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VASILE STAICU

WELL POSEDNESS FOR DIFFERENTIAL INCLUSIONS

Ph. D. Thesis

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WELL POSEDNESS FOR DIFFERENTIAL INCLUSIONS

CONTENTS

1. INTRODUCTION

According to a generally accepted terminology in the theory of differential equations, an initial (resp. a boundary) value problem whose solutions exist, are unique and depend continuously on the initial data (parameters) (resp. boundary data) is called well posed.

The great importance of this type of properties in mathematical modelling, numerical methods and other fields motivate the large number of existing results characterizing the well posed problems.

The intense development of the theory of differential inclusions in the last 30 years contains as essential part a corresponding qualitative theory in which the well posedness properties are of major interest.

The aim of this work is to present in a unified manner certain results concerning well posed problems and other qualitative properties of differential inclusions insisting on those to which the author has recently contributed ([18], [19], [24], [52], [53], [54], [55]).

A natural extension of well posedness for problems without uniqueness, hence also for differential inclusions, is expressed by saying that any solution can be embedded in a continuous, single valued, family of solutions depending on the initial data (parameter, or boundary data). In other words, the multivalued map assigning to an initial data (parameter, or boundary data) the set of solutions of the corresponding initial value (or boundary value) problem admits a continuous selection, passing trough a given point of the graph of this map. We shall study this kind of well posedness for Lipschitzean differential inclusions on open and closed sets in Chapter 3, for evolution equations in Chapter 4 and for a class of Darboux problems in Chapter 5.

Another extension of well posedness, which will be not considered in this work, is expresed in terms of different type of continuity of the above multivalued map ([4], [25], [41], [45]).

Any chapter of this thesis contains, as a first section, an introduction comparing the results in that chapter with known results and giving the appropriate references. In this section we shall present the main results and their location in this thesis and in the literature.

In chapter 2 we recall some notations, definitions and technical lemmas used in what follows. Chapter 3 is devoted to Lipschitzean differential inclusions on open and closed sets.

Section 3.2 contains a result obtained in the joint paper with Colombo, Fryszkowski and Rzezuchowski [24]. Denoting by $\mathcal{T}(s)$ is the set of solutions of the Cauchy problem

(1.1) $$\dot{x} \in F(t, x,s) \ , \ \ x(0) = \xi(s),$$

where F Lipschitzean in x, lower semicontinuous in s and $\xi(.)$ is continuous, we have proved the existence of a continuous selection from the multivalued map $s \to \mathcal{T}(s)$ passing by a given point of graph $(\mathcal{T})$. This result generalizes the continuous selections theorems from solution sets by Cellina [14] and Cellina and Ornelas in [16]. A kind of continuous selection from solution sets of Lipschitzean differential inclusions has been obtained by Polovinkin and Smirnov in [50].

Remark that the existence of such a selection from $s \to \mathcal{T}(s)$ implies that the multivalued map $s \to \mathcal{T}(s)$ is lower semicontinuous ([4], p.80), hence it implies the result of Naselli Ricceri and Ricceri [45]. On the other hand it is known that the sets $\mathcal{T}(s)$ are in general nonclosed and nonconvex ([17], [34]).

Our result contains also as particular cases the selection theorems by Antosiewicz and Cellina [1], Bressan and Colombo [9] and Fryszkowski [31] related to multivalued maps with decomposable values. The idea of regarding solution sets of Lipschitzean differential inclusions as generalizations of decomposable sets is due to Cellina and it has been developed in [55].

The proof of this result is different from the one in [14] and [16] and it is based on a continuous selection theorem by Bressan and Colombo [9].

In section 3.3 we present an analogue for solution sets $\mathcal{T}(\xi)$ of the Cauchy problem

(1.2) $$x' \in F(t,x), \ \ x(0)=\xi,$$

with F Lipschitzian in x, of Michael's extension theorem [42]. We have obtained this result in [52], and we have used for this a result by Bressan, Cellina and Fryszkowski [8].

Section 3.4 contains a result proved in the joint paper with Cellina [19]. Here we have considered a multifunction F defined on $R \times C$, where C is a closed subset of R^n, Lipschitzean in x and satisfying a tangentiality condition, and we have proved the existence of a continuous selection from the map assigning to an initial point ξ in C the set of solutions (with values in C) of the Cauchy problem (1.2).

Remark that the (generalized) successive approximations process that is the base of the construction, in the case under consideration requires at each step a projection over the (in general, non convex) set C , since F is not defined outside C , and that this projection is not continuous. Moreover, the lack of an argument allowing the extension of a multivalued Lipschitzean map from a closed set to an open set containing it, prevents the possibility of exploiting the available techniques for the present case.

As a side result we obtain the convergence of the sequence of generalized successive approximations for any initial function x_0. Taking the initial function to be a solution of (1.2) for $\xi=\xi_0$, we obtain a continuous selection from $\xi\to\mathcal{T}(\xi)$ passing by (ξ_0,x_0).

Section 3.5 concerns the arcwise connectedness of solution sets $\mathcal{T}(\xi)$ of the Cauchy problem

(1.3) $$x'\in F(x),\ \ x(0)=\xi,$$

where F is Lipschitzean. By the results in sections 3.2 and 3.4 we have that there exists a continuous selection from the multivalued map $\xi\to\mathcal{T}(\xi)$ passing by any given point of the graph of $\mathcal{T}$. Using this result we shall prove that the set $\mathcal{T}(\xi)$ is arcwise connected. We have obtained this result in the joint paper with Wu [54], for F defined on an open set and in the joint paper with Cellina [19], for F defined on closed sets. Moreover, following [19], we shall prove that any two continuous selections from the map assigning to the initial point the set of solutions, are linked by a continuous homotopy with values in the solution sets.

As it is well known the solution sets to ordinary differential equations without uniqueness defined on open sets are connected but not (in general) arcwise connected. In the case of solutions to differential inclusions on closed sets the difference between continuity and Lipschitz continuity is even more striking: example in ([4], pp. 203) shows that the solution set to (1.3) for $x\to F(x)$ single valued, continuous (and independent on t), may consist of exactly two solutions, a disconnected set.

Chapter 4 is dedicated to existence and well posedness for evolution equations, that is differential inclusions having in the right-hand side a maximal monotone map or a generator of a semigroup.

In section 4.3 we present a result obtained in the joint paper with Cellina [18], concerning the local existence of solutions for the Cauchy problem

(1.4) $$\dot{x}(t)\in-\partial V(x(t))+F(x(t))\,,\ \ F(x)\subset\partial W(x)$$
$$x(0)=x_0\,,$$

where V is a lower semicontinuous proper convex function (hence its subdifferential ∂V, is a maximal monotone map), W is a lower semicontinuous convex function and F is an upper semicontinuous compact valued map defined over some neighborhood of x_0. Remark that at the right hand side of (4.1.5) is defined by a maximal monotone map with a minus sign and a bounded monotone but not maximal monotone map with a plus sign. This result is a generalization of the one by Bressan, Cellina and Colombo [7] asserting the existence of a local solution of the Cauchy problem

$$(1.5) \qquad \dot{x}(t) \in F(x(t)) \subset \partial W(x(t)),\ x(0) = x_0,$$

where, as before, F is a monotonic upper semicontinuous (not necessarily convex-valued, hence not maximal) map contained in the subdifferential of a locally bounded convex function. Existence of solutions depends on arguments of convex analysis.

Sections 4.4 and 4.5 contain two results proved in [52]. In the first one we prove the existence of a continuous selection from the map assigning to ξ in the closure of the domain D(A) of a maximal monotone map A, in a Hilbert space, the set of weak solutions of the Cauchy problem

$$(1.6) \qquad \dot{x}(t) \in -Ax(t) + F(t, x(t)) \ ,\ x(0) = \xi,$$

where F is a multivalued map Lipschitzean in x. In the second one we prove a similar result for the map assigning to ξ in a Banach space the set of mild solutions of the Cauchy problem

$$(1.7) \qquad \dot{x}(t) \in Ax(t) + F(t, x(t)) \ ,\ x(0) = \xi$$

where A is infinitesimal generator of a C_0-semigroup on a Banach space and F is as before.

The last part of this thesis, chapter 5, contains a result wich we have proved in [53] concerning well posedness of a Darboux problem

$$(1.8) \qquad u_{xy} \in F(x,y,u) \ ,\ u(x,0)=\alpha(x),\ u(0,y)=\beta(y).$$

where F is Lipschitzian with respect to u, without convexity assumptions on the values of F. Denoting by $\mathcal{T}(\alpha,\beta)$ the set of all solutions of (1.8) we prove the existence of a continuous selection from the map $(\alpha,\beta) \to \mathcal{T}(\alpha,\beta)$. Properties of the multivalued map $(\alpha,\beta) \to \mathcal{T}(\alpha,\beta)$ have been also obtained in [27] and [56] for convex valued F and in [41] for F Lipschitzean, not necessarily convex valued.

2. NOTATIONS, BASIC DEFINITIONS AND PRELIMINARY RESULTS.

Let X be a separable metric space with distance d(.,.). For $x \in X$ and A, B any two closed subsets of X we define the distance from x to A by $d(x,A) = \inf \{d(x,y): y \in A\}$; the separation of A from B by $d^*(A,B) = \sup \{d(a,B): a \in A\}$ and the Hausdorff-Pompeiu distance from A to B by $\mathbf{d}(A,B) = \max \{d^*(A,B), d^*(B,A)\}$.

The following properties may be easily proved and are widely used

(2.1) $d(x,B) \leq d(x,A) + \mathbf{d}(A,B)$

(2.2) $|d(x,A) - d(y,B)| \leq d(x,y) + \mathbf{d}(A,B)$.

Denote by $B(x,r) := \{y \in X: d(x,y) < r\}$ (resp. $B[x,r] := \{y \in X: d(x,y) \leq r\}$) the open (resp. the closed) ball of center $x \in X$ and radius $r > 0$. By $B(A,r) = \{y \in X: d(y,A) < r\}$ we denote the open r-neighborhood of $A \subset X$, and by cl A the closure of A in X.

Let C be a closed nonempty subset of X. For $x \in X$ let $\pi_C(x) = \{y \in C: d(x,y) = d(x,C)\}$ be the projection of x onto C. If X is a Hilbert space and C is closed and convex then $\pi_C(x)$ has an unique element. In this case we denote by m(C) the unique element of $\pi_C(0)$.

Let I be the interval [0,T], let $\mathcal{L}$ be the σ-algebra of Lebesgue measurable subsets of I and let μ be the Lebesgue measure. For $A \in \mathcal{L}$ we denote by χ_A the characteristic function of A, that is $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise.

If X is a Banach space with norm $\|.\|$ then we denote by C(I,X) the Banach space of continuous functions $x: I \rightarrow X$ with the norm

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in I\};$$

by $L^\infty(I,X)$ the Banach space of essentially bounded measurable functions $x: I \rightarrow X$ with the norm

$$\|x\|_\infty = \inf \{r \geq 0 : \|x(t)\| \leq r \text{ a.e. in } I\};$$

by $L^1(I,X)$ the Banach space of Bochner integrable functions $x: I \rightarrow X$ with the norm

$$\|x\|_1 = \int_0^T \|x(t)\| dt;$$

and by AC(I,X) the Banach space of absolutely continuous functions $x : I \to X$ with the norm

$$\|x\|_{AC} = \|x(0)\| + \|\dot{x}\|_1 ,$$

where $\dot{x}$ stands for the derivative of x.

Let X and Y be two metric spaces. Denote by 2^Y the family of all nonempty subsets of Y and by $\mathcal{B}(Y)$ the σ-algebra of Borel subsets of Y. If $\mathcal{A}$ be a σ-algebra of subsets of X then we denote by $\mathcal{A}\otimes\mathcal{B}(Y)$ the product σ-algebra on X×Y, generated by all the sets of the form A×B with $A\in\mathcal{A}$ and $B\in\mathcal{B}(Y)$.

A map $F: X\to 2^Y$ is said to be a *multifunction* or a *multivalued map* or a *set-valued map* from X into the subsets of Y. We recall in the following some definitions and basic properties related to the continuity and measurability of a multivalued map.

Definition 2.1. A multivalued map $F: X\to 2^Y$ is called:

(a) *lower semicontinuous* (l.s.c.) in X if for any $x_0\in X$, $y_0\in F(x_0)$ and any $\varepsilon>0$ there exists $\delta>0$ such that

$$d(x,x_0) < \delta \text{ implies } F(x)\cap B(y_0,\varepsilon)\neq\emptyset,$$

(b) *upper semicontinuous* (u.s.c.) in X if for any open subset U of Y containing $F(x_0)$ there exists $\delta>0$ such that

$$d(x,x_0) < \delta \text{ implies } F(x)\subset U ,$$

(c) *continuous* if it is both l.s.c. and u.s.c..

It is easy to see that the following equivalent formulations of (a) and (b) hold:

(a) ⟺ the set $F^+(C) := \{x\in X : F(x)\subset C\}$ is closed in X for any closed subset C of Y;

(b) ⟺ the set $F^-(C) := \{x\in X : F(x)\cap C\neq\emptyset\}$ is closed in X for any closed subset C of Y.

Definition 2.2. A multivalued map $F: X\to 2^Y$ is called:

(α) *Hausdorff-lower semicontinuous* (h-l.s.c.) in X if for any $x_0\in X$, and any $\varepsilon > 0$ there exists $\delta>0$ such that

$$d(x,x_0) < \delta \text{ implies } d^*(F(x_0), F(x)) < \varepsilon,$$

(β) *Hausdorff-upper semicontinuous* (h-u.s.c.) in X if for for any $x_0 \in X$, and any $\varepsilon>0$ there exists $\delta>0$ such that

$$d(x,x_0) < \delta \text{ implies } d^*(F(x), F(x_0)) < \varepsilon,$$

(γ) *Hausdorff-continuous* if it is both h-l.s.c. and h-u.s.c..

If $F: X \to 2^Y$ is a multivalued map then its *graph* is defined by

$$\text{graph}(F) := \{(x,y) : x \in X, y \in F(x)\}$$

The following proposition gives some relations between the above concepts and its proof can be found in [4, § 1.1].

Proposition 2.3. Let $F: X \to 2^Y$ be a multivalued map. Then

(i) (α) ⟹ (a) and (b) ⟹ (β) ;

(ii) if F has compact values then (a) ⟺ (α) and (b) ⟺ (β), so that (c) ⟺ (γ);

(iii) if F is u.s.c. with closed values then F has closed graph. Conversely, if F has closed graph and Y is compact then F is u.s.c..

Let $(X, \mathcal{A})$ be a measure space, where $\mathcal{A}$ is a σ-algebra of subsets of X, and let Y be a metric space.

Definition 2.4. A multivalued map $F: X \to 2^Y$ is called *$\mathcal{A}$-measurable* if $F^-(C)$ belongs to $\mathcal{A}$ for each closed subset C of Y.

If $X \subset R^n$ and $\mathcal{A}=\mathcal{L}$ is the σ-algebra of Lebesgue measurable subsets of X then we say *measurable* instead of *$\mathcal{L}$ - measurable.*

The following proposition is a consequence of Theorems 3.5 and 5.6 in [36]

Proposition 2.5. Let F: $X \to 2^Y$ be a multivalued map. Then:

(i) if F is $\mathcal{A}$-measurable with closed values then graph $(F) \in \mathcal{A} \otimes \mathcal{B}(Y)$ and $x \to d(y,F(x))$ is measurable for any $y \in Y$.

(ii) if Y is complete and separable then: F is $\mathcal{A}$-measurable $\Longleftrightarrow$ graph $(F) \in \mathcal{A} \otimes \mathcal{B}(Y)$ $\Longleftrightarrow$ $x \to d(y,F(x))$ is measurable for any $y \in Y$ $\Longleftrightarrow$ there exists a countable family $\mathcal{F}$ of measurable selections from F(.) such that $F(x) = \text{cl}\,\{f(x) : f \in \mathcal{F}\}$ for each $x \in X$.

Definition 2.6. Let F: $X \to 2^Y$ be a multivalued map. A function f: $X \to Y$ is called a *selection* from F if $f(x) \in F(x)$ for every $x \in X$.

We recall some results concerning the measurable selections from a measurable multivalued map.

Proposition 2.7.(Proposition 1 in [37]) Let u : $I \to R^n$ be a measurable function and let $F: I \to 2^{R^n}$ be a measurable multifunction with closed nonempty values. Then there exists a measurable selection f: $I \to R^n$ from F such that

$$\|u(t)-f(t)\| = d(u(t), F(t)) \quad \text{a.e. in I.}$$

The following proposition, which is the infinite dimensional counterpart of the previous one, follows from Proposition 2 in [9], and it has been stated also in [45].

Proposition 2.8. Let X be a separable Banach space, u : $I \to X$ be a measurable function and let F: $I \to 2^X$ be a measurable multifunction with closed nonempty values. Then for every $\varepsilon > 0$ there exists a measurable selection f: $I \to X$ from F such that

$$\|u(t)-f(t)\| < d(u(t), F(t)) + \varepsilon \quad \text{a.e. in I.}$$

The following Lemma was proved in [18] and will be used to prove Theorem 4.11.

Lemma 2.9. Let $\{\delta_n(\cdot) : n \in N\}$ be a sequence of measurable functions, $\delta_n : I \to R^n$ and assume that there exists $\alpha \in L^1(I, R)$ such that for a.e. $t \in I$

$$\|\delta_n(t)\| \leq \alpha(t) .$$

Then:

(i) $t \to \psi_i(t) := \text{cl}\,(\bigcup_{n \geq i} \{\delta_n(t)\})$ is measurable, $\forall i \in N$.

(ii) for a.e. $t \in [0, T]$, $\psi_*(t) = \bigcap_{i \in N} \psi_i(t)$ is nonempty compact and $t \to \psi_*(t)$ is measurable.

(iii) For any multifunction $G: I \to 2^{R^n}$ with closed nonempty values such that, for a.e. $t \in [0,T]$, $d(\delta_n(t), G(t)) \to 0$ as $n \to \infty$ we have that $\psi_*(t) \subset G(t)$.

Proof. (i) Since $t \to \{\delta_n(t)\}$ is measurable and since $\psi_i(t)$ is compact, by Proposition III.4 in [13] it follows that $t \to \psi_i(t)$ is measurable.

(ii) $\{\psi_i(t) : i \in N\}$ is a decreasing sequence of compact subsets of R^n hence $\psi_*(t)$ is compact nonempty. The measurability of $t \to \psi_*(t)$ follows from (i) and Proposition III.4 in [13].

(iii) Fix $\varepsilon > 0$. Then there exists $n_\varepsilon \in N$ such that for $n \geq n_\varepsilon$, $\delta_n(t) \in \text{cl}\, B(G(t), \varepsilon)$, hence

$$\psi_{n_\varepsilon}(t) \subset \text{cl}\, B(G(t), \varepsilon) \text{ and } \psi_*(t) \subset \text{cl}\, B(G(t), \varepsilon) .$$ ♦

Let X be a separable Banach space, and let $F: I \to 2^X$ be a measurable multifunction with closed nonempty values. Let

$$S^1_F = \{v \in L^1(I,X) : v(t) \in F(t) \text{ a.e. in } I\}$$

be the set of integrable selections from F. Then S^1_F has the following property : for every u, v in S^1_F and $A \in \mathcal{L}$ we have $u\chi_A + v\chi_{I \setminus A} \in S^1_F$. This property characterizes a important class of subsets of $L^1(I,X)$, the so called *decomposable sets*.

Definition 2.10.([35]) A subset K of $L^1(I,X)$ is called *decomposable* if for every u, v in K and $A\in \mathcal{L}$ we have

$$u\chi_A + v\chi_{I\setminus A} \in K.$$

It follows that the set of integrable selections of a measurable multifunction is decomposable. Moreover, we have the following proposition:

Proposition 2.11. ([35]) A closed nonempty subset K of $L^1(I,X)$ is decomposable if and only if there exists a measurable multifunction $F: I\to 2^X$ with closed nonempty values such that $K=S^1_F$.

We denote by $\mathcal{D}$ the family of all closed nonempty decomposable subsets of $L^1(I,X)$.

Assume that S is a separable metric space, X is a separable Banach space and let $F:I\times S\to 2^X$ be a multivalued map. Define the multivalued map $s\to G_F(s)$ from S into the subsets of $L^1(I,X)$ by

(2.3) $$G_F(s) = \{v\in L^1(I,X) : v(t)\in F(t,s) \text{ a.e. in } I\}.$$

The following result proved in [24] will be used several times in what follows:

Lemma 2.12. Assume that $F: I\times S \to 2^X$ is $\mathcal{L}\otimes\mathcal{B}(S)$-measurable. Then the map $s\to G_F(s)$ defined by (2.3) is l.s.c. from S into $\mathcal{D}$ if and only if there exists a continuous map $\beta:S\to L^1(I,R)$ such that for every $s\in S$

(2.4) $$d(0, F(t,s)) \le \beta(s)(t) \text{ a.e. in } I.$$

Proof. If $G_F(.)$ is l.s.c. from S into $\mathcal{D}$ then, by Theorem 3 in [9], there exists $g:S\to L^1(I,X)$ a continuous selection from $G_F(.)$ and (2.4) is satisfied for $\beta(s)(t)=\|g(s)(t)\|$.

Assume that there exists a continuous $\beta:S\to L^1(I,R)$ satisfying (2.4). Let C be a closed subset of $L^1(I,X)$ and let $\{s_n\}_{n\in N}\subset S$ convergent to some $s_0\in S$ such that $G_F(s_n)\subset C$. We have to prove that $G_F(s_0)\subset C$. Let $v_0\in G_F(s_0)$. Then by Proposition 2.8 we obtain that for any $n\in N$ there exists $v_n(.)$, a measurable selection from $t\to F(t,s_n)$, such that

(2.5) $$\|v_n(t) - v_0(t)\| < d(v_0(t), F(t,s_n)) + \frac{1}{n} \quad \text{a.e. in I.}$$

Since $s\to F(t,s)$ is l.s.c. for every $t\in I$, it follows that for every $x\in X$ the map $s\to d(x, F(t,s))$ is u.s.c. and by (2.5) it follows that : $v_n(t)$ converges to $v_0(t)$ a.e. in I.
By (2.4) and (2.5) we obtain that

(2.6) $$\|v_n(t) - v_0(t)\| < \|v_0(t)\| + \beta(s_n)(t) + \frac{1}{n} \quad \text{a.e. in I.}$$

Denote by $a_n(t)$ the right hand side of (2.6) and remark that $a_n(.)$ converges strongly in $L^1(I,R)$, hence it is bounded in $L^1(I,R)$ and uniformly integrable, so the same holds for the sequence of functions $s \to \|v_n(t) - v_0(t)\|$. Then by (2.6) and the dominated convergence theorem we obtain that $v_n(.)$ converges to $v_0(.)$ in $L^1(I,X)$. Since $v_n\in C$ and C is closed we have that $v_0\in C$ and, since v_0 is arbitrary in $G_F(s_0)$, it follows that $G_F(s_0)\subset C$. ♦

The next lemma has been stated in [24] and it is a direct consequence of Proposition 4 and Theorem 3 in [9].

Lemma 2.13. Let $G: S \to \mathcal{D}$ be a l.s.c. map and let $\varphi:S\to L^1(I,X)$, $\psi:S\to L^1(I,R)$ be continuous and such that for every $s\in S$ the set

(2.7) $$H(s) = \text{cl}\,\{u\in G(s): \|u(t) - \varphi(s)(t)\| < \psi(s)(t) \text{ a.e. in I}\}$$

is nonempty. Then the multivalued map $s\to H(s)$ defined by (2.7) is l.s.c. from S into $\mathcal{D}$, hence it admits a continuous selection.

Remark 2.14. Lemma 2.12 and Lemma 2.13 hold true in the more general case when instead of $(I,\mathcal{L},\mu)$ consider a measure space $(T,\mathcal{F},\nu)$ where $\mathcal{F}$ is a σ-algebra of subsets of T and ν is a nonatomic probability measure (see [9]).

Consider the Cauchy problem

$$\dot{x} \in F(t,x), \quad x(0)=\xi, \tag{2.8}$$

where $F: I \times X \to 2^X$ is a multifunction and $\xi \in X$.

Definition 2.15. By a solution of (2.8) we mean any absolutely continuous function $x: I \to X$ satisfying $x(0)=\xi$ and $\dot{x}(t) \in F(t,x(t))$ a.e. in I.

Definition 2.16. By an approximate solution of (2.8) we mean any absolutely continuous function $x: I \to X$ such that $t \to \rho(t) := d(\dot{y}(t), F(t,y(t))$ is integrable.

Several results in the next chapters concern multifunctions satisfying the following:

Assumption 2.17: $F: I \times X \to 2^X$ takes closed nonempty values and satisfies:

(H_1) $t \to F(t, x)$ is $\mathcal{L}$-measurable, for all $x \in X$,

(H_2) there exists $k \in L^1(I,R)$ such that $d(F(t,x), F(t,y)) \le k(t)\|x - y\|$, for all $x, y \in X$, a.e. in I,

(H_3) there exists $\beta \in L^1(I, R)$ such that $d(0, F(t, 0)) \le \beta(t)$, $t \in I$ a.e..

We say that F is *Lipschitzean in x* if it satisfies (H_2).

Between solutions and approximate solutions of (2.8) when F is Lipschitzean in x we have the following relation proved by Filippov [28]:

Proposition 2.18 If F is Hausdorff-continuous in (t,x) and Lipschitzean in x then :

(i) there exists a solution x(.) of (2.8) if and only if there exists an approximate solution

(ii) if y(.) is a given approximate solution then there exists a solution x(.) of (2.8) such that

(2.9) $$\int_0^t \|\dot{y}(u) - \dot{x}(u)\| \, du \leq \int_0^t e^{K(t)-K(s)} \rho(s) ds$$

where $K(t) = \int_0^t k(s)ds$.

The above result was improved by Himmelberg and Van Vleck in [37], replacing continuity of F by measurability.

3. WELL POSEDNESS FOR LIPSCHITZEAN DIFFERENTIAL INCLUSIONS.

3.1 INTRODUCTION.

Consider the Cauchy problem

(3.1.1) $$\dot{x} \in F(t, x) \ , \quad x(0) = \xi$$

where F is a multivalued map from $I \times R^n$ into the closed nonempty subsets of R^n, Lipschitzean in x, without any convexity or boundedness assumptions, and I=[0,T].

The existence of a solution of (3.1.1) has been proved by Filippov [28] and improved by Himmelberg and Van Vleck in [37]. A continuous version of Filippov's result is due to Ornelas [47] and asserts that if $\dot{y}(.,\xi)$ varies continuously in $L^1(I,R^n)$, with respect to ξ in a compact subset Ξ of R^n and if $\delta>0$ is given then a solution $x(.,\xi)$ of (3.1.1) verifies the folloving (relaxed) inequality

(3.1.2) $$\int_0^t \|\dot{y}(u,\xi) - \dot{x}(u,\xi)\| \, du \le \delta + \int_0^t e^{K(t)-K(s)} \rho(s,\xi) ds$$

where $\rho(t,\xi) := d(\dot{y}(t,\xi), F(t,y(t,\xi)))$. Clearly this implies the existence of a solution $x(.,\xi)$ of (3.1.1) which is continuous with respect to ξ in Ξ.
For ξ in Ξ let $\mathcal{T}(\xi)$ be the set of solutions of (3.1.1) and let $\mathcal{A}_T(\xi) := \{x(T) : x(.) \in \mathcal{T}(\xi)\}$ be the *attainable set* at time T. In this way we associate to F two new multifunctions $\mathcal{T}(.)$ and $\mathcal{A}_T(.)$ from Ξ into the subsets of $AC(I,R^n)$ and R^n respectivelly, with values $\mathcal{T}(\xi)$ and $\mathcal{A}_T(\xi)$ which are in general nonclosed, nonconvex [34]. Moreover, in general, the map $\mathcal{T}(.)$ is neither upper nor lower semicontinuos. When the map F is upper semicontinuous with compact convex values then the map $\mathcal{T}(.)$ is known to be upper semicontinuous with compact connected values ([4], [25]).

The first result asserting the existence of a continuous selection from $\mathcal{T}(.)$ and $\mathcal{A}_T(.)$ was proved by Cellina [14], under the assumption that Ξ is a compact subset of R^n and F is Lipschitzean in x with compact (non necessarily convex) values contained in a bounded subset of R^n. The proof is based on the Liapunov's theorem on the range of nonatomic vector measures. The assumption on F was relaxed and the proof was simplified by Cellina and Ornelas in [16], proving the same result for F Lipschitzean in x with closed nonempty values.

A generalisation of this results has been obtained in the joint paper with Colombo, Fryszkowski and Rzezuchowski [24] where the Cauchy problems

$$\dot{x} \in F(t, x, s) \ , \quad x(0) = \xi(s) \tag{3.1.3}$$

was considered, with F Lipschitzean in x and l.s.c. in s, and $\xi(.)$ continuous. Denoting by $\mathcal{T}(s)$ the set of solutions of (3.1.3) we have proved the existence of a continuous selection from the multivalued map $s \to \mathcal{T}(s)$, passing by a fixed point of graph $(\mathcal{T})$. Remark that the existence of such a selection implies that the multivalued map $s \to \mathcal{T}(s)$ is lower semicontinuous ([4], p.80), hence it implies the result by Naselli Ricceri and Ricceri [45]. Remark that our result contains also as particular cases the selection theorems by Antosiewicz and Cellina [1], Bressan and Colombo [9] and Fryszkowski [31]. Using this result Fryszkowski and Rzezuchowski have obtained in [33] a continuous version of Fillipov-Wazewski relaxation theorem for the problem (3.1.3).

The proof of this result, which we present in section 3.2, is based on a continuous selection theorem obtained by Bressan and Colombo in [9].

Following [52], in section 3.3 we prove an analogue of Michael's extension theorem [42] for solution sets.

Section 3.4 contains a result proved in [19]. Here we consider a multifunction F defined on $R \times C$, where C is a closed subset of R^n, and satisfying a tangentiality condition, and we prove the existence of a continuous selection from the map assigning to an initial point the set of solutions (with values in C) of the Cauchy problem (3.1.1).

Remark that the (generalized) successive approximations process that is the base of the construction, in the case under consideration requires at each step a projection over the (in general, non convex) set C, since F is not defined outside C, and that this projection is not continuous. Moreover, the lack of an argument allowing the extension of a multi-valued Lipschitzean map from a closed set to an open set containing it, prevents the possibility of exploiting the available techniques for the present case.

As a side result we obtain the convergence of the sequence of generalized successive approximations for any initial function x_0.

In the last section of this chapter we shall use the selection theorems in Sections 3.2 and 3.4 to prove the arcwise connectedness of the set of solutions and of the attainable set. We show also that any two continuous selections from the map assigning to the initial point the set of solutions, are linked by a continuous homotopy with values in the solution sets.

3.2 WELL POSEDNESS FOR DIFFERENTIAL INCLUSIONS ON OPEN SETS.

In this section I is the interval [0, 1], S is a separable metric space and X is a separable Banach space. Consider a multifunction $F: I \times X \times S \to 2^X$ satisfying

Assumption 3.1: F takes closed nonempty values and:

(A_1) F is $\mathcal{L}\otimes\mathcal{B}(X\times S)$-measurable,

(A_2) there exists a continuous map $k : S \to L^1(I, R)$ such that $k(s)(t)>0$ and for all $x, y \in X$ and $s \in S$:

$$\mathbf{d}(F(t, x,s), F(t, y,s)) \leq k(s)(t) \, \|x - y\| \text{ a.e. in } I,$$

(A_3) for any $(t,x) \in I \times X$ the multivalued map $s \to F(t,x,s)$ is l.s.c..

(A_4) there exists a continuous map $\beta : S \to L^1(I, R)$ such that for every s:

$$d(0, F(t,0,s)) \leq \beta(s)(t) \text{ a.e. in } I.$$

For s in S consider the Cauchy problem

(3.2.1) $$\dot{x} \in F(t, x,s) \;,\; x(0) = \xi(s)$$

where $\xi(.)$ satisfies

Assumption 3.2: $\xi : S \to X$ is continuous.

Denote by $\mathcal{T}(s)$ the set of solutions of (3.2.1).

The main result of this section is the following

Theorem 3.3. Let F and ξ satisfy Assumptions 3.1 and 3.2, respectively, let $s_0 \in X$ and let $x_0 \in \mathcal{T}(s_0)$ be given. Then there exists $x(\cdot,\cdot) : I \times S \to X$ with the following properties:

(i) $x(\cdot, s) \in \mathcal{T}(s)$ for each $s \in S$;
(ii) $s \to x(\cdot, s)$ is continuous from S into AC(I, X) ;
(iii) $x(\cdot, s_0) = x_0(\cdot)$.

Proof. Remark that we may assume $\xi(s)=0$ and $x_0(t)=0$ for all t in I and s in S. In fact, set

$$F^*(t,z,s)=F(t,z+\xi(s)-\xi(s_0)+x_0(t),s) - \dot{x}_0(t)$$

and consider the Cauchy problem

(3.2.2) $$\dot{z} \in F^*(t, z,s) \ , \ z(0) = 0.$$

Then F* satisfies Assumption 3.1 with

$$\beta^*(s)(t) = \|\dot{x}_0(t)\| + \beta(s)(t) + k(s)(t)\|\xi(s)-\xi(s_0)+x_0(t)\|,$$

and $0 \in F^*(t,0,s_0)$. Moreover $x(.,.)$ given by

$$x(t,s) = z(t,s) + \xi(s) - \xi(s_0) + x_0(t),$$

satisfies assertions (i), (ii) and (iii) in the theorem whenever $z(.,.)$ do so for the problem (3.2.2) and $x_0 = 0$.
In the following we assume that $x_0=0$, hence $0 \in F^*(t,0,s_0)$, and we prove the theorem for the problem

(3.2.3) $$\dot{x} \in F(t, x,s) \ , \ x(0) = 0.$$

Let $F_* : I \times X \times S \to 2^X$ be given by

$$F_*(t,x,s) = \begin{cases} F(t,x,s) & \text{if } s \neq s_0 \\ \{0\} & \text{if } s = s_0 \end{cases}$$

and remark that $d(0, F_*(t,x,s)) = d(0, F(t,x,s))$ and, clearly, F_* satisfies Assumption 3.1.
Fix $\varepsilon > 0$ and, set $\varepsilon_n = \varepsilon \frac{n+1}{n+2}$, $n \in N$.
Let $x_0(t,s) := 0$ and define the multivalued maps G_0: $S \to 2^{L^1(I,X)}$ and H_0: $S \to 2^{L^1(I,X)}$ by

$$G_0(s) = \{v \in L^1(I, X) : v(t) \in F_*(t, x_0(t, s), s) \text{ a.e. in } I\},$$

$$H_0(s) = cl\{v \in G_0(s) : \|v(t)\| < \beta(s)(t) + \varepsilon_0 \text{ a.e. in } I\}.$$

Then by (A_4) in Assumption 3.1 and by Lemma 2.12 we obtain that $G_0(\cdot)$ is l.s.c. from S into $\mathcal{D}$ and for all $s \in S$ the set $H_0(s)$ is nonempty. Then by Lemma 2.13 it follows that there exists $h_0: S \to L^1(I, X)$, a continuous selection from $H_0(\cdot)$, hence such that:

$$h_0(s)(t) \in F(t, x_0(t, s), s),$$

$$h_0(s_0)(t) = 0,$$

and

$$\|h_0(s)(t)\| \leq \beta(s)(t) + \varepsilon_0 \text{ a.e. in } I.$$

Define

$$x_1(t, s) = \int_0^t h_0(s)(u)du$$

and notice that

$$\|x_1(t,s) - x_0(t, s)\| \leq \int_0^t \|h_0(s)(u)\|du \leq \int_0^t \beta(s)(u)du + \varepsilon_0 < \beta_1(s)(t),$$

where, by definition,

$$(3.2.4) \qquad \beta_n(s)(t) = \int_0^t \beta(s)(u) \frac{[K(t,s)-K(u,s)]^{n-1}}{(n-1)!} du + \frac{[K(t,s)]^{n-1}}{(n-1)!} \varepsilon_n$$

and

$$K(t,s) = \int_0^t k(s)(u)du.$$

Remark that since β is continuous from S into $L^1(I,R)$, by (3.2.4) it follows that also β_n is continuous from S into $L^1(I, R)$.

We claim that we can construct a sequence of successive approximation $\{x_n(t,s)\}_{n\in N}$ such that $x_n(.,s)\in AC(I,X)$, $x_n(0,s)=0$ and, for each $n \geq 1$:

(a) $s \to x_n(\cdot, s)$ is continuous from S into $AC(I,X)$;

(b) $\dot{x}_n(t,s)\in F(t, x_{n-1}(t,s),s)$ a.e. in I;

(c) $\|\dot{x}_n(t,s) - \dot{x}_{n-1}(t,s)\| \leq k(s)(t)\, \beta_{n-1}(s)(t)$, a.e. in I;

(d) $x_n(., s_0)=0$.

Remark that the above holds for n=1 and also that by (c) we obtain

$$\|x_n(t, s) - x_{n-1}(t,s)\| \leq \int_0^t \|\dot{x}_n(u,s) - \dot{x}_{n-1}(u,s)\|\, du \leq \int_0^t k(s)(u)\beta_{n-1}(s)(u)du =$$

(3.2.5) $$= \int_0^t \beta(s)(u) \int_u^t k(s)(\tau)\frac{[K(t,s)-K(\tau,s)]^{n-2}}{(n-2)!}d\tau du + \varepsilon_{n-1} \int_0^t k(\tau)\, \frac{[K(\tau,s)]^{n-2}}{(n-2)!}d\tau =$$

$$= \int_0^t \beta(s)(u)\, \frac{[K(t,s)-K(u,s)]^{n-1}}{(n-1)!}du) + \varepsilon_{n-1}\frac{[K(t,s)]^{n-1}}{(n-1)!} < \beta_n(s)(t) .$$

hence

(e) $$\|x_n(t, s) - x_{n-1}(t,s)\| < \beta_n(s)(t) \text{ a.e. in I.}$$

Assume we have constructed $x_1, \ldots, x_n$ satisfying (a)-(d) and let us construct x_{n+1}. Observe that by (b) and (A_2) in Assumption 3.1 we obtain

$$d(\dot{x}_n(t,s), F(t, x_n(t, s),s)) \leq d(F(t, x_{n-1}(t, s),s), F(t, x_n(t, s),s)) \leq k(s)(t)\|x_n(t, s) - x_{n-1}(t,s)\|$$

and by (e) it follows that

$$(3.2.6)\qquad d(\dot{x}_n(t,s), F(t, x_n(t, s),s)) < k(s)(t)\ \beta_n(s)(t) \text{ a.e. in } I$$

Define the multivalued maps $G_n: S \to 2^{L^1(I,X)}$ and $H_n: S \to 2^{L^1(I,X)}$ by

$$(3.2.7)\qquad G_n(s) = \{v \in L^1(I, X) : v(t) \in F_*(t, x_n(t, s),s) \text{ a.e. in } I\},$$

and

$$(3.2.8)\qquad H_n(s) = cl\{v \in G_n(s) : \|v(t) - \dot{x}_n(t,s)\| < k(s)(t)\ \beta_n(s)(t) \text{ a.e. in } I\}.$$

Then by (3.2.6) and Lemma 2.12 it follows that $G_n(.)$ is l.s.c. from S into $\mathcal{D}$ and for all $s \in S$ the set $H_0(s)$ is nonempty. Moreover, by Lemma 2.13 it follows that there exists $h_n: S \to L^1(I,X)$, a continuous selection from $H_n(\cdot)$, hence such that:

$$h_n(s)(t) \in F(t, x_n(t, s),s),$$

$$h_n(s_0)(t)=0,$$

and

$$\|h_n(s)(t) - \dot{x}_n(t,s)\| \leq k(s)(t)\ \beta_n(s)(t) \text{ a.e. in } I.$$

Define

$$x_{n+1}(t, s) = \int_0^t h_n(s)(u)du$$

and notice that it satisfies (a)-(d) in our claim.
By (c) and (3.2.5) we obtain that

$$(3.2.9)\qquad \|x_{n+1}(.,s) - x_n(., s)\|_{AC} \leq \beta_{n+1}(s)(1) \leq \frac{[\|k(s)\|_1]^n}{n!}(\|\beta(s)\|_1 + \varepsilon).$$

Since the functions $s \to \|k(s)\|_1$ and $s \to \|\beta(s)\|_1$ are continuous they are locally bounded, hence, by (3.2.9), the sequence $\{x_n(., s)\}_{n \in N}$ satisfies the Cauchy condition locally uniformly with respect to s. Hence if we set $x(t,s) = \lim x_n(t, s)$ we obtain that $s \to x(.,s)$ is continuous from S into AC(I,X). Moreover since $x_n(., s_0) = 0$ for any $n \in N$ we have that $x(., s_0) = 0$.
To see that $t \to x(t,s)$ is a solution of (3.2.3) it is enough to notice that

$$d(\dot{x}_n(t,s), F(t, x(t, s),s)) \leq k(s)(t)\|x_n(t, s) - x(t,s)\|$$

and F has closed values. ♦

Corollary 3.4. Let $F: I\times X \to 2^X$ satisfies Assumption 2.17 and for ξ in X let $\mathcal{T}(\xi)$ be the set of the Cauchy problem

(3.2.10) $$\dot{x} \in F(t, x) \ , \quad x(0) = \xi.$$

Let $\xi_0 \in X$ and let $x_0 \in \mathcal{T}(\xi_0)$ be given. Then there exists $x(\cdot, \cdot) : I\times X \to X$ with the following properties:

(i) $x(\cdot, \xi) \in \mathcal{T}(\xi)$ for each $\xi \in X$;
(ii) $\xi \to x(\cdot, \xi)$ is continuous from X into AC(I, X) ;
(iii) $x(\cdot, \xi_0) = x_0(\cdot)$.

Proof. Since F satisfies Assumption 2.17 it follows that Assumptions 3.1 and 3.2 are satisfied for $F(t,x,s)=F(t,x)$, $S=X$ and $\xi(.)$ = the identity. Therefore the Corollary follows from Theorem 3.3. ♦

Remark 3.5. By Theorem 3.3 it follows that the multivalued map $s \to \mathcal{T}(s)$ is locally selectionable (Definition 1 in [4, p. 80]), hence by Proposition 1 in [4, p. 80], it is lower semicontinuous. Therefore the result of Naselli Ricceri and Ricceri in [45] is a consequence of Theorem 3.3.

Corollary 3.4 contains as particular cases the selection theorems by Cellina [14], Cellina and Ornelas [16] and Ornelas [47].

As a corollary of Theorem 3.3 we obtain also the continuous selection theorem by Bressan and Colombo [9], which generalises the selection theorems by Antosiewicz and Cellina [1] and by Fryszkowski [31].

Corollary 3.6. Let F be a lower semicontinuous multivalued map from S into $\mathcal{D}$. Then it admits a continuous selection.

Proof. Obviously $F(t,x,s)=F(s)$ satisfies Assumption 3.1. Let $\mathcal{T}(s)$ be the set of solutions of the Cauchy problem

$$\dot{x} \in F(s) \quad , \quad x(0) = 0,$$

and let $\mathcal{T}'(s) = \{\dot{x} : x \in \mathcal{T}(s)\}$. Then by Theorem 3.3 there exists a continuous selection from $s \to \mathcal{T}(s)$, hence also from $s \to \mathcal{T}(s)$ and $s \to F(s)$. ♦

The folloving proposition was proved in [52] and gives the existence of solutions for a class of boundary value problems.

Proposition 3.7. Let F as in Corollary 3.4, let K be a nonempty compact convex subset of X and assume that $\mathcal{T}(K)(T) \subset K$, where $\mathcal{T}(K)(T)=\{x(T): x \in \mathcal{T}(\xi), \xi \in K\}$. Then the boundary value problem

$$\dot{x} \in F(t, x) \quad , \quad x(0) = x(T) \in K, \tag{3.2.11}$$

admits a solution.

Proof. Let $\varphi: X \to AC(I,X)$ be a continuous selection from $\mathcal{T}(\xi)$ given by Corollary 3.4 and define $\psi: X \to X$ by $\psi(\xi)=\varphi(\xi)(T)$. Then ψ is continuous and $\psi(K) \subset K$, hence by Schauder's Theorem there exists $\xi_0 \in K$, a fixed point of ψ. Then $\varphi(\xi_0)(T)=\xi_0= \varphi(\xi_0)(0)$ and so $x=\varphi(\xi_0)$ is a solution of (3.2.11) . ♦

3.3 AN ANALOGUE FOR SOLUTION SETS OF MICHAEL'S EXTENSION THEOREM

In this section $I=[0,T]$, X is a separable Banach space, $F: I\text{x}X \to 2^X$ is a multifunction satisfying Assumption 2.17 and $\mathcal{T}(\xi)$ stands for the set of solutions of the Cauchy problem

$$(3.3.1) \qquad \dot{x} \in F(t, x) \ , \ x(0) = \xi.$$

The Michael's theorem [42] asserts that if A is a closed subset of X, Z is a Banach space, $G:X\to 2^Z$ is a lower semicontinuous multivalued map with closed convex nonempty values and $g: A\to Z$ is a continuous selection from G on A (i.e. $g(x)\in G(x)$ for all $x\in A$) then there exists a continuous extension $g^*: X\to Z$ of g such that $g(x)\in G(x)$ for all $x\in X$

We present here an analogue of the Michael's result for the map $\xi\to\mathcal{T}(\xi)$, obtained in [52]. To prove it we shall use a result in [8].

Theorem 3.8. If Y is a closed nonempty subset of X and $\varphi:Y\to AC(I,X)$ is a continuous map such that $\varphi(\xi)\in\mathcal{T}(\xi)$ for all $\xi\in Y$ then there exists $\varphi^*: X\to AC(I,X)$, a continuous extension of φ, such that $\varphi^*(\xi)\in\mathcal{T}(\xi)$ for all $\xi\in X$.

Proof. Let $\mathcal{T}'(\xi)=\{\dot{x}: x\in\mathcal{T}(\xi)\}$ and set $\varphi'(\xi)=[\dot{\varphi}(\xi)]$. Then $\varphi': Y\to L^1(I, X)$ is continuous and satisfies $\varphi'(\xi)\in\mathcal{T}'(\xi)$ for all $\xi\in Y$. By Theorem 1 in [9] there exists $\lambda: X\to L^1(I, X)$ a continuous extension of φ' and by Theorem 2 in [8] there exists a continuous map $\psi:X\text{x}L^1(I,X)\to L^1(I,X)$ such that :

(i) $\psi(\xi,u)\in\mathcal{T}'(\xi)$ for each $u\in L^1(I,X)$,

(ii) $\psi(\xi,u) = u$ for each $u\in\mathcal{T}'(\xi)$.

Define $\eta: X\to L^1(I, X)$ by $\eta(\xi)=\psi(\xi,\lambda(\xi))$ and, by (i) we obtain that $\eta(\xi)\in\mathcal{T}'(\xi)$ for each $\xi\in X$. Moreover, since for $\xi\in Y$ we have $\lambda(\xi)=\varphi'(\xi)\in\mathcal{T}'(\xi)$, by (ii) it follows that for all $\xi\in Y$ we have $\eta(\xi)=\psi(\xi,\varphi'(\xi))=\varphi'(\xi)$. Therefore $\eta(.)$ is a continuous extension of $\varphi'(.)$ and $\eta(\xi)\in\mathcal{T}'(\xi)$ for each $\xi\in X$.

Setting

$$\varphi^*(\xi)(t) = \xi + \int_0^t \eta(\xi)(\tau)d\tau$$

we obtain that $\varphi^*(.)$ is a continuous extension of $\varphi(.)$ and $\varphi^*(\xi)\in \mathcal{T}(\xi)$ for every $\xi\in X$. ♦

Corollary 3.9. Let $\xi_0, \xi_1 \in X$, $\xi_0 \neq \xi_1$, and let $x_0\in \mathcal{T}(\xi_0)$, $x_1\in \mathcal{T}(\xi_1)$. Then there exists a continuous map $\lambda:[0,1]\to AC(I,X)$ such that: $\lambda(0)=x_0$, $\lambda(1)=x_1$ and, for $\alpha\in[0,1]$, $\lambda(\alpha)\in \mathcal{T}(\xi_\alpha)$, where $\xi_\alpha=(1-\alpha)\xi_0+\alpha\xi_1$.

Proof. Let $Y=\{\xi_0, \xi_1\}$ and $\varphi:Y\to AC(I,X)$ be given by $\varphi(\xi_0) = x_0$, $\varphi(\xi_1) = x_1$. By Theorem 3.8, there exists a continuous extension $\varphi^*(.)$ of $\varphi(.)$ such that $\varphi^*(\xi)\in \mathcal{T}(\xi)$ for every $\xi\in X$. Then the map $\lambda:[0,1]\to AC(I,X)$ defined by $\lambda(\alpha)=\varphi^*(\xi_\alpha)$, has the properties stated in the corollary. ♦

3.4 WELL POSEDNESS FOR DIFFERENTIAL INCLUSIONS ON CLOSED SETS.

Let I be the interval [0, T] and let C be a closed nonempty subset of R^n. For $x\in C$ set $d_C(x):=d(x,C)$ and let

$$T_C(x) = \{v\in R^n : \liminf_{h\to 0^+} d_C(x + hv) = 0\}$$

be the contingent cone to C at x. Let $F:I\times C\to 2^{R^n}$ be a multivalued map satisfying the Assumption 2.17 and

Assumption 3.10: $F(t, x)\subset T_C(x)$ for all $(t, x)\in I\times C$.

For ξ in C consider the Cauchy problem

(3.4.1) $$\dot{x} \in F(t, x) \ , \ x(0) = \xi$$

and denote by $\mathcal{T}(\xi)$ the set of solutions (with values in C) of (3.4.1).

Let $\xi_0 \in C$ and let $x_0 \in AC(I, R^n)$ be such that $x_0(0) = \xi_0$. Let $z_0(\cdot)$ be a measurable selection from $t \to \pi_C(x_0(t))$; let $v_1(\cdot)$ be a measurable selection from $t \to F(t, z_0(t))$ such that

$$\|v_1(t) - \dot{x}_0(t)\| = d(\dot{x}_0(t), F(t, z_0(t))) \text{ a.e. in } I,$$

and let $x_1(\cdot)$ be given by

$$x_1(t) = \xi_0 + \int_0^t v_1(s)ds .$$

We call $x_1(\cdot)$ a (first) *successive approximation* from x_0. Remark that, when x_0 is already a solution to (P_{ξ_0}) (with values in C), we have $z_0 \equiv x_0$ and $v_1 \equiv \dot{x}_0$, so that the sequence of successive approximations consists of the point x_0 only.

We shall prove the following

Theorem 3.11. Let $C \subset R^n$ be compact, $\xi_0 \in C$ and let $x_0 \in AC(I, R^n)$ be such that $x_0(0) = \xi_0$. Let $F : I \times C \to 2^{R^n}$ satisfy Assumptions 2.17 and 3.10. Then there exists $x(\cdot, \cdot) : I \times C \to C$ with the following properties:

(i) $x(\cdot, \xi) \in \mathcal{T}(\xi)$ for each $\xi \in C$;

(ii) $\xi \to x(\cdot, \xi)$ is continuous from C into $AC(I, R^n)$;

(iii) $x(\cdot, \xi_0)$ is limit of a sequence of successive approximations $\{x_n(\cdot, \xi_0)\}_{n \geq 0}$ with $x_0(\cdot, \xi_0) = x_0(\cdot)$. In particular if $x_0 \in \mathcal{T}(\xi_0)$ then $x(\cdot, \xi_0) = x_0(\cdot)$.

Proof.

Set $x_0(t, \xi)$ to be $\xi + \int_0^t \dot{x}_0(s)ds$, so that $x_0(t, \xi_0) = x_0(t)$. For ξ in C let $z_0(\cdot, \xi)$ be a measurable selection from $t \to \pi_C(x_0(t, \xi))$. Then

$$d(\dot{x}_0(t, \xi), F(t, z_0(t, \xi))) \leq \|\dot{x}_0(t)\| + d(0, F(t, 0)) + k(t)\, \|z_0(t, \xi)\| \leq$$

$$\leq \|\dot{x}_0(t)\| + \beta(t) + Mk(t) =: \delta_0(t) ,$$

where $M := \sup\{\|\xi\| : \xi \in C\}$.

By Proposition 1 in ([4], p. 202) we have that

$$\frac{d}{dt}[d_C(x_0(t, \xi))] \leq d(\dot{x}_0(t, \xi), T_C(\pi_C(x_0(t, \xi)))) \leq$$

$$\leq d(\dot{x}_0(t, \xi), T_C(z_0(t, \xi))) \leq d(\dot{x}_0(t, \xi), F(z_0(t, \xi))) \leq \delta_0(t)$$

and, since $d_C(x_0(0, \xi)) = 0$, we obtain

$$d_C(x_0(t, \xi)) \leq \int_0^t \delta_0(s)ds .$$

Following ideas in [1] and [16], let $\eta := \frac{1}{2}$ and, for ξ in C, set

$$\rho_1(\xi) = \min\{\eta, \|\xi - \xi_0\|/2\} \text{ if } \xi \neq \xi_0 , \rho_1(\xi_0) = \eta .$$

Cover C with balls $B(\xi, \rho_1(\xi))$ and let $\{B(\xi_j^1, \rho_1(\xi_j^1)) : 0 \leq j \leq N_1\}$ be a finite subcovering with $\xi_0^1 = \xi_0$. Let $\{p_j^1 : 0 \leq j \leq N_1\}$ be a continuous partition of unity subordinate to this subcovering and define

$$I_0^1(\xi) = [0, Tp_0^1(\xi)] ,$$

$$I_j^1(\xi) = \left]T\sum_{k=0}^{j-1} p_k^1(\xi) \ , \ T\sum_{k=0}^{j} p_k^1(\xi)\right] \text{ for } 0 < j \leq N_1 .$$

Since ξ_0 belongs only to $B(\xi_0, \rho_1(\xi_0))$ we have that $I_0^1(\xi_0) = [0, T]$.

Let $v_1(\cdot, \xi)$ be a measurable selection from $t \to F(t, z_0(t, \xi))$ such that

$$\|v_1(t, \xi) - \dot{x}_0(t, \xi)\| = d(\dot{x}_0(t, \xi) , F(t, z_0(t, \xi))) ,$$

and set

$$x_1(t,\xi) = \xi + \int_0^t \sum_{j=0}^{N_1} \chi_{I^1_j(\xi)}(s) v_1(s, \xi^1_j) ds .$$

Remark that $x_1(t, x_0) = \xi_0 + \int_0^t v_1(s,\xi_0)ds$, hence it is a (first) successive approximation from x_0. Moreover,

$$\|x_1(t,\xi) - x_0(t,\xi)\| \le \int_0^t \|\dot{x}_1(s,\xi) - \dot{x}_0(s,\xi)\| ds \le$$

$$\le \int_0^t \sum_{j=0}^{N_1} \chi_{I^1_j(\xi)}(s) \|v_1(s,\xi^1_j) - \dot{x}_0(s,\xi)\| ds \le$$

$$\le \int_0^t \sum_{j=0}^{N_1} \chi_{I^1_j(\xi)}(s) \|v_1(s,\xi^1_j) - \dot{x}_0(s,\xi^1_j)\| +$$

$$+ \int_0^t \sum_{j=0}^{N_1} \chi_{I^1_j(\xi)}(s) \|\dot{x}_0(s,\xi^1_j) - \dot{x}_0(s,\xi)\| ds \le \int_0^t \delta_0(s) ds .$$

Let $z_1(\cdot,\xi)$ be a measurable selection from $t \to \pi_C(x_1(t,\xi))$. Fix $t \in I$ and let j be such that $t \in I^1_j(\xi)$. Then :

$$d(\dot{x}_1(t,\xi), F(t, z_1(t,\xi))) = d(v_1(t,\xi^1_j), F(t, z_1(t,\xi))) \le$$

$$\le d(F(t, z_0(t,\xi^1_j)), F(t, z_1(t,\xi))) \le k(t) \|z_0(t,\xi^1_j) - z_1(t,\xi)\| \le$$

$$\le k(t)[2d_C(x_0(t,\xi)) + 2\|x_0(t,\xi^1_j) - x_0(t,\xi)\|] +$$

$$+ k(t)[d_C(x_0(t,\xi)) + d_C(x_1(t,\xi)) + \|x_1(t,\xi) - x_0(t,\xi)\|] \le$$

$$\le k(t)[d_C(x_1(t,\xi)) + 3d_C(x_0(t,\xi)) + \int_0^t \delta_0(s)ds + 2\eta] \le$$

$$\le k(t)[d_C(x_1(t,\xi)) + 4\int_0^t \delta_0(s)ds + 3\eta] ,$$

and this estimate is independent of j, hence it holds on I.

As before $d_C(x_1(0, \xi)) = 0$ and

$$\frac{d}{dt}[d_C(x_1(t, \xi))] \le d(\dot{x}_1(t, \xi), F(t, x_1(t, \xi))) \le$$

$$\le k(t)\, d_C(x_1(t, \xi) + k(t)\, [4\int_0^t \delta_0(s)\, ds + 3\eta] ,$$

so that

$$d_C(x_1(t, \xi)) \le \int_0^t e^{\int_s^t k(\tau)d\tau} k(s)[4\int_0^s \delta_0(u)du + 3\eta]ds =$$

$$= 4\int_0^t \delta_0(s)[e^{\int_s^t k(\tau)d\tau} - 1]ds + 3\eta[e^{\int_s^t k(\tau)d\tau} - 1]$$

and

$$d(\dot{x}_0(t, \xi) , F(t, z_0(t, \xi))) \le k(t)[4\int_0^t \delta_0(s)\, e^{\int_s^t k(\tau)d\tau} ds + 3\eta e^{\int_s^t k(\tau)d\tau}] .$$

Set

$$K(t) = \int_0^t k(\tau)d\tau$$

and, as in [46, pag. 121], set $\delta_1(t)$ to be the essential supremum of the family

$$\{t \to d(\dot{x}_1(t, \xi), F(t, z(t, \xi)) : \xi \in C , z(\cdot, \xi) \text{ measurable, } z(t, \xi) \in \pi_C(x_1(t, \xi))\} .$$

Then, for a.e. $t \in I$,

$$\delta_1(t) \le k(t)e^{K(t)}[4\int_0^t \delta_0(s)e^{-K(s)}ds + 3\eta] .$$

Finally set

$$\alpha_1(t) = \sup\{\|v_1(t,\xi_j^1) - v_1(t,\xi_i^1)\| : 0 \le i, j \le N_1\}$$

and remark that $\alpha_1 \in L^1(I, R)$ and, since all of the functions $\xi \to \sum_{k=0}^{j} p_k^1(\xi)$ are uniformly continuous on C, for every $\varepsilon > 0$ there exists $\rho > 0$ such that

$$\|\xi' - \xi\| < \rho \text{ implies } \|\dot{x}_1(t,\xi) - \dot{x}_1(t,\xi')\| \le \alpha_1(t)\,\chi_E(t)$$

for some $E \subset I$ with $\mu(E) < \varepsilon$.

We claim we can define sequences of functions $\{x_n\}_{n\ge1}$, $\{\delta_n\}_{n\ge1}$ and $\{\alpha_n\}_{n\ge1}$ with the following properties:

(i) $\alpha_n \in L^1(I, R)$ is such that for every $\varepsilon > 0$ there exists $\rho > 0$ such that

$$\|\xi' - \xi\| < \rho \text{ implies } \|\dot{x}_n(t,\xi) - \dot{x}_n(t,\xi')\| \le \alpha_n(t)\chi_E(t)$$

for some $E \subset I$ with $\mu(E) < \varepsilon$;

(ii) $x_n(\cdot, \xi_0)$ is a n-successive approximation from $x_0(\cdot)$;

(iii) $\int_0^t \|\dot{x}_n(s,\xi) - \dot{x}_{n-1}(s,\xi)\| ds \le \int_0^t \delta_{n-1}(s)ds + \eta^n$

where

$$\delta_n(t) = \text{ess}\sup\{d(\dot{x}_n(t,\xi), F(t, z(t,\xi))) : \xi \in C,\ z(\cdot,\xi) \text{ measurable},$$
$$z(t,\xi) \in \pi_C(x_n(t,\xi)) \text{ a.e.}\};$$

(iv) $d_C(x_n(t,\xi)) \le \int_0^t \delta_n(s)ds$;

(v) $\delta_n(t) \le k(t)e^{K(t)}\{4^n \int_0^t \delta_0(s)e^{-K(s)}\frac{[K(t) - K(s)]^{n-1}}{(n-1)!}ds + 3\eta^n \sum_{i=0}^{n-1} \frac{1}{i!}\left[\frac{4K(t)}{\eta}\right]^i$.

For $n = 1$ the above holds. Assume it holds up to m and let us show it holds for $m + 1$. By (i) of the inductive assumption there exist $\alpha_m \in L^1(I, R)$ and $\rho_{m+1} > 0$ such that

$$\|\xi' - \xi\| < \rho_{m+1} \text{ implies } \|\dot{x}_m(t,\xi) - \dot{x}_m(t,\xi)\| \le \alpha_m(t)\chi_E(t)$$

with $\int_E \alpha_m(s)ds < \eta^{m+1}$.

Define, for ξ in C,

$$\rho_{m+1}(\xi) = \min\{\rho_{m+1}, \|\xi - \xi_0\|/2\} \text{ if } \xi \neq \xi_0, \rho_{m+1}(\xi_0) = \rho_{m+1}$$

and let $\{B(\xi_j^{m+1}, \rho_{m+1}(\xi_j^{m+1})) : 0 \leq j \leq N_{m+1}\}$ be a finite covering of C with $\xi_0^{m+1} = \xi_0$.

Set for simplicity $N := N_{m+1}$ and $\xi_j = \xi_j^{m+1}$, $0 \leq j \leq N$. Let $\{p_j : 0 \leq j \leq N\}$ be a continuous partition of unity subordinate to this covering and define

$$I_0(\xi) = [0, Tp_0(\xi)],$$

$$I_j(\xi) =]T\sum_{k=0}^{j-1} p_k(\xi), T\sum_{k=0}^{j} p_k(\xi)], \; 0 < j \leq N.$$

Since ξ_0 belongs only to $B(\xi_0, \rho_{m+1}(\xi_0))$ we have that $I_0(\xi_0) = I$.
Let $z_m(\cdot, \xi)$ be any measurable selection from $t \to \pi_C(x_m(t, \xi))$ and let $v_{m+1}(\cdot, \xi)$ be a measurable selection from $t \to F(t, z_m(t, \xi))$ such that

$$\|v_{m+1}(t, \xi) - \dot{x}_m(t, \xi)\| = d(\dot{x}_m(t, \xi), F(t, z_m(t, \xi))).$$

Set

$$x_{m+1}(t, \xi) = \xi + \int_0^t \sum_{j=0}^{N} \chi_{I_j(\xi)}(s) \; v_{m+1}(s, \xi_j)ds$$

and remark that

$$x_{m+1}(t, \xi_0) = \xi + \int_0^t v_{m+1}(s, \xi_0)ds$$

so that $x_{m+1}(\cdot, \xi_0)$ is a (m+1)-successive approximation from $x_0(\cdot)$.
By the construction, $x_{m+1}(\cdot, \xi)$ is absolutely continuous and $\dot{x}_{m+1}$ satisfies (i) of the inductive assumption with

$$\alpha_{m+1}(t) := \sup\{\|v_{m+1}(t, \xi_i) - v_{m+1}(t, \xi_j)\| : 0 \leq i, j \leq N\}.$$

Moreover,

$$\|x_{m+1}(t, \xi) - x_m(t, \xi)\| \le \int_0^t \|\dot{x}_{m+1}(s, \xi) - \dot{x}_m(s, \xi)\| ds \le$$

$$\le \int_0^t \sum_{j=0}^{N} \chi_{I_j(\xi)}(s) \, [\, \|v_{m+1}(s, \xi_j) - \dot{x}_m(s, \xi_j)\| + \|\dot{x}_m(s, \xi_j) - \dot{x}_m(s, \xi)\| \,] \, ds \le$$

$$\le \int_0^t \delta_m(s) ds + \int_E \alpha_m(s) ds \le \int_0^t \delta_m(s) ds + \eta^{m+1} .$$

Let $z_{m+1}(\cdot, \xi)$ be a measurable selection from $t \to \pi_C(x_{m+1}(t, \xi))$. Let $t \in I$ be fixed and let j be such that $t \in I_j(\xi)$. Then

$$d(\dot{x}_m(t, \xi) , F(t, z_{m+1}(t, \xi))) = d(v_{m+1}(t, \xi_j) , F(t, z_{m+1}(t, \xi))) \le$$

$$\le d(F(t, z_m(t, \xi_j)) , F(t, z_{m+1}(t, \xi))) \le k(t) \, \|z_m(t, \xi_j) - z_{m+1}(t, \xi)\| \le$$

$$\le k(t) \, [\, \|z_m(t, \xi_j) - z_m(t, \xi)\| + \|z_m(t, \xi) - z_{m+1}(t, \xi)\| \,] \le$$

$$\le k(t) \, [2 d_C(x_m(t, \xi)) + 2\|x_m(t, \xi) - x_m(t, \xi_j)\| \,] + k(t) \, [d_C(x_m(t, \xi)) +$$

$$+ d_C(x_{m+1}(t, \xi)) + \|x_{m+1}(t, \xi) - x_m(t, \xi)\| \,] \le k(t) \, [d_C(x_{m+1}(t, \xi)) +$$

$$+ 3 d_C(x_m(t, \xi)) + \int_0^t \delta_m(s) ds + 3\eta^{m+1} \,]$$

and, since

$$d_C(x_m(t, \xi)) \le \int_0^t \delta_m(s) ds,$$

we have, for t in I,

$$d(\dot{x}_{m+1}(t, \xi) , F(t, z_{m+1}(t, \xi))) \le k(t) \; d_C(x_{m+1}(t, \xi)) + k(t) \, [4 \int_0^t \delta_m(s) ds + 3\eta^{m+1}].$$

Since $d_C(x_{m+1}(0, \xi)) = 0$ and

$$\frac{d}{dt}[d_C(x_{m+1}(t,\xi))] \le d(\dot{x}_{m+1}(t,\xi)\,,\, T_C(\pi_C(x_{m+1}(t,\xi))) \le d(\dot{x}_{m+1}(t,\xi)\,,\, F(t, z_{m+1}(t,\xi)))\,,$$

we obtain

$$d_C(x_{m+1}(t,\xi)) \le \int_0^t e^{\int_s^t k(\tau)d\tau} k(s)[4\int_0^s \delta_m(u)du +$$

$$+ 3\eta^{m+1}]ds = 4\int_0^t \delta_m(u)\,(e^{K(t)-K(u)} - 1)du + 3\eta^{m+1}[e^{K(t)} - 1]$$

and

$$d(\dot{x}_{m+1}(t,\xi)\,,\, F(t, z_{m+1}(t,\xi)) \le k(t)\, d_C(x_{m+1}(t,\xi)) +$$

$$+ k(t)\,[4\int_0^t \delta_m(u)du + 3\eta^{m+1}] \le k(t)\, e^{K(t)}\,[4\int_0^t \delta_m(u)\, e^{-K(u)}\, du + 3\eta^{m+1}]\,.$$

Therefore, setting

$$\delta_{m+1}(t) = \text{ess sup}\{d(\dot{x}_{m+1}(t,\xi)\,,\, F(t, z(t,\xi))) : \xi\in C,\ z(\cdot,\xi) \text{ measurable},$$

$$z(t,\xi)\in \pi_C(x_{m+1}(t,\xi)) \text{ a.e.}\},$$

we have that $d_C(x_{m+1}(t,\xi)) \le \int_0^t \delta_{m+1}(s)ds$ and

$$\delta_{m+1}(t) \le k(t)e^{K(t)}\,[4\int_0^t \delta_m(u)\, e^{-K(u)}\, du + 3\eta^{m+1}]\,.$$

Finally, by using (v) of the inductive assumptions, we obtain

$$\delta_{m+1}(t) \le k(t)e^{K(t)}\,\Big\{4\int_0^t e^{-K(s)}\, k(s)\, e^{K(s)}\Big[4^m \int_0^s \delta_0(u)e^{-K(u)}\,\frac{(K(s)-K(u))^{m-1}}{(m-1)!}\, du +$$

$$+ 3\eta^m \sum_{i=0}^{m-1} \frac{1}{i!}\Big[\frac{4K(s)}{\eta}\Big]^i\Big]\, ds + 3\eta^{m+1}\Big\} = k(t)\, e^{K(t)}\,\Big\{4^{m+1} \int_0^t \delta_0(u)e^{-K(u)}\,.$$

$$\cdot \Big[\int_u^t \frac{(K(s)-K(u))^{m-1}}{(m-1)!} k(s)ds \Big] du + 4 \cdot 3\eta^m \int_0^t k(s) \sum_{i=0}^{m-1} \frac{1}{i!} \Big[\frac{4K(s)}{\eta}\Big]^i ds + 3\eta^{m-1} \Big\} =$$

$$= k(t)e^{K(t)} \Big\{ 4^{m+1} \int_0^t \delta_0(u) e^{-K(u)} \frac{[K(t)-K(u)]^m}{m!} du + 3\eta^{m+1} \sum_{i=0}^{m} \frac{1}{i!} \Big[\frac{4K(t)}{\eta}\Big]^i \Big\}$$

and the proof of our claim is complete.

Remark that as a consequence of (i) each map $\xi \to x_n(\cdot, \xi)$ is continuous from C into $AC(I,R^n)$. Also, from (v),

$$\delta_n(t) \le 4k(t)e^{K(t)} \frac{(4K(T))^{n-1}}{(n-1)!} \|\delta_0\|_1 + 3\eta^n e^{\frac{4K(T)}{\eta}} \tag{3.4.2}$$

so that, by (iii)

$$\|x_{n+1}(\cdot, \xi) - x_n(\cdot, \xi)\|_{AC} \le \int_0^T \delta_n(t)dt + \eta^{n+1} \le$$

$$\le 4e^{K(T)} \frac{(4K(T))^{n-1}}{(n-1)!} \|\delta_0\|_1 + 3T\eta^n e^{\frac{4K(T)}{\eta}} + \eta^{n+1} .$$

The sequence $\{x_n(\cdot, \xi)\}_n$ is Cauchy in $AC(I, R^n)$ uniformly with respect to ξ and it converges to the map $\xi \to x(\cdot,\xi)$ continuous from C into $AC(I, R^n)$. Since, from (3.4.2), $\{\delta_n(t)\}_n$ converges to 0 for t in I and $d_C(x_n(t, \xi)) \le \int_0^t \delta_n(s)ds$, from the dominated convergence theorem we infer that $x(\cdot, \xi)$ takes values in C.

The function $x(\cdot, \xi_0)$ is a limit of a sequence of successive approximations from $x_0(\cdot)$ (in particular if $x_0(\cdot)$ is a solution to (P_{ξ_0}), $x(\cdot, \xi_0) = x_0(\cdot)$). To show that $x(\cdot, \xi)$ is a solution for every ξ choose $z_n(\cdot, \xi)$ a measurable selection from $t \to \pi_C(x_n(t, \xi))$ and remark that

$$d(\dot{x}_n(t, \xi), F(t, x(t, \xi))) \le d(\dot{x}_n(t, \xi), F(z_n(t, \xi))) +$$

$$+ k(t) [d_C(x_n(t, \xi)) + \|x_n(t, \xi) - x(t, \xi)\| \le \tag{3.4.3}$$

$$\leq \delta_n(t) + k(t) \left[\int_0^t \delta_n(s)ds + \|x_n(t, \xi) - x(t, \xi)\| \right] .$$

Since $\{\dot{x}_n(\cdot, \xi)\}_n$ is Cauchy in $L^1(I, R^n)$, a subsequence converges pointwise a.e. to $\dot{x}(\cdot,\xi)$. Passing to the limit in (3.4.3) we obtain that x is a solution, since F has closed values. ♦

3.5 ARCWISE CONNECTEDNESS OF SOLUTION SETS.

Solution sets to ordinary differential equations without uniqueness defined on open sets are known to be connected but not (in general) arcwise connected. In the case of solutions to differential inclusions on closed sets the difference between continuity and Lipschitz continuity is even more striking: example in ([4], pp. 203) shows that the solution set to (3.5.3) for $x \to F(x)$ single valued, continuous and independent on t, may consist of exactly two solutions, a disconnected set. In this section we show that for Lipschitzean and time independent maps, the set of solutions are arcwise connected.

Assume to have a multifunction F satisfying one of the following assumptions :

(3.5.1) $F : X \to 2^X$ is Lipschitzean with nonempty closed values and X is a separable Banach space

(3.5.2) $F : C \to 2^{R^n}$ is Lipschitzean with nonempty closed values and C is a compact subset of R^n.

For F satisfying (3.5.1) (resp. (3.5.2)) and for ξ in X (resp. in C) consider the Cauchy problem

(3.5.3) $$\dot{x} \in F(x) \ , \ x(0) = \xi \ ,$$

and denote by $\mathcal{T}(\xi)$ the set of solutions of (3.5.3) defined on the interval I=[0,T], with values in X (resp. in C). Fix $(\xi_0,x_0)\in$ graph $(\mathcal{T})$. Then by Corollary 3.4 (resp. Theorem 3.11) there exists a continuous selection $\varphi:X\to AC(I,X)$ (resp. $\varphi:C\to AC(I,R^n)$) from $\xi\to\mathcal{T}(\xi)$ such that $\varphi(\xi_0)=x_0$. We derive from this the arcwise connectedness of the solution sets $\mathcal{T}(\xi)$.

Theorem 3.12. Let F satisfy (3.5.1) (resp. (3.5.2)). Let $\xi\to x_0(\xi)$ and $\xi\to x_1(\xi)$ be two continuous selections from $\xi\to\mathcal{T}(\xi)$. Then there exists a map H from $[0,1]\times X$ into AC(I,X) (resp. from $[0,1]\times C$ into $AC(I,R^n)$) with the following properties:

(i) H is continuous,
(ii) $H(0,\xi)=x_0(\xi)$ and $H(1,\xi)=x_1(\xi)$,
(iii) for λ in [0,1], $H(\lambda,\xi)$ is in $\mathcal{T}(\xi)$.

Proof. Set $x_0(\xi,t) = x_0(\xi)(t)$ and $x_1(\xi,t) = x_1(\xi)(t)$ and define H(.,.) by

$$H(\lambda,\xi)(t) = \begin{cases} x_1(\xi,t) & \text{if } 0\le t\le\lambda T \\ x_0(x_1(\xi,\lambda T),t-\lambda T) & \text{if } \lambda T\le t\le T \end{cases}$$

Then H(.,.) is well defined, $H(0,\xi)=x_0(\xi)$, $H(1,\xi)=x_1(\xi)$ and $H(\lambda,\xi)$ is in $\mathcal{T}(\xi)$.
Fix : $\varepsilon>0$, λ_0 in [0,1], ξ_0 in X (resp C). Then

$$(3.5.4)\qquad \|H((\lambda,\xi)-H((\lambda_0,\xi_0)\|_{AC}=\|\xi-\xi_0\|+\int_0^T\|H'(\lambda,\xi)(t)-H'(\lambda_0,\xi_0)(t)\|dt,$$

where ' means derivative with respect to t.
We perform the estimates for the case $\lambda<\lambda_0$.

$$\int_0^T\|H'(\lambda,\xi)(t)-H'(\lambda_0,\xi_0)(t)\|dt\le\int_0^{\lambda T}\|x_1'(\xi,t)-x_1'(\xi_0,t)\|dt\ +$$

$$(3.5.5)\qquad + \int_{\lambda T}^{\lambda_0 T} \|x_0'(x_1(\xi,\lambda T),t-\lambda T) - x_1'(\xi_0,t)\|\,dt \;+$$

$$+ \int_{\lambda_0 T}^{T} \|x_0'(x_1(\xi,\lambda T),t-\lambda T) - x_0'(x_1(\xi_0,\lambda_0 T),t-\lambda_0 T)\|\,dt\ .$$

Denote by U, V, W the first, the second and the third term of the right hand side of (3.5.5) respectivelly.

By the continuity of $\xi\to x_1'(\xi)$ in ξ_0, it follows that U is less than $\frac{\varepsilon}{4}$ for $\|\xi-\xi_0\| < \delta_1$.

For V we have the following estimations

$$V \le \int_{\lambda T}^{\lambda_0 T} \|x_1'(\xi_0,t)\|\,dt + \int_{0}^{(\lambda_0-\lambda)T} \|x_0'(x_1(\xi_0,\lambda_0 T),s)\|\,ds \;+$$

$$+ \int_{\lambda T}^{\lambda_0 T} \|x_0'(x_1(\xi_0,\lambda_0 T),\ t-\lambda T) - x_0'(x_1(\xi_0,\lambda T),\ t-\lambda T)\|\,dt \;+$$

$$+ \int_{\lambda T}^{\lambda_0 T} \|x_0'(x_1(\xi_0,\lambda T),\ t-\lambda T) - x_0'(x_1(\xi,\lambda T),\ t-\lambda T)\|\,dt \le$$

$$\le \int_{\lambda T}^{\lambda_0 T} \|x_1'(\xi_0,t)\|\,dt + \int_{0}^{(\lambda_0-\lambda)T} \|x_0'(x_1(\xi_0,\lambda_0 T),s)\|\,ds \;+$$

$$+ \|x_0'(x_1(\xi_0,\lambda_0 T)) - x_0'(x_1(\xi_0,\lambda T))\|_1 + \|x_0'(x_1(\xi_0,\lambda T)) - x_0'(x_1(\xi,\lambda T))\|_1 .$$

Hence, by the integrability of $x_1'(\xi_0)$ and of $x_0'(x_1(\xi_0,\lambda T))$, the continuity of the maps $(t,\xi)\to x_1(\xi,t)$ and $\eta\to x_0'(\eta)$ we obtain that V is bounded by $\frac{\varepsilon}{4}$ whenever $\|\lambda-\lambda_0\| < \delta_2$ and $\|\xi-\xi_0\| < \delta_2$. Define the translation operator $U_\tau\colon L^1\to L^1$ by $U_\tau(x(t)) = x(t+\tau)$. Then W is bounded by

$$\int_{(\lambda_0-\lambda)T}^{T-\lambda T} \|(x_0'(x_1(\xi,\lambda T), s) - x_0'(x_1(\xi,\lambda_0 T), s)\|)ds. +$$

$$+ \int_0^{T-\lambda_0 T} \|x_0'(x_1(\xi,\lambda_0 T), s+(\lambda_0-\lambda)T) - x_0'(x_1(\xi,\lambda_0 T), s)\|ds \leq$$

$$\leq \|x_0'(x_1(\xi,\lambda T)) - x_0'(x_1(\xi,\lambda_0 T))\|_1 +$$

$$+ \int_0^{T-\lambda_0 T} \|U_{(\lambda_0-\lambda)T}(x_0'(x_1(\xi,\lambda_0 T)))(s) - x_0'(x_1(\xi,\lambda_0 T))(s)\|ds.$$

By the same argument as before and recalling that $\|U_\tau(x)-x\|_1 \to 0$ as $\tau \to 0$, term W can made smaller than $\frac{\varepsilon}{4}$ for $\|\lambda-\lambda_0\| < \delta_3$. Hence by choosing $\delta = \min\{\frac{\varepsilon}{4}, \delta_1, \delta_2, \delta_3\}$, by (3.5.4) and (3.5.5) we obtain that $\|H((\lambda,\xi) - H((\lambda_0,\xi_0)\|_{AC} < \varepsilon$ whenever $\|\lambda-\lambda_0\| < \delta$ and $\|\xi-\xi_0\| < \delta$. ♦

Corollary 3.13. For every $\xi \in X$ the set $\mathcal{T}(\xi)$ and the attainable set $\mathcal{A}_T(\xi)$ are arcwise connected.

Proof. Fix $\xi_0 \in X$ and let x, y be in $\mathcal{T}(\xi_0)$. Let φ be a continuous selection from $\xi \to \mathcal{T}(\xi)$ such that $\varphi(\xi_0)=x$ and define for $\lambda \in [0,1]$.

$$x_\lambda(t) = \begin{cases} y(t) & \text{if } 0 \leq t \leq \lambda T \\ \varphi(y(\lambda T))(t-\lambda T) & \text{if } \lambda T \leq t \leq T \end{cases}$$

Remark that $x_0(.)=x(.)$, $x_1(.)=y(.)$ and $x_\lambda \in \mathcal{T}(x_\lambda)$. From the proof of Theorem 3.12 it follows that $\lambda \to x_\lambda$ is continuous from [0,1] into AC. ♦

Theorem 3.12 and Corollary 3.13 have been proved in [19] for F satisfying assumption (3.5.2). For F satisfying assumption (3.5.1) the statement of Corollary 3.13 has been proved in [54].

4. EXISTENCE AND WELL POSEDNESS FOR EVOLUTION EQUATIONS.

4.1. INTRODUCTION.

Differential inclusions of the form

(4.1.1) $$\dot{x}(t) \in -Ax(t) + f(t)$$

where A is a maximal monotone (in general, unbounded) map, have been extensively studied (see Brezis [11]), both in finite and in infinite dimensional spaces.

Existence of solutions follows, to some extents, from the basic relation

(4.1.2) $$\frac{d}{dt}\|x(t)\|^2 = 2 < \dot{x}(t), x(t)>$$

(whenever meaningful), that applied to two solutions of (4.1.1), by the monotonicity of A and the minus sign on the right hand side, yields that their distance is nonincreasing. This reasoning allows the construction of a Cauchy sequence of approximate solutions, converging to a solution.

The existence of the right approximate solutions is supplied by the maximality of A , that permits the use of the Yosida approximations. Hence existence is a result of completeness, of having the sign minus at the right hand side, and of maximality.

The same conditions have allowed to prove existence for several classes of perturbations of (4.1.1) to

(4.1.3) $$\dot{x}(t) \in -Ax(t) + F(t, x(t))$$

by Benilan-Brezis [6], Attouch-Damlamian [3], Cellina-Marchi [15], Colombo-Fonda-Ornelas [21], Mitidieri-Vrabie [44], Mitidieri-Tosques [43], Colombo-Tosques [22], Kravvaritis-Papageorgiou [39], Papageorgiou [49],Tolstonogov [57].

On the other hand, in [7] the problem

(4.1.4) $$\dot{x}(t) \in F(x(t)) \subset \partial V(x(t)),\ x(0) = x_0$$

has been considered, where F is a monotonic upper semicontinuous (not necessarily convex-valued, hence not maximal) map contained in the subdifferential of a locally bounded convex function, and x is in a finite dimensional space.

Here the basic relation (4.1.2) (and the lack of the minus sign in (4.1.4)) yields that the distance among solutions increases and typically there is no uniqueness. Existence of solutions depends on arguments of convex analysis.

This result has been generalized by Ancona-Colombo [2] to cover perturbations of the kind

$$\dot{x}(t) \in F(x(t)) + f(t, x(t))$$

with f satisfying Carathéodory conditions.

After some definitions and preliminary results on evolution equations recalled in section 4.2, in section 4.3 we present a result, proved in [18], concerning local existence of solutions for the Cauchy problem

$$\text{(4.1.5)} \qquad \dot{x}(t) \in -\partial V(x(t)) + F(x(t)) \,, \quad F(x) \subset \partial W(x)$$
$$x(0) = x_0 \,,$$

where x is in a finite dimensional space, V is a lower semicontinuous proper convex function (hence its subdifferential ∂V, is a maximal monotone map), W is a lower semicontinuous convex function and F is an upper semicontinuous compact valued map defined over some neighborhood of x_0 (this last assumption implies, [7], that locally W is Lipschitzean). Remark that at the right hand side of (4.1.5) is defined by a maximal monotone map with a minus sign and a bounded monotone but not maximal monotone map with a plus sign.

Sections 4.4 and 4.5 are devoted to the well posedness for evolution equation and contain two results proved in [52]. In the first one we consider the Cauchy problem

$$\text{(4.1.6)} \qquad \dot{x}(t) \in -Ax(t) + F(t, x(t)) \,, \quad x(0) = \xi \in \text{cl}\, D(A),$$

where A is a maximal monotone map, with domain D(A), on a Hilbert space, and in the second one we consider the Cauchy problem

$$\text{(4.1.7)} \qquad \dot{x}(t) \in Ax(t) + F(t, x(t)) \,, \quad x(0) = \xi$$

where A is infinitesimal generator of a C_0-semigroup on a Banach space. In both this sections $F: I\mathrm{x}X \to 2^X$ is assumed to satisfy Assumption 2.17.

Denoting by $\mathcal{T}(\xi)$ the set of all weak (resp. mild) solutions of (4.1.6) (resp. (4.1.7)) we shall prove the existence of a continuous selection of the multivalued map $\xi \to \mathcal{T}(\xi)$. To do this we adapt the construction presented in Section 3.2. For properties of the solution sets $\mathcal{T}(\xi)$ we refer also to [30] and [57].

4.2. SOME DEFINITIONS AND PRELIMINARY RESULTS.

Let X be a real separable Hilbert space with scalar product $<.,.>$ and norm $\|\,.\,\|$ and let $A: X \to 2^X$ be a multivalued map. Let $Ax := A(x)$, the value of A at x, and let $D(A) = \{x \in X: Ax \neq \emptyset\}$ be the *domain* of A.

Definition 4.1 A is said to be *maximal monotone* on X if:

(i) for all $x_1, x_2 \in D(A)$ and all $y_1 \in Ax_1$, $y_2 \in Ax_2$ we have $< y_1 - y_2, x_1 - x_2 > \geq 0$;

(ii) for every $y \in X$ there exists $x \in D(A)$ such that $x + Ax = y$.

We say that A is a *monotone* map if it satisfies (i) in the previous definition.

As it is well known (see [11]) if A is maximal monotone then cl D(A) is convex, and for each $x \in D(A)$ the set Ax is closed and convex.

Let $V : X \to (-\infty, +\infty]$ be a proper convex lower semicontinuous function, let $D(V) = \{x \in R^n : V(x) < \infty\}$, and let $\partial V : X \to 2^X$ be the *subdifferential* of V defined by

$$\partial V(x) = \{\xi \in X : V(y) - V(x) \geq <\xi, y - x>, \quad \forall y \in X\}. \tag{4.2.1}$$

It is known (see [11], p. 21) that $x \to \partial V(x)$ is a maximal monotone map and $D(\partial V) \subset D(V)$.

Definition 4.2. A multivalued map $A : X \to 2^X$ is called *cyclically monotone* if for every cyclical sequence

$$x_0, x_1, \dots, x_N = x_0 \quad (N \text{ arbitrary})$$

in D(A) and every sequence $y_i \in F(x_i)$, $i = 1, \dots, N$, we have

$$\sum_{i=1}^{N} \langle y_i, x_i - x_{i-1} \rangle \geq 0 .$$

About cyclically monotone maps we have the following result (Theorem 2.5 in [11]):

Proposition 4.3. Let $A : X \to 2^X$ be a monotone map. Then A is cyclically monotone if and only if there exists a proper convex lower semicontinuous function $V : R^n \to (-\infty, +\infty]$, such that for every $x : F(x) \subset \partial V(x)$.

Let $A : X \to 2^X$ be a maximal monotone map, $\xi \in \text{cl } D(A)$, and let $f \in L^1(I,X)$, where $I=[0,T]$. Consider the Cauchy problem

(P_f) $$\dot{x}(t) \in -Ax(t) + f(t), \quad x(0) = \xi .$$

Definition 4.4. A function $x : I \to X$ is called a *strong solution* of the Cauchy problem (P_f) if it is continuous on I, absolutely continuous on every compact subset of $]0,T[$, $x(0) = \xi$ and for almost all $t \in I$ we have $x(t) \in D(A)$ and $\dot{x}(t) \in -Ax(t) + f(t)$.

A function $x : I \to X$ is called a *weak solution* of the Cauchy problem (P_f) if there exist two sequence $\{f_n\}_{n\in N} \subset L^1(I, X)$ and $\{x_n\}_{n\in N} \subset C(I, X)$ such that: x_n is a strong solution of (P_{f_n}), f_n converges to f in $L^1(I, X)$, and x_n converges to x in $C(I, X)$.

By Theorem 3.4 and Lemma 3.1 in [11] we obtain the following

Lemma 4.5. Let A be a maximal monotone on X. Then for each $\xi \in \mathrm{cl}\, D(A)$ and $f \in L^1(I, X)$ there exists a unique weak solution $x^f(.,\xi)$ of the Cauchy problem (P_f) . Moreover if f, $g \in L^1(I, X)$, and $x^f(.,\xi)$, $x^g(.,\xi)$ are the weak solutions of the Cauchy problems (P_f), (P_g) then, for any $0 \le s \le t \le T$, we have

(4.2.2) $$\|x^f(t,\xi) - x^g(t,\xi)\| \le \|x^f(s,\xi) - x^g(s,\xi)\| + \int_s^t \|f(u) - g(u)\|du \ .$$

Remark 4.6 In the particular case when A is the subdifferential of a convex function and $f \in L^2([I,X)$ any weak solution of (P_f) is also a strong solution of (P_f) (Theorem 3.6 in [11]).

Set $X_0 = \mathrm{cl}\, D(A)$.

Remark 4.7. The map $\xi \to x^f(.,\xi)$ is continuous from X_0 into C(I,X). Indeed, since

$$\frac{d}{dt}\|x^f(t,\xi_1)-x^f(t,\xi_2)\|^2 = 2 < \dot{x}^f(t,\xi_1)-f(t)+f(t) - \dot{x}^f(t,\xi_2),\ x^f(t,\xi_1) - x^f(t,\xi_2)> \le 0,$$

we have $\|x^f(t,\xi_1)-x^f(t,\xi_2)\| \le \|x^f(0,\xi_1)-x^f(0,\xi_2)\| = \|\xi_1-\xi_2\|$ for all $t \in I$, which implies the continuity of the map $\xi \to x^f(.,\xi)$. ♦

Remark 4.8. Let $x^0(.,\xi)$ be the unique weak solution of (P_f) with f=0. Then, by Theorem 3.2.1 in [4] :

$$\frac{d}{dt}x^0(t,\xi) = -m(Ax^0(t,\xi)) \ \text{ and } \ t \to \|m(Ax^0(t,\xi))\| \ \text{ is nonincreasing,}$$

(where m(Ax) is the element of minimal norm of the set Ax).

Therefore, for any $t \in [0, T]$,

$$\|x^0(t,\xi) - \xi\| = \|\int_0^t \dot{x}^0(s,\xi)ds\| \le \int_0^t \|m(Ax^0(s,\xi))\|ds \le \int_0^t \|m(Ax^0(0,\xi))\|ds ,$$

hence

(4.2.3) $$\|x^0(t,\xi) - \xi\| \le t\,\|m(A\xi)\|.$$

On the other hand if $f \in L^1(I, X)$ then by (4.2.2) and (4.2.3) it follows that

(4.2.4) $$\|x^f(t,\xi) - \xi\| \le \int_0^t \|f(s)\|ds + t\,\|m(A\xi)\|.$$ ♦

If the maximal monotone map is the subdifferential of a convex function then we have the following result (Theorem 1.3 in [3]) which will be used in the next section.

Lemma 4.9. Let $V : X \to (-\infty, +\infty]$ be a proper convex lower semicontinuous function and let ∂V be its subdifferential. Then:

(i) for every $x_0 \in D(\partial V)$ and $f \in L^2(I, X)$ there exists a unique strong solution $x^f: I \to X$ to the problem

(P_f) $$\dot{x}(t) \in -\partial V(x(t)) + f(t),\ x(0) = x_0.$$

(ii) $t \to V(x^f(t))$ is absolutely continuous on I (hence it is differentiable a. e. on I)

(iii) $$\left\|\frac{dx^f(t)}{dt}\right\|^2 = -\frac{d}{dt}V(x^f(t)) + \left\langle f(t), \frac{dx^f(t)}{dt}\right\rangle$$

(iv) $$\frac{dx^f}{dt} \in L^2(I, X) \text{ and, if } V \ge 0,$$

(4.2.5) $$\left[\int_0^T \left\|\frac{dx^f(t)}{dt}\right\|^2 dt\right]^{1/2} \le \left(\int_0^T \|f(t)\|^2 dt\right)^{1/2} + \sqrt{V(x_0)}.$$

Consider now the Cauchy problem

(4.2.6) $$\dot{x}(t) \in -Ax(t) + F(t, x(t)),\ x(0) = \xi,$$

where $F: I \times X \to 2^X$ is a multivalued map and $\xi \in X_0$.

Definition 4.10. A function $x(\cdot,\xi) : I \to X$ is called a *strong* (*weak*) solution of the Cauchy problem (4.2.6) if there exists $f(\cdot,\xi) \in L^1(I, X)$, a selection of $F(., x(., \xi))$, such that $x(\cdot,\xi)$ is a strong (weak) solution of the Cauchy problem $(P_{f(\cdot,\xi)})$.

4.3 EXISTENCE OF SOLUTIONS TO EVOLUTION EQUATIONS HAVING MONOTONICITIES OF OPPOSITE SIGN.

In this section consider the space R^n endowed with the Euclidian norm $\|\,.\,\|$ and the scalar product $<.,.>$. Consider the Cauchy problem

(4.3.0) $$\dot{x}(t) \in -\partial V(x(t)) + F(x(t)) , x(0) = x_0 .$$

where $V : R^n \to (-\infty, +\infty]$ is a proper lower semicontinuous function and F is a multivalued map defined on a neighborhood of x_0 .

Given any compact set K containing x_0 , $\inf \{V(x): x \in K\} = V(x^*)$ for some x^* in K. Since $\partial(V(x)-V(x^*))=\partial V(x)$ we will assume that $V \geq 0$.

The following result gives the local existence of solutions of (4.3.0).

Theorem 4.11. Let V be a proper lower semicontinuous function and x_0 be in $D(\partial V)$; let F be an upper semicontinuous cyclically monotone map with compact nonempty values defined on a neighborhood of x_0. Then there exist $T > 0$ and $x : [0, T] \to R^n$, a strong solution to the Cauchy problem (4.3.0).

Proof. Let $x_0 \in D(\partial V)$ and let $W: R^n \to (-\infty, +\infty]$ be a proper convex lower semicontinuous function such that for every $x : F(x) \subset \partial W(x)$. As in [7] we can assume that there exist $r > 0$ and $M < \infty$ such that W is Lipschitzean with Lipschitz constant M on $B(x_0, r)$. It follows that F is bounded by M on $B(x_0, r)$.

Let $m(\partial V(x_0))$ be the element of minimal norm of $\partial V(x_0)$ and set

$$T < \frac{r}{M + \|m(\partial V(x_0))\|}.$$

Our purpose is to prove that there exists $x : [0, T] \to B(x_0, r)$, a solution to the Cauchy problem (4.3.0).

Let $n \in N$ and set for $k = 0, 1, \dots, n$, $t_k^n = k\frac{T}{n}$. Take $y_0^n \in F(x_0)$ and define $f_1^n : (t_0^n , t_1^n] \to R^n$ by $f_1^n(t) = y_0^n$. Then $f_1^n \in L^2([t_0^n , t_1^n], R^n)$ and by Lemma 4.9, (ii), there exists $x_1^n: [t_0^n, t_1^n] \to R^n$, the unique solution to the problem

$$(P_1^n) \qquad \dot{x}(t) \in -\partial V(x(t)) + f_1^n(t) \ , \ x(0) = x_0 .$$

By (4.2.4) we obtain that for any $t \in [t_0^n , t_1^n]$, $\|x_1^n(t) - x_0\| \leq \int_0^t \|f_1^n(s)\| ds + t \, \|m(\partial V(x_0))\| \leq$

$\leq \frac{T}{n}(M + \|m(\partial V(x_0))\|) < \frac{r}{n}$, hence $x_1^n(t) \in B(x_0, \frac{r}{n})$.

Analogously for $k = 2, \dots , n$ take $y_{k-1}^n \in F(x_{k-1}^n(t_{k-1}^n))$ set $I_k^n = (t_{k-1}^n , t_k^n]$; define $f_k^n : I_k^n \to R^n$ by $f_k^n(t) = y_{k-1}^n$ and set $x_k^n : I_k^n \to B(x_0, k\frac{r}{n})$ to be the unique solution to the problem

$$(P_k^n) \qquad \dot{x}(t) \in -\partial V(x(t)) + f_k^n(t) \ , \ x(t_{k-1}^n) = x_{k-1}^n(t_{k-1}^n) .$$

Remark that from $x_{k-1}^n(t_{k-1}^n) \in B(x_0, (k-1)\frac{r}{n})$, (4.2.3) and from (4.2.2) applied for $s = t_{k-1}^n$ it follows that $x_k^n(I_k^n) \subset B(x_0, k\frac{r}{n})$.

Define for $t \in [0, T]$:

$$x_n(t) = \sum_{k=1}^{n} x_k^n(t) \, \chi_{I_k^n}(t) \ , \ f_n(t) = \sum_{k=1}^{n} f_k^n(t) \, \chi_{I_k^n}(t) \ , \ a_n(t) = \sum_{k=1}^{n} t_{k-1}^n \, \chi_{I_k^n}(t) .$$

By construction we have

$$(4.3.1) \qquad \dot{x}_n(t) \in -\partial V(x_n(t)) + f_n(t) \quad \text{a.e. on } [0, T] ,$$

(4.3.2) $$f_n(t) \in F(x_n(a_n(t))) \quad \text{a.e. on } [0, T]$$

(4.3.3) $$x_n(t) \in D(\partial V) \cap B(x_0, r) \quad \text{a.e. on } [0, T] .$$

and by (4.2.5) we obtain

$$\Big(\int_0^T \| \frac{dx_n(t)}{dt} \|^2 dt\Big)^{1/2} \le \Big(\int_0^T \| f_n(t) \|^2 dt\Big)^{1/2} + \sqrt{V(x_0)} \le$$

$$\le M\sqrt{T} + \sqrt{V(x_0)} =: N .$$

It follows that $\| \frac{dx_n}{dt} \|_2 \le N$ and since $\|x_n\|_\infty \le r + \|x_0\|$, we can assume that $(x_n, \dot{x}_n)$ is precompact in $C([0, T], R^n) \times L^2([0, T], R^n)$, the first space with the sup norm and the second with the weak topology. Therefore there exists a subsequence (again denoted by) x_n and an absolutely continuous function $x : [0, T] \to B[x_0, r]$ such that

(4.3.4) x_n converges to x uniformly on compact subsets on $[0, T]$

(4.3.5) $\dot{x}_n$ converges weakly in L^2 to $\dot{x}$.

Since $\|f_n(t)\| \le M$ on $[0, T]$, we can assume that

(4.3.6) f_n converges weakly in L^2 to f .

By (4.3.2) we have that

$$d((x_n(t) , f_n(t)) , \text{graph } F) \le \|x_n(a_n(t)) - x_n(t)\|$$

and, since $a_n(t) \to t$ and $x_n \to x$ uniformly , we obtain that

$$d((x_n(t) , f_n(t)) , \text{graph } F)) \to 0 \text{ for } n \to \infty .$$

Then by (4.3.4), (4.3.6) and the Convergence theorem ([4], p. 60) it follows that $f(t) \in coF(x(t)) \subset \partial V(x(t))$ and by Lemma 3.3 in ([11], p. 73) we obtain that $\frac{d}{dt} W(x(t)) = \langle \dot{x}(t), f(t) \rangle$, i.e.,

$$(4.3.7) \qquad \int_0^T \langle \dot{x}(s), f(s)\rangle \, ds = W(x(T)) - W(x_0) .$$

By the definition of ∂W,

$$W(x_n(t_k^n)) - W(x_n(t_{k-1}^n)) \geq \langle y_{k-1}^n , \int_{t_{k-1}^n}^{t_k^n} \dot{x}_n(s) \, ds\rangle =$$

$$= \int_{t_{k-1}^n}^{t_k^n} \langle f_n(s), \dot{x}_n(s)\rangle \, ds \quad .$$

Adding for $k = 1, \ldots, n$ we obtain

$$(4.3.8) \qquad W(x_n(T)) - W(x_0) \geq \int_0^T \langle f_n(s) \, , \, \dot{x}_n(s)\rangle \, ds \; .$$

Comparing (4.3.7) and (4.3.8), using the continuity of W in $x(T)$ and the convergence of x_n to x, it follows

$$(4.3.9) \qquad \limsup_{n\to\infty} \int_0^T \langle \dot{x}_n(s), f_n(s)\rangle \, ds \leq \int_0^T \langle \dot{x}(s), f(s)\rangle \, ds \; .$$

By (4.3.1) and Lemma 4.9 (iii) we obtain

$$(4.3.10) \qquad \int_0^T \|\dot{x}_n(s)\|^2 \, ds = \int_0^T \langle f_n(s), \dot{x}_n(s)\rangle \, ds - V(x_n(T)) + V(x_0) \, .$$

Let

$$(4.3.11) \qquad \delta_n(t) := f_n(t) - \dot{x}_n(t)$$

$$(4.3.12) \qquad \delta(t) := f(t) - \dot{x}(t).$$

Then δ_n converges to δ weakly in L^2 (by (4.3.5) and (4.3.6)), $\delta_n(t) \in \partial V(x_n(t))$ and, since x_n converges to x, it follows that $\delta(t) \in \partial V(x(t))$, hence, by Lemma 3.3 in [11],

$$\frac{d}{dt} V(x(t)) = \langle f(t), \dot{x}(t) \rangle - \|\dot{x}(t)\|^2.$$

By integrating we obtain

$$(4.3.13) \qquad \int_0^T \|\dot{x}(s)\|^2 \, ds = \int_0^T \langle f(s), \dot{x}(s) \rangle - V(x(T)) + V(x_0)\,.$$

By (4.3.10), (4.3.9) and the lower semicontinuity of V it follows that

$$\limsup_{n\to\infty} \int_0^T \|\dot{x}_n(s)\|^2 \, ds \le \limsup_{n\to\infty} \int_0^T \langle f_n(s), \dot{x}_n(s) \rangle \, ds -$$

$$- \liminf_{n\to\infty} V(x_n(T)) + V(x_0) \le \int_0^T \langle f(s), \dot{x}(s) \rangle \, ds -$$

$$- V(x(T)) + V(x_0) = \int_0^T \|\dot{x}(s)\| \, ds\ .$$

Hence

$$\limsup_{n\to\infty} \|\dot{x}_n\|_2 \le \|\dot{x}\|_2\,.$$

Since, by the weak convergence of $\dot{x}_n$ to $\dot{x}$, $\liminf_{n\to\infty} \|\dot{x}_n\|_2 \ge \|\dot{x}\|_2$ we obtain that

$$\lim_{n\to\infty} \|\dot{x}_n\|_2 = \|\dot{x}\|_2.$$

Hence $\dot{x}_n$ converges to $\dot{x}$ in L^2-norm, and (Theorem IV.9 in ([12], p. 58)) a subsequence (denoted again by) $\dot{x}_n$ converges pointwise almost everywhere on $[0, T]$ to $\dot{x}$ and there exists $\lambda \in L^2([0, T], R^n)$ such that $\|\dot{x}_n(t)\| \le \lambda(t)$.

Now we apply Lemma 2.13 for δ_n given by $\delta_n(t) = f_n(t) - \dot{x}_n(t)$. By construction, $\delta_n(t) \in F(x_n(a_n(t))) - \dot{x}_n(t)$, hence $\|\delta_n(t)\| \le M + \lambda(t) =: \alpha(t)$.

Set $G(t) := F(x(t)) - \dot{x}(t)$ and obtain

$$d(\delta_n(t), G(t)) = d(\delta_n(t) + \dot{x}(t), F(x(t)) \le$$

$$\le \|\dot{x}_n(t) - \dot{x}(t)\| + d^*(F(x_n(a_n(t))), F(x(t))),$$

Since $\dot{x}_n(t) \to \dot{x}(t)$, $x_n(a_n(t)) \to x(t)$ and F is upper semicontinuous we have that

$$d(\delta_n(t), G(t)) \to 0 \text{ for } n \to \infty .$$

Then, Lemma 2.9 implies that

$$\psi_*(t) = \bigcap_{i \in N} cl(\bigcup_{n \ge i} \{\delta_n(t)\})$$

is nonempty, compact, contained in $G(t)$ and $t \to \psi_*(t)$ is measurable.
Taking $G^*(t) = \partial V(x(t)) \cap B(0, \alpha(t))$ we have that $\delta_n(t) \in \partial V(x_n(t)) \cap B(0, \alpha(t))$ and since $x \to \partial V(x) \cap B(0, \alpha(t))$ is upper semicontinuous, it follows that

$$d(\delta_n(t), G^*(t)) \to 0 \text{ for } n \to \infty$$

and, by Lemma 2.9, $\psi_*(t) \subset \partial V(x(t)) \cap B(0, \alpha(t))$.
Let $\sigma(\cdot)$ be a measurable (hence in $L^1([0, T], R^n)$) selection of $\psi_*(\cdot)$. Set $g(t) := \dot{x}(t)+\sigma(t)$. By definition of G, $g(t) \in F(x(t))$. Therefore $\dot{x}(t) = -\sigma(t)+g(t) \in -\partial V(x(t)) + g(t)$ and the proof is complete. ♦

Example 4.12. As an illustration of the previous theorem in the case n=2, let V be indicator function of the closed unit disk D ; let $F(x_1,x_2)$ be $\{(\text{sign } x_1,0)\}$ where

$$\text{sign } x = \begin{cases} -1 & \text{if } x<0 \\ \{-1,1\} & \text{if } x=0 \\ 1 & \text{if } x>0. \end{cases}$$

Since F is uniformly bounded on D solutions exist on $[0,+\infty[$, and converge to either (-1,0) or (1,0). So the two invariant points (-1,0) and (1,0) attract solutions from every initial point in the disk D. ♦

4.4 WELL POSEDNESS FOR EVOLUTION EQUATIONS IN HILBERT SPACES.

Consider the Cauchy problem

(4.4.0) $\dot{x}(t)\in -Ax(t) + F(t, x(t))$, $x(0) = \xi$,

where $A: X \to 2^X$ is a maximal monotone map, X is a real separable Hilbert space, $F: I\times X\to 2^X$ satisfies Assumption 2.17 and $\xi\in X_0 := \mathrm{cl}\, D(A)$. Denote by $\mathcal{T}(\xi)$ the set of all weak solutions of (4.4.0) (in the sense of Definition 4.10).

We shall prove the following theorem.

Theorem 4.13. Let A be a maximal monotone map on X and let $F: I\times X \to 2^X$ satisfy Assumption 2.17. Then there exists a function $x(\cdot, \cdot) : I \times X_0 \to X$ such that :

(i) $x(\cdot,\xi)\in \mathcal{T}(\xi)$ for every $\xi\in X_0$

(ii) $\xi \to x(\cdot, \xi)$ is continuous from X_0 into $C(I, X)$.

Proof. Fix $\varepsilon > 0$ and, set $\varepsilon_n = \frac{\varepsilon}{2^{n+1}}$, $n\in N$. For $\xi\in X_0$ let $x_0(.,\xi): I \to X$ be the unique weak solution of the Cauchy problem

(P_0) $\dot{x}(t)\in -Ax(t)$, $x(0) = \xi$,

and, for k and β given by (H_2) and (H_3) in Assumption 2.17, define $\alpha: X_0 \to L^1(I, R)$ by

(4.4.1) $\alpha(\xi)(t) = \beta(t) + k(t)\, \|x_0(t,\xi)\|$.

Since, by Remark 4.7, the map $\xi\to x_0(.,\xi)$ is continuous from X_0 into $C(I, X)$, from (4.4.1) it follows that $\alpha(\cdot)$ is continuous from X_0 into $L^1(I, R)$. Moreover, as consequence of (H_2) and (H_3) in Assumption 2.17, for each $\xi\in X_0$ we have:

(4.4.2) $d(0, F(t, x_o(t,\xi)) \leq \alpha(\xi)(t)$ a.e. in I .

Define $G_0:X_0 \to 2^{L^1(I,X)}$ and $H_0:X_0 \to 2^{L^1(I,X)}$ by

(4.4.3) $G_0(\xi) = \{v \in L^1(I, X) : v(t) \in F(t, x_0(t, \xi))$ a.e. $t \in I\}$,

(4.4.4) $H_0(\xi) = cl\{v \in G_0(\xi) : \|v(t)\| < \alpha(\xi)(t) + \varepsilon_0$ a.e. $t \in I\}$.

Clearly, by virtue of (4.4.2) and Lemma 2.12, $G_0(\cdot)$ is l.s.c. from X_0 into $\mathcal{D}$ and $H_0(\xi) \neq \emptyset$ for each $\xi \in X_0$. Hence, by Lemma 2.13, there exists $h_0 : X_0 \to L^1(I, X)$, a continuous selection of $H_0(\cdot)$. Set $f_0(t, \xi)=h_0(\xi)(t)$. Then $f_0(.,\xi)$: $X_0 \to L^1(I,X)$ is continuous, $f_0(t,\xi) \in F(t,x_0(t,\xi))$ and $\|f_0(t,\xi)\| \leq \alpha(\xi)(t) + \varepsilon_0$ for $t \in I$ a.e..

Set $K(t) = \int_0^t k(u)du$ and define, for $\xi \in X_0$, $n \geq 1$,

(4.4.5) $$\beta_n(\xi)(t) = \int_0^t \alpha(\xi)(u) \frac{[K(t)-K(u)]^{n-1}}{(n-1)!} du + T\left(\sum_{i=0}^{n} \varepsilon_i\right) \frac{[K(t)]^{n-1}}{(n-1)!}, \quad t \in I.$$

Since $\alpha(\cdot)$ is continuous from X_0 into $L^1(I,R)$, by (4.4.5) it follows that also $\beta_n(.)$ is continuous from X_0 into $L^1(I, R)$.

Let $x_1(.,\xi)$: $I \to X$ be the unique weak solution of the Cauchy problem $(P_{f_0(\cdot,\xi)})$. By (4.2.2) we have

$$\|x_1(t,\xi) - x_0(t, \xi)\| \leq \int_0^t \|f_0(u,\xi)\| du \leq \int_0^t \alpha(\xi)(u)du + \varepsilon_0 T < \beta_1(\xi)(t),$$

for each $\xi \in X_0$ and $t \in I \setminus \{0\}$.

We claim that there exist two sequences $\{f_n(.,\xi)\}_{n\in N}$ and $\{x_n(.,\xi)\}_{n \in N}$ satisfying, for each $n \geq 1$, the following properties:

(a) $\xi \to f_n(\cdot, \xi)$ is continuous from X_0 into $L^1(I, X)$,

(b) $f_n(t, \xi) \in F(t, x_n(t, \xi))$ for each $\xi \in X_0$ and a.e. $t \in I$,

(c) $\|f_n(t,\xi) - f_{n-1}(t, \xi)\| \leq k(t)\, \beta_n(\xi)(t)$, for a.e. $t \in I$,

(d) $x_n(\cdot, \xi)$ is the unique weak solution of the Cauchy problem $(P_{f_{n-1}(\cdot,\xi)})$.

Suppose we have constructed $f_1, \ldots, f_n$ and $x_1, \ldots, x_n$ satisfying (a)-(d).
Let $x_{n+1}(\cdot,\xi): I \to X$ be the unique weak solution of the Cauchy problem $(P_{f_n(\cdot,\xi)})$. Then, by (4.2.2) and (c), we have for $t \in I \setminus \{0\}$,

$$\|x_{n+1}(t,\xi) - x_n(t,\xi)\| \le \int_0^t \|f_n(u,\xi) - f_{n-1}(u,\xi)\| du \le \int_0^t k(u)\beta_n(\xi)(u)du =$$

$$(4.4.6) \qquad = \int_0^t \alpha(\xi)(u) \int_u^t k(\tau) \frac{[K(t)-K(\tau)]^{n-1}}{(n-1)!} d\tau du + T\left(\sum_{i=0}^n \varepsilon_i\right) \int_0^t k(\tau) \frac{[K(\tau)]^{n-1}}{(n-1)!} d\tau =$$

$$= \int_0^t \alpha(\xi)(u) \frac{[K(t)-K(u)]^n}{n!} du + T\left(\sum_{i=0}^n \varepsilon_i\right) \frac{[K(t)]^n}{n!} < \beta_{n+1}(\xi)(t) .$$

Hence, by (H_2) in Assumption 2.17,

$$(4.4.7) \qquad d(f_n(t,\xi), F(t, x_{n+1}(t,\xi))) \le k(t) \|x_{n+1}(t,\xi) - x_n(t,\xi)\| < k(t)\, \beta_{n+1}(\xi)(t) .$$

By (4.4.7) and Lemma 2.12, we have that the multivalued map $G_{n+1}: X_0 \to 2^{L^1(I,X)}$ defined by

$$(4.4.8) \qquad G_{n+1}(\xi) = \{v \in L^1(I, X) : v(t) \in F(t, x_{n+1}(t,\xi)) \text{ a.e. in } I\} ,$$

is l.s.c. with decomposable closed nonempty values and, by (4.4.7),

$$(4.4.9) \qquad H_{n+1}(\xi) = cl\{v \in G_{n+1}(\xi) : \|v(t) - f_n(t,\xi)\| < k(t).\beta_{n+1}(\xi)(t) \text{ a.e. in } I\} .$$

is a nonempty set. Then, by Lemma 2.13, there exists $h_{n+1} : X_0 \to L^1(I, X)$ a continuous selection of $H_{n+1}(.)$. Setting $f_{n+1}(t,\xi) = h_{n+1}(\xi)(t)$, for $\xi \in X_0$, $t \in I$, we have that f_{n+1} satisfies properties (a), (b) and (c) of our claim.
By virtue of (c) and (4.4.6), we have

$$(4.4.10) \qquad \|f_n(\cdot,\xi) - f_{n-1}(\cdot,\xi)\|_1 = \int_0^T \|f_n(u,\xi) - f_{n-1}(u,\xi)\| du \le \int_0^T \alpha(\xi)(u) \frac{[K(T)-K(u)]^n}{n!} du +$$
$$+ T\left(\sum_{i=0}^n \varepsilon_i\right) \frac{[K(T)]^n}{n!} \le \frac{[\|k\|_1]^n}{n!} (\|\alpha(\xi)\|_1 + T\varepsilon) .$$

Since $\xi \to \|\alpha(\xi)\|_1$ is continuous it is locally bounded. Therefore (4.4.10) implies that for every $\xi \in X_0$ the sequence $(f_n(\cdot, \xi'))_{n\in N}$ satisfies the Cauchy condition uniformly with respect to ξ' on some neighborhood of ξ. Hence, if $f(\cdot, \xi)$ is the limit of $(f_n(\cdot, \xi))_{n\in N}$ then $\xi \to f(\cdot,\xi)$ is continuous from X_0 into $L^1(I, X)$.

On the other hand, using (4.4.6) and (4.4.10), we have

$$\|x_{n+1}(\cdot, \xi) - x_n(\cdot, \xi)\|_\infty \le \|f_n(\cdot, \xi) - f_{n-1}(\cdot, \xi)\|_1 \le \frac{[\|k\|_1]^n}{n!} (\|\alpha(\xi)\|_1 + T\,\varepsilon) .$$

and so, as before, $(x_n(\cdot, \xi))_{n\in N}$ is Cauchy in $C(I, X)$ locally uniformly with respect to ξ. Then, denoting by $x(\cdot,\xi)$ its limit, it follows that the map $\xi \to x(\cdot, \xi)$ is continuous from X_0 into $C(I,X)$.

Since $x_n(\cdot,\xi)$ converges to $x(\cdot,\xi)$ uniformly and

$$d(f_n(t, \xi), F(t, x(t,\xi)) \le k(t)\, \|x_n(t, \xi)-x(t,\xi)\|$$

passing to the limit along a subsequence $(f_{n_k})_{k\in N}$ of $(f_n)_{n\in N}$ converging pointwise to f, we obtain that

(4.4.11) $\qquad f(t, \xi)\in F(t, x(t, \xi))$ for each $\xi \in X_0$ and $t\in I$ a.e..

Let $x^*(\cdot, \xi)$ be the unique weak solution of the Cauchy problem

$(P_{f(\cdot,\xi)})$ $\qquad \dot{x}(t)\in -Ax(t) + f(t, \xi)\ ,\ x(0) = \xi.$

By (4.2.2) we have

$$\|x_{n+1}(t, \xi) - x^*(t, \xi)\| \le \int_0^t \|f_n(u,\xi) - f(u,\xi)\|\, du$$

from which, letting $n \to \infty$, we get $x^*(.,\xi) \equiv x(.,\xi)$. Therefore $x(\cdot, \xi)$ is the weak solution of $(P_{f(\cdot,\xi)})$ and, by (4.4.11), it follows that $x(\cdot,\xi)\in \mathcal{T}(\xi)$ for every $\xi \in X_0$. ♦

4.5 WELL POSEDNESS FOR EVOLUTION EQUATIONS IN BANACH SPACES.

In this section X is a separable Banach space and $\{G(t) : t \geq 0\} \subset L(X,X)$ is a strongly continuous semigroup of bounded linear operators from X to X having infinitesimal generator A.

Consider the Cauchy problem

$$\dot{x}(t) \in Ax(t) + F(t, x(t)) \ , \ x(0) = \xi \ , \tag{4.5.0}$$

where $F: I \times X \to 2^X$ is a multivalued map satisfying Assumption 2.17 and $\xi \in X$.

Definition 4.14. A function $x(\cdot, \xi) : I \to X$ is called a *mild solution* of the Cauchy problem (4.5.0) if there exists $f(\cdot, \xi) \in L^1(I, X)$ such that

(i) $f(t,\xi) \in F(t, x(t, \xi))$ for almost all $t \in I$,

(ii) $x(t, \xi) = G(t)\xi + \int_0^t G(t-\tau)f(\tau,\xi)d\tau$, for each $t \in I$.

Remark 4.15. If X is finite dimensional and G(.) is the identity, then every mild solution of (4.5.0) is an absolutely continuous function satisfying

$$\dot{x}(t,\xi) \in F(t, x(t,\xi)) \ , \ x(0,\xi) = \xi \ .$$

We denote by $\mathcal{T}(\xi)$ the set of all mild solutions of (4.5.0).

Theorem 4.16. Let A be the infinitesimal generator of a C_0-semigroup $\{G(t): t \geq 0\}$, and let F satisfy Assumption 2.17. Then there exists a function $x(\cdot, \cdot) : I \times X \to X$ such that

(i) $x(\cdot, \xi) \in \mathcal{T}(\xi)$ for every $\xi \in X$

(ii) $\xi \to x(\cdot,\xi)$ is continuous from X into $C(I, X)$.

Proof. Let $\varepsilon > 0$ be fixed and, for $n\in N$, set $\varepsilon_n = \frac{\varepsilon}{2^{n+1}}$. Let $M = \sup\{\|G(t)\| : t\in I\}$, and for $\xi\in X$, define $x_0(\cdot,\xi) : I \to X$ by $x_0(t,\xi)=G(t)\xi$.
Since

$$\|x_0(t,\xi_1) - x_0(t,\xi_2)\| \le \|G(t)\|.\|\xi_1-\xi_2\| \le M.\|\xi_1-\xi_2\|$$

we have that $\xi \to x_0(\cdot,\xi)$ is continuous from X into C(I,X). For each $\xi\in X$, let $\alpha(\xi): I \to R$ be given by

$$\alpha(\xi)(t) = \beta(t) + k(t)\, \|x_0(t,\xi)\|.$$

Clearly $\alpha(.)$ is continuous from X into $L^1(I, R)$. Moreover, for each $\xi\in X$,

(4.5.1) $$d(0, F(t, x_0(t,\xi))) \le \alpha(\xi)(t) \text{ for a.e. } t\in I .$$

Let $G_0:X\to 2^{L^1(I,X)}$ and $H_0:X\to 2^{L^1(I,X)}$ be defined by (4.4.3) and (4.4.4). Then as in Theorem 4.13 one finds $h_0 : X \to L^1(I, X)$, a continuous selection of $H_0(\cdot)$.

Set $K(t) = \int_0^t k(\tau)d\tau$ and, for $n\ge1$, define $\beta_n: X \to L^1(I, R)$ by

(4.5.2) $$\beta_n(\xi)(t) = M^n \int_0^t \alpha(\xi)(u) \frac{[K(t)-K(u)]^{n-1}}{(n-1)!} du + M^nT \left(\sum_{i=0}^n \varepsilon_i\right) \frac{[K(t)]^{n-1}}{(n-1)!}, \quad t\in I.$$

Set $f_0(t,\xi) = h_0(\xi)(t)$, and define

$$x_1(t, \xi)=G(t)\xi + \int_0^t G(t-\tau)f_0(\tau,\xi)d\tau , \quad t\in I.$$

Then $f_0(t,\xi)\in F(t, x_0(t, \xi))$, $\|f_0(t, s)\| \le \alpha(s)(t) + \varepsilon_0$ and, for $t\in I\setminus\{0\}$,

$$\|x_1(t, \xi) - x_0(t, \xi)\| \le \int_0^t \|G(t-\tau)\|.\|f_0(\tau,\xi)\|d\tau \le M \int_0^t \|f_0(\tau,\xi)\|d\tau \le M \int_0^t \alpha(\xi)(\tau)d\tau + MT\varepsilon_0 <$$
$$< \beta_1(\xi)(t) .$$

We claim that there exist two sequences $\{f_n(.,\xi)\}_{n\in N}$ and $\{x_n(.,\xi)\}_{n\in N}$ satisfying for each $n \ge 1$ the following properties:

(a) $\xi \to f_n(\cdot, \xi)$ is continuous from X into $L^1(I, X)$

(b) $f_n(t, \xi) \in F(t, x_n(t, \xi))$ for each $\xi \in X$ and a.e. $t \in I$,

(c) $\|f_n(t,\xi) - f_{n-1}(t, \xi)\| \leq k(t)\, \beta_n(\xi)(t)$, for a.e. $t \in I$,

(d) $x_{n+1}(t, \xi) = G(t)\xi + \int_0^t G(t-\tau) f_n(\tau,\xi) d\tau$ for $t \in I$.

Suppose we have already constructed $f_1, \ldots, f_n$ and $x_1, \ldots, x_n$ satisfying (a)-(d). Define $x_{n+1}(\cdot,\xi) : I \to X$ by

$$x_{n+1}(t, \xi) = G(t)\xi + \int_0^t G(t-\tau) f_n(\tau,\xi) d\tau , \quad t \in I.$$

Then by (d), (c) we have, for $t \in I \setminus \{0\}$,

$$\|x_{n+1}(t, \xi) - x_n(t, \xi)\| \leq \int_0^t \|G(t-u)\|.\|f_n(u,\xi) - f_{n-1}(u,\xi)\| du \leq M \int_0^t \|f_n(u,\xi) - f_{n-1}(u,\xi)\| du \leq$$

$$(4.5.3) \quad \leq M \int_0^t k(u) \beta_n(\xi)(u) du = M^{n+1} \int_0^t \alpha(s)(\tau) \frac{[K(t)-K(\tau)]^n}{n!} d\tau + M^{n+1} T \left(\sum_{i=0}^n \varepsilon_i \right) \frac{[K(t)]^n}{n!} <$$

$$< \beta_{n+1}(\xi)(t) ,$$

and

$$(4.5.4) \quad d(f_n(t,\xi) , F(t, x_{n+1}(t, \xi))) \leq k(t) \|x_{n+1}(t, \xi) - x_n(t,\xi)\| < k(t)\, \beta_{n+1}(\xi)(t) .$$

Let $G_{n+1}: X \to 2^{L^1(I,X)}$, $H_{n+1}: X \to 2^{L^1(I,X)}$ be defined by (4.4.8), (4.4.9) respectively.
By (4.5.4) and Lemma 2.12, $G_{n+1}(\cdot)$ is l.s.c. from S into $\mathcal{D}$ and $H_{n+1}(\xi) \neq \emptyset$ for each $\xi \in X$. Hence by Lemma 2.13, there exists $h_{n+1} : X \to L^1(I, X)$ a continuous selection of $H_{n+1}(.)$. Then $f_{n+1}(t,\xi) = h_{n+1}(\xi)(t)$ satisfies the properties (a), (b) and (c) of our claim.
By (c) and (4.5.3) it follows that

$$\|x_{n+1}(\cdot, \xi) - x_n(\cdot,\xi)\|_\infty \leq M \|f_n(\cdot, \xi) - f_{n-1}(\cdot, \xi)\|_1 \leq \frac{[M.\|k\|_1]^n}{n!} (M \|\alpha(\xi)\|_1 + MT\varepsilon) .$$

Therefore $(f_n(\cdot, \xi))_{n\in N}$ and $(x_n(\cdot, \xi))_{n\in N}$ are Cauchy sequences in $L^1(I, X)$ and $C(I,X)$ respectively. Let $f(\cdot,\xi)\in L^1(I, X)$ and $x(\cdot, \xi)\in C(I,X)$ be their limits. Then as in the proof of Theorem 4.13 one can show: $\xi\rightarrow f(\cdot, \xi)$ is continuous from X into $L^1(I, X)$, $\xi \rightarrow x(\cdot, \xi)$ is continuous from X into $C(I, X)$ and, for all $\xi\in X$ and almost all $t\in I$, $f(t, \xi)\in F(t, x(t, \xi))$.
Passing to the limit in (d) we obtain

$$x(t, \xi)=G(t)\xi + \int_0^t G(t-\tau)f(\tau,\xi)d\tau \text{ for each } t\in I.$$

completing the proof. ♦

5. WELL POSEDNESS FOR A CLASS OF NON-CONVEX HYPERBOLIC DIFFERENTIAL INCLUSION.

5.1. INTRODUCTION.

Let $Q=[0,1]\times[0,1]$, $I=[0,1]$ and denote by $\mathcal{L}$ the σ-algebra of the Lebesgue measurable subsets of Q.

By $C(Q, R^n)$ (resp. $L^1(Q, R^n)$) we mean the Banach space of all continuous (resp. Bochner integrable) functions $u: Q \to R^n$ with the norm $\|u\|_\infty = \sup\{\|u(x,y)\| : (x,y)\in Q\}$ (resp. $\|u\|_1=\int_0^1\int_0^1 \|u(x,y)\|dxdy$), where $\|.\|$ is the norm in R^n. Denote by $\mathcal{D}$ the family of decomposable subsets of $L^1(Q, R^n)$. Consider the Banach space

$$\mathbb{S} =\{(\alpha,\beta)\in C(I,R^n)\times C(I,R^n): \alpha(0)=\beta(0)\}$$

with the norm

$$\|(\alpha,\beta)\|=\|\alpha\|_\infty+\|\beta\|_\infty.$$

Let $F : Q\times R^n \to 2^{R^n}$ be a multivalued map satisfying the following assumption:

Assumption 5.1: F takes closed nonempty values and:

(H_1) F is $\mathcal{L}\otimes\mathcal{B}(R^n)$-measurable

(H_2) there exists $L > 0$ such that $d(F(x,y,u), F(x,y,v)) \le L \|u-v\|$ for all $u, v\in R^n$, a.e. in Q.

(H_3) there exists a function $\delta\in L^1(Q, R)$ such that $d(0, F(x,y,0)) \le \delta(x,y)$ a.e. in Q.

For such an F and for $(\alpha,\beta)\in \mathbb{S}$, consider the Darboux problem

$(D_{\alpha\beta})$ $\qquad u_{xy}\in F(x,y,u)\ ,\ u(x,0)=\alpha(x),\ u(0,y)=\beta(y).$

Definition 5.2. $u(.,.;\alpha,\beta)\in C(Q,R^n)$ is said to be a *solution* of the Darboux problem $(D_{\alpha\beta})$ if there exists $v(.,.;\alpha,\beta)\in L^1(Q, R^n)$ such that

(i) $v(x,y;\alpha,\beta)\in F(x,y,u(x,y;\alpha,\beta))$ a.e. in Q,

(ii) $u(x,y;\alpha,\beta) = \alpha(x)+\beta(y)-\alpha(0) + \int_0^x \int_0^y v(\xi,\eta;\alpha,\beta)d\xi d\eta$, for every $(x,y)\in Q$.

Denote by $\mathcal{T}(\alpha,\beta)$ the set of all solutions of $(D_{\alpha\beta})$.

As it is well known (see [20]) if $F(x,y,u)=\{f(x,y,u)\}$, and $f: QxR^n\to R^n$ is continuous, Lipschitzean with respect to u then there exists a unique solution of the problem $(D_{\alpha\beta})$. Moreover, in [27] it has been proved that if f is measurable, continuous with respect to u and integrably bounded, then the set $\mathcal{T}(\alpha,\beta)$, of all solutions of $(D_{\alpha\beta})$, is a R_δ-set in $C(Q,R^n)$. The case when F is a multivalued map with convex values case has been considered in [27] and [56]. Properties of lower semicontinuity and Lipschitzeanity of the multivalued map $(\alpha,\beta)\to\mathcal{T}(\alpha,\beta)$, for F Lipschitzea, have been obtained in [41].

In what follows we present a result in [53] concerning the existence of a continuous selection $(\alpha,\beta)\to u(.,.;\alpha,\beta)$ from the multivalued map assigning to (α,β) in $\mathcal{S}$ the set $\mathcal{T}(\alpha,\beta)$ of osultions of the Darboux problem $(D_{\alpha\beta})$.

5.2. CONTINUOUS SELECTIONS FROM SOLUTION SETS.

Theorem 5.3. If F satisfies Assumption 5.1 then there exists $u:Q\times\mathcal{S} \to R^n$ such that

(i) $u(.,.;\alpha,\beta)\in \mathcal{T}(\alpha,\beta)$ for every $(\alpha,\beta)\in\mathcal{S}$

(ii) $(\alpha,\beta) \to u(.,.;\alpha,\beta)$ is continuous from $\mathcal{S}$ into $C(Q,R^n)$.

To prove this theorem we shall use Lemma 2.12 and Lemma 2.13 in the following setting (see Remark 2.14):

Lemma 5.4. Assume S is a separable metric space and $F_*: Q\times S \to 2^{R^n}$ is $\mathcal{L}\otimes\mathcal{B}(S)$-measurable, l.s.c. with respect to $s\in S$. Then the map $s\to G_*(s)$ given by

$$G_*(s)=\{v\in L^1(Q, R^n) : v(x,y)\in F_*(x,y,s) \text{ a.e. in } Q\},\ s\in S,$$

is l.s.c. with decomposable closed nonempty values if and only if there exists a continuous function $\sigma: S\to L^1(Q,R)$ such that $d(0, F(x,y,s)) \leq \sigma(s)(x,y)$ a.e. in Q.

Lemma 5.5. Let $G:S\to \mathcal{D}$ be a l.s.c. multifunction and let $\varphi:S\to L^1(Q, R^n)$ and $\psi:S\to L^1(Q,R)$ be continuous maps. If for every $\xi\in S$ the set

$$H(\xi) = \text{cl}\{v\in G(\xi) : \|v(x,y) - \varphi(\xi)(x,y)\| < \psi(\xi)(x,y) \text{ a.e. in } Q\} \tag{5.1}$$

is nonempty then the map $H: S \to \mathcal{D}$ defined by (5.1) admits a continuous selection.

Proof of the theorem. Fix $\varepsilon > 0$ and set $\varepsilon_n = \frac{\varepsilon}{2^{n+1}}$, $n\in N$. For $(\alpha,\beta)\in \mathbb{S}$ define $u_0(.,.;\alpha,\beta):Q\to R^n$ by $u_0(x,y;\alpha,\beta)=\alpha(x)+\beta(y)-\alpha(0)$ and observe that for all $(x,y)\in Q$ we have

$$\|u_0(x,y;\alpha_1,\beta_1)-u_0(x,y;\alpha_2,\beta_2)\| \leq \|\alpha_1(x)-\alpha_2(x)\| + \|\beta_1(y)-\beta_2(y)\| + \|\alpha_1(0)-\alpha_2(0)\| \leq$$

$$\leq 2\ \|(\alpha_1,\beta_1)-(\alpha_2,\beta_2)\|.$$

This implies that $(\alpha,\beta)\to u_0(.,.;\alpha,\beta)$ is continuous from $\mathbb{S}$ into $C(Q,R^n)$. Setting $\sigma(\alpha,\beta)(x,y)=\delta(x,y)+L\|u_0(x,y;\alpha,\beta)\|$ we obtain that σ is a continuous map from $\mathbb{S}$ into $L^1(Q,R)$ and

$$d(0,F(x,y,u_0(x,y;\alpha,\beta)))\leq\sigma(\alpha,\beta)(x,y) \text{ a.e. in } Q. \tag{5.2}$$

For $(\alpha,\beta)\in \mathbb{S}$, define

$$G_0(\alpha,\beta) = \{v\in L^1(Q, X) : v(x,y)\in F(x,y,u_0(x,y;\alpha,\beta)) \text{ a.e. in } Q\},$$

and

$$H_0(\alpha,\beta) = \mathrm{cl}\{v\in G_0(\alpha,\beta) : \|v(x,y)\| < \sigma(\alpha,\beta)(x,y) + \varepsilon_0 \text{ a.e. in } Q\} .$$

Then, by (5.2) and Lemma 5.4, it follows that $G_0(\cdot)$ is l.s.c. from $\mathcal{S}$ into $\mathcal{D}$ and, by (5.2), $H_0(\alpha,\beta) \neq \emptyset$ for each $(\alpha,\beta)\in\mathcal{S}$. Therefore by Lemma 5.5, there exists $h_0 : \mathcal{S}\to L^1(Q, R^n)$, which is a continuous selection of $H_0(\cdot)$. Set $v_0(x,y;\alpha,\beta)=h_0(\alpha,\beta)(x,y)$ and observe that $v_0(x,y;\alpha,\beta)\in F(x,y,u_0(x,y;\alpha,\beta))$ and $\|v_0(x,y)\| \leq \sigma(\alpha,\beta)(x,y) + \varepsilon_0$, for a.e. $(x,y)\in Q$.
Define

$$u_1(x,y,s)=u_0(x,y;\alpha,\beta)+\int_0^x\int_0^y v_0(\xi,\eta;\alpha,\beta)d\xi d\eta ,$$

and, for $n\geq 1$, set

(5.3) $$\sigma_n(\alpha,\beta)(x,y) =L^{n-1}\Big[\int_0^x\int_0^y \frac{(x-\xi)^{n-1}}{(n-1)!}\frac{(y-\eta)^{n-1}}{(n-1)!}\sigma(\alpha,\beta)(\xi,\eta)\, d\xi d\eta+\Big(\sum_{i=0}^n\varepsilon_i\Big) \frac{(x+y)^n}{n!}\Big].$$

Then, for every $(x,y)\in Q\setminus\{(0,0)\}$, we have

$$\|u_1(x,y;\alpha,\beta)-u_0(x,y;\alpha,\beta)\| \leq \int_0^x\int_0^y \|v_0(\xi,\eta,\alpha,\beta)\|d\xi d\eta \leq \int_0^x\int_0^y \sigma(\alpha,\beta)(\xi,\eta)d\xi d\eta+\varepsilon_0(x+y)<$$
$$< \sigma_1(\alpha,\beta)(x,y),$$

and so

$$d(v_0(x,y;\alpha,\beta),F(x,y,u_1(x,y;\alpha,\beta)) \leq L\|u_1(x,y;\alpha,\beta)-u_0(x,y;\alpha,\beta)\| < L\sigma_1(\alpha,\beta)(x,y).$$

We claim that there exist two sequences $\{v_n(x,y;\alpha,\beta)\}_{n\in N}$ and $\{u_n(x,y;\alpha,\beta)\}_{n\in N}$ such that for each $n \geq 1$ we have:

(a) $(\alpha,\beta) \to v_n(.,.;\alpha,\beta)$ is continuous from $\mathcal{S}$ into $L^1(Q, R^n)$

(b) $v_n(x,y;\alpha,\beta)\in F(x,y,u_n(x,y;\alpha,\beta))$ for any $(\alpha,\beta)\in\mathcal{S}$ and a.e. $(x,y)\in Q$,

(c) $\|v_n(x,y;\alpha,\beta) - v_{n-1}(x,y;\alpha,\beta)\| \leq L\sigma_n(\alpha,\beta)(x,y)$ a.e. in Q.

(d) $u_n(x,y;\alpha,\beta) = u_0(x,y;\alpha,\beta) + \int_0^x\int_0^y v_{n-1}(\xi,\eta;\alpha,\beta)d\xi d\eta$.

Suppose we have constructed $v_1, \ldots, v_n$ and $u_1, \ldots, u_n$ satisfying (a)-(d). Then define

$$u_{n+1}(x,y;\alpha,\beta) = u_0(x,y;\alpha,\beta)+ \int_0^x \int_0^y v_n(\xi,\eta;\alpha,\beta)d\xi d\eta \; .$$

Let $(x,y)\in Q\setminus\{(0,0)\}$. Using (c) we have

$$\|u_{n+1}(x,y;\alpha,\beta)-u_n(x,y;\alpha,\beta)\| \le \int_0^x \int_0^y \|v_n(\xi,\eta;\alpha,\beta)-v_{n-1}(\xi,\eta;\alpha,\beta)\|d\xi d\eta \le$$

$$\le L \int_0^x \int_0^y \sigma_n(\alpha,\beta)(\xi,\eta)d\xi d\eta = L^n \int_0^x \int_0^y \sigma(\alpha,\beta)(\xi,\eta)\Big(\int_\xi^x \frac{(x-u)^{n-1}}{(n-1)!}du \int_\eta^y \frac{(y-v)^{n-1}}{(n-1)!}dv\Big)d\xi d\eta +$$

(5.4)

$$+L^n\Big(\sum_{i=0}^n \varepsilon_i\Big)\int_0^x \int_0^y \frac{(\xi-\eta)^n}{n!}d\xi d\eta = L^n \int_0^x \int_0^y \frac{(x-\xi)^n}{n!}\frac{(y-\eta)^n}{n!}\sigma(\alpha,\beta)(\xi,\eta)\, d\xi d\eta +$$

$$+ \frac{L^n}{n!}\Big(\sum_{i=0}^n \varepsilon_i\Big) \frac{(x+y)^{n+2}-x^{n+2}-y^{n+2}}{(n+1)(n+2)} \le L^n\Big[\int_0^x \int_0^y \frac{(x-\xi)^n (y-\eta)^n}{n!\;\; n!}\sigma(\alpha,\beta)(\xi,\eta)\, d\xi d\eta +$$

$$+ \Big(\sum_{i=0}^n \varepsilon_i\Big) \frac{(x+y)^{n+1}}{(n+1)!}\Big] < \sigma_{n+1}(\alpha,\beta)(x,y),$$

Then, by virtue of (5.4) and of the (H_2) in Assumption 5.1, it follows that

(5.5)

$$d(v_n(x,y;\alpha,\beta),\, F(x,y,u_{n+1}(x,y;\alpha,\beta))) \le L\|u_{n+1}(x,y;\alpha,\beta),s)-u_n(x,y;\alpha,\beta)\| <$$

$$< L\sigma_{n+1}(\alpha,\beta)(x,y).$$

Since σ is continuous from $\mathfrak{S}$ into $L^1(Q,R)$, by (5.3) it follows that also σ_n is continuous from $\mathfrak{S}$ into $L^1(Q,R)$. Therefore, by (5.5) and Lemma 5.4, we have that the multivalued map G_{n+1} defined by

$$G_{n+1}(\alpha,\beta) = \{v\in L^1(Q, X) : v(x,y)\in F(x,y,u_{n+1}(x,y;\alpha,\beta)) \text{ a.e. in } Q\}$$

is l.s.c. from $\mathfrak{S}$ into $\mathcal{D}$. Moreover, by (5.5), it follows that that

$$H_{n+1}(\alpha,\beta) = \mathrm{cl}\{v\in G_{n+1}(\alpha,\beta): \|v(x,y)-v_n(\xi,\eta,\alpha,\beta)\| < L\sigma_{n+1}(\alpha,\beta)(x,y) \text{ a.e. in } Q\}$$

is nonempty. Then, by Lemma 5.5, there exists a continuous selection $h_{n+1} : \mathcal{S} \to L^1(Q, R^n)$ of H_{n+1}. Set $v_{n+1}(x,y;\alpha,\beta) = h_{n+1}(\alpha,\beta)(x,y)$ and observe that v_{n+1} satisfies the properties (a)-(d).

By (c) and the computations in (5.4) it follows that

(5.6) $$\|v_n(.,.;\alpha,\beta)-v_{n-1}(.,.;\alpha,\beta)\|_1 \le \frac{L^n}{n!}\|\sigma(\alpha,\beta)\|_1+\varepsilon\frac{[2L]^n}{n!},$$

and

(5.7) $$\|u_{n+1}(.,.;\alpha,\beta)-u_n(.,.;\alpha,\beta)\|_\infty \le \|v_n(.,.;\alpha,\beta)-v_{n-1}(.,.;\alpha,\beta)\|_1 \le \frac{L^n}{n!}\|\sigma(\alpha,\beta)\|_1+\varepsilon\frac{[2L]^n}{n!}.$$

Therefore $\{u_n(.,.;\alpha,\beta)\}_{n\in N}$ and $\{v_n(.,.;\alpha,\beta)\}_{n\in N}$ are Cauchy sequences in $C(Q,R^n)$, $L^1(Q,R^n)$, respectively. Moreover since $(\alpha,\beta)\to\|\sigma(\alpha,\beta)\|_1$ is continuous, it is locally bounded hence the Cauchy condition is satisfied locally uniformly with respect to (α,β). Let $u(.,.;\alpha,\beta)\in C(Q,R^n)$ and $v(.,.;\alpha,\beta)\in L^1(Q,R^n)$ be the limit of $\{u_n(.,.;\alpha,\beta)\}$ and $\{v_n(.,.;\alpha,\beta)\}$ respectively. Then $(\alpha,\beta)\to u(.,.;\alpha,\beta)$ is continuous from $\mathcal{S}$ into $C(Q,X)$ and $(\alpha,\beta)\to v(.,.;\alpha,\beta)$ is continuous from $\mathcal{S}$ into $L^1(Q, R^n)$. Letting $n\to\infty$ in (d) we obtain that

(5.8) $$u(x,y;\alpha,\beta) = u_0(x,y;\alpha,\beta)+\int_0^x\int_0^y v(\xi,\eta;\alpha,\beta)d\xi d\eta \quad \text{for any } (x,y)\in Q.$$

Furthermore, since

$$d(v_n(x,y;\alpha,\beta),F(x,y,u(x,y;\alpha,\beta)) \le L\|u_{n+1}(x,y;\alpha,\beta)-u(x,y;\alpha,\beta)\|$$

and F has closed values , letting $n\to\infty$ we have

(5.9) $$v(x,y;\alpha,\beta)\in F(x,y,u(x,y;\alpha,\beta)) \quad \text{a.e. in } Q.$$

By (5.8) and (5.9) it follows that $u(.,.,s)$ is a solution of $(D_{\alpha\beta})$, which completes the proof.

Remark 5.6. Theorem 1 remains true (with the same proof) if R^n is replaced by a separable Banach space X and F is a multifunction from QxX into the closed bounded nonempty subsets of X satisfying the Assumption 5.1.

Remark 5.7. If (H_2) in Assumption 5.1 is relaxed, allowing F to be merely continuous, the conclusion of the theorem is in general no longer true. To see this consider the Darboux problem

$$(D_{\alpha\beta}) \qquad u_{xy}=\sqrt[3]{u}\,, \qquad u(x,0)=\alpha(x),\ u(0,y)=\beta(y),\ (x,y)\in Q$$

Remark that $f(u)=\sqrt[3]{u}$ is continuous but not Lipschitzean in a neighbourhood of 0 and, for $\alpha_0(x)=0=\beta_0(y)$, the problem $(D_{\alpha_0\beta_0})$ admits as solutions:

$$u_0^+(x,y)=\left(\frac{2}{3}\right)^3 x^{3/2}\, y^{3/2} \quad \text{and} \quad u_0^-(x,y)=-\left(\frac{2}{3}\right)^3 x^{3/2}\, y^{3/2}\ .$$

Let

$$\alpha_n^+(x)=\left(\frac{2}{3\sqrt{n}}\right)^3 x^{3/2},\ \alpha_n^-(x)=-\left(\frac{2}{3\sqrt{n}}\right)^3 x^{3/2},\ \beta_n^+(y)=0=\beta_n^-(y).$$

Then $(\alpha_n^+, \beta_n^+), (\alpha_n^-, \beta_n^-)\in \mathbb{S}$ and $\|(\alpha_n^+, \beta_n^+)\|=\|(\alpha_n^-, \beta_n^-)\|=\left(\frac{2}{3\sqrt{n}}\right)^3$, therefore (α_n^+,β_n^+) and (α_n^-, β_n^-) converge to $(\alpha_0,\beta_0)=(0,0)$ in the space $\mathbb{S}$.

On the other hand the unique solution of the Darboux problem $(D_{\alpha_n^+,\beta_n^+})$ (resp. of $(D_{\alpha_n^-,\beta_n^-})$) is given by

$$u_n^+(x,y)=\left(\frac{2}{3}\right)^3 x^{3/2}\left(\frac{1}{n}+y\right)^{3/2}$$

(resp.

$$u_n^-(x,y)=-\left(\frac{2}{3}\right)^3 x^{3/2}\left(\frac{1}{n}+y\right)^{3/2}).$$

which for $n\to\infty$ converges to u_0^+ (resp. u_0^-).

Suppose that there exists $r : \mathcal{S} \to C(Q,X)$ a continuous selection of the solution map $(\alpha,\beta) \to \mathcal{T}(\alpha,\beta)$. Then, for $n \to \infty$, we have that $r((\alpha_n^+, \beta_n^+)) = u_n^+$ converges to u_0^+ and $r((\alpha_n^-, \beta_n^-)) = u_n^-$ converges to u_0^-. This is a contradiction to the continuity of r. ♦

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