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Global Aspects In Quantum Field Theories.

Thesis submitted for the degree of
Doctor Philosophiae.

Academic year 1988-1989

Candidate
S.P.Sorella

Advisor
Prof. C.Becchi

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Contents.

1. Introduction.

Nowadays, the understanding of the global aspects of the rich class of the models which populate the world of theoretical physics is one of the most fascinating problems which requires hard effort and common work between physicists of different areas.

By global aspects are meant all the features which, in some way, are related to the global properties of the considered model; as it happens, for example, in the two-dimensional conformal theories on a Riemann surface, in the recently proposed topological gauge theories in four-dimensions and in the three-dimensional Chern-Simons model.

The aim of this thesis is to show how these global aspects can be explored by using the methods and the techniques of the quantum field theory.

We begin by considering a two-dimensional bosonic sigma model defined on a generic Riemannian space M. Here the global aspect is represented by the Friedan operator [1] which characterizes the transition functions between different charts and which can be used to define global quantities on M.

From a field theory viewpoint the Friedan operator corresponds to a non-linear symmetry which, by the help of the BRS technique, can be translated into a Slavnov-Taylor identity.

This identity will be used to discuss the ultraviolet stability of the model and to identify the parameters which affect the metric tensor.

As a second example, we consider the recently proposed topological σ-model [7]. Here the idea is to build a model in which the observables are of a global topological nature; for instance, in the case of the topological σ-model the observables are given by the De Rham cohomology of the manifold on which the model is defined.

In a field theory realization, the observables of the topological σ-model can be recovered through the so-called basic cohomology [10], which is a kind of restriction of the usual BRS cohomology and which can be defined by combining the original BRS symmetry [7] with the Friedan symmetry.

The third example considered is the Chern-Simons model in three dimensions.

The Chern-Simons action, being a three-form, is metric independent and it is not only the starting point for computing topological quantities of a three manifolds, but it is also expected to be strongly related with the two-dimensional conformal models.

However, also at the perturbative level [14], the Chern-Simons model has numerous aspects which render it interesting.

This is due to the expected finiteness [15] of the perturbative expansion and to its rich class of symmetries. We will see, indeed, that these symmetries form a supersymmetry algebra and can be used to characterize the perturbative expansion of the model. Also if a complete proof of the finiteness of the Chern-Simons model has not yet been established, it is interesting to note that these symmetries, being anomaly-free, impose quite strong conditions on the possible counterterms.

The bosonic σ-model.

1. The Friedan symmetry.

Let Ξ be the two-dimensional flat space-time and M a compact Riemannian n-dimensional manifold. The field of the model is defined by the map :

$$\Phi : \Xi \rightarrow M \tag{1.1}$$

and described locally by the functions $\phi^i(x)$.

The action of the model reads:

$$S(\phi) = \frac{1}{2}\int d^2x \; g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j \tag{1.2}$$

Introducing the normal coordinates [1] , the field ϕ^i splits as:

$$\phi^i(x) = \varphi^i + \pi^i(\varphi,\xi) \tag{1.3}$$

where φ^i is the classical constant background and $\xi^i(x)$ is taken as the quantum field. The functions $\pi^i(\varphi,\xi)$ are formal power series in ξ^i :

$$\pi^i(\varphi,\xi) = \xi^i - \sum_{n=2}^{\infty}\frac{1}{n!}\xi^{j_1}.....\xi^{j_n}\Gamma^i_{j_1....j_n}(\varphi) \tag{1.4}$$

with

$$\Gamma^i_{j_1....j_n}(\varphi) = \tilde{\nabla}_{j_1}.....\tilde{\nabla}_{j_n-2}\Gamma^i_{j_{n-1},j_n}(\varphi) \tag{1.5}$$

where $\tilde{\nabla}$ indicates that the covariant derivative is taken only with respect to the lower indices of the Riemann connection $\Gamma^i_{jk}(\varphi)$. The non-linear splitting (1.3) allows us to introduce a non-linear shift symmetry. This symmetry derives from the possibility of varying the fields φ^i and ξ^i in a way such that the total variation of ϕ^i vanishes. Let us consider, indeed, the transformations :

$$\begin{aligned} \delta_v\varphi^i &= v^i \\ \delta_v\xi^i &= v^k F^i_k(\varphi,\xi) \end{aligned} \tag{1.6}$$

where v^i are constant parameters and $F^i_k(\varphi,\xi)$ is determined by the requirement that ϕ^i

be invariant, i.e. :

$$\delta^i_k + \frac{\partial \pi^i}{\partial \varphi^k} + F^l_k \frac{\partial \pi^i}{\partial \xi^l} = 0 \tag{1.7}$$

The equation (1.7) implies that :

$$\left(\frac{\partial F^i_k}{\partial \varphi^j} - \frac{\partial F^i_j}{\partial \varphi^k}\right) + \left(\frac{\partial F^i_k}{\partial \xi^l} F^l_j - \frac{\partial F^i_j}{\partial \xi^l} F^l_k\right) = 0 \tag{1.8}$$

and, by (1.4) :

$$F^i_k(\varphi, 0) = -\delta^i_k \tag{1.9}$$

The condition (1.8) shows that the transformations (1.6) are Abelian. The field $\xi^i(x)$ is a vector belonging to the tangent space T_φ in the point of M whose coordinates are given by φ^i. From $\delta_v \phi^i = 0$ it follows that

$$\delta_v S(\phi) = 0 \tag{1.10}$$

It is advantageous to rewrite the action $S(\phi)$ as:

$$S(\phi) = \frac{1}{2} \int d^2x \; \tilde{g}_{ij}(\varphi, \xi) \partial_\mu \xi^i \partial^\mu \xi^j \tag{1.11}$$

where

$$\tilde{g}_{ij}(\varphi, \xi) = g_{mn}(\phi) \frac{\partial \phi^m}{\partial \xi^i} \frac{\partial \phi^n}{\partial \xi^j} \tag{1.12}$$

The metric $\tilde{g}_{ij}(\varphi, \xi)$ satisfies the equation:

$$\frac{\partial \tilde{g}_{mn}}{\partial \varphi^i} + F^k_i \frac{\partial \tilde{g}_{mn}}{\partial \varphi^k} + \tilde{g}_{kn} \frac{\partial F^k_i}{\partial \xi^m} + \tilde{g}_{mk} \frac{\partial F^k_i}{\partial \xi^n} = 0 \tag{1.13}$$

As shown by Friedan [1], $F^i_k(\varphi, \xi)$ has a geometrical interpretation as a " non-linear connection " in the tangent space T_φ and can be used to define global quantities on M [1,2]. Indeed, in general, if a tensor-valued function $\tilde{\vartheta}_{i_1 \ldots i_n}(\varphi, \xi)$ satisfies the condition :

$$\frac{\partial \tilde{\vartheta}_{i_1 \ldots i_n}}{\partial \varphi^k} + F^q_k \frac{\partial \tilde{\vartheta}_{i_1 \ldots i_n}}{\partial \xi^q} + \tilde{\vartheta}_{q \ldots i_n} \frac{\partial F^q_k}{\partial \xi^{i_1}} + \ldots\ldots\ldots + \tilde{\vartheta}_{i_1 \ldots i_{n-1} q} \frac{\partial F^q_k}{\partial \xi^n} = 0 \tag{1.14}$$

then $\tilde{\vartheta}_{i_1 \ldots i_n}$ is the expression in coordinates of a unique globally defined tensor $\tilde{\vartheta}$ on M.

This can be understood in the following way: the equation (1.14) is a first-order differential equation and its solutions are uniquely determined once an initial condition is specified. If one imposes that

$$\tilde{\vartheta}_{i_1\ldots i_n}(\varphi,\xi=0)=\vartheta_{i_1\ldots i_n}(\varphi) \tag{1.15}$$

then, as can be easily verified, the unique solution of (1.14) is given by [1]:

$$\tilde{\vartheta}_{i_1\ldots i_n}(\varphi,\xi)=\frac{\partial\phi^{j_1}}{\partial\xi^{i_1}}\;\cdots\cdots\;\frac{\partial\phi^{j_n}}{\partial\xi^{i_n}}\vartheta_{j_1\ldots j_n}(\phi) \tag{1.16}$$

and, if we require that $\vartheta_{i_1\ldots i_n}(\varphi)$ is a tensor on M, then (1.16) is nothing other than the transformation law of a tensor under the change of coordinates given by (1.3).

One sees that the Friedan equation (1.14) represents the infinitesimal version of the transition functions; equations (1.13) and (1.10) tell us that the metric $\bar{g}(\varphi,\xi)$ and the action S are globally defined.

2. BRS transformations and Slavnov-Taylor identities.

The transformations in (1.6) being non-linear, we need to translate them into a BRS identity [2,3] in order to discuss their quantum implementability.

This is easily achieved by introducing the anticommuting dimensionless constant ghost field c^i and the operator s, defined by

$$
\begin{aligned}
s\phi^i &= c^i \\
s\xi^i &= F^i_j c^j \\
sc^i &= 0
\end{aligned}
\tag{2.1}
$$

which, due to (1.8) is nilpotent. By adding to the action $S(\phi)$ a source term

$$
\Sigma = S(\phi) + \int d^2x \gamma_i F^i_k c^k \tag{2.2}
$$

one has the Slavnov-Taylor identity:

$$
c^i \frac{\partial \Sigma}{\partial \varphi^i} + \int d^2x \frac{\delta \Sigma}{\delta \gamma_i} \frac{\delta \Sigma}{\delta \xi^i} = 0 \tag{2.3}
$$

The sources γ_i have canonical dimension two, and c^i have a positive unit of Faddeev-Popov charge, so that γ_i carry a negative unit of Faddeev-Popov charge. The quantum implementability of equation (2.3) is ensured if we can prove that the action is stable and the symmetry is anomaly free. In this case, the absence of an anomaly is automatically ensured by the existence of an invariant regularization scheme, namely the dimensional one, which reduces the problem of the quantum extension of equation (2.3) to the stability analysis, i.e. to the characterization of the cohomology of the linearized BRS nilpotent operator

$$
\begin{aligned}
D_\Sigma &= c^i \frac{\partial}{\partial \varphi^i} + \int d^2x \left(\frac{\delta \Sigma}{\delta \gamma_i} \frac{\delta}{\delta \xi^i} + \frac{\delta \Sigma}{\delta \xi^i} \frac{\delta}{\delta \gamma_i} \right) \\
D_\Sigma D_\Sigma &= 0
\end{aligned}
\tag{2.4}
$$

in the space of the integrated dimension-two local functionals with zero Faddeev-Popov charge.

One has to note that, due the ill-defined nature of the propagator, the model needs an infrared regulator which breaks the split symmetry. However, in a minimal renormalization scheme, the breaking introduced by the mass term remains soft.

A convenient infrared term is given by

$$\frac{1}{2}g_{ij}(\varphi)\xi^i\xi^j \tag{2.5}$$

where

$$g_{ij}(\varphi) = \tilde{g}_{ij}(\varphi, \xi = 0) \tag{2.6}$$

The term (2.5) is not globally defined, however, thanks to the properties of $F^i_k(\varphi, \xi)$ (see app. A); its BRS variation is

$$s(\frac{g_{ij}(\varphi)\xi^i\xi^j}{2}) = -c^i g_{ij}(\varphi)\xi^j \tag{2.7}$$

If one considers the action

$$\Sigma_m = \Sigma - \int d^2x \; \frac{m^2}{2} g_{ij}(\varphi)\xi^i\xi^j \tag{2.8}$$

one has the modified Slavnov-Taylor identity:

$$c^i \frac{\partial \Sigma_m}{\partial \varphi^i} + \int d^2x \frac{\delta \Sigma_m}{\delta \gamma_i}\frac{\delta \Sigma_m}{\delta \xi^i} = m^2 c^i \int d^2x \; g_{ij}(\varphi)\xi^j \tag{2.9}$$

This identity can be used to study the existence of the infrared limit of quantities invariant under the Friedan symmetry. Without entering into details, one can say that, up to now, the existence of the infrared limit has been proved only for σ-models defined on compact homogeneous coset spaces [4]. For a general Riemannian σ-model the situation is quite difficult; there the infrared limit has to be taken after averaging on the background field φ; there is some evidence for a possible factorization of the infrared divergences [5], but a final result has not yet been established.

3. From functionals to functions.

To begin the stability analysis of the Slavnov-Taylor identity (2.3) let us study the action of the nilpotent operator D_Σ on the space of the integrated dimension-two local functionals which depend on the variables ξ^i, φ^i, c^i and γ_i. A generic element of this space reads:

$$X = \int d^2x(\ \gamma_i P^i(\varphi, c, \xi) + B_{ij}(\varphi, c, \xi)\partial_\mu \xi^i \partial^\mu \xi^j \ ) \tag{3.1}$$

where P^i and B_{ij} are dimensionless formal power series in the quantum field ξ^i. Choosing as independent variables:

$$c^i, \ \gamma_i, \ \xi^i, \quad \xi^{i\mu} = \partial^\mu \xi^i \tag{3.2}$$

the operator D_Σ induces in the space $X(x)$ of local functions:

$$X(x) = \gamma_i P^i(\varphi, c, \xi) + B_{ij}(\varphi, c, \xi)\xi^i_\mu \xi^{j\mu} \tag{3.3}$$

the ordinary differential operator:

$$\begin{aligned} d_\Sigma \ &= c^i \frac{\partial}{\partial \varphi^i} + F^i_k c^k \frac{\partial}{\partial \xi^i} + \frac{\partial F^i_k}{\partial \xi^l} \xi^l_\mu c^k \frac{\partial}{\partial \xi^i_\mu} \\ &+ \gamma_l \frac{\partial F^l_k}{\partial \xi^i} c^k \frac{\partial}{\partial \gamma_i} + \frac{1}{2} \frac{\partial \tilde{g}_{mn}}{\partial \xi^i} \xi^m_\mu \xi^{n\mu} \frac{\partial}{\partial \gamma_i} + \tilde{g}_{mn} \xi^{n\mu} \xi^i_\mu \frac{\partial}{\partial \xi^i} \frac{\partial}{\partial \gamma_m} \end{aligned} \tag{3.4}$$

Now the integrated cohomology equation

$$D_\Sigma X = 0 \tag{3.5}$$

becomes equivalent to

$$d_\Sigma X(x) = \partial^\mu (Y_i(\varphi, c, \xi)\xi^i_\mu) \tag{3.6}$$

We see that the right-hand side of (3.6) always contains a term $\partial^\mu \partial_\mu \xi^i$ which does not appear on the left-hand side. Therefore (3.6) implies $Y_i = 0$ and

$$d_\Sigma X(x) = 0 \tag{3.7}$$

To study the cohomology of d_Σ we introduce the counting operator

$$N = \xi^i \frac{\partial}{\partial \xi^i} + \xi^{i\mu} \frac{\partial}{\partial \xi^{i\mu}} + c^i \frac{\partial}{\partial c^i} \tag{3.8}$$

N decomposes the linear space $X(x)$ into subspaces $X^{(\nu)}(x)$ according to its eigenvalues

$\nu = 0, 1, 2, ...$; and induces a separation in d_Σ

$$d_\Sigma = \sum_{n=0}^{\infty} d_\Sigma^{(n)} \tag{3.9}$$

with

$$[N, d_\Sigma^{(n)}] = n \; d_\Sigma^{(n)} \tag{3.10}$$

From (3.4) (see app. A) $d_\Sigma^{(0)}, d_\Sigma^{(1)}, d_\Sigma^{(2)}$ are given by:

$$d_\Sigma^{(0)} = -c^i \frac{\partial}{\partial \xi^i} \tag{3.11}$$

$$\begin{aligned} d_\Sigma^{(1)} &= c^i \frac{\partial}{\partial \varphi^i} - c^k \Gamma^i_{kl}(\varphi) \xi^l \frac{\partial}{\partial \xi^i} - c^k \Gamma^i_{kl}(\varphi) \xi^l_\mu \frac{\partial}{\partial \xi^i_\mu} \\ &+ g_{mn}(\varphi) \xi^{n\mu} \xi^i_\mu \frac{\partial}{\partial \xi^i} \frac{\partial}{\partial \gamma_m} + c^k \Gamma^l_{ki}(\varphi) \gamma_l \frac{\partial}{\partial \gamma_i} \end{aligned} \tag{3.12}$$

$$\begin{aligned} d_\Sigma^{(2)} &= -\frac{1}{3} c^k R^i_{mnk}(\varphi) \xi^m \xi^n \frac{\partial}{\partial \xi^i} - \frac{1}{3} c^k (R^i_{lmk}(\varphi) + R^i_{mlk}(\varphi)) \xi^m \xi^l_\mu \frac{\partial}{\partial \xi^i_\mu} \\ &+ \frac{1}{3} c^k (R^i_{lmk}(\varphi) + R^i_{mlk}(\varphi)) \xi^m \gamma_i \frac{\partial}{\partial \gamma_l} \end{aligned} \tag{3.13}$$

The nilpotency of d_Σ implies that:

$$\sum_{p=0}^{n} d_\Sigma^{(p)} d_\Sigma^{(n-p)} = 0 \qquad n = 0, 1, ... \tag{3.14}$$

from which it follows that $d_\Sigma^{(0)}$ is nilpotent:

$$d_\Sigma^{(0)} d_\Sigma^{(0)} = 0 \tag{3.15}$$

According to the general theory [6], the cohomology of d_Σ is isomorphic to a subspace of the cohomology of $d_\Sigma^{(0)}$, so the first step consists of the identification of the cohomology space of $d_\Sigma^{(0)}$ (see app. B).

4. Cohomology of $d_\Sigma^{(0)}$.

To discuss the cohomology of $d_\Sigma^{(0)}$ we recall that $d_\Sigma^{(0)}$ commutes with the filtering operator N, so that the equation

$$d_\Sigma^{(0)} X(x) = 0 \tag{4.1}$$

can be separately analyzed in each eigenspace $X^{(\nu)}$ belonging to the eigenvalue ν of N. $X^{(\nu)}$ is a space of polynomials in the variables $\xi^i, \xi^{i\mu}, \gamma_i, c^i$ with coefficients which are functions of the constant background φ^i, therefore we can embed it in a Fock space with respect to these variables. This allows us to define an adjoint operator (see app. B):

$$d_\Sigma^{(0)\dagger} = -\xi^i \frac{\partial}{\partial c^i} \tag{4.2}$$

so that the cohomology space of $d_\Sigma^{(0)}$ in each subspace $X^{(\nu)}(x)$ coincides with the kernel of the Laplace-Beltrami operator (see app. B):

$$\Delta_\Sigma = d_\Sigma^{(0)\dagger} d_\Sigma^{(0)} + d_\Sigma^{(0)} d_\Sigma^{(0)\dagger} = c^i \frac{\partial}{\partial c^i} + \xi^i \frac{\partial}{\partial \xi^i} \tag{4.3}$$

It is therefore apparent that the cohomology of $d_\Sigma^{(0)}$ does not contain the variables c^i, ξ^i and hence it can be identified with:

$$\delta = \omega_{ij}(\varphi)\xi^{i\mu}\xi^j_\mu + \gamma_i \omega^i(\varphi) \tag{4.4}$$

where ω_{ij} and ω^i are functions of the sole background constant field φ^i. One sees that the cohomology $d_\Sigma^{(0)}$ decomposes into two subspaces according to the eigenvalues $\nu = 0$ and $\nu = 2$ of N, i.e.

$$\delta = \delta^{(0)} + \delta^{(2)} \tag{4.5}$$

with

$$\begin{aligned} \delta^{(0)} &= \gamma_i \omega^i(\varphi) \\ \delta^{(2)} &= \omega_{ij}(\varphi)\xi^{i\mu}\xi^j_\mu \end{aligned} \tag{4.6}$$

5. Cohomology of d_Σ.

Having determined the cohomology space of $d_\Sigma^{(0)}$, we now turn to the study of d_Σ. We shall show iteratively that any solution of the equation $d_\Sigma X(x) = 0$ can be written as:

$$X(x) = l + d_\Sigma \hat{X}(x) \tag{5.1}$$

where

$$l = \delta^{(2)} + \sum_{n=3}^{\infty} l^{(n)} \tag{5.2}$$

is a d_Σ-cocycle with zero ghost-number uniquely determined, modulo a d_Σ-coboundary, by $\delta^{(2)}$. It is easily shown that, $\delta^{(2)}$ being given by (4.6) a possible form for l is:

$$\begin{aligned} l &= \omega_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j \\ &= \tilde{\omega}_{ij}(\varphi,\xi)\xi^i_\mu\xi^{j\mu} \end{aligned} \tag{5.3}$$

with

$$\tilde{\omega}_{ij}(\varphi,\xi) = \omega_{mn}(\phi)\frac{\partial\phi^m}{\partial\xi^i}\frac{\partial\phi^n}{\partial\xi^j} \tag{5.4}$$

It will be shown that the functions ω_{ij} identify a cohomology class of d_Σ if they cannot be written as:

$$\omega_{ij}(\varphi) = \nabla_i v_j(\varphi) + \nabla_j v_i(\varphi) \tag{5.5}$$

Let us now turn to the proof of eqs. (5.1). Decomposing $X(x)$ according to the eigenvalues of the operator N, i.e.:

$$X(x) = \sum_{n=0}^{\infty} X^{(\nu)}(x) \tag{5.6}$$

and using (3.9) one has the equations:

$$\sum_{p=0}^{n} d_\Sigma^{(p)} X^{(n-p)}(x) = 0 \qquad n = 0,1,.... \tag{5.7}$$

We proceed inductively on the index n. At the level n=0:

$$d_\Sigma^{(0)} X^{(0)}(x) = 0 \tag{5.8}$$

whose general solution is:

$$X^{(0)}(x) = \delta^{(0)} + d_\Sigma^{(0)}\hat{X}^{(0)}(x) \tag{5.9}$$

Next, at n=1,

$$d_\Sigma^{(0)}X^{(1)}(x) + d_\Sigma^{(1)}X^{(0)}(x) = 0 \tag{5.10}$$

which, by (5.9) and (3.14), gives:

$$d_\Sigma^{(0)}(X^{(1)} - d_\Sigma^{(1)}\hat{X}^{(0)}) = -d_\Sigma^{(1)}\delta^{(0)} \tag{5.11}$$

Since

$$d_\Sigma^{(1)}\delta^{(0)} = c^i(\nabla_i\omega^l)\gamma_l \tag{5.12}$$

it follows that, modulo a $d_\Sigma^{(0)}$-coboundary

$$\begin{aligned} d_\Sigma^{(1)}\delta^{(0)} &= -\, d_\Sigma^{(0)}\hat{Y}^{(1)} \\ \hat{Y}^{(1)} &= \xi^i(\nabla_i\omega^{(l)})\gamma_l \end{aligned} \tag{5.13}$$

and (5.11) becomes

$$d_\Sigma^{(0)}(X^{(1)} - d_\Sigma^{(1)}\hat{X}^{(0)} - \hat{Y}^{(1)}) = 0 \tag{5.14}$$

whose general solution reads:

$$X^{(1)} = \hat{Y}^{(1)} + d_\Sigma^{(1)}\hat{X}^{(0)} + d_\Sigma^{(0)}\hat{X}^{(1)} \tag{5.15}$$

One more step at n=2, we find

$$d_\Sigma^{(0)}X^{(2)} + d_\Sigma^{(1)}X^{(1)} + d_\Sigma^{(2)}X^{(0)} = 0 \tag{5.16}$$

which, by (5.9) and (5.15), becomes:

$$d_\Sigma^{(0)}(X^{(2)} - d_\Sigma^{(1)}\hat{X}^{(1)} - d_\Sigma^{(2)}\hat{X}^{(0)}) = -d_\Sigma^{(1)}\hat{Y}^{(1)} - d_\Sigma^{(2)}\delta^{(0)} \tag{5.17}$$

From (3.12), (3.13) one has, modulo a $d_\Sigma^{(0)}$ coboundary:

$$\begin{aligned} d_\Sigma^{(1)}\hat{Y}^{(1)} + d_\Sigma^{(2)}\delta^{(0)} &= (\nabla_m\omega_n)\xi^{m\mu}\xi^n_\mu - d_\Sigma^{(0)}\hat{Y}^{(2)} \\ \hat{Y}^{(2)} &= \xi^i\xi^m(\frac{1}{2}\nabla_i\nabla_m\omega^l + \frac{1}{6}R^l_{mpi}\omega^p)\gamma_l \end{aligned} \tag{5.18}$$

and (5.17) becomes

$$d_\Sigma^{(0)}(X^{(2)} - d_\Sigma^{(1)}\hat{X}^{(1)} - d_\Sigma^{(2)}\hat{X}^{(0)} - \hat{Y}^{(2)}) = -(\nabla_m\omega_n)\xi^{m\mu}\xi^n_\mu \tag{5.19}$$

Now the right-hand side belongs to the cohomology space of $d_\Sigma^{(0)}$ and thus (5.19) is inconsistent, unless

$$(\nabla_m \omega_n + \nabla_n \omega_m) = 0 \tag{5.20}$$

Equation (5.20) implies the existence of isometries for the action S. In such a case, we have to use a modified BRS operator [4] which properly takes into account the isometries and which yields a quite different renormalization analysis. Therefore we consider a generic model with no additional symmetry, so that (5.20) implies $\omega_n(\varphi) = 0$. It then follows that the general solution of (5.16) is

$$X^{(2)} = \delta^{(2)} + d_\Sigma^{(1)} \hat{X}^{(1)} + d_\Sigma^{(2)} \hat{X}^{(0)} + d_\Sigma^{(0)} \hat{X}^{(2)} \tag{5.21}$$

Proceeding by induction it is easy to implement the equation (5.2) to all orders. Let us suppose indeed that (5.7) can be solved up to the order $\overline{n}$, i.e. that the equation

$$\sum_{p=0}^{\overline{n}} d_\Sigma^{(p)} X^{(\overline{n}-p)} = 0 \qquad \overline{n} > 2 \tag{5.22}$$

can be solved for $X^{(\overline{n})}$ and let us show that at the order $(\overline{n}+1)$, the equation

$$\sum_{p=0}^{\overline{n}+1} d_\Sigma^{(p)} X^{(\overline{n}+1-p)} = 0 \tag{5.23}$$

does not contain any obstruction. Using (5.22) one has

$$\begin{aligned} d_\Sigma^{(0)} \Big(\sum_{p=1}^{\overline{n}+1} d_\Sigma^{(p)} X^{(\overline{n}+1-p)} \Big) &= - \sum_{p=1}^{\overline{n}+1} \sum_{\nu=1}^{p} d_\Sigma^{(\nu)} d_\Sigma^{(p-\nu)} X^{(\overline{n}+1-p)} \\ &= - \sum_{\nu=1}^{\overline{n}+1} d_\Sigma^{(\nu)} \sum_{p=\nu}^{\overline{n}+1} d_\Sigma^{(p-\nu)} X^{(\overline{n}+1-p)} = 0 \end{aligned} \tag{5.24}$$

This equation shows that

$$\sum_{p=1}^{\overline{n}+1} d_\Sigma^{(p)} X^{(\overline{n}+1-p)} \tag{5.25}$$

is a trivial cocycle of $d_\Sigma^{(0)}$ since it belongs to the subspace with filtration number equal to $(\overline{n}+1)$. However, the cohomology space of $d_\Sigma^{(0)}$ with filtration number equal to $(\overline{n}+1)$ is empty, and thus there exists $\hat{X}^{(\overline{n}+1)}$ such that

$$\sum_{p=1}^{\overline{n}+1} d_{\Sigma}^{(p)} X^{(\overline{n}+1-p)} = d_{\Sigma}^{(0)} \hat{X}^{(\overline{n}+1)} \tag{5.26}$$

This equation ensures the absence of obstructions in solving (5.23). It is not difficult now, using the previous argument, to prove equation (5.5), thereby concluding our proof.

6. Renormalization.

The results in (5.1) and (5.2) show that the general solution of the integrated cohomology equation $D_\Sigma X = 0$ can be written as:

$$X = \int d^2x \; \tilde{\omega}_{ij}(\varphi, \xi)\partial_\mu \xi^i \partial^\mu \xi^j \quad + \quad D_\Sigma \hat{X} \tag{6.1}$$

From this expression we can read the renormalizations needed for the stability of the model. In particular the function $\tilde{\omega}_{ij}$ in (6.1) corresponds to a redefinition of the metric tensor $\tilde{g}_{ij}$, provided that it cannot be written as in (5.5). This redefinition really means that, on a generic Riemannian manifold, the stability requires an infinite number of parameters. In this situation the renormalizability of the model has to be understood in the generalized sense of Friedan [1]. Up to now the only non-linear σ-models which are really renormalizable (i.e. only a finite number of parameters are needed for the stability), are those for which the Riemannian manifold can be identified with a coset space [4]. Concerning the other renormalizations, one has to note that no terms proportional to the external sources $\gamma_i(x)$ appear in the cohomology of D_Σ, which implies that the corresponding counterterms are generated by a D_Σ variation, i.e.

$$\begin{aligned} D_\Sigma \int d^2x \gamma_i R^i(\varphi, \xi) \;=\; & \int d^2x \frac{\delta S}{\delta \xi^i} R^i \;+ \\ & \int d^2x \gamma_k \left(\frac{\partial F^k_j}{\partial \xi^i} R^i - F^i_j \frac{\partial R^k}{\partial \xi^i} - \frac{\partial R^k}{\partial \varphi^j}\right) c^j \end{aligned} \tag{6.2}$$

They correspond to a non-linear renormalization of the quantum field:

$$\xi^i \rightarrow \xi^i + R^i(\varphi, \xi) \tag{6.3}$$

together with a redefinition of the split transformation F^i_j according to

$$\hat{F}^i_j \rightarrow F^i_j - \frac{\partial R^i}{\partial \xi^l} F^l_j + \frac{\partial F^i_j}{\partial \xi^l} R^l - \frac{\partial R^i}{\partial \varphi^j} \tag{6.4}$$

The terms in (6.4) cannot be viewed as γ_i sources and ξ^i quantum field renormalizations; in particular, while the first two terms on the right-hand side of (6.4) can be reabsorbed by a (φ, ξ) -dependent renormalization of the sources γ_i and of the fields ξ^i, the last one cannot. Notice that the renormalized $\hat{F}^i_j$ satisfies again the condition (1.8).

The topological σ-model.

1. The model.

2. Basic cohomology.

3. BRS cohomology.

1. The model.

The topological non-linear σ-model has been proposed by Witten [7] in the context of the topological field theories [8]. The model is built by means of a BRS symmetry which leads to a cohomological trivial classical Lagrangian; i.e., it is a pure BRS variation. In spite of this, the model possesses a non-trivial class of observables [7] which, as indicated by Witten, are of a global topological nature and are given by the De Rham cohomology of the target manifold. However, as shown recently by Ouvry, Stora and Van Baal [9] in the case of topological gauge theories, the definition of the observables involves what is called the basic cohomology of the BRS operator rather than its usual cohomology. This is due to the fact that the BRS operator has trivial cohomology while, as shown in [9] the Witten observables belong to its basic cohomology. A similar situation is met in the case of the topological σ-model [10], where, on the space of forms, the BRS operator acts like the exterior derivative and hence the Poincare lemma ensures that locally its cohomology is trivial; in other words, to recover the De Rham cohomology of the target manifold, it seems necessary to introduce some global information. This is achieved by combining the Witten BRS symmetry with the Friedan non-linear shift symmetry [10]. Indeed, as we shall see, the use of the Friedan symmetry will allow us to define the basic cohomology, which will be shown to be the De Rham cohomology of the target manifold.

The bosonic field of the model is defined by the map:

$$\Phi : \Xi \rightarrow K \tag{1.1}$$

where K is a compact n-dimensional Kahlerian manifold.

As in the previous example, the field ϕ splits as:

$$\phi^i(x) = \varphi^i + \pi^i(\varphi, \xi) \tag{1.2}$$

where φ^i is the classical constant background and $\xi^i(x)$ is the quantum field.

The non-linear splitting (1.2) induces the Riemannian metric $\tilde{g}_{ij}(\varphi, \xi)$ and the complex structure $\tilde{J}_{ij}(\varphi, \xi)$ which are globally defined on K:

$$\frac{\partial \tilde{g}_{ij}}{\partial \varphi^k} + F_k^q \frac{\partial \tilde{g}_{ij}}{\partial \xi^q} + \tilde{g}_{qj} \frac{\partial F_k^q}{\partial \xi^i} + \tilde{g}_{iq} \frac{\partial F_k^q}{\partial \xi^j} = 0 \tag{1.3}$$

$$\frac{\partial \tilde{J}_{ij}}{\partial \varphi^k} + F_k^q \frac{\partial \tilde{J}_{ij}}{\partial \xi^q} + \tilde{J}_{qj} \frac{\partial F_k^q}{\partial \xi^i} + \tilde{J}_{iq} \frac{\partial F_k^q}{\partial \xi^j} = 0 \tag{1.4}$$

where F^q_k is the Friedan connection [1]:

$$F^i_k(\varphi,0) = -\delta^i_k \tag{1.5}$$

$$\left(\frac{\partial F^i_k}{\partial\varphi^j} - \frac{\partial F^i_j}{\partial\varphi^k}\right) + \left(\frac{\partial F^i_k}{\partial\xi^l}F^l_j - \frac{\partial F^i_j}{\partial\xi^l}F^l_k\right) = 0 \tag{1.6}$$

One introduces a fermionic field λ^i :

$$\lambda^i = \eta^i + \chi^i(x) \tag{1.7}$$

where η^i is, in analogy with (1.2), a constant fermionic background and $\chi^i(x)$ is the corresponding quantum fluctuation, and the self-dual fields

$$\begin{aligned} \rho^i_\alpha &= \varepsilon^\beta_\alpha \tilde{J}^i_j(\varphi,\xi)\rho^j_\beta \\ H^i_\alpha &= \varepsilon^\beta_\alpha \tilde{J}^i_j(\varphi,\xi)H^j_\beta \end{aligned} \tag{1.8}$$

where ε^β_α is the complex structure of Ξ:

$$\varepsilon^\alpha_\beta\varepsilon^\beta_\gamma = -\delta^\alpha_\gamma \tag{1.9}$$

The mass dimensions of the fields φ^i, ξ^i, $\lambda^i, \rho^i_\alpha, H^i_\alpha$ are respectively 0, 0, 0, 1, 1 and the assigned ghost charges 0, 0, +1, -1, 0 . The field content of the model allows us to write the local BRS symmetry

$$\begin{aligned} s\varphi^i &= i\eta^i \\ s\xi^i &= i(\eta^i + \chi^i) + iF^i_l\eta^l \\ s\eta^i &= 0 \\ s\chi^i &= -i\frac{\partial F^i_l}{\partial\xi^m}(\eta^m + \chi^m)\eta^l \\ s\rho^i_\alpha &= H^i_\alpha - i\tilde{\Gamma}^i_{jk}\lambda^j\rho^k_\alpha + i\frac{\partial F^i_m}{\partial\xi^k}\eta^m\rho^k_\alpha \\ sH^i_\alpha &= -\frac{1}{2}\tilde{R}^i_{tkl}\rho^t_\alpha\lambda^k\lambda^l - i\tilde{\Gamma}^i_{jk}\lambda^j H^k_\alpha + i\frac{\partial F^i_m}{\partial\xi^k}\eta^m H^k_\alpha \end{aligned} \tag{1.10}$$

where

$$\tilde{\Gamma}^i_{jk}(\varphi,\xi) = \frac{1}{2}\tilde{g}^{il}\left(\frac{\partial\tilde{g}_{kl}}{\partial\xi^j} + \frac{\partial\tilde{g}_{jl}}{\partial\xi^k} - \frac{\partial\tilde{g}_{jk}}{\partial\xi^l}\right) \tag{1.11}$$

and

$$\tilde{R}^i_{qjp}(\varphi,\xi) = \frac{\partial\tilde{\Gamma}^i_{pq}}{\partial\xi^j} - \frac{\partial\tilde{\Gamma}^i_{jq}}{\partial\xi^p} + \tilde{\Gamma}^i_{jk}\tilde{\Gamma}^k_{pq} - \tilde{\Gamma}^i_{pk}\tilde{\Gamma}^k_{jq} \tag{1.12}$$

are the Riemann connection and the Riemann tensor associated to the background metric

$\tilde{g}_{ij}(\varphi,\xi)$ which satisfy the conditions:

$$\frac{\partial \tilde{\Gamma}^l_{jm}}{\partial \varphi^i} + F^k_i \frac{\partial \tilde{\Gamma}^l_{jm}}{\partial \xi^k} + \tilde{\Gamma}^l_{km}\frac{\partial F^k_i}{\partial \xi^j} + \tilde{\Gamma}^l_{jk}\frac{\partial F^k_i}{\partial \xi^m} - \tilde{\Gamma}^k_{jm}\frac{\partial F^l_i}{\partial \xi^k} + \frac{\partial^2 F^l_k}{\partial \xi^j \partial \xi^m} = 0 \tag{1.13}$$

$$\frac{\partial \tilde{R}_{imnl}}{\partial \varphi^k} + F^p_k \frac{\partial \tilde{R}_{imnl}}{\partial \xi^p} + \tilde{R}_{pmnl}\frac{\partial F^p_k}{\partial \xi^i} + \tilde{R}_{ipnl}\frac{\partial F^p_k}{\partial \xi^m} + \tilde{R}_{impl}\frac{\partial F^p_k}{\partial \xi^n} + \tilde{R}_{imnp}\frac{\partial F^p_k}{\partial \xi^l} = 0 \tag{1.14}$$

It is not difficult to show, using (1.6), (1.13), (1.14) and the Bianchi identity for $\tilde{R}^{ijkl}$, that s is nilpotent.

A Lagrangian for such a model is given by taking the BRS variation of

$$\int d^2x(\tilde{g}_{ij}\rho^{i\alpha}\partial_\alpha \xi^j - \frac{1}{4}\tilde{g}_{ij}\rho^{i\alpha}H^j_\alpha) \tag{1.15}$$

i.e.

$$S = \int d^2x(\tilde{g}_{ij}H^{i\alpha}\partial_\alpha \xi^j - \frac{1}{4}\tilde{g}^{ij}H^{i\alpha}H^j_\alpha - i\tilde{g}_{ij}\rho^{i\alpha}\tilde{D}_\alpha \lambda^j - \frac{1}{8}\tilde{R}_{itkl}\rho^{i\alpha}\rho^t_\alpha \lambda^k \lambda^l) \tag{1.16}$$

where

$$\tilde{D}_\alpha \lambda^j = \partial_\alpha \lambda^j + \tilde{\Gamma}^j_{km}\lambda^k \partial_\alpha \xi^m \tag{1.17}$$

One has to note that the fields η^i and χ^i enter into (1.16) only through the combination λ^i and that the derivative of $H^{i\alpha}$ does not appear in (1.16), so $H^{i\alpha}$ is an auxiliary field which can be eliminated by using the equations of motion

$$H^i_\alpha = \partial_\alpha \xi^i + \varepsilon^\beta_\alpha \tilde{J}^i_j \partial_\beta \xi^j \tag{1.18}$$

As in the standard cases it is necessary to ensure the-off shell nilpotency of the BRS operator.

One can easily verify that the action S is invariant under the Friedan shift symmetry δ_v defined as:

$$\begin{aligned}
\delta_v \varphi^i &= v^i \\
\delta_v \xi^i &= v^k F^i_k(\varphi,\xi) \\
\delta_v \eta^i &= 0 \\
\delta_v \chi^i &= v^k \frac{\partial F^i_k}{\partial \xi^l}(\eta^l + \chi^l) \\
\delta_v \rho^i_\alpha &= v^k \frac{\partial F^i_k}{\partial \xi^l}\rho^l_\alpha \\
\delta_v H^i_\alpha &= v^k \frac{\partial F^i_k}{\partial \xi^l}H^l_\alpha
\end{aligned} \tag{1.19}$$

i.e.

$$\delta_v S = 0 \tag{1.20}$$

meaning that the action S is globally defined.

We now introduce the operator ι_v whose action on the fields of the model is :

$$\begin{aligned}
\iota_v \varphi^i &= 0 \\
\iota_v \xi^i &= 0 \\
\iota_v \rho^i_\alpha &= 0 \\
\iota_v H^i_\alpha &= 0 \\
\iota_v \eta^i &= -iv^i \\
\iota_v \chi^i &= iv^i
\end{aligned} \tag{1.21}$$

From (1.21) it follows that

$$\iota_v \iota_\mu + \iota_\mu \iota_v = 0 \tag{1.22}$$

and the operator δ_v decomposes as

$$\delta_v = s\iota_v + \iota_v s \tag{1.23}$$

The operators s, ι_μ, δ_v satisfy the algebra :

$$\begin{aligned}
[\delta_v, \delta_\mu] &= 0 \\
[\delta_v, \iota_\mu] &= 0 \\
[\delta_v, s] &= 0 \\
\iota_v \iota_\mu + \iota_\mu \iota_v &= 0 \\
\delta_v &= s\iota_v + \iota_v s
\end{aligned} \tag{1.24}$$

and, on the action S

$$\delta_v S = \iota_\mu S = sS = 0 \tag{1.25}$$

The algebra (1.24) is the Abelian version of that introduced in [9] and, as we will see in the next section, it allows us to define a basic cohomology for the BRS operator.

2. Basic cohomology.

To recover the Witten observables [7] , we study the operators s , ι_μ and δ_v in the class of integrated local dimensionless functionals which depend on the variables $\varphi^i, \eta^i, \xi^i, \chi^i$, i.e:

$$X = \int d^2x A(\varphi, \eta, \xi(x), \chi(x)) \tag{2.1}$$

where analyticity in η, ξ, χ has to be understood. On this space there is no differeence between the local and integrated cohomology, so we can consider s, ι_μ and δ_v as ordinary differential operators acting on the local space A.

As ordinary differential operators s, ι_μ and δ_v read :

$$s = i\eta^i \frac{\partial}{\partial \varphi^i} + i(\eta^i + \chi^i)\frac{\partial}{\partial \xi^i} + i\eta^l F_l^i \frac{\partial}{\partial \xi^i} - i\frac{\partial F_l^i}{\partial \xi^m}(\eta^m + \chi^m)\eta^l \frac{\partial}{\partial \chi^i} \tag{2.2}$$

$$\iota_\mu = i\mu^i \frac{\partial}{\partial \chi^i} - i\mu^i \frac{\partial}{\partial \eta^i} \tag{2.3}$$

$$\delta_v = v^i \frac{\partial}{\partial \varphi^i} + v^i F_i^l \frac{\partial}{\partial \xi^l} + v^i \frac{\partial F_i^p}{\partial \xi^l} \lambda^l \frac{\partial}{\partial \chi^p} \tag{2.4}$$

It is not difficult to show that, on the local space A, the cohomology of s is trivial (see next chapter). Now we define the basic cohomology of s as :

$$H^{basic} = (\theta \in A; s\theta = \iota_\mu\theta = \delta_v\theta = 0 \text{ and } \exists \text{ no } \gamma \in A \text{ such that } \theta = s\gamma \text{ and } \iota_\mu\gamma = \delta_v\gamma = 0) \tag{2.5}$$

One has to note that, due to the triviality of the cohomology of s, every quantity invariant under s, δ_v, ι_μ can be written as an exact cocycle of s. However, it is not true that every s-exact cocycle is invariant under δ_v, ι_μ, meaning that quantities which are trivial in the cohomology of s can be non-trivial in H^{basic}. Let us examine first the condition

$$\iota_\mu\theta = 0 \tag{2.6}$$

From (2.3) it follows that (2.6) implies that θ depends only on the combination $\lambda^i = \eta^i + \chi^i$, i.e $\theta = \theta(\varphi, \xi, \lambda)$. Analyticity in λ implies that

$$\begin{aligned} \theta &= \tilde{\theta}_{i.\dots i_n}(\varphi, \xi)\lambda^{i_1}\dots\dots\lambda^{i_n} \\ n &= 0, 1, \dots\dots, dimM \end{aligned} \tag{2.7}$$

where $\tilde{\theta}_{i_1\dots i_n}$ is a form-valued function on T_φ. The condition $\delta_v\theta = 0$ implies that

$$\frac{\partial\tilde{\theta}_{i_1\dots i_n}}{\partial\varphi^k} + F^q_k\frac{\partial\tilde{\theta}_{i_1\dots i_n}}{\partial\xi^q} + \tilde{\theta}_{qi_2\dots i_n}\frac{\partial F^q_k}{\partial\xi^{i_1}} + \dots\dots + \tilde{\theta}_{i_1\dots i_{n-1}q}\frac{\partial F^q_k}{\partial\xi^{i_n}} = 0 \tag{2.8}$$

i.e. $\tilde{\theta}_{i_1\dots i_n}$ is a globally defined form-valued function. Finally the condition $s\theta = 0$ means that $\tilde{\theta}_{i_1\dots i_n}$ is closed since, by (2.7) and (2.8),

$$s\theta = i\lambda^i\frac{\partial\tilde{\theta}_{i_1\dots i_n}}{\partial\xi^i}\lambda^{i_1}\dots\dots\lambda^{in} \tag{2.9}$$

Then H^{basic} is given by globally defined closed forms which cannot be written in a global way as exact form-valued functions, i.e H^{basic} can be identified with the De Rham cohomology of the target manifold M, so leading to the Witten observables of the model.

3. Cohomology of the BRS operator.

To study the cohomology of the operator s on the local space A we introduce the counting operator

$$N = \xi^k \frac{\partial}{\partial \xi^k} + \eta^k \frac{\partial}{\partial \eta^k} + \chi^k \frac{\partial}{\partial \chi^k} \tag{3.1}$$

N decomposes the local space A into subspaces $A^{(\nu)}$ according to its eigenvalue $\nu = 0,1,2,....$; and induces a separation in s

$$s = \sum_{n=0}^{\infty} s^{(n)} \tag{3.2}$$

with

$$[N, s^{(n)}] = n s^{(n)} \tag{3.3}$$

From (2.2) (see App. A) $s^{(0)}$ and $s^{(1)}$ are given by

$$s^{(0)} = i\chi^i \frac{\partial}{\partial \xi^i} \tag{3.4}$$

$$s^{(1)} = i\eta^i \frac{\partial}{\partial \varphi^i} - i\eta^l \Gamma^i_{lm}(\varphi)\xi^m \frac{\partial}{\partial \xi^i} + i\Gamma^i_{lm}(\varphi)(\eta^m + \chi^m)\eta^l \frac{\partial}{\partial \chi^i} \tag{3.5}$$

The nilpotency of s implies that

$$\sum_{p=0}^{n} s^{(p)} s^{(n-p)} = 0 \quad , \quad n = 0, 1, ... \tag{3.6}$$

from which it follows that $s^{(0)}$ is nilpotent :

$$s^{(0)} s^{(0)} = 0 \tag{3.7}$$

According to the general theory [6] the cohomology of s is isomorphic to a subspace of the cohomology of $s^{(0)}$, so the first step consists of the identification of the cohomology space of $s^{(0)}$. From (3.4) it is apparent that the cohomology space of $s^{(0)}$ does not contain the variables χ^i and ξ^i and hence it can be parametrized by a function $\Omega(\varphi,\eta)$ of the variables φ^i , η^i. Analyticity in η^i implies that

$$\begin{aligned} \Omega =&\Omega_{i_1.....i_n}(\varphi)\eta^{i_1}......\eta^{i_n} \\ n =&0,1,.........., \; dimM \end{aligned} \tag{3.8}$$

Having determined the cohomology of $s^{(0)}$ we turn to the equation

$$sY(\varphi,\eta,\xi,\chi) = 0 \tag{3.9}$$

We shall show that the general solution of (3.9) is given by

$$Y = const + s\hat{Y} \tag{3.10}$$

which means that the cohomology of s is trivial. To prove (3.10) we proceed by induction with respect to the eigenvalues of N. Decomposing Y according to the operator N,

$$Y = \sum_{n=0}^{\infty} Y^{(n)} \tag{3.11}$$

and using (3.2) one has the equation

$$\sum_{p=0}^{n} s^{(p)}Y^{(n-p)} = 0 \quad , \quad n = 0,1,....... \tag{3.12}$$

At the level n=0 (3.12) gives

$$s^{(0)}Y^{(0)} = 0 \tag{3.13}$$

whose general solution is

$$Y^{(0)} = \tau(\varphi) + s^{(0)}\hat{Y}^{(0)} \tag{3.14}$$

where $\tau(\varphi)$ is an arbitrary function of the background variables φ^i . Next at n=1

$$s^{(0)}Y^{(1)} + s^{(1)}Y^{(0)} = 0 \tag{3.15}$$

Inserting (3.14) and using the equation (3.6) with n=1

$$s^{(0)}(Y^{(1)} - s^{(1)}\hat{Y}^{(0)}) = -s^{(1)}\tau \tag{3.16}$$

From (3.5)

$$s^{(1)}\tau = i\eta^i \frac{\partial\tau}{\partial\varphi^i} \tag{3.17}$$

This equation shows that $(s^{(1)}\tau)$ belongs to the cohomology of $s^{(0)}$ in the N=1 sector and then (3.16) implies

$$\frac{\partial \tau}{\partial \varphi^i} = 0 \tag{3.18}$$

i.e. τ is a constant. From (3.18) it follows that

$$Y^{(1)} = \eta^l v_l(\varphi) + s^{(1)}\hat{Y}^{(0)} + s^{(0)}\hat{Y}^{(1)} \tag{3.19}$$

where $v_l(\varphi)$ is a vector valued-function on M. At n=2

$$s^{(2)}Y^{(0)} + s^{(1)}Y^{(1)} + s^{(0)}Y^{(2)} = 0 \tag{3.20}$$

Inserting the expression for $Y^{(0)}$ and $Y^{(1)}$ and using (3.6) with n=2

$$s^{(0)}(Y^{(2)} - s^{(1)}\hat{Y}^{(1)} - s^{(2)}\hat{Y}^{(0)}) = -s^{(1)}\eta^l v_l(\varphi) \tag{3.21}$$

Since

$$s^{(1)}\eta^l v_l(\varphi) = i\eta^m \eta^l \frac{\partial v_l}{\partial \varphi^m} \tag{3.22}$$

it follows that $(s^{(1)}\eta^l v_l)$ belongs to the cohomology of $s^{(0)}$ in the N=2 sector; (4.21) implies that

$$\frac{\partial v_l}{\partial \varphi^m} - \frac{\partial v_m}{\partial \varphi^l} = 0 \tag{3.23}$$

and the Poincare lemma ensures that, locally,

$$v_l = \frac{\partial \omega(\varphi)}{\partial \varphi^l} \tag{3.24}$$

The expression for $Y^{(1)}$ in (3.19) becomes

$$Y^{(1)} = s^{(1)}(\hat{Y}^{(0)} - i\omega(\varphi)) + s^{(0)}\hat{Y}^{(1)} \tag{3.25}$$

meaning that the sector N=1 is completely trivial. It is not difficult now to iterate the procedure and to show that all the sectors with filtration number ≥ 2 are trivial, recovering then the expression (3.10) for Y .

The Chern-Simons model.

1. The Chern-Simons action and its symmetries.
2. Stability.
3. Anomalies.

1. The Chern-Simons action and its symmetries.

Among the topological field theories the Chern-Simons action in three dimensions seems to play a quite interesting role due to its relation with the topology of the three-manifolds [11], and with some conformal models in two dimensions [12].

However, also at the perturbative level [13] the Chern-Simons theory has quite peculiar features, due to its very rich symmetry structure in a covariant gauge, namely the Landau gauge. Indeed, the Chern-Simons action possesses a set of symmetries which form a supersymmetry-like algebra [14].

The Chern-Simons action in three dimensions reads:

$$S_{inv}(A) = -\frac{1}{2}\int d^3x\ \varepsilon^{\mu\nu\rho}(\ A^a_\mu \partial_\nu A^a_\rho + \frac{1}{3} f^{abc} A^a_\mu A^b_\nu A^c_\rho\ ) \tag{1.1}$$

Including the gauge fixing and the ghost terms, the full action in the Landau gauge takes the form

$$S = S_{inv} + \int d^3x\ d^a \partial^\mu A^a_\mu + b^a \partial^\mu (D_\mu c)^a \tag{1.2}$$

where d^a, b^a, c^a are respectively the Lagrangian multiplier, the antighost, the ghost, and

$$(D_\mu c)^a = \partial_\mu c^a + f^{abc} A^b_\mu c^c \tag{1.3}$$

The action (1.2) is invariant under the BRS transformation

$$\begin{aligned} sA^a_\mu &= -(D_\mu c)^a \\ sc^a &= \frac{1}{2} f^{abc} c^b c^c \\ sb^a &= d^a \\ sd^a &= 0 \end{aligned} \tag{1.4}$$

and, as shown in [13], under a set of anticommuting global symmetries carriyng a Lorentz vector-index:

$$\begin{aligned} \delta_\rho A^a_\mu &= \varepsilon_{\mu\rho\nu} \partial^\nu c^a \\ \delta_\rho c^a &= 0 \\ \delta_\rho b^a &= A^a_\rho \\ \delta_\rho d^a &= (D_\rho c)^a \end{aligned} \tag{1.5}$$

Both s and δ_ρ increase the ghost number by one unit and, altogether, these symmetries form an Abelian superalgebra:

$$\{s\ ,\ \delta_\rho\} = 0 \qquad \{\delta_\sigma\ ,\ \delta_\rho\} = 0 \tag{1.6}$$

As is well known, integration by parts allows us to rewrite the action (1.2) as

$$S = S_{inv} + \int d^3x\ (d^a - f^{abc}b^b c^c)\partial^\mu A^a_\mu - c^a \partial^\mu (D_\mu b)^a \tag{1.7}$$

which expression may be viewed as resulting from (1.4) by the substitutions

$$\begin{aligned} A^a_\mu &\longrightarrow A^a_\mu \\ c^a &\longrightarrow b^a \\ b^a &\longrightarrow -c^a \\ d^a &\longrightarrow d^a - f^{abc}b^b c^c \end{aligned} \tag{1.8}$$

Considering the same replacements in eqs. (1.4) and (1.5) leads to the anti-BRS transformations

$$\begin{aligned} \bar{s}A^a_\mu &= -(D_\mu b)^a \\ \bar{s}b^a &= \frac{1}{2} f^{abc}b^b b^c \\ \bar{s}c^a &= -d^a + f^{abc}b^b c^c \\ \bar{s}d^a &= f^{abc}b^b d^c \end{aligned} \tag{1.9}$$

and to a new set of anticommuting global symmetries:

$$\begin{aligned} \bar{\delta}_\rho A^a_\mu &= \varepsilon_{\mu\rho\nu}\partial^\nu b^a \\ \bar{\delta}_\rho c^a &= -A^a_\rho \\ \bar{\delta}_\rho b^a &= 0 \\ \bar{\delta}_\rho d^a &= \partial_\rho b^a \end{aligned} \tag{1.10}$$

For their part, the transformations (1.9) and (1.10) describe an Abelian superalgebra:

$$\{\bar{s}\ ,\ \bar{\delta}_\rho\} = 0 \qquad \{\bar{\delta}_\sigma\ ,\ \bar{\delta}_\rho\} = 0 \tag{1.11}$$

However, some of the anti-commutation relations with the former symmetries (1.4), (1.5) are non-trivial.

One finds

$$
\begin{aligned}
\{s\ ,\ \bar{s}\} =&\ 0 \\
\{\bar{s}\ ,\ \delta_\rho\}(b^a, c^a, d^a) =&\ -\partial_\rho(b^a, c^a, d^a) \\
\{\bar{s}\ ,\ \delta_\rho\}A^a_\mu =&\ -\partial_\rho A^a_\mu + \varepsilon_{\mu\rho\nu}\frac{\delta S}{\delta A^a_\nu} \\
\{\delta_\rho\ ,\ \bar{\delta}_\sigma\}(b^a, c^a) =&\ \varepsilon_{\rho\sigma\nu}\partial^\nu(b^a, c^a) \\
\{\delta_\rho\ ,\ \bar{\delta}_\sigma\}d^a =&\ \varepsilon_{\rho\sigma\nu}\partial^\nu d^a + \varepsilon_{\rho\sigma\nu}\frac{\delta S}{\delta A^a_\nu} \\
\{\delta_\rho\ ,\ \bar{\delta}_\sigma\}A^a_\mu =&\ \varepsilon_{\rho\sigma\nu}\partial^\nu A^a_\mu + \varepsilon_{\rho\mu\sigma}\frac{\delta S}{\delta d^a} \\
\{s\ ,\ \bar{\delta}_\rho\}(b^a, c^a, d^a) =&\ \partial_\rho(b^a, c^a, d^a) \\
\{s\ ,\ \bar{\delta}_\rho\}A^a_\mu =&\ \partial_\rho A^a_\mu + \varepsilon_{\rho\mu\nu}\frac{\delta S}{\delta A^a_\nu}
\end{aligned}
\tag{1.12}
$$

Thus, it is clear that these generators define, up to field equations, a supersymmetry algebra. Using the notation

$$
\begin{aligned}
\delta^1 &= s \\
\delta^2 &= \bar{s} \\
\delta^1_\rho &= \delta_\rho \\
\delta^2_\rho &= \bar{\delta}_\rho
\end{aligned}
\tag{1.13}
$$

the on-shell algebra can be summarized as follows:

$$
\begin{aligned}
\{\delta^i_\rho\ ,\ \delta^j_\sigma\} &= \epsilon^{ij}\varepsilon_{\rho\sigma\nu}\partial^\nu \qquad (\epsilon^{12} = 1) \\
\{\delta^i\ ,\ \delta^j_\rho\} &= \epsilon^{ij}\partial_\rho
\end{aligned}
\tag{1.14}
$$

A few remarks concerning the presented invariance algebras are in order. Notice that in the algebra (1.12), and contrarily to ordinary supersymmetry, equations of motion of bosonic fields appear. This is not the only place where the usual roles of commuting and anticommuting fields are interchanged. In fact, this also applies to the order of the equations of motion and to the counting of the degrees of freedom. The free equations of motion for the ghost c and the antighost b are just Klein-Gordon equations and thereby each of these variables corresponds to one degree of freedom. On the other hand, the bosonic fields, i.e. the vector A_μ and the Lagrange multiplier d satisfy a set of first-order differential equations which is analogous to the Dirac equation for free, massless fermions. For the same reason as in the case of Dirac's equation, the number of on-shell degrees of freedom is one-half the number of off-shell degrees of freedom, and thus equal to 2. As usual in supersymmetry, the number of commuting and anti-commuting degrees of freedom match. However, from a physical point of view, ghost degrees of freedom are to be counted negatively and therefore the theory has no on-shell degrees of freedom.

2. Stability.

Let us now discuss the consequences of the BRS and of the $\bar{\delta}_\rho$ symmetries on the stability of the action (1.2).

To discuss at the quantum level these symmetries we introduce the source term

$$S_s = \int d^3x \left(-\Omega^{a\mu}(D_\mu c)^a + L^a \frac{f^{abc}}{2} c^b c^c \right) \tag{2.1}$$

which allows us to write the Slavnov-Taylor identity

$$\int d^3x \left(\frac{\delta\Sigma}{\delta\Omega^{a\mu}} \frac{\delta\Sigma}{\delta A^a_\mu} + \frac{\delta\Sigma}{\delta L^a} \frac{\delta\Sigma}{\delta c^a} + d^a \frac{\delta\Sigma}{\delta b^a} \right) = 0 , \tag{2.2}$$

where

$$\Sigma = S + S_s \tag{2.3}$$

The mass dimensions of the fields $A^a_\mu, c^a, b^a, d^a, L^a, \Omega^{a\mu}$ are respectively 1, 0, 1, 1, 3, 2 and the assigned ghost charges 0, 1, -1, -0, -2, -1.

The source term (2.1) is BRS invariant but not $\bar{\delta}_\rho$ invariant; however it is possible to write a Ward identity corresponding to the $\bar{\delta}_\rho$-symmetry; one gets indeed

$$\tilde{W}_\rho \Sigma = \int d^3x (-\Omega^{a\mu}\partial_\rho A^a_\mu + \varepsilon_{\mu\rho\nu}\Omega^{a\mu}\partial^\nu d^a + L^a \partial_\rho c^a) \tag{2.4}$$

and

$$\tilde{W}_\rho = \int d^3x (\varepsilon_{\mu\rho\nu}(\Omega^{a\nu} + \partial^\nu b^a)\frac{\delta}{\delta A^a_\mu} - A^a_\rho \frac{\delta}{\delta c^a} + \partial_\rho b^a \frac{\delta}{\delta d^a} - L^a \frac{\delta}{\delta \Omega^{a\rho}}) \tag{2.5}$$

The equation (2.4) shows that Σ is not invariant under the action of the Ward operator $\tilde{W}_\rho$; however, the breaking term which appears on the right-hand side of (2.4) is linear in the quantum fields and then it is present only at the classical level.

To discuss the stability we limit ourselves to the perturbations of the classical action Σ which are integrated local functionals of dimension 3, Faddeev-Popov charge 0 and are invariant under global transformations of the gauge group.

The perturbation $\bar{\Sigma}$ satisfies the linearized equations:

$$
\begin{aligned}
D_\Sigma \bar{\Sigma} &= 0 \\
\tilde{W}_\rho \bar{\Sigma} &= 0
\end{aligned}
\tag{2.6}
$$

where

$$
D_\Sigma = \int d^3x \left(\frac{\delta\Sigma}{\delta\Omega^{a\mu}} \frac{\delta}{\delta A^a_\mu} + \frac{\delta\Sigma}{\delta A^{a\mu}} \frac{\delta}{\delta\Omega^a_\mu} + \frac{\delta\Sigma}{\delta L^a} \frac{\delta}{\delta c^a} + \frac{\delta\Sigma}{\delta c^a} \frac{\delta}{\delta L^a} + d^a \frac{\delta}{\delta b^a} \right) \tag{2.7}
$$

is the linearized operator corresponding to the Slavnov-Taylor identity (2.2).

The operators D_Σ and $\tilde{W}_\rho$ satisfy the algebra

$$
\{D_\Sigma \ , \ D_\Sigma\} = 0 \tag{2.8}
$$

$$
\{\tilde{W}_\rho \ , \ \tilde{W}_\sigma\} = 0 \tag{2.9}
$$

$$
\begin{aligned}
\{D_\Sigma \ , \tilde{W}_\rho\} \ = \\
\int d^3x(\ \partial_\rho c^a \frac{\delta}{\delta c^a} + \partial_\rho d^a \frac{\delta}{\delta d^a} + \partial_\rho A^a_\mu \frac{\delta}{\delta A^a_\mu} + \partial_\rho b^a \frac{\delta}{\delta b^a} + \partial_\rho \Omega^a_\mu \frac{\delta}{\delta \Omega^a_\mu} + \partial_\rho L^a \frac{\delta}{\delta L^a} \ )
\end{aligned}
\tag{2.10}
$$

The equation (2.10) shows that the anticommutator between D_Σ and $\tilde{W}_\rho$ gives a translation, so, on the space of the local integrated functionals, D_Σ and $\tilde{W}_\rho$ can be considered as anticommuting operators.

To make the discussion more simple one can impose also the equation of motion of the Lagrangian multiplier d^a:

$$
\frac{\delta\Sigma}{\delta d^a} = \partial^\mu A^a_\mu \tag{2.11}
$$

which implies that the antighost b^a appears in the action Σ only through the combination

$$
\gamma^{a\mu} = \partial^\mu b^a + \Omega^{a\mu} \tag{2.12}
$$

The stability equations (2.6) become

$$
\begin{aligned}
D_{\hat{\Sigma}} \bar{\Sigma} &= 0 \\
\tilde{W}_\rho \bar{\Sigma} &= 0
\end{aligned}
\tag{2.13}
$$

where

$$D_{\hat{\Sigma}} = \int d^3x \left(\frac{\delta\hat{\Sigma}}{\delta\gamma^{a\mu}}\frac{\delta}{\delta A^a_\mu} + \frac{\delta\hat{\Sigma}}{\delta A^{a\mu}}\frac{\delta}{\delta\gamma^a_\mu} + \frac{\delta\hat{\Sigma}}{\delta L^a}\frac{\delta}{\delta c^a} + \frac{\delta\hat{\Sigma}}{\delta c^a}\frac{\delta}{\delta L^a} \right) \tag{2.14}$$

$$\tilde{W}_\rho = \int d^3x(\ \varepsilon_{\mu\rho\nu}\gamma^{a\nu}\frac{\delta}{\delta A^a_\mu} - A^a_\rho\frac{\delta}{\delta c^a} + -L^a\frac{\delta}{\delta\gamma^{a\rho}} \ ) \tag{2.15}$$

$$\hat{\Sigma}(A,c,L,\gamma) = \ S_{inv}(A) + \int d^3x \left(-\gamma^{a\mu}(D_\mu c)^a + L^a\frac{f^{abc}}{2}c^b c^c \right) \tag{2.16}$$

and $\bar{\Sigma}$ is a local functional of A, c, L, γ with dimension 3 and zero Faddeev-Popov charge. The most general form for $\bar{\Sigma}(A,c,L,\gamma)$ reads:

$$\begin{aligned} \bar{\Sigma}(A,c,L,\gamma) = &-\frac{1}{2}\int d^3x \ \varepsilon^{\mu\nu\rho}(\ \delta A^a_\mu\partial_\nu A^a_\rho + \frac{1}{3}\alpha f^{abc}A^a_\mu A^b_\nu A^c_\rho \ ) \\ &+ \int d^3x(\ \frac{1}{2}l^{abc}L^a c^b c^c + \omega^{abc}\gamma^{a\mu}A^b_\mu c^c + \beta\gamma^{a\mu}\partial_\mu c^a \ ) \end{aligned} \tag{2.17}$$

where δ, α, β are arbitrary parameters and l^{abc}, ω^{abc} are invariant tensors.

The Slavnov-Taylor condition

$$D_{\hat{\Sigma}}\bar{\Sigma} = \ 0 \tag{2.18}$$

gives

$$\begin{aligned} l^{abc} &= (\alpha - \beta - \delta)f^{abc} \\ \omega^{abc} &= \ -(\alpha - \beta - \delta)f^{abc} \end{aligned} \tag{2.19}$$

Finally, from

$$\tilde{W}_\rho\bar{\Sigma} = \ 0 \tag{2.20}$$

one has

$$\delta = \ \beta = \ 0 \tag{2.21}$$

Then, the most general counterterm is

$$\begin{aligned} \bar{\Sigma}_{count}(A,c,L,\gamma) = &-\frac{1}{6}\alpha\int d^3x \ \varepsilon^{\mu\nu\rho}f^{abc}A^a_\mu A^b_\nu A^c_\rho \\ &+ \alpha\int d^3x(\ \frac{1}{2}f^{abc}L^a c^b c^c - f^{abc}\gamma^{a\mu}A^b_\mu c^c \ ) \end{aligned} \tag{2.22}$$

and corresponds to a possible renormalization of the coupling constant of the model.

It is interesting to note that in (2.22) there are no quadratic terms. An explicit computation up to two loops [15] shows that α vanishes; however, a complete proof of the finiteness of the Chern-Simons model has not yet been established.

3. Anomalies.

To complete the discussion of the renormalization of the Chern-Simons model let us show that the symmetries considered in the previous chapter are not anomalous.

By using the fact that in three dimensions there are no gauge anomalies [15] we focus on the operator $\tilde{W}_\rho$ in (2.15).

The Quantum Action Principle tells us that the breaking term $\Delta_\rho(A,c,\gamma,L)$ associated to $\tilde{W}_\rho$ is an integrated local functional which carries negative unit of Faddeev-Popov charge, dimensions 4 and which satisfies the consistency conditions

$$\begin{aligned} \tilde{W}_\rho \Delta_\sigma + \tilde{W}_\sigma \Delta_\rho &= 0 \\ D_{\hat{\Sigma}} \Delta_\rho &= 0 \end{aligned} \tag{3.1}$$

The most general form for Δ_ρ reads:

$$\begin{aligned} \Delta_\rho = &\int d^3x(\ \alpha L^a \partial_\rho c^a + \beta \gamma^{a\mu} \partial_\rho A^a_\mu + \omega \gamma^{a\mu} \partial_\mu A^a_\rho \ ) \\ &\int d^3x(\ \lambda^{abc} L^a A^b_\rho c^c + \mu^{(ab)c} \varepsilon_{\mu\nu\rho} \gamma^{a\mu} \gamma^{b\nu} c^c + \\ &\qquad \chi^{abc} \gamma^{a\mu} A^b_\mu A^c_\rho + \eta^{a(bc)} \gamma^a_\rho A^{b\mu} A^c_\mu \ ) \end{aligned} \tag{3.2}$$

where α, β, ω are arbitrary parameters and $\lambda^{abc}, \mu^{(ab)c}, \chi^{abc}, \eta^{a(bc)}$ invariant tensors.

The condition

$$\tilde{W}_\rho \Delta_\sigma + \tilde{W}_\sigma \Delta_\rho = \ 0 \tag{3.3}$$

gives

$$\begin{aligned} \eta^{a(bc)} &= 0 \\ \lambda^{a(bc)} &= - \ \chi^{a(bc)} \\ \mu^{(ab)c} &= \chi^{(ab)c} \\ \omega &= - \ (\alpha + \beta) \end{aligned} \tag{3.4}$$

and

$$D_{\hat{\Sigma}} \Delta_\rho = \ 0 \tag{3.5}$$

gives

$$\begin{aligned} \lambda^{a[bc]} &= 0 \\ \chi^{a(bc)} &= 0 \\ \chi^{abc} &= - \ (\alpha + \beta) f^{abc} \end{aligned} \tag{3.6}$$

Thus, using (3.4), (3.6), the expression (3.2) can be written as

$$\Delta_\rho = \tilde{W}_\rho(\alpha\hat{\Sigma} - (\alpha+\beta)S_{inv}(A)) \tag{3.7}$$

showing that the possible breaking term (3.2) can be reabsorbed in an invariant Slavnov way.

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APPENDIX A. Normal coordinates expansion.

We give here the explicit expansion in normal coordinates of the Friedan connection $F^i_k(\varphi,\xi)$ and of the metric $\tilde{g}_{ij}(\varphi,\xi)$.

The normal coordinates expansion can be computed by using the formulas:

$$\phi^i(x) = \varphi^i + \pi^i(\varphi,\xi) \qquad (a.1)$$

$$\pi^i(\varphi,\xi) = \xi^i - \sum_{n=2}^{\infty} \frac{1}{n!}\xi^{j_1}.....\xi^{j_n}\Gamma^i_{j_1....j_n}(\varphi) \qquad (a.2)$$

$$\Gamma^i_{j_1....j_n}(\varphi) = \tilde{\nabla}_{j_1}.....\tilde{\nabla}_{j_n-2}\Gamma^i_{j_{n-1},j_n}(\varphi) \qquad (a.3)$$

The expansion of the Friedan connection F^i_k follows from

$$\delta^i_k + \frac{\partial \pi^i}{\partial \varphi^k} + F^l_k \frac{\partial \pi^i}{\partial \xi^l} = 0 \qquad (a.4)$$

and gives:

$$\begin{aligned} F^j_i =& - \delta^j_i - \Gamma^j_{ik}(\varphi)\xi^k - \frac{1}{3}R^j_{kli}(\varphi)\xi^k\xi^l - \frac{1}{12}(\nabla_p R^j_{kli}(\varphi))\xi^p\xi^k\xi^l \\ &- (\ \frac{1}{60}\nabla_p\nabla_q R^j_{kli}(\varphi) - \frac{1}{45}R^j_{klm}(\varphi)R^m_{pqi}(\varphi)\ )\xi^k\xi^l\xi^p\xi^q + ... \end{aligned} \qquad (a.5)$$

For the metric $\tilde{g}_{ij}(\varphi,\xi)$ one has

$$\tilde{g}_{ij}(\varphi,\xi) = g_{mn}(\phi)\frac{\partial \phi^m}{\partial \xi^i}\frac{\partial \phi^n}{\partial \xi^j} \qquad (a.6)$$

from which

$$\begin{aligned} \tilde{g}_{ij}(\varphi,\xi) =& \ g_{ij}(\varphi) + \frac{1}{3}R_{iklj}(\varphi)\xi^k\xi^l + \frac{1}{6}(\nabla_k R_{ilnj}(\varphi))\xi^k\xi^l\xi^n \\ &+ \frac{1}{20}(\ \nabla_m\nabla_n R)_{iklj}(\varphi) + \frac{8}{9}R^j_{klm}(\varphi)R_{pmnj}(\varphi)\ )\xi^k\xi^l\xi^m\xi^n + ... \end{aligned} \qquad (a.7)$$

Finally, for the Riemann connection associated to $\tilde{g}_{ij}(\varphi,\xi)$

$$\tilde{\Gamma}^i_{jk}(\varphi,\xi) = \frac{1}{2}\tilde{g}^{il}(\frac{\partial \tilde{g}_{kl}}{\partial \xi^j} + \frac{\partial \tilde{g}_{jl}}{\partial \xi^k} - \frac{\partial \tilde{g}_{jk}}{\partial \xi^l}) \qquad (a.8)$$

one finds:

$$\tilde{\Gamma}^i_{jk}(\varphi,\xi) = \frac{1}{3}(R^i_{jnk}(\varphi) + R^i_{knj}(\varphi))\xi^n + ... \qquad (a.9)$$

In the equations (a.5), (a.7), (a.9) $\Gamma^j_{ik}(\varphi)$ and $R^i_{jnk}(\varphi)$ are the Riemann connection and the Riemann tensor built with the metric $g_{ij}(\varphi)$.

APPENDIX B. Cohomology of nilpotent operators.

In this Appendix we present a general method [6] for computing the cohomology of a nilpotent BRS functional operator whose linearized form may be written as

$$\begin{aligned} D_\Sigma &= \int d^\lambda x \left(\frac{\delta\Sigma}{\delta\phi}\frac{\delta}{\delta\gamma} + \frac{\delta\Sigma}{\delta\gamma}\frac{\delta}{\delta\phi} \right) \\ D_\Sigma D_\Sigma &= 0 \end{aligned} \tag{b.1}$$

In (b.1) ϕ denotes a set of fields, γ a set of external sources, λ the dimensions of the flat space-time and Σ the integrated local functional which represents the classical action of the considered model. The expression (b.1) is the typical form of the linearized BRS operator which enters into the characterization of the counterterms and of the possible anomalies associated to a given classical symmetry.

As a functional differential operator D_Σ acts on a linear space L whose generic element is the integral of a local polynomial in the external sources γ, in the fields ϕ and their derivatives.

The problem is to find the general solution of the integrated equation

$$D_\Sigma Y = 0 \tag{b.2}$$

where

$$Y = \int d^\lambda x \; Y(\phi(x), \gamma(x)) \tag{b.3}$$

is an integrated local functional of the fields and the sources.

The first step in the analysis of D_Σ consists of its reduction to an ordinary differential operator together with the reduction of a linear space L to an ordinary space of functions, i.e., the equation (b.2) is translated into a local equation:

$$\begin{aligned} D_\Sigma Y &= 0 \quad \longrightarrow \\ d_\Sigma Y(\phi(x), \gamma(x)) &= 0 \quad \text{" } modulo\ d \text{ "} \end{aligned} \tag{b.4}$$

where d_Σ is an ordinary differential nilpotent operator and " *modulo d* " takes into account the surface terms which may appear when one translates an integrated cohomology to a local one.

There are cases, like the bosonic σ-model without torsion, where one can forget about the right-hand side of (b.4); so we focus on the simpler local equation

$$d_\Sigma Y(\phi(x), \gamma(x)) = 0 \tag{b.5}$$

A detailed discussion of the general case (b.4) for a bosonic σ-model with torsion has been done in [3].

The elements of $Y(\phi(x), \gamma(x))$ are local polynomials spanned by the variables:

$$\gamma(x), \quad \phi(x), \quad \phi_{\mu_1....\mu_k}(x) = \partial_{\mu_1}......\partial_{\mu_k}\phi(x) \tag{b.6}$$

To the local space $Y(\phi(x), \gamma(x))$ can be given the structure of a Fock space H_F.

One introduces a vacuum state denoted by $|\psi\rangle$; the elements of H_F are generated by the action of $\gamma(x)$, $\phi(x)$ and $\phi_{\mu_1....\mu_k}(x)$ on $|\psi\rangle$. In practice $\gamma(x)$, $\phi(x)$ and $\phi_{\mu_1....\mu_k}(x)$ become equivalent to creation operators.

One introduces also the adjoint operators $\gamma^\dagger(x)$, $\phi^\dagger(x)$ and $\phi^\dagger_{\mu_1....\mu_k}(x)$, with the following properties:

$$[\phi^\dagger_{\nu_1....\nu_j}(x), \phi_{\mu_1....\mu_l}(x)] = \delta_{\nu_1....\nu_j, \mu_1....\mu_l}\delta_{jl} \tag{b.7}$$

if ϕ is bosonic and:

$$\{\phi^\dagger_{\nu_1....\nu_j}(x), \phi_{\mu_1....\mu_l}(x)\} = \delta_{\nu_1....\nu_j, \mu_1....\mu_l}\delta_{jl} \tag{b.8}$$

if ϕ is an anticommuting field.

$$\begin{aligned} \delta_{\nu_1....\nu_k, \mu_1....\mu_k} &= \frac{1}{k!}\sum_{(1,...,k)} \delta_{\nu_1\mu_{i_1}}.....\delta_{\nu_k\mu_{i_k}} \\ \sum_{(1,...,k)} &= \textit{sum over the } k! \textit{ permutations of } (1,...,k) \textit{ into } (i_1,...,i_k) \end{aligned} \tag{b.9}$$

The operators $\gamma^\dagger(x)$, $\phi^\dagger(x)$ and $\phi^\dagger_{\mu_1....\mu_k}(x)$ annihilate the vacuum $|\psi\rangle$, i.e.

$$\gamma^\dagger(x)\,|\psi\rangle = \phi^\dagger(x)\,|\psi\rangle = \phi^\dagger_{\mu_1....\mu_k}(x)\,|\psi\rangle = 0 \tag{b.10}$$

As an example the adjoint of $\phi_\mu = \partial_\mu\phi$ is:

$$\phi^\dagger_\mu = \frac{\partial}{\partial\phi_\mu} = \frac{\partial}{\partial\partial_\mu\phi} \tag{b.11}$$

The relations (b.7), (b.8) induce in H_F the standard scalar product that can be defined in a usual Fock space. We will denote this scalar product by $<\,|\,>$.

On the space H_F one can define the adjoint operator $d^\dagger_\Sigma$ and the Laplace-Beltrami operator Δ_Σ defined as:

$$\Delta_\Sigma = d_\Sigma d^\dagger_\Sigma + d^\dagger_\Sigma d_\Sigma \qquad (b.12)$$

The Fock space H_F can be decomposed into subspaces according to the eigenvalues q of the Faddeev-Popov charge $Q_{\phi\pi}$

$$H_F = \bigoplus_q \ H_F^{(q)} \qquad (b.13)$$

One defines the q-cohomology of d_Σ:

$$H^q(d_\Sigma) = \{ \ \omega \in H_F^{(q)}, \ d_\Sigma \omega = 0 \ \ and \ \ \omega \neq d_\Sigma \eta \ \} \qquad (b.14)$$

As the second step in the analysis of d_Σ we introduce in H_F a self-adjoint operator N with integer non-negative eigenvalues $\nu = 0, 1, ...$; which commutes with $Q_{\phi\pi}$.

The operator N induces a separation in d_Σ

$$d_\Sigma = \sum_{n=0}^{\overline{n}} d_\Sigma^{(n)} \qquad (b.15)$$

($\overline{n}$ can eventually be infinite), and

$$[N, d_\Sigma^{(n)}] = n \ d_\Sigma^{(n)} \qquad (b.16)$$

From $d_\Sigma d_\Sigma = 0$ it follows that

$$\sum_{p=0}^{n} d_\Sigma^{(p)} d_\Sigma^{(n-p)} = 0 \qquad n = 0, 1....... \qquad (b.17)$$

From (b.17) one has, for $d_\Sigma^{(0)}$

$$d_\Sigma^{(0)} d_\Sigma^{(0)} = 0$$

The operator N decomposes the space $Y(\phi(x), \gamma(x))$ according to its eigenvalues:

$$Y(\phi(x), \gamma(x)) = \sum_{\nu=0}^{\infty} Y^{(\nu)}(\phi(x), \gamma(x)) \qquad (b.18)$$

and the equation $d_\Sigma Y(\phi(x), \gamma(x)) = 0$ becomes

$$\sum_{p=0}^{n} d_\Sigma^{(p)} Y^{(n-p)}(\phi(x), \gamma(x)) = 0 \qquad n = 0, 1, ... \qquad (b.19)$$

The following theorem holds:

Theorem.

In every eigenspace q of $Q_{\phi\pi}$, $H^q(d_\Sigma)$ is trivial if $H^q(d_\Sigma^{(0)})$ is empty.

The theorem is an almost trivial consequence of the following lemma:

Lemma.

$H^q(d_\Sigma)$ is isomorphic to the kernel of the operator Δ_Σ.

Proof.

$\text{Ker}(\Delta_\Sigma)$ is spanned by the vectors ω satisfying

$$\begin{aligned} d_\Sigma \omega &= 0 \\ d_\Sigma^\dagger \omega &= 0 \end{aligned} \qquad (b.20)$$

since:

$$< \omega \mid \Delta_\Sigma \omega > = < d_\Sigma \omega \mid d_\Sigma \omega > + < d_\Sigma^\dagger \omega \mid d_\Sigma^\dagger \omega > \qquad (b.21)$$

Now if ω belongs to the intersection of $\ker(d_\Sigma)$ and $\ker(d_\Sigma^\dagger)$ the equation $\omega = d_\Sigma \eta$ has no solution for η since otherwise:

$$< \omega \mid \omega > = < \omega \mid d_\Sigma \eta > = < d_\Sigma^\dagger \omega \mid \eta > = 0 \qquad (b.22)$$

and hence ω is trivial. Vice versa, if $d_\Sigma \omega = 0$ and if the equation $\omega = d_\Sigma \eta$ has no solution for η, $d_\Sigma^\dagger \omega$ must vanish, since otherwise

$$\eta = \frac{1}{\Delta_\Sigma} d_\Sigma^\dagger \omega \qquad (b.23)$$

would contradict the second hypothesis.

Notice that $d_\Sigma^\dagger \omega$ is orthogonal to $\ker(\Delta_\Sigma)$ since $< \alpha \mid d_\Sigma^\dagger \omega > \; = \; < d_\Sigma \alpha \mid \omega > \; = 0$ if $\alpha \in \ker(\Delta_\Sigma)$. This completes the proof of the lemma.

Now concerning the theorem, let us write

$$d_\Sigma = d_\Sigma^{(0)} + d_\Sigma^{(R)} \qquad (b.24)$$

An element τ of $\ker(\Delta_\Sigma)$ with q Faddeev-Popov charge $Q_{\phi\pi}$ satisfies:

$$d_\Sigma^{(0)} \tau + d_\Sigma^{(R)} \tau = d_\Sigma^{(0)\dagger} \tau + d_\Sigma^{(R)\dagger} \tau = 0 \qquad (b.25)$$

Decomposing τ according to the different eigenspaces of N

$$\tau = \sum_{\nu=\overline{\nu}} \tau^{(\nu)} \tag{b.26}$$

where $\tau^{(\overline{\nu})}$ is the first non-trivial component, we have

$$d_{\Sigma}^{(0)} \tau^{(\overline{\nu})} = d_{\Sigma}^{(0)\dagger} \tau^{(\overline{\nu})} = 0 \tag{b.27}$$

hence $\tau^{(\overline{\nu})}$ belongs to the $\ker(\Delta_{\Sigma}^{(0)})$

$$\Delta_{\Sigma}^{(0)} = d_{\Sigma}^{(0)} d_{\Sigma}^{(0)\dagger} + d_{\Sigma}^{(0)\dagger} d_{\Sigma}^{(0)} \tag{b.28}$$

Thus if $\ker(\Delta_{\Sigma}^{(0)})$ is trivial, that of Δ_{Σ} is empty. This proves the theorem. The same argument proves the following corollary:

Corollary.

$H^q(d_{\Sigma})$ is isomorphic to a subspace of $H^q(d_{\Sigma}^{(0)})$.

Indeed the projection of any element τ of $\ker(\Delta_{\Sigma})$ on its first non-trivial component $\tau^{(\overline{\nu})}$ defines an element of $\ker(\Delta_{\Sigma}^{(0)})$ and hence the isomorphism under consideration.

APPENDIX C. Stability of the Chern-Simons action.

It is interesting to discuss the stability of the Chern-Simons model in a way which is different from the previous analysis.

The operators s and $\bar{\delta}_\rho$ in (1.4), (1.10) can be combined in a unique operator $\Im$ by means of the introduction of two global commuting parameters α, $\bar{\alpha}^\mu$.

$\Im$ is then defined as

$$\Im = \bar{\alpha}^\nu \bar{\delta}_\nu + \alpha s \tag{c.1}$$

and its action on the fields $A^{a\mu}, c^a, d^a, b^a$ reads:

$$\begin{aligned} \Im A^a_\mu &= \varepsilon_{\mu\nu\rho}\bar{\alpha}^\nu \partial^\rho b^a - \alpha(D_\mu c)^a \\ \Im c^a &= -\bar{\alpha}^\nu A^a_\nu + \alpha \frac{1}{2} f^{abc} c^b c^c \\ \Im b^a &= \alpha d^a \\ \Im d^a &= \bar{\alpha}^\nu \partial_\nu b^a \end{aligned} \tag{c.2}$$

One has

$$\begin{aligned} \Im^2 A^a_\mu &= \alpha\bar{\alpha}^\nu \partial_\nu A^a_\mu + \alpha\bar{\alpha}^\nu \varepsilon_{\nu\mu\rho} \frac{\delta S}{\delta A^a_\rho} \\ \Im^2 c^a &= \alpha\bar{\alpha}^\nu \partial_\nu c^a \\ \Im^2 b^a &= \alpha\bar{\alpha}^\nu \partial_\nu b^a \\ \Im^2 d^a &= \alpha\bar{\alpha}^\nu \partial_\nu d^a \end{aligned} \tag{c.3}$$

where

$$S = S_{inv} + \int d^3x \; d^a \partial^\mu A^a_\mu + b^a \partial^\mu (D_\mu c)^a \tag{c.4}$$

The equations (c.3) show that the operator $\Im$ is not nilpotent and that its square gives, on-shell, a translation.

Introducing the source term

$$\begin{aligned} S_s = \int d^3x (\; &\Omega^{a\mu} (\; \varepsilon_{\mu\nu\rho}\bar{\alpha}^\nu \partial^\rho b^a - \alpha(D_\mu c)^a \;) \; + \; L^a (\; - \bar{\alpha}^\nu A^a_\nu + \alpha \frac{f^{abc}}{2} c^b c^c \;) \\ &- \frac{1}{2} \varepsilon_{\mu\nu\rho} \Omega^{a\mu} \Omega^{a\nu} \alpha \bar{\alpha}^\rho \;) \end{aligned} \tag{c.5}$$

one has the Slavnov identity

$$\int d^3x\Big(\frac{\delta\Sigma}{\delta\Omega^{a\mu}}\frac{\delta\Sigma}{\delta A^a_\mu}+\frac{\delta\Sigma}{\delta L^a}\frac{\delta\Sigma}{\delta c^a} + \bar{\alpha}^\rho\partial_\rho b^a\frac{\delta\Sigma}{\delta d^a} + \alpha d^a\frac{\delta\Sigma}{\delta b^a} \Big) = \alpha\bar{\alpha}^\nu \int d^3x\Big(-\Omega^{a\mu}\partial_\nu A^a_\mu + L^a\partial_\nu c^a \Big) \tag{c.6}$$

where

$$\Sigma = S + S_s \tag{c.7}$$

The presence of the non linear source term in (c.5) takes into account the fact that the operator $\Im$ forms with the translations a closed algebra only on-shell.

Acting with $\frac{\delta}{\delta d^a}$ on the Slavnov identity (c.6) and using the equation of motion of the Lagrangian multiplier

$$\frac{\delta\Sigma}{\delta d^a} = \partial^\mu A^a_\mu \tag{c.8}$$

one gets

$$\partial^\mu\frac{\delta\Sigma}{\delta\Omega^{a\mu}} + \alpha\frac{\delta\Sigma}{\delta b^a} = 0 \tag{c.9}$$

which implies that Σ depends on the variables $\Omega^{a\mu}, b^a$ only through the combination

$$\alpha\gamma^{a\mu} = \alpha\Omega^{a\mu} + \partial^\mu b^a \tag{c.10}$$

Introducing the local functional $\hat{\Sigma}(A,c,\gamma,L,\alpha,\bar{\alpha})$

$$\hat{\Sigma}(A,c,\gamma,L,\alpha,\bar{\alpha}) = S_{inv} - \int d^3x\Big(\alpha\gamma^{a\mu}(D_\mu c)^a + \frac{1}{2}\varepsilon_{\mu\nu\rho}\gamma^{a\mu}\gamma^{a\nu}\alpha\bar{\alpha}^\rho \Big) + \int d^3x L^a\Big(-\bar{\alpha}^\nu A^a_\nu + \alpha\frac{f^{abc}}{2}c^b c^c \Big) \tag{c.11}$$

the Slavnov identity (c.6) becomes

$$\int d^3x \left(\frac{\delta\hat{\Sigma}}{\delta\gamma^{a\mu}}\frac{\delta\hat{\Sigma}}{\delta A^a_\mu} + \frac{\delta\hat{\Sigma}}{\delta L^a}\frac{\delta\hat{\Sigma}}{\delta c^a} \right) = \alpha\bar{\alpha}^\nu \int d^3x \Big(-\gamma^{a\mu}\partial_\nu A^a_\mu + L^a\partial_\nu c^a \Big) \tag{c.12}$$

and for the linearized operator $D_{\hat{\Sigma}}$

$$D_{\hat{\Sigma}} = \int d^3x \left(\frac{\delta\hat{\Sigma}}{\delta\gamma^{a\mu}}\frac{\delta}{\delta A^a_\mu} + \frac{\delta\hat{\Sigma}}{\delta A^{a\mu}}\frac{\delta}{\delta\gamma^a_\mu} + \frac{\delta\hat{\Sigma}}{\delta L^a}\frac{\delta}{\delta c^a} + \frac{\delta\hat{\Sigma}}{\delta c^a}\frac{\delta}{\delta L^a} \right) \tag{c.13}$$

one gets

$$D_{\hat{\Sigma}} D_{\hat{\Sigma}} = \alpha \bar{\alpha}^{\nu} \int d^3x \left(\partial_\nu A^a_\mu \frac{\delta}{\delta A^a_\mu} + \partial_\nu \gamma^a_\mu \frac{\delta}{\delta \gamma^a_\mu} + \partial_\nu c^a \frac{\delta}{\delta c^a} + \partial_\nu L^a \frac{\delta}{\delta L^a} \right) \tag{c.14}$$

Equation (c.14) shows that, on the integrated functional, $D_{\hat{\Sigma}}$ can be considered nilpotent. The mass dimensions of the fields $A^{a\mu}, c^a, \gamma^{a\mu}, L^a, \alpha, \bar{\alpha}^\rho$ are respectively 1, 0, 2, 3, 0, -1, and the assigned ghost charges 0, 1, -1, -2, 0, 2.

One has to note that the action $\hat{\Sigma}$ is invariant also under the transformations:

$$\begin{aligned} N_g^{'} A^a_\mu &= 0 \\ N_g^{'} c^a &= 0 \\ N_g^{'} \gamma^{a\mu} &= -\gamma^{a\mu} \\ N_g^{'} L^a &= -L^a \\ N_g^{'} \alpha &= \alpha \\ N_g^{'} \bar{\alpha}^\rho &= \bar{\alpha}^\rho \end{aligned} \tag{c.15}$$

Equations (c.15) describe a descrete transformation which commutes with the ghost number and which allows us to control the global parameters $\alpha, \bar{\alpha}^\rho$.

To discuss the stability of the Slavnov identity (c.12) we limit ourselves to the perturbations $\bar{\Sigma}(A, c, \gamma, L, \alpha, \bar{\alpha})$ of the classical action $\hat{\Sigma}$ which are integrated local functionals of dimension three, ghost number zero, invariant under the global transformations of the gauge group, under the transformations (c.15), and satisfying the equation

$$D_{\hat{\Sigma}} \bar{\Sigma} = 0 \tag{c.16}$$

The most general form for $\bar{\Sigma}(A, c, L, \gamma)$ reads:

$$\begin{aligned} &\bar{\Sigma}(A, c, L, \gamma, \alpha, \bar{\alpha}) = \\ &\int d^3x \, (\, a_1 \varepsilon^{\mu\nu\rho} A^a_\mu \partial_\nu A^a_\rho + a_2 \alpha \gamma^{a\mu} \partial_\mu c^a + a_3 \bar{\alpha}^\rho L^a A^a_\rho \, + a_4 \varepsilon_{\mu\nu\rho} \alpha \bar{\alpha}^\rho \gamma^{a\mu} \gamma^{a\nu} \,) \\ &+ \int d^3x \, (\, \mu^{abc} \varepsilon^{\mu\nu\rho} A^a_\mu A^b_\nu A^c_\rho + \lambda^{abc} \alpha L^a c^b c^c + \eta^{abc} \alpha \gamma^{a\mu} A^b_\mu c^c \,) \end{aligned} \tag{c.17}$$

where a_1, a_2, a_3, a_4 are arbitrary parameters and $\lambda^{abc}, \mu^{abc}, \eta^{abc}$ are invariant tensors.

The Slavnov-Taylor condition (c.16) gives

$$
\begin{aligned}
\lambda^{abc} &= \beta f^{abc} \\
\eta^{abc} &= -2\beta f^{abc} \\
\mu^{abc} &= \frac{1}{3}\left(\frac{a_3}{2} - a_1 - \beta\right) f^{abc} \\
a_1 &= -a_4 \\
a_2 &= -a_3
\end{aligned}
\tag{c.18}
$$

then, the most general counterterm can be written

$$
\begin{aligned}
&\bar{\Sigma}_{count}(A, c, L, \gamma, \alpha, \bar{\alpha}) = \\
&\quad (2\beta + a_3 + a_4) \int d^3x \left(-\frac{1}{6}\varepsilon^{\mu\nu\rho} f^{abc} A^a_\mu A^b_\nu A^c_\rho + \frac{\alpha}{2} f^{abc} L^a c^b c^c - \alpha f^{abc} \gamma^{a\mu} A^b_\mu c^c \right) \\
&\quad + D_{\hat{\Sigma}} \int d^3x \left(a_4 \gamma^{a\mu} A^a_\mu - (a_3 + a_4) L^a c^a \right)
\end{aligned}
\tag{c.19}
$$

and is equivalent, a part from a trivial cocycle of $D_{\hat{\Sigma}}$, to the counterterm of the previous analysis.

REFERENCES.

[1] D.Friedan, Phys. Rev. Lett. 45(1980)1057.

D.Friedan, Ann. Phys. (N.Y.) 163(1985)318.

[2] C.Becchi, " The non-linear sigma model. " in
" Renormalizaztion of Quantum Field-Theories with non-linear Field transformations. " Lecture Notes in Physics, 303(1988), P.Breitenlohner, D.Maison, K.Sibold (Eds.) Springer-Verlag.

[3] A.Blasi, F.Delduc and S.P.Sorella, Nucl. Phys. B314(1989)409.

S.P.Sorella " Notes on the BRS renormalization:
the non-linear sigma model. ",
Universidade Catolica de Petropolis, RJ, Brasil, UCP-89/2.

[4] F.David, Phys. Lett. 96B(1980)371.

F.David, Comm. Math. Phys. 81(1981)149.

S.Elitzur, Nucl. Phys. B212(1983)536.

C.Becchi, A.Blasi, G.Bonneau, R.Collina and F.Delduc,
Comm. Math. Phys. 120(1988)121.

[5] F.David, Nucl. Phys. B190(1981)205.

G.Curci and G.Paffuti, Preprint IFUP-TH 9/88.

[6] J.Dixon, " Cohomology and renormalization of gauge fields. ",
Imperial College Preprint (1977)unpub.

[7] E.Witten, Comm. Math. Phys. 118(1988)601.

[8] E.Witten, Comm. Math. Phys. 117(1988)353.

[9] S.Ouvry, R.Stora and P.Van Baal, Phys. Lett. B220(1989).

[10] S.P.Sorella, Phys. Lett. B228(1989)69.

[11] E.Witten, IAS Preprint, IASSNS-HEP-88/33.

[12] T.P.Killingback, Phys. Lett. B219(1989)448.

[13] D.Birmingham, M.Rakowski and G.Thompson, ICTP Preprint IC/88/387.

[14] F.Delduc, F.Gieres and S.P.Sorella, Phys. Lett. B225(1989)367.

[15] E.Guadagnini, M.Martellini and M.Mintchev, Preprint CERN-TH.5324/89.