FREE CONFORMAL FIELD THEORIES ON ALGEBRAIC CURVES AND THEIR APPLICATIONS

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I touched in this work the mostly unexplored world of conformal field theories on nonhyperelliptic algebraic curves. This was a risky and thankless task: inevitably new problems arise and the old ones appear from a different point of view: by definition, if there is not a previous knowledge in a field, then there is also not so much interest in it. Therefore I was tempted many times to expect results serving more established fields. I am for this reason bound to gratitude to my supervisor, Galen Sotkov, who followed my work with interest and who perused the entire manuscript. The results I present here owe so much to his exceptional combination of expertise in conformal field theory, algebraic geometry and superstring theories. Finally and most of all, I thank him for his friendship.

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INTRODUCTION

In this thesis we present some results of our studies on free conformal field theories defined on algebraic curves. Despite their simplicity, free fields cover a relevant role in contemporary physics. Their most important and genial application are probably superstrings [Sch, Gre], which can be expressed in terms of free scalar fields and free fermions. We deal essentially with the Polyakov formulation of superstrings [Pol] which encountered the favour of a large part of the scientific community. In the last few years we assisted to a rapid progress in its development and to a considerable amount of promising results. As it is well known superstrings are the only theory which at the moment seems to conciliate the gravitational forces with the strong and electroweak forces. Moreover there are basic clues which indicate that superstrings are finite at any order of their perturbative expansion [Gre]. Even if in the future new theories will arise and they will be closer to the "true" description of the high energy world, surely this will not spoil our current belief that they should contain something of fundamental in this description. For example just a fundamental constant appears in string theories, namely the string tension, in terms of which, via the various schemes of compactification, it is possible to derive all the other constants. Also, they satisfy the axiom of self-duality of the amplitudes which is not verified in the usual pointlike field theories. Unfortunately, or fortunately depending on the points of view, the perturbative expansion in the theory of strings involves the definition of free fields on Riemann surfaces. This entails the use of mathematical quantities like theta functions and prime forms [Fay] which are not perfectly under control from the physical point of view. Therefore, despite many efforts, we have still far from a thoroughly understanding of string perturbation theory at high loop. In this sense we are compelled to investigate new techniques for the development of two dimensional field theories on Riemann surfaces.

The most widely used approach certainly consists in expressing the partition functions and the correlators of the fields through theta functions and prime forms [AS, Ver, BI]. The big problem in this case lies in the fact that the geometrical

informations about the Riemann surface, the so called moduli, appear in the theta functions through the period matrix. In the space of period matrices, known as the Siegel upper half plane, the moduli space is embedded in a non trivial way when the genus of the surface increases beyond $g = 3$. This makes it hard to find the moduli's dependence inside the period matrix. This is the so called Shottky problem wich has been solved only recently by mathematicians [Mul] in a very unexplicit way. In superstrings we are faced moreover with the demand to have explicitly modular invariant amplitudes and, besides, it would be nice to study their factorization properties when a non-trivial homology cycle is pinched. A satisfactory development of these points exists until now only up to genus two, using a simple class of algebraic curves [GIS,CG]. Finally a non-negligible trouble in using theta functions is the "computability" because, after all, we wish to extract numbers form a physical theory. We focus the problem with an example. In both the correlation functions [Knil] and Green functions of free conformal field theories appear the holomorphic differentials. They can be only implicitly defined through theta functions by means of the prime form [Fay] or equivalent mathematical constructions. But prime form contains in its definition the holomorphic differentials as well and so an endless cycle starts. An alternative approach to theta functions is provided by the Shottky parametrization of moduli space. First attempts of physical applications where performed by Mandelstam and al. [Man] in deriving an expression for the off-shell vertices in string field theory. Shottky parametrization is well suited for studying the factorization properties of scattering amplitudes [Mar], but it is much less brilliant in producing results which lead to a clear physical interpretation. A very promising formalism which makes use of the Shottky parametrization of moduli space is provided by N-reggeonic amplitudes [PS,DiV]. The approach however contains many technical points such as the procedure of paints sewing and the connection with theta functions approach is very unclear. Moreover it produces cumbersome results which do not lean to a good computability. Finally in both cases the region of integration over the Shottky parameters which should give the sum over the classes of conformally inequivalent Riemann surfaces is not known.

Other methods, perturbative or not, have been applied during the story of

strings. Unfortunately, either they are limited to one loop [Ale] or they are lacking in solving the problems exposed above. We refer the interested reader to the existing literature [FS,AGMV,KN]. In any case there no one of them gets rid of the Shottky problem in perturbation theory or it is able to yield numbers.

For all these reasons in this thesis we study free field theories on Riemann surfaces from a completely different perspective, i.e. using algebraic curves [GH] and parametrizing the space of moduli in terms of branch points. This approach takes origin from ref. [SJM] in which statistical models, holonomic quantum fields and the Riemann Hilbert problem were treated. In this work a set of differential equations for the partition functions at loop g in terms of branch points was obtained in a very general and mathematical way for certain statistical models. The physical applications of these ideas were firstly developed by Zamolodchikov [Zam] who showed the equivalence between the determinant of the scalar laplacian on a hyperelliptic curve and a correlation function of some spinning conformal fields. Soon after Bershadsky and Radul [BR1,BR2,BR3] and Knizhnik [Kni] formulated simultaneously a string-suited method to compute the Green functions and correlation functions of free fields on hyperelliptic curves and Z_n symmetric curves. The main idea behind their approach is that the branch points can be simulated by certain primary fields called twist fields with relatively simple operator product expansions (OPE) with the fields.

Their results are not only relevant for string theories on higher genus surfaces but also for conformal field theories on the complex plane. In fact we know that the conformal blocks of these theories have non-trivial monodromy properties around certain critical points as much as the correlation functions of b-c systems on algebraic curves around branch points. For example there is a correspondence between the Green functions of free fields on Z_n symmetric surfaces and correlation functions on Z_n orbifolds [DFMS]. In view of a classification and construction of conformal field theories it may be therefore useful to study free fields on surfaces of general monodromy even if, in this respect, the monodromy groups provides a coarser description than the braid groups [Arn]. Moreover in [CSS] it is showed how to obtain (in the hyperelliptic case) the correlation functions of minimal models using the

Coulomb gas representation. Unfortunately the method of twist fields, apart from the hyperelliptic and Z_n symmetric curves, is not valid. It fails when we apply it to more general cases, like fermions or b-c system coupled with an external gauge field which gives non trivial monodromy properties around the homology cycles of the Riemann surface. It is not very well understood for example how to construct the two-point functions for the twisted bosons of [Mik] or the usual fermions even on Z_n symmetric surfaces.

The motivations to go beyond Z_n symmetric surfaces despite the serious difficulties, stem out partly from the works of [GIS,IZ,CS,CGS,Mon,LM] in which perturbative string theory is systematically developed up to genus two included with the modular invariance explicitly realized, partly from the asymptotic expansion method of [GM] to derive high energy string amplitudes. In addition we assisted in the last two years to a warm interest in the subject of algebraic curves with the most various applications: [SU,MMS,Mor,La,Gav,Smi,Zve] and many others.

The generalization from hyperelliptic to general curves has revealed unfortunately non-trivial and still a full application of the formalism we developed requires further developments. The main complication lies in the fact that on arbitrary curves the monodromy properties at the branch points become involved and it is difficult even to establish the form of their Riemann representation in terms of sheets and branch points. Consequently also the twist fields which simulate the branch points must be complicated. Unfortunately this fact prevent us from finding the twist fields explicitly [Fer3]. The same considerations are valid for twisted bosons and fermions: in this case, even on Z_n symmetric curves, the non-ramified covering of the curve on which these fields are single-valued is no longer Z_n symmetric [Fer 1]. Consequently its representation in terms of sheets and branch points is not simple and the twist fields, which are still primary fields, cannot be computed. Finally, despite Riemann surfaces represented as n-sheeted coverings of CP_1 were invented 150 years ago by Riemann, still many aspects of the theory are under study. This is for example the case of non-ramified coverings of algebraic curves which are so important for the understanding of spin structures and fermions (see for example [Hor] for the hyperelliptic curves and refs. therein and [Fay] for the

general case). Also the problem of determining the number of moduli on curves and the dependence of the period matrix to them was solved just thirty years ago by Rauch [Rau]. Despite these difficulties, we hope that the good results obtained in [IGS,IZ,CG,CGS,Mon,ML] and [GM] will stimulate further research in thhis direction. Our work is an attempt to generalize the resultss obtained by the authors above on hyperelliptic curves only, which unfortunately cover the entire moduli space only up to genus two. Perhaps too optimistically, we believe that we succedeed in this task for bosonic fields on higher genus Riemann surfaces ($g = 3, 4$ at least) and for the superversion of the two dimensional electrostatic of [GM]. Moreover the studies performed by us on the spin structures carried by fermions seems to be promising in the direction of a full development of superstrings and fermionic (free) conformal field theories on general algebraic curves. Since the latter are conformally equivalent to all abstract compact and oriented Riemann surfaces, these ideas are appealing for applications in string perturbation theory [Fer2].

The picture emerging from our work is that free conformal field theories on general algebraic curves present an intermediate complexity between the simple hyperelliptic curves and the formalism of theta functions. The amplitudes we obtain are free from Shottky problem but is difficult to go beyong genus 4. This is because it is not easy to find explicit algebraic equations for curves of genus $g > 4$ which cover a submanifold of dimension $3g-3$ in the moduli space. The computability we obtain is very good and using numerical algorithms it is possible to extract numbers from the theory. In expressing analytically the scattering amplitudes we often encounter the problem of solving polynomials of high degree, whose solutions is possible just numerically. This is not however stumbling: also in the usual pointlike field theories in fact very complicated functions appear [Thl]. Moreover algebraic curves provide the simplest possible representation of Riemann surfaces known up to now and therefore they seem to be a good laboratory to study remarkable properties of two dimensional conformal field theories like bosonization and modular invariance of amplitudes. As was remarked previously, the study of free fields on general Riemann surfaces with involved monodromy properties can also provide links between them and non-trivial conformal field theories on the plane. Finally our formalism allows

a good comparison with analogous results expressed through theta functions.

Our approach in treating free conformal field theories on general algebraic curves is the following:

Chapter 1 is an introduction of the classical theory of algebraic curves following standard treatises on the subject like [Osg,For]. We apply triangulation [San] as an algorithm to find the homology cycles of generic algebraic curves with explicit examples for physically relevant Z_n surfaces. We provide also a detailed explanation of some Čech cohomological techniques [Fer1] which will be useful in the study of relative spin structures (this denomination is mutuated from [ABMN]). In chapter 2 we show how to derive the two point functions of free fields [Fer2] on a general algebraic curve following refs. [Ver,BI]. The procedure is valid on any curve on which it is possible to construct the third kind differentials. If the surface is non-degenerate, we can use for this task the algorithm of Weierstrass [Wei]. Using the two point functions we can derive a system of differential equations for the chiral determinants using the stress energy tensor method [AS,Ver]. In this way it is possible to compute the vacuum expectation value of the stress energy tensor and also to show that the branch points can be represented by primary fields even on curves which are no longer Z_n symmetric [Fer2]. We do the calculation explicitly in a particular case. In chapter 3 we study b-c systems on Z_n symmetric surfaces using bosonization techniques on the complex plane and the twist fields as in [BR1,Kni]. We discover that the twist fields can be constructed once we know a basis of n j-differentials with the appropriate monodromy properties near the branch points. It turns out that the two point functions of b-c systems are polynomial combinations of the elements of this basis. In this way we link the properties of the twist fields with the geometrical properties of the surface like the existence of certain kind of j- differentials. The method of j-differentials can be extended also to more general curves [Fer3]. We provide a simple but non-trivial example of a surface with non-abelian monodromy group for which we compute the two point functions. It is also shown as these j-differentials provide the OPE between the fields b-c and the twist fields inside the vacuum expectation values. Finally we show that there are strong connections between the j-differentials here derived explicitly and those of

refs. [BR3, SJM] which are defined through Painleve differential equations.

Chapter 4 is devoted to twisted bosons and fermions. We analyse the problem of finding the twist fields which realize the various Z_m spin structures on Z_n symmetric curves. Using the cohomological methods of chapter 1, we study what m-unramified coverings respect the Z_n symmetry of the surface and are Z_n symmetric. This is a powerful mathematical tool which can be applied to the study of the m-unramified coverings of algebraic curves. We are non-experts on the field of algebraic surfaces but we believe tha this point was never investigated in the mathematical literature apart hyperelliptic curves (See for example [Hor] and ref. therein for some review on this problem on hyperelliptic curves.) In this way we show that only the Z_n symmetric surfaces of ref. [BR2] can be expressed in terms of twist fields because in the other cases the coverings destroy the Z_n symmetry of the surface and the usual twist fields cannot take into account of this feature. Therefore we have to use more involved primary fields whose expression is still not known or the methods outlined in chapters 2 and 3. We give also some examples on how the j-differentials introduced in the previous chapter can be used to compute the two-point functions and the correlation functions of twisted bosons. Finally we derive the two point functions and chiral determinants for the important class of rational curves introduced in [GM].

Chapter 5 is a trial to develope the theory of free field theories on algebraic curves in its full generality. In the first section we show how it is possible to derive the chiral determinants explicitly using the prescription of [Knil]. We do not treat the chiral determinants of fermions because in this case we need the knowledge of theta constants. The problem is that it is not easy to find the dependence of theta functions on branch points. Section two is a long discussion of the variational methods of Rauch [Rau] in which we try to construct the partition functions at least for bosonic fields.

In appendix A we give an example of a general procedure to provide the Riemann representation of algebraic curves in terms of sheets and branch points

Appendix B contains a short review on the formalism of theta functions.

Appendix C reviews the formula of Cardano to express analytically the roots

of a polynomial of degree 3.

In appendix D we express the prime form in the language of branch points following [Fay].

CHAPTER 1.
RIEMANN REPRESENTATION OF 2-D SURFACES

1.1 ALGEBRAIC SURFACES,BRANCH POINTS AND BRANCH LINES

The Riemann surfaces have been widely used in current high energy physics. To get explicit results avoiding Shottky problem or to check factorization properties in string theories , one needs an explicit representation of such surfaces. The simplest representation is provided by algebraic surfaces. In general algebraic surfaces are projective varieties on the space CP_n [GH]. The algebraic surfaces considered in this work are instead n-fold ramified coverings of the complex plane $\mathrm{C} \cup \{\infty\} \equiv CP_1$. The symbol $\equiv$ denotes conformal equivalence between CP_1 and the complex plane with the point at infinity. It is known that an abstract (compact and orientable) Riemann surface is conformally equivalent to an algebraic surfece Σ associated to an algebraic equation of the kind:

$$F(z,y) = y^n + P_{n-1}(z)y^{n-1} + \ldots + P_0(z) = 0 \qquad (1.1).$$

w, z are complex variables, $P_i(z)$ represent rational functions of z only:

$$P_i(z) = \frac{Q_i(z)}{R_i(z)} \; Q_i, R_i \text{ being polynomials.}$$

A simpler but very important class of algebraic surfaces, (see chapters 3 and 4), is given by the Z_n symmetric surfaces Σ_n associated to the equation:

$$y^n - P_0(z) = 0. \qquad (1.2)$$

When $n = 2$ we obtain the hyperelliptic surfaces.

If we consider z as the independent variable, $y(z)$ turns out to be a multivalued function of z. Let us denote with $y^{(l)}(z)$, $(l = 0,\ldots,n-1)$, its branches. The **branch points** of $y(z)$, in which two or more branches are identified, are provided by the system of equations:

$$\begin{cases} F(z,y) = 0 \\ \partial_y F(z,y) = 0 \end{cases} \qquad (1.3).$$

When eq. (1.3) is verified for a given value of z together with the relation:

$$\partial_z F(z,y) = 0,$$

then z is called a **singular branch point**. We will suppose throughout this thesis that all branch points are regular. Eq. (1.3) implies that two or more branches of $y(z)$ become equal at branch points. The multiplicity ν of a branch point indicates the number of branches which coincide at that point. Transporting $y(z)$ along a small circle surrounding a branch point, the ν branches get interchanged. Σ is determined by the requirement that $y(z)$ is singlevalued on it. The Riemann representation of Σ consists in n replicas of the complex plane z, he so called **sheets**, joined in a appropriate way along the **branch cuts**. The latter are obtained cutting the sheets along certain **branch lines** which connect the branch points. The actual construction of Σ starting from eq. (1.1) is in the general case a very difficult problem. We refer the interested reader to classical treatises on complex analysis such as [For,Osg]. See also appendix A for an example. However let us mention at least two important classes of algebraic surfaces in which the problem is simplified. The first class consists in the Z_n symmetric surfaces. In figs. (1.1)-(1.4) we show the system of cuts valid for the most general Σ_n. As we see, comparing fig. (1.1) with fig. (1.2), the same algebraic equation leads to different systems of cuts. The fact is that once we have a system of cuts, the branch lines can still be deformed over the branch points, yielding in this way a different system of cuts. Stated in another way, let us build the following vector:

$$\vec{y}^{(l)} = (0,\ldots,1,\ldots,0)$$

with the l-th component equal to 1 and all the other put equal to zero. The change of branches when $y^{(l)}(z)$ is moved around a branch point a is then expressed by a matrix in the following way:

$$M_a y^{(l)} = y^{(s)} \qquad (1.4).$$

Eq. (1.4) means that if $y^{(l)}(z)$ is transported along a small circle enclosing the branch point a on the l-th sheet, the branch l interchanges with the branch s. Now

we can build a vector from the branches $y^{(l)}(z)$:

$$\vec{y}(z) = \sum_l \vec{y}^{(l)} y^{(l)}(z).$$

After performing a complete cycle around a branch point the transformation law of $\vec{y}(z)$ is $\vec{y}(z) \rightarrow M_a \vec{y}(z)$. The matrix M_a is called **the monodromy matrix** at the branch point a. The monodromy matrices at all branch points form the **monodromy group** of Σ. Since a circle which surrounds all the branch points on any sheet can be shrinked to a point because it is homotopically trivial, the monodromy matrices should satisfy the following constraint:

$$\prod_{i=1}^{n_{bp}} M_{a_i} = 1 \tag{1.5}$$

where n_{bp} is the total number of branch points. Now the action of deforming a branch line over a branch point a' (fig.(1.5)) entails a transformation of the monodromy matrices M_{a_i} for the branch points a_i belonging to that line of the folowing kind:

$$M_{a_i} \rightarrow M_{a'} M_{a_i} M_{a'}^{-1} = M'_{a_i}.$$

Clearly this transformation leaves condition (1.5) unaltered. The conclusion of this discussion is that once we have a reference system of cuts, all kind of transformations preserving condition (1.5) are allowed. An algebraic surface Σ is determined completely once the positions of the branch points and the monodromy matrices are specified. Let us go back to the problem of finding a system of branch cuts for Σ. The second case in which this problem turns out to be easy to solve consists in the surfaces for in which all the branch points are simple, i.e. their multiplicity ν is always equal to two. For this class of algebraic surfaces, which we denote with the symbol Σ_s, the following theorem of Lüroth [Lü] holds:

theorem: when the branch points of a Riemann surface with n sheets and genus g are all simple, the surface can be taken in such a form that there is a single branch line between consecutive sheets except for the last two sheets; and between the last two sheets there are $g+1$ branch lines fig.(1.6).

The relevance of the class of surfaces with all simple branch points lies in the fact that a "general" algebraic surface of genus g can be always put in this form via

a birational transformation [For]. The meaning of the adjective "general" will be explained in chapter 5. Here we anticipate just that a general surface depends on $3g-3$ parameters, functions of the coefficients of the algebraic eq. (1.1) called the moduli.

1.2 TRIANGULATION OF ALGEBRAIC SURFACES, REDUCTION TO THE CANONICAL POLYGON, HOMOLOGY CYCLES

In this section we wish to illustrate some useful technics in dealing with algebraic surfaces. First of all we provide a method to find the homology cycles once the branch cuts are given. This is useful in order to check the properties of monodromy of fermions and twisted bosons. Moreover in the case of scalar fields the Green functions involve integrals over the homology cycles (see chapter 2). The strategy consists in reducing the Riemann surface represented as an n sheeted covering to the canonical polygon (fig. (1.7)) using **triangulation**; from this configuration it is then easy to deduce the homology cycles.

A basic property of Riemann surfaces is their triangulability. Roughly speaking triangulation consists in dividing the surface in a finite (for closed Riemann surfaces) number of open sets isomorphic to a triangle [San]. Then we have to cut this triangles and to join them again like in a puzzle to build the canonical polygon. The tiangulation process takes place in three steps:

Step A): each sheet with cuts (fig. (1.8a))can be viewed as a sphere with holes (fig. (1.8b)).

When we stretch it on a plane, we get a polygon with vertices $\alpha_1 \dots \alpha_{n_{bp}-1}$ (fig. (1.8c)). In this figure we have opened the sphere at the point $\alpha_{n_{bp}}$ which in the polygon of fig. (1.8c) appears as n_{bp} identified points.

Step B): after repeating the procedure of step A) for all the n sheets, we obtain n of such polygons and we can join them along the identified cycles in an appropriate way. After a deformation, we can bring this new figure again to a polygon (fig. (1.9)). This is not of curse the canonical polygon. First of all we have to simplify the

adiacent sides as in fig. (1.10). Moreover the sides have as vertices the various points $\alpha_1 \ldots \alpha_{n_{b_p}-1}$. Instead they should not appear in the canonical polygon where all the vertices are identified. We cope with this situation applying the first kind of cuts showed in fig. (1.11) till all the unwanted vertices will disappear from the external sides.

step C): At the end of step B) it may happen that the polygon is still not canonical since the sides are not in the correct sequence $a_1 b_1 a_2 b_2 \ldots a_g b_g$ as in fig. (1.7). The second kind of cuts gets rid of this shortcoming fig. (1.12).

After a finite number of cutting and glueing we get therefore the canonical polygon and hence the a and b cycles. Examples of a and b cycles are given in figs. (1.13).

This procedure becomes complicated when the genus of the surface increases or there are many sheets and branch point. Fortunately this task is oversimplified in the case of surfaces in which the branch points are simple. Using Lüroth theorem they can be brought always in the form of fig.(1.14) with the system of a and b cycles showed.

Matrix representation of homology cycles

The purpose of this section is to set up a convenient notation to represent the homology cycles. In fact often the necessity arises to subtract and to sum the homology cycles, for example when discussing the modular invariance of the partition function. This becomes difficult when the Riemann representation of the surface becomes involved with many sheets and branch points.

The method we introduce is general but it is better suited for Z_n symmetric surfaces. To fix the ideas we use the curve $\Sigma(1,3,1,0)$ (see the notation of chapter 3, section 1) with the system of cuts of fig. (1.13). We draw all the possible oriented paths encircling the branch points as in fig. (1.15). These will be called elementary paths. All the homology cycles can be decomposed in terms of these elementary paths. To each path we associate a matrix A in the following way: the columns represent the branch points in the order c_1, c_2, c_3. The rows correspond to the

possible branches of $y(z)$ at those branch points. All the branches of a multivalued function can be characterized by a set of numbers which are roots of the unity. For example the branches of y such that $y^3 = (z - c_1)(z - c_2)(z - c_3)$ are characterized by

$$y^{(l)}(z) = \epsilon^l \sqrt[3]{(z - c1)(z - c2)(z - c3)}$$

with $\epsilon^3 = 1$. The elements of the matrix A associated to a closed path (in CP_1) are the increments or decrements to the variable l at each of the branch points which $y(z)$ encircles when transported along the path (see fig. (1.15)). Having a closed path C it is easy to construct the related matrix M_C. With the help of appropriate continuous deformations of C we can individuate all the elementary paths composing C and therefore:

$$M_C = \sum_{elem.paths} M_{C_{elem}}.$$

The converse is also true. Starting from M_C we can draw all the elementary paths encircling the branch points and then join them. In some cases there is an apparent ambiguity because the connection can be done in different ways. As a matter of fact all closed cycles coming from the same matrix M_C are equivalent and can be reduced to a common reference cycle. This matricial representation is additive, i.e. $M_{C_1+C_2} = M_{C_1} + M_{C_2}$ modulo trivial cycles. The zero element of this algebra is not unique: all cycles which can be deformed to a point are zero elements and the associated matrices need not to ber zero. The closed cycles (on Σ) are individuated by imposing the condition that the number of the lines entering branch lines at each sheet is equal to the number of outgoing lines.This fact allows to determine all independent matrices for a given Riemann surface and to construct the other cycles as linear combinations of them.

1.3 CECH COHOMOLOGY AND SPIN STRUCTURES

Let us divide the algebraic surface in the Riemann representation into a set of local patches $\{U_\alpha\}$. A prototype of this set is given in fig. (fig. (1.16)). This

covering can be easily generalized to all algebraic surfaces constructed in terms of sheets and branch points. To U_α we assign a local set of coordinates z_α and consider local trivializations $\phi_\alpha(z_\alpha)dz_\alpha^j$ of a line bundle L. A physical meaning to this mathematical language will be given in chapter 4 in which the $\phi_\alpha(z_\alpha)dz_\alpha^j$ will be the b-c fields with lagrangian $L = b\bar{\partial}c$. $j \in \mathbf{Z}$ for twisted bosons while $j \in \frac{\mathbf{Z}}{2}$ for twisted fermions. Here we are interested in sections of line bundles of the kind [BR2,GH]: $L_{\bar{u}} = K^j \otimes L^0_{\bar{u}}$, where $L^0_{\bar{u}}$ is a flat line bundle with degree 0 and K denotes the canonical line bundle. When j is half-integer $\bar{u}$ is a $2x2g$ matrix of phases corresponding to a definite behaviour of the sections $\phi_\alpha(z_\alpha)dz_\alpha^j$ when they are transported along the homology cycles. For example when $j = 0$ these sections are multivalued functions on the Riemann surfaces and a representations of them is provided by theta functions (see appendix B for a short review):

$$f_{\bar{u}} = \frac{\theta[\bar{u}](z-a)}{\theta[0](z-a)}.$$

The sections for the case $j \neq 1$ are simply the usual tensors $\phi_\alpha(z_\alpha)dz^j$, singlevalued on the Riemann surface Σ, multiplied by functions of this kind [AMV]. In particular when

$$\bar{u} = \begin{bmatrix} \frac{q_1}{n} & \cdots & \frac{q_g}{n} \\ \\ \frac{p_1}{n} & \cdots & \frac{p_g}{n} \end{bmatrix}$$

we have the Z_n spin structures of chapter 4. For now on the characteristics $\bar{u}$ will be only of this kind. The transition functions $g_{\alpha\beta}$ relating the trivializations $\phi_\alpha(z_\alpha)dz_\alpha^j$ and $\phi_\beta(z_\beta)dz_\beta^j$ at the intersections $U_\alpha \cap U_\beta$ are very simple to find on algebraic surfaces (see fig. (1.16)):

$$\phi_\alpha(z_\alpha) = \phi_\beta(z_\beta)\left(\frac{dz_\beta}{dz_\alpha}\right)^j \eta_{\alpha\beta}.$$

$\eta_{\alpha\beta}$ are the transition functions corresponding to the line bundle $L^0_{\bar{u}}$. They should satisfy the usual consistency conditions on $U_\alpha \cap U_\beta \cap U_\chi$:

$$\eta_{\alpha\beta} + \eta_{\beta\chi} - \eta_{\alpha\chi} = 0 \qquad (3.1).$$

These transition functions are of the form:$\eta_{\alpha\beta} = e^{\frac{2\pi i k_{\alpha\beta}}{n}}$, $k_{\alpha\beta}$ being integers. We wish now to compute their logarithms $\gamma_{\alpha\beta} = log(\eta_{\alpha\beta})$; this will turn out to be useful

in chapter 4 where we are faced with the problem of cataloguing all the possible Z_n spin structures leaving on an algebraic surface. The result of the calculation will be n^{2g} sets of $\gamma_{\alpha\beta}$ corresponding to the Z_n spin structures.

Since $L^0_{\vec{u}}$ is a flat line bundle, its transition functions $\gamma_{\alpha\beta}$ belong to the first cohomology group of Čech $\check{H}^1(\Sigma, Z_n)$ [GH]. In this case we represent Z_n as the set

$$\{\alpha \in \{0, \frac{1}{n}, \ldots, \frac{n-1}{n} \mod \mathrm{Z}\}\}.$$

To compute the elements of such group it is sufficient to have a good covering $\{U_\alpha\}$ on Σ, i.e. such that the intersections $U_\alpha \cap U_\beta$ are contractible open sets [BT]. Again a generalizable example of a good covering is in fig.(1.16). At this point we assign a number $\eta_\alpha \in Z_n$ and we define the usual cohomology operator $\delta_n : \check{H}^n(\Sigma, Z_m) \rightarrow \check{H}^{n+1}(\Sigma, Z_m)$ [GH]:

$$\begin{aligned} \delta_0 &: \eta_\alpha \rightarrow \eta_\alpha - \eta_\beta \\ \delta_1 &: \eta_{\alpha\beta} \rightarrow \eta_{\alpha\beta} + \eta_{\beta\chi} - \eta_{\alpha\chi} \end{aligned} \tag{3.2}$$

and so on. Now we recall that:

$$\begin{aligned} \check{H}^0(\Sigma, Z_n) &= ker\delta_0 \\ \check{H}^1(\Sigma, Z_n) &= \tfrac{ker\delta_1}{Im\delta_0} \\ \check{H}^2(\Sigma, Z_n) &= \tfrac{ker\delta_2}{Im\delta_1} \end{aligned} \tag{3.3}$$

where ker and Im denote respectively the kernel and the image of the operators δ_n. From the first of eqs. (3.2) it should be clear that the only possible values of η_α belonging to $ker\delta_0$ are those for which $\eta_\alpha = a = constant$. So $ker\delta_0$ has only one degree of freedom $a \in Z_n$ implying $\check{H}^0(\Sigma, Z_n) = Z_n$. From the second of eqs. (3.2) we get the linear system of equations (3.1) which yields $ker\delta_1$. Finally, since on algebraic surfaces there there are no four-fold intersections between the open sets U_α, $ker\delta_2 = \bigoplus_1^{n_s} Z_n$ where n_s is the total number of open sets forming the good covering. For example in fig. (1.16) there are forty-five intersections $U_\alpha \cap U_\beta$ and thirty intersections of the kind $U_\alpha \cap U_\beta \cap U_\chi$. The former correspond to 45 transition functions $\gamma_{\alpha\beta} \in Z_n$ while the latter are related to 30 elements of the kind $\gamma_{\alpha\beta\chi} \in Z_n$. Obviously $ker\delta_2 = \bigoplus_1^{30} Z_n$ and we have n^{30} possible choices of values for the variables $\gamma_{\alpha\beta\gamma}$ which give $\delta_2\gamma_{\alpha\beta\chi=0}$. Analogously from the second of eqs. (3.2)

we get that $ker\delta_1 = \bigoplus_1^{16} Z_n$. Since $Im\delta_1 = \frac{\bigoplus_1^{45} Z_n}{ker\delta_1} = \bigoplus_1^{29} Z_n$ we are able to compute all the cohomology groups of eq. (3.3). The result for the worked out example is:

$$\breve{H}^0(\sigma, Z_n) = \breve{H}^2(\sigma, Z_n) = Z_n$$

and

$$\breve{H}^1(\Sigma, Z_n) = Z_n \bigoplus Z_n.$$

As a matter of fact the surface of fig. (1.16) is of genus one and therefore there are n^2 spin structures corresponding to the n^2 possibilities to choose the transition functions $\gamma_{\alpha\beta} \in \breve{H}^1(\Sigma, Z_n)$ This method allows also to determine the transition functions $\eta_{\alpha\beta}$ explicitly. In fact after solving eqs. (3.1) we get an expression of $\gamma_{\alpha\beta} \in Im(\delta_1)$ in terms of the independent variables $\gamma_{\alpha\beta} \in ker(\delta_1)$. We denote the latter with a prime to distinguish them from the former. To find out what are the transition functions corresponding to non trivial spin structures we set up the linear system of equations:

$$\gamma'_{\alpha\beta} = \gamma_\alpha - \gamma_\beta.$$

If we consider the γ_α as unknown and the $\gamma_{\alpha\beta}$ as coefficients, it turns out that in general this system is overdetermined and there are $2g$ consistency relations that the coefficients should satisfy to become soluble. For example for the surface in fig. (1.16) we get:

$$\begin{array}{l} \gamma'_{07} - \gamma'_{17} + \gamma'_{16} - \gamma'_{06} = 0 \\ \gamma'_2 6 - \gamma'_{28} + \gamma'_{08} - \gamma'_{06} = 0 \end{array} \qquad (3.4).$$

We choose now the values of these γ' in such a way that these equations are violated. For example when we deal with Z_2 spin structures there are only 2^2 possibilities:

$$\begin{cases} i)\gamma'_{06} = \frac{1}{2} \text{and all other } \gamma'_{\alpha\beta} = 0 \\ ii)\gamma'_{07} = \frac{1}{2} \text{ and all other } \gamma'_{\alpha\beta} = 0 \\ iii)\gamma'_{26} = \frac{1}{2} \text{and all other } \gamma'_{\alpha\beta} = 0 \\ iv)\text{all coefficients set equal to } 0 \end{cases} \qquad (3.5).$$

The last equation represents the trivial spin structure. In the other equations we have simply to use the values of the independent $\gamma'_{\alpha\beta}$ to compute all the other $\gamma_{\alpha\beta}$.

We proceed now with the analysis of algebraic curves generated by the Weierstrass polynomials of eq. (1.1). One of their most striking characteristic is that it is possible to get explicit expressions in terms of known functions of the differentials of first, second and third kinds. As it is known all the other differentials are simply linear combinations of them. As well we can explicitly derive all the j-differentials of chapter 3. The importance of these j-differentials lies in the fact that the Green functions of scalar fields X and b-c systems are just j-differentials. Moreover prime form and theta functions (see appendix B) are expressible in terms of third kind differentials and viceversa. The advantages in the use of j-differentials instead of theta functions are their expliciteness which allows to extract numbers from physical amplitudes and moreover the fact that moduli are parametrized automatically by branch points. In this way we avoid the problem of Shottky. The rest of this section is devoted to generalize to algebraic curves the construction of **local uniformizer** (see below) and of the divisors of the differential dz and the functions $F_y(z,y)$, y. These are the building blocks in the derivation of j-differentials. In part of this analysis we follow standard book on the subject [Wei,Osg,FK,For]. Since explicit examples are worked out in the mathematical literature just for Z_n simmetric surfaces (eq. (1.1)) we do here the exercise to extend these results to arbitrary surfaces.

Local analysis:

The branch points of Σ are the solutions of the system of equations:

$$\begin{cases} F(z,y) = 0 \\ \partial_y F(z,y) = 0 \end{cases} \qquad (1.3).$$

For each regular branch point a , we can take y as a local uniformizer. By definition [Zve] a local uniformization is a pair of single-valued analytic functions of a complex variable t, given by a power series convergent in a neighborhood U_a of the branch point a:

$$\begin{cases} z = \varphi(t) = a + a_k t^k + a_{k+1} t^{k+1} + \ldots, k > 0, a_k \neq 0, \\ y = \psi(t) = y(a) + b_j t^j + b_{j+1} t^{j+1} + \ldots, j > 0, b_j \neq 0 \end{cases}$$

and such that $F(\varphi(t), \psi(t)) = 0$ in the common domain of convergence of the two series. More roughly, a local uniformizer provides a system of local transition

functions for the b-c tensor fields in the neigborhoods of branch points. To show that z and y provide a local uniformization, we expand eq (1.1) near a using the following change of variables:

$$z = a + z', y = y(a) + y'.$$

The result is an algebraic equation in z' and y':

$$\sum_r \sum_s A_{rs} z'^r y'^s = 0 \tag{4.1}$$

with

$$A_{rs} = \partial_z^r \partial_y^s F(z,y)_{|_{\substack{y=y(a)\\ z=a}}}.$$

The general form of eq.(4.1) is:

$$Az' + higher \quad orders \quad in \quad z' +$$

$$+By'^m + higher \quad orders \quad in \quad y' + mixed \quad terms \quad involving \quad z' \quad and \quad y' = 0.$$

With the substitutions: $z' = \xi^m$, $w' = v\xi$, this equation becomes:

$$(A + Bv^m)\xi^m + higher \quad orders \quad in \quad \xi = 0.$$

At the lowest order we have the m solutions:

$$z' = \xi^m, y' = v_r \xi \tag{4.1a},$$

where $v_r = exp\frac{2\pi i r}{m} \sqrt[n]{\frac{A}{B}}, (r = 0, ..., m-1)$. It is possible to conclude from here that w' plays the role of the local uniformizer.

A similar procedure is also valid for the points at infinity, performing the change of variables $z \rightarrow z' = \frac{1}{z}$ in eq.(1.1) and then expanding it near the point $z' = 0$ as in eq.(4.1). From this local analysis we have seen that m different solutions of y correspond to the branch point a as we said in section (1.1).

The genus of Σ is determined by the multiplicities ν_i of the branch points via the Riemann Hurwitz formula:

$$2g - 2 = -2n + \sum_{b.p.} (\nu_i - 1) \tag{4.2}.$$

The sum in eq.(4.2) is extended over the whole set of branch points.

The algebraic equation of a general Riemann Surface of genus g.

A general algebraic equation for a Riemann Surface of genus g should depend on $3g$ free parameters [GH]. Three of them are fixed by the automorphisms of the z complex plane, the others are related to the $3g-3$ moduli. These parameters appear in eq. (2.1) as the coefficients of the polynomial in y and z. Solving the system (1.3) we can express the branch points in terms of these coefficients. The maximum number of free parameters (or equivalently of free branch points) allowed by eq. (4.2) is $N = 2n + 2g - 2$, obtained when all branch points have multiplicity two. It is clear that for large n we have $N > 3j$. This does not imply that the number of moduli of a surface of genus g can be higher than $3g-3$. The apparent contradiction is solved in the mathematical literature [GH]. (see also chapter 6)

We proceed with the construction of a general algebraic equation for $g = 3$. In this case the Riemann Hurwitz formula predicts $n \geq 3g$ and therefore Σ cannot be covered by hyperelliptic curves. We choose $n = 3$ because it is advisable to keep the number of sheets as low as possible.

With $g = 3$ and $n = 3$ eq.(4.2) is consistent when the following possibilities are verified: there are 10 branch points with multiplicity 2 or 9 branch points one of which with multiplicity 3 and the other with multiplicity 2. In the first case the algebraic equation is:

$$y^3 + 3P(z)y - 2Q(z) = 0 \tag{4.3}$$

with

$$P(z) = \alpha_0 \prod_{i=1}^{3}(z - \alpha_i),$$

$$Q(z) = \beta_0 \prod_{i=1}^{4}(z - \beta_i).$$

P and Q take into account 9 parameters: the last parameter is the point at infinity. In the second case the equation is more complicated and an expression can be found in [SU].

An analogous analysis repeated for the case $g = 4$ yields $n = 3$ with 12 branch points of multiplicity 2 and the equation:

$$y^3 + 3p(z)y - 2q(z) = 0 \qquad (4.4).$$

Now:

$$p(z) = \alpha_0 \prod_{i=1}^{4}(z - \alpha_i),$$

$$q(z) = \beta_0 \prod_{i=1}^{6}(z - \beta_i)$$

and there are exactly 12 parameters.

Eqns. (4.3) and (4.4) are strong candidates to represent the general Riemann Surfaces of $g = 3, 4$ respectively. Their regularity can be tested using Baker's method [For] and the homology cycles can be found with the methods of section 1.2. Moreover these surfaces have all simple branch points and therefore their Riemann representation is given in fig. (1.6). Both of them have been derived with the aid of the exact formulas for the roots of a general polynomial of degree three listed in appendix C. Such formulas exist only for polynomials of degree less or equal to 4. This limits the validity of the method exposed to surfaces with $g \leq 6$. Beyond this value of g, the problem of constructing the algebraic equation of a Riemann Surface knowing the positions of branch points and their multiplicities is a special case of the classical converse problem of Galois theory. No algebraic method to solve it exists yet, so we can just make a guess of the solution. It is not our intention to treat this problem. We will always assume that the algebraic eq.(1.1) is given together with its solution $y(z)$ and starting from this we will construct the relevant correlation functions for (super)string calculations.

1.5 DIFFERENTIALS ON A GENERAL ALGEBRAIC CURVE

In [Fer1] and above we have stressed the role of differentials in the computation of correlation functions for b-c systems on surfaces with abelian group of symmetry.

Here we construct the differentials of first and third kind for more general surfaces. These differentials turn out to be necessary in order to express the correlation functions of bosonic fields. The building blocks of differentials are the quantities dz, $z-a$, y, $F_y(z,w)$. We need to specify their divisor, i.e. the positions and the order of their zeros and poles. The zeros of dz occur only at branch points. The order of these zeros at a branch point of multiplicity ν is $\nu-1$. In fact using the local uniformizer ξ of eq.(4.1a) we have $dz = dz' = \xi^{\nu-1}d\xi$. Next we consider the behaviour of dz at infinity. The projection of the generic point $z=a$ on the sheet k is denoted by $a_{(k)}$. Let us suppose that from the local analysis $z=\infty$ turns out to be a branch point of order m. Then m sheets are joined at this point and correspondingly m of the points $\infty_{(k)}$are identified to an unique point ∞. If $\infty_{(i_1)},..,\infty_{(i_{n-m})}$ are the points for which $y(z)$ is not branched, we can write the divisor of dz as:

$$div(dz) = \frac{a_1^{\nu_1-1}...a_l^{\nu_l-1}\infty^{m-1}}{\infty_{i_1}^2...\infty_{(i_{n-m})}^2\infty^{2m}} \qquad (5.1).$$

In eq.(5.1) l represents the total number of branch points and m_j their multiplicities.

To find the divisor of y we look at eq.(1.1) which we rewrite here:

$$y^n + P_{n-1}(z)y^{n-1} + ... + P_1(z)y + P_0(z) = 0 \qquad (1.1).$$

The zeros of y occur when $P_0(z)=0$. The order of these zeros is the order of the zeros of $P_0(z)$, but can change if the roots of P_0 coincide with the branch points or if P_0 has some roots in common with the other P_i. The infinities of y are given by the singularities of the P_i's [For]. The order of these infinities is computed performing the change of variable $z \to z' = \frac{1}{(z-a)}$ and then proceeding as in the case of the point at infinity. To show how the method works, we apply it to two interesting perticular cases.

Case 1) $g=3$, eq.(4.3).

There are 9 branch points a_i of multiplicity 2. For $z=\infty$ we put $z' = \frac{1}{z}$ and we search for a solution of eq.(4.3) of the form: $y(z') = \gamma z'^{-p} f(z') = \gamma z'^{-p}(1+..)$. At the leading order eq. (4.3) becomes then:

$$\gamma^3 z'^{4-3p} + ... + \gamma z'^{1-p} + ... + 1 + ... = 0 \qquad (5.2).$$

Consistency requires that:

$$4 - 3p = 1 - p \Longrightarrow p = \frac{3}{2}.$$

This means that $z = \infty$ is a point of multiplicity 2. Since the leading order terms are divergent, they should cancel simultaneously requiring: $\gamma^3 + \gamma = 0$. This equation has three solutions. The first, $\gamma = 0$, corresponds to the behaviour $w \sim \frac{1}{z^l}$ because if $\gamma = 0$ this is the only way to satisfy eq.(5.2). This branch of y is therefore singlevalued at the point $z = \infty$. We will call ∞_1 the point at infinity of the related sheet.

The second solution is given by $\gamma^2 = -1$. These are the two branches of y which have a branch point at ∞. The behaviour of y in this case is that of a multivalued function:

$$y \sim \begin{cases} \frac{i}{z^{\frac{l3}{2}}} + \frac{a_1}{z^l} + \frac{a_2}{z^{\frac{l1}{2}}} + .. \\ \frac{-i}{z^{\frac{l3}{2}}} + \frac{b_1}{z^l} + \frac{b_2}{z^{\frac{l1}{2}}} + .. \end{cases}$$

The polynomial coefficients of eq.(4.3) have no other singularities except $z = \infty$, so we are now in the position to write the divisors of dz and y:

$$div(dz) = \frac{a_1 ... a_9}{\infty_1^2 \infty^3} \qquad (5.3a)$$

$$div(y) = \frac{\beta_1 ... \beta_4}{\infty_1 \infty^3} \qquad (5.3b)$$

These divisors have the correct degree, $2g - 2$ and 0 respectively. $\beta_1 ... \beta_4$ are the zeros of $Q(z)$ in eq.(4.3). In general we have to specify in which sheet these zeros occur, but for our purposes this is not necessary. In the same way it is possible to compute the divisors of $F_y(z, y)$ and of $z - a$:

$$div(F_y(z,y)) = \frac{a_1 ... a_9}{\infty_1^3 \infty^6} \qquad (5.3c);$$

$$div(z - a) = \frac{a_{(1)} a_{(2)} a_{(3)}}{\infty_1 \infty^2} \qquad (5.3d).$$

Case 2)$g = 4$, eq(4.4) The procedure to find the divisors is analogous to that applied in the previous case. The results are:

$$div(dz) = \frac{a_1 ... a_{12}}{\infty_{(0)}^2 \infty_{(1)}^2 \infty_{(2)}^2} \qquad (5.4a);$$

$$div(y) = \frac{\beta_1..\beta_6}{\infty_{(0)}^2 \infty_{(1)}^2 \infty_{(2)}^2} \qquad (5.4b);$$

where $\beta_1...\beta_6$ are the zeros of $q(z)$ in eq.(4.4);

$$div(F_y(z,y)) = \frac{a_1...a_{12}}{\infty_{(0)}^4 \infty_{(1)}^4 \infty_{(2)}^4} \qquad (5.4c);$$

$$div(z-a) = \frac{a_{(0)}a_{(1)}a_{(2)}}{\infty_{(0)}\infty_{(1)}\infty_{(2)}} \qquad (5.4d).$$

<u>I^{st} kind differentials:</u>

on a given surface of genus g there are g holomorphic differentials.

Case 1)$g = 3$.

$$\omega_i = \frac{z^{(i-1)}dz}{F_y(z,y)}, \quad i = 1,2; \qquad \omega_3 = y\omega_1.$$

case 2) $g = 4$.

$$\omega_i = \frac{z^{(i-1)}dz}{F_y(z,y)}, \quad i = 1,2,3; \qquad \omega_4 = y\omega_1.$$

III^{rd} kind differentials.

A third kind differential $\omega_{ab}(z)$ has two simple poles at a and b with residues +1 and -1 respectively. On a general surface Σ with equation (1.1) ω_{ab} is constructed starting from the Weierstrass kernel [Wei]:

$$G(z)_a = \frac{F(a,y(z))}{(y(z)-y(a))F_y(z,y(z))}\frac{dz}{z-a} \qquad (5.5).$$

It is easy to proof that, once a branch $y^{(l)}(a)$ has been chosen, $G(z)_a$ has a simple pole in $a_{(l)}$ with residue +1. Therefore $G(z)_a - G(z)_b = \omega_{ab}$ is a good candidate for a third kind differential. Nevertheless this differential can still have spurious poles. We have to eliminate these poles subtracting from ω_{ab} other terms with the same singularities without changing the behavior in a and b.

Explicit construction of III^{rd} kind differentials

Case 1) $g = 3$.

In this case we get from eq.(5.5):

$$G(z)_a = \frac{y^3(z) + 3P(a)y(z) - 2Q(a)}{3(y(z) - y(a))(y^{(2)}(z) + P(z))} \frac{dz}{z-a} \qquad (5.6).$$

Let us suppose that a lies on the first sheet. Then, since $F(a, y(z)) = (y(z) - y^{(0)}(a))(y - y^{(1)}(a))(y - y^{(2)}(a))$ $G(z)_a$ becomes:

$$G(z)_a = \frac{(y - y^{(1)}(a))(y - y^{(2)}(a))}{3(y^2(z) + P(z))} \frac{dz}{(z-a)}.$$

It is easy to check with the aid of eqs. (5.3) that the divisor of $G(z)_a$ is given by:

$$div(G(z)_a) = \frac{\gamma_3 \cdots \gamma_8}{\infty a_{(0)}} \qquad (5.7).$$

In eq. (5.7) $\gamma_3 ... \gamma_8$ are six zeros of the function $(y - y^{(1)}(a))(y - y^{(2)}(a))$. The other two are of course $a_{(1)}$ and $a_{(2)}$. From eq. (5.7) we see that $G(z)_a$ has the correct pole in $a_{(0)}$ and a pole at ∞. In doing the difference $G(z)_a - G(z)_b$ this extra pole at infinity is changed to a pole in b. Therefore

$$\omega_{ab}(z) = G(z)_a - G(z)_b \qquad (5.8)$$

is the wanted third kind differential. In the $g = 4$ case we get for the third kind differentials the same formula (5.8). Also here there is no need to subtract spurious poles. Of course $G(z)_a$ is built starting from eq(4.4).

Notice that in both cases ω_{ab} is given by an expression in which the dependence on the free parameters of the curve is explicit since from appendix C we know the form of $y^{(l)}(z)$.

CHAPTER 2
FERMIONIC CONSTRUCTION OF CORRELATION FUNCTIONS ON ALGEBRAIC CURVES

2.1 TWO POINT FUNCTIONS FOR b-c SYSTEMS

We realize on Σ a conformal field theory in terms of b-c systems with conformal weight j and the usual action:

$$S^{(l)} = \int d^2z b^{(l)}(z)\bar{\partial}c^{(l)}(z),$$

where $b^{(l)}(z)dz^j$ and $c^{(l)}dz^{1-j}$ are the branches of the multivalued fields $b(z)dz^j$ and $c(z)dz^{1-j}$ defined on the sheet l. The boundary conditions of the fields b and c near the branch points are dictated by the monodromy matrices of eq.(2.4) of chapter 1. On a general algebraic curve the monodromy matrices are not simultaneously diagonalizable. Therefore the methods of [Kni,BR1,Fer1] to compute the correlation functions are no longer applicable [Fer1].

We concentrate in the propagator $G_{j,1-j}(z,a)$ for b-c systems of central weight j. The other correlation functions can be obtained with the Wick theorem.

$G_{j,1-j}(z,a)$ is determined by a pole in $z=a$ and by the zeros and poles at the location of the zero modes. There is still the freedom to add a term containing the zero modes but this turns out to be irrelevant in the physical amplitudes [BI]. Our strategy consists in reconstructing the structure of zeros and poles of $G_{j,1-j}(z,a)$ by means of differentials of the first and third kind.

We start with the case $j=1$, $g=3$. The Riemann Roch theorem predicts three zero modes for b and one for c. Therefore $G_{1,0}(z,a)$ is given by:

$$G_{1,0}(z,a) = \frac{< b(z)c(a)\prod_{i=1}^{3} b(z_i)c(z_4) >}{< \prod_{i=1}^{3} b(z_i)c(z_4) >} \qquad (1.1).$$

$G_{1,0}(z,a)$ should be a differential in z and a function in a. Roughly speaking the

position of the relevant zeros and poles for $G_{1,0}(z,a)$ is:

$$G_{1,0}(z,a) \sim \frac{(a-z_4)(z-z_1)(z-z_2)(z-z_3)(z_1-z_2)(z_1-z_3)(z_2-z_3)}{(z-a)(a-z_1)(a-z_2)(a-z_3)(z-z_4)(z_1-z_4)(z_2-z_4)(z_3-z_4)}.$$

This behavior is exactly reproduced by:

$$G'_{1,0}(z,a) = \frac{det \begin{vmatrix} \omega(z)_{az_4} & \omega_1(z) & \omega_2(z) & \omega_3(z) \\ \omega(z_1)_{az_4} & \omega_1(z_1) & \omega_2(z_1) & \omega_3(z_1) \\ \omega(z_2)_{az_4} & \omega_1(z_2) & \omega_2(z_2) & \omega_3(z_2) \\ \omega(z_3)_{az_4} & \omega_1(z_3) & \omega_2(z_3) & \omega_3(z_3) \end{vmatrix}}{det\,|\omega_i(z_j)|} \qquad (1.2).$$

We notice that ω_{ab} has singularities at infinity as a function of a and b. Nevertheless these extra poles do not appear in the divisor of $G'_{1,0}(z,a)$. In fact the divergent terms at $a=\infty$ of $G(z)_a$ are always factorized in the form $f^i(a)\omega_i(z)$ where the $\omega_i(z)$ are holomorphic differentials and $f^i(a)$ are singular functions of a. Therefore these terms will not contribute to $G'_{1,0}(z,a)$. Since there are no other sources of extra poles, we can identify $G'_{1,0}(z,a)$ with $G_{1,0}(z,a)$ apart from an irrelevant arbitrariness due to zero modes.

Now we treat the case $j=2$, $g=3$ corresponding to the ghosts of string theories. There are six zero modes related to the fields b. To build the propagator $G_{2,-1}(z,a)$ we need a meromorphic tensor $\Omega_a(z)$, with a simple pole at the point a with residue $+1$ and with the tensorial structure dz^2da^{-1}. The following expression for $\Omega_a(z)$ fulfills all of these requirements:

$$\Omega_a(z) = \frac{F(a,y(z))}{(y(z)-y(a))(F_y(z,y(z))^2}\frac{dz^2}{z-a}\frac{F_y(a,y(a))}{da} \qquad (1.3).$$

Moreover the singularities in a occurring at $a=\infty$ are of the form $f^j(a)\Omega_j(z)$. Now $\Omega_j(z)$, $j=1,...3g-3$, are holomorphic quadratic differentials. A basis for $\Omega_j(z)$ can be constructed performing all possible products of two holomorphic differentials. As before it is possible to write the propagator $G_{2,-1}(z,a)$ in terms of determinants:

$$G_{2,-1}(z,a) = \frac{< b(z)c(a)\prod_{i=1}^{6} b(z_i) >}{< \prod_{i=1}^{6} b(z_i) >} = \frac{det \begin{vmatrix} \Omega_a(z) & \Omega_1(z) & \dots & \Omega_6(z) \\ \Omega_a(z_1) & \Omega_1(z_1) & \dots & \Omega_6(z_1) \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_a(z_6) & \Omega_1(z_6) & \dots & \Omega_6(z_6) \end{vmatrix}}{det\,|\Omega_j(z_i)|} \qquad (1.4).$$

Analogous expressions can be written in the case $g = 4$.

$$G_{1,0}(z,a) = \frac{det \begin{vmatrix} \omega(z)_{az_5} & \omega_1(z) & \dots & \omega_4(z) \\ \omega(z_1)_{az_5} & \omega_1(z_1) & \dots & \omega_4(z_1) \\ \vdots & \vdots & \ddots & \vdots \\ \omega(z_4)_{az_5} & \omega_1(z_4) & \dots & \omega_4(z_4) \end{vmatrix}}{det\,|\omega_i(z_j)|} \qquad (1.5).$$

$$G_{2,-1}(z,a) = \frac{det \begin{vmatrix} \Omega_a(z) & \Omega_1(z) & \dots & \Omega_9(z) \\ \Omega_a(z_1) & \Omega_1(z_1) & \dots & \Omega_9(z_1) \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_a(z_9) & \Omega_1(z_9) & \dots & \Omega_9(z_9) \end{vmatrix}}{det\,|\Omega_j(z_i)|} \qquad (1.6).$$

All the correlation functions of eqns. (1.2), (1.4), (1.5), (1.6) have an explicit dependence on the parameters of the algebraic equations (4.3) and (4.4) of chapter 1. It is easy to show that for general j the generalization $\Omega_{a,(j)}(z)$ of eq.(1.3) is provided by:

$$\Omega_{a,(j)}(z) = \frac{F(a,y(z))}{(y(z)-y(a))(z-a)} \left[\frac{dz}{F_y(z,y(z))}\right]^j \left[\frac{da}{F_w(a,w(a))}\right]^{1-j}$$

2.2 TWO POINT FUNCTIONS FOR THE SCALAR FIELDS

Let's consider on Σ the scalar fields X(z) with the action:

$$S^{(l)} = \int d^2z \partial X^{(l)}(z) \bar{\partial} X^{(l)}(z).$$

These fields are functions on Σ and therefore they obey the monodromy properties of eq.(1.4) of chapter 1.

We start from the correlation functions:

$$G(z;a,b) = < \partial X(z)(X(a) - X(b)) > \qquad (2.1).$$

This is a differential in z and a function in a and b. Moreover it has a zero in $a = b$ and two poles in $z = a$, $z = b$ with residues +1 and -1 respectively. Therefore $G(z;a,b)$ is a third kind differential and its general form will be:

$$G(z;a,b) = \omega_{ab}(z) + \sum_{i=1}^{g} A_i \omega_i(z)$$

where ω_i are holomorphic differentials and the constants A_i depend on a and b. Clearly when $a = b$ we have $A_i = 0$. The coefficients A_i are computed with the method of [BR], that is imposing the condition that $G(z;a,b)$ is single-valued on Σ. This condition provides a system of equations in which A_i are the unknowns. From this system we get:

$$A_i = \frac{det \begin{vmatrix} T_a & \bar{A} \\ T_b & \bar{B} \end{vmatrix}}{det \begin{vmatrix} \oint_{a_i} \omega_j & \oint_{\bar{a}_i} \bar{\omega}_j \\ \oint_{b_i} \omega_j & \oint_{\bar{b}_i} \bar{\omega}_j \end{vmatrix}} \qquad (2.2).$$

We used here the following notations:

$$T_a = \begin{vmatrix} \oint_{a_1} \omega_1(z) & \dots & \oint_{a_1} \omega_{i-1}(z) & \oint \omega_{ab}(z) & \dots & \oint_{a_1} \omega_g(z) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \oint_{a_g} \omega_1(z) & \dots & \oint_{a_g} \omega_{i-1}(z) & \oint \omega_{ab}(z) & \dots & \oint_{a_g} \omega_g(z) \end{vmatrix};$$

$$\bar{A}_{ij} = \left| \oint_{\bar{a}_i} \bar{\omega}(\bar{z}) \right|$$

and T_b, $\bar{B}$ are the analogue for the b-cycles. Again $G(z;a,b)$ has an explicit dependence on the parameters of the curve, though in a very complicated way. From $G(z;a,b)$ we can find:

$$< \partial X(z) \partial X(a) >= \partial_a G(z;a,b) \qquad (2.3).$$

and

$$< (X(c) - X(d))(X(a) - X(b)) >= \int_d^c G(z;a,b)dz \qquad (2.4).$$

These correlation functions are sufficient to compute the physical amplitudes of bosonic strings. The chiral determinants can be derived using the stress energy tensor method [AS,Ver] which in this general case is still applicable. Finally we give an expression for the prime form $E(z,a)$ which is suitable for Riemann Surfaces realized as branched coverings of the complex plane [Fay]. We suppose, without losing generality, that the zero of the prime form occurs in the first sheet. Therefore:

$$E^2(z,a) = \frac{(z-a)^2}{dzda} \exp\left[\int_z^a \sum_{i=1}^{n-1} \tilde{\omega}_{a_{(i)} z_{(i)}} + \sum_{j=1}^{g} \int_z^a m_j \tilde{\omega}_j\right] \qquad (2.5),$$

where:

$$m_j = \frac{1}{2\pi i}\int_{A_j} d[ln(\frac{(t-z)}{t-a})].$$

In eq. (2.5) $\tilde{\omega}_{ab}(z)$ and $\tilde{\omega}_j$ are the normalized third kind and first kind differentials respectively. It is easy to see that in terms of the old basis we have:

$$\tilde{\omega}_j(z) = \frac{det\begin{vmatrix} 0 & \omega_1(z) & \dots & \omega_g(z) \\ -\delta_{j1} & \oint_{a_1}\omega_1 & \dots & \oint_{a_1}\omega_g \\ \vdots & \vdots & \ddots & \vdots \\ -\delta_{jg} & \oint_{a_g}\omega_1 & \dots & \oint_{a_g}\omega_g \end{vmatrix}}{det\begin{vmatrix} \oint_{a_1} & \dots & \oint_{a_1}\omega_g \\ \vdots & \ddots & \vdots \\ \oint_{a_g}\omega_1 & \dots & \oint_{a_g}\omega_g \end{vmatrix}}$$

and:

$$\tilde{\omega}_{ab}(z) = \frac{det\begin{vmatrix} \omega_{ab}(z) & \omega_1(z) & \dots & \omega_g(z) \\ \oint_{a_1}\omega_{ab}(z) & \oint_{a_1}\omega_1 & \dots & \oint_{a_1}\omega_g \\ \vdots & \vdots & \ddots & \vdots \\ \oint_{a_1}\omega_{ab}(z) & \oint_{a_g}\omega_1 & \dots & \oint_{a_g}\omega_g \end{vmatrix}}{det\begin{vmatrix} \oint_{a_1}\omega_1 & \dots & \oint_{a_1}\omega_g \\ \vdots & \ddots & \vdots \\ \oint_{a_g}\omega_1 & \dots & \oint_{a_g}\omega_g \end{vmatrix}}$$

In appendix C we will prove eq.(5.5) following ref. [Fay].

2.3 CHIRAL DETERMINANTS

The explicit expression of the two point functions enables us to get the expectation value of the stress-energy tensor for b-c systems by means of the well known formula [Ver]:

$$< T(z) >= \lim_{a\to z}[-j\partial_a G_{j,1-j}(z,a) + (1-j)\partial_z G_{j,1-j}(z,a) - \frac{1}{(z-a)^2}] \quad (3.1).$$

In analogy with the Z_n symmetric surfaces we suppose that on general algebraic curves the equivalence between branch points a_i and certain primary fields $V(a_i)$ called twist fields still holds. Under this hypothesis, to be checked a posteriori, we

use the stress-energy tensor method [AS,Ver] to compute the chiral determinants:

$$det\bar{\partial}_j = \frac{< \prod_{i=1}^{N} b(z_i) \prod_{k=1}^{M} c(z'_k) V(a_1) \dots V(a_{n_{bp}}) >}{det\,|\Omega_i(z_j)|} \tag{3.2}$$

In this equation Ω_i represent the zero modes, n_{bp} is the total number of branch points and:

$$\begin{cases} N = (2j-1)(g-1) \\ M = 0 \end{cases} \qquad for\ j \neq 1,$$

$$\begin{cases} N = g \\ M = 1 \end{cases} \qquad for\ j = 1.$$

We review briefly the stress-energy tensor method to compute the chiral determinants. From the operator product expansion of the stress-energy tensor with a primary field $V(a_i)$:

$$T(z)V(a_i) = \left(\frac{h_i}{(z-a_i)^2} + \frac{\partial_{a_i}}{(z-a_i)} \right) V(a_i) \tag{3.3}$$

and the definition of the Green functions:

$$G_{j,1-j}(z,a) = \frac{< b(z)c(a) \prod_{i=1}^{N} b(z_i) \prod_{k=1}^{M} c(z'_k) V(a_1) \dots V(a_{n_{bp}}) >}{< \prod_{i=1}^{N} b(z_i) \prod_{k=1}^{M} c(z'_k) V(a_1) \dots V(a_{n_{bp}}) >} \tag{3.4}$$

it descends that:

$$< T(z) > = \sum_{i=1}^{n_{bp}} \frac{h_i}{(z-a_i)^2} + \frac{\partial_{a_i} < \prod_{i=1}^{N} b(z_i) \prod_{k=1}^{M} c(z'_k) V(a_1) \dots V(a_{n_{bp}}) >}{(z-a_i)} \tag{3.5}.$$

Therefore the coefficients of the second order poles in a_i of $< T(z) >$ represent the conformal weight of $V(a_i)$ while the residue of $< T(z) >$ at the branch points provides a set of differential equations for the numerator of eq. (3.2). If, following [Gav], we pick up from $G_{j,1-j}(z,a)$ just the part independent from the zero modes, the result of these differential equations will be automatically $det\bar{\partial}_j$ with a "natural" normalization for the basis of zero modes.

Let us apply this method to the b-c systems with $j = 1$ and $g = 3$. The computation can be easily extended to the case $g = 4$. The part of Green functions (1.2) and (1.5) which is independent on the zero modes is represented by the Weierstrass

kernel $G(z)_a$ of eq. (5.5) of chapter 1. Inserting $G(z)_a$ in the correlation functions of eq. (3.2) we get the three branches of $< T(z) >$ which are:

$$< T(z) >_{(0)} = \frac{1}{3} \frac{\left[\frac{1}{2} w^{(1)\prime\prime}(w^{(0)} - w^{(2)}) + \frac{1}{2} w^{(2)\prime\prime}(w^{(0)} - w^{(1)}) + w^{(1)\prime} w^{(2)\prime}\right]}{(w^{(0)^2} + P(z))} \qquad (3.6a)$$

$$< T(z) >_{(1)} = \frac{1}{3} \frac{\left[\frac{1}{2} w^{(0)\prime\prime}(w^{(1)} - w^{(2)}) + \frac{1}{2} w^{(2)\prime\prime}(w^{(1)} - w^{(0)}) + w^{(0)\prime} w^{(2)\prime}\right]}{(w^{(1)^2} + P(z))} \qquad (3.6b)$$

$$< T(z) >_{(2)} = \frac{1}{3} \frac{\left[\frac{1}{2} w^{(1)\prime\prime}(w^{(2)} - w^{(0)}) + \frac{1}{2} w^{(0)\prime\prime}(w^{(2)} - w^{(1)}) + w^{(1)\prime} w^{(0)\prime}\right]}{(w^{(2)^2} + P(z))} \qquad (3.6c).$$

In these equations $w^{(l)}$ is defined in Appendix B and w', w'' denote the first and second derivatives of $w(z)$. Unlike the case of Z_n symmetric surfaces, the poles of $< T(z) >$ are distributed in a non symmetrical way on the sheets. In order to proceed, we need the position of the branch points a_i. Unfortunately it is not possible to know simultaneously the branch points and the parameters α_i, β_i of eqs. (4.3) and (4.4). In fact the positions of the branch points are roots of the polynomial of degree 9:

$$Q^2 + P^3 = 0 \qquad (C.1)$$

(see App. C) for which an explicit solution does not exist. Nevertheless the relations between branch points and parameters of the algebraic equation (4.3) are known (eq.(C.1)). Rewriting P and Q in the following way:

$$P = \sum_{i=1}^{3} b_i z^i \qquad , \quad q = \sum_{i=0}^{4} c_i z^i$$

and equating the coefficients of the different powers of z in the equation:

$$Q^2 + P^3 = \prod_{i=0}^{9} (z - a_i) \qquad (3.7)$$

we yield a set of algebraic equations $f_m(a_i; b_j, c_k) = 0$, $m = 0, \ldots, 9$ determining the dependence of b_j, c_k on a_i. For the moment we suppose that the branch points are independent variables and that the parameters are expressed in terms of them trough these equations. With this assumption it is possible to evaluate the poles of $< T(z) >$ at a_i:

$$< T(z) > \sim -\frac{1}{8} \sum_i \frac{1}{(z - a_i)^2} + 1/4 \sum_{k>i} \frac{1}{(z - a_i)} \frac{1}{(a_i - a_k)} -$$

$$-\sum_i \frac{\left(\frac{1}{3}\frac{Q(a_i)'}{Q(a_i)}+\frac{1}{27Q(a_i)^2}\right)}{(z-a_i)}+\frac{5}{36}\sum_i \frac{1}{(z-a_i)Q^2(a_i)}\prod_{k\neq i}(a_i-a_k) \qquad (3.8)$$

Eq. (3.8) proofs our previous assumption that a branch point a_i is equivalent to a twist field concentrated at that point and moreover tells us that the conformal dimension of such fields is $h_i = -\frac{1}{8}$. Our result is quite reasonable if we remember that on the algebraic curve defined by eq. (4.3) all branch points connect just two sheets, so that in the neighborhood of a branch point our curve closely resembles an hyperelliptic one. This result is also in agreement with [BR3] in which it is said that the branch points can be still simulated by primary field whose conformal dimension depends just on the multiplicity of the branch point with the formula (1.12a) of chapter 3. Next we consider the residue of $< T(z) >$ from which we set up the differential equations determining $det\bar{\partial}_j$:

$$\partial_{a_i}(det\bar{\partial}_j) = \frac{1}{4}\sum_{k>i}\frac{1}{(a_i-a_k)}-$$

$$\left(\frac{1}{3}\frac{Q'(a_i)}{Q(a_i)}+\frac{1}{27(Q^2(a_i))}\right)+\frac{5}{36}\frac{1}{(z-a_i)Q^2(a_i)}\prod_{k\neq i}(a_i-a_k) \qquad (3.9)$$

The first term in the RHS of eq. (3.9) is "hyperelliptic" but the other two represent nontrivial corrections which prevent us to give an easy analytic solution to this equation.

Comments

We have seen how to explicitly construct the 2-point correlation functions for the bosonic string on surfaces defined by a general algebraic equation. Examples have been worked out for the genus three case, while for genus 4 we still have to prove that eq. (4.4) is a good representative. On the other hand, we have shown that on general algebraic curves the behavior of the fields b and c at branch points can be simulated by twist fields with definite conformal dimension h. Moreover our analysis allows the computation of h. At least in our particular case our analysis

shows that the conformal dimension of the twist fields at a branch point is the same one which is obtained if we consider a Z_n surface approximating the general curve in a neighborhood of the branch point. This result could shed some light in the search for the explicit form of twist fields on nonhyperelliptic curves. In practice it gives a link between the fermionic method exposed above and the bosonization techniques using twist fields using [BR,Kni,Ferl]. Concerning the partition function, two obstacles prevent us from computing the chiral determinants $det\bar{\partial}_1$ and $det\bar{\partial}_2$. The first lies in the complexity of the differential equations (6.9). The second consists in the difficulty to express the branch points in terms of the parameters of the starting algebraic equation and viceversa. For these reasons we believe that in the case of nonhyperelliptic surfaces the Beilinson-Manin formula [BM] is better suited than the stress-energy tensor method for computing the partition function of bosonic string theory. The Beilinson Manin formula in fact requires just the knowledge of a half differential corresponding to an odd spin structure together with its $g-1$ zeros. This problem is well known in the mathematical literature (see for example [Fay] and references therein). A more difficult problem is to extend the fermionic construction of correlation functions to superstrings due to the presence of fermions. In [Ferl] we pointed out that even in the case of Z_n symmetric surfaces the spin structures are troublesome in the language of branch points. There are two main difficulties:

1) the derivation of the 2^{2g} differentials with quadratic divisors such that their square root represents well defined half differentials. There exist algorithms to obtain these quadratic divisors in terms of branch points [Fay] but they involve the solution of a nontrivial algebraic equation.

2) The square root of a differential with quadratic divisor is defined only up to a sign. When we choose a sign, it is not easy to know how it will change when crossing a nontrivial homology cycle apart from relatively simple cases [Ferl].

We will show in section [4.4] that one can overcome these troubles at least for the particular Z_n symmetric surfaces of ref. [GM] appearing in high energy string calculations. Finally we stress that the formalism that we have introduced can be

applied to a larger class of conformal theories, for example the minimal models and the bosonic WZW models.

CHAPTER 3

BOSONIZATION ON ALGEBRAIC SURFACES

3.1 CONFORMAL FIELDS ON Z_N SURFACES

BOSONIZATION AND TWIST FIELDS

We consider here the family of Riemann surfaces of genus g $\Sigma(g,n,m,q)$ expressed as n-sheeted coverings of the sphere CP_1 with associated algebraic equation:

$$y^n = \prod_{i=1}^{mn}(z-c_i)\prod_{j=1}^{q}\frac{(z-a_j)}{(z-b_j)} \quad c_i \neq a_j \neq b_j \ ; \ m,n \ integers \tag{1.1}$$

The genus g of these surfaces is given by the Riemann Hurwitz formula (eq. (A.2)):

$$g = (1-n) + \frac{mn(n-1)}{2} + q(n-1) \tag{1.2}$$

When $m = 0$ and $q \neq 0$ we get an important class of Riemann surfaces which will be called rational surfaces. The surfaces in [GM] represent the subset with $q = 2$. It is possible to see that each Σ is invariant under a Z_n symmetry generated by the operator which interchanges the sheets and and that the point at infinity is not a branch point. Slightly more general surfaces can be obtained allowing two or more branch points to coincide: the results presented here can be straightforwardly extended also to these curves. All mathematical details used here about R.S. are contained in appendix A. We consider on Σ a CFT of b-c systems with conformal weight j and the usual action (see chapter 2):

$$S^{(l)} = \int d^2z \; b^{(l)}(z)\bar{\partial}c^{(l)}(z) \tag{1.3}$$

where $b(z)dz^j$, $c(z)dz^{1-j}$ depend on the variable $z \in CP_1$ and are multi-valued fields over the sphere whose branches are denoted by $b^{(l)}(z)$ and $c^{(l)}(z)$, $l = 0,...,n-1$. Let us briefly review what is known up to now about bosonization on Z_n symmetric R.S. The basic references [Kni,Br1,Br2,Gav] deal with the case $\Sigma(g,n,m,0)$: we add here the treatment of the 'rational' points b_j.

Local Monodromy Matrix

We introduce here the monodromy matrices M_a for fields as we did in chapter 1 for the function $y(z)$. To a branch $\phi^{(l)}(z)$ of a general multivalued tensor field it corresponds a vector $\vec{\phi}^{(l)} = (0, \ldots, 1, \ldots, 0)^T$ with the $l-th$ component equal to 1 and all other put equal to 0. The change of branches when $\phi^{(l)}(z)$ is moved around a branch point a is then expressed by an $n \cdot n$ matrix in the following way:

$$M_a \vec{\phi}^{(l)}(z) = \vec{\phi}^{(s)}(z) \qquad (1.4).$$

If we put $\vec{\phi}(z) = \sum_l \vec{\phi}^{(l)} \phi^{(l)}(z)$, then $\vec{\phi}(z) \to M_a \vec{\phi}(z)$. Since the action of M_a correspond to encircling a branch point, we have:

$$\phi^{(s)}(z) = \phi^{(l)}((z-a)e^{2\pi i} + a) = M_a \phi^{(l)}(z)) \qquad (1.4a).$$

M_a is called the monodromy matrix at the branch point a of the fields $\phi^{(l)}(z)dz^j$. This monodromy matrix do not coincide in general with the monodromy matrices of chapter 1, which are instead refered to the surface Σ. See for example the note below and chapter 4. Nevertheless, since a circle which surrounds all the branch points α_i at a given sheet can be shrunk to a point being homotopically trivial, the monodromy matrices of the fields should again satisfy the following constraint:

$$\prod_{i=1}^{n_{bp}} M_{\alpha_i} = 1 \qquad (1.5)$$

where n_{bp} is the total number of the branch points α_i. From eq. (1.1), going clockwise around a branch point, the $l-th$ sheet is interconnected only with the $(l+1)$-th sheet when $a = a_j, c_i$ and with the $(l-1)$-th sheet when $a = b_j$. Therefore eq. (1.4) can be rewritten as:

$$M_a \phi^{(l)} = \phi^{(l+1)} \ for \ c_i, a_j \quad and \quad M_a \phi^{(l)} = \phi^{(l-1)} \ for \ b_j \qquad (1.6).$$

These equations represent the local boundary conditions of the field $\phi^{(l)}(z)$ at the branch points. Clearly these are just local relations. The spin structures we will introduce below, involve non trivial transition functions relating the fields at the intersections of local patches. Therefore they induce global effects consisting in an

additional multivaluedness of the fields carrying spin structures when they cross the homology cycles. In this case eq. (1.6) does not hold in general. As we will see, for certain spin structures, their global effect on the fields can be still expressed using modified local boundary conditions. If s branch points coincide, the boundary conditions become $M_a\Phi^{(l)}(z) = \Phi^{(l\pm s)}$, where the minus sign is valid when a is a rational point. In all cases described by eq. (1.1), it is easy to conclude that monodromy matrix should be of the form:

$$M_a = \begin{pmatrix} 0 & . & . & . & 1 \\ 1 & . & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & . & 1 & 0 \end{pmatrix} \tag{1.7}$$

or some integer power of it (M^n = identity).

Conformal Properties of the b-c Fields and Twist Fields.

It is convenient to choose a basis which diagonalizes the monodromy matrices. Its eigenvalues are $1, \alpha, ..., \alpha^{n-1}$, ($\alpha^n = 1$) and the eigenvectors are:

$$b_k = \sum_l \alpha^{kl} b^{(l)}, \quad c_k = \sum_l \alpha^{-kl} c^{(l)} \quad (k = 0, ..., n-1) \tag{1.8}.$$

The local behavior of the b-c fields near a generic branch point a of multiplicity ν is constrained by the requirement that $b_k(z)dz^j$ and $c_k(z)dz^{1-j}$ must be finite in a neighborhood of a. This condition is necessary to avoid punctures on Σ located at the branch points and to preserve the usual equations of motion: $\bar{\partial} b_k = \bar{\partial} c_k = 0$.

Going to the local uniformizer $t^\nu = (z - a)$ (see the definition of chapter 1, section 1.4), we have:

$$b_k(z) \sim b_k(t) \left(\frac{dz}{dt}\right)^j dt^j = n b_k(t) t^{j(n-1)} dt \tag{1.9}$$

$$c_k(z) \sim c_k(t) \left(\frac{dz}{dt}\right)^{1-j} dt^{1-j} = n c_k(t) t^{(1-j)(n-1)} dt \tag{1.9a}.$$

From these equations the behavior of b_k, c_k near the branch points should be:

$$b_k(z) \sim (z - a)^{-q_k} \qquad q_k \geq \frac{j(i-n)}{n} \tag{1.10}$$

$$c_k(z) \sim (z-a)^{-q'_k} \qquad q'_k \geq \frac{(1-j)(i-n)}{n} \tag{1.10a}$$

Therefore

$$q_k \in \{\frac{l+j(1-n)}{n},\ l=0,\ldots,n-1\}\ mod\ n \tag{1.11}$$

$$q'_k \in \{\frac{l+(1-j)(1-n)}{n},\ l=0,\ldots,n-1\}\ mod\ n \tag{1.11a}$$

Moreover the definition of b_k band c_k (eq. 1.8) entails $q_k = -q'_k\ mod\ n$. Eqs. (1.10), (1.10a), (1.11), (1.11a) determine the fractional values of q_k and q'_k apart from the addition of an integer compatible with eqs. (1.10) and (1.10a). This is obvious because the fields b and c can have also zeros of different orders at the branch points. Moreover the local analysis above just depends on the multiplicity of the branch point but does not distinguish for example between "rational" branch points b_j and "normal" branch points a_j, c_i. For example if a normal branch point has a monodromy matrix M as in eq. (1.7), a rational branch point has the monodromy matrix M^{-1}. If we wish to diagonalize simultaneously both these matrices, we can use the same basis b_k, c_k of eq.(1.8) but the corresponding eigenvalues will be assigned in a different way to the eigenvectors b_k, c_k. In other words, if at the first branch point we act with the monodromy operator, we get:

$$Mb_k = e^{(2\pi i q_k)} b_k$$

but at the second branch point we have:

$$Mb_k = e^{(-2\pi i q_k)} b_k$$

with $q_k = k/n\ mod\ \mathbf{Z}$. It is nevertheless important to derive uniquely the charges q_k at all branch points of the Riemann surface. The knowledge of the $q'_k s$ allows in fact the computation of the n-point functions and of the correlation functions of b-c systems on Z_n symmetric surfaces. To show this, we follow the analysis of [Kni] extended here to general algebraic surfaces. The algebraic surface is imagined as an n-fold covering of the complex plane $C \cup \{\infty\}$ or, equivalently, of the sphere CP_1. A quantum field $\phi(z)dz^j$ singlevalued on this surface, is multivalued on CP_1. Its general expression consists in n independent fields $\phi^{(l)}(z)dz^j$ defined on CP_1,

each one assigned to one of the n replicas of CP_1 which build the covering. $\phi^{(l)}(z)$ play the role of the branches of $\phi(z)$. The fields $\phi^{(l)}(z)$ are locally subjected to the boundary conditions of eqs. (1.4) dictated by the monodromy properties of the surface. Moreover in a neighborhood of a branch point a of multiplicity ν, we can always introduce a set of fields $\phi_k(z)$, linear combinations of the $\phi^{(l)}(z)$, which diagonalizes the monodromy matrix M_a. Going to the local uniformizer $t^\nu = (z-a)$ which provides local coordinates in the neighborhood of the branch point, we can show [Kni] that the stress energy tensor of the b-c systems:

$$T(t) = T(z)\left(\frac{dz}{dt}\right)^2 + \frac{C}{12}\left(\frac{(d^3z/d^3 3t)}{(dz/dt)^2} - \frac{(d^2z/d^2t)}{(dz/dt)}\right) \tag{1.12}$$

has a second order pole at the branch point of the form $\frac{\Delta}{(z-a)^2}$ with

$$\Delta = \frac{\nu(6j^2-6j-1)}{24}\left(1-\frac{1}{\nu^2}\right) \tag{1.12a}.$$

An analogous procedure repeated for the conserved current of the ghost number $J_k =: b_k c_k :$, shows that J_k has a first order pole at a with residue $\tilde{q}_k$. This is sufficient to conclude that a branch point can be represented by a primary field $V(a)$ also called a twist field. Introducing the bosonization rules $b_k \sim e^{i\phi_k}$, $c_k \sim e^{-i\phi_k}$ and

$$< \phi_k(z)\phi_{k'}(z') >= \delta_{kk'}\,|ln(z-z')|\,,$$

$V(a)$ has the following simple form near a branch point:

$$V(a) \sim e^{(i\sum_{k=0}^{n-1}\tilde{q}_k\phi_k(a))} = \tag{1.13}.$$

Accordingly:

$$\Delta = \sum_k \left(\frac{1}{2}\tilde{q}_k^2 + (j-\frac{1}{2})\tilde{q}_k\right) \tag{1.14}.$$

In eq. (1.14) $\tilde{q}_k$ is defined as in eqs. (1.11) and (1.11a) but it is denoted by a different symbol to remember that this is the charge associated to the branch point and which we have to determine. We impose now the following requirement A), which is typical of Z_n symmetric surfaces:

A) the monodromy matrices M_a are simultaneously diagonalizable at all branch points.

In this way we obtain everywhere on the surface a basis b'_k, c'_k of fields single-valued on CP_1. They diagonalize the monodromy matrices at all the branch points and the related twist fields have the form of eq. (1.13). The consistency with the boundary conditions (1.10), (1.10a) is provided by the OPE:

$$b'_k(z)V(a) \sim (z-a)^{-q_k} : b'_k(z)V(a) : \tag{1.15}$$

$$c'_k(z)V(a) \sim (z-a)^{-q'_k} : c'_k(z)V(a) : \tag{1.15a}$$

with $q_k = \tilde{q}_k + m$, $q'_k = \tilde{q}'_k + m'$, m, m' integers determined by the vanishing properties of $b'_k(z)$ and $c'_k(z)$ at a. From these considerations it descends that it is possible to describe b-c systems on Z_n symmetric surfaces in the following way:

i) introduce $2n$ independent fields b'_k, c'_k single valued on CP_1. We used a prime to distinguish these fields from the multivalued ones of eqs. (1.15), (1.15a).

ii) Their behavior near the branch points is simulated by the twist fields $V(a)$ which, under requirement A), assume the form shown in eq. (1.13).

iii) The two point functions and chiral determinants $det\bar{\partial}_j$ are given respectively by:

$$G_{j,1-j}(z-a)_{k,k} = \frac{< 0|b_k(z)c_k(w)(zero\ \ modes)\prod_{i=1}^{mn} V(c_i)\prod_{j=1}^{q} V(a_j)V(b_j)|0>}{< 0|(zero\ \ modes)\prod_{i=1}^{mn} V(c_i)\prod_{j=1}^{q} V(a_j)V(b_j)|0>} \tag{1.16}$$

and

$$det\bar{\partial}_j = \frac{< 0|(zero\ \ modes)\prod_{i=1}^{mn} V(c_i)\prod_{j=1}^{q} V(a_j)V(b_j)|0>}{det|\omega_\sigma(z_r)|\ \ det|\mu_{\sigma'}(z_{r'})|} \tag{1.17}$$

where $|0>$ is the usual $SL_2(C)$ invariant vacuum on the sphere,

$$(zero\ \ modes) = \prod_{r=1}^{N} b_k(z_r)\prod_{r'=1}^{M} c_k(z_s)$$

and $\mu_i(z_2)$, $\nu_j(z_{r'})$ are the zero modes in b and c respectively ($N - M = (2j-1)(g-1)$. Wick theorem and standard bosonization techniques provide all

the other correlation functions once the charges $\tilde{q}_k$ are known. This rises the problem of determining the charges $\tilde{q}_k$. As we said previously, this can be done only taking into account also the global properties of the fields. Before to show our method to compute the $\tilde{q}_k'$s for the most general case in which requirement A is valid, let us point out that also on an arbitrary Riemann surface (eq. A.1) there is evidence that analogs of relations (1.16) and (1.17) still hold (see [Fer2] and below). Unfortunately we do not know so far an explicit expression of the twist fields in this case.

3.2 A GENERAL METHOD TO DERIVE THE CHARGES $\tilde{q}_k$

Definition: a faithful j-differential [FK] ψ_0 is a j-differential which transforms non trivially under the symmetry group of the surface: for j half- integer the divisor D of ψ_0^2 is a quadratic divisor of degree $2g-2$, i.e. $D=\xi^2$ where ξ is a divisor of degree $g-1$. More explicitly, a faithful differential should transform non trivially under the local monodromy matrices. For example dz/y is faithful differential (y as in eq. (1.1)) but not dz/y^n. As we will see in the half integer case the spin structures change the monodromy properties of the fields. In this case the faithful $2j$-differentials can transform trivially at some branch points and we need the concept of a j differential faithful at single branch points. This means that at these points the local monodromy transformations of ψ_0 are not trivial. If we take a field $b^{(l)}dz^j$ satisfying eq. (1.6), go to the diagonal basis $b_k dz^j$ and expand b_k as in eq. (1.10) at each branch point, then it is always possible to associate a j-differential to b_k with the same behavior at the branch points:

$$\psi dz^j = \prod_i (z-c_i)^{\nu_i} \prod_j (z-a_j)^{\mu_j}(z-b_j)^{k_j}\, dz^j \qquad (2.1).$$

Here ν_i, μ_j, k_j are the charges $\tilde{q}_k$ at the branch points a_j, b_j anbd c_i. If ψ is a well defined j-differential on the Riemann surface, i.e. it has the correct monodromy properties on it, then some constraints on the phases ν_i, μ_j, k_j are in order. Since these phases come out from general quantum field, it is not immediate to conclude that they should obey such constraints. Let us proof that this statement is indeed true. First of all ψ should be branched as the surface on which it is defined. So

for example at a branch point of multiplicity ν, $\psi \sim (z-a)^{\frac{m}{\nu}}$, m being an integer. This condition is clearly fulfilled: a glance at the form of q_k in eq. (1.9) is sufficient. Secondly eq. (1.5) implies that the sum over all phases ν_i, μ_j, k_j must be an integer number. This requirement is nothing but the law of balance of fermionic charge on each sheet proved in ref. [Kni]:

$$N_{b_k} - N_{c_k} = 1 - 2j - \sum_{i=1}^{n_{bp}} \tilde{q}_{\alpha_i,k} \tag{2.2}$$

where $\tilde{q}_{\alpha_i,k}$ represent the charge at the branch points α_i and N_{b_k}, N_{c_k} denote the numbers of zero modes in b_k and c_k respectively. Therefore we conclude that to a given field b_k it corresponds a well defined j differential of the kind shown in eq. (2.1). Conversely a well defined differential has the same local behavior at branch points of a class of fields $b_k dz^j$. Due to this correspondence, we can read the charges $\tilde{q}_k$ at all the branch points directly from j-differentials using the consistency of eq. (1.10) with eq. (1.15). To decide what what are the right j-differentials carrying the charges $\tilde{q}_k$, we consider eqs. (1.16) and (1.17). Here we see that the twist fields modify the definition of their left vacuum state $|0>$. Since the branch points are sources of stress energy, the insertion of twist fields rises the energy of the system. If we choose a set of differentials with higher zeros at branch points, we get the charges $\tilde{q}_k + n_k$, n_k being a positive integer. These values are allowed by eq. (2.2), therefore leading to nonzero physical amplitudes, but in this way we ignore the spectrum of the theory corresponding to lower energy states. On the other side the $\tilde{q}_k'$s are bounded from below. In fact the two point functions of eq. (1.16) are tensors for which conditions (1.10)-(1.10a) apply as well. At this point we can determine the charges $\tilde{q}_k$ using the following recipe:

starting from a faithful j-differential ψ, we multiply it by integer powers of y, $(z-a_j)$, $(z-b_j)$, $(z-c_i)$ to get the first n_j differentials ψ_k, $(k = 0,\ldots,n-1)$ which are finite at the branch points and with the lowest degree of zeros at these points. On a Z_n symmetric surface $n-1$ of these differentials should be faithful for j integer. If j is half integer and there are spin structures or j is integer but fields have nontrivial twisted boundary conditions (twisted bosons), then at each branch point $n-1$ of the j-differentials ψ_k are faithful at those points. These last

two conditions guarantee that the local monodromy group of the surface is fully represented. The changes $\tilde{q}_k$ at the points a_j, b_j, c_i are then simply the powers $\mu_{a_j,k}$, $k_{b_j,k}$ and $\nu_{c_i,k}$ with which the factors $(z-a_j)$, $(z-b_j)$, $(z-c_i)$ appear in ψ_k (see eq. (2.1)). Let us notice that the only hypothesis needed in the above procedure is requirement A), which is unavoidable if we want to use twist fields of the form (1.13) to simulate the effect of branch points. Our method for the computation of the charges $\tilde{q}_k$ depends just on the geometrical properties of the surface such as the zeros of the n j differentials ψ_k at branch points. It takes automatically into account the distinction between rational and normal branch points. It is therefore well suited in the case of twisted bosons and fermions in which spin structures can reshuffle the eigenvalues corresponding to the basis b_k, c_k at different branch points as we will see.

3.3 APPLICATIONS TO SOME SIMPLE CLASSES OF SURFACES CORRELATION FUNCTIONS AND GREEN FUNCTIONS

We give some applications of the method just explained through explicit examples. The characteristics of spin structures are denoted here by the symbol: $\begin{bmatrix} a_1 & \dots & a_g \\ b_1 & \dots & b_g \end{bmatrix}$. When a_i and b_i $(i = 1, \dots, g)$ are rational numbers of the kind p/n, we say that the spin structure has Z_n characteristic. The charges of twist fields at a generic branch point α are denoted with $\tilde{q}_{\alpha,k}$ We will often use also the vectorial notation for the charges $\bar{\tilde{q}} = (\tilde{q}_0, \dots, \tilde{q}_{n-1})$. Moreover the divisor of a j-differential ψ is written in the following form:

$$div\psi = \frac{z_1 \dots z_M}{p_1 \dots p_N}.$$

Of course $N - M = j(2g-2)$. Finally when dealing with curves with many branch points and sheets, it is convenient to use a matricial representation of the homology basis because the graphical picture turns out to be often not clear. The details of this representation are in appendix B.

Bosons with Conformal Weight j

Let us consider the surface $\Sigma(g,n,m,0)$. The holomorphic differentials for this surface are:

$$\omega_{l,\lambda}dz = \frac{z^\lambda dz}{y(z)^l} \begin{cases} 1+\delta_{1,m} \leq l \leq n-1 \\ \lambda = ml-2, l-3, \ldots, 0 \end{cases}.$$

Since the divisor of dz is:

$$divdz = \frac{c_1^{n-1}\ldots c_{mn}^{n-1}}{\infty_1^2 \ldots \infty_n^2}$$

the j-differentials ψ_k of the previous section are given by:

$$\psi_k = \frac{dz}{y^{n-1-k}}.$$

The powers with which the factors $(z-c_i)$ appear in ψ_k are $\nu_{i,k} = \frac{k+j(1-n)}{n}$. Therefore we conclude that the charges $\tilde{q}_{c_i,k}$ are:

$$\tilde{q}_{c_i,k} = \frac{k+(1-n)}{n}.$$

At this point we can compute the two point functions and the correlation functions for these surfaces using the bosonization techniques on the sphere. Let us consider $j>1$. A basis of the holomorphic j differentials is provided by:

$$\omega_{l,\lambda}dz^j = \frac{z^\lambda dz^j}{y(z)^l} \begin{cases} l = \left\{\frac{2j}{m}\right\}, \ldots, j(n-1) \\ \lambda = ml-2j, ml-2j-1, \ldots, 0 \end{cases}.$$

where $\{x\}$ denotes the integer part of x. Not all these differentials are independent when $j(n-1) \geq mn + \left\{\frac{2j}{m}\right\}$. In this case the values with $\lambda \geq mn$ are not to be taken into account. Relation (2.2) implies that there are $M_0 = 1-2j-mj(1-n)$ zero modes for b_0, $M_1 = 1-2j-mj(1-n)-m$ zero modes for $b_1 \ldots$ *and* $M_{n-1} = 1-2j-mj(1-n)-\frac{n(n-1)}{2}m$ for b_{n-1}. Therefore the correlation function has the following form:

$$< 0| \prod_{i_0=1}^{M_0} \prod_{i_1=M_0+1}^{M_0+M_1} b_i(z_{i_1}) \ldots$$

$$\ldots \prod_{i_{n-1}=M_0+M_1+\ldots M_{n-2}+1}^{M_0+\ldots+M_{n-1}} b_{n-1}(z_{i_{n-1}}) V(c_1)\ldots V(c_{mn})|0> =$$

$$\prod_{\lambda=1}^{M_0+\ldots+M_{n-1}} \prod_{\mu>\lambda} (z_\lambda - z_\mu) y(z_\lambda)^{-q_{k(\lambda)}} \prod_{\rho=1}^{mn} \prod_{\sigma>\rho} (c_\rho - c_\sigma)^{-\vec{q}_{r_{k\rho}}\cdot\vec{q}_\sigma} \qquad (3.2).$$

Here $\vec{q}_\rho \cdot \vec{q}_\sigma = \sum_k q_{c_\rho} \cdot q_{c_\sigma}$. For the Green functions (here we consider just $j > 1$) we get an analogous expression:

$$G_{j,1-j}(z,w)_{k,k} = \frac{dz^j dw^{1-j}}{z-w} \left[\frac{y(z)}{y(w)}\right]^k \prod_i \frac{z-z_i}{w-z_i} \qquad (3.2a)$$

Here Z_i are the locations of the zero modes of the b fields. All the results obtained are in agreement with refs. [Kni,BR1].

Let us study now the class of surfaces $\Sigma(g,n,0,q)$. A basis for first kind differentials is given by:

$$\omega_{l,i} = \frac{dz}{y^l} \frac{1}{(z-b_1)(z-b_i)} \qquad i = 2,\ldots,q.$$

The divisors of dz and y are:

$$div dz = a_1^{(n-1)} \ldots a_q^{(n-1)} b_1^{(n-1)} \ldots b_q^{(n-1)} \infty_1^2 \ldots \infty_n^2 \quad ; \quad div y = \frac{a_1 \ldots a_q}{b_1 \ldots b_q} \qquad (3.3).$$

The second of these equations shows that the charges $\tilde{q}_{a_j,k}$ will be in general different from their counterparts $\tilde{q}_{b_j,k}$. It is also clear that all the $a's$ (and the $b's$ separately) should be treated in a symmetric way. The strategy in order to find the charges is the following: let us consider the j-differentials:

$$\Omega_{l,\lambda} dz^j \qquad (3.4)$$

with divisor:

$$div \Omega_{l,\lambda} dz^j = \frac{a_1^{(n-1)j-l-n\lambda} \ldots a_q^{(n-1)j-l-n\lambda} b_1^{(n-1)j+l} \ldots a_q^{(n-1)j+l\lambda}}{\infty_1^{2j-q\lambda} \ldots \infty_{n-1}^{2j-q\lambda}}$$

The lowest order of the zeros at the rational branch points b occur when $l = j(n-1)$. This implies that one of the phases defined in eq. (1.18) is $k_i = \frac{j(1-n)}{n}$, $(i = 1,\ldots,q)$. In order that also the order of the zeros at the branch point a is the lowest possible, we have to adjust λ in such a way that:

$$\lambda = \left\{\frac{2j(1-n)}{n}\right\}.$$

This choice corresponds to

$$\mu_i = -\left[\left\{\frac{2j(1-n)}{n}\right\} + \frac{j(1-n)}{n}\right] \qquad i = 1,\ldots,q.$$

These values of k and μ determine $\psi_0 dz^j$. The second possible choice of l is now $l = j(1-n)+1$ which implies $k_i = \frac{j(1-n)+1}{n}$. Again the zeros in a are minimal if we set:

$$\lambda = \left\{ \frac{2j(1-n)-1}{n} \right\}$$

This implies:

$$\mu_i = -\left[\left\{ \frac{2j(1-n)-1}{n} \right\} + \frac{1+j(1-n)}{n} \right] \quad i = 1, \ldots, g.$$

Repeating this procedure for $n-3$ times, we get all the j-differentials ψ_k, $(k = 0, \ldots, n-1)$ from which we can read the charges $\tilde{q}_{a,k}$ and $\tilde{q}_{b,k}$:

$$\begin{aligned} \tilde{q}_{b,k} &= \tfrac{k+j(1-n)}{n} \\ \tilde{q}_{a,k} &= -\left[\left\{ \tfrac{2j(1-n)-k}{n} \right\} + \tfrac{k+j(1-n)}{n} \right] \end{aligned} \qquad (3.5).$$

We consider just an example because the general case entails a complicated notation: $j = 3$ and $\Sigma(6,4,0,3)$. Then: $\vec{\tilde{q}_b} = \left(\frac{-9}{4}, -2, \frac{-7}{4}, \frac{-3}{2}\right)$ and $\vec{\tilde{q}_a} = \left(\frac{-7}{4}, -2, \frac{-9}{4}, \frac{-3}{2}\right)$. Putting these values in eq.(2.2) we obtain that $j = 3$ differentials admit 7 zero modes in b_0, b_1 and b_2 respectively and 4 zero modes in b_3. The total number of zero modes is 25 in agreement with Riemann Roch theorem. The correlation functions can now be computed in the same way as we did for the case of surfaces $\Sigma(g,n,m,0)$.

We discuss now a somewhat different class of surfaces, those associated with the multivalued functions:

$$y(z) = \sqrt[n]{q(z)} + \sqrt[m]{p(z)} \qquad (3.6).$$

Let us consider for example $j = 1$, $n = 2$, $m = 3$ and $q(z), p(z)$ polynomials of degrees n and m respectively. More general cases do not present in fact any further conceptual complication. $y(z)$ satisfies an algebraic equation of the form:

$$y^6 - 3q(z)y^4 - 2p(z)y^3 + 3q^2(z)y^2 - 6p(z)q(z)y + p^2(z) - q^3(z) = 0 \qquad (3.7).$$

The branch points are obviously the zeros of $p(z)$, α_1, α_2, α_3 and the zeros of $q(z)$ β_1 and β_2. The Riemann surface constructed in terms of sheets and branch points

is shown in fig. (1.17). There are two different projections of the branch points α_i on the sheets which we have denoted with p_i and p'_i. The branch points β_j, $(j = 1, 2)$ have instead three different projections q_j, q'_j, q''_j. The algebraic function $q(z)$ of eq.(3.6) is strictly related with $y'(z) = \sqrt[2]{q(z)}\sqrt[3]{p(z)}$. In fact the monodromy matrices are:

$$M_{\alpha_i} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad M_{\beta_j} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore requirement A of section 2 is still fulfilled. The genus of this algebraic surface is four. Accordingly we have four independent holomorphic differentials:

$$\omega_1 = \frac{dz}{\sqrt[3]{p^2(z)}\sqrt{q(z)}} \qquad \omega_2 = \frac{dz}{\sqrt[3]{p(z)}\sqrt{q(z)}}$$

$$\omega_3 = \frac{dz}{\sqrt[3]{p^2(z)}} \qquad \omega_4 = \frac{dz}{\sqrt[3]{p(z)}\sqrt{q(z)}}$$

with:

$$div(dz) = \frac{p_1^2 p_2^2 p_3^2 p_1'^2 p_2'^2 p_3'^2 q_1^2 q_2^2 q_1'^2 q_2'^2 q_1''^2 q_2''^2}{\infty_1^2 \ldots \infty_6^2}.$$

Four of the ψ_k are the holomorphic differentials. The remaining ψ_k are:

$$\psi_4 dz = \frac{dz}{\sqrt[3]{p(z)}} \quad ; \quad \psi_5 dz = dz.$$

These j-differentials yield the following charges:

$$\vec{q}_{\alpha_i} = (-\tfrac{2}{3}, \tfrac{1}{3}, 0, -\tfrac{2}{3}, -\tfrac{1}{3}, 0)$$

$$\vec{q}_{\alpha_i} = (-\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, 0, 0, 0$$

Substituting these values in eq. (2.2) we obtain $\sum_k N_{b_k} - N_{c_k} = 3$ as it is expected from Riemann Roch theorem. Finally we compute the part of the Green function $G_{1,0}(z, w)$ independent on the zero modes: this is the relevant part for the computation the expectation value of the stress energy tensor [Gav]:

$$G_{1,0}(z, w) =$$

$$\frac{1}{6(z-w)}\left(\frac{p^{\frac{2}{3}}(w)q^{\frac{1}{2}}(w)}{p^{\frac{2}{3}}(z)q^{\frac{1}{2}}(z)}+\frac{p^{\frac{1}{3}}(w)q^{\frac{1}{2}}(w)}{p^{\frac{2}{3}}(w)q^{\frac{1}{2}}(w)}+\frac{q^{\frac{1}{2}}(w)}{q^{\frac{1}{2}}(z)}+\frac{p^{\frac{2}{3}}(w)}{p^{\frac{2}{3}}(z)}\frac{p^{\frac{1}{3}}(w)}{p^{\frac{1}{3}}(w)}+1\right).$$

$G_{1,0}(z,w)$ is a third kind differential as it should be as pointed out in ref. [Fer2]. As we have seen the b-c systems on the surfaces as in eq. (3.6) correspond to correlation functions of a conformal field theory with the insertion of spin fields of the kind:

$$V(z)=e^{i\frac{k}{n}\varphi(z)} \text{ and } \mathrm{V(z)}=\mathrm{e}^{\mathrm{i}\frac{k'}{m}\varphi(z)}.$$

The Green functions of b-c systems with conformal weight j obtained with bosonization can be compared with their analogs constructed by means of the fermionic construction of ref. [Fer2, Fer4]. Let us apply this method in the case of the surfaces $\Sigma(g,n,m,0)$. it is known from refs. [Ver,Bon] that the green functions $G_{j,1-j}(z,w)$ are determined by their poles and zeros when z, w approach the coordinates z_i in which the zero modes are located. Moreover the pole at $z=w$ should have residue equal to 1. Therefore ($j>1$):

$$G_{j,1-j}(z,w) = \frac{det \begin{vmatrix} \Omega_w(z) & \Omega_1(z) & \dots & \Omega_N(z) \\ \Omega_w(z_1) & \Omega_1(z_1) & \dots & \Omega_N(z_1) \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_w(z_N) & \Omega_1(z_N) & \dots & \Omega_N(z_N) \end{vmatrix}}{det\,|\Omega_j(z_i)|} \qquad (4.1).$$

In eq. (4.1) $N=(2j-1)(g-1)$, $\Omega_i(z)dz^j$, $(i=1,\dots,n)$ are the zero modes determined as in the previous section and $\Omega_w(z)dz^j$ is a tensor satisfying the following equation on Σ:

$$\bar{\partial}\Omega_w(z) = \delta(z,w) \qquad (4.2).$$

In the case of $j=1$,which we do not discuss, there exist an analogous expression with a third kind differential at the place of $\Omega_w(z)dz^j$ [Ver]. The only unknown quantity of eq. (4.1) is the meromorphic j-differential $\Omega_w(z)dz^j$. To compute it explicitly, we assign to the sheet l, $(l=0,\dots,n-1)$, the branch $y^{(l)}(z) = e^{\frac{(2\pi i l}{n}} \sqrt[n]{(z-c_1)\dots(z-c_{mn}}$. Hence the differential:

$$G_w(z)dz = \frac{1}{n}\frac{\prod_{i=1}^{n-1}(y(z)-\alpha^i y^{(l)}(w))}{(z-w)y(z)^{n-1}} \qquad \alpha^n = 1$$

exhibits a first order pole with residue 1 at the projection of the point $z=w$ on the $l-th$ sheet. Besides this, $G_w(z)$ has also n spurious poles at the points $z=\infty_1\dots\infty_{n-1}$ due to the fact that $g_w(z)dz \sim \frac{dz}{(z-w)}$ near infinity. These extra singularities cancel in the final expression of $\Omega_w(z)dz^j$ which is:

$$\Omega_w(z)dz^j = \frac{1}{n}\frac{\prod_{i=1}^{n-1} y(z)-\alpha_i y^{(l)}(w)}{(z-w)}\left[\frac{dz}{y(z)^{n-1}}\right]^j\left[\frac{dw}{y(w)^{n-1}}\right]^{1-j} \qquad (4.3).$$

Moreover all the singularities in w occur when $w = \infty$ and come out from terms of the kind $f(w)\Omega(z)$, where $f(w)$ is a meromorphic function in w and $\Omega(z)$ a linear combination of the holomorphic j-differentials. Obviously these terms automatically drop out from the Green functions of eq. (4.1).

3.5 BOSONIZATION IN THE CASE OF ALGEBRAIC SURFACES WITH NON ABELIAN MONODROMY GROUP

The fermionic construction method discussed in the previous section can be generalized to arbitrary algebraic surfaces [Fer3]. The natural question arises, if it is possible to extend also the "bosonized" version which makes use of the twist fields to more general surfaces. The answer is yes in the examples we have worked out. In fact from the Green functions computed by means of fermionic construction, we can compute the mean value of the stress energy tensor and of the current $J(z) =: b(z)c(z) :$. From all the examples worked out, (see e.g. [Fer2] and below), looking at the pole structure of the stress energy tensor and of the current $J(z)$ at the branch points, it turns out that the branch points are sources of stress energy and have a well defined conformal weight. In particular relation (1.12a) seems to hold. Therefore we can imagine also on a general surface a basis of fields b'_k, c'_k, singlevalued on CP_1 and a set of twist fields $V(a_i)$ which simulate the branch points. The OPE of b', c' with $V(a_i)$ will be involved, but in a neighborhood of the branch points becomes as in eqs. (1.15)-(1.15a). The charges $\tilde{q}_k$, $\tilde{q}'_k$ can be considered as " asymptotic charges". Moreover the geometrical method outlined at the end of section two to find the basis ψ_k is also true because we did not use any peculiar property of the Z_n symmetric curves to proof it. The demonstration that given a certain behavior of the multivalued fields b_k and c_k at the branch points there is a j-differential with the same behavior and conversely still holds in the general case. So we can find the j-differentials ψ_k (and the $1-j$ differentials ψ'_k)regular at the branch points and with the smallest possible degree of zeros at these points. They are related to a "natural" basis of singlevalued fields b'_k and c'_k.

Of course this basis does not simultaneously diagonalize the monodromy matrices because requirement A no longer holds. The behavior of these differentials at the branch points is again the behavior of the Green functions themselves. Nevertheless the outlined picture entails that b-c systems on higher genus Riemann surfaces are equivalent to (indeed involved) conformal field theories on the plane. In particular the Green functions of the former correspond to multipoint correlation functions of the former (eq. (1.16)). The twist fields can be imagined as insertions of spin fields localized at the branch points. Hence we can say that the behavior of the ψ_k (ψ'_k) at these points is nothing but the result of the OPE between the twist fields nd the b'_k (c'). The only difference from the case of Z_n symmetric surfaces is that now the j-differentials ψ_k are no longer of the simple form (2.1). In fact a general j-differential is always of the form $\psi dz^j = R(z,y(z))dz^j$, with $R(z,y(z))$ a rational function in the variables z and y. R involves in general nested roots from which it is difficult to extract the informations like the asymptotic behavior near a branch point. Therefore to read the asymptotic charges $\tilde{q}_k$, $\tilde{q}'_k$ from the j-differentials we need the following integral expression of a general functions $R(z,y)$ [Mum]:

$$R(z,y(z)) = exp\left[\int^z dt \sum_i \omega_{z_i q_i} + \sum_{j=1}^{g} m_j \omega_j\right] \qquad (5.1).$$

Here ω_j denotes a basis of holomorphic differentials and $\omega_{z_i q_i}$ is a meromorphic differential with first order poles at z_i and q_i. z_i and q_i turns out to be the zeros and the poles of R respectively. Finally the m'_js are numbers determined by the condition:

$$\oint_{A_k} dt \sum_i \omega_{z_i q_i} + \sum_{j=1}^{g} m_j \omega_j = 0 \qquad k = 1, \ldots, g \qquad (5.2)$$

over all the homology cycles of the kind A (fig. 1.7). Now the points z_i, q_i in the case of the j-differentials are branch points or infinity points. In eq. (5.1) the behavior of R at each of these points is transparent. Since $\int^z \omega_{z_i,q_i} \sim \mu_i log(z - z_i) - \nu_i log(z - q_i) + \ldots$ it is also possible to obtain easily an expansion of $R(z,y)$ in their neighborhood. Let us illustrate the procedure with a simple example provided by the surface associated to the multivalued function:

$$y = \sqrt[3]{q(z) + \sqrt{p(z)}} \qquad (5.3).$$

$p(z)$ is a polynomial of degree 6 and $q(z)$ has degree less than three. The algebraic equation satisfied by y is:

$$y^6 - 2q(z)y^3 + q^2(z) - p(z) = 0.$$

The branch points are the six zeros of $p(z)$ α_i, $(i = 1, \ldots, 6)$, for which $y^3 = q(z)$ and the zeros of y q_j, $(j = 1, \ldots, 6)$ which occur when $q^2(z) = p(z)$. The Riemann representation of this surface in terms of sheets is shown in fig. (1.18). In this figure we have denoted with p_i, p_i', p_i'' the projections on the sheets of the branch points α_i. The surfaces of fig. (1.18) has genus 10. The monodromy matrices are:

$$M_{\alpha_i} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \qquad M_{q_i} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

and do not commute. A basis of holomorphic differentials is:

$$\begin{aligned} \omega_{1,j}dz &= \frac{z^j dz}{\sqrt{p(z)}y^2} \qquad j = 0,1,2,3 \\ \omega_{2,j}dz &= \frac{z^j dz}{\sqrt{p(z)}y} \qquad j = 0,1,2 \\ \omega_{3,j}dz &= \frac{z^j dz}{\sqrt{p(z)}} \qquad j = 0,1 \\ \omega_4 dz &= \frac{dz}{y^2} \end{aligned}$$

with

$$div(dz) = \frac{p_1 \cdots p_6 p_1' \cdots p_6' p_1'' \cdots p_6'' q_1^2 \cdots q_6^2}{\infty_1^2 \cdots \infty_6^2}.$$

The j-differentials ψ_k with minimal zeros at branch points are given by:

$$\begin{aligned} \psi_0 dz = \omega_{1,0}dz \ ; \ \psi_1 dz = \omega_{2,0}dz \\ \psi_2 dz = \omega_{3,0}dz \ ; \ \psi_0 dz = \omega_4 dz \\ \psi_4 = \frac{dz}{y} \ ; \ \psi_5 = dz \end{aligned}$$

We need also the functions ψ_k' corresponding to the fields c_k'. In fact in the non abelian case the general expression of the twist field is not known. It is just possible, through the j-differentials, to guess the OPE of the twist fields with the

b'_k. Nevertheless also the fields c'_k enter in the computation of the Green functions and therefore we must know also their OPE with the twist fields. The ψ' are the following functions:

$\psi'_0 = 1$, $\psi'_1 = y$, $\psi'_2 = y^2$, $\psi'_3 = \sqrt{p}$, $\psi'_4 = \sqrt{p}y$, $\psi'_5 = \sqrt{p}y^2$. As we see, even in this simple example it is not easy to find the behavior of ψ_k and ψ'_k at branch points because $y = \sqrt[3]{q + \sqrt{p}}$. Therefore we use the representation of eq. (5.1). It is just sufficient to express in this way $p(z)$ and y. First of all we need the third kind differentials at the branch points. It is possible to show that:

$$\omega_{p_i,\infty_1 \ldots \infty_6} = \frac{1}{6} \frac{(y - \sqrt[3]{q(\alpha_i)})(y^2 + \sqrt[3]{q(\alpha_i)}y + \sqrt[3]{q^2(\alpha_i)})^2}{(z - \alpha_i)} \frac{dz}{\sqrt{p}y^2}$$

has exactly a pole of first order and residue $+1$ at p_i and a pole of first order and residue $-\frac{1}{6}$ at the projections of $z = \infty$ on the various sheets. Analogous expressions are valid for p'_i and p''_i substituting $\alpha \sqrt[3]{q(\alpha_i)}$ and $\alpha^2 \sqrt[3]{q(\alpha_i)}$ to $\sqrt[3]{q(\alpha_i)}$. As well we can write the third kind differentials at the points q_i:

$$\omega_{q_i,\infty_1 \ldots \infty_6} = \frac{1}{6} \frac{q + \sqrt{(p(z) - 2q(q_i)}}{(z - q_i)} \frac{dz}{\sqrt{(p(z)}}.$$

Again ω_{q_i} has a pole of first order with residue $+1$ at q_i and a pole of first order with residue $\frac{1}{6}$ at $\infty_1 \ldots \infty_6$. Therefore we can write:

$$p(z) = exp \int^z dt \left[\sum_i (\omega_{p_i,\infty_1 \ldots \infty_6} + (p_i \leftrightarrow p'_i) + (p_i \leftrightarrow p''_i) + \sum_{j=1}^{g} m_j \omega_j \right] =$$

$$= exp \int^z dt \left[\sum_i \frac{(q(t) + \sqrt{p(t)} - q(\alpha_i)}{t - \alpha_i} \frac{1}{\sqrt{(p(t)}} + \sum_{j=1}^{g} m_j \omega_j \right] \qquad (5.4)$$

and:

$$y(z) = exp \int^z dt \left[\sum_i \frac{(q(t) + \sqrt{p(t)} - 2q(q_i)}{t - q_i} \frac{1}{\sqrt{(p(t)}} + \sum_{j=1}^{g} m'_j \omega_j(t) \right] \qquad (5.4a).$$

m and m' depend on the branch points. Using eqs. (5.4) and (5.4a) we can study the behavior of ψ_k near the branch points. For example:

$$\psi_0 =$$

$$exp\int^z dt\left[-\tfrac{1}{2}\sum_i \tfrac{(q(t)+\sqrt{p(t)}-q(\alpha_i)}{t-\alpha_i}\tfrac{1}{\sqrt{(p(t)}}-\tfrac{1}{3}\sum_i \tfrac{(q(t)+\sqrt{p(t)}-2q(q_i)}{t-q_i}\tfrac{1}{\sqrt{(p(t)}}-\sum_{j=1}^{g}(\tfrac{m_j}{2}+2m'_j)\omega_j(t)\right] \tag{5.5}$$

The equation above, when z approaches a branch point a, yields the OPE b'_0 with the twist field $V(a)$. As a matter of fact, studying the third kind differentials appearing in the exponent of ψ_0, we can compute the asymptotic charges which are:

$$\psi_0 \sim (z-\alpha_i)^{\tilde{q}_{\alpha_i,0}} \qquad with\ \tilde{q}_{\alpha_i} = -\frac{1}{2}$$

$$\psi_0 \sim (z-q_i)^{\tilde{q}_{q_i,0}} \qquad with\ \tilde{q}_{q_i,0} = \begin{cases} -\frac{2}{3}\ at\ the\ sheets\ 3,4,5 \\ 0\ at\ the\ sheets\ 0,1,2 \end{cases}$$

Analogously we find:

$$\tilde{q}_{\alpha_i,1} = -\frac{1}{2} \qquad \tilde{q}_{q_i,1} = \begin{cases} -\frac{1}{3}\ at\ the\ sheets\ 3,4,5 \\ 0\ at\ the\ sheets\ 0,1,2 \end{cases}$$

$$\tilde{q}_{\alpha_i,2} = -\frac{1}{2} \qquad \tilde{q}_{q_i,2} = 0$$

$$\tilde{q}_{\alpha_i,3} = 0 \qquad \tilde{q}_{q_i,3} = \begin{cases} -\frac{2}{3}\ at\ the\ sheets\ 3,4,5\ . \\ 0\ at\ the\ sheets\ 0,1,2 \end{cases}$$

$$\tilde{q}_{\alpha_i,4} = 0 \qquad \tilde{q}_{q_i,4} = \begin{cases} -\frac{1}{3}\ at\ the\ sheets\ 3,4,5 \\ 0\ at\ the\ sheets\ 0,1,2 \end{cases}$$

$$\tilde{q}_{\alpha_i,5} = 0 \qquad \tilde{q}_{q_i,1} = 0$$

As can be expected, the charges $\tilde{q}_k$ are dependent from the sheets in the non abelian case. In fact, according to [BR3], in this case the charges are matrices $\tilde{q}_k = \tilde{q}_k(l)$ depending also on the various different projections of the branch points on the sheets. Following [BR3], the Riemann Roch theorem is now given by:

$$N_b - N_c = \sum_{sheets}(1-2j) - \sum_{i=1}^{n_{b.p.}} tr\tilde{q}_{i,k}(l) \tag{5.6}$$

where:

$$\tilde{q}_{q_i,k}(l) = \begin{pmatrix} 0 & 0 & 0 & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \tilde{q}_{\alpha_i,k}(l) = \begin{pmatrix} -\frac{1}{2} & \dots & -\frac{1}{2} \\ 0 & \dots & 0 \\ -\frac{1}{2} & \dots & -\frac{1}{2} \\ 0 & \dots & 0 \\ -\frac{1}{2} & \dots & -\frac{1}{2} \\ 0 & \dots & 0 \end{pmatrix}.$$

Putting these matrices on eq. (5.6) we get $N_b - N_c = 9$ as expected. Moreover we know also for which values of k the zero modes of the fields b'_k and c'_k occur. In fact

from the definition of ψ_k and ψ'_k we see that there are 4 zero modes in b'_0, 3 in b'_1, 2 in b'_2, 1 in b'_3 and 1 in c'_5. Finally we test our hypothesis that the ψ_k and ψ'_k provide the OPE of the fields with the twist fields. Of course in this non abelian case we do not know the OPE between the twist fields themselves, but it is still possible to compute the Green functions using the "bosonization" rules induced by ψ_k, ψ'_k in eq. (2.16). The nonvanishing Green functions are:

$$G_{kk}(z,w) = \frac{< b_k(z)c_k(w)(zero\ \ modes)_k \prod_{i=1}^{6} V(\alpha_i) \prod_{i=1}^{6} V(q_i) >}{< (zero\ \ modes)_k \prod_{i=1}^{6} V(\alpha_i) \prod_{i=1}^{6} V(q_i) >}.$$

Let us compute for example $G_{00}(z,w)$. In this case there are four zero modes $b_0(z_j)$, $(j = 1,\ldots,4)$, so we have:

$$G_{00}(z,w) = \frac{< b_0(z)c_0(w)b_0(z_1)\ldots b_0(z_4) \prod_{i=1}^{6} V(\alpha_i) \prod_{i=1}^{6} V(q_i) >}{< b_0(z_1)\ldots b_0(z_4) \prod_{i=1}^{6} V(\alpha_i) \prod_{i=1}^{6} V(q_i) >}.$$

From eq. (4.5) and considering the fact that ψ'_0 is exactly the inverse of ψ_0, we see that:

$$G_{00}(z,w) = \frac{dz}{z-w} \frac{\sqrt{p(w)}y^2(w)}{\sqrt{p(z)}y^2(z)} \frac{(z-z_1)\ldots(z-z_4)}{(w-z_1)\ldots(w-z_4)}.$$

Analogously we obtain all the other Green functions. From eq. (1.16) they are just determined by the finiteness at $z,w = \infty$ and their poles at the locations of the zero modes. Finally the residue at the pole $z = w$ should be 1. Let us now recover the Riemann Roch theorem from the residues of the conserved charges $J_k(z)$ at the branch points:

$$< J_k(z) >= \lim_{z\to w} G_{kk}(z,w) - \frac{1}{z-w}.$$

For example

$$< J_0(z) >\sim \frac{d}{dz} ln(\sqrt{p(z)}y^2(z)) + \sum_j \frac{1}{z-z_j}.$$

Substituting the expression of $\sqrt{p(z)}y^2(z)$ in terms of third kind differentials (eqs. (5.4) and (5.4a)) we get:

$$< J_0(z) >\sim -[\frac{1}{2} \sum_i \frac{q(z) + \sqrt{p(z)} - q(\alpha_i)}{(z-\alpha_i)\sqrt{p(z)}} dz -$$

$$-\frac{1}{3}\sum_i \frac{q(z)+\sqrt{p(z)}-2q(q_i)}{(z-q_i)\sqrt{p(z)}}dz+\sum_j \frac{dz}{w-z_j}].$$

The contribution to the residues at branch points depends now also on the sheet l. Summing these contributions over all the sheets we have:

$$J_0(z) >\sim -\frac{1}{2}\sum_i \frac{dz}{(z-\alpha_i)}-\frac{1}{3}\sum_i \frac{1}{(z-q_i)}+\sum_j \frac{1}{w-z_j}.$$

Therefore:

$$N_{b_0}-N_{c_0}=\oint_C < J_0(z) > dz = -1$$

Here C is a contour enclosing all the singularities of $< J_0(z) >$. This result can be expected from the charge conservation for the fields b_0', c_0'.In fact from conformal field theory we know that all amplitudes in which the inserted operators do not annihilate the charge $(1-2j)$ vanish. This confirms eq. (5.6) at least on this particular class of Riemann surfaces provided by eq. (5.3) and $j=1$. Moreover we can now use the knowledge of the Green functions $G_{kk}(z,w)$ to derive the usual Green functions. The latter are a linear combination of the former as is shown in ref [BR3]. The coefficients are chosen in such a way that the residue at the pole $z=w$ is the unity. The result, limiting ourselves just to the part independent on the zero modes which is the relevant contribution to the vacuum expectation value of the stress energy tensor, is:

$$G_{1,0}(z,w)=$$

$$\frac{dz}{6(z-w)}\left(\frac{\sqrt{p(w)}y(w)^2}{\sqrt{p(z)}y(z)^2}+\frac{\sqrt{p(w)}y(w)}{\sqrt{p(z)}y(z)}+\frac{\sqrt{p(w)}}{\sqrt{p(z)}}+\frac{y(w)^2}{y(z)^2}+\frac{y(w)}{y(z)}+1\right) \quad (5.7).$$

As we can see, this is a third kind differential with a pole in $z-w$. The sheets in which this pole occurs depends on the branches of $y(w)$ and $\sqrt{p(w)}$ we have chosen. Eqs. (5.5) and (5.6) are in agreement with ref.[BR3]. Let us recall that in this paper analogs of our j-differentials ψ_k and ψ_k' are defined through a system of differential equations:

$$\frac{d}{dz}\psi_k(z)=\sum_{i=1}^{n_{b.p.}}\frac{q_{\alpha_i,k}(l)}{z-\alpha_i}\psi_l(z) \quad (5.7)$$

$$\frac{d}{d\alpha_j}\psi_l(z)=-\frac{q_{\alpha_j,k}(l)}{z-\alpha_j}\psi_l(z) \quad (5.8)$$

Here we denote with α_i the branch points of a general surface ($i = 1, \ldots, n_{b.p.}$). Similar equations define also the ψ'_k. The charges $q_{\alpha_i,k}(l)$ should satisfy the Schlesinger equation [SJM] in order that eqs.(5.7) and (5.8) are compatible. Our method relates the zeros of the j-differentials ψ_k and ψ'_k at branch points to physical considerations on the total energy of the left state $\overline{|0>} = V(\alpha_1)\ldots V(\alpha_{n_{b.p.}}|0>$ of the Green functions of eq. (1.16). This allows an explicit computation of the j-differentials ψ_k and ψ'_k. In this way we get also the OPE of the related fields b'_k and c'_k with the twist fields. From this it is possible to find the asymptotic charges $\tilde{q}_{\alpha_i,k}(l)$. Finally using the OPE between fields and twist fields we are able to express the Green functions in the bosonized form. For the case $j > 1$ we proceed in an analogous way. The new ψ_k are those of $j = 1$ multiplied by the factor $\frac{dz^{j-1}}{(\sqrt{p(z)}y^2)^{j-1})}$. In this way the zeros at branch points are kept to a minimal degree. Since:

$$div\frac{1}{\sqrt{p(z)}y^2)} = \frac{\infty_1^5 \ldots \infty_6^5}{p_1 \cdots p_6 p'_1 \cdots p'_6 p''_1 \cdots p''_6 q_1^2 \cdots q_6^2}$$

and

$$div(\frac{\sqrt{p(z)}y^2}{dz}) = \infty_1^3 \ldots \infty_6^3$$

the ψ'_k are obtained multiplying the ψ'_k of the case $j = 0$ with the factor:

$$\frac{dz^{1-j}}{(\sqrt{p(z)}y^2)^{1-j})}$$

. Again we verify that the asymptotic charges satisfy the condition: $\tilde{q}_k = -\tilde{q}'_k$. The charges $\tilde{q}_k$ are simply those of $j = 1$ shifted by a constant term:

$$\tilde{q}_{q_i,k}(l) = \begin{pmatrix} 0 & 0 & 0 & -\frac{2}{3} - \frac{2(j-1)}{3} & -\frac{2}{3-\frac{2(j-1)}{3}} & -\frac{2}{3} - \frac{2(j-1)}{3} \\ 0 & 0 & 0 & -\frac{1}{3} - \frac{2(j-1)}{3} & -\frac{1}{3} - \frac{2(j-1)}{3} & -\frac{1}{3} - \frac{2(j-1)}{3} \\ 0 & 0 & 0 & -\frac{2(j-1)}{3} & -\frac{2(j-1)}{3} & -\frac{2(j-1)}{3} \\ 0 & 0 & 0 & -\frac{2}{3} - \frac{2(j-1)}{3} & -\frac{2}{3} - \frac{2(j-1)}{3} & -\frac{2}{3} - \frac{2(j-1)}{3} \\ 0 & 0 & 0 & -\frac{1}{3} - \frac{2(j-1)}{3} & -\frac{1}{3} - \frac{2(j-1)}{3} & -\frac{1}{3} - \frac{2(j-1)}{3} \\ 0 & 0 & 0 & -\frac{2(j-1)}{3} & -\frac{2(j-1)}{3} & -\frac{2(j-1)}{3} \end{pmatrix}$$

$$\tilde{q}_{\alpha_i,k}(l) = \begin{pmatrix} -\frac{1}{2} - \frac{(j-1)}{2} & \cdots & -\frac{1}{2} - \frac{(j-1)}{2} \\ -\frac{(j-1)}{2} & \cdots & -\frac{(j-1)}{2} \\ -\frac{1}{2} - \frac{(j-1)}{2} & \cdots & -\frac{1}{2} - \frac{(j-1)}{2} \\ -\frac{(j-1)}{2} & \cdots & -\frac{(j-1)}{2} \\ -\frac{1}{2} - \frac{(j-1)}{2} & \cdots & -\frac{1}{2} - \frac{(j-1)}{2} \\ -\frac{(j-1)}{2} & \cdots & -\frac{(j-1)}{2} \end{pmatrix}.$$

The Riemann Roch theorem stated in the form of eq. (5.6) is again verified. All the above analysis can be extended in a straightforward way to classes of Riemann surfaces associated to the multivalued functions:

$$y = \sqrt[n]{q + \sqrt[m]{p}}$$

We believe that it is valid for all the simple surfaces of standard textbook of complex analysis for which we can compute the third kind differentials at branch points. Besides b-c systems on higher genus surfaces examples of applications are the non-abelian orbifolds. Also it should be possible to study the bosonization on a general surface of genus three generated by the multivalued function:

$$y = \sqrt[3]{q + \sqrt{q^2 + p^3}} + \sqrt[3]{q - \sqrt{q^2 + p^3}}.$$

However in this case the two members of the RHS represent a degenerate curve with singular branch point [For]. These singularities cancel in a subtle way when we put the two terms together and the behavior of fields on this surface is very complicated. For example [Fer2] in the expansion of $< T(z) >$ powers of the kind $(z - \alpha_i)^{-k/3}$ occur in some of the sheets. Of course these pieces cancel after summation over all the sheets but it seems anyway that one cannot use blindly in this case the techniques here exposed or those of ref. [BR3]. The same problems exist when we have to bosonize fermionic fields. In fact as we will show, in this case the twist fields are defined on a non-branched covering of the surface detemined by the monodromy properties of the fields. Again this covering has a complicated equation in terms of the branch point of the starting surface for which it is not simple even to find the way in which the sheets are joined. Moreover in this case we need to find the quadratic divisors corresponding to the $\frac{1}{2}$-differentials ψ_k and this can be done just solving an algebraic equation of degree $2^{g-1}(2^g + 1)$ [Sta]. In the following we will introduce the example of the surfaces of [GM]. Besides to have a physical application in the computation of high energy superstring amplitudes, they also provide a good laboratory to study spin structures on non-hyperelliptic surfaces.

CHAPTER 4

TWISTED BOSONS AND FERMIONS ON ALGEBRAIC SURFACES

4.1 TWISTED BOSONS AND FERMIONS

In this section we consider the Riemann surface Σ divided into a set of local patches $\{U_\alpha\}$ with a local system of coordinates z_α. An example of local covering $\{U_\alpha\}$ is shown in fig. (1.16). Generalizations to arbitrary Riemann surfaces are straightforward. The analytic local tensor fields $\bar{\partial}\phi_\alpha(z_\alpha)$ of conformal weight j are related at $U_\alpha \bigcup U_\beta$ by the holomorphic transition functions $g_{\alpha\beta}$:

$$\phi_\alpha(z_\alpha) = g_{\alpha\beta}\phi_\beta(z_\beta) \qquad (1.1).$$

The fields discussed in the previous section have for example transition fuctions:

$$g_{\alpha\beta} = \left(\frac{dz_\beta}{dz_\alpha}\right)^j$$

defining the complex line bundle K^j [GH], where K denotes the canonical line bundle. In this case $\phi_\alpha(z_\alpha)dz^j$ is called a trivialization of K^j. The chiral operator $\bar{\partial}_j$ is the holomorphic connection on K^j. This becomes clear if we think to the b-c fields as the left and right components of a free massless dirac field Ψ coupled with an external gauge field A_μ, $(\mu = 1, 2)$ and with the action [AMV]:

$$S = \int e d^2 z \bar{\Psi}\gamma^a e_a^\alpha(\partial_\mu + A_\mu)\Psi \qquad (1.2).$$

Here e_a^α is the usual zweibein and $\partial_z = \partial_1 + i\partial_2$. Moreover the field strength of A_μ is zero. Since the fields are massless, the Dirac equation for Ψ splits into two equations for left and right movers:

$$(\partial_{\bar{z}} + A_{\bar{z}})b = 0 \qquad (\partial_z + A_z)\bar{c} = 0 \qquad (1.3)$$

and with

$$\Psi = \begin{pmatrix} b \\ \bar{c} \end{pmatrix}.$$

The classical solutions are:

$$b = b_0 \exp \int^{\bar{z}} A_{\bar{z}'} d\bar{z}' \qquad \bar{c} = \bar{c}_0 \exp \int^{z} A_{z'} dz' \tag{1.4}$$

with $\bar{\partial} b_0 = \bar{\partial} c_0$. The action of eq. (1.2) represents a gauge theory with $U(1)$ group of transfomations. Eqs. (1.3) induce a complex structure on the line bundle K^j (and K^{1-j} for c-fields). Therefore the fields b and c of eq. (1.4) admit also a holomorphic representation. In this case a gauge tranformation $A_\mu \rightarrow A_\mu - \partial_\mu \varphi$ and

$$\Psi \rightarrow e^{\int \bar{\partial} \varphi} \Psi$$

corresponds to a local change of coordinates $z_\alpha \rightarrow z'_\alpha$ on the open sets U_α. The trivializations $\phi_\alpha(z_\alpha)$ and the gauge field respectively transform in the following way:

$$\phi_\alpha(z_\alpha) = g_{\alpha\alpha'} \phi_{\alpha'}(z'_\alpha) \tag{1.5}$$

and

$$A_{z_\alpha}(z_\alpha) = A_{z'_\alpha}(z'_\alpha) - d(log g_{\alpha\alpha'}) \tag{1.6}.$$

Eqs. (1.4) and (1.5) describe the same gauge group of transformations. In the first case the fields are not holomorphic because Σ has the structure of a $\mathbf{R}^2$ manifold. The two representations are nevertheless related apart from a gauge transformation. In fact from eq. (1.4) a gauge transformation acting on the fields is given by:

$$\Psi \rightarrow U(P)\Psi = \exp \int_{(P_{0,1}, P_{0,2})}^{(P_1, P_2)} A_\mu dP_\mu \Psi \tag{1.7}$$

where A_μ is constructed in terms of the properly normalized harmonic differentials α_i and β_i, $(i = 1, \ldots, g)$:

$$A = 2\pi i \sum_1^g \theta^i \alpha_i - 2\pi i \sum_1^g \phi^i \beta_i \tag{1.8}.$$

In the usual b-c systems the only allowed gauge transformations are those in which $\theta^i, \phi^i \in \mathrm{Z}$. If this condition is not true Ψ gets twisted boundary conditions along the nontrivial homology cycles. Now we are interested in the case in which θ^i, ϕ^i are general. Accordingly $U(P)$ becomes a multivalued function. Expressing the

harmonic forms α_i, β_i in terms of holomorphic and anti-holomorphic abelian differentials we get:

$$A_{\bar{z}} = 2\pi i \bar{u}_k (T - \bar{T})^{-1}_{kj} \omega_j(z)$$

where $\bar{u}_k = \theta^k + T_{kk'}$. We can give at this point a holomorphic representation of

$$b(z) = b_0(z) exp \int^{\bar{z}} A_{\bar{z}'} d\bar{z}'$$

replacing the exponential with a theta function having the same transformation properties along the homology cycles.

From now on we will consider just holomorphic trivializations of b-c fields. A general U(1) gauge transformation (1.5), (1.6) implies that a section $b \in K^j$ moves to a section b' belonging to the line bundle $L_{\bar{u}} = K^j \otimes L^0_{\bar{u}}$ with:

$$u = \begin{bmatrix} \theta_1 \dots \theta_g \\ \phi_1 \dots \phi_g \end{bmatrix}.$$

Here $L^0_{\bar{u}}$ is an holomorphic line bundle of degree 0. In fact the sections of $L^0_{\bar{u}}$are just multivalued functions $f_{\bar{u}}(z)$ which can be expressed in terms of theta functions.For example we can write:

$$f_{\bar{u}}(z) = \frac{\theta[\bar{u}](z)}{\theta[0](z)}.$$

Following [BR2] we impose the unitarity of the monodromy factors of $f_{\bar{u}}(z)$. This means that the changes of the branches of $f_{\bar{u}}(z)$ when a non-trivial cycle γ is crossed are of the kind:

$$f_{\bar{u}}(z) \rightarrow e^{2\pi i \varphi(\gamma)} f_{\bar{u}}(z)$$

with $\varphi(\gamma)$ a real number. The sections of $L_{\bar{u}}$ are therefore b-c fields belonging to K^j multiplied by a multivalued function. Their monodromy properties along the boundary values are therefore changed and become:

$$b(z) \rightarrow e^{2\pi i \varphi(\gamma)} b(z) \qquad c(z) \rightarrow e^{2\pi i \varphi(\gamma)} c(z) \qquad (1.9).$$

In the next section we restrict ourselves to the case in which $\varphi(\gamma)$ are rational numbers of the kind: $\varphi(\gamma) = p/n$. These spin structures will be called throughout Z_n spin structures.

Finally let us discuss the spin statistics of b-c systems. When j is an integer they are ghost, i.e bosons with the wrong statistics. In the case of twisted boundary conditions ($L_{\bar{u}} \neq K^j$) we call them twisted bosons. When j is half-integer we have the usual fermions defined on the line bundle K^j_R, where R now denotes a reference spin structure. A remarkable exception is constituted by the $\beta-\gamma$ ghosts related to supersymmetric transformations in superstring theories. In fact they are commuting fermions. This makes the bosonization rules more complicated [Ver].

In the case of hyperelliptic curves the bosonization of the $\beta-\gamma$ system is studied in [CS,La]:

$$\beta_k = i : \partial\xi e^{i\varphi_k} : \qquad \gamma_k =: \eta e^{-i\varphi_k} :$$

It is possible to show again that the branch points retain a definite conformal dimension given by the stress energy tensor:

$$T_k = -j\beta_k\partial\gamma_k + (1-j)(\partial\beta_k)\gamma_k.$$

Moreover the fields η and ξ, of conformal dimension 0 and 1 respectively, can interact with spin fields producing spurious poles in the correlation functions[Ver]. Since the twist fields of eq. (1.13) are spin fields, such extra poles are likely to occur on more general surfaces then the hyperelliptic ones. To conclude this discussion on $\beta-\gamma$ systems, let us recal that the $\frac{3}{2}$ differentials do not represent the zero modes of the $\bar{\partial}_{\frac{3}{2}}$ operator acting on the ghosts β and γ. They correspond instead to anticommuting fields of the same conformal dimension. This becomes evident when we use the prescription of eq. (1.17) of chapter 3 to compute the chiral determinant. The dependence on the zero modes of the numerator does not cancel with that of the denominator if we use $det|\omega_\sigma(z_r)|$ contructed in terms of $\frac{3}{2}$-differentials. In order to obtain the correct result We have to impose the different statistics by hand taking the inverse of such determinant in the definition of eqs. (1.16) and (1.17) of chapter 3.

According to the previous section, we want to construct twisted bosons and fermions deriving the multivalued functions $f_{\bar{u}}(z) \in L^0_{\bar{u}}$. The strategy consists in constructing meromorphic functions single-valued on Σ and with zeros and poles of order m. Acting on them with the monodromy operators, they pick up trivial phases of the kind $e^{2\pi i l_i}$ and $e^{2\pi i l'_i}$, $(i = 1, \ldots, g)$, along the homology cycles A_i and B_i respectively. It is clear that when we take the m-th root of these functions we get the so called root functions [see Sta, Krz] which are section of $L_{\bar{u}}$ with:

$$\bar{u} = \begin{bmatrix} \frac{l_1}{m} & \cdots & \frac{l_g}{m} \\ \frac{l'_1}{m} & \cdots & \frac{l'_g}{m} \end{bmatrix}$$

Looking at eq. (1.1) of the previous chapter we see that the meromorphic functions $(z - a_j)$, $(z - b_j)$ and $(z - c_i)$ have zeros of order n at branch points. Thus their n-th roots are indeed root functions corresponding to Z_n spin structures apart from the fact that the poles at infinity generate extra branches at $z = \infty$. We can get rid of this shortcoming multiplying them together in order to cancel these spurious branches:

$$f_{\bar{u}}(z) = \sqrt[n]{\prod_{k=1}^{n_{b.p.}} (z - \alpha_{i_k})^{j_k}} \quad \sum_k j_k = mn, \;\; m \text{ integer} \tag{2.1}$$

with $\{\alpha_i, i = 1, \ldots, n_{bp}\}$ denoting the set of branch points of the surface. The root functions obtained in this way correspond to ratios of theta functions as discussed above. Since the branch points are also Weierstrass points on Z_n symmetric curves, (this needs not to be true on general algebraic surfaces), the theta functions involved will be very special. It is natural to consider firstly the subset of Z_n spin structures associated with root functions with zeros at branch points. We will call them Z_n symmetric spin structures for reasons which will be clear below. This strategy has been applied for the first time in [BR2]. We start to notice here that the Z_2 symmetric surfaces look very different from the other surfaces with $n > 2$. In fact in the first case there are $2g + 2$ root functions $\sqrt{z - \alpha_i}$ corresponding to some of the 2^{2g} Z_2 spin structures. Now let us recall that the periods of Z_n spin structures form a group of n^{2g} elements. The special feature of the Z_2 symmetric surfaces lies in the fact that Using the $2g + 2$ root functions $\sqrt{z - \alpha_i}$ we can generate all the 2^{2g} elements of the group. This does not happen in all the other surfaces, partly

because the number of root functions in this case is always less than $2g+2$, partly because the number of the elements of the group is much bigger.

The second step is to catalogue all the possible b-c systems with Z_n spin structures realized in this way. In order to do this we use the elements of the Čech cohomology group $\check{H}^1(\Sigma, Z_m)$ (see chapter 1). They give the transition functions $\eta_{\alpha\beta}$ of the line bundle $L^0_{\bar{u}}$. Obviously in the case of fermions they represent just "differences" of spin structures [ABMN] because we do not know the reference spin structure R. Strictly speaking the $\eta_{\alpha\beta}$ are transition functions of twisted bosons only and therefore we will consider here just twisted bosons. Afterward we will extend the discussion to fermions. The word spin structures is an abuse in the case of twisted bosons. It is better to use the denomination Z_n twisted characteristics or Z_n twisted boundary conditions. Let us consider, to fix ideas, the surface $\Sigma(1,3,1,0)$ in fig. (1.16) and Z_3 twisted characteristics on it. For the general Z_3 surface the generalization is straightforward because, form the cuts of fig. (1.1), it is evident that they are roughly builded from a basic cell $\Sigma(1,3,1,0)$ repeated m times. Analogous is the case of rational surfaces with $q \neq 0$. Moreover the cases with general n, apart from $n=2$, do not introduce any new remarkable case. From table 1 we can read the solutions of eq. (3.4) of chapter one corresponding to the various twisted boundary conditions. (They are referred to the system of homology cycles given by fig (1.19). Now we can transport the fields b-c along all possible cycles enclosing two or more points. Soon one realizes that for the characteristics:

$$\bar{u}_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \quad ; \quad \bar{u}_2 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \tag{2.2}$$

the phases which the fields take along the cycles do not depend on the sheets. In other words, the phases which the fields undergo over two cycles obtained shifting one of them by one sheet (fig 1.20) are identical. This means that the three-fold covering of $\Sigma(1,3,1,0)$ corresponding to the characteristics of eq. (2.2) retains the Z_3 symmetry of the surface. There are no other characteristics except those in eq. (2.2) for which this is true. For example if we repeat the above procedure for Z_2 twisted boundary conditions or for the remnants Z_3 twisted boundary conditions, we see immediately that they do not conserve the Z_3 symmetry of Σ. The same analysis

shows that for the rational surface $\Sigma(2,3,0,2)$, just the following characteristics are Z_3 symmetric:

$$\begin{aligned} \bar{u}_1 &= \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix} \bar{u}_2 = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix} \\ \bar{u}_3 &= \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 1/3 \end{bmatrix} \bar{u}_4 = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 2/3 \end{bmatrix} \end{aligned} \qquad (2.3).$$

Finally on hyperelliptic curves all Z_2 characteristics respect the Z_2 symmetry of the curve. Therefore we can conclude that on Z_n symmetric surfaces there is a subset of Z_n symmetric characteristics (apart for $n = 2$ in which we have the whole set), whose representatives j-differentials, sections of the line bundle $L_{\bar{u}}$, are expressed in terms of the root functions of eq. (2.1). This descends also from Abel theorem [Fay,GM]. On a Z_n surface the branch points α_i, α_j are such that under the Abel map:

$$I[D] = n \int_{\alpha_i}^{\alpha_j} \omega \equiv 0 \qquad (2.4).$$

They correspond therefore to Z_n periods in the Jacobi variety (see appendix B) related to Z_n characteristics. Actually the analysis using the Čech cohomology provides more informations than Abel theorem. First of all it tells us that it is not possible to express general Z_m characteristics (or spin structures) on Z_n surfaces in terms of the twist fields of eq. (1.13)-chapter 3. In fact the m-fold non ramified covering is no more Z_m symmetric and the phases taken by b-c fields on cycles like those in fig. (1.20) are different. This implies that, as for the surface of eqs. (4.3)-chapter 1 and (5.3)-chapter 3, the asymptotic charges $\tilde{q}_{i,k}$ of the twist fields depend on the sheets and are matrices. Secondly the Čech cohomology method allows to find the monodromy group of the m-fold covering, solving a problem posed in ref. [BR2].

Let us consider a cycle $\gamma(\alpha_1 \dots \alpha_s^l)$ enclosing s branch points $\alpha_1 \dots \alpha_s$ and starting on the sheets l. The branch points $\alpha_1, \dots, \alpha_s$ are encircled by the cycle $\nu_1, \dots, \nu_s$ times respectively. The phases by which the fields are multiplied when transported around γ are denoted by $\varphi(\gamma(\alpha_1 \dots \alpha_s^l))$. The monodromy matrices associated with the cycle γ satisfy therefore the following equations:

$$(M_{\alpha_1}^{\nu_1} \cdot M_{\alpha_2}^{\nu_2} \dots \cdot M_{\alpha_s}^{\nu_s})_{ij} = \delta_{ij} e^{2\pi\varphi(\gamma(\alpha_1 \dots \alpha_s^l))} \qquad (2.5).$$

Now from the transition functions $\eta_{\alpha\beta} \in \check{H}^1(\Sigma, Z_m)$, we know all the phases φ for any γ and therefore we are able to provide enough equations as (2.5) in order to derive the monodromy matrices at each branch point of Σ. As an example let us consider $\Sigma(1,3,1,0)$ with the Riemann representation of fig. (1.16). Again these techniques are generalizable to all Z_n symmetric surfaces and eved to arbitrary algebraic surfaces paing the price of ackward calculations. For each of the spin structures of table 1, we compute the phases corresponding to the all possible cycles enclosing just the branch points. Thi is enough to compute the monodromy matrices. The twisted characteristic $\bar{u}_1$ of eq. (2.2) yields for example the following matrices:

$$M_{c_1} = \begin{pmatrix} 0 & 0 & \alpha \\ \alpha & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix} \quad M_{c_2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad M_{c_3} = \begin{pmatrix} 0 & 0 & \alpha^2 \\ \alpha^2 & 0 & 0 \\ 0 & \alpha^2 & 0 \end{pmatrix}.$$

($\alpha = e^{\frac{2\pi i}{3}}$). while for u_2 we have:

$$M_{c_1} = \begin{pmatrix} 0 & 0 & \alpha^2 \\ \alpha^2 & 0 & 0 \\ 0 & \alpha^2 & 0 \end{pmatrix} \quad M_{c_2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad M_{c_3} = \begin{pmatrix} 0 & 0 & \alpha \\ \alpha & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix}.$$

As we see, in both cases property A of chapter 3 holds also for those monodromy matrices which, in a certain sense, describe 3-coverings of Σ. They are therefore simultaneously diagonalizable and we expect that the bosonization with the twist fields of eq. (1.13)-chapter 3 is possible. Instead the remaining Z_3 characteristics and the Z_2 twisted boundary conditions produce matrices which do not satisfy property A (see table 2) and the standard method of twist fields is no longer valid for them. One can see that these monodromy matrices form non abelian groups, D_9 for the Z_3 characteristics and D_6 for the Z_2 characteristics. A treatment of these characteristics can be done using the methods of section 5-chapter 3. Nevertheless a further complication is introduced by the fact that for twisted bosons we need an algorithm to find the j-differentials ψ_k. This is possible, as we will see, but it requires the solution of a polynomial of order n^{2g} [Sta] which is, in the language of branch points, the analogous of the Jacobi inversion problem.

Z_n twisted fermions

Here we consider half-integer j. Some of the root functions of eq. (2.1), combined with the differentials dz^{2j} ($2j$ even) yield $2j$-differentials with quadratic divisors as required in section 3.2. They should correspond therefore to fermionic fields with Z_n symmetric spin structures after taking their root. With this respect the Z_n symmetric surfaces have a different behavior if n is odd or even. In the first case $n = 3, 5, \ldots$ we have $2j$-differentials corresponding to some Z_n symmetric twisted boundary condition. When we take its square root we get a j differential which, apart the reference spin structure R, corresponds to a Z_n spin structure again. This is because in this case $Z_n^2 = Z_n$, where Z_n^2 is a shorthand to indicate the group $G = \{1^2, \alpha^2, \ldots, \alpha^{2(n-1)} | \alpha^n = 1\}$. As an example let us consider on a general Z_3 surface the $2j$ differential:

$$\psi^2 = \left(\frac{z - c_i}{z - c_j}\right)^{\frac{2}{3}} dz.$$

(see eq. (1.1) of chapter 3 for the notation). Going around the homology cycles, some of them will encircle the branch points c_i and c_j and the differential becomes multiplied by a phase of the kind $\varphi = 2\pi i 2/3$. Since the divisor of dz on a Z_3 symmetric surface is already quadratic, it is not possible to have phases of the form: $\varphi = 2\pi i 1/3$ because they would correspond to factors like $(z - c_i)^{\frac{1}{3}}$ which are no longer perfect squares on the surface. Taking the square root of ψ we get therefore always Z_3 boundary conditions along the homology cycles. Conversely this is not true for curves with n even. In this case the Z_n twisted characteristics become Z_{2n} spin structures because now $Z_n^2 = Z_{2n}$. Since in the case of even n the group of the characteristics is enlarged, this phenomenon should be reflected in the local properties of the fields. As a matter of fact going to the local uniformizer in the neighborhood of a branch point a and acting on the fields with the monodromy matrix n times we have:

$$M_a^n \phi^{(l)}(z) = \pm \phi^{(l)}(z).$$

The minus sign is valid for odd n and allows to enlarge the group of spin structures from Z_n to Z_{2n}. Looking at the expression of a j-differential as in eq. (2.1) of chapter 3, we see that the condition that it corresponds to the square root of a

$2j$-differential is equivalent to the following linear system of equations [Fer1]:

$$\begin{cases} \frac{n\nu_i+n-1}{2} = integer \\ \frac{n\mu_j+n-1}{2} = integer \\ \frac{nk_i+n-1}{2} = integer \\ \frac{\sum_i \nu_i + \sum_j (\mu_j+k_j)}{2} = integer \end{cases} \qquad (2.6).$$

Substituting the solutions of this system of equations in eq. (2.1) of chapter 3 we can construct the faithful j-differentials ψ_k and from these the charges $\tilde{q}_k$ are derived via the procedure explained before.

Finally we have to notice that to perform the square root of a $2j$-differential ψ is not so simple. In fact we have to determine the changes of signs when $\sqrt{\psi}$ crosses the homology cycles. If we do not do this we get an absurd. For example let us consider the hyperelliptic curve $\Sigma(1,2,1,0)$ of genus 1. On this surface we have just an holomorphic differential $\omega = \frac{dz}{y}$. Its square root should correspond to the only odd spin structure $[\frac{0}{0}]$. However if we transport it naively along the homology cycles we discover that it changes the sign on both the cycles a and b. This shortcoming is of course not important in string theories where in general summation on the even spin structures is considered.

4.3 APPLICATIONS TO TWISTED BOSONS AND FERMIONS

A)$\Sigma(g,3,m,0)$

We use the system of cuts of fig (1.2). The solutions of eq. (2.6) correspond to twisted bosons with Z_3 symmetric boundary conditions. Their square root are j-differentials corresponding to fermionic fields. In order to keep the notation as simple as possible, we refer to the Riemann surface $\Sigma(1,3,1,0)$ but the formalism is easily generalized to all other cases. Using triangulation we find the two cycles A and B which, in the notation of section (1.2), assume the form:

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad B = - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The minus sign in the definition of the B cycle provides the correct intersection numbers for this homology basis. There are three solutions to eq. (2.6), the first of them is the $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ odd spin structure. The remaining two are:

$$(\nu_1, \nu_2, \nu_3) = (-1/3, -2/3, 0)$$

$$(\nu_1, \nu_2, \nu_3) = (0, -2/3, -1/3).$$

They correspond to the divisors $D_1 = \frac{c_3}{c_2}$, $D_2 = \frac{c_1}{c_2}$ and do not admit any holomorphic section. In the general case of $\Sigma(g, 3, m, 0)$ we can always manipulate the $2j$-differentials using the method of [Lew] in such a way that their quadratic divisor is composed of b.p. only and has a double pole in c_2. The square root is a faithful j-differential defined only up to a sign. Nevertheless the Z_n boundary conditions are privileged on Z_n surfaces and it is possible to know to which characteristic a j-differential ψ belongs together with its reference spin structure. Its divisor will be given by:

$$div(\psi) = \frac{c_{i_1} \dots c_{i_g}}{c_{i_k}} \quad k \neq 1, \dots g \ enskip; \quad i_k = 1, \dots, 3m.$$

We relate ψ to its analogous constructed in terms of theta functions [Fay]:

$$\psi_\theta = \frac{\theta[\beta](\int_{c_{i_k}}^{z} \omega)\eta_\gamma(z)}{\theta[\gamma](\int_{c_{i_k}}^{z} \omega)} \tag{3.1}$$

where $[\beta]$ is the spin structure corresponding to the divisor of ψ, $[\gamma]$ is an arbitrary odd spin structure and $\eta_\gamma(z)$ is the holomorphic 1/2-differential corresponding to γ. Using Riemann theorem it is easy to see that if the zeros of ψ_θ are $c_{i_1}, \dots, c_{i_g}$ then:

$$\beta = \sum_{m=1}^{g} \int_{c_{i_k}}^{c_{i_m}} \vec{\omega} - \vec{K} \tag{3.2}$$

In eq. (3.2) $\vec{K}$ is the vector of Riemann constants and β is the half period corresponding to the characteristic $[\beta]$. It is easy to show that the integrals in the

RHS, whose extrema are branch points, correspond to Z_3 characteristics. In fact, turning to our example $\Sigma(1,3,1,0)$ the periods are given by:

$$\oint_A \omega = (1-\alpha)\int_{c_1}^{c_2} \omega \quad ; \quad \oint_B \omega = (\alpha-\alpha^2)\int_{c_1}^{c_2} \omega.$$

Hence:

$$2\oint_A \omega + \oint_B \omega = 3\int_{c_1}^{c_2} \omega \qquad (3.3).$$

and the divisor D_1 correspond to the characteristic $\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$. Analogously we can infer that D_2 correspond to the characteristic $\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$. From eq. (3.1) it turns out (considering D_1 that $[\beta] = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} + [K]$. The characteristic $[K]$ associated to the period of the Riemann constant K is always a Z_2 characteristics on $\Sigma(g,3,m,0)$. When $g=1$ this is clear because $K = \frac{2\pi i+\tau}{2}$, τ being the period matrix. In the other cases we use as basepoint of K a branch point and the fact [FK] that $-2K$ is the image of the divisor of a meromorphic differential under the Abel map. Since on $\Sigma(g,3,m,0)$ an holomorphic differential with a single zero of degree $6m-6$ at one of the branch points always exist, we can take this as basepoint and conclude that $-2K$ is equivalent to the trivial period and consequently that K correspond to an half period. From this analysis it follows that the characteristics of the faithful j-differentials which are square roots of bosons with Z_3 symmetric boundary conditions correspond to Z_6 characteristics; in particular the branch points do not form any partition as happens in the case of the hyperelliptic surface [Fay]. Moreover in the case discussed above the choice of the spin structure related with the period K as the reference spin structure seems to be natural.

B) $\Sigma(g,3,0,q)$

A system of cuts is provided in fig 1.2 . As before it is convenient to fix a particular case, for example $\Sigma(2,3,0,2)$ with equation:

$$y^3 = \frac{(z-a_1)(z-a_2)}{(z-b_1)(z-b2)}.$$

The projection of this curve, viewed on CP_2, over the z axis is an hyperelliptic curve. Due to this fact y has an additional Z_2 symmetry. With respect to the system of cuts in fig. (1.2) the homology cycles are:

$$A_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

There are four solutions to eq. (2.6) given by:

$$(\mu_1, \mu_2, \nu_1, \nu_2 2) \equiv (0, 1/3, 1/3, 1/3).$$

The symbol $\equiv$ here means that the four solutions are given by all possible permutations of the R.H.S. The corresponding j-differentials admit holomorphic representatives with divisors

$$D_1 = a_1, D_2 = a_2, D_3 3 = b_1, D_4 = b_2.$$

In terms of theta functions they are of the form:

$$\psi_\theta = \frac{\theta[\beta](\int_{p_0}^{z} \vec{\omega})}{E(z, p_0)}$$

$E(z, p_0)$ being the prime form. Using Riemann theorem we have:

$$\beta = \int_{p_0}^{D_i} \vec{\omega} - \vec{K}^{p_0} \qquad (3.4).$$

$\vec{K}$ corresponds to an half period. This time it is no more possible to have an holomorphic differential with a zero of degree $2g-2$ at a branch point to show this. Instead we have to find those points p which solve the system of equations:

$$\begin{cases} \omega_1(z) + A\omega_2(z)|_{z=p} = 0 \\ \partial_z(\omega_1(z) + A\omega_2(z)|_{z=p} = 0 \end{cases}$$

ω_1, ω_2 being two independent holomorphic differentials and a a coefficient to be-determined. The solutions have rather cumbersome expressions. For our purposes

it is sufficient to fix the branch points conveniently. For example: $a_1 = -a_2 = 1$ and $b_1 = -b_2 = a$. In this case we have $A = -exp(2\pi\frac{k}{3})\sqrt[3]{\frac{p^2-a^2}{p^2-1}}$, $(k = 1,2,3)$ and six values of p given by the infinity points ∞_k and the zero points 0_k at each sheet k. This is in agreement with the fact that for genus two surfaces there are six odd spin structures. Choosing $0_1 \equiv 0$ as basepoint p_0 we get $-2\vec{K} = 0$ and therefore $\vec{K}$ corresponds to an half period. We are left to compute the integral $\int_0^{D_i} \vec{\omega}$. Let us choose $D_1 = a$. Then:

$$\int_0^a \vec{\omega} = \int_0^1 \vec{\omega} + \int_1^a \vec{\omega}.$$

The second term on the right can be evaluated as we did for

$$\Sigma(g,3,m,0)$$

and gives the Z_3 spin structure $\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 1/3 \end{bmatrix}$.

The first integral is related to $\begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}$. In fact the B1 cycle is given by:

$$\oint_{B_1} \omega(z^2)dz = (1-\alpha^2)\int_{-1}^1 \vec{\omega}(z^2)dz.$$

and the R.H.S. is, using the symmetry $z \to -z$ of the integrand: $2(1-\alpha^2)\int_0^1 \vec{\omega}(z^2)dz$. In conclusion β is a Z_6 spin structure. The demonstration given here for D_1 is similar for all other D_i's. Neglecting the effects of transition functions our method provides the charges $\tilde{q}_k$. For the Z_3 characteristic $\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 1/3 \end{bmatrix}$ and $j = 1/2$ we have:

$$\vec{\tilde{q}}_{a_1,k} = (-1/3,1/3,0) \quad \vec{\tilde{q}}_{a_2,k} = (0,-1/3,1/3)$$

$$\vec{\tilde{q}}_{b_1,k} = (-1/3,0,1/3) \quad \vec{\tilde{q}}_{b_2,k} = (-1/3,0,1/3).$$

c) Finally we consider the Z_4 surface $\Sigma(3,4,4,0)$ with equation:

$$y^4 = (z-c_1)(z-c_2)(z-c_3)(z-c_4).$$

With respect to the system of cuts in fig (1.3) a basis for the homology cycles is given by:

$$A_1 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using the same methods as in the previous examples we find that on this surface there are 64 solutions of the system in eq. (2.6). They correspond to Z_4 spin structures and six of them correspond strictly to Z_2 characteristics. Anyway it is not possible to obtain the other 58 Z_2 spin structures using the method of twist fields. We write here the charges for the particular solution of eqs. (2.6):$\nu_1 = \nu_4 = 1/4, \nu_2 = \nu_3 = 3/4$ and $j = 2$:

$$\vec{\tilde{q}}_{c_1,k} = \vec{\tilde{q}}_{c_4,k} = (-1/8, 1/8, 3/8, 3/8)$$

$$\vec{\tilde{q}}_{c_2,k} = \vec{\tilde{q}}_{c_3,k} = (-3/8, 1/8, 3/8, 3/8).$$

The charges of the other spin structures can be derived using the same procedure.

The aim of this section is the computation of the two point functions and chiral determinants for fermionic fields defined on the class of rational algebraic curves $\Sigma(g,n,0,2)$:

$$y^n = \frac{(z-a_1)(z-a_2)}{(z-b_1)(z-b_2)} \tag{4.1}$$

with $g = n+1$. These curves were studied in the previous section, where we learned that the Z_2 spin structures cannot be realized in terms of simple twist fields as in eq. (1.13) of chapter 3. For example in the case $n = 3$ the odd spin structures are solutions of the non-trivial system of equations (3.5). Nevertheless these curves have a physical importance because they provide the leading terms of the high energy string amplitudes [GM]. Moreover they are simple and constitute a first step in understanding fermionic fields theories on algebraic curves. One of the reason of their simplicity is that they possess an extra Z_2 symmetry and belong to the hyperelliptic subvariety of the moduli space. As a matter of fact, performing the change of variables $(z,y) \rightarrow (y,z)$, eq. (4.1) becomes:

$$z^2(y^n-1) + z((a_1+a_2) - y^n(b_1+b_2)) + (b_1b_2y^n - a_1a_2) = 0 \tag{4.2}.$$

This polynomial equation is of the kind $az^2+bz+c = 0$ and represent an hyperelliptic curve. As a matter of fact a linear transformation yield the equation:

$$4x^2(y^n-1)^2 = y^{2n}(b_1-b_2)^2 + 2y^n s + (a_1-a_2)^2 \tag{4.3}$$

with $s = (2a_1a_2 - a_1b_1 - a_1b_2 - a_2b_1 - a_2b_2 + 2b_1b_2)$.

Now it is easy to find from eq (4.2')the $2g+2$ root functions in terms of which it is possible to construct sections corresponding to the whole group of the Z_2 spin structures. The strategy is therefore to map these sections from the curve of eq. (4.2) to the curve (4.1). Both of them are different projections of the same curve. The mapping from (4.2) to (4.1) is provided by:

$$y \rightarrow y(z) = \sqrt[n]{\frac{z(z-1)}{z-a}}; \quad z(y) \rightarrow z \tag{4.4}.$$

We simplify equations (4.1),(4.2) and (4.3) using the $Sl(2,\mathrm{C})$ group of automorphisms of the sheets to fix three of the branch points:

$$a_1 = 0,\ \ a_2 = 1,\ \ b_1 = \infty, \infty b_2 = a.$$

Consequently these equations become:

$$y^n = \frac{z(z-1)}{z-a} \tag{4.1'}$$

$$z^2 - z(y^n+1) - ay^n = 0 \tag{4.2'}$$

$$z'^2 = \frac{(y^n+1)^2}{4} - ay^n \tag{4.3'}.$$

Notice that the branch points of eqs. (4.2') and (4.3') are the same and are the solutions of:

$$y^{2n} + y^n(2-4a) + 1 = 0.$$

The roots of this polynomial are:

$$\alpha_i = e^{2\pi i}\sqrt[n]{(1-2a \pm \sqrt{a^2-a})},\ \ i = 0,\ldots,n-1 \tag{4.5}.$$

As it is well known [Fay] the even spin structures on hyperelliptic surfaces are in one to one correspondence with the partitions of the branch points into two sets with the same number of elements. We denote the branch points of the first set A_i and those of the second set B_i. Let us discuss the case of fields with conformal weight $j = \frac{1}{2}$ on the surface (4.1'). We start from the Szego kernel $S(y_1, y_2)$ for the hyperelliptic surface of eq. (4.2) which is given by [Fay]:

$$S(y_1,y_2) = \frac{\sqrt{dy_1}\sqrt{dy_2}}{2(y_1 - y_y 2)}\left[\frac{u(y_1)+u(y_2)}{\sqrt{u(y_1)u(y_2)}}\right] \tag{4.6}.$$

The function $u(y)$ has the following expression:

$$u(y) = \sqrt{\prod_{i=0}^{n-1}\frac{y-A_i}{y-B_i}} \tag{4.6a}.$$

The fact that the curves of equations (4.2) and (4.3) have the same branch points and Szego kernels is not surprising. As a matter of fact the two curves are related

by a linear transformation involving z but not y. Of course the square of the Szego kernel should be a single-valued differential in both the variables y_1 and y_2 on the starting surface (4.2). We can check this taking the square of eq. (4.6). At this point we perform the transformation (4.4) on this Szego kernel. After some manipulations we have:

$$S(z_1,z_2)=\frac{1}{2n}(\prod_{i=0}^{n-1}(y_1-A_i)^{\frac{1}{2}}(y_2-B_i)^{\frac{1}{2}}+$$

$$\prod_{i=0}^{n-1}(y_1-B_i)^{\frac{1}{2}}(y_2-A_i)^{\frac{1}{2}})\frac{(y_2y_1)^{\frac{1-n}{2}}}{\sqrt{(z_1-a)(z_2-a)}}\frac{\sqrt{dz_1}\sqrt{dz_2}}{y_1-y_2} \tag{4.7}$$

with $y_1\equiv y(z_1)$ and $y_2\equiv y(z_2)$. It is easy to show that $S(z_1,z_2)$ has just a pole of residue 1 at the point $z_1=z_2$ when the branches of y_1 and y_2 coincide. Besides, $S(z_1,z_2)$ is finite when its two variables approach infinity. Moreover its square is single-valued on the Z_3 symmetric surface of eq.(4.1). For example when $n=3$ it assumes the form:

$$S^2(z_1,z_2)=$$

$$\frac{1}{6}\left[\prod_{i=0}^{n-1}(y_1-A_i)(y_2-B_i)+\prod_{i=0}^{n-1}(y_1-B_i)(y_2-B_i)+2(z_1-a)(z_2-a)\right]\cdot$$

$$\frac{(y_1^2+y_1*y_2+y_2^2)^2}{(z_1-z_2)^2(z_1z_2+a(z_2+z_1)+a)^2}\frac{dz_1dz_2}{y_1^2y_2^2} \tag{4.8}.$$

It is easy to proof that the differential in eq. (4.8) is a single-valued second kind differential on $\Sigma(2,3,0,2)$ as expected. Since $S^2(z_1,z_2)$ is related to a spin structure, its zeros and poles are quadratic. Intuitively this is true also for eq. (4.8). In fact, after the transformation (4.4) double zeros are stil double zeros using the derivative of composed functions. Let us consider for example a differential $f(y)dy$ such that, when $y=y_0$ it has a double zero:

$$\begin{cases} f(y)dy|_{y=y_0)}=0 \\ \partial_y f(y)dy|_{y=y_0}=0 \end{cases}.$$

After a transformation $y\rightarrow y(z)$, $f(y(z))dz$ has still a second order zero at the solutions p_i of the equation $p_i=y^{-1}(y_0)$ because:

$$\begin{cases} f(y(z))dz|_{z=p_i}=0 \\ \partial_z(f(y(z)dz)|_{z=p_i}=(\partial_y f(y(z))\frac{dy}{dz}dz|_{z=p_i})=0 \end{cases}.$$

However, despite the fact that the the half-differential in eq. (4.7) represents the Szego kernel of an even spin structure on $\Sigma(n-1,n,0,2)$, it is not possible to check it transporting naively $S(z_1,z_2)$ along the homology cycles of fig. (1.21) and looking at its changes of sign. In fact, as we show in appendix A, even for the simplest multivalued functions it is very difficult to guess the way in which they change their branches. Now $S(z_1,z_2)$ contains a very complicated rational function R depending on $y(z_1)$, $y(z_2)$, z_1 and z_2. We have therefore to study this function using the methods of chapter 5, i.e. computing the algebraic equation to which R satisfy. This algebraic equation is very complicated and requires an algebraic manipulator on a powerful computer as MACSYMA. We give here just the form of this equation in the variable z_1:

$$R^{2n} + P_{2n-2}(y_2,z_1,z_2)R^{2n-2} + \dots P_0(y_2,z_1,z_2) \tag{4.9}$$

where the $P_{2i}(y_2,z_1,z_2)$ are rational functions. In some sense, the Riemann representation of the algebraic curve corresponding to the equation above describe in terms of sheets and branch point the double coverings of the surfaces of [GM]. Since only even powers of R appear, we can do the substitution $R^2=\rho$ and to solve the algebraic equation in ρ. Therefore $R=\pm\sqrt{\rho}$ and it changes sign as is expected from a Szego kernel. To find in which way the signs interchange when transported around the homology cycles is a too difficult problem. Eq. (4.9) shows us again the unique features of hyperelliptic surfaces for the usual spin structures. In fact when $n=2$ the equation can be reduced to a form similar to that of eq. (4.2) and it is again hyperelliptic. Even if we do not proof it, eq. (4.9) is the general form of the algebraic equations satisfied by Szego kernels on hyperelliptic surfaces. We can compute explicitly the zeros of the Szego kernels, which are the zeros of the corresponding theta functions, just looking at the zeros of $P_0(y_2,z_1,z_2)$.

We are still left with the problem of finding the chiral determinants $det\partial_{\frac{1}{2}}$. For this task we can use the stress energy tensor method [AS,Ver] introduced in chapter 1. As for the Green functions the strategy is to start from the hyperelliptic stress-energy tensor

$$< T(y) >= \frac{1}{32}(\sum_i \frac{1}{z-A_i} - \sum \frac{1}{z-B_i}) \tag{4.10}.$$

and then to use the mapping (4.4) to get the expression of the stress energy tensor on the Z_n surface associated to eq.(4.1). Now it is known that the effects of a general coordinate transformation on the stress energy tensor is given by:

$$T(z) = T(y)\left(\frac{dy}{dz}\right)^2 + \frac{C}{12}\left[\frac{d^3y/dz^3}{dy/dz} - \frac{3}{2}\left(\frac{d^2y/dz}{dz/dy}\right)^2\right] \qquad (4.11).$$

The second term in the RHS is the Schwarzian connection $\{y, z\}$. Under the transformation (4.4) it yields:

$$\{y, z\} = \frac{1}{9}\frac{1}{(z-a)^2} - \frac{2}{9}\frac{2a-1}{(a^2-a)}\frac{1}{z-a} \qquad (4.12).$$

The other term, containing $T(y)$ is more troublesome. In fact when we expand the vacuum expectation value of $T(y(z))$ around the branch point a we get also powers of the form:

$$< T(y) > \left(\frac{dy}{dz}\right)^2 \sim \frac{h_1}{(z-a)^{\frac{n-1}{n}}} + \ldots.$$

These irrational powers are not physical because we know that terms of that kind are not allowed in conformal field theories. Here we experience the effect about which we spoke in section two. The global effects of the spin structures destroy the abelian symmetry of the surface and the contributions to the residues given by the various branches of $y(z)$ in the expression of $< T(y(z)) >$ do not appear in a symmetrical way. Something similar happened in the computation of the stress energy tensor for the $\{1,0\}$ b-c system on the surfacegiven by eq. (4.3)-chapter 1. As a matter of fact the irrational term cancel when we sum over the branches of y. Using the stress energy tensor method we get a differential equation for the determinant $\partial_{\frac{1}{2}}$ which can be easily solved explicitly taking an integral because the surfaces corresponding to eq. (4.1') depend just on one branch point. As an example we provide the differential equation for the chiral determinant corresponding to the spin structure:

$$\{A_0 = (+), A_1 = \epsilon(+), A_2 = (-) | B_0 = \epsilon^2(+), B_1 = \epsilon(-), B_2 = \epsilon^2(-)\}.$$

In this equation ϵ represent the third root of unity while

$$(\pm) = (1 - 2a \pm \sqrt{a^2 - a})^{\frac{1}{3}}.$$

The result is:

$$< T(z) >= \frac{1}{9}\frac{1}{(z-a)^2} - [\frac{2}{9}\frac{2a-1}{(a^2-a)} +$$

$$\frac{1}{32(a^2-a)}(10(+)+10(-)-2(\epsilon(+)+\epsilon^2(-)+3(+)+3\epsilon^2(-))-4(1-2a))]\frac{1}{z-a}$$

and therefore:

$$\partial_a det(\partial_{\frac{1}{2}}) = \frac{2}{9}\frac{2a-1}{(a^2-a)}\frac{1}{32(a^2-a)}\cdot$$

$$(10(+)+10(-)-2(\epsilon(+)+\epsilon^2(-)+$$

$$3(+)+3\epsilon^2(-))-4(1-2a)).$$

As we can see the conformal dimension of the twist fields is $1/9$, as it can be expected on a Z_3 symmetric curve.

CHAPTER 5
FURTHER DEVELOPMENTS

5.1 THE PROBLEM OF SPIN STRUCTURES ON GENERAL ALGEBRAIC SURFACES

Let us briefly review what was done in the previous chapter, section (4.3) about the problem of spin structures on algebraic surfaces. We considered b-c systems with conformal weight j on Z_n symmetric surfaces:

$$w^n = \prod_i (z - a_i)^{L_i} \qquad L_i = \pm 1 \tag{2.1}$$

Using the method of twist fields $V(a_i)$ which simulate the behavior of the fields b and c at branch points, we have catalogued all the possible b-c systems expressibile in this way.

1) j = integer

When $n = 2$, we have the 2^{2g} bosons with Z_2 twisted boundary conditions. Representatives of this class are of the form:

$$\sqrt{(z - a_i)(z - a_k)}dz^j \qquad i, k \quad arbitrary.$$

When $n \geq 3$ we have a subset of the n^{2g} twisted boundary conditions, the socalled Z_n symmetric twisted boundary conditions. They owe their name to the fact that the monodromy matrices expressing the boundary conditions of these fields, when they are transported around a branch point along a small circle surrounding it, are elements of the group Z_n. The j-differentials which, with various degrees of meromorphicity, represent these twisted bosons, are of the kind:

$$\sqrt[n]{\prod_{k=1}^{n_b} (z - a_{i_k})^{j_k} dz^j} \qquad \sum_k j_k = mn, \; m \; integer.$$

In this formula all products with n_b points a_{i_k} are to be considered with n_b such that the right monodromy properties at ∞ are assured.

2) j = half-integer.

In the case $n = 2$ all the 2^{2g} spin structures are expressed as a partition of the $2g + 2$ branch points [Fay]. In the other cases we have instead that a part of the bosons with Z_n twisted boundary conditions are $2j$-differentials with a quadratic divisor, so that it is possible to take their square root. The resulting j-differentials correspond to Z_{2n} spin structures [Fer1].

The reason for which it is so natural to realize these b-c systems with twist fields is that their transition functions respect the Z_n symmetry of the surface. This implies that the phases taken by these fields after they are transported around a non-trivial homology cycle, are dependent on the branch points inside the cycle and on how many times the branch linesoutgoing from the branch points are crossed [Fer1]. It is nevertheless independent on the sheets in which the cycle meets the branch lines. This behavior can be simulated by simple twist fields $V(a_i) \sim e^{iq_k\varphi_k(a_i)}$, $\varphi_k \quad (k = 1, ..., n-1)$ being a basis of fields for which all monodromy matrices are diagonalized and q_k being the charge of the branch point a_i with respect to the current $J_k(z) =: b_k(z)c_k(z) :$. As a matter of fact the simple vertex $V(a_i)$ above does not distinguish a sheet from the other.

All the other spin structures and twisted boundary conditions are singlevalued on a unramified n-covering with nonabelian group of local monodromy. Therefore the monodromy matrices cannot be simultaneously diagonalizable and the twist fields are not expressed in the form they have in the case of Z_n characteristics. Moreover the phases taken by fields after going around homology cycles depend on the sheets on which these cycles meet the branch lines. Even if we would be able to compute the monodromy matrices for the non-abelian case, we were then faced with the problem to construct twist fields for a nonabelian group of symmetry which is so far unsolved. Anyway each meromorphic j-differential leaving on an algebraic surface obeys as well an algebraic equation [For]. For this equation a previous analysis [Fer2] shows that on general curves with nonabelian local monodromy group, the twist fields are again primary fields concentrated at the branch points as in the case of surfaces with abelian group of symmetry. Their conformal weight for a branch point of multiplicity m is the same they would have for a branch point of a Z_m symmetric surface. Whith respect to the searching of sections related to

a given spin structure, general algebraic surfaces do not present much more complications than Z_n symmetric surfaces, so that we will consider the problem in its generality. Anyway a look at eq.(2.1) teaches us a good lesson: all meromorphic functions $(z - a_i)$ have zeros of the $n - th$ order in z and their poles are all at ∞ . Therefore factors of these functions with appropriate weights are associated to products of theta functions with periods of the kind $\frac{1}{n}$ [Krz,Sta]. We have to find a way which reproduces this situation also on more general surfaces.

5.2 THE ODD SPIN STRUCTURES

We consider first the odd spin structures on a general algebraic surface Σ associated with the equation $F(z,w) = 0$. We treat z as the independent variable. $F(z,w)$ is a polynomial at most of degree n in the complex coordinates z and w. The holomorphic differentials for these surfaces are of the kind:

$$\omega_i = \frac{\Phi_i(z,w)}{F_w(z,w)}dz \qquad i = 1,\ldots,g \tag{2.2}$$

In this equation $\Phi_i(z,w)$ are polynomials of degree $n-3$ in z and w while $F_w(z,w) = \frac{\partial}{\partial_w}F(z,w)$. The square of a $\frac{1}{2}$ differential ν_* corresponding to an odd spin structure $*$ is given by an appropriate combination of the holomorphic differentials ω_i:

$$\nu_*^2 = \sum_{i=1}^{g} A_i\omega_i \tag{2.3}$$

The coefficients A_i are to be chosen in such a way that ν_*^2 has $g-1$ quadratic zeros. Explicitly the holomorphic differentials are written as a differential $\frac{dz}{F_w(z,w)}$ multiplied by a rational function $R_i(z) = \Phi_i(z,w)$ which is singlevalued on Σ. It is known that such functions obey a polynomial equation as well as $w(z)$ and that this equation describes a surface with the same genus of Σ. Putting $\tilde{y} = \sum_{i=1}^{g} A_i\Phi_i(z,w)$ we have to compute a finite number of powers of $\tilde{y}$ (in general n) and then to do a linear combination of them with coefficients T_i ($i = 1, ..., n$). These coefficients are determined by the condition that the result of the sum should be free from radicals of functions of z. At the end of the procedure the equation for $\tilde{y}$ will be of the kind:

$$\tilde{y}^n + \tilde{P}_{n-1}(z)\tilde{y}^{n-1} + ... + \tilde{P}_1(z)\tilde{y} + \tilde{P}_0(z) = 0 \tag{2.4}$$

$\tilde{P}_i(z)$ being polynomials in z whose coefficients consists in polynomials of A_i. From this equation it is easy to realize that the zeros of $\tilde{y}$, i.e. the zeros of ν_*^2 because the divisor of $\frac{dz}{F_w(z,w)}$ can be considered known, are the zeros of $P_0(z)$ [For, Fer2]. In fact explicit examples show that $\tilde{P}_0(z)$ has degree $2g-2$. If we insert in eq. (2.4.) $\tilde{y}$ with arbitrary coefficients A_i all the zeros of $\tilde{P}_0(z)$ will be in general simple. We wish instead $g-1$ quadratic zeros. The best way to determine the correct coefficients A_i is to impose the condition $\tilde{P}_0(z) = \prod_{i=1}^{g-1}(z-z_i)^2$. This relation gives constraints on the coefficients of $\tilde{P}_0(z)$ which determine all the $A_i's$ corresponding to the $2^{g-1}(2^g-1)$ odd spin structures. A similar algorithm is written in [Sta] but we prefer this procedure because in this way we are able to separate automatically the calculation of odd spin structures from that of the even spin structures. Let us study for example the surface of genus 3:

$$w^3 + p(z)w + q(z) = 0 \tag{2.5}$$

where:

$$p(z) = \sum_{i=1}^{4} a_i z^{4-i}; \qquad q(z) = \sum_{i=1}^{5} b_i z^{5-i}.$$

The holomorphic differentials are in this case:

$$\omega_1 = \frac{dz}{F_w(z,w)}; \quad \omega_2 = dz\frac{z}{F_w(z,w)}; \quad \omega_3 = dz\frac{w}{F_w(z,w)} \tag{2.6}.$$

Using eq. (2.4.) we get:

$$\nu_*^2 = \frac{A_1 w + A_2 z + 1}{F_w(z,w)} \tag{2.7}$$

ν_*^2 is defined apart from a multiplicative constant so that we can drop A_3. $\tilde{y}$ obeys in this case a polynomial equation of the third degree in $\tilde{y}$ with:

$\tilde{P}_2(z) = 3A_2 z - 3;$

$\tilde{P}_1(z) = A_1^2 p(z) + 3A_2^2 z^2 + 6A_2 z + 3;$

$\tilde{P}_0(z) = A_1^3 q(z) + z A_1^2 A_2 p(z) + + A_1^2 p(z) + z^3 A_2^3 + z^2 A_2^2 + 3zA_2 + 1.$

In particular $\tilde{P}_0(z)$ has degree $2g-2=4$ as we expected. The relations that the coefficients of a polynomial $\sum_{i=0}^{4} c_i z^{4-i}$ $(c_0 = 1)$ of a fourth degree should satisfy in order that the roots have multiplicities two are:

$$\begin{cases} c_4 = \frac{1}{4}(c_2 - \frac{c_1^2}{4})^2 \\ c_3 = -\frac{1}{2}(c_2 - \frac{c_1^2}{4})c_1 \end{cases} \tag{2.8}$$

Plugging the coefficients of $\tilde{P}_0(z)$ in these equations we obtain a system of two equations between the A_i's which provides the solutions corresponding to the whole set of spin structures. This system is the analog of the inverse Jacobi problem in the language of branch points. Even if it is impossible to solve eqs. (2.8) explicitly, we can get at least the coefficients A_i for a single odd spin structure. This is sufficient in order to construct an expression for chiral determinants in the case of bosonic string [Fer2]. The idea is that we can change the parametrizations of the curve of eq. (2.5) so that $\tilde{P}_0(z)$ becomes exactly of the form:

$$\tilde{P}_0(z) = (z - z_0)^2 * (z - z_1)^2,$$

ν_*^2 being as in eq. (2.7). This is a very easy achievement which does not involve any solution of higher order polynomials. The result is:

$b_2 = -A_1^2 A_2(2a_1 z_0 + 2a_1 z_1 - a_2) - A_1^2 a_1 - A_2^3 - 2b_1 z_0 - 2b_1 z_1$;

$b_3 = A_1^2 A_2(a_1 z_0^2 + 4a_1 z_0 z_1 + a_1 z_1^2 - a_3) - A_1^2 a_2 - 3A_2 - 3A - 2 = b_1 z_0^2 + 4b_1 z_0 z_1 + b_1 z_1^2$;

$b_4 = -A_1^2 A - 2(+2a_1 z_0^2 z_1 + 2a_1 z_0 z_1^2 + a_4) - A_1^2 a_3 - 3A_2 - 2b_1 z_0^2 z_1 - 2b_1 z_0 z_1^2$;

$b_5 = A_1^2 A_2 a_1 z_0^2 z_1^2 - A_1^2 a_4 + b_1 z_0^2 z_1^2 - 1$

Let us notice that with these substitutions, b_2, b_3, b_4, and b_5 become functions of the other parameters of the surface (2.5), of $A_i's$ and of the zeros z_i. The total number of free parameters remains in all cases equal to nine. Actually the choice of the parameters b_i $(i = 2, .., 5)$ was arbitrary and we can use any other four coefficients of eq. (2.5) instead of them. The method proposed here is valid for each curve $F(z, w) = 0$. Another way to obtain explicitly odd spin structures is to rearrange the starting algebraic equation in such a way that some functions of w and z play the same role of $(z - a_i)$ in the hyperelliptic case. An example is provided by the surface of genus 3:

$$wzt - f^2 = 0 \tag{2.10}$$

with

$t = c_1 z + c_2 w + c_3$ and

$f = a_{00} z^2 + a_{11} w^2 + a_{22} + a_{10} zw + a_{20} z + a_{21} w$

This is a general surface of genus 3 [Rie] with 9 free parameters and it is easy to proof that z, w and t correspond to three different odd spin structures. Unfortunately this method is not appliable for all surfaces of genus g. The Z_n symmetric curves are for example very difficult to rewrite in a symmetric form as in eq. (2.10).

5.3 CHIRAL DETERMINANTS

The knowledge of an odd spin structure enables us to write an expression for the special combination of chiral determinants [Kni,Leb]:

$$\Lambda_j = (det\bar{\partial}_0)^{\frac{1}{2}}(det\bar{\partial}_j).$$

In particular we are interested in:

$$\Lambda_0 = \frac{\Theta_{*,i}\tilde{\omega}_i''(z_1)}{det[\tilde{\omega}_i''(z_1)\tilde{\omega}_i(z_1)...\tilde{\omega}_i(z_{g-1})]} \tag{2.11}$$

and:

$$\Lambda_2 = \frac{\Theta_{*,i}\tilde{\omega}_i''(z_1)}{det[\varphi_\alpha(z_1)\varphi_\alpha''(z_1)\varphi_\alpha''''(z_1)\varphi_\alpha(z_2)\varphi_\alpha'(z_2)\varphi_\alpha''(z_2)...\varphi_\alpha'(z_{g-1})\varphi_\alpha''(z_{g-1})]} \tag{2.12}$$

In these formulas:

$$\Theta_{\alpha,i} = \frac{\partial}{\partial u_i}\Theta[\alpha](u_1...u_g|T)\|_{u_1=...=u_g=0};$$

$\tilde{\omega}_i$ are the normalized holomorphic differentials;

$\varphi_\alpha(\alpha = 1,...,3g-3)$ form a basis of the holomorphic quadratic differentials and:

$z_1,...,z_{g-1}$ are the zeros of ν_*^2 which, in the theta function formalism, is given by:

$$\nu_*^2 = \Theta_{*,i}\tilde{\omega}_i(z).$$

The dependence on the spin structure $*$ and on the normalization of the $\varphi_\alpha{}'s$ should cancel from physical amplitudes.

Let us consider the curve of genus 3 of eq. (2.10). This is a polynomial of the fourth degree in w and its branch points can be derived from the following system of equations:

$$\begin{cases} wzt - f^2 = 0 \\ zt + c_2 wz - 2f\frac{df}{dw} = 0 \end{cases}$$

Using the elimination theory of algebraic equations we get a polynomial in z of order 12 which gives twelf branch points $\alpha_1, ..., \alpha_{12}$.

Apart certain configurations of the parameters of eq. (2.10) which correspond to degenerate surfaces, these branch points are simple, i.e. at these points just two sheets are interconnected. Since $z = 0$ is a branch point, we put $\alpha_1 = 0_1$, $\alpha_2 = 0_2$; at each of these zeros two sheets of the surface are connected. With the aid of Riemann Hurwitz formula [Far] it is easy to proof that all branch points have multiplicity two. Therefore we can write the relevant divisors for the construction of tensors as in [Fer2]:

$$div\ dz = \frac{O_1 O_2 \alpha_3 ... \alpha_{12}}{\infty_1^2 ... \infty_4^2}$$

$$div\ F_w(z,w) = \frac{O_1 O_2 \alpha_3 ... \alpha_{12}}{\infty_1^3 ... \infty_4^3}$$

$$div\ z = \frac{O_1^2 O_2^2}{\infty_1 ... \infty_4}; \qquad div\ w = \frac{\gamma_1^2 \gamma_2^2}{\infty_1 ... \infty_4}$$

Here γ_1, γ_2 are the zeros of w and are easily computed from equation (2.10):

$$a\gamma_{1,2} = \frac{-a_{20} \pm \sqrt{a_{20}^2 - 4a_{00}a_{22}}}{2a_{00}}$$

A basis for the holomorphic differentials is provided by:

$$\omega_1 = \frac{dz}{F_w(z,w)}; \quad \omega_2 = \frac{zdz}{F_w(z,w)}; \quad \omega_3 = \frac{wdz}{F_w(z,w)}$$

Their divisor is:

$$\begin{aligned} div\ w_1 &= \infty_1 ... \infty_4 \\ div\ w_2 &= O_1^2 O_2^2 \\ div\ w_3 &= \gamma_1^2 \gamma_2^2 \end{aligned} \tag{2.12}$$

As an upshot of our choice of parametrization of the surface, ω_2 and ω_3 correspond automatically to two odd spin structures. Let us call them $*$ and $*'$ respectively. Then:

$$\omega_2 = \Theta_{*,i}\tilde{\omega}_i, \quad \omega_3 = \Theta_{*',i}\tilde{\omega}_i.$$

Finally a basis for the quadratic differentials φ_α is obtained from the product of all the pairs of holomorphic differentials:

$$\varphi_1 = \omega_1^2, \quad \varphi_2 = \omega_1\omega_2, \quad \varphi_3 = \omega_1\omega_3, \quad \varphi_4 = \omega_2^2, \quad \varphi_5 = \omega_2\omega_3, \quad \varphi_6 = \omega_3^2.$$

At this point we have at disposal all the ingradients which eqs. (2.11) and (2.12) need.

FINAL CONCLUSIONS

The reader who is arrived at this point through the difficulties of the matter exposed may be wondered or disappointed that, after all our conclamations to set up an explicit formalism for superstrings and conformal field theories on higher genus Riemann surfaces, so a few applications are possible analytically without the aid of a computer. If this is the case, we are very sorry, but we think that this is just like if, making the debt proportions, after having studied path integrals, one gets disappointed because nothing can be computed except some simple physical systems or because there are not analytical solutions of the Navier Stokes equations. Part of the embarassement we had here in dealing with the cumbersome problem to avoid as much as possible the solutions of tremendous algebraic equations or the difficulties we had, being not mathematicians, in developing topics of algebraic geometry in which the literature at our disposal was unexplicit or even lacking, are not our fault. The matter is that even if many scientist are telling us that string theories are now old, nobody studied seriously the problem of how to get numbers from a theory which can't be tested experimentally. Many of the results gotten for superstrings make use of abstract theorems like those concerning the existence of the solution for the inverse Jacobi problems without thinking how these solutions concretely are. Unfortunately we were faced with this lack of knowledge and experience in the field and, even worse, at least for us, there exists the theorem atating that where there is no knowledge, there is also no interest. Despite all the difficulties, we think we have succeeded in these three topics:

1) The bosonic string at low genus $g \leq 4$.

We provided an algorithm which is able to compute all two-point functions on a general algebraic curve for which the third kind differentials exists. The latter, moreover, can be easily computed using the Weierstrass kernel shown in chapter 1. Less brilliantly, we can compute also the partition functions of bosonic strings. The chiral determinants are provided by the method exposed in chapter 5, while the necessary normalization of the quadratic differentials appearing in $det\partial_2$ can be

given using the expression of the Beltrami differentials. The Beltrami differentials corresponding to the parametrization of moduli space given by branch points can be found using the variational techniques of Rauch [Rau]. The fact that we are limited to surfaces of genus $g \leq 4$ is due to the difficulty to find algebraic equations which cover the whole moduli space apart sets of zero measures when genus is higher than 4.

2) Superstrings on the surfaces of Gross and Mende.

This is our central result. In fact for this case it is possible to solve explicitly the differential equations leading to the chiral determinants of fermion because they are just integrals. Moreover we have an expression for the Green functions. Superstring on Gross and Mende surfaces are therefore the first step in developing string theories on general algebraic curves. It is an important step since, as we previously remarked, the amplitudes computed on these surfaces give the leading behavior of superstrings at very high energies. Unfortunately also in this simple case the expressions becomes involved and there is the need of a computing machine to perform the calculations.

3) We have found a cohomological technique to find what Z_m coverings of algebraic curves with Z_m symmetry still respect this symmetry. Being non experts in the field, we know just the works on hyperelliptic curves of ref. [Hor] and of the other references contained therein. Therefore we do not know if Čech cohomology was already applied to this problem or even if some result on non-hyperelliptic curves was already published.

We have presented in our thesis also some more qualitative results:

A) we knew from [JMS] and successively from [BR3] that the correlation functions of b-c systems and scalar fields can be computed in the bosonized formalism by means of certain j-differentials which were defined implicitly through an involved systems of differentia equations. With their basis of j-differentials the authors of [BR3] were able to compute the residue at branch points of the stress energy tensor and of the current related to fermionic number conservation in a quite (at lest for us) mysterious way. Using simple considerations on the

energy of physical states and on the behavior of the Green functions near the branch points, we havr shown how to find explicitly a basis of j-differentials with analogous characteristics to that of the references above quoted. Our basis provides correct results in all the developed cases, including a surface with non-abelian monodromy group. The results, like for example the fact that the asymptotic charges becomes matrices, are in agreement with the conclusions of [BR3]. We believe that still a lot of work should be done to extend these formalisms to very complicated curves as that in eq. (4.3) of chapter 2. For example we are not in agreement with ref [BR3] that the difficulty in solving their equations for the charges $\tilde{q}_{\alpha_i,k}$ is just due to the fact that they depend on the branch points α_i. As we have shown here this happens even in the Z_n symmetric case. The problem on general surfaces is that these charges depend also on the index which denotes the sheets.

B) Modular invariance.

As we say in chapter 1, all algebraic curves are conformally equivalent to algebraic curves in which all branch points are simple. For example eq. (4.3) in chapter 2 is related to a such curve. Now a glance at fig. (1.6) shows that the system of cuts and the canonical homology cycles are very similar to the hyperelliptic ones. The only difference is that now the branch points are to be found solving a polynomial of high degree and this is possible just numerically. This obstacle forbids the proof that modular invariance can be explicitly implemented to these surfaces in the same way as in the hyperelliptic surfaces. Anyway it is promising to proceed in this direction because as we said all Riemann surfaces are conformally equivalent to curves in which all branch points are simple.

Many applications of the formalism exposed above can be pursued, not only in superstring theories but also in conformal field theories defined on higher genus Riemann surfaces. For examples it would be interesting to study the Coulomb gas representation on algebraic curves. Moreover we are interested to some possible generalizations of the formalism here introduced of which we provide an example. We have already said that in certain cases the monodromy properties of the Green

functions of free field on Riemann surfaces coincide with the Green functions of non-free conformal field theories on the plane. So, what theories are corresponding to the following integro-algebraic equations:

$$F(z, \frac{dw}{dz}) = 0 \ ?$$

To conclude, the best reason for us in studying algebraic curves despite all the difficulties and the unsolved questions was amusement. It was delightfully to look again at our old university books of complex analysis and to do field theory on the simple algebraic equations they give as examples of Riemann surfaces.

APPENDIX A

In this appendix we want to show an example worked out by Klein [Osg] in which he construct the Riemann repesentation for the curve associated with the simple algebraic equation:

$$y^3 - 3y - z = 0 \qquad (A.1).$$

The example is instructive because it shows that the way in which the branches of a multi-valued function are interchanged is not always intuitive. The solutions of equations (A.1) are a subcase of the solutions shown in appendix C. The branch points of this surface are all real: $z = \pm 2$ and $z = \infty$, corresponding to the values of y: $y = \mp 1$, $y = \infty$. From the local analysis of chapter 1 we can easily check yjat the first two branch points have multiplicity two, while the other has multiplicity 3. It is clear that at $z = \infty$ the three branches of $y(z)$ are interchanged but at the other points it seem naively that just two branches, $y^{(1)}$ and $y^{(2)}$ are interchanged. But if it is so, this violates clearly the fact that the product of the monodromy matrices should be the identity matrix (see chapter 1). We have therefore to use a different method to find what the Riemann reprsentation of this curve is. The general method consists in drawing a line C connecting all the branch points in the z plane and then to map it in the y plane, The projection consists in a set of curves C_i' because the function y is multivalued, which will divide the y plane in n areas which are nothing but the sheets. Finally the junctions between the sheets are done along those pieces of the curves C_i' which come from the same piece on the z plane. See for this fig. (1.22) in which to each segment or half line of the C_i' indicate the part of C to which they correspond with the notation $[a, b]$. Thus these segments or half lines of the urves C_i' transform into the branch lines. Since the branch points are all real, we can take the real axis R as the line connecting all the branch points. Now we split eq. (A.1) in its real and immaginary part:

$$\begin{cases} u^3 - 3v^2u - 3u = s \\ 3u^2v - v^3 - 3v = t \end{cases}$$

with $z = s + it$ and $y = u + iv$. At this point we map the line $t = 0$ on the u, v-plane.

To $t = 0$, i.e. the line R, there correspond the solutions:

$$\begin{cases} v = 0 \\ 3u^2 - v^2 - 3 = 0 \end{cases}.$$

The first is the line of the real axis on the u, v-plane, the other is an hyperbola. These curves divide the plane in six parts, each one corresponding to the upper half plane or the lower half plane of a sheet. This is shown in fig. (1.22a). We denote with $I^{\pm}$, $II^{\pm}$, $III^{\pm}$ the upper (lower) half plane of the sheets I, II, III respectively. The assignement of upper and lower half planes we give to the sheets in fig. (1.22a) is arbitrary [Osg]. For example the assignement of fig. (1.22b) is also good. Of course the monodromy matrices will be differentin the two cases, however they respect always the condition that their product is equal to the unity matrix. In the case of fig. (1.22a) we have the following monodromy matrices:

$$M_{-2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} M_{\infty} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

For the choice of fig. (1.22b) we have instead:

$$M_{-2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} M_{\infty} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

As we have seen here it is not trivial to find in which way a multi-valued function hanges its branches going around the branch points.

APPENDIX B

Theta Functions

We review here some known properties of the theta functions. Let us call Ω the period matrix and a, b two vectors belonging to $\mathbf{R}^g$. Then the theta function $\theta \begin{bmatrix} a \\ b \end{bmatrix} (z|\Omega)$ with characteristic $\begin{bmatrix} a \\ b \end{bmatrix}$ is defined by:

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z|\Omega) = \sum_{n \in \mathbf{Z}^g} e^{i\pi(n+a)\cdot\Omega\cdot(n+a)+2\pi i(n+a)(z+b)} =$$

$$e^{i\pi a\cdot\Omega\cdot a+2\pi i a(z+b)}\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z|\Omega) \qquad (B.1).$$

After a transformation $z \longrightarrow z + \Omega \cdot n + m$ the behavior of the theta functions is given by:

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z + \Omega \cdot n + m|\Omega) = e^{-i\pi n\cdot\Omega\cdot n-2\pi i n\cdot(z+b)+2\pi i a\cdot m}\theta \begin{bmatrix} a \\ b \end{bmatrix} (z|\Omega) \qquad (B.2)$$

Now we can Riemann surface Σ in the Siegel upper half plane, i.e. the space of all the period matrices Ω_{ij}, by means of the Jacobi map:

$$I_i(p) = \int_{p0}^{p} \omega_i \qquad (B.3)$$

with $p, p_0 \in \Sigma$ and ω being the usual holomorphic differential normalized in such a way that:

$$\oint_{a_i} \omega_j = \delta_{ij}.$$

In this way the theta function becomes a multivalued function on Σ:

$$f(p) = \theta \begin{bmatrix} a \\ b \end{bmatrix} (z|\Omega).$$

At this point we can state the Riemann vanishing theorem for theta functions:

If $\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z|\Omega) \neq 0$, then there exist a divisor $\theta = p_1, \ldots, p_g$ such that:

$$z = \sum_{i=1}^{g} \int_{p_0}^{p_i} -K^{p_0} \qquad (B.4)$$

where p_0 is also called basepoint and K^{p_0} denotes the vector of Riemann constants given by:

$$K_j^a = \frac{2\pi i - \Omega_{jj}}{2} + \frac{1}{2\pi i} \sum_{\substack{i=1 \\ i \neq j}}^{g} \oint_{a_j} (\omega_i(z) \int_a^z \omega_j) \qquad (B.5).$$

The divisor θ of the zeros of the theta function are with multiplicity 0, i.e. the theta has just g simple zeros. If $\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z|\Omega) = 0$, the it exists some positive divisor $\theta_s = p_1, \ldots, p_{g-1}$ with multiplicity $i(\theta_s) \geq 1$ such that:

$$z = \sum_{i=1}^{g-1} \int_{p_0}^{p_i} - K^{p_0}.$$

Because of the properties descending from the definition of the theta function, the above formulas are valid also for a general characteristic $\begin{bmatrix} a \\ b \end{bmatrix}$.

APPENDIX C

In this section we give the solutions of a general polynomial of order three:

$$y^3 + P_2 y^2 + P_1 y + P_0 = 0.$$

With the trasformation $y \rightarrow y' = y + P_2$ it is always possible to reduce this equation to the canonical form:

$$y'^3 + 3py - 2q = 0.$$

The roots of this equation are :

$$y^{(0)} = \sqrt[3]{q + \sqrt{q^2 + p^3}} + \sqrt[3]{q - \sqrt{q^2 + p^3}};$$

$$y^{(1)} = \alpha \sqrt[3]{q + \sqrt{q^2 + p^3}} + \alpha^2 \sqrt[3]{q - \sqrt{q^2 + p^3}};$$

$$y^{(2)} = \alpha^2 \sqrt[3]{q + \sqrt{q^2 + p^3}} + \alpha \sqrt[3]{q - \sqrt{q^2 + p^3}}$$

with $\alpha = \exp \frac{2\pi i}{3}$.

The branch points are given by:

$$q^2 + p^3 = 0 \qquad (C.1).$$

Even for eq. (2.7) this is a polynomial of degree 9 and it is not possible to solve it. In eq. (B.1) the branch points are simple if $q, p \neq 0$. If $p = q = 0$ the branch point is of order three and connects all the three sheets.

APPENDIX D

The argument of this appendix is the proof of eq. (2.5) in chapter 2 [Fay]. To construct the prime form we start from the following meromorphic function:

$$\frac{f(y)}{f(x)} = \frac{z(y) - z(q)}{z(y) - z(p)} \frac{z(x) - z(p)}{z(x) - z(q)} \qquad (D.1).$$

This function is clearly single-valued and its divisor is

$$D = \frac{q_{(0)} \cdots q_{(n-1)}}{q_{(0)} \cdots q_{(n-1)}}$$

in y and D^{-1} in x. We suppose that neither q nor p are branch points and that the surface is n-sheeted. We can express a meromorphic function in terms of normalized third kind differentials in the following way:

$$\frac{f(y)}{f(x)} = \exp\left[\int_x^y \sum_{i=0}^{n-1} \tilde{\omega}_{(q_i)(p_i)} - \sum_{i=1}^{g} m_j \tilde{\omega}_j\right].$$

Since the function defined in eq. (D.1) is single-valued, we need the following condition over all the A and B cycles:

$$\oint_{A,B} \left(\sum_{i=0}^{n-1} \tilde{\omega}_{(q_i)(p_i)} - \sum_{i=1}^{g} m_j \tilde{\omega}_j\right) = 0 (modulo \quad integers) \qquad (D.2).$$

Since

$$\sum_{i=0}^{n-1} \tilde{\omega}_{(q_i)(p_i)} + \sum_{i=1}^{g} m_j \tilde{\omega}_j = \frac{d}{dy} ln\left(\frac{f(y)}{f(x)}\right)$$

eq. (D.2) implies [FK]:

$$m_i = \frac{1}{2\pi i} \int_{A_i} d[ln(f(t))] = integer.$$

Therefore only the imaginary part of $dln[f(t)]$ contributes to this integral. Now we use in eq. (D.1) the following properties of third kind differentials:

$$\int_a^b \tilde{\omega}_{dc} = \int_c^d \tilde{\omega}_{ab}$$

and

$$\int_a^b \tilde{\omega}_{dc} = \frac{E(c,b)E(d,a)}{E(d,b)E(c,a)}.$$

Supposing without loss of generality that x and y belong to the sheet (0) and using the first property, eq (D.1)becomes:

$$\frac{f(y)}{f(x)} = \exp\left[\sum_{i=1}^{n-1} \int_{p_{(i)}}^{q_{(i)}} \tilde{\omega}_{y_{(0)}x_{(0)}} + \int_{p_{(0)}}^{q_{(0)}} \tilde{\omega}_{y(0)x(0)} + \sum_{i=1}^{g} m_j \tilde{\omega}_j\right].$$

Applying the second property and performing the limit $q \to y$, $p \to x$ we get:

$$\lim_{q\to y}\lim_{p\to x}\frac{z(y)-z(q)}{z(y)-z(p)}\frac{z(x)-z(p)}{z(x)-z(q)}\frac{E(x,q)E(y,p)}{E(y,q)E(x,p)} =$$

$$= \lim_{q\to y}\lim_{p\to x}\exp\,[\sum_{i=1}^{n-1}\int_{p_{(i)}}^{q_{(i)}}\tilde{\omega}_{y_{(0)}x_{(0)}} + \sum_{i=1}^{g} m_j\tilde{\omega}_j].$$

Finally, since:

$$\lim_{q\to y}\frac{z(y)-z(q)}{E(y,q)} = const. \quad dy,$$

we get eq. (2.5)-chapter 3:

$$\frac{E^2(y,x)}{(z(y)-z(x))^2} = \frac{1}{dydx}\; exp[\sum_{i=1}^{n-1}\int_{x_{(i)}}^{y_{(i)}}\tilde{\omega}_{y_{(0)}x_{(0)}} + \sum_{i=1}^{g} m_j\tilde{\omega}_j].$$

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TABLE 1

transition functions $\gamma_{\alpha\beta}$ for Z_3 spin structures

for the torus $y^3 = (z - c_1)(z - c_2)(z - c3)$

spin structure $\frac{1}{3}$, $\frac{2}{3}$

$$\gamma'_{06} = 1/3$$

$\gamma_{18} = 2/3$	$\gamma_{48} = 1/3$	$\gamma_{38} = 1/3$
$\gamma_{57} = 1/3$	$\gamma_{27} = 2/3$	$\gamma_{36} = 1/3$
$\gamma_{58} = 1/3$	$\gamma_{2D} = 1/3$	$\gamma_{5C} = 1/3$
$\gamma_{01} = 1/3$	$\gamma_{23} = 2/3$	$\gamma_{69} = 2/3$
$\gamma_{7B} = 2/3$	$\gamma_{8D} = 1/3$	$\gamma_{8E} = 2/3$

spin structure $\frac{1}{3}$, 0

$$\gamma'_{07} = 1/3$$

$\gamma_{48} = 2/3$	$\gamma_{38} = 2/3$	$\gamma_{36} = 2/3$
$\gamma_{58} = 2/3$	$\gamma_{39} = 2/3$	$\gamma_{5C} = 2/3$
$\gamma_{23} = 1/3$	$\gamma_{79} = 2/3$	$\gamma_{7C} = 2/3$
$\gamma_{8E} = 1/3$		

spin structure $\frac{2}{3}$, $\frac{1}{3}$

$$\gamma'_{06} = 2/3$$

$\gamma_{18} = 1/3$	$\gamma_{48} = 2/3$	$\gamma_{38} = 2/3$
$\gamma_{57} = 2/3$	$\gamma_{27} = 1/3$	$\gamma_{36} = 2/3$
$\gamma_{58} = 2/3$	$\gamma_{2D} = 2/3$	$\gamma_{5C} = 2/3$
$\gamma_{01} = 2/3$	$\gamma_{23} = 1/3$	$\gamma_{69} = 1/3$
$\gamma_{7B} = 1/3$	$\gamma_{8D} = 2/3$	$\gamma_{8E} = 1/3$

spin structure $\frac{2}{3}$, 0

$$\gamma'_{07} = 2/3$$
$$\gamma_{48} = 1/3 \quad \gamma_{38} = 1/3 \quad \gamma_{36} = 1/3$$
$$\gamma_{58} = 1/3 \quad \gamma_{39} = 1/3 \quad \gamma_{5C} = 1/3$$
$$\gamma_{23} = 2/3 \quad \gamma_{79} = 1/3 \quad \gamma_{7C} = 1/3$$
$$\gamma_{8E} = 2/3$$

spin structure $\frac{1}{3}$, $\frac{2}{3}$

$$\gamma'_{17} = 1/3 \quad \gamma'_{08} = 1/3$$
$$\gamma_{18} = 1/3 \quad \gamma_{48} = 1/3 \quad \gamma_{47} = 1/3$$
$$\gamma_{58} = 1/3 \quad \gamma_{2D} = 2/3 \quad \gamma_{4E} = 1/3$$
$$\gamma_{7A} = 2/3 \quad \gamma_{7D} = 2/3 \quad \gamma_{8C} = 2/3$$
$$\gamma_{8D} = 2/3$$

spin structure $\frac{2}{3}$, $\frac{1}{3}$

$$\gamma'_{17} = 2/3 \quad \gamma'_{08} = 2/3$$
$$\gamma_{18} = 2/3 \quad \gamma_{48} = 2/3 \quad \gamma_{47} = 2/3$$
$$\gamma_{58} = 2/3 \quad \gamma_{2D} = 1/3 \quad \gamma_{4E} = 2/3$$
$$\gamma_{7A} = 1/3 \quad \gamma_{7D} = 1/3 \quad \gamma_{8C} = 1/3$$
$$\gamma_{8D} = 1/3$$

spin structure $\frac{2}{3}$, $\frac{2}{3}$

$$\gamma'_{26} = 1/3$$
$$\gamma_{48} = 2/3 \quad \gamma_{38} = 2/3 \quad \gamma_{57} = 2/3$$
$$\gamma_{58} = 2/3 \quad \gamma_{5B} = 2/3 \quad \gamma_{5C} = 2/3$$
$$\gamma_{23} = 1/3 \quad \gamma_{6B} = 2/3 \quad \gamma_{8E} = 1/3$$

spin structure $\frac{1}{3}, \frac{1}{3}$

$$\gamma'_{26} = 2/3$$
$$\gamma_{48} = 1/3 \quad \gamma_{38} = 1/3 \quad \gamma_{57} = 1/3$$
$$\gamma_{58} = 1/3 \quad \gamma_{5B} = 1/3 \quad \gamma_{5C} = 1/3$$
$$\gamma_{23} = 2/3 \quad \gamma_{6B} = 1/3 \quad \gamma_{8E} = 2/3$$

TABLE 2

Monodromy matrices associated to the Z_3 twisted bosons on $\Sigma(1,3,1,0)$.

Twisted characteristic $\begin{bmatrix} 0 \\ 1/3 \end{bmatrix}$

$$M_{c_1} = \begin{pmatrix} 0 & 0 & \alpha \\ \alpha^2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} M_{c_2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} M_{c_3} = \begin{pmatrix} 0 & 0 & \alpha^2 \\ \alpha & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Twisted characteristic $\begin{bmatrix} 0 \\ 2/3 \end{bmatrix}$

$$M_{c_1} = \begin{pmatrix} 0 & 0 & \alpha^2 \\ \alpha & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} M_{c_2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} M_{c_3} = \begin{pmatrix} 0 & 0 & \alpha \\ \alpha^2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Twisted characteristic $\begin{bmatrix} 1/3 \\ 0 \end{bmatrix}$

$$M_{c_1} = \begin{pmatrix} 0 & 0 & 1 \\ \alpha & 0 & 0 \\ 0 & \alpha^2 & 0 \end{pmatrix} M_{c_2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} M_{c_3} = \begin{pmatrix} 0 & 0 & 1 \\ \alpha^2 & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix}$$

Twisted characteristic $\begin{bmatrix} 2/3 \\ 0 \end{bmatrix}$

$$M_{c_1} = \begin{pmatrix} 0 & 0 & 1 \\ \alpha^2 & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix} M_{c_2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} M_{c_3} = \begin{pmatrix} 0 & 0 & 1 \\ \alpha & 0 & 0 \\ 0 & \alpha^2 & 0 \end{pmatrix}$$

Twisted characteristic $\begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$

$$M_{c_1} = \begin{pmatrix} 0 & 0 & 1 \\ \alpha^2 & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix} M_{c_2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} M_{c_3} = \begin{pmatrix} 0 & 0 & 1 \\ \alpha & 0 & 0 \\ 0 & \alpha^2 & 0 \end{pmatrix}$$

Twisted characteristic $\begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}$

$$M_{c_1} = \begin{pmatrix} 0 & 0 & 1 \\ \alpha & 0 & 0 \\ 0 & \alpha^2 & 0 \end{pmatrix} M_{c_2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} M_{c_3} = \begin{pmatrix} 0 & 0 & 1 \\ \alpha^2 & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix}$$

Twisted characteristic $\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$

$$M_{c_1} = \begin{pmatrix} 0 & 0 & \alpha^2 \\ \alpha^2 & 0 & 0 \\ 0 & \alpha^2 & 0 \end{pmatrix} M_{c_2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} M_{c_3} = \begin{pmatrix} 0 & 0 & \alpha \\ \alpha & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix}$$

Twisted characteristic $\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$

$$M_{c_1} = \begin{pmatrix} 0 & 0 & \alpha \\ \alpha & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix} M_{c_2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} M_{c_3} = \begin{pmatrix} 0 & 0 & \alpha^2 \\ \alpha^2 & 0 & 0 \\ 0 & \alpha^2 & 0 \end{pmatrix}$$

FIGURE CAPTION

fig. (1.1): a system of cuts for the algebraic surface $y^n = (z - c_1)\dots(z - c_{mn})$. Going clockwise (anticlockwise) around a branch point $y^{(l)}(z)$ goes in $y^{(l+1)}(z)$ $(y^{(l-1)}(z)$.

fig. (1.2): an alternative system of cuts for $y^n = (z - c_1)\dots(z - c_{mn})$. As in the previous figure the branch number of $y(z)$ increases (decreases) by one encircling clockwise (anticlockwise) a branch point.

fig. (1.3): a system of cuts for the equation $y^n = \prod_{j=1}^{q} \frac{(z-a_j)}{(z-b_j)}$.

fig. (1.4): a system of cuts for a general Z_n symmetric surface Σ_n associated to the equation $y^n = \prod_{(i=1)}^{mn} (z - c_i) \prod_{j=1}^{q} \frac{(z-a_j)}{(z-b_j)}$.

fig. (1.5): deformation of a branch line $O - a_2$ over the branch point a_1 and relative transformation of the monodromy matrix of a_2. The point 0 is not a branch point.

fig. (1.6): a system of cuts for a surface with all simple branch points.

fig. (1.7): canonical polygon for a genus 2 surface. The a and b cycles are shown.

figs. (1.8a-c): triangulation of an algebraic curve, step A. The procedure should be repeated over all the sheets.

fig. (1.9): the glueing of two sheets reduced in the polygonal form.

fig. (1.10): the identifying of two adjacent equal sides.

fig. (1.11): triangulation of an algebraic curve, step B, first kind of cuts.

fig. (1.12): triangulation of an algebraic curve, step C, second kind of cuts.

fig. (1.13): a and b cycles for $\Sigma(1,3,1,0)$ (fig. 1.13a) and for $\Sigma(2,3,0,2)$ (fig. 1.13b).

fig. (1.14): a and b cycles for surfaces in which all branch points are simple.

fig. (1.15): elementary cycles around branch points in the case of the Z_3 symmetic algebraic curve $\Sigma(1,3,1,0)$.

fig. (1.16): a good cover $\{U_\alpha\}$ for the algebraic curve $\Sigma(1,3,1,0)$.

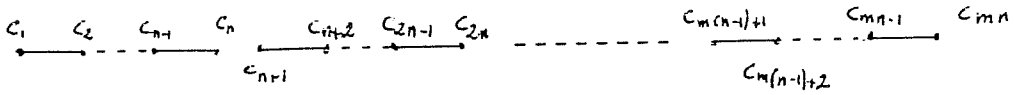


Fig. 1.1

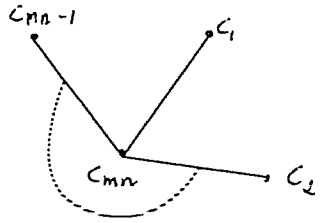


fig. 1.2

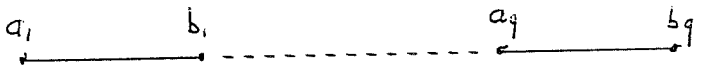


fig. 1.3

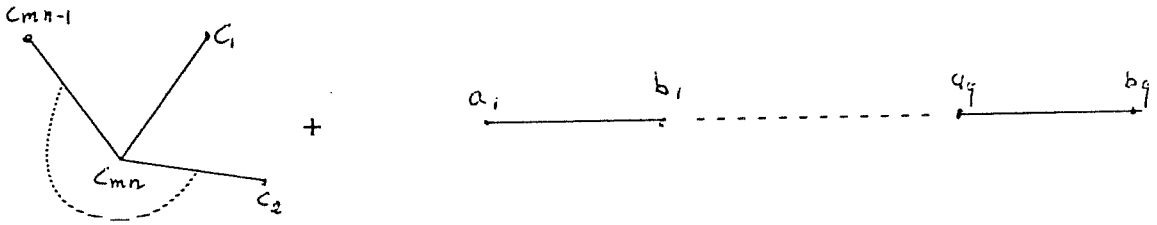


fig. 1.4

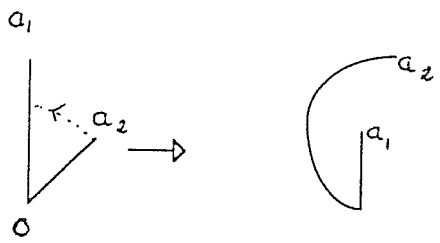


Fig. 1.5

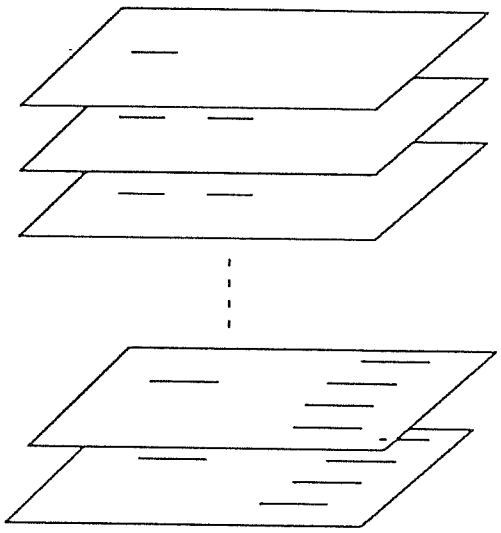


Fig. 1.6

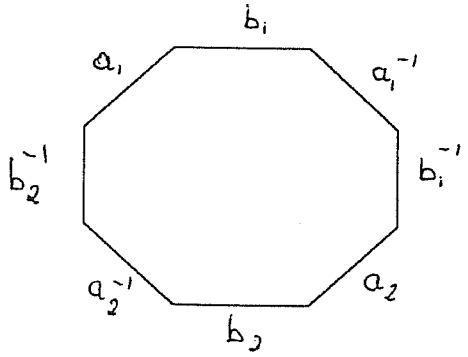


Fig. 1.7

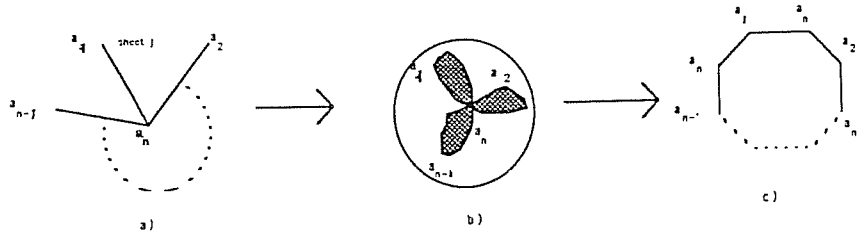


fig a.3

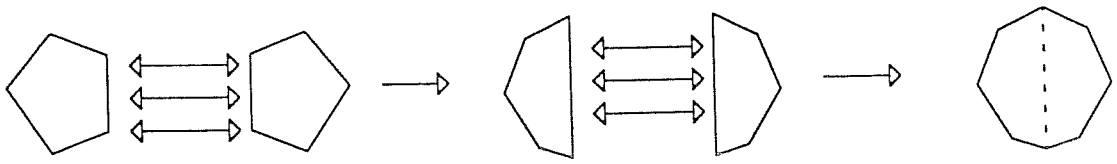


Fig. 1.9

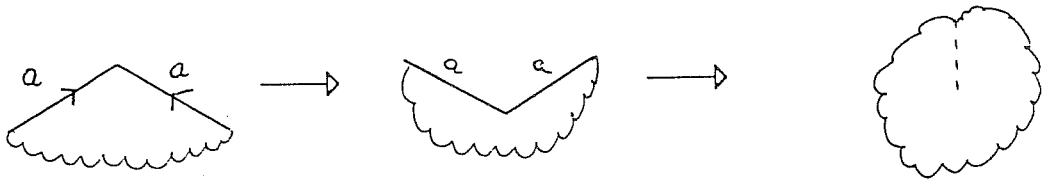


Fig. 1.10

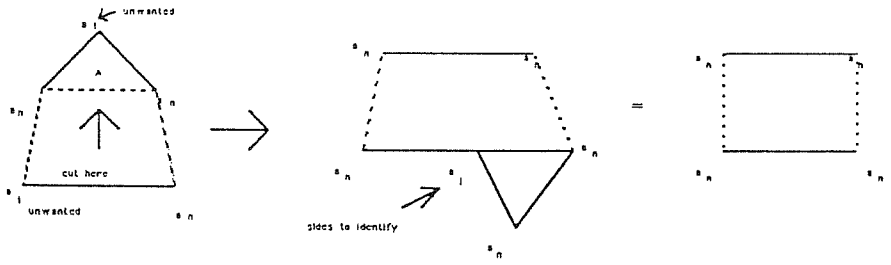
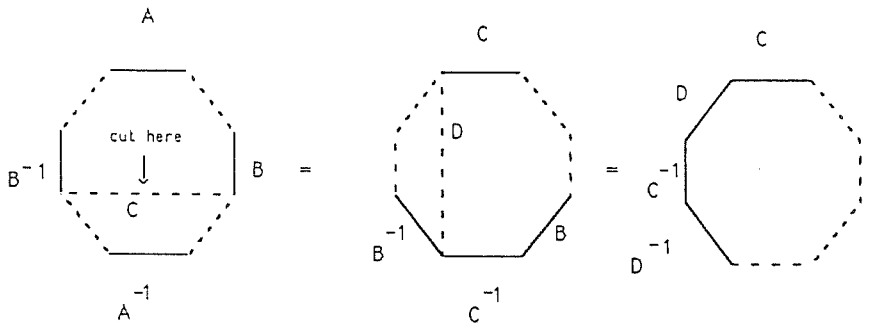


Fig. 1.11



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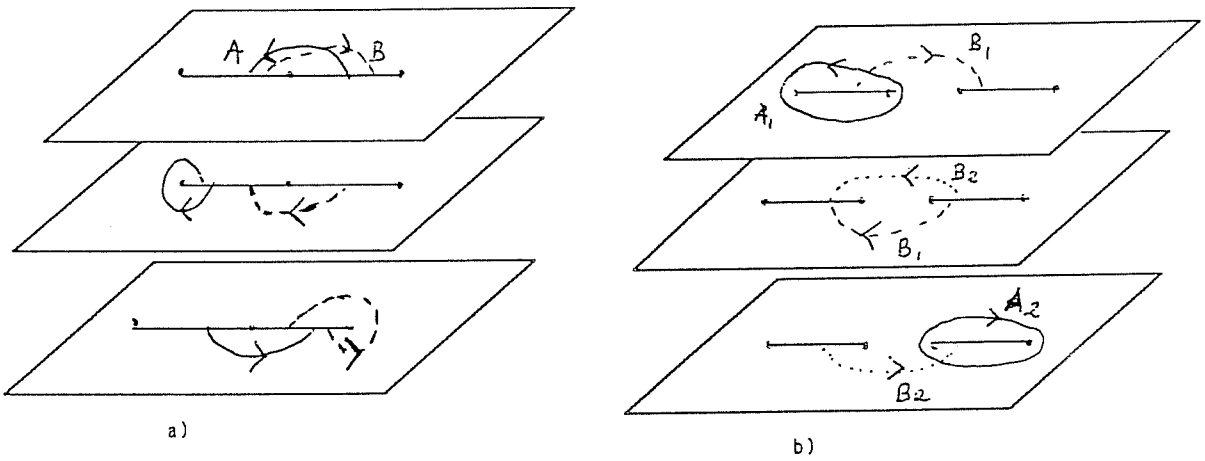

Fig. 1.13

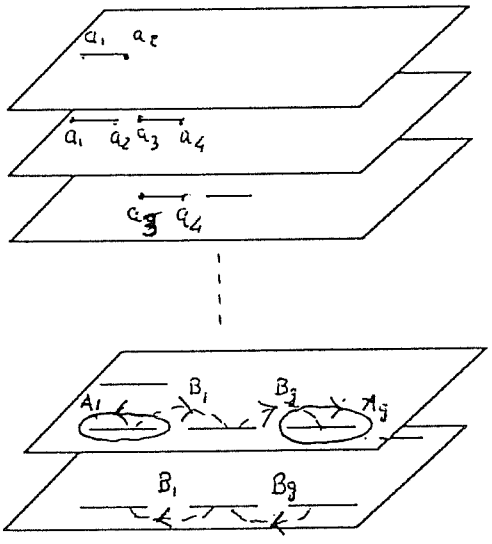

Fig. 1.14

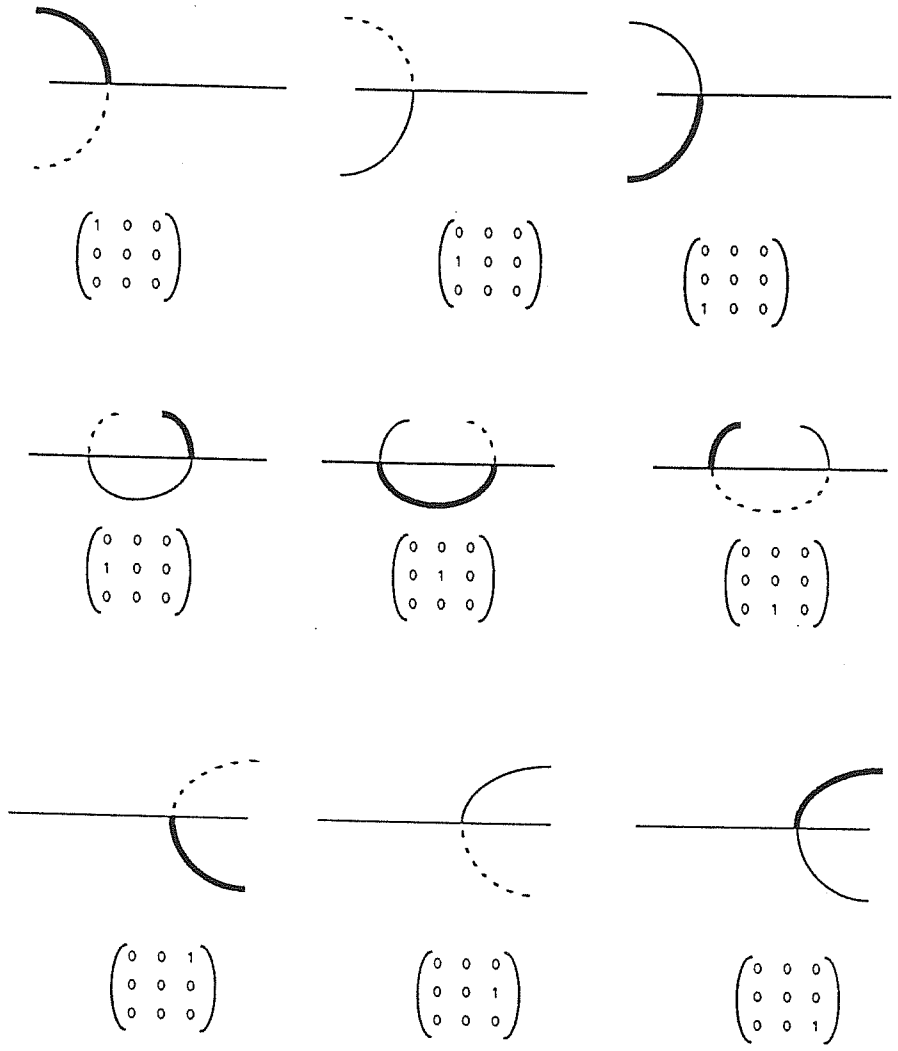
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Fig. 1.15

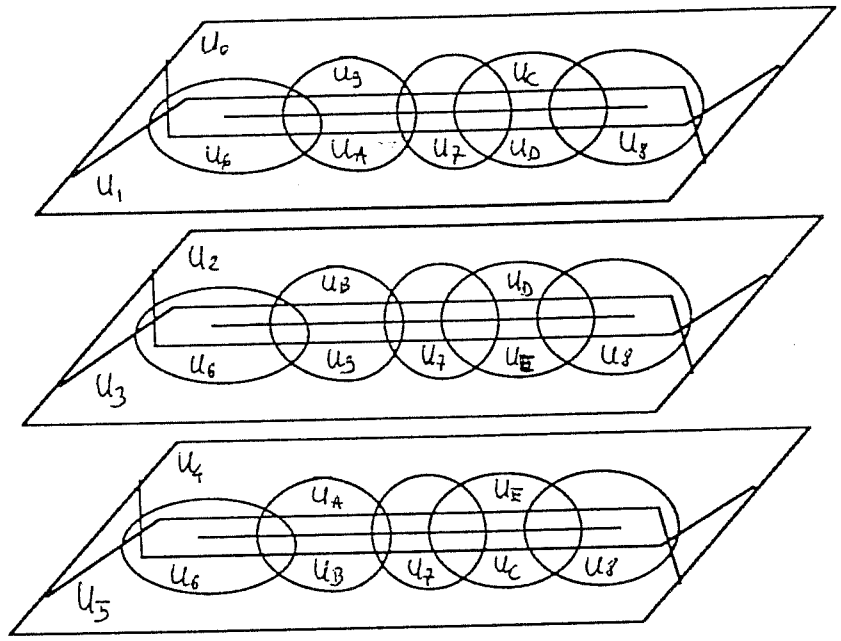
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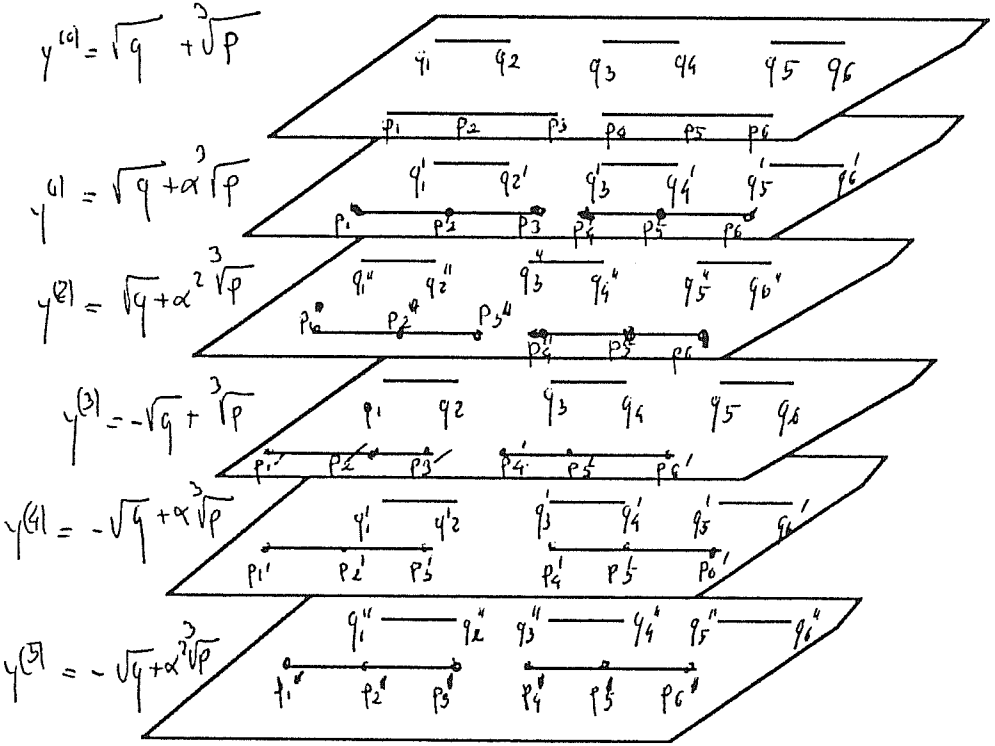
Fig. 1.16



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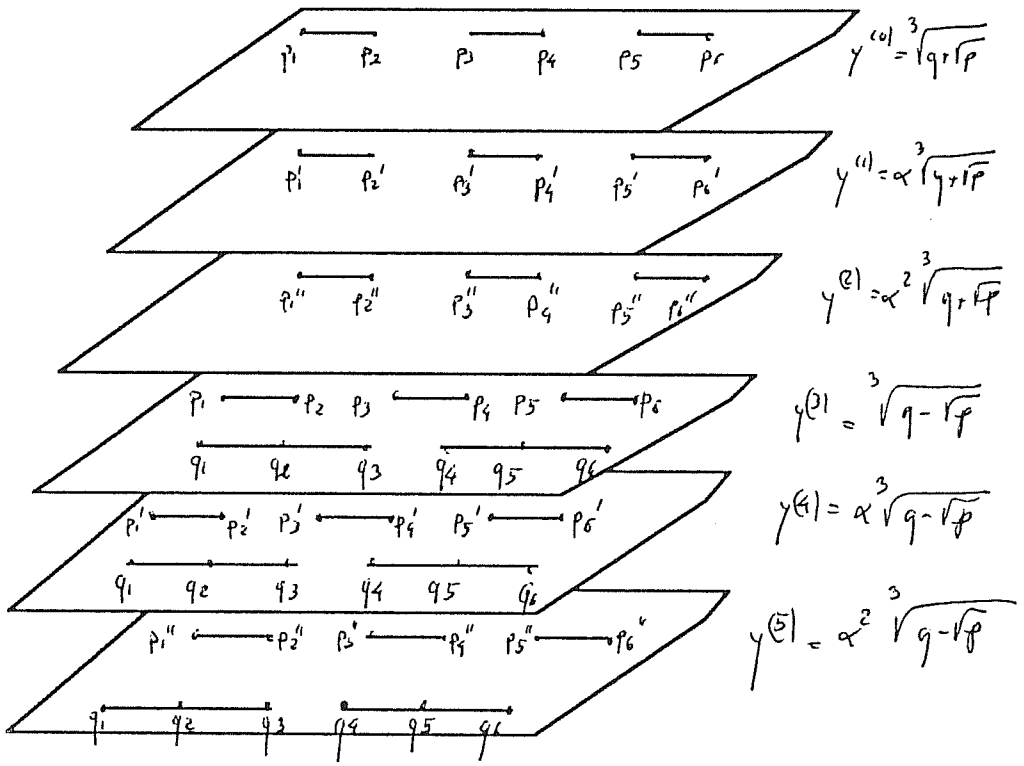

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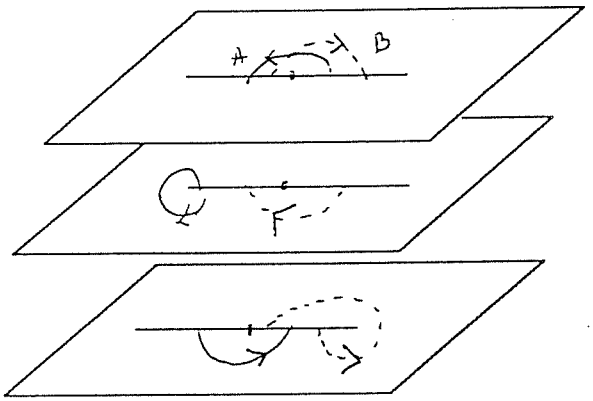

Fig. 1.19

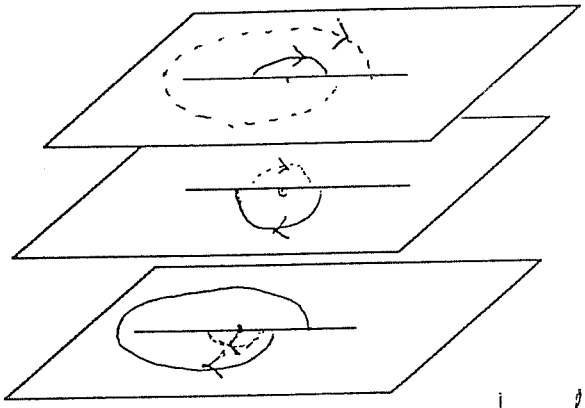

Fig. 1.20

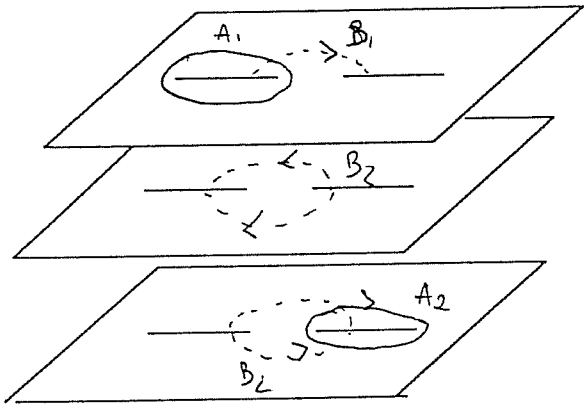

Fig. 1.21

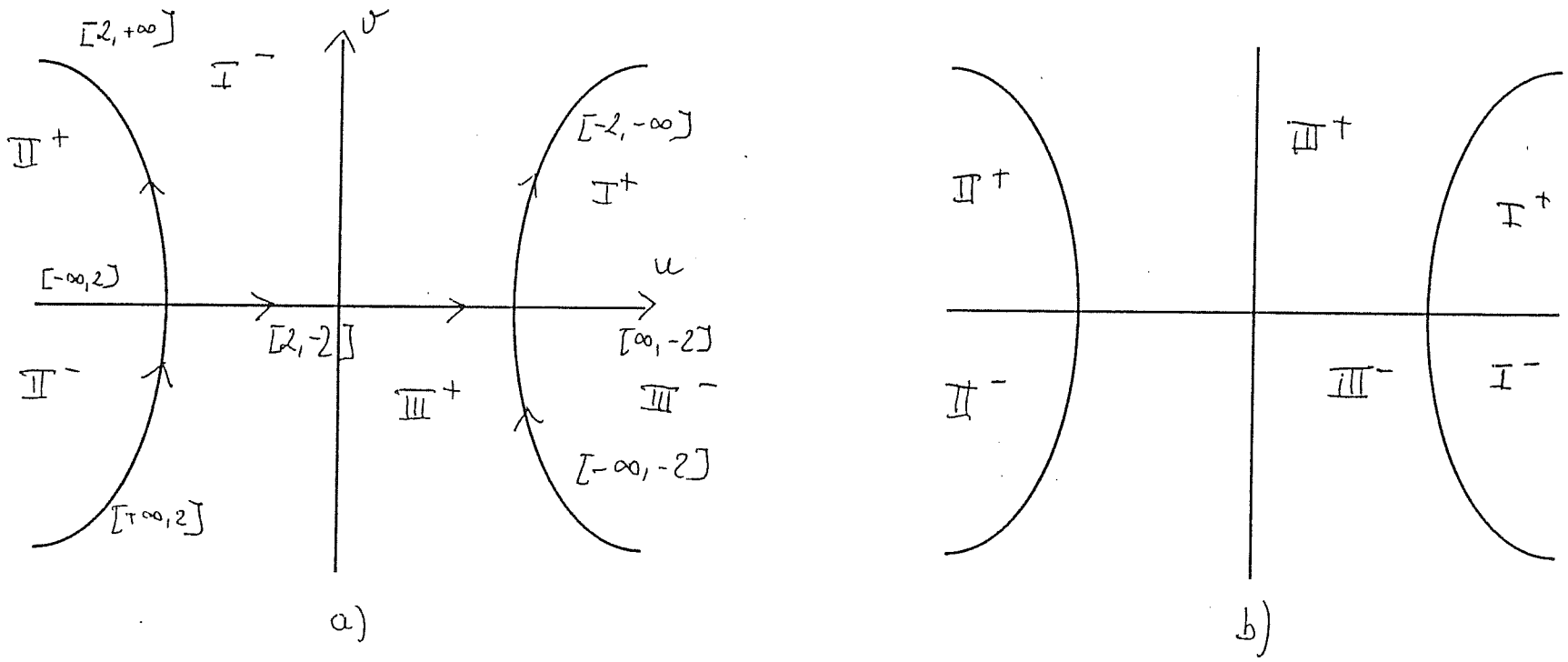

Fig. 1.22