(\bar{p}_X)_*[\overline{M}_{0,n}(\tilde{X},\tilde{\beta})]^{\text{virt}} = [\overline{M}_{0,n}(X,\beta)]^{\text{virt}}.$$

□

The projection formula gives the following Corollary.

Corollary 2.3.7. *Let X and Y as above, and let $\gamma \in A^*(X)^{\otimes n}$ be any n-tuple of classes such that $\sum codim(\gamma_i) = vdim \overline{M}_{0,n}(X,\beta)$. Then, $I^{\tilde{X}}_{0,n,\tilde{\beta}}(p_X^*\gamma) = I^X_{0,n,\beta}(\gamma)$.*

Remark 2.3.8. This result was obtained in [24] in a more general context. Lai starts with X and Y such that $N_{Y/X}$ is convex and he analyzes the map $\overline{M}_{0,n}(\tilde{X},\tilde{\beta}) \xrightarrow{\bar{p}} \overline{M}_{0,n}(X,\beta)$. Under this hypothesis the relative obstruction theory induced by p is perfect that in our language means that p admits a virtual pull-back. We should stress however, that we cannot use the usual relative obstruction theories to $\mathfrak{M}$ in order to obtain the previous result because the diagram in Corollary 1.3.7 is not commutative. In [24], Lai uses the *absolute* obstruction theories and he shows they are compatible. In our language this means that $\bar{p}_X^![\overline{M}_{0,n}(X,\beta)]^{\text{virt}} = [\overline{M}_{0,n}(\tilde{X},\tilde{\beta})]^{\text{virt}}$. Under these assumptions Lai analyzes the normal cones of $\overline{M}_{0,n}(\tilde{X},\tilde{\beta})$ and $\overline{M}_{0,n}(X,\beta)$ and uses the relation between them in order to obtain that $(\bar{p}_X)_*[\overline{M}_{0,n}(\tilde{X},\tilde{\beta})]^{\text{virt}} = [\overline{M}_{0,n}(X,\beta)]^{\text{virt}}$ (see [24] Theorem 4.11).

Remark 2.3.9. If X is the zero-locus of a section $s \in H^0(\mathbb{P}, V)$, for some convex vector bundle V on $\mathbb{P}$ and Y respects the hypothesis of Proposition 1.3.7, then the equality $(\bar{p}_X)_*[\overline{M}_{0,n}(\tilde{X}, \tilde{\beta})]^{\mathrm{virt}} = [\overline{M}_{0,n}(X, \beta)]^{\mathrm{virt}}$ follows from the "Conjecture" proved in [19] as described below. In notations of [ibid.] we have

$$\bar{i}_*[\overline{M}_{0,n}(X, \beta)]^{\mathrm{virt}} = c_{top}(\mathcal{R}^0(\pi_{\mathbb{P}})_* ev_{\mathbb{P}}^* V) \cdot [\overline{M}_{0,n}(\mathbb{P}, i_*\beta)]^{\mathrm{virt}}.$$

Again, using equality Proposition 2.1.4 we get the same relation with blow-ups, namely,

$$\bar{j}_*[\overline{M}_{0,n}(\tilde{X}, \tilde{\beta})]^{\mathrm{virt}} = c_{top}(\bar{p}^* \mathcal{R}^0(\pi_{\tilde{\mathbb{P}}})_* ev_{\tilde{\mathbb{P}}}^* V) \cdot [\overline{M}_{0,n}(\tilde{\mathbb{P}}, \widetilde{i_*\beta})]^{\mathrm{virt}}.$$

Now, the equality follows from the projection formula.

2.3.1 Blow-ups of points and curves

Although this section is not a relevant application to virtual pull-backs we decided to include this result since we find it useful. The main technique is the degeneration method which is combined with a very weak version of the virtual push-forward property.

Proposition 2.3.10. *Let n be some fixed natural number, let $\beta \in H_2(\mathbb{P})$ be a fixed curve class and $\gamma \in \bigotimes H^*(\mathbb{P})$ any n-tuple of cohomology classes such that $\sum codim(\gamma_i) = vdim \overline{M}_{g,n}(\mathbb{P}, \beta)$. If $\mathbb{P}$ is convex, then $p_*[\overline{M}_{0,n}(\tilde{X}, \tilde{\beta})]^{\mathrm{virt}} = [\overline{M}_{0,n}(X, \beta)]^{\mathrm{virt}}$ and in particular, $I^{\tilde{X}}_{0,n,\tilde{\beta}}(p^*\gamma) = I^X_{0,n,\beta}(\gamma)$.*

Corollary 2.3.11. (Hu [15]) *If Y is either a point or a curve of positive genus, then $p_*[\overline{M}_{0,n}(\tilde{X}, \tilde{\beta})]^{\mathrm{virt}} = [\overline{M}_{0,n}(X, \beta)]^{\mathrm{virt}}$. In particular,*

$$I^{\tilde{X}}_{0,n,\tilde{\beta}}(p^*T) = I^X_{0,n\beta}(T),$$

for every $T \in \bigotimes H^*(X)$.

Proof. By our assumption on Y, $\mathbb{P}$ is convex (see for instance [9], §0.4) and hence the conditions of the above proposition are fulfilled. □

Construction 2.3.12. *Li's degeneration formula*
Let $\mathcal{F} \to \mathbb{P}^1$ be a family of projective schemes with $0 \in \mathbb{P}^1$ a distinguished point, so that the general fiber $\mathcal{F}_t$ over $t \neq 0$ are smooth varieties, and

the special fib re $\mathcal{F}_0$ has two irreducible components $\mathcal{F}_0^+$, $\mathcal{F}_0^-$ intersecting transversely along a connected smooth divisor E of $\mathcal{F}_0$.
By [27] we have that the moduli space of stable maps to $\mathcal{F}_t$ is expressible as a product of moduli spaces of maps to $\mathcal{F}_0^+$ and $\mathcal{F}_0^-$ and moreover the following formula holds

$$[\overline{M}_{g,n}(\mathcal{F}_t,\Gamma)]^{\text{virt}} = \sum_{\Gamma_1,\Gamma_2 \in \bar{\Omega}^H_{(g,n,d)}} m(\Gamma_1\Gamma_2)\cdot[\overline{M}^E_{\Gamma_1}(\mathcal{F}_0^-)]^{\text{virt}} \boxtimes [\overline{M}^E_{\Gamma_2}(\mathcal{F}_0^+)]^{\text{virt}} \quad (2.6)$$

where we have used the following notation. The space $\overline{M}^E_{\Gamma_1}(\mathcal{F}_0^+)$, $(\overline{M}^E_{\Gamma_2}(\mathcal{F}_0^-))$ is a moduli space of stable relative maps to $\mathcal{F}_0^+$ (respectively $\mathcal{F}_0^-$) relative E. Here Γ stands for the triple (g,n,d), with $d := \beta \cdot H$, (β is any homology class in $H_2(X)$, H is a fixed relative ample line bundle on $\mathcal{F}$) and Γ_1, Γ_2 denote the collection of the following data:

(i) the number r of connected components of the stable relative maps,
(ii) the (non-zero) homology classes of all connected components,
(iii) for every connected component a subset $\{1,...,n^i\}$ of $\{1,...,n\}$ of the marked points lying on it, where all these points have multiplicity 0,
(iv) for every connected component of class $\beta^{+,i} \in H_2(\mathcal{F}_0^+)$, $i \in \{1,...,r^+\}$, a collection of additional marked points $\{y^i_j\}$ lying on X with associated positive multiplicities α^i_j, such that $\sum_{i,j} \alpha_j^{+,i} = i_*\beta^+ \cdot E$ and analogously for $\mathcal{F}_0^-$.

We denote by $\Omega^H_{(g,n,d)}$ in the sum above, the set of all pairs of data (Γ_1,Γ_2) such that

- the glued stable map is connected and has the correct homology class, and
- the additional marked points y_i are labeled on both the $\mathcal{F}_0^+$ and the $\mathcal{F}_0^-$ side by the same index set $\{1,...,m\}$ for some m, and the multiplicities α_i associated to these points agree on both sides.

and by $\bar{\Omega}^H_{(g,n,d)}$ the set of equivalence classes in $\Omega^H_{(g,n,d)}$ from re-ordering of the points y_i.

The coefficient $m(\Gamma_1\Gamma_2)$ is defined to be $\frac{\alpha_1\cdots\alpha_r}{r!}$. The notation $\boxtimes$ means that we take the moduli spaces of collapsed stable relative maps on both sides and take their fiber product over the r-fold evaluation map to X at the points y_i (see [13]).

Construction 2.3.13. We are interested in the following two particular situations. Let $\mathcal{F}$ be the blow up of $X \times \mathbb{P}^1$ along $Y \times \{0\}$. We obtain a family with general fibre X and special fibre equal the union of $\tilde{X}$ and $\mathbb{P}$ glued along the exceptional divisor of $\tilde{X}$. In this case we know by [28] gives us even more.
Let us fix the notations and restate the result. The family $\mathcal{F}$ induces a morphism $p : \mathcal{F}_0 \to X$. We denote the restrictions of p to the irreducible components of $\mathcal{F}_0$ by $p_1 : \mathbb{P} \to X$ and $p_2 : \tilde{X} \to X$. If $\beta \in H_2(X)$ is a fixed homology class, then consider

$$\Omega^H_{(g,n,\beta)} := \{(\Gamma_1, \Gamma_2) \in \Omega^H_{(g,n,d)} |\ p_{1*}(\beta^+) + p_{2*}(\beta^-) = \beta\}.$$

Let us now consider s the strict transform of $Y \times \mathbb{P}^1$ in $\mathcal{F}$. A direct verification shows that $s \cap \mathcal{F}_0 = Y_\infty$. If $\tilde{\mathcal{F}}$ is the blow up of $\mathcal{F}$ along s, then the general fiber of this new family is $\tilde{X}$ and the special one is the union of $\tilde{X}$ and $\tilde{\mathbb{P}}$ glued along the E.

As before, we have morphisms $p_1 : \tilde{\mathbb{P}} \to X$ and $p_2 : \tilde{X} \to X$. If $\bar{\beta} \in H_2(\tilde{X})$ is a fixed homology class, then consider

$$\Omega^H_{(g,n,\bar{\beta})} := \{(\Gamma_1, \Gamma_2) \in \Omega^H_{(g,n,d)} |\ p_{1*}(\bar{\beta}^+) + p_{2*}(\bar{\beta}^-) = \bar{\beta}\}.$$

Lemma 2.3.14. (Liu,Yau, [28]) *Li's result holds with $\Gamma = (g, n, d)$ replaced (g, n, β) and the homology classes of curves in $\mathbb{P}$ and $\tilde{X}$ replaced by $\Gamma_1,\ \Gamma_2 \in \bar{\Omega}^H_{(g,n,\beta)}$.*

Lemma 2.3.15. *If we take $\bar{\beta}$ to be $\tilde{\beta}$ the lifting of some fixed $\beta \in H_2(X)$, then*

$$\bar{\beta}^+_0 = \widetilde{\beta^+_0} \text{ and}$$
$$\bar{\beta}^-_0 = \beta^-_0$$

Proof. Let us fix $\beta^- \in H_2(\tilde{X})$ and take $\bar{\beta}^- := \beta^-$. Then, the conditions $p_{1*}(\bar{\beta}^+) + p_{2*}(\bar{\beta}^-) = \bar{\beta}$ and $p_{1*}(\beta^+) + p_{2*}(\beta^-) = \bar{\beta}$ give $\bar{\beta}^+_0 = \widetilde{\beta^+_0} - mf$, where $f \in H_2(\tilde{\mathbb{P}})$ is the class of a fibre in E and $m \in \mathbb{Z}$ some integer number. To see that m must be zero, all we have to do is to notice that the degree of the glued curve in $\tilde{\mathcal{F}}_0$ cannot be independent of m. Take for instance H to be $A - E$ in the fibers, where A is a sufficiently ample line bundle in X. This concludes the proof. □

2.4 Projective bundles

The this section we are interested in the situation which appeared in Section 1.4, in the special case of the map between the moduli space of genus zero stable maps to a projective bundle $p_X : \mathbb{P}_X(V) \to X$ and the moduli space of genus zero stable maps to the base X. The virtual machinery applied to this case gives rise to a way to compute the scalar which appears in the statement of the Theorem 1.4.6. More precisely, we show that the scalar we are interested in is equal to the scalar corresponding to a projective bundle over a projective line. This reduces the computations to Elezi's computations of Gromov-Witten invariants of splitted bundles over a toric base (see [7], [8]). Having computed this scalar and assuming we know the genus zero Gromov-Witten theory of X, we can compute certain genus-zero Gromov-Witten invariants of arbitrary projective bundles.

Let X be a a smooth projective variety of dimension n. Let $\mathbb{C}_X$ be the trivial line bundle on X and $V := W \oplus \mathbb{C}_X$ be a rank r vector bundle on X with non-zero Chern roots $\{c_1, ..., c_{r-1}\}$. We denote by $p_X : \mathbb{P}_X(V) \to X$ the associated projective bundle. It is well known that there exists an isomorphism $\varphi : A_1(\mathbb{P}_X(V)) \to A_1(X) \oplus f \cdot \mathbb{Z}$, where f_X denotes the class of a curve in a fibre of p_X. Let us fix such an isomorphism. For this, let us consider the following exact sequence

$$0 \to A_1(\mathbb{P}^{r-1}) \xrightarrow{i} A_1(\mathbb{P}_X(V)) \xrightarrow{p} A_1(X) \to 0$$

where i is the map induced by the inclusion of a fiber of p_X and p denotes the push-forward by p_X. Then, taking $s_X : X \to \mathbb{P}_X(V)$ the zero section of the projective bundle $\mathbb{P}_X(V)$ we see that the map $s : A_1(X) \to A_1(\mathbb{P}_X(V))$ induced by s_X splits the sequence above. This fixes φ.

Definition 2.4.1. Let $\beta \in A_1(X)$, then we call $\tilde{\beta} := s(\beta)$ the lifting of β.

In these notations any class of a curve in $\mathbb{P}_X(V)$ can be written uniquely as $\tilde{\beta} + qf_X$, for some $\beta \in A_1(X)$ and some $q \in \mathbb{Z}$. Let us consider $\alpha \in$

$A^k(\overline{M}_{0,n}(\mathbb{P}_X(V),\tilde{\beta}+qf_X))$ such that

$$\mathrm{vdim}\overline{M}_{0,n}(\mathbb{P}_X(V),\tilde{\beta}+qf_X)-k=\mathrm{vdim}\overline{M}_{0,n}(X,\beta). \qquad (2.7)$$

We will say that (β,W,q,α) satisfies condition (2.7) or when there is no risk of confusion that (W,q,α) satisfies condition (2.7).
In the same way as in the case of blow-ups, we will relate genus-zero GW invariants of $\mathbb{P}_X(V)$ to genus-zero GW invariants of X.

Remark 2.4.2. Let X be a convex variety, $\beta\in A_1(X)$, W a rank-$(r-1)$ vector bundle on X with Chern roots $c_1,...,c_{r-1}$, $q\in\mathbb{Z}$ and $\alpha\in A^*(\overline{M}_{0,n}(\mathbb{P}_X(V),\tilde{\beta}+qf_X))$ satisfying condition (2.7). Let $Z_1,...Z_k$ be the connected components of $\overline{M}_{0,n}(X,\beta)$. As X is convex, by dimensional reasons

$$(\bar{p}_X)_*\left(\alpha\cdot[\overline{M}_{0,n}(\mathbb{P}_X(V),\tilde{\beta}+qf_X)]^{\mathrm{virt}}\right)=\sum_{i=1}^{k}N_i[Z_i],$$

for some $N_i\in\mathbb{Q}$, possibly zero.
In particular, if $X=\mathbb{P}^1$, then $\overline{M}_{0,n}(X,\beta)$ is smooth and irreducible and therefore

$$(\bar{p}_X)_*\left(\alpha\cdot[\overline{M}_{0,n}(\mathbb{P}_X(V),\tilde{\beta}+qf_X)]^{\mathrm{virt}}\right)=N[\overline{M}_{0,n}(X,\beta)]$$

for some $N\in\mathbb{Q}$.

Definition 2.4.3. (i) In notations as above, let us consider the locally constant function

$$N_{X,W,\beta,q}(\alpha):\overline{M}_{0,n}(X,\beta)\to\mathbb{Q}$$

defined by the formula

$$(\bar{p}_X)_*\left(\alpha\cdot[\overline{M}_{0,n}(\mathbb{P}_X(V),\tilde{\beta}+qf_X)]^{\mathrm{virt}}\right)=N_{X,W,\beta,q}(\alpha)[\overline{M}_{0,n}(X,\beta)].$$

(ii) Let $X=\mathbb{P}^1$, $d=(d_1,...,d_{r-1})\in\mathbb{Z}^{r-1}$, $V=\mathcal{O}\oplus\mathcal{O}_{\mathbb{P}^1}(d_1)\oplus...\oplus\mathcal{O}_{\mathbb{P}^1}(d_{r-1})$, and $k=(k_1,...,k_n)\in\mathbb{N}^n$. Let $\xi_{\mathbb{P}_X(V)}=c_1(\mathcal{O}_{\mathbb{P}_X(V)}(1))$ and assume that $\alpha=ev_i^*\xi^{k_i}$ satisfies the dimension condition (2.7). We define $N(q,d,k)\in\mathbb{Q}$ by the formula

$$(\bar{p}_X)_*\left(\alpha\cdot[\overline{M}_{0,n}(\mathbb{P}_X(V),\tilde{1}+qf_X)]^{\mathrm{virt}}\right)=N(q,d,k)[\overline{M}_{0,n}(X,1)].$$

Remark 2.4.4. Let X and Y be smooth projective varieties, let $f : Y \to X$ be a morphism and $\beta_Y \in A_1(Y)$ such that $f_*\beta_Y = \beta$. Let W be a vector bundle on X. Then there exists an induced map $h : \mathbb{P}_Y(f^*V) \to \mathbb{P}_X(V)$. This induces a map $\bar{h} : \overline{M}_{0,n}(\mathbb{P}_Y(f^*V), \tilde{\beta}_Y + qf_Y) \to \overline{M}_{0,n}(\mathbb{P}_X(V), \tilde{\beta} + qf_X)$

Proposition 2.4.5. *Let X and Y be smooth projective convex varieties, let $f : Y \to X$ be a morphism and $\beta_Y \in A_1(Y)$ such that $f_*\beta_Y = \beta$. Let W be a vector bundle on X and $\alpha \in A^*(\overline{M}_{0,n}(\mathbb{P}_X(V), \tilde{\beta} + qf_X)$ satisfying the dimension condition (2.7). Then*
(i) $(\beta_Y, f^*W, q, \bar{h}^*\alpha)$ satisfies condition 2.7.
(ii) Let us assume that f is a closed embedding and let $Z_1, ..., Z_k$ be the connected components of $\overline{M}_{0,n}(X, \beta)$. For any $i \in \{1, ..., k\}$ we denote by V_i the open and closed component of $\overline{M}_{0,n}(Y, \beta_Y)$ which maps to Z_i. Then we have an equality

$$N_{Y,f^*W,\beta_Y,q}(\bar{h}^*\alpha)|_{V_i} = N_{X,W,\beta,q}(\alpha)|_{Z_i},\ \forall i \in \{1, ..., k\}.$$

Proof. (i) Let us consider the following diagram

$$\begin{array}{ccc} \overline{M}_{0,n}(\mathbb{P}_Y(f^*V), \tilde{\beta}_Y + qf_Y) & \xrightarrow{\bar{h}} & \overline{M}_{0,n}(\mathbb{P}_X(V), \tilde{\beta} + qf_X) \\ \downarrow{\scriptstyle \bar{p}_Y} & & \downarrow{\scriptstyle \bar{p}_X} \\ \overline{M}_{0,n}(Y, \beta_Y) & \xrightarrow{\bar{f}} & \overline{M}_{0,n}(X, \beta). \end{array}$$

Applying Proposition 2.1.5 we obtain that the relative obstruction theories $E_{\bar{f}}$ and $E_{\bar{h}}$ are compatible. This shows that $(\beta_Y, f^*W, q, \bar{h}^*\alpha)$ satisfies condition 2.7.
(ii)Without loss of generality we may assume that $\overline{M}_{0,n}(X, \beta)$ is irreducible. Let us denote its unique connected component by Z_1. By Remark 2.4.2 we have that

$$(\bar{p}_X)_*\left(\alpha \cdot [\overline{M}_{0,n}(\mathbb{P}_X(V), \tilde{\beta} + qf_X)]^{\text{virt}}\right) = N_{X,W,\beta,q}(\alpha)|_{Z_1}[Z_1].$$

Applying Proposition 1.3.1 (iii) to the above diagram we obtain that

$$\bar{f}^!((\bar{p}_X)_*\alpha\cdot[\overline{M}_{0,n}(\mathbb{P}_X(V), \tilde{\beta}+qf_X)]^{\text{virt}}) = (\bar{p}_Y)_*\bar{f}^!(\alpha\cdot[\overline{M}_{0,n}(\mathbb{P}_X(V), \beta+qf_X)]^{\text{virt}}). \tag{2.8}$$

As the obstruction theories $\bar{p}_Y^* E_{\bar{f}}$ and $E_{\bar{h}}$ are compatible by Proposition 2.1.5, we obtain that

$$\bar{f}^!_{\mathfrak{E}_{\bar{f}}} = \bar{h}^!_{\mathfrak{E}_{\bar{h}}}. \tag{2.9}$$

This shows that

$$\bar{f}^!\left(\alpha\cdot[\overline{M}_{0,n}(\mathbb{P}_X(V),\tilde{\beta}+qf_X)]^{\mathrm{virt}}\right)=\bar{h}^*\alpha\cdot\bar{f}^![\overline{M}_{0,n}(\mathbb{P}_X(V),\tilde{\beta}+qf_X)]^{\mathrm{virt}}.$$

By Corollary 1.3.7 we obtain $\bar{f}^![\overline{M}_{0,n}(X,\beta)]^{\mathrm{virt}}=[\overline{M}_{0,n}(Y,\beta_Y)]$ and using moreover relation (2.9) we get

$$\bar{f}^![\overline{M}_{0,n}(\mathbb{P}_X(V),\tilde{\beta}+qf_X)]^{\mathrm{virt}}=[\overline{M}_{0,n}(\mathbb{P}_Y(f^*V),\beta_Y+qf_Y)]^{\mathrm{virt}}. \qquad (2.10)$$

From equations (2.8) and (2.10) we see that

$$N_{X,W,\beta,q}(\alpha)|_{Z_1}=(\bar{p}_Y)_*\left((\bar{h}^*\alpha)\cdot\overline{M}_{0,n}(\mathbb{P}_Y(f^*V),\tilde{\beta_Y}+qf_Y)\right). \qquad (2.11)$$

This concludes the proof. □

Corollary 2.4.6. *We follow the notations of Proposition 2.4.5. Let*

$$\alpha:=\prod_{i=1}^{n}ev_i^*\xi^{k_i}\in A^*(\overline{M}_{0,n}(\mathbb{P}_X(V),\tilde{\beta}+qf_X))$$

satisfy condition (2.7). Then

$$(\bar{p}_X)_*\left(\alpha\cdot[\overline{M}_{0,n}(\mathbb{P}_X(V),\tilde{\beta}+qf_X)]^{\mathrm{virt}}\right)=N(q,d,k)[\overline{M}_{0,n}(X,\beta)]^{\mathrm{virt}},$$

where $d=(\beta\cdot c_1,...,\beta\cdot c_n)$.

Proof. In notations as above, let us fix j and look at $(C,x_1,...,x_n,f)\in Z_j$. As X is convex, it suffices to look at irreducible curves. We apply Proposition 2.4.5 with $Y=C$ and $\beta=1$, $f:C\to X$ and $h:\mathbb{P}_C(f^*V)\to\mathbb{P}_X(V)$ the map induced by f. As $\bar{i}$ does not stabilize any curve, the induced diagram is cartesian. Since C is isomorphic to $\mathbb{P}^1$, $f^*V\simeq\oplus_{i=1}^r\mathcal{O}(e_i)$, where $e_i:=g^*c_i$. By the projection formula $g_*c_1(\mathcal{O}(e_i))=\beta\cdot c_i$, $\forall i\in\{1,...,r\}$ and therefore $e_i=d_i$. It can be easily seen that $h^*\xi_{\mathbb{P}_X(V)}=\xi_{\mathbb{P}^1}$ and therefore $N_j=N(q,d,k)$. □

Let us now extend the result to a more general base X.

Setting 2.4.7. As before, we consider a convex space $\mathbb{P}$ and $g:X\to\mathbb{P}$ be a closed embedding of a smooth projective variety X in $\mathbb{P}$. Let V be a vector bundle on X such that there exists a vector bundle $W\oplus\mathbb{C}_{\mathbb{P}}$ on $\mathbb{P}$ with $V=g^*(W\oplus\mathbb{C}_{\mathbb{P}})$ and let $p:\mathbb{P}_X(V)\to X$ be the associated projective bundle. In notations as above we have an induced map $\bar{p}_X:\overline{M}_{0,n}(\mathbb{P}_X(V),\tilde{\beta}+qf_X)\to\overline{M}_{0,n}(X,\beta)$.

Corollary 2.4.8. *In notations as in 2.4.7, let*

$$\alpha := \prod_{i=1}^{n} ev_i^* \xi^{k_i} \in A^*(\overline{M}_{0,n}(\mathbb{P}_X(V), \tilde{\beta} + qf_X))$$

satisfy condition (2.7). Then we have that

$$(\bar{p}_X)_* \left(\alpha \cdot [\overline{M}_{0,n}(\mathbb{P}_X(V), \tilde{\beta} + qf_X)]^{\mathrm{virt}}\right) = N(q,d,k)[\overline{M}_{0,n}(X,\beta)]^{\mathrm{virt}},$$

where $d = (\beta \cdot c_1, ..., \beta \cdot c_n)$.

Proof. The proof follows from Proposition 2.1.5 and Corollary 2.4.6. □

Remark 2.4.9. In Corollary 2.4.8 we have shown that we can compute GW invariants of projective bundles in terms of GW invariants of the base X and GW invariants of a projective bundle over $\mathbb{P}^1$. The latter can be analyzed using toric methods (see [7], [8]). More precisely, we can compute in this way GW-invariants of $\mathbb{P}_X(V)$ with at least $\mathrm{vdim}\overline{M}_{0,n}(X,\beta)$ insertions that are pull-backs from X.

2.5 Conservation of number for virtually smooth morphisms

Let us recall Fulton's principle of conservation of number (see Proposition 10.2 [10]).

Proposition 2.5.1. *Let* $f : F \to G$ *be a proper morphism,* G *an* m*-dimensional irreducible scheme. Let* $i_P : P \to G$ *be a point in* G *and* α *be an* m*-dimensional cycle on* F*. Then the cycle classes* $\alpha_P := i_P^* \alpha$ *have the same degree.*

In this section we will give a version of this principle in the situation when $f : F \to G$ is a virtually smooth morphisms.
As a consequence of the conservation of number principle we give a proof of the fact that the virtual Euler characteristic is constant in virtually smooth families (see Definition 1.4.4). This statement is a generalization of Proposition 4.14 in [11] of Fantechi and Göttsche.
As in the section on virtual push-forwards we work with *homology* rather than with Chow groups.

Let us now state the conservation of number principle for virtually smooth morphisms.

Proposition 2.5.2. *Let G be a connected stack of pure dimension and let $f : F \to G$ be a proper virtually smooth morphism of stacks (see Definition 1.4.4) of virtual relative dimension d. Let $i : P \to X$ be a point in X and let us consider $\alpha \in A^d(F)$. Then, the number*

$$i^*\alpha \cdot [X_P]$$

is constant.

Proof. Let P be any point of G and let us consider the following cartesian diagram

$$\begin{array}{ccc} X_P & \xrightarrow{j} & F \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ P & \xrightarrow{i} & G \end{array}$$

where X_P is the fiber of X over P and $g : X_P \to P$ is the map induced by f. By Theorem 1.4.6 we have that there exists $n \in \mathbb{Q}$ such that

$$f_*\alpha \cdot [F]^{\mathrm{virt}} = n[G]. \tag{2.12}$$

By Proposition 1.3.1 (i) we have that

$$i^! f_*\alpha \cdot [F]^{\mathrm{virt}} = g_* j^! \alpha \cdot [F]^{\mathrm{virt}}.$$

Using equation (2.12) we obtain that

$$i^! f_*\alpha \cdot [F]^{\mathrm{virt}} = n[P]$$

and by the commutativity of virtual pull-backs (see Theorem 1.3.9) we have that $j^![F]^{\mathrm{virt}} = [X_P]^{\mathrm{virt}}$ and therefore

$$j^!\alpha \cdot [F]^{\mathrm{virt}} = \alpha \cdot [X_P]^{\mathrm{virt}}.$$

The three relations above show that the the number

$$i^*\alpha \cdot [X_P]$$

is constant. ☐

Definition 2.5.3. Let $f : F \to G$ be a morphism of proper stacks with a 1-perfect obstruction theory $E_{F/G}$ which admits a global resolution of $E_{F/G}$ as a complex of vector bundles $[E^1 \to E^0]$ (e.g. if F can be embedded as closed substack in a separated stack which is smooth over G.) We denote by $[E_0 \to E_1]$ the dual complex and by d the expected dimension $d := rkE_{F/G} = rkE^0 - rkE^1$. We denote the class $[E_0] - [E_1] \in K^0(F)$ by $T^{\text{virt}}_{F/G}$ and we call it the virtual relative tangent of f.

Definition 2.5.4. Let $f : F \to G$ be a morphism of stacks as before. We define the relative virtual Euler characteristic of f to be the virtual Chern number $e^{\text{virt}}(F/G) := c_d(T^{\text{virt}}_{F/G})$.

Remark 2.5.5. The definition is coherent with Definition 4.2 in [11] by Corollary 4.8 (Hopf index theorem) in [11].

Corollary 2.5.6. *Let G be a connected stack of pure dimension and let $f : F \to G$ be a proper virtually smooth morphism of stacks (see Definition 1.4.4). Then, all the fibers of f have the same virtual Euler characteristic.*

Proof. We use the above proposition with $\alpha := c_d(T^{\text{virt}}_{F/G})$. □

Remark 2.5.7. Taking G to be smooth we obtain the conservation of number principle in families of virtually smooth schemes (see definition 3.14 in [11]) which is Corollary 3.16 in [11].
Taking G to be smooth we obtain that the virtual Euler characteristic is constant in a family of virtually smooth schemes which is the statement of Proposition 4.14 in [11].

2.6 Quantum hyperplane section principle for projective bundles

Let us briefly recall the Quantum hyperplane section principle. Let us consider a smooth projective variety Y and smooth complete intersection $i : X \hookrightarrow Y$ which is obtained by cutting out Y by r hyperplanes $H_1, ..., H_r$. Let us denote by $V := \mathcal{O}(H_1) \oplus ... \oplus \mathcal{O}(H_r)$ the splitted vector bundle on $\mathbb{P}$ associated to these hyperplanes. If $V_{X/Y}$ is convex (i.e. $H^1(C, f^*V_{X/Y}) = 0$ for any stable map $f : C \to X$) of fixed homology type β, then by the work of Kim, Kresch and Pantev ([19]) we have that

$$c_d(\mathcal{R}^0\pi_* ev^* V_{X/Y}) \cdot [\bar{M}_{0,n}(Y, i_*\beta)]^{\text{virt}} = i_*[\bar{M}_{0,n}(X, \beta)]^{\text{virt}},$$

where d is the rank of the bundle $\mathcal{R}^0\pi_*ev^*V_{X/Y}$. The Quantum Lefschetz formula of Coates and Givental ([5]) gives a precise formula which computes the Gromov-Witten invariants of X in terms of Gromov-Witten invariants of Y.
In this thesis we are intersted in a similar situation. We consider a smooth morphism $p : \mathbb{P} \to X$ from a projective bundle to its base. Let $\Omega_{\mathbb{P}/X}$ be the relative cotangent of p. Then we relate Gromov-Witten invariants of $\mathbb{P}$ twisted by the relative cotangent bundle with Gromov-Witten invariants of X. The basic idea is to apply the virtual machinery in order to reduce the computation to a smaller relative toric variety.

Construction 2.6.1. Let X be a smooth projective variety and E a rank r bundle on X. In notations as in the beginning of Section 2.1 we consider the K-theoretic element $E_{g,n,\beta}$ on $\overline{M}_{g,n}(X,\beta)$ defined by

$$E_{g,n,\beta} := [\mathcal{R}^0\pi_*ev^*E] - [\mathcal{R}^1\pi_*ev^*E].$$

Let r_0 denote the rank of $E^0 := \mathcal{R}^0\pi_*ev^*E$ and similarly, let r_1 denote the rank of $E^1 := \mathcal{R}^1\pi_*ev^*E$. Using the fiberwise $\mathbb{C}^*$-action on $E^i_{g,n,\beta}$ we define

$$e_t(E^i) := t^{r_i} + t^{r_i-1}c_1(E^i) + ... + c_{r_i}(E^i)$$

for any $i \in \{0,1\}$. We call $e_t(E^i)$ the $\mathbb{C}^*$-equivariant Euler class of E^i. Moreover, we define the the $\mathbb{C}^*$-equivariant Euler class of $E_{g,n,\beta}$ as

$$e_t(E_{g,n,\beta}) := \frac{e_t(E^0)}{e_t(E^1)} \in A^*(\overline{M}_{g,n}(X,\beta))(t).$$

If $\mathbb{T}$ is another torus which acts equivariantly on $\overline{M}_{g,n}(X,\beta)$, E^0, E^1, then we define

$$e^{\mathbb{T}}_t(E^i) := t^{r_i} + t^{r_i-1}c^{\mathbb{T}}_1(E^i) + ... + c^{\mathbb{T}}_{r_i}(E^i)$$

for any $i \in \{0,1\}$. We call $e_t(E^i)$ the $\mathbb{C}^*$-equivariant Euler class of E^i. Moreover, we define the the $\mathbb{C}^* \times \mathbb{T}$-equivariant Euler class of $E_{g,n,\beta}$ as

$$e_t(E_{g,n,\beta}) := \frac{e^{\mathbb{T}}_t(E^0)}{e^{\mathbb{T}}_t(E^1)} \in A^*_{\mathbb{T}}(\overline{M}_{g,n}(X,\beta))(t).$$

Convention 2.6.2. Let X be a smooth projective variety, and let V a vector bundle over X. We denote by $\mathbb{P}_X(V)$ the associated projective bundle. Let $\beta \in H_2(X)$. By abuse of notation, we indicate a lifting of β again by β. We recall that a lifting of β is a curve class in $\mathbb{P}_X(V)$ which maps to β.

Proposition 2.6.3. *(B. Kim [18]) Let $\mathbb{P}_X(V)$ be a splitted projective bundle of projective rank r over X. Then*

$$\bar{p}_* \sum_{\tilde{\beta}\mapsto\beta} (e_t(T_p)_{g,n,\tilde{\beta}} \cdot [M_{g,n}(\mathbb{P}_X(V), \tilde{\beta})]^{\text{virt}}) = (r+1)[M_{g,n}(X,\beta)]^{\text{virt}}$$

modulo t.

Proof. Let $\mathbb{T} := (\mathbb{C}^*)^{r+1}$ and let $E := T_p$. Then $\mathbb{T}$ induces an equivariant action on $\overline{M}_{g,n}(\mathbb{P}_X(V),\beta)$, E^0, E^1. As the $\mathbb{T}$-fixed components of $\mathbb{P}_X$ are $X_i = \mathbb{P}(L_i)$, for $i = 0, ..., r$ we see that the induced action of $\mathbb{T}$ on $\overline{M}_{g,n}(\mathbb{P}_X(V),\beta)$ fixes maps $(C, x_1, ..., x_n, f)$ such that $f(C)$ is a curve with components included in X_i, glued to components included in the fibers of p. All the marked points map to X_i. To such maps we can associate a marked graph Γ as in [14] with the additional marking $\beta_v \in H_2(\mathbb{P}_X(V))$ for each vertex v. For any graph Γ we have an evaluation map

$$\prod_{flags} e_F : \prod_{vertices} M_{g(v),val(v)}(\mathbb{P}_X(V), \beta_v) \to \prod_{flags} (\mathbb{P}_X(V))$$

Let us consider

$$\bar{M}_\Gamma := (\prod_{flags} e_F)^{-1} (\prod_{edges} \Delta_{\mathbb{P}_X(V)})$$

where $\Delta_{\mathbb{P}_X(V)}$ is the diagonal associated to an edge e is the diagonal of $\mathbb{P}_X(V) \times \mathbb{P}_X(V)$ associated to two flags of e. By the virtual localization formula we have that

$$e_t^{\mathbb{T}}(T_p)_{g,n,\tilde{\beta}} \cdot [\overline{M}_{g,n}(\mathbb{P}_X(V),\tilde{\beta})]^{\text{virt}} = \sum_{\Gamma} e_t^{\mathbb{T}}(T_p)_{g,n,\tilde{\beta}} \cdot \frac{[M_\Gamma]^{\text{virt}}/\mathbf{A}_\Gamma}{e^{\mathbb{T}}(N_\Gamma^{\text{virt}})}$$

in $A_{\mathbb{T}}^*(\overline{M}_{g,n}(X,\beta))(t) \otimes \mathbb{C}[[\lambda^{-1}]]$, where $\mathbb{C}[\lambda] = A_{\mathbb{T}}^*(pt)$. In the above formula $\mathbf{A}_\Gamma$ indicates the semidirect product of $Aut(\gamma)$ and $\prod_{edges} \mathbb{Z}/d_e$.
Let E be a vector bundle over $\mathbb{P}_X(V)$, then we formulate $e_t^{\mathbb{T}}(E_{g,n,\tilde{\beta}})_{|\bar{M}_\Gamma}$ in terms of Euler classes of vector bundles on $\overline{M}_{g,n}(\mathbb{P}_X(V),\tilde{\beta})$ which we indicate by their fibers:

$$\frac{e_t^{\mathbb{T}}(H^0(C,f^*E))}{e_t^{\mathbb{T}}(H^1(C,f^*E))} = \prod_{vertices} \frac{e_t^{\mathbb{T}}(H^0(C_v,f^*E))}{e_t^{\mathbb{T}}(H^1(C_v,f^*E))} \prod_{edges} \frac{e_t^{\mathbb{T}}(H^0(C_e,f^*E))}{e_t^{\mathbb{T}}(H^1(C_e,f^*E))} \prod_{flags} \frac{1}{e_t^{\mathbb{T}}(e_F^*E)}. \tag{2.13}$$

Using the Euler sequence of the relative tangent bundle we see that $e^{\mathbb{T}}(T_p)_{g,n,\beta}$ has zero $\mathbb{T}$-weight from $e^{\mathbb{T}}(H^0(C_e, T_p)$. All the other terms from vertices and flags which appear on the right hand side of equation (2.13) have non-zero $\mathbb{T}$-weights. This shows that the left hand side of equation (2.13) is an element in $A^*_{\mathbb{T}}(\overline{M}_{g,n}(X,\beta))[t] \otimes \mathbb{C}[[\lambda^{-1}]]$ and modulo t it vanishes, unless Γ has one vertex. The contribution of each of the $r+1$ one vertex graphs is modulo t equal to $\overline{M}_{g,n}(X,\beta)$. This concludes the proof. □

Proposition 2.6.4. *Let X be a smooth projective variety such that $M_{0,n}(X,\beta)$ is connected for some $\beta \in H_2(X)$. Let $p : \mathbb{P}_X(V)$ be a projective bundle of projective rank r over X. Then*

$$\bar{p}_* \sum_{\tilde{\beta} \mapsto \beta} (c_d(T_p^{\mathrm{virt}}) \cdot [M_{0,n}(\mathbb{P}_X(V), \tilde{\beta})]^{\mathrm{virt}}) = (r+1)[M_{0,n}(X,\beta)]^{\mathrm{virt}}. \qquad (2.14)$$

Proof. By Proposition 2.5.6 it is enough to show the claim for a smooth map $(C, x_1, ...x_n, f) \in M_{0,n}(X,\beta)$. As is in the proof of Corollary 2.4.6 we have a Cartesian diagram

$$\begin{array}{ccc} \overline{M}_{0,n}(\mathbb{P}_C(f^*V), 1 + qf_C) & \xrightarrow{\bar{h}} & \overline{M}_{0,n}(\mathbb{P}_X(V), \beta + qf_X) \\ \downarrow{\scriptstyle \bar{q}} & & \downarrow{\scriptstyle \bar{p}} \\ \overline{M}_{0,n}(C,1) & \xrightarrow{\bar{f}} & \overline{M}_{0,n}(X,\beta) \end{array}$$

and therefore for any β and $q \in \mathbb{Z}$

$$\bar{p}_*(c_d(T_p^{\mathrm{virt}}) \cdot [M_{0,n}(\mathbb{P}_X(V), \beta + qf_X)]^{\mathrm{virt}}) =$$
$$\bar{q}_*(c_d(T_q^{\mathrm{virt}}) \cdot [M_{0,n}(\mathbb{P}_C(f^*V), 1 + qf_C)]^{\mathrm{virt}}).$$

Summing on both sides over all curve classes $\tilde{1} := 1 + qf_C$ and respectively $\tilde{\beta} := \beta + qf_X$ we obtain that

$$\bar{p}_* \sum_{\tilde{\beta} \mapsto \beta} (c_d(T_p^{\mathrm{virt}}) \cdot [M_{0,n}(\mathbb{P}_X(V), \tilde{\beta})]^{\mathrm{virt}}) = \bar{q}_* \sum_{\tilde{1} \mapsto 1} (c_d(T_q^{\mathrm{virt}}) \cdot [M_{0,n}(\mathbb{P}_C(f^*V), \tilde{1})]^{\mathrm{virt}}).$$

But over a smooth map C, the vector bundle f^*V splits as a direct sum of line bundles and therefore the conclusion holds by Proposition 2.6.3 for C. □

Remark 2.6.5. If X is a flag variety, then $M_{0,n}(X, \beta)$ is connected for any $\beta \in H_2(X)$ (see [20]).

Corollary 2.6.6. *Let $i : Y \to X$ be a closed embedding of smooth projective varieties and let V be any vector bundle on X. Then*

$$\bar{p}_* \sum_{\beta \mapsto d} \sum_{\tilde{\beta} \mapsto \beta} (c_d(T_p^{\text{virt}}) \cdot [M_{0,n}(\mathbb{P}_Y(i^*V), \tilde{\beta})]^{\text{virt}}) =$$
$$\bar{q}_* \sum_{\tilde{d} \mapsto d} (c_d(T_q^{\text{virt}}) \cdot [M_{0,n}(\mathbb{P}_X(V), \tilde{d})]^{\text{virt}}).$$

Proof. The proof follows from Proposition 2.1.5 and Corollary 2.4.6. □

Remark 2.6.7. The scalar on the right hand side of equation 2.14 is equal to the Euler characteristic of the fiber of p. We do not know if this phenomenon apprears for any other smooth morphism $p : \mathbb{P} \to X$.

Remark 2.6.8. Let us note that although this statement is different from the one of Proposition 2.5.6. For a general choice of a curve class $\tilde{\beta}$, the virtual relative tangent is not equal to $(T_p)_{g,n,\tilde{\beta}}$. This happens because, in general the map $p : \mathbb{P}_X \to X$ stabilizes the curve and therefore the virtual relative tangent has a "contribution" from the deformation of the curve.

2.7 Costello's push-forward formula

We can use the basic properties of virtual pull-backs (push-forward and functoriality) to give a short proof of Costello's push-forward formula in [6]. We recall the set-up.

Let us consider a cartesian diagram

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ \mathfrak{M}_1 & \xrightarrow{g} & \mathfrak{M}_2 \end{array}$$

such that

1. f is a proper morphism;

2. $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are Artin stacks of the same pure dimension;

3. g is DM-type morphism of degree d;

4. F and G are DM-stacks equipped with perfect relative obstruction theories $E_{F/\mathfrak{M}_1}$ and $E_{G/\mathfrak{M}_2}$ inducing virtual classes $[F]^{\text{virt}}$ and $[G]^{\text{virt}}$;

5. $E_{F/\mathfrak{M}_1} \simeq f^* E_{G/\mathfrak{M}_2}$.

Proposition 2.7.1. *(Costello, [6], Theorem 5.0.1.) Under the assumptions above,* $f_*[F]^{\text{virt}} = d[G]^{\text{virt}}$.

Proof. As $E_{F/\mathfrak{M}_1}$ and $E_{G/\mathfrak{M}_2}$ are perfect, p_1 and p_2 induce pull-back morphisms and $E_{F/\mathfrak{M}_1} = f^* E_{G/\mathfrak{M}_2}$ implies $p_1^!$ is induced by $p_2^!$. Applying Theorem 1.3.1 (i) we get $f_* p_1^! [\mathfrak{M}_1] = p_2^! g_* [\mathfrak{M}_1]$. Using the fact that $g_*[\mathfrak{M}_1] = d[\mathfrak{M}_2]$ and the definition of virtual classes we get

$$f_*[F]^{\text{virt}} = d[G]^{\text{virt}}.$$

$\square$

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