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A Generic Classification of Time Optimal Planar Stabilizing Feedbacks

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Introduction

Let F, G be smooth vector fields on the plane, with $F(0) = 0$, and consider the problem of reaching the origin in minimum time, for the control system

$$\dot{x} = F(x) + G(x)u \qquad \qquad |u(t)| \leq 1. \tag{1}$$

Under generic assumptions on F, G, we prove that the optimal control admits a regular feedback synthesis. Namely, for any given $\tau > 0$, on the set A_τ of points which can be steered to the origin within time τ, one can define a feedback control $u = \varphi(x)$ with the following properties:

(i) The set A_τ can be partitioned into finitely many submanifolds V_i such that the restriction of φ to each V_i is smooth.

(ii) Every trajectory of the feedback equation

$$\dot{x} = F(x) + G(x)\varphi(x) \tag{2}$$

starting inside A_τ reaches the origin in minimum time.

For generic $F, G \in \mathcal{C}^3$, the optimal feedback $\varphi = \varphi_{F,G}$ is essentially unique. We thus regard (2) as a differential equation with discontinuous right hand side, uniquely determined by the vector fields F, G. Aim of this thesis is to provide a global classification of the flow generated by (2), in the generic case. This program can be outlined as follows:

1. Introduce an equivalence relation between couples of vector fields: $(F, G) \sim (F', G')$, determined by the topological equivalence of the corresponding flows (2). Roughly speaking, if $u = \varphi_{F,G}(x)$ and $u = \varphi_{F',G'}(x)$ are the corresponding time-optimal stabilizing feedbacks, the above equivalence should imply the existence of a homeomorphism, defined on a suitable subset of the plane, which maps oriented arcs of trajectories of

$$\dot{x} = F(x) + G(x)\varphi_{F,G}(x) \tag{3}$$

onto oriented arcs of trajectories of

$$\dot{x} = F(x) + G(x)\varphi_{F,G}(x). \tag{4}$$

2. With each equivalence class associate a discrete topological graph, with some additional structure, in such a way that two systems are equivalent if and only if they correspond to the same graph.

3. Show that, generically, the time-optimal feedbacks are structurally stable. In other words, given a couple of generic smooth vector fields F, G, prove that $(F', G') \sim (F, G)$ whenever F' is sufficiently close to F and G' is sufficiently close to G, in the $\mathcal{C}^3$ norm. A small perturbation of the fields F, G will not change the global structure of the optimal feedback flow.

4. Characterize the family of all graphs which arise in connection with an optimal feedback, for some system of the form (1). In practice, given a suitable graph $\mathcal{G}$, this requires the construction of vector fields F, G such that the corresponding optimal feedback flow (2) has precisely the topological structure specified by $\mathcal{G}$.

For convenience, throughout this thesis we consider the entirely equivalent problem of reaching points $x \in I\!R^2$ in minimum time, starting from the origin.

In the first chapter we introduce the problem of time optimal stabilization and prove that under generic conditions on the vector fields F, G, the system (1) admits a piecewise smooth feedback optimal control (2). First, we complete the local analysis, developed in [14]. We then determine some generic conditions in such a way that every point has a neighborhood on which every time optimal trajectory is concatentation of a finite number of arcs. Each arc either corresponds to a constant control $u \equiv \pm 1$ or is a singular arc determined by the equation (3.18) of the first chapter. Therefore, using the compactness of the reachable set R, we can obtain a uniform bound N on the number of the arcs. Obviously, N depends on the systems. In section 5, we describe an algorithm to construct a synthesis for such systems. The algorithm constructs at step n, precisely the trajectories that are concatenations of n arcs. The uniform bound N ensures that the algorithm terminates its construction in a finite number of steps. At the end of the algorithm, we obtain a family of optimal trajectories, from which we extract an optimal synthesis. Indeed, generically there are a finite number of one dimensional manifolds whose points can be reached by two distinct optimal trajectories. Some of them form the set where the minimum time function has empty subgradient. These are called overlap curves because two families of extremal trajectories, one with constant control $+1$ and one with constant control -1, *overlap* along these curves. The other curves arise as consequence of the ending of an *overlap*. See Example 3 of the second chapter. In the last section there are some examples of systems for which the algorithm works.

The boundaries of the manifolds V_i, described in (i), constitute the singularities of the feedback synthesis, that is the points where the feedback is discontinuous. A second step is the classification of these singularities, called frame curves, if they are one dimensional manifolds, and frame points. Indeed, this means to classify the local behaviour of the feedback flow, which is trivial on the regions where the feedback is constant (this is the case outside frame curves). Restricting ourselves to the generic singularities, we perform this local classification in the second chapter. First, in the third section, we exhibit some explicit examples. These and the ones in the first chapter cover all possibilities for frame curves and points. We then perfom the classification using geometric methods and genericity assumptions. In this way, we establish the existence of 6 types of frame curves and 22 types of frame points. In doing this, we introduce an equivalence relation that does not distinguish the sign of a bang-bang control. This means that trajectories with control $+1$ or -1 are not considered different. The definition of equivalence is not the same as the one used later for systems, but a bit less restrictive. In this way, we are able to generate a smaller number of equivalence classes. We avoid confusion, since locally the interchange of control $+1$ with -1 make no significant difference. On the contrary, the global synthesis can change structure after such a substitution, see Remark 4.1 of chapter three.

We have that a frame curve is of one of the following types. The trajectories $\gamma^{\pm}$ that start from the origin and correspond to the constant controls $u \equiv \pm 1$, restricted to the maximal interval on which they are extremal, are frame curves. The singular arcs called turnpikes, see section 3 of the first chapter, are frame curves. The overlap curves described above are frame curves. The frontier of the reachable set is a frame curve. Finally, there are some curves on which the extremal trajectories change the control from $+1$ to -1 or viceversa. These are called switching curves (the change of control is called switching), and are frame curves.

For every type of curve and of point it is indicated the corresponding explicit example. We then produce a first *dictionary* for the local structure of a feedback synthesis.

The third chapter complete the research program, operating a global classification and proving the structural stability of classified systems. We first describe a new algorithm that works for a smaller class of systems, but describes accurately the synthesis. A generic set of systems is singled out using the generic conditions given in the first chapter and some additional stability conditions, still valid for a generic system. In the first step of the algorithm we consider the trajectories γ^+, γ^- starting from the origin and corresponding, respectively, to constant controls $+1, -1$. These play a key role in the construction of the synthesis. We only need to consider γ^+, the study of γ^- being entirely similar. Let I^+ be the maximal interval on which γ^+ is extremal. We can divide I^+ into a finite number of subintervals I_i^+ such that the following holds. From each point of $\gamma^+(I_i^+)$ with i odd, a trajectory with constant control -1 bifurcates. Moreover, these trajectories leave going to the left or to the right of $\gamma^+(I^+)$ according to the sign of the function (3.1) of the third chapter. On the contrary, if i is even then no trajectory can arise.

Each endpoint of $\gamma^+(I^+)$ is characterized by the presence of a frame curve. More precisely, as described in Proposition 3.1 of chapter 3, if $x = \gamma^+(\inf I_i)$, where i is odd, $i > 1$, then at x it originates either a turnpike or a switching curve. If $x = \gamma^+(\sup I_i)$, i as above, then at x it originates either an overlap curve or a switching curve. We use the letters C, K, S to indicate the type of frame curves as in chapter 2. That is we use C to indicate a switching curve, K and overlap curve and S a turnpike. Moreover, we use the symbol 0 to indicate the origin and $\pm$ to express the fact that some curves originate going to the right $(+)$ or to the left $(-)$ of γ^+. With these conventions, we can express the behaviour along γ^+ giving a word formed by the symbols $0, \pm, C, K, S$. The word must start with 0, the second sign is $\pm$ and the third is S or C. Then we can put any sequence of three letters having C or K at the first place, $+$ or $-$ at the second and S or C at the third. This correspond to describe the synthesis on I_3. We can choose again a sequence of three letters with the above rules, that describe the synthesis on I_i, i odd. Given such a word, we can construct a system having the corresponding behaviour, as it is proved in section 4.

Thus from the sets $\gamma^+(I_i^+)$ some *strips* formed by trajectories having the same constant control ± 1, originate. We give the sign $\pm$ to the strip if ± 1 is the corresponding control. These strips then develop having a finite number of possible behaviours. These behaviours correspond to the types of frame curves that the strip generates or encounters. A strip can duplicate at the beginning of a turnpike, thus giving rise to two attached strips. If the strip exhibits a switching curve then it changes its sign. Finally, if the strip encounters an overlap curve then either it ends on it or it glues with another strip when the overlap terminates. For the last situation see the Example 3 of chapter two. The strips then fill the reachable set duplicating, gluing toghether and ending on overlap curves or on the frontier of the reachable set. If we know the *history* of each strip then we can recover the synthesis. Obviously, there are some topological conditions on the behaviour of the strips, due to the fact that we have to allocate them on a planar region. For these reason, we will use a topological graph to describe the synthesis, instead of describing the history of each strip.

In section 4 we give a definition of topological graph, which includes some additional structure. Defining some admissibility conditions, we single out a set of graphs which are in bijective correspondence with the equivalence classes of systems. The countable family of graphs which we obtain can be regarded as a "dictionary" of all possible structures of global optimal feedbacks, in connection with generic vector fields F, G. For the flows generated by these discontinuous feedbacks, the present classification is analogous to the classical work of Peixotò [9,10] for smooth dynamical systems. We conclude proving the structural stability of classified systems.

Chapter 1

1.Introduction

This chapter is concerned with the standard problem of reaching the origin in minimum time, for the control system

$$\dot{x} = F(x) + G(x)u, \qquad u(t) \in [-1,1] \quad \text{a.e.}, \tag{1.1}$$

where F, G are $\mathcal{C}^3$ vector fields on the plane, with $F(0) = 0$. Calling $A(\tau)$ the set of points which can be steered to the origin within a fixed time τ, by a regular optimal feedback synthesis we mean a partition of $A(\tau)$ into finitely many embedded manifolds $\mathcal{M}_i$ and a feedback control law $u = u(x)$, whose restriction to each $\mathcal{M}_i$ is smooth, such that every Carathéodory trajectory of the (usually discontinuous) O.D.E.

$$\dot{x} = F(x) + G(x)u(x)$$

starting within $A(\tau)$ reaches the origin in minimum time. In general, the attainable set $A(\tau)$ will thus be divided into finitely many open regions where $u(x) \equiv \pm 1$, separated either by switching curves, or by singular arcs, or else by "overlap curves", consisting of points which can be steered to the origin optimally by two distinct control functions.

In the case of analytic vector fields, the existence of a regular optimal feedback was established in [15,16]. In this chapter, using the Pontryagin Maximum Principle, we single out a set of generic assumption on F, G in the $\mathcal{C}^3$ norm which imply that all optimal trajectories are a finite concatenation of integral curves of the three flows

$$\dot{x} = F(x) + G(x), \qquad \dot{x} = F(x) - G(x), \qquad \dot{x} = F(x) + G(x)\varphi_S(x), \tag{1.2}$$

where the feedback function $u = \varphi_S(x)$ is defined at (3.18), in terms of F, G and of the Lie bracket $[F, G]$. A uniform bound on the number of arcs forming these trajectories is derived. Relying on this "finite dimensional reduction" of the time optimal problem, we present here an inductive algorithm for constructing a time optimal synthesis, valid for generic vector fields $F, G \in C^3$.

2.Basic definitions

Troughout this thesis, $I\!R$ denotes the real line and $I\!R^n$ the n-dimensional Euclidean space, whose points are regarded as column vectors. We write $]a,b[$, $[a,b]$, respectively, for the open and the closed interval with endpoints a,b. If $x \in I\!R^n$ and $r \geq 0$, by $B(x,r)$, $\bar{B}(x,r)$ we denote respectively the open and the closed ball centered at x with radius r. Given a set $C \subset I\!R^n$ we write $Int(C)$, $Cl(C)$ and $Fr(C)$ for the interior, the closure and the topological frontier of C. If C is a submanifold with boundary, we will use the symbol ∂C to denote the boundary of C.

A *curve* in $I\!R^n$ is a continuous map $\gamma : I \mapsto I\!R^n$, where I is some real interval. Its domain is thus $Dom(\gamma) = I$. If $x \in \{\gamma(t) : t \in Dom(\gamma)\}$, we simply write $x \in \gamma$. The symbol $\gamma \restriction J$, where $J \subset Dom(\gamma)$ is an interval, denotes the restriction of γ to J. More generally, the same symbol will be used to indicate the restriction of a mathematical object to some subset of his domain of definition (e.g. if Σ is a control system defined in $I\!R^2$ and $U \subset I\!R^2$ is an open set then $\Sigma \restriction U$ is the restriction of Σ to U).

A C^3 *vector field* on $I\!R^2$ is a C^3 map $F : I\!R^2 \mapsto I\!R^2$. As usual, the space of vector fields $F = (F_1, F_2)$ whose partial derivatives of order ≤ 3 are bounded on $I\!R^2$, will be endowed with the norm:

$$\|F\|_{C^3} = \sup \ \{D^\alpha F_i(x) : x \in I\!R^2, |\alpha| \leq 3, i = 1,2\}$$

where $D^\alpha = D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2}$ is a differential operator of order $|\alpha| = \alpha_1 + \alpha_2$.

A vector field F on $I\!R^2$ can be written in the form:

$$F \ = \ \alpha \, \partial_{x_1} \ + \ \beta \, \partial_{x_2} \tag{2.1}$$

where ∂_{x_1}, ∂_{x_2} are the constant vector fields with components (1,0), (0,1), respectively. By ∇F we denote the Jacobian matrix of first order partial derivatives of the vector field F in (2.1). We write $e^{tF}(\bar{x})$ for the value at time t of the solution of the Cauchy problem:

$$\dot{x} = F(x) \qquad x(0) = \bar{x}, \tag{2.2}$$

while $\left(e^{tF}\right)_*$ will denote the Jacobian matrix of the map

$$x \mapsto e^{tF}(x). \tag{2.3}$$

We recall that the *Lie bracket* of two vector fields F, G is the vector field

$$[F, G] \doteq \nabla G \cdot F - \nabla F \cdot G \tag{2.4}$$

This Lie bracket can be interpreted as the derivative of the vector field G along the flow of F. Indeed,

$$[F, G](x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(e^{-\varepsilon F}\right)_* G\left(e^{\varepsilon F}(x)\right). \tag{2.5}$$

We use the symbol Ξ to denote the subspace of all pairs of vector fields on the plane $(F, G) \in \mathcal{C}^3 \times \mathcal{C}^3$ with $F(0) = 0 \in \mathbb{R}^2$. From now on, we fix a pair $(F, G) \in \Xi$ and consider the control system:

$$\dot{x} = F(x) + uG(x), \qquad |u| \leq 1. \tag{2.6}$$

We shall usually write $\Sigma = (F, G)$ to indicate the control system (2.6).

A *control* is a measurable function $u : [a, b] \mapsto [-1, 1]$ where $-\infty < a \leq b < +\infty$. A *trajectory of* Σ corresponding to u is an absolutely continuous curve $\gamma : Dom(u) \mapsto \mathbb{R}^2$ which satisfies the equation:

$$\dot{\gamma}(t) = F(\gamma(t)) + u(t)G(\gamma(t)) \tag{2.7}$$

for almost every t in the domain of u. The set of all traiectories of Σ is denoted by $Traj(\Sigma)$. If $\gamma : [a, b] \mapsto \mathbb{R}^2$ is a trajectory of Σ we define the initial and terminal points of γ as $In(\gamma) \doteq \gamma(a)$ and $Term(\gamma) \doteq \gamma(b)$. The time along γ is defined as

$$T(\gamma) = b - a. \tag{2.8}$$

A trajectory $\gamma \in Traj(\Sigma)$ is *time optimal* if, for every trajectory $\gamma' \in Traj(\Sigma)$ with $In(\gamma') = In(\gamma)$ and $Term(\gamma') = Term(\gamma)$, one has $T(\gamma') \geq T(\gamma)$.

If $u_1 : [a, b] \mapsto [-1, 1]$ and $u_2 : [b, c] \mapsto [-1, 1]$ are controls, their *concatenation* $u_2 * u_1$ is the control:

$$(u_2 * u_1)(t) \doteq \begin{cases} u_1(t) & \text{for } t \in [a, b], \\ u_2(t) & \text{for } t \in (b, c]. \end{cases}$$

If $\gamma_1 : [a,b] \mapsto \mathbb{R}^2$, $\gamma_2 : [b,c] \mapsto \mathbb{R}^2$ are trajectories of Σ for u_1 and u_2 such that $\gamma_1(b) = \gamma_2(b)$, then the *concatenation* $\gamma_2 * \gamma_1$ is the trajectory:

$$(\gamma_2 * \gamma_1)(t) = \begin{cases} \gamma_1(t) & \text{for } t \in [a,b], \\ \gamma_2(t) & \text{for } t \in [b,c]. \end{cases}$$

For convenience, we also define the vector fields

$$X = F - G, \qquad Y = F + G.$$

We use $Traj(X)$ $[Traj(Y)]$ to denote the set of all trajectories of Σ which correspond to the constant control $u \equiv -1$ $[u \equiv 1]$. Elements of $Traj(X)$, $Traj(Y)$ will be called *X-trajectories* and *Y-trajectories*, respectively. A *bang-bang trajectory* is a trajectory obtained as a finite concatenation of X- and Y-trajectories. We write $Traj(\xi_1 * \cdots * \xi_n)$, where $\xi_i = X$ or $\xi_i = Y$, to denote the set of all concatenations $\gamma = \gamma_1 * \cdots * \gamma_n$ where $\gamma_i \in Traj(\xi_i)$ and is not trivial, i.e. its domain is not a single point. We also say that γ is of type $\xi_1 * \cdots * \xi_n$ (see [14] for complete description of this notation).

Instead of steering the system to the origin in minimum time, throughout the following we shall consider the entirely equivalent problem of reaching points in $\mathbb{R}^2$ in minimum time, starting from the origin. If $\tau \geq 0$, we denote by $R(\tau)$ the *reachable set* within time τ:

$$R(\tau) = \{x : \exists \gamma \in Traj(\Sigma) \quad \text{s.t. } \gamma(0) = 0 \in \mathbb{R}^2, \ \gamma(t) = x, \ \text{ for some } t \leq \tau\}. \tag{2.9}$$

The *minimum time function*, $T : \mathbb{R}^2 \mapsto [0, +\infty]$ is defined by

$$T(x) \doteq \inf\{\tau : x \in R(\tau)\}. \tag{2.10}$$

Recalling (2.9) we have:

$$T(x) = \inf \{T(\gamma) : \gamma \in Traj(\Sigma), \ In(\gamma) = 0, \ Term(\gamma) = x\}.$$

Clearly, a trajectory $\gamma \in Traj(\Sigma)$, with $Dom(\gamma) = [0,b]$, $In(\gamma) = 0$, is optimal if and only if $T(\gamma(b)) = b$. In this case we write $\gamma \in Opt(\Sigma)$.

The convexity of the set $\{F(x) + uG(x) : |u| \leq 1\}$ and the bound on the derivatives of F and G imply the following:

Lemma 2.1 *If $0 \leq \tau < +\infty$ then the set $R(\tau)$ is compact.*

Lemma 2.2 *For any* $x \in I\!R^2$*, if* $T(x) = \tau$ *then there exists* $\gamma \in Traj(\Sigma)$ *such that* $\gamma(0) = 0,\ \gamma(\tau) = x$.

For the proof see [7] Th. 20.1 p.107.

The control system Σ is *locally controllable* if, for each $\tau > 0$, the set $R(\tau)$ contains a neighborhood of the origin. The following results are well known [8, p.366]:

Lemma 2.3 *If the system* Σ *is locally controllable then the minimum time function is continuous and, for every* $\tau > \sigma > 0$*, one has*

$$R(\sigma) \subseteq Int(R(\tau)). \tag{2.11}$$

Lemma 2.4 *If* $F(0) = 0$ *and the vector fields* G*,* $[F, G]$ *are linearly independent at the origin, then the system* $\Sigma = (F, G)$ *in (2.6) is locally controllable.*

A *synthesis* for the control system Σ at time τ is a family $\Gamma = \{\gamma_x : [0, b_x] \mapsto I\!R^2 | x \in R(\tau)\}$ of trajectories satisfying the following conditions:

a) For each $x \in R(\tau)$ one has $\gamma_x(0) = 0,\ \gamma_x(b_x) = x$

b) If $y = \gamma_x(t)$ where $t \in Dom(\gamma)$ then $\gamma_y = \gamma_x \restriction [0, t]$.

A synthesis for the system Σ is *time optimal* if, for each $x \in R(\tau)$, one has $\gamma_x(T(x)) = x$, where T is the minimum time function defined at (2.10).

3.Pontryagin maximum principle and special curves

An *admissible pair* for the system Σ is a couple (u, γ) such that u is a control and γ is a trajectory corresponding to u. We use the symbol $Adm(\Sigma)$ to denote the set of admissible pairs and we say that $(u, \gamma) \in Adm(\Sigma)$ is optimal if γ is optimal.

A *variational vector field along* $(u, \gamma) \in Adm(\Sigma)$ is a vector-valued absolutely continuous function $v : Dom(\gamma) \mapsto I\!R^2$ that satisfies the equation:

$$\dot{v}(t) = \Big((\nabla F)(\gamma(t)) + u(t)(\nabla G)(\gamma(t)) \Big) \cdot v(t) \tag{3.1}$$

for almost all $t \in Dom(\gamma)$. Observe that the system (3.1) is linear, homogeneus, with bounded, measurable coefficients. Therefore, it admits one and only one solution, for any initial condition $v(t_0) = v_0$.

A *variational covector field along* $(u,\gamma) \in Adm(\Sigma)$ is an absolutely continuous function $\lambda : Dom(\gamma) \mapsto I\!R^2_*$ that satisfies the equation:

$$\dot{\lambda}(t) = -\lambda(t) \cdot \Big((\nabla F)(\gamma(t)) + u(t)(\nabla G)(\gamma(t)) \Big) \tag{3.2}$$

for almost all $t \in Dom(\gamma)$. Here $I\!R^2_*$ denotes a space of row vectors. It is well known [14] that variational vector and covector fields are invariant under changes of coordinates.

Lemma 3.1 *Let* $(u,\gamma) \in Adm(\Sigma)$ *and let* $\lambda : Dom(\gamma) \mapsto I\!R^2_*$ *be absolutely continuous. Then* λ *is a variational covector field along* (u,γ) *if and only if the function* $t \mapsto \lambda(t) \cdot v(t)$ *is constant for every variational vector field* v *along* (u,γ).

The Hamiltonian $\mathcal{H} : I\!R^2_* \times I\!R^2 \times I\!R \mapsto I\!R$ is defined as

$$\mathcal{H}(\lambda, x, u) = \lambda \cdot (F(x) + uG(x)). \tag{3.3}$$

If λ is a variational covector field along $(u,\gamma) \in Adm(\Sigma)$, we say that λ is *maximizing* if:

$$\mathcal{H}(\lambda(t), \gamma(t), u(t)) = \max \{\mathcal{H}(\lambda(t), \gamma(t), w) : |w| \leq 1\} \tag{3.4}$$

for almost all $t \in Dom(\gamma)$.

The *Pontryagin Maximum Principle* (PMP) states that, if $(u,\gamma) \in Adm(\Sigma)$ is time optimal, then there exists:

(PMP1) *A non trivial maximizing variational covector field* λ *along* (u,γ)

(PMP2) *A constant* $\lambda_0 \leq 0$ *such that :* $\mathcal{H}(\lambda(t), \gamma(t), u(t)) + \lambda_0 = 0$ for almost all $t \in Dom(\gamma)$.

In this case λ is called an *adjoint covector field along* (u,γ) or simply an *adjoint variable*, and we say that (γ, λ) satisfies the PMP, or that γ is an *extremal trajectory*.

A pair $(u,\gamma) \in Adm(\Sigma)$ is *extremal* if γ is extremal. A synthesis Γ for a control system Σ is *extremal* if every $\gamma \in \Gamma$ is extremal.

If λ is an adjoint covector field along $(u,\gamma) \in Adm(\Sigma)$, the corresponding *switching function* is defined as

$$\phi_\lambda(t) = \lambda(t) \cdot G(\gamma(t)). \tag{3.5}$$

From the above definition it follows

Lemma 3.2 *Let* $(u,\gamma) \in Adm(\Sigma)$ *be extremal and let* λ *be an adjoint covector field along* (u,γ). *Then:*

a) The switching function ϕ_λ *is continuous.*

b) If $\phi(t) > 0$ *for all* t *in some interval* I*, then* $u(t) \equiv 1$ *for almost all* $t \in I$ *and* $\gamma \restriction I$ *is an* Y*-trajectory.*

c) If $\phi(t) < 0$ *for all* t *in some interval* I*, then* $u(t) \equiv -1$ *for almost all* $t \in I$ *and* $\gamma \restriction I$ *is a* X*-trajectory.*

A time $t \in Dom(\gamma)$ is called a *switching time* for γ if, for each $\varepsilon > 0$, $\gamma \restriction [t-\varepsilon,\ t+\varepsilon]$ is neither an X-trajectory nor a Y-trajectory. If t is a switching time for γ then we say that $\gamma(t)$ is a *switching point for* γ, or that γ has a switching at $\gamma(t)$.

Lemma 3.3 *If* $(u,\gamma) \in Adm(\Sigma)$ *is extremal and* λ *is a adjoint covector field along* (u,γ) *then* $\phi_\lambda(t) = 0$ *at every switching time* t*.*

Consider $(u,\gamma) \in Adm(\Sigma)$, $t_0 \in Dom(\gamma)$ and $v_0 \in I\!R^2$. We write $v(v_0, t_0; t)$ to denote the value at time t of the variational vector field along (u,γ) satisfying (3.1) together with the boundary condition $v(t_0) = v_0$. If $t_0, t_1 \in Dom(\gamma)$ we say that t_0 and t_1 are *conjugate* along γ if the vectors $v\big(G\big(\gamma(t_1)\big), t_1; t_0\big)$ and $G(\gamma(t_0))$ are linearly dependent. Let D, D' be two C^3 connected one-dimensional embedded submanifolds of $I\!R^2$. We say that D' is a *conjugate curve* to D along the X-trajectories if there is a bijective function $\psi : D \mapsto D'$ with the following properties. If γ_x is the X-trajectory satisfying $\gamma_x(0) = x$, then $\psi(x) = \gamma_x(t(x))$ for some time t depending continuously on x, and the times 0, $t(x)$ are conjugate along γ_x. Conjugate curves along the Y-trajectories are defined similarly.

Lemma 3.4 *If* $\gamma \in Traj(\Sigma)$ *is extremal and* t_0, t_1 *are switching times for* γ*, then* t_0 *and* t_1 *are conjugate along* γ*.*

For the proof of this lemma see [14]. A straightforward computation yields:

Lemma 3.5 *If* $(u,\gamma) \in Adm(\Sigma)$ *is extremal and* λ *is an adjoint covector field along* (u,γ)*, then the switching function* ϕ_λ *is* C^1 *and its derivative is given by:*

$$\dot{\phi}_\lambda(t) = \lambda(t) \cdot [F, G](\gamma(t)). \tag{3.6}$$

For each $x \in I\!R^2$, one can form the 2×2 matrices whose columns are the vectors F, G, or $[F, G]$. As in [14], we shall use the following scalar functions on $I\!R^2$:

$$\Delta_A(x) \doteq det\big(F(x), G(x)\big) = F(x) \wedge G(x), \tag{3.7}$$

$$\Delta_B(x) \doteq det\big(G(x), [F,G](x)\big) = G(x) \wedge [F,G](x), \tag{3.8}$$

where *det* stands for determinant and $\wedge$ denotes an exterior product. A point $x \in I\!R^2$ is called an *ordinary point* if

$$\Delta_A(x) \cdot \Delta_B(x) \neq 0. \tag{3.9}$$

On the set of ordinary points we define the scalar functions f, g as the coefficients of the linear combination

$$[F,G](x) = f(x)F(x) + g(x)G(x). \tag{3.10}$$

In [5, p.447] it was shown that

$$f(x) = -\frac{\Delta_B(x)}{\Delta_A(x)}. \tag{3.11}$$

In the following, given two nonzero vectors $v, v' \in I\!R^2$, by $\arg(v,v') \in [-\pi,\pi]$ we denote the angle between them, oriented from v to v'. If v_0 is a constant vector and $v(t) \neq 0$, one has

$$\frac{d}{dt}\Big\{ \arg\big(v_0,\ v(t)\big)\Big\} = \frac{v(t) \wedge \dot v(t)}{\|v(t)\|^2}. \tag{3.12}$$

Lemma 3.6. *Let $(u,\gamma) \in Adm(\Sigma)$, $t_0 \in Dom(\gamma)$. For every t such that $G(\gamma(t)) \neq 0$, define the angle*

$$\alpha(t) = \arg\Big(G(\gamma(t_0)),\ \ v\big(G(\gamma(t)), t;\ t_0\big)\Big), \tag{3.13}$$

Then, one has

$$\operatorname{sgn}(\dot\alpha(t)) = \operatorname{sgn}\Big(\Delta_B\big(\gamma(t)\big)\Big). \tag{3.14}$$

Indeed, for any t at which $G(\gamma(t))$ does not vanish, one has

$$\begin{aligned} \frac{d}{dt} v\big(G(\gamma(t)),\ t;\ t_0\big) &= \lim_{\varepsilon\to 0} \frac{v\big(G(\gamma(t+\varepsilon)),\ t+\varepsilon;\ t_0\big) - v\big(G(\gamma(t)),\ t,\ t_0\big)}{\varepsilon} \\ &= M(t_0,t) \cdot \lim_{\varepsilon\to 0} \frac{v\big(G(\gamma(t+\varepsilon),\ t+\varepsilon,\ t\big) - G(\gamma(t))}{\varepsilon} \\ &= M(t_0,t) \cdot \big[F + u(t)G,\ G\big](\gamma(t)) = M(t_0,t) \cdot [F,G](\gamma(t)), \end{aligned} \tag{3.15}$$

where the matrix $M(t_0,t)$ is defined by

$$M(t_0,t)w = v\big(w,\ t;\ t_0\big). \tag{3.16}$$

Since $M(t_0,t)$ preserves orientation, by (3.12) we have

$$\begin{aligned}\operatorname{sgn}(\dot{\alpha}(t)) &= \operatorname{sgn}\Big(M(t_0,t)G(\gamma(t)) \ \wedge \ M(t_0,t)[F,G](\gamma(t))\Big)\\ &= \operatorname{sgn}\Big(G(\gamma(t)) \ \wedge \ [F,G](\gamma(t))\Big),\end{aligned}$$

proving (3.14).

Theorem 3.7 *Let $U \subset \mathbb{R}^2$ be an open set such that each $x \in U$ is an ordinary point. Then all optimal trajectories γ of $\Sigma \restriction U$ are bang-bang with at most one switching. Moreover if $f > 0$ throughout U then γ is an $X-, Y-$ or $Y * X-$trajectory, if $f < 0$ throughout U then γ is an $X-, Y-$ or $X * Y-$trajectory.*

For the proof see [14] Theorem 3.9 p.443.

A point x at which $\Delta_A(x)\Delta_B(x) = 0$ is called a *nonordinary point.* A *nonordinary arc* is a $\mathcal{C}^3$ one-dimensional connected embedded submanifold S of $\mathbb{R}^2$, with the property that every $x \in S$ is a nonordinary point. A nonordinary arc will be said *isolated* if there exists a set U satisfying the following conditions:

(C1) U is an open connected subset of $\mathbb{R}^2$
(C2) S is a relatively closed subset of U
(C3) If $x \in U - S$ then x is an ordinary point
(C4) The set $U - S$ has exactly two connected components.

An *open turnpike* is an isolated nonordinary arc that satisfies the following conditions:

(S1) For each $x \in S$ the vectors $X(x)$ and $Y(x)$ are not tangent to S and point to opposite sides of S
(S2) For each $x \in S$ one has $\Delta_B(x) = 0$ and $\Delta_A(x) \neq 0$
(S3) Let U be an open set which satisfies (C1-4) above. If U_X and U_Y are the connected components of $U - S$ labelled in such a way that $X(x)$ points into U_X and $Y(x)$ points into U_Y, then the function f in (3.10) satisfies

$$f(x) > 0 \quad \text{on} \quad U_Y \qquad\qquad f(x) < 0 \quad \text{on} \quad U_X.$$

A $\mathcal{C}^3$ one-dimensional connected, embedded submanifold with boundary $S \subset \mathbb{R}^2$ is a *turnpike* if $S \setminus \partial S$ is an open turnpike. Next, consider a turnpike S and a point $x_0 \in S$. We wish to construct a trajectory $\gamma \in Traj(\Sigma)$ such that $\gamma(t_0) = x_0$ and $\gamma(t) \in S$ for each $t \in Dom(\gamma) \doteq [t_0, t_1]$. Clearly, one should have $\Delta_B(\gamma(t)) \equiv 0$ for all t. Since $\Delta_B(\gamma(t_0)) = 0$, it suffices to verify that:

$$\frac{d}{dt}\Delta_B(\gamma(t)) = (\nabla\Delta_B \cdot \dot{\gamma})(t) = 0.$$

The above holds provided that

$$(\nabla\Delta_B \cdot uG)(\gamma(t)) + (\nabla\Delta_B \cdot F)(\gamma(t)) = 0.$$

From (S1) we have that:

$$(\nabla\Delta_B \cdot G)(x) \neq 0 \qquad \forall x \in S, \tag{3.17}$$

the values of the control u are thus uniquely determined by

$$u = \varphi_S(x) \doteq -\frac{\nabla\Delta_B \cdot F(x)}{\nabla\Delta_B \cdot G(x)}. \tag{3.18}$$

The turnpike S is said to be *regular* if the function φ_S in (3.18) satisfies:

$$|\varphi_S(x)| < 1 \qquad x \in S \setminus \partial S. \tag{3.19}$$

A curve $\gamma \in Traj(\Sigma)$ is said to be a *Z-trajectory* if there exists a regular turnpike S such that $\{\gamma(t) : t \in Dom(\gamma)\} \subset S$, in this case we write $\gamma \in Traj(Z)$.

A point $x_0 \in I\!R^2$ will be called an *exit checkpoint* for a turnpike S if there exist a solution of

$$\dot{\gamma}(t) = F(\gamma(t)) + G(\gamma(t))\varphi_S(\gamma(t))$$

such that $x_0 = \gamma(t_0)$ for some t_0 and

$$|\varphi_S(\gamma(t))| < 1 \quad \text{for } t < t_0, \qquad |\varphi_S(\gamma(t))| > 1 \quad \text{for } t > t_0.$$

A an isolated nonordinary arc (or INOA) S is said to be of the *turnpike type* if it verifies (S1),(S3) and:

(S2') Each of the function Δ_A, Δ_B is either identically zero on S or nowhere zero on S

then every turnpike is of the turnpike type but not viceversa.
A an isolated nonordinary arc (or INOA) is said to be of the *antiturnpike type* if verifies (S1),(S2') and the following condition:

(S3') Let U be an open set which satisfies (C1-4) above. If U_X and U_Y are the connected components of $U - S$ labelled in such a way that $X(x)$ points into U_X and $Y(x)$ points into U_Y, then the function f in (3.10) satisfies

$$f(x) < 0 \quad \text{on} \quad U_Y \qquad\qquad f(x) > 0 \quad \text{on} \quad U_X.$$

An INOA S of the antiturnpike type is said to be *nondegenerate* if it verifies:

(SN) If $\Delta_B \equiv 0$ on S then $X \cdot \nabla\Delta_B$ or $Y \cdot \nabla\Delta_B$ never vanish on S.

A point $x \in I\!R^2$ is a *near ordinary point* if it is an ordinary point or belongs to an INOA that is either of the turnpike type or nondegenerate of the antiturnpike type. In [5,p.459] was proved the following:

Theorem 3.8 *Let x be a near ordinary point. Then there exists a neighborhood U of x such that every optimal trajectory γ of $\Sigma \restriction U$ is concatenation of at most five trajectories each of which is an $X-, Y-$ or $Z-$trajectory.*

It can happen that $R(\tau)$ contains curves whose points can be reached in minimum time using different optimal controls. An *overlap curve* is a $\mathcal{C}^3$ one-dimensional connected embedded submanifold K of $I\!R^2$, with the property that for each point of K there exist two distinct time optimal trajectories $\gamma_1, \gamma_2 : [0, b] \mapsto I\!R^2$, and $\varepsilon > 0$ such that:

$$\gamma_1(0) = \gamma_2(0) = 0 \qquad \gamma_1(b) = \gamma_2(b) = x, \qquad T(x) = b,$$

and $\gamma_1 \restriction [b-\varepsilon, b]$ is an X-trajectory, while $\gamma_2 \restriction [b-\varepsilon, b]$ is a Y-trajectory.

4.Bound on the number of switchings

The aim of this section is to prove, given $\tau > 0$, the existence of generic conditions on F, G assuring that every time optimal trajectory in $R(\tau)$ is a finite concatenation of $X-, Y-$ and $Z-$trajectories; more precisely for each Σ in a generic subset of Ξ there exists $N(\Sigma)$ that bounds the number of these trajectories. With an abuse of language we call switching points the points in which two of these trajectories meet each other forming the concatenation. Indeed, fron the previous definition it follows that every point of a Z-trajectory is a switching point.

Given a trajectory $\gamma \in Traj(\Sigma)$ we denote by $n(\gamma)$ the smallest integer such that there exist $\gamma_i \in Traj(X) \cup Traj(Y) \cup Traj(Z)$, $i = 1, \ldots, n(\gamma)$ verifying:

$$\gamma = \gamma_{n(\gamma)} * \cdots * \gamma_1.$$

We call $n(\gamma) - 1$ the number of switchings of γ.
Given $\tau > 0$ let define Π_τ to be the class of systems having an a priori bound on the number of switchings of optimal trajectories:

$$\Pi_\tau = \{\Sigma \in \Xi : \exists N(\Sigma) \ s.t. \ \forall \gamma \in Opt(\Sigma \restriction R(\tau)) \ \ n(\gamma) \leq N(\Sigma)\}.$$

A subset of Ξ is said to be *generic* if it contains an open and dense subset of Ξ. A *condition* for $\Sigma = (F, G) \in \Xi$ is a set of equalities and unequalities involving the components of the vector fields (F, G), their derivatives or set and functions that can be defined using them. Given a condition P for $\Sigma \in \Xi$ we write $P(\Sigma)$ if the system satisfies the condition P. A condition P is said to be *generic* if $\{\Sigma \in \Xi : P(\Sigma)\}$ is generic. If $P_1, \ldots, P_n$ are generic condition then it is easy to verify that $\{\Sigma \in \Xi : P_1(\Sigma), \ldots, P_n(\Sigma)\}$ is generic.

We now give a finite number of generic conditions $P_1, \ldots, P_n$ that assure the genericity of Π_τ, this means:

$$\{\Sigma \in \Xi : P_1(\Sigma), \ldots, P_n(\Sigma)\} \subset \Pi_\tau. \tag{4.1}$$

From now on we consider a fixed time $\tau > 0$ and a fixed system $\Sigma = (F, G) \in \Xi$ and we describe the conditions of (4.1). The first condition is:

(P_1) $F(0)$ and $[F, G](0)$ are linearly independent.

From P_1 and Lemma 2.4 it follows that the system Σ is locally controllable. The second condition is:

(P_2) Zero is a regular value for the functions Δ_A and Δ_B

this means:

$$\forall x \in I\!R^2 \quad \Delta_A(x) = 0 \Rightarrow \nabla \Delta_A(x) \neq 0 \tag{4.2}$$

and the similar condition with Δ_B rather than Δ_A in (4.2). From P_2 we have that the sets $\Delta_A^{\leftarrow}(0) = \{x \in R(\tau) : \Delta_A(x) = 0\}$ and $\Delta_B^{\leftarrow}(0) = \{x \in R(\tau) : \Delta_B(x) = 0\}$ are C^2 one-dimensional compact embedded submanifold of $I\!R^2$. So we can give the following generic condition:

(P_3) The set $\Delta_A^{\leftarrow}(0) \cap \Delta_B^{\leftarrow}(0)$ is finite.

Let Tan_A be the set of points $x \in \Delta_A^{\leftarrow}(0)$ such that $X(x)$ or $Y(x)$ is tangent to $\Delta_A^{\leftarrow}(0)$. Define Tan_B in the same way using Δ_B rather than Δ_A.

(P_4) Tan_A and Tan_B are finite sets.

We will call *bad points* the elements of the set:

$$Bad(\tau) = \left(\Delta_A^{\leftarrow}(0) \cap \Delta_B^{\leftarrow}(0)\right) \cup \left(Tan_A \cup Tan_B\right). \tag{4.3}$$

It's easy to verify that if $x \in R(\tau) \setminus Bad(\tau)$ then x is a nearordinary point. From (P_3), (P_4) we obtain:

(P_5) $Bad(\tau)$ is finite.

From Theorems 3.7 and 3.8 we know the structure of time opimal trajectories in a neighborhood of a near ordinary point; it remains to consider the case of bad points.

Lemma 4.1 *If $x \in Bad(\tau)$, $G(x) \neq 0$ then $x \in \left(\Delta_A^{\leftarrow}(0) \cap \Delta_B^{\leftarrow}(0)\right)$ if and only if $x \in Tan_A$.*

Proof. Being $G(x) \neq 0$ we can choose a local system of coordinates such that $G = (1,0)$, then we have $\alpha F(x) = G(x)$ ($\alpha \in \mathbb{R}$) and:

$$\nabla(\Delta_A)(x) = \nabla(F_1 G_2 - G_1 F_2)(x) = -\nabla F_2(x)$$

$$[F,G](x) = -\nabla F \cdot G.$$

Assume, first, that $x \in \Delta_A^{\leftarrow}(0) \cap \Delta_B^{\leftarrow}(0)$. From $\Delta_B(x) = 0$ we have $[F,G](x), G(x)$ are prarllel and then:

$$0 = (\partial_1 F_2)(x)\ G_1(x) + (\partial_2 F_2)(x)\ G_2(x) = (\partial_1 F_2)(x)$$

finally:

$$\nabla(\Delta_A)(x) \cdot G(x) = 0.$$

We conclude that $x \in Tan_A$.
In the same way if $x \in Tan_A$ then $\Delta_B(x) = 0$.

We can now prove the following:

Theorem 4.2 *Under generic conditions for every $x \in Bad(\tau)$ there exist U_x, neighborhood of x, and $N_x \in I\!N$ such that if $\gamma \in Opt(\Sigma)$ and $\{\gamma(t) : t \in [b_0, b_1]\} \subset U_x$ then:*

$$n(\gamma \restriction [b_0, b_1]) \leq N_x.$$

Proof. Consider x and γ satisfying the assumptions above. For sake of simplicity in the proof we write γ instead of $\gamma \restriction [b_0, b_1]$. We have three cases:

(1) $G(x) = 0$
(2) $G(x) \neq 0 \qquad x \in (\Delta_A^{\leftarrow}(0) \cap \Delta_B^{\leftarrow}(0)) \cap Tan_A$
(3) $x \in Tan_B$.

Case (1). We suppose that:

(P_6) $F(x) \cdot \nabla(\Delta_A)(x) \neq 0$, $F(x) \cdot \nabla(\Delta_B)(x) \neq 0$.

Take U_x open connected such that x is the only bad point in U_x and $U_x - (\Delta_A^{\leftarrow}(0) \cap \Delta_B^{\leftarrow}(0))$ has four connected components $U_1, \ldots, U_4$. Assume that, say, $F(x)$ points into U_1 and $-F(x)$ into U_2. Then it is clear that for U_x sufficiently small the same happens for $X(y), Y(y)$ $y \in U_x$. Following γ, we can move from U_2 into any other component, while from U_3 or U_4 we can move into U_1; there are no other possibilities. From Theorem 3.7 we have that for each $Cl(U_i)$ $(i = 1, \ldots, 4)$, $n(\gamma \restriction \{t : \gamma(t) \in U_i\}) \leq 5$ therefore the conclusion holds, with $N_x = 15$.

Case (2). Let y range on $\Delta_A^{\leftarrow}(0)$ and assume that:

(P_7) $\frac{\partial}{\partial y} (X \cdot \nabla \Delta_A)|_{y=x} \neq 0, \qquad X(x) \neq 0, Y(x) \neq 0.$

If X and Y have the same orientation, then we can proceed as in case (1). In the following, we thus study the case where they have opposite orientations.
Take U_x open connected such that x is the only bad point in U_x and $U_x \setminus (\Delta_A^{\leftarrow}(0) \cap \Delta_B^{\leftarrow}(0))$ has four connected components $U_1, \ldots, U_4$. Let A_1 be the connected component of $(\Delta_A^{\leftarrow}(0) \cap U_x) \setminus \{x\}$ that comes before x in the orientation of $X(x)$ and A_2 the other component. We label $U_1, \ldots, U_4$ such that:

$X(y)$ points into U_1, $Y(y)$ points into U_2 for $y \in A_1$
$X(y)$ points into U_3, $Y(y)$ points into U_4 for $y \in A_2$.

Choosing U_x smaller if necessary, we can assume that $X \neq 0 \neq Y$ on U_x. Moreover, we can assume that $X(y)$ points into U_3 for each $y \in Fr(U_3) \cap U_x$ and $Y(y)$ points into U_2 for each $y \in Fr(U_2) \cap U_x$. Suppose first that $f < 0$ (see (3.11)) on U_1 then $f > 0$ on U_2. From Theorem 3.7 it follows that if γ enters U_3 then it remains in U_3 and the same happens for U_2.
The set $B_1 \doteq \Delta_B^{\leftarrow}(0) \cap Fr(U_1)$ is an INOA of the antiturnpike type and there exists k, depending on U_x, such that:

$$|(X\Delta_B)(y)| \geq k > 0, \ |(Y\Delta_B)(y)| \geq k > 0 \qquad \forall y \in B_1. \tag{4.4}$$

Following the proof of Theorem 3.7 in [14,pp.459–465] we obtain that, for U_x sufficiently small, every time optimal trajectory in $U_1 \cup U_4 \cup B_1$ does not contain any trajectory of the type $X * Y * X$ or $Y * X * Y$. The proof relies essentially on the construction of an envelope for such trajectories. For the theory of envelopes we refer to [17] and [18]. The uniform bound in (4.4) assure the admissibility of these envelopes.
Then it is clear that $N(\gamma)$ is finite unless γ has a switching point on $\Delta_A^{\leftarrow}(0)$. In this case γ switches precisely at the points where it meets $\Delta_A^{\leftarrow}(0)$ (see Lemma 3.1). Assume that X, Y points toward $\Delta_A^{\leftarrow}(0)$ otherwise consider γ with reversed time. Choose a local coordinate system such that $x = (0,0)$, $\Delta_A^{\leftarrow}(0) = \{(p,0) : p \in I\!R\}$ and $\Delta_B^{\leftarrow}(0) = \{(0,q) : q \in I\!R\}$. Suppose that $\gamma(t_0) = (0, q_0)$, $\gamma(t_1) = (0, q_1)$ and that γ switches at $(p_0, 0) = \gamma(s_0)$, $s_0 \in]t_0, t_1[$. We have that:

$$q_0 = ap_0^2 + bp_0^3 + o(p_0^3), \qquad q_1 = ap_0^2 + cp_0^3 + o(p_0^3) \tag{4.5}$$

$$X(0) = (p_X, 0) \qquad t_1 - t_0 \simeq 2\frac{p_0}{p_X} \simeq 2\frac{\sqrt{q_0}}{p_X} \tag{4.6}$$

in fact X, Y are parallel along $\Delta_A^{\leftarrow}(0)$. From (4.5),(4.6) we have that:

$$q_1 = q_0 + (c-b)p_0^3 + o(p_0^3) = q_0 + (c-b)q_0^{\frac{3}{2}} + O(q_0^2)$$

and if we call t_2 the next time in which γ touches the ordinate axe we have:

$$t_2 - t_1 = 2\frac{\sqrt{q_1}}{p_X} = \frac{2}{p_X}\sqrt{q_0 + (c-b)q_0^{\frac{3}{2}} + O(q_0^2)} =$$

$$= (t_1 - t_0)\sqrt{1 + \frac{p_X\,(c-b)}{2}(t_1 - t_0) + o(t_1 - t_0)}.$$

Then calling t_n the time of the $n-th$ crossing of the ordinate axis we obtain:

$$\sum_n t_n - t_{n-1} = +\infty \qquad (4.7).$$

If $\bar{t}$ is the time between the first two switchings t_0, t_1 on $\Delta_A^{\leftarrow}(0) \cap U_x$ then we have a lower bound on $\bar{t}$. Indeed, if moving backwards along γ we intersect $\Delta_A^{\leftarrow}(0) \cap U_x$ again, then t_0 is not the first switching. Hence, given $\tau > 0$, from this lower bound and from (4.7) we have a bound on the number of switchings for γ.

It remains to consider the case in which $f > 0$ on U_1 and then $f < 0$ on U_2. Let γ_X, γ_Y be, respectively, the $X-, Y-$trajectory with $In(\gamma_X) = In(\gamma_Y) = x$; let V_1 be the connected component of $U_x \setminus (\gamma_X \cup \gamma_Y)$ containing U_2, U_3, and let V_2 be the other connected component. If $\Delta_A > 0$ on U_2 and γ move from V_1 to V_2, then it can not return back to V_1 remaining in U_x; the opposite happens if $\Delta_A < 0$ on U_2.
If γ is contained in V_1, we can use (4.4) to obtain, as above, a bound on $n(\gamma)$.
Suppose that γ is contained in V_2. If γ either has a switching in U_1 or in U_4, or switches at t_0 on $\Delta_B^{\leftarrow}(0)$, then it can not switch again; in fact from Lemma 3.6 we have that $\alpha(t_0) = 0$, α defined in (3.13), and α is monotone since the sign of his derivative is equal to the sign of Δ_B. Hence, for U_x sufficiently small γ can not have another switching in U_x.
If γ switches on $\Delta_A^{\leftarrow}(0)$ we proceed as above. We obtain a bound on the number of switching unless γ has a sequence of switching on $V \doteq (V_2 \cap (U_2 \cup U_3))$. In this case by (4.4) and (P_7) we can use the same construction of Theorem 3.7 in [14,pp.459–465], i.e. we can construct an admissible envelope. This concludes the proof of the second case.

Case(3). We assume that:

(P_8) $\Delta_A(x) \neq 0$.

Suppose , for example, that $X(x) \cdot \nabla\Delta_B(x) = 0$. Take U_x open connected such that x is the only bad point in U_x and $\Delta_A \neq 0$ on U_x. Let γ_Y be the maximal Y-trajectory passing through x and let U_1, U_2 be the connected component of $U_x \setminus \gamma_Y$. For U_x sufficiently small $\gamma_Y \cap \Delta_B^{\leftarrow}(0) \cap U_x = \{x\}$, and $X(y)$ points to the same side of $\gamma_Y \cap U_x$ for every $y \in \gamma_Y \cap U_x$. If $X(x)$ points into U_1 then γ can not cross from U_1 into U_2, and viceversa if $X(x)$ points into U_2. Since in U_1, U_2 we have an a priori bound on the number of switchings of γ, as in the preceding cases, the proof is completed.

By the previous analysis, under the generic assumptions $(P_6), (P_7), (P_8)$ the conclusion of the theorem holds.

Using Theorem 4.2 for each $x \in R(\tau)$ we can select an neighborhood U_x such that every optimal trajectory remaining in U_x is the concatenation of $\leq N_x$ regular arcs. Choose $\varepsilon_x > 0$ such that $B(x, 2\varepsilon_x) \subset U_x$. Since $R(\tau) \subset \cup_{x \in R(\tau)} B(x, \varepsilon_x)$, by compactness we can extract a finite subcover $B(x_i, \varepsilon_i)$, $i = 1, \ldots, n$, $\varepsilon_i \doteq \varepsilon_{x_i}$. Consider an extremal pair (u, γ), $\gamma : [0, \tau] \mapsto \mathbb{R}^2$, $In(\gamma) = 0$. Define:

$$\varepsilon \doteq \min_{i=1,\ldots,n} \varepsilon_i \qquad N = \max_{i=1,\ldots,n} N_{x_i}. \tag{4.8}$$

Choose i_1 such that $0 \in B(x_{i_1}, \varepsilon)$. Let t_1 be either the first time such that:

$$\gamma(t_1) \notin B(x_{i_1}, 2\varepsilon)$$

or $t_1 = \tau$ if γ remains in $B(x_{i_1}, 2\varepsilon)$. Then there exists $i_2 \neq i_1$ such that $\gamma(t_1) \in B(x_{i_2}, \varepsilon)$. Let t_2 be either the first time for which $\gamma(t_2) \notin B(x_{i_2}, 2\varepsilon)$ or $t_2 = \tau$ if γ remains in $B(x_{i_2}, 2\varepsilon)$. We proceed in the same way defining a set of increasing times $\{t_0 \doteq 0, t_1, \ldots, t_\nu = \tau\}$. If $M \doteq \max\{|F(x)| + |G(x)| : x \in R(\tau)\}$ denotes the maximum speed of trajectories inside $R(\tau)$, it's clear that $t_j - t_{j-1} \geq \frac{\varepsilon}{M}$. Therefore:

$$\nu \leq \frac{M\,\tau}{\varepsilon}. \tag{4.9}$$

By definition, for each t_j, $j = 1, \ldots, \nu$, we have that $\{\gamma(t) : t \in [t_{j-1}, t_j[\}$ is contained in $B(x_{i_j}, 2\varepsilon)$. Using Theorems 3.7,3.8,4.2 toghether with (4.9) we obtain:

$$n(\gamma) \leq N(\Sigma) \doteq N\,\frac{M\,\tau}{\varepsilon}.$$

We have thus proved the following:

Corollary 4.3 *For every $\tau > 0$ the set Π_τ is a generic subset of Ξ.*

5. An algorithm for the synthesis

In this section we describe an inductive procedure for constructing a regular synthesis for the time optimal problem. At step N, the algorithm will construct precisely those trajectories $x(\cdot)$ which are concatenation of N bang- or singular arcs and satisfy the Pontryagin maximum principle. The endpoints of the arcs forming these trajectories, corresponding to the switching times of the control, are determined by certain nonlinear equations. Under generic conditions such equations can be solved by the implicit function theorem, thus determining a smooth switching locus. Eventually the algorithm will partition the reachable set $R(\tau)$ into finitely many open regions (where the optimal feedback control is either $u = 1$ or $u = -1$), separated by boundary curves and points, here called *frame curves* and *frame points*, respectively.

At each step, it may happen that distinct extremal trajectories reach the same point x_0, at different times. It is therefore necessary to delete from the synthesis those trajectories which are not globally optimal. This procedure will usually produce new "overlap curves", consisting of points reached in minimum time by two distinct trajectories, one ending with the control value $u = 1$, the other with $u = -1$.

In the following, we fix a locally controllable system $\Sigma = (F, G) \in \Xi$, satisfying the generic assumptions $(P_1), \ldots, (P_8)$, and a time $\tau > 0$.

Remark 5.1 If τ is sufficiently small all optimal trajectories in $R(\tau)$, starting from the origin, are bang-bang with at most one switching. Indeed those are the only trajectories which satisfy the Pontryagin maximum principle. It follows that for each optimal trajectory γ there exists a time $b > 0$ such that $\gamma \restriction [0, b]$ is an $X-$ or a $Y-$trajectory.

We will use the symbol $\gamma^{\pm}$ to denote the trajectory $\gamma^{\pm} : [0, \tau] \mapsto I\!R^2$, $In(\gamma^{\pm}) = (0)$ corresponding to the constant control $u^{\pm}(t) = \pm 1$. At each step the algorithm constructs trajectories $\gamma \in Traj(\Sigma)$ such that $Dom(\gamma) = [0, b]$ $(b \leq \tau)$, $In(\gamma) = 0$. Moreover, if a trajectory $\gamma \in Traj(\Sigma)$ with $Dom(\gamma) = [0, b]$ is constructed as part of the synthesis, we then regard all trajectories $\gamma \restriction [0, a]$ with $a < b$ as constructed trajectories. Similarly we regard as frame curve every connected subset of a frame curve , whose endpoints are frame points.

Algorithm $\mathcal{A}$, STEP 1.

The origin is a frame point. We construct all trajectories γ that are $X-$ or $Y-$trajectory and satisfies the PMP. The trajectories $\gamma^{\pm} \restriction [0, t_0]$ are frame curves and also constructed trajectories, where $[0, t_0]$ is the maximal interval on which they are extremal.

End of STEP 1.

We now define by induction the Step N (N> 1).

Algorithm $\mathcal{A}$, STEP N.
We construct all trajectories γ, $Dom(\gamma) = [0,b]$, for which there exists $a \in [0,b[$ such that:

(1) $\gamma \restriction [0,a]$ is a trajectory constructed by Step N-1
(2) $\gamma \restriction [a,b] \in Traj(X) \cup Traj(Y) \cup Traj(Z)$
(3) γ satisfies the PMP.

At this stage, may be points reached by two or more extremal trajectories, at different times. Trajectories which are not globally optimal must be deleted from the synthesys. In the same way frame curves and points that lie on deleted trajectories will be deleted. Let $x \in \mathbb{R}^2$ be a point reached by some trajectory constructed in this step. There are a finite number of constructed trajectories (not necessarily constructed in this step) $\gamma_1, \ldots, \gamma_n$ that reach x at a certain time $t_i \in Dom(\gamma_i)$, $i = 1, \ldots, n$. Let $\bar{t} = \min_i t_i$. If $t_i > \bar{t}$ then delete the trajectories γ_i after the time t_i, i.e. consider only $\gamma_i \restriction [0, t_i]$.

Define the following new frame curves:

a) Regular turnpikes S such that $S \subset \{\gamma(t) : t \in Dom(\gamma)\}$ for some constructed trajectory γ
b) Overlap curves
c) Conjugate curves to frame curves, constructed in the previous step, along $X-$ or $Y-$trajectories.

The curves of type c) are also called *switching curves*, because if a trajectory reaches one of these curves then it has to switch.

We define the intersections between frame curves to be frame point.

End of STEP N.

If at step N the algorithm $\mathcal{A}$ does not construct any new trajectory then we say that *$\mathcal{A}$ stops at step N (for Σ at time τ)*. From Corollary 4.3 it is clear that under generic assumptions, there exists $N(\Sigma)$ such that $\mathcal{A}$ stops before step $N(\Sigma)$ and, by construction, we have that $\{\gamma : \gamma \text{ is constructed by } \mathcal{A}\} = Opt(\Sigma)$. In this case we define $R_{\mathcal{A}}(\tau)$ to be the set of points reached by the trajectories constructed by $\mathcal{A}$; notice that $R_{\mathcal{A}}(\tau) = R(\tau)$. We let $Fr\big(R_{\mathcal{A}}(\tau)\big)$ be a frame curve and let its intersections with other frame curves be frame points.

If $\mathcal{A}$ stops then for each $x \in R(\tau)$ there exists a set of constructed trajectories that reach x. Define $\Gamma_x \doteq \{\gamma : \gamma \text{ is constructed by } \mathcal{A},\ Term(\gamma) = x\}$.

We want to select, for each $x \in R(\tau)$, a trajectory from Γ_x to form a synthesis. Define K_k to be the set of points $x \in R(\tau)$ reached by at least one constructed trajectory γ satisfying $n(\gamma) \leq k$. Notice that K_k is compact for each k and $K_{N(\Sigma)} = R(\tau)$. We now construct, by induction, an optimal synthesis on K_k associating to each x a trajectory $\gamma_x \in \Gamma_x$.

STEP 1. Let $x \in K_1$. If $\gamma \in \Gamma_x^1 \doteq \{\gamma \in \Gamma_x,\ n(\gamma) = 1\}$ then γ is an $X-$trajectory or a $Y-$trajectory (see Remark 5.1). Therefore Γ_x^1 consists either of one or of two trajectories. If there exists $\gamma \in \Gamma_x^1 \cap Traj(X)$ let define $\gamma_x \doteq \gamma$ otherwise let $\gamma_x = \Gamma_x^1 \cap Traj(Y)$.

STEP k. Let $x \in K_k$. If $x \in K_{k-1}$ then we have already defined γ_x. Hence, in the following, we assume that $x \in K_k \setminus K_{k-1}$. Let $\gamma_{_X}, \gamma_{_Y}$ be, respectively, the $X-, Y-$trajectory such that $Term(\gamma_{_X}) = \gamma_{_X}(T(x)) = x$ (see (2.10) for the definition of T) and the same condition for $\gamma_{_Y}$. If there exists a regular turnpike passing through x we define $\gamma_{_Z}$ to be the maximal $Z-$trajectory such that $Term(\gamma_{_Z}) = \gamma_{_Z}(T(x)) = x$. We have that at least one of the trajectories $\gamma_{_X}, \gamma_{_Y}, \gamma_{_Z}$ intersects K_{k-1}. If $\gamma_{_V} \cap K_{k-1} \neq \emptyset$, $V = X, Y, Z$, we define x_V in the following way:

$$x_V \doteq \gamma_{_V}(t_V), \qquad t_V \doteq \max\{t \in Dom(\gamma_{_V}) : \gamma_{_V}(t) \in K_{k-1}\}.$$

To x_V, by induction, we have associated a trajectory γ_{x_V}. If $t_V = T(x_V)$ we define:

$$\gamma'_V = \gamma_{_V} \restriction [t_V, T(x)] * \gamma_{x_V}$$

therefore all γ'_V defined are optimal. If we have defined more than one γ'_V, we choose one, say according to the preference order X, Y, Z, and let $\gamma_x \doteq \gamma'_V$.

In this way, at step $N(\Sigma)$, we have constructed a synthesis for Σ at time τ. We use the symbol $\Gamma_{\mathcal{A}}(\Sigma, \tau)$ to denote this synthesis and we call it *the synthesis generated by the algorithm* $\mathcal{A}$.

Theorem 5.2 *Consider $\Sigma \in \Xi$ and $\tau > 0$. If $\mathcal{A}$ stops for Σ at time τ then $\Gamma_{\mathcal{A}}(\Sigma, \tau)$ is an optimal synthesis.*

6.Examples

In this section we give some examples of systems $\Sigma \in \Xi$ for which the algorithm $\mathcal{A}$ stops at a time $\tau > 0$.

Example 1. Let $\tau > 2$ and consider the control system:

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 + \frac{1}{2}x_1^2 \end{cases} \tag{6.1}$$

The $X-$ and $Y-$trajectories can be described giving x_2 as a function of x_1 and are, respectively, cubic polynomials of the following type:

$$x_2 = -\frac{x_1^3}{6} - \frac{x_1^2}{2} + \alpha \qquad \alpha \in I\!R \tag{6.2}$$

$$x_2 = \frac{x_1^3}{6} + \frac{x_1^2}{2} + \alpha \qquad \alpha \in I\!R. \tag{6.3}$$

With a straightforward computation we obtain:

$$[F,G] = \begin{pmatrix} 0 \\ -1 - x_1 \end{pmatrix}$$

then

$$\Delta_B(x) = \det \begin{pmatrix} 0 & 1 \\ -1 - x_1 & 0 \end{pmatrix} = 1 + x_1 \tag{6.4}.$$

Equation (6.4) tells us that if there is a turnpike than it is a subset of $\{(x_1, x_2) \in I\!R^2 : x_1 = -1\}$. Indeed at the second step the algorithm $\mathcal{A}$ constructs the turnpike:

$$S = \left\{ (x_1, x_2) : x_1 = -1, x_2 \leq -\frac{1}{3} \right\}. \tag{6.5}$$

The algorithm $\mathcal{A}$ constructs the trajectories $\gamma_1 : [0, b] \mapsto I\!R^2$ for which there exists $t_0 \in [0, b]$ such that $\gamma_1 \restriction [0, t_0]$ is a $Y-$trajectory and $\gamma_1 \restriction [t_0, b]$ is an $X-$trajectory; $\mathcal{A}$ also constructs trajectories $\gamma_2 : [0, b] \mapsto I\!R^2$, $b > 2$, for which there exists $t_1 \in [2, b]$ such that $\gamma_2 \restriction [0, t_1]$ is an $X-$trajectory and $\gamma_2 \restriction [t_1, b]$ is a $Y-$trajectory. For every $b > 2$, these trajectories cross each other in the region of the plane above the cubic (6.3) with $\alpha = 0$ and determine an overlap curve K that starts from the point $(-2, -\frac{2}{3})$. We use the symbols $x^{+-}(b, t_0)$ and $x^{-+}(b, t_1)$ to indicate, respectively, $\gamma_1(b)$ and $\gamma_2(b)$. Explicitly we have:

$$x_1^{+-} = 2t_0 - b \qquad x_2^{+-} = -\frac{(2t_0 - b)^3}{6} - \frac{(2t_0 - b)^2}{2} + t_0^2 + \frac{t_0^3}{3} \tag{6.6}$$

$$x_1^{-+} = b - 2t_1 \qquad x_2^{-+} = \frac{(b-2t_1)^3}{6} + \frac{(b-2t_1)^2}{2} - t_1^2 + \frac{t_1^3}{3}. \tag{6.7}$$

Now the equation:

$$x^{+-}(b, t_0) = x^{-+}(b, t_1) \tag{6.8}$$

as b varies in $[2, +\infty[$ describes K. From (6.6),(6.7) and (6.8) we have:

$$t_0 = b - t_1 \qquad t_1\Big(-2t_1^2 + (2+3b)t_1 + (-b^2 - 2b)\Big) = 0.$$

Solving for t_1 we obtain three solutions:

$$t_1 = 0, \qquad \text{or } t_1 = b \qquad \text{or } t_1 = 1 + \frac{b}{2}. \tag{6.9}$$

The first two of (6.9) are trivial, while the third describes K:

$$K = \left\{ (x_1, x_2) : x_1 = -2, x_2 \geq -\frac{2}{3} \right\}.$$

The set $R(\tau)$ is portrayed in Fig.1.

Example 2. Consider a time $\tau > \pi$ and the control system:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + u. \end{cases}$$

This example is well described in [8,pp.11–14] and in [7,p.80].

The $X-$ and $Y-$trajectories are circle centered in $(-1, 0)$ and in $(1, 0)$, respectively. We use, as in section 5, the symbols $\gamma^{\pm} : [0, \tau] \mapsto I\!R^2$ to denote the $X-, Y-$trajectories with $In(\gamma^{\pm}) = 0$. The algorithm $\mathcal{A}$ contructs $\gamma^{\pm}$ only up to time π, indeed after π they are not extremal.

At step n+1 the algorithm constructs the following switching curves:

a) All the semicircle of radius 1 centered in $(2n+1, 0)$ and contained in the half plane $\{(x_1, x_2) : x_2 \geq 0\}$.

b) All the semicircle of radius 1 centered in $(-2n-1, 0)$ and contained in the half plane $\{(x_1, x_2) : x_2 \leq 0\}$.

Along the switching curves described in a) the constructed trajectories arrive as Y-trajectories and leave as $X-$trajectories, i.e. the controls switch from $+1$ to -1. The opposite happens along the switching curves described in b).

According to the definitions of section 4, the points $(1,0)$ and $(-1,0)$ are bad points. Indeed, $\Delta_A^{\leftarrow}(0) = \{(x_1,x_2) : x_2 = 0\}$ and $Y(1,0) = 0$, $X(-1,0) = 0$.
In Fig.2 it is represented $R(4)$.

Example 3. The following provides an example of an exit checkpoint for a turnpike. Let $\tau > \frac{7}{3} + \sqrt[3]{4}$ and consider the system:

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = \Big(x_1 + \psi(x_2)\Big) + \frac{1}{2}\Big(x_1 + \psi(x_2)\Big)^2 \end{cases} \tag{6.10}$$

where:

$$\psi(x_2) = \begin{cases} 0 & x_2 > -1 \\ (x_2+1)^4 & x_2 \leq -1. \end{cases} \tag{6.11}$$

For $x_2 > -1$ the system is the same as the first example one. There is a turnpike S that lies on the line $x_1 = -1$ between the points $(-1, -\frac{1}{3})$ and $(-1,-1)$. Moreover, for $x_2 \leq -1$, S is represented by the equations:

$$x_1 + (x_2+1)^4 + 1 = 0 \qquad x_2 \leq -1. \tag{6.12}$$

Recalling (3.18), from (6.10)-(6.,12) we have that the control φ_S is:

$$\varphi_S(x_1,x_2) = 0 \quad if\ x_2 \geq -1 \qquad\qquad \varphi_S(x_1,x_2) = 2(x_2+1)^3 \quad if\ x_2 \leq -1. \tag{6.13}$$

By (6.13), the turnpike S is regular up to the point:

$$(\bar{x}_1, \bar{x}_2) = \left(-1 - \frac{1}{2\sqrt[3]{2}}, -1 - \frac{1}{\sqrt[3]{2}}\right) \tag{6.14}$$

indeed:

$$\varphi_S(\bar{x}_1, \bar{x}_2) = -1.$$

The algorithm $\mathcal{A}$ then constructs a turnpike that ends at the the point (6.14).
In Fig.3 it is represented $R(\tau)$ near the exit checkpoint.

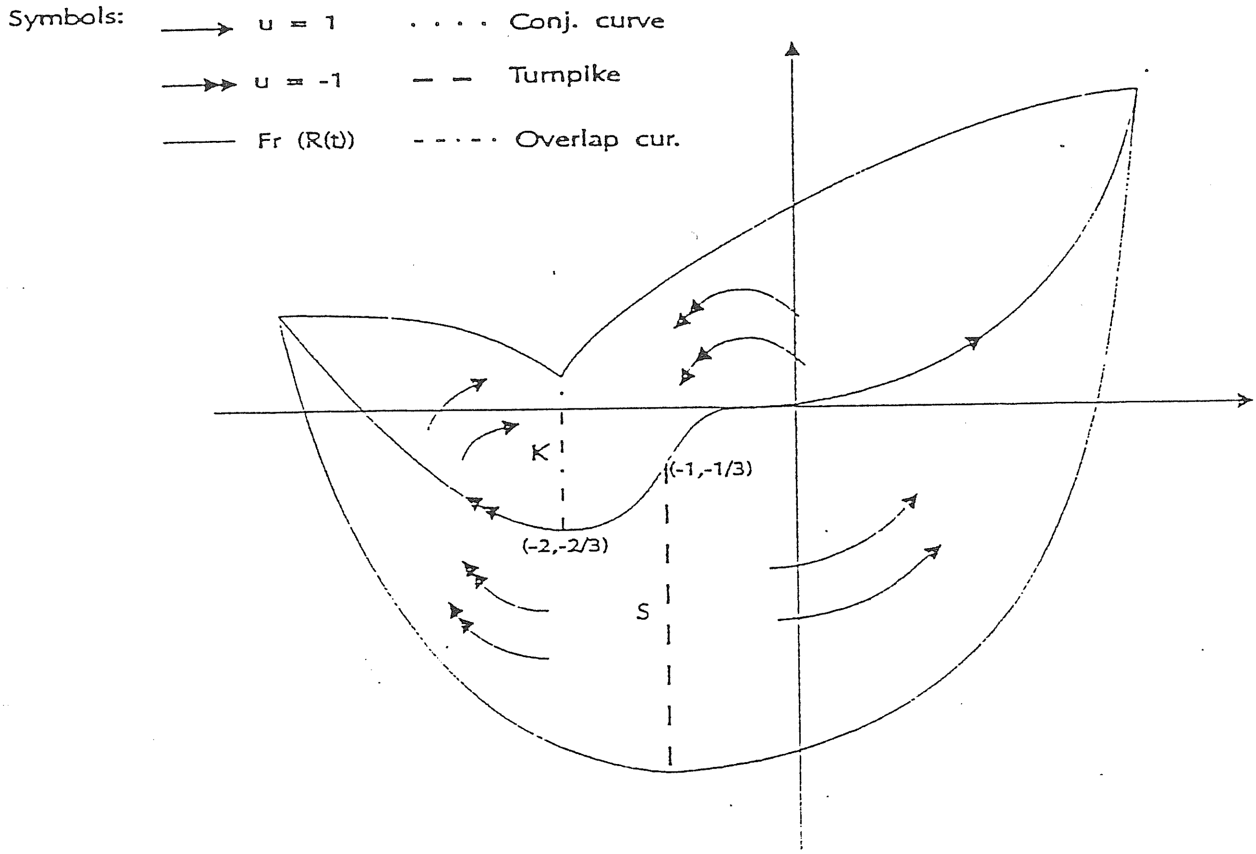
Symbols:
u = 1
u = -1
Fr (R(t))
Conj. curve
Turnpike
Overlap cur.
K
(-2,-2/3)
(-1,-1/3)
S

Fig.1

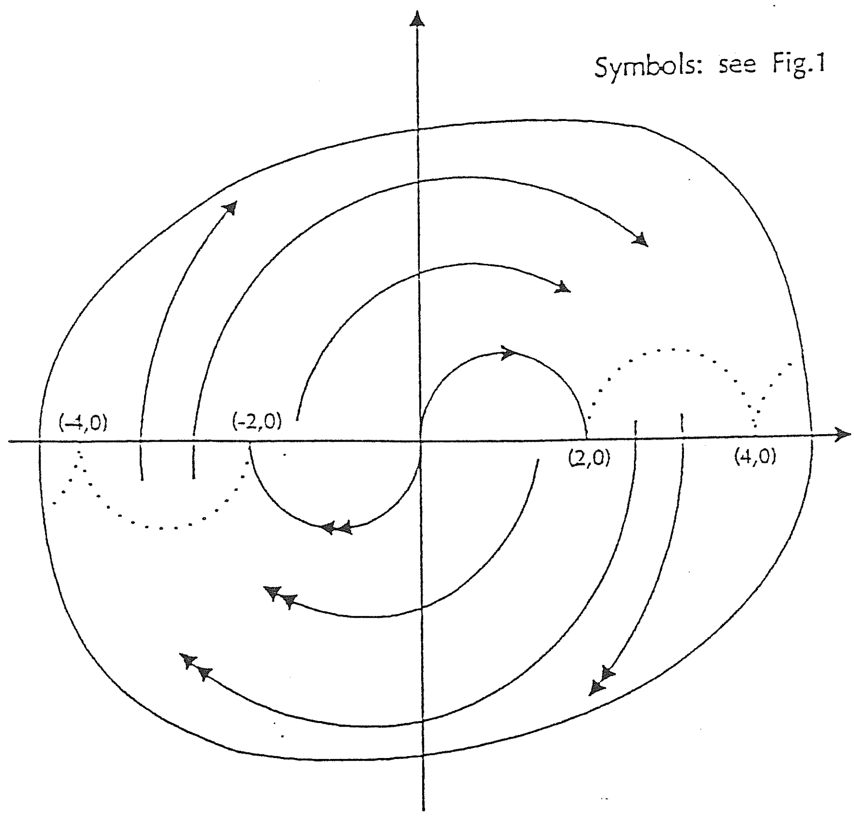
Symbols: see Fig.1
(-4,0)
(-2,0)
(2,0)
(4,0)

Fig.2

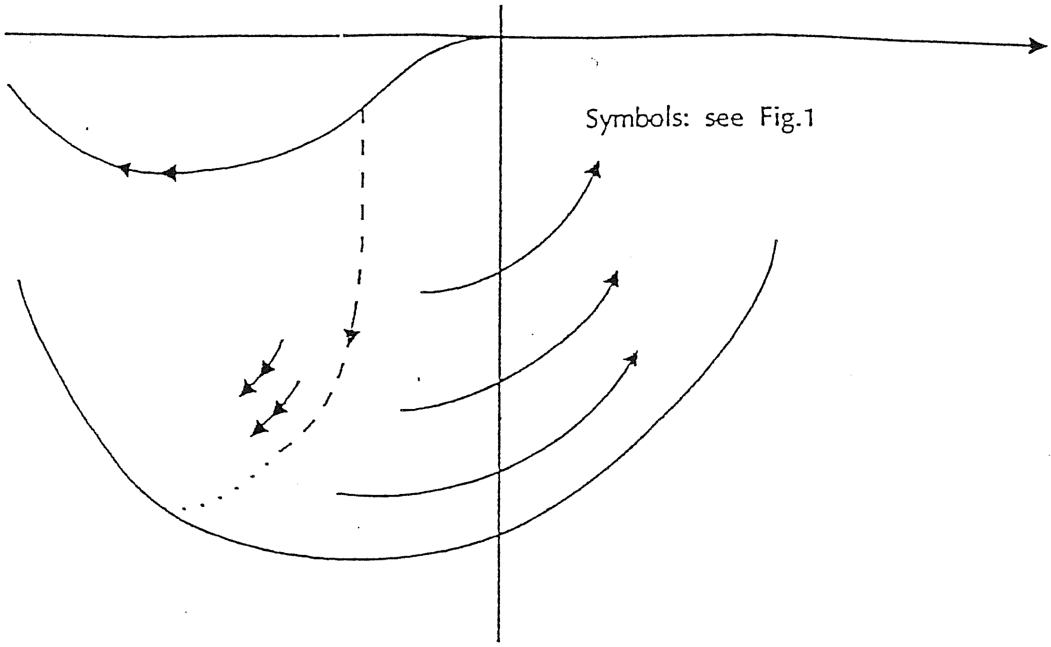

Fig.3

Chapter 2

1.Introduction

Given the control system:

$$\dot{x} = F(x) + u\, G(x) \qquad |u| \leq 1 \tag{1.1}$$

with $F(0) = 0$, let $R(\tau)$ be the reachable set for (1.1) at a given time τ. In the first chapter it was proved that, under generic conditions on the vector fields F, G, the problem of reaching points $x \in R(\tau)$ from the origin in minimum time admits a regular synthesis. Indeed, one can partition the set $R(\tau)$ into finitely many embedded submanifolds $\mathcal{M}_i$ such that, on each $\mathcal{M}_i$, the corresponding optimal control is either $u \equiv \pm 1$, or singular, i.e. $u = \varphi_S(x) \notin \{-1, +1\}$ where:

$$\varphi_S(x) = -\frac{\nabla \Delta_B(x) \cdot F(x)}{\nabla \Delta_B(x) \cdot G(x)}.$$

In paricular, the minimum time function is piecewise smooth on $R(\tau)$, possibly non-differentiable on the boundaries of the submanifolds $\mathcal{M}_i$. These boundaries, of dimension 1 or 0, were called frame curves and frame points respectively.

The pourpose of the present chapter is to provide a classification of all types of frame curves and points, arising under generic conditions on F, G. In section 3, we illustrate how the various singularities can occur, constructing explicit examples of each one of them. In sections 4 and 5 we classify all generic types of singularities, up to a relation of topological equivalence. We refer to the previous examples as representations of the various equivalence classes.

A classification of dynamics of planar control systems, independently of optimization problems, was considered in [4].

2.Basic definitions

For each point $x \in R(\tau)$ we use the symbols γ_x and u_x to indicate the associated trajectory $\gamma_x \in \Gamma_{\mathcal{A}}(\Sigma, \tau)$, such that $Term(\gamma_x) = \gamma(t_x) = x$, and the corresponding optimal control steering the origin to x in minimum time. If x does not belong to any frame curve then we define the feedback control:

$$u_{\mathcal{A}}(x) = u_x(t_x). \tag{2.1}$$

From the description of $\mathcal{A}$ it is easy to verify that $u_{\mathcal{A}}$ is constant on every connected set that does not intersect any frame curve, therefore the frame curves divide $R(\tau)$ in a finite number of connected components on which the control $u_{\mathcal{A}}$ is constant.

The algorithm constructs only six types of frame curves:

(F1) The trajectory γ^-, $In(\gamma^-) = \gamma^-(0) = 0$, corresponding to the control $u^- \equiv -1$

(F2) The trajectory γ^+, $In(\gamma^+) = \gamma^+(0) = 0$, corresponding to the control $u^+ \equiv 1$

(F3) The topological frontier of the reachable set: $Fr(R(\tau))$

(F4) Conjugate curves to other frame curves, also called switching curves

(F5) Regular turnpikes

(F6) Overlap curves.

To denote these types of curves we use, respectively, the symbols: X, Y, F, C, S and K. Therefore we say that a frame curve D, or briefly a FC, is an $X-$curve if $D \subset \gamma^-(Dom(\gamma^-))$ and similarly for the other types of curves. We write $D \in \Gamma_{\mathcal{A}}(\Sigma, \tau)$ to denote the fact that D is a FC constructed by $\mathcal{A}$.

Now consider two systems Σ_1, Σ_2, a time $\tau \geq 0$ and two open sets $U_1 \subset R_1(\tau)$ and $U_2 \subset R_2(\tau)$; here R_1 and R_2 denote the reachable sets of Σ_1 and Σ_2 respectively. Assume that $\mathcal{A}$ succeedes for Σ_1 and for Σ_2 at time τ. We will say that $\Gamma_1 = \Gamma_{\mathcal{A}}(\Sigma_1, \tau) \restriction U_1$ and $\Gamma_2 = \Gamma_{\mathcal{A}}(\Sigma_2, \tau) \restriction U_2$ are *equivalent* if there exists an homeomorphism $\psi : U_1 \mapsto U_2$ such that:

(E1) ψ induces a bijection on Γ_i: $\{\psi(\gamma_x(t)) : t \in Dom(\gamma_x)\} \cap U_1 = \{\gamma_{\psi(x)}(t) : t \in Dom(\gamma_{\psi(x)})\} \cap U_2$ for each $x \in U_1$; if the two set are oriented for increasing t then ψ preserves the orientation

(E2) ψ induces a bijection on frame curves, i.e. for each FC D_1 of Γ_1 we have that $\psi(D_1)$ is a FC of Γ_2 of the same type and viceversa, assuming that the types $X-, Y-$ are equivalent.

In this case we write $\Gamma_1 \restriction U_1 \equiv \Gamma_2 \restriction U_2$.

Remark 2.1

Note that in the definition of equivalence there are no request about the time along γ_x, in fact there are no condition of the type $\psi(\gamma_x(t)) = \gamma_{\psi(x)}(t)$. It is necessary to give a not too strict definition of equivalence to have a discrete set of equivalence classes. The same problem occurs in the definition of equivalence for a singular point of a dynamical system. In this case the orbital equivalence was introduced, see [1].

Given x_1, x_2 we say that $\Gamma_1 \restriction x_1$ and $\Gamma_2 \restriction x_2$ are equivalent, or we write $\Gamma_1 \restriction x_1 \equiv \Gamma_2 \restriction x_2$, if there exists U_1, U_2 neighborhoods of x_1, x_2, respectively, such that $\Gamma_1 \restriction U_1 \equiv \Gamma_2 \restriction U_2$. We say that two FC's D_i of Γ_i, $i = 1, 2$, are *equivalent* if for each $y_1 \in D_1 \setminus \partial D_1$, $y_2 \in D_2 \setminus \partial D_2$ we have that $\Gamma_1 \restriction y_1 \equiv \Gamma_2 \restriction y_2$. Similarly two frame points x_i of Γ_i, $i = 1, 2$, are *equivalent* if $\Gamma_1 \restriction x_1 \equiv \Gamma_2 \restriction x_2$.

If $\mathcal{A}$ succeedes for Σ at time τ then, under generic conditions, all frame points are intersection of two frame curves. Using the same notation used for frame curves, we will say that the origin is an $(X, Y)-$point, in fact $0 \in \gamma^+ \cap \gamma^-$. Similarly if a frame point x is intersection of two frame curves D_1, D_2 of type, respectively, V_1, V_2 then we say that x is a $(V_1, V_2)-$point. As for frame curves we write $x \in \Gamma_{\mathcal{A}}(\Sigma, \tau)$ to denote the fact that x is a frame point constructed by $\mathcal{A}$.

Given $\varepsilon > 0$ we say that two systems $\Sigma_1 = (F_1, G_1)$, $\Sigma_2 = (F_2, G_2)$ are $\varepsilon-$*near* if:

$$max\ \{\| F_1 - F_2 \|_{C^3}, \| G_1 - G_2 \|_{C^3}\} \leq \varepsilon. \tag{2.2}$$

Consider a system Σ for which $\mathcal{A}$ succeedes at a time τ and a frame point x of $\Gamma_{\mathcal{A}}(\Sigma, \tau)$. We will say that x is *structurally stable* if there exist $\varepsilon > 0$, $\delta > 0$ such that for each system Σ', $\varepsilon-$near to Σ, there exists a unique frame point x' of the same type verifying:

$$\| x - x' \| \leq \delta \tag{2.3}$$

$$\Gamma_{\mathcal{A}}(\Sigma, \tau) \restriction x \equiv \Gamma_{\mathcal{A}}(\Sigma', \tau) \restriction x'. \tag{2.4}$$

We are interested only in structurally stable frame point: from conditions (2.3,4) we know that these are the only points that are *observable*, i.e. a small perturbation of the system does not change the structure of the synthesis near these points.

3.Examples of frame curves and of frame points

In this section we give some examples of systems for which $\mathcal{A}$ succeedes at a given time $\tau > 0$ and some local examples of frame points. These examples, in connection with the examples of the first chapter cover all possibilities for frame curves and frame points: we will prove it in the following sections.

Example 1.

Consider ε, $0 < \varepsilon < 1$, $\tau > \frac{\pi}{\sqrt{1-\varepsilon}}$ and the system Σ:

$$\begin{cases} \dot{x}_1 = \varepsilon x_2 + u x_2 \\ \dot{x}_2 = u(1 - x_1) \end{cases} . \tag{3.1}$$

It is easy to check that $\Sigma \in \Xi$ and:

$$[F, G] = \begin{pmatrix} -\varepsilon(1 - x_1) \\ -\varepsilon x_2 \end{pmatrix} . \tag{3.2}$$

From (3.2) and Lemma 2.4 of chapter 1 we have that the system is locally controllable. The $X-$trajectory passing through the point (x_1^0, x_2^0) at time 0 is:

$$x_1(t) = (x_1^0 - 1)\ cos(\sqrt{1-\varepsilon}\ t) + x_2^0\sqrt{1-\varepsilon}\ sin(\sqrt{1-\varepsilon}\ t) + 1 \tag{3.3}$$

$$x_2(t) = \frac{(x_1^0 - 1)}{\sqrt{1-\varepsilon}}\ sin(\sqrt{1-\varepsilon}\ t) - x_2^0\ cos(\sqrt{1-\varepsilon}\ t). \tag{3.4}$$

The $Y-$trajectory passing through the point (x_1^0, x_2^0) at time 0 is:

$$x_1(t) = (x_1^0 - 1)\ cos(\sqrt{1+\varepsilon}\ t) + x_2^0\sqrt{1+\varepsilon}\ sin(\sqrt{1+\varepsilon}\ t) + 1 \tag{3.5}$$

$$x_2(t) = -\frac{(x_1^0 - 1)}{\sqrt{1+\varepsilon}}\ sin(\sqrt{1+\varepsilon}\ t) + x_2^0\ cos(\sqrt{1+\varepsilon}\ t). \tag{3.6}$$

The equation for the turnpikes is:

$$0 = \Delta_B(x_1, x_2) = -\varepsilon x_2^2 + \varepsilon(1 - x_1)^2. \tag{3.7}$$

From (3.7) it follows that every turnpike is a subset of $S = \{(x_1, x_2) : x_2 = \pm(1-x_1)\}$. Now let $\gamma^{\pm}$, $In(\gamma^{\pm}) = \gamma^{\pm}(0) = 0$, be the trajectories corresponding to the controls $u^{\pm} = \pm 1$. Using (3.5,6) it's easy to verify that the trajectory γ^{+} intersects the set S in a point (x_1^{+}, x_2^{+}) of the first quadrant. The algorithm $\mathcal{A}$ constructs the turnpike $S_1 = \{(x_1, x_2) :$

$x_2 = 1 - x_1, x_1^+ \leq x_1 < 1\}$. The singular control φ_S^1 along the turnpike S_1, cfr. (3.18) of chapter 1, is:

$$\varphi_S^1(x_1, x_2) = -\frac{\varepsilon x_2}{1 - x_1 + x_2} > -1. \tag{3.8}$$

From (3.8) we have:

$$\dot{x}_1(\varphi_S^1) = \frac{\varepsilon}{2}(1 - x_1) \tag{3.9}$$

hence the point $(1, 0)$ is not reached in finite time.
Similarly using (3.3,4) it's easy to verify that the trajectory γ^- intersects the set S in a point of the fourth quadrant:

$$(x_1^-, x_2^-) \doteq \gamma^-\left(\frac{1}{\sqrt{1-\varepsilon}}\, arccos\left(\sqrt{\frac{1}{2-\varepsilon}}\right)\right). \tag{3.10}$$

The algorithm $\mathcal{A}$ constructs the turnpike $S_2 = \{(x_1, x_2) : x_2 = x_1 - 1, x_1 \leq x_1^-\}$.The control φ_S^2, cfr. (3.18) of chapter 1, is:

$$\varphi_S^2(x_1, x_2) = \frac{\varepsilon x_2}{1 - x_1 - x_2}.$$

The trajectories $\gamma^\pm$ are very close to the circle A of centre $(1, 0)$ and radius 1; γ^+ runs clockwise and γ^- counterclockwise. From (3.3-6) we have that γ^+ lies inside A, γ^- outside, and:

$$\gamma^+ \cap \gamma^- \cap A = \{(0, 0), (2, 0)\}.$$

The two trajectories do not meet each other at $(2, 0)$, in fact:

$$(2, 0) = \gamma^+\left(\frac{\pi}{\sqrt{1+\varepsilon}}\right) = \gamma^-\left(\frac{\pi}{\sqrt{1-\varepsilon}}\right) \tag{3.11}$$

but the trajectories constructed by the algorithm give rise to an overlap curve K and $\gamma^\pm$ end on it.
In fig.1 it is represented $R(\tau)$.

Example 2.

Consider $\tau > \frac{1}{3} ln(4)$ and the system Σ:

$$\begin{cases} \dot{x}_1 = 3x_1 + u \\ \dot{x}_2 = x_1^2 + x_1 \end{cases} . \tag{3.12}$$

We have that $\Sigma \in \Xi$ and

$$[F, G] = \begin{pmatrix} -3 \\ -2x_1 - 1 \end{pmatrix}$$

hence the system is locally controllable.
The $X-$trajectory passing through the point (x_1^0, x_2^0) at time 0 is:

$$x_1(t) = \left(x_1^0 - \frac{1}{3}\right) e^{3t} + \frac{1}{3} \tag{3.13}$$

$$x_2(t) =$$

$$\frac{1}{6}\left(x_1^0 - \frac{1}{3}\right)^2 e^{6t} + \frac{5}{9}\left(x_1^0 - \frac{1}{3}\right) e^{3t} + \frac{4}{9}t + x_2^0 - \frac{1}{6}\left(x_1^0 - \frac{1}{3}\right)^2 - \frac{5}{9}\left(x_1^0 - \frac{1}{3}\right). \tag{3.14}$$

The $Y-$trajectory passing through the point (x_1^0, x_2^0) at time 0 is:

$$x_1(t) = \left(x_1^0 + \frac{1}{3}\right) e^{3t} - \frac{1}{3} \tag{3.15}$$

$$x_2(t) =$$

$$\frac{1}{6}\left(x_1^0 + \frac{1}{3}\right)^2 e^{6t} + \frac{1}{9}\left(x_1^0 + \frac{1}{3}\right) e^{3t} - \frac{2}{9}t + x_2^0 - \frac{1}{6}\left(x_1^0 + \frac{1}{3}\right)^2 - \frac{1}{9}\left(x_1^0 + \frac{1}{3}\right). \tag{3.16}$$

The equation for the turnpikes is:

$$0 = \Delta_B(x_1, x_2) = -(2x_1 + 1)$$

then every turnpike is subset of $S = \{(x_1, x_2) : x_1 = -\frac{1}{2}\}$. We have that the control φ_S to stay on S, cfr. (3.18) of chapter 1, is:

$$\varphi_S(x_1, x_2) = \frac{3}{2}$$

then there are no regular turnpike.

Now consider the pairs $(\gamma_s, u_s) \in Adm(\Sigma)$, $In(\gamma_s) = 0$, $Dom(\gamma_s) = [0, s + \varepsilon_s]$ $(\varepsilon_s \geq 0)$, that start as $X-$trajectories and switch at time s going on as $Y-$trajectories up to the time $s + \varepsilon_s$. Let $(\gamma^*, u^*) = (\gamma_{s^*}, u_{s^*})$ be the pair that verifies:

$$\gamma^*(s^*) = \left(-\frac{1}{2}, -\frac{13}{72} + \frac{4}{9} ln\left(\sqrt[3]{\frac{5}{2}}\right)\right). \tag{3.17}$$

Let $\varepsilon^* = \varepsilon_{s^*}$ and suppose that γ^* satisfies the PMP with adjoint variable λ^*. We know that:

$$\lambda^*(s^*) \cdot G(\gamma^*(s^*)) = 0, \qquad \Delta_B(\gamma^*(s^*)) = 0 \tag{3.18a}$$

then:

$$\frac{d}{dt}\left(\lambda^*(t)\cdot G(\gamma^*(t))\right)\Big|_{t=s^*}=\lambda^*(s^*)\cdot[F,G](\gamma^*(s^*))=0. \qquad (3.18b)$$

From (3.18a,b) and straightforward calculations we have in $\gamma^*(s^*)$ the situation of fig.2. Now it's easy to verify that:

$$\forall t\in[s^*,s^*+\varepsilon^*] \qquad \Delta_B(\gamma^*(t))>0 \qquad (3.19)$$

then, for each $t\in[s^*,s^*+\varepsilon^*]$, the couple of vectors $(G(t),[F,G](t))$ must form a positive oriented base of $I\!R^2$. From $u^*(t)\restriction[s^*,s^*+\varepsilon^*]\equiv 1$ it follows that the two functions $\lambda^*(t)\cdot G(\gamma^*(t))$, $\lambda^*(t)\cdot[F,G](\gamma^*(t))$ have to be positive in a right neighborhood of s^*. Therefore G and $[F,G]$ lies in the signed zone of fig.2 but this is prohibited by (3.19) as observed above. It follows that $\varepsilon^*=0$.
Similarly, if the trajectories γ_s, $s\in[ln(\sqrt[3]{2}),s^*]$, satisfy the PMP, then ε_s has a bound. More precisely the algorithm $\mathcal{A}$ constructs trajectories that have a second switching point and these switching points form a switching curve C_1.

The geometric reasoning above is very general, however in our case we can calculate all directly. Suppose γ_s satisfies the PMP with adjoint variable λ^s. The quation for λ^s is:

$$\dot\lambda^s(t)=-\lambda^s(t)\cdot\left(\nabla F(\gamma_s(t))+u_s(t)\ \nabla G(\gamma_s(t))\right)=-\lambda^s(t)\cdot\left(\nabla F(\gamma_s(t))\right). \qquad (3.20)$$

The equation (3.20) for time $t\geq s$ becomes:

$$(\dot\lambda_1^s,\dot\lambda_2^s)(t)=\left(-3\lambda_1^s(t)-\lambda_2^s(t)\left[2\left(\gamma_s(s)+\frac{1}{3}\right)\ e^{3(t-s)}+\frac{1}{3}\right],0\right). \qquad (3.21)$$

Denote by ϕ_s the switching function along (γ_s,u_s,λ^s). The solution to (3.21) with initial condition:

$$\lambda^s(s)\cdot G(\gamma_s(s))=0$$

is:

$$\lambda_2^s(t)\equiv\lambda_2^s(0)$$

$$\phi_s(t)=\lambda_1^s(t)=\lambda_2^s(t)\left[\frac{1}{3}\left(\gamma_s(s)+\frac{2}{3}\right)\ e^{-3(t-s)}-\frac{1}{3}\left(\gamma_s(s)+\frac{1}{3}\right)\ e^{3(t-s)}-\frac{1}{9}\right].$$

The equation $\phi_s(t)=0$ has two solutions:

$$t_1^s=s \qquad t_2^s=ln\left(\sqrt[3]{\frac{-3\gamma_s(s)-2}{3\gamma_s(s)+1}}\right)$$

and we obtain:

$$\gamma_s(t_2^s) = -\gamma_s(s) - 1$$

that describes C_1.

Let γ^- be, as before, the X−trajectory verifying $In(\gamma^-) = \gamma^-(0) = 0$. The point $\gamma^-(ln\ \sqrt[3]{4})$ is conjugate to the origin along γ^-. Consider the trajectories γ_r, $In(\gamma_r) = 0$, $Dom(\gamma_r) = [0, b_r]$ $(b_r \geq r)$, that start as Y−trajectories and have a switching at time r going on as X−trajectories. Again we can make direct calculations and we obtain the existence of a second switching time (if $r < ln\ \sqrt[3]{2}$):

$$t_r = ln\left(\sqrt[3]{-1 - \frac{10}{6\ a_r}}\right) \tag{3.22}$$

where $a_r = \left(\gamma_r(r) - \frac{1}{3}\right)$. These switching points form another switching curve C_2 that starts at $\gamma^-(ln\sqrt[3]{4})$. Again we obtain from (3.22):

$$\gamma_r(t_r) = -\gamma_r(r) - 1.$$

In fig.3 it is represented $R(\tau)$.

Remark 3.1

For a conjugate point x_0 to the origin along $\gamma^{\pm}$ we have one of these two cases:
(CP1) x_0 belongs to a switching curve
(CP2) x_0 belongs to an overlap curve.
It is useful to notice that, in this example, if the trajectories γ_r go on as X−trajectories not doing the second switching, they form an overlap curve with the $Y * X$−trajectories. On the contrary, in the first example of chapter 1 if the same type of trajectories satisfy the PMP and go on, after the overlap curve, they form a switching curve. Generically at a conjugate point there are both curves but one *comes before* the other.

Example 3.

Consider $0 < \varepsilon << 1$, $\tau > 1$ and the system Σ:

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1^3 - \varepsilon x_1 \end{cases} \tag{3.23}$$

We have that $\Sigma \in \Xi$ and:

$$[F, G] = \begin{pmatrix} 0 \\ \varepsilon - 3x_1^2 \end{pmatrix}$$

then the system is locally controllable. The $X, Y-$trajectories are quartic polynomials of the following types, respectively:

$$x_2 = -\frac{x_1^4}{4} + \varepsilon\,\frac{x_1^2}{2} + \alpha \qquad \alpha \in I\!R$$

$$x_2 = \frac{x_1^4}{4} - \varepsilon\,\frac{x_1^2}{2} + \alpha \qquad \alpha \in I\!R.$$

The equation for turnpikes is:

$$0 = \Delta_B(x_1, x_2) = \varepsilon - 3x_1^2$$

whose set of solutions is:

$$\left\{(x_1, x_2) : x_1 = \pm\sqrt{\frac{\varepsilon}{3}}\right\}. \tag{3.24}$$

Every turnpike is subset of (3.24). The algorithm constructs the turnpikes:

$$S_1' = \left\{(x_1, x_2) : x_1 = \sqrt{\frac{\varepsilon}{3}}, x_2 \le -\frac{5}{36}\varepsilon^2\right\} \cap R(\tau)$$

$$S_2' = \left\{(x_1, x_2) : x_1 = -\sqrt{\frac{\varepsilon}{3}}, x_2 \ge \frac{5}{36}\varepsilon^2\right\} \cap R(\tau).$$

The points $(\pm\sqrt{\varepsilon}, \mp\frac{\varepsilon^2}{4})$ are conjugate to the origin along $\gamma^{\pm}$. Two overlap curves K_2, K_1, respectively, start at these points. The algorithm cuts partially the turnpikes S_1', S_2' determining two turnpikes $S_1 \subset S_1'$, $S_2 \subset S_2'$. The turnpikes S_1, S_2 end on K_1, K_2 respectively. In fig.4 it is rapresented $R(\tau)$

Remark 3.2

Let Σ'' be the system $(\dot{x}_1, \dot{x}_2) = (u, x_1^3 - \psi(x_1)\varepsilon x_1)$ where $\psi \in \mathcal{C}^\infty(I\!R, [0, +\infty[)$, $\{x : \psi(x) \neq 0\} \subset B(0,1)$ and $\psi \restriction B(0, \frac{1}{2}) \equiv 1$. Note that this system is obtained by a small perturbation of the system Σ': $(\dot{x}_1, \dot{x}_2) = (u, x_1^3)$. The synthesis $\Gamma_{\mathcal{A}}(\Sigma', \tau)$ is formed by bang-bang trajectories with at most one switching. It's clear that Σ' is not structurally stable (in a sense that will be stated more precisely in the following paper), in fact we have that for ε small the system Σ'' is $\varepsilon-$near (cfr. (2.2)) to Σ' but $\Gamma_{\mathcal{A}}(\Sigma'', \tau)$ has a structure different from $\Gamma_{\mathcal{A}}(\Sigma', \tau)$.

Remark 3.3

Note that the overlap curves K_1, K_2 are tangent to γ^-, γ^+ respectively. This is the general case and the first example of chapter 1 is degenerate. Consider x_0 of Remark 3.1. At x_0 we can have a switching curve and also this curve generically is tangent to $\gamma^{\pm}$.

The following will be local examples of other types of singular points. We will consider a control system Σ and some manifolds, from which some trajectories of Σ start with a given control and possibly a given adjoint variable. We apply the algorithm $\mathcal{A}$ with the above conditions. These are examples of what can be the local structure of $\Gamma_{\mathcal{A}}$ and their local character let them to be simpler.

Example 4.

Consider the system (cfr. Example 1 of chapter 1):

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = \frac{1}{2}x_1^2 + x_1 \end{cases} \tag{3.25}$$

and the two embedded submanifold:

$$M_1 = \left\{(x_1, x_2) : x_1 = 0, -\frac{2}{3} \le x_2 \le 0\right\}$$

$$M_2 = \left\{(x_1, x_2) : x_1 = \sqrt{2}, -\frac{2}{3} \le x_2 \le \frac{2}{3}\right\}.$$

We assume that from each $y \in M_1$ it leaves, with initial time 0, a $Y-$trajectory $\gamma_1(y)$ and from each $x \in M_2$ it leaves, with initial time 0, an $X-$trajectory $\gamma_2(x)$. We assume also that from each point of the $Y-$trajectory $\gamma_1(0)$ it leaves an $X-$trajectory, i.e. $\gamma_1(0)$ is the trajectory γ^+ of a given system.
At the point $(1, \frac{2}{3})$ the trajectories $\gamma_1(0)$, $\gamma_2\big((\sqrt{2}, 0)\big)$ meet each other. After this point $\gamma_1(0)$ is not constructed by $\mathcal{A}$ because the trajectories $\gamma_2\big((\sqrt{2}, c)\big)$, $c \ge 0$, are faster. The trajectories $\gamma_1\big((0, -c)\big)$ and $\gamma_2\big((\sqrt{2}, -c)\big)$ give raise to an overlap curve:

$$K = \left\{(x_1, x_2) : x_1 = -1, 0 \le x_2 \le \frac{2}{3}\right\}.$$

In fig.5 it is portrayed this local example.

Example 5.

Consider the system (3.25) and the manifold:

$$M = \left\{(x_1, x_2) : x_1 = 0, -\frac{1}{3} \le x_2 \le \frac{1}{3}\right\}.$$

We assume that from each $(x_1, x_2) \in M$ it starts, with initial time 0, an $X-$trajectory $\gamma(x_1, x_2) = \gamma(x_2)$ with adjoint varable $\lambda(x_2)$ that satisfies:

$$[\lambda_1(x_2)](0) = \frac{-1 - 4\ sgn(x_2)\ x_2^2}{2} \qquad [\lambda_2(x_2)](0) = -1$$

where $sgn(x) = x \mid x \mid^{-1}$ if $x \neq 0$ and $sgn(0) = 0$. Now the switching function along $(\gamma(x_2), -1, \lambda(x_2))$ is:

$$[\phi(x_2)](t) = \lambda_1(t) = -\frac{t^2}{2} + t - \frac{1 + 4\ sgn(x_2)\ x_2^2}{2}.$$

If $x_2 \leq 0$ the equation $[\phi(x_2)](t) = 0$ has the following solutions:

$$t_1(x_2) = 1 + 2 \mid x_2 \mid \qquad t_2(x_2) = 1 - 2 \mid x_2 \mid$$

otherwise there are no solutions. Then every trajectory $\gamma(x_2)$, $x_2 \leq 0$, switches at the point:

$$[\gamma(x_2)](t_2) = \left(2 \mid x_2 \mid -1, -\frac{(2 \mid x_2 \mid -1)^3}{6} - \frac{(2 \mid x_2 \mid -1)^2}{2} + x_2\right).$$

These switching points give rise to a switching curve C.
The trajectory $\gamma(0)$ crosses the set $\{(x_1, x_2) : \Delta_B(x_1, x_2) = 0\} = \{(x_1, x_2) : x_1 = -1\}$ at a switching point, hence the algorithm $\mathcal{A}$ constructs the turnpike:

$$S = \left\{(x_1, x_2) : x_1 = -1, x_2 \leq -\frac{1}{3}\right\}.$$

In fig.6 it is represented this local example.

Example 6.

Now consider the same system and the same manifold of the previous example and define S in the same way. We assume that from each $(0, x_2) \in M$ it leaves an $X-$trajectory $\gamma(x_2)$ with initial time:

$$t_0(x_2) = -\frac{2}{3}\ x_2$$

and with adjoint variable satisfying:

$$[\lambda_1(x_2)](0) = \frac{-1 - \alpha\ sgn(x_2)\ x_2^2}{2} \qquad [\lambda_2(x_2)](0) = -1$$

where $\alpha > 0$ and sgn is defined as above. We have again a switching curve C.

The $X-$trajectories starting from $(0, x_2)$, $x_2 \leq 0$, reach $(-1, -\frac{1}{3} - x_2) \in S$ at time:

$$1 + \frac{2}{3}\ |x_2|.$$

On the other hand, the $Z * X-$trajectories, see chapter 1 for the notations, starting from 0 reach the same point at time:

$$1 + \frac{|x_2|}{2}.$$

Therefore the $Y * Z * X-$trajectories starting from the origin and the $X-$trajectories starting from $(0, x_2)$, $x_2 \leq 0$, give rise to an overlap curve K having $(-1, -\frac{1}{3})$ as endpoint. Let $(s, k(s))$ be a parametrization of K in a neighborhood of $(-1, -\frac{1}{3})$ and define:

$$\beta \doteq \left. \frac{dk(s)}{ds} \right|_{s=-1^+} .$$

Now we have a parametrization $(c_1(x_2), c_2(x_2))$ of the switching curve C. After straightforward calculations we have:

$$\left. \frac{dc_2}{dc_1} \right|_{c_1=-1^+} = -\frac{3}{2} - \frac{1}{\alpha}.$$

Then if α is sufficiently small:

$$\beta \geq -\frac{3}{2} - \frac{1}{\alpha}$$

and the overlap curve K *comes before* the switching curve C. Therefore the curve C is cut by the algorithm.
In fig.7 it is portrayed this local example.

Example 7.

Consider the system (3.12) of Example 2 and the manifold:

$$M = \{(x_1, x_2) : x_1 = 0, |x_2| \leq 1\}.$$

We assume that from every point $(0, x_2) \in M$ it leaves, with initial time 0, an $X-$trajectory $\gamma(x_2)$ with adjoint variable satisfying:

$$[\lambda_1(x_2)](0) = -\frac{9}{36} - \frac{1}{36}\, sgn(x_2)\, x_2 \qquad [\lambda_2(x_2)](0) = -1.$$

With simple calculations we obtain the solutions to the equation $[\phi(x_2)](t) = 0$, where $\phi(x_2)$ is the switching function along $(\gamma(x_2), -1, \lambda(x_2))$:

$$t^{\pm}(x_2) = ln \left(\sqrt[3]{\frac{5}{2} \pm \frac{1}{2}\sqrt{9 + 36\, [\lambda_1(x_2)](0)}} \right).$$

Hence the trajectories $\gamma(x_2)$, $x_2 \leq 0$, has a switching at time $t^-(x_2)$, instead the trajectories $\gamma(x_2)$, $x_2 > 0$, do not switch. These switching points form a switching curve C_1 having the point (3.17) as endpoint.

Now the equation $\phi(x_2) = 0$, where $\phi(x_2)$ denote again the switching function along $\gamma(x_2)$, after the time $t^-(x_2)$ has another solution:

$$t'(x_2) = t^-(x_2) + ln\left(\sqrt[3]{\frac{-3\,x_1(x_2) - 2}{3\,x_1(x_2) + 1}}\right)$$

where $x_1(x_2)$ is the first coordinate of the first switching point of $\gamma(x_2)$. These switching points form another switching curve C_2 that meet C_1 at the point (3.17).
In fig.8 this local example is portrayed.

Example 8.

Consider the system:

$$\begin{cases} \dot{x}_1 = \frac{x_1+1}{2} + u\,\frac{x_1-1}{2} \\ \dot{x}_2 = \frac{x_2}{2} + u\,\frac{x_2}{2} \end{cases} \tag{3.26}$$

and the manifold:

$$M = \{(x_1, x_2) : x_1 = 0, 0 < x_2 < 1\}.$$

We assume that from every point $(0, x_2) \in M$ it leaves, with initial time 0, an $X-$trajectory $\gamma(x_2)$ with adjoint variable satisfying:

$$[\lambda_1(x_2)](0) = \frac{x_2}{1 - \sqrt{1 - x_2^2}} \qquad [\lambda_2(x_2)](0) = -1. \tag{3.27}$$

It is easy to verify, from (3.26,27), that every $\gamma(x_2)$ switches at time:

$$t(x_2) = 2 - \sqrt{1 - x_2^2}$$

and the corresponding switching points form a switching curve:

$$C = \{(x_1, \psi(x_1)) : 1 < x_1 < 2\} \qquad \psi(x_1) \doteq \sqrt{1 - (2 - x_1)^2}$$

that is an arc of circle.

Observe that for ε small, $Y(\varepsilon, \psi(\varepsilon))$ points to the right of C and $Y(2 - \varepsilon, \psi(2 - \varepsilon))$ points to the left of C. Then there exists $(\bar{x}_1, \bar{x}_2) \in C$ such that $Y(\bar{x}_1, \bar{x}_2)$ is tangent to C. Define $C' \doteq \{(x_1, \psi(x_1)) \in C : x_1 \geq \bar{x}_1\}$. The trajectories $\gamma(x_2)$ that reach C' meet other trajectories $\gamma(x_2)$ giving rise to an overlap curve K. We can move along C' with a trajectory of the system, hence we can construct an envelope for the curves $\gamma(x_2) \restriction [0, t(x_2)]$

that reach C'; see [17],[18] for envelope theory. We have that the subset C' of C is cut by the algorithm.
In fig.9 this local example is represented.

Example 9.

Consider the system (cfr. Example 2 of chapter 1):

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + u \end{cases} \tag{3.28}$$

and the two manifolds:

$$M_1 = \{(x_1, x_2) : x_1 = 0, 1 \le x_2 \le 2\}$$

$$M_2 = \{(x_1, x_2) : 1 \le x_1 \le 2, x_2 = 3\}.$$

The algorithm $\mathcal{A}$ succeedes for the system (3.28) at time 4. A $Y-$trajectory $\gamma'(x) \in \Gamma_{\mathcal{A}}(\Sigma, 4)$, with an associate adjoint variable $\lambda'(x)$, passes through each point $x \in M_1$. We suppose that from each $x \in M_1$ it leaves, with initial time 0, a $Y-$trajectory $\gamma(x)$ with adjoint variable $\lambda(x)$ such that $(\gamma(x), \lambda(x))$ is obtained from $(\gamma'(x), \lambda'(x))$ shifting the time.
Consider the line given by the equation:

$$x_2 = (\sqrt{2} - 3)\ x_1 + 8 - \sqrt{8}. \tag{3.29}$$

For each $s \in [1, \frac{2}{3}\sqrt{6}]$ the trajectory $\gamma^s = \gamma\big((0,s)\big)$ intersects the line (3.29) in a point, say (x_1^s, x_2^s).
Let $r(s) \in [1,2]$ be such that the $X-$trajectory passing through $(r(s), 3)$ intersects the line (3.29) in (x_1^s, x_2^s). We assume that from each $(r, 3) \in M_2$ it starts an $X-$trajectory γ_r with initial time $t(r)$. If $r = r(s)$ for some s we define, denoting by d the euclidean distance:

$$t_s \doteq t\big(r(s)\big) = 2\ arcsin\left(\frac{d\big((0,s),(x_1^s,x_2^s)\big)}{2\sqrt{1+s^2}}\right) - 2\ arcsin\left(\frac{d\big((r(s),3),(x_1^s,x_2^s)\big)}{2\sqrt{(r(s)+1)^2+9}}\right). \tag{3.30}$$

otherwise:

$$t(r) = max\ \{t_s : s \in [1,2]\}.$$

We associate to every γ_r an adjoint variable λ_r verifying:

$$\lambda_1^r\big(t(r)\big) = -1 \qquad \lambda_2^r\big(t(r)\big) = 0. \tag{3.31}$$

The equation (3.31) implies that $\lambda^r\big(t(r)\big)\cdot G\big((r,3)\big) = 0$ then to satisfy the PMP it remains an $X-$trajectory for an interval of time of length π.
The trajectories γ^s, $s \in [1, \frac{2}{3}\sqrt{6}]$, form a switching curve:

$$C = \left\{(x_1, x_2) : x_2 = \sqrt{1-(x_1-3)^2}, 2 \le x_1 \le \frac{8}{3}\right\}.$$

By direct calculations one can verify that (3.30) assures that the trajectories γ^s and $\gamma_{r(s)}$, $s \in [\frac{2}{3}\sqrt{6}, 2]$, meet each other giving rise to an overlap curve:

$$K = \left\{(x_1, x_2) : (x_1, x_2) \text{ satisfies } (3.29), 2 \le x_1 \le \frac{8}{3}\right\}.$$

The curves K and C meet each other at the point:

$$(\frac{8}{3}, \frac{\sqrt{8}}{3}). \qquad (3.31)$$

In fig.10 it is rapresented this local example.

Example 10.

Consider the system:

$$\begin{cases} \dot{x}_1 = 3\ x_1 - u \\ \dot{x}_2 = x_1^2 + x_1 \end{cases} \qquad (3.32)$$

that is obtained from the system (3.12) substituting G with $-G$, and the manifold:

$$M = \{(x_1, x_2) : x_1 = 0, |x_2| < \varepsilon\}.$$

We assume that from every point $(0, x_2) \in M$ it leaves, with initial time $t_0(x_2)$, a $Y-$trajectory $\gamma(x_2)$ with adjoint variable satisfying:

$$[\lambda_1(x_2)](0) = -\frac{9}{36} + 25\ sgn(x_2)\ x_2 \qquad [\lambda_2(x_2)](0) = -1.$$

With simple calculations we obtain the solutions to the eqaution $[\phi(x_2)](t) = 0$, where $\phi(x_2)$ is the switching function along $(\gamma(x_2), +1, \lambda(x_2))$:

$$t^{\pm}(x_2) = t_0(x_2) + ln\left(\sqrt[3]{\frac{5}{2} \pm \frac{1}{2}\sqrt{9+36\,[\lambda_1(x_2)](0)}}\right). \qquad (3.33)$$

Hence the trajectories $\gamma(x_2)$, $x_2 \ge 0$, has a switching at time $t^-(x_2)$, instead the trajectories $\gamma(x_2)$, $x_2 < 0$, do not switch. Let $x^-(x_2) \doteq [\gamma(x_2)](t^-(x_2))$, $x_2 \ge 0$. From (3.33) we have:

$$x_1^- = -\frac{1}{2} + 5\ x_2^2 \qquad (3.34)$$

and these switching points form a switching curve C_1 having the point (3.17) as endpoint. Now the equation $\phi(x_2) = 0$, where $\phi(x_2)$ denote again the switching function along $\gamma(x_2)$, after the time $t^-(x_2)$ has another solution:

$$t'(x_2) = t^-(x_2) + ln\left(\sqrt[3]{\frac{-3\, x_1^-(x_2) - 2}{3\, x_1^-(x_2) + 1}}\right) \qquad (3.35).$$

These switching points form another switching curve C_2 that meet C_1 in the point (3.17). It is easy to verify that the $X-$trajectories leaving from C_1 cross the trajectory $\gamma_0 \doteq \gamma(0)$ before reaching the switching curve C_2. Hence we can define $P(x_2)$, $x_2 \geq 0$, to be the point in which $\gamma(x_2)$ meets γ_0.
Let $r(x_2)$, $x_2 \geq 0$, be such that the trajectory $\gamma(r(x_2))$ meet $\gamma(x_2)$ at the point $Q(x_2) \doteq [\gamma(x_2)](t'(x_2))$, i.e. they meet each other on C_2.
Now define $t_1(x_2), t_2(x_2)$ to be the time in which, respectively, $\gamma(x_2), \gamma_0$ reaches $P(x_2)$ and $t_3(x_2), t_4(x_2)$ to be the time in which, respectively, $\gamma(x_2), \gamma(r(x_2))$ reaches $Q(x_2)$. If x_2 is sufficiently small, from (3.34,35) we have that:

$$t_4 - t_3 < t_2 - t_1$$

then, taking ε sufficiently small, we can define

$$t_0(x_2) = 0 \qquad if\ x_2 < 0$$

$$t_0(x_2) = \frac{(t_2 - t_1) + (t_4 - t_3)}{2} \qquad if\ x_2 \geq 0.$$

Therefore the trajectories $\gamma(x_2)$, $x_2 \geq 0$, and $\gamma(x_2)$, $x_2 < 0$, meet each other forming an overlap curve K that meet C_1 at (3.17). The curve C_2 is cut by the algorithm.
In fig.11 this local example is portrayed.

4. Frame curves

In this section we give a complete description of the frame curves that can be generated by the algorithm $\mathcal{A}$. We use the notation introduced in section 2 for the six types $(F1)-(F6)$ of frame curves. Throughout this section we consider a fixed $\tau \geq 0$ and a fixed system Σ for which $\mathcal{A}$ succeedes at time τ.

We say that a FC D is *simple* if $D \setminus \partial D$ does not contain any frame point. Every FC can be divided into a finite number of simple FC's. The classification of simple FC's in connection with the classification of frame points, described in the following section, give a complete classification of FC's. In fact two FC's D_1, D_2 are equivalent if we can divide them in two families $D_1^1, \dots, D_1^n$ $D_2^1, \dots, D_2^n$ such that:

$$D_1^i \equiv D_2^i \qquad D_1^i \cap D_1^j \equiv D_2^i \cap D_2^j \quad \forall\, i,j \in \{1, \dots, n\}$$

where we assume, by definition, that $\emptyset \equiv \emptyset$. Therefore throughout this section we consider only simple FC's.

X-curve.

Consider an $X-$curve D and $x \in D \setminus \partial D$. There exists a neighborhood U of x such that the control $u_{\mathcal{A}}$ is constant in each one of the two connected components U_1, U_2 of $U \setminus D$. If, for example, $u_{\mathcal{A}} = 1$ on U_1 then $Y-$trajectories leave from D entering U_1. It's clear that there are only two possibilities:

(X1) $u_{\mathcal{A}} = -1$ on $U_1 \cup U_2$

(X2) $u_{\mathcal{A}} = 1$ on U_1 and $u_{\mathcal{A}} = -1$ on U_2, or viceversa.

Consider the system Σ_1 of Example 1 at time $\frac{\pi}{2\sqrt{1-\varepsilon}}$. If (X1) holds true then:

$$\Gamma_{\mathcal{A}}(\Sigma, \tau) \restriction x \equiv \Gamma_{\mathcal{A}}(\Sigma_1, \tau_1) \restriction \gamma^-(t_0) \tag{4.1a}$$

where, see (3.10):

$$\frac{1}{\sqrt{1-\varepsilon}}\ arccos\left(\sqrt{\frac{1}{2-\varepsilon}}\right) < t_0 < \frac{\pi}{2\sqrt{1-\varepsilon}}. \tag{4.1b}$$

Then D is equivalent to $\gamma^- \restriction [\frac{1}{\sqrt{1-\varepsilon}}\ arccos\left(\sqrt{\frac{1}{2-\varepsilon}}\right), \frac{\pi}{2\sqrt{1-\varepsilon}}]$. In this case we say that D is of type X_1 or that D is an X_1-curve.
If (X2) holds true then:

$$\Gamma_{\mathcal{A}}(\Sigma, \tau) \restriction x \equiv \Gamma_{\mathcal{A}}(\Sigma_1, \tau_1) \restriction \gamma^-(t_0) \tag{4.2a}$$

where, see (3.10):

$$0 < t_0 < \frac{1}{\sqrt{1-\varepsilon}}\ arccos\left(\sqrt{\frac{1}{2-\varepsilon}}\right). \tag{4.2b}$$

Then D is equivalent to $\gamma^- \restriction [0, \frac{1}{\sqrt{1-\varepsilon}}\ arccos\left(\sqrt{\frac{1}{2-\varepsilon}}\right)]$. In this case we say that D is of type X_2 or that D is an X_2−curve.

Y-curve.

This case can be treated as the previous one and we have the same equivalences. In this case the only difference is the sign of $u_{\mathcal{A}}$.

F-curve.

Consider an F−curve D and $x \in D \setminus \partial D$. There exists a neighborhood U of x in $R(\tau)$ such that $u_{\mathcal{A}}$ is constant on $U \setminus Fr(R(\tau))$. Consider the system Σ_2 at time τ_2 of Example 2. Let x_1 be such that γ_{x_1} corresponds to the control:

$$u_1 = 1\ on\ \left[0, \frac{1}{3}ln(2)\right], \qquad u_1 = -1\ on\ \left[\frac{1}{3}ln(2), \tau\right].$$

We have:

$$\Gamma_{\mathcal{A}}(\Sigma, \tau) \restriction x \equiv \Gamma_{\mathcal{A}}(\Sigma_2, \tau_2) \restriction x_1. \tag{4.3}$$

C-curve.

Consider a C−curve D and $x \in D \setminus \partial D$. There exists a neighborhood U of x such that the control $u_{\mathcal{A}}$ is constant in each one of the two connected components of $U \setminus D$. From the description of the switching curves it is clear that $u_{\mathcal{A}}$ is equal to 1 on one component and equal to −1 on the other component. Consider the system Σ_2 of Example 2 of the first chapter at time $\tau_2 > \frac{3}{2}\pi$. We have:

$$\Gamma_{\mathcal{A}}(\Sigma, \tau) \restriction x \equiv \Gamma_{\mathcal{A}}(\Sigma_2, \tau_2) \restriction (-3, -1). \tag{4.4}$$

S-curve.

Consider an S−curve D and $x \in D \setminus \partial D$. There exists a neighborhood U of x such that $u_{\mathcal{A}}$ is constant in each connected component of $U \setminus D$. From the definition of turnpike we have that $u_{\mathcal{A}}$ has different signs on the two components. Consider the system Σ_1 at time τ_1 of Example 1 of the first chapter. We have that:

$$\Gamma_{\mathcal{A}}(\Sigma, \tau) \restriction x \equiv \Gamma_{\mathcal{A}}(\Sigma_1, \tau_1) \restriction \left(-1, -\frac{1}{2}\right). \tag{4.5}$$

K-curve.

Consider a K−curve D and $x \in D \setminus \partial D$. There exists a neighborhood U of x such that $u_{\mathcal{A}}$ is constant in each connected component of $U \setminus D$. From the definition of overlap curve we have that $u_{\mathcal{A}}$ has different signs on the two components. Consider the system Σ_1 of Example 1 of the first chapter, at time $\tau_1 > 4$. We have that:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \upharpoonright x \equiv \Gamma_{\mathcal{A}}(\Sigma_1,\tau_1) \upharpoonright (-2,0). \tag{4.6}$$

We have immediately the following:

Theorem 4.1 *Consider $\Sigma \in \Xi$ and $\tau > 0$. If $\mathcal{A}$ succeedes for Σ at time τ and D is a simple FC of $\Gamma_{\mathcal{A}}(\Sigma,\tau)$ then D is of one of the following 6 types:*

$$X_1, \quad X_2, \quad F, \quad C, \quad S, \quad K$$

and we have, respectively, one of the equivalences (4.1-6).

Recalling sections 4,5 of chapter 1 we obtain for each $\tau > 0$ a classification of simple FC's for a generic set of systems, namely $\Pi_\tau \subset \Xi$.

5.Frame points

In this section we give a complete description of the local structure of $\Gamma_{\mathcal{A}}$ in a neighborhood of a frame point. More precisely we consider only structurally stable frame points. Therefore we consider only frame points that are intersections of no more than two frame curves. In fact, an intersection of three or more frame curves can be destroyed by a small perturbation, see (2.2), of the system.
For the rest of the section we consider a fixed $\tau > 0$ and a fixed system Σ for which $\mathcal{A}$ succeedes at time τ. In particular Σ is locally controllable. For each type of frame point there are only a finite number of equivalence classes.

Before starting to examine frame points, case by case, we make a general observation. Consider a frame point x and two frame curves D_1, D_2 such that $\{x\} = D_1 \cap D_2$. We have one of the following possibilities:

(FP0) $x \in D_1 \setminus \partial D_1$, $x \in D_2 \setminus \partial D_2$

(FP1) $x \in D_1 \setminus \partial D_1$, $x \in \partial D_2$

(FP2) $x \in \partial D_1$, $x \in D_2 \setminus \partial D_2$

(FP3) $x \in \partial D_1$, $x \in \partial D_2$.

It's easy to check that, by construction, (FP0) can never occur. However, for each point we have to examine the other three possibilities.
The classification of frame points will be based on the types of the two intersecting curves D_1, D_2. We will use the notation, introduced in section 2, for frame points and the symbols: $\gamma^{\pm}$, F, C, S, K to indicate the curve of type $(F1), \ldots, (F6)$, respectively.

(X,Y)-point.

Consider an (X,Y)–point x of $\Gamma_{\mathcal{A}}(\Sigma,\tau)$. If $x = (0,0)$ then it is a structurally stable (X,Y)–frame point. In fact if Σ' is ε–near to Σ and ε is sufficiently small, then Σ' is locally controllable and $\Gamma_{\mathcal{A}}(\Sigma,\tau) \restriction (0,0) \equiv \Gamma_{\mathcal{A}}(\Sigma',\tau) \restriction (0,0)$, see Remark 5.1 of chapter 1. Let Σ_1 be the system of Example 1 at time $\tau_1 > 0$, then:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \restriction (0,0) \equiv \Gamma_{\mathcal{A}}(\Sigma_1,\tau_1) \restriction (0,0). \tag{5.1}$$

Now suppose that $x \neq (0,0)$. It follows that $x = \gamma^-(t^-) = \gamma^+(t^+)$, $t^- > 0$, $t^+ > 0$. We have that $t^- = t^+$ otherwise one of the two trajectories would have been cut by $\mathcal{A}$. Since the condition $t^- = t^+$ can be destroyed by a small perturbation, x is not structurally stable. In fact in this case x belongs to an overlap curve, hence it is the intersection of at least three frame curves.

(X,F)-point.

Consider an (X,F)–frame point x of $\Gamma_{\mathcal{A}}(\Sigma,\tau)$. The cases (FP1), (FP3) can't occur because $\partial\big(Fr(R(\tau))\big) = \emptyset$. Therefore we are in the case (FP2). There exists a neighborhood U of x (in $R(\tau)$) such that $u_{\mathcal{A}}$ is constant in each one of the two connected components U_1, U_2 of $U \setminus (\gamma^- \cup F)$. We have one of the two following cases:

(XF1) $u_{\mathcal{A}} = -1$ on $U_1 \cup U_2$

(XF2) $u_{\mathcal{A}} = 1$ on U_1 and $u_{\mathcal{A}} = -1$ on U_2, or viceversa.

Let Σ_2 at time τ_2 be the system of Example 2 and R_2 the reachable set of Σ_2. Let $x_2 \in Fr(R_2(\tau_2))$ be the endpoint of the curve γ^- of Σ_2. If (XF1) holds true and:

$$\frac{1}{3}\, ln\left(\frac{5}{2}\right) < \tau_2 < \frac{1}{3}\, ln(4) \tag{5.2a}$$

then:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \restriction x \equiv \Gamma_{\mathcal{A}}(\Sigma_2,\tau_2) \restriction x_2. \tag{5.2b}$$

In this case we say that x is a frame point of type $(X,F)_1$.
If (XF2) holds true $Y-$trajectories start from γ^-. If these trajectories does not reach F then at x arrives a $Y-$trajectory, hence x belongs to an overlap curve but this is not generic. Let:

$$\tau_2 > \frac{1}{3}\, ln(4) \tag{5.3a}$$

then:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \restriction x \equiv \Gamma_{\mathcal{A}}(\Sigma_2,\tau_2) \restriction x_2. \tag{5.3b}$$

In this case we say that x is a frame point of type $(X,F)_2$.

(X,C)-point.

Consider an $(X,C)-$frame point x. Suppose that (FP1) holds true. There exists a neighborhood U of x such that $u_{\mathcal{A}}$ is constant in each one of the three connected components U_1,U_2,U_3 of $U \setminus (\gamma^- \cup C)$. We suppose that U_1,U_2,U_3 are labelled in such a way that: U_3 is the connected components of $U \setminus \gamma^-$ that does not contain $C \cap U$; U_1 comes before U_2 along γ^-. Because of the definition of $C-$curve we have one of the following:

(XC1) $u_{\mathcal{A}} = 1$ on U_1

(XC2) $u_{\mathcal{A}} = 1$ on U_2.

Consider again the system Σ_2 at time τ_2 of Example 2. If (XC1) holds true then $u_{\mathcal{A}} = -1$ on $U_2 \cup U_3$. It is generic that the $Y-$trajectories leaving γ^- reach C. In fact, if this does not happen then $X(x)$, $Y(x)$ are parallel but this is not generic. We have:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \restriction x \equiv \Gamma_{\mathcal{A}}(\Sigma_2,\tau_2) \restriction \gamma^-\left(\frac{1}{3}\, ln\, \frac{5}{2}\right). \tag{5.4}$$

In this case we say that x is of type $(X,C)_1$.
If (XC2) holds true then $u_{\mathcal{A}} = -1$ on $U_1 \cup U_3$. We have:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \restriction x \equiv \Gamma_{\mathcal{A}}(\Sigma_2,\tau_2) \restriction \gamma^-\left(\frac{1}{3}\, ln\; 4\right). \tag{5.5}$$

In this case we say that x is of type $(X,C)_2$.

Remark 5.1

Consider the equivalence (5.4). In the Example 2, $\gamma^-(\frac{1}{3}\ ln\frac{5}{2})$ belongs to an INOA, see chapter 1 for definition, that is not a turnpike. This happens for every frame point of type $(X,C)_1$. Suppose $x = \gamma^-(t_x)$ and let (γ_r, u_r), $r \in [t_x - \varepsilon, t_x + \varepsilon]$ $(\varepsilon > 0)$, be the pair such that $\gamma_r(0) = \gamma^-(r)$ and $u_r \equiv 1$. Let λ_r be the covector field along (γ_r, u_r) satisfying:

$$\lambda_r(0) \cdot G(\gamma_r(0)) = 0 \qquad det\,[\lambda_r(0), G(\gamma_r(0))] > 0 \qquad \| \lambda_r(0) \| = 1.$$

Consider the function:

$$\psi(r,s) = \lambda_r(s) \cdot G(\gamma_r(s)).$$

From Lemma 3.4 of chapter 1 it follows that the equation $\psi(r,s) = 0$ has two branches of solutions in $(t_x, 0)$ then we have, from Lemma 3.5 of chapter 1:

$$0 = \left.\frac{\partial \psi}{\partial s}\right|_{(t_x,0)} = \lambda_{t_x}(0) \cdot [F,G](\gamma^-(t_x)).$$

Now $0 = \lambda_{t_x}(0) \cdot G(x) = \lambda_{t_x}(0) \cdot [F,G](x)$ and $\lambda_{t_x}(0) \neq 0$ then $\Delta_B(x) = det(G(x), [F,G](x)) = 0$. It follows that if $\nabla(\Delta_B(x)) \neq 0$ then x belongs to an INOA. This INOA can not be a turnpike otherwise it would have been constructed by the algorithm $\mathcal{A}$.

The case (FP2) is not generic. In fact, suppose that (FP2) holds true. Then there exists a neighborhood U of x in C such that for each $y \in U$ there exists a trajectory γ_y that switches at $y = \gamma_y(t_y)$. One side of C with respect to x is reached by trajectories γ_y that start from a FC D_1. The other side is reached by trajectories that start from a different FC, say D_2. Then at x meet two different switching curves and x is not stable. Suppose that (FP3) holds true. If C lays on the left (right) of γ^- then $u_{\mathcal{A}} \equiv -1$ to the left (right) of γ^-. Consider the system Σ_2 of Example 2 of chapter 1 at time $\tau_2 > \pi$. Then we have that:

$$\Gamma_{\mathcal{A}}(\Sigma, \tau) \restriction x \equiv \Gamma_{\mathcal{A}}(\Sigma_2, \tau_2) \restriction (2,0). \tag{5.6}$$

In this case we say that x is of type $(X,C)_3$.

(X,S)-point.

Consider an (X,S)–point x. Suppose that (FP1) holds true. There exists a neighborhood U of x such that $u_{\mathcal{A}}$ is constant on each one of the three connected components U_1, U_2, U_3 of $U \setminus (\gamma^- \cup S)$. We suppose that U_1, U_2, U_3 are labelled in such a way that: U_3 is the connected components of $U \setminus \gamma^-$ that does not contain $S \cap U$; U_1 comes before U_2 along γ^-. From the definition of turnpike it follows that $u_{\mathcal{A}} = 1$ on U_1 and $u_{\mathcal{A}} = -1$ on

$U_2 \cup U_3$. Consider the system Σ_1 at time τ_1 of the first example of chapter 1. In this case we have:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \restriction x \equiv \Gamma_{\mathcal{A}}(\Sigma_1,\tau_1) \restriction \left(-1,-\frac{1}{3}\right). \tag{5.7}$$

The cases (FP2),(FP3) can't occur because from the description of turnpikes, it follows that γ^- cannot terminate at x.

(X,K)-point.

Consider an (X,K)−point x. Suppose that (FP1) holds true. There exists a neighborhood U of x such that $u_{\mathcal{A}}$ is constant in each one of the three connected components U_1, U_2, U_3 of $U \setminus (\gamma^- \cup K)$. We suppose that U_1, U_2, U_3 are labelled in such a way that: U_3 is the connected components of $U \setminus \gamma^-$ that does not contain $K \cap U$; U_1 comes before U_2 along γ^-. From the definition of overlap curve it follows that $u_{\mathcal{A}} = 1$ on U_2 and $u_{\mathcal{A}} = -1$ on $U_1 \cup U_3$. Generically, the Y−trajectories leaving from γ^- reach K. In fact, if this does not happen then $X(x)$ and $Y(x)$ are parallel, but this is not generic. Consider again the system Σ_1 at time τ_1 of the first example of chapter 1. In this case we have:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \restriction x \equiv \Gamma_{\mathcal{A}}(\Sigma_1,\tau_1) \restriction \left(-2,-\frac{2}{3}\right) \tag{5.8}$$

and we say that x is of type $(X,K)_1$.

Suppose that (FP2) holds true. There exists a neighborhood U of x such that $u_{\mathcal{A}}$ is constant in each one of the two connected components U_1, U_2 of $U \setminus K$. If, for example, U_1 contains $\gamma^- \cap U$ then $u_{\mathcal{A}} = -1$ on U_1 and $u_{\mathcal{A}} = 1$ on U_2. Consider the system Σ_1 at time τ_1 of Example 1 and let x_1 be the point in which γ^- intersects the overlap curve. We have:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \restriction x \equiv \Gamma_{\mathcal{A}}(\Sigma_1,\tau_1) \restriction x_1 \tag{5.9}$$

and we say that x is of type $(X,K)_2$.

Suppose that (FP3) holds true. There exists a neighborhood U of x such that $u_{\mathcal{A}}$ is constant in each one of the two connected components U_1, U_2 of $U \setminus (\gamma^- \cup K)$. Suppose that U_1, U_2 are labelled in such a way that the vector $X(x)$ points into U_2. It's clear that $u_{\mathcal{A}} = -1$ on U_1 and $u_{\mathcal{A}} = 1$ on U_2. The Y−trajectories leaving from γ^- do not reach K, otherwise the nongeneric condition $Y(x) = 0$ is verified. Consider the synthesis Γ_4 of Example 4. We have that:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \restriction x \equiv \Gamma_4 \restriction \left(1\ ,\ \frac{2}{3} \right) \tag{5.10}$$

and we say that x is of type $(X,K)_3$.

(Y,F),(Y,C),(Y,S),(Y,K)-point.

These points can be treated as the preceeding points and we have the same equivalences. In this case the only difference is the sign of $u_{\mathcal{A}}$.

(X,X),(Y,Y),(F,F)-point.

It's easy to verify that points of these types can not exist.

(F,C)-point.

Consider an $(F,C)-$point x. The cases (FP2),(FP3) can't occur because $\partial(Fr(R(\tau))) = \emptyset$. Then (FP1) holds true. There exists a neighborhood U of x in $R(\tau)$ such that $u_{\mathcal{A}}$ is constant in each one of the two connected components of $U \setminus (F \cup C)$. It's clear that $u_{\mathcal{A}} = 1$ on one connected component and $u_{\mathcal{A}} = -1$ on the other. It is generic that the trajectories leaving from C reach F, otherwise x belongs to an overlap curve. Consider the system Σ_2 of Example 2 of chapter 1 at time π. We have:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \restriction x \equiv \Gamma_{\mathcal{A}}(\Sigma_2,\pi) \restriction (-3,-1). \tag{5.11}$$

(F,S)-point.

Consider an $(F,S)-$point x. As for the preceeding type, only the case (FP1) can hold. There exists a neighborhood U of x in $R(\tau)$ such that $u_{\mathcal{A}}$ is constant in each one of the two connected components of $U \setminus (F \cup S)$. It's clear that $u_{\mathcal{A}} = 1$ on one connected component and $u_{\mathcal{A}} = -1$ on the other. As before it is generic that the trajectories leaving from S reach F. Consider the system Σ_1 at time τ_1 of Example 1 of chapter 1. Let x_1 be the point in which the turnpike intersects the frontier of the reachable set, namely $x_1 = \left(-1, -\frac{1}{3} - \frac{1}{2}(\tau_1 - 1)\right)$. We have that:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \restriction x \equiv \Gamma_{\mathcal{A}}(\Sigma_1,\tau_1) \restriction x_1. \tag{5.12}$$

(F,K)-point.

Consider an $(F,K)-$point x. As above only the case (FP1) can hold. There exists a neighborhood U of x in $R(\tau)$ such that $u_{\mathcal{A}}$ is constant in each one of the two connected components of $U \setminus (F \cup K)$. It's clear that $u_{\mathcal{A}} = 1$ on one connected component and $u_{\mathcal{A}} = -1$ on the other. Consider again the system Σ_1 at time τ_1 of Example 1 of chapter

1. Let x_1 be the point in which the overlap curve intersects the frontier of the reachable set, namely:

$$x_1 = \left(-2, \frac{2}{3} + \frac{1}{3}\left(1+\frac{\tau}{2}\right)^3 - \left(1+\frac{\tau}{2}\right)^2\right).$$

We have that:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \upharpoonright x \equiv \Gamma_{\mathcal{A}}(\Sigma_1,\tau_1) \upharpoonright x_1. \tag{5.13}$$

(C,C)-point.

Consider a (C,C)–point x. From the definition of switching curve we have that the cases (FP1),(FP2) can not occur. Therefore (FP3) holds.
There exist two switching curves C_1, C_2 verifying $x = C_1 \cap C_2$ and a neighborhood U of x such that $u_{\mathcal{A}}$ is constant in each connected component of $U \setminus (C_1 \cup C_2)$. We have that $u_{\mathcal{A}}$ has different sign on the two connected components. Consider the cases:

(CCa) The curves leaving from C_1 reach C_2

(CCb) The curves leaving from C_2 reach C_1.

It is easy to show that (CCa),(CCb) cannot hold at the same time. Hence we have two cases:

(CC1) (CCa) holds and (CCb) does not, or viceversa

(CC2) (CCa) and (CCb) do not hold.

Consider the synthesis Γ_7 of Example 7. If (CC1) holds true then:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \upharpoonright x \equiv \Gamma_7 \upharpoonright \left(-\frac{1}{3}, -\frac{13}{72} + \frac{4}{9}\, ln\left(\sqrt[3]{\frac{5}{2}}\right)\right) \tag{5.14}$$

and we say that x is of type $(C,C)_1$.
Consider the system Σ_2 at time $\tau_2 > 3\pi$ of Example 2 of chapter 1. If (CC2) holds true then:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \upharpoonright x \equiv \Gamma_{\mathcal{A}}(\Sigma_2,\tau_2) \upharpoonright (4,0) \tag{5.15}$$

and we say that x is of type $(C,C)_2$.

Remark 5.2

Proceeding as in Remark 5.1 one can prove that if x is a frame point of type $(C,C)_1$ then $\Delta_B(x)=0$.
The frame points of type $(C,C)_2$ are not *effective* singular points. In fact, the optimal synthesis at these points is equivalent to the synthesis at a point x of a simple FC of type C, verifying $x \in C \setminus \partial C$.

(C,S)-point.

Consider a $(C,S)-$point x. There exists a neighborhood U of x such that $u_{\mathcal{A}}$ is constant in each connected component of $U \setminus (C \cup S)$.
The cases $(FP1),(FP2)$ can not occur because the control $u_{\mathcal{A}}$ must change sign when we cross S (or C), but on the other side it has to be constant on each side of C (or S).
Therefore $(FP3)$ holds true. There exists a C^1 diffeomorphism $\alpha : [0,\varepsilon] \mapsto I\!R^2$, $\varepsilon > 0$, such that $\alpha(t) \in C$, $\alpha(0) = x$. Consider the vectors:

$$C(x) = \lim_{t\to 0} \dot\alpha(t) \qquad S(x) = F(x) + \varphi_S(x)G(x)$$

where φ_S is the control to stay on S (cfr. (3.18) of chapter 1). Suppose that $C(x)$ and $S(x)$ are not parallel. Let U_X, U_Y be the connected component of $U \setminus \{x + tS(x) : t \in I\!R\}$ labelled in such a way that $X(x), Y(x)$ point into U_X, U_Y respectively. Let U_1, U_2 be the connected component of $U \setminus (C \cup S)$ labelled in such a way that the angle with vertex x and sides $C(x), S(x)$ contained in U_1 is smaller than that one contained in U_2. If U_1 is contained in U_Y then $u_{\mathcal{A}} = 1$ on U_1 otherwise $u_{\mathcal{A}} = -1$ on U_1.
There exists $\gamma_S \in Traj(\Sigma)$ such that $\gamma_S(Dom(\gamma_S)) = S \cap U$. We have two cases:

(CS1) $In(\gamma_S) = x$

(CS2) $Term(\gamma_S) = x$.

Suppose that (CS1) holds. We have two subcases:

(CSa) Constructed trajectories arrive onto C from U_2

(CSb) Constructed trajectories arrive onto C from U_1.

If (CSb) holds, then no nontrivial trajectory reaches x but this is not possible, hence (CSa) holds true. For the same reason the trajectories leaving from S and entering U_2 can not reach C. Consider the synthesis Γ_5 of Example 5, we have:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \restriction x \equiv \Gamma_5 \restriction \left(-1, -\frac{1}{3}\right) \tag{5.16}$$

and we say that x is of type $(C,C)_1$.
Suppose that (CS2) holds. We have again the subcases (CSa),(CSb).
Suppose (CSa) holds. If the trajectories leaving from S and entering U_2 reach C, then no non trivial trajectory arrive onto x. But if the opposite happens, from the direction of $X(x)$, $Y(x)$ we have that $In(\gamma_S) = x$ contradicting (CS2). We conclude that (CSa) can not hold.
Suppose (CSb) hold. We have that the trajectories leaving from S and entering U_1 reach C. From Theorem 3.7 of chapter 1 it follows that Δ_B cannot have constant sign on $V \cap U_1$ for any neighborhood V of x. Hence we have the nongeneric condition $\nabla\Delta_B(x) = 0$.

Suppose now that $C(x)$ and $S(x)$ are parallel. The trajectories arriving onto C have to come from S. Consider the system Σ_3 at time τ_3 of Example 3 of chapter 1, we have:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \restriction x \equiv \Gamma_{\mathcal{A}}(\Sigma_3,\tau_3) \restriction \left(-1-\frac{1}{2\sqrt[3]{2}}, -1-\frac{1}{\sqrt[3]{2}}\right) \tag{5.17}$$

and we say that x is of type $(C,S)_2$.

(C,K)-point.

Consider a $(C,K)-$point x. There exists a neighborhood U of x such that $u_{\mathcal{A}}$ is constant in each connected component of $U \setminus (C \cup K)$.
The cases $(FP1)$, $(FP2)$ can not occur because the control $u_{\mathcal{A}}$ must change sign when we cross K (or C), but on the other side it has to be constant on each side of C (or K). Therefore $(FP3)$ holds true. There exist two $\mathcal{C}^1$ diffeomorphisms $\alpha_{1,2} : [0,\varepsilon] \mapsto I\!R^2$, $\varepsilon > 0$, such that $\alpha_1(t) \in C$, $\alpha_2(t) \in K$, $\alpha_{1,2}(0) = x$. Consider the vectors:

$$C(x) = \lim_{t\to 0} \dot\alpha_1(t) \qquad K(x) = \lim_{t\to 0} \dot\alpha_2(t).$$

Suppose that $C(x)$ and $K(x)$ are not parallel. Let U_X, U_Y be the connected component of $U \setminus \{x + tK(x) : t \in I\!R\}$ labelled in such a way that $X(x), Y(x)$ point into U_X, U_Y respectively. Let U_1, U_2 be the connected component of $U \setminus (C \cup K)$ labelled in such a way that the angle with vertex x and sides $C(x), K(x)$ contained in U_1 is smaller than that one contained in U_2. If U_1 is contained in U_X then $u_{\mathcal{A}} = 1$ on U_1 otherwise $u_{\mathcal{A}} = -1$ on U_1. We have two cases:

(CK1) Constructed trajectories arrive onto C from U_1

(CK2) Constructed trajectories arrive onto C from U_2.

Suppose that (CK1) holds. The trajectories leaving from C can not reach K, otherwise we obtain one of the not generic condition $Y(x)=0$, $X(x)=0$. Consider the synthesis Γ_9 of Example 9, we have:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \restriction x \equiv \Gamma_9 \restriction \left(\frac{8}{3}, \frac{\sqrt{8}}{3}\right) \tag{5.18}$$

and we say that x is of type $(C,K)_1$.
Suppose that (CK2) holds. Consider the synthesis Γ_{10} of Example 10, we have:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \restriction x \equiv \Gamma_{10} \restriction \left(-\frac{1}{3}, -\frac{13}{72} + \frac{4}{9}\, ln\left(\sqrt[3]{\frac{5}{2}}\right)\right) \tag{5.19}$$

and we say that x is of type $(C,K)_2$.

Now suppose that $C(x)$ and $K(x)$ are parallel. If the trajectories leaving from C do not reach K then we have the equivalence (5.18) and a not stable tangency between C and K.
If the opposite happens, then we have a stable tangency between C and K. Consider the sybthesis Γ_8 of Example 8 and define $(\bar{x}_1, \bar{x}_2)$ in the same way. We have:

$$\Gamma_{\mathcal{A}}(\Sigma,\tau) \restriction x \equiv \Gamma_8 \restriction (\bar{x}_1, \bar{x}_2)$$
$$\equiv \Gamma_{10} \restriction \left(-\frac{1}{3}, -\frac{13}{72} + \frac{4}{9}\, ln\left(\sqrt[3]{\frac{5}{2}}\right)\right).$$

Remark 5.3

The point $(\bar{x}_1, \bar{x}_2)$ of Example 8 and (3.17) of Example 10 are equivalent but they are in some sense different. In fact proceeding as in Remark 5.1 one can prove that if $K(x), C(x)$ are linearly independent and (CK2) hold then $\Delta_B(x)=0$. If, instead, $K(x), C(x)$ are parallel we can have that $\Delta_B(x) \neq 0$ as in Example 10.

(S,S)-point.

It is easy to verify that these points can not exist.

(S,K)-point.

Consider an $(S,K)-$point x. There exists a neighborhood U of x such that $u_{\mathcal{A}}$ is constant in each connected component of $U \setminus (S \cup K)$.
The cases $(FP1), (FP2)$ can not occur because the control $u_{\mathcal{A}}$ must change sign when we cross K (or S), but on the other side it has to be constant on each side of S (or K).

Therefore $(FP3)$ holds true. The cases in which every trajectory leaving from S reaches K or no trajectory leaving from S reaches K are not generic. In fact, in these cases $X(x)$, $Y(x)$ are linearly dependent. Therefore the trajectories leaving from one side of S reach K and those leaving from the other side do not. There exists $\gamma_S \in Traj(\Sigma)$ such that $\gamma_S(Dom(\gamma_S)) = S \cap U$. We have two cases:

(SK1) $In(\gamma_S) = x$

(SK2) $Term(\gamma_S) = x$.

If (SK1) holds true then consider the synthesis Γ_6 of Example 6. We have:

$$\Gamma_{\mathcal{A}}(\Sigma, \tau) \restriction x \equiv \Gamma_6 \restriction \left(-1, -\frac{1}{3}\right) \tag{5.20}$$

and we say that x is of type $(S, K)_1$.
If (SK2) holds than consider the system Σ_3 at time τ_3 and the curves S_1, K_1 of Example 3. Let $x_1 = S_1 \cap K_1$. We have:

$$\Gamma_{\mathcal{A}}(\Sigma, \tau) \restriction x \equiv \Gamma_{\mathcal{A}}(\Sigma_3, \tau_3) \restriction x_1 \tag{5.21}$$

and we say that x is of type $(S, K)_2$.

(K,K)-point.

Consider a $(K, K)-$point x. From the definition of overlap curve we have that the cases (FP1), (FP2) can not occur, then (FP3) holds. Consider the system Σ_3 at time τ_3 of Example 3. The overlap curve K_1 is union of two overlap curves K_1', K_1''; onto K_1' arrive $Y * X-$ and $X * Y-$trajectories, onto K_1'' arrive $Y * X-$ and $X * S * Y-$trajectories. Let $x_1 = K_1' \cap K_1''$. We have:

$$\Gamma_{\mathcal{A}}(\Sigma, \tau) \restriction x \equiv \Gamma_{\mathcal{A}}(\Sigma_3, \tau_3) \restriction x_1 \tag{5.22}$$

Remark 5.4

As for $(C, C)_2$ points (see Remark 5.2), we have that frame points of type (K, K) are not *effective* singular points.

From the preceeding analysis we have immediately the following:

Theorem 5.2 *Consider $\Sigma \in \Xi$ and $\tau > 0$. If $\mathcal{A}$ succedes at time τ for Σ and x is a frame point then x is of one of the following 22 types:*

$$(X,Y),\ (X,F)_{1,2},\ (X,C)_{1,2,3},\ (X,S),\ (X,K)_{1,2,3},\ (F,C)$$

$$(F,S),\ (F,K),\ (C,C)_{1,2},\ (C,S)_{1,2},\ (C,K)_{1,2},\ (S,K)_{1,2},\ (K,K)$$

and we have, respectively, one of the equivalences (5.1-22).

Recalling sections 4 and 5 of chapter 1 we obtain, for each $\tau > 0$, a classification of frame points for a generic set of systems.

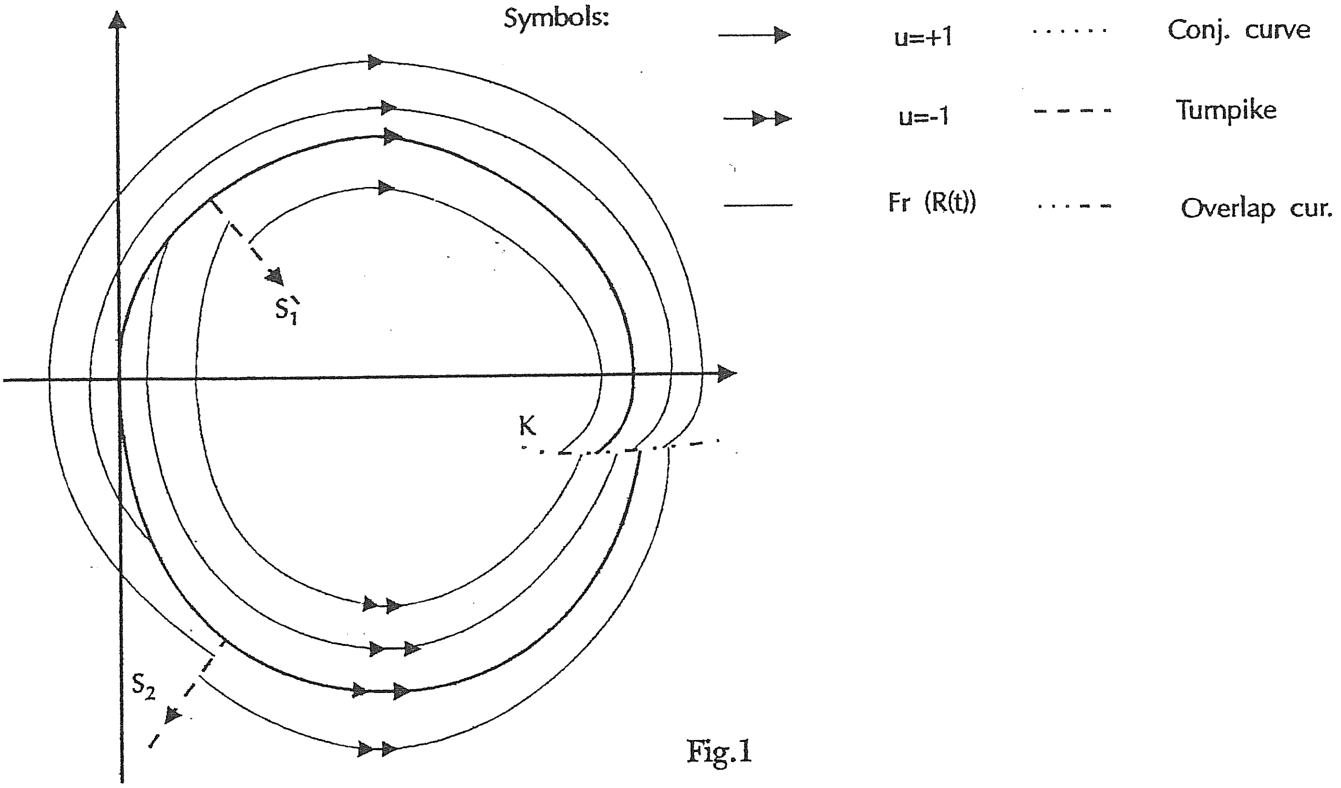
Symbols:
u=+1
Conj. curve
u=-1
Turnpike
Fr (R(t))
Overlap cur.
S1
K
S2

Fig.1

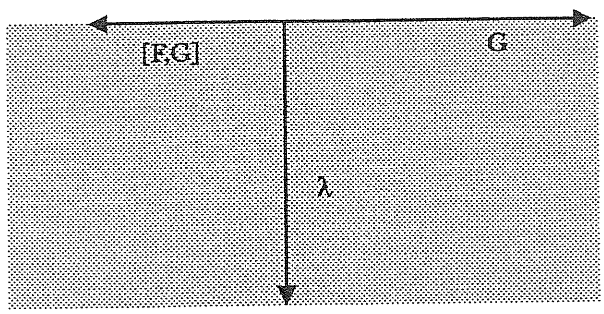
[F,G]
G
λ

Fig.2

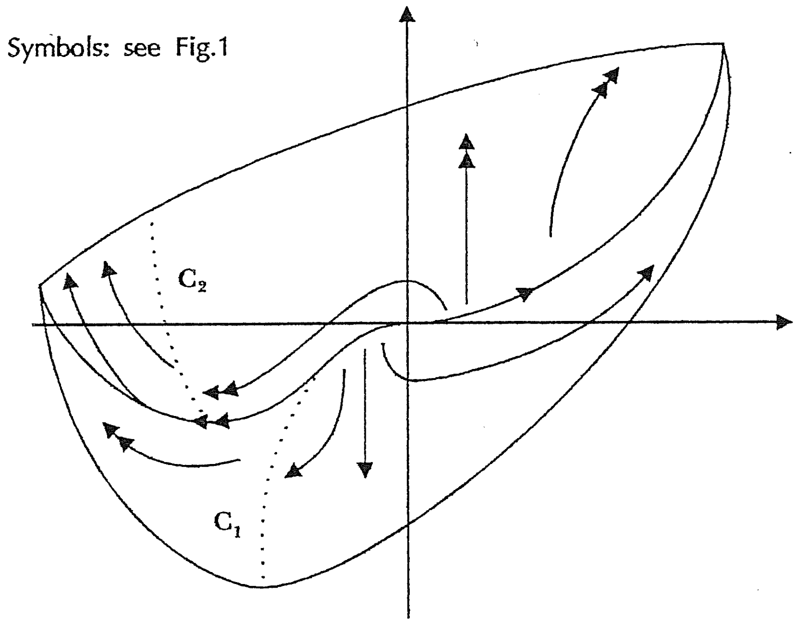

Fig.3

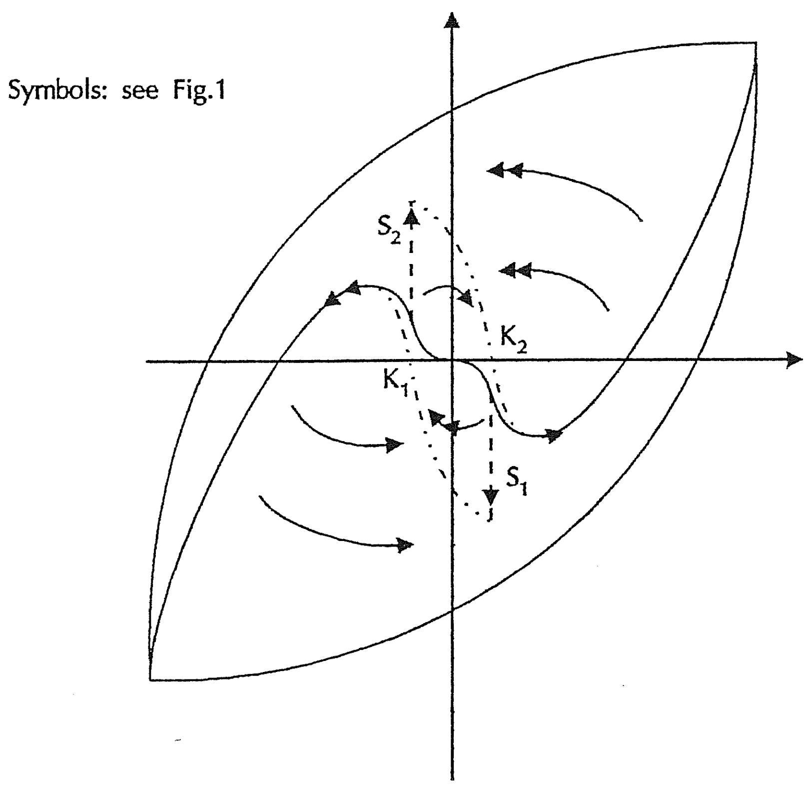

Fig.4

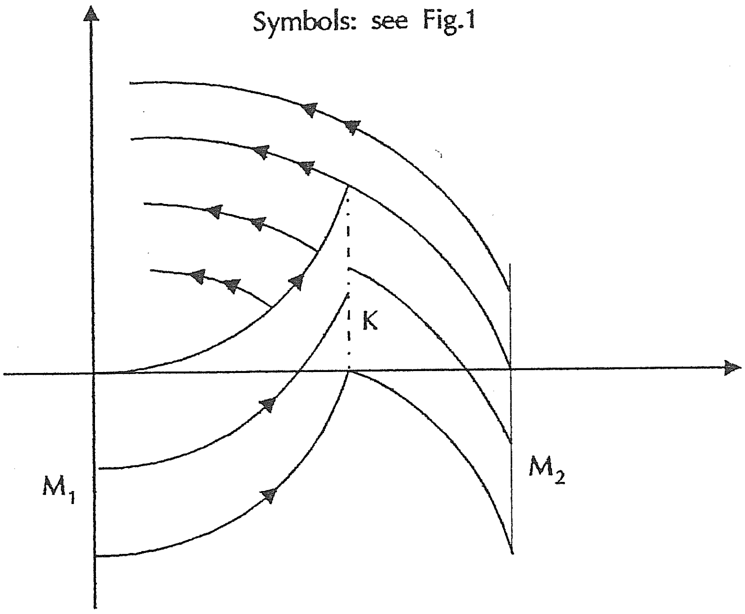

Fig.5

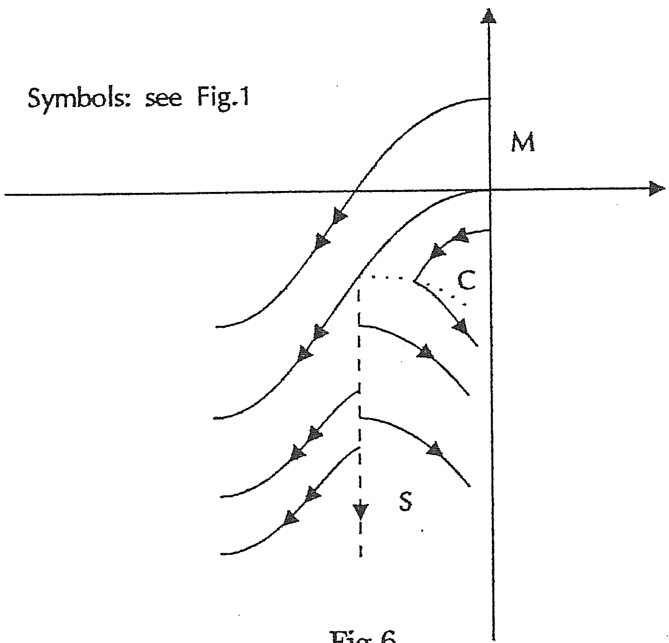

Fig.6

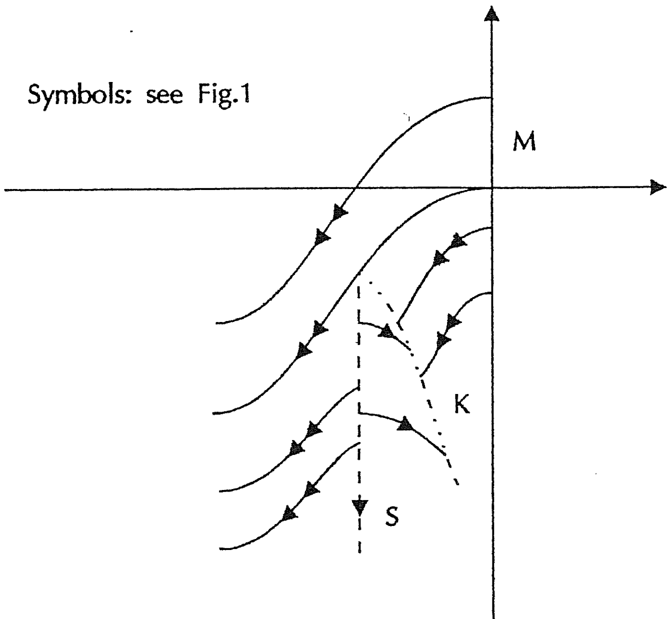

Fig.7

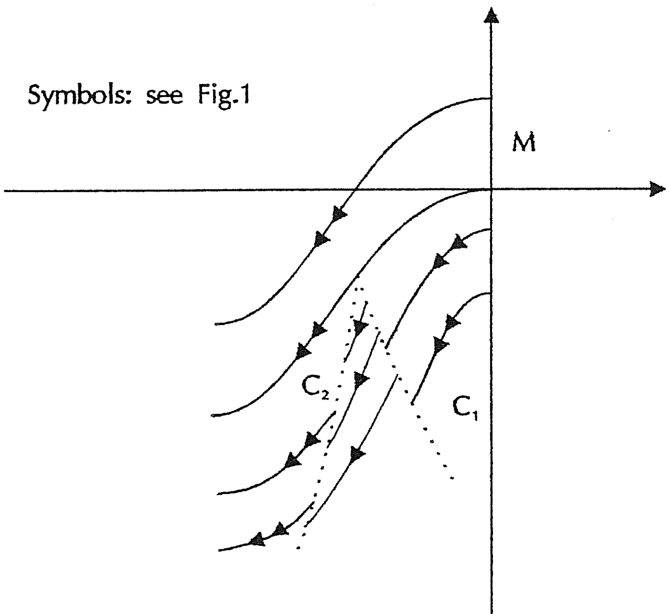

Fig.8

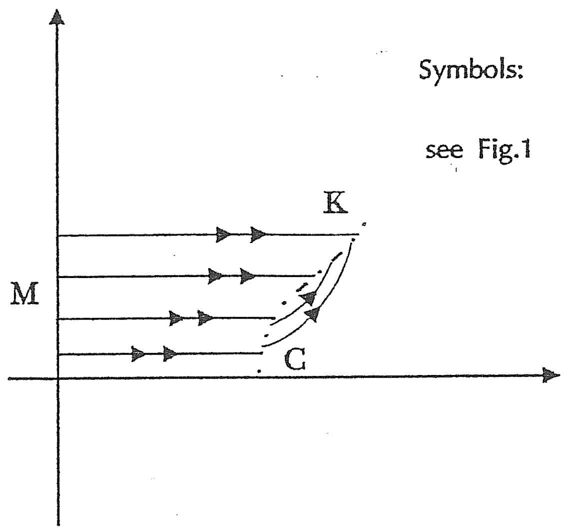

Fig.9

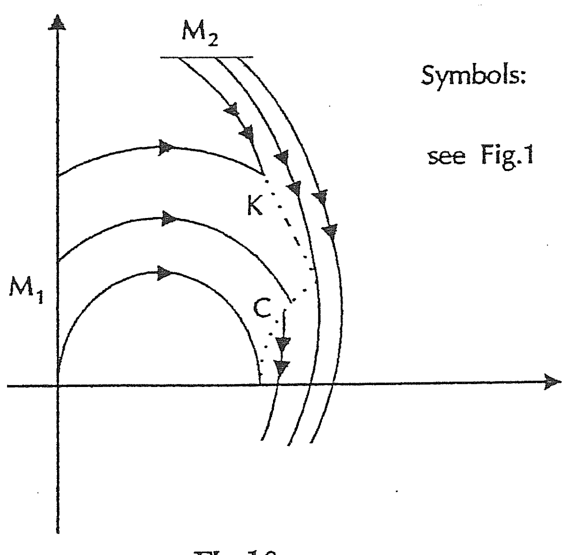

Fig.10

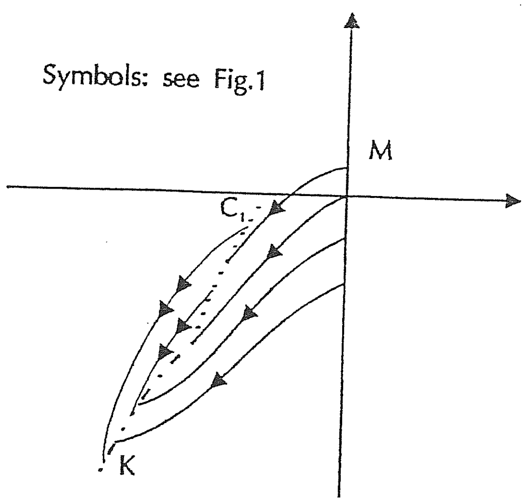

Fig.11

Chapter 3

1.Introduction

Let F, G be smooth vector fields on the plane, with $F(0) = 0$. For the control system

$$\dot{x} = F(x) + G(x)u \qquad |u(t)| \leq 1, \tag{1.1}$$

under generic assumptions on F, G, we have proved that the optimal control admits a regular feedback synthesis . In the previous chapter we have considered the reachable set in time τ: $R(\tau)$. It is not restrictive to assume $\tau = 1$ and to study the global feedback synthesis on the set $R = R(1)$ of points reachable from the origin within unit time.

In section 2 we introduce a relation of topological equivalence between optimal feedback flows, which plays a key role in the sequel. We also define the structural stability for a system.

In Section 3 we describe an algorithm which constructs a feedback synthesis for the time-optimal problem. The algorithm terminates in a finite number of steps, in connection with the system (1.1) for a generic set Ξ^* of vector fields $F, G \in \mathcal{C}^3$. As a by-product of this construction, one also obtains a discrete graph, which can be regarded as a "label" for the topological structure of the feedback flow.

The correspondence between equivalence classes of feedbacks and graphs, together with the question of structural stability, is then studied in Section 4.

A general introduction to structural stability and topological equivalence can be found in [1].

2.Basic Definitions

Recall that for a fixed $\tau \geq 0$, the *reachable set* within time τ is

$$R(\tau) = \{x : \exists \gamma \in Traj(\Sigma) \quad \text{such that} \quad \gamma(0) = 0 \in I\!R^2, \quad \gamma(t) = x, \quad \text{for some } t \leq \tau\}.$$

We reduce ourselves to consider the case $\tau = 1$, being similar the case of every $\tau > 0$. For every system $\Sigma = (F, G)$ in a suitable open dense subset $\Xi^* \subset \Xi$, the analysis in chapter one shows that the set $R \doteq R(1)$ of points reachable from the origin within unit time can be partitioned in a natural way into a finite number of open regions, covered by Y- or X-trajectories, separated by curves called frame curves. The intersections of these frame curves are called frame points. In connection with the above partition, there exists a piecewise smooth feedback control $u = \varphi(x)$, with the property that the (Carathéodory) solutions of

$$\dot{x} = F(x) + G(x)\varphi(x), \qquad x(0) = 0 \tag{2.1}$$

are precisely the time-optimal trajectories. Each optimal trajectory is a fine concatenation of X-, Y- and Z-trajectories.

All frame curves and frame points were classified in chapter two. In particular, only five types of frame curves can generically occur:

(F1) The trajectory $\gamma^+ : t \mapsto e^{tY}(0)$ corresponding to the constant control $u^- \equiv +1$

(F2) The trajectory $\gamma^- : t \mapsto e^{tX}(0)$ corresponding to the constant control $u^- \equiv -1$

(F3) The topological frontier of the reachable set: $Fr(R)$,

(F4) Curves of points conjugate to points of other frame curves, also called *switching curves*,

(F5) Regular turnpikes,

(F6) Overlap curves.

We now introduce an equivalence relation between systems, expressing the fact that their time-optimal feedback flows have similar structure. Consider two systems $\Sigma_1, \Sigma_2 \in \Xi^*$. For $i = 1, 2$ let R_i be the reachable set for Σ_i at time $t = 1$. Define

$$\mathcal{K}_i = \{x \mid x \in K \setminus \partial K, \ K \text{ is an overlap curve of } \Gamma_{\mathcal{A}}(\Sigma_i)\},$$

and set $R_i' = R_i \setminus \mathcal{K}_i$, $i = 1, 2$. In the following, for each $x \in R_i$, we denote by $t \mapsto \gamma_x^i(t)$ a trajectory of Σ_i which reaches x from the origin in minimum time.

Definition 1 (Equivalence of Feedback Flows). We say that the time-optimal feedback flows for Σ_1 on R_1 and Σ_2 on R_2 are *equivalent*, or simply that $\Sigma_1 \sim \Sigma_2$, if there exists a homeomorphism $\Psi : R_1' \mapsto R_2'$ such that:

(E1) Ψ maps arcs of optimal trajectories for Σ_1 onto arcs of optimal trajectories for Σ_2. More precisely, for every $x \in R_1'$ one has $\{\Psi(\gamma_x^1(t)) : t \in Dom(\gamma_x^1)\} = \{\gamma_{\Psi(x)}^2(t) : t \in Dom(\gamma_{\Psi(x)}^2)\}$.

(E2) Ψ induces a bijection on frame curves that are not overlap curves, i.e. for each frame curve D_1, which occurs in the construction of the optimal feedback for Σ_1 and is not a K-curve, we have that $\varphi(D_1)$ is a frame curve of the same type corresponding to Σ_2, and viceversa.

(E3) If A is an open region of R_1' enclosed by frame curves and entirely covered by Y- or X-trajectories, then $\Psi(A)$ is enclosed by the corresponding frame curves and is covered by Y- or X-trajectories, respectively.

Definition 2 (Structural Stability). We say that a system $\Sigma \in \Xi^*$ is *structurally stable* if there exists a neighborhood $\mathcal{N}$ of Σ in the space Ξ (endowed with the $\mathcal{C}^3$ norm), such that the feedback flows for Σ and Σ' are equivalent, for every $\Sigma' \in \mathcal{N}$.

Remark 1. In the above definition, we deliberately chose to exclude the overlap curves from the domain of the homeomorphism Ψ.

As an alternative, one could define the two systems Σ_1, Σ_2 to be *strongly equivalent* if there exists a homeomorphism Ψ, defined on the entire reachable set R_1, which maps arcs of optimal trajectories for Σ_1 onto arcs of optimal trajectories for Σ_2. This new equivalence relation generates a greater number of equivalence classes, whose classification becomes a much more difficult task. Yet, it does not seem that the fine distinctions between feedback flows, introduced by this strong equivalence relation, reflect any additional relevant property at the level of feedback flows. This point is best illustrated by the following examples.

Example 1. Consider the system Σ:

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = sin\ x_1. \end{cases}$$

The two trajectories corresponding to the constant controls $u \equiv \pm 1$ are $\gamma^+(t) = (t,\ 1 - cost)$ and $\gamma^-(t) = (-t,\ cost - 1)$. There are two turnpikes, having equations $S_1 = \{(x_1, x_2) : x_1 = 3\pi/2, x_2 \leq 1\}$, $S_2 = \{(x_1, x_2) : x_1 = -\pi/2, x_2 \leq -1\}$, while the set $K = \{(x_1, x_2) : x_1 =$

$\pi, x_2 \leq 2\}$ describes an overlap curve. The X-trajectory γ_1 starting at the (Y, S) frame point $(3\pi/2, 1)$ and the Y-trajectory γ_2 starting at the (X, S) frame point $(-\pi/2, -1)$, meet each other at $(\pi, 0) \in K$ (see fig.1). On the other hand, by an arbitrarily small C^3 perturbation of the system Σ, we can let the corresponding trajectories γ_1, γ_2 reach K at distinct points. In this case, the optimal feedback flow for Σ is structurally stable (according to our definition), but not strongly structurally stable.

Example 2. Consider the system with equations:

$$\begin{cases} \dot{x}_1 = \varepsilon x_2 + x_2 u \\ \dot{x}_2 = (1 - x_1) u \end{cases}$$

treated in Example 1 of the second chapter. Consider the $X-$trajectory γ_1 that start from $\gamma^+ \cap S_1$ (we use the same notation of chapter two), and let x_1 be the first point in which γ_1 intersect K. There exists a constructed trajectory γ_2 that arrive to x_1 from the other side of K. Let x_2 be the last switching point of γ_2. If x_2 belongs to S_2 then we consider the constructed trajectory γ_3 that switches at x_2 and then goes on as $X-$trajectory, and we define x_3 to be the intersection of γ_3 with K. Proceeding in this way, for every frame point x we define a curve $D(x)$ that connects the points $x_1, x_2, \ldots$. If Σ' is strongly equivalent to Σ and $\varphi(x) = x'$ then we must have $\varphi(D(x)) = D(x')$. This condition can be very restrictive and also very hard to be verified. We can construct a great number of strong equivalence classes varying the position of these curves $D(x)$. Moreover the strong equivalence classes become smaller as the number of overlap curves increases.

3. An algorithm for the synthesis.

In this section we describe an algorithm $\mathcal{A}$, similar to the one in the first chapter, which contructs an optimal synthesis for a generic system $\Sigma \in \Xi$. This new algorithm succedes for a class of systems smaller than the class considered in chapter one. On the other hand, it describes more accurately the optimal synthesis and ensures the structural stability of these systems.

In the first step, the algorithm constructs the two trajectories $\gamma^{\pm}(t) = e^{t(F \pm G)}(0)$, and marks some special points along these curves, from which additional frame curves bifurcate.

At step N, the algorithm will construct precisely those trajectories which are concatenation of N bang- or singular arcs and satisfy the Pontryagin Maximum Principle.

The endpoints of the arcs forming these trajectories, corresponding to the switching times of the control, are determined by certain nonlinear equations. Under generic conditions, which will be listed below, such equations can be solved by the implicit function theorem, thus determining a smooth switching locus. Eventually the algorithm will partition the reachable set R into finitely many open regions (where the optimal feedback control is either $u \equiv 1$ or $u \equiv -1$), separated by boundary curves and points, called *frame curves* and *frame points*, respectively. Appropriate stability conditions for these curves and points will be formulated.

At each step, it may happen that distinct extremal trajectories reach a same point x_0. In this case, the trajectories which are not globally optimal must be discarded. This procedure will produce new "overlap curves", consisting of points reached in minimum time by two distinct trajectories, one ending with the control value $u = 1$, the other with $u = -1$.

In the following, we fix a locally controllable system $\Sigma = (F, G) \in \Xi$. We assume that Σ verifies the generic properties $(P_1), \ldots, (P_8)$ in Section 4 of chapter one. These assumptions imply that, for some integer N_0, every extremal trajectory γ starting from the origin, with $Dom(\gamma) \subseteq [0,1]$, is a concatenation of at most N_0 bang- or singular arcs. Moreover, they imply that at every step only a finite number of frame curves and frame points are constructed by the algorithm. Additional conditions, which assure the structural stability of these curves and points, will be formulated in the description of the algorithm.

In the following, if a trajectory γ with $Dom(\gamma) = [0, b]$ is constructed as part of the synthesis, we then regard all trajectories $\gamma \restriction [0, a]$ with $a < b$ as constructed trajectories. Similarly, we regard as frame curve every connected subset of a frame curve, having frame points as endpoints. To every constructed trajectory we will associate a covector field.

Algorithm $\mathcal{A}$, STEP 1. Consider first the trajectory $t \mapsto e^{tY}(0)$ and assume that $G(e^{tY}(0)) \neq 0$ for all $t \in [0,1]$. Transporting this vector back to the origin along the flow of Y we thus obtain another nonzero vector, which will form some angle θ with $G(0)$. More precisely, set

$$\theta(t) \doteq \arg\left(G(0), \quad (e^{-tY})_* G(e^{tY}(0))\right). \tag{3.1}$$

We then define

$$t^+ \doteq \min\left\{t \in [0,1]; \quad |\theta(s_1) - \theta(s_2)| = \pi \ \text{ for some } \ s_1, s_2 \in [0,t]\right\}, \tag{3.2}$$

with the understanding that $t^+ = 1$ if $|\theta(s_1) - \theta(s_2)| < \pi$ for all $s_1, s_2 \in [0,1]$. Next, we single out times $t_i, t_i' \in [0, t^+]$ where the function θ assumes increasingly large local maxima, and increasingly small local minima, respectively. By induction, define

$$\begin{aligned} t_1 &\doteq \inf\{t > 0;\quad \theta \text{ has a local max at } t,\ \ \theta(t) > 0\}, \\ t_i &\doteq \inf\{t > t_{i-1};\quad \theta \text{ has a local max at } t,\ \theta(t) > \theta(t_{i-1})\}, \\ t'_1 &\doteq \inf\{t > 0;\quad \theta \text{ has a local min at } t,\ \ \theta(t) < 0\}, \\ t'_i &\doteq \inf\{t > t'_{i-1};\quad \theta \text{ has a local min at } t,\ \ \theta(t) < \theta(t'_{i-1}\}. \end{aligned} \tag{3.3}$$

We also set $t_0 = t'_0 = 0$, and define the intermediate times for $i \geq 1$

$$\begin{aligned} s_i &\doteq \max\{t \in [t_{i-1},\ t_i];\quad \theta(t) = \theta(t_{i-1})\}, \\ s'_i &\doteq \max\{t \in [t'_{i-1},\ t'_i];\quad \theta(t) = \theta(t'_{i-1})\}. \end{aligned} \tag{3.4}$$

In the first step of the algorithm, we construct the trajectory $\gamma^+ : [0, t^+] \mapsto \mathbb{R}^2$ and regard it also as frame curve. On γ^+ we define as frame points: the origin, $\gamma^+(t^+)$, and all points $\gamma^+(s_i)$, $\gamma^+(t_i)$, $\gamma^+(s'_i)$, $\gamma^+(t'_i)$.

The following stability assumptions on the function θ will imply that the sequences $\{t_i\}$, $\{t'_i\}$ are finite, strictly increasing, and also stable with respect to small perturbations of the vector fields F, G.

(SA1) $G(e^{tY}(0)) \neq 0 \quad \text{for all} \quad t \in [0,1]$,
(SA2) $\dot\theta(0) \neq 0,\ \ \dot\theta(t^+) \neq 0$.
(SA3) If $\dot\theta(t) = 0$, then $\theta(t) \neq 0$, $\ddot\theta(t) \neq 0$.
(SA4) If $t \neq s$ and $\dot\theta(s) = \dot\theta(t) = 0$, then $\theta(s) \neq \theta(t)$.
(SA5) $(\nabla\Delta_B \cdot W)(e^{tY}(0)) \neq 0$, $W = X, Y$, at all points $t \in \{t_i,\ t'_i;\ \ i \geq 1\}$.
(SA6) If $t^+ = 1$, then $\max\{|\theta(t) - \theta(1)|;\ \ t \in [0,1]\} < \pi$.

Observe that t^+ is the first time which is negatively conjugate along γ^+ to some previous time, while s_i, s'_i are positively conjugate to t_{i-1}, t'_{i-1}, respectively. Moreover, θ is monotonically increasing on each interval $[s_i,\ t_i]$ and decreasing on $[s'_i,\ t'_i]$.

We then perform exactly the same construction for the trajectory γ^-, replacing G with $-G$ and interchanging X with Y throughout.

The following results motivate our construction.

Proposition 3.1. *Let the function θ be defined as above, and let the stability assumptions* (SA1)-(SA6) *hold.*

(I) The trajectory $\gamma^+ : t \mapsto e^{tY}$ is extremal up to the time $t = t^+$.

(II) If t^* lies in one of the open intervals $]s_i,\ t_i[$ or $]s'_i,\ t'_i[$, then there exists an extremal control of the form

$$u(t) = \begin{cases} 1 & \text{if} \quad t \in [0, t^*], \\ -1 & \text{if} \quad t \in [t^*,\ t^* + \varepsilon], \end{cases} \tag{3.5}$$

for some $\varepsilon > 0$. In the first case the trajectory bifurcates to the right of γ^+ (clockwise), in the second case to the left (counterclockwise). On the other hand, no extremal control of the form (3.5) exists if t^* is not contained in any one of the closed intervals $[s_i,\ t_i]$ or $[s'_i,\ t'_i]$.

(III) To the right of each point $e^{t_i Y}(0)$, $t_i > 0$, and to the left of each point $e^{t'_i Y}(0)$, $t'_1 > 0$, originates either a turnpike, when the inner products $\nabla\Delta_B \cdot X$ and $\nabla\Delta_B \cdot Y$ have opposite signs, that is when the two vector fields X, Y points to opposite sides of the set of zeros of Δ_B, or a switching curve of conjugate points, when the signs are equal.

(IV) If $t^+ < 1$, then a switching curve of conjugate points originates from $e^{t^+ Y}(0)$. Such curve bifurcates to the left of γ^+ when $\dot\theta(t^+) > 0$, and to the right of γ^+ when $\dot\theta(t^+) < 0$.

Proof. The trajectory $t \mapsto e^{tY}$ is extremal on some interval $[0,\ t^*]$ iff there exists a nonzero adjoint vector which satisfies

$$\dot\lambda(t) = -\lambda(t) \cdot \nabla Y\big(e^{tY}(0)\big), \qquad \lambda(t) \cdot G\big(e^{tY}(0)\big) \geq 0, \tag{3.6}$$

for all $t \in [0,\ t^*]$. From (3.6) it follows

$$\lambda(t) \cdot G\big(e^{tY}(0)\big) = \lambda(0) \cdot \big(e^{-tY}\big)_* G\big(e^{tY}(0)\big) \geq 0. \tag{3.7}$$

Observe that a vector $\lambda(0)$ satisfying (3.7) for all $t \in [0,\ t^*]$ can exist if and only if the vectors $\big(e^{-tY}\big)_* G\big(e^{tY}(0)\big)$ range within an angle $\leq \pi$. By the definition of t^+ and by (SA2), this happens if and only if $t^* \leq t^+$. This establishes (I).

To prove (II), call γ the trajectory corresponding to the control u in (3.5). Then γ is extremal iff there exists an adjoint vector λ which satisfies (3.2) of chapter one together with

$$\begin{cases} \lambda(t) \cdot G(\gamma(t)) \geq 0 & \text{if} \quad t \in [0,\ t^*], \\ \lambda(t) \cdot G(\gamma(t)) \leq 0 & \text{if} \quad t \in [t^*\ t^* + \varepsilon]. \end{cases} \tag{3.8}$$

The inequalities in (3.8) are equivalent to

$$\lambda(0) \cdot \big(e^{-tY}\big)_* G\big(e^{tY}(0)\big) \geq 0 \qquad \text{if} \qquad t \in [0,\ t^*], \tag{3.9}$$

$$\lambda(0)\cdot\left(e^{-t^*Y}\right)_*\left(e^{(t^*-t)X}\right)_*G\left(e^{(t-t^*)X}e^{t^*Y}(0)\right)\le 0 \qquad \text{if} \qquad t\in[t^*,\ t^*+\varepsilon]. \tag{3.10}$$

From the properties of the angular function θ it follows that, if t^* does not belong to any closed interval $[s_i,\ t_i]$ or $[s_i',\ t_i']$, there can be no vector $\lambda(0)\neq 0$ wich satisfies (3.9) together with

$$\lambda(0)\cdot\left(e^{-t^*Y}\right)_*G\left(e^{t^*Y}(0)\right)=0. \tag{3.11}$$

On the other hand, if t^* is contained in one of the open intervals $]s_i,\ t_i[$ or $]s_i',\ t_i'[$, then some vector $\lambda(0)$ does exist, such that (3.9), (3.11) hold. Using the notation adopted in Lemma 3.6 of chapter one, we have to show that there exists $\varepsilon>0$ sufficiently small so that

$$\lambda(0)\cdot v\big(G(\gamma(t)),t;0\big)<0 \qquad \forall t\in\,]t^*,\ t^*+\varepsilon]. \tag{3.12}$$

The above inequality will be a consequence of (3.10), (3.11) if we show that, as t ranges in a suitably small neighborhood of t^*, the vector $v\big(G(\gamma(t)),t;0\big)$ rotates in a constant direction. This is indeed the case, because, by Lemma 3.6 of chapter one,

$$\text{sgn}\left(\frac{d}{dt}\arg\Big(G(0),\ v\big(G(\gamma(t)),t;0\big)\Big)\right)=\text{sgn}\big(\Delta_B(\gamma(t))\big)=\text{sgn}\big(\Delta_B(e^{t^*Y}(0))\big),$$

for t sufficiently close to t^*. Now (SA3) implies $\dot\theta(t^*)\neq 0$, with

$$\text{sgn}\big(\Delta_B(e^{t^*Y}(0))\big)=\text{sgn}\big(\dot\theta(t^*)\big)=\text{sgn}\big(\theta(t^*)\big)=\begin{cases}1 & \text{if}\quad t^*\in]s_i,\ t_i[\ ,\\ -1 & \text{if}\quad t^*\in]s_i',\ t_i'[\ .\end{cases} \tag{3.13}$$

As the control u switches from 1 to -1 at time t^*, the corresponding trajectory will bifurcate from the the curve $\gamma^+ : t\mapsto e^{tY}(0)$ to the right (i.e., clockwise) or to the left (counterclockwise) depending on whether the determinant $X\wedge Y=2\,F\wedge G$ at the point $e^{t^*Y}(0)$ is positive or negative. Since the Jacobian matrix $(e^{-tY})_*$ preserves orientation,

$$\text{sgn}\ F\big(e^{tY}(0)\big)\wedge G\big(e^{tY}(0)\big)=\text{sgn}\ Y\big(e^{tY}(0)\big)\wedge G\big(e^{tY}(0)\big)$$

$$=\text{sgn}\ \left(e^{-tY}\right)_*Y\big(e^{tY}(0)\big)\wedge\left(e^{-tY}\right)_*G\big(e^{tY}(0)\big)=\text{sgn}\ Y(0)\wedge\left(e^{-tY}\right)_*G\big(e^{tY}(0)\big)$$

$$=\text{sgn}\ G(0)\wedge\left(e^{-tY}\right)_*G\big(e^{tY}(0)\big)=\text{sgn}\ \theta(t).$$

Recalling (3.13), this completes the proof of (II).

To prove (III), fix some $t_i\in]0,t^+[$. The analysis for a time t_i' is entirely similar. By Lemma 3.6 of chapter one, we have

$$\dot\theta(t_i)=0=\Delta_B\big(e^{t_iY}(0)\big). \tag{3.16}$$

From (3.12) of chapter one, (3.16) and the stability assumption (SA3) it follows

$$0 > \ddot{\theta}(t_i) = \frac{v(t_i) \wedge \ddot{v}(t_i)}{\|v(t_i)\|^2}, \tag{3.17}$$

with $v(t) \doteq \left(e^{-tY}\right)_* G\left(e^{tY}(0)\right)$, so that

$$\dot{v}(t) = \left(e^{-tY}\right)_* [F, G]\left(e^{tY}(0)\right), \qquad \ddot{v}(t) = \left(e^{-tY}\right)_* [Y,\ [F, G]]\left(e^{tY}(0)\right). \tag{3.18}$$

Indeed $(v \wedge \dot{v})(t_i) = \Delta_B(\gamma^+(t_i)) = 0$. At the point $p_i \doteq e^{t_i Y}(0)$ we now have

$$\begin{aligned} \nabla\Delta_B \cdot Y &= (\nabla G)Y \wedge [F, G] + G \wedge (\nabla[F, G])Y \\ &= [Y, G] \wedge [F, G] - (\nabla Y)G \wedge [F, G] + G \wedge [Y, [F, G]] - G \wedge (\nabla Y)[F, G] \\ &= G \wedge [Y, [F, G]]. \end{aligned} \tag{3.19}$$

Indeed, $G \wedge [F, G] = 0$ implies $G = k[F, G]$ for some k, hence

$$(\nabla Y)G \wedge [F, G] + G \wedge (\nabla Y)[F, G] = (\nabla Y)G \wedge kG + G \wedge (\nabla Y)kG = 0.$$

Together, (3.17)–(3.19) yield

$$\nabla\Delta_B \cdot Y(p_i) < 0. \tag{3.20}$$

In particular, this implies $\nabla\Delta_B(p_i) \neq 0$. By the implicit function theorem, the equation $\Delta_B = 0$ locally defines a $\mathcal{C}^1$ curve S, passing through the point $p_i = e^{t_i Y}(0)$. If now $\nabla\Delta_B \cdot X(p_i) > 0$, then we must have $\nabla\Delta_B \cdot G(p_i) \neq 0$, hence the function φ_S in (3.18) of chapter one is well defined and satisfies $|\varphi_S(x)| < 1$ in a neighborhood of p_i. For $\varepsilon > 0$ suitably small, the solution of the Cauchy problem

$$x(0) = 0, \qquad \dot{x}(t) = \begin{cases} Y(x) & \text{if} \quad t \in [0,\ t_i], \\ F(x) + \varphi_S(x)G(x) & \text{if} \quad t \in [t_i,\ t_i + \varepsilon], \end{cases}$$

is thus an admissible, extremal trajectory of the control system. To show that S is a turnpike, it remains to check the sign of the function f in (3.11) of chapter one. Here $\Delta_A = F \wedge G = \frac{1}{2} X \wedge Y > 0$ because of (3.15). Hence, if U is a small open ball centered at p_i, divided by S into the connected components U_X, U_Y, recalling (3.11) of chapter one, we have

$$\operatorname{sgn}\left(f(x)\right) = -\operatorname{sgn}\left(\nabla\Delta_B \cdot Y(p_i)\right) > 0 \qquad \forall x \in U_Y,$$

$$\operatorname{sgn}\left(f(x)\right) = -\operatorname{sgn}\left(\nabla\Delta_B \cdot X(p_i)\right) < 0 \qquad \forall y \in U_X.$$

Now consider the case where $\nabla\Delta_B \cdot X$ and $\nabla\Delta_B \cdot Y$ are both negative at the point $p_i \doteq e^{t_iY}(0)$. For $\varepsilon_1, \varepsilon_2$ in a neighborhood of the origin, define the function

$$\alpha(\varepsilon_1,\varepsilon_2) \doteq \arg\left(G(0),\ \left(e^{(\varepsilon_1-t_i)Y}\right)_*\left(e^{-\varepsilon_2 X}\right)_* G\left(e^{\varepsilon_2 X}e^{(t_i-\varepsilon_1)Y}(0)\right)\right). \tag{3.21}$$

Since α is twice continuously differentiable, we can define the C^1 function β by setting

$$\beta(\varepsilon_1,\varepsilon_2) \doteq \begin{cases} \frac{\alpha(\varepsilon_1,\varepsilon_2)-\alpha(\varepsilon_1,0)}{\varepsilon_2} & \text{if} \quad \varepsilon_2 \neq 0, \\ \frac{\partial\alpha(\varepsilon_1,\varepsilon_2)}{\partial\varepsilon_2} & \text{if} \quad \varepsilon_2 = 0. \end{cases}$$

By (3.12) of chapter one, at $(\varepsilon_1,\varepsilon_2) = (0,0)$, we have

$$\beta = \frac{\partial\alpha}{\partial\varepsilon_2} = \frac{\left(e^{-t_iY}\right)_* G(p_i) \wedge \left(e^{-t_iY}\right)_*[F,G](p_i)}{\|\left(e^{-t_iY}\right)_* G(p_i)\|^2},$$

$$\frac{\partial\beta}{\partial\varepsilon_1} = \frac{\partial^2\alpha}{\partial\varepsilon_1\partial\varepsilon_2} = -\frac{\left(e^{-t_iY}\right)_* G(p_i) \wedge \left(e^{-t_iY}\right)_*[Y,[F,G]](p_i)}{\|\left(e^{-t_iY}\right)_* G(p_i)\|^2},$$

$$\frac{\partial\beta}{\partial\varepsilon_2} = \frac{\partial^2\alpha}{\partial\varepsilon_2^2} = \frac{\left(e^{-t_iY}\right)_* G(p_i) \wedge \left(e^{-t_iY}\right)_*[X,[F,G]](p_i)}{\|\left(e^{-t_iY}\right)_* G(p_i)\|^2}.$$

From (3.19) we have $\nabla\Delta_B \cdot Y = G \wedge [Y,[F,G]]$ and similarly we obtain $\nabla\Delta_B \cdot X = G \wedge [X,[F,G]]$. Since the matrix $(e^{-t_iY})_*$ preserves the orientation it follows $\beta = 0$ and:

$$\operatorname{sgn}\left(\frac{\partial\beta}{\partial\varepsilon_1}\right) = \operatorname{sgn}((-Y)\cdot\nabla\Delta_B(p_i)), \qquad \operatorname{sgn}\left(\frac{\partial\beta}{\partial\varepsilon_2}\right) = \operatorname{sgn}(X\cdot\nabla\Delta_B(p_i)).$$

By the implicit function theorem, we can now locally solve the equation $\beta(\varepsilon_1,\varepsilon_2) = 0$ and determine a function $\varepsilon_2 = \psi(\varepsilon_1)$, with

$$\operatorname{sgn}\left(\frac{\partial\psi}{\partial\varepsilon_1}(0)\right) = \operatorname{sgn}\left(\frac{\nabla\Delta_B\cdot X(p_i)}{\nabla\Delta_B\cdot Y(p_i)}\right).$$

From the previous analysis, it follows that for $\varepsilon \geq 0$ suitably small, there exists $t^\dagger > t_i - \varepsilon + \psi(\varepsilon)$ such that the trajectory corresponding to the control

$$u(t) = \begin{cases} 1 & \text{if} \quad t \in [0,\ t_i-\varepsilon] \cup (t_i-\varepsilon+\psi(\varepsilon),\ t^\dagger], \\ -1 & \text{if} \quad t \in (t-\varepsilon,\ t-\varepsilon+\psi(\varepsilon)], \end{cases}$$

is extremal. The parametrized curve

$$\varepsilon \mapsto e^{\psi(\varepsilon)X}e^{(t_i-\varepsilon)Y}(0) \tag{3.22}$$

is the switching curve of conjugate points, originating to the right of γ^+.

To prove (IV), assume $\dot{\theta}(t^+) > 0$, the other case being entirely similar. By Lemma 3.6 of chapter one, this implies $\Delta_B(x) > 0$ for all x in a neighborhood of $\left(e^{t^+Y}(0)\right)$. Let j be the largest index for which t'_j is defined. By the definitions of t^+ and the points t'_i, we thus have $\theta(t^+) - \theta(t'_j) = \pi$; hence t^+ and t'_j are negatively conjugate. If $j > 1$ then by (III), from $p'_j = e^{t'_j Y}(0)$ originates either a turnpike or a curve of conjugate points. To set the ideas, consider the turnpike case. Then a left neighborhood of the arc $\gamma^+ \upharpoonright [t'_j, t^+]$ is covered by extremal trajectories of the form

$$t \mapsto e^{tY} e^{\varepsilon(F+\varphi_S G)}(p'_j).$$

Since t^+ is conjugate to t'_j and $\Delta_B(x) > 0$ near $\gamma^+(t^+)$, by the implicit function theorem for each $\varepsilon > 0$ sufficiently small there exists a unique $t(\varepsilon)$ close to $t^+ - t'_j$ such that the points

$$\Lambda'(\varepsilon) \doteq e^{\varepsilon(F+\varphi_S G)}(p'_j), \qquad \Lambda''(\varepsilon) \doteq e^{t(\varepsilon)Y} e^{\varepsilon(F+\varphi_S G)}(p'_j)$$

are conjugate along an integral curve of Y. The map Λ'' now parametrizes the desired curve of conjugate point. If $j = 1$ we can repeat the same argument using the extremal trajectories:

$$t \mapsto e^{tY} e^{\varepsilon X}(0).$$

This completes the proof of the Proposition 3.1.

In order to study the behavior of optimal trajectories in a neighborhood of the points $q_i \doteq e^{s_i Y}(0)$, $q'_i \doteq e^{s'_i Y}(0)$, and complete our description of the time optimal feedback in a neighborhood of the curve γ^+, an additional stability condition will be needed. As a preliminary, observe that, for each i, a curve of conjugate points starting at q_i can be defined as follows.

Let $i \geq 2$ and assume first that at $p_{i-1} = e^{t_{i-1}Y}(0)$ originates a turnpike. For $\varepsilon \geq 0$, define the time $t(\varepsilon)$ by requiring that the point

$$\Gamma_i(\varepsilon) \doteq e^{t(\varepsilon)Y} e^{\varepsilon(F+\varphi_S G)} e^{t_{i-1}Y}(0) \tag{3.23}$$

be conjugate to $e^{\varepsilon(F+\varphi_S G)} e^{t_{i-1}Y}(0)$ along the integral curve of Y. On the other hand, if at p_{i-1} originates the curve of conjugate points given in (3.22), define $t(\varepsilon)$ by requiring that

$$\Gamma_i(\varepsilon) \doteq e^{t(\varepsilon)Y} e^{\psi(\varepsilon)X} e^{(t_{i-1}-\varepsilon)Y}(0) \tag{3.24}$$

be conjugate to $e^{\psi(\varepsilon)X}e^{(t_{i-1}-\varepsilon)Y}(0)$ along the integral curve of Y. Finally, if $i=1$, $s_1>0$, we define $t(\varepsilon)$ by requiring that

$$\Gamma_1(\varepsilon) \doteq e^{t(\varepsilon)Y}e^{\varepsilon X}(0) \tag{3.25}$$

be conjugate to $e^{\varepsilon X}(0)$ along the integral curve of Y. The conjugate curves Γ'_i, originating from the points s'_i, can be defined in an entirely similar manner.

Observe that, in all three of these cases, one has $t(0) = s_i - t_{i-1}$, and that the local existence of the function $t(\cdot)$ is provided by the implicit function theorem. Indeed, from (SA3) and (SA4) it follows $\dot{\theta}(s_i) \neq 0$, and hence $\nabla\Delta_B(q_i) \neq 0$, because of Lemma 3.6 of chapter one.

By a result of Sussmann [17,18], the trajectories that undergo a switching along a curve Γ_i can afterwards remain optimal only if the curve Γ_i itself is not a trajectory of the control system, i.e., if X and Y do not point to opposite sides of Γ_i. This motivates the following stability assumptions, which ensure that X is not tangent to Γ_i at points close to q_i. Here $\dot{\Gamma}_i = d\Gamma_i/d\varepsilon$ provides a tangent vector to Γ_i.

(SA7) At every point $q_i \doteq e^{s_iY}(0)$, the conjugate curve Γ_i satisfies $\dot{\Gamma}_i(0) \wedge X(q_i) \neq 0 \quad \forall i \geq 2$. The same holds for the conjugate curves Γ'_i, at the points $q'_i \doteq e^{s'_iY}(0)$.

Proposition 3.2. *In addition to the assumptions of Proposition 3.1, let (SA7) hold. Then, to the right of every point $q_i \doteq e^{s_iY}(0)$ with $i \geq 2$, the time optimal synthesis contains either the curve of conjugate points Γ_i defined at (3.23)-(3.25), or an overlap curve, starting at q_i. The first case occurs precisely when the vector fields X, Y point to the same side of Γ_i, in a neighborhood of q_i. The analogous results hold for the points $q'_i \doteq e^{s'_iY}(0)$.*

Proof. Fix some time s_i with $i \geq 2$. By (III), at the point $p_{i-1} \doteq e^{t_{i-1}Y}(0)$ initiates either a turnpike, or a curve of conjugate points.

We study the turnpike case first. Consider the equation

$$\Psi(\sigma_1,\sigma_2,\sigma_3) \doteq e^{(\sigma_3-\sigma_1)Y}e^{\sigma_1(F+\varphi_S G)}(p_{i-1}) - e^{\sigma_2 X}e^{(\sigma_3-\sigma_2)Y}(p_{i-1}) = 0. \tag{3.26}$$

A trivial branch of solutions is $\sigma_1 = \sigma_2 = 0$. Observing that

$$\frac{\partial\Psi}{\partial\sigma_1}(0,0,\sigma_3) = \left(e^{\sigma_3 Y}\right)_*\left(\varphi_s G - G\right)(p_{i-1}), \qquad \frac{\partial\Psi}{\partial\sigma_2}(0,0,\sigma_3) = 2G\left(e^{\sigma_3 Y}(p_{i-1})\right), \tag{3.27}$$

since $|\varphi_S(p_{i-1})| < 1$, it is clear that a nontrivial branch of solutions of (3.27) can bifurcate only when $t_{i-1} + \bar{\sigma}_3$ is conjugate to t_{i-1} along γ^+. At $\sigma_3 = s_i - t_{i-1}$ we have

$$\frac{d}{d\sigma_3}\Big(G(p_{i-1}) \wedge \left(e^{-\sigma_3 Y}\right)_* G\left(e^{\sigma_3 Y}(p_{i-1})\right)\Big) > 0$$

because of the stability assumptions (SA3), (SA4). Therefore,

$$\frac{\partial}{\partial \sigma_3}\left(\frac{\partial \Psi}{\partial \sigma_1} \wedge \frac{\partial \Psi}{\partial \sigma_2}\right) \neq 0.$$

A standard result in bifurcation theory [3] now implies the existence of a $\mathcal{C}^1$ function $\varepsilon \mapsto \big(\sigma_1(\varepsilon), \sigma_2(\varepsilon), \sigma_3(\varepsilon)\big)$ such that

$$\big(\sigma_1, \sigma_2, \sigma_3\big)(0) = (0, 0, s_i - t_{i-1}), \qquad \Psi\big(\sigma_1(\varepsilon), \sigma_2(\varepsilon), \sigma_3(\varepsilon)\big) = 0 \qquad \forall \varepsilon,$$

and such that the nontrivial vector

$$\left(\frac{\partial \sigma_1}{\partial \varepsilon}, \frac{\partial \sigma_2}{\partial \varepsilon}\right)$$

is in the kernel of the 2×2 matrix

$$A = \left(\frac{\partial \Psi}{\partial \sigma_1}, \frac{\partial \Psi}{\partial \sigma_2}\right).$$

Because of (3.27), the vectors $\partial\Psi/\partial\sigma_1$, $\partial\Psi/\partial\sigma_2$ have opposite orientations at $(0, 0, s_i - t_{i-1})$. We can thus assume that the nontrivial branch of solutions is parametrized so that the maps $\varepsilon \mapsto \sigma_1(\varepsilon)$, $\varepsilon \mapsto \sigma_2(\varepsilon)$ are both increasing. The assignment

$$\varepsilon \mapsto e^{\sigma_2(\varepsilon)X} e^{\big(\sigma_3(\varepsilon) - \sigma_2(\varepsilon)\big)Y}(p_{i-1}) \qquad \varepsilon \in [0, \varepsilon_0] \tag{3.28}$$

locally parametrizes the overlap curve, for ε_0 small enough.

At this stage, two lines through q_i have been constructed: the curve Γ_i of conjugate points at (3.23) and the overlap curve Λ_i in (3.28). Of these two, only one actually occurs in the time optimal synthesis. To decide which one, observe that by (SA7) the vector field X is not tangent to Γ_i. If X, Y point to the same side of Γ_i, then there is a neighborhood $\mathcal{N}$ of q_i such that all points in $\mathcal{N}$ to the right of γ^+ can be covered by extremal trajectories which either make a switching on the curve Γ_i, or else follow γ^+ up to some time $t \geq s_i$ and then make a switching. By a dynamic programming argument, these trajectories are optimal. On the other hand, if X, Y point to opposite sides of Γ_i, then the trajectories of the form

$$t \mapsto e^{tY} e^{\varepsilon(F + \varphi_S G)} e^{t_{i-1}Y}(0) \tag{3.29}$$

cross the curve

$$t \mapsto e^{tX} e^{s_i Y}(0) \tag{3.30}$$

before hitting the curve Γ_i. In this case, the curves (3.29) remain optimal for a short time beyond the crossing of the trajectory (3.30). This implies that in (3.28) one has

$$\frac{d}{d\varepsilon}\big(\sigma_3(\varepsilon) - \sigma_2(\varepsilon)\big) > 0,$$

hence the trajectories that reach the overlap curve make their switching after time s_i, and are thus extremal. By a dynamic programming argument, such a local feedback is optimal.

In the case where at p_{i-1} starts a curve of conjugate points, let

$$\varepsilon \mapsto e^{\psi(\varepsilon)X} e^{-\varepsilon Y}(p_{i-1}) \tag{3.31}$$

be a parametrization of such curve, with ψ as in (3.22). Consider the equation

$$\Psi(\sigma_1,\sigma_2,\sigma_3) \doteq e^{(\sigma_3-\psi(\sigma_1)+\sigma_1)Y} e^{\psi(\sigma_1)X} e^{-\sigma_1 Y}(p_{i-1}) \;-\; e^{\sigma_2 X} e^{(\sigma_3-\sigma_2)Y}(p_{i-1}) = 0. \tag{3.32}$$

Again, $\sigma_1 = \sigma_2 = 0$ is a trivial branch of solutions. We now have

$$\frac{\partial\Psi}{\partial\sigma_1}(0,0,\sigma_3) = \big(e^{\sigma_3 Y}\big)_*(-2\psi'(0))G(p_{i-1}), \qquad \frac{\partial\Psi}{\partial\sigma_2}(0,0,\sigma_3) = 2G\big(e^{\sigma_3 Y}(p_{i-1})\big).$$

Therefore, when $\sigma_3 = s_i - t_{i-1}$, the assumptions (SA3), (SA4) imply

$$\frac{\partial\Psi}{\partial\sigma_1}\wedge\frac{\partial\Psi}{\partial\sigma_2} = 0, \qquad \frac{\partial}{\partial\sigma_1}\left(\frac{\partial\Psi}{\partial\sigma_1}\wedge\frac{\partial\Psi}{\partial\sigma_2}\right) \neq 0.$$

As in the previous case, standard bifurcation theory now yields the existence of a nontrivial branch of solutions $\varepsilon \to (\sigma_1,\sigma_2,\sigma_3)(\varepsilon)$ of (3.32). The assignment

$$\varepsilon \mapsto e^{\sigma_2(\varepsilon)X} e^{\left(\sigma_3(\varepsilon)-\sigma_2(\varepsilon)\right)Y}(p_{i-1}) \qquad\qquad \varepsilon \in [0,\varepsilon_0]$$

locally parametrizes the overlap curve.

As in the turnpike case, this overlap curve is actually present in the optimal feedback synthesis if the trajectories

$$t \mapsto e^{tY} e^{\psi(\varepsilon)X} e^{-\varepsilon Y}(p_{i-1})$$

cross the curve (3.30) before reaching Γ_i. This completes the proof.

It remains to consider the points $q_1 = e^{s_1 Y}(0)$, $q_1' = e^{s_1' Y}(0)$. From the definition, we have that only one of the two numbers s_1, s_1' is different from zero. Assume that $s_1 \neq 0$. The case $s_1' \neq 0$ can be treated in an entirely similar manner.

We introduce a new system of coordinates in such a way that:

$$Y \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad X(s_1, 0) = \begin{pmatrix} a \\ 0 \end{pmatrix}.$$

Moreover we define:

$$b = -\frac{\partial X_2}{\partial x_1}(0) < 0 \qquad c = \frac{\partial X_2}{\partial x_1}(s_1, 0) < 0. \tag{3.33}$$

The signs of b, c follows from the stability conditions (SA2),(SA3). The trajectories:

$$\sigma_2 \mapsto e^{\sigma_2 X} e^{(s_1+\sigma_1)Y}(0), \quad \tau_2 \mapsto e^{\tau_2 Y} e^{\tau_1 Y}(0),$$

for σ_1, τ_1 sufficiently small, cross each other near the point $(s_1, 0)$. An overlap curve at $(s_1, 0)$ is determined by a system of three equations of the form

$$\begin{cases} \sigma_1 + \sigma_1 + \sigma_2 = \tau_1 + \tau_2, \\ s_1 + \sigma_1 + a\sigma_2 + O(1)\sigma_1\sigma_2 + O(1)\sigma_2^2 = -\tau_1 + \tau_2 + O(1)\tau_1^2, \\ b\tau_1^2 + O(1)\tau_1^3 = c\sigma_1\sigma_2 + ac\sigma_2^2 + O(1)\sigma_2^3 + O(1)\sigma_1^2\sigma_2 + O(1)\sigma_1\sigma_2^2). \end{cases} \tag{3.34}$$

Here with the Landau symbol $O(1)$ we indicate some function which is uniformly bounded, together with its derivatives. The first equation yields

$$\tau_2 = s_1 + \sigma_1 + \sigma_2 - \tau_1.$$

Hence:

$$\begin{cases} (a-1)\sigma_2 + 2\tau_1 = O(1)\sigma_1\sigma_2 + O(1)\sigma_2^2 + O(1)\tau_1^2, \\ c\sigma_1\sigma_2 + ac\sigma_2^2 - b\tau_1^2 = O(1)\sigma_2^3 + O(1)\sigma_1^2\sigma_2 + O(1)\sigma_1\sigma_2^2. \end{cases}$$

A trivial branch of solutions is $\tau_1 = \sigma_2 = 0$. When σ_1, τ_1 are sufficiently small, one can solve the first equation for τ_1 as a function of (σ_1, σ_2). This yields:

$$\frac{\partial \tau_1}{\sigma_2}(0,0) = \frac{1-a}{2}, \qquad \frac{\partial \tau_1}{\sigma_1}(0,0) = 0.$$

Then the third equation, for σ_1, τ_1 sufficiently small, is equivalent, up to higher order terms, to:

$$\left[c\sigma_1 + \left(ac - b\left(\frac{1-a}{2}\right)^2\right)\sigma_2\right]\sigma_2 = 0. \tag{3.35}$$

We have a nontrivial branch of solutions if the following stability condition holds:

$$(SA8) \qquad a \neq 0, \qquad \delta \doteq ac - b\left(\frac{1-a}{2}\right)^2 \neq 0.$$

Proposition 3.3. *In addition to the assumption of Proposition 3.1, assume $s_1 > 0$ and let (SA8) holds. Then, to the right of $q_1 \doteq e^{s_1 Y}(0)$, the time optimal synthesis contains either the curve of conjugate points Γ_1 defined at (3.25), or an overlap curve, starting at q_1. The first case occurs precisely when $\delta > 0$. The analogous results hold for the point $q_1' \doteq e^{s_1' Y}(0)$.*

Proof. To determine the switching curve Γ_1 of (3.25) consider the trrajectories:

$$t_2 \mapsto e^{t_2 Y} e^{t_1 X}(0) \tag{3.36}$$

and the system for the dual variables:

$$\begin{cases} \lambda(t_1) \cdot G\big(-t_1 + O(1)t_1^2,\ bt_1^2 + O(1)t_1^3\big) = 0, \\ \lambda(t_2) \cdot G\big(-t_1 + t_2 + O(1)t_1^2,\ bt_1^2 + O(1)t_1^3\big) = 0. \end{cases} \tag{3.37}$$

From $Y \equiv (1,0)$ it follows $\lambda(t_1) = \lambda(t_2)$. Moreover, we can normalize λ defining $\lambda_2 \doteq \sqrt{1-\lambda_1^2}$. Then (3.37) has two equations for the variables (λ_1, t_1, t_2) and $(1, 0, s_1)$ is a solution. We now compute the 2×2 Jacobian matrix of partial derivatives of the left hand sides of (3.37), with respect to the variables (λ_1, t_2), and check that its determinant, at the point $(1, 0, s_1)$, is equal to $c/2 \neq 0$. Therefore we can solve the system (3.37) in a neighborhood of $(1, 0, s_1)$ expressing (λ_1, t_2) as functions of t_1. This yields

$$\frac{\partial t_2}{\partial t_1}(1, 0, s_1) = 1 + \frac{b(1-a)}{2c}.$$

Hence Γ_1 is given by:

$$x_2 = \frac{4c^2}{b(1-a)^2}(x_1 - s_1)^2 + o(|x_1 - s_1|). \tag{3.38}$$

Notice that $a = 1$ gives $G(s_1, 0) = 0$ but this contradicts (SA1), hence $a \neq 1$. Indeed $a < 1$, otherwise $s_1 = t^+$. From (3.35), being $\delta \neq 0$, we can express σ_1 as a function of σ_2. After straightforward calculations one obtains the same expression (3.38) also for the overlap curve K. The image of the X-trajectory γ through $(s_1, 0)$ is given by the expression:

$$x_2 = \frac{c}{a}(x_1 - s_1)^2 + o(|x_2 - s_2|^2).$$

Therefore Γ_1, K and γ have a first order tangency at $(s_1, 0)$. Moreover, Γ_1 and K have a second order tangency and the condition $\delta \neq 0$ means precisely that γ does not have a second order tangency with Γ_1, K. Now, if $\delta > 0$ then $a > 0$, σ_1 and σ_2 have opposite signs and the system (3.34) does not describe an admissible overlap curve. On the other side the curve γ lies between γ^+ and Γ_1. Therefore the trajectories (3.36) switch along Γ_1 before crossing γ. By a dynamic programming argument these trajectories are optimal.
If $\delta < 0$ then σ_1, σ_2 have the same sign, hence the trajectories:

$$\sigma_2 \mapsto e^{\sigma_2 X} e^{(s_1+\sigma_1)Y}(0)$$

that reaches K switches after s_1 and thus are extremal. The curve Γ_1 lies between γ and γ^+. We can conclude again by a dynamic programming argument.

Of course, entirely similart results hold for the curve γ^-. This completes the analysis of STEP 1 of the algorithm. We now define by induction the step N, where $N > 1$.

STEP N. Consider all frame curves generated by the previuos step which are not overlap curves. Let D be such a frame curve. Consider the constructed trajectories $\gamma \in Traj(\Sigma)$, $\gamma : [0, t_0] \mapsto I\!R^2$, $In(\gamma) = 0$, such that $\gamma(t_0) = x \in D$. For each one of these trajectories, calling u the corresponding control, let construct two new trajectories $\gamma_1, \gamma_2 : [0, 1] \mapsto I\!R^2$, $In(\gamma_1) = In(\gamma_2) = 0$, corresponding , respectively, to the following controls u_1 and u_2:

$$u_1(t) = u_2(t) = u(t) \qquad \forall t \in [0, t_0]$$

$$u_1(t) = +1 \quad u_2(t) = -1 \qquad \forall t \in [t_0, 1].$$

If there exists a conjugate point $\gamma_1(t_1)$ $(t_1 > t_0)$ to $\gamma_1(t_0)$ along (u_1, γ_1) then construct only $\gamma_1 \restriction [0, t_1]$, and do the same for γ_2.

Consider γ_1, as defined above, and indicate with $\bar{\lambda} : [0, t_0] \mapsto I\!R^2_*$ the covector field associate by induction to $\gamma_1 \restriction [0, t_0]$. Let construct γ_1 only if there exists a covector field λ along (u_1, γ_1) such that:

$$\lambda \restriction [0, t_0] = \bar{\lambda} \qquad\qquad \lambda(t_0) \cdot G(\gamma_1(t_0)) = 0 \tag{3.39}$$

and, if $\gamma_1(t_1)$ is a conjugate point to $\gamma_1(t_0)$ $(t_1 > t_0)$ along (u_1, γ_1), then:

$$\lambda(t) \cdot G(\gamma_1(t)) > 0 \quad \forall t \in]t_0, t_1[\qquad\qquad \lambda(t_1) \cdot G(\gamma_1(t_1) = 0 \tag{3.40}$$

otherwise:

$$\lambda(t) \cdot G(\gamma_1(t)) > 0 \qquad \forall t \in]t_0, 1]. \tag{3.41}$$

If there exist such a covector field then associate it to γ_1. Proceed similarly for the other case defined above (γ_2) changing signes in the (3.39-41).

Consider the connected components of $\{x : \Delta_B(x) = 0, |\varphi_S(x)| \leq 1\}$ (φ_S was defined (3.18) of chapter one) that intersect some contructed trajectories. Assume that S is such a component, γ is the constructed trajectory with associated covector field λ and t_0 is the first time of intersection of γ with S. If $\lambda(t_0) \cdot G(\gamma(t_0)) = 0$ then we consider the trajectory γ_1 that verifies:

$$\gamma_1 \restriction [0, t_0] \equiv \gamma, \qquad \gamma_1(t) \in S \ \forall t > t_0.$$

We construct the trajectory $\gamma_1 \restriction ([0,1] \cap Dom(\gamma_1))$. If u is the control corresponding to γ, we have that γ_1 correspond to the control u_1 verifying:

$$u_1 \restriction [0, t_0[\equiv u, \qquad u_1(t) = \varphi_S(\gamma_1(t)) \ \forall t \geq t_0.$$

We associate to γ_1 the covector field λ_1 that verifies $\lambda_1 \equiv \lambda$ on $[0, t_0]$.

Now the algorithm will cut some trajectories. If a trajectory γ will be cut then all frame point on it will be cut. In the same way, a frame curve (or part of it), consisting of point belonging to cut trajectories, will be cut. Let $x \in I\!R^2$ be a point reached by some trajectories constructed in this step. There are a finite number of constructed trajectories (not necessarily constructed in this step) $\gamma_1, \ldots, \gamma_n$ that reach x at a certain time $t_i \in Dom(\gamma_i)$, $i = 1, \ldots, n$. Let $\bar{t} = \min_i t_i$. If $t_i > \bar{t}$ then cut the trajectory γ_i after the time t_i, i.e. consider only $\gamma_i \restriction [0, t_i]$. It can happen that $\bar{t} = t_i$ for more than one i, in this case the algorithm constructs an overlap curve.

Define the following new frame curves:

a) Maximal regular turnpikes S for which there exists a trajectory γ, constructed in this step, that verifies $\gamma(I) = S$ for some $I \subset Dom(\gamma)$

b) Overlap curves constructed in this step

c) Conjugate curves to frame curves, constructed in the previous step, along $X-$ or $Y-$trajectories.

The curves of type c) are also called *switching curves*, because all trajectories, reaching one of these curves, switch. We give now the stability conditions for the frame curves. Consider a turnpike S as in a), and let $\gamma, I = [t_0, t_1]$ be as in a). Let φ_S be the control defined in (3.18) of chapter one, then the stability condition is:

$$| \varphi_S(\gamma(t)) | < 1 \quad \forall t \in [t_0, t_1[. \tag{3.42}$$

Consider an overlap curve K. It is defined by a system of equations of the following type:

$$(B)\begin{cases} (exp \quad tX) & x_1(s) \\ (exp \quad t'Y) & x_2(s') \\ s+t & = \quad s'+t' \end{cases}$$

where $x_1(s)$ and $x_2(s')$, $s \in [0,1]$, are parametrizations of two frame curves defined in the previous steps. The stability condition is that the system (B) has rank 3 uniformly on K except the posibble inersection with $\gamma^{\pm}$. It is easy to verify that (B) has rank 2 on $K \cap \gamma^{\pm}$.

Pass now to the stability conditions for a conjugate curve. Let D be a frame curve, D' the conjugate curve along the $Y-$trajectories. From the definition, for each $x \in D$ there exist a trajectory $\gamma_x : [0, \sigma_x] \mapsto I\!R^2$, constructed in the previuos step, and $t_x \in]0, \sigma_x[$ such that $\gamma_x(t_x) = x$, $\gamma_x \restriction [t_x, \sigma_x]$ is a $Y-$trajectory and $\gamma_x(\sigma_x) \in D'$ ($\gamma_x(\sigma_x)$ is conjugate to x). Denote by u_x the control corresponding to γ_x and by v_x the vector fields along (u_x, γ_x), i.e. the solutions to (3.1) of chapter one. The stability conditions are the following:

$$det\left[v_x\Big(G(x), t_x; t\Big), G\Big(\gamma_x(t)\Big)\right] \neq 0 \qquad \forall x \in D \quad \forall t \in]t_x, \sigma_x[\tag{3.43}$$

$$\frac{\partial}{\partial t}\, det\left[v_x\Big(G(x), t_x; t\Big), G\Big(\gamma_x(t)\Big)\right]\Bigg|_{t=\sigma_x} \neq 0 \quad \forall x \in D. \tag{3.44}$$

We define in the same way the stability condition for conjugate curve along the $X-$trajectories.

We define the intersections between frame curves to be frame points. Now, we give the stability cnditions for frame points. We assume that all frame points are generic, hence we have to consider only the cases listed in chapter two. We will also use the same notations of chapter two. For example, if x is of (C, K) type then we call C, resp. K, the switching curve, resp. overlap curve, to which x belongs.

The frame points that belongs to $\gamma^{\pm}$ have been considered in the first step. We have not yet constructed the frontier of the reachable set, hence we have to consider the remaining nine cases.

Let x be a frame point, $\gamma : [0.b] \to I\!R^2$ be one constructed trajectory that satisfies $\gamma(b) = x$ and λ the associated covector field. Let t_0 be the last switching time of γ before b and let D be the frame curve to which $\gamma(t_0)$ belongs. For each $y \in D$ there exists a constructed trajectory γ_y, with associated covector field λ_y, that switches at y.

Consider first the case of a frame point of type $(C, C)_1$. We have $\Delta_B(x) = 0$ and $\nabla\Delta_B(x) \neq 0$ (from condition (P_2) of section 4 of chapter one). The first stability condition is:

$$X(\gamma(b)) \cdot \nabla\Delta_B(\gamma(b)) \neq 0, \qquad Y(\gamma(b)) \cdot \nabla\Delta_B(\gamma(b)) \neq 0. \tag{3.45}$$

If (3.45) holds, then by the implicit function theorem, for each y in a neighborhood of $\gamma(t_0)$ in D, there exists a time σ_y such that $\Delta_B(\gamma_y(\sigma_y)) = 0$. Let $y(s)$ be an arclegth parametrization of D in a neighborhood of $\gamma(t_0) = y(s_0)$. Define:

$$\psi(y(s)) \doteq \lambda_y(\sigma_y) \cdot G(\gamma_y(\sigma_y)),$$

the second stability condition is:

$$\left.\frac{\partial \psi}{\partial s}\right|_{s=s_0} \neq 0. \tag{3.46}$$

Consider the switching curve C_2 as in the description of the (C, C) frame points given in chapter two. Let $C_2(x)$ be a tangent vector to C_2 at x. The last stability condition is:

$$Y(x) \wedge C_2(x) \neq 0, \qquad X(x) \wedge C_2(0) \neq 0. \tag{3.47}$$

If x is of type $(C, S)_1$, resp. $(S, K)_1$, then we have the same stability conditions, replacing in (3.47) the vector $C_2(x)$ with $C(x)$, resp. $K(x)$, tangent to C, resp. K, at x.
If x is of type $(C, K)_2$ and $\Delta_B(x) = 0$ then we assume that C and K are not tangent at x and we have the same stability conditions, replacing the vector $C_2(x)$ with $K(x)$ in (3.47).
If x is of type $(C, C)_2$ or of type (K, K) then there is no stability condition.
If x is of type $(C, S)_2$ let $\varphi_S(y)$ be the control defined in (3.18) of chapter one for every $y \in S$. Let $y(s)$ be an arclength parametrization of S in a neighborhood of $x = y(s_0)$. The stability condition is:

$$\left.\frac{\partial \varphi_S}{\partial s}\right|_{s=s_0} \neq 0 \tag{3.48}.$$

If x is of $(C, K)_1$ type then, as above, let $C(x)$, resp. $K(x)$, be the vector tangent to C, resp. K, at x. The stability condition is:

$$C(x) \wedge K(x) \neq 0. \tag{3.49}$$

If x is of type $(C, K)_2$ and $\Delta_B(x) \neq 0$, let $C(y)$ be the vector tangent to C at y, where y ranges in a neighborhood of x in C. The stability condition is:

$$\left.\frac{\partial}{\partial y}(Y(y) \wedge C(y))\right|_{y=x} \neq 0 \qquad \left.\frac{\partial}{\partial y}(X(y) \wedge C(y))\right|_{y=x} \neq 0. \tag{3.50}$$

Finally let x be of type $(S, K)_2$. If φ_S is the control of (3.18) of chapter one then the stability condition is:

$$|\varphi_S(x)| < 1. \tag{3.51}$$

End of step N.

If at step N the algorithm $\mathcal{A}$ does not construct any frame curve, then at step $N+1$ the algorithm $\mathcal{A}$ does no operation. Therefore, in this case, we will say that *the algorithm $\mathcal{A}$ stops at step N.*

The conclusions of section 4 of chapter one ensure the existence of $N \in I\!N$ such that $\mathcal{A}$ stops at step N. Consider the set $R_{\mathcal{A}}(1)$ of points reached by constructed trajectories of $\mathcal{A}$. We define $Fr(R_{\mathcal{A}}(1))$ to be a frame curve and its intersections with other frame curves to be frame points. We assume that (3.41) holds for every trajectory reaching $Fr(R_{\mathcal{A}}(1))$. We now give the stability conditions for these frame points. Let x be such a frame point. If x is not of type (F,S) then we assume that $\Delta_B(x) \neq 0$. If x is of type $(X,F)_{1,2}$ then $x = \gamma^{\pm}(1)$ and we assume that $1 \notin \{t_i, t'_i, s_i, s'_i\}$ (see the first step for definitions). If x is of type (F,C) then we assume that $X(x), Y(x)$ are not tangent to C at x. If x is of type (F,S) then we assume $|\varphi_S(x)| < 1$, where φ_S is defined in (3.18) of chapter one. Finally, if x is of type (F,K) then we assume that x is not a switching point for the constructed trajectories arriving at x (this condition is yet ensured by (3.41)).

We define an equivalence relation $\sim$ between frame points. If x_1, x_2 are two frame points, that are not of type $(C,C)_2, (K,K)$, we let $x_1 \sim x_2$ if and only if there exist some points $y_0 = x_1, y_2, \ldots, y_n = x_2$ such that the following holds. Every y_i belong to a frame curve D_i. If y_i is a frame point then it is not of (K,K) type, if y_i is not a frame point then D_i is not of K type. For every y_i, $i = 1, \ldots, n-1$, there exist a contructed trajectory γ_i and $a_i, b_i \in Dom(\gamma_i)$ verifying $\gamma_i(a_i) = y_i, \gamma_i(b_i) = y_{i+1}, \gamma_i \restriction [a_i, b_i]$ is an X or Y trajectory and $\gamma_i(]a_i, b_i[) \cap \gamma^{\pm}([0, t^{\pm}]) = \emptyset$. That is there exists a curve, connecting x_1 to x_2, formed by X and Y arcs of constructed trajectories and that does not intersect the frame curves $\gamma^{\pm}$. To understand the meaning of this equivalence relation see also Remark 2.1.

Remark 2. Notice that if we let γ_i intersect $\gamma^{\pm}$, then the origin is obviously in relation with every frame point reached by a bang-bang constructed trajectory. We have to exclude the points $(C,C)_2, (K,K)$ in the definition of $\sim$. It is not a generic situation for two frame points to be in relation unless one of them is of the above types. Indeed, the points $(C,C)_2$ and (K,K) were constructed by $\mathcal{A}$ exactly because they are in relation with another frame point. But locally the synthesis generated by $\mathcal{A}$ does not present a singularity, see Remarks 5.2,5.4 of chapter two.

If $\mathcal{A}$ stops at step N and verifies the following conditions:

($\mathcal{A}1$) All frame curves and points verify the stability conditions

($\mathcal{A}2$) If x_1, x_2 are two frame points and $x_1 \sim x_2$ then $x_1 = x_2$

then we say that *the algorithm $\mathcal{A}$ succeedes for Σ at time 1.*

Proceeding as in section 5 of chapter one, we construct a synthesis $\Gamma_{\mathcal{A}}(\Sigma, 1)$ and we call it the synthesis generated by the algorithm $\mathcal{A}$. It is easy to see that the conditions $(\mathcal{A}_1), (\mathcal{A}_2)$ are generic then from Corollary 4.3 of chapter one we have that $\mathcal{A}$ succedes for a generic set of systems. Finally, from Theorem 5.2 of chapter one, $\Gamma_{\mathcal{A}}(\Sigma, 1)$ is an optimal synthesis. Therefore we have the following:

Theorem 3.4 *There exists a generic set $\Pi \subset \Xi$ such that for every $\Sigma \in \Pi$ the algorithm $\mathcal{A}$ succedes for Σ at time 1 and $\Gamma_{\mathcal{A}}(\Sigma, 1)$ is an optimal synthesis.*

4. Classification and structural stability

In this section we describe a classification method for a generic set of systems. The systems for which the algorithm $\mathcal{A}$ succedes are classified assigning to each system a planar topological graph with some additional structure.. The points and edges of this graph correspond to frame points and curves of the system.

From now on, we fix the time $\tau = 1$ and consider only the systems of Ξ for which the algorithm $\mathcal{A}$ succeedes at time 1.

A graph $\mathcal{G}$ is a finite set of points of $\mathbb{R}^2$ and curves connecting the points, called edges. Moreover, inside each region enclosed by edges there are possibly some lines connecting points and edges. We assume that edges and lines does not cross each other.

Every edge can be of one of the following type: X, Y, F, S, C, K; corresponding to the types of frame curves. An edge of type X, Y or S has an orientation and hence an initial and a terminal point. The edges of type C have a positive side, corresponding to the fact that constructed trajectories cross a frame curve of type C passing from one side to another.

To every region enclosed by edges, that are not all of F type, has a sign $+$ or $-$. This correspond to the fact that a region of the rechable set R, that contains no frame curve, is covered by X- or by Y-trajectories. On each region we can have some curves connecting points to edges. These correspond to constructed trajectories that pass through frame points. See Remark 4.1 for the necessity of such definition of graph.

We say that two edges E_1, E_2 are related and we write $E_1 \sim E_2$ if they have in common a point of the graph.

We now describe a canonical way of associating a graph to a system. Given a system Σ (for which $\mathcal{A}$ succeedes at time 1) we associate a graph $\mathcal{G}$ to Σ in the following way. For every frame point we construct a point of $\mathcal{G}$ having the same coordinate in $\mathbb{R}^2$. For every frame curve D, with no frame point in $D \setminus \partial D$, $\partial D = \{x_1, x_2\}$, we construct an edge E of G of the same type connecting the points of $\mathcal{G}$ corresponding to x_1, x_2. If D is an X, Y or S-curve then D has the orientation of increasing time and we endow E with the corresponding orientation. If D is of type C, then constructed trajectories arrives one side of D. We define the corresponding side of E to be positive.

For every region $A \subset R$ enclosed by frame curves there is a region A', in the plane of the graph, enclosed by the corresponding edges. If A is covered by Y-trajectories, we assign to A' the positive sign, otherwise we assign to A' the negative sign.

Consider a frame point x of $Cl(A)$, which is not of (K, K) type (recall the terminology of chapter two), and the constructed trajectory γ_x verifying $\gamma_x(t_x) = x$ for some t_x. Assume that $\gamma_x(I) \subset A$, for some $I = [a, b] \subset Dom(\gamma)$, $t_x \in I$. Notice that it can happen $a \neq t_x \neq b$, e.g. if x is of type $(X, K)_3$. If $t_x \neq a$ and $\gamma_x(a) \in D$ frame curve, then we construct a line in A' going from an point y of the edge E, corresponding to D, to the point x' of $\mathcal{G}$ corresponding to x. If $\gamma_x(a)$ is a frame point then we choose y to be the corresponding point of E, otherwise we choose y in $E \setminus \partial E$. If D is of C type, and $\gamma_x(a) \in D \setminus \partial D$, then we consider the last switching point z of γ_x before $\gamma_x(a)$. If D is of S type then there exists a constructed trajectory γ_1 that switches at $\gamma_x(a)$ and enters the region on the opposite side, with respect to D, of the region entered by γ_x. Indeed, a Y and an X constructed trajectories start from every point of a turnpike. We let, in this case, z to be the first switching point of γ_1 after $\gamma_x(a)$. If z belong to a frame curve D_1 then we construct a line going from a point z' of the edge E_1, corresponding to D_1, to the point y. Again if z is a frame point we let z' be the corresponding point of $\mathcal{G}$. If D_1 is a C or S frame curve then we proceed in the same way. We continue until we reach a frame curve not of C or S type. We do the same if $t_x \neq b$.

We can contruct these lines in such a way that they do not cross each other. If $\mathcal{G}$ is associated to Σ in this way then we say that $\mathcal{G}$ is *canonically associated* to Σ.

Remark 4.1 Consider the system Σ_2 of Example 2 of chapter two and the graph $\mathcal{G}_2$ canonically associated to Σ_2. If we do not specify a sign for every region of $\mathcal{G}_2$ then the two (X, C) frame points are not distinguishable. Hence, for some system Σ with a frame point of type $(X, C)_1$ or $(X, C)_2$, we can construct a system with the same graph, except the signs of the regions, but not equivalent to Σ. This show the necessity of specifying a sign for every region.

Consider the system Σ_1 of Example 1 of chapter two. There is a region A that is a connected component of the complement of the reachable set and is limited. In the corresponding graph, we have no necessity of giving a sign to the region corresponding to A. The regions enclosed by edges all of F type correspond exactly to the *holes* of the reachable set and there is no necessity to give a sign to these regions.

Consider now a frame point x of $(C,S)_2$ type. If we do not specify, in the corresponding graph, a positive side for the edge corresponding to the switching curve then we do not know, from the graph, if the Y or the X trajectories enter the switching curve. Again there exists two not equivalent systems corresponding to the same graph.

The lines divide the graphs into subregions in such a way that the trajectories, contained in the same subregion, have the same *story*, i.e. cross the same frame curves in the same order. If the lines are not constructed then, in some cases, we can not decide the story of every trajectory and then we can not recognize equivalent systems. For example, consider the system Σ_3 of the third example of chapter one. Let γ be the constructed trajectory that pass through the (X,S) point $(-1,-1/3)$ and then goes on as Y trajectory. If we do not consider the lines, from the graph associated to Σ_3 we can not know if γ reaches the switching curve or the frontier of the reachable set.

We now give some admissibility conditions that characterize a class of graphs. This class will be proved to be the class of graphs associated to systems canonically.

To every system of the Examples of chapter one and two we can associate a topological graph in the canonical way. We consider these examples restricted to a neighborhood of a frame point, then we obtain a set of graphs $\mathcal{E}$, whose elements are defined locally and each one corresponds to a type of frame point. A point x' of a graph $\mathcal{G}$ is said to be *admissible* if there exist a graph $\mathcal{G}' \in \mathcal{E}$ such that $\mathcal{G}$ contains a copy of $\mathcal{G}'$ to which x' belongs. We use the same terminology for the points of $\mathcal{G}$, e.g. (X,Y) point. The first condition is:

($\mathcal{G}1$) All points of $\mathcal{G}$ are admissible.

We consider graphs that contain exactly one point of the type (X,Y) and we call this point the *origin* of the graph. Assume that ($\mathcal{G}1$) holds. Let E be a Y-edge and let x be the initial point of E. If x is not the origin then there exists a Y-edge E_1 for which x is the terminal point. We consider the initial point x_1 of E_1 and do the same considerations. Since $\mathcal{G}$ is finite, proceeding by induction, we find a finite collection $E_1,\dots,E_n$ of Y-edges such that $E_i \sim E_{i+1}$, $i=1,\dots,n-1$, and the initial point of E_n is the origin. Then, since there is only one origin, the Y-edges form a set $\{E_1,\dots,E_m\}$ such that the initial point of

E_1 is the origin and for each $i = 1, \dots, m-1$ the terminal point of E_i is the initial point of E_{i+1}. We call η^+ the union of these edges. Analogously we define η^- for the X-edges. In the first step of $\mathcal{A}$ we have described all the possibilities for the sequence of frame points on a curve γ^+ of a system Σ. We say that η^+ is *admissible* if there exists a system Σ such that the curve γ^+ correspond to η^+ canonically. That is there is a correspondence defined for points, edges of η^+, for lines intersecting η^+ and for the regions to which η^+ belongs, that follows the rules of canonical correspondence. This happens exactly when η^+ and γ^+ have an ordered sequence of corresponding points. The second condition is:

($\mathcal{G}2$) $\mathcal{G}$ has exactly one (X, Y) point, called the origin. The collections of edges $\eta^{\pm}$ are admissible.

Let E be a C-edge, x'_1, x'_2 be the endpoints of E and A' a region on one side of E. There exist two frame points x_1, x_2 corresponding to x'_1, x'_2. Consider the correspondence between x'_1 and x_1. Let D be the frame curve that correspond to E and $\tilde{A}_1$ the region correponding to A'. We define A_1 to be the connected component of $\tilde{A}_1 \setminus \{x : \Delta_B(x) = 0\}$ that contains D. Similarly we define the region A_2. We say that E is *admissible* if there exist x_1, x_2 such that the function Δ_B has the same sign on A_1 and A_2.

Remark 4.2 If, for example, E connect two points of (C, S) type then A_1, A_2 are both covered by Y-trajectories or both covered by X-trajectories. In this case, since the two vector fields must point to opposite side of turnpikes, it follows that Δ_A have a different sign on the two regions A_1, A_2. From Theorem 3.7 of chapter one we have that along C the function f, see (3.11) of chapter one, does not change sign. Hence there exists at least one curve, intersecting C, on which $\Delta_A = 0, \Delta_B = 0$. This is clearly a not generic situation and is not compatible with the condition (P_3) of section 4 of chapter one.

Another admissibility condition is:

($\mathcal{G}3$) Every C-edge is admissible.

The relation $\sim$ divide the set of F-edges into a finite number of equivalence classes. If ($\mathcal{G}1$) holds then the union of the elements of an equivalence class form a closed curve.

($\mathcal{G}4$) Only one closed curve, that is union of the elements of an equivalence class of F-edges, encloses a region in which there are points and edges.

Notice that we can have some situation in which there are more than one equivalence class of F-edges, e.g. the system in Example 1 of chapter two where $R(\tau)$ has one hole.

Consider now a region A' enclosed by edges of $\mathcal{G}$. If one edge E is of X type if A' is positive, of Y type if A' is negative, of C type with the negative side on A' or of S type then we say that E is an *entrance*. If E is of K, F or C type with positive side on A' then we say that E is an *exit*. Otherwise, we say that E is a *side*, i.e. if it is of Y type and A' is positive or of X type and A' is negative. The definitions are justified by the fact that if D is a frame curve corresponding to E canonically, then through each point of D pass a constructed trajectory that enters, resp. exit from, the region corresponding to A' if and only if E is an entrance, resp. exit.

We say that the set of lines of $\mathcal{G}$ is admissible if the followings hold. Every line connects an entrance to an exit. If a point x' belong to two entrances, resp. exits, then there is a line connecting x' with an exit, resp. entrance. Let x' be a point of one of the types $(X,C)_3, (X,K)_3, (C,C)_1, (C,S)_2, (C,K)_1, (S,K)$, and let A', B' be the two regions such that $x' \in Cl(A'), Cl(B')$. There are two lines l_1, l_2, both contained in A' or both in B', passing through x'; l_1 connect x' to an entrance and l_2 connect x' to an exit. If x' is of $(C,C)_2$ points then there are two lines arriving at x' from different region and at least one of them reaches another point. These are the only lines that connect two points. If $x' \in E \setminus \partial E$, E is of C or S type and there is a line l arriving at x' from a region A' then there is a line arriving at x' from the other region B' such that $x' \in Cl(B')$.

There are no other lines. If all these conditions are satisfied then we say that the set of lines of $\mathcal{G}$ is admissible.

Remark 4.3 The conditions given for the set of lines follow directly from the canonical way of associating a graph to a system and from the description of frame points given in chapter two. If we do not assume that the set of lines is admissible, then we can not expect that there exists a system corresponding to $\mathcal{G}$.

Consider the closed curve $\tilde{F}$, union of F-edges, described in ($\mathcal{G}4$). Let U be the connected component of the complement of the union of F-edges, that is enclosed by $\tilde{F}$ and verifies $\tilde{F} \subset Cl(U)$. If A' is a region contained in U we define $L(A')$ to be the set of lines contained in A'. The last condition is:

($\mathcal{G}5$) The set of lines of $\mathcal{G}$ is admissible. If $A' \subset U$ and A'_1 is a connected component of $A' \setminus L(A')$ then $Cl(A'_1)$ contains exactly one entrance and one exit.

If a graph $\mathcal{G}$ satisfies the conditions $(\mathcal{G}1),\ldots,(\mathcal{G}5)$ then we say that $\mathcal{G}$ is *admissible.* It is easy to check that if $\mathcal{G}$ corresponds to a system Σ then $\mathcal{G}$ is admissible. In the following we will prove the converse.

To ensure that the canonical way of associating a graph to a system is well defined we have to prove that two systems are equivalent if and only if the associated graphs are equivalent.

Given two admissible graphs $\mathcal{G}_1, \mathcal{G}_2$, we say that they are equivalent and we write $\mathcal{G}_1 \sim \mathcal{G}_2$ if there is a correspondence ψ between edges and lines such that the following hold. We let ψ be multivalued and not injective on the set of K-edges, but it has to be a bijective function restricted to the edges not of K type. Moreover, ψ is a bijective function restricted to the set of lines. Finally the following holds:

(H1) For every edge E, not of K type, $\psi(E)$ is an edge of the same type; $E_1 \sim E_2$ if and only if $\psi(E_1) \sim \psi(E_2)$, when E_1, E_2 are not both K-edges; ψ preserves orientations and positive sides

(H2) If l is a line that connect E_1 with E_2 then $\psi(l)$ connect $\psi(E_1)$ with $\psi(E_2)$. The same holds for line connecting points. If l_1, l_2 arrive to the same point than the same happens for $\psi(l_1), \psi(l_2)$.

(H3) If A' is a region enclosed by edges $E_1, \ldots, E_n$ then the region enclosed by $\psi(E_1), \ldots, \psi(E_n)$ has the same sign.

(H4) If $\mathcal{K}_1, \mathcal{K}_2$ are the set of equivalence classes of K-edges (for the relation $\sim$) of $\mathcal{G}_1, \mathcal{G}_2$, then ψ induces a bijective correspondence between $\mathcal{K}_1, \mathcal{K}_2$.

We have the following:

Theorem 4.1 *If Σ_1, Σ_2 are two systems and $\mathcal{G}_1, \mathcal{G}_2$ the corresponding graphs then $\Sigma_1 \sim \Sigma_2$ if and only if $\mathcal{G}_1 \sim \mathcal{G}_2$.*

Proof. Assume first that $\Sigma_1 \sim \Sigma_2$ and let φ be as in the definition of equivalnce. For simplicity we will use the symbols Γ_1, Γ_2 for $\Gamma_{\mathcal{A}}(\Sigma_1, 1), \Gamma_{\mathcal{A}}(\Sigma_2, 1)$ respectively.
Given a frame curve D of $\Gamma_{\mathcal{A}}(\Sigma_1, 1)$ that is not a K-curve let E_1, E_2 be the edges corresponding respectively to D and $\varphi(D)$. We define $\psi(E_1) = E_2$. We can proceed in the same way to define ψ on the set of lines. From (E1),(E2) it follows that (H1) and (H2)

hold, and from (E3) it follows that (H3) holds.
Now, if K_1, K_2 are two K frame curves of Γ_1, or of Γ_2, then we set $K_1 \sim K_2$ if they have a point in common. The union of the elements of an equivalence class of Γ_1 is a connected curve K. If we extend by continuity, φ then $\varphi(K)$ is the union of elements of an equivalence class of Γ_2. Therefore we can define ψ on K-edges in such a way that (H4) holds.

Assume now that $\mathcal{G}_1 \sim \mathcal{G}_2$. Let E_1 be an X, Y or S-edge of $\mathcal{G}_1$, $E_2 = \psi(E_1)$ and D_1, D_2 the frame curves corresponding to E_1, E_2 respectively. From (H1) we have that D_1, D_2 are of the same type. Assume that $x_1, \dots, x_n$ are the points of $D_1 \setminus \partial D_1$, ordered for increasing time, that are in relation with a frame point, not of (K, K) type, for the definition given before Remark 3.1. There are exactly n lines if D_1 is of X or Y and $2n$ lines if D_1 is of S type, starting at x_i. From (H2) it follows that there exists $y_1, \dots, y_n \in D_2 \setminus \partial D_2$, ordered for increasing time, from which some lines of $\mathcal{G}_2$ start. We necessarily have that if l is a line passing through x_i and $\psi(l)$ pass through y_j then $i = j$. Indeed, if the opposite happens there must be a crossing between lines, but this is a contradiction with the definition of graph. We define φ on D_1 in such a way that φ is an homeomorphism, $\varphi(D_1) = D_2$ and $\varphi(x_i) = y_i$, $i = 1, \dots, n$. For every $y \in D_1$ consider the constructed trajectories $\gamma_y \in \Gamma_1$ for which $y = \gamma_y(b_y)$ is a switching point. If D_1 is of X or Y type there is at most one such trajectory, if D_1 is of S type then there are two such trajectories. If D_1 is of type X or Y and there exists γ_y then from (H3) there exists a trajectory $\gamma_{\varphi(y)} \in \Gamma_2$ having the same property. Let $c_y > b_y$ be the first time in which γ_y reaches another frame curve. We define:

$$\varphi\big(\gamma_y(t)\big) \doteq \gamma_{\varphi(y)}\left(b_{\varphi(y)} + \frac{c_{\varphi(y)} - b_{\varphi(y)}}{c_y - b_y}(t - b_y)\right) \quad \forall t \in [b_y, c_y].$$

In this way we have defined φ also on the frame curves that are reached by the trajectories γ_y. We proceed in the same way, defining φ on the images of the constructed trajectories that switch at the point of these new frame curves. After a finite number of steps we define φ on the whole reachable set R_1. Notice that we can have two different definitions of φ on the K frame curves, but φ restricted to R_1' (see the definition of equivalence) is well defined. From conditions (H1)-(H4), we have that φ is a well defined homeomorphism that satisfies (E1)-(E3).

Assume now that $\mathcal{G}$ is an admissible graph. We want to find a system Σ such that $\mathcal{G}$ is associated to Σ in the canonical way. This and Theorem 4.1 establish that the correspondence $\Sigma \leftrightarrow \mathcal{G}$ is a bijection between the set of equivalence classes of systems, for which $\mathcal{A}$ succeedes at time 1, and the set of equivalence classes of admissible graphs.

Theorem 4.2 *If $\mathcal{G}$ is an admissible graph then there exists a system Σ to which $\mathcal{G}$ is canonically associated.*

Proof. We construct Σ defining it on a finite collection of open sets that cover $\mathcal{G}$ and then gluing toghether on the intersections. We proceed defining Σ and a synthesis Γ for Σ at the same time. Moreover, every trajectory $\gamma \in \Gamma$ will be endowed with an adjoint covector. At the end of the construction, we will have $\Gamma \equiv \Gamma_{\mathcal{A}}(\Sigma, 1)$.

Let $\tilde{F}$ be the union of the elements of the equivalence class of F-edges described in ($\mathcal{G}4$). Consider the connected components of the complement, in $\mathbb{R}^2$, of the union of F-edges of $\mathcal{G}$. There is only one such component R that is contained in the region enclosed by $\tilde{F}$ and such that $\tilde{F} \subset Cl(R)$. We have to construct Σ only on R.

From ($\mathcal{G}2$), we have that there is one origin O and O will be also the origin for Σ. It is clear that, possibly traslating $\mathcal{G}$, we can assume that O is the origin of $\mathbb{R}^2$. Consider a differentiable change of coordinate such that η^+ corresponds, in the new coordinates, to the line $\{(x_1, x_2) : x_2 = 0, 0 \leq x_1 \leq a\}$ for some $a > 0$. We define the field Y to be the constant field $(1, 0)$ on a neighborhood N^+ of η^+ that contains only the points of $\mathcal{G}$ that are in η^+. Since η^+ is admissible there exists a function $\tilde{\theta}$ such that the points $\{t_i, t'_i, s_i, s'_i\}$ of (3.3) determine the same sequence of frame points of η^+. We can define a vector field $G(x_1)$ on N^+ such that the function θ of (3.1) verifies $\theta(t) = \tilde{\theta}(t)$. This is easy because from the definition of Y we have that $\theta(t) = \theta(x_1) = arg(G(0), G(x_1))$. Since the synthesis is determined by the sequence of maxima and minima of θ and not by the values at these points, we can assume that $|\theta| < \pi/2$. Therefore, if $G(x_1) = (\alpha(x_1), \beta(x_1))$ then $\alpha > 0$ and $\nabla\Delta_B \cdot X = (1 - 2\alpha)\ \nabla\Delta_B \cdot Y$. Choosing θ we determine the direction of the vector G, but not its module. Hence, we can choose α in such a way that $\nabla\Delta_B \cdot Y, \nabla\Delta_B \cdot X$ have the same, resp. the opposite, sign at the points $p_i = \gamma^+(t_i), p'_i = \gamma^+(t'_i)$ if at the corresponding points of η^+ there is a C-edge, resp. a S-edge. From (III) of Proposition 3.1, we have that there is a canonical correspondence at the points p_i, p'_i. We can again modify θ, G in such a way that $\dot{\Gamma}_i(0)$, $i > 1$, see (3.23,24),(SA7), lies in the cone determined by $Y(q_i), G(q_i)$, $q_i = \gamma^+(s_i)$. Choosing the module of $G(q_i)$ in a suitable way, we can assume that X, Y point to the same side, resp. to opposite side, of Γ_i if at the corresponding points of η^+ there is a C-edge, resp. a K-edge. We can repeat the same construction for $q'_i = \gamma^+(s'_i)$. Finally, possibly changing θ, G, we can assume that $\delta > 0$, resp. < 0, see (SA8) for the definition of δ, if at the point of η^+ corresponding to q_1 there is a C-edge, resp. a K-edge. We repeat the same argumentations for q'_1. Now, from Proposition 3.1,3.2,3.3, we have that η^+ corresponds to γ^+ in the canonical way. Since we have defined Y and G the system Σ is determined.

Now consider η^- and a change of coordinate as for η^+. Possibly restricting N^+, we can define X and G on a neighborhood N^- of η^-, that coincide on N^+ with the previous definition and such that γ^- correspond to η^- in the canonical way. In this way, we have defined Σ on $N^+ \cup N^-$, that is a neighborhood of $\eta^+ \cup \eta^-$. We define $\Gamma = \Gamma_{\mathcal{A}}(\Sigma, 1)$ on $N^+ \cup N^-$ and to every $\gamma \in \Gamma$ we associate the covector field constructed by $\mathcal{A}$.

Now, let x' be a point of $\mathcal{G}$ that is not in $N^+ \cup N^-$. From ($\mathcal{G}1$), there exists a frame point x, corresponding to x', that is of one of the types classified in chapter two. We have shown, in chapter two, an example for every classified point, hence there exist a system $\Sigma(x')$, a synthesis $\Gamma(x')$ both defined on an open set $U(x')$ and a frame point $x \in \Gamma(x')$ that correspond to x' in the canonical way. Consider an open neighborhood U' of x' that does not contain any other frame point and define a diffeomorphism $\varphi : U(x') \to U'$ in such a way that φ sends frame points and curves to corresponding points and edges. Moreover, φ sends constructed trajectories to corresponding lines. Using φ, we define Σ and Γ on U' and we associate a covector field to every $\gamma \in \Gamma$.

From ($\mathcal{G}3$) it follows that every C-edge E is admissible. However, it can happen that if x', y' are the points belonging to E, the function Δ_A, Δ_B do not have the right signs on $U'(x'), U'(y')$. If $\Sigma(x') = (F, G)$ is one of the system of the examples of chapter two, we can consider the systems $\Sigma_1 = (F, -G), \Sigma_2 = (-F, G), \Sigma_3 = (-F, -G)$. Let Δ_A^i, Δ_B^i be the functions Δ_A, Δ_B for Σ_i. We have that $\Delta_A^1, \Delta_A^2 = -\Delta_A; \Delta_A^3 = \Delta_A; \Delta_B^1 = \Delta_B; \Delta_B^2, \Delta_B^3 = -\Delta_B$. The systems Σ_i have the same kind of synthesis of Σ (choosing the dual vectors in a suitable way). Therefore we can define $\Sigma(x'), \Sigma(y')$ in such a way that the functions Δ_A, Δ_B have the right signs.

Now, we define Σ on neighborhoods of frame curves. Let E be a frame curve, not of X or Y type, connecting the points x', y'. We choose a differentiable change of coordinates φ in such a way that E correspond to the line $\{(x_1, x_2) : x_2 = 0, 0 \leq x_1 \leq a\}$ for some $a > 0$. If E is of C, S or K type then we define φ in such a way that the vector field Y (defined on $U'(x') \cup U'(y')$) correspond to the vector field $(0,1)$. If E is of F type and the region on one side of F is positive then again we let Y correspond to $(0,1)$, otherwise we let X correspond to $(0,1)$. For each type of curve we have shown an example in chapter two. We choose the system $\Sigma(E)$ that gives an example of frame curve D of the same type of E and is defined on an open set $U(E)$. If E is of C type, we can choose $\Sigma(E)$ in such a way that Δ_A, Δ_B have the right sign, i.e. compatible with the systems $\Sigma(x'), \Sigma(y')$. We define a diffeomorphism $\varphi' : U(E) \to U'(E)$, where $U'(E)$ is a neighborhood of E, in such a way that φ' establish a canonical correspondence between D and E and its differential $d\varphi'$ send the vector fiels Y or X onto the vector field $(0,1)$, following the same rules used for φ.

We now glue toghether the systems defined near points and edges. Let V_1, V_2 be two open neighborhoods of x' verifying $Cl(V_1) \subset V_2 \subset Cl(V_2) \subset U'(x')$ and consider a function smooth $h_{x'}$ defined on $U = U'(x') \cup U'(y') \cup U'(E)$ such that $h_{x'} \restriction V_1 \equiv 1$, $h_{x'} \restriction U \setminus V_2 \equiv 0$. We define $h_{y'}$ in the same way for y'. Let $(F', G'), (F'', G'')$ be the vector fields already defined on $U'(x') \cup U'(y'), U'(E)$ respectively and define them to be zero elsewhere in U. We set:

$$\tilde{F} \doteq (h_{x'} + h_{y'})F' + (1 - h_{x'} - h_{y'})F'', \qquad \tilde{G} \doteq (h_{x'} + h_{y'})G' + (1 - h_{x'} - h_{y'})G''.$$

In this way we have defined a system $\tilde{\Sigma} = (\tilde{F}, \tilde{G})$ on U. Since the syntheses corresponding to $\Sigma(x'), \Sigma(y')$ and $\Sigma(E)$ coincide in the set of intersections, Γ is well defined on U. However, if E is of C or of S type, it can happen that in the set where $h_{x'}, h_{y'} \neq 0, 1$, $\tilde{\Sigma}$, in particular $\tilde{\Delta}_A, \tilde{\Delta}_B$, have not the right properties.
Consider first the case in which E is an S-edge. Assume that, in some coordinate chart, $E = \{(x_1, x_2) : x_2 = 0, 0 \leq x_1 \leq a\}$ has the orientation of increasing x_1 and that from E start Y-trajectories that enter the half plane $\{(x_1, x_2) : x_2 > 0\}$. In this case we have: $\tilde{X}_2 < 0, \tilde{X}_1 > 0, \tilde{G}_1 < 0, \tilde{\Delta}_A > 0$. We define a new system Σ by setting $Y \doteq \tilde{Y} + (0, \alpha), X \doteq \tilde{X}$ where $|\alpha| < 1, supp(\alpha) \subset (U'(E) \cap U'(x')) \cup (U'(E) \cap U'(y'))$. We have $\Delta_A = (1/2)(1 + \alpha)\tilde{X}_1 > 0$. If $\alpha(x_1, 0) \equiv 0$ then, after straightforward calculations, we obtain:

$$\Delta_B(x_1, 0) = \frac{1}{2}\left(2\tilde{\Delta}_B + \frac{\partial \alpha}{\partial x_2}\tilde{G}_1\tilde{X}_2\right) \tag{4.1}$$

and then we can choose $(\partial\alpha/\partial x_2)(x_1, 0)$ in such a way that $\Delta_B(x_1, 0) \equiv 0$. Moreover:

$$\nabla\Delta_B(x_1, 0) = \nabla\tilde{\Delta}_B(x_1, 0) + \Theta_1 + \Theta_2 \qquad \Theta_1 = \begin{pmatrix} 0 \\ \frac{\partial^2 \alpha}{\partial x_2^2}\tilde{G}_1\tilde{X}_2 \end{pmatrix}$$

$$\Theta_2 = \frac{\partial \alpha}{\partial x_2}\left[\nabla(\tilde{G}_1\tilde{X}_2) + \begin{pmatrix} 0 \\ \tilde{G}_2\frac{\partial \tilde{X}_1}{x_2} - \tilde{G}_1\frac{\partial \tilde{X}_2}{x_2} - \frac{1}{2}[\tilde{X}, \tilde{Y}]_1 \end{pmatrix}\right] + \begin{pmatrix} \frac{\partial^2 \alpha}{\partial x_1 \partial x_2}\tilde{G}_1\tilde{X}_2 \\ \frac{\partial^2 \alpha}{\partial x_2 \partial x_1}\tilde{G}_1\tilde{X}_1 \end{pmatrix}$$

hence Θ_2 is determined by the previous choices but we can define α choosing:

$$\frac{\partial^2 \alpha}{\partial x_2^2}(x_1, 0)$$

in such a way that $\nabla\Delta_B(x_1, 0) \neq 0$. From the compactness of E, it follows that there exists a neighborhood U' of E such that $\{x \in U' : \Delta_B(x) = 0\} = \{(x_1, x_2) : x_2 = 0\}$. Then we consider Σ restricted to U'.
Consider now the case in which E is a C-edge. Assume that from E start Y-trajectories that enter the half plane $\{(x_1, x_2) : x_2 > 0\}$ and that $\tilde{X}_1 > 0$ ($\tilde{X}_2 > 0$ follows from $\tilde{Y}_2 > 0$).

Again we define $Y = \tilde{Y} + (0, \alpha), X = \tilde{X}$. If we set $\alpha(x_1, 0) = 0$ then (4.1) holds and we can choose $(\partial\alpha/\partial x_2)$ in such a way that $\Delta_B(x_1, 0) \neq 0$. Again by the compactness of E, there exists a neighborhood U' of E in which Δ_B does not vanish. We consider Σ restricted to U'.

Finally, we want to associate to every trajectory γ of Γ a covector field. If γ is contained in $V_1(x')$ or $V_1(y')$ or in $U'(E) \setminus (V_2(x') \cup V_2(y'))$ (see the definitions above), we can associate a dual variable to γ using φ or φ', because γ corresponds to a trajectory of the synthesis of $\Sigma(x')$ or $\Sigma(y')$ or $\Sigma(E)$. Otherwise assume that γ verifies $\gamma(t_x) = x \in E \setminus \partial E$. If E is an F- or K-edge and γ is a Y-trajectory, resp. X-trajectory, then we choose λ_γ such that $\lambda_\gamma \cdot G(x) > 0$, resp. < 0. If E is an S- or a C-edge and γ is a Y-trajectory, resp. X-trajectory, after t_x then we choose λ_γ in such a way that $\lambda_\gamma \cdot G(x) = 0$ and $\lambda_\gamma \cdot [F, G](x) > 0$, resp. < 0. We associate to γ the adjoint variable that verifies $\lambda(t_x) = \lambda_\gamma$. It is clear, from Lemma 3.2 of chapter one, that if $\gamma(I)$ is not a turnpike for every $I \subset Dom(\gamma)$ then (γ, λ) satisfies the PMP on some neighborhood of t_x. Assume now that $\gamma(I)$ is a turnpike, $I = [a, b]$. Let φ_S be the control defined in (3.18) of chapter one and consider the system:

$$\begin{cases} \dot{x} = F(x) + \varphi_S(x) G(x) \\ \dot{\lambda} = -\lambda \cdot (\nabla F(x) + \varphi_S(x) \nabla G(x)) \end{cases} \tag{4.2}$$

and the following submanifold of $\mathbb{R}^4$:

$$Z = \{(x, \lambda) : \lambda \cdot G(x) = 0\}.$$

From the definition of λ, we have $\lambda(b) \cdot G(\gamma(b)) = 0$. Since $\Delta_B(\gamma(t)) = 0$ fot $t \in [a, b]$, from Lemma 3.5 of chapter one, we have:

$$\lambda(t) \cdot G(\gamma(t)) = 0 \quad \Rightarrow \quad \frac{d}{ds}\Big(\lambda(s) \cdot G(\gamma(s))\Big)\Big|_{s=t} = 0.$$

From the theory of existence on closed set, we obtain the existence of a solution (x, μ) that verifies $x(b) = \gamma(b), \mu(b) = \lambda(b)$ and $(x(t), \mu(t)) \in Z$ for every $t \in [a, b]$. Since the system is lipschitzian, there is a unique solution for every initial data. Hence $\lambda(t) \cdot G(\gamma(t)) = 0$ for every $t \in [a, b]$. We conclude that (γ, λ) satisfies the PMP.

From the compactness of E there exists a neighborhood U'' of E such that every $\gamma \in \Gamma$ restricted to U'' is extremal. We consider Σ restricted to U''. In this way we have defined Σ, Γ on an open set that contains all frame points and curves.

For every region $A \subset R$ let $B_i(A)$, $i = 1, \ldots, n(A)$, be the connected components of $A \setminus L(A)$, where $L(A)$ is the union of lines in A. Let $\mathcal{B}$ be the set of all $B_i(A)$, $i = 1, \ldots, n(A)$, as A ranges over the set of regions contained in R. We will define Σ on every B

by induction. From $(\mathcal{G}5)$ we have that every $Cl(B)$, $B \in \mathcal{B}$, contains exactly one entrance $E(B)$. The induction hypotheses is that for every $x \in E(B)$ there exists $\gamma_x : [0, t_x] \to I\!R^2$, $\gamma_x \in \Gamma$, such that $\gamma_x(t_x) = x$, i.e. the system Σ is constructed along γ_x backward in time. We start defining Σ on the regions B for which $E(B)$ is of X or Y type. Then we consider the regions B such that on the region B', that lies on the other side of $E(B)$, the system Σ is already defined. If $E(B)$ is of S type and x is the initial point of $E(B)$, then we consider B if there is a trajectory γ_x that verifies the induction hypothesis. In a finite number of steps we define Σ on every $B \in \mathcal{B}$.

Fix, now, a region $B \in \mathcal{B}$ and assume that the induction hypothesis holds. From $(\mathcal{G}5)$ we have that $Cl(B)$ contains exactly one entrance E_1 and one exit E_2. If $E_1 \sim E_2$ then B is enclosed by E_1, E_2 and a line l or a side E_3. Otherwise, B is enclosed by E_1, E_2, a line l_1 and another line l_2 or a side E_3. We define Σ on B defining Y or X, and G. Indeed, we define Σ also on a neighborhood of the lines in B if the system is not already defined near these lines. Consider the case $E_1 \sim E_2$ and assume that B is positive, being similar the other case. Possibly using a change of coordinates, we can assume that $E_1 = \{(x_1, x_2) : x_1 = 0, 0 \leq x_2 \leq a\}$, $E' = \{(x_1, x_2) : x_2 = 0, 0 \leq x_1 \leq b\}$, where $E' = l$ or $E' = E_3$, and that Y is the constant vector field $(1, 0)$. Obviuosly we can define $Y \equiv (1, 0)$ on B and let Γ be formed by Y-trajectories, but we have to make some modifications to ensure that every $\gamma \in \Gamma$ is extremal.

Consider $\gamma_y \in \Gamma$ that verifies $\gamma_y(t_1) = (0, y)$, $\gamma_y(t_2) \in E_2$. By the induction hypothesis such a trajectory γ_y exists defined on $[0, t_2]$ for every $y \in [0, a]$. Since we have already defined Σ on a neighborhood of $E_1 \cup E_2$, there is a covector field λ_y associated to γ_y that is defined on $I = [t_1, t_1 + \mu_1] \cup [t_2 - \mu_2, t_2]$ for some positive μ_1, μ_2. It can happen that $t_1 + \mu_1 = t_2 - \mu_2$, e.g. if we are near the point $E_1 \cap E_2$. We want to define Y in such a way that we can associate to γ_y a covector field, defined om $Dom(\gamma_y)$, that coincide with λ_y on I. This will ensure, choosing G in a suitable way, that every γ_y is extremal.

Consider a region $\Omega = [\delta_1, \delta_2] \times [\varepsilon, a - \delta]$, $\delta > 0$, $0 < \delta_1 < \delta_2$ such that the following holds. We have $\Omega \subset A$, where A is the region containing B, and $\Omega \cap (E_1 \cup E_2) = \emptyset$. Let B' be the region on the other side of E'. If Σ is already defined on B' then $\varepsilon > 0$ otherwise $\varepsilon < 0$. Notice that if E' is a side then it is of X or of Y type and the former hold. We choose ε, δ in such a way that Σ is already defined on $B \cap \{(x_1, x_2) : a - 2\delta \leq x_2 \leq a\}$ and if $\varepsilon > 0$ then Σ is already defined on $B \cap \{(x_1, x_2) : 0 \leq x_2 \leq 2\varepsilon\}$. For every $y \in [\varepsilon, a - \delta]$, let $\gamma_y^1 \in \Gamma$ be the trajectory that verify $\gamma_y^1(t_1(y)) = (0, y)$ and let λ_y^1 be the covector field associated to γ_y^1. We have that γ_y^1, λ_y^1 are defined on a neighborhood of $t_1(y)$. Consider a

Mayer problem with final target E_2 and a cost function:

$$\psi(T, x(T)) = -T + \psi_0(x(T)),$$

depending on terminal point and time. We want to maximize ψ. For every $y \in [\varepsilon, a-\delta]$ let $x(y)$ be such that $(x(y), y) \in E_2$. There exists $\gamma_y^2 \in \Gamma$ that reach $(x(y), y)$ with an associated covector field λ_y^2. Let $t_2(y)$ be such that $\gamma_y^2(t_2(y)) = (x(y), y)$. We have that γ_y^2, λ_y^2 are defined on a neighborhood of $t_2(y)$. We define ψ in such a way that $(\gamma_y^2, \lambda_y^2)$ satisfies the PMP and the final transversality condition for the Mayer problem, see [5]. Choose ν_1, ν_2, T_1, T_2 such that $\delta_1 < \nu_1 < \nu_2 < \delta_2$ and:

$$T_1 > \sup\{t_1(y) : y \in [\varepsilon, a-\delta]\} \qquad T_2 < \inf\{\psi_0((x(y), y)) : y \in [\varepsilon, a-\delta]\}.$$

We define $Y = (\alpha, 0)$ on Ω, α continuous and positive, $\alpha \equiv 1$ on $\partial\Omega \cup [\nu_1, \nu_2] \times [\varepsilon, a-\delta]$, and we let $Y = (0, 1)$ outside Ω. We choose α in such a way that the following holds. For every y we have $\gamma_y^1(T_1) = (\nu_1, y)$. If $T_2(y) < t_2(y)$ is the time in which γ_y^2 reaches, backward in time, the point (ν_2, y) then $\psi(t_2(y) - T_2(y), (x(y), y)) = T_2$. With this definition of Y we prolongue $\gamma_y^{1,2}, \lambda_y^{1,2}$ defining them on the whole B. Consider the reachable set $R(T_1)$, we have that $\{(\nu_1, y) : \varepsilon \leq y \leq a - \delta\} \subset \partial R(T_1)$. Since $\lambda_y^1(T_1)$ has to be perpendicular to $\partial R(T_1)$, it follows that $\lambda_y^1(T_1)$ has the second component equal to zero. From Theorem 8.2 of Chapter IV of [5], we have that λ_y^2 has to be perpendicular to the level set of the function:

$$\psi'(x, y) = \psi(t_2(y) - t(x, y), (x(y), y)),$$

where $t(x, y)$ is defined by $\gamma_y^2(t(x, y)) = (x, y)$. Hence also the second component of $\lambda_y^2(T_2(y))$ has to be zero. From the PMP, since the Hamiltonian is positive (see chapter one, section 3), we have that the first components of $\lambda_y^1(T_1), \lambda_y^2(T_2(y))$ have the same sign. Since $\alpha = 1$ on $[\nu_1, \nu_2] \times [\varepsilon, a-\delta]$, we obtain that $\lambda_y^{1,2}$ coincide up to the moltiplication by a scalar. We associate to every γ_y^1 the covector field λ_y^1. We can define G in such a way that G is of class C^3 and every γ_y^1 is extremal. It can happen, however, that α is not C^3 and then Σ is not C^3. Since α is continuous there exists a sequence $\alpha_n \in C^3$ converging uniformly to α. Let Σ_n, Γ_n be the system and synthesis associated to α_n. If E_2 is of K or F type then for n sufficiently large every $\gamma \in \Gamma_n$ is extremal and we are done. If E_2 is of C type then Σ_n has a switching curve E_n near to E_2. Since Σ has not already been defined on the region B' that lyes on the other side of E_2, we can define $\Sigma = \Sigma_n$ for n sufficiently large.

The other case, that is when B is enclosed by E_1, E_2 and two lines or one line and one side, can be treated in an entirely similar manner. This concludes the construction on the regions $B \in \mathcal{B}$ and then we have defined Σ and Γ on R.

We can again modify Σ on the regions $B \in \mathcal{B}$, using the same techniques described above, in such a way the following holds. If $\gamma \in \Gamma$ reaches $Fr(R)$ then it reaches $Fr(R)$ at time 1. If x belongs to an overlap curve, $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1(t_1) = x = \gamma_2(t_2)$ then $t_1 = t_2 \leq 1$, with equality holding only if $x \in Fr(R)$. By a dynamic programming argument, we can conclude that every $\gamma \in \Gamma$ is optimal and then $R = R(1)$, the reachable set in time 1 for Σ. We can construct a synthesis from Γ, that we call again Γ, following the procedure described in section 5 of chapter one. We obtain $\Gamma = \Gamma_{\mathcal{A}}(\Sigma, 1)$. From the construction it is clear that $\mathcal{G}$ corresponds to Σ in the canonical way.

We conclude proving structural stability of classified systems. This implies that the classified systems are *observable*, that is a small perturbation does not change the topological structure of the optimal synthesis.

Theorem 4.3 *If $\mathcal{A}$ succedes for Σ at time 1 then Σ is structurally stable.*

Proof. We have that Σ is locally controllable and satisfies $(P_1), \ldots, (P_8)$ of section 4 of chapter one. Let Σ' be a system in a small neighborhood of Σ. More precisely, assume that:

$$\|F - F'\|_{\mathcal{C}^3} + \|G - G'\|_{\mathcal{C}^3} < \varepsilon$$

where $\Sigma' = (F', G')$. From Lemma 2.4 of chapter one, we know that Σ' is locally controllable if and only if:

$$F'(0) \wedge [F', G'](0) \neq 0.$$

But we have that:

$$F(0) \wedge [F, G](0) \neq 0$$

and

$$|F'(0) \wedge [F', G'](0) - F(0) \wedge [F, G](0)| \leq 2\varepsilon(\|F\| + \|G\|),$$

hence for ε sufficiently small we have that Σ' is locally controllable.
The conditions $(P_1), \ldots, (P_8)$ involve the components of the vector fields (F, G) and their derivatives and then can be established for Σ', if ε is sufficiently small, in the same way. Now, we can apply the algorithm $\mathcal{A}$ to Σ'. The conditions (SA1)-(SA8) and the stability conditions for frame points and curves hold for Σ. An iterative application of the inverse function theorem guarantees that, for ε sufficiently small, $\mathcal{A}$ produces the same frame curves, except for K-curves, and the same frame points, except (K, K) points, for Σ' and these frame curves and points verify the stability conditions as well.

For each frame curve $D \in \Gamma_{\mathcal{A}}(\Sigma, 1)$ of type C or S, we have an ordered sequence of points $x_1, \ldots, x_n \in D$ that are in relation with some not (K, K) frame point. These are exactly the points considered in the construction of lines of $\mathcal{G}$. From $(\mathcal{A}_2)$, we have that for ε sufficiently small the corresponding frame curve D' of Σ' have the same number of distinct points y_i that have the same property of x_i and are in the same order. Hence we have that $(\mathcal{A}_1), (\mathcal{A}_2)$ are fullfilled and then $\mathcal{A}$ succedes for Σ' at time 1. Moreover, the graphs $\mathcal{G}, \mathcal{G}'$ canonically associated to Σ, Σ' respectively, are equivalent and then by Theorem 4.1 Σ and Σ' are equivalent.

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