Correlation functions of integrable flows

Thesis submitted for the degree of
"Doctor Philosophiæ"

CANDIDATE

Prospero Simonetti

SUPERVISOR

Dr. Giuseppe Mussardo

October 1994

Contents

Introduction

The explicit analogy between the path integral formulation of quantum field theory and the statistical mechanical formulation of thermodynamics becomes a true correspondence within the concept of renormalization group [1, 2, 3, 4]. In the realm of two dimensional physics this opportunity has led to the consequence of a quite unified and very powerful approach. For the sake of its clarity and beauty we embrace this attitude totally [5]. The basic concept is the infinite dimensional space of models. A model has to be considered both as an euclidean two dimensional classical statistical system and as a minkowskian $(1+1)$ quantum field theory. This equivalence serves also as a definition of generalized quantum field theory for those cases in which it strictly could not be defined (for example, perfectly legitimate statistical models like the Yang-Lee model admit an elementary particle interpretation, although at price of a generalization of unitarity requirements [5, 6, 7]). Each point of the space of models represents a state of equilibrium of a statistical system, or a quantum field theory at a fixed value of energy. Each direction represents a thermodynamical variable, or a quantum field theory coupling constant, as well as their conjugated partners, i.e. a statistical observable or a hermitean operator (with some characteristics of locality, possibly). This space is structured by the action of the renormalization group: each point is mapped into another point by a dilatation transformation, according to the dynamics of the theory. Therefore, this picture of a renormalization group flow between short distance and large distance emerges as an elegant "classical" representation of the non-linear dynamics of the quantum field theory. The knowledge of the correlation functions of the (local) observables at a point in the space of models is equivalent to the complete determination of the flow of the renormalization group passing through that point. The object of this thesis is mainly the description of a method, the bootstrap approach, for the computation of the correlation functions of two-dimensional integrable models.

A fixed point is a point of the space of models which is invariant with respect to the action of the renormalization group. There are fixed points of massless and massive nature. The first kind corresponds to the scaling invariant theories, i.e. conformally invariant [8]. In two dimensions the conformal group is infinite-dimensional, so that this huge invariance allows an almost complete classification and solution of the conformal field theories as described in refs. [9, 10, 5]. The massless fixed points live on the so-called critical surface, the locus of the points representing systems with infinite correlation length, and they distinguish themselves from generic critical points by the complete absence of nonzero massive parameters. The second kind of fixed points corresponds to completely decoupled systems: they can be reached, for example, by an infinite mass limit of any massive theory. They belong to the surface of zero correlation length and distinguish themselves from generic points of this surface by the absence of finite massive parameters. Any flow starts from a fixed point and ends up into another one [1] , since at very short and large distance any dimensionful parameter vanishes. The neighbourhood of a fixed point is thus characterized by the trajectories of the renormalization group flowing in or out of it. A useful choice of coordinates near the fixed point is naturally chosen along the directions which are eigenstates of the linearization of the renormalization group action. A direction is called relevant if it flows outside the fixed point, and it corresponds to a positive eigenvalue or anomalous dimension. A negative anomalous dimension characterizes an irrelevant direction, a flow inside the fixed point. The direction of null anomalous dimension is called marginal. In this case the flow will be outside or inside the fixed point according to the higher orders of the renormalization group, and it can also depend on the orientation of the marginal direction. The limit case of persistent marginality describes the situation in which the direction is a line of fixed points, such that the renormalization group flow is trivial at any point.

Among all the possible flows, the simplest are the integrable ones. Integrable flows may be generated either by perturbation along suitable directions of conformal field theories [13] or by the quantization of lagrangian field theories already integrable at classical level [14, 15, 16]. Integrability means the existence of an infinity of conserved currents which generate conserved charges in involution, a fact which implies a strong simplification of the infrared properties. The infrared properties can be generally encoded in a spectrum of fundamental particles, free at large distances in a generalized sense, i.e. generated by creation operators obeying infrared generalized

[1] We follow the Wilson convention in running along the renormalization group flow from the ultraviolet region to the infrared region.

statistics[2]. The kinematics of two dimensions makes the statistics properties of any number of these particles at any value of individual momenta coincide with the scattering amplitudes. For an integrable theory the scattering amplitudes of any number of particles are obtained by the two particle amplitudes simply by factorization [11, 12]. This allows us to encode the scattering or infrared properties of the flow in the non-commutativity of an associative algebra of creation and destruction operators assigned to the fundamental excitations of the theory. When these excitations correspond to massive (massless) particles, we are describing a massive (massless) integrable flow. The physical Hilbert space is thus generated by the asymptotic bases of "in" and "out" particles built 'a la' Fock by repeatedly acting on the physical vacuum with the creation algebra. Within this scattering approach, the observables are defined in terms of the form factors, the matrix elements on the asymptotic states of the corresponding operators. The correlation functions are then reconstructed by resumming the intermediate states series.

The first chapter of this thesis is devoted to the detailed explanation of the method just mentioned of computing the correlation functions for integrable models. It contains the starting sum rule formula for a correlation function in terms of form factors, and their definition in terms of the Zamolodchikov-Faddeev algebra, together with a short survey of factorized scattering properties. General recursive equations for the form factors are obtained from the analytic properties of the scattering amplitudes. A description of the principal features in the approach of form factors of two privileged operators then will follow. They are the trace of the stress-energy tensor, which drives the renormalization group flow, and the elementary interpolating field, which plays the role of the lagrangian field creating the particles. These properties are then shown for the very easy example of the free massive boson.

The second chapter describes the implementation of this method for lagrangian massive integrable theories. The form factors of the most significant fields of the sinh-Gordon and the Bullough-Dodd models are computed. The ultraviolet behaviour of the theories is discussed in terms of the c-theorem. Very interesting exact non-perturbative features of these theories are described. The two quantum equations of motion are established, and are identified at a particular value of the coupling constants. It is then shown that a line of ultraviolet fixed points can be described in terms of a persistent marginal perturbation of these models, that is within the same quantum dynamics.

The form factor program described in the first chapter and carried on in the second chapter

[2] With statistics we mean the behaviour under exchange of particles.

for massive flows shows a flexibility which allows us a natural generalization to the massless case. It is mainly based on disentangling the massless particles into right and left movers. The massless form factor program is proposed in the third chapter, where it is also worked out for the massless flow from the tricritical Ising model to the Ising model. The form factors of the principal operators flowing from a Kac-table to the other are obtained. The expected infrared divergency problem emerges only in the last step of this massless program, that is computing the correlation functions. However, due to the special feature of the infrared theory, which is free, the correlation functions of the operators in the energy sector of the Kac-table are not plagued by the occurence of infrared divergencies. So they can be computed without the introduction of a volume cut-off and with the same efficiency of the massive case. The computed correlation functions confirm the conjectured identification of the operator content.

In the fourth chapter we face the problem of integrable theories for inhomogeneous systems. The general scattering theory of integrable particles on lines of integrable defects is settled. It is treated by means of a very natural generalization of the Zamolodchikov-Faddeev algebra. It is shown how the only massive non-diagonal bulk theories which admit non trivial reflection-transmission amplitudes are the free fermionic (Ising) and bosonic ones. The detailed description of these two theories is developed. Some physical features are exhibited for the case of many defect lines. The correlation functions for the energy and the magnetization of Ising model with one defect are computed by taking advantage of the complete knowledge of their form factors in the bulk and of the scattering amplitudes on the defect. Our results are in complete agreement with the results obtained in the lattice approach.

Chapter 1

Correlation functions and form factors

The complete characterization of a renormalization group flow is possible through the computation of the correlation functions of the theory. The standard definition of the correlation function of the observables $O_i(x)$ is

$$< O_1(x_1)O_2(x_2)\dots O_n(x_n) > = \frac{1}{Z}\int \prod_{i,y} d\varphi_i(y)\, e^{-S_E[\varphi]} O_1(x_1)O_2(x_2)\dots O_n(x_n) \tag{1.1}$$

where the total statistical energy S_E is the euclidean continuation of the minkwoskian action $S[\varphi] = \int dx L(\varphi, \partial_\mu \varphi)$. The partition function Z is

$$Z = \int \prod_{i,y} d\varphi_i(y)\, e^{-S_E[\varphi]} \; . \tag{1.2}$$

The minkowskian Feynmann correlation function

$$G^F_{i_1,i_2,\dots,i_n}(x_1, x_2, \dots, x_n) = < 0|\mathcal{T}\left(\hat{O}_1(x_1)\hat{O}_2(x_2)\dots\hat{O}_n(x_n)\right)|0 > , \tag{1.3}$$

being $\mathcal{T}$ the symbol of the time ordering of a product of operators, can be also obtained as a suitable boundary value of the Schwinger function, the analytic continuation of the euclidean correlation function [17, 18].

Our method of computation of a correlation function is based on the intermediate state sum[1]

$$< 0|\hat{O}_1(x_1)\hat{O}_2(x_2)|0 > = \sum_s < 0|\hat{O}_1(0)|s >< s|\hat{O}_2(0)|0 > e^{-iP_s\cdot(x_1-x_2)} \; . \tag{1.4}$$

[1] For the sake of simplicity we will restrict ourselves in this thesis to the correlation functions of two operators. The general case is obtained straightforwardly with multiple insertions of (1.5) between nearby operators in (1.4).

It is obtained by insertion of the completeness relation of the energy-momentum eigenstates

$$\sum_s |s><s| = 1 \ , \quad \hat{P}^\mu|s> = P^\mu_s|s> \ . \tag{1.5}$$

The quantities

$$F^{O_i}(s) \ = <0|\hat{O}_i(0)|s> \tag{1.6}$$

where the base vectors $|s>$ are the asymptotic in-states, are called the form factors of the operator O_i. In two-dimensional integrable models the form factors satisfy monodromy equations and residue recursive equations which depend only on the two particle scattering amplitudes [19, 20]. For many of these models the scattering matrices are exactly known [11, 21], and we will show in the sequel the technique for the solution also of the form factors problem.

The price one has to pay in this route to the computation of the correlation function is obviously the infinite sum (1.4) which remains to be calculated. Actually, for massive flows the sum (1.4) exhibits a very fast convergence: usually the first two or three contributions are enough to obtain with high precision the correlation functions. On the contrary, for the massless case resummation techniques have in general to be considered in order to get rid of the unavoidable infrared cutoff. However, for practical purposes the form factor approach turns out to be very efficient.

A very useful formula for the correlation function of scalar operators [22] can be obtained by considering the Lehmann representation

$$G^F_{O_1,O_2}(x_1,x_2) \ = \ \int dp e^{ip\cdot(x_1-x_2)} \int d\mu^2 \frac{\rho_{O_1,O_2}(\mu^2)}{\mu^2-p^2-i\varepsilon} \ , \tag{1.7}$$

where the spectral function is

$$\rho_{O_1,O_2}(\mu^2) \ = \ \frac{1}{2\pi}\sum_s <0|\hat{O}_1(0)|s><s|\hat{O}_2(0)|0> \ \delta^{(2)}(P_s-\mu) \ . \tag{1.8}$$

Now rotating to the euclidean plane and making use of the integral

$$\int dp\, e^{ip\cdot x}/(p^2+\mu^2) = 2\pi K_0(\mu r) \ , \tag{1.9}$$

we have the useful representation

$$<O_1(x)O_2(0)> = \ 2\sum_s \left(F^{O_1}(s)\right)^* F^{O_2}(s)\, P^0_s\, K_0(P^0_s r)\, \delta(P^1_s) \tag{1.10}$$

where $r = \sqrt{(x_1)^2+(x_2)^2}$ is the euclidean invariant length of x.

1.1 The scattering matrix and the Zamolodchikov-Faddeev algebra

The fundamental starting point for our analysis is the knowledge of the spectrum of asymptotic particles, which is assumed a priori. For the first two chapters, we will consider the massive case. However, formulas will be written as much as possible in a general fashion in terms of the momenta, or of the kinematical invariants, so that we will be allowed an easy generalization to the massless case.

Let's consider a model whose excitations are constituted by a set of N kind of particles with masses m_a. These particles do not interact at very large distances, and so their dispersion relation is the free one $p^2 = m_a^2$. It is useful to introduce the parameterization of the on shell momenta of the particle a

$$p^0 = m_a \cosh\beta \,,\; p^1 = m_a \sinh\beta \tag{1.11}$$

in term of the rapidity variable β.

The behaviour of these particles in the extreme past is described by the base of in states, in the extreme future by the out states. This terminology refers to scattering experiments of incoming and outgoing particles. The scattering amplitudes are then given by the transition probabilities

$$S(\{p'_j, a'_j\}, \{p_i, a_i\}) = {}_{out} < \{p'_j, a'_j\}|\{p_i, a_i\} >_{in} \;. \tag{1.12}$$

The hypothesis of integrability, that is the existence of an infinite number of conserved currents and related conserved charges I_s, has strong implications on the scattering, namely elasticity and factorization [11, 12]. Elasticity states that each individual momenta is conserved; therefore the only nondiagonal interaction can occur between equal mass particles. The two-dimensional kinematics then forces the scattering amplitudes of n particles to factorize in a product of $n(n-1)/2$ two particles scattering amplitudes

$$S^{(n)}{}^{\{a'_j\}}_{\{a_i\}}(\{p_i\}) = {}_{out} < \{p_j, a'_j\}|\{p_i, a_i\} >_{in} = \sum \prod S^{(2)}{}^{a'_l a'_k}_{a_l a_k}(p_l, p_k) \tag{1.13}$$

$$|p_1 a_1, p_2 a_2 >_{in} = S^{(2)}{}^{a'_1 a'_2}_{a_1 a_2}(p_1, p_2)|p_1 a'_1, p_2 a'_2 >_{out} \;. \tag{1.14}$$

This simplification is at the origin of the solvability of the quantum theory. The consistency equations imposed by these hypotheses are so restrictive that many exact scattering amplitudes have been associated to integrable models in the last fifteen years [11, 21]. As a first step, we take the observation that elasticity and factorization properties imply the existence of an associative algebra of destruction-creation operators

$$
\begin{aligned}
Z^{a_1}(p_1)Z^{a_2}(p_2) &= S^{(2)}{}^{a_1a_2}_{a'_1a'_2}(p_1,p_2)Z^{a'_2}(p_2)Z^{a'_1}(p_1) \\
Z^{\dagger}_{a_1}(p_1)Z^{\dagger}_{a_2}(p_2) &= S^{(2)}{}^{a'_1a'_2}_{a_1a_2}(p_1,p_2)Z^{\dagger}_{a'_2}(p_2)Z^{\dagger}_{a'_1}(p_1) \\
Z^{a_1}(p_1)Z^{\dagger}_{a_2}(p_2) &= S^{(2)}{}^{a'_2a_1}_{a_2a'_1}(p_1,p_2)Z^{\dagger}_{a'_2}(p_2)Z^{a'_1}(p_1) + 2\pi p_1^0\delta(p_1^1-p_2^1)\,\delta^{a_1}_{a_2} \qquad (1.15)
\end{aligned}
$$

called Zamolodchikov-Faddeev (ZF) algebra. In term of this algebra we thus define the two bases of asymptotic states

$$
\begin{aligned}
|p_1a_1,p_2a_2,\ldots,p_na_n>_{in} &= Z^{\dagger}_{a_1}(p_1)Z^{\dagger}_{a_2}(p_2)\ldots Z^{\dagger}_{a_n}(p_n)|0> \\
|p_1a_1,p_2a_2,\ldots,p_na_n>_{out} &= Z^{\dagger}_{a_n}(p_n)Z^{\dagger}_{a_{n-1}}(p_{n-1})\ldots Z^{\dagger}_{a_1}(p_1)|0> \qquad (1.16)
\end{aligned}
$$

where the momenta are ordered according to $p_1^1 > p_2^1 > \ldots > p_n^1 >$ and the Fock vacuum $|0>$ is annihilated by the "destructors" $Z_a(p)$

$$
Z_a(p)|0> = 0\ . \qquad (1.17)
$$

The operators $Z^{\dagger}_a(p)$ of the ZF-algebra carry a representation of the commuting algebra of conserved charges $I_{\pm s}$, with $s \in \mathcal{S} \subset \mathcal{Z}_+$ labeling the Lorentz spin,

$$
\begin{aligned}
[\Sigma, I_{\pm s}] &= \pm s I_{\pm s} \\
\left[I_{\pm s}, Z^{\dagger}_a(p)\right] &= \chi^{(s)}_a p^{\pm s} Z^{\dagger}_a(p)\ . \qquad (1.18)
\end{aligned}
$$

Σ is the generator of the Lorentz boost and $p^{\pm} = p^0 \pm p^1$ are the light-cone coordinates of the momentum vector p. The pure numbers $\chi^{(s)}_a$ are fixed functions of the mass ratios, normalized by $\chi^{(1)}_a = 1$. The $s=1$ charges are the momentum vector operators in light-cone coordinates. The energy operator is $H = \frac{I_1+I_{-1}}{2}$ and the linear momentum $P = \frac{I_1-I_{-1}}{2}$. The structure (H,P,Σ) constitutes the basic Poincarè algebra of traslations and boost. The corresponding Poincarè group $x' = x+\xi$, $x'^{\pm} = e^{\pm\Lambda}x^{\pm}$ has the following unitary representation on the operators $Z^{\dagger}_a(p)$

$$
\begin{aligned}
U(\xi)\,Z^{\dagger}_a(p)\,U(\xi)^{-1} &= e^{ip\cdot\xi}Z^{\dagger}_a(p) \\
U(\Lambda)\,Z^{\dagger}_a(p)\,U(\Lambda)^{-1} &= Z^{\dagger}_a(p')\ . \qquad (1.19)
\end{aligned}
$$

The first consistency equation the ZF-algebra has to satisfy results when one applies the first line of the definition (1.15) twice

$$
S^{(2)}{}^{a''_1a''_2}_{a_1a_2}(p_1,p_2)S^{(2)}{}^{a'_2a'_1}_{a''_2a''_1}(p_2,p_1) = \delta^{a'_1}_{a_1}\delta^{a'_2}_{a_2}\ . \qquad (1.20)
$$

This equation translates the quantum unitarity constraint of the scattering operator S, $SS^\dagger = 1$. The quantum field theory property of real analyticity of the scattering amplitude is obtained by comparing the hermitean conjugate of the first line with the second line of (1.15), $S^{(2)}{}^{a_1 a_2}_{a_1' a_2'}(p_1, p_2) = \left(S^{(2)}{}^{a_2' a_1'}_{a_2 a_1}(p_2, p_1) \right)^*$.

The factorized dynamics of the theory is encoded in the associativity of the algebra. The three particle scattering amplitude has to be independent on the different order of the application of the ZF algebra (1.15) to obtain it. Compatible dynamics have thus to be found among the solutions of the Yang-Baxter triangle equation

$$
\begin{aligned}
S^{(2)}{}^{a_1'' a_2''}_{a_1\, a_2}(p_1, p_2) S^{(2)}{}^{a_1' \, a_3''}_{a_1''\, a_3}(p_1, p_3) S^{(2)}{}^{a_2' \, a_3'}_{a_2'' \, a_3''}(p_2, p_3) \; = \\
S^{(2)}{}^{a_2'' \, a_3''}_{a_2\, a_3}(p_2, p_3) S^{(2)}{}^{a_1'' \, a_3'}_{a_1 \, a_3''}(p_1, p_3) S^{(2)}{}^{a_1' \, a_2'}_{a_1'' \, a_2''}(p_1, p_2) \; .
\end{aligned}
\tag{1.21}
$$

The two particle scattering amplitude $S^{(2)}{}^{a_1' a_2'}_{a_1 a_2}(p_1, p_2)$ is the boundary value of an analytic function of the unique independent Mandelstam variable $s = (p_1 + p_2)^2$ [22, 23]. This function, denoted with the same symbol, must exhibit square root[2] branch points located at $Ims = 0$, $Res = (m_{a_1} - m_{a_2})^2, (m_{a_1} + m_{a_2})^2$ and satisfy a crossing symmetry relation

$$
S^{a_1' a_2'}_{a_1 a_2}(s) = S^{a_1' \bar{a}_2'}_{a_1 \bar{a}_2}(2m^2_{a_1} + 2m^2_{a_2} - s) \tag{1.22}
$$

where we indicate with $\bar{a}$ the antiparticle of the particle a. The physical sheet is the upper plane in the Riemann surface described by s. The physical sheet poles can be located only in the segment $Ims = 0$, $(m_{a_1} - m_{a_2})^2 < Res < (m_{a_1} + m_{a_2})^2$ and represent bound state poles. Since they are stable states of the theory, they must be associated to some of the particles in the spectrum. Taking the residues of a three particles amplitude on the pole in the s-channel of two of them, one has to obtain the amplitude of the third particle with the bound state particle. This observation leads to a finite set of nested quadratic equations which leads to a classification of possible integrable scattering systems. This method of finding the scattering amplitudes has been called the bootstrap approach [24, 25, 11, 21]. The possible unphysical sheets poles represent resonances.

The on shell parameterization (1.11) is very useful because it uniformizes the amplitudes. The Mandelstam variable becomes $s = m^2_{a_1} + m^2_{a_2} + 2\, m_{a_1} m_{a_2} \cosh(\beta_1 - \beta_2)$ and the two particle scattering amplitude becomes an analytic function of the rapidity variable $\beta_{12} \equiv \beta_1 - \beta_2$. The physical sheet is mapped into the physical strip $0 < Im\beta_{12} < \pi$, the first unphysical sheet into

[2] The branchings are of square root type since a contour which runs twice around one of the branch point is closed due to the unitarity equation (1.20).

the strip $\pi < Im\beta_{12} < 2\pi$. If it is diagonal, the scattering function is $2\pi i$-periodic and there is only one unphysical sheet-strip. Possible poles are mapped into the segment $Re\beta_{12} = 0$ inside the physical strip. The branches are mapped as in Fig. 1.1. The ZF-algebra becomes

$$
\begin{aligned}
Z^{a_1}(\beta_1)Z^{a_2}(\beta_2) &= S^{a_1 a_2}_{a_1' a_2'}(\beta_{12})Z^{a_2'}(\beta_2)Z^{a_1'}(\beta_1) \\
Z^{\dagger}_{a_1}(\beta_1)Z^{\dagger}_{a_2}(\beta_2) &= S^{a_1' a_2'}_{a_1 a_2}(\beta_{12})Z^{\dagger}_{a_2'}(\beta_2)Z^{\dagger}_{a_1'}(\beta_1) \\
Z^{a_1}(\beta_1)Z^{\dagger}_{a_2}(\beta_2) &= S^{a_2' a_1}_{a_2 a_1'}(\beta_{12})Z^{\dagger}_{a_2'}(\beta_2)Z^{a_1'}(\beta_1) + 2\pi\delta(\beta_1-\beta_2)\,\delta^{a_1}_{a_2} \,.
\end{aligned}
\tag{1.23}
$$

The conserved charges spectrum reads as

$$\left[I_{\pm s}, Z^{\dagger}_a(\beta)\right] = \chi^{(s)}_a m^s_a e^{\pm s\beta} Z^{\dagger}_a(\beta) \,, \tag{1.24}$$

while the Lorentz boost action becomes

$$U(\Lambda)\, Z^{\dagger}_a(\beta)\, U(\Lambda)^{-1} = Z^{\dagger}_a(\beta+\Lambda) \,. \tag{1.25}$$

The in and out bases are now

$$
\begin{aligned}
|\beta_1 a_1, \beta_2 a_2, \ldots, \beta_n a_n >_{in} &= Z^{\dagger}_{a_1}(\beta_1)Z^{\dagger}_{a_2}(\beta_2)\ldots Z^{\dagger}_{a_n}(\beta_n)|0> \\
|\beta_1 a_1, \beta_2 a_2, \ldots, \beta_n a_n >_{out} &= Z^{\dagger}_{a_n}(\beta_n)Z^{\dagger}_{a_{n-1}}(\beta_{n-1})\ldots Z^{\dagger}_{a_1}(\beta_1)|0>
\end{aligned}
\tag{1.26}
$$

where the rapidities are ordered according to $\beta_1 > \beta_2 > \ldots > \beta_n >$. The completeness relation (1.5) can thus be written

$$
\begin{aligned}
1 &= \sum_s |s><s| = \sum_n \int \left(\prod_i^n \tfrac{d\beta_i}{2\pi}\right) |\beta_1,\ldots,\beta_n>_{in\ in}<\beta_1,\ldots,\beta_n| \\
&= \sum_n \tfrac{1}{n!}\int_{-\infty}^{+\infty} \left(\prod_i^n \tfrac{d\beta_i}{2\pi}\right) |\beta_1,\ldots,\beta_n><\beta_1,\ldots,\beta_n| \,.
\end{aligned}
\tag{1.27}
$$

The unitarity equation now reads as

$$S^{a_1'' a_2''}_{a_1 a_2}(\beta) S^{a_2' a_1'}_{a_2'' a_1''}(-\beta) = \delta^{a_1'}_{a_1}\delta^{a_2'}_{a_2} \tag{1.28}$$

while the crossing symmetry equation turns into

$$S^{a_1' a_2'}_{a_1 a_2}(\beta) = S^{a_1' \bar{a}_2'}_{a_1 \bar{a}_2}(i\pi - \beta) \,. \tag{1.29}$$

The Yang-Baxter equation (1.21) is practically the same also in the new variables, so we do not rewrite it.

Let's consider in some detail the bootstrap equations. If the scattering amplitude of the particles a and b exhibites a pole located at $\beta = iu^c_{ab}$, then this bound state singularity is associated to the particle c of the spectrum whose mass is related to the other masses by the equation

$$m_a^2 + m_b^2 + 2m_a m_b \cos u_{ab}^c \;=\; m_c^2 \,. \tag{1.30}$$

According to the bootstrap argument, the particle a will appear as a pole of the amplitude of the scattering of the particles b and c and similarly for the particle b. Corresponding equations like (1.30) involving the other two pole angles u_{bc}^a, u_{ca}^b hold true. These poles have to show themselves also in the multiparticle scattering amplitudes. Two particle amplitudes can thus also be recovered as residues of three particle amplitudes. Denoting with f_{ab}^c the coupling costant[3] of the three particle a, b, c interaction, we end up with the bootstrap equations

$$f_{ab}^{c'} S_{c'd}^{cd'}(\beta) \;=\; f_{a'b'}^{c} S_{ad}^{a'd''}(\beta + i(\pi - u_{ac}^b)) S_{bd''}^{b'd'}(\beta - i(\pi - u_{bc}^a)) \tag{1.31}$$

where it is summed over every d'' and over the particles a', b', c' which satisfy the kinematical equations (1.30) with m_a, m_b, m_c fixed. The consistency of this bootstrap system is so restrictive that it gives rise to constraints between mass ratios, scattering angles and spin of conserved charges. These constraints have been exploited to solve completely many scattering problems [21]. In addition, starting with one exact two particle amplitude it is possible to find all the others simply studying the location of its poles and solving the related bootstrap equations, and then iterating the procedure.

For theories with no degeneracy in the particles spectrum, the scattering occurs in a purely elastic way and thus the amplitudes are diagonal. A generic diagonal scattering amplitude is denoted by the function $S_{ab}(\beta)$ for the particles a, b. For these theories charge conjugation symmetry requires $S_{ab}(\beta) \;=\; S_{\bar{a}\bar{b}}(\beta)$ and parity invariance implies $S_{ab}(\beta) \;=\; S_{ba}(\beta)$ so that unitarity and crossing symmetry become

$$S_{ab}(\beta) S_{ab}(-\beta) \;=\; 1 \tag{1.32}$$

$$S_{a\bar{b}}(\beta) \;=\; S_{\bar{a}b}(\beta) \;=\; S_{ab}(i\pi - \beta) \,. \tag{1.33}$$

As a first consequence the scattering amplitude is $2\pi i$-periodic. Wightman axioms [26] constrain the scattering amplitude to be polynomially bounded in the momenta. One can thus show [27] that the most general meromorphic, real analytic[4], $2\pi i$-periodic solution to the unitarity equation

[3] It is defined as the mass-shell limit of the three particles a, b, c vertex function $\Gamma_{ab}^c(p_a, p_b, p_c)$.

[4] Real analyticity here means to be real on the imaginary axis.

has the form

$$S_{ab} = \prod_{\alpha \in A_{ab}} s_\alpha(\beta) \tag{1.34}$$

with

$$s_\alpha(\beta) = \frac{\sinh\frac{1}{2}(\beta + i\alpha\pi)}{\sinh\frac{1}{2}(\beta - i\alpha\pi)} \tag{1.35}$$

where α is real $-1 < \alpha < 1$ so that the pole at $\beta = i\alpha\pi$ is in the physical strip. The building blocks s_α satisfy a lot of useful functional relations which can be found in [28, 21]. If the particle a is equal to its antiparticle $\bar{a}$, a fact which occurs for every particle in non-degenerate spectra, then crossing symmetry becomes $S_{ab}(\beta) = S_{ab}(i\pi - \beta)$. The solution will be in this case of the form

$$S_{ab} = \prod_{\alpha \in A'_{ab}} S_\alpha(\beta) \tag{1.36}$$

with

$$S_\alpha(\beta) = \frac{\tanh\frac{1}{2}(\beta + i\alpha\pi)}{\tanh\frac{1}{2}(\beta - i\alpha\pi)} = s_\alpha(\beta)s_\alpha(i\pi - \beta)\ . \tag{1.37}$$

The determination of the sets A_{ab} and A'_{ab} corresponds to the solution of the scattering problem for a given theory. As mentioned before, it can be done implementing the bootstrap method through the equations

$$S_{cd}(\beta) = S_{ad}(\beta + i(\pi - u^b_{ac}))S_{bd}(\beta - i(\pi - u^a_{bc}))\ . \tag{1.38}$$

1.2 Definition of form factors and Smirnov's axioms

Specializing the definition (1.6) of the form factors of an operator $O(x)$ to the Zamolodchikov-Faddeev base, we have

$$F^O_{a_1 a_2 \ldots a_n}(\beta_1, \beta_2, \ldots, \beta_n) = < 0|O(0)|\beta_1 a_1, \beta_2 a_2, \ldots, \beta_n a_n >_{in}\ . \tag{1.39}$$

Strictly speaking, these functions are defined only in the subset $\beta_1 > \beta_2 > \ldots \beta_n$ of $\mathcal{R}^n$. However, it is natural to extend them to the whole $\mathcal{R}^n$ making use of the ZF-algebra (1.23)

$$\begin{aligned} &F^O_{a_1 \ldots a_j a_{j+1} \ldots a_n}(\beta_1, \ldots, \beta_j, \beta_{j+1}, \ldots, \beta_n) = \\ &\quad S^{a'_j a'_{j+1}}_{a_j a_{j+1}}(\beta_{jj+1}) F^O_{a_1 \ldots a_{j+1} a_j \ldots a_n}(\beta_1, \ldots, \beta_{j+1}, \beta_j, \ldots, \beta_n)\ . \end{aligned} \tag{1.40}$$

With reference to their relativistic properties, the form factors of an operator of Lorentz spin s satisfy the relation

$$F^O_{a_1 a_2 \ldots a_n}(\beta_1 + \Lambda, \beta_2 + \Lambda, \ldots, \beta_n + \Lambda) = e^{s\Lambda} F^O_{a_1 a_2 \ldots a_n}(\beta_1, \beta_2, \ldots, \beta_n)\ . \tag{1.41}$$

Hence, the form factors of a scalar or spinless operator are functions of the difference of the rapidities. The above Lorentz behaviour (1.41), in the case of spin $s \neq 0$, can be taken into account by factorizing any $F^O_{a_1 a_2 \dots a_n}(\beta_1, \beta_2, \dots, \beta_n)$ into a product of a kinematical function $I^s(\beta_1, \beta_2, \dots, \beta_n)$ satisfying (1.41) and a 'scalar' dynamical function $f^O_{a_1 a_2 \dots a_n}(\{\beta_i - \beta_j\})$. As shown in refs. [29, 30, 31], the function I^s has to be a symmetric homogeneous function of degree s of the variables $x_i = e^{\beta_i}$, and furthermore it is related to the positions of the bound state poles in a consistent way, in a one-to-one correspondence with the set of the conserved charges. This suggests a structure of the space of local operators in terms of "primary" and "descendant" fields, in strict analogy with the conformal field theories.

Similarly to the case of the scattering amplitudes, the analytic properties of the form factors lead to very restrictive equations. The 'physical' form factors defined in (1.39) are boundary values of their analytic continuation in the planes of the invariant Mandelstam kinematical variables. These variables in two dimensions can be mapped into the planes of the difference of rapidities. As well as the Mandelstam variables, the difference variables are obviously not independent. We choose as independent ones the differences in the subset $\{\beta_{1i}\}$, so that the 'physical' boundary values (1.39) are reached for positive values of their arguments. This is strictly true for a scalar operator. For a definite spin operator, we also have to add the dependence on the variable β_1. In practice, the analytic properties will be generally established considering $F^O_{a_1 a_2 \dots a_n}(\beta_1, \beta_2, \dots, \beta_n)$ as true functions of the rapidities.

As an example, let us restrict ourselves to the form factor $F(s)$ of a scalar operator on two incoming particles with equal mass

$$F(s) = < 0|O(0)|p_1, p_2 >_{in} \quad . \tag{1.42}$$

It can be rigorously shown [22] with dispersion techniques that this form factor is in general an analytic function in the variable $s = (p_1 + p_2)^2$ with a branching point at $s = 4m^2$ and possible bound state poles in the segment $0 < s < 4m^2$. The point $s = 0$ is not a branching point, so that, contrary to the scattering amplitude case, crossing the half-line $s < 0$ does not lead to a discontinuity. As depicted in Fig. 1.2, the boundary value of this analytic function at $s + i\epsilon, s > 4m^2$ is the 'physical' form factor (1.42), while its boundary value at $s - i\epsilon, s > 4m^2$ corresponds to the 'physical' form factor $< 0|O(0)|p_1, p_2 >_{out}$. For $s < 0$, $F(s)$ measures the last 'physical' form factor $_{out} < p_1|O(0)|p_2 >_{in}$ whose kinematical variable is $t = (p_1 - p_2)^2 < 0$. The absence of the cut for $s < 0$, implies that under the parameterization in term of the rapidities $s = 2m^2(1 + \cosh\beta_{12})$ the physical 'incoming' halfline $z = s + i\epsilon$, $s > 4m^2$ is mapped into the two halflines $Re\beta_{12} > 0, Im\beta_{12} = 0$ and $Re\beta_{12} < 0, Im\beta_{12} = 2\pi i$. We thus end up with the property

$$F_2^O(\beta_{12}) = F_2^O(2\pi i - \beta_{12}) \,. \tag{1.43}$$

The exact solution for the form factors of the current of the massive Thirring model, within the quantum inverse scattering method [16, 32], allowed F.A. Smirnov to find the generalization of (1.43) [19, 20]. The form factors of a general operator therefore satisfy

$$F^O_{a_1 a_2 \ldots a_n}(\beta_1, \beta_2, \ldots, \beta_n + 2\pi i) = F^O_{a_n a_1 \ldots a_{n-1}}(\beta_n, \beta_1, \ldots, \beta_{n-1}) \,. \tag{1.44}$$

This property can be mixed to (1.40) in order to yield

$$\begin{aligned} F^O_{a_1 a_2 \ldots a_n}(\beta_1, \beta_2, \ldots, \beta_n + 2\pi i) \;=\; & S^{a_1' b_1}_{a_1 a_n}(\beta_1 - \beta_n) S^{a_2' b_2}_{a_2 b_1}(\beta_2 - \beta_n) \ldots S^{a_{n-1}' a_n'}_{a_{n-1} b_{n_2}}(\beta_{n-1} - \beta_n) \\ & F^O_{a_1' a_2' \ldots a_n'}(\beta_1, \beta_2, \ldots, \beta_n) \,. \end{aligned} \tag{1.45}$$

Therefore, in the diagonal case, contrary to scattering amplitudes, the form factors are not $2\pi i$-periodic functions of rapidities.

Besides the monodromy properties just described, also the poles structure for a generic form factor can be stated with precision. The dynamical bound state poles located in the segment $Re\beta_{12} = 0, 0 < Im\beta_{12} < \pi$ are not the only singularities of a form factor when the number of particles exceeds 2. Kinematical poles necessarily appear when the difference of rapidities of a particle and its antiparticle approaches $\beta_{ij} = i\pi$. Their presence is due to the fact that the incoming 'physical' form factor $< 0|O(0)|\tilde{p}\bar{a}, pa, p_1 a_1, \ldots, p_n a_n >_{in}$ analytically continued to $-\tilde{p}$ describes also the 'physical' form factor $_{out} < \tilde{p}a|O(0)|pa, p_1 a_1, \ldots, p_n a_n >_{in}$ which has a well-known forward scattering singularity when $\tilde{p} = p$. The corresponding residues [19] give rise to a recursive equation between the n-particle and the $(n+2)$-particle form factors

$$\begin{aligned} & -i \lim_{\tilde{\beta} \to \beta} (\tilde{\beta} - \beta) F^O_{\bar{a} a a_1 \ldots a_n}(\tilde{\beta} + i\pi, \beta, \beta_1, \beta_2, \ldots, \beta_n) = \\ & \left(\delta^{a_1'}_{a_1} \delta^{a_2'}_{a_2} \ldots \delta^{a_n'}_{a_n} - S^{\bar{a} a_1'}_{b_1 a_1}(\beta - \beta_1) S^{b_1 a_2'}_{b_2 a_2}(\beta - \beta_2) \ldots S^{b_{n-1} a_n'}_{a a_n}(\beta - \beta_n) \right) \\ & F^O_{a_1' \ldots a_n'}(\beta_1, \beta_2, \ldots, \beta_n) \,. \end{aligned} \tag{1.46}$$

In corrispondence with bound states in the spectrum, there is another set of recursive equations obtained by looking at their poles in the matrix elements. Let $\beta_{ij} = i\, u^c_{ab}$ be the location of the pole in the two-particle a, b scattering amplitude corresponding to the bound state c. Then the corresponding residue in the form factors is given by

$$-i \lim_{\epsilon \to 0} \epsilon\, F^O_{a b a_1 \ldots a_{n-1}}(\beta + i\, u^c_{ab} - \frac{\epsilon}{2}, \beta + \frac{\epsilon}{2}, \beta_1, \ldots, \beta_{n-1}) = f^c_{ab}\, F^O_{c a_1 \ldots a_{n-1}}(\beta, \beta_1, \ldots, \beta_{n-1}) \;, \tag{1.47}$$

where f_{ab}^c is the on-shell three-particle vertex. This equation establishes a recursive structure between the $(n+1)$- and n-particle form factors.

The two chains of recursive equations,

$$\begin{array}{c} \ldots \to F_{n+4} \to F_{n+2} \to F_n \to F_{n-2} \to \ldots \\ \ldots \to F_{n+4} \to F_{n+3} \to F_{n+2} \to F_{n+1} \to \ldots \end{array} \tag{1.48}$$

(and the consistency conditions associated to them) are quite effective for the explicit determination of the form factors of a given theory. In principle, given a pair of initial conditions F_1 and F_2, if the form factors diverges at most polynomially with an uniform degree[5], then the two chains (1.48) have a finite set of solutions. The solutions are unique if the form factors do not diverge with the momenta. The polynomial asymptotic behaviour is also a necessary condition in order to prove that an operator is local, i.e. satisfies the relativistic property of microcausality [19, 20]

$$[O(x), O(y)] = 0 \quad ,(x-y)^2 < 0 \,. \tag{1.49}$$

This concept of locality coincide with the analyticy property of the correlation functions on the whole euclidean plane except the finite number of possible ultraviolet singularities [5]. A generalization is possible in order to include the so-called semilocal operators. These are operators for which some polidromy is allowed. Each pairs of operators is characterized by an index of mutual locality which has to be taken into account when turning one around the other. This leads to slight modifications of the Smirnov's axioms (1.45), (1.46) which can be found in [33, 34].

1.3 Parameterization of form factors and recursive equations

As a first step towards the characterization of the space of local fields we analyse the monodromy properties (1.40), (1.44) for the $n = 2$ external particles form factor of a spinless operator[6]

$$\begin{aligned} F_{a_1a_2}(\beta_{12}) &= S_{a_1a_2}^{a_1'a_2'}(\beta_{12})F_{a_2'a_1'}(-\beta_{12}) \\ F_{a_1a_2}(i\pi+\beta_{12}) &= F_{a_2a_1}(i\pi-\beta_{12}) \,. \end{aligned} \tag{1.50}$$

We denote with $F_{a_1a_2}^{min}(\beta)$ the solutions of the so-called [35] Watson equations (1.50) with the minimality request of analyticity in the physical strip $0 < Im\beta < \pi$. These solutions allows to

[5] This means with a degree independent of the numbers of external particles.

[6] Hereinafter we drop the upper index O unless some ambiguity may occur.

write the generic form factor as

$$F_{a_1 a_2 \ldots a_n}(\beta_1, \beta_2, \ldots, \beta_n) = K_{a_1 a_2 \ldots a_n}(\beta_1, \beta_2, \ldots, \beta_n) \prod_{i<j} F^{min}_{a_i a_j}(\beta_{ij}) \,, \tag{1.51}$$

with the functions K totally symmetric and $2\pi i$-periodic

$$K_{a_1 a_2 \ldots a_n}(\beta_1, \beta_2, \ldots, \beta_n) = K_{a_2 a_1 \ldots a_n}(\beta_2, \beta_1, \ldots, \beta_n) = K_{a_1 a_2 \ldots a_n}(\beta_1 + 2\pi i, \beta_2, \ldots, \beta_n) \,. \tag{1.52}$$

Therefore, while the minimal functions $F^{min}_{a_1 a_2}$ take into account the monodromy properties (1.40) and (1.44), the K functions have to match with the pole structure prescribed by the residue equations (1.46) and (1.47).

For the diagonal case, in which the scattering amplitudes assume the form (1.34), the easiest way to compute $F^{min}_{a_1 a_2}(\beta)$ is to exploit an integral representation of each building block $s_\alpha(\beta)$ [36, 37]

$$s_\alpha(\beta) = -\exp\left[\int_0^\infty \frac{dx}{x} g_\alpha(x) \sinh\left(\frac{x\beta}{i\pi}\right)\right] \,, \tag{1.53}$$

with

$$g_\alpha(x) = 2\frac{\sinh x(1-\alpha)}{\sinh x} \,. \tag{1.54}$$

Then, a solution of (1.50) is given by

$$F^{min}_{a_1 a_2}(\beta) = \prod_{\alpha \in A_{a_1 a_2}} f_\alpha(\beta) \,, \tag{1.55}$$

with the minimal building block given by

$$f_\alpha(\beta) = \frac{-i}{\sinh \beta/2} \exp\left[\int_0^\infty \frac{dx}{x} g_\alpha(x) \frac{\sin^2\left(\frac{x\hat{\beta}}{2\pi}\right)}{\sinh x}\right] \tag{1.56}$$

($\hat{\beta} \equiv i\pi - \beta$). The factor $\frac{-i}{\sinh \beta/2}$ takes into account the minus sign in front of the Fourier representation (1.53) of the building block of the scattering amplitudes. These minimal building blocks are the solutions of the Watson equations $f_\alpha(\beta) = s_\alpha(\beta) f_\alpha(-\beta)$ and $f_\alpha(i\pi + \beta) = f_\alpha(i\pi - \beta)$, with no poles and zeroes in the whole strip $0 < Im\beta < \pi$ and with an asymptotic behaviour given by

$$f_\alpha(\beta) \sim e^{-\frac{\alpha}{2}|\beta|}, \text{ as } |\beta| \to \infty \,. \tag{1.57}$$

The resulting asymptotic behaviour of F^{min} can be changed by multiplication of even powers of $\frac{-i}{\sinh \beta/2}$.

The required pole structure of a generic form factor can now be exploited to write it as follows

$$
\begin{aligned}
F_{a_1 a_2 \ldots a_n}(\beta_1, \beta_2, \ldots, \beta_n) \;=\;& Q_{a_1 a_2 \ldots a_n}(x_1, x_2, \ldots, x_n) \\
& \prod_{i<j} \frac{F^{min}_{a_i a_j}(\beta_{ij})}{\prod'_k (x_i - \omega^k_{ij} x_j)(x_i - \omega^{k\,-1}_{ij} x_j)} \prod_{i<j}{}'' \frac{1}{x_i + x_j} \,,
\end{aligned}
\tag{1.58}
$$

where we have introduced the variables

$$
x_i = e^{\beta_i} \,, \; \omega^k_{ij} = e^{i u^{a_k}_{a_i a_j}} \,. \tag{1.59}
$$

The Q's are symmetric functions of the variables x_i. The denominator of the first fraction has to be included only in case of bound state poles; the primed product means product only over the particles k which are bound states of the particles i, j. The second fraction takes into account the kinematical poles at $\beta_{ij} = i\pi$. They only occur when we have more than two external particles and the double primed product means product only over couples of a particle and the respective antiparticle.

The residue equations (1.46) and (1.47) become recursive equations for the symmetric functions $Q_{a_1 \ldots a_n}(x_1, \ldots, x_n)$ which we write in complete generality as

$$
Q_{a\bar{a}a_1 \ldots a_n}(-x, x, x_1, \ldots, x_n) = U^{a'_1 \ldots a'_n}_{a\bar{a}|a_1 \ldots a_n}(x|x_1, \ldots, x_n) Q_{a'_1 \ldots a'_n}(x_1, \ldots, x_n) \tag{1.60}
$$

and

$$
Q_{aba_1 \ldots a_n}(\omega^{c\;1/2}_{ab} x, \omega^{c\;-1/2}_{ab} x, x_1, \ldots, x_n) = f^c_{ab} D_{ab|a_1 \ldots a_n}(x|x_1, \ldots, x_n) Q_{ca_1 \ldots a_n}(x, x_1, \ldots, x_n) \,, \tag{1.61}
$$

where the auxiliary functions U and D can be found once the spectrum of the fundamental particles and their scattering amplitudes are known.

The asymptotic condition that the form factors of the local operators behave at most polynomially in the momenta implies that the Q's have to be rational symmetric functions of the variables x_i. Since the pole structure has been already taken into account, they can exhibit singularities only at the points $x_i = 0$ and $x_i = +\infty$. Lorentz invariance implies that, for a definite spin operator, they have to be homogeneous functions of degree determined by the spin of the operator and by the spectrum of the bound states, since the latter fixes the degree of the polynomial in the denominator of the parameterization (1.58). A very useful base for the symmetric homogeneous functions of n variables is that of the polynomials $\sigma^{(n)}_k$ defined by the generating function [38]

$$\prod_{i=1}^{n}(x + x_i) = \sum_{k=0}^{n} x^{n-k}\sigma_k^{(n)}(x_1,\ldots,x_n) \; . \tag{1.62}$$

They are explicitly given by

$$\begin{aligned} \sigma_0^{(n)} &= 1 \; , \\ \sigma_1^{(n)} &= x_1 + x_2 + \cdots + x_n \; , \\ \sigma_2^{(n)} &= x_1x_2 + x_1x_3 + \cdots + x_1x_n + \cdots + x_{n-1}x_n \; , \\ &\ldots \\ \sigma_n^{(n)} &= x_1x_2\ldots x_n \; . \end{aligned} \tag{1.63}$$

1.4 The trace of the stress-energy tensor and the characterization of the flow

One of the most relevant steps towards the physical interpretation of a given field dynamics consists in the identification of the stress-energy tensor $T_{\mu\nu}(x)$. In order to ensure the Poincarè symmetry of the system it has to be a symmetric conserved current. The conserved generator of translations and the generators of Lorentz transformations are therefore expressed out of it. In this section we will prove that these fundamental features strongly restrict the space of possible form factors for this special operator.

However, the role of the stress-energy tensor in a quantum theory is more fundamental, since it also rules the response of the system under (local) scale transformations. Contrary to the translations and the Lorentz transformations, dilatations do not constitute a fundamental symmetry of the system we want to describe. As a consequence of this difference, while it is very natural to assign to each field of the theory a transformation rule under translations and boosts, which is independent whether we are in a classic or in a quantum theory, this is no more true in general for the dilatations. It is only possible for the scale invariant theories, i.e. for the theories represented by the renormalization group fixed points. Hence, the renormalization group analysis gives us the tool to encode the whole dynamics of a theory into non-trivial properties of transformations of the system under local scale transformations. This picture is only possible after an unambiguos determination of the stress-energy tensor.

As a basic starting point, we therefore assume to have in the space of fields of the theory a local (see (1.49)), well behaved at infinity, second rank tensor $T_{\mu\nu}(x)$ which rules the behaviour of correlation functions under a general coordinates transformation $x'^{\mu} = x^{\mu} - \varepsilon^{\mu}(x)$ according to the equation

$$\sum_i < \varphi_1(x_1)\ldots\delta\varphi_i(x_i)\ldots\varphi_n(x_n) > \;=\; -\frac{1}{2\pi}\int d^2y\partial_\mu\varepsilon_\nu(y) < T^{\mu\nu}(y)\varphi_1(x_1)\ldots\varphi_n(x_n) >$$
$$= \frac{1}{2\pi}\int d^2y\varepsilon_\nu(y)\,\partial_\mu < T^{\mu\nu}(y)\varphi_1(x_1)\ldots\varphi_n(x_n) > \;. \tag{1.64}$$

For a translation $\varepsilon^{\mu}(x) = \epsilon^{\mu}$, the transformation of any field is given by $\delta\varphi_i(x) = \epsilon^{\mu}\partial_\mu\varphi_i(x)$, so that the property $\partial_\mu T^{\mu\nu}(x) = 0$ follows as well as the invariance equation

$$\sum_i \left(\frac{\partial}{\partial x_i^{\mu}}\right) < \varphi_1(x_1)\ldots\varphi_i(x_i)\ldots\varphi_n(x_n) > = 0 \;. \tag{1.65}$$

For the boost $x'^{\mu} = \Lambda^{\mu}{}_{\nu}(\omega)x^{\nu} =$, $\Lambda^{\mu}{}_{\nu}(\omega) = ((\cosh\omega, -\sinh\omega), (-\sinh\omega, \cosh\omega))$, the transformation of a field of Lorentz spin s_i is $\delta\varphi_i(x) = \omega\,[\partial^0\partial^1 - \partial^1\partial^0 + s_i]\,\varphi_i(x)$. The spin number s_i has a geometrical meaning which is valid for any scale of the theory, or equivalently for any assignment to the set of the coupling constants, and it is a feature which contributes to define the operator $\varphi_i(x)$. The Lorentz transformation properties of any operator with non-definite spin is understood to be defined by (1.64). The Lorentz invariance is then recovered postulating the symmetry of the stress-energy tensor $T^{\mu\nu} = T^{\nu\mu}$

$$\sum_i \left(x_i^0\partial_i^1 - x_i^1\partial_i^0 + s_i\right) < \varphi_1(x_1)\ldots\varphi_i(x_i)\ldots\varphi_n(x_n) > = 0 \;. \tag{1.66}$$

Since the dilatations $x'^{\mu} = e^{-\epsilon}x^{\mu}$ constitute a further space-time symmetry only for the fixed point theories, we can classify operators $\varphi_i(x)$ of a flow according to their conformal (anomalous) dimension d_i^{g*} at the fixed points g_*, $\delta\varphi_i(x) = \epsilon\,(x^{\mu}\partial_\mu + d_i^{g*})\,\varphi_i(x)$. This definition has a true geometrical meaning only at the fixed points g_*, the effective scaling behaviour of a correlation function at any point of the flow being determined by

$$\sum_i (x_i^{\mu}\partial_{\mu i} + d_i^{g*}) < \varphi_1(x_1)\ldots\varphi_i(x_i)\ldots\varphi_n(x_n) > = -\frac{1}{2\pi}\int d^2y < T_\mu^{\mu}(y)\varphi_1(x_1)\ldots\varphi_n(x_n) > \;. \tag{1.67}$$

The fields with definite conformal dimension are called 'conformal' operators; the others transform according to (1.64). The vanishing of the trace of the stress-energy tensor $T_\mu^{\mu}(x)$, hereinafter denoted by $\Theta(x)$, is thus the condition of scale invariance. As it is well-known, scale invariance implies invariance under conformal transformations $x'^{\pm} = x^{\pm} - \epsilon^{\pm}(x^{\pm})$, with $\epsilon^{\pm}$ analytic functions. The operator structure of the fixed points is completely classified by means of the primary operators of the theory. They are the lowest weight states of the representations of the Virasoro algebra, the algebra of the Fourier components of the stress-energy tensor [9].

In summary, we can understand any flow from a fixed point to another as a perturbation of a conformal field theory by means of the trace of the stress-energy tensor. This operator is thus anambiguosly defined once the conformal data are given, i.e. the conformal charge, the set of the anomalous dimensions and the direction of the perturbation. The ultraviolet conformal data can thus be recovered once the stress energy-tensor is known by extracting the short distance singularities of the euclidean correlators

$$\begin{aligned} < T_{zz}(z,\bar{z})\, \mathcal{O}_k(0) > &\overset{mR\to 0}{\simeq} \frac{\Delta(\mathcal{O}_k)}{z^2} , \\ < T_{zz}(z,\bar{z})\, T_{zz}(0) > &\overset{mR\to 0}{\simeq} \frac{c}{2z^4} , \end{aligned} \tag{1.68}$$

($R^2 = z\bar{z}$ and $z = x^2 + ix^1$, $\bar{z} = x^2 - ix^1$); here $T_{zz}, T_{z\bar{z}}, T_{\bar{z}\bar{z}}$ are the ligth-cone coordinates of the stress-energy tensor, in the complex euclidean notation. The infrared data are on the contrary obtained studying the same correlators in the long distance limit.

A metric characterization of the space of models is possible through the c-theorem by A.B.Zamolodchikov [39]. It states that along a unitary flow, the central charge has to decrease. In its integral formulation [40] the c-theorem precisely states that the difference [7] $\Delta c = c_{uv} - c_{ir}$ is given by

$$\Delta c = \frac{3}{4\pi} \int_{R>\epsilon} dx\, x^2 < \Theta(x)\Theta(0) >_c . \tag{1.69}$$

where the connected correlation function is Wick-rotated to the euclidean plane. Inserting the representation (1.10) for an euclidean correlator in the equation above we end up with the following sum rule for the difference of the central charges [41, 42]

$$\Delta c = 12 \sum_s \left| F_s^{\Theta} \right|^2 \frac{1}{(p_s^0)3} \delta(p_s^1) . \tag{1.70}$$

This formula provides a very useful check for the correct identification of the form factors of the trace of the stress-energy tensor.

Let us consider now the consequence coming from the conservation law $\partial^{\mu} T_{\mu\nu} = 0$. Being conserved, $T_{\mu\nu}(x)$ may be expressed in terms of an auxiliary scalar field $A(x)$ as [43]

$$T_{\mu\nu}(x) = (\partial_\mu \partial_\nu - g_{\mu\nu}\Box)\, A(x) . \tag{1.71}$$

[7] An infinite massive fixed point has central charge $c = 0$ since it does not contain any massless degree of freedom.

In light-cone coordinates its components are given by

$$T_{++} = \partial_+^2 A \quad , T_{--} = \partial_-^2 A \ , \tag{1.72}$$

$$\Theta = T_\mu^\mu = -\Box A = -4\,\partial_+\partial_- A \ . \tag{1.73}$$

In terms of the symmetric polynomials $\sigma_i^{(n)}$ of the variables $x_j = e^{\beta_j}$ introduced in (1.62) it is easy to see that, for a single massive particle spectrum,

$$\begin{aligned} F_n^{T_{++}}(\beta_1,\ldots,\beta_n) &= -\frac{1}{4}\,m^2 \left(\frac{\sigma_{n-1}^{(n)}}{\sigma_n^{(n)}}\right)^2 F_n^A(\beta_1,\ldots,\beta_n) \ , \\ F_n^{T_{--}}(\beta_1,\ldots,\beta_n) &= -\frac{1}{4}\,m^2 \left(\sigma_1^{(n)}\right)^2 F_n^A(\beta_1,\ldots,\beta_n) \ , \\ F_n^{\Theta}(\beta_1,\ldots,\beta_n) &= m^2\,\frac{\sigma_1^{(n)}\sigma_{n-1}^{(n)}}{\sigma_n^{(n)}}\,F_n^A(\beta_1,\ldots,\beta_n) \ . \end{aligned} \tag{1.74}$$

Solving for F_n^A, we have

$$\begin{aligned} F_n^{T_{++}}(\beta_1,\ldots,\beta_n) &= -\frac{1}{4}\,\frac{\sigma_{n-1}^{(n)}}{\sigma_1^{(n)}\sigma_n^{(n)}}\,F_n^{\Theta}(\beta_1,\ldots,\beta_n) \ , \\ F_n^{T_{--}}(\beta_1,\ldots,\beta_n) &= -\frac{1}{4}\,\frac{\sigma_1^{(n)}\sigma_n^{(n)}}{\sigma_{n-1}^{(n)}}\,F_n^{\Theta}(\beta_1,\ldots,\beta_n) \ . \end{aligned} \tag{1.75}$$

Hence the complete knowledge of $T_{\mu\nu}$ is encoded into the form factors of the trace Θ. As any spinless operator, its form factors $F_n^\Theta(\beta_1,\ldots,\beta_n)$ depend only on the difference of the rapidities $\beta_{ij} = \beta_i - \beta_j$. Moreover, since the form factors of T_{--} and T_{++} have the same singularity structure of the form factors of Θ, $F_n^\Theta(\beta_1,\ldots,\beta_n)$ (for $n > 2$) has to be proportional to the combination of symmetric polynomials $\sigma_1^{(n)}\sigma_{n-1}^{(n)}$ which corresponds to the invariant total energy-momentum. The more general spectrum case, which we do not meet in our applications, can be treated along the same line.

Additional constraints on F_n^Θ are obtained from the knowledge of their asymptotic behaviour in each variable β_i. This behaviour generally depends on the particular model under consideration. For the case of massive lagrangian theories discussed in third chapter, we have

$$F_n^\Theta(\beta_1+\Delta,\beta_2,\ldots,\beta_n) \overset{\Delta\to\infty}{\longrightarrow} o(1) \ , \tag{1.76}$$

i.e. they become constant for large values of the individual momenta. This condition can be easily checked by analysing the asymptotic behaviour of the Feynman diagrams entering the perturbative definition of these matrix elements [43, 44].

With reference to their normalization, the recursive structure of the space of form factors reduces the problem of finding the normalization of the matrix elements of $\Theta(x)$ to the initial conditions of the double chain (1.48), i.e. the two-particle form factor $F_2^{\Theta}(\beta_{12})$ and the one-particle form factor $F_1^{\Theta}(\beta)$, besides the vacuum expectation value $F_0^{\Theta} =< \Theta >$. The vacuum expectation value does not enter the computation of the difference of the central charge but it is an important quantity which can be independently determined with the thermodynamical Bethe ansatz [45].

The normalization of the two-particle form factors $F_2^{\Theta}(\beta_{12})$ can be fixed by making use of the definition of the energy operator [43, 7, 46]

$$E = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx^1 \, T^{00}(x) \ . \tag{1.77}$$

In fact, computing the matrix element of both terms of this equation on the asymptotic states $< \beta'|$ and $|\beta >$, for the left hand side we have

$$< \beta'| \, E \, |\beta > = \, 2\pi \, m \, \cosh \beta \, \delta(\beta' - \beta) \ . \tag{1.78}$$

On the other hand, taking into account that $T^{00} = \partial_1^2 A$ and using the relation

$$< \beta'| \, \mathcal{O}(x) |\beta > = \, e^{i(p^{\mu}(\beta') - p^{\mu}(\beta)) x_{\mu}} \, F_2^{\mathcal{O}}(\beta, \beta' \, - \, i\pi) \tag{1.79}$$

valid for any hermitian operator $\mathcal{O}$, we obtain

$$F_2^{\partial_1^2 A}(\beta_1, \beta_2) \; = \; - \, m^2 \, (\sinh \beta_1 \, + \sinh \beta_2)^2 \, F_2^A(\beta_{12}) \ . \tag{1.80}$$

Then, from eqs. (1.74) and (1.77), the normalization of F_2^{Θ} is given by

$$F_2^{\Theta}(i\pi) \; = \; 2 \, \pi \, m^2 \ . \tag{1.81}$$

However, no special constraint exists for the matrix element of $\Theta(x)$ on the one-particle state

$$F_1^{\Theta} \; = < 0 \mid \Theta(0) \mid \beta > \ . \tag{1.82}$$

Notice that from Lorentz invariance, it does not depend on the rapidity variable β. Since higher form factors of Θ are obtained as solutions of the recursive equations (1.60) and (1.61) (with initial condition given by the one-particle and two-particle matrix elements), the arbitrariness of F_1^{Θ} propagates in the recursive structure (1.48) of the form factors and therefore gives rise to a one-parameter family of possible stress-energy tensors $T_{\mu\nu}$. The one particle form factor F_1^{Θ} appears then as a labeling parameter of different theories. Since these theories are distinguished one from the other only in the interpretation of the stress-energy tensor, sharing the same local operator structure, they must obey the same dynamics.

1.4.1 The free massive boson

A simple example of the above discussion is provided by the free massive bosonic theory, with equation of motion

$$(\Box + m^2)\,\varphi \,=\, 0 \ . \tag{1.83}$$

The S-matrix in this case is simply $S = 1$ and therefore the form factors have trivial monodromy properties and simple analytic structure. Among the local operators of the theory, the elementary field $\varphi(x)$ is identified by the set of form factors

$$F_n^{\varphi}(\beta_1,\beta_2,\ldots,\beta_n) \; = < 0|\varphi(0)|\beta_1,\beta_2,\ldots,\beta_n > = \frac{1}{\sqrt{2}}\,\delta_{1,n} \ . \tag{1.84}$$

This operator creates the particle; we will see in the next section the definition of this field for the interacting theories in the form factors approach. Its two-point euclidean correlator reduces to a Bessel function

$$\begin{aligned} < \varphi(R)\,\varphi(0) >_E \;&=\; \sum_{n=0}^{\infty}\frac{1}{n!}\int_{-\infty}^{+\infty}\frac{d\beta_1}{2\pi}\cdots\int_{-\infty}^{+\infty}\frac{d\beta_n}{2\pi}\,|F_n(\beta 1,\ldots,\beta_n)|^2\,e^{-mR\sum_i \cosh\beta_i} \\ &=\; \frac{1}{2\pi}K_0(mR) \ . \end{aligned} \tag{1.85}$$

The absence of interaction implies that the composite operators $\varphi^k/k!$ are simply defined by the following form factors

$$< 0|\frac{\varphi^k(0)}{k!}|\beta_1,\beta_2,\ldots,\beta_n > = \left(\frac{1}{\sqrt{2}}\right)^k\,\delta_{n,k} \ . \tag{1.86}$$

The equation of motion is compatible with a class of stress-energy tensors labelled by the free parameter Q appearing in the definition of $\Theta(x)$

$$\Theta(x) \,=\, 2\pi\left(m^2\varphi^2 + \frac{Q}{\sqrt{\pi}}\Box\varphi\right) \ . \tag{1.87}$$

In terms of form factors we have

$$\begin{aligned} F_0^{\Theta} \;&=\; 0 \ , \\ F_1^{\Theta} \;&=\; -\sqrt{2\pi}m^2\,Q \ , \\ F_2^{\Theta} \;&=\; 2\pi m^2 \ , \\ F_k^{\Theta} \;&=\; 0 \ \ , k > 2 \ . \end{aligned} \tag{1.88}$$

The meaning of Q becomes clear once we analyse the ultraviolet limit of this massive theory. The central charge of the underlying conformal field theory which rules the ultraviolet properties of the model may be computed by using the integral form of the c-theorem (1.69). The euclidean correlator is given by

$$
\begin{aligned}
< \Theta(R)\Theta(0) >_E &= \sum_{k=0}^{\infty} \frac{1}{k!} \int_{-\infty}^{+\infty} \frac{d\beta_1}{2\pi} \cdots \int_{-\infty}^{+\infty} \frac{d\beta_k}{2\pi} \left|F_k(\beta 1, \ldots, \beta_k)\right|^2 e^{-mR\sum_i \cosh\beta_i} \\
&= m^4 \left(2\left(K_0(mR)\right)^2 + 2Q^2 K_0(mR)\right) \ , \qquad (1.89)
\end{aligned}
$$

and the result of the integral (1.69) is simply

$$
c = 1 + 12\, Q^2 \ . \qquad (1.90)
$$

Hence the one-particle form factors of Θ is related to the background charge of the conformal field theory reached in the ultraviolet limit [47, 48]. We could have obtained the same result by directly analysing the ultraviolet limit of the holomorphic component of the stress-energy tensor. Indeed

$$
\begin{aligned}
< T_{zz}(z,\bar{z}) T_{zz}(0,0) >_E &= \left(\frac{\bar{z}}{z}\right)^2 < T_{zz}(R,R) T_{zz}(0,0) >_E \\
&= m^4 \left(\frac{\bar{z}}{z}\right)^2 \left(2\left(K_2(mR)\right)^2 + 2Q^2 K_4(mR)\right) \ , \qquad (1.91)
\end{aligned}
$$

and in the limit $(mR) \to 0$ we have

$$
< T_{zz}(z,\bar{z}) T_{zz}(0,0) >_E \sim \frac{c}{2z^4} = \frac{1 + 12\, Q^2}{2z^4} \ . \qquad (1.92)
$$

To complete our discussion on the free theory, let us compute the conformal dimensions $\Delta(\alpha)$ characterizing the scaling properties of the exponential operators $V_a = e^{\alpha\varphi}$ in the ultraviolet limit. This will be identified as the coefficient of the most singular term in the ultraviolet limit of the correlator $< T_{zz}(z,\bar{z})\, V_\alpha(0) >$. Using the form factors of $V_\alpha(0)$ given by

$$
< 0|V_\alpha(0)|\beta_1, \ldots, \beta_n > = \left(\frac{\alpha}{\sqrt{2}}\right)^n \ , \qquad (1.93)
$$

we have

$$
\begin{aligned}
< T_{zz}(z,\bar{z})\, V_\alpha(0) >_E &= \left(\frac{\bar{z}}{z}\right) < T_{zz}(R,R)\, V_\alpha(0) >_E \\
&= m^2 \left(\frac{\bar{z}}{z}\right) \left(-\frac{\alpha^2}{2\pi}\left(K_1(mR)\right)^2 + \alpha\frac{Q}{\sqrt{\pi}} K_2(mR)\right) \ ,
\end{aligned}
$$

and for $(mR) \to 0$

$$
< T_{zz}(z,\bar{z})\, V_\alpha(0) >_E \sim \frac{1}{z^2}\left(-\frac{\alpha^2}{8\pi} + \frac{\alpha\, Q}{\sqrt{4\pi}}\right) \ , \qquad (1.94)
$$

i.e.

$$
\Delta(\alpha) = -\frac{\alpha^2}{8\pi} + \frac{\alpha\, Q}{\sqrt{4\pi}} \ . \qquad (1.95)
$$

Due to the background charge, they differ from the gaussian value $\Delta(\alpha) = -\alpha^2/8\pi$ [47, 48, 49].

1.5 The elementary interpolating field

The usual canonical approach to the quantum field theory considers the creation and destruction operators ${b_a}^{in\dagger}(k), {b_a}^{out\dagger}(k), {b_a}^{in}(k), {b_a}^{out}(k)$ of the particle a in the "in" and "out" states. They are defined as the Fourier components of the asymptotic fields, the weak limits $t \to \mp\infty$ of the interpolating fields φ_a which appear in the complete lagrangian $L(\varphi) = L_{free}(\varphi) + L_{int}(\varphi)$. The interpolating fields φ_a evolve according to the dynamics dictated by L, while the asymptotic fields ${\varphi_a}^{in}$, ${\varphi_a}^{out}$ are free. The Lehmann-Symanzik-Zimmermann derivation of the reduction formulas provides us with the following representation of the scattering amplitudes (1.12) in terms of the Green functions (1.3)

$$\begin{aligned} S(\{p'_j, a'_j\}, \{p_i, a_i\}) &= \text{disconnected terms } + \\ &\left(\tfrac{1}{\sqrt{2}}\right)^{m+n} \lim_{(p'_j)^2 \to m^2_{a_j}} \lim_{(p_i)^2 \to m^2_{a_i}} \prod_{j=1}^m \left(G^{(2)}_{a'_j a'_j free}(p'_j)\right)^{-1} \prod_{i=1}^n \left(G^{(2)}_{a_i a_i free}(p_i)\right)^{-1} \\ &(2\pi)^2 \delta\left(\textstyle\sum_j p'_j + \sum_i p_i\right) G^{(m+n)}_{a'_1 \ldots a'_m, a_1 \ldots a_n}(p'_1, \ldots, p'_m, p_1, \ldots, p_n)\,, \end{aligned} \tag{1.96}$$

where

$$(2\pi)^2 \delta\left(\sum_i q_i\right) G^{(k)}_{a_1 \ldots a_k}(q_1, \ldots, q_k) = \int \prod dx_j e^{-i\sum_j q_j \cdot x_j} < 0|T\left(\varphi_{a_1}(x_1) \ldots \varphi_{a_k}(x_k)\right)|0 > \tag{1.97}$$

and $G^{(2)}_{aa\,free}(p)$ is the free propagator of the particle a. The factor of powers of $\frac{1}{\sqrt{2}}$ is there since we insist, for the bosonic case, in considering as phase space canonical coordinates the couple $(\varphi_a(x^1), \partial_0\varphi_a(x^1))$, together with the normalization chosen of (1.15).

Similar reduction formulas exist also for the form factor of an operator $O(x)$

$$F^O_{a_1 \ldots a_n}(p_1, \ldots, p_n) = \left(\frac{1}{\sqrt{2}}\right)^n \lim_{(p_i)^2 \to m^2_{a_i}} \prod_{i=1}^n \left(G^{(2)}_{a_i a_i free}(p_i)\right)^{-1} G^{(O,n)}_{a_1 \ldots a_n}\left(q = -\sum p_i, p_1, \ldots, p_n\right) \tag{1.98}$$

where

$$(2\pi)^2 \delta\left(q + \sum_i p_i\right) G^{(O,n)}_{a_1 \ldots a_n}(q, p_1, \ldots, p_n) = \tag{1.99}$$

$$\int \prod dx_j dy e^{-i\sum_i p_i \cdot x_i} e^{-iq\cdot y} < 0|T\left(O(y)\varphi_{a_1}(x_1) \ldots \varphi_{a_k}(x_k)\right)|0 > \,. \tag{1.100}$$

The usefulness of this kind of formulas for lagrangian theories is threefold. Firstly, they may allow to fix the initial conditions of the equations (1.48). Secondly, they permit to study the asymptotic behavior of the form factors, with a corresponding restriction on the space of solutions. Finally, they provide a tool to check the validity of the form factor solutions through perturbation theory.

When $O(x)$ is the field $\varphi_a(x)$ itself, the application of eq. (1.98) for $n = 1$ gives

$$F_a^{\varphi_a}(p) = < 0|\varphi_a(0)|a\,p> = \frac{1}{\sqrt{2}} \lim_{p^2 \to m_a^2} \left(G^{(2)}_{aa\,free}(p)\right)^{-1} G^{(2)}_{aa}(p) \tag{1.101}$$

which provides the initial condition

$$F_a^{\varphi_a}(p) = \frac{1}{\sqrt{2}} \,. \tag{1.102}$$

Note the remarkable fact that this initial condition is not a property of the interpolating fields for lagrangian theories only. F.A. Smirnov has proved, [19, 20], the general result that the local operator, i.e. whose form factors satisfy the axioms (1.40), (1.44), (1.46), with initial condition $F_a^{\varphi_a} = \frac{1}{\sqrt{2}}$ goes in the weak limits $t \to \mp\infty$ to the free asymptotic fields ${\varphi_a}^{in}$ and ${\varphi_a}^{out}$ whose Fourier components creates the 'in' and 'out' asymptotics scattering states. The connection between the Zamolodchikov-Faddeev operators of (1.23) and the asymptotic creators $b_a^\dagger$ and destructors b_a is the following

$${b_a}^{in}(\beta) Z^\dagger_{a_1}(\beta_1) \ldots Z^\dagger_{a_n}(\beta_n)|0> = \sum_i \delta_{aa_i} \delta(\beta - \beta_i) Z^\dagger_{a_1}(\beta_1) \ldots \hat{Z}^\dagger_{a_i}(\beta_i) \ldots Z^\dagger_{a_n}(\beta_n)|0> \tag{1.103}$$

$${b_a}^{out}(\beta) Z^\dagger_{a_n}(\beta_n) \ldots Z^\dagger_{a_1}(\beta_1)|0> = \sum_i \delta_{aa_i} \delta(\beta - \beta_i) Z^\dagger_{a_n}(\beta_n) \ldots \hat{Z}^\dagger_{a_i}(\beta_i) \ldots Z^\dagger_{a_1}(\beta_1)|0> \tag{1.104}$$

where the rapidities are ordered as $\beta_1 > \ldots > \beta_n$ and the hat over an operator means its absence in the product.

As a final remark of this section, we now establish that the form factors $F^{\varphi_a}_{aa_1\ldots a_n}$ of the elementary interpolating field φ_a are proportional to the symmetric polynomial $\sigma^{(2n+1)}_{2n+1}$. The reason is that we can factorize from any Feynman diagram which enters $F^{\varphi_a}_{aa_1\ldots a_n}$ the free boson propagator

$$\frac{i}{q^2 - m^2}\Big|_{q=-\sum p_i,\; p_i^2 = m^2} \tag{1.105}$$

Written in terms of the variables x_i, it becomes proportional to $\sigma^{(2n+1)}_{2n+1}$

$$\frac{i}{q^2 - m^2}\Big|_{q=-\sum p_i,\; p_i^2 = m^2} = \frac{i}{m^2} \frac{\sigma^{(2n+1)}_{2n+1}}{\sigma^{(2n+1)}_1 \sigma^{(2n+1)}_{2n} - \sigma^{(2n+1)}_{2n+1}} \,. \tag{1.106}$$

Chapter 2

Massive flows

2.1 The sinh-Gordon massive boson

As a first example of a non-trivial integrable massive lagrangean theory we consider the sinh-Gordon model defined by the action

$$\mathcal{S} = \int d^2x \left[\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m^2}{g^2} \cosh g\varphi(x)\right] . \tag{2.1}$$

It is the simplest example of an affine Toda field theories ([15], [49]-[53]), possessing a Z_2 symmetry $\varphi \to -\varphi$. By an analytic continuation in g, i.e $g \to ig$, it can formally be mapped to the sine-Gordon model.

There are numerous alternative viewpoints for the sinh-Gordon model. First, it can be regarded either as a perturbation of the free massless conformal action by means of the relevant operator $\cosh g\varphi(x)$. Alternatively, it can be considered as a perturbation of the conformal Liouville action[1]

$$\mathcal{S} = \int d^2x \left[\frac{1}{2}(\partial_\mu\varphi)^2 - \lambda e^{g\varphi}\right] \tag{2.2}$$

by means of the relevant operator $e^{-g\varphi}$ or as a conformal affine A_1-Toda theory [51] in which the conformal symmetry is broken by setting the free field to zero. We mention also that the sinh-Gordon model has been described also by means of the quantum inverse scattering method [16], [54]-[56].

In a perturbative approach to the quantum field theory defined by the action (2.1), the only ultraviolet divergences which occur in any order in g come from tadpole graphs and can be

[1] Or as the conformal Liouville $e^{-g\varphi}$ perturbed by $e^{g\varphi}$.

removed by a normal ordering prescription with respect to an arbitrary mass scale M. All other Feynman graphs are convergent and give rise to finite renormalizations.

An essential feature of the sinh-Gordon theory is its integrability, which in the classical case can be established by means of the inverse scattering method [14]. In order to obtain the expressions of the (classical) conserved currents, let us consider the Euclidean version of the model in terms of the complex coordinates z and $\overline{z}$

$$z = (x^0 + ix^1) \; ; \quad \overline{z} = (x^0 - ix^1) \; , \tag{2.3}$$

and define a field $\hat{\varphi}(z, \overline{z}, \epsilon)$ which satisfies the following (Bäcklund) equations

$$\begin{aligned} \frac{\partial}{\partial z}(\hat{\varphi} + \varphi) &= \frac{m}{g}\epsilon \sinh\left(\frac{g}{2}(\hat{\varphi} - \varphi)\right) \; , \\ \frac{\partial}{\partial \overline{z}}(\hat{\varphi} - \varphi) &= \frac{m}{g\epsilon} \sinh\left(\frac{g}{2}(\hat{\varphi} + \varphi)\right) \; . \end{aligned} \tag{2.4}$$

Given that $\varphi(z, \overline{z})$ is a solution of the equation of motion originated by (2.1), eqs. (2.4) define a new solution $\hat{\varphi}(z, \overline{z}, \epsilon)$ and imply as well the following conservation laws

$$\epsilon^{-1} \partial_z \left(\cosh \frac{g}{2}(\hat{\varphi} + \varphi)\right) - \epsilon \, \partial_{\overline{z}} \left(\cosh \frac{g}{2}(\hat{\varphi} - \varphi)\right) = 0 \; . \tag{2.5}$$

$\hat{\varphi}(z, \overline{z}, \epsilon)$ can be expressed in terms of a power series in ϵ

$$\hat{\varphi}(z, \overline{z}, \epsilon) = \sum_{n=0}^{\infty} \varphi^{(n)}(z, \overline{z}) \, \epsilon^n \tag{2.6}$$

with the fields $\varphi^{(n)}(z, \overline{z})$ calculated by using eqs. (2.4). Placing (2.6) into (2.5) and matching equal power in ϵ, one obtains an infinite set of conservation laws

$$\partial_{\overline{z}} T_{s+1} = \partial_z \Theta_{s-1} \; . \tag{2.7}$$

The corresponding charges Q_s are given by

$$Q_s = \oint [T_{s+1} \, dz + \Theta_{s-1} \, d\overline{z}] \; . \tag{2.8}$$

The integer-valued index s which labels the integrals of motion is the spin of the operators. Non-trivial conservation laws[2] are obtained for odd values of s

$$s = 1, 3, 5, 7, \ldots \tag{2.9}$$

[2]This means conservation laws generated by currents which cannot be put in the form $T_{s+1} = \partial_z K_s$, $\Theta_{s-1} = \partial_{\overline{z}} K_s$.

In analogy to the sine-Gordon theory [57], an infinite set of conserved charges I_s with spin s given in (2.9) also exists for the quantized version of the sinh-Gordon theory. The existence of these higher integrals of motion precludes the possibility of production processes and hence guarantees that the n-particle scattering amplitudes are purely elastic and factorized into $n(n-1)/2$ two-particle S-matrices. The exact expression for the sinh-Gordon theory is given by [58]

$$S(\beta, B) = \frac{\tanh \frac{1}{2}(\beta - i\frac{\pi B}{2})}{\tanh \frac{1}{2}(\beta + i\frac{\pi B}{2})}, \tag{2.10}$$

where B is the following function of the coupling constant g

$$B(g) = \frac{2g^2}{8\pi + g^2} \quad . \tag{2.11}$$

This formula has been checked against perturbation theory in ref. [58], [53] and can also be obtained by analytic continuation of the S-matrix of the first breather of the Sine-Gordon theory [11]. For real values of g the S-matrix has no poles in the physical sheet and hence there are no bound states. The absence of bound states in the sinh-Gordon model is also supported by the general fusing rule of affine Toda field theories [59].

An interesting feature of the S-matrix is its invariance under the map [52]

$$B \to 2 - B \tag{2.12}$$

i.e. under the *strong-weak* coupling constant duality

$$g \to \frac{8\pi}{g}. \tag{2.13}$$

This symmetry will be respected by all the form factors of manifestly self-dual operators.

2.1.1 The space of form factors

The form factors of the sinh-Gordon model have been investigated in [44, 60]. In this subsection we briefly recall the basic results obtained in refs. [44, 60] which are relevant for our subsequent considerations.

As explained in chapter 1, in order to compute the form factors of a theory, the first step is to take into account their monodromy properties dictated by the S-matrix through the solution of the Watson equations (1.50) $F_{\rm SG}^{min}(\beta, B)$. The integral representation (1.53) of the scattering amplitude (2.10) gives for the minimal form factor the explicit expression

$$F_{\rm SG}^{min}(\beta, B) = \mathcal{N} \exp \left[8 \int_0^\infty \frac{dx}{x} \frac{\sinh\left(\frac{xB}{4}\right) \sinh\left(\frac{x}{2}(1-\frac{B}{2})\right) \sinh\frac{x}{2}}{\sinh^2 x} \sin^2\left(\frac{x\hat{\beta}}{2\pi}\right)\right] \quad . \tag{2.14}$$

We choose our normalization to be

$$\mathcal{N} = \exp\left[-4\int_0^\infty \frac{dx}{x}\frac{\sinh\left(\frac{xB}{4}\right)\sinh\left(\frac{x}{2}(1-\frac{B}{2})\right)\sinh\frac{x}{2}}{\sinh^2 x}\right] . \tag{2.15}$$

With this normalization the asymptotic behaviour is given by

$$\lim_{\beta\to\infty} F_{\rm SG}^{min}(\beta,B) = 1 . \tag{2.16}$$

The analytic structure of $F_{\rm SG}^{min}(\beta,B)$ can be easily read from its infinite product representation in terms of Γ functions

$$F_{\rm SG}^{min}(\beta,B) = \prod_{k=0}^{\infty}\left|\frac{\Gamma\left(k+\frac{3}{2}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+\frac{1}{2}+\frac{B}{4}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+1-\frac{B}{4}+\frac{i\hat\beta}{2\pi}\right)}{\Gamma\left(k+\frac{1}{2}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+\frac{3}{2}-\frac{B}{4}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+1+\frac{B}{4}+\frac{i\hat\beta}{2\pi}\right)}\right|^2 . \tag{2.17}$$

$F_{\rm SG}^{min}(\beta,B)$ has no poles in the physical strip and a simple zero at the threshold $\beta=0$ since $S(0)=-1$. As discussed in [61], this generally induces a suppression of all higher thresholds in the spectral representation of the correlation functions and gives rise to very fast convergent series. It satisfies the functional equation

$$F_{\rm SG}^{min}(i\pi+\beta,B)F_{\rm SG}^{min}(\beta,B) = \frac{\sinh\beta}{\sinh\beta+\sinh\frac{i\pi B}{2}} \tag{2.18}$$

which is useful in order to find a convenient form for the recursive equations of the form factors.

A useful expression for the numerical evaluation of $F_{\rm SG}^{min}(\beta,B)$ is given by

$$F_{\rm SG}^{min}(\beta,B) = \mathcal{N}\prod_{k=0}^{N}\left[\frac{\left(1+\left(\frac{\hat\beta/2\pi}{k+\frac{1}{2}}\right)^2\right)\left(1+\left(\frac{\hat\beta/2\pi}{k+\frac{3}{2}-\frac{B}{4}}\right)^2\right)\left(1+\left(\frac{\hat\beta/2\pi}{k+1+\frac{B}{4}}\right)^2\right)}{\left(1+\left(\frac{\hat\beta/2\pi}{k+\frac{3}{2}}\right)^2\right)\left(1+\left(\frac{\hat\beta/2\pi}{k+\frac{1}{2}+\frac{B}{4}}\right)^2\right)\left(1+\left(\frac{\hat\beta/2\pi}{k+1-\frac{B}{4}}\right)^2\right)}\right]^{k+1} \tag{2.19}$$
$$\times\exp\left[8\int_0^\infty\frac{dx}{x}\frac{\sinh\left(\frac{xB}{4}\right)\sinh\left(\frac{x}{2}(1-\frac{B}{2})\right)\sinh\frac{x}{2}}{\sinh^2 x}(N+1-N\,e^{-2x})\,e^{-2Nx}\sin^2\left(\frac{x\hat\beta}{2\pi}\right)\right] .$$

The rate of convergence of the integral may be improved substantially by increasing the value of N.

In terms of $F_{\rm SG}^{min}(\beta,B)$, the parameterization of the n-particle form factors of the sinh-Gordon model is given by

$$F_n(\beta_1,\ldots,\beta_n) = H_n\,Q_n(x_1,\ldots,x_n)\prod_{i<j}\frac{F_{\rm SG}^{min}(\beta_{ij},B)}{(x_i+x_j)} , \tag{2.20}$$

Here H_n are normalization constants, which can be conveniently chosen as

$$H_{2n+1} = H_1\mu^{2n}(B) \;,\; H_{2n} = H_2\mu^{2n-2}(B) , \tag{2.21}$$

with

$$\mu(B) \equiv \left(\frac{4\sin(\pi B/2)}{\mathcal{N}(B)}\right)^{\frac{1}{2}} \quad . \tag{2.22}$$

The functions $Q_n(x_1,\ldots,x_n)$ are symmetric polynomials in the variables $x_1,\ldots,x_n$, solutions of the recursive equations

$$Q_{n+2}(-x,x,x_1,\ldots,x_n) = U_n(x|x_1,x_2,\ldots,x_n)\,Q_n(x_1,x_2,\ldots,x_n) \ , \tag{2.23}$$

with

$$U_n(x|x_1,\ldots,x_n) = (-)^n x \sum_{k=1}^{n} \sum_{m=1,odd}^{k} [m]\, x^{2(n-k)+m} \sigma_k^{(n)} \sigma_{k-m}^{(n)} (-1)^{k+1} \ , \tag{2.24}$$

and

$$[n] \equiv \frac{\sin(n\pi\frac{B}{2})}{\sin(\pi\frac{B}{2})} \ . \tag{2.25}$$

For form factors of spinless operators, their total degree is equal to $n(n-1)/2$ whereas their partial degree in each variable x_i is fixed by the asymptotic behaviour of the operator $\mathcal{O}_k$ which is under investigation.

The unique elementary interpolating field $\varphi(x)$, according to the discussion of chapter 1, is given by the form factors (2.20) solution of (2.23) with initial condition $F_1 = \frac{1}{\sqrt{2}}$ and factorization property $Q_n = \sigma_n^n P_n$. Being this field odd by definition under the $\mathcal{Z}_2$-symmetry $\varphi \to -\varphi$, the reduction formula (1.98) suggests that it has non-vanishing form factors only with odd number of external particles. The presence of the propagator (1.105) in front of any form factor of the elementary interpolating field φ which implies the above mentioned factorization has also as a consequence that F_{2n+1} behaves asymptotically as

$$F_{2n+1}(\beta_1,\beta_2,\ldots,\beta_{2n+1}) \to 0 \quad \text{as } \beta_i \to +\infty \quad \beta_{j\neq i} \text{ fixed.} \tag{2.26}$$

In fact, the propagator (1.105) goes to zero in this limit whereas the remaining expression of the Feynman graphs entering F_{2n+1} is a perturbative series which starts from the tree level vertex diagram shown in Fig. 2.1, which is a constant. Other tree level contributions at the lowest order and higher order corrections are either finite or they vanish in the limit (2.26). In fact, by dimensional analysis they must have external momenta in the denominator in order to compensate the increasing power of the mass in the coupling constants. With initial conditions $H_1 = \frac{1}{\sqrt{2}}, H_2 = 0$ the solution for the elementary interpolating field φ becomes

$$F^{\varphi}_{2n+1}(\beta_1,\ldots,\beta_{2n+1}) = \frac{1}{\sqrt{2}}\,(\mu(B))^{2n}\, \sigma_{2n+1}^{(2n+1)} P_{2n+1}(x_1,\ldots,x_{2n+1}) \prod_{i<j} \frac{F_{\min}(\beta_{ij})}{x_i + x_j} \tag{2.27}$$

where for the first values of n we have

$$
\begin{aligned}
P_3(x_1,\ldots,x_3) &= 1 \\
P_5(x_1,\ldots,x_5) &= \sigma_2\sigma_3 - c_1^2\sigma_5 \\
P_7(x_1,\ldots,x_7) &= \sigma_2\sigma_3\sigma_4\sigma_5 - c_1^2(\sigma_4\sigma_5^2 + \sigma_1\sigma_2\sigma_5\sigma_6 + \sigma_2^2\sigma_3 - c_1^2\sigma_2\sigma_5) + \\
&\quad -c_2(\sigma_1\sigma_6\sigma_7 + \sigma_1\sigma_2\sigma_4\sigma_7 + \sigma_3\sigma_5\sigma_6) + c_1c_2^2\sigma_7^2
\end{aligned}
\tag{2.28}
$$

where $c_1 = 2\cos(\pi B/2)$, $c_2 = 1 - c_1^2$ and we have dropped the upper index of the elementary polynomials σ_k^n in the righthand side being equal to the lower index of the P_n of the lefthand side.

As shown in [60], the problem to classify all possible scalar operators of the sinh-Gordon model reduces to find the most general class of solutions of eq. (2.23). Since this is a homogeneous equation, its solutions span a linear space whose base may be written in terms of the so-called *elementary solutions* given by

$$
Q_n(k) = \det M_{ij}(k) \ , \tag{2.29}
$$

where $M_{ij}(k)$ is an $(n-1)\times(n-1)$ matrix with entries

$$
M_{ij}(k) = \sigma_{2i-j}\,[i-j+k] \ . \tag{2.30}
$$

These polynomials depend on an arbitrary integer k and satisfy

$$
Q_n(k) = (-1)^{n+1}Q_n(-k) \ . \tag{2.31}
$$

Therefore the structure of the form factors of the SGM consists in a sequence of finite linear spaces whose dimensions grow linearly as n increasing the number $2n-1$ or $2n$ of external particles. The reason is that, at each level of the recursive process, the space of the form factors is enlarged by including the kernel solutions of the recursive equation (2.23), i.e. $Q_n(-x, x, x_1, \ldots, x_{n-2}) = 0$. With the constraint that the total order of the polynomials is $\frac{n(n-1)}{2}$, the kernel is unique and given by $\Sigma_n(x_1,\ldots,x_n) = \det\sigma_{2i-j}$. These solutions then gives rise to the half-infinite chains under the recursive pinching $x_1 = -x_2 = x$

$$
\begin{array}{cccccccccccccccc}
\ldots & \to & Q^{(1)}_{n+4} & \to & Q^{(1)}_{n+2} & \to & Q^{(1)}_{n} & \to & Q^{(1)}_{n-2} & \to & \ldots & \to & Q^{(1)}_{3} & \to & 1 \\
\ldots & \to & Q^{(2)}_{n+4} & \to & Q^{(2)}_{n+2} & \to & Q^{(2)}_{n} & \to & Q^{(2)}_{n-2} & \to & \ldots & \to & \Sigma_2 & & \\
 & & & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \\
 & & & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \\
\ldots & \to & Q^{(n-2)}_{n+4} & \to & Q^{(n-2)}_{n+2} & \to & Q^{(n-2)}_{n} & \to & \Sigma_{n-2} & & & & & & \\
\ldots & \to & Q^{(n)}_{n+4} & \to & Q^{(n)}_{n+2} & \to & \Sigma_{n} & & & & & & & & \\
\ldots & \to & Q^{(n+2)}_{n+4} & \to & \Sigma_{n+2} & & & & & & & & & &
\end{array}
\tag{2.32}
$$

The explicit expressions of such solutions can be found by determining the linear combination of $Q_n(k)$ which reduces to Σ_n at the level n.

2.1.2 Cluster operators and fundamental exponentials

The *fundamental exponential operators* $E_{\pm}(x) = e^{\pm g\varphi(x)}$ define the sinh-Gordon model and in general appear in the expression of the stress-energy tensor. In order to calculate their matrix elements, let us consider initially those form factors which satisfy the requirements

- To be asymptotically constant for $\beta_i \to \infty$, i.e.

$$F_n(\beta_1 + \Delta, \beta_2, \ldots, \beta_n) \overset{\Delta\to\infty}{\longrightarrow} o(1) \ .$$

- To be proportional to the combination[3] $\sigma_1 \sigma_{n-1}$ (for $n > 2$).

- To be the solution of the *cluster equations*

$$\lim_{\Delta\to+\infty} F_{k+l}\left(\beta_1 + \Delta, \ldots, \beta_k + \Delta, \beta_{k+1}, \ldots, \beta_{k+l}\right) = F_k\left(\beta_1, \ldots, \beta_k\right) F_l\left(\beta_{k+1}, \ldots, \beta_{k+l}\right)$$

  with initial condition $F_0 = 1$.

There are two classes of form factors which fulfill the three above conditions. Their expressions are given by

$$F^{(\pm)}_n(\beta_1, \ldots, \beta_n) = H^{(\pm)}_n(B)\, Q_n(1) \prod_{i<j}^{n} \frac{F^{min}_{\rm SG}(\beta_{ij}, B)}{(x_i + x_j)} \ , \tag{2.33}$$

where

$$H^{(+)}_n(B) = (\mu(B))^n \quad , \quad H^{(-)}_n = (-1)^n\,(\mu(B))^n \ . \tag{2.34}$$

[3] As discussed in sect. 2, this factorization property is shared by the general form factors of Θ.

The corresponding operators, which are self-dual by construction, will be called *cluster operators* and denoted as $V_{\pm}(x,B)$. We conjecture that the fundamental exponentials of the sinh-Gordon model may be written as[4]

$$
\begin{aligned}
E_{+}(x,B) &\equiv \theta(1-B)\,V_{+}(x,B) + \theta(B-1)\,V_{-}(x,B) \ , \\
E_{-}(x,B) &\equiv \theta(1-B)\,V_{-}(x,B) + \theta(B-1)\,V_{+}(x,B) \ .
\end{aligned}
\tag{2.35}
$$

Postponing a non-trivial check of this position until when we will study the UV-behaviour of the model, let us in the meantime discuss the properties of the operators $E_{\pm}(x,B)$ so defined.

First of all, they satisfy the cluster property by construction, in agreement with the perturbative analysis for the matrix elements of the operators $e^{\pm g\varphi(x)}$. Secondly, the form factors of $E_{\pm}(x,B)$ are not individually invariant under the duality transformation but each operator is mapped onto the other under the mapping $B \rightarrow 2-B$, i.e.

$$
E_{\pm}(x,B) = E_{\mp}(x,2-B) \ . \tag{2.36}
$$

Therefore they form a bidimensional representation of the duality symmetry. However, this mapping becomes degenerate at the self-dual point $B=1$ where an identification occurs between the two exponential operators $E_{\pm}(x,B)$. Namely, at $B=1$ the matrix elements of the two fundamental exponentials become indistinct and denoting by $E(x)$ the resulting operator, its form factors are given by

$$
F_n^E(\beta_1,\ldots,\beta_2) = \begin{cases} (\mu(1))^n \, Q_n(1) \prod_{i<j}^n F_{\rm SG}^{min}(\beta_{ij})/(x_i+x_j) & n \text{ even} \ , \\ 0 & n \text{ odd} \ . \end{cases} \tag{2.37}
$$

The identification of $E_{\pm}(x)$ at the self-dual point has the additional consequence that the resulting field $E(x)$ is an even operator under the Z_2 parity of the sinh-Gordon model, as it is evident from the vanishing of its matrix elements on all $2n+1$ particle states.

Using the form factors of the fundamental exponentials and those of the elementary interploating field given in terms of the elementary solutions (2.29) by

$$
F_n^{\varphi}(\beta_1,\ldots,\beta_n) = H_n^{\varphi}\, Q_n(0) \prod_{i<j}^{n} \frac{F_{\rm SG}^{min}(\beta_{ij})}{x_i+x_j} \tag{2.38}
$$

$$
H_{2n+1}^{\varphi} = \frac{1}{\sqrt{2}}\,(\mu(B))^n \quad , \quad H_{2n}^{\varphi} = 0 \ ,
$$

[4]For the value of the step function at the origin we use $\theta(0) = 1/2$.

in terms of which we can easily obtain the form factors of the operator $\Box\varphi$

$$F_n^{\Box\varphi} = -m^2 \frac{\sigma_1 \sigma_{n-1}}{\sigma_n} F_n^{\varphi} , \tag{2.39}$$

the quantum version of the equation of motion may be written as

$$\begin{aligned} \Box\varphi(x) &= \frac{m^2}{2\sqrt{2}\mu(B)} \left(\theta(1-B) - \theta(B-1)\right) \left(e^{-g\varphi(x)} - e^{g\varphi(x)}\right) \\ &= \frac{m^2}{2\sqrt{2}\mu(B)} \left(V_-(x,B) - V_+(x,B)\right) . \end{aligned} \tag{2.40}$$

This equation has to be understood as an identity satisfied by the form factors of the operators appearing on the left and right sides of this relation. It is then natural to define as renormalized coupling the function $g(B) = \sqrt{2}\mu(B)(\theta(1-B) - \theta(B-1))$.

2.1.3 Class of stress-energy tensors

In this section we show the already mentioned fact that within the same dynamics, or set of local operators, we can define different stress-energy tensors, i.e. we can describe different renormalization group flows. The quantum equation of motion (2.40) is compatible with a one-dimensional space of stress-energy tensors given by

$$\Theta(x) = F_0^{\Theta}(B) \left(a\, E_+(x,B) + (1-a)\, E_-(x,B)\right) . \tag{2.41}$$

The normalization constant $F_0^{\Theta}(B)$ may be fixed by means of the thermodynamical Bethe ansatz [45]

$$F_0^{\Theta}(B) = \frac{\pi m^2}{2\sin(\pi B/2)} . \tag{2.42}$$

The variable a, on the contrary, is a free parameter. Varying its value, we can weight differently the two fundamental exponentials in the trace and, consequently, we can interpolate between different scaling regimes of the sinh-Gordon model in its ultraviolet limit.

The case $a = 1$

The trace of the stress-energy tensor is given in this case by

$$\Theta(x) = F_0^{\Theta}(B)\, E_+(x,B) . \tag{2.43}$$

With such definition of Θ, we expect that the massive theory will flow in the ultraviolet regime to a conformal field theory defined by the bare action

$$S_- = \int d^2x \left[\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m_0^2}{2g^2}\, e^{-g\varphi}\right] . \tag{2.44}$$

In order to support this conclusion, let us compare the central charge associated to the conformal field theory (2.44) with the central charge obtained, on the contrary, in terms of the form factors by using the c-theorem sum rule (1.69).

Assuming eq. (2.44) as definition of the ultraviolet theory, the corresponding central charge is given by (see, for instance, [49])

$$c(g) = 1 + 12\, Q_-^2(g) \tag{2.45}$$

where

$$Q_-(g) = -\left(\frac{\sqrt{4\pi}}{g} + \frac{g}{2\sqrt{4\pi}}\right) . \tag{2.46}$$

Using eq. (2.11), it may be written as

$$c(B) = 1 + 6\left(\frac{2-B}{B} + \frac{B}{2-B} + 2\right) . \tag{2.47}$$

Notice that this is a self-dual function of the coupling constant, i.e. invariant under $B \to 2-B$.

On the other hand, we may compute the central charge associated to the ultraviolet limit of the massive theory in terms of the second moment of the two-point function of the trace $\Theta(x)$ (1.69). According to eq. (2.43), the two-point function of the trace $\Theta(x)$ has to be computed in terms of the form factors of the operator $E_+(x,B)$ defined in eq. (2.35). The data reported in Table 2.1 and plotted in Fig. 2.2 show that the first two form factors of $E_+(x)$ are sufficient to saturate the sum-rule (1.70) and to reproduce with high percentage of precision the expression (2.47).

An additional check that the ultraviolet limit induced by this choice of Θ is ruled by the conformal field theory (2.44), is given by the computation of the conformal dimensions of the fundamental exponentials. This can be done in two different ways, using directly conformal field theory method or analyzing the ultraviolet behaviour of massive correlators.

For the conformal field theory defined by eq. (2.44), the conformal dimensions of the primary fields corresponding to the operators $e^{\alpha\varphi}$ are given by

$$\Delta_-(\alpha) = -\frac{\alpha^2}{8\pi} + \frac{\alpha Q_-(g)}{\sqrt{4\pi}} , \tag{2.48}$$

and therefore, for the fundamental exponentials we have

$$\begin{aligned} \Delta_-(E_-) &= 1 , \\ \Delta_-(E_+) &= -1 - \tfrac{g^2}{4\pi} . \end{aligned} \tag{2.49}$$

On the other hand, we may compute the conformal dimensions of the fundamental exponentials by investigating the UV-limit of the correlators

$$< T_{zz}(z,\bar{z})\, E_{\pm}(0) >_E \;=\; \frac{\bar{z}}{z}\sum_{n=0}^{\infty}\frac{1}{n!}\int_{-\infty}^{+\infty}\frac{d\beta_1}{2\pi}\cdots\int_{-\infty}^{+\infty}\frac{d\beta_n}{2\pi}\left(F_n^{T_{++}}(\beta 1,\ldots,\beta_n)\right)^* \times F_n^{E_{\pm}}(\beta 1,\ldots,\beta_n) e^{-mR\sum_i \cosh\beta_i} \;. \tag{2.50}$$

At order $O(g^4)$, it is sufficient to truncate the series to the first two terms and also use the perturbative expansion

$$\mathcal{N}(B) = 1 - \frac{g^2}{8\pi^2} + O(g^4) \;. \tag{2.51}$$

Since for small values of g

$$F_n^{T_{++}}(x_1,\ldots,x_n) \;=\; -F_0^{\Theta}(B)\,\frac{\sigma_{n-1}^{(n)}}{4\sigma_1^{(n)}\sigma_n^{(n)}}\,F_n^{V_+}(x_1,\ldots,x_n) \;, \tag{2.52}$$

and

$$< T_{zz}(z,\bar{z})\, E_{\pm}(0) >_E \;\simeq\; \frac{\bar{z}}{z}\Big\{ - F_0^{\Theta}(B)\, F_1^{V_+} F_1^{V_\pm}\frac{1}{4\pi}K_2(mR) - \frac{F_0^{\Theta}(B)}{2\pi^2}\int_{-\infty}^{+\infty} d\beta\, K_2(2mR\cosh\beta)\, F_2^{V_+}(2\beta)F_2^{V_\pm}(2\beta)\Big\} \;, \tag{2.53}$$

in the limit $mR \to 0$, we have

$$< T_{zz}(z,\bar{z})\, E_{\pm}(0) >_E \;\simeq\; \frac{\Delta_-(\Phi_\pm)}{z^2} \;, \tag{2.54}$$

with

$$\begin{aligned} \Delta_-(E_-) &= 1 + O(g^4) \;, \\ \Delta_-(E_+) &= -1 - \tfrac{g^2}{4\pi} + O(g^4) \;, \end{aligned} \tag{2.55}$$

in agreement with eq. (2.49).

The case $a = 0$

The trace of stress-energy tensor is given in this case by

$$\Theta(x) \;=\; F_0^{\Theta}(B)\,\Phi_-(x,B) \;. \tag{2.56}$$

and we expect that the massive model will flow in the ultraviolet limit to a conformal field theory defined by the bare action

$$\mathcal{S}_+ \;=\; \int d^2x\left[\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m^2}{2g^2}\,e^{g\varphi}\right] \;. \tag{2.57}$$

The corresponding background charge is given by [49]

$$Q_+(g) = \left(\frac{\sqrt{4\pi}}{g} + \frac{g}{2\sqrt{4\pi}}\right) . \tag{2.58}$$

This conformal field theory differs from that analyzed in the previous subsection for the exchange of the role of the two fundamental exponentials.

According to the conformal field theory defined by the bare lagrangian (2.57), the anomalous dimensions of the primary fields corresponding to the operators $e^{\alpha\varphi}$ are given by

$$\Delta_+(\alpha) = -\frac{\alpha^2}{8\pi} + \frac{\alpha Q_+(g)}{\sqrt{4\pi}} , \tag{2.59}$$

and for the fundamental exponentials we have in this case

$$\begin{aligned} \Delta_+(E_+) &= 1 , \\ \Delta_+(E_-) &= -1 - \tfrac{g^2}{4\pi} . \end{aligned} \tag{2.60}$$

Repeating the same kind of computations of the previous subsection, it is easy to check that these expressions coincide, at order $O(g^4)$, with the conformal dimensions extracted from the ultraviolet behaviour of the correlators $< T_{zz}(z,\overline{z})\, E_\pm(0) >$.

For what concerns the central charge, since it depends quadratically on $Q_\pm$, its value is given as before by the self-dual function (2.47). Analogous computations for the second moment of the Θ computed in terms of the form factors of $E_-(x,B)$ (which is the dual operator of $E_+(x,B)$), lead therefore to the same results of Table 2.1 (see also Fig. 2.2).

General case

We are now able to give the conformal dimension $\Delta(\alpha)$ of the exponential operator $e^{\alpha\varphi}$ and the central charge of the conformal field theory reached in the ultraviolet regime for generic value of the parameter a.

Since $\Delta(\alpha)$ is the coefficient of the most singular term obtained in the UV-limit of the correlation function $< T_{zz}(z,\overline{z})\, e^{\alpha\varphi(0)} >$, and the form factors of $T_{zz}(z,\overline{z})$ depends linearly on the parameter a, the conformal dimension is given by

$$\begin{aligned} \Delta(\alpha) &= a\,\Delta_-(\alpha) + (1-a)\,\Delta_+(\alpha) = \\ &= -\frac{\alpha^2}{8\pi} + \frac{\alpha}{\sqrt{4\pi}}(1-2a)\left(\frac{\sqrt{4\pi}}{g} + \frac{g}{2\sqrt{4\pi}}\right) , \end{aligned} \tag{2.61}$$

with $\Delta_\pm(\alpha)$ defined in eqs. (2.48) and (2.59). The coefficient in front of the linear term in α in eq. (2.61) identifies the background charge and therefore the central charge of the conformal field theory reached in the ultraviolet limit is given by

$$c = 1 + 24 \frac{(1-2a)^2}{B(2-B)} \,. \tag{2.62}$$

We have checked the validity of this result with the computation of the central charge in terms of the first form factors of the operator (2.41). The comparison between them is shown in Fig. 2.4, varying a at fixed B.

As last example of possible choices of the stress-energy tensor, observe that for $a = 1/2$ we have $c = 1$, independent of the coupling constant. The corresponding expression of Θ is given by [44]

$$\Theta(x) = \frac{F_0^{\Theta}(B)}{2} \left(e^{g\varphi} + e^{-g\varphi}\right) \,. \tag{2.63}$$

This operator is manifestly self-dual and Z_2-even. With this choice of a, the anomalous dimensions of the fundamental exponentials coincide, at lowest order, with their gaussian values

$$\Delta(\pm g) = -\frac{g^2}{8\pi} + o(g^4) \,. \tag{2.64}$$

The check of the central charge obtained in this case is already satisfactory in the two-particle approximation [44] and is reported here in Table 2.2.

2.2 The Bullough-Dodd massive boson

As a last example of the possibility to characterize massive integrable flows with the form factors technique, stress-energy tensor can be easily we discusse another lagrangian integrable theory involving an interacting bosonic field, the so-called Bullough-Dodd (BD) model.

2.2.1 Basic properties

The Bullough-Dodd (BD) model is defined by the equation of motion [62, 63, 15]

$$\Box\varphi = \frac{m_0^2}{3\lambda} \left(e^{-2\lambda\varphi} - e^{\lambda\varphi}\right) \,. \tag{2.65}$$

At the quantum level, the integrability of the model leads to the elasticity and factorization of the scattering processes. The spectrum of the model consists of a massive particle state A created by the elementary interpolating field φ. This particle appears as bound state of itself in the scattering process

$$A \times A \to A \to A \times A \,. \tag{2.66}$$

This property is called "φ^3"-property referring to the cubic interaction which makes it possible for the bound state to be created. The bootstrap dynamics of the model is supported by an infinite set of local conserved charges I_s where s is an odd integer but multiple of 3, $s = 1, 5, 7, 11, 13, \ldots$. The absence of spin s multiple of 3 is consistent with the bootstrap process (2.66) [11, 21]. The corresponding S-matrix is given by [58]

$$S(\beta, \mathcal{B}) = f_{\frac{2}{3}}(\beta)\, f_{\frac{\mathcal{B}}{3}-\frac{2}{3}}(\beta)\, f_{-\frac{\mathcal{B}}{3}}(\beta) \ . \tag{2.67}$$

We recall the building block of self-conjugated diagonal theories

$$f_x(\beta) = \frac{\tanh\frac{1}{2}(\beta + i\pi x)}{\tanh\frac{1}{2}(\beta - i\pi x)} \ , \tag{2.68}$$

and the coupling constant dependence of the model is encoded into the function

$$\mathcal{B}(\lambda) = \frac{\lambda^2}{2\pi}\frac{1}{1+\frac{\lambda^2}{4\pi}} \ . \tag{2.69}$$

Like the sinh-Gordon model, the S-matrix of the Bullough-Dodd model is invariant under the mapping

$$\mathcal{B}(\lambda) \to 2 - \mathcal{B}(\lambda) \ , \tag{2.70}$$

i.e. under the weak-strong coupling constant duality $\lambda \to 4\pi/\lambda$.

The minimal part of the S-matrix, i.e. the term $f_{\frac{2}{3}}(\beta)$, contains the physical pole $\beta = 2\pi i/3$ of the bound state and, as matter of fact, it coincides with the S-matrix of the Yang-Lee model [6]. Taking into account the coupling constant dependence of the S-matrix, the residue at the pole is given by

$$\Gamma^2(\mathcal{B}) = 2\sqrt{3}\frac{\tan\left(\frac{\pi\mathcal{B}}{6}\right)}{\tan\left(\frac{\pi\mathcal{B}}{6}-\frac{2\pi}{3}\right)}\frac{\tan\left(\frac{\pi}{3}-\frac{\pi\mathcal{B}}{6}\right)}{\tan\left(\frac{\pi\mathcal{B}}{6}+\frac{\pi}{3}\right)} \ . \tag{2.71}$$

This function, that corresponds to the three-particle vertex on mass-shell, vanishes for $\mathcal{B} = 0$ and $\mathcal{B} = 2$ (which are the free theory limits) with the corresponding scattering amplitude $S = 1$. However, it also vanishes at the self-dual point $\mathcal{B} = 1$, with the corresponding scattering amplitude $S(\beta) = f_{-2/3}$. This coincides with the S-matrix of the sinh-Gordon model computed at $B = 2/3$. As analyzed in [64], this equality between the S-matrices of the two models implies that at the self-dual point the Bullough-Dodd model dynamically develops a Z_2-symmetry which is a non-perturbative property of the model.

2.2.2 The recursive equations

Taking into account the bound state pole in the two-particle channel at $\beta_{ij} = 2\pi i/3$ and the one-particle pole in the three-particle channel at $\beta_{ij} = i\pi$, the general form factors of the Bullough-Dodd model can be parameterized as

$$F_n^O(\beta_1,\ldots,\beta_n) = Q_n^O(x_1,\ldots,x_n) \prod_{i<j} \frac{F_{\rm BD}^{min}(\beta_{ij})}{(x_i+x_j)(\omega x_i+x_j)(\omega^{-1}x_i+x_j)} , \tag{2.72}$$

where we have introduced the variables

$$x_i = e^{\beta_i} , \ \omega = e^{i\pi/3} . \tag{2.73}$$

$F_{\rm BD}^{min}(\beta)$ is an analytic function without zeros and poles in the physical sheet, whose explicit expression is given by

$$F_{\rm BD}^{min}(\beta,B) = \prod_{k=0}^{\infty} \left| \frac{\Gamma\left(k+\frac{3}{2}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+\frac{7}{6}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+\frac{4}{3}+\frac{i\hat\beta}{2\pi}\right)}{\Gamma\left(k+\frac{1}{2}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+\frac{5}{6}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+\frac{2}{3}+\frac{i\hat\beta}{2\pi}\right)} \right. \tag{2.74}$$
$$\times \left. \frac{\Gamma\left(k+\frac{5}{6}-\frac{B}{6}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+\frac{1}{2}+\frac{B}{6}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+1-\frac{B}{6}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+\frac{2}{3}+\frac{B}{6}+\frac{i\hat\beta}{2\pi}\right)}{\Gamma\left(k+\frac{7}{6}+\frac{B}{6}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+\frac{3}{2}-\frac{B}{6}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+1+\frac{B}{6}+\frac{i\hat\beta}{2\pi}\right)\Gamma\left(k+\frac{4}{3}-\frac{B}{6}+\frac{i\hat\beta}{2\pi}\right)} \right|^2$$

($\hat\beta = i\pi - \beta$). Its normalization is fixed by requiring the asymptotic behaviour

$$\lim_{\beta\to\infty} F_{\rm BD}^{min}(\beta,B) = 1 . \tag{2.75}$$

Notice that at the self-dual point $B = 1$, the above function coincides with the $F_{\rm min}^{\rm SG}\left(\beta,\frac{2}{3}\right)$ of the sinh-Gordon model, i.e.

$$F_{\rm BD}^{min}(\beta,1) = F_{\rm min}^{\rm SG}\left(\beta,\frac{2}{3}\right) . \tag{2.76}$$

The kernels of the residue recursive equations for the symmetric polynomials $Q_n^O(x_1,\ldots,x_n)$ are obtained using the functional relations satisfied by $F_{\rm min}(\beta,B)$

$$\begin{aligned} F_{\rm BD}^{min}(i\pi+\beta,B)\,F_{\rm BD}^{min}(\beta,B) &= \frac{\sinh\beta\left(\sinh\beta+\sinh\frac{i\pi}{3}\right)}{\left(\sinh\beta+\sinh\frac{i\pi B}{3}\right)\left(\sinh\beta+\sinh\frac{i\pi(1+B)}{3}\right)} , \\ F_{\rm BD}^{min}(\beta+\frac{i\pi}{3},B)\,F_{\rm BD}^{min}(\beta-\frac{i\pi}{3},B) &= \frac{\cosh\beta+\cosh\frac{2i\pi}{3}}{\cosh\beta+\cosh\frac{i\pi(2+B)}{3}}\,F_{\rm BD}^{min}(\beta,B) . \end{aligned} \tag{2.77}$$

The kinematical and bound state residue conditions give rise to the following recursive equations satisfied by the functions $Q_n(x_1,\ldots,x_n)$ [64]

$$Q_{n+2}(-x,x,x_1,x_2,\ldots,x_n) = U_n(x|x_1,x_2,\ldots,x_n)Q_n(x_1,x_2,\ldots,x_n) , \tag{2.78}$$

where

$$U_n(x|x_1,\ldots,x_n) = (-1)^n \frac{2}{F_{\rm BD}^{min}(i\pi,\mathcal{B})} x^3 \sum_{k_1,\ldots,k_6=0}^{n} (-1)^{k_2+k_3+k_5}\, x^{6n-(k_1+\cdots+k_6)}$$
$$\times \sigma_{k_1}^{(n)}\sigma_{k_2}^{(n)}\cdots\sigma_{k_6}^{(n)} \sin\left[\tfrac{\pi}{3}\left[2(k_2+k_4-k_1-k_3)+\mathcal{B}(k_3+k_6-k_4-k_5)\right]\right] , \qquad (2.79)$$

and

$$Q_{n+2}(\omega x,\omega^{-1}x,x_1,\ldots,x_n) = D_n(x|x_1,\ldots,x_n)\, Q_{n+1}(x,x_1,\ldots,x_n) , \qquad (2.80)$$

where

$$\begin{aligned} D_n(x|x_1,\ldots,x_n) &= -\frac{\sqrt{3}\Gamma(\mathcal{B})}{F_{\rm BD}^{min}\left(\frac{2\pi i}{3},\mathcal{B}\right)} x^3 \prod_{i=1}^{n}(x+x_i)(x\omega^{2+\mathcal{B}}+x_i)(x\omega^{-\mathcal{B}-2}+x_i) \\ &= \sum_{k_1,k_2,k_6=0}^{n} x^{3n-(k_1+k_2+k_6)}\,\omega^{(2+\mathcal{B})(k_2-k_3)}\,\sigma_{k_1}^{(n)}\sigma_{k_2}^{(n)}\sigma_{k_3}^{(n)} . \end{aligned} \qquad (2.81)$$

Unlike the sinh-Gordon model, we do not know presently a close solution for the recursive equations satisfied by the form factors of the Bullough-Dodd model at generic value of the coupling constant. However, as it will become clear in the following, all we need for our consideration is the explicit computation of the first representative form factors of the elementary interpolating field φ and of the so-called cluster operators $\mathcal{V}_{\pm}(x)$.

2.2.3 The elementary interpolating field

For the elementary interpolating field φ, identified as that operator that creates one-particle state

$$F_1^{\varphi} = <0|\varphi(0)|\beta> = \frac{1}{\sqrt{2}} \qquad (2.82)$$

the next form factors, with vanishing asymptotic behavior (2.26), are given by

$$F_2^{\varphi}(\beta) = -\frac{\Gamma(\mathcal{B})}{\sqrt{2}} \frac{\sin\frac{\pi}{6}(2+\mathcal{B})}{\sqrt{F_{\rm BD}^{min}(i\pi,\mathcal{B})}} \frac{F_{\rm BD}^{min}(\beta,\mathcal{B})}{\cosh\beta+\frac{1}{2}} , \qquad (2.83)$$

and

$$\begin{aligned} F_3^{\varphi}(\beta_1,\beta_2,\beta_3) = & \left(\prod_{i<j}^{3}\frac{F_{\rm BD}^{min}(\beta_i-\beta_j,\mathcal{B})}{(x_i+x_j)(e^{i\pi/3}x_i+x_j)(e^{-i\pi/3}x_i+x_j)}\right)\frac{2\sqrt{2}}{F_{\rm BD}^{min}(i\pi,\mathcal{B})}\sigma_3 \times \\ & \left\{2\sin^2\frac{\pi}{6}(2+\mathcal{B})\,\Gamma^2(\mathcal{B})\left(\cos\frac{\pi}{3}(2+\mathcal{B})-1\right)\sigma_3\sigma_2\sigma_1 - \right. \\ & \left. W(\mathcal{B})\left(\sigma_3\sigma_1^3+\sigma_2^3\right) + \left(\sin^2\frac{\pi}{6}(2+\mathcal{B})\,\Gamma^2(\mathcal{B})+W(\mathcal{B})\right)\sigma_2^2\sigma_1^2\right\} , \end{aligned} \qquad (2.84)$$

where

$$W(\mathcal{B}) = 2\sqrt{3}\sin\left(\frac{\pi\mathcal{B}}{6}\right)\sin\left(\frac{\pi}{6}(2-\mathcal{B})\right) . \qquad (2.85)$$

In terms of them, we can easily obtain the first form factors of the operator $\Box\varphi$

$$F_n^{\Box\varphi} = -m^2 \frac{\sigma_1\sigma_{n-1}}{\sigma_n} F_n^{\varphi} \ . \tag{2.86}$$

2.2.4 Cluster operators and fundamental exponentials

In order to define the form factors of the two fundamental exponential operators $E_1(x,\mathcal{B}) \equiv e^{\lambda\varphi(x)}$ and $E_2(x,\mathcal{B}) \equiv e^{-2\lambda\varphi(x)}$ of the Bullough-Dodd model, let us consider initially the definition of the cluster operators $\mathcal{V}_{\pm}(x,\mathcal{B})$. As for the sinh-Gordon model, we are looking for these operators in the class of form factors which are asymptotically constant for $x_i \to \infty$ and proportional to the invariant combination of symmetric polynomials $\sigma_1\sigma_{n-1}$ for $n > 2$. The first representative of such form factors are given by

$$F_2(\beta) = F_{\rm BD}^{min}(\beta,\mathcal{B}) \left\{ H_2 - H_1 \frac{\sin\frac{\pi}{6}(\mathcal{B}+2)\,\Gamma(\mathcal{B})}{\sqrt{F_{\rm BD}^{min}(i\pi,\mathcal{B})}} \frac{1}{\cosh\beta + \frac{1}{2}} \right\} , \tag{2.87}$$

and

$$\begin{aligned} F_3(\beta_1,\beta_2,\beta_3) &= \left(\prod_{i<j}^{3} \frac{F_{\rm BD}^{min}(\beta_i-\beta_j,\mathcal{B})}{(x_i+x_j)(\omega\, x_i + x_j)(\omega^{-1}\, x_i + x_j)} \right) \frac{4}{F_{\rm BD}^{min}(i\pi,\mathcal{B})} \sigma_1\sigma_2 \times \\ & \left\{ q_1(\mathcal{B})\sigma_3^2 + q_2(\mathcal{B})\sigma_3\sigma_2\sigma_1 - \right. \\ & \left. H_1\, W(\mathcal{B}) \left(\sigma_3\sigma_1^3 + \sigma_2^3\right) + q_3(\mathcal{B})\sigma_2^2\sigma_1^2 \right\} , \end{aligned} \tag{2.88}$$

where

$$\begin{aligned} q_1(\mathcal{B}) &= 2\Gamma(\mathcal{B}) \sin\frac{\pi}{6}(2+\mathcal{B}) \left[\cos\frac{\pi}{3}(2+\mathcal{B}) - 1\right] \left[\Gamma(\mathcal{B})H_1 \sin\frac{\pi}{6}(2+\mathcal{B}) + \sqrt{F_{\rm BD}^{min}(i\pi)}\frac{H_2}{2}\right] \\ q_2(\mathcal{B}) &= \Gamma(\mathcal{B}) \sin\frac{\pi}{6}(2+\mathcal{B}) \left[\Gamma(\mathcal{B})H_1 \sin\frac{\pi}{6}(2+\mathcal{B}) - \sqrt{F_{\rm BD}^{min}(i\pi)}H_2 \left(\cos\frac{\pi}{3}(2+\mathcal{B}) - \frac{3}{2}\right)\right] \\ q_3(\mathcal{B}) &= H_1\, W(\mathcal{B}) - \sqrt{F_{\rm BD}^{min}(i\pi)}\,\frac{H_2}{2}\,\Gamma(\mathcal{B}) \sin\frac{\pi}{6}(2+\mathcal{B}) \ . \end{aligned} \tag{2.89}$$

H_1 and H_2 are two arbitrary parameters. The important point is that all the higher form factors obtained by solving the recursive equations will depend on the constants H_1 and H_2 appearing in the equations (2.87) and (2.88), which play the role of arbitrary initial conditions of the recursive structure. Their relative value can be fixed though, if we require that the above form factors satisfy an additional condition, i.e. the cluster property

$$\lim_{\Delta\to+\infty} F_{k+l}\left(\beta_1+\Delta,\ldots,\beta_k+\Delta,\beta_{k+1},\ldots,\beta_{k+l}\right) = F_k\left(\beta_1,\ldots,\beta_k\right) F_l\left(\beta_{k+1},\ldots,\beta_{k+l}\right) \tag{2.90}$$

with $F_0 = 1$. In this case we have

$$
\begin{aligned}
H_1^{\pm}(\mathcal{B}) &= \frac{1}{\sqrt{F_{\rm BD}^{min}(i\pi,\mathcal{B})}}\left\{-\sin\left(\frac{\pi}{6}(\mathcal{B}+2)\right)\Gamma(\mathcal{B}) \pm \sqrt{\sin^2\left(\frac{\pi}{6}(\mathcal{B}+2)\right)\Gamma^2(\mathcal{B})+4W(\mathcal{B})}\right\} ,\\
H_2^{\pm}(\mathcal{B}) &= (H_1^{\pm})^2(\mathcal{B}) . \qquad (2.91)
\end{aligned}
$$

With this choice of $H_1^{\pm}(\mathcal{B})$ and $H_2^{\pm}(\mathcal{B})$, the infinite tower of form factors with (2.87) and (2.88) as first representatives, define two cluster operators $\mathcal{V}_{\pm}(x,\mathcal{B})$. By construction, the matrix elements of such operators are invariant under the duality transformation $\mathcal{B} \to 2-\mathcal{B}$. Also in this case, we conjecture that the fundamental exponential operators are given by

$$
\begin{aligned}
E_1(x,\mathcal{B}) &= \equiv \theta(1-B)\,\mathcal{V}_+(x,\mathcal{B}) + \theta(B-1)\,\mathcal{V}_-(x,\mathcal{B}) ,\\
E_2(x,\mathcal{B}) &= \equiv \theta(1-B)\,\mathcal{V}_-(x,\mathcal{B}) + \theta(B-1)\,\mathcal{V}_+(x,\mathcal{B}) .
\end{aligned} \qquad (2.92)
$$

This definition is in agreement with the perturbative analysis of the matrix elements of the two exponential operators $e^{\lambda\varphi(x)}$ and $e^{-2\lambda\varphi(x)}$. Concerning their properties under the duality mapping, as far as $\mathcal{B} \neq 1$, the fundamental exponentials are mapped each into the other under the mapping $\mathcal{B} \to 2-\mathcal{B}$, i.e.

$$
E_{1,2}(x,\mathcal{B}) = E_{2,1}(x,2-\mathcal{B}) . \qquad (2.93)
$$

However, this mapping becomes degenerate at the self-dual point $\mathcal{B} = 1$ where, similarly to the sinh-Gordon model, the two operators $E_1(x,\mathcal{B})$ and $E_2(x,\mathcal{B})$ collapse into a single operator $E(x)$. Moreover, as already noticed in [64], at the self-dual point a Z_2-symmetry is dynamically implemented in the Bullough-Dodd model. Due to the fact that at $\mathcal{B} = 1$ the three-particle vertex $\Gamma(\mathcal{B})$ vanishes and $H_1^+ = -H_1^-$, the resulting operator $E(x)$ will have non-zero matrix elements only on the $2n$ particle states. Its form factors are entirely expressed in terms of the form factors of the sinh-Gordon model at $B = 2/3$

$$
F_{2n}^E = \left(\mu\left(\frac{2}{3}\right)\right)^{2n} Q_n(1) \prod_{i<j} \frac{F_{\rm SG}^{min}\left(\beta_{ij},\frac{2}{3}\right)}{x_i + x_j} , \qquad (2.94)
$$

with $\mathcal{N}(B)$ and $\mu(B)$ defined in eqs. (2.15) and (2.22).

Comparing the form factors of $\Box\varphi$ and the form factors of the fundamental exponentials the quantum equation of motion can be cast in the form

$$
\begin{aligned}
\Box\varphi(x) &= \frac{m^2}{2\sqrt{2}}\sqrt{\frac{F_{\rm BD}^{min}(i\pi,\mathcal{B})}{\sin^2\left(\frac{\pi}{6}(\mathcal{B}+2)\right)\Gamma^2(\mathcal{B})+4W(\mathcal{B})}}\,(\theta(1-\mathcal{B}) - \theta(\mathcal{B}-1))\left(e^{-2\lambda\varphi(x)} - e^{\lambda\varphi(x)}\right)\\
&= \frac{m^2}{2\sqrt{2}}\sqrt{\frac{F_{\rm BD}^{min}(i\pi,\mathcal{B})}{\sin^2\left(\frac{\pi}{6}(\mathcal{B}+2)\right)\Gamma^2(\mathcal{B})+4W(\mathcal{B})}}\,(V_-(x,\mathcal{B}) - V_+(x,\mathcal{B})) . \qquad (2.95)
\end{aligned}
$$

Again it is very natural to define the renormalized coupling constant

$$\lambda(\mathcal{B}) = \frac{2\sqrt{2}}{3}\sqrt{\frac{\sin^2\left(\frac{\pi}{6}(\mathcal{B}+2)\right)\Gamma^2(\mathcal{B})+4W(\mathcal{B})}{F_{\rm BD}^{min}(i\pi,\mathcal{B})}}\,(\theta(1-\mathcal{B}) - \theta(\mathcal{B}-1)) \ . \tag{2.96}$$

2.2.5 Class of stress-energy tensors

The most general expression of the trace of the stress-energy tensor compatible with the quantum equation of motion of the Bullough-Dodd model can be expressed in terms of the fundamental exponentials as

$$\Theta(x) \ = \ F_0^{\Theta}(\mathcal{B})\,(a\,E_1(x,\mathcal{B}) + (1\,-\,a)\,E_2(x,\mathcal{B}))\ , \tag{2.97}$$

where $F_0^{\Theta}(\mathcal{B})$ is its vacuum expectation value

$$F_0^{\Theta}(\mathcal{B}) = \frac{\pi m^2}{2W(\mathcal{B})}\ , \tag{2.98}$$

(as computed by the Thermodynamical Bethe Ansatz [45]), whereas a is a free parameter. Varying the value of a, we may reach different ultraviolet limit of the Bullough-Dodd model. Before considering the general case, let us analyse separately the two cases $a = 1$ and $a = 0$.

The case $a = 1$

For this value of a, the trace of the stress-energy tensor is given entirely by the operator $E_1(x,\mathcal{B})$ and therefore we expect that the ultraviolet behaviour will be described by a conformal field theory with bare action given by

$$\mathcal{S}_2 = \int d^2x\left[\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m^2}{6\lambda^2}\,e^{-2\lambda\varphi}\right]\ , \tag{2.99}$$

and background charge [49]

$$Q_2(\lambda) = -\left(\frac{\sqrt{\pi}}{\lambda}+\frac{\lambda}{2\sqrt{\pi}}\right)\ . \tag{2.100}$$

Using eq. (2.69), the corresponding central charge is given by

$$c(\mathcal{B}) = 1+12Q_2^2(\lambda) = 1+12\left(\frac{2-\mathcal{B}}{4\mathcal{B}}+\frac{\mathcal{B}}{2-\mathcal{B}}+1\right)\ , \tag{2.101}$$

This is confirmed by the computation of the central charge in terms of the c-theorem by using the form factors of the operator $E_1(x,\mathcal{B})$ which defines in this case the trace of the stress-energy tensor. The result of this computation is reported in Table 2.3 (see also Fig. 2.3) and the sum rule turns out to be saturated with high percentage of precision by using just the first two form factors of $E_1(x, \mathcal{B})$.

According to the conformal field theory (2.99), the conformal dimensions of the primary fields $e^{\alpha\varphi}$ are given by

$$\Delta_2(\alpha) = -\frac{\alpha^2}{8\pi} + \frac{\alpha Q_2(\lambda)}{\sqrt{4\pi}} \ . \tag{2.102}$$

and for the fundamental exponential operators of the Bullough-Dodd model we have

$$\begin{aligned} \Delta_2(E_2) &= 1 \\ \Delta_2(E_1) &= -\frac{1}{2} - \frac{3}{8\pi}\lambda^2 \ . \end{aligned} \tag{2.103}$$

It is easy to see that these expressions are in agreement with those extracted by looking at the short-distance behaviour of the correlators $< T_{zz}(z, \overline{z})\, E_1(0) >$ and $< T_{zz}(z, \overline{z})\, E_2(0) >$. The computation are similar to that of the sinh-Gordon model, the only difference being the perturbative expansion

$$\begin{aligned} F_{\rm BD}^{min}(i\pi, \mathcal{B}) &= \exp\left[-8\int_0^\infty \frac{dx}{x} \frac{\sinh\left(\frac{x\mathcal{B}}{6}\right)\sinh\left(\frac{x}{6}(2-\mathcal{B})\right)\sinh\frac{x}{2}\cosh\frac{x}{6}}{\sinh^2 x}\right] \sim \\ &\sim 1 - \left(\frac{1}{\pi} + \frac{1}{6\sqrt{3}}\right)\frac{\lambda^2}{6} + o(\lambda^4) \ , \end{aligned} \tag{2.104}$$

and therefore we will not repeat them here.

The case $a = 0$

Since the trace of the stress-energy tensor is given in this case by the operator $E_2(x, \mathcal{B})$, the ultraviolet limit will be ruled by a conformal field theory with a bare action given by

$$\mathcal{S}_1 = \int d^2x \left[\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m^2}{3\lambda^2}\, e^{\lambda\varphi}\right] \ , \tag{2.105}$$

and a background charge given by

$$Q_1(\lambda) = \left(\frac{\sqrt{4\pi}}{\lambda} + \frac{\lambda}{2\sqrt{4\pi}}\right) \ . \tag{2.106}$$

For the corresponding value of the central charge we have

$$\begin{aligned} c(\mathcal{B}) &= 1 + 12\, Q_1^2(\lambda) \\ &= 1 + 12\left(\frac{2-\mathcal{B}}{\mathcal{B}} + \frac{1}{4}\frac{\mathcal{B}}{2-\mathcal{B}} + 1\right) \ , \end{aligned} \tag{2.107}$$

whereas for the conformal dimension of the primary operators $e^{\alpha\varphi}$

$$\Delta_1(\alpha) = -\frac{\alpha^2}{8\pi} + \frac{\alpha Q_1(\lambda)}{\sqrt{4\pi}} .. \tag{2.108}$$

Hence, for the fundamental exponential operators we have in this case

$$\begin{aligned} \Delta_1(E_1) &= 1 \\ \Delta_1(E_2) &= -2 - \frac{3}{4\pi}\lambda^2 . \end{aligned} \tag{2.109}$$

These conformal data are again confirmed by the form factors approach, as shown for instance in Fig. 2.3 for the central charge.

The general case

It is now easy to write down the conformal dimension $\Delta(\alpha)$ of the exponential operator $e^{\alpha\varphi}$ and the central charge of the conformal field theory reached in the ultraviolet regime for generic value of the parameter a appearing in the definition of the stress-energy tensor of the Bullough-Dodd model. The argument is similar to that already employed for the sinh-Gordon model. Since $\Delta(\alpha)$ is the coefficient of the most singular term obtained in the UV-limit of the correlation function $< T_{zz}(z,\bar{z})\, e^{\alpha\varphi(0)} >$ and the form factors of $T_{zz}(z,\bar{z})$ depends linearly on the parameter a, the conformal dimension is given by

$$\begin{aligned} \Delta(\alpha) &= a\,\Delta_2(\alpha) + (1-a)\,\Delta_1(\alpha) = \\ &= -\frac{\alpha^2}{8\pi} + \frac{\alpha}{\sqrt{4\pi}}\left[\sqrt{\frac{2-\mathcal{B}}{\mathcal{B}}}\left(1 - \frac{3}{2}a\right) + \sqrt{\frac{\mathcal{B}}{2-\mathcal{B}}}\left(\frac{1}{2} - \frac{3}{2}a\right)\right] , \end{aligned} \tag{2.110}$$

where $\Delta_1(\alpha)$ and $\Delta_2(\alpha)$ are given in eq. (2.102) and (2.108) respectively. The linear term in α in (2.110) identifies the background charge of the corresponding Coulomb gas. Therefore the central charge of the conformal field theory reached in the ultraviolet limit is given by

$$c = 1 + 3\frac{(\mathcal{B} + 6a - 4)^2}{\mathcal{B}(2-\mathcal{B})} . \tag{2.111}$$

The check of this formula in terms of the form factors approach is shown in Fig. 2.4.

Observe that, with the choice

$$a = \frac{4-\mathcal{B}}{6} , \tag{2.112}$$

we have identically $c = 1$, a result that is confirmed by the form factors approach within the usual accuracy of few percents. The corresponding trace of the stress-energy tensor is given by

$$\Theta(x) = F_0^{\Theta}(\mathcal{B})\left(\frac{4-\mathcal{B}}{6}\, e^{\lambda\varphi(x)} + \frac{2+\mathcal{B}}{6}\, e^{-2\lambda\varphi(x)}\right) . \tag{2.113}$$

This operator is manifestly self-dual. In the limit $\lambda \to 0$, it reduces to

$$\Theta(x) \quad = \quad \frac{2\pi m^2}{3\lambda^2}\left(2e^{\lambda\varphi(x)} + e^{-2\lambda\varphi(x)}\right) \quad , \tag{2.114}$$

which is the classical expression of $\Theta(x)$ for the Bullough-Dodd model.

Chapter 3

Massless flows

In the first two chapters, we have dealt with deformations of the critical point actions which produce finite correlation lengths such that the systems are then driven in a massive phase. In this phases all the correlation functions decay exponentially at large distance scales. Using relativistic invariance and analytic properties, it has been shown how to accomplish the difficult task of computing on-shell quantities, as for instance matrix elements of local operators, the form factors. The reconstruction of the full theory in the massive regime, i.e. the computation of all set of correlation functions, is then obtained through spectral representation methods. In addition to the theoretical insights gained by this approach, the success of this method relies on the extremely fast convergent expressions obtained for the correlation functions, such that a little analytic work is in many case sufficient to extract physical quantities with arbitrary precision [7, 33, 66].

There are however several important physical situations where the deformation destroys the scale invariance of the critical action although the resulting spectrum of the excitations presents no mass gap. In this case, the corresponding quantum field theory is that associated to the renormalization group flow responsible for the massless crossover between two different fixed points. Important examples of this kind are provided by the Kondo problem [67], by the $O(3)$ non-linear sigma model with the $\theta = \pi$ topological term [68] and by the massless flows between two consecutive conformal minimal models [39, 40], the first one being the flow that links the tricritical Ising model to the critical Ising model [69]. The most relevant feature of these quantum field theories is that the absence of the mass gap as well as of the higher mass thresholds implies a power law behaviour of the correlation functions with different critical exponents both in the ultraviolet and infrared regions, separated by a non trivial crossover in between. The analysis of such theories along the massless renormalization group flow is notoriously difficult partly because it needs the resummation of the perturbation series to obtain reliable results.

Under such circumstances, we are therefore left with the spectral representation methods. However, even assuming the integrability of the model along the flow, due to the absence of mass gap it is no *a-priori* obvious that they will be of any help to deal successfully with such theories. In fact, there are several aspects that need a careful investigation. The most important of these problems are the analytic structure of the form factors and the convergent properties of the spectral representation series. In order to approach these questions, we analyse in details in this chapter the simplest massless non-scaling invariant theory, i.e. that one associated to the massless flow between the tricritical Ising model to the critical Ising model. This analysis will provide the basic strategy to approach the massless integrable quantum field theory and may be useful, once generalized, to investigate a concrete physical problem as for instance the computation of correlation functions entering the flow of the Kondo problem between the two fixed points. As shown in the following, in the massless case there is a natural generalization of the monodromy and recursive equations satisfied by the form factors based on the analytic properties of the massless scattering amplitudes. In particular, the presence of left and right massless excitations give rise to a double chain of recursive equations which are quite effective in order to fix the matrix elements of local operators. Concerning the problem of the computation of the correlation functions, the expected infrared divergencies naturally occur in the evaluation of each contribution of the sums (1.4). Therefore resummation techniques need to be developed to grasp from the cutoff dependent correlation functions the physical ones. We will show, for example, how to recover the infrared anomalous dimension of the magnetization operator. However, the particular feature of the flow under investigation of being a free theory in the infrared limit exceptionally allows the direct computation of the correlation functions of the fields in the energy sector.

The chapter is organized as follows. In section 1 we will analyse the relevant features arising in the massless bidimensional integrable quantum field theories. In section 2 we make use of this analysis in order to understand the analytic structure of the form factors for massless quantum field theories. Section 3 focalizes on the massless deformation of the tricritical Ising Model that drives the system in the infrared limit to the critical Ising model. We take advantage of the non-linear supersymmetric formulation of the model [70, 71] and we compute the matrix elements of the trace of the stress-energy tensor, the spin energy density, and of the magnetization and disorder operators. By making a numerical integration we then extract the scaling laws of the two point functions of the first two of these operators in the short and large distance regions. The sum rule relative to the c-theorem [39, 40] is also employed to check the flow between the two models. Finally the anomalous dimension of the magnetization is recovered with a thermodynamic analogy.

3.1 Massless scattering

The scattering theory for massless integrable models has been recently developed in [69, 72]. In this section we briefly review the points of this construction necessary for our successive considerations. In a massless bidimensional theory the excitations consist in a set of left and right mover particles. These are the lowest energy states which propagate along all the renormalization group flow and constitute the relevant degrees of freedom of the problem. Higher massive states are thought to decouple from the theory, their influence being eventually seen in the analytic structure of the physical amplitudes of the massless particles. For the sake of simplicity we restrict our attention to self-conjugate particles without additional internal indices. We denote these elementary excitations by $\mid A_L(\beta) >$ and $\mid A_R(\beta) >$. The rapidity variable β parameterizes the dispersion relations of the two set of massless particles, given by

$$\begin{aligned} p^{(0)} &= -p^{(1)} = \frac{M}{2}\, e^{-\beta} \quad \text{for left movers} \\ p^{(0)} &= p^{(1)} = \frac{M}{2}\, e^{\beta} \quad \text{for right movers} \end{aligned} \tag{3.1}$$

where M is a mass parameter.

The dynamics of the massless particles is encoded into the set of scattering processes which take place in the system during its time evolution. They can be formally described by the following Zamolodchikov-Faddeev algebra

$$A_R(\beta_1)A_R(\beta_2) = S_{RR}(\beta_1 - \beta_2)\, A_R(\beta_2)A_R(\beta_1)\,, \tag{3.2}$$

$$A_L(\beta_1)A_L(\beta_2) = S_{LL}(\beta_1 - \beta_2)\, A_L(\beta_2)A_L(\beta_1)\,, \tag{3.3}$$

$$A_R(\beta_1)A_L(\beta_2) = S_{RL}(\beta_1 - \beta_2)\, A_L(\beta_2)A_R(\beta_1)\,. \tag{3.4}$$

The two-particle scattering amplitudes S_{RR}, S_{LL} and S_{RL} introduced above are constrained by a set of functional relations expressing the basic principles of unitarity and crossing symmetry. The best way to derive them for the massless case is to consider the theory as the massless limit of some massive integrable quantum field theory (see chapter 1) with $A(\theta)$ as elementary excitation of mass m. The dispersion relation is given in this case by

$$p^{(0)} = m\cosh\theta \quad ; \quad p^{(1)} = m\sinh\theta \ . \tag{3.5}$$

and the two-particle elastic S-matrix is defined by

$$A(\theta_1)A(\theta_2) = S(\theta_1-\theta_2)\,A(\theta_2)A(\theta_1)\ . \tag{3.6}$$

As function of the Mandelstam variable $s = (p_1+p_2)^2 = 2m^2\,(1+\cosh(\theta_1-\theta_2))$, we recall that the massive scattering amplitude is defined on the two sheets of a Riemann surface with square root branch cuts starting at $s=0$ and $s=4m^2$. In terms of rapidities, $S(\theta_1-\theta_2)$ becomes a meromorphic, $2\pi i$-periodic function satisfying the following unitarity and crossing symmetry equations:

$$\begin{aligned} &S(\theta_1-\theta_2)S(\theta_2-\theta_1)=1\\ &S(\theta_1-\theta_2)=S(i\pi+\theta_2-\theta_1)\ . \end{aligned} \tag{3.7}$$

The massless limit is obtained by sending to zero the mass m while shifting to infinity the rapidity of the particles $A(\theta)$ in order to keep energy and momentum finite. It is immediately seen that, after defining $M = me^{\beta_0/2}$ and taking the limit $\beta_0 \to +\infty$, the massive parameterization (3.5) reproduces the first of (3.1) for $\theta \equiv \beta + \beta_0/2$ and the second for $\theta \equiv \beta - \beta_0/2$.

Two qualitatively different possibilities arise when we take the massless limit of the two-particle scattering amplitude $S(\theta_1-\theta_2)$.

a) One of the particles becomes a right-mover and the other one a left-mover. As m goes to zero the two cuts in the s-plane join in the origin and the Riemann surface splits into two distinct sheets: the upper (lower) one contains the half of the physical (unphysical) sheet with $\mathrm{Im}s > 0$ and the half of the unphysical (physical) sheet with $\mathrm{Im}s < 0$. The relation $s = M^2e^{\beta_1-\beta_2}$ valid for the right-left scattering shows that, contrary to the massive case, we need two $2\pi i$-periodic, meromorphic functions of the rapidity difference in order to represent the double-valued function S_{RL}. We denote by $S_{RL}(\beta_1-\beta_2)$ ($\tilde{S}_{RL}(\beta_1-\beta_2)$) the value of the scattering amplitude on the upper (lower) sheet. One easily realizes that in the limit $A(\theta_1)\to A_R(\beta_1)$, $A(\theta_2)\to A_L(\beta_2)$ the functions $S_{RL}(\beta_1-\beta_2)$ and $\tilde{S}_{RL}(\beta_1-\beta_2)$ are obtained as follows from the massive scattering amplitude $S(\theta_1-\theta_2)$:

$$S_{RL}(\beta_1-\beta_2) = \lim_{\beta_0\to+\infty} S(\beta_1-\beta_2+\beta_0)\ , \tag{3.8}$$

$$\tilde{S}_{RL}(\beta_1-\beta_2) = \lim_{\beta_0\to+\infty} S(\beta_2-\beta_1-\beta_0)\ . \tag{3.9}$$

These relations can be obtained requiring that the two values $S(\theta_1 - \theta_2)$ and $S(\theta_2 - \theta_1)$ corresponding in the massive case to the different boundary values[1] of $S(s)$ at $s + i\epsilon$ and $s - i\epsilon$ are mapped into $S_{RL}(\beta_1 - \beta_2)$ and $\tilde{S}_{RL}(\beta_1 - \beta_2)$ respectively in the massless limit under consideration. Taking the limit of eqs.(3.7) one immediately gets unitarity and crossing relations for the right-left scattering amplitude:

$$S_{RL}(\beta_1 - \beta_2)\tilde{S}_{RL}(\beta_1 - \beta_2) = 1 \, , \tag{3.10}$$

$$S_{RL}(\beta_1 - \beta_2) = \tilde{S}_{RL}(\beta_1 - \beta_2 + i\pi) \, . \tag{3.11}$$

Comparing with (3.4), we learn immediately that the lower sheet amplitude $\tilde{S}_{RL}(\beta_1 - \beta_2)$ coincides with the upper sheet one $S_{LR}(\beta_2 - \beta_1)$.

b) Both the particles become right-movers or left-movers. In this case the Mandelstam variable s identically vanishes and the standard analyticity arguments are not useful [72]. Thus the only available criterion to establish the analyticity properties of S_{RR} and S_{LL} relies on the massless limit of the amplitude $S(\theta_1 - \theta_2)$. Since in such limit the rapidity shift β_0 cancels one concludes that in terms of the rapidity variables $S_{RR}(\beta_1 - \beta_2)$ and $S_{LL}(\beta_1 - \beta_2)$ actually satisfy the same unitarity and crossing relation as in the massive case

$$\begin{aligned} S_{RR}(\beta_1 - \beta_2)S_{RR}(\beta_2 - \beta_1) = 1 \\ S_{RR}(\beta_1 - \beta_2) = S_{RR}(i\pi + \beta_2 - \beta_1) \, . \end{aligned} \tag{3.12}$$

The difference of rapidities parameterizes the ratio of linear momenta $p_R^1/p_R^2 = e^{\beta_1 - \beta_2}$ which is the only relativistic invariant in the RR-channel. So the first equation of (3.12) can be understood as the usual unitarity property which establishes the transformation of the scattering amplitudes under the exchanging of the 'in' and 'out' states. The fact that, without other internal indices, the right and left movers are singlet of charge conjugation leads to the usual interpretation of the second equation of (3.12) as true a crossing symmetry relation. We assume to deal with parity invariant theories, which is assured if we take $S_{RR}(\beta) = S_{LL}(\beta)$. To conclude this section we remark that S_{RR} and S_{LL} are Lorentz and scale-invariant amplitudes since they are functions of the ratios of momenta. Then the mass parameter M appearing in (3.1) breaks scale invariance only in presence of a rapidity dependent right-left scattering. This fact can also lead to the treatment of the conformal field theories in terms of the scattering of their massless

[1] This two different boundary values obviously correspond to the description of the same scattering process from the point of view of the incoming and outgoing particles, resp.

excitations. We will use this possibility in the determination of the anomalous dimensions of operators given in terms of their massless form factors.

3.2 Massless form factors

As we have shown in chapters 1 and 2, the knowledge of the S-matrix in a massive integrable quantum field theory allows the reconstruction of the off-shell physics through the computation of the matrix elements of local operators on asymptotic states, the form factors. In this section we show how this program can be carried out also in the massless case. We define the n-particle form factor of a local operator $O(x)$ as

$$F_{\alpha_1\alpha_2\dots\alpha_n}(\beta_1,\beta_2,\dots,\beta_n) = <0|O(0)|\beta_1,\beta_2,\dots,\beta_n>_{\alpha_1\alpha_2\dots\alpha_n} , \tag{3.13}$$

where the index α_i is R for right-movers and L for left-movers and

$$|\beta_1,\beta_2,\dots,\beta_n>_{\alpha_1\alpha_2\dots\alpha_n} = A_{\alpha_1}(\beta_1)A_{\alpha_2}(\beta_2)\dots A_{\alpha_n}(\beta_n)\,|0> \ . \tag{3.14}$$

Among the states defined in (3.14) the asymptotic 'in' and 'out' bases respectively correspond to the following orderings

$$A_R(\beta_1)A_R(\beta_2)\dots A_R(\beta_r)A_L(\beta_{r+1})A_L(\beta_{r+2})\dots A_L(\beta_n)\,|0> \tag{3.15}$$

$$\beta_1 > \beta_2 > \dots > \beta_r, \qquad \beta_{r+1} > \beta_{r+2} > \dots > \beta_n \ , \tag{3.16}$$

$$A_L(\beta_1)A_L(\beta_2)\dots A_L(\beta_l)A_R(\beta_{l+1})A_R(\beta_{l+2})\dots A_R(\beta_n)\,|0> \tag{3.17}$$

$$\beta_1 < \beta_2 < \dots < \beta_r, \qquad \beta_{r+1} < \beta_{r+2} < \dots < \beta_n \ . \tag{3.18}$$

This prescription is immediately obtained by performing the massless limit described in the previous section on the massive asymptotic bases (1.26). The completeness relation becomes in this parameterization

$$\begin{aligned} 1 &= \sum_s |s><s| = \sum_n \int \left(\prod_i^n \frac{d\beta_i}{2\pi}\right) |\beta_1,\dots,\beta_n>_{in\ in}<\beta_1,\dots,\beta_n| \\ &= \sum_{r,l} \frac{1}{r!l!} \int_{-\infty}^{+\infty} \left(\prod_i^n \frac{d\beta_i}{2\pi}\right) |\beta_1,\dots\beta_r,\beta_{r+1},\dots,\beta_n><\beta_1,\dots\beta_r,\beta_{r+1},\dots,\beta_n| \ . \end{aligned} \tag{3.19}$$

The determination of the form factors relies on the existence of a set of functional relations which fix their analytic structure. In particular the monodromy properties are ruled by the so-called Watson equations (1.50) which follow from unitarity and crossing symmetry and in the rapidity parameterization exhibit the same form as in the massive case:

$$F_{\alpha_1\ldots\alpha_i\alpha_{i+1}\ldots\alpha_n}(\beta_1,\ldots,\beta_i,\beta_{i+1},\ldots,\beta_n) = S_{\alpha_i\alpha_{i+1}}(\beta_i-\beta_{i+1}) \; F_{\alpha_1\ldots\alpha_{i+1}\alpha_i\ldots\alpha_n}(\beta_1,\ldots,\beta_{i+1},\beta_i,\ldots,\beta_n) \tag{3.20}$$

$$F_{\alpha_1\ldots\alpha_n}(\beta_1,\ldots,\beta_n+2\pi i) = F_{\alpha_n\alpha_1\ldots\alpha_{n-1}}(\beta_n,\beta_1,\ldots,\beta_{n-1})\,. \tag{3.21}$$

In the massive case the form factors are meromorphic functions of the rapidity variables whose only singularities are simple poles of two different kinds:

a) bound states poles. They show up whenever the massive scattering amplitude $S(\theta)$ exhibits a pole with residue Γ in the physical strip $0<\theta<i\pi$ at $\theta=iu$;

b) kinematical poles. They signal the occurrence of the particular kinematical configuration in which a particle and its antiparticle enter the form factor with a rapidity difference equal to $i\pi$, which corresponds to opposite bidimensional momenta.

The residue equations corresponding to these singularities give rise to two recursive relations which remains unchanged in the massless limit as far as the RR and LL channels are concerned

$$i\lim_{\varepsilon\to 0}\varepsilon F_{\alpha_1\ldots\alpha_n\alpha\alpha}(\beta_1,\ldots,\beta_n,\beta+i\bar{u}-\frac{\varepsilon}{2},\beta-i\bar{u}+\frac{\varepsilon}{2}) = \Gamma F_{\alpha_1\ldots\alpha_n\alpha}(\beta_1,\ldots,\beta_n,\beta) \tag{3.22}$$

$$-\,i\lim_{\tilde{\beta}\to\beta}(\tilde{\beta}-\beta)\,F_{\alpha\alpha\alpha_1\ldots\alpha_n}(\tilde{\beta}+i\pi,\beta,\beta_1,\ldots,\beta_n) = \left(1-\prod_{i=1}^{n}S_{\alpha\alpha_i}(\beta-\beta_i)\right)F_{\alpha_1\ldots\alpha_n}(\beta_1,\ldots,\beta_n)\,. \tag{3.23}$$

Concerning the RL channel, we simply observe that no such singularities can occur since there is no place on the real s-axis where such bound states could be located, according to the massless limit argument. This impossibility translates the physical fact that a ϕ^3 property can not be accomplished between a right and a left mover without breaking parity. Furthermore a pole on the first sheet of the massless s-plane would automatically violate unitarity (resonances are second sheet poles). Kinematical poles also are absent in this channel because a RL pair is not a particle-antiparticle pair.

Under a Lorentz transformation, corresponding to the rapidity shift $\beta_i\to\beta_i+\Lambda$, the form factors of a spin s operator transform as

$$F_{\alpha_1\ldots\alpha_n}(\beta_1+\Lambda,\ldots,\beta_n+\Lambda) = e^{s\Lambda}F_{\alpha_1\ldots\alpha_n}(\beta_1,\ldots,\beta_n)\,. \tag{3.24}$$

According to this, the form factors of a scalar operator will only depend on the difference of the rapidities.

The parity transformation reverse the direction of the particles: therefore a left mover becomes a right mover and vice versa. Therefore parity invariance requires right-left symmetry

$$F_{\mathcal{P}[\alpha_n]\ldots\mathcal{P}[\alpha_2]\mathcal{P}[\alpha_1]}(-\beta_n,\ldots,-\beta_2,-\beta_1) \;=\; \pm F_{\alpha_1\ldots\alpha_n}(\beta_1,\ldots,\beta_n)\;, \tag{3.25}$$

where $\mathcal{P}[R]=L$, $\mathcal{P}[L]=R$ and the $\pm$ refers to the intrinsic parity of the operator.

3.3 A massless flow: from tricritical Ising model to Ising model

It is well known that the perturbation of the minimal unitary model $\mathcal{M}_p$ with central charge $c_p = 1-\frac{6}{p(p+1)}$ by the relevant operator $\Phi_{1,3}$ leads (for positive values of the coupling constant) to a massless integrable theory flowing to the model $\mathcal{M}_{p-1}$ in the infrared limit [39]. The simplest among these flows connects the tricritical Ising model ($p=4$) to the critical Ising model ($p=3$). The tricritical Ising model anomalous dimensions are shown in Table 3.1. The corresponding closed algebra of semilocal primary fields is

$$\mathcal{A}_{TIM} \;=\; [I]\oplus[\sigma]\oplus[\bar{\sigma}]\oplus[\mu]\oplus[\bar{\mu}]\oplus[\psi]\oplus[\bar{\psi}]\oplus[G]\oplus[\bar{G}]\oplus[\varepsilon]\oplus[\bar{\varepsilon}]\oplus[X] \tag{3.26}$$

where the anomalous dimension assignments can be found in Table 3.2 , together with the fusion rules. As one can check by inspection, this fusion rules satisfy two $\mathcal{Z}_2$ discrete symmetries. The first, which we call the $\mathcal{Z}_2$-spin symmetry, is implemented by the following transformations

$$\sigma\to-\sigma,\;\; \bar{\sigma}\to-\bar{\sigma},\;\; \mu\to\mu,\;\; \bar{\mu}\to\bar{\mu}, \tag{3.27}$$

$$\varepsilon\to\varepsilon,\;\; \bar{\varepsilon}\to\bar{\varepsilon},\;\; X\to X, \tag{3.28}$$

$$\psi\to-\psi,\;\; \bar{\psi}\to-\bar{\psi},\;\; G\to-G,\;\; \bar{G}\to-\bar{G}\;. \tag{3.29}$$

Therefore to each primary field is associated an even or odd chirality. Even chiral fields are $\mu,\bar{\mu},\varepsilon,\bar{\varepsilon},X$. Odd chiral fields are $\sigma,\bar{\sigma},\psi,\bar{\psi},G,\bar{g}$. The $\mathcal{Z}_2$-dual symmetry is given by the rules

$$\sigma\to\mu,\;\; \bar{\sigma}\to\bar{\mu},\;\; \mu\to\sigma,\;\; \bar{\mu}\to\bar{\sigma}, \tag{3.30}$$

$$\varepsilon\to-\varepsilon,\;\; \bar{\varepsilon}\to\bar{\varepsilon},\;\; X\to-X, \tag{3.31}$$

$$\psi\to\bar{\psi},\;\; \bar{\psi}\to\psi,\;\; G\to-\bar{G},\;\; \bar{G}\to-G\;. \tag{3.32}$$

The tricritical semilocal algebra of primaries (3.26) admits maximal sections of mutually local fields. The Neveau-Schwarz section [73] is

$$\mathcal{A}_{NS} \;=\; [I]\oplus[G]\oplus[\bar{G}]\oplus[X]\oplus[\varepsilon]\oplus[\psi]\oplus[\bar{\psi}]\oplus[\bar{\varepsilon}] \tag{3.33}$$

while the so called "spin models" are

$$\mathcal{A}_\sigma = [I] \oplus [\sigma] \oplus [\tilde{\sigma}] \oplus [\varepsilon] \oplus [\tilde{\varepsilon}] \oplus [X] \tag{3.34}$$

$$\mathcal{A}_\mu = [I] \oplus [\mu] \oplus [\tilde{\mu}] \oplus [\varepsilon] \oplus [\tilde{\varepsilon}] \oplus [X] \,. \tag{3.35}$$

As one can learn from the fusion rules, these three local sections, are not mutually local. While the $\mathcal{Z}_2$-spin symmetry leaves each of the above local sections invariant, the $\mathcal{Z}_2$-duality maps the spin models (3.34), (3.35) one into each other.

The tricritical Ising model is also the first model of the discrete unitary series of superconformal field theories [73]. The superconformal symmetry is generated by the superconformal stress-energy tensor

$$\mathcal{T}(z) = G(z) + \theta T(z) \tag{3.36}$$

$$\bar{\mathcal{T}}(\bar{z}) = \bar{G}(\bar{z}) + \bar{\theta}\bar{T}(\bar{z}) \,. \tag{3.37}$$

The θ and $\bar{\theta}$ denote the grassmannian part of the superspace coordinates. The Fourier components of (3.36) in cylindrical coordinates with periodic boundary conditions[2], generate the holomorphic Ramond algebra. The antiperiodic boundary conditions generate the holomorphic Neveau-Schwarz algebra. The same is obviously true for the antiholomorphic component (3.37).

The spectrum of anomalous dimensions of the superprimary fields is shown in Table 3.3 . The NS-multiplets are

$$\begin{aligned} \{I\}_{NS} &= [I] \oplus [G] \oplus [\bar{G}] \oplus [X] \\ \{\Phi\}_{NS} &= [\varepsilon] \oplus [\psi] \oplus [\bar{\psi}] \oplus [\tilde{\varepsilon}] \,. \end{aligned} \tag{3.38}$$

The Ramond algebra is represented on the combinations of order and disorder operators

$$\begin{aligned} \left\{\frac{3}{80}\right\}_R &= [\sigma] \oplus [\mu] \\ \left\{\frac{7}{16}\right\}_R &= [\tilde{\sigma}] \oplus [\tilde{\mu}] \,. \end{aligned} \tag{3.39}$$

Since the supersymmetric charge $Q \equiv G_0^2 = L_0 - c/24$ acts on the Ramond lowest weight vectors $|\sigma >= \sigma(0,0)|0>$, $|\mu >= \mu(0,0)|0>$ as $Q|\sigma >= |\mu>$, there is the possibility for spontaneous supersymmetry breaking [74].

[2] Periodic and anti-periodic boundary conditions refers to the space direction. In the time direction we consider periodic boundary conditions.

It was shown in [71] that the flow we want to describe is actually associated to the spontaneous breaking of the supersymmetry of $\mathcal{M}_4$ and it is described by an effective lagrangian whose infrared expansion is

$$\mathcal{L} = \psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi} - \frac{4}{M^2}(\psi\partial\psi)(\bar{\psi}\bar{\partial}\bar{\psi}) + \dots \quad (3.40)$$

When the mass scale M goes to infinity (infrared limit) the lagrangian of a free massless majorana fermion characteristic of Ising model is recovered. The massless majorana fermion plays the role of the goldstino of the supersymmetry breaking. The universality class of the tricritical Ising model depicted in the phase diagram of Fig. 3.1 is described by the explicitly supersymmetric action

$$\begin{aligned} S_{LG} &= \int d^2z d^2\theta \left[\frac{1}{2}D\Phi\bar{D}\Phi + \frac{g}{3}\Phi^3 + \lambda\Phi\right] \\ &= \int d^2z \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\bar{\psi}\partial\bar{\psi} + \frac{1}{2}\psi\bar{\partial}\psi - \frac{1}{2}(\lambda + g\phi^2)^2 - g\bar{\psi}\psi\phi\right] , \end{aligned} \quad (3.41)$$

where the superfield is $\Phi = \phi + \theta\psi + \bar{\theta}\bar{\psi} + \theta\bar{\theta}F$. A fine tuning of the couplings allows us to describe an effective potential $V_{eff} = 1/2(\lambda + g\phi^2)^2$ that, for[3] $\lambda/g > 0$, has a unique vacuum $\phi = 0$ which preserves the $\mathcal{Z}_2$-spin symmetry $\phi \to -\phi$ and breaks supersymmetry $m_\phi = \sqrt{2g\lambda}$, $m_\psi = 0$. Integrating out the massive boson degree of freedom we end up with the lagrangian (3.40). It looks as a non-renormalizable perturbation of the Ising model. The predictivity of the theory is guaranteed by the integrability requirement: it eliminates the infinite arbitrariness of the renormalization counterterms. In Table 3.4 we have listed the Virasoro anomalous dimensions of the Ising model. The algebra of semilocal primary fields is

$$\mathcal{A}_{IM} = [I] \oplus [\sigma] \oplus [\mu] \oplus [\psi] \oplus [\bar{\psi}] \oplus [\varepsilon] \quad (3.42)$$

where the primary fields identification is in Table 3.5, together with their operator product expansion rules. Analogously to the tricritical Ising model the fusion rules are invariant under two $\mathcal{Z}_2$ transformations. They are the $\mathcal{Z}_2$-spin symmetry

$$\sigma \to -\sigma, \ \mu \to \mu, \ \psi \to -\psi, \ \bar{\psi} \to -\bar{\psi}, \ \varepsilon \to \varepsilon \quad (3.43)$$

and the $\mathcal{Z}_2$-dual antiunitary symmetry

$$\sigma \to \mu, \ \mu \to \sigma, \ \psi \to \bar{\psi}, \ \bar{\psi} \to \psi, \ \varepsilon \to -\varepsilon \quad (3.44)$$

[3] The other direction of the perturbation $\Phi_{(1,3)}$, that is $\lambda/g < 0$, leads to the massive integrable flow describing the scattering of three kinks [75, 13, 70].

which is the famous Kramers-Wanniers duality. Therefore the integrable flow between the tricritical Ising model and the Ising model must exhibit the same set of discrete symmetries. The analogy between the two conformal models extends also to the operator structure. Also the Ising model semilocal algebra (3.42) admits maximal local sections which are not mutually local. The majorana subalgebra is

$$\mathcal{A}_M = [I] \oplus [\psi] \oplus [\bar{\psi}] \oplus [\varepsilon] \tag{3.45}$$

while the so called "spin models" are

$$\mathcal{A}_\sigma = [I] \oplus [\sigma] \oplus [\varepsilon] \tag{3.46}$$

$$\mathcal{A}_\mu = [I] \oplus [\mu] \oplus [\varepsilon] \ . \tag{3.47}$$

The massless scattering theory for the flow between $\mathcal{M}_4$ and $\mathcal{M}_3$ has been derived in [69]. The particle spectrum coincide with the one of the infrared fixed point and consists of a single type of right and left mover neutral fermions. As a consequence

$$S_{RR}(\beta_1 - \beta_2) = S_{LL}(\beta_1 - \beta_2) = -1 \tag{3.48}$$

while conformal invariance is broken by the rapidity dependent RL scattering amplitude

$$S_{RL}(\beta_1 - \beta_2) = \tanh\left(\frac{\beta_1 - \beta_2}{2} - \frac{i\pi}{4}\right) \tag{3.49}$$

which also reduces to -1 in the infrared limit $\beta_1 - \beta_2 \to -\infty$. The pole at $\beta_1 - \beta_2 = -i\pi/2$ lies on the unphysical sheet and can be thought of as a resonance at $s = -iM^2$. The scattering amplitude (3.49) has been checked in [69] by means of perturbation theory with the lagrangian (3.40).

We now turn to the computation of form factors using the general properties exposed in the previous chapters. In order to simplify the notation, let us consider the following subset of form factors:

$$\begin{aligned} F_{r,l}(\beta_1, \beta_2, \ldots, \beta_r; \beta'_1, \beta'_2, \ldots, \beta'_l) &= \\ < 0|O(0)\, A_R(\beta_1) A_R(\beta_2) \ldots A_R(\beta_r) A_L(\beta'_1) A_L(\beta'_2) \ldots A_L(\beta'_l)\, |0> \ . \end{aligned} \tag{3.50}$$

Any other form factor of the local operator $O(x)$ can be obtained from these using eq.(3.20).

We parameterize the functions (3.50) as

$$\begin{aligned} F_{r,l}(\beta_1, \beta_2, \ldots, \beta_r; \beta'_1, \beta'_2, \ldots, \beta'_l) &= H_{r,l} Q_{r,l}(x_1, x_2, \ldots, x_r; y_1, y_2, \ldots, y_l) \\ \prod_{1 \le i < j \le r} \frac{f_{RR}(\beta_i - \beta_j)}{x_i + x_j} \prod_{i=1}^{r} \prod_{j=1}^{l} f_{RL}(\beta_i - \beta'_j) \prod_{1 \le i < j \le l} &\frac{f_{LL}(\beta'_i - \beta'_j)}{y_i + y_j} \ , \end{aligned} \tag{3.51}$$

where $x_i \equiv e^{\beta_i}$, $y_i \equiv e^{-\beta_i'}$ and $H_{r,l}$ are normalization constants introduced for later convenience. The auxiliary functions f_{RR}, f_{LL} and f_{RL} satisfy the following equations obtained combining eqs. (3.20), (3.21) for a scalar operator with $n = 2$

$$f_{\alpha_1\alpha_2}(\beta) = S_{\alpha_1\alpha_2}(\beta) f_{\alpha_1\alpha_2}(\beta + 2\pi i) \tag{3.52}$$

Given the scattering amplitudes (3.48), (3.49) we choose the following solutions with neither poles nor zeroes in the strip $0 < \text{Im}\beta < 2\pi$

$$f_{RR}(\beta) = f_{LL}(\beta) = \sinh\frac{\beta}{2} \tag{3.53}$$

$$f_{RL}(\beta) = \exp\left(\frac{\beta}{4} - \int_0^\infty \frac{dt}{t} \frac{\sin^2\left(\frac{(i\pi-\beta)t}{2\pi}\right)}{\sinh t \cosh\frac{t}{2}}\right) . \tag{3.54}$$

The function $f_{RL}(\beta)$ satisfies the equation

$$f_{RL}(\beta \pm i\pi) f_{RL}(\beta) = \frac{i\gamma}{1 \pm ie^{-\beta}} , \tag{3.55}$$

where $\gamma = \sqrt{2}e^{2G/\pi}$, G being the Catalan costant. The functions (3.53), (3.54) completely take into account the monodromy properties in (3.51). Since the scattering amplitudes S_{RR} and S_{LL} in this theory are free of poles, only kinematical poles appear in the form factors and are explicitely inserted in the parameterization (3.51) through the factors $x_i + x_j$ and $y_i + y_j$ in the denominator. As a consequence, after requiring the form factors to be power bounded in the momenta, $Q_{r,l}$ have to be rational functions separately symmetric in the $\{x_i\}$ and $\{y_i\}$ with at most poles located at $x_i = 0$ or $y_i = 0$. Furthermore, parity invariance (3.25) will require

$$Q_{r,l}(\{x_i\};\{y_i\}) = \pm Q_{l,r}(\{y_i\};\{x_i\}) . \tag{3.56}$$

By filling the parameterization (3.51) into the residue equations (3.23) we find the recursive relations satisfied by $Q_{r,l}$

$$Q_{r+2,l}(-x,x,x_1,\ldots,x_r;y_1,\ldots,y_l) = x^{r-l+1}\frac{\rho_r(\{x_i\})}{\lambda_l(\{y_i\})} \sum_{k=0}^{l\,\prime} (-ix)^k \lambda_k(\{y_i\}) Q_{r,l}(x_1,\ldots,x_r;y_1,\ldots,y_l) \tag{3.57}$$

$$Q_{r,l+2}(x_1,\ldots,x_r;y_1,\ldots,y_l,y,-y) = y^{l-r+1}\frac{\lambda_l(\{y_i\})}{\rho_r(\{x_i\})} \sum_{k=0}^{r\,\prime} (-iy)^k \rho_k(\{x_i\}) Q_{r,l}(x_1,\ldots,x_r;y_1,\ldots,y_l) , \tag{3.58}$$

where the primed sums run over odd indices if $(r+l)$ is even and vice versa, and $\rho_k(\{x_i\})$ is the base of symmetric polynomials in the variables $\{x_i\}$ generated by

$$\prod_{j=1}^{r}(x+x_j)=\sum_{k=0}^{r}x^{r-k}\rho_k(\{x_i\}) \tag{3.59}$$

(analogously for $\lambda_k(\{y_i\})$). In writing eqs. (3.57), (3.58) we have chosen

$$H_{r,l}=-\frac{\gamma^l}{i^r 2^{2r+1}}H_{r+2,l}=-\frac{\gamma^r}{i^l 2^{2l+1}}H_{r,l+2}\ . \tag{3.60}$$

So far we have made no reference to any particular operator; therefore the general solution of eqs. (3.57), (3.58) and (3.60) would provide a complete classification of the operators of the theory local with respect to the field which creates the particles. In the following we will restrict our attention to some operators of particular physical relevance, namely the trace of the stress-energy tensor $\Theta(x)=T^{\mu}_{\mu}(x)=\tilde{\varepsilon}(x)$ and the energy field $\varepsilon(x)$, the magnetization (order) operator $\sigma(x)$ and the disorder operator $\mu(x)$ for the spin sector. With these symbols we denote the fields which are denoted with the same symbol in the tricritical Ising model and which are supposed to flow in the Ising model fields with the same name. This hypothesys is positively checked through the comparison between the short (long) distance behaviour of the correlation functions and the prescribed anomalous dimensions. Since all these fields are spinless, eq.(3.24) implies that under a Lorentz transformation the functions $Q_{r,l}(\{x_i\};\{y_i\})$ behave as

$$Q_{r,l}(\{e^{\Lambda}x_i\};\{e^{-\Lambda}y_i\}) = e^{\left(\frac{r(r-1)}{2}-\frac{l(l-1)}{2}\right)\Lambda}\,Q_{r,l}(\{x_i\};\{y_i\})\ . \tag{3.61}$$

3.3.1 The stress-energy tensor: form factors and the correlation function

The form factors of the trace of the stress-energy tensor are selected by the conservation law $\partial_\mu T^{\mu}_{\nu}=0$ which implies the factorization (see chapter 1)

$$Q_{r,l}(\{x_i\};\{y_i\}) = \rho_1\lambda_1 T_{r,l}(\{x_i\};\{y_i\})\ . \tag{3.62}$$

Since the lagrangian (3.40) is Z_2-symmetric in both ψ and $\bar{\psi}$, Θ will have nonvanishing form factors $F_{r,l}$ only for even r and l, starting from $r=l=2$. Inserting eq. (3.62) into (3.57), (3.58) we get the recursive equations for $T_{r,l}$

$$\begin{aligned}T_{r+2,l}(-x,x,\{x_i\};\{y_i\}) &= x^{r-l+1}\frac{\rho_r}{\lambda_l}\sum_{k=0}^{l}{}'(-ix)^k\lambda_k\,T_{r,l}(\{x_i\};\{y_i\})\\ T_{r,l+2}(\{x_i\};\{y_i\},y,-y) &= y^{l-r+1}\frac{\lambda_l}{\rho_r}\sum_{k=0}^{r}{}'(-iy)^k\rho_k\,T_{r,l}(\{x_i\};\{y_i\})\ .\end{aligned} \tag{3.63}$$

The leading infrared contribution to $F_{2,2}$ is easily computed from (3.40)

$$F_{2,2}(\beta_1,\beta_2;\beta_1',\beta_2') \simeq -4\pi M^2 \sinh\frac{\beta_1-\beta_2}{2} \sinh\frac{\beta_1'-\beta_2'}{2}\, e^{\beta_1+\beta_2-\beta_1'-\beta_2'} . \qquad (3.64)$$

With this information we fix the exact $F_{2,2}$ to be

$$F_{2,2}(\beta_1,\beta_2;\beta_1',\beta_2') = \frac{4\pi M^2}{\gamma^2} \sinh\frac{\beta_1-\beta_2}{2} \prod_{i,j=1,2} f_{RL}(\beta_i-\beta_j') \sinh\frac{\beta_1'-\beta_2'}{2} . \qquad (3.65)$$

The recursive equations (3.63), (3.60) are then iteratively solved by using (3.65) as initial condition. With the normalization constant

$$H_{2n,2m} = \pi M^2 (-1)^{n+m} i^{n(n-1)+m(m-1)} 2^{2(n^2+m^2)-n-m} \gamma^{-2nm} \qquad (3.66)$$

the first right chains are determined to be

$$T_{2n,2}(\{x_i\};\{y_i\}) = (i)^{n^2-1} \left(\rho_{2n}\frac{\lambda_1}{\lambda_2}\right)^{n-1} \qquad (3.67)$$

$$T_{2n,4}(\{x_i\};\{y_i\}) = (i)^{n^2-2n} \left(\frac{\rho_{2n}}{\lambda_4}\right)^{n-2} \sum_{k=0}^{n-1} \rho_{2k+1}\lambda_1^{n-1-k}\lambda_3^k . \qquad (3.68)$$

Using the right-left symmetry relation $T_{r,l}(\{x_i\};\{y_i\}) = T_{l,r}(\{y_i\};\{x_i\})$ one can immediately obtain the solution for the corresponding left chains.

As a first check of the solution we give for the trace of the stress-energy tensor we have computed the difference of central charges $c_{uv} - c_{ir} = 0.2$ with Cardy's formula (1.69). The first nonvanishing contribution coming from the form factor $F_{2,2}$ is $\Delta_4 = 0.196001$ and is already very near to the expected value. The second contribution $\Delta_6 = 0.00347$ gives the indication that the expected value 0.2 can be saturated with few contributions. This fast convergence is not surprising: in Cardy's formula (1.69) the ultraviolet contributions are suppressed in the integration.

We have also computed te correlation function $< \Theta(x)\Theta(0) >$. Again the first two contribution are enough to exhibit the good scaling dimensions at short distance $h = 3/5$ and at large distance $h = 2$, as one can see from Fig. 3.2 . This operator at large distance is easily expressed in terms of the Ising free majorana fermion. This is the reason why the computation of the correlation function in this case is not plagued by infrared divergencies.

3.3.2 The energy operator: form factors and the correlation function

The energy operator $\varepsilon(x)$ has the same characteristic of the trace operator of having an easy infrared expression in terms of the free Ising field, $\varepsilon \simeq \bar{\psi}\psi$. Therefore the correlation function can be computed with no difficulties. Since it is odd under duality transformation its non-vanishing form factors will be only the $F_{2n+1,2m+1}$. The initial condition is thus $F_{1,1}(\beta) = H_{1,1} f_{RL}(\beta)$. The normalization constant $H_{1,1}$ can be computed in the infrared limit with the lagrangian (3.40). The next form factors are

$$Q_{3,1} = iH_{1,1}\rho_3 \, , \; Q_{1,3} = iH_{1,1}\lambda_3 \, . \tag{3.69}$$

In terms of these the correlation function is computed and is shown in Fig. 3.3 . The infrared $h = 1/2$ and ultraviolet $h = 1/10$ are thus recovered.

3.3.3 The magnetization and disorder operators: form factors and the correlation function

The magnetization (order) operator $\sigma(x)$ is odd under the trasformation $\psi \to -\psi$; thus it has nonvanishing form factors only on an odd number of particles. Taking $F_{1,0} = F_{0,1} = 1$ as initial conditions for the recursive equations, we get

$$H_{r,l} = i^{\frac{r^2+l^2+2r+2l-3}{4}} 2^{\frac{r^2+l^2-r-l}{2}} \gamma^{-\frac{rl}{2}} \tag{3.70}$$

$$Q_{r,0} = (i)^{\frac{r^2-1}{4}} \rho_r^{(r-1)/2} \tag{3.71}$$

$$Q_{r,1} = (i)^{\frac{r^2-2r}{4}} \frac{\rho_r^{r/2-1}}{\lambda_1^{r/2}} \tag{3.72}$$

$$Q_{r,2} = (i)^{\frac{r^2-4r+3}{4}} \rho_r^{(r-3)/2} \sum_{k=0}^{r}{}' \rho_k \lambda_2^{(k-r+1)/2} \tag{3.73}$$

$$Q_{r,3} = (i)^{\frac{r^2-6r+8}{4}} \frac{\rho_r^{(r/2-2)}}{\lambda_3^{r/2-1}} \sum_{k=0}^{r}{}' \rho_k \lambda_2^{k/2} \, , \tag{3.74}$$

where the prime denotes sum over even indices only and r should be chosen such that $r + l$ is odd.

The disorder operator μ is nonlocal with respect to the magnetization. This circumstance induces a slight modification in equation (3.23): the minus sign in front of the product becomes plus [33, 34]. One can easily check that the functions $Q_{r,l}$ for $\mu(x)$ derived from the modified recursive equation are still given by the formulas (3.74) with $r + l$ even ($\mu(x)$ is Z_2-spin even).

Since these operators are very complicated also in the infrared, due to the nonlocal relation with the free fermion field, the computation of the correlation function is not so direct as for the "energy" operators. Infrared divergencies naturally occur and so physical properties can be recovered after resummation of the form factor series. As an example we compute the infrared anomalous dimension $h_\sigma = h_\mu = 1/16$. In order to be concrete, first let us note that the infrared limit of the form factors of these two operators are

$$F^{ir}_{r,l} = \prod_{i<j}^{r} \tanh\frac{\beta_{ij}}{2} \prod_{i<j}^{l} \tanh\frac{\beta_{ij}}{2} \equiv F_r F_l \tag{3.75}$$

so that the sum of the σ and the μ correlation functions looks like the product of grandcanonical partition functions of two identical gases [33]

$$G_\sigma(t) + G_\mu(t) = Z_R Z_L \tag{3.76}$$

$$Z_R = \sum_r \frac{1}{r!} \int_{-1/\epsilon}^{+\infty} \prod_i^r \frac{d\beta_i}{2\pi} e^{-t(e^{\beta_1}+\ldots+e^{\beta_r})} |F_r|^2 \tag{3.77}$$

$$Z_L = \sum_l \frac{1}{l!} \int_{-\infty}^{1/\epsilon} \prod_i^l \frac{d\beta_i}{2\pi} e^{-t(e^{-\beta_1}+\ldots+e^{-\beta_l})} |F_l|^2 \tag{3.78}$$

$$Z_R = Z_L = \sum_n \frac{1}{n!} \int_{-\infty}^{+\infty} \prod_i^n \frac{d\beta_i}{2\pi} e^{-te^{-\frac{1}{\epsilon}}(e^{\beta_1}+\ldots+e^{\beta_n})} |F_n|^2 , \tag{3.79}$$

where $t = mr/2$. The gas is of one-dimensional particles which interact with the potential energy $V(\beta_{ij}) = -\log(\tanh^2(\beta_{ij}))$ and with a coordinate dependent chemical potential $z = \frac{1}{2\pi} e^{-te^{-\frac{1}{\epsilon}}(e^{\beta_1})}$. The limit $\epsilon \to 0$ in which we remove the infrared cutoff coincides with the thermodynamic limit of the gas in the volume $V = \log 1/t + 1/\epsilon$ with the constant chemical potential $z = 1/2\pi$. The cumulant expansion for the thermodynamic limit of $\log Z$ can be exactly summed to yield [33]

$$Z = \frac{e^{\frac{1}{8\epsilon}}}{t^{1/8}} \tag{3.80}$$

thus giving through (3.79) the correct behaviour $t^{1/4}$ of the magnetization (and of the disorder operator) correlation function.

As a final remark, we note that the infrared limit of the form factors (3.75) are solutions of the recursive equations at the Ising conformal point where the infrared RL-scattering amplitude is obviously $S^{ir}_{RL}(\beta) = -1$. Applying the same logic to the ultraviolet fixed point, we obtain as ultraviolet limits of the form factors of $\sigma(x)$ the functions

$$\begin{aligned} F^{uv}_{r,0} &= \prod_{i<j}^{r} \tanh\frac{\beta_{ij}}{2} \ \text{, r is odd ,} \\ F^{uv}_{0,l} &= \prod_{i<j}^{l} \tanh\frac{\beta_{ij}}{2} \ \text{, l is odd ,} \\ F^{uv}_{r,l} &= 0 \ \text{, otherwise ,} \end{aligned} \tag{3.81}$$

and for the disorder operator $\mu(x)$

$$\begin{aligned} F^{uv}_{r,l} &= \prod_{i<j}^{r} \tanh\frac{\beta_{ij}}{2} \prod_{i<j}^{l} \tanh\frac{\beta_{ij}}{2} \quad \text{,r and l are both even ,} \\ F^{uv}_{r,l} &= 0 \quad \text{, otherwise .} \end{aligned} \tag{3.82}$$

One can show that these form factors are solutions of the recursive equations at the conformal points with the RL scattering amplitude given by $S^{uv}_{RL}(\beta) = 1$. Recalling that the asymptotic particles are fermions, this scattering amplitude is not a free one, at any scale. So probably the conformal tricritical Ising point should allow a better form factor description in a different base. The computation of the ultraviolet anomalous dimension $h = 3/80$ along the same line used for the Ising point becomes here a hard task since the correlation function $G_\sigma + G_\mu$ is the grandcanonical partition function Z of a one-dimensional gas with interaction energy which is different whether there is an odd or an even number of particles. As a consequence, taking the thermodynamical limit with the usual cumulant expansion for the free energy $-\log Z$ is not enough, because it gives a series in the infinite volume which needs to be resummed in order to extract the effective leading linear term.

Chapter 4

Integrable flows with defect lines

In the past three chapters, methods and concepts of quantum field theory have been successfully applied to the analysis of homogeneous statistical models at and away from criticality. Homogeneous systems, however, are in many cases a mathematical idealization of the real physical samples which may present instead boundary effects and various types of inhomogeneity or defects. It is an interesting problem in statistical mechanics to estimate the influence of the inhomogeneities on the results obtained in pure cases and to develop the corresponding theory. With reference to systems with boundaries, they have been the subject of a wide investigation which has employed a large variety of techniques ([77]-[82]). The bootstrap approach recently developed has brought new light on the topic and has provided quite remarkable achievements in the understanding of quantum field theories with boundary ([83]-[89]). As we show in this chapter, bootstrap methods are also extremely efficient to describe integrable statistical models with extended lines of defects. Before developing the bootstrap theory, it is worth to briefly discuss general aspects of statistical models with lines of defect in order to gain some insight to their properties ([88]-[95]).

One of the main reasons for considering extended lines of inhomogeneities is that only such kind of defects may affect the critical properties of the pure systems. Indeed, in the opposite case where there are only a finite number of localized inhomogeneities in the lattice, they would be eventually neutralized by iterating the renormalization group transformations so that the regime of the pure model will definitely take over.

With reference to the discussion in the introduction of this thesis, these extended lines of inhomogeneities can also be considered as infinite-mass degrees of freedom. In other words they are the residual quantum degrees of freedom which are left in the infinite-mass limit along a massive direction of the space of models. This direction consists of a physical space dependent coupling constant, such that translation invariance is spoiled at the end of the limiting procedure. This limit is taken such that these degrees of freedom are prevented to propagate in physical space. Thus we are left with localized quantum degrees of freedom which can affect the dynamics of usual bulk particles.

Scaling considerations are also useful to understand in simple terms the continuum version of the models with an infinite line of defect and to show that they may interpolate between a bulk or a boundary statistical behaviour. For the sake of clarity, let us consider the simplest physical realization given by a system at temperature T in the bulk but heated at a different temperature $\tilde{T}$ along a line placed at the y axis. This system may be equally regarded as two semi-infinite copies of the model at temperature T coupled together through the energy density at the defect line. Its continuum properties are described by the euclidean action

$$\mathcal{A} = \mathcal{A}_B + g \int d^2r \, \delta(x) \, \epsilon(r) \,, \tag{4.1}$$

where $\mathcal{A}_B$ is the action relative to the bulk and $\epsilon(r)$ is the energy density with scaling dimension ν. The scaling dimension of the coupling constant $g = (\tilde{T} - T)$ is then given by $y_g = 1 - \nu$. Consequently, all those systems with an irrelevant energy operator of scaling dimension $\nu > 1$ will exhibit the bulk critical behaviour near a defect line. On the contrary, those models which have a relevant energy operator with $\nu < 1$ will present a surface critical behaviour. The reason is that, in the former case the effective coupling constant may become arbitrarly small and then the action reduces to that of the bulk theory, whereas, in the latter case it may take arbitrarly large values suppressing all the fluctuations across the defect line between the two semi-infinite copies which will eventually decouple.

An exception to the above pictures is given by the purely marginal case, i.e. $\nu = 1$ which is realized in the Ising model. The interesting result obtained in the past by Bariev [90] and McCoy and Perk [91] is that the model presents a non universal critical behaviour, with the critical indices of the magnetization operators continuously dependent on the parameter g of the action (4.1). The energy operator on the contrary remains a purely marginal operator for all values of the coupling constant g since its critical exponent ν is fixed at the Ising value of 1 [90, 91, 92, 93].

Let us now turn our attention to the bootstrap theory of the integrable statistical models with extended line of defect which was originally proposed in [97]. In the continuum limit and away from criticality, such a theory can be formulated in terms of scattering processes of the massive excitations which take place either in the bulk or on the defect line. Hence, in addition to the bulk scattering amplitudes, we have to consider a new set of amplitudes relative to the interaction of the particles with the defect line. Because of the integrability condition, they only reduce to reflection and transmission processes. It is then convenient to associate an extra operator D to the defect line and to formulate the dynamics in terms of algebraic relations which involve D and the operators $A_a^\dagger$ of the massive excitations. The consistency of this algebra provides the unitary equations for the transmission and reflection amplitudes while the associativity condition gives rise to a set of cubic relations called transmission-reflection equations. Although the general solution of these equations is still lacking, the Ising model with a line of defect can be identified as one particular solution of them. For this model, the particles in the bulk are given by the Ising massive majorana fermion. The availability of an exact resummation of the perturbative series in powers of the strength of the defect permits to test the complementary algebraic approach referred above. A similar approach is also used to discuss the case of free boson theory with a line of defect, which constitutes another solution of the reflection-transmission equations. The novelty of the model consists in the presence of resonance states and instability properties by varying the strength of the defect coupling.

As for the integrable theories in the bulk, we will show that also for the statistical models with a line of defect, the knowledge of the total set of scattering amplitudes permit the full reconstruction of the theory, i.e. the computation of the multipoint correlators. In particular, the aforementioned non-universal critical behaviour of the Ising model can be easily recovered by looking at the ultraviolet behaviour of the correlation functions.

The layout of this chapter is as follows. In the first section, we introduce the defect operator D and the relative algebra with the operators of the asymptotic particles $A_a^\dagger$. We initially derive the unitarity and crossing equations for the scattering amplitudes and then the transmission-reflection equations which express their factorization properties. In section 4.2, we discuss in detail one of the solutions of these equations, i.e. the Ising model with a line of defect. In section 4.3 we consider the behaviour of the bosonic theory when coupled to a defect line. Geometrical configurations with richer structure of inhomegeneities are considered in section 4.4. The computation of correlation functions of the model, based on the knowledge of the form factors of the theory in the bulk and on the matrix elements of the defect operator $\mathcal{D}$, is presented in section 4.5. Our conclusions are then summarized in section 4.6.

4.1 Defect algebra

In the bulk, the theory of two-dimensional integrable statistical models with a finite correlation length can be elegantly formulated in terms of an ensemble of particle excitations in bootstrap interaction. Although the bootstrap principle alone is in many cases sufficient to solve the dynamics, it is also useful to rely on a more conventional approach and to introduce an action that describes the interactions in the bulk

$$\mathcal{A}_B = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \, \mathcal{L}_B \left(\partial_\mu \phi_i, \phi_i\right) \quad . \tag{4.2}$$

In the following we assume we have completely solved the dynamics of the corresponding theory in the Minkoswki space and as a result, we know both the mass spectrum $\{m_a\}$ and the bulk scattering amplitudes $S_{ab}^{cd}(\beta_{ab})$ (see chapter 1). Let us now consider the presence of a defect line in the system, placed along the y-axis. The general action of the system is given in this case by

$$\mathcal{A} = \mathcal{A}_B + \int d^2 r \, \delta(x) \mathcal{L}_D \left(\phi_i, \frac{d\phi_i}{dy}\right) \quad . \tag{4.3}$$

The new interaction, responsible for scattering processes which take place on the defect line, will generally spoil the original integrability of the theory: particles which hit the defect with sufficient energy may excite internal degrees of freedom of the defect (being eventually absorbed by it), or may give rise to production processes with multiparticle states propagating through the two semi-infinite systems placed on the two sides of the defect line. However, assuming that the additional interaction along the defect line is still compatible with the existence of an infinite number of conserved charges in involution, the dynamics drastically simplifies and consequently is suitable for an exact analysis, as we show in the sequel.

By translation invariance along the y-direction (which we here identify with the time axis in the Minkowski space), for the theory described by the action (4.3) we still have the conservation of the energy but not of the momentum. Therefore we may have scattering processes with an exchange of momentum on the defect line, compatible though with the conservation of the energy. If in addition to the energy other higher charges are also conserved, the scattering processes at the defect line must be completely elastic. In particular, this means that a particle which hits the defect line with rapidity β can only proceed forward with the same rapidity or reverses its motion acquiring a rapidity of $-\beta$. A further effect of the interaction with the defect line may be a change of the label of the particle inside its multiplet of degeneracy. The interactions of the particles $\mid \beta; i >$ with the defect line will then be described in terms of the transmission and reflection amplitudes, denoted respectively by $T_{ij}(\beta)$ and $R_{ij}(\beta)$ (Fig. 4.1).

The interaction of the particles at the line of inhomogeneity may be encoded in a set of algebraic relations analogous to those which describe the scattering processes in the bulk. In order to illustrate this, an additional operator D associated to the defect line should be introduced in the theory[1]. This operator may be considered in relation to an additional particle state with zero rapidity in the entire time evolution of the system. Its commutation relations with the creation operators $A_i^\dagger(\beta)$ associated to the asymptotic particles in the bulk are given by

$$\begin{aligned} A_i^\dagger(\beta)\, D &= R_{ij}(\beta)\, A_j^\dagger(-\beta)\, D + T_{ij}(\beta)\, D\, A_j^\dagger(\beta) \;; \\ D\, A_i^\dagger(\beta) &= R_{ij}(-\beta)\, D\, A_j^\dagger(-\beta) + T_{ij}(-\beta)\, A_j^\dagger(\beta)\, D \;. \end{aligned} \tag{4.4}$$

The first of these equations expresses the scattering of a particle that hits the defect coming from the semi-infinite system on the left hand side with rapidity β. The second of (4.4) is obtained by an analytic continuation $\beta \to -\beta$ of the scattering amplitudes of a particle that approaches the defect coming from the semi-infinite system on the right hand side. The consistency condition of this algebra requires the unitarity equations

$$\begin{aligned} R_{ij}(\beta)\, R_{jk}(-\beta) + T_{ij}(\beta)\, T_{jk}(-\beta) &= \delta_{ik} \;; \\ R_i^j(\beta)\, T_j^k(-\beta) + T_i^j(\beta)\, R_j^k(-\beta) &= 0 \;. \end{aligned} \tag{4.5}$$

Additional constraints emerge from the crossing relations

$$\begin{aligned} R_{ij}(\beta) &= S_{\bar{j}l}^{k,\bar{i}}(2\beta)\, R_{kl}(i\pi - \beta) \;; \\ T_{i\bar{j}}(\beta) &= T_{ij}(i\pi - \beta) \;. \end{aligned} \tag{4.6}$$

The first equation in (4.6) is obtained according to the argument proposed in [83, 84] which exploits the quantization of the theory in the scheme where the time axis is placed along the x-axis. With reference to the second equation in (4.6), the transmission channel of the process shares the same properties of ordinary scattering in the bulk, the only difference being the occurrence of the particle D with zero rapidity. Thus, it is natural to assume that in the transmission channel the crossing symmetry is implemented in the usual way. We will assume the validity of eqs. (4.6) and will check that they are actually satisfied each time we will provide explicit solutions of the scattering theories with a line of defect.

Usually the presence of an infinite number of integrals of motion implies not only the elasticity of all scattering processes but also their complete factorization, i.e. an n-particle scattering amplitude can be entirely expressed in terms of the elementary two-body interactions [11, 12].

[1] For simplicity, we discuss the case of a defect without internal degrees of freedom and therefore carrying no additional indices. The formulation of the more generale case is straightforward.

A crucial step for proving the factorization property of the total S-matrix is to impose the associativity condition of the algebra (4.4). In terms of physical process, this means that we prepare initially an asymptotic two-particle state consisting of $| A_i^\dagger(\beta_1)\, A_j^\dagger(\beta_2) >$ with $\beta_1 > \beta_2$, and we let it scatter with the defect particle D with zero rapidity. The final output of the process should be independent from the temporal sequence of the elementary two-body interactions. Although what we have just described looks like an ordinary three-body process of the type that occurs in the bulk, there is however one distinguishing feature. In fact, in the three-body processes which take place in the bulk, given an initial state $| A_i^\dagger(\beta_1) A_j^\dagger(\beta_2) A_k^\dagger(\beta_3) >$ identified by a set of three ordered rapidities $\beta_1 > \beta_2 > \beta_3$, there is an *unique* final state given by the reverse ordering of the rapidities and possible exchange of the internal indices among the particles. On the contrary, for the scattering processes on the defect line we may have four possible final states, namely: (a) the state with both particles reflected by the defect line; (b) the state with both particles transmitted; (c and d) the states with one particle reflected whereas the other one transmitted. The final states may also differ from the initial one for the exchange of the internal indices and the above four possibilities give rise to a set of reflection-transmission equations shown in Fig. 4.2.

The first of these (Fig. 4.2.a) coincides with the well-known boundary equations already analysed in [81, 82, 83],

$$S_{ac}^{ef}(\beta_1 - \beta_2) R_{fg}(\beta_2) S_{ge}^{dh}(\beta_1 + \beta_2) R_{gb}(\beta_1) \;=\; R_{ah}(\beta_2) S_{ch}^{fe}(\beta_2 + \beta_1) R_{fg}(\beta_2) S_{eg}^{bd}(\beta_1 - \beta_2) \;. \quad (4.7)$$

The reflection-transmission equations associated with the configurations of Figs. (4.2.b), (4.2.c) and (2.d) are given respectively by

$$\begin{aligned} S_{ac}^{lm}(\beta_1 - \beta_2)\, T_{lb}(\beta_1)\, T_{md}(\beta_2) \;&=\; S_{ml}^{bd}(\beta_1 - \beta_2)\, T_{cm}(\beta_2)\, T_{dl}(\beta_1) \;; \\ S_{ac}^{fe}(\beta_1 - \beta_2)\, T_{fb}(\beta_1)\, R_{ed}(\beta_2) \;&=\; R_{ce}(\beta_2)\, S_{ae}^{fd}(\beta_1 + \beta_2)\, T_{fb}(\beta_1) \;; \\ S_{ac}^{fe}(\beta_1 - \beta_2)\, R_{fg}(\beta_2)\, S_{ge}^{dh}(\beta_1 + \beta_2)\, T_{hb}(\beta_1) \;&=\; T_{ab}(\beta_1)\, R_{cd}(\beta_2) \;. \end{aligned} \quad (4.8)$$

Although a general solution of these equations is still lacking, it is easy to see that they become extremely restrictive once applied to quantum field theories with a non-degenerate spectrum, i.e. those which have a diagonal S-matrix in the bulk. In fact, whereas eq. (4.7) and the first in (4.8) are identically satisfied, the last two equations in (4.8) become in this case

$$\begin{aligned} S_{ab}(\beta_a + \beta_b) \;&=\; S_{ab}(\beta_b - \beta_a) \;, \\ S_{ab}(\beta_a + \beta_b)\, S_{ab}(\beta_a - \beta_b) \;&=\; 1 \;, \end{aligned} \quad (4.9)$$

Hence, from the first equation in (4.9) we see that the S-matrix in the bulk has to be a constant and from the second equation (or equivalently from the unitarity condition) this constant is fixed to be ± 1. Thus we conclude that the only integrable quantum field theories with diagonal S-matrix in the bulk and factorizable scattering in the presence of the defect line are those associated to generalized-free theories.

Obviously this restriction on the bulk S-matrix does not apply when one considers purely reflective theories because they are ruled only by equations (4.7). Non-trivial solutions of these equations have been analysed for several models and they provide explicit examples of quantum field theories with boundary ([83]-[89]), some of them of relevant importance in statistical mechanics.

4.2 Ising model with a line of defect

As we have seen at the end of the previous section, the validity of the transmission-reflection equations in the case of non-degenerate mass spectrum selects $S = -1$ as a possible scattering matrix in the bulk. This solution can be identified as the scattering amplitude of the particle excitations of the Ising model, given by the massive Majorana fermions [98]. The Lagrangian density of the continuum theory in the bulk is given by

$$\mathcal{L}_B = \overline{\Psi}(x,t)\,(i\gamma^\mu \partial_\mu - m)\,\Psi(x,t) \ . \tag{4.10}$$

In the Majorana representation, given by $\gamma^0 = \sigma_2$, $\gamma^1 = -i\sigma_1$, the fermionic field $\Psi(x,t)$ is real, i.e. $\Psi^\dagger(x,t) = \Psi(x,t)$. The physical content of the model, as defined by the Lagrangian (4.10), does not depend on the sign in front of the mass term since it can be altered by the transformation $\Psi \rightarrow \Psi$; $\overline{\Psi} \rightarrow -\overline{\Psi}$ of the fermionic field. As it is well known, the mass m is a linear measurement of the deviation of the temperature with respect to the critical one

$$m = 2\pi(T - T_c) \ , \tag{4.11}$$

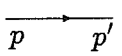
and the symmetry $m \rightarrow -m$ simply expresses the self-duality of the model. In the high temperature phase given by $m > 0$, the vacuum expectation value of the magnetization operator σ vanishes, whereas, the corresponding quantity of the disorder operator μ is different from zero. Under the duality transformation, the role of order and disorder operators is reversed whereas for the energy operator ϵ, given by $\epsilon = i\overline{\Psi}\Psi$, we simply have a change in its sign.

On a square lattice, the Ising model with a line of defect can be realized in two different ways (Fig. 4.3). The first is the chain geometry with bulk coupling constants J and modified couplings $\tilde{J}$ parallel to the defect line. The second one is the ladder geometry, with the modified set of couplings placed in the perpendicular direction. Since the two geometric realizations are related by Kramer-Wannier duality symmetry, from now on we can restrict our attention to one of them, say the chain geometry. In the continuum formulation, the defect line introduces the additional term[2]

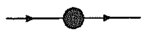
$$\mathcal{L}_D = -g\,\delta(x)\overline{\Psi}(x,t)\Psi(x,t) \tag{4.12}$$

to the lagrangian (4.10). The new interaction is purely marginal and therefore the beta-function associated to the coupling constant g is identically zero. The marginality of the interaction has as a consequence that the theory presents a non-universal ultraviolet behaviour in the magnetization sector, with the critical exponent of the magnetization operators which depend continuously on the parameter g, whereas the energy operator always keeps its original value of 1 of the Ising model [90, 91, 92, 93].

In this section we are interested in determining the reflection $R(\beta)$ and transmission $T(\beta)$ amplitudes for the scattering of the fermion with the defect line, i.e. the S-matrix elements between initial and final states $u(p_i)$ and $\overline{u}(p_f)$ with $p_f = \pm p_i$. To this aim, let us consider the perturbative series of the Green function of the fermion field Ψ based on the Feynman rules

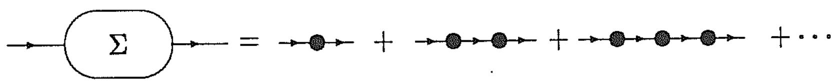


$$= i(2\pi)^2\,\delta^2(p-p')\,\frac{\not p+m}{p^2-m^2+i\epsilon}$$

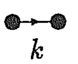


$$= -\,ig\,2\pi\,\delta(p_0-p_0')$$

For the self-energy entering the exact propagator we have the following series of diagrams



where we have to integrate on the spatial component of the momentum running in the internal lines. The integral on the intermediate state is given by



$$= (-ig)^2\,i\,\delta(k_0-p_0)\int\frac{dk^1}{2\pi}\,\frac{\not k+m}{k^2-m^2+i\epsilon} = -g^2\,\delta(k_0-p_0)\,\frac{p_0\gamma^0+m}{2\omega}\ .$$

[2]The exact relationship between g and the lattice coupling constants will be established in section 4.5 by comparing correlation functions computed in the lattice and in the continuum formulation.

In the above quantity we have discarded by parity the infinity related to the linear term in k. With this prescription, the geometric series for Σ is finite and can be expressed in a closed form as

$$\Sigma(p_0) \;=\; 2\pi\, i\, \delta(p_0 - p_0')\, \sin\chi\, \frac{\omega - i\frac{g}{2}(p_0\gamma^0 - m)}{\omega - im\,\sin\chi}\;, \tag{4.13}$$

where

$$\omega \;=\; \sqrt{p_0^2 - m^2}\;\;, \quad \sin\chi \;=\; -\,\frac{g}{1+\frac{g^2}{4}}\;.$$

We can now apply the usual LSZ reduction formulae, and for the transmission and reflection amplitudes defined by

$${}_{out} < \beta' \mid \beta >_{in} = \; 2\pi\delta(\beta - \beta')\, T(\beta, g) + 2\pi\delta(\beta + \beta')\, R(\beta, g)\;,$$

we have

$$\begin{aligned} T(\beta, g) \;&=\; \frac{\cos\chi\,\sinh\beta}{\sinh\beta - i\sin\chi}\;, \\ R(\beta, g) \;&=\; i\,\frac{\sin\chi\,\cosh\beta}{\sinh\beta - i\sin\chi}\;. \end{aligned} \tag{4.14}$$

The transmission amplitude also contains the disconnected part relative to the free motion.

Before commenting on the properties of these amplitudes, it is interesting to present an alternative derivation of (4.14). This is obtained by implementing the algebra (4.4) on the creation operators of the fermion field. Let $\Psi_{\pm}(x,t)$ be the solutions of the free Dirac equation in the two intervals $x > 0$ and $x < 0$, i.e.

$$\Psi(x,t) \;=\; \theta(x)\,\Psi_+(x,t) \;+\; \theta(-x)\,\Psi_-(x,t)\;, \tag{4.15}$$

with the value at the origin given by $\Psi(0,t) \;=\; \frac{1}{2}\,(\Psi_+(0,t) \;+\; \Psi_-(0,t))$. The mode expansion of the two components of the fields $\Psi_{\pm}(x,t)$ is expressed as

$$\begin{aligned} \psi^{(1)}_{(\pm)}(x,t) \;&=\; \int \tfrac{d\beta}{2\pi}\left[\omega e^{\frac{\beta}{2}}\, A_{(\pm)}(\beta)\, e^{-im(t\cosh\beta - x\sinh\beta)} \;+\; \overline{\omega} e^{\frac{\beta}{2}}\, A^{\dagger}_{(\pm)}(\beta)\, e^{im(t\cosh\beta - x\sinh\beta)}\right] \\ \psi^{(2)}_{(\pm)}(x,t) \;&=\; -\int \tfrac{d\beta}{2\pi}\left[\overline{\omega} e^{-\frac{\beta}{2}}\, A_{(\pm)}(\beta)\, e^{-im(t\cosh\beta - x\sinh\beta)} \;+\; \omega e^{-\frac{\beta}{2}}\, A^{\dagger}_{(\pm)}(\beta)\, e^{im(t\cosh\beta - x\sinh\beta)}\right]\;, \end{aligned} \tag{4.16}$$

with $\omega \;=\; \exp(i\pi/4), \overline{\omega} \;=\; \exp(-i\pi/4)$. The operators $A_{\pm}(\beta)$ and $A^{\dagger}_{\pm}(\beta)$ satisfy the usual anti-commutation relations of a free fermion

$$\left\{A_{\pm}(\beta), A^{\dagger}_{\pm}(\theta)\right\} \;=\; 2\pi\,\delta(\beta - \theta)\;, \tag{4.17}$$

although they are not all independent. They are related to each other by the conditions at $x = 0$ which arise from applying the eqs. of motion to (4.15), i.e.

$$\begin{aligned} (\psi_+^{(2)} - \psi_-^{(2)})(0,t) &= \tfrac{g}{2}(\psi_+^{(1)} + \psi_-^{(1)})(0,t) \ ; \\ (\psi_+^{(1)} - \psi_-^{(1)})(0,t) &= \tfrac{g}{2}(\psi_+^{(2)} + \psi_-^{(2)})(0,t) \ . \end{aligned} \tag{4.18}$$

These equations are equivalent to the relationship between the modes

$$M \begin{pmatrix} A_-^\dagger(\beta) \\ A_+^\dagger(-\beta) \end{pmatrix} = N \begin{pmatrix} A_-^\dagger(-\beta) \\ A_+^\dagger(\beta) \end{pmatrix} , \tag{4.19}$$

where

$$M = \begin{pmatrix} \omega e^{-\frac{\beta}{2}} + \frac{g}{2}\overline{\omega} e^{\frac{\beta}{2}} & -\omega e^{\frac{\beta}{2}} + \frac{g}{2}\overline{\omega} e^{-\frac{\beta}{2}} \\ \omega e^{-\frac{\beta}{2}} + \frac{2}{g}\overline{\omega} e^{\frac{\beta}{2}} & \omega e^{\frac{\beta}{2}} - \frac{2}{g}\overline{\omega} e^{-\frac{\beta}{2}} \end{pmatrix} ;$$

$$N = \begin{pmatrix} -\omega e^{\frac{\beta}{2}} - \frac{g}{2}\overline{\omega} e^{-\frac{\beta}{2}} & \omega e^{-\frac{\beta}{2}} - \frac{g}{2}\overline{\omega} e^{\frac{\beta}{2}} \\ -\omega e^{\frac{\beta}{2}} - \frac{2}{g}\overline{\omega} e^{-\frac{\beta}{2}} & -\omega e^{-\frac{\beta}{2}} + \frac{2}{g}\overline{\omega} e^{\frac{\beta}{2}} \end{pmatrix} .$$

Hence,

$$\begin{pmatrix} A_-^\dagger(\beta) \\ A_+^\dagger(-\beta) \end{pmatrix} = M^{-1} N \begin{pmatrix} A_-^\dagger(-\beta) \\ A_+^\dagger(\beta) \end{pmatrix} = \begin{pmatrix} R(\beta,g) & T(\beta,g) \\ T(\beta,g) & R(\beta,g) \end{pmatrix} \begin{pmatrix} A_-^\dagger(-\beta) \\ A_+^\dagger(\beta) \end{pmatrix} \tag{4.20}$$

with $R(\beta,g)$ and $T(\beta,g)$ given in (4.14). Note that, although the boundary conditions (4.18) are both linear in g, there is however a feedback between the two components of the fermionic field. The final dependence from the coupling constant is then expressed in terms of trigonometric functions of the auxiliary angle χ.

Given the explicit expressions of the amplitudes (4.14), it is easy to check that they satisfy the unitarity and crossing equations (4.5) and (4.6). They present several interesting features. Firstly, by taking their sum and difference we obtain

$$e^{2i\delta_0} \equiv T(\beta,g) + R(\beta,g) = \frac{\sinh\frac{1}{2}(\beta + i\chi)}{\sinh\frac{1}{2}(\beta - i\chi)} ;$$

$$e^{2i\delta_1} \equiv T(\beta,g) - R(\beta,g) = \frac{\cosh\frac{1}{2}(\beta - i\chi)}{\cosh\frac{1}{2}(\beta + i\chi)} ,$$

which can be considered as partial-wave phase shifts, with δ_0 and δ_1 crossed functions of each other. Secondly, notice that as functions of the coupling constant g, they satisfy a strong-weak duality given by

$$T\left(\beta, \frac{4}{g}\right) = -T(\beta,g) \ , \quad R\left(\beta, \frac{4}{g}\right) = R(\beta,g) \ . \tag{4.21}$$

At the self-dual points $g = \pm 2$ the transmission amplitude vanishes and therefore the defect line behaves as a pure reflecting surface. From the unitarity equations (4.5), the corresponding reflection amplitudes $R(\beta, \pm 2)$ become pure phases, as can be explicitly seen by their equivalent expressions

$$R(\beta, \pm 2) = -\frac{\cosh\frac{\beta}{2} \pm i\sinh\frac{\beta}{2}}{\cosh\frac{\beta}{2} \mp i\sinh\frac{\beta}{2}} \ . \tag{4.22}$$

They coincide with the reflection amplitudes of the Ising model with fixed and free boundary conditions respectively, as determined in [83]. To establish directly the pure reflecting properties of the defect line at the self-dual points, let us analyse more closely the decoupling which occurs in the boundary conditions when $g = \pm 2$. For $g = 2$, the boundary conditions (4.18) become

$$\begin{aligned} (\psi_+^{(2)} - \psi_-^{(2)})(0,t) &= (\psi_+^{(1)} + \psi_-^{(1)})(0,t) \ ; \\ (\psi_+^{(1)} - \psi_-^{(1)})(0,t) &= (\psi_+^{(2)} + \psi_-^{(2)})(0,t) \ , \end{aligned} \tag{4.23}$$

and taking their sum and difference, they can be written as

$$\begin{aligned} (\psi_+^{(2)} - \psi_+^{(1)})(0,t) &= 0 \ ; \\ (\psi_-^{(2)} + \psi_-^{(1)})(0,t) &= 0 \ . \end{aligned} \tag{4.24}$$

For $g = -2$, the original boundary conditions (4.18) are reduced instead to

$$\begin{aligned} (\psi_+^{(2)} + \psi_+^{(1)})(0,t) &= 0 \ ; \\ (\psi_-^{(2)} - \psi_-^{(1)})(0,t) &= 0 \ . \end{aligned} \tag{4.25}$$

Equations (4.24) and (4.25) explicitly show that the two semi-infinite systems across the defect line are completely decoupled, and each of them can be treated as a quantum field theory in the presence of pure reflecting surface whose role is to supply the appropriate boundary conditions [83]. At first sight, though, one may be surprised by the asymmetric form assumed by the eqs. (4.24) and (4.25) which treat differently the two fermionic fields $\Psi_\pm(x,t)$. However, this asymmetry has a physical origin. By means of the mode expansion (4.16), the first equation in (4.24) and (4.25) can be used to determine directly the reflection amplitudes $R(\beta, \pm 2)$. By the same token, using the second equation in (4.24) and (4.25) we find $R(-\beta, \pm 2)$, instead. But, this is physically correct, the reason being that, in order to have a reflection of a particle described by $\Psi_+(x,t)$ with the defect (boundary) line, this particle must approach the origin with positive rapidity β. On the contrary, a reflection of a particle described by $\Psi_-(x,t)$ with the defect (boundary) line is only realized for negative values of its rapidity.

Further support of the identification of $R(\beta, \pm 2)$ with the reflection amplitudes of the Ising model with fixed and free boundary conditions comes from the analysis of the relationship between the lattice and the continuum formulations of the chain geometry, which will be established in section 4.5. Anticipating the result, this is provided by the formula

$$\sin\chi = \tanh 2(J - \tilde{J}) \ . \tag{4.26}$$

Hence, the condition $\sin\chi = -1$ corresponds to a coupling constant $\tilde{J}$ along the defect line infinitely larger (and positive) than the coupling constant J of the bulk. As a consequence, the spins along the defect line are frozen into a fixed boundary condition. On the other hand, the condition $\sin\chi = 1$ is obtained in the anti-ferromagnetic limit $\tilde{J} \to -\infty$ where the spins along the defect line are aligned in antiparallel configurations. Since the nearby spins couple to a surface with vanishing magnetization, this situation corresponds to the free boundary conditions [89].

Let us now turn our attention to the analytic structure of the reflection and transmission amplitudes. For negative values of g, the interaction with the defect line is attractive and consequently the theory presents a bound state with binding energy $e_b = m\cos\chi$. It is quite instructive to calculate the transmission and reflection amplitudes $T_b(\beta)$ and $R_b(\beta)$ relative to the scattering of the fermion with the excited state of the defect line. The first thing to observe is that both amplitudes $R(\beta)$ and $T(\beta)$ present a pole singularity at $\beta = i\chi$ and $\beta = i(\pi - \chi)$. The reflection amplitude $R(\beta)$ has positive residue at both locations, given by $i\sin\chi$. On the other hand, $T(\beta)$ presents a positive residue with the same value as $R(\beta)$ at $\beta = i\chi$ and a negative residue $-i\sin\chi$ at the other pole $\beta = i(\pi - \chi)$. The problem of identifying which one of the two poles corresponds to the bound state is solved by selecting the singularity with positive residue in both amplitudes. This is the pole at $\beta = i\chi$. The relative binding energy is positive, as it should be. To recover the transmission and reflection amplitudes relative to the excited state, we have to impose the commutativity of the graphs shown in Fig. 4.4. Since the S-matrix in the bulk is -1, the reflection amplitude $R_b(\beta)$ coincides with the original one i.e. $R_b(\beta) = R(\beta)$ whereas the transmission amplitude is given by $T_b(\beta) = -T(\beta)$. If we again identify the singularity associated to a bound state as that pole with a positive residue in both channels, we see that for the defect bound state amplitudes the role of the two poles has been reversed! Namely, the pole which corresponds to the bound state in $R_b(\beta)$ and $T_b(\beta)$ is now located at $\beta = i(\pi - \chi)$ and is relative to the original ground state of the defect line[3].

[3]Note that the presence of the transmission amplitude has been quite crucial in order to discriminate which

As a final remark of this section, the marginal nature of the defect interaction in the Ising model can be also inferred by looking at the high-energy limit of the amplitudes. For large values of β we have

$$T(\beta) \sim \cos\chi \ , \quad R(\beta) \sim i \sin\chi \ . \tag{4.27}$$

Hence, except for the special values of the coupling constant g where one of the two quantities vanish, both amplitudes are always simultaneously present. Since this limit probes the short distance scales of the model, we see that its critical properties of bulk and surface behaviour are simultaneously present[4].

4.3 Bosonic theory with a line of defect

Another solution of the reflection-transmission equations is provided by the massive free bosonic theory with the S-matrix in the bulk given by $S = 1$. As an example of a bosonic theory with a line of defect, we consider the model described by the lagrangian

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\varphi)^2 - m^2\varphi^2 - g\,\delta(x)\varphi^2\right] \ . \tag{4.28}$$

For the equation of motion we have

$$\left[\Box + m^2 + g\,\delta(x)\right]\varphi = 0 \ . \tag{4.29}$$

As for the Ising model, one can obtain the reflection and transmission amplitudes by an exact resummation of the perturbative series in the coupling constant g. The calculations are analogous to the fermionic case, and rather than repeating them here, we prefer to exploit the algebraic approach directly. The solution of the equation of motion may be written as

$$\varphi(x,t) = \theta(x)\varphi_+(x,t) + \theta(-x)\varphi_-(x,t) \ , \tag{4.30}$$

one of the two poles with positive residue in the reflection channel corresponds to the bound state. In the pure reflecting situation, as for instance may be the case of the Ising model with a boundary magnetic field considered in [83], the occurrence of positive residue at both poles in the reflection amplitude and a misinterpretation of their role could in fact lead to a paradoxical hierarchy of bound states obtained by applying iteratively the boundary bootstrap equations.

[4] For an irrelevant interaction which leads to a bulk critical behaviour near the defect line, we expect in fact a vanishing of the reflection amplitude in the high-energy limit. On the contrary, for a relevant interaction, the system should show a purely surface critical behaviour characterized by the vanishing of the transmission amplitude.

where the mode decomposition of the two fields $\varphi_{\pm}(x,t)$ is given by

$$\varphi_{(\pm)}(x,t) = \int \frac{d\beta}{2\pi} \left[A_{(\pm)}(\beta)\, e^{-im(t\cosh\beta - x\sinh\beta)} + A^{\dagger}_{(\pm)}(\beta)\, e^{im(t\cosh\beta - x\sinh\beta)} \right] . \tag{4.31}$$

The operators $A_{\pm}(\beta)$ and $A^{\dagger}_{\pm}(\beta)$ satisfy the usual commutation relations of a free massive boson

$$\left[A_{\pm}(\beta), A^{\dagger}_{\pm}(\theta) \right] = 2\pi\, \delta(\beta - \theta) . \tag{4.32}$$

The interaction along the defect however makes them not linearly independent. In fact, substituting eq. (4.30) into the equation of motion, the latter is equivalent to the boundary conditions

$$\begin{aligned} \varphi_{+}(0,t) - \varphi_{-}(0,t) &= 0 ; \\ \frac{\partial}{\partial x}(\varphi_{+}(0,t) - \varphi_{-}(0,t)) &= \frac{g}{4m}(\varphi_{+}(0,t) + \varphi_{-}(0,t)) , \end{aligned} \tag{4.33}$$

which, in terms of the mode, can be written as

$$\begin{pmatrix} A^{\dagger}_{-}(\beta) \\ A^{\dagger}_{+}(-\beta) \end{pmatrix} = \begin{pmatrix} R(\beta,g) & T(\beta,g) \\ T(\beta,g) & R(\beta,g) \end{pmatrix} \begin{pmatrix} A^{\dagger}_{-}(-\beta) \\ A^{\dagger}_{+}(\beta) \end{pmatrix} . \tag{4.34}$$

The transmission and reflection amplitudes in the above formula are given by

$$\begin{aligned} T(\beta,g) &= \frac{\sinh\beta}{\sinh\beta + ig/4m} , \\ R(\beta,g) &= -\frac{ig/4m}{\sinh\beta + ig/4m} . \end{aligned} \tag{4.35}$$

These amplitudes satisfy the unitarity and crossing equations (4.5) and (4.6). It is easy to see that by substituting $\sinh\beta$ in (4.35) with the linear momentum k, the two resulting amplitudes are the same as those obtained in one-dimensional quantum mechanics for the scattering in a δ-function potential (see, for instance, [99]). However, due to the relativistic nature of the quantum field theory, there is an important difference between the two cases, as shown by the analysis which follows on the pole structure of the amplitudes (4.35).

For the $2\pi i$ periodicity of the amplitudes, we can restrict our attention to the strip $-i\pi \leq \beta \leq i\pi$. Let us consider initially the case when g is a positive quantity. As long as g satisfies the condition $0 < g < 4m$, there are two poles on the negative imaginary axis relative to the unphysical sheet. By increasing the value of g they approach each other, and there is a critical value $g_{c_1} = 4m$ where they collide at position $\beta = -i\pi/2$. Additional increment of the coupling constant causes the poles to move in the complex strip keeping their imaginary part equal to

$-i\pi/2$ but acquiring a real part (Fig. 4.5). In terms of quantum field theory, this means that the bosonic theory with a coupling constant of the defect line larger than $4m$ presents two resonance states. As g grows, these poles move to infinity, and in the limit $g \to \infty$, the defect line acts as a pure reflecting surface. Indeed, the transmission amplitude vanishes, whereas the reflection amplitude expresses the fixed boundary condition $\varphi(0,t) = 0$.

Let us now analyse the case when g is a negative quantity. In the range $-4m < g < 0$, the amplitudes present two poles placed on the positive imaginary axis relative to the physical sheet. The closest one to the origin can be interpreted as a defect bound state. By decreasing g, these two poles approach each other until they finally collide at $\beta = i\pi/2$ for the critical value $g_{c_2} = -4m$. Further decrement of the coupling constant makes them move in the complex strip with an imaginary part equals to $i\pi/2$ and with a real component which increases by decreasing g. However, these poles are now located in the physical strip and therefore the theory presents instability properties. The easiest way to explicitly illustrate this instability is to consider the analytic continuation $\beta \to (i\frac{\pi}{2} - \beta)$ in $R(\beta)$. As discussed in section 4.5, the resulting quantity $\hat{R}(\beta)$, given by

$$\hat{R}(\beta, g) = -\frac{g/4m}{\cosh\beta + g/4m}, \tag{4.36}$$

can be interpreted as the amplitude relative to the emission of a pair of particles with momentum β and $-\beta$ from the defect line placed along the x-axis [83]. Then, for $g < -4m$, $\hat{R}(\beta)$ presents a pole for real values of β that induces a spontaneous emission of pairs of particles. The occurrence of such processes obviously spoils the stability of the theory.

In light of the above results, we can summarize the discussion by saying that the quantum field theory associated to the lagrangian (4.28) makes sense only for values of g in the range $-4m < g \leq \infty$. In a path integral approach to the problem, it is easy to see that there may be a competition in the lagrangian (4.28) between the genuine mass term and the defect interaction. Adopting the interpretation of the δ-function interaction as a suitable limit of a constant potential in the strip $(-\epsilon, \epsilon)$ around the origin, when g is sufficiently positive in this interval, we may have an effective mass of the field φ in this strip higher than the threshold mass m in the bulk. This produces the resonance poles in the transmission and reflection amplitudes. Viceversa, for negative values of g, the effective mass of the field φ in the tiny interval around the origin is smaller than the mass gap in the bulk and it decreases until it vanishes at $g = -4m$. After this value it becomes imaginary, giving rise to the instability property previously discussed.

It is likewise interesting to understand the different behaviour of the bosonic and the fermionic theories in terms of the coupling constant. The reason is that the physical content of the fermionic model does not depend on the sign of the mass term, which enters linearly in the action. Therefore, by varying the coupling constant g, there is no a real competition with the genuine mass term in the action, so that the fermionic model cannot present instabilities or resonance states. In fact, crossing the critical values $g = \pm 2$, the poles simply interchange their positions, i.e. the weak coupling regime swaps with the strong coupling one.

As a last comment on the bosonic theory analysed in this section, the defect interaction is associated to an irrelevant operator and therefore the defect line should be completely transparent in the ultraviolet limit. Indeed, taking the high-energy limit $\beta \to \infty$ of the amplitudes (4.35), the reflection amplitude vanishes whereas the transmission amplitude is identically equal to 1.

4.4 Models with multi-defect lines

The solutions so far determined for the fermionic and bosonic theories in the presence of a single line of defect can be generalized and geometrical situations with a richer structure of defect lines can be also included. In this section, we analyse the case of two parallel lines of defect, and then the quantization conditions induced by a periodic array of defects. Due to the different behaviour of the fermionic and bosonic theories, it is convenient to discuss them separately.

4.4.1 Fermionic theory

Let us initially consider the Ising model with two parallel lines of defect, one placed at the origin along the y-axis with strength g_1, the other shifted by a and with strength g_2. In the fermionic formulation of the model, the field $\Psi(x,t)$ has a free motion in each of the three intervals $I_- \equiv (-\infty, 0)$, $I_0 \equiv (0,a)$ and $I_+ \equiv (a, +\infty)$ separated by the two defect lines. Therefore in each of the three intervals the field $\Psi(x,t)$ admits the usual decomposition in modes and the role of the defect lines is to provide the boundary conditions at the edges of the intervals. The first of them is at $x = 0$ and is given by

$$\begin{aligned} (\psi_0^{(2)} - \psi_-^{(2)})(0,t) &= \tfrac{g_1}{2}(\psi_0^{(1)} + \psi_-^{(1)})(0,t) \ ; \\ (\psi_0^{(1)} - \psi_-^{(1)})(0,t) &= \tfrac{g_1}{2}(\psi_0^{(2)} + \psi_-^{(2)})(0,t) \ , \end{aligned} \tag{4.37}$$

whereas for the second boundary condition at $x = a$ we have

$$
\begin{aligned}
(\psi_+^{(2)} - \psi_0^{(2)})(a,t) &= \tfrac{g_2}{2}(\psi_+^{(1)} + \psi_0^{(1)})(a,t) \ ; \\
(\psi_+^{(1)} - \psi_0^{(1)})(a,t) &= \tfrac{g_2}{2}(\psi_+^{(2)} + \psi_0^{(2)})(a,t) \ .
\end{aligned}
\tag{4.38}
$$

In these equations the intervals are labelled by the subscript of the fields while their components by the upper indices. By using the notation R_i and T_i ($i = 1,2$) for the reflection and the transmission amplitudes relative to the defect line with strength g_i, it is easy to see that eliminating the intermediate modes relative to the interval I_0, there is a linear relationship between the modes of the fields in the intervals I_- and I_+ given by

$$
\begin{pmatrix} A_-^\dagger(\beta) \\ A_+^\dagger(-\beta) \end{pmatrix} = \begin{pmatrix} R(\beta,g_1,g_2,a) & T(\beta,g_1,g_2,a) \\ T(\beta,g_1,g_2,a) & R(\beta,g_1,g_2,a) \end{pmatrix} \begin{pmatrix} A_-^\dagger(-\beta) \\ A_+^\dagger(\beta) \end{pmatrix} , \tag{4.39}
$$

where

$$
\begin{aligned}
T(\beta,g_1,g_2,a) &= \frac{T_1(\beta)T_2(\beta)}{1-\eta(\beta,a)R_1(\beta)R_2(\beta)} \ , \\
R(\beta,g_1,g_2,a) &= \frac{R_1(\beta)+\eta(\beta,a)R_2(\beta)[T_1^2(\beta)-R_1^2(\beta)]}{1-\eta(\beta,a)R_1(\beta)R_2(\beta)} \ .
\end{aligned}
\tag{4.40}
$$

In the above expressions $\eta(\beta,a)$ is a pure phase given by $\eta(\beta,a) = \exp[-2ima\sinh\beta]$.

The above amplitudes satisfy the unitarity and crossing equations (4.5) and (4.6). They describe the physical situation of a particle coming from the interval I_- with rapidity β which hits the first defect line and, as result of this interaction, it can be either reflected or transmitted. When it is reflected, it appears as an asymptotic particle with rapidity $-\beta$ whereas when it is transmitted it approaches the next defect and can again be reflected or transmitted. As shown in Fig. 4.6, these two types of process may be repeated an arbitrary number of times at the two defect lines.

Due to the existence of the fixed points $g = \pm 2$ of a single defect line, it is interesting to analyse some special limits of the expressions (4.40). To begin with, note that, at the values $g_1 = \pm 2$ where $T_1 = 0$, the total transmission amplitude $T(\beta,g_1,g_2,a)$ vanishes as well, whereas the reflection amplitude reduces to a pure phase given by $R(\beta,\pm 2,g_2,a) = R_1(\beta,\pm 2)$. In this case, the first defect acts as a pure reflecting surface which therefore completely screens the presence of the second defect. The total transmission amplitude also vanishes when $g_2 = \pm 2$. Concerning the reflection amplitude, it becomes a pure phase given by

$$
R(\beta,g_1,\pm 2,a) = \eta(\beta,a)\,R_2(\beta,\pm 2)\frac{\sinh\beta(1+\eta^{-1}(\beta,a)\sin\chi_1)+i\sin\chi(1-\eta^{-1}(\beta,a))}{\sinh\beta(1+\eta(\beta,a)\sin\chi_1)-i\sin\chi(1-\eta(\beta,a))} \ . \tag{4.41}
$$

The total reflection process is now the result of an infinite sequence of elementary transmission and reflection scatterings at the first defect line combined with pure reflecting processes at the second defect line. Hence it is not surprising that the final expression depends on both $R_2(\beta, \pm 2)$ and the separation distance a.

Except for the values of g when the defects behave as mirror surfaces, the possibility for the fermion to go back and forth between the two defect lines produces typical resonance phenomena which are illustrated for instance by plotting the absolute value of $T(\beta, g_1, g_2, a)$. An example is shown in Fig. 4.7.

Finally, by taking the limit $a \to 0$, the two defect lines behave as a single one but with an effective strength g given by

$$g = \frac{g_1 + g_2}{1 + g_1 g_2/4} \ . \tag{4.42}$$

This composition law of the defect strengths is similar to the addition of velocities in relativistic dynamics. The effective coupling constant g has as critical values $g = \pm 2$ and reaches these limits when either g_1 or g_2 are equal to ± 2. This can be also seen by analysing the fixed points of the composition law defined by the iterative map

$$g_{n+1} = \frac{g_n + g}{1 + g_n g/4} \ , \tag{4.43}$$

for some initial value g.

The natural generalization of the situation with two defect lines is to consider a periodic array of defects all with equal strength g and separated by a distance a. The fermionic field satisfies in this case the equation

$$\left[i\gamma^\mu \partial_\mu - m - g \sum_{n=-\infty}^{\infty} \delta(x + na) \right] \Psi(x, t) = 0 \ , \tag{4.44}$$

and admits the decomposition

$$\Psi(x, t) = \sum_{n=-\infty}^{\infty} \theta(x - na)\theta(-x + (n+1)a)\Psi_n(x, t) \ , \tag{4.45}$$

with $\Psi_n(x, t)$ solutions of the free Dirac equation. The dynamics of the model is entirely encoded into an infinite set of linear equations relative to the boundary conditions between the interval na and $(n-1)a$, i.e.

$$\begin{aligned} (\psi_{n-1}^{(2)} - \psi_n^{(2)})(na, t) &= \tfrac{g}{2}(\psi_{n-1}^{(1)} + \psi_n^{(1)})(na, t) \ ; \\ (\psi_{n-1}^{(1)} - \psi_n^{(1)})(na, t) &= \tfrac{g}{2}(\psi_{n-1}^{(2)} + \psi_n^{(2)})(na, t) \ . \end{aligned} \tag{4.46}$$

The simplest way to solve these equations is to employ a relativistic generalization of the Bloch theorem [100], i.e to associate a wave vector k to the spinor field Ψ such that

$$\Psi(x+a,t) \;=\; e^{ika}\,\Psi(x,t)\ . \tag{4.47}$$

Equivalently,

$$\Psi_n(na,t) \;=\; e^{ika}\,\Psi_{n-1}((n-1)a,t)\ . \tag{4.48}$$

The wave vector k can always be confined to the first Brillouin zone $-\pi/a \leq k \leq \pi/a$. Plugging (4.48) into eqs. (4.46), the resulting system is compatible provided that the equation

$$\cos ka \;=\; \frac{1}{\cos\chi}\left[\cos(ma\sinh\beta) - \sin\chi\frac{\sin(ma\sinh\beta)}{\sinh\beta}\right] \tag{4.49}$$

is valid. This equation gives rise to a band structure in the energy levels of the Majorana fermion of the Ising model, completely analogous to the periodic potentials considered in condensed matter physics. In fact, eq. (4.49) can be satisfied for real k if and only if the right hand side of the equation is less than unity. Consequently, there will be allowed and forbidden regions of β and the corresponding spectrum of the energy, given by $E = m\cosh\beta$, consists of a family of energy bands. A characteristic form of the spectrum is plotted in Fig. 4.8. For the pure reflecting values $g = \pm 2$, the above equation reduces to the quantization condition of the rapidity variable β

$$\sinh\beta \;=\; \pm\tan(ma\sinh\beta)\ , \tag{4.50}$$

which arises by considering the fermionic field defined in a strip of width a with fixed (+) or free (−) boundary conditions at the edge of the interval.

4.4.2 Bosonic theory

The discussion of the bosonic theory largely follows the previous one and eqs. (4.40) is valid as it stands on the condition that we insert the bosonic amplitudes instead. Also in this case there are typical resonance phenomena produced by the trapping of the bosonic particle between the two defect lines. There is however a significant difference with respec. to the fermionic case and this concerns the composition law relative to two defect lines with a separation $a \to 0$. In this limit, the two defect lines behave as a single one with an effective strength g given by

$$g \;=\; g_1 + g_2\ . \tag{4.51}$$

Due to the peculiar properties of the bosonic system discussed in section 4.3, this composition law implies that a system with two defect lines in the limit $a \to 0$ may become unstable although each of the defect lines taken individually does not present any instability property. Viceversa, one can obtain a well-defined bosonic system as a result of the limit $a \to 0$ of a system which presents instability properties at one defect line and resonance states at the other.

Taking the limit $g_1 \to +\infty$, the first defect line becomes a pure reflecting surface and the total transmission amplitude vanishes. In this case the reflection amplitude reduces to $R(\beta, +\infty, g_2, a) = -1$. The total transmission amplitude also vanishes when the second defect line acts as a pure reflecting surface. The corresponding reflection amplitude is a pure phase given by

$$R(\beta, g_1, +\infty, a) = -\eta(\beta, a) \frac{\sinh\beta - i\frac{g}{4m}(1 - \eta^{-1}(\beta, a))}{\sinh\beta + i\frac{g}{4m}(1 - \eta(\beta, a))} . \tag{4.52}$$

As in the fermionic case, the presence of an infinite periodic array of defect lines of strength g and separation a gives rise to a band structure described by a Kronig-Penney type equation

$$\cos ka = \cos(ma \sinh\beta) + \frac{g}{m} \frac{\sin(ma \sinh\beta)}{\sinh\beta} . \tag{4.53}$$

The pure reflective case $g \to +\infty$ gives rise to the quantization condition

$$ma \sinh\beta = \pi n , \quad (n = 0, \pm 1, \ldots) \tag{4.54}$$

relative to the bosonic field in a strip of width a with fixed boundary conditions $\varphi(0, t) = \varphi(a, t) = 0$ at the end points of the interval.

4.5 Correlation functions

In the bulk, the scattering theory and the bootstrap approach -in addition to yield a clear understanding on the physical content of the continuum limit of the integrable statistical models- have the added advantage of providing a powerful method for the computation of the correlation functions of the order parameters. Once the bulk scattering amplitudes are known, there are well-defined techniques for computing matrix elements of the local fields $\phi_i(x, t)$ on the set of asymptotic states $< \beta_1, \ldots \beta_n \mid \phi_i(x, t) \mid \beta_{n+1}, \ldots \beta_m >$ and for reconstructing their correlation functions through spectral representation method based on the completeness relation of the asymptotic states, as we have shown in the other chapters. In this section we intend to prove that the spectral methods are also suitable for computing the correlation functions of the Ising model and the bosonic theory in the presence of a defect line.

The easiest way to approach the problem is to use a formalism which takes full advantage of the solution of the theory in the bulk. To this aim, it is convenient to interchange the original role of the x and the t axes by the transformation $x \to -it$, $t \to ix$. The new space has a Minkowski structure with the defect line placed now at $t = 0$. In this new geometry, the space of the states is the same as in the bulk, and therefore, even in the presence of the defect line, the local operators ϕ_i can be completely characterized by their known form factors. The presence of the defect line can be taken into account by defining an operator $\mathcal{D}$ placed at $t = 0$, acting on the bulk states. This operator plays the role of the S-matrix of the problem, and therefore, standard formulas of quantum field theory allow the correlation functions to be expressed as [101]

$$< \Phi_1(x_1,t_1)\ldots\Phi_n(x_n,t_n) > = \frac{< 0 \mid T[\phi_1(x_1,t_1)\ldots\mathcal{D}\ldots\phi_n(x_n,t_n)] \mid 0 >}{< 0 \mid \mathcal{D} \mid 0 >} \ . \tag{4.55}$$

In the above formula, $\Phi_i(x_i,t_i)$ are the fields in the Heisenberg representation, i.e. the representation where the time evolution is ruled by the exact Hamiltonian of the problem, including the defect interaction. On the other hand, $\phi_i(x_i,t_i)$ are the field operators of the bulk theory and, as such, their time evolution operator is the bulk Hamiltonian[5]. The main advantage of eq. (4.55) is that, using the completeness relation of the bulk states, its right hand side can be entirely expressed in terms of the Form Factors of the bulk fields and the matrix elements of the operator $\mathcal{D}$ which are determined as follows.

The defect operator $\mathcal{D}$ encodes all information relative to the physical processes which take place at the defect line. To examine them, we have to initially realize that the first effect of the interchange of the x and the t axes consists in an analytic continuation of the original rapidity $\beta \to (i\frac{\pi}{2} - \beta)$, the reason being that, to preserve the Minkowski structure in the new set of axes, we have to interchange correspondingly the momentum and the energy role. The rapidities are now measured as in Fig. 4.9. For convenience, it is useful to introduce the new transmission and reflection amplitudes, given by

$$\hat{T}(\beta) = T\left(i\frac{\pi}{2} - \beta\right) \quad , \quad \hat{R}(\beta) = R\left(i\frac{\pi}{2} - \beta\right) \ . \tag{4.56}$$

They enter the expression of the simplest matrix elements of the operator $\mathcal{D}$, given by $\mathcal{D}_{1,1} =< \beta \mid \mathcal{D} \mid \theta >$, $\mathcal{D}_{2,0} =< \beta_1,\beta_2 \mid \mathcal{D} \mid 0 >$ and $\mathcal{D}_{0,2} =< 0 \mid \mathcal{D} \mid \beta_1,\beta_2 >$. For the fermionic and the

[5] An equivalent way to look at eq. (4.55) is to consider a transfer matrix approach in the euclidean space. The transfer matrix may be written as $\mathcal{T} = \exp[-H_B t]$ for all t but $t = 0$, where it is placed the defect line. Hence $\mathcal{D}$ in (4.55) can be interpreted as the continuum limit of the transfer matrix operator which connects the states below and above the defect line.

bosonic theory analysed in the previous sections, the first matrix element is easily computed by resumming the perturbative series with the defect interaction now localized at $t = 0$ and the result is

$$< \beta \mid \mathcal{D} \mid \theta > = 2\pi \, \hat{T}(\beta) \, \delta(\beta - \theta) \ . \tag{4.57}$$

By the same means, for the other two matrix elements, we have respectively

$$< \beta_1, \beta_2 \mid \mathcal{D} \mid 0 > = 2\pi \, \hat{R}(\beta_1) \, \delta(\beta_1 + \beta_2) \ , \tag{4.58}$$

and

$$< 0 \mid \mathcal{D} \mid \theta_1, \theta_2 > = 2\pi \, \hat{R}(\theta_1) \, \delta(\theta_1 + \theta_2) \ . \tag{4.59}$$

Hence, $\hat{T}(\beta)$ describes the process where a particle with rapidity β hits the defect line and is transmitted through it, keeping the same value of the rapidity (Fig. 4.9.a). On the contrary, $\hat{R}(\beta)$ may be interpreted as the amplitude for the creation or the annihilation of a pair of particles with equal and opposite rapidity β (Fig. 4.9.b). These three processes are compatible with the dynamics of the model because in a situation where the defect line is placed at $t = 0$, the processes are constrained by the conservation of the momentum but not of the energy.

For the general matrix elements of the operator $\mathcal{D}$, we can exploit the factorization property of the scattering theory and write down a set of recursive equations which involve the elementary two-body interactions considered above. For the bosonic case, the recursive equations are expressed by

$$\begin{aligned} &< \beta_1, \ldots, \beta_i, \ldots, \beta_m, \beta \mid \mathcal{D} \mid \theta_1, \ldots \theta_n > = \\ &= 2\pi \sum_{i=1}^{m} \hat{R}(\beta)\delta(\beta + \beta_i) \; < \beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots \beta_m \mid \mathcal{D} \mid \theta_1, \ldots \theta_n > + \\ &+ 2\pi \sum_{j=1}^{n} \hat{T}(\beta) \, \delta(\beta - \theta_j) \; < \beta_1, \ldots \beta_m \mid \mathcal{D} \mid \theta_1, \ldots \theta_{j-1}, \theta_{j+1}, \ldots, \theta_n > \ ; \end{aligned} \tag{4.60}$$

$$\begin{aligned} &< \beta_1, \ldots, \beta_m, \mid \mathcal{D} \mid \theta_1, \ldots \theta_n, \theta > = \\ &= 2\pi \sum_{i=1}^{n} \hat{R}(\theta)\delta(\theta + \theta_i) \; < \beta_1, \ldots, \beta_m \mid \mathcal{D} \mid \theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n > + \\ &+ 2\pi \sum_{j=1}^{m} \hat{T}(\theta)\delta(\theta - \beta_j) \; < \beta_1, \ldots, \beta_{j-1}, \beta_{j+1}, \ldots, \beta_m \mid \mathcal{D} \mid \theta_1, \ldots, \theta_n > \ . \end{aligned} \tag{4.61}$$

For the fermionic case, taking into account the anti-commutation relations of the fields, they can be written as

$$< \beta_1, \ldots, \beta_i, \ldots, \beta_m, \beta \mid \mathcal{D} \mid \theta_1, \ldots \theta_n > = \tag{4.62}$$
$$= 2\pi \sum_{i=1}^{m} (-1)^{m+1-i} \hat{R}(\beta)\delta(\beta+\beta_i) \; < \beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots \beta_m \mid \mathcal{D} \mid \theta_1, \ldots \theta_n > +$$
$$+ 2\pi \sum_{j=1}^{n} (-1)^j \hat{T}(\beta)\, \delta(\beta-\theta_j) \; < \beta_1, \ldots \beta_m \mid \mathcal{D} \mid \theta_1, \ldots \theta_{j-1}, \theta_{j+1}, \ldots, \theta_n > \; ;$$

$$< \beta_1, \ldots, \beta_m, \mid \mathcal{D} \mid \theta_1, \ldots \theta_n, \theta > =$$
$$= 2\pi \sum_{i=1}^{n} (-1)^i \hat{R}(\theta)\delta(\theta+\theta_i) \; < \beta_1, \ldots, \beta_m \mid \mathcal{D} \mid \theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n > + \tag{4.63}$$
$$+ 2\pi \sum_{j=1}^{m} (-1)^{m-j} \hat{T}(\theta)\delta(\theta-\beta_j) \; < \beta_1, \ldots, \beta_{j-1}, \beta_{j+1}, \ldots, \beta_m \mid \mathcal{D} \mid \theta_1, \ldots, \theta_n > \; ,$$

These recursive equations can be graphically represented as in Fig. 4.10 and express the exact resummation of the perturbative series associated to the scattering matrix elements $< m \mid \mathcal{D} \mid n >$. Since the particles are created or destroyed in couples, the non-vanishing matrix element $< m \mid \mathcal{D} \mid n >$ are only those with $m - n = 0$ (mod 2). They are proportional to $< 0 \mid \mathcal{D} \mid 0 >$ (which, for convenience, is set equal to 1) and the recursive equations permit to express all of them in terms of the elementary matrix elements $\mathcal{D}_{1,1}, \mathcal{D}_{2,0}$ and $\mathcal{D}_{0,2}$ as previously determined.

A useful method for solving the recursive equations is to introduce a generating functional of the matrix elements of $\mathcal{D}$ by the formula

$$\mathcal{G}(\eta,\gamma) = \exp\left\{\int d\beta \left(\frac{\hat{R}(\beta)}{2}\left[\eta(-\beta)\eta(\beta)+\gamma(\beta)\gamma(-\beta)\right] + \hat{T}(\beta)\eta(\beta)\gamma(\beta)\right)\right\} \; . \tag{4.64}$$

$\mathcal{G}$ depends on the two currents $\eta(\beta)$ and $\gamma(\beta)$, which commute or anti-commute, depending on whether we are considering the bosonic theory or the fermionic one. The matrix elements of $\mathcal{D}$ are then given by

$$< \beta_1, \ldots, \beta_m, \mid \mathcal{D} \mid \theta_1, \ldots \theta_n > = (2\pi)^{\frac{m+n}{2}} \; \frac{\partial}{\partial\gamma(\theta_n)} \cdots \frac{\partial}{\partial\gamma(\theta_1)} \frac{\partial}{\partial\eta(\beta_1)} \cdots \frac{\partial}{\partial\eta(\beta_m)} \mathcal{G}\Big|_{\eta=\gamma=0} \; . \tag{4.65}$$

.

We are now in the position to compute correlation functions of local operators of the Ising model and the bosonic theory with a line of defect. Note that in computing the left hand side of eq. (4.55) we should consider two different cases, namely: (a) the case where some of the operators Φ_i are in the upper half-plane and the others are in the lower one, or (b) the case where the operators Φ_i are all in one semi-plane, for example the upper one. In the former case, one has to use the general matrix elements $< i \mid \mathcal{D} \mid j >$, and consequently both transmission and reflection amplitudes will enter the final expression of the correlation functions. In the latter case, on the contrary, the correlation functions will depend only on the reflection amplitudes $\hat{R}(\beta)$ because, in virtue of the time ordering in eq. (4.55), the defect operator $\mathcal{D}$ is in this case the last in the row and so, it acts directly on the vacuum state $\mid 0 >$. Hence, the only matrix elements which enter the computation are $\mathcal{D}_{i,0} =< i \mid \mathcal{D} \mid 0 >$. Those describe the creation of the particle pairs and therefore only depend on $\hat{R}(\beta)$.

In the remaining part of this section, using the form factors of the Ising model determined in [36, 33, 29], and the matrix elements of the defect operator we compute some correlation functions of this model in the presence of the defect line[6]. The simplest one is the one-point function of the energy operator $\epsilon(x,t)$ which can be computed through the formula

$$\epsilon_0(t) = \sum_{n=0}^{\infty} < 0 \mid \epsilon(x,t) \mid n >< n \mid \mathcal{D} \mid 0 > \quad . \tag{4.66}$$

The energy operator couples the vacuum only to the two particle state, as can be easily checked by the fermionic representation of this operator, and for its matrix element we have

$$\begin{aligned} < 0 \mid \epsilon(x,t) \mid \beta_1, \beta_2 > = 2\pi i\, m \sinh \frac{\beta_1 - \beta_2}{2} \times \\ \times \exp\left[-mt\left(\cosh\beta_1 + \cosh\beta_2\right) + imx\left(\sinh\beta_1 + \sinh\beta_2\right)\right] \;, \end{aligned} \tag{4.67}$$

Hence the above sum (4.66) consists of only one term (Fig. 4.11) and using eq. (4.58), it can be expressed as

$$\epsilon_0(t) = m \sin\chi \int_0^{\infty} d\beta \frac{\sinh^2\beta}{\cosh\beta - \sin\chi}\, e^{-2mt\cosh\beta} \quad . \tag{4.68}$$

The one-point function does not depend on x, as it can be equivalently argued by translation invariance along this axis. The above integral reduces to closed expressions in terms of Bessel functions when the defect line acts as pure reflecting surface. In the case of fixed boundary conditions, we have

$$\epsilon(t) = -m\left[K_1(2mt) - K_0(2mt)\right] \;, \tag{4.69}$$

whereas for free boundary conditions

$$\epsilon(t) = m\left[K_1(2mt) + K_0(2mt)\right] \;, \tag{4.70}$$

[6] All correlation functions will be computed in the euclidean space obtained by the analytic continuation $t \to it$.

In the general case, the one-point function interpolates between the two curves. The critical exponent of the energy operator in the presence of the defect line can be extracted by looking at the ultraviolet limit $t \to 0$ of its one-point function. For this limit we have

$$\epsilon_0(t) \sim \frac{\sin\chi}{2t} \ . \tag{4.71}$$

From this expression, we see that the defect line does not influence the critical exponent of the energy operator, which is the same as in the bulk, but rather enters the universal amplitude of the one-point function. For the pure reflecting case, the universal amplitudes coincide with those calculated in [80].

The relationship between the coupling constant in the continuum theory and in the discrete formulation can be extracted by comparing eq. (4.71) with the analogous lattice computation, which reads [96]

$$\epsilon_0(t) \sim \frac{\tanh 2(J-\tilde{J})}{2t} \ . \tag{4.72}$$

Hence, we have the following identification

$$\sin\chi \ = \ \tanh 2(J-\tilde{J}) \ . \tag{4.73}$$

In addition to the one-point function of the energy operator, it is also interesting to compute its two-point function. To simplify calculations, it is convenient to define the function

$$F(x,t) \ = \ \frac{1}{2}\int_{-\infty}^{\infty} d\beta\, \frac{\exp[-t\cosh\beta + ix\sinh\beta]}{\cosh\beta - \sin\chi} \ . \tag{4.74}$$

Let us initially consider the situation where the energy density operators are on opposite sides of the defect line, i.e. $t_2 > 0$ and $t_1 < 0$. The relevant expression in this case is given by[7]

$$G_1(\rho_1,\rho_2) \ = \ \sum_{i,j} < 0 \mid \epsilon(\rho_2) \mid i >< i \mid \mathcal{D} \mid j >< j \mid \epsilon(\rho_1) \mid 0 > \ . \tag{4.75}$$

As before, the above series terminate. To explicitly evaluate it, in addition to the matrix elements $\mathcal{D}_{2,0}$ and $\mathcal{D}_{0,2}$, we also need the matrix element $\mathcal{D}_{2,2}$ given by

$$\begin{aligned} < \beta_1,\beta_2 \mid \mathcal{D} \mid \theta_1,\theta_2 > \ = \ & (2\pi)^2 \Big[\hat{R}(\beta_1)\,\hat{R}(\theta_1)\delta(\beta_1+\beta_2)\delta(\theta_1+\theta_2) + \\ & + \hat{T}(\beta_1)\,\hat{T}(\beta_2)\delta(\beta_1-\theta_1)\,\delta(\beta_2-\theta_2) + \\ & - \hat{T}(\beta_1)\,\hat{T}(\beta_2)\,\delta(\beta_1-\theta_2)\,\delta(\beta_2-\theta_1)\Big] \ . \end{aligned} \tag{4.76}$$

[7] To simplify the notation, in the sequel we denote the couple of coordinate (x_i, t_i) simply by ρ_i.

With the notation $t \equiv t_2 - t_1$ and $x \equiv x_2 - x_1$, eq. (4.75) can be expressed as

$$\begin{aligned} G_1(\rho_1, \rho_2) &= \cos^2\chi \left[\left(\frac{\partial^2}{\partial x \partial t} F(x,t) \right)^2 + \left(\frac{\partial^2}{\partial t^2} F(x,t) \right)^2 - \left(\frac{\partial}{\partial t} F(x,t) \right)^2 \right] + \\ &\quad + \epsilon_0(t_1)\, \epsilon_0(t_2) \ . \end{aligned} \tag{4.77}$$

When the defect line acts as a pure reflecting surface, all fluctuations across it are suppressed and this formula correctly reduces to the vacuum expectation values of the energy densities.

Let us consider now the situation where the two energy operators are on the same side of the defect line, with $t_2 \geq t_1 > 0$. For convenience, let us introduce the notation $t \equiv t_2 - t_1$, $\bar{t} \equiv t_2 + t_1$, $x \equiv x_2 - x_1$ and $r \equiv \sqrt{x^2 + t^2}$. The two point function can be written in this case as

$$G_2(\rho_1, \rho_2) = \sum_{i,j} < 0 \mid \epsilon(\rho_2) \mid i >< i \mid \epsilon(\rho_1) \mid j >< j \mid \mathcal{D} \mid 0 > \ . \tag{4.78}$$

There are only a finite number of non-vanishing matrix elements of the energy density and therefore the series truncates. It can be written as

$$G_2(\rho_1, \rho_2) = I_1 + I_2 + I_3 \ , \tag{4.79}$$

where

$$\begin{aligned} I_1 &= \frac{1}{2!} \int_{-\infty}^{+\infty} \frac{d\beta_1}{2\pi} \frac{d\beta_2}{2\pi} < 0 \mid \epsilon(\rho_2) \mid \beta_1, \beta_2 >< \beta_1, \beta_2 \mid \epsilon(\rho_1) \mid 0 >< 0 \mid \mathcal{D} \mid 0 > \\ I_2 &= \frac{1}{2!2!} \int_{-\infty}^{+\infty} \frac{d\beta_1}{2\pi} \cdots \frac{d\beta_4}{2\pi} < 0 \mid \epsilon(\rho_2) \mid \beta_1, \beta_2 >< \beta_1, \beta_2 \mid \epsilon(\rho_1) \mid \beta_3, \beta_4 >< \beta_3, \beta_4 \mid \mathcal{D} \mid 0 > \\ I_3 &= \frac{1}{2!4!} \int_{-\infty}^{+\infty} \frac{d\beta_1}{2\pi} \cdots \frac{d\beta_6}{2\pi} < 0 \mid \epsilon(\rho_2) \mid \beta_1, \beta_2 > \\ &\quad \times < \beta_1, \beta_2 \mid \epsilon(\rho_1) \mid \beta_3, \cdots, \beta_6 >< \beta_3, \cdots, \beta_6 \mid \mathcal{D} \mid 0 > \ . \end{aligned} \tag{4.80}$$

I_1 coincides with the two-point function of the energy operator in the bulk,

$$I_1 = m^2 \left[\left(\frac{\partial}{\partial x} K_0(mr) \right)^2 + \left(\frac{\partial}{\partial t} K_0(mr) \right)^2 - (K_0(mr))^2 \right] \ .$$

The quantities which appear in I_2 and I_3 are the higher matrix elements of the energy density (which may be directly computed by the fermionic representation of this operator, $\epsilon = i\overline{\Psi}\Psi$) and the matrix elements of the defect operator $\mathcal{D}$, given by (4.65). Considering that the computation of these quantities is lengthy but straightforward, we shall only present the final result

$$\begin{aligned} I_2 &= 2\, m^2 \sin\chi \left[\left(\frac{\partial}{\partial x} K_0(r) \right) \left(\frac{\partial}{\partial x} F(x, \bar{t}) \right) - K_0(r) \left(\frac{\partial^2}{\partial x^2} F(x, \bar{t}) \right) \right] \ , \\ I_3 &= m^2 \sin^2\chi \left[\left(\frac{\partial}{\partial x} F(x, \bar{t}) \right)^2 - \left(\frac{\partial^2}{\partial x^2} F(x, \bar{t}) \right)^2 - \left(\frac{\partial^2}{\partial x \partial \bar{t}} F(x, \bar{t}) \right)^2 \right] + \\ &\quad + \epsilon_0(t_1) \epsilon_0(t_2) \ . \end{aligned}$$

Returning to eq. (4.78), the two-point function can be cast in the form

$$
\begin{aligned}
G_2(\rho_1,\rho_2) &= \epsilon_0(t_1)\epsilon_0(t_2) + \left[\frac{\partial}{\partial x}K_0(r) + \sin\chi\,\frac{\partial}{\partial x}F(x,\bar{t})\right]^2 + \left[\frac{\partial}{\partial t}K_0(r)\right]^2 \\
&- \left[\sin\chi\,\frac{\partial}{\partial x\partial \bar{t}}F(x,\bar{t})\right]^2 - \left[K_0(r) + \sin\chi\,\frac{\partial^2}{\partial x^2}F(x,\bar{t})\right]^2 \quad . \qquad (4.81)
\end{aligned}
$$

It is now easy to verify that the expressions (4.77) and (4.81) coincide with those obtained in the lattice calculation [96].

As our last example, we discuss the one-point function of the magnetization operator $\sigma(\rho)$ in the low temperature phase in the presence of the defect line. It can be calculated through the formula

$$
\sigma_0(t) = \sum_{n=0}^{\infty} < 0 \mid \sigma(\rho) \mid n >< n \mid \mathcal{D} \mid 0 > \quad . \qquad (4.82)
$$

The magnetization operator couples the vacuum to all states with an even number of particles and its form factors are given by [36, 33, 29]

$$
< 0 \mid \sigma(0,0) \mid \beta_1, \ldots \beta_{2n} > = (-i)^n \prod_{i<j} \tanh\frac{\beta_i - \beta_j}{2} \quad . \qquad (4.83)
$$

Since the matrix elements of $\mathcal{D}$ in (4.82) are different from zero only for pairs of particles of opposite momentum, we are lead to consider the matrix elements of the magnetization operator given by $< 0 \mid \sigma(0) \mid -\beta_1, \beta_1, \ldots, -\beta_n, \beta_n >$. They can be conveniently written as

$$
< 0 \mid \sigma(0,0) \mid -\beta_1, \beta_1, \ldots - \beta_n, \beta_n > = i^n \left(\prod_{i=1}^{n} \tanh\beta_i\right) \times \det W(\beta_i,\beta_j) \,, \qquad (4.84)
$$

where $W(\beta_i,\beta_j)$ is the $n \times n$ matrix given by

$$
W(\beta_i,\beta_j) = \left(\frac{2\sqrt{\cosh\beta_i\cosh\beta_j}}{\cosh\beta_i + \cosh\beta_j}\right) \quad . \qquad (4.85)
$$

Hence, the one-point function is the sum of an infinite number of terms shown in Fig. 4.12 and it can be expressed as a Fredholm determinant

$$
\begin{aligned}
\sigma_0(t) &= \sum_{n=0}^{\infty}\frac{1}{n!}\int_{-\infty}^{\infty} d\beta_1\ldots d\beta_n \left(\prod_{k=0}^{n} i\,\tanh\beta_k\,\hat{R}(\beta_k)\,e^{-2mt\cosh\beta_k}\right) \det W(\beta_i,\beta_j) = \\
&= \mathrm{Det}\,(1 + z\,\mathcal{W}) \quad . \qquad (4.86)
\end{aligned}
$$

The explicit form of the kernel is given by

$$
\mathcal{W}(\beta_i,\beta_j,\chi) = \frac{E(\beta_i,mt,\chi)E(\beta_j,mt,\chi)}{\cosh\beta_i + \cosh\beta_j} \,, \qquad (4.87)
$$

where

$$E(\beta, mt, \chi) = \sinh\beta \, e^{-mt\cosh\beta} (\cosh\beta - \sin\chi)^{-1/2} \quad , \quad z = \frac{\sin\chi}{2\pi} \; . \tag{4.88}$$

In terms of the eigenvalues of the integral operator and their multiplicity, $\sigma_0(t)$ can be also expressed as

$$\sigma_0(t) = \prod_{i=1}^{\infty} (1 + z\,\lambda_i)^{a_i} \tag{4.89}$$

As far as mt is finite, the kernel is square integrable and therefore all results valid for bounded symmetric integral operators apply (see, for instance [102]). In particular, for $mt \to \infty$, $\sigma_0(t)$ falls off exponentially to the bulk vacuum expectation value. However, when $mt \to 0$, the integral operator becomes unbounded. The multiplicity of the eigenvalues grows logarithmically as $a \sim \frac{1}{\pi} \ln \frac{1}{mt}$ whereas the eigenvalues become dense in the interval $(0, \infty)$ according to the distribution

$$\lambda(p) = \frac{2\pi}{\cosh \pi p} \; . \tag{4.90}$$

Hence, for the critical exponent of the magnetization operator, defined by

$$\sigma_0(t) \sim \frac{C}{(2t)^{x_\sigma}} \; , \; t \to 0 \; , \tag{4.91}$$

we have

$$x_\sigma(\chi) = -\frac{1}{\pi} \int_0^\infty dp \ln\left(1 + \frac{2\pi z}{\cosh p}\right) = -\frac{1}{8} + \frac{1}{2\pi^2} \arccos^2(-\sin\chi) \; . \tag{4.92}$$

This expression agrees with the lattice calculations [90, 91] and since it depends on the coupling constant, it explicitly shows the non-universality of the model.

4.6 Concluding remark

The main purpose of this chapter was to prove that the bootstrap approach can be successfully extended to integrable models with linear inhomogeneities and that the computation of the correlation functions for those systems can be achieved by means of a suitable generalization of the form factor techniques. We have analysed the general situation in which translation invariance is broken by the presence of defect lines allowing reflection and transmission processes. While at the moment it is still an open problem to see whether there are other solutions of the transmission-reflection equations in addition to the fermionic and the bosonic theories analysed in the text, it is worth to stress that the method for the computation of the correlation functions exposed in section 4.5 is expected to work without limitation in the pure reflecting case. This corresponds to the boundary field theories which have recently received a lot of attention in view of their potential application to a wide class of physical situations.

Conclusion

The principal aim of this thesis was the exposition of the usefulness of the bootstrap method in the description of the integrable renormalization group flows. A quantum field theory or a statistical model is solved once we know its correlation functions. The bootstrap approach to the solution of a given integrable theory can be summarized into three steps.

i) Solution of the scattering problem: the identification of the spectrum of 'infrared free' particles and the computation of their scattering amplitudes, which according to integrability reduces to the problem of finding the two particle scattering amplitudes.

ii) Solution of the form factors equations for local operators.

iii) Computation of the correlation functions by spectral representation, i.e. resummation of the intermediate state sum.

Homogeneous systems, i.e. systems with usual symmetry properties under translations and rotations (Euclidean or Minkowskian), can be divided into two classes, the massive and the massless ones, according to the dispersion relations of their asymptotic particles.

This program has been originally developed in the study of the massive integrable flows and it has been carried out very intensively in the last fifteen years. The scattering amplitudes of a large variety of models have been exactly calculated [11, 21]. The form factors problem also has been solved for some of these theories [33, 7, 19, 32, 103], although such solutions are often expressed by means of implicit integral representations. In this thesis we have presented some explicit solutions to the form factors equation of two lagrangian massive integrable models, the sinh-Gordon and the Bullough-Dodd ones. Many interesting features of these models can be described making only use of the knowledge of the form factors of some local operators. The main result of this analysis is the description of a one parameter family of renormalization group flows sharing the same quantum dynamics. Quantum dynamics in our approach means a complete set of solutions to the form factors equations. Thus, given a scattering system, the corresponding set of form factors of local operators is in principle fixed. The freedom we have in the specification of the renormalization group flows is associated to the freedom in the identification of the stress-energy tensor among the local operators. This arbitrariness, is very naturally understood in the renormalization group language since it corresponds to a persistently marginal direction. Nevertheless, it opens an interesting two-dimensional quantum field theory problem, namely the problem of the effective description of the different resulting ultraviolet conformal field theories as massless limit of a given massive integrable set of correlation functions. A satisfactory answer to this problem may be relevant for the description of conformal field theories with central charge greater than one [104].

The computation of the massive correlation functions – item iii) of the bootstrap method – has exhibited a strong efficiency of this approach as shown in refs. [33, 7, 66]. The reason for the success of the form factor method is the following. Firstly, the large distance regime is well described with a low number of particles in the spectral sum, due to the exponential suppression typical of massive theories. This spectral argument is enhanced by the dynamical fact that the form factors vanish at the n-particle thresholds [8]. As a consequence, the correlation function increases smoothly as we decrease the scale of distances crossing those thresholds. This dynamical suppression phenomenon [66] is at the origin of the validity of the form factor representations of the correlation functions also in the crossover regime between the infrared and the ultraviolet region.

The problem of the computation of the correlation functions can also be faced with other techniques. For some quantum integrable models [98, 106], it has been shown that the correlation functions are solutions of classically integrable differential equations. This reduction of the quantum problem to a classical one is in principle very far reaching, although the derivation of this equivalence is very involved in its original formulation. However, in ref. [105] that result is obtained by showing that the spectral sum on the form factors of the correlation functions can be directly interpreted as a Fredholm determinant. Therefore, once the form factors problem is solved, there is some hope that the above identification can be derived more directly. Since the general n^{th}-form factor of the sinh-Gordon model can be given a matrix-determinant form, it is natural to try the application of the functional techniques of [105] in order to gain the Fredholm kernels. A hint in this direction could also come from the quantum equations of motion we

[8] This is strictly true for the theories which we consider, since they have the property that $S(0) = -1$.

have found for our models simply by comparing different form factors solutions – one of which corresponds to the d'alambertian of the elementary interpolating field.

The investigation of the massless set of theories is much less developed. In principle, dealing with massless infrared degrees of freedom in terms of scattering particles is a quite complex and involved approach. However, the integrability condition in two dimensions allows us to use a straightforward generalization of the massive scattering formalism even to the massless case, as shown in some examples analysed in [69, 72]. In those references the form factors part of the bootstrap program had not been developed. The original contribution contained in the sector of this dissertation devoted to the massless flows is the setting up of the form factors bootstrap program. This method has been applied to the very rich example constituted by the flow between the tricritical Ising model and the Ising model, the last massless flow of the minimal unitary series $M_p \to M_{p-1}$. With regard to the solution of the form factors equations, everything goes through in complete analogy with the massive situation. Explicit solutions have been found for the most important fields of the theory: the stress-energy tensor, the spin energy density, the order and disorder operators. For the operators in the energy sector, the correlation functions computation is as effective as for the massive case. The c-theorem sum rule is successfully checked, as well as the ultraviolet and infrared behaviour of the correlation functions. However, the massless nature of the spectrum prevents in general the computation of the correlation functions. The expected infrared divergencies occur in the computation of each contribution to the spectral sum of the magnetization and disorder operators, since the infrared behaviour of their form factors is not 'soft' enough. Nevertheless, it is possible to recover the correct infrared anomalous dimensions by a resummation technique over the spectral sum of the correlation function with a cut-off. Therefore, the idea to pursue in the next future is the attempt of finding a paradigma for a generic massless flow in order to classify operators with respect to their infrared behaviour, or in other words with respect to the computability of their correlation functions; likewise, it would be interesting the investigation of methods to extract some physical information such as ultraviolet and infrared conformal data also from the 'diverging' operators[9].

[9] Note that a positive solution to this problem would imply as a by-product the possibility to describe conformal field theories in terms of massless particles. Each primary field could, for example, be associated to a convenient base of left and right 'parafermions', such that the spectral resummations as well as the infrared problem become trivial. Anyway, the comparison of the usual conformal field theory formulation with the desirable one we have alluded to could be relevant with respect to the study of the structure of quantum field theories.

The study of integrable theories with boundary has recently received a great impulse [79, 80, 83, 84, 85, 86, 87]. The methods of the bootstrap approach turned out to be very fruitful. In addition to the scattering amplitudes of the particles in the bulk, the introduction of the reflection amplitudes of the particles on the boundary provides with an effective description of these systems. Many examples of solutions are given in the references cited above. Treating the boundary as an impenetrable obstacle to the motion of the particles on the line, it is natural the attempt to generalize the above results also to the case in which some transmission across the obstacle is allowed. In the last part of this thesis we have derived the implementation of the bootstrap method for a general line of dishomogeneity. We have been therefore led to the introduction of the transmission amplitudes of the particles across the defect. An immediate and rather strong restriction emerges when the integrability condition is imposed: for bulk-diagonal theories, the only theories with non-vanishing reflection and transmission amplitudes are those which are quasi-free in the bulk, the free fermionic and bosonic ones. A detailed derivation and a physical interpretation of the reflection and transmission amplitudes of these two kind of theories are given, in presence both of a single defect and of a multiple series of defects. The possibility to understand the defect, by means of a Wick rotation, also as a sudden widespread perturbation at the instant of time $t = 0$ is exploited to develop a method of computation of the correlation functions based on the bulk form factors. The physical properties of the impurity is now encoded into a defect operator whose matrix elements are given in terms of the creation (annihilation) amplitudes of pairs of particles and of the time delays in the motion of the particles. The main correlation functions of the Ising model in presence of a line of defect have been computed with this technique; they reproduce correctly the lattice model results. Our restrictive result on bulk-diagonal theories implies that the reflection-transmission integrable theory we propose could become a true new interesting subject only if nontrivial solutions to the Yang-Baxter equations will be found. Nevertheless, for the pure reflecting case many non trivial solutions have been found. Hence, the defect (boundary) operator is known and the form factors problem in the bulk can be solved along the line described in this thesis. Therefore, the technique we have developed for the computation of the correlation functions by means of the form factors in the bulk can reveal itself powerful for the integrable theories with boundary.

As a last possible future development that our results can have, we mention that our methods could perhaps be useful also in the description of the interpolating regime between fixed points on critical surfaces of particular condensed matter systems, such as the quantum wires [107] or the Kondo effect [67], which are examples of massless flows in presence of a defect. For such systems the conformal field theory description [67] of the fixed points as well as the spectrum of the infrared particles [108] are already been exhibited. Even experimentally measurable quantities like the conductance through different quantum Hall edges [109] has been computed within scattering methods! Therefore it seems quite urgent to address ourselves to the problem of the computation of the correlation functions also for these theories in the framework of form factors.

Acknowledgements

It is a pleasure to express my gratitude to Giuseppe Mussardo. His scientific help has friendly guided me throughout all my permamence at Sissa.

Many thanks also go to the colleagues and friends with whom I have had the opportunity to collaborate or to have fruitful discussions: Luca Griguolo, Pietro Donatis, Marco Billó, Vlad Inculetz, Andreas Fring, Anni Koubek, Marta Nolasco, Adam Schwimmer, Loriano Bonora, Franco Strocchi, Andrea Cavagna and Gesualdo Delfino. Aldo, working together this year has been really exciting.

Finally, I wish to thank the staff and the elementary particle sector. They have contributed to make my long stay at Sissa very pleasant.

Bibliography

[1] K.G. Wilson and J. Kogut, *Phys. Rep.* **12** (1974), 200.

[2] P. Pfeuty and G. Toulouse, *Introduction to the Renormalization Group and to Critical Phenomena*, (J. Wiley) 1967.

[3] D.J. Amit, *Field Theory, the Renormalization Group and Critical Phenomena*, (Mc-Graw Hill Inc.) 1978.

[4] S.K. Ma, *Modern Theory of Critical Phenomena*, (W.A. Benjamin Inc.) 1976.

[5] A.B. Zamolodchikov and Al.B. Zamolodchikov, *Conformal Field Theory and 2d Critical Phenomena (I-VI)*, ITEP-112/89-173/89-31/90-83/90-91/90-103/90.

[6] J.L. Cardy, G. Mussardo, *Phys. Lett.* **B225** (1989) , 275.

[7] Al.B. Zamolodchikov, *Nucl. Phys.* **B348** (1991), 619.

[8] C. Itzykson and J.B. Zuber, *Quantum Field Theory*, (McGraw-Hill Inc.) 1985.

[9] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, *Nucl. Phys.* **B241** (1984), 333.

[10] P. Ginsparg, Applied Conformal Field Theory, in *Fields, Strings and Critical Phenomena*, proc. Les Houches XLIX 1988, (Elsevier) 1989; J.L. Cardy, Conformal Invariance and Statistical Mechanics, in *Fields, Strings and Critical Phenomena*, proc. Les Houches XLIX 1988, (North-Holland, Amsterdam) 1989; C. Itzykson, H. Saleur and J.B. Zuber, *Conformal Invariance and Applications to Statistical Mechanics*, (World Scientific) 1988; P. Christe and M. Henkel, *Introduction to Conformal Invariance and Its Applications to Critical Phenomena*, (Springer-Verlag) 1993.

[11] A.B. Zamolodchikov, Al.B. Zamolodchikov, *Ann.Phys.* **120** (1979), 253.

[12] D. Iagolnitzer, *The S-Matrix*, (North-Holland, Amsterdam) 1978; D. Iagolnitzer, *Phys. Rev.* **D 18** (1978), 1275.

[13] A.B. Zamolodchikov, in *Advanced Studies in Pure Mathematics* **19** (1989), 641; *Int. J. Mod. Phys.***A3** (1988), 743.

[14] L.D. Faddeev and L.A. Takhtajan, *Hamiltonian Method in the Theory of Solitons*, (Springer, N.Y.) 1987.

[15] A.V. Mikhailov, M.A. Olshanetsky and A.M. Perelomov, *Comm. Math. Phys.* **79** (1981), 473; G. Wilson, *Ergod. Th. Dyn. Syst.* **1** (1981), 361.

[16] E.K. Sklyanin, *Lett. Math. Phys.* **15** (1988), 357; L. Faddeev, Integrable Models in (1+1)-dimensions Quantum Field Theory, in *Recent Advances in Field Theory and Statistical Mechanics*, proc. Les Houches 1982, (North-Holland, Amsterdam) 1984; V.E. Korepin, A.G. Izergin and N.N. Bogoliubov, *Quantum inverse scattering method and correlation functions*, (Cambridge Univ. Press) 1994.

[17] R.F. Streater and A.S. Wightman, *PCT, Spin, Statistics and All That*, (W.A. Benjamin Inc.) 1964.

[18] F. Strocchi, *A Short Introduction to the Foundations of Quantum Field Theory*, Lecture Notes ISAS (Trieste) 1985-1990.

[19] F. A. Smirnov, *Form Factors in Completely Integrable Models of Quantum Field Theory*, (World Scientific) 1992, and references therein.

[20] F. A. Smirnov, in *Introduction to Quantum Group and Integrable Massive Models of Quantum Field Theory*, Nankai Lectures on Mathematical Physics, (World Scientific) 1990.

[21] G. Mussardo, *Phys. Reports* **218** (1992), 215 and references therein.

[22] G. Barton, *Introduction to Dispersion Techniques in Field Theory*, (W.A. Benjamin Inc.) 1965.

[23] R.J. Eden, P.V. Landshoff, D.I. Olive and J.C. Polkinghorne, *The Analytic S-Matrix*, (Cambridge Univ. Press) 1966.

[24] M. Karowski, *Nucl. Phys.* **B 153** (1979), 244.

[25] R. Köberle and J.A. Swieca, *Phys. Lett.* **86B** (1979), 209.

[26] K. Hepp, *Helv. Phys. Acta* **32** (1964), 639.

[27] P. Mitra, *Phys. Lett.* B **72** (1977), 62.

[28] T.R. Klassen and E. Melzer, *Nucl. Phys.* **B 338** (1990), 485.

[29] J.L. Cardy and G. Mussardo, *Nucl. Phys.* **B340** (1990), 387.

[30] P. Christe, *Int. J. Mod. Phys.* **A6** (1991), 5271.

[31] A. Koubek, DAMTP-HEP-94/15.

[32] F.A. Smirnov, *J. Phys.* **A17** (1984), L873; F.A. Smirnov, *J. Phys.* **A19** (1984), L575; A.N. Kirillov and F.A. Smirnov, *Phys. Lett.* **B198** (1987), 506; A.N. Kirillov and F.A. Smirnov, *Int. J. Mod. Phys.* **A3** (1988), 731.

[33] V.P. Yurov and Al. B. Zamolodchikov, *Int. J. Mod. Phys.* **A6** (1991), 3419.

[34] M.Yu. Lashkevich, LANDAU-94-TMP-4 (hep-th/9406118).

[35] K.M. Watson, *Phys. Rev.* **95** (1954), 228.

[36] B. Berg, M. Karowski, P. Weisz, *Phys. Rev.* **D19** (1979), 2477; M. Karowski, P. Weisz, *Nucl. Phys.* **B139** (1978), 445; M. Karowski, *Phys. Rep.* **49** (1979), 229;

[37] C. Destri and H.J. De Vega, *Nucl. Phys.* **B 358** (1991) 251.

[38] I.G. MacDonald, *Symmetric Functions and Hall Polynomials*, (Clarendon Press, Oxford) 1979.

[39] A.B. Zamolodchikov, *JETP Lett.*, **43** (1986), 730; A.B. Zamolodchikov, *Sov. J. Nucl. Phys.*, **46** (1987), 1090; J.L. Cardy and A.W.W. Ludwig, *Nucl. Phys.*, **B 285** (1987), 687.

[40] J.L. Cardy, *Phys. Rev. Lett.*, **60** (1988), 2709.

[41] A. Cappelli, D. Friedan and J.L. Latorre, *Nucl. Phys.* **B352** (1991), 616.

[42] D.Z. Freedman, J.I. Latorre and X. Vilasis, *Mod. Phys. Lett.* **A6** (1991), 531.

[43] F.A. Smirnov, *Nucl. Phys.* **B337** (1989), 156; *Int. J. Mod. Phys.* **A4** (1989), 4213.

[44] A. Fring, G. Mussardo and P. Simonetti, *Nucl. Phys.* **B 393** (1993), 413.

[45] Al. B. Zamolodchikov, *Nucl. Phys.* **B342** (1990), 695; *Nucl. Phys.* **B358** (1990), 497; *Nucl. Phys.* **B358** (1990), 524.

[46] G. Mussardo and P. Simonetti, *Int. J. Mod. Phys.*A **19** (1994), 3307.

[47] Vl.S. Dotsenko and V.A. Fateev, *Nucl. Phys.* **B240** [FS 12] (1984), 312; *Nucl. Phys.* **B251** [FS 13] (1985), 691; *Phys. Lett.* **B 154** (1985), 291.

[48] G. Felder, *Nucl. Phys.* **B317** (1989), 215.

[49] T.J. Hollowood and P. Mansfield, *Phys. Lett.* **B226** (1989), 73.

[50] P. Mansfield, *Nucl. Phys.* **B208** (1982), 277; P. Mansfield, *Nucl. Phys.* **B222** (1983), 419.

[51] O. Babelon and L. Bonora, *Phys. Lett.* **B244** (1990), 220.

[52] P. Christe and G. Mussardo, *Nucl.Phys. B* **B330** (1990), 465; P. Christe and G. Mussardo *Int. J. Mod. Phys.* **A5** (1990), 1025; H. W. Braden, E. Corrigan, P. E. Dorey, R. Sasaki, *Nucl. Phys.* **B338** (1990), 689; H. W. Braden, E. Corrigan, P. E. Dorey, R. Sasaki, *Nucl. Phys.* **B356** (1991), 469; V. A. Fateev and A.B. Zamolodchikov, *Int. J. Mod. Phys.* **A5** (1990), 1025.

[53] H.W. Braden and R. Sasaki, *Phys. Lett* **B255**(1991) 343.

[54] E.K. Sklyanin, *Nucl. Phys.* **B326** (1989), 719.

[55] I.M Khamitov, *J. Sov. Math.* **40** (1988), 115.

[56] A.N. Kirillov, *J. Sov. Math.* **40** (1988), 6.

[57] R. Sasaki and I. Yamanaka, in *Advanced Studies in Pure Mathematics* **16** (1988), 271.

[58] A.E. Arinshtein, V.A. Fateev and A.B. Zamolodchikov, *Phys. Lett.* **87B** (1979), 389.

[59] P.E. Dorey, *Nucl. Phys* **B358** (1991) 654; A. Fring, H.C. Liao and D.I. Olive, *Phys. Lett.* **B266** (1991) 82; A. Fring and D.I. Olive, *Nucl. Phys.* **B379** (1992), 429.

[60] A. Koubek and G. Mussardo, *Phys. Lett.* **B 311** (1993), 193.

[61] J.L. Cardy and G. Mussardo, *Nucl. Phys.* **B 410** (1993), 451 .

[62] R.K. Dodd and R.K. Bullough, *Proc. Roy. Soc. Lond* **A352** (1977), 481.

[63] A.V. Zhiber and A.B. Shabat, *Dokl. Akad. Nauk URSS* **247** (1979), No. 5.

[64] A. Fring, G. Mussardo and P. Simonetti, *Phys. Lett.* **B 307** (1993), 83.

[65] C. Ahn, G. Delfino and G. Mussardo, *Mapping between the Sinh-Gordon and Ising Models*, IC/93/143, ISAS/EP/93/89.

[66] G. Delfino and G. Mussardo, *Phys. Lett.* **B 324** (1994), 40; G. Mussardo, *Form Factor Approach to Correlation Functions in Integrable QFT*, in Lecture Notes of High Energy Summer School, ICTP (Trieste), 1993;.

[67] I. Affleck, UBCTP-93-25; I. Affleck and A.W.W. Ludwig, *Nucl. Phys.* **B 352** (1991), 849; I. Affleck and A.W.W. Ludwig, *Nucl. Phys.* **B 360** (1991), 641; P. Fendley, *Phys. Rev. Lett.* **71** (1993), 2485; E. Wong and I. Affleck, UBCTP-93-20.

[68] A.B. Zamolodchikov and Al.B. Zamolodchikov, *Nucl. Phys.* **B 379** (1992), 602.

[69] Al.B. Zamolodchikov, *Nucl. Phys.* **B 358** (1991), 524.

[70] A.B. Zamolodchikov, *Sov. J. Nucl. Phys.* 44 (1986), 529; M. Lassig, G. Mussardo and J.L. Cardy, *Nucl. Phys* **B 348** (1991), 591.

[71] D.A. Kastor, E.J. Martinec and S.H. Shenker, *Nucl. Phys* **B 316** (1989), 590.

[72] P. Fendley, H. Saleur and Al.B. Zamolodchikov, BUHEP-93-5, USC-93/003, LPM-93-07; P. Fendley, H. Saleur and Al.B. Zamolodchikov, BUHEP-93-6, USC-93/004, LPM-93-08; N.Yu Reshetikhin and H. Saleur, USC-93-020, LPM-93-07; P. Fendley and H. Saleur, USC-93-022.

[73] D. Friedan, Z. Qiu and S. Shenker, *Phys. Lett.* **B 151** (1985), 37.

[74] E. Witten, *Nucl. Phys.* **B 202** (1982), 253.

[75] Al.B. Zamolodchikov, *Nucl. Phys.* **B 358** (1991), 497.

[76] K. Binder, in *Phase Transitions and Critical Phenomena*, vol. 8 ed. C. Domb and J. Lebowitz (Academic Press, London 1983).

[77] H.W. Diehl, S. Dietrich and E. Eisenriegler, *Phys. Rev.* **B 27** (1983), 2937.

[78] H.W. Diehl, in *Phase Transitions and Critical Phenomena*, vol. 10 ed. C. Domb and J. Lebowitz (Academic Press, London 1983).

[79] J.L. Cardy, *Nucl. Phys.* B **240** [FS 12] (1984), 514; *Nucl. Phys.* B **324** (1989), 581.

[80] J.L. Cardy and D.C. Lewellen, *Phys. Lett.* B **259**, (1991), 274.

[81] I. Cherednik, *Theor. Math. Phys.* **61** (1984), 977.

[82] E. Sklyanin, *Journ. Phys.* A: Math. Gen. **21** (1988), 2375.

[83] S. Ghoshal and A. Zamolodchikov, *Int. J. Mod. Phys.* A **9** (1994), 3841.

[84] A. Fring and R. Köberle, *Nucl. Phys.* B **421** (1994), 159; *Nucl. Phys.* B **419** (1994), 647; "Boundary Bound States in Affine Toda Field Theory" USP-ISQSC/TH/94-03, hep-th/9404188.

[85] R. Sasaki, "Reflection Bootstrap Equations for Toda Field Theory", preprint hep-th/9401008; E. Corrigan, P.E. Dorey, R.H. Rietdijk and R. Sasaki, *Phys. Lett.* B **333** (1994), 83; E. Corrigan, P.E. Dorey, R.H. Rietdijk, "Aspects of Affine Toda Field Theory on a half-line", hep-th/9407148.

[86] P. Fendley and H. Saleur, "Deriving boundary S-matrices", preprint USC-94-001, hep-th/9402045; "Exact Theory of Polymer Adsorption in Analogy with the Kondo Problem", USC-94-006, cond-mat/9403095; M.T. Grisaru, L. Mezincescu and R.I. Nepomechie, "Direct Calculation of the Boundary S-matrix for the Open Heisenberg Chain", UMTG-177, hep-th/9407089.

[87] S. Ghoshal, "Boundary S-Matrix of the $O(n)$ Symmetric Nonlinear Sigma Model", preprint RU-94-02, hep-th/9401008; L. Chim, "Boundary S-matrix for the Integrable q-Potts model", preprint RU-94-93, hep-th/9404118.

[88] T.W. Burkhardt and E. Eisenriegler, *Phys. Rev.* B **24** (1981), 1236; M.E. Fisher and A.E. Ferdinand, *Phys. Rev. Lett.* **19** (1967), 169; M.P. Nightingale and H.W.J. Blöte, *J. Phys. A: Math. Gen.* **15** L33.

[89] F. Igloi, I. Peschel and L. Turban, *Adv. in Phys.* **42** (1993), 683.

[90] R.Z. Bariev, *Sov. Phys. JETP* **50** (1979), 613.

[91] B.M. McCoy and J.H.H. Perk, *Phs. Rev. Lett.* **44** (1980), 840.

[92] L.P. Kadanov, *Phys. Rev.* **B 24** (1981), 5382.

[93] A.C. Brown, *Phys. Rev.* **B 25** (1982), 331.

[94] M. Henkel and A. Patkos, *J. Phys.* **A 20** (1987), 2199; *Nucl. Phys.* **B 285** [FS 19] (1987), 29; U. Grimm, *Nucl. Phys.* **B 340** (1990), 633.

[95] T.W. Burkhardt and J.Y. Choi, *Nucl. Phys.* **B 376** (1992), 447.

[96] L.F. Ko, H. Au-Yang and J.H.H. Perk, *Phys. Rev. Lett.* **54** (1985), 1091.

[97] G. Delfino, G. Mussardo and P. Simonetti, *Phys. Lett.* **B 328** (1994), 123.

[98] T.D. Schultz, D.C. Mattis and E.H. Lieb, *Rev. Mod. Phys.* **36** (1964), 856; M. Sato, T. Miwa and M. Jimbo, *Proc. Japan Acad.* **A 53** (1977), 147, 153, 183; A. Luther and I. Peschel, *Phys. Rev.* **B 12** (1975), 3908; M. Bander and C. Itzykson, *Phys. Rev.* **D 15** (1977), 463; B.M. McCoy and T.T. Wu, *The Two-Dimensional Ising Model* (Harvard Univ. Press, Cambridge, 1982) and references therein.

[99] H.J. Lipkin, *Quantum Mechanics: new approaches to selected topics* (North Holland, Amsterdam 1973), Chapter 8.

[100] E.I. Blount, in *Solid State Physics*, vol. 13, ed. H. Ehrenreich, F. Seitz and D. Turnbull, (Academic, New York, 1962).

[101] E.M. Lifshitz and L.P. Pitaevskii, *Relativistic Quantum Theory* (Pergamon Press, Oxford 1973), Chapter 11.

[102] M. Bocher, *An Introduction to the Study of Integral Equations*, Hafner Publishing Co. N.Y.

[103] C. Efthimiou and A. LeClair, CLNS-93-1263, hep-th/9312121; J. Balog, *Phys. Lett.* **B 300** (1992), 145; J. Balog and T. Hauer, KFKI-1994-10-A, hep-th/9406155; S. Lukyanov, hep-th/9307196.

[104] N. Seiberg, *Prog. Theor. Phys.* **Suppl. 102** (1990), 319.

[105] O. Babelon and D. Bernard, *Phys. Lett.* **B 288** (1992), 113; D. Bernard and A. LeClair, CLNS-94-1276, hep-th/9402144.

[106] V.E. Korepin, A.G. Izergin and N.N. Bogoliubov, *Quantum inverse scattering method and correlation functions*, Cambridge University Press, 1994.

[107] C.L. Kane and M.P.A. Fisher, *Phys. Rev. Lett.* **68** (1992), 1220; C.L. Kane and M.P.A. Fisher, *Phys. Rev.* B **46** (1992), 15233.

[108] P. Fendley, *Phys. Rev. Lett.* **71** (1993), 2485.

[109] P. Fendley, A.W.W. Ludwig and H. Saleur,USC-94-12 PUPT-94-1491.

B	C_{UV}^{num}	C_{UV}^{Liouv}
$\frac{1}{10}$	127.28994	127.31579
$\frac{1}{5}$	67.61695	67.66667
$\frac{3}{10}$	47.98763	48.05882
$\frac{2}{5}$	38.40998	38.5
$\frac{1}{2}$	32.89395	33.
$\frac{3}{5}$	29.45222	29.57143
$\frac{7}{10}$	27.24418	27.37363
$\frac{4}{5}$	25.86323	26.
$\frac{9}{10}$	25.10126	25.24242
1	24.85738	25.

Table 2.1 . The first two-particle term entering the sum rule of the c-theorem for the sinh-Gordon model with the choice $a = 1$ (second column) compared with the central charge (2.47) of conformal field theory.

B	C_{UV}^{num}
$\frac{1}{500}$	0.9999995
$\frac{1}{100}$	0.9999878
$\frac{1}{10}$	0.9989538
$\frac{3}{10}$	0.9931954
$\frac{2}{5}$	0.9897087
$\frac{1}{2}$	0.9863354
$\frac{2}{3}$	0.9815944
$\frac{7}{10}$	0.9808312
$\frac{4}{5}$	0.9789824
1	0.9774634

Table 2.2 . The first two-particle term entering the sum rule of the c-theorem for the sinh-Gordon model with the choice $a = 1/2$. It must be compared with the claimed free boson $C_{UV} = 1$ ultraviolet behaviour.

B	C_{UV}^{num}	C_{UV}^{Liouv}
$\frac{1}{10}$	70.63001	70.63158
$\frac{1}{5}$	41.32883	41.33333
$\frac{3}{10}$	32.10844	32.11765
$\frac{2}{5}$	27.98391	28.
$\frac{1}{2}$	25.97441	26.
$\frac{3}{5}$	25.10474	25.14286
$\frac{7}{10}$	24.97886	25.03297
$\frac{4}{5}$	25.42607	25.5
$\frac{9}{10}$	26.38691	26.48485
1	27.87364	28.
$\frac{11}{10}$	29.96195	30.12121
$\frac{6}{5}$	32.80360	33.
$\frac{13}{10}$	36.66406	36.90110
$\frac{7}{5}$	42.00619	42.28571
$\frac{3}{2}$	49.67928	50.
$\frac{8}{5}$	61.39520	61.75
$\frac{17}{10}$	81.15833	81.52941
$\frac{9}{5}$	120.98370	121.33333
$\frac{19}{10}$	240.90584	241.15789

Table 2.3 . The first two-particle term entering the sum rule of the c-theorem for the Bullough-Dodd model with the choice $a = 1$ (second column) compared with the central charge (2.101) of conformal field theory.

3/2	3/5	1/10	0
7/16	3/80	3/80	7/16
0	1/10	3/5	3/2

Table 3.1 . The table of Virasoro anomalous dimensions $\Delta_{m,n}$ for the tricritical Ising model, $c = 7/10$. The row index is $m = 1, 2, 3$. The column index is $n = 1, 2, 3, 4$.

Field	Anomalous dimensions (Δ, Δ)
$I = \Phi_{(1,1)}(z) \otimes \Phi_{(1,1)}(\bar{z})$	$(0, 0)$
$\sigma(x) = \Phi_{(2,2)}(z) \otimes \Phi_{(2,2)}(\bar{z})$	$(3/80, 3/80)$
$\bar{\sigma}(x) = \Phi_{(2,1)}(z) \otimes \Phi_{(2,1)}(\bar{z})$	$(7/16, 7/16)$
$\varepsilon(x) = \Phi_{(1,2)}(z) \otimes \Phi_{(1,2)}(\bar{z})$	$(1/10, 1/10)$
$\bar{\varepsilon}(x) = \Phi_{(1,3)}(z) \otimes \Phi_{(1,3)}(\bar{z})$	$(3/5, 3/5)$
$\psi(x) = \Phi_{(1,3)}(z) \otimes \Phi_{(1,2)}(\bar{z})$	$(3/5, 1/10)$
$\bar{\psi}(x) = \Phi_{(1,2)}(z) \otimes \Phi_{(1,3)}(\bar{z})$	$(1/10, 3/5)$
$G(x) = \Phi_{(1,4)}(z) \otimes \Phi_{(1,1)}(\bar{z})$	$(3/2, 0)$
$\bar{G}(x) = \Phi_{(1,1)}(z) \otimes \Phi_{(1,4)}(\bar{z})$	$(0, 3/2)$
$X(x) = \Phi_{(1,4)}(z) \otimes \Phi_{(1,4)}(\bar{z})$	$(3/2, 3/2)$
$\mu(x) = G_0\sigma(x) = G_0\sigma(x)$	$(3/80, 3/80)$
$\bar{\mu}(x) = G_0\bar{\sigma}(x) = G_0\bar{\sigma}(x)$	$(7/16, 7/16)$

$$
\begin{aligned}
G \cdot G &= [I] & \sigma \cdot \sigma &= [I] + [\varepsilon] + [\bar{\varepsilon}] + [X] \\
\bar{G} \cdot \bar{G} &= [I] & \bar{\sigma} \cdot \bar{\sigma} &= [I] + [X] \\
G \cdot \bar{G} &= [X] & \sigma \cdot \bar{\sigma} &= [\varepsilon] + [\bar{\varepsilon}] \\
\psi \cdot \psi &= [I] + [\bar{\varepsilon}] & \varepsilon \cdot \sigma &= [\sigma] + [\bar{\sigma}] \\
\psi \cdot \bar{\psi} &= [\varepsilon] + [X] & \varepsilon \cdot \bar{\sigma} &= [\sigma] \\
\varepsilon \cdot \psi &= [\bar{\psi}] + [G] & \bar{\varepsilon} \cdot \sigma &= [\sigma] + [\bar{\sigma}] \\
\bar{\varepsilon} \cdot \psi &= [\psi] + [\bar{G}] & \bar{\varepsilon} \cdot \bar{\sigma} &= [\sigma] \\
\varepsilon \cdot \varepsilon &= [I] + [\bar{\varepsilon}] & X \cdot \sigma &= [\sigma] \\
\bar{\varepsilon} \cdot \bar{\varepsilon} &= [I] + [\bar{\varepsilon}] & \varepsilon \cdot \mu &= [\mu] \\
\varepsilon \cdot \bar{\varepsilon} &= [\varepsilon] + [X] & \sigma \cdot \mu &= [\psi] + [\bar{\psi}] \\
X \cdot X &= [I] & & . \\
G \cdot \varepsilon &= [\psi] & & . \\
G \cdot \psi &= [\varepsilon] & & .
\end{aligned}
$$

Table 3.2 . Primary fields of the maximal semilocal algebra of the tricritical critical Ising model and their fusion rules. The disorder fields $\mu(x)$ and $\bar{\mu}(x)$ are defined in terms of the Fourier zero modes of the fields $G(z) = \sum_{n=0}^{+\infty} z^{n-3/2} G_{-n}$, $\bar{G}(\bar{z}) = \sum_{n=0}^{+\infty} \bar{z}^{n-3/2} \bar{G}_{-n}$.

7/16	1/10	3/80	0
0	3/80	1/10	7/16

Table 3.3 . The table of superconformal anomalous dimensions $h_{p,q}$ for the tricritical Ising model, $c = 7/10$. The row index is $p = 1, 2$. The column index is $q = 1, 2, 3, 4$.

1/2	1/16	0
0	1/16	1/2

Table 3.4 . The table of Virasoro anomalous dimensions $\Delta_{m,n}$ for the critical Ising model, $c = 1/2$. The row index is $m = 1, 2$. The column index is $n = 1, 2, 3$.

Field	Anomalous dimensions (Δ, Δ)
$I = \bar{\Phi}_{(1,1)}(z) \otimes \bar{\Phi}_{(1,1)}(\bar{z})$	$(0, 0)$
$\sigma(x) = \bar{\Phi}_{(1,2)}(z) \otimes \bar{\Phi}_{(1,2)}(\bar{z})$	$(1/16, 1/16)$
$\psi(x) = \bar{\Phi}_{(2,1)}(z) \otimes \bar{\Phi}_{(1,1)}(\bar{z})$	$(1/2, 0)$
$\psi(x) = \bar{\Phi}_{(1,1)}(z) \otimes \bar{\Phi}_{(2,1)}(\bar{z})$	$(0, 1/2)$
$\varepsilon(x) = \bar{\Phi}_{(2,1)}(z) \otimes \bar{\Phi}_{(2,1)}(\bar{z})$	$(1/2, 1/2)$
$\mu(x) = a_0\sigma(x) = \bar{a}_0\sigma(x)$	$(1/16, 1/16)$

$$
\begin{aligned}
\psi \cdot \psi &= [I] \\
\bar{\psi} \cdot \bar{\psi} &= [I] \\
\psi \cdot \bar{\psi} &= [\varepsilon] \\
\varepsilon \cdot \psi &= [\bar{\psi}] \\
\varepsilon \cdot \bar{\psi} &= [\psi] \\
\varepsilon \cdot \varepsilon &= [I] \\
\sigma \cdot \sigma &= [I] + [\varepsilon] \\
\varepsilon \cdot \sigma &= [\sigma] \\
\mu \cdot \mu &= [I] + [\varepsilon] \\
\varepsilon \cdot \mu &= [\mu] \\
\sigma \cdot \mu &= [\psi] + [\bar{\psi}]
\end{aligned}
$$

Table 3.5 . Primary fields of the maximal semilocal algebra of the critical Ising model and their fusion rules. The disorder field $\mu(x)$ is defined in terms of the Fourier zero modes of the fields $\psi(z) = \sum_{n=0}^{+\infty} z^{n-1/2} a_{-n}$, $\bar{\psi}(\bar{z}) = \sum_{n=0}^{+\infty} \bar{z}^{n-1/2} \bar{a}_{-n}$.

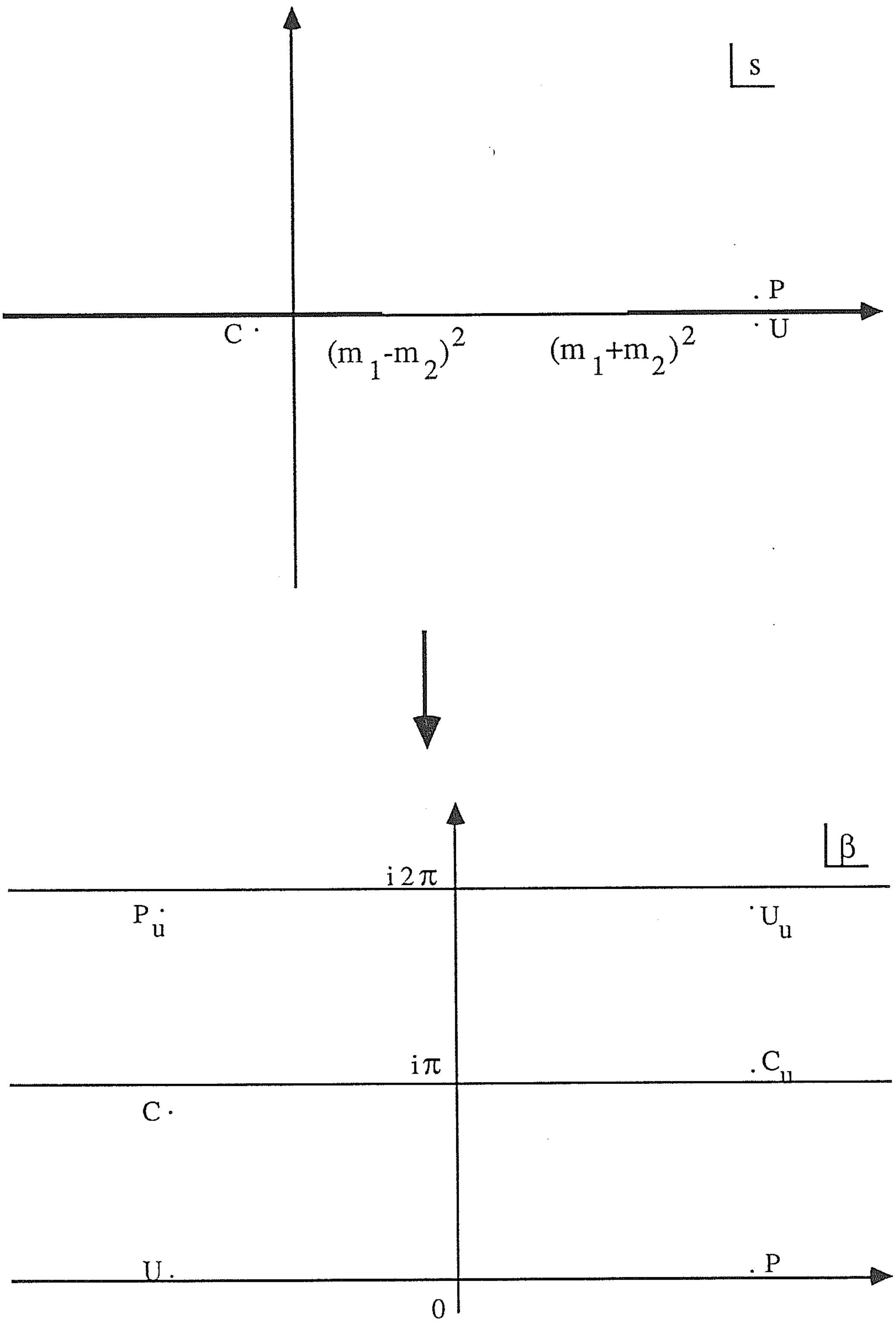


Figure 1.1 . The mapping of the s Riemann surface into the rapidity β plane for the two particle scattering amplitude. The points P, U, and C correspond to the physical (upper) sheet. The points P_u, U_u, and C_u correspond to the first unphysical (lower) sheet.

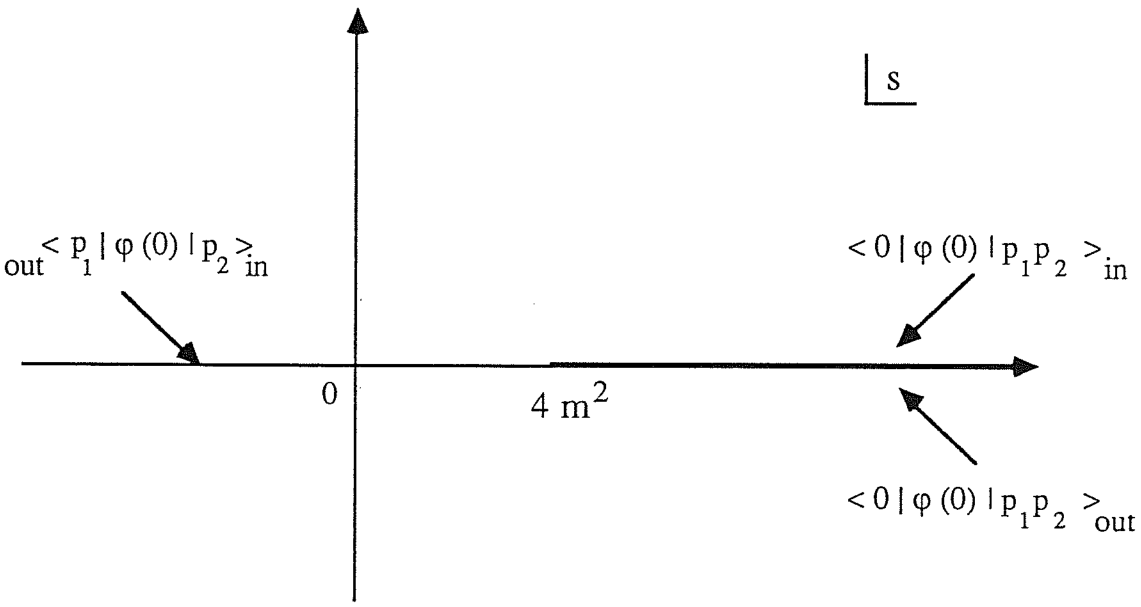


Figure 1.2 . The s Riemann surface for the two particle form factors $F_2(s)$. The boundary value on the cut from above gives the physical 'in' form factor $< 0|\varphi(0)|p_1, p_2 >_{in}$. The boundary value on the cut from below gives the physical 'out' form factor $< 0|\varphi(0)|p_1, p_2 >_{out}$. The boundary value on the negative real axis gives the physical 'crossed' form factor ${}_{out} < p_1|\varphi(0)|p_2 >_{in}$.

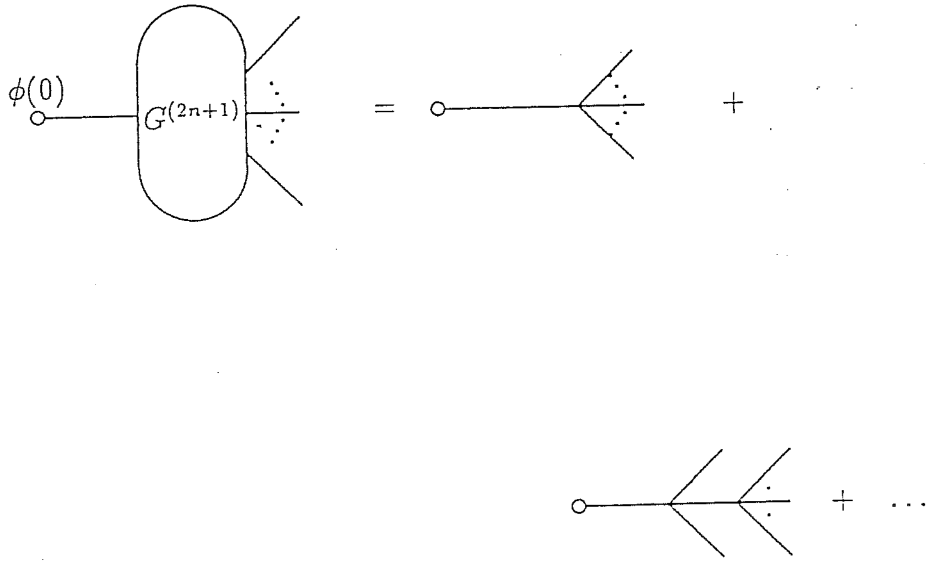


Figure 2.1 . Diagrammatic perturbative expansion for the $2n+1$-particle form factor of the elementary interpolating field $\varphi(x)$.

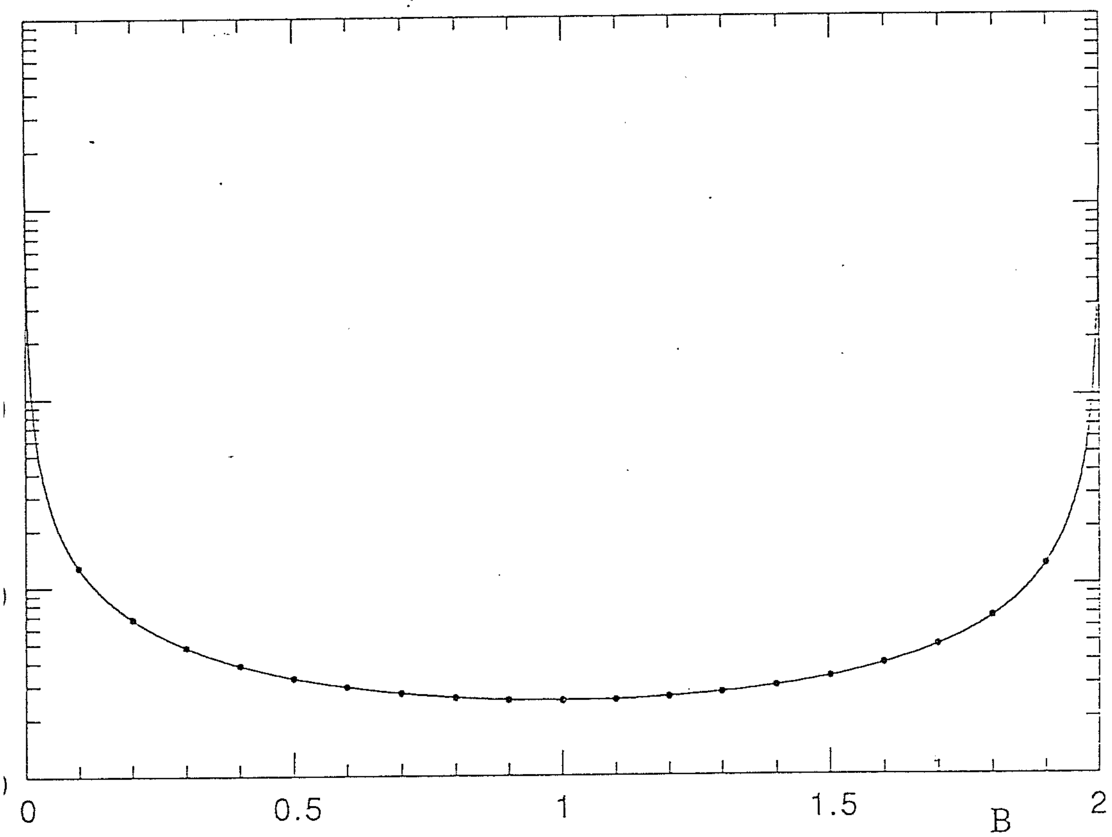


Fig. 1

Figure 2.2 . The first two-particle term entering the sum rule of the c-theorem for the sinh-Gordon model with the choice $a = 1, 0$ (dots), compared with the self-dual central charge (2.47) of conformal field theory (solid line).

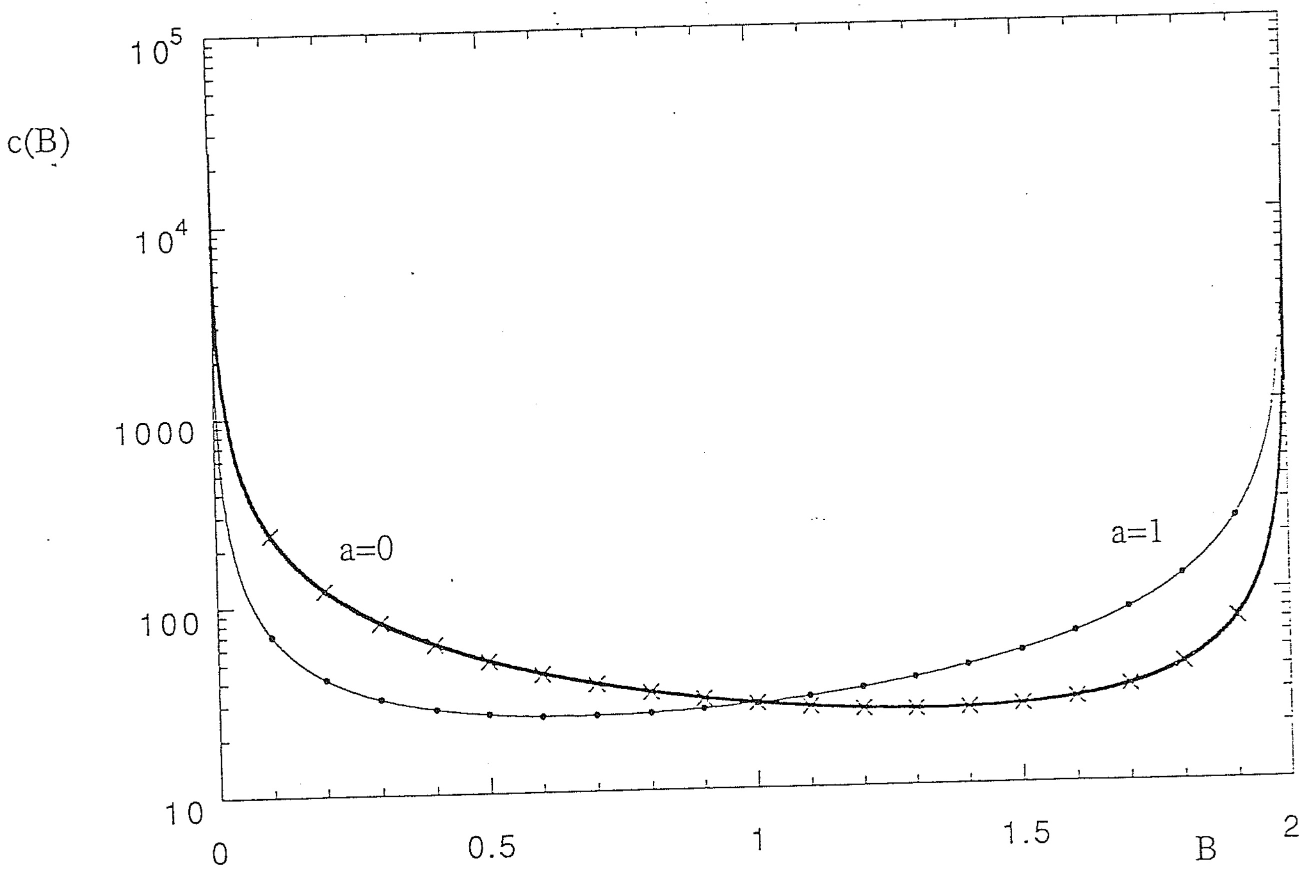


Fig. 2

Figure 2.3 . The first two-particle term entering the sum rule of the c-theorem for the Bullough-Dodd model with the choice $a = 1, 0$ (dots and crosses, resp.), compared with the non self-dual central charges (2.101), (2.107) of conformal field theory (solid thin line and solid thick line, resp.).

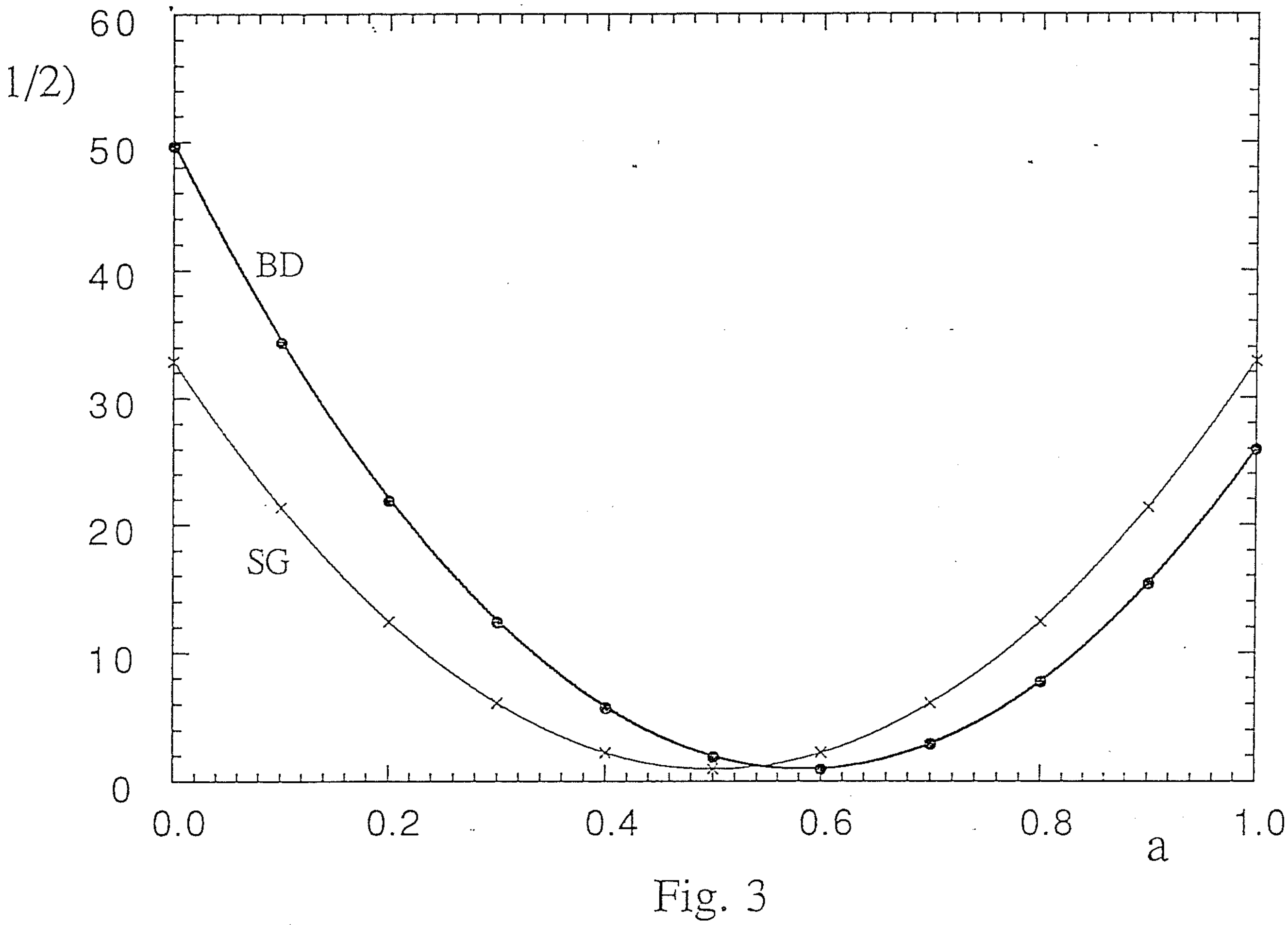


Figure 2.4 . The first two-particle term entering the sum rule of the c-theorem for the sinh-Gordon and Bullough-Dodd models with the coupling constants fixed at $B = 1/2$ for different values of a (crosses and dots, resp.), compared with the central charges (2.62), (2.111) of conformal field theory (solid thin line and solid thick line, resp.).

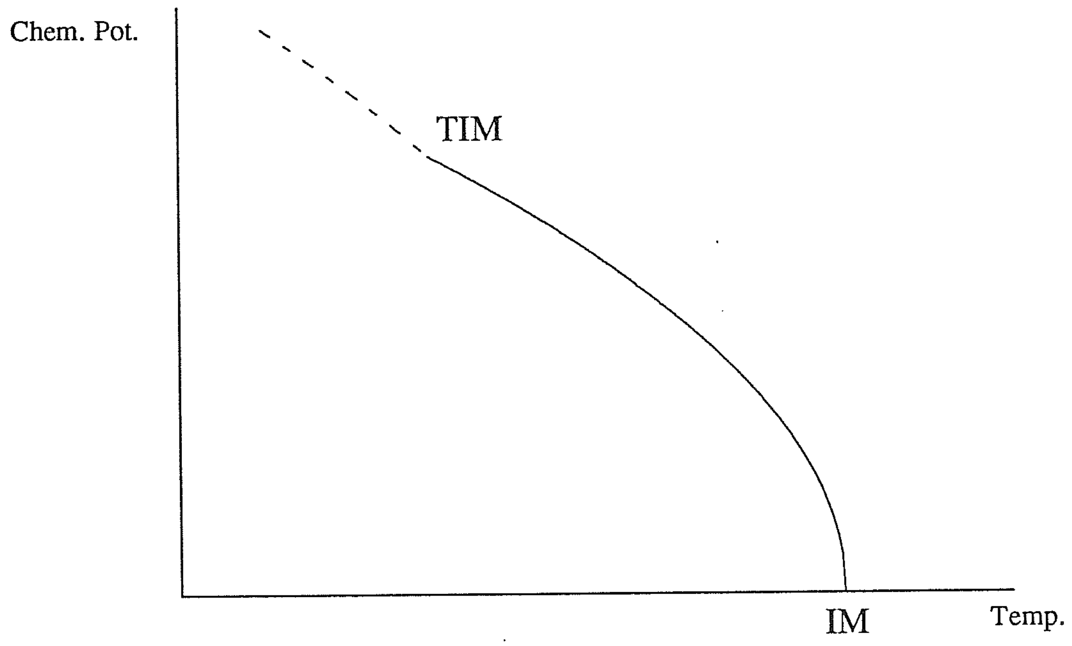


Figure 3.1 . The phase diagram of the universality class of the tricritical Ising model. Its lattice energy is $H = -J \sum_{<i,j>} \sigma_i \sigma_j + \mu \sum_i s_i^2$, where the spin variables can assume the values $s_i = 0, \pm 1$. The value $s_i = 0$ takes into account the possibility of vacancy.

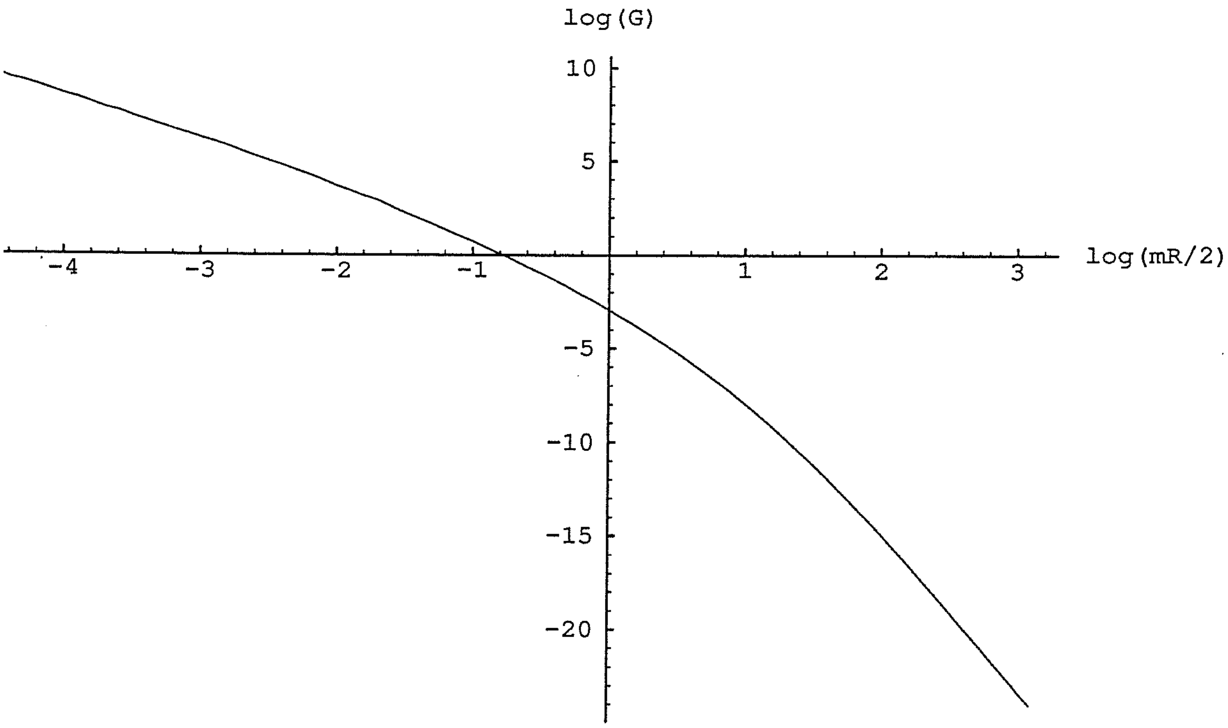


Figure 3.2 . Plot of the correlation function (first two non-vanishing contributions) of the trace of the stress-energy tensor $G =< \tilde{\varepsilon}(x)\tilde{\varepsilon}(0) >$, for the massless flow from the tricritical Ising model to the Ising model. The ultraviolet and the infrared slopes confirm the conformal data $h_{uv} = 3/5$, $h_{ir} = 2$.

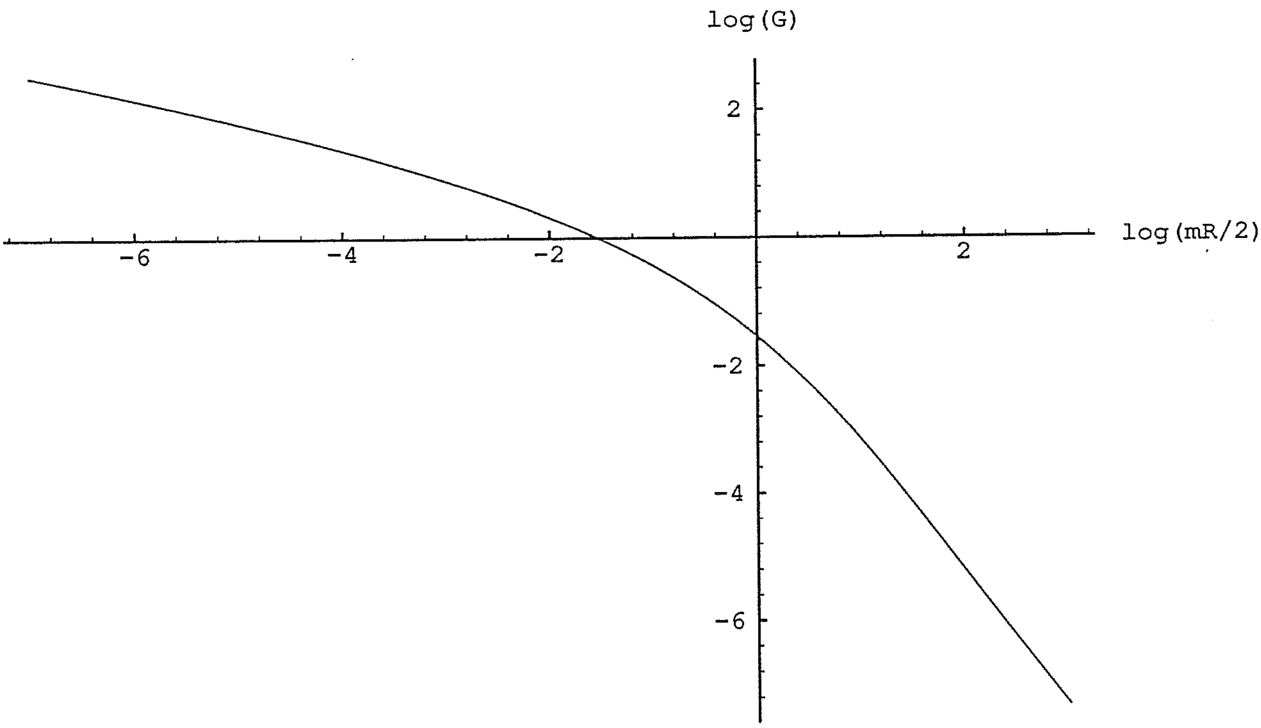


Figure 3.3 . Plot of the correlation function (first two non-vanishing contributions) of the spin energy density $G = < \varepsilon(x)\varepsilon(0) >$, for the massless flow from the tricritical Ising model to the Ising model. The ultraviolet and the infrared slopes confirm the conformal data $h_{uv} = 1/10$, $h_{ir} = 1/2$.

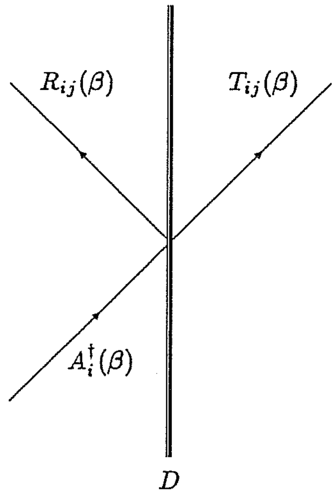


Figure 4.1 . Reflection and transmission amplitudes.

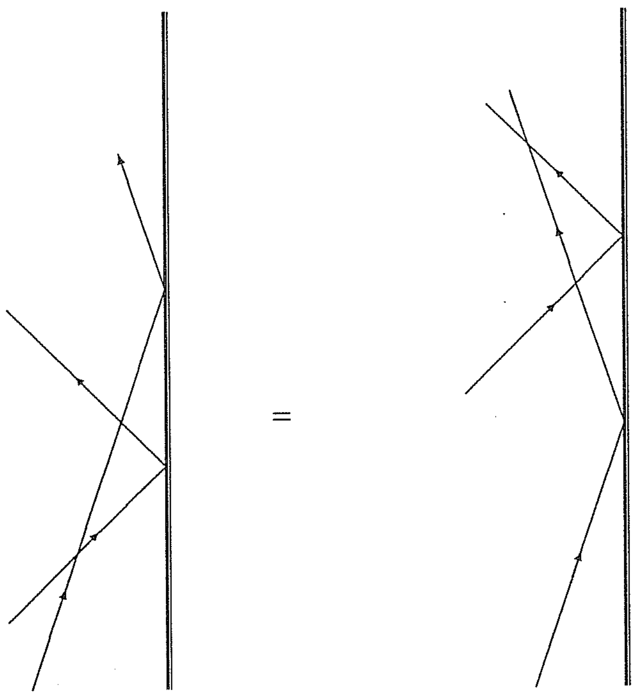


Figure 4.2a . Reflection-transmission equations: the case of two reflections.

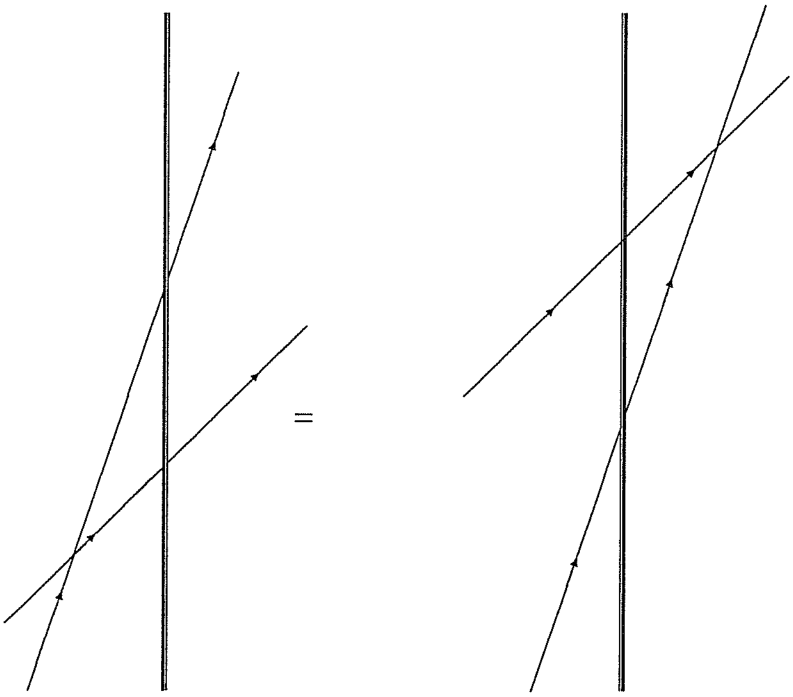


Figure 4.2b . Reflection-transmission equations: the case of two transmissions.

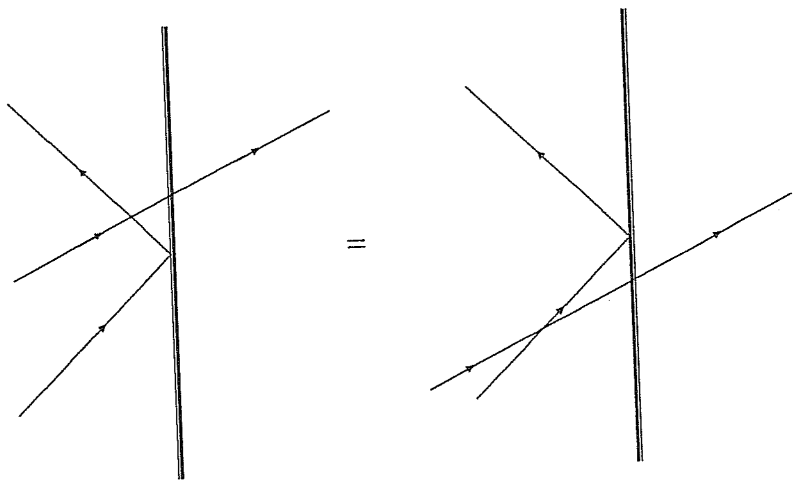


Figure 4.2c . Reflection-transmission equations: the first possibility of one reflection and one transmission.

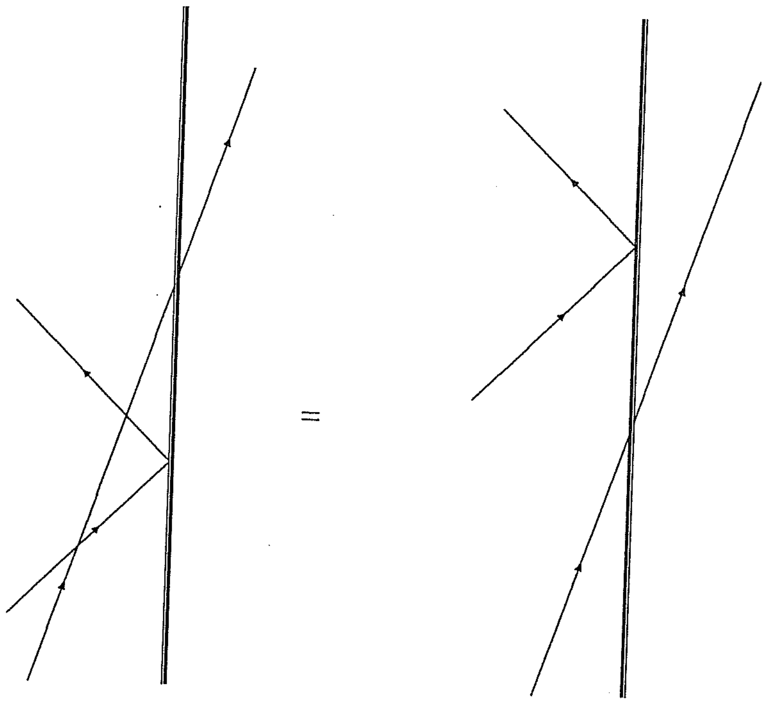


Figure 4.2d . Reflection-transmission equations: the second possibility of one reflection and one transmission.

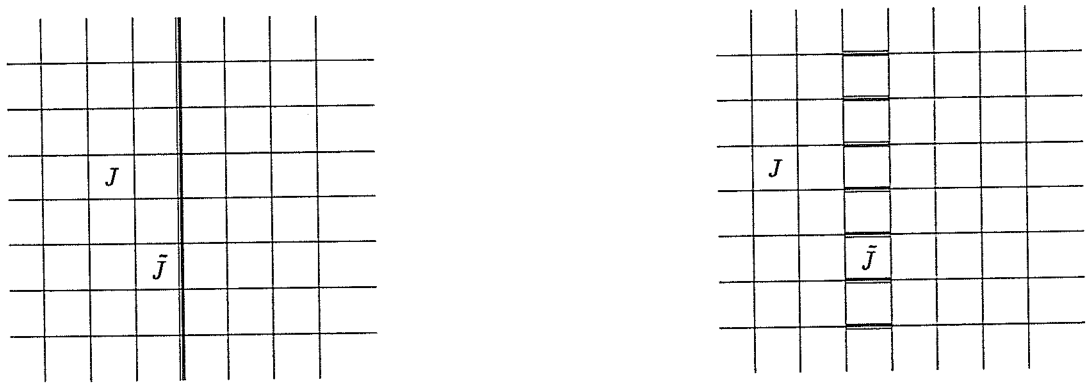


Figure 4.3 . Chain and ladder geometry of a defect line lattice.

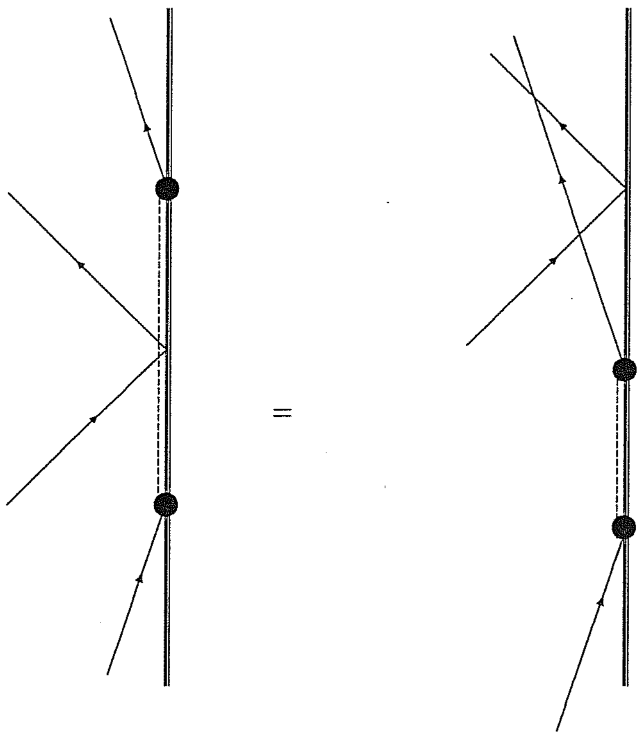


Figure 4.4a . Bootstrap equations for the reflection on the defect bound state in the reflection channel.

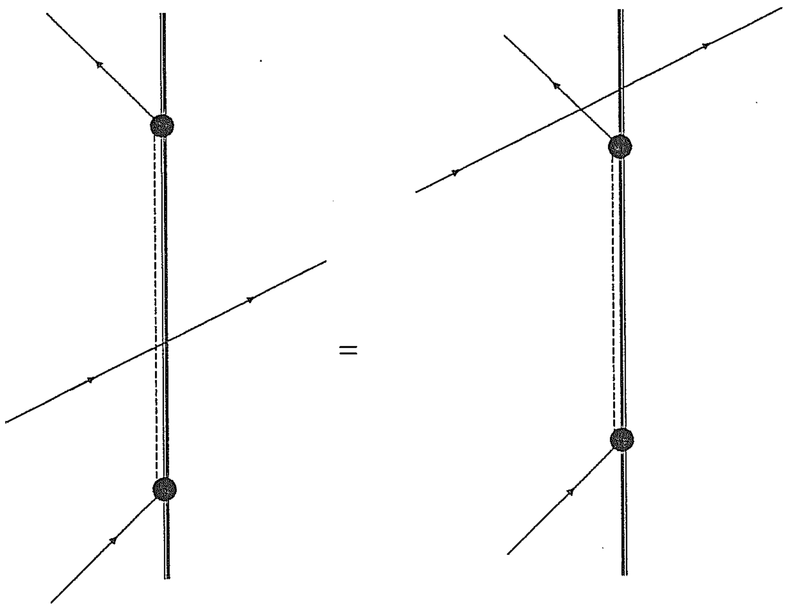


Figure 4.4b . Bootstrap equations for the transmission across the defect bound state in the reflection channel.

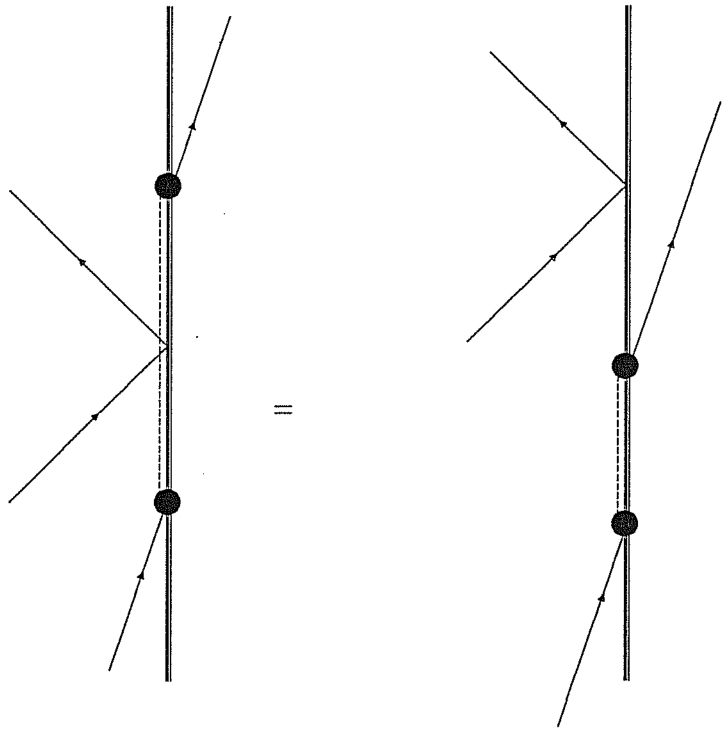


Figure 4.4c . Bootstrap equations for the reflection on the defect bound state in the transmission channel.

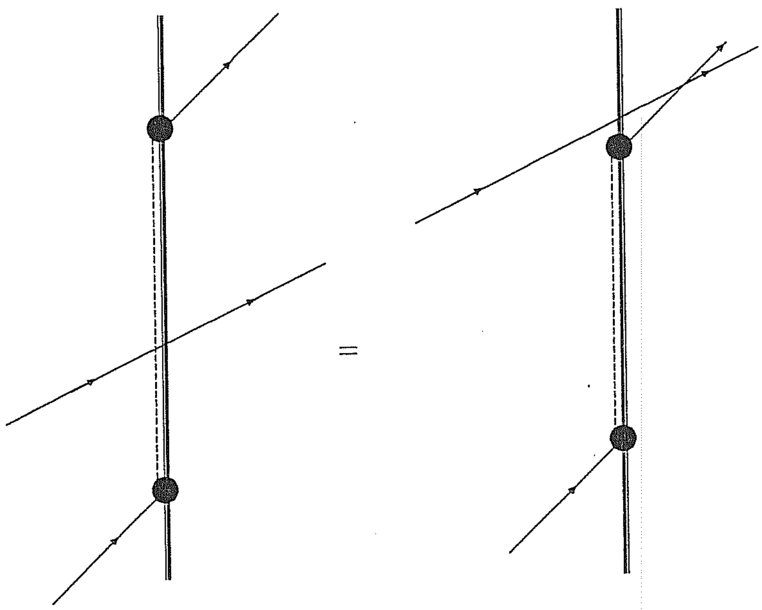


Figure 4.4d . Bootstrap equations for the transmission across the defect bound state in the transmission channel.

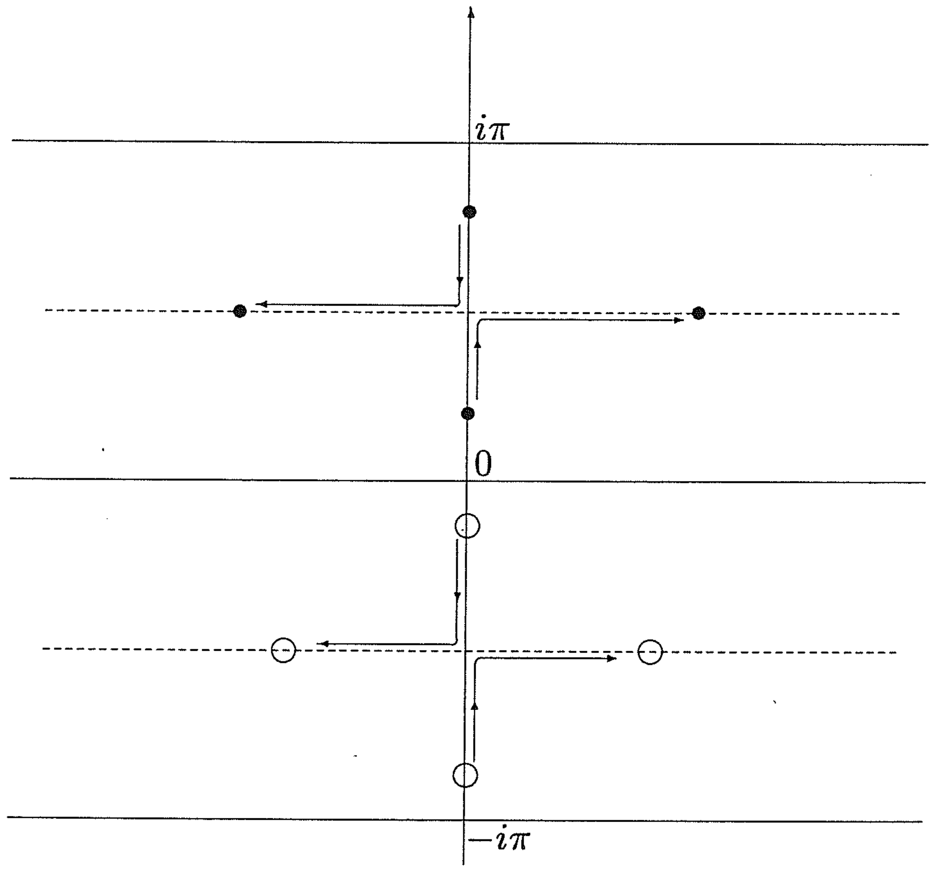


Figure 4.5 . Pole structure of the bosonic amplitudes for positive values of the coupling constant (empty circles) and for negative ones (filled circles).

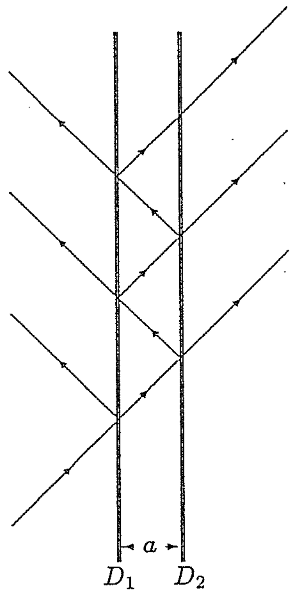


Figure 4.6 . Scattering processes at the two defect lines.

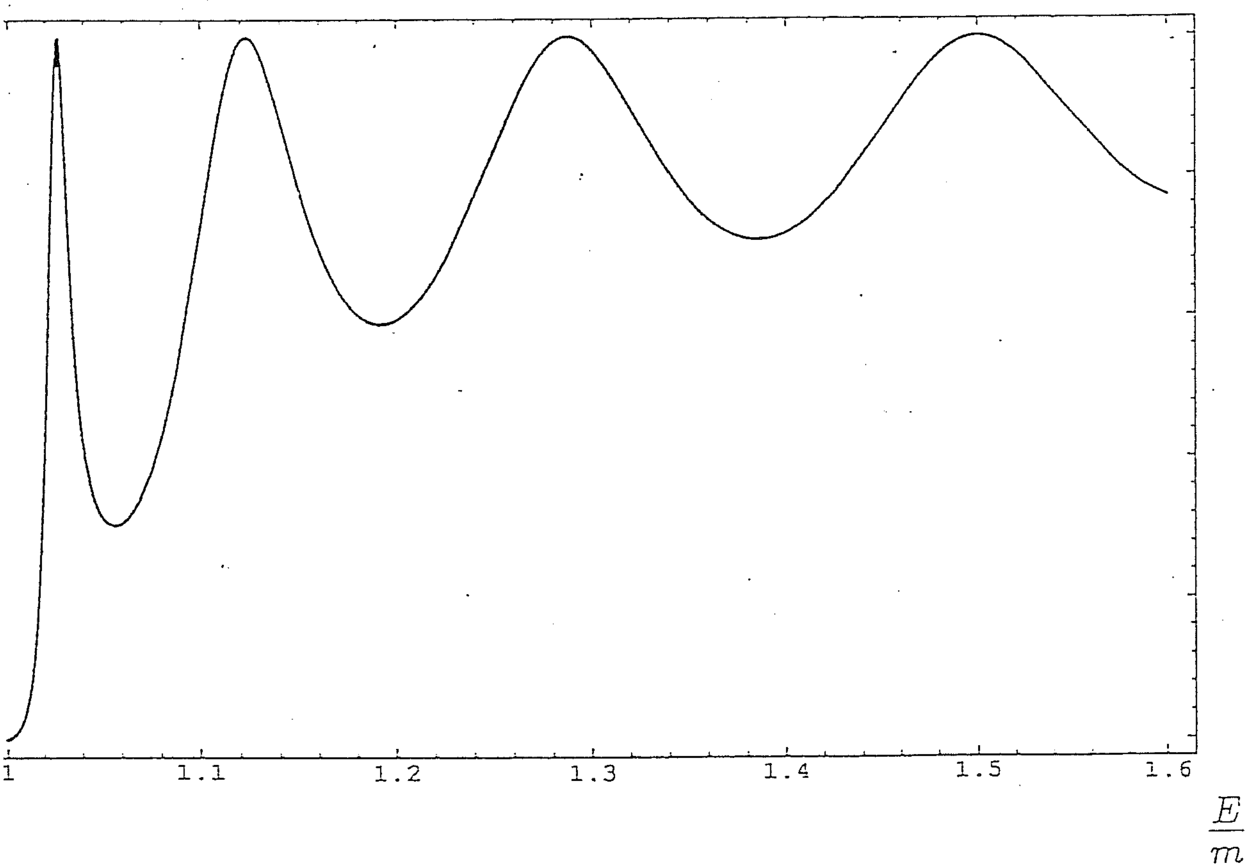


Figure 4.7 . Resonances in the amplitude of transmission through two defect lines for $\chi_1 = \chi_2 = \pi/15$ and $ma = 20$.

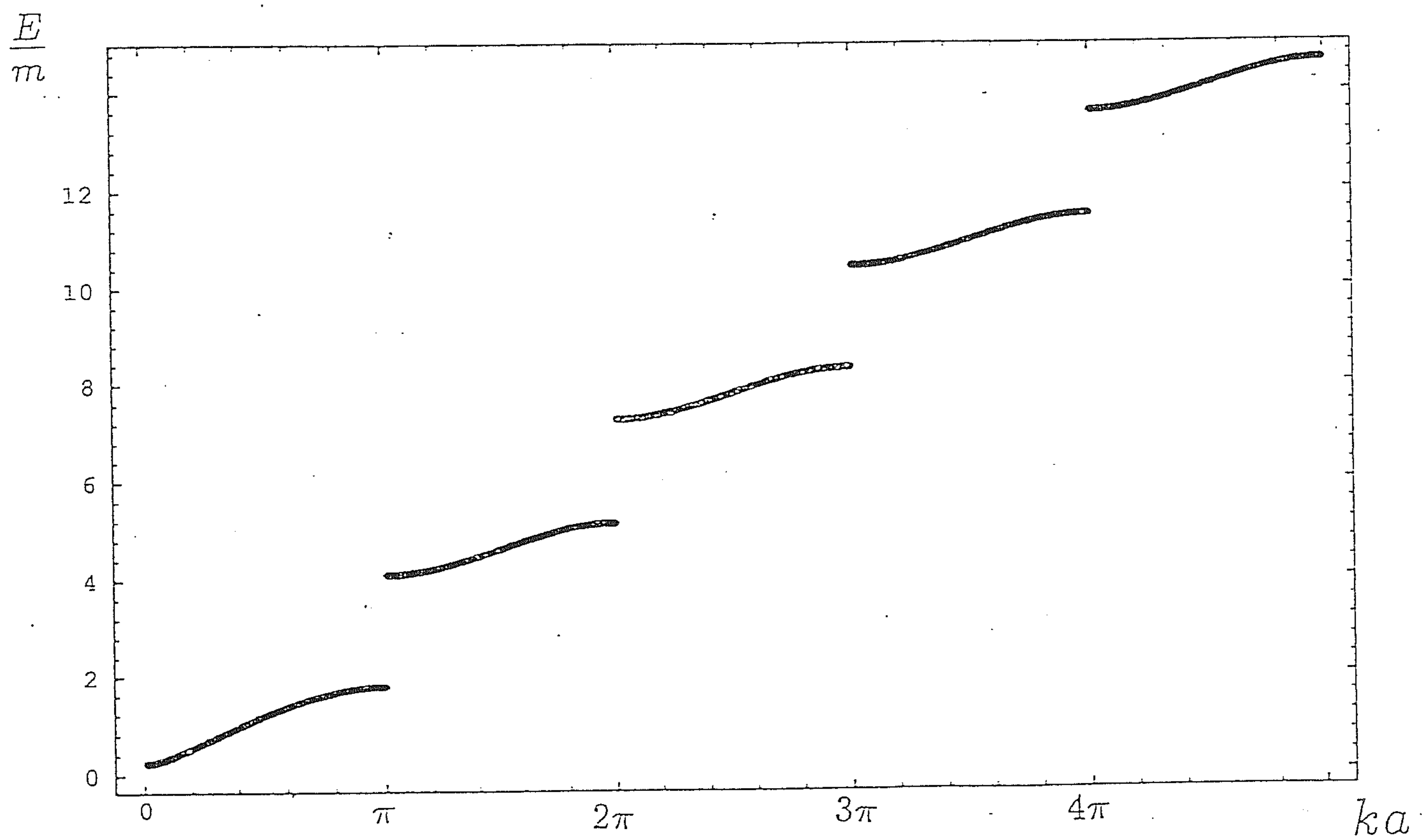


Figure 4.8 . Band structure of the energy levels of the majorana fermion with an infinite array of defects for $\chi = \pi/3$ and $ma = 1$. The first band includes also the states lying below the threshold $E = m$.

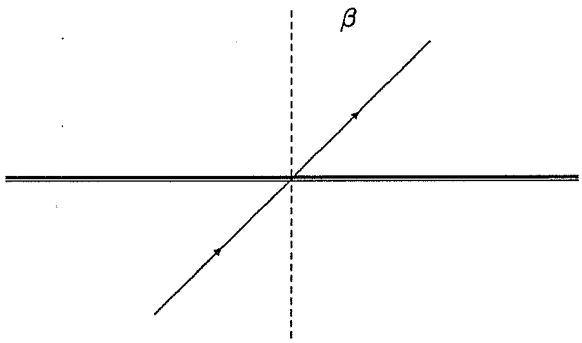


Figure 4.9a . Defect line at $t = 0$. Process of transmission.

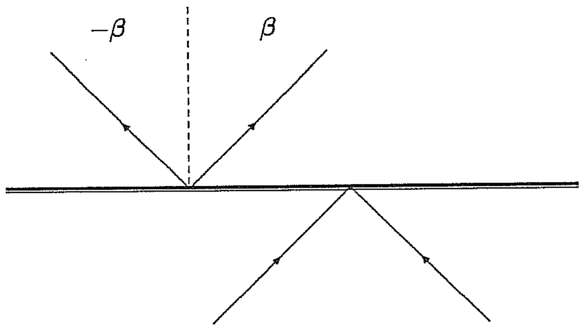


Figure 4.9b . Defect line at $t = 0$. Processes of pair creation and annihilation.

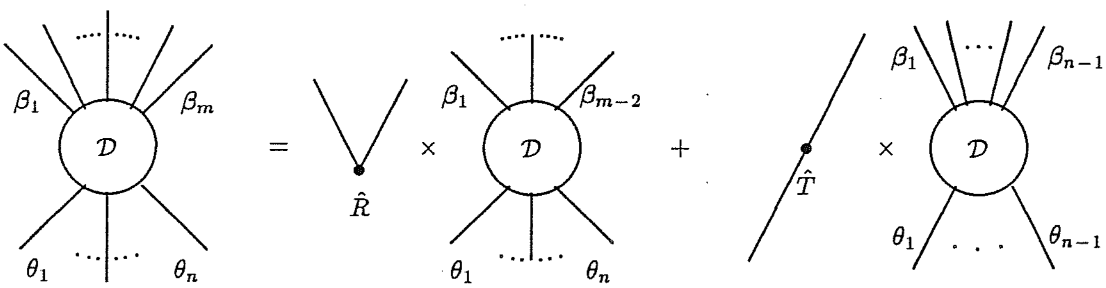


Figure 4.10a . Diagrammatic picture of the first recursive equation for the matrix elements of the operator $\mathcal{D}$.

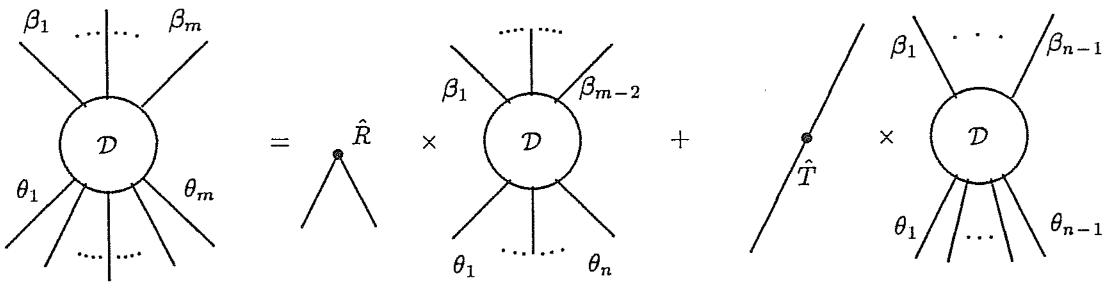


Figure 4.10b . Diagrammatic picture of the second recursive equation for the matrix elements of the operator $\mathcal{D}$.

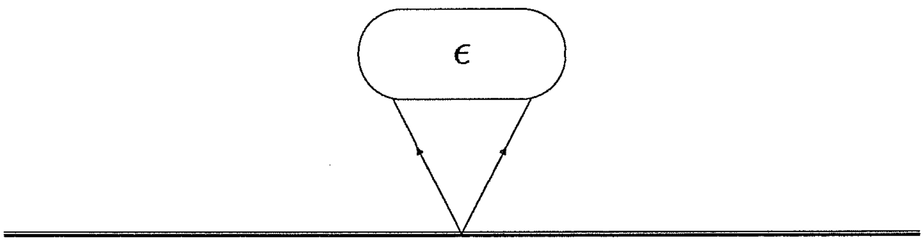


Figure 4.11 . One point function of the energy operator of the Ising model in the presence of the defect line.

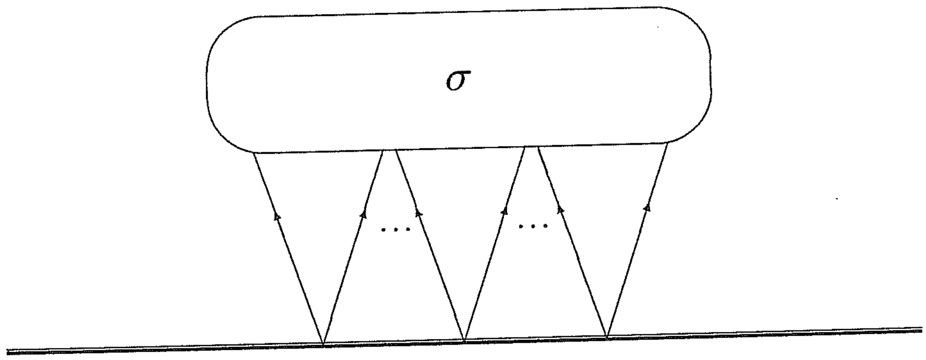


Figure 4.12 . One point function of the magnetization operator of the Ising model in the presence of the defect line.