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PERIODIC SOLUTION TO SINGULAR DYNAMICAL SYSTEMS WITH KEPLERIAN TYPE POTENTIALS

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INTRODUCTION

Since this work has been written as a Phd thesis, one of its main purposes is to present the author's researches; on the other hand, it contains an attempt of systematizing the present state of art. Of course, this work is mostly dedicated to the author's results. The author apologizes for this with the reader in advance.

The objective of this work is to discuss problems related to the existence and multiplicity of periodic solutions to conservative systems of the type

$$-\ddot{x} = \nabla F(x) \; ;$$

in the case when the potential F presents a singularity at zero of attractive type; in order to fix your mind, think of the model potential $F(x) = \frac{-a}{|x|^\alpha}$.

The first example of singular potential is the Kepler potential $F(x) = \frac{-1}{|x|}$ related to Newton's law of gravitation. In the early 17th century, Newton himself solved the associated motion equation and found that every orbit with negative energy is periodic, the period depending on the energy, the mass and a universal constant. From then on, the interest in this problem and the related N-bodies problem has always been alive. More recently a new method of approaching the problem has been provided by the use of variational techniques. A new development of the critical points theory has been necessary for the application to this field.

We shall mainly deal with the following problems:

The *fixed period problem*

given a period $T > 0$, look for solutions to

$$(P_T) \qquad \begin{cases} -\ddot{x} = F(x) & \\ x(t+T) = x(t) & \forall t \in \mathbb{R} \\ x(t) \neq 0 & \forall t \in \mathbb{R} \; ; \end{cases}$$

and the *fixed energy problem*

given an energy $E \in \mathbb{R}$, look for solutions to

$$(P_E) \qquad \begin{cases} -\ddot{x} = F(x) & \\ \frac{1}{2}|\dot{x}|^2 + F(x) = E & \\ x(t+\lambda) = x(t) & \forall t \in \mathbb{R} \\ x(t) \neq 0 & \forall t \in \mathbb{R} \end{cases}$$

(in the fixed energy problem the unknowns are both the function x and its period λ).

In this thesis, existence and multiplicity results are proved by means of variational methods. To this end, we shall associate to (P_T) and (P_E) suitable functionals, whose critical points correspond to the solutions to each problem. Then, existence and multiplicity of solutions to both (P_T) and (P_E) are provided by proving existence and multiplicity of critical points of the associated functionals. Generally speaking, our assumptions on the potential F take the form

$$\text{(H1)} \qquad \frac{a}{|x|^\alpha} \leq -F(x) \leq \frac{b}{|x|^\alpha}$$

$$\text{(H2)} \qquad -\alpha_1 F(x) \leq \nabla F(x) \cdot x \leq -\alpha_2 F(x) \, .$$

Different variational principles will be applicable, depending on the value of the exponent α. We shall treat separately the following cases:

$\alpha \geq 2$ (existence and multiplicity of solutions for both (P_T) and (P_E));
$1 < \alpha < 2$ (existence and multiplicity of solutions for both (P_T) and (P_E));
$0 < \alpha < 1$ (existence of solutions for both (P_T) and (P_E)).

As far as the case $\alpha = 1$ is concerned, we shall see that a structural degeneracy occurs and that the variational approach under assumptions (H1)-(H2) fails. Therefore, pinching conditions on the coefficients a, b (and sometimes on α_1, α_2) are expected when α approaches the value 1. (for pinching condition we mean additional assumptions to (H1)-(H2), of the type $\frac{b}{a} \leq \Phi(\alpha)$, with $\lim_{\alpha \to 1} \Phi(\alpha) = 1$).

In recent years, many results have been obtained by several authors about problems with singular potentials in a variational framework (mainly about the fixed period problem); we quote the most relevant to this work. Concerning (P_T), we first recall Gordon's works [**27**],[**28**], about the planar case ($\alpha \geq 2$ and $\alpha = 1$); next Ambrosetti and Coti Zelati, [**1**], treated the higher dimensional case, when $\alpha \geq 2$, and found an infinity of solutions, by means of a result by Fadell and Husseini about the category of the noncollision loop space in itself; more recently the same result has been used by Majer to generalize the theorem of Ambrosetti and Coti Zelati. A simpler approach was introduced by Greco, [**29**], in proving the existence of at least one solution to (P_T) in the case $\alpha \geq 2$, and by Bahri and Rabinowitz, [**10**]. The case $1 < \alpha < 2$, under assumption (H1), has been treated by Degiovanni and Giannoni in [**25**], where they proved an existence result under a pinching condition on the coefficients a and b; for this case, other results (with different assumptions from (H1)) are provided in [**21**], [**22**]. A counterexample has been provided by Capozzi Solimini and the author in [**19**], where a strong limit to the variational approach to (P_T) when $\alpha = 1$ is shown. An entirely different approach has been used by Ambrosetti and Coti Zelati in [**5**], where they obtained a multiplicity of solutions to (P_T) in the case of small perturbations of the radial potential $F(x) = \frac{-a}{|x|^\alpha}$ (for every $\alpha \neq 1$), by means of a bifurcation argument. Multiplicity results for both cases $\alpha \geq 2$ and $1 < \alpha < 2$ have been obtained by the author in [39]. Finally, the cases $0 < \alpha < 1$ has been successfully discuss in a variational framework by Ramos and the author [34].

Concerning the fixed energy problem, existence results have been obtained by Benci and Giannoni [15], when $\alpha > 2$ (in [15] also the case $\alpha \le 2$ was examined, under assumptions very far from (H1)-(H2)). The problem of the existence of at least one solution to (P_E) has been treated, under assumptions (H1)-(H2), by Ambrosetti and Coti Zelati in [6] where a new functional has been introduced; the results there cover the case $\alpha > 2$ and the case $0 < \alpha < 2$, when the generalized solutions (i.e. possibly passing through the singularity) are considered. Existence and multiplicity of noncollision solutions in both cases $\alpha \ge 2$ and $1 < \alpha < 2$ are provided by the author in [40] and [41]. The case $0 < \alpha < 1$ has been examined by Ramos and the author in [34]. Concerning the case $\alpha = 1$, existence and multiplicity results have been obtained by Moser [32], in the case when the potential F has some symmetry properties.

In the last part of this thesis we shall be concerned with a three body type problem. This topic has been so widely studied that it is impossible to give an exhaustive bibliography, we refer to [33] and the references therein contained. Concerning the variational approach, we quote the recent works of Coti Zelati and Bessi and Coti Zelati [23],[24],[17] where symmetrical potentials are considered; in these papers, existence results are given for solutions free of triple collisions (actually the N-body problem was examined, and existence results are given for this more general situation). In the non symmetrical case we recall a very recent work of Bahri and Rabinowitz, [11], about the three body problem in $\mathbb{R}^3$. In Part 6 we shall give the results about this problem obtained in collaboration with Serra in [35] and [36].

This work is organized as follows:

Part 1 is an introductive part, where the solutions of the motion equations

$$-\ddot{x} = \frac{\alpha a x}{|x|^{\alpha+2}}$$

are discussed. The phenomenology provided in Part 1 will be taken as a guide in the following discussion of (P_T) and (P_E) when the potential F does not have a radial symmetry.

Part 2 is devoted to the study of (P_T) when the problem of the existence of at least one solution is considered. We look there for solutions of (P_T) as critical points of the action integral

$$I(x) = \int_0^T \frac{1}{2}|\dot{x}|^2 - F(x) \, .$$

It is known that the solutions of (P_T) are critical points of I in the subset of $H = \{x \in H^1_{loc}(\mathbb{R}, \mathbb{R}^N) \; / \; x(t+T) = x(t) \; , \forall t \in \mathbb{R}\}$.

$$\Lambda = \{x \in H \;/\; x(t) \neq 0 \,, \forall t \in \mathbb{R}\}$$

Let us consider the assumption (H1): then the main difficulties we find in looking for critical points of I are the following

a) I is not coercive

b) Λ is open.

Moreover, of course, the minimization approach fails, since $\inf_\Lambda I = 0$ is not attained. Therefore, some minimax arguments are needed in order to find critical points of I.

The first chapter of Part 2 is devoted to the case $\alpha > 1$. We collect there some of the results contained in [**1**], [**29**],[**10**] and [**25**]. Let us roughly explain the common abstract setting of these works.

We first give a definition: *we say that a subset $A \subseteq \Lambda$ is contractible if there exists a continuous homotopy $h : A \times [0,1] \to H$, (homotopically equivalent to the identity) such that $h(A,[0,1]) \cap \partial\Lambda = \emptyset$ and $h(A,1)$ is contained in the spaces of constant functions.*

Then the properties one requires for I are the following.

(i)	I has a positive l.s.c. extension to the whole of H ;
(ii)	0 is the only level where the (PS) condition fails;
(iii)	there is $\varepsilon > 0$ such that the sublevel $I^\varepsilon = \{x \in \Lambda \;/\; I(x) < \varepsilon\}$ is contractible
(iv)	there is a non contractible compact set $A \subseteq \Lambda$ such that: $\sup_A I \leq \inf_{\partial\Lambda} I$.

Critical points in Λ can the be shown to appear at the level

$$c = \inf_{A \in \mathcal{A}} \sup_A I$$

where

$$\mathcal{A} = \{A \subset \Lambda \text{ compact} \;/\; A \text{ is not contractible} \,, \sup_A \leq \inf_{\partial\Lambda} I\} \,.$$

The fundamental step consists in showing that (iv) is fulfilled. Under assumption (H1), this procedure succeeds when $\alpha \geq 2$ without any additional hypotheses on a and b ([**1**], [**29**]) and when $1 < \alpha < 2$, provided that the coefficients a and b satisfy a condition of the type $\frac{b}{a} \leq \Psi(\alpha)$, with $\lim_{\alpha\to 1} \Psi(\alpha) = 1$ and $\lim_{\alpha \to 2} \Psi(\alpha) = +\infty$ ([**25**]).

On the other hand, whenever $0 < \alpha < 1$, this method is not available ((iv) is in general false). However, critical points are found via the application of Rabiniwitz's Saddle Point Theorem at levels where no solution crossing the singularity is allowed. The results about this case are contained in [**34**].

In **Part 3** the existence of at least one periodic solution with prescribed energy is investigated. We are concerned there with the following cases

$\alpha > 2$ and $E > 0$
$1 < \alpha < 2$ and $E < 0$
$0 < \alpha < 1$ and $E < 0$.

The conditions on the energy sign are necessary for the solvability of (P_E) when (H1)-(H2) hold with $\alpha_2 > 2$ (case $\alpha > 2$) or $\alpha_1 < 2$ (case $\alpha < 2$).
The functional associated to (P_E) is then[1]

$$(2) \qquad I(x) = \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)\left(\int_0^1 E - F(x)\right).$$

In fact, up to the rescaling of the period, each critical point of I in Λ at a positive level solves (P_E) (and conversely). Depending on the sign of the energy, the functional I is positive or unbounded (above and below).

When $\alpha > 2$ and $E > 0$, the features of I are such that the same argument (minimax over classes of non contractible sets) as the one used for the fixed period problem is applicable (except for some technical difficulties arising from the fact that $\liminf_{x\to\partial\Lambda} I = 0$). This fact has already been pointed out in [**15**]; the existence result given in 3.I is just a slight modification of a result there.

When $\alpha < 2$ and $E < 0$, the main problems arise from the fact that the functional I is not definite; for this reason one has $\liminf_{x\to\partial\Lambda} = -\infty$ as well as $\limsup_{x\to\partial\Lambda} = +\infty$. On the other hand, looking at the restriction of the functional to a set of the type $S_\rho = \{x \in H \ / \ \int_0^1 |\dot{x}|^2 = \rho\}$, the same properties as those of the functional in (1) hold. A natural approach seems to be then to combine intersection properties with the techniques used for the fixed period problem.

First, the case $1 < \alpha < 2$ is discussed. Taking into account the features of I restricted to S_ρ, a new inf-sup class is introduced, joining intersection properties with noncontractibility arguments. Roughly speaking, the main property one requires of the elements of this class is that their intersections with the set S_ρ are not contractible sets. This method has been developed by the author in [**39**].

As far as the case $0 < \alpha < 1$ is concerned, the existence result is obtained by a linking type argument.

Part 4 and **Part 5** are devoted to the problem of finding multiple solutions to (P_T) and (P_E). To this end, the variational methods used in proving the existence of at least one solution are combined with the theory of the geometrical indices. A geometrical index related to a group of symmetries gives, in some sense, a way of measuring the size of a set; this concept has been widely used in searching multiple critical point for coercive functionals invariant under a group of unitary transormations ([**31**],[**13**],[**12**]). In

[1] In [6] the functional associated to (P_E) was

$$I(x) = \left(\int_0^1 \nabla F(x)\cdot x\right)\left(\int_0^1 E - F(x)\right)$$

particular, we shall exploit the invariance of both the functionals in (1) and (2) under the compact group $G = \{T_s, P_s\}_{s\in[0,T]}$, where $T_s(x)(t) = x(s+t)$ and $P_s(x)(t) = x(s-t)$.

The homotopical index of a set $A \subseteq \Lambda$ measures (in terms of the geometrical index) the minimal size of the subset one has to remove from A in order to make it contractible. This theory is a convenient tool for finding multiple critical points of bounded functionals having a singularity and a lack of coercivity at the level of the large constant functions. This is the situation when dealing with the fixed period problem ($\alpha > 1$) and the fixed energy problem when $\alpha > 2$ and $E > 0$.

In the fixed energy problem when $1 < \alpha < 2$, this approach fails, since the associated functional is not bounded from below. However, we have already pointed out that positive critical point can be found by joining intersection set properties with non contractibility arguments. These techniques can also be combined with the geometrical index theory: an homotopical pseudo-index can be defined. Roughly speaking, the homotopical pseudo-index of a set is the homotopical index of its intersection with a given closed subset of the function space (in our case with S_ρ). This procedure has been applied in proving a multiplicity result for (P_E), when $1 < \alpha < 2$ and (H1)-(H2) hold in addition to a pinching condition involving a, b, α_1 and α_2.

In **Part 6** problem (P_T) is examined in the case when the potential F is even (i.e.$F(x) = F(-x)$). In this case, as it has been already shown out in [DGM][2], the degeneracy ocurring at $\alpha = 1$ can be overcome by the introduction of some symmetry constraints on the function space. We shall be concerned there with the problem of finding noncollision solutions under local assumptions on the behaviour of the potential in a neighbourhood of the singularity. We next consider a three body problem:

$$
(3b) \qquad \begin{cases} -m_i\ddot{x}_i = \sum_{\substack{j=1 \\ j\neq i}}^{3} \nabla F_{i,j}(x_i - x_j) & i = 1,2,3 \\ x_i(t+T) = x(t) & i = 1,2,3\ ,\ \forall t \\ x_i(t+\frac{T}{2}) = -x_i(t) & i = 1,2,3\ ,\ \forall t \\ x_i(t) \neq x_j(t) & i \neq j\ ,\ i,j = 1,2,3\ ,\ \forall t \in \mathbb{R} \end{cases}
$$

Existence results are given in the locally radial symmetric case. As a by-product of our approach we will also be able to prove existence of periodic solutions under pinching conditions of the type

$$
\frac{a_{i,j}}{|x|^\alpha} \leq -F_{i,j} \leq C\frac{a_{i,j}}{|x|^\alpha}
$$

with $C \leq \Psi(\alpha, \frac{a_{i,j}}{a_{i,k}})$. Finally we will perform a theorem on the existence of infinitely many solutions when $F_{i,j}(x) = \frac{-a_{i,j}}{|x|^\alpha} + U_{i,j}(x)$, $\lim_{x\to 0} |x|^\alpha U_{i,j}(x) = 0$. The results of Part 6 have been obtained by the author in collaboration with Serra in [35], [36].

[2] in that paper, existence results have been proved under the assumption (H1), with a pinching condition on a and b

NOTATIONS: Throughout this work we shall adopt the following notations:

$$B^N = \{x \in \mathbb{R}^N \ / \ |x| \leq 1\} \ ;$$

$$S^{N-1} = \{x \in \mathbb{R}^N \ / \ |x| = 1\} \ .$$

H denotes the Sobolev space of the 1-periodic functions:

$$H = \{y \in H^1_{loc}(\mathbb{R}; \mathbb{R}^N) \ / \ y(t+1) = y(t) \ , \forall t \in \mathbb{R}\} \ ,$$

endowed with the Hilbertian norm

$$||x||^2 = \int_0^1 |\dot{x}|^2 + \int_0^1 |x|^2 \ .$$

we shall also denote by H the space of the T-periodic functions in H^1_{loc}. This might look ambigous but at the beginning of every part we shall specify the exact definition.

We shall consider the open subset of H defined by

$$\Lambda = \{y \in H \ / \ y(t) \neq 0 \ , \forall t \in \mathbb{R}\} \ ,$$

and we shall denote

$$\partial\Lambda = \{y \in H \ / \ \exists t \in \mathbb{R} \ , \ y(t) = 0\} \ .$$

E_N denotes the subspace of H of all the constant functions. Moreover, in any metric space X, and for every subset $K \subseteq X$,

$$V_\varepsilon(K) = \{x \in X \ / \ dist(x, K) < \varepsilon\}$$
$$N_\varepsilon(K) = \{x \in X \ / \ dist(x, K) \leq \varepsilon\}$$

Finally, since we shall deal with critical points of the functional I, we shall denote

$$K_c = \{x \in H \ / \ I(x) = c \ , \ dI(x) = 0\}$$

NOTA BENE: Each part of this thesis is divided in chapter each of one can be divided in sections when necessary. The system of numbering formulas, theorems, corollaries, lemmas, propositions refers, if not otherwise specified, to the part in which it appears; the roman numerals indicate the chapter, the secod cipher indicates the section and the third cipher indicate the formula itself. For example, (II.3.27) is the refence number of the 27th formula appearing in the third section of chapter II (the part is omitted).

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PART 1. THE CASE $F(x) = \frac{-a}{|x|^\alpha}$

In this preliminary part we shall study, by means of classical mechanics methods, the basic properties of the solutions to the motion equation

$$(1.1) \qquad -\ddot{x} = \frac{a\alpha x}{|x|^\alpha},$$

where a,α are positive numbers. Our objective is to provide a phenomenology for the case $F(x) = \frac{a}{|x|^\alpha}$ that we shall use in the sequel as a guide in discussing problems when the potential F is similar, in some sense, to one of that form. The class of (periodic) solutions of (1.1) we are mostly interested in is the one of the circular solutions (which always exist in the case of radial forces); indeed, throughout all this work, our purpose will be the search of periodic solutions to problems without radial symmetry, and the solutions we shall look for will correspond (and possibly approximate) to the circular ones of the case here.

1.I. Discussion of the Cauchy Problem

In what follows, for each pair of initial data $(x_0, \dot{x}_0) \in \mathbb{R}^{2N}$, with $x_0 \neq 0$, the solution of the Cauchy Problem

$$(CP) \qquad \begin{cases} -\ddot{x} = \frac{a\alpha x}{|x|^\alpha} \\ x(0) = x_0 \\ \dot{x}(0) = \dot{x}_0, \end{cases}$$

is the unique function $x \in C^\infty((\omega_1, \omega_2); \mathbb{R}^N)$ satisfying (CP) and such that (ω_1, ω_2) is the maximal interval of existence of x. In other words, ω_i can be finite if and only if $lim_{t\to\omega_i} x(t) = 0$. Of course, the interval ω_1, ω_2 is never empty when $x_0 \neq 0$. Because of the radial symmetry of the equation, the solution of (CP) lies on the plane spanned by x_0 and $\dot{x}_0$; for the same reason, it lies on the straight line spanned by x_0 is x_0 and $\dot{x}_0$ are parallel. We can hence restrict our discussion to planar equations.

Since the equation is time-independent, x verifies the energy integral:

$$(I.2) \qquad \frac{1}{2}|\dot{x}(t)|^2 - \frac{a}{|x(t)|^\alpha} = \frac{1}{2}|\dot{x}_0|^2 - \frac{a}{|x_0|^\alpha} = E, \quad \forall t \in (\omega_1, \omega_2)$$

and the radial symmetry of the force also implies the conservation of the angular momentum:

(I.3) $$\dot{x}(t) \times x(t) = \dot{x}_0 \times x_0 = B\,, \qquad \forall t \in (\omega_1, \omega_2)\,.$$

Therefore, if $(\rho(t), \theta(t))$ are the polar coordinates of x, by means of the above first integrals, system (1.1) reduces to the system

(I.4) $$\begin{cases} \frac{1}{2}\dot{\rho}^2 + \frac{1}{2}\frac{B^2}{\rho^2} - \frac{a}{\rho^\alpha} = E \\ \rho^2\dot{\theta} = B\,. \end{cases}$$

(here $B = |B|$). Also remark that ρ satisfies

(I.5) $$\ddot{\rho} = \frac{B^2}{\rho^3} - \frac{a\alpha}{\rho^{\alpha+1}}\,.$$

1.II. Some useful changes of coordinates

We first observe that, from (1.1) and (I.2) we have that

$$\frac{1}{2}\ddot{|x|^2} = |\dot{x}|^2 + \ddot{x} \cdot x = |\dot{x}|^2 - \frac{a\alpha}{|x|^\alpha} = 2E + (2-\alpha)\frac{a}{|x|^\alpha}\,,$$

that is

(II.1) $$\frac{1}{2}\ddot{\rho^2} = 2E + (2-\alpha)\frac{a}{\rho^\alpha}\,.$$

Now let $B \neq 0$; then, because of the second equation in (I.4), the dependence $\theta(t)$ can be reverted. From (I.5) we then obtain

(II.2) $$-\frac{d^2}{dt^2}\frac{1}{\rho} = \frac{1}{\rho} - \frac{\alpha}{B^2}\frac{1}{\rho^{\alpha-1}}\,.$$

Now we assume that $\alpha \neq 2$, and we set

$$\mu = \rho^{2-\alpha}$$
$$s(t) = \int_0^t \frac{1}{\rho^\alpha}$$
$$\beta = \frac{\alpha}{2-\alpha}\,.$$

Then, by (II.1), μ satisfies

$$\text{(II.3)} \qquad -\frac{d^2}{dt^2}\mu = -2(2-\alpha)E\mu^\beta - (2-\alpha)^2 a\,, \qquad \forall s \in \left(\int_{\omega_1}^{0}\frac{1}{\rho^\alpha}, \int_{0}^{\omega_2}\frac{1}{\rho^\alpha}\right).$$

1.III. The case $\alpha = 1$

When $\alpha = 1$, in addition to the one of (I.2) and (I.3), we have the further first integral

$$\text{(III.1)} \qquad (x \times \dot{x}) \times \dot{x} + \frac{ax}{|x|} = P\,,$$

called the Lenz's vector. It is well known since the early 17th century that each solution of (1.1), with $\alpha = 1$, lies on the path of a conic section. The direction of vector P is the symmetry axis of this path.

Indeed, if θ is the angle between x and P, from (1.9) and (I.3) we obtain

$$x \cdot P = -B^2 + a|x|\,,$$

that is,

$$(a - |P|\cos\theta)\rho = B^2\,,$$

which is, in facts, the equation of a conic section in polar coordinates. An easy computation shows that $|P|^2 = a^2 - 2EB^2$, so that three cases are possible:

a) $E > 0$ (and therefore $|P| > a$): then the trajectory is a hyperbola;

b) $E = 0$ (and therefore $|P| = a$): then the trajectory is a parabola;

c) $E < 0$ (and therefore $|P| < a$): then the trajectory is an ellipse with semiaxis $(\frac{a}{-2E}, \frac{B}{\sqrt{-2E}})$; moreover the motion is periodic of period $T = 2\pi\frac{a}{(-2E)^{\frac{3}{2}}}$.

The computation of the period follows from the fact that $\frac{1}{2}B$ is the areal speed; hence, if A is the area of the interior of the ellipse, we have

$$\frac{1}{2}BT = A = \pi\frac{a}{-2E}\frac{B}{\sqrt{-2E}}\,.$$

As a consequence of the above discussion, the following proposition holds:

PROPOSITION 1.III.1.. *When $\alpha = 1$, each solution of (1.1) having negative energy E is periodic of period $T = 2\pi\frac{a}{(-2E)^{\frac{3}{2}}}$.*

REMARK 1.III.2: Here the collision solution obtained when $B = 0$ is extended by periodicity. Indeed, is not difficult to see that its maximal interval of existence (ω_1, ω_2) has length $T = 2\pi \frac{a}{(-2E)^{\frac{3}{2}}}$.

The remarkable property that every solution having negative energy is periodic, the period just depending on the value of the energy, is peculiar of the case $\alpha = 1$; a direct proof of it (without integrate the equation) can be made by means of the one-to-one change of phase coordinates

$$\text{(III.2)} \qquad \begin{cases} \phi(\rho,\dot\rho) = \cos\left(\frac{\sqrt{-2E}}{a}\rho\dot\rho\right)\left(\frac{\sqrt{-2E}}{a^{\frac{2}{3}}}\rho - \frac{a^{\frac{1}{3}}}{\sqrt{-2E}}\right) - \sin\left(\frac{\sqrt{-2E}}{a}\rho\dot\rho\right)\frac{1}{a^{\frac{2}{3}}}\rho\dot\rho \\ \psi(\rho,\dot\rho) = \sin\left(\frac{\sqrt{-2E}}{a}\rho\dot\rho\right)\left(\frac{\sqrt{-2E}}{a^{\frac{2}{3}}}\rho - \frac{a^{\frac{1}{3}}}{\sqrt{-2E}}\right) + \cos\left(\frac{\sqrt{-2E}}{a}\rho\dot\rho\right)\frac{1}{a^{\frac{2}{3}}}\rho\dot\rho \,. \end{cases}$$

Indeed, by virtue of (1.1) and (I.2), (ϕ,ψ) satisfies the linear first order system

$$\text{(III.3)} \qquad \begin{cases} \dot\phi = \frac{(-2E)^{\frac{3}{2}}}{a}\psi \\ \dot\psi = -\frac{(-2E)^{\frac{3}{2}}}{a}\phi \,. \end{cases}$$

1.IV. Existence of periodic orbits

As a first straightforward consequence of formula (II.1) we can state the following

PROPOSITION 1.IV.1. *Necessary condition for the periodicity of the solutions of (1.1) are*

(a)	$E > 0$	*if* $\alpha > 2$;
(b)	$E = 0$	*if* $\alpha = 2$;
(c)	$E < 0$	*if* $0 < \alpha < 2$.

PROOF: Indeed, arguing by contradiction, (II.1) implies that $\ddot{\rho^2}$ should have constant sign, contradicting the periodicity of ρ.◇

Next propositions show an important difference between the case $\alpha \geq 2$ and the case $0 < \alpha < 2$. In the first case the only periodic solutions are the circular ones (with constant angular speed and positive energy), and every other solution either is unbounded or crosses the origin. In the case $0 < \alpha < 2$, all solution having negative energy is either periodic or quasi-periodic (hence bounded); moreover a solution passes through the origin only if its angular momentum vanishes. Thus we shall refer to the first case as the *strong force case*, while the second will be called the *weak force case*.

PROPOSITION 1.IV.2. *Assume that* $\alpha \geq 2$. *The only noncollision solution of (1.1) are of the form* $x(t) = Re^{i\omega t}$ *(with* $\omega^2 = \frac{a\alpha}{R^{\alpha+2}}$*) in some 2-plane of* $\mathbb{R}^N$. *Every other solution either crosses the origin or it is unbounded.*

PROOF: It is easily to see that a function of that form solves (1.1). Conversely, if a periodic function x solves (1.1) then it is planar and, from (II.1), it has constant radius. One then easily concludes that its angular speed is constant.
Now, from the first part of (I.4) we deduce that the motion of a solution is constrained in the set

$$\{\rho \geq 0 \;/\; E - \frac{1}{2}\frac{B^2}{\rho^2} + \frac{a}{\rho^\alpha} \geq 0\} \;;$$

the proof is then complete by the discussion of the inequality $E\rho^\alpha - \frac{1}{2}B^2\rho^{\alpha-2} + a \geq 0$, when $\alpha \geq 2$.◇

PROPOSITION 1.IV.3. *Assume that $0 < \alpha < 2$. Then every noncollision solution of (1.1) having negative energy is either periodic or quasi-periodic. Every other solution is unbounded. Moreover a solution can cross the origin if and only if its angular momentum vanishes.*

PROOF: It follows from (II.1) that all solution having nonnegative energy is unbounded. On the other hand, if the energy is negative, from the first equation in (I.4), the motion is constrained in the bounded set

$$\{\rho \; / \; E - \frac{1}{2}\frac{B^2}{\rho^2} + \frac{a}{\rho^\alpha} \geq 0\} \, ,$$

so that the trajectory can cross the origin only if $B = 0$. As in the case $\alpha \geq 2$, all function of the form $x(t) = Re^{i\omega t}$ (with $\omega^2 = \frac{a\alpha}{R^{\alpha+2}}$) in some 2-plane of $\mathbb{R}^N$ is a periodic solution of (1.1). Moreover, if $E < 0$ is the energy, and B is the value of the angular momentum of a non circular solution, a necessary and sufficient condition for the periodicity is that

$$\frac{1}{\pi}\int_{\rho_-}^{\rho_+} \frac{B}{\rho\sqrt{2E\rho^2 + 2a\rho^{2-\alpha} - B^2}} \in \mathbb{Q} \, ,$$

where (ρ_-, ρ_+) is the unique pair of distinct solutions of the equation $2E\rho^2 + 2a\rho^{2-\alpha} - B^2 = 0$.◇

1.V. Variational properties of the periodic solutions in the planar case

As we have shown in Chapter III, when $\alpha = 1$, all solution of (1.1) having negative energy is periodic with (minimal) period $T = \frac{2a\pi}{(-2E)^{\frac{3}{2}}}$. We enclose in this set also the collision solutions appearing for zero angular momenta. Indeed, in this case, the energy integral reduces to

$$\frac{1}{2}\dot{\rho}^2 - \frac{a}{\rho} = E \, ,$$

so that $0 \leq \rho \leq \frac{a}{-E}$, and the time $\frac{T}{2}$ needed in going from $\frac{a}{-E}$ to 0 can be computed as

$$\frac{T}{2} = \int_0^{\frac{a}{-E}} \frac{1}{\sqrt{2E + \frac{2a}{\rho}}} d\rho = \frac{a\pi}{(-2E)^{\frac{3}{2}}} \, ;$$

therefore the maximal existence interval of the solution has length $T = \frac{2a\pi}{(-2E)^{\frac{3}{2}}}$.

It turns out from the discussion of Cap III that, whenever $\alpha = 1$, the set of all the solutions of (1.1) having a fixed energy $E < 0$ is homeomorphic to the ball B^2 of $\mathbb{R}^2$ of radius a, up to shift of the time parameter. Indeed, to every value of the vector P, such that $|P| \leq a$, there corresponds a solution of (1.1) satisfying (1.9). The circular solution

corresponds to the value $P = 0$, and the collision solutions ($B = 0$) to the values of P on the sphere ($|P| = a$).
Of course, because of the relation $T = \frac{2a\pi}{(-2E)^{\frac{3}{2}}}$, for each fixed $T > 0$, the same characterization holds for the set of all the solutions of (1.1) having T as *minimal* period. We shall call a T-periodic *Kepler orbit* an element of this set.

Now we fix a period T; it is a well known fact that each noncollision solution of (1.1) is a critical point of the action integral

$$(V.1) \qquad I_a^1(x) = \int_0^T \frac{1}{2}|\dot{x}|^2 + \frac{a}{|x|} \, ,$$

defined on the open subset $\Lambda = \{x \in H \ / \ x(t) \neq 0 \ \forall t \in \mathbb{R}\}$ of the Sobolev space of T-periodic functions $H = \{x \in H^1_{loc}(\mathbb{R}; \mathbb{R}^2) \ / \ x(t+T) = x(t) \ \forall t \in \mathbb{R}\}$. We can extend I_a^1 over the whole of H, taking the value of the integral in (V.1) if $\frac{1}{|x|}$ is integrable, and $+\infty$ otherwise. As such, I_a^1 is (weakly) lower semicontinous. Observe that $\partial\Lambda$ in H is actually the set of all the collision functions of H: $\partial\Lambda = \{x \in H \ / \ \exists t_0 \ , \ x(t_0) = 0\}$.

The following result has been proved by Gordon, [**27**]:

PROPOSITION 1.V.1. (GORDON). *Let $\alpha = 1$. For every fixed period $T > 0$, each noncollision T-periodic Kepler orbit minimizes the action integral I_a^1 over the open subset of Λ,*

$$\Lambda_0 = \{x \in \Lambda \ / \ deg_0 x \neq 0\} \, .$$

Moreover, the collision T-periodic Kepler orbits minimize I_a^1 over $\partial\Lambda$. The two minima are equal and

$$(V.2) \qquad \min_{\Lambda_0} I_a^1 = \min_{\partial\Lambda} I_a^1 = \frac{3}{2}(2a\pi)^{\frac{2}{3}} T^{\frac{1}{3}} \, .$$

PROOF: First of all, since I_a^1 is coercive on Λ_0, the infimum is positive and it is attained by a function $x \in \Lambda_0 \cup \partial\Lambda$. Now, if $x \in \Lambda_0$, then it solves (1.1), so that it is a noncollision Kepler orbit (one can easily prove that T is the minimal period). If not, then x satisfies (1.1) for every noncollision time, so that it is a collision Kepler orbit. In both cases we obtain

$$(V.3) \qquad I_a^1(x) = \frac{3}{2}(2a\pi)^{\frac{2}{3}} T^{\frac{1}{3}} \, .$$

On the other hand, every T-periodic Kepler orbit satisfies (1.14), and hence minimizes I_a^1 over $\Lambda_0 \cup \partial\Lambda$.$\diamond$

Hence, whenever $\alpha = 1$ we have a countinuum of T-periodic local minima, connecting the circular solution to the collision ones. Now we turn to the case $\alpha > 0$. We recall that, for every $\alpha > 0$, there always exists a circular T-periodic solution: indeed, the function $x(t) = Re^{\frac{2i\pi t}{T}}$ with $R^{\alpha+2} = a\alpha\left(\frac{T}{2\pi}\right)^2$ solves (1.1). Moreover, if we consider the action integral corresponding to the problem,

$$I_a^\alpha(x) = \int_0^T \frac{1}{2}|\dot{x}|^2 + \frac{a}{|x|^\alpha}\,, \tag{V.4}$$

a collision T-periodic solution can be found by minimizing I_a^α over $\partial\Lambda$.

Let us denote

$$x_a^\alpha(t) = \left(a\alpha\left(\frac{T}{2\pi}\right)^2\right)^{\frac{1}{2+\alpha}} e^{\frac{2i\pi t}{T}} \tag{V.5}$$

$$c_a^\alpha(T) = I_a^\alpha(x_a^\alpha) = \left(\frac{2+\alpha}{2\alpha}\right)(2\pi)^{\frac{\alpha}{2+\alpha}}\alpha^{\frac{2}{2+\alpha}}a^{\frac{2}{2+\alpha}}T^{\frac{2-\alpha}{2+\alpha}}\,. \tag{V.6}$$

The following proposition has been proved by Degiovanni and Giannoni in [**25**]:

PROPOSITION 1.V.2. (DEGIOVANNI-GIANNONI). *Let $\alpha > 1$. For every fixed period $T > 0$, the circular solution x_a^α is the strict minimum of I_a^α over Λ_0. Define*

$$\Psi(\alpha) = \frac{\min_{\partial\Lambda} I_a^\alpha}{c_a^\alpha(T)}\,, \tag{V.7}$$

then $\Psi : (0, +\infty) \to (0, +\infty)$ just depend on α and moreover:

$$\begin{aligned} &\Psi \textit{ is increasing if } \alpha \geq 1\\ &\Psi(1) = 1\\ &\lim_{\alpha\to 2^-}\Psi(\alpha) = +\infty\\ &\Psi(\alpha) = +\infty \qquad \textit{if } \alpha \geq 2\,. \end{aligned}$$

PROOF: We first prove that

$$\inf_{\partial\Lambda} I_a^\alpha > \min_{\Lambda_0} I_a^\alpha\,, \quad \forall\alpha > 1\,. \tag{V.8}$$

Indeed, by the strict convexity of the function s^α for $\alpha > 1$, we have that

(V.9)
$$I_a^\alpha(x) \geq \int_0^T \frac{1}{2}|\dot{x}|^2 + aT^{1-\alpha}\left(\int_0^T \frac{1}{|x|}\right)^\alpha = J_a^\alpha(x) \ ,$$

and the strict inequality holds unless x has constant modulus. By arguments similar to those used on the proof of Proposition 1.V.1, one can prove that

(V.10)
$$\min_{\Lambda_0} J_a^\alpha = \min_{\partial\Lambda} J_a^\alpha = c_a^\alpha(T) \ .$$

and it is attained by a set of T-periodic Kepler orbits of an equation of the type of (1.1) with $\alpha = 1$, and the circular solution of this set is x_a^α. Moreover, for the circular function x_a^α we have

$$I_a^\alpha(x_a^\alpha) = J_a^\alpha(x - A\alpha) = c_a^\alpha(T) \ ,$$

while, for the collision solution, we have

$$\min_{\partial\Lambda} I_a^\alpha > \min_{\partial\Lambda} J_a^\alpha = c_a^\alpha(T) \ .$$

Hence (V.8) is proved. We deduce from it that x_a^α is the strict minimum of I_a^α in Λ_0. Indeed it is the only solution of (1.1) having constant modulus and the strict inequality holds in (V.9) holds unless x has constant modulus.

It has been proved in Proposition 1.V.1 that $\Psi(1) = 1$. Ψ is well defined (just depends on α) because of the same homogeneity of both terms $\min_{\partial\Lambda} I_a^\alpha$ and $c_a^\alpha(T)$ in a and T. Moreover one can prove that Ψ is continuous whenever it is finite valued, and that it is increasing for each $\alpha \geq 1$. Therefore, since $\Psi(\alpha) = +\infty$, $\forall\alpha \geq 2$, we have

$$\lim_{\alpha \to 2^-} \Psi(\alpha) = +\infty$$

(see also [25]). The proof is then complete.◇

Now we turn to the case $0 < \alpha < 1$. First of all, let us remark that, because of the strict concavity of the function s^α for $0 < \alpha < 1$, the same arguments used in proving Proposition 1.V.2 can be applied to prove that the function Ψ of (V.7) satisfies

$$\Psi(\alpha) < 1 \ , \qquad \forall 0 < \alpha < 1 \ .$$

Hence, of course, the circular solutions x_a^α do not minimize I_a^α over Λ_0.
On the other hand, we are going to show an other variational characterization of the Kepler orbits. To this aim, let us fix some notations.

In the following, we are going to identify the space E_2 of all the constant functions of H with $\mathbb{R}^2$. For a fixed (large) $R > 0$, consider the class

$$\Gamma = \{\gamma : B^2 \to H \text{ continuous } / \ \gamma_{|S^1} = R \cdot id_{|E_2}\} \ , \tag{V.11}$$

of all the continous deformations of the unit ball of E_2 in H, preserving the value at the boundary. Let us consider $H_0 = \{x \in H \ / \ \int_0^T x = 0\}$. Thanks to Brower's fixed point Theorem we have both

$$\gamma(B^2) \cap H_0 \neq \emptyset \ , \forall \gamma \in \Gamma \ ; \tag{V.12}$$

and

$$\gamma(B^2) \cap \partial\Lambda \neq \emptyset \ , \forall \gamma \in \Gamma \ . \tag{V.13}$$

For any functional I, we say that $x \in H$ *has the saddle point type with respect to I and Γ* if

$$I(x) = \inf_{\gamma \in \Gamma} \sup_{\gamma(B^2)} I \ ,$$

and

$$\exists \gamma \in \Gamma \text{ such that } x \in \gamma(B^2) \text{ and } I(x) = \sup_{\gamma(B^2)} I \ .$$

Then one can easily prove the following fact:

PROPOSITION 1.V.3. *Let $\alpha = 1$. For every period T, all the T periodic Kepler orbits have the saddle point type with respect to I_a^1 and Γ (provided R is chosen sufficiently large). Moreover*

$$c_a^1 = \inf_{\gamma \in \Gamma} \sup_{y \in \gamma(B^2)} I_a^1(y) = \inf_{H_0} I_a^1 = \min_{\partial\Lambda} I_a^1 \ . \tag{V.14}$$

PROOF: As we have pointed out before, the set of all the T-periodic kepler orbits can be parametrized by means of the vector P of (1.9) as the image of a continuous function $f : B^2(\frac{1}{2}) \to H$ (here $B^2(\frac{1}{2})$ is the ball of radius $\frac{1}{2}$), in such a way that $f(0)$ is the circular solution and $f(S^1)$ is the set of all the collision T-periodic Kepler orbits. This function can be obviously extended to an element of Γ (if R is large) in such a way that $I_a^1(\gamma(x)) < c_a^1(T)$ whenever $|x| > \frac{1}{2}$. The proof just follows from this fact, since, obviously

$$\min_{H_0} I_a^\alpha \geq \min_{x \in H_0} \left(\int_0^T |\dot{x}|^2 + \frac{aT^{\frac{2+\alpha}{2}}}{\left(\int_0^T |x|^2\right)^{\frac{\alpha}{2}}} \right) = c_a^\alpha(T) \ , \quad \forall \alpha > 0 \ . \tag{V.15}$$

◇

Next Proposition follows from Proposition 1.V.3 and from the strict concavity of the function s^α for $0 < \alpha < 1$:

PROPOSITION 1.V.4. *Let $0 < \alpha < 1$. Then for every period T, the circular solutions x_a^α have the saddle point type with respect to I_a^α and Γ. Moreover:*

$$c_a^\alpha(T) = \inf_{E_0} I_a^\alpha = \inf_{\gamma\in\Gamma} \sup_{\gamma(B^2)} I_a^\alpha > \inf_{\partial\Lambda} I_a^\alpha \,. \tag{V.16}$$

VI. Collision solutions

In this chapter we shall deal with the case $0 < \alpha < 2$. We say that x is a T periodic collision solution of (1.1) if $x \in H_0^1([0,T];\mathbb{R}^N)$ solves

$$\begin{cases} -\ddot{x}(t) = \frac{a\alpha x(t)}{|x(t)|^{\alpha+2}} & \forall t \in [0,T] \setminus x^{-1}(0) \\ \frac{1}{2}|\dot{x}(t)|^2 - \frac{a}{|x(t)|^\alpha} = E & \forall t \in [0,T] \setminus x^{-1}(0) \\ x(0) = x(T) = 0 & \\ x(t) \neq 0 \quad \text{a.e.} & \end{cases} \tag{VI.1}$$

We are going to show that a collision solution can actually collide just a finite number of times, depending on the value of the energy. Indeed, let T_1 and $T_2 \in [0,T]$ such that $x(T_1) = x(T_2) = 0$ and $x(t) \neq 0$, $\forall t \in (T_1, T_2)$. From the energy integral, since the angular momentum of a collision solution vanishes, we obtain

$$\begin{aligned} \frac{|T_2 - T_1|}{2} &= \int_0^{\left(\frac{a}{-E}\right)^{\frac{1}{\alpha}}} \frac{1}{\sqrt{2}\sqrt{E + \frac{a}{\rho^\alpha}}} d\rho \\ &= \frac{a^{\frac{1}{\alpha}}}{(-E)^{\frac{2+\alpha}{2\alpha}}} \frac{2}{\alpha} \int_0^{\frac{\pi}{2}} |\sin\theta|^{\frac{2}{\alpha}} d\theta \end{aligned} \tag{VI.2}$$

(remember that, from (II.1), the energy E has to be negative).

Moreover, formula (II.1) implies that $|x|^2 \in C^1$ (indeed, from the fact that $x \in H_0^1$ and the conservation of the energy, we have $\frac{1}{|x|^\alpha} \in L^1$). Therefore a T-periodic collision solution satisfies

$$\int_0^T |\dot{x}|^2 = \alpha \int_0^T \frac{a}{|x|^\alpha} \,,$$

so that

$$-E = \left(\frac{2-\alpha}{2+\alpha}\right) \frac{I_a^\alpha(x)}{T} \,. \tag{VI.3}$$

Therefore, for each period T, all the T-periodic collision solutions having T as *minimal* period have the same energy and the same value of the action integral. Since the collision solution minimizing I_a^α over $\partial\Lambda$ has minimal period T, we can conclude from (VI.2) and (1.29) that

$$\text{(VI.4)} \qquad \min_{\partial\Lambda} I_a^\alpha = \left(\frac{2+\alpha}{2-\alpha}\right) \frac{1}{(\sqrt{2}\alpha)^{\frac{2\alpha}{2+\alpha}}} \left(\int_0^{2\pi} |\sin\theta|^{\frac{2}{\alpha}}\right)^{\frac{2\alpha}{2+\alpha}} a^{\frac{2}{2+\alpha}} T^{\frac{2-\alpha}{2+\alpha}} \; .$$

From (V.6) and (V.7) we then deduce that

$$\text{(VI.5)} \qquad \Psi(\alpha) = \frac{4^{1-\alpha}}{(2-\alpha)\alpha^{\frac{\alpha}{2+\alpha}}} \left(\frac{1}{\pi}\int_0^{2\pi} |\sin\theta|^{\frac{2}{\alpha}}\right)^{\frac{2\alpha}{2+\alpha}} \; .$$

Finally, for every period $T > 0$, let us consider the set of all the collision critical levels of I_a^α (i.e. the set of the values of the functional over the T-periodic collision solutions): then this set consists in a sequence, each term of it corresponding to the minimum of I_a^α over the subset of $\partial\Lambda$ of all the T-periodic functions having $\frac{T}{k}$ as minimal period. More precisely, denoting by c_k^0 the kth term of this sequence, we have

$$\text{(VI.6)} \qquad c_k^0 = k^{\frac{2\alpha}{2+\alpha}} c_1^k = k^{\frac{2\alpha}{2+\alpha}} \min_{\partial\Lambda} I_a^\alpha \; .$$

Let us remark that, for each $\alpha > 0$, we have

$$\text{(VI.7)} \qquad c_a^\alpha(T) = \min_{H_0} I_a^\alpha < \min_{H_0\cap\partial\Lambda} I_a^\alpha = c_2^0 = 2^{\frac{2\alpha}{2+\alpha}} \min_{\partial\Lambda} I_a^\alpha \; .$$

PART 2. THE FIXED PERIOD PROBLEM: EXISTENCE RESULTS

In this part we are concerned with the problem of the existence of one solution to the problem

$$(P_T) \qquad \begin{cases} -\ddot{x} = \nabla_x F(x,t) & \\ x(t+T) = x(t) & \forall t \in \mathbb{R} \\ x(t) \neq 0 & \forall t \in \mathbb{R}\,, \end{cases}$$

where the period $T > 0$ is prescribed. To this end, we shall study the critical points of the functional

$$(2.1) \qquad I(x) = \int_0^T \frac{1}{2}|\dot{x}|^2 - F(x,t)\,.$$

Since the potential F will be singular at the origin, the natural domain of I is the subset

$$(2.2) \qquad \Lambda = \{x \in H \;/\; x(t) \neq 0 \;\forall t \in \mathbb{R}\}\,,$$

of the Sobolev space of T-periodic functions $H = \{x \in H^1_{loc}(\mathbb{R};\mathbb{R}^N) \;/\; x(t+T) = x(t) \;\forall t \in \mathbb{R}\}$.

Let us first consider our model potential $F(x,t) = \frac{-a}{|x|^\alpha}$: then the variational approach to (P_T) presents two kinds of problems:

(a) the functional I is not coercive;

(b) the domain Λ is open in H
(the Palais-Smale's sequences can converge to $\partial\Lambda$).

As far as the cases $\alpha > 1$ and $0 < \alpha < 1$ are concerned, we are going to show that suitable minimax techniques can be introduced in order to overcome problem (a). Moreover, in both cases critical points will appear at levels where no collision solution is allowed (problem (b)). On the other hand, whenever $\alpha = 1$, all these methods fail. In facts, we shall see in chapter III that the counter part to the existence of a continuum of periodic solution is its instability under perturbations.

I.1 The strong force case ($\alpha \geq 2$).

The notion of Strong Force has been introduced by Gordon ([28]):

$$(SF) \qquad \begin{cases} \exists U \in C^1(\mathbb{R}^N \setminus \{0\};\mathbb{R}) \,, \exists r > 0 \text{ such that} \\ \lim_{x\to 0} U(x) = +\infty \,, \\ -F(x,t) \geq |\nabla U(x)|^2 \,, \qquad \forall |x| < r \,. \end{cases}$$

When the potential F satisfies the (SF) condition, the associated functional I can not be finite in a neighbourhood of $\partial\Lambda$. More precisely we have the

PROPOSITION I.1.1. (GORDON). *Let $F \in C(\mathbb{R}^N \setminus \{0\} \times [0,T];\mathbb{R})$ satisfy the (SF) condition in addition to*

$$(I.1.1) \qquad F(x,t) \leq 0 \,, \qquad \forall (x,t) \in \mathbb{R}^N \setminus \{0\} \times [0,T]$$

Then

$$\lim_{x\to\partial\Lambda} I(x) = +\infty$$

PROOF: Assume on the contrary that there exists a sequence $(x_n)_n$ in Λ such that

$$x_n \to \overline{x} \in \partial\Lambda \,,$$

and

$$\limsup_{n\to\infty} I(x_n) < +\infty \,.$$

Then we first deduce from (I.1.1) that $\int_0^T |\dot{x}_n|^2 \leq 2I(x_n)$, so that $\limsup_{n\to\infty} \int_0^T |\dot{x}|^2 < +\infty$. Moreover, the limit function $\overline{x}$ can not be identically zero.
Now let t_0, $t_1 \in [0,T]$ be such that $\overline{x}(t_0) = 0$ and $\overline{x}(t_1) \neq 0$. From (SF) we deduce that

$$|U(x_n(t_1)) - U(x_n(t_0))| = \int_{t_0}^{t_1} \frac{d}{dt} U(x_n(t))$$
$$\leq \left(\int_{t_0}^{t_1} |\nabla U(x_n(t))|^2\right)^{\frac{1}{2}} \left(\int_{t_0}^{t_1} |\dot{x}_n|^2\right)^{\frac{1}{2}} \leq \left(\int_{t_0}^{t_1} |\nabla U(x_n(t))|^2\right)^{\frac{1}{2}} (2I(x_n))^{\frac{1}{2}} \,,$$

hence

$$\int_0^T -F(x_n,t) \geq \int_{t_0}^{t_1} -F(x_n,t) \geq \frac{|U(x_n(t_1)) - U(x_n(t_0))|^2}{2I(x_n)} \to +\infty \,,$$

since x_n converges uniformly to $\overline{x}$. Remark that the same proof works whenever x_n converges to $\overline{x}$ in the weak topology of H.◇

REMARK: When F is of the form $F(x) = \frac{-a}{|x|^\alpha}$, the (SF) condition holds if and only if $\alpha \geq 2$.◇

We are going to deal with potentials satisfying (SF) in addition to the following assumptions

$$(H) \qquad \begin{aligned} &F(x,t) < 0 \qquad \forall (x,t) \in \mathbb{R}^N \setminus \{0\} \times [0,T] \\ &\lim_{|x|\to+\infty} F(x,t) = 0 , \qquad \text{uniformly in } t \\ &\lim_{|x|\to+\infty} \nabla_x F(x,t) = 0 , \qquad \text{uniformly in } t . \end{aligned}$$

Let us recall a basic definition in critical point theory. We say that I fulfils the *Palais-Smale condition* at the level c if

$$(PS_c) \qquad \begin{aligned} &\text{every sequence satisfying} \\ &\lim_{n\to+\infty} I(x_n) = c \\ &\lim_{n\to+\infty} dI(x_n) = 0 \\ &\text{possesses a converging subsequence.} \end{aligned}$$

We shall call a Palais-Smale's at the level c a sequence (x_n) such that

$$\lim_{n\to+\infty} I(x_n) = c$$
$$\lim_{n\to+\infty} dI(x_n) = 0 .$$

A Palais-Smale sequence is a sequence of *almost critical points* of I. This notion is motivated by the fact that the minimax arguments we are going to apply provide the existence of Palais-Smale's sequences.

Assume that (H) holds; then, of course, I does not satisfy (PS_0) condition. Indeed, just take a sequence of diverging constant functions. However, 0 is the only level where the (PS) condition fails:

PROPOSITION I.1.2. *Let F satisfies (SF) and (H). Then I fulfils the (PS) condition in Λ at any positive level.*

PROOF: The proof is contained in [**1**]. We sketch it here for the reader convenience. Let $(x_n)_n$ be a Palais-Smale sequence in Λ at level $c > 0$, that is

$$
\begin{aligned}
&\text{(I.1.2)} \qquad && I(x_n) \to c \\
&\text{(I.1.3)} \qquad && -\ddot{x}_n - \nabla_x F(x_n, t) \to 0\,, \qquad \text{in } H^{-1}\,.
\end{aligned}
$$

We first remark that $||\dot{x}||_{L^2}$ is bounded, so that if the sequence of the mean values of x_n is bounded, then (I.1.3), together with standard compactness arguments, leads to the existence of a converging subsequence (remember that, by Proposition 2.I.1, a sequence at bounded level can not converge to $\partial\Lambda$). On the other hand, assuming on the contrary that the sequence $\int_0^T x_n$ is unbounded, we obtain that $\liminf_{n\to+\infty} \inf_t |x_n(t)| = +\infty$; from this fact and (I.1.2), (I.1.3) we can conclude that $I(x_n) \to 0$.◇

A more detailed description of the behavior of the functional near the level of lack of compactness is given by the following

PROPOSITION I.1.3. *Assume that (H) holds. Then there exists $\varepsilon > 0$ such that the sub-level $I^\varepsilon = \{x \in \Lambda \;/\; I(x) < \varepsilon\}$ can be contracted in a continuous way into the set of large constant functions, without crossing the boundary of Λ.*

PROOF: From (H), we deduce that for every $M > 0$ there exists $\varepsilon > 0$ such that $I^\varepsilon \subseteq \{x \in \Lambda \;/\; |x(t)| > M\;, \forall t \in [0,T]\}$. Therefore, for large values of M (and hence small εs), the homotopy $h(x,\sigma) = (1-\sigma)x + \sigma\frac{1}{T}\int_0^T x$ can be performed, contracting I^ε in a set of large constant functions, without crossing the boundary $\partial\Lambda$.

One of the first results about existence of periodic solution to singular dynamical systems is the following Theorem 2.I.1, due to Ambrosetti and Coti Zelati ([**1**]). They make use of a result of Fadell and Husseini about the category of Λ in Λ. We shall not enter in the details of their approach here; however, we wish to point out that from the result of Fadella nd Husseini one deduces the existence of sets which cannot be contracted in a continuous way into sets of large constant functions, without crossing the boundary of Λ. One then obtains the existence of Palais-Smales's sequence at positive levels by deformations techniques.

THEOREM I.1.1. (AMBROSETTI-COTI ZELATI). *Assume that $F \in C^1(\mathbb{R}^N \setminus \{0\} \times \mathbb{R}; \mathbb{R})$ is T periodic in t and satisfies (SF) as well as (H). Then (P_T) has infinitely many solutions.*

I.2. The case $\alpha > 1$

The approach used in proving Theorem I.1.1 just work when $\lim_{x\to\partial\Lambda} I(x) = +\infty$. On the other hand, when $\alpha < 2$, the functional associated to a potential of the form $F(x) = \frac{-a}{|x|^\alpha}$ has always finite infimum near the boundary of Λ (cfr 1.VI). In order to overcome this difficulty, let us state an abstract result.

The following definition has been implicitly given by Greco ([29]):

DEFINITION I.2.1. *A closed subset A of Λ is contractible (into a set of large constant functions) if there exists a continuous homotopy $h : A \times [0,1] \to \Lambda$ such that $h(\cdot, 0) = id_{|A}$ and $h(A,1) \subseteq E_N$, where $E_N \subseteq H$ is the subset of the constant functions. A subset of Λ is non contractible if it is not contractible, that is if for every continuous homotopy $h : A \times [0,1] \to H$ such that $h(\cdot,0) = id_{|A}$ and $h(A,1) \subseteq E_N$, then $h(A \times [0,1]) \cap \partial\Lambda \neq \emptyset$. We shall denote*

$$\mathcal{H}_0(A) = \{h : A \times [0,1] \to H \text{ continuous } / \ h(\cdot,0) = id_{|A} \ , \ h(A,1) \subseteq E_N\} \ . \tag{I.2.1}$$

THEOREM I.2.1. *Let $I \in C^1(\Lambda, \mathbb{R})$ admit a lower semicontinuous extension $\overline{I} : H \to \mathbb{R} \cup \{+\infty\}$. Assume that:*

(*i*) $\overline{I}(x) > 0 \qquad \forall x \in H$;

(*ii*) *0 is the only level where the (PS) condition fails;*

(*iii*) *there is $\varepsilon > 0$ such that the sublevel I^ε is contractible;*

(*iv*) *there is a non contractible compact set $A \subseteq \Lambda$ such that:*

$$\sup_A I \le \inf_{\partial\Lambda} \overline{I} \ .$$

Then I has at least one critical point in Λ at positive level.

PROOF: The proof can be carry out by the usual deformation techniques (see [29]). Indeed, assuming that I does not admit positive critical levels, an homotopy (gradient flow) could be performed contracting the sublevel $\{x \in \Lambda \ / \ I(x) \le \inf_{\partial\Lambda} \overline{I}\}$ into I^ε, in such a way that the value of the functional strictly decreases along the homotopy. Therefore, by (iii), the compact set of (iv) should be contracted into I^ε and then into a set of constant functions, without crossing $\partial\Lambda$, in contradiction with Definition 2.I.1.
Let us point out that a positive critical level is expressed by

$$\inf_{A\in\mathcal{A}} \sup_A I \ , \tag{I.2.2}$$

where

(I.2.3) $$\mathcal{A} = \{A \subseteq \Lambda \text{ compact } / \ A \text{ is non contractible }, \sup_A I \leq \inf_{\partial\Lambda} I\} .$$

◇

Now, natural questions are if Λ contains non contractible sets and how to compute their maximal level. To this aim we prove the following

PROPOSITION I.2.1. *Let $(e_1, ..., e_N)$ be an orthonormal system of $\mathbb{R}^N$ and let $S^{N-2}(e_1) = \{x \in \mathbb{R}^N \ / \ |x - e_1| = 1 \ , \ x \cdot e_N = 0\}$. Define*

(I.2.4) $$A_0 = \bigcup_{x \in S^{N-2}(e_1)} \{4x + (x - e_1) \cos \frac{2\pi}{T} t + e_N \sin \frac{2\pi}{T} t\}$$

Then A_0 is a noncontractible set.

PROOF: Let $f : S^{N-2}(e_1) \to \Lambda$ be the continuous parametrization of A_0 defined by

$$f(x)(t) = 4x + (x - e_1) \cos \frac{2\pi}{T} t + e_N \sin \frac{2\pi}{T} t \ , \quad \forall x \in S^{N-2}(e_1) \ ,$$

and let $F : S^{N-2}(e_1) \times S^1 \to \mathbb{R}^N \setminus \{0\}$ be defined as

$$F(x, e^{it}) = f(x)(t) \ .$$

F is continuous and has $deg_0 F = 1$ (or $deg_0 F = -1$, depending on the orientation chosen for $\mathbb{R}^N$). Corresponding to each $h \in \mathcal{H}_0(A_0)$, there are the extensions of f and F, respectively $\tilde{f} : S^{N-2}(e_1) \times [0,1] \to H$ and $\tilde{F} : S^{N-2}(e_1) \times B^1 \to \mathbb{R}^N$, given by

$$\tilde{f}(x, \sigma) = h(f(x), \sigma) \ ,$$
$$\tilde{F}(x, \tau e^{it} = \tilde{f}(x, 1 - \tau) \ .$$

The second definition makes sense since, from the definition of $\mathcal{H}_0(A_0)$, one has $\tilde{f}(x,1) \in E_N, \forall x \in S^{N-2}(e_1)$. From the fact that $deg_0 F = 1$ one deduces that $0 \in \tilde{F}(S^{N-2}(e_1) \times B^1)$, that is that $h(A, [0,1]) \cap \partial\Lambda \neq \emptyset$. Since h was arbitrarily chosen in $\mathcal{H}_0(A_0)$, one finally proves that A_0 is a non contractible set.◇

REMARK I.2.1: One obviously has that ρA_0 is non contractible for every $\rho > 0$. Remark that, in the radial case, we have

(I.2.5) $$\inf_{\rho > 0} \sup_{\rho A_0} I_a^\alpha = \sup_{4 R_a^\alpha A_0} I_a^\alpha = c_a^\alpha(T) \ ,$$

where $(R_a^\alpha)^{2+\alpha} = a\alpha \left(\frac{T}{2\pi}\right)^2$.◇

The following theorem has been proved by Degiovanni and Giannoni ([**25**]).

THEOREM I.2.2. (DEGIOVANNI-GIANNONI). *Let $F \in C^1(\mathbb{R}^N \setminus \{0\} \times \mathbb{R}; \mathbb{R})$ be T-periodic in t and satisfy (H) in addition to*

$$\exists \alpha \; 1 < \alpha < 2 \, , \exists b \geq a > 0 \, ,$$

$$\frac{a}{|x|^\alpha} \leq -F(x,t) \leq \frac{b}{|x|^\alpha} \qquad \forall (x,t) \in \mathbb{R}^N \setminus \{0\} \times \mathbb{R} \, , \tag{I.2.6}$$

$$\left(\frac{b}{a}\right)^{\frac{2}{2+\alpha}} \leq \Psi(\alpha) \, . \tag{I.2.7}$$

(the function Ψ is defined in (V.7) of Part 1. Then (P_T) has at least one solution.

PROOF: The proof is a direct consequence of Theorem I.2.1. We proved in the previous section that (H) implies that (i), (ii) and (iii) of Theorem 2.I.2 are fulfilled. Moreover, by Proposition I.2.1 and the Remark, the set $A = 4RA_0$, with $R^{2+\alpha} = a\alpha\left(\frac{T}{2\pi}\right)^2$ is non contractible and satisfies (by (I.2.5)):

$$\min_{\partial\Lambda} I \geq \min_{\partial\Lambda} I_a^\alpha \geq c_b^\alpha(T) = \sup_A I_b^\alpha \geq \sup_A I \, , \tag{I.2.8}$$

indeed, (I.2.7) leads to $\min_{\partial\Lambda} I_a^\alpha \geq c_b^\alpha(T)$. Therefore , the sublevel $\{x \in \Lambda \; / \; I(x) \leq \inf_{\partial\Lambda} I\}$ contains at least one non contractible set; hence Theorem I.2.1 can be applied, finding a critical point of I in Λ, that is a solution to (P_T).$\diamond$

2.II. The case $0 < \alpha < 1$

As we have remarked in 1.V, when $F(x) = \frac{-a}{|x|^\alpha}$, $a > 0$, and $0 < \alpha < 1$, the variational problem associated to (P_T) can be handled bay means of saddle point theorem type arguments. Let us start this chapter by the variational characterization of the circular solutions in the case $0 < \alpha < 1$:

PROPOSITION II.1. *Let $F(x) = \frac{-a}{|x|^\alpha}$, with $a > 0$ and $0 < \alpha < 1$. For a fixed $R > 0$, consider the class*

$$\Gamma = \{\gamma : B^N \to H \text{ continuous } / \ \gamma_{|S^{N-1}} = Rid_{E_N}\} .$$

Then, for any large R,

$$c_a^\alpha(T) = \inf_{\gamma \in \Gamma} \sup_{\gamma(B^N)} I_a^\alpha .$$

Here E_N is the subset of all the constant functions of H: up to an isomorphism we have $E_N = \mathbb{R}^N$.

PROOF: It is a generalization in higher dimension of the planar case of Proposition V.4. of Part 1 First, by Brower's Theorem we have

$$\gamma(B^N) \cap H_0 \neq \emptyset , \qquad \forall \gamma \in \Gamma ,$$

(H_0 is the subspace of H of all the function having zero mean value) and hence, by (V.15) of part 1,

$$\inf_{\gamma \in \Gamma} \sup_{\gamma(B^N)} I_a^\alpha \geq \inf_{H_0} I_a^\alpha = c_a^\alpha(T) .$$

Now we have to prove the reversed inequality, that is, we have to find γ such that $c_a^\alpha(T) = \sup_{\gamma(B^N)} I_a^\alpha$. If $N = 2$, such a $\gamma : B^2 \to H$ was found using the property of the set of the T-periodic Kepler orbits of the case $\alpha = 1$ to be a 2-manifold homeomorphic to the ball (cfr Proposition 1.V.4). More precisely, let us consider the set of all the K-orbits of the problem

$$\text{(II.1)} \qquad \begin{cases} -\ddot{x} = \alpha a \left(\alpha a \left(\frac{T}{2\pi}\right)^2\right)^{\frac{1-\alpha}{2+\alpha}} \frac{x}{|x|^3} & x \in \mathbb{R}^2 \\ x(t+T) = x(t) . \end{cases}$$

Taking into account of the results of Part 1, chapter III, we know that the set of all the solutions (possibly crossing the origin) of (II.1) is (up to the S^1 symmetry) a 2-manifold homeomorphic to the ball B^2. Moreover, each K-orbit x verifies

$$\text{(II.2)} \qquad \int_0^T |\dot{x}|^2 = T\left(\frac{2\pi}{T}\right)^{\frac{2\alpha}{2+\alpha}} (\alpha a)^{\frac{2}{2+\alpha}}$$

$$\text{(II.3)} \qquad \int_0^T \frac{1}{|x|} = T\left(\frac{1}{\alpha a}\left(\frac{2\pi}{T}\right)^2\right)^{\frac{1}{2+\alpha}} .$$

Now, let us consider the continuous $\gamma : B^2 \to H$ defined as

$$\text{(II.4)} \quad \gamma(z)(t) = \begin{cases} \left(\alpha a\left(\frac{T}{2\pi}\right)^2\right)^{\frac{1}{2+alpha}} e^{2\pi i t} & \text{if } z = 0 \\ \text{elliptic T-periodic Kepler orbit with symmetry axis } \frac{z}{|z|} & \text{if } 0 < |z| < \frac{1}{2} \\ \text{collision T-periodic Kepler orbit} & \text{if } |z| = \frac{1}{2} \\ 2(1-|z|)\gamma\left(\frac{z}{2|z|}\right)(t) + (2|z|-1)\frac{z}{|z|}2\left(\alpha a\left(\frac{T}{2\pi}\right)^2\right)^{\frac{1}{2+\alpha}} & \text{if } \frac{1}{2} \leq |z| \leq 1 ; \end{cases}$$

We then have

$$I(\gamma(z)) \leq \int_0^T |\dot{\gamma(z)}|^2 + aT^{1-\alpha}\left(\int_0^T \frac{1}{|\gamma(z)|}\right)^\alpha \leq c_a^\alpha(T)$$

Moreover, γ admits a continuous extension $\overline{\gamma} : B^2 \times B^{N-2}$ as

$$\text{(II.5)} \qquad \overline{\gamma}(z,y) = \begin{cases} \gamma(z) + Cy & \text{if } |y| \leq \frac{1}{2} \\ 2(1-|y|)g(z, \frac{y}{2|y|}) + (2|y|-1)C\frac{y}{|y|} & \text{if } \frac{1}{2} \leq |y| \leq 1 . \end{cases}$$

It is clear that, when the constant C is chosen sufficiently large, we have, for every $(z,y) \in B^2 \times B^{N-2}$,

$$\text{(II.7)} \qquad I(\gamma(z,y)) \leq \int_0^T |\gamma(\dot{z},y)|^2 + aT^{1-\alpha}\left(\int_0^T \frac{1}{|\gamma(z,y)|}\right)^\alpha \leq c_a^\alpha(T)$$

The proof is then complete, since $B^2 \times B^{N-2}$ is homeomorphic to the ball B^N.◇

The following results is on the line of Theorem 2.I.3; it has been proved in [**34**]:

THEOREM 2.II.1. *Let $F \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfy (H) in addition to*

$$\exists\alpha\,,\alpha_1\,,\alpha_2 > 0\ \ \alpha_1 \le \alpha \le \alpha_2 < 1\,, \exists b \ge a > 0\,,$$

$$\frac{a}{|x|^\alpha} \le -F(x) \le \frac{b}{|x|^\alpha} \qquad \forall x \in \mathbb{R}^N \setminus \{0\}\,, \tag{II.7}$$

$$-\alpha_1 \le \nabla F(x)\cdot x \le -\alpha_2 F(x) \qquad \forall x \in \mathbb{R}^N \setminus \{0\};. \tag{II.8}$$

Then there exists a function $\overline{\Psi} : (0,1) \to \mathbb{R}$ such that if

$$\frac{b(2-\alpha_1)(2+\alpha_2)}{a(2+\alpha_1)(2-\alpha-2)} < \overline{\Psi}(\alpha)\,, \tag{II.9}$$

then (P_T) has at least one solution. Moreover $\overline{\Psi}$ fulfils the properties:

$$\overline{\Psi}(\alpha) > 1\,, \qquad \forall\, 0 < \alpha < 1$$
$$\overline{\Psi}(0) = \overline{\Psi}(1) = 1\,.$$

PROOF: We first apply Rabinowitz's Saddle Point Theorem ([8]) with Γ as infsup class, obtaining the existence of either a solution to (P_T) or a *generalized* solution of (P_T). We say that x is a generalized solution of (P_T) if $x \in H^1_0([0,T]; \mathbb{R}^N)$ satisfies

$$\begin{cases} -\ddot{x}(t) = \nabla F(x(t))\,, & \forall t \notin x^{-1}(0) \\ \frac{1}{2}|\dot{x}(t)| + F(x(t)) = E\,, & \forall t \notin x^{-1}(0) \\ x(t) \ne 0\,, & \text{a.e. in } [0,T]\,. \end{cases} \tag{II.10}$$

Indeed, although I is singular, we apply the Saddle Point Theorem to the one parameter family of functionals of the form

$$I_\varepsilon(x) = \int_0^T \frac{1}{2}|\dot{x}|^2 - F_\varepsilon(x)\,,$$

where $F_\varepsilon \in C^1(\mathbb{R}^N; \mathbb{R})$ are regular truncations of F such that:

$$F_\varepsilon(x) \;=\; F(x) \qquad \text{if } \ |x| \ge \varepsilon \tag{II.11}$$

$$0 \le \nabla F_\varepsilon(x)\cdot x \;\le\; -\alpha_2 F_\varepsilon(x) \qquad \forall\, x \in \mathbb{R}^N \tag{II.12}$$

$$-F_\varepsilon(x) \;\le\; \frac{b}{|x|^\alpha} \qquad \forall\, x \in \mathbb{R}^N \tag{II.13}$$

$$\begin{cases} \text{there is a non decreasing } f : \mathbb{R} \to \mathbb{R}\,, \text{ with} \\ f(|x|) \ge \frac{a}{\varepsilon^\alpha} \qquad \text{if } |x| \le \varepsilon \\ \text{such that } -F_\varepsilon(x) \;\ge\; f(|x|) \qquad \forall x \in \mathbb{R}^N\,. \end{cases} \tag{II.14}$$

Since the critical levels corresponding to Γ are positive and, by Proposition 2.I.2 (since (H) holds)), I_ε fulfils the (PS) condition at any positive level, the Saddle Point Theorem can be applied to I_ε, for small values of ε.
Hence, for every ε sufficiently small, a critical point x_ε of I_ε is found such that

$$c_a^\alpha(T) \leq I_\varepsilon(x_\varepsilon) \leq I(x_\varepsilon) \leq c_b^\alpha(T)\ . \tag{II.15}$$

Now, since $(x_\varepsilon)_\varepsilon$ is uniformly bounded, it possesses a sequence $(x_{\varepsilon_n})_n$ weakly convergent to some limit x in H. Since the x_ε satisfy

$$\begin{cases} -\ddot{x}_\varepsilon = \nabla F_\varepsilon(x_\varepsilon) \\ \frac{1}{2}|\dot{x}_\varepsilon|^2 + F_\varepsilon(x_\varepsilon) = E_\varepsilon \\ x(t+T) = x(t)\ , \end{cases} \tag{II.16}$$

and the energies E_ε have a bound independent of ε, we can conclude that the weak limit x is either a solution or a generalized solution of (P_T), and it satisfies (II.10) with $E = \lim_{n\to+\infty} E_{\varepsilon_n}$. Of course, by (II.15) and the weak lower semicontinuity of I, we have

$$I(x) \leq c_b^\alpha(T)\ . \tag{II.17}$$

We claim that the convergence actually hold in the strong topology of H, so that, from (II.16), we can conclude that

$$\begin{aligned} \int_0^T |\dot{x}|^2 &= \lim_{n\to+\infty} \int_0^T |\dot{x}_{\varepsilon_n}|^2 \\ \int_0^T F(x) &= \lim_{n\to+\infty} \int_0^T F_{\varepsilon_n}(x_{\varepsilon_n})\ . \end{aligned} \tag{II.18}$$

In order to prove the claim, we are going to prove that, setting $\Omega_\varepsilon = \{t \in [0,T]\ /\ |x(t)| < \varepsilon\}$, there are constants C_1 and C_2 independent of ε such that

$$\begin{aligned} \int_{\Omega_\varepsilon} |\dot{x}|^2 &\leq C_1\varepsilon^\alpha + C_2\varepsilon^{\frac{2-\alpha}{2}} \\ \int_{\Omega_\varepsilon} |\dot{x}_\varepsilon|^2 &\leq C_1\varepsilon^\alpha + C_2\varepsilon^{\frac{2-\alpha}{2}}\ . \end{aligned} \tag{II.19}$$

This fact, together with the uniform convergence of x_{ε_n} to x in $[0,T]\setminus\Omega_\varepsilon$, will prove the claim. From (II.7), (II.17) and Tchebicev's inequality we have

$$meas\ \Omega_\varepsilon \leq C_3\varepsilon^\alpha\ ; \tag{II.20}$$

Now, from (II.8) (II.10), and (II.12), (II.16) we deduce

$$\text{(II.21)}\qquad \begin{aligned} \frac{d^2}{dt^2}|\dot{x}|^2 &\geq 2E - (2-\alpha_2)F(x) \\ \frac{d^2}{dt^2}|\dot{x}_\varepsilon|^2 &\geq 2E_\varepsilon - (2-\alpha_2)F_\varepsilon(x_\varepsilon) \ , \end{aligned}$$

so that local maxima can not be in Ω_ε (if ε is small). Since the kinetic part are bounded, we can then deduce that the number of connected components of Ω_ε has an upper bound independent of ε; moreover, the energy integral implies that

$$\text{(II.22)}\qquad |x(t)\cdot\dot{x}(t)| \leq |x(t)||\dot{x}(t)| \leq C_4\varepsilon^{\frac{2-\alpha}{2}} \ , \quad \forall t \in \Omega_\varepsilon \ ,$$

(and the same estimate holds for x_ε). Therefore, from (II.20), the integration of (II.21) leads to

$$\text{(II.23)}\qquad \int_{\Omega_\varepsilon} -F(x) \leq C_5\varepsilon^\alpha + C_7\varepsilon^{\frac{2-\alpha}{2}} \ ,$$

and the same estimate holds for x_ε. By the energy integrals and (II.23), (II.19) is easily proved.

Hence we have found a generalized solution x to (P_T) satisfying the further estimate

$$\text{(II.24)}\qquad c_a^\alpha(T) \leq I(x) \leq c_b^\alpha(T) \ .$$

Next step consists in showing that a function $\overline{\Psi}$ exists such that if $\frac{b}{a}\frac{\alpha_2}{\alpha_1} < \overline{\Psi}(\alpha)$, then (P_T) does not admit generalized solutions satisfying (II.24).
Let us call $\rho(t) = |x(t)|$: then ρ satisfies

$$\text{(II.25)}\qquad \begin{cases} \frac{1}{2}\ddot{\rho^2} = 2E + (2-\alpha)a\frac{c(t)}{\rho^\alpha} & \\ \rho(t+T) = \rho(t) & \forall t \in \mathbb{R} \\ \rho(t) \geq 0 & \forall t \in \mathbb{R} \ , \end{cases}$$

where

$$\text{(II.26)}\qquad c(t) = (-2F(x(t)) - \nabla F(x(t))\cdot x(t))\left(\frac{|x(t)|^\alpha}{a(2-\alpha)}\right) \ ;$$

c is T-periodic and, by (II.7) and (II.8), has the bounds

$$\text{(II.27)} \qquad \frac{2-\alpha_2}{2-\alpha} \leq c(t) \leq \frac{b(2-\alpha_1)}{a(2-\alpha)} \qquad \forall t \in \mathbb{R} .$$

We point out that, from the energy integral, $\dot{\rho}^2$ is absolutely continuous. Therefore, from (II.10), x satisfies

$$\int_0^T |\dot{x}|^2 = \int_0^T \nabla F(x) \cdot x ,$$

hence, from (II.8) and (II.24) we obtain

$$\text{(II.28)} \qquad \frac{2-\alpha_2}{2+\alpha_2}\frac{c_a^\alpha(T)}{T} \leq -E \leq \frac{2-\alpha_1}{2+\alpha_1}\frac{c_b^\alpha(T)}{T} .$$

Setting $\mu(t) = \left(\frac{2-\alpha}{2+\alpha}\frac{c_1^\alpha(1)}{-ET^2}\right)^{\frac{1}{2}} \rho(Tt)$, we find that μ solves

$$\text{(II.29)} \qquad \begin{cases} \frac{1}{2}\ddot{\mu^2} = -2\frac{2-\alpha}{2+\alpha}c_1^\alpha(1) + \frac{(2-\alpha)h(t)}{\mu^\alpha} \\ \mu(t+1) = \mu(t) \\ \mu(t) \geq 0 , \end{cases}$$

where

$$h(t) = a\left(\frac{2-\alpha}{2+\alpha}\frac{c_1^\alpha(1)}{-ET^2}\right)^{\frac{2+\alpha}{2}} T^2 c(tT) ;$$

therefore, from (II.27) and (II.28) we obtain

$$\text{(II.30)} \qquad \frac{(2-\alpha_1)(2+\alpha_2)b}{(2-\alpha_2)(2+\alpha_1)a} \leq h(t) \leq \frac{(2-\alpha_2)(2+\alpha_1)a}{(2-\alpha_1)(2+\alpha_2)b} .$$

Finally, since $\dot{\rho}^2$ is absolutely continuous, so is $\dot{\mu}^2$. Next proposition will end the proof:

PROPOSITION II.2. *For every $0 < \alpha < 1$ there exists $h_0(\alpha) > 0$ such that, if $h \in L^\infty([0,1])$ has $||h|_\infty - 1| < h_0(\alpha)$, then the problem*

$$\text{(II.31)} \qquad \begin{cases} \frac{1}{2}\ddot{\mu^2} = -2\frac{2-\alpha}{2+\alpha}c_1^\alpha(1) + \frac{(2-\alpha)h(t)}{\mu^\alpha} \\ \mu(0) = \mu(1) = 0 \\ \mu(t) \geq 0 \\ \dot{\mu}^2 \text{ absolutely continuous;} , \end{cases}$$

has no solution.

PROOF OF PROPOSITION II.2: First of all, the following problem has no solution with absolutely continuous $\dot{\mu}^2$:

$$
\text{(II.32)} \qquad \begin{cases} \frac{1}{2}\ddot{\mu}^2 = -2\frac{2-\alpha}{2+\alpha}c_1^\alpha(1) + \frac{(2-\alpha)}{\mu^\alpha} \\ \mu(t+1) = \mu(1) = 0 \\ \mu(0) = \mu(1) = 0 \\ \mu(t) \geq 0 \\ \dot{\mu}^2 \text{ absolutely continuous }, \end{cases}
$$

Indeed, if a solution exists, it should be the modulus of a generalized solution of

$$
\text{(II.33)} \qquad \begin{cases} -\ddot{x} = \frac{\alpha x}{|x|^\alpha} \\ \frac{1}{2}|\dot{x}| - \frac{1}{|x|^\alpha} = -\frac{2-\alpha}{2+\alpha}c_1^\alpha(1) \\ x(0) = x(1) = 0 \\ x(t) \neq 0 \qquad \text{a.e. in } [0,1] \, . \end{cases}
$$

at level

$$
\int_0^1 |\dot{x}|^2 + \frac{1}{|x|^\alpha} = c_1^\alpha(1) \, .
$$

On the other hand, we know from 1.VI that 1-periodic collision solutions satisfying (II.33) can appear only at the levels

$$
k^{\frac{2\alpha}{2+\alpha}} \min_{\partial\Lambda} \left(\int_0^1 |\dot{x}|^2 + \frac{1}{|x|^\alpha} \right) \ , \qquad k = 1, 2, \cdots \, .
$$

Moreover, as we have pointed out in 1.VI, we have

$$
\min_{\partial\Lambda} \left(\int_0^1 |\dot{x}|^2 + \frac{1}{|x|^\alpha} \right) < c_1^\alpha(1) < 2^{\frac{2\alpha}{2+\alpha}} \min_{\partial\Lambda} \left(\int_0^1 |\dot{x}|^2 + \frac{1}{|x|^\alpha} \right) \, .
$$

Therefore (II.32) does not admit any solutions.
Now we turn to the proof of Proposition 2.II.2; to this aim, we assume on the contrary that there exists a sequence h_n in L^∞ such that

$$
\text{(II.34)} \qquad h_n \to 1 \text{ uniformly}
$$

and such that, for every $n \in \mathbb{N}$, (II.31) has a solution μ_n with absolutely continuous $\dot{\mu}_n^2$. Of course the proof will be complete when we shall prove that μ_n has a converging subsequence such that $\ddot{\mu}_n^2$ converges in the strong L^1 topology: as a matter of facts the limit should satisfy (II.33). We first deduce that $\ddot{\mu}_n^2$ is bounded in L^1, and therefore μ_n has a subsequence uniformly converging to some μ. By arguments similar to the ones of the Proof of Theorem 2.II.1, we can prove that, setting $\Omega_\varepsilon = \{t \in [0,1] \,/\, \mu(t)| < \varepsilon\}$, there are constants C_1 and C_2 independent of ε such that

$$\int_{\Omega_\varepsilon} \frac{1}{\mu^\alpha} \leq C_1\varepsilon^\alpha + C_2\varepsilon^{\frac{2-\alpha}{2}} \ ;$$

and this fact implies the convergence of $\ddot{\mu}_n^2$ in L^1.
We finally define h_0 as

$$h_0(\alpha) = \sup\{C > 0 \,/\, \text{(II.31) has no solution for } ||h|_\infty - 1| < C\} \ . \tag{II.34}$$

◇

End of the proof of Theorem 2.II.1: Let $h_0(\alpha)$ be given by (II.34). Define

$$\overline{\Psi}(\alpha) = 1 + h_0(\alpha 0 \ ,$$

so that (II.9) implies both

$$\left(\frac{(2-\alpha_1)(2+\alpha_2)b}{(2-\alpha_2)(2+\alpha_1)a} - 1\right) < h_0(\alpha) \ \text{, and } \left(1 - \frac{(2-\alpha_2)(2+\alpha_1)a}{(2-\alpha_1)(2+\alpha_2)b}\right) < h_0(\alpha) \ .$$

From (II.29), (II.30) and Proposition 2.II.2, we can conclude that the solution x can not interact with the singularity, that is that x solves (P_T).◇

2.III. The case $\alpha = 1$

In this chapter we are going to show that results in the line of Theorem 2.II.2 (case $1 < \alpha < 2$) or Theorem 2.II.2 (case $0 < \alpha < 1$) can not be proved when $\alpha = 1$.

A sequence of planar potentials potentials $F_n \in C^1(\mathbb{R}^2 \setminus \{0\}; \mathbb{R})$ can be defined in such a way that

$$\frac{1-\varepsilon_n}{|x|} \le -F_n(x) \le \frac{1+\varepsilon_n}{|x|}\,, \qquad \forall x \in \mathbb{R}^2 \setminus \{0\}$$
$$\nabla F_n(x) \cdot x = -F_n(x)\,, \qquad \forall x \in \mathbb{R}^2 \setminus \{0\}$$
$$\lim_{n\to+\infty} \varepsilon_n = 0\,,$$

and such that

$$\lim_{n\to+\infty} \inf_{x \text{ solves } (P_T)} I(x) = +\infty\,.$$

Therefore, in the following counterexample, both the variational methods used in proving Theorem 2.I.2 (local minimization in the planar case) and Theorem 2.II.1 (infsup arguments) lead to the existence of *collision* solutions.

In other words, the set of T-periodic Kepler orbits is not stable under some kinds of perturbations.

DEFINITION 2.III.1. *In a polar system of coordinates (ρ, θ) of $\mathbb{R}^2$, we say that F is a K-type potential if*

$$F(x) = \frac{-M(\theta)}{\rho}\,,$$

and M is defined in the following way. Consider the partition of $\mathbb{R}^2$ induced by two straight lines. We denote $0 < 2\mu < \pi$, $\nu = \pi - 2\mu$ the amplitude of the angles between the two straight lines. We denote by I_- and I_+ the two sectors of amplitude 2μ and we set $M(\theta) = M_+$ in I_+ and $M(\theta) = M_-$ in I_-, where M_+ and M_- are positive constants such that $M_- < M_+$. In the sectors I_1 and I_2 of amplitude ν, the function M varies in a continuous and monotonic way between the values M_- and M_+.

The following results have been proved in [**1'9**]:

PROPOSITION 2.III.1. *If F is a K-type potential, there are no non collision solutions of the motion equation*

$$-\ddot{x} = \nabla F(x)$$

with angular speed of constant sign, when ν is sufficiently small.

THEOREM 2.III.1. *There exists a sequence of K-type potentials F_n such that the corresponding sequence M_n converges uniformly to the constant function one and such that, for every fixed period $T > 0$,:*

every sequence x_n of T-periodic solutions of the problems

$$(P_T^n) \qquad \begin{cases} \ddot{x}_n = \nabla F_n(x_n) \\ x_n(t+T) = x_n(t)\ , \qquad \forall t \in \mathbb{R} \\ x(t) \neq 0\ , \qquad \forall t \in \mathbb{R}\ , \end{cases}$$

(if any exists) converges uniformly to zero. Moreover

$$\lim_{n\to+\infty} I_n(x_n) = +\infty\ ,$$

$$\lim_{n\to+\infty} E_n = -\infty\ ,$$

where I_n is the functional associated to (P_T^n) and E_n is the sequence of the energies of x_n. Moreover, large values of n, (P_T^n) has no solution having non zero topological degree, with respect to the origin.

The phenomenon described in Theorem 2.III.1 shows the strong limitation to the variational approach to the fixed period problem, in the case $\alpha = 1$. The same degeneracy holds whenever the fixed energy problem is considered:

THEOREM 2.III.2. *There exists a sequence of K-type potentials F_n such that the corresponding sequence M_n converges uniformly to the constant function one and such that, for every fixed energy $E < 0$,:*

every sequence x_n of periodic solutions of the problems

$$(P_E^n) \qquad \begin{cases} \ddot{x}_n = \nabla F_n(x_n) \\ \frac{1}{2}|\dot{x}_n|^2 + F_n(x_n) = E \\ x(t) \neq 0\ , \qquad \forall t \in \mathbb{R}\ , \end{cases}$$

(if any exists) converges uniformly to zero. Moreover

$$\lim_{n\to+\infty} T_n = +\infty\ ,$$

where T_n is the associated sequence of the minimal periods. Moreover, for large values of n, (P_E^n) has no solution having non zero topological degree, with respect to the origin.

The degeneracy occurring at $\alpha = 1$ can be overcame when F possesses some symmetry properties ([26]), for example when F is even. We shall discuss this case in Part 6.
The proof of these results is quite long and consists in a very detailed analysis of the solutions to the associated Cauchy problem. We refer to [19] for proofs and comments.

PART 3. The fixed energy problem

In this part we are concerned with the problem of finding one solution to

$$(P_E) \qquad \begin{cases} -\ddot{x} = \nabla F(x) & \\ \frac{1}{2}|\dot{x}|^2 + F(x) = E & \\ x(t+\lambda) = x(t)\ , & \forall t \in \mathbb{R} \\ x(t) \neq 0\ , & \forall t \in \mathbb{R}\ , \end{cases}$$

where the energy $E \in \mathbb{R}$ is prescribed. Here the unknowns are both the function x and its period λ.
The variational approach to (P_E) can consist[1] in looking for critical points of the associated functional

$$(3.1) \qquad I(x) = \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)\left(\int_0^1 E - F(x)\right)\ ,$$

at positive levels. As a matter of facts, if a 1-periodic function x is a critical point of I at level $I(x) > 0$, then, setting

$$\lambda^2 = \frac{\frac{1}{2}\int_0^1 |\dot{x}|^2}{\int_0^1 E - F(x)}\ ,$$

the function $y(t) = x(\lambda^{-1}t)$ is a λ-periodic solution to (P_E).
Since the potential F is assumed to be singular at the origin, the suitable domain of I will be the function set

$$(3.2) \qquad \Lambda = \{x \in H\ /\ x(t) \neq 0\ , \forall t \in \mathbb{R}\}\ ,$$

where H is the Sobolev space of all the 1-periodic functions of $H^1_{loc}(\mathbb{R};\mathbb{R}^N)$.

As usual, the behavior of our potential will be assumed to be similar to the one of a model potential of the form $F(x) = \frac{-a}{|x|^\alpha}$. Thus , taking into account of the results of 1.IV and 2.III, we shall be concerned with the following cases:

[1] Another variational approach to (P_E) has been used in [6]

(I)	$\alpha > 2$	(and $E > 0$) ;
(II)	$1 < \alpha < 2$	(and $E < 0$) ;
(III)	$0 < \alpha < 1$	(and $E < 0$) .

As far as the case $\alpha = 1$ is concerned, we have seen in 2.III that a structural degeneracy occurs. Concerning the case $\alpha = 2$, Proposition 1.IV.1 leads to the choice $E = 0$, so that the functional in (3.1) has the invariance $I(\rho x) = I(x)$, for every $\rho > 0$ and $x \in \Lambda$. Therefore the problem is not well posed.

I. The case $\alpha > 2$

When $E > 0$, the functional of (3.1) has more or less the same features as the functional associated to the fixed period problem (2.1). Thus we are going to prove the existence result by the application of Theorem 2.I.2. On this line, an additional difficulty arises from the fact that, since the functional I vanishes identically on the space of constant functions, we have $\lim_{x \to \partial\Lambda} I(x) = 0$. Therefore we are going to apply Theorem 2.I.2 to a family of functionals of the form

$$I_\varepsilon(x) = I(x) + \int_0^1 V_\varepsilon(x) \ ,$$

where the term V_ε induces the strong force. By virtue of an a priori estimate we shall be able to insure that, for small values of ε, the critical points of I_ε are actually critical points of I. This approach is just a slight modification of the one of [**15**], where a result very similar to the following Theorem I.1 was proved.
Let us consider the following assumptions:

$$\exists \alpha \geq \alpha_1 > 2 \ \exists b \geq a \geq 0 \text{ such that}$$

$$(H1) \qquad \frac{a}{|x|^\alpha} \leq -F(x) \leq \frac{b}{|x|^\alpha}/; , \qquad \forall x \in \mathbb{R}^N \setminus \{0\}$$

$$(H2) \qquad \nabla F(x) \cdot x \geq -\alpha_1 F(x) \qquad \forall x \in \mathbb{R}^N \setminus \{0\}$$

The main goal of this chapter is in the following result:

THEOREM I.1 (BENCI-GIANNONI). *Assume that (H1) and (H2) hold. Then (P_E) admits at least one solution if and only if $E > 0$.*

Before proving the theorem, we need some preliminary results.

DEFINITION I.1. *For any $\varepsilon > 0$, $V_\varepsilon \in C^2(\Lambda;\mathbb{R})$ denotes a function such that*

$$V_\varepsilon(x) \geq \frac{1}{|x|^2} \qquad \text{if } 0 < |x| \leq \varepsilon \tag{I.1}$$

$$V_\varepsilon(x) > 0 \qquad \text{if } 0 < |x| < 2\varepsilon \tag{I.2}$$

$$V_\varepsilon(x) = 0 \qquad \text{if } |x| \geq 2\varepsilon \tag{I.3}$$

$$\nabla V_\varepsilon(x)\cdot x \leq -2V_\varepsilon(x) \qquad \forall\, x \in \mathbb{R}^N - \{0\}\,. \tag{I.4}$$

Define

$$I_\varepsilon(x) = I(x) + \int_0^1 V_\varepsilon(x)$$

PROPOSITION I.1. *(A priori estimate). Assume (H1) and (H2) hold, and let $x \in \Lambda$ be a critical point of I_ε, such that $I_\varepsilon(x) > 0$. If*

$$(2\varepsilon)^\alpha \leq \frac{(\alpha_1 - 2)a}{2E} \tag{I.5}$$

then

$$|x(t)| \geq 2\varepsilon \qquad \forall\, t \in [0,1]\,. \tag{I.6}$$

PROOF: Let x be such a critical point. We have

$$\left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)\left(\int_0^1 E - F(x)\right) + \int_0^1 V_\varepsilon(x) = C > 0 \tag{I.7}$$

$$-\int_0^1 E - F(x)\ddot{x} = \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)\nabla F(x) - \nabla V_\varepsilon(x)\,. \tag{I.8}$$

First, we deduce from (I.7), (I.8), (I.2) and (I.4) that x is not constant. Associated with the time-independent equation (I.8), we have the energy integral

$$\frac{1}{2}|\dot{x}|^2\left(\int_0^1 E - F(x)\right) + F(x)\left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right) - V_\varepsilon(x) = h\,, \tag{I.9}$$

and by integrating it we find the upper bound to h,

$$h = E\left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right) - \int_0^1 V_\varepsilon(x) \leq E\left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right), \tag{I.10}$$

since $V_\varepsilon(x) \geq 0$.

Let $t_m \in \mathbb{R}$ such that $|x(t_m)| = \min_{t\in\mathbb{R}} |x(t)|$, we have

$$\text{(I.11)} \qquad \frac{d^2}{dt^2}\frac{1}{2}|x(t_m)|^2 \;=\; |\dot{x}(t_m)|^2 + x(t_m)\cdot\ddot{x}(t_m) \;\geq\; 0\,.$$

From (H2), (I.4), (I.8) and (I.11) we obtain

$$\begin{aligned}
0 \;\leq\; & |\dot{x}(t_m)|^2 - \frac{\left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)}{\left(\int_0^1 E - F(x)\right)} \nabla F(x(t_m))\cdot x(t_m) \\
& + \frac{1}{\left(\int_0^1 E - F(x)\right)} \nabla V_\varepsilon(x(t_m))\cdot x(t_m) \\
\leq\; & |\dot{x}(t_m)|^2 + \alpha_1 \frac{\left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)}{\left(\int_0^1 E - F(x)\right)} F(x(t_m)) \\
& - \frac{2}{\left(\int_0^1 E - F(x)\right)} V_\varepsilon(x(t_m))\,,
\end{aligned}$$

and by substituting in (I.9),

$$\text{(I.12)} \qquad h \;\geq\; -(\frac{\alpha_1}{2} - 1)\left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right) F(x(t_m))\,.$$

Taking into account of (H1) and (I.10), (I.12) leads to

$$\frac{a}{|x(t_m)|^\alpha} \;\leq\; (\frac{2}{\alpha_1 - 2})E\,,$$

that is (from (I.5)),

$$|x(t)| \;\geq\; |x(t_m)| \;\geq\; \left(\frac{(\alpha_1 - 2)a}{2E}\right)^{1/\alpha} \;\geq\; 2\varepsilon\,.$$

◇

By virtue of Proposition I.1, looking for critical points of I is equivalent to looking for critical points of I_ε, provided that (I.5) holds. We are going to show that I_ε satisfies the assumptions of Theorem I.2.1 of Part 2.

PROPOSITION I.2. *Let $E > 0$, and let I_ε be defined in Definition I.1. Then*

$$\lim_{x \to \partial\Lambda} I_\varepsilon(x) = +\infty . \tag{I.13}$$

PROOF: Indeed, by definition, we have

$$I_\varepsilon(x) \geq \frac{E}{2} \int_0^1 |\dot{x}|^2 + \int_0^1 V_\varepsilon(x) , \tag{I.14}$$

and, from (3.3), V_ε satisfies the "strong force" condition. This fact (see Proposition 2.I.1 and Remark 2.I.1) implies (I.13).◇

PROPOSITION I.3. *Assume (H1), (H2) hold, and let $E > 0$. Then, for any $c > 0$, I_ε satisfies the $(PS)_c$ condition.*

PROOF: Let $(x_n)_n$ be a Palais- Smale sequence in Λ, that is

$$x_n \in \Lambda$$

$$I_\varepsilon(x_n) = c_n \to c > 0 \tag{I.15}$$

$$-\left(\int_0^1 E - F(x_n)\right)\ddot{x}_n = \left(\frac{1}{2}\int_0^1 |\dot{x}_n|^2\right) \nabla F(x_n) - \nabla V_\varepsilon(x_n) + h_n \quad \text{with} \quad h_n \to 0 \quad \text{in} \quad H^{-1} . \tag{I.16}$$

From Proposition I.2 we deduce that

$$\exists\, M > 0 \quad \text{such that} \quad d(x_n, \partial\Lambda) \geq M \quad (\forall\, n \in \mathbb{N}) . \tag{I.17}$$

From (I.14) and (I.15) we then deduce that a constant $C > 0$ exists such that

$$\int_0^1 |\dot{x}_n|^2 \leq \frac{4C}{E} \qquad \text{for } n \text{ large} . \tag{I.18}$$

Let $m_n = \min_{t \in \mathbb{R}} |x_n(t)|$. Now, if $(m_n)_n$ is bounded, then, up to a subsequence, it converges uniformly to some limit $x \in H$. Moreover, from (I.16), we deduce that $(x_n)_x$

actually converges strongly in H, so from (I.17) we conclude that $x \in \Lambda$.

Assuming by the contrary that $(m_n)_n$ is unbounded, then (up to a subsequence) we can assume that

$$\lim_{n\to+\infty} m_n = +\infty ,$$

so that both $(\nabla F(x_n))_n$ and $(\nabla V_\varepsilon(x_n))_n$ converge uniformly (and hence in H^{-1}) to zero. Moreover, since

$$\lim_{n\to+\infty} \int_0^1 E - F(x_n) = E > 0 ,$$

(I.6) implies that $(-\ddot{x}_n)_n$ converges to zero in H^{-1}. Hence $\|\dot{x}_n\|_{L^2} \to 0$ and therefore

$$I(x_n) \to 0$$

(indeed, for n large $\int_0^1 V_\varepsilon(x_n) = 0$), which contradicts (I.15).◇

PROPOSITION I.4. *For every $\varepsilon > 0$, there exists $\delta > 0$ such that the sublevel $I_\varepsilon^\delta = \{x \in \Lambda \;/\; I_\varepsilon(x) < \delta\}$ is contractible.*

PROOF: When δ is sufficiently small, then $\sup_{x\in I_\varepsilon^\delta} \int_0^1 |\dot{x}|^2$ has to be very small and moreover $\inf_{x\in I_\varepsilon^\delta} \inf_{t\in\mathbb{R}} |x(t)|$ has to be very large. The homotopy $h(x,\sigma) = (1-\sigma)x + \sigma \int_0^1 x$ can then be performed, contracting the sublevel into the space of constant functions, without crossing the boundary of Λ.◇

PROOF OF THEOREM I.1: We first prove the existence of one solution for positive values of the energy. To carry out the proof, we are going to apply Theorem I.2.1 of Part 2 to a functional of the family I_ε, for a suitably small ε. The proof will be then complete after the application of Proposition I.1. We first extend the functional I_ε to the whole of H by setting $\overline{I}_\varepsilon(x) = +\infty$, for $x \in \partial\Lambda$. Of course $\overline{I}_\varepsilon$ is (weakly) lower semicontinuous, and $\inf_{\partial\Lambda} \overline{I}_\varepsilon = +\infty$. Finally, the above Propositions make all the assumptions of Theorem I.2.1 of Part 2 be fulfilled.
Now we prove the reversed implication. Assume on the contrary that a negative energy $E < 0$ exists such that (P_E) admits a solution x. From (H2) and the conservation of the energy, we deduce that

$$\frac{1}{2}\frac{d^2}{dt^2}|x(t)|^2 \leq 2E + (2-\alpha_1)F(x(t)) < 0 , \qquad \forall t \in \mathbb{R} .$$

This fact contradicts the periodicity of x.◇

3.II. The case $1 < \alpha < 2$

We have seen in Proposition IV.1 of Part 1 that, for potentials of the form $F(x) = \frac{-a}{|x|^\alpha}$ with $0 < \alpha < 2$, a necessary condition for the solvability of (P_E) is that the energy E is negative. The natural variational setting of (P_E), in this case, then leads to work with unbounded (above and below) functionals. A further difficulty arises from the fact that we have $\liminf_{x\to\partial\Lambda} I(x) = -\infty$ as well as $\limsup_{x\to\partial\Lambda} I(x) = +\infty$. Therefore, results on the line of Theorem I.2 .1 of Part 2 (used for the case $\alpha > 2$) are not applicable in the case $0 < \alpha < 2$. On the other hand, we are going to show that, whenever $\alpha > 1$, the restriction of I over a set of the type $S_\rho = \{x \in H \ / \ \int_0^1 |\dot{x}|^2 = \rho\}$ satisfies (for suitable values of ρ) the assumptions of Theorem I.2.1 of Part 2. A new infsup argument is then introduced, joining set intersection properties with non contractibility properties. Roughly speaking, we are going to work with classes of sets whose intersection with a given closed subset subset of H can not be contracted in a continuous way into sets of constant functions, without crossing the boundary of Λ.

We shall consider the following assumptions on F:

$$\exists\ \alpha, \alpha_1, \alpha_2,\ , 1 < \alpha_1 \leq \alpha \leq \alpha_2 < 2,\ \exists b \geq a > 0 \text{ such that:}$$

(H1)
$$\frac{a}{|x|^\alpha} \leq -F(x) \leq \frac{b}{|x|^\alpha} \qquad \forall x \in \mathbb{R}^N \setminus \{0\}\,.$$

(H2)
$$-\alpha_1 F(x) \leq \nabla F(x) \cdot x \leq -\alpha_2 F(x) \qquad \forall x \in \mathbb{R}^N \setminus \{0\}\,.$$

(H3)
$$\frac{b}{a} < \frac{2}{2-\alpha}$$

The following theorem can be proved by means of the above mentioned infsup argument:

THEOREM II.1. *Let $F \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfying (H1),(H2),(H3). Then there exists a function $\Phi : [1,2) \to \mathbb{R}$ such that, when*

(H4)
$$\frac{b}{a}\left(\frac{(2-\alpha_1)\alpha}{(2-\alpha)\alpha_1}\right)^{\frac{\alpha}{2}} \frac{2-\alpha}{2-\alpha_2} < \Phi(\alpha)\,;$$

then (P_E) has at least one solution if and only if $E < 0$. Moreover, Φ fulfils the following properties:

$$\Phi(1) = 1$$
$$\Phi \text{ is increasing}$$
$$\lim_{\alpha \to 2} \Phi(\alpha) = +\infty \; .$$

This result is on the line of Theorem I.2.2 of Part 2, where the function Ψ defined in Part 1 (V.7) was used as bound for the pinching condition. We wish to point out that the functions Ψ and Φ are related by

$$\Phi(\alpha) \; = \; \big(\Psi(\alpha)\big)^{\frac{2+\alpha}{2}} \; ,$$

so that, by (V.7) of Part 1, Φ can be computed as

$$\Phi(\alpha) = \frac{1}{2^{\alpha-1}\alpha^{\frac{\alpha}{2}}(2-\alpha)^{\frac{2+\alpha}{2}}}\Big\{\frac{1}{\pi}\int_0^{2\pi} |\sin\theta|^{\frac{2}{\alpha}}\, d\theta\Big\}^{\alpha} \; .$$

Since $\Phi(1) = 1$, condition (H4) is never satisfied for $\alpha = 1$. On the other hand, we know from 2.III that a result similar to the one here can not be proved in this case. However, the pinching condition is not empty for each $1 < \alpha < 2$ and its field becomes larger and larger when α converges to 2.

REMARK II.1: The fact that $E < 0$ is a necessary condition for the solvability of (P_E) can be proved as follows: let $t_0 \in \mathbb{R}$ be such that $|x(t_0)|^2 = \min_{t\in\mathbb{R}} |x(t)|^2$. Since x is regular, we have $\frac{d^2}{dt^2}|x|(t_0) \leq 0$. Thus (H3) leads to $0 \geq |\dot{x}(t_0)|^2 + \alpha_2 F(x(t_0)) = 2E - (2-\alpha_2)F(x(t_0))$, so that, from (H1), we can conclude that $E < 0$.$\diamond$

REMARK II.2: From (H1), the trajectory of every solution of (P_E) is constrained in the set $\{x \in \mathbb{R}^N\setminus\{0\} \;/\; -F(x) \geq -E\;\} \subseteq \{x \in \mathbb{R}^N\setminus\{0\} \;/\; \frac{b}{|x|^\alpha} \geq -E\;\}$. Therefore, hypotheses (H1) and (H2) can always be assumed to hold true just for each x in this set.$\diamond$

Therefore, the following result easily follows from Theorem II.1

COROLLARY II.1. *Let $U \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfy*

$$\lim_{|x|\to 0} |x|^\alpha |\nabla U(x)| = 0 \; ,$$

and let

$$F(x) = \frac{-a}{|x|^\alpha} + U(x) \; , \qquad \forall x \in \mathbb{R}^N \setminus \{0\} \; .$$

Then there exists $\overline{E} < 0$ such that for any $E < \overline{E}$, (P_E) has at least one solution.

We shall see in the proof of Theorem II.1 that (H4) implies an a priori lower bound for the minima of the modulus of the solutions at suitable levels of the functional. In the model problem with $F(x) = \frac{-a}{|x|^\alpha}$, one can easily prove that (P_E) always admits a solution $(\omega_a, x(t) = \rho_a e^{i\omega_a t})$ with $\rho_a = \left(\frac{(2-\alpha)a}{-2E}\right)^{\frac{1}{\alpha}}$ and $\omega_a = \frac{4\pi^2}{\alpha a}\left(\frac{(2-\alpha)a}{-2E}\right)^{\frac{2+\alpha}{\alpha}}$. Now we wonder whether assumptions (H1) and (H2) allow estimates of the minima of the modulus of the solutions in terms of $\left(\frac{(2-\alpha_2)a}{-2E}\right)^{\frac{1}{\alpha}}$.

We shall prove the following estimate:

COROLLARY II.2. *Under the assumptions of Theorem II.1, there are functions* $\Phi^* : [0,2) \to \mathbb{R}$ *and* $\sigma : (1,2) \times (1,+\infty) \to (0,1)$ *such that, if*

$$\text{(H5)} \qquad \left(\frac{(2-\alpha)b}{(2-\alpha_2)a}\right)^2 < \Phi^*(\alpha)\,,$$

holds, then (P_E) admits at least one solution x with

$$|x(t)| \geq \sigma\left(\alpha, \frac{(2-\alpha_1)b}{(2-\alpha_2)a}\right)\left(\frac{(2-\alpha)a}{-2E}\right)^{\frac{1}{\alpha}} \qquad \forall t \in \mathbb{R}\,.$$

Moreover, σ fulfils the properties:

$$\lim_{\alpha\to 2} \sigma\left(\alpha, \frac{(2-\alpha_1)b}{(2-\alpha_2)a}\right) = 1 \qquad \textit{if } \frac{(2-\alpha_1)b}{(2-\alpha_2)a} \textit{ remains bounded;}$$

and

$$\lim_{\frac{(2-\alpha_1)b}{(2-\alpha_2)a}\to 1} \sigma\left(\alpha, \frac{(2-\alpha_1)b}{(2-\alpha_2)a}\right) = 1 \qquad \textit{for each fixed } \alpha \in (1,2)\,.$$

The function Φ^* of (H5) can be estimate as

$$\Phi^*(\alpha) \geq \frac{1}{4\alpha}\left(\frac{2}{2-\alpha}\right)^{\frac{2}{\alpha}} \left\{\frac{1}{\pi}\int_0^{2\pi} |\sin\theta|^{\frac{2}{\alpha}}\, d\theta\right\}^2 .$$

The value of σ is defined in (II.5.25).

REMARKII.3: One can easily see that $\Phi^*(1) = 1$, so that (H5) can never be satisfied at $\alpha = 1$. However, since $\lim_{\alpha\to 2} \Omega^*(\alpha) = +\infty$, the field of condition (H5) is nonempty. It is possible that $\Phi^*(\alpha) > 1$, $\forall\, 1 < \alpha < 2$, but we were not able to prove it.$\diamond$

REMARK II.4: In the model problem with $F(x) = \frac{-a}{|x|^\alpha}$, one can easily prove that (P_E) always admits a solution $x(t) = \rho_a e^{i\omega_a t}$ with $\rho_a = \left(\frac{(2-\alpha)a}{-2E}\right)^{\frac{1}{\alpha}}$ and $\omega_a = \frac{4\pi^2}{\alpha a}\left(\frac{(2-\alpha)a}{-2E}\right)^{\frac{2+\alpha}{\alpha}}$.
The meaning of Corollary 3.II.2 is that the solution of (P_E) found by Theorem 3.II.1 corresponds (in a variational sense) and approximates the circular solution of the radial case, in a way just depending on α and on the fraction $\frac{(2-\alpha_1)b}{(2-\alpha_2)a}$.◇

The proof of the results is organized as follows:
II.1. The variational principle
II.2. Some lemmas about the case $F(x) = \frac{-a}{|x|^\alpha}$
II.3. A first a priori estimate
II.4. Proof of Theorem 3.II.1
II.5. Proof of Corollary 3.II.2
II.6. Appendix

3.II.1. The variational principle.

The first purpose of this section is to state an abstract min-max theorem (Theorem II.2.1) concerning existence of critical points for indefinite functionals.
Roughly speaking, we are going to work with a class of subsets of H having the property that their intersection with a given closed $\Sigma_1 \subseteq H$ cannot be contracted in a continuous way into a set of constant functions, without crossing the boundary of Λ.
The second goal of this section (Theorem II.2.2) consists in showing that this min-max class is not empty.

Let us fix up some notations. We denote by $\mathcal{H}$ the set of all the continuous homotopies of H homotopically equivalent to the identity:

$$\mathcal{H} \;=\; \{h \in C(H \times [0,1]; H) \;/\; h(\cdot, 0) = id\}.$$

$E_N \equiv \mathbb{R}^N$ denotes the space of all the constant functions of H. For any $A \subseteq \Lambda$ we denote

$$\mathcal{H}_0(A) \;=\; \{h \in \mathcal{H} \;/\; h(A,1) \subseteq E_N\},$$

and, according with the notations of I.2 of Part 2, we denote

$$\mathcal{A}_0 \;=\; \{A \subseteq \Lambda \;/\; h(A,[0,1]) \cap \partial\Lambda \neq \emptyset \;,\; \forall h \in \mathcal{H}_0(A)\}.$$

Let $\Sigma_1, \Sigma_2 \subseteq H$ be closed such that $\Sigma_1 \cap \Sigma_2 = \emptyset$, and let

$$
\begin{aligned}
\text{(II.1.1)} \qquad \mathcal{H}^*_{\Sigma_1,\Sigma_2} = \{h \in \mathcal{H} \;/\; & h(\cdot, s) \text{ is an homeomorphism},\\
& h(x, s_1 + s_2) = h(h(x, s_1), s_2)\\
& h(x, s) = x \;\; \forall x \in (\Sigma_1 \cap \partial\Lambda) \cup \Sigma_2, \forall s \in [0,1]\},
\end{aligned}
$$

Define

$$
\text{(II.1.2)} \qquad \mathcal{A}^*_{\Sigma_1,\Sigma_2} = \{A \in H \text{ compact} \;/\; h(A, 1) \cap \Sigma_1 \in \mathcal{A}_0 \;,\; \forall h \in \mathcal{H}^*_{\Sigma_1,\Sigma_2}\} \,.
$$

The following Theorem holds:

THEOREM II.1.1. *Let $I \in C^1(H;\mathbb{R})$. Assume that there exist $\Sigma_1, \Sigma_2 \subseteq H$ closed, and $c > 0$ such that*

$$
\text{(II.1.3)} \qquad \mathcal{A}^*_{\Sigma_1,\Sigma_2} \neq \emptyset.
$$

$$
\text{(II.1.4)} \qquad \sup_A I \geq c > 0 \,, \qquad \forall A \subseteq \Sigma_1 \,, A \in \mathcal{A}_0.
$$

Then the number

$$
c^* = \inf_{A \in \mathcal{A}^*_{\Sigma_1,\Sigma_2}} \sup_A I
$$

is well defined. Assume moreover that

$$
\text{(II.1.5)} \qquad \inf_{\Sigma_1 \cap \partial\Lambda} I > c^* > \sup_{\Sigma_2} I
$$

(II.1.6) *I satisfies the condition $(C)_{c^*}$ of Cerami; namely every sequence $(u_n)_n$ in H such that*

$$
\lim_{n\to\infty} I(u_n) = c^*
$$

$$
\lim_{n\to\infty} (1 + ||u_n||)||dI(u_n)|| = 0
$$

possesses a converging subsequence.

Then there exists a critical point of I at the level c^*.

PROOF: The theorem can be proved by the usual deformation techniques (see for example [**37**]). As a matter of fact, one can construct a pseudogradient flow η such that $\eta \in \mathcal{H}^*_{\Sigma_1,\Sigma_2}$. Therefore the class $\mathcal{A}^*_{\Sigma_1,\Sigma_2}$ is invariant under this flow and, by the deformation lemma one proves that the set $K_{c^*} = \{x \in H \ / \ I(x) = c^* \ , \ dI(x) = 0\}$ has to be nonempty.◇

We shall apply Theorem II.1.1 in the following situation:

Let $\rho > 0$ be fixed: as Σ_1 we shall take

$$\Sigma_1 \ = \ \{x \in H \ / \ \frac{1}{2}\int_0^1 |\dot{x}|^2 = 2\pi^2\rho^2\} \tag{II.1.7}$$

Let $0 < \varepsilon < 1$; let $(e_1, \ldots, e_N)$ be an orthonormal system of $\mathbb{R}^N$ and let $S^{N-2}(e_1) = \{x \in \mathbb{R}^N \ / \ |x - e_1| = 1 \ , \ x \cdot e_N = 0\}$. Define

$$\Sigma_2 \ = \ \bigcup_{x \in S^{N-2}(e_1)} \ \bigcup_{\theta \in \{\varepsilon, \varepsilon^{-1}\}} \{\theta\rho(4x + (x - e_1)\cos 2\pi t + e_N \sin 2\pi t)\}. \tag{II.1.8}$$

One easily sees that $\Sigma_1 \cap \Sigma_2 = \emptyset$. Consider

$$A_0 = \bigcup_{x \in S^{N-2}(e_1)} \{\varepsilon\rho(4x + (x - e_1)\cos 2\pi t + e_N \sin 2\pi t)\}. \tag{II.1.9}$$

$$A^* \ = \ \bigcup_{x \in S^{N-2}(e_1)} \ \bigcup_{\theta \in [\varepsilon, \varepsilon^{-1}]} \{\theta\rho(4x + (x - e_1)\cos 2\pi t + e_N \sin 2\pi t)\}. \tag{II.1.10}$$

The second goal of this section is the following result:

THEOREM II.2.2. *For any $\rho > 0$ and $0 < \varepsilon < 1$, let Σ_1,Σ_2 and A be defined as before. Then*

(i) $\quad A_0 \in \mathcal{A}_0$;

(ii) $\quad A^* \in \mathcal{A}^*_{\Sigma_1,\Sigma_2}$.

PROOF: i) was proved in Proposition I.2.3 of Part 2

ii) According to the definitions of the above mentioned Proposition, in order to prove the claim, one has to prove that, for any $h_1 \in \mathcal{H}^*_{\Sigma_1,\Sigma_2}$ and for every $h_2 \in \mathcal{H}_0(h_1(A^*,1) \cap \Sigma_1)$, then

$$h_2(h_1(A^*,1) \cap \Sigma_1, [0,1]) \cap \partial\Lambda \neq \emptyset \,. \tag{II.1.11}$$

First remark that h_2 can be extended to an $\bar{h}_2 \in \mathcal{H}_0(h_1(A^*,1))$ by the application of Tietze-Dugundji's extension Theorem.
Moreover, since $A^* \cap (\Sigma_1 \cap \partial\Lambda) = \emptyset$, one has

$$h_1(A^*, [0,1]) \cap (\Sigma_1 \cap \partial\Lambda) = \emptyset \,. \tag{II.1.12}$$

Let $g : S^{N-2}(e_1) \times [0,1] \to \Lambda$ the continuous parametrization of A^* defined by

$$g(x,\theta) = (\varepsilon\rho + \theta\rho(\varepsilon^{-1} - \varepsilon))(4x + (x - e_1)\cos 2\pi t + e_N \sin 2\pi t)$$

and let $\tilde{g} : S^{N-2}(e_1) \times [0,1] \times [0,2] \to H$ be defined as

$$\tilde{g}(x,\theta,s) = \begin{cases} h_1(g(x,\theta),s) & \text{if } 0 \le s \le 1; \\ \bar{h}_2(h_1(g(x,\theta),1), s-1) & \text{if } 1 \le s \le 2. \end{cases}$$

$\tilde{g}$ is continuous and has $\tilde{g}(x,\theta,s) \equiv \tilde{g}(x,s)$ when $\theta \in \{0,1\}$ and $s \in [0,1]$, because of the fact that of $h_1 \in \mathcal{H}^*_{\Sigma_1,\Sigma_2}$. Moreover, since $\tilde{g}(x,\theta,2) \in E_N$, $\forall (x,\theta) \in S^{N-2}(e_1) \times [0,1]$, it makes sense to define the continuous $G : S^{N-2}(e_1) \times [0,1] \times B^1 \to \mathbb{R}^N$ as

$$G(x,\theta,\tau e^{it}) = \tilde{g}(x,\theta,2(1-\tau))(t).$$

Note that, by definition, $0 \notin G(S^{N-2}(e_1) \times [0,1] \times S^1)$. Up to a small C^0 perturbation, we can assume that G is C^1 and that G, $G_{|S^{N-2}(e_1)\times\{0\}\times B^1}$ and $G_{|S^{N-2}(e_1)\times\{1\}\times B^1}$ are non degenerate at the value zero. Hence one has

a) $G^{-1}(0)$ is a finite union of 1-manifolds either homeomorphic to S^1 or connecting two points of $S^{N-2}(e_1) \times \{0,1\} \times B^1$. One can assume without loss of generality that each of them intersect the boundary in a transversal way.

b) both $deg_0 G_{|S^{N-2}(e_1)\times\{0\}\times B^1}$ and $deg_0 G_{|S^{N-2}(e_1)\times\{1\}\times B^1}$ are equal to one (or minus one, depending on the orientation chosen for $\mathbb{R}^N$). Anyway, both $\left(G_{|S^{N-2}(e_1)\times\{0\}\times B^1}\right)^{-1}(0)$ and $\left(G_{|S^{N-2}(e_1)\times\{1\}\times B^1}\right)^{-1}(0)$ consist in an odd number of points.

Hence one deduces that there is at least one continuous functions $\xi : [0,1] \to S^{N-2}(e_1) \times [0,1] \times B^1$, $\xi(\lambda) = (x(\lambda), \theta(\lambda), \tau(\lambda)e^{it(\lambda)})$ such that $\xi(0) \in S^{N-2}(e_1) \times \{0\} \times B^1$, $\xi(1) \in S^{N-2}(e_1) \times \{1\} \times B^1$ and $G(\xi(\lambda)) = 0$.

Now consider the function $\psi : S^{N-2} \times [0,1] \times [0,2] \to \mathbb{R}$ defined as

$$\psi(x,\theta,s) = \begin{cases} \frac{1}{2}\int_0^{2\pi} |\frac{d}{dt}\tilde{g}(x,\theta,s)|^2, & \text{if } 0 \le s \le 1; \\ \frac{1}{2}\int_0^{2\pi} |\frac{d}{dt}\tilde{g}(x,\theta,1)|^2, & \text{if } 1 \le s \le 2, \end{cases}$$

and let $\tilde{\psi} : S^{N-2}(e_1) \times [0,1] \times B^1 \to \mathbb{R}$ be defined as $\tilde{\psi}(x,\theta,\tau e^{it}) = \psi(x,\theta,2(1-\tau))$. One has $\tilde{\psi}(\xi(0)) = 2\pi^2\varepsilon^2\rho^2 < 2\pi^2\rho^2 < 2\pi^2\varepsilon^{-2}\rho^2 = \tilde{\psi}(\xi(1))$, so that there is a $\lambda_* \in [0,1]$ such that $\tilde{\psi}(\xi(\lambda_*)) = 2\pi^2\rho^2$. From (II.1.12) on deduces that $\tau(\lambda_*) < \frac{1}{2}$.
Finally one can conclude that the function $g(x(\lambda_*),\theta(\lambda_*))$ satisfies $h_1(g(x(\lambda_*),\theta(\lambda_*)),1) \in \Sigma_1$ and $h_2[h_1(g(x(\lambda_*),\theta(\lambda_*)),1),1-\tau(\lambda_*)](t(\lambda_*)) = 0$ and this fact proves (II.1.11).◇

3.II.2. Some lemmas about the case $F(x) = \frac{-a}{|x|^\alpha}$.

In this section we are going to show that, when $F(x) = \frac{-a}{|x|^\alpha}$, $1 < \alpha < 2$, then the associates functional fulfils the assumptions (II.1.4) and (II.1.5) of Theorem II.1.1. To this end, let us introduce some more notations:

For every $\rho > 0$

$$S_\rho = \{x \in H \;/\; \frac{1}{2}\int_O^1 |\dot{x}|^2 = \rho^2\}$$

$$\mathcal{A}_\rho^* = \{A \subseteq S_\rho \cap \Lambda \text{ compact} \;/\; A \in \mathcal{A}_0\}$$

$$J(\alpha,\rho) = \inf_{x \in S_\rho \cap \partial\Lambda} \int_0^1 \frac{1}{|x|^\alpha}$$

$$K(\alpha,\rho) = \inf_{A \in \mathcal{A}_\rho^*} \sup_{x \in A} \int_0^1 \frac{1}{|x|^\alpha}$$

$$\Phi(\alpha) = \frac{J(\alpha,1)}{K(\alpha,1)}.$$

Proposition II.2.1. *For every α, $1 \le \alpha < 2$, and for every $\rho > 0$, we have*

(i) $$J(\alpha,\rho) = J(\alpha,1)\rho^{-\alpha};$$

(ii) $$K(\alpha,\rho) = K(\alpha,1)\rho^{-\alpha};$$

(iii) $$J(\alpha,1) = \frac{(\sqrt{2}\pi)^\alpha}{2^{\alpha-1}\alpha^{\frac{\alpha}{2}}(2-\alpha)^{\frac{2+\alpha}{2}}}\{\frac{1}{\pi}\int_0^{2\pi} |\sin\theta|^{\frac{2}{\alpha}} d\theta\}^\alpha$$

(v) $$\Phi(1) = 1;$$

(vi) $$\Phi(\alpha) > 1\,, \qquad if\, 1 < \alpha < 2;$$

(vii) $$\lim_{\alpha\to 2} \Phi(\alpha) = +\infty\,.$$

PROOF: i) and ii) are easily verified.

iii) Let us denote, for any $\lambda > 0$

(II.2.1) $$\bar{c}_\lambda^\alpha(1) \;=\; \inf_{x\in\partial\Lambda}\{\frac{1}{2}\int_0^1 |\dot{x}|^2 + \lambda\int_0^1 \frac{1}{|x|^\alpha}\}.$$

It is shown in 1.V that $\bar{c}_\lambda^\alpha(1)$ is homogeneous of degree $\frac{2}{2+\alpha}$ in the variable λ. Moreover, for any constant vector e_1 of unit norm, there is a minimizer $x \in \Lambda$ of (II.2.1) such that

$$\begin{aligned} &x(t) = \rho(t)e_1\\ &\rho(0) = \rho(1) = 0\\ &\rho(t) > 0\,, \qquad \forall t \in (0,1)\\ &-\ddot{\rho} = \frac{\lambda\alpha}{\rho^\alpha} \qquad \text{in } (0,1) \end{aligned}$$

(II.2.2) $$\frac{1}{2}(\dot{\rho})^2 - \frac{\lambda}{\rho^\alpha} = -\frac{2-\alpha}{2+\alpha}\bar{c}_\lambda^\alpha(1)$$

Thus we find

$$\frac{1}{2}\int_0^1 |\dot{x}|^2 = \frac{\alpha}{2+\alpha}\bar{c}_\lambda^\alpha(1)\;;$$

moreover, we obviously have that

$$\bar{c}_\lambda^\alpha(1) = \frac{1}{2}\int_0^1 |\dot{x}|^2 + \lambda J(\alpha, \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)^{\frac{1}{2}})\,,$$

and hence

$$\overline{c}_\lambda^\alpha(1) = \frac{\alpha}{2+\alpha}\overline{c}_\lambda^\alpha(1) + \lambda J\Big(\alpha, \Big(\frac{\alpha}{2+\alpha}\overline{c}_\lambda^\alpha(1)\Big)^{\frac{1}{2}}\Big),$$

so that

$$J(\alpha,1) = \Big(\frac{2}{2+\alpha}\Big)\Big(\frac{\alpha}{2+\alpha}\Big)^{\frac{\alpha}{2}}\left(\overline{c}_\lambda^\alpha(1)\right)^{\frac{2+\alpha}{2}}.$$

Therefore, by definitions of Ψ (V.7 of Part 1) and Ψ, we deduce

$$\Phi(\alpha) = (\Psi(\alpha))^{\frac{2+\alpha}{2}}. \tag{II.2.3}$$

We can then conclude by the direct computation of $\overline{c}(\alpha,1)$.Indeed, from (II.2.2) we find that

$$\frac{1}{2} = \int_0^{\left(\frac{2-\alpha}{2+\alpha}\theta(\alpha,1)\right)^{-\alpha}} \frac{1}{\rho}d\rho = \int_0^{\left(\frac{2-\alpha}{2+\alpha}\theta(\alpha,1)\right)^{-\alpha}} \frac{1}{\sqrt{2}\sqrt{-\frac{2-\alpha}{2+\alpha}\theta(\alpha,1)+\frac{1}{\rho^\alpha}}}d\rho$$

hence, by a change of variables in the above integral, we can calculate the value of $\overline{c}(\alpha,1)$.

iv) Let us consider

$$\varphi(\alpha) = \inf_{A\in\mathcal{A}_0}\sup_A\Big\{\frac{1}{2}\int_0^1|\dot{x}|^2 + \frac{2}{\alpha\left(\sqrt{2}\pi\right)^\alpha}\int_0^1\frac{1}{|x|^\alpha}\Big\}. \tag{II.2.4}$$

Following the arguments of Theorem I.2.1 of Part 2, we can prove that, whenever $\alpha \geq 1$, $\varphi(\alpha)$ is a critical value for the functional in the right hand side of (II.2.4).
Moreover, as we have seen in I.V, the critical value (minimum) corresponds to the circular solution, so that we have

$$\varphi(\alpha) = \inf_{R>0}\Big[\frac{1}{2}(2\pi)^2R^2 + \frac{2}{\alpha\left(\sqrt{2}\pi\right)^2R^\alpha}\Big] = \frac{2+\alpha}{2};$$

and it is achieved (for example) on the set

$$A_0 = \bigcup_{x\in S^{N-2}(e_1)}\Big\{\frac{4}{\sqrt{2}\pi} + \frac{1}{\sqrt{2}\pi}((x-e_1)\cos 2\pi t + e_N\sin 2\pi t)\Big\},$$

where $(e_1,\cdots,e_n)$ is an othonormal system of $\mathbb{R}^N$ and $S^{N-2}(e_1) = \{x\in\mathbb{R}^N \,/\, |x-e_1| = 1, x\cdot e_N = 0\}$. It follows from Theorem 2.2.i) that $A \in \mathcal{A}_0\cap\mathcal{A}_1^*$, so that

$$\begin{aligned}\varphi(\alpha) &= \inf_{A\in\mathcal{A}_1^*}\sup_A\Big\{\frac{1}{2}\int_0^1|\dot{x}|^2 + \frac{2}{\alpha\left(\sqrt{2}\pi\right)^\alpha}\int_0^1\frac{1}{|x|^\alpha}\Big\}\\ &= 1 + \frac{2}{\alpha\left(\sqrt{2}\pi\right)^\alpha}K(\alpha,1).\end{aligned}$$

From (II.2.3), v), vi) and vii) are direct consequences of the results of Proposition 2.I.?..◇

PROPOSITION II.2.2. *Let F be of the form $F(x) = \frac{-a}{|x|^\alpha}$ with $1 < \alpha < 2$ and $a > 0$. Let $\rho^\alpha = \frac{(2-\alpha)a}{-2E}$, Σ_1, Σ_2 be defined in (II.1.9) and (II.1.10) and let $\mathcal{A}^*_{\Sigma_1,\Sigma_2}$ be defined in (II.1.2). Then:*

$$\text{(II.2.5)} \qquad \inf_{\Sigma_1 \cap \partial\Lambda} I > \inf_{A \in \mathcal{A}^*_{\Sigma_1,\Sigma_2}} \sup_A I = \pi^2 a^{2/\alpha} \alpha \left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} > \sup_{\Sigma_2} I .$$

PROOF: Indeed, from the results of Proposition II.2.1, the infsup term is achieved on the set $A^* \in \mathcal{A}^*_{\Sigma_1,\Sigma_2}$ of (II.1.10), as well as on $A_0 = A^* \cap \Sigma_1$. Therefore we have

$$\frac{\inf_{\Sigma_1 \cap \partial\Lambda} I}{\inf_{A \in \mathcal{A}^*_{\Sigma_1,\Sigma_2}} \sup_A I} = \Phi(\alpha) .$$

On the other hand, we can easily verify that

$$\sup_{A^* \cap \Sigma_1} I \geq \sup_{A^* \cap \theta\Sigma_1} I , \qquad \forall \theta \in [\varepsilon, \varepsilon^{-1}] ,$$

and the strict inequality holds for each $\theta \neq 0$.◇

PROPOSITION II.2.3. *For every $\rho_2 \geq \rho_1 > 0$, for every $\gamma > 0$, there exists $\varepsilon(\gamma)$ such that, for every $0 \leq \varepsilon \leq \varepsilon(\gamma)$ and for every $\rho \in [\rho_1, \rho_2]$, then*

$$\text{(II.2.6)} \qquad \inf_{\substack{x \in S_\rho \\ |x(0)|=|x(1)|=\varepsilon}} \int_0^1 \frac{1}{|x|^\alpha} \geq (J(\alpha,1) - \gamma) \frac{1}{\rho^\alpha} .$$

PROOF: It is just a consequence of the weakly lower semicontinuity of the functional $I(x) = \int_0^1 \frac{1}{|x|^\alpha}$ and the weak compactness of the constraints.◇

The following Proposition will play an important role in the proof of Corollary 1.

PROPOSITION II.2.4. *Let $E_T^0 = \{x \in H^1(\mathbb{R};\mathbb{R}) \ / \ x(t+T) = x(t) \ , \ \forall t \in \mathbb{R} \ , \ \int_0^T x = 0\}$, endowed with the Hilbertian norm $\left(\int_0^T |\dot{x}|^2\right)^{\frac{1}{2}}$. For any $\gamma \geq 1$, let $S(\gamma, T)$ denotes the best Sobolev constant of the injection $E_T^0 \to L_T^{2\gamma}$. Then*

$$S^2(\gamma,T) = \min_{x\in E^0_T} \frac{\int_0^T |\dot{x}|^2}{\left(\int_0^T |x|^{2\gamma}\right)^{\frac{1}{\gamma}}}$$

$$= \frac{1}{T^{\frac{1+\gamma}{\gamma}}} \frac{(1+\gamma)^{\frac{1+\gamma}{\gamma}}}{\gamma} \{\int_0^{2\pi} |\sin\theta|^{\frac{1+\gamma}{\gamma}} d\theta\}^2 .$$

PROOF: It is a classical result. However, since we did not find a reference, let us sketch the proof. It is clear that $S^2(\gamma,T)$ is homogeneous of degree $\frac{-(1+\gamma)}{\gamma}$ in the variable T. First remark that one actually has

$$S^2(\gamma,1) = \min_{x\in E^S_1} \frac{\int_0^1 |\dot{x}|^2}{\left(\int_0^1 |x|^{2\gamma}\right)^{\frac{1}{\gamma}}} ,$$

where $E^S_1 = \{x \in E^0_1 \ / \ x(t+\frac{T}{2}) = -x(t) \ , \forall t \in \mathbb{R}\}$, so that the minimum is attaint and satisfies

$$-\ddot{x} = \lambda |x|^{2(\gamma-1}$$

for some Lagrange multiplier $\lambda \in \mathbb{R}$. Assuming that $\int_0^1 |x|^{2\gamma} = 1$, by taking the L^2 product of the above equality, one finds that

$$S^2(\gamma,1) = \lambda .$$

From the energy integral one has

$$\frac{1}{2}|\dot{x}| + \frac{\lambda}{2\gamma}|x|^{2\gamma} = c ,$$

and by integrating one finds

$$c = \frac{S^2}{2}\left(\frac{\gamma+1}{\gamma}\right) .$$

One can easily see that, up to a shift of the parameter,x is increasing in $[0,\frac{1}{2}]$, because of the fact that it is a minimizer. Therefore one has

$$\frac{1}{2} = \int_{-(\gamma+1)^{\frac{1}{2\gamma}}}^{(\gamma+1)^{\frac{1}{2\gamma}}} \frac{1}{\sqrt{2}\sqrt{c - \frac{\lambda}{2\gamma}|x|^{2\gamma}}} .$$

By the above equality and by a change of variable on the above integral one can conclude the proof.◇

DEFINITION II.2.4. *Let us define*

$$\Omega(\gamma) = \frac{1}{4\pi^2}\frac{(\gamma+1)S^2(\gamma,1)}{2} \tag{II.2.7}$$

One can easily prove the following estimates:

PROPOSITION II.2.4. $\Omega : [1,+\infty] \to \mathbb{R}$ *has the following properties:*

(i)
$$\Omega(1) = 1;$$

(ii)
$$\lim_{\gamma\to+\infty} \Omega(\gamma) = +\infty$$

Remark that $\Omega(\alpha) = \Omega(\frac{\alpha}{2-\alpha})$ (cfr. Assumption (H5)).

II.3. A FIRST A PRIORI ESTIMATE

In order to overcome the discontinuity of I, we first introduce a class of truncated functionals I_ϵ which are defined and regular on the whole space H. Next step consists in showing that the critical points of the I_ϵ at suitable levels are actually critical points of I.

DEFINITION II.3.1. *Assume that F satisfies (H1),(H2). For any $\epsilon > 0$, F_ϵ denotes a function such that*

$$F_\epsilon \in C^2(\mathbb{R}^N;\mathbb{R}) \tag{II.3.1}$$

$$F_\epsilon(x) = F(x) \qquad \text{if } |x| \geq \epsilon \tag{II.3.2}$$

$$0 \leq \nabla F_\epsilon(x)\cdot x \leq -\alpha_2 F_\epsilon(x) \qquad \forall\, x \in \mathbb{R}^N \tag{II.3.3}$$

$$-F_\epsilon(x) \leq \frac{b}{|x|^\alpha} \qquad \forall\, x \in \mathbb{R}^N \tag{II.3.4}$$

$$\begin{cases} \text{there is a non decreasing } f:\mathbb{R}\to\mathbb{R}\text{, with} \\ f(|x|) \geq \frac{a}{\epsilon^\alpha} \qquad \text{if } |x| \leq \epsilon \\ \text{such that } -F_\epsilon(x) \geq f(|x|) \qquad \forall x \in \mathbb{R}^N . \end{cases} \tag{II.3.5}$$

We set

$$I_\epsilon(x) = \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)\left(\int_0^1 E - F_\epsilon(x)\right) \qquad \forall\, x \in H .$$

PROPOSITION II.3.1 (A PRIORI ESTIMATE). *Let F satisfy (H1), (H2), (H4), and let F_ε, I_ε as in Definition II.3.1. Let $c_1 > 0$ be fixed. There exists $\bar{\varepsilon} > 0$ such that, for every $0 < \varepsilon \leq \bar{\varepsilon}$, if x is a critical point of I_ε at level*

$$c_1 \leq I_\varepsilon(x) \leq c_2 \leq \pi^2 b^{2/\alpha} \alpha \left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}}, \tag{II.3.6}$$

then

$$|x(t)| \geq \varepsilon \qquad \forall t \in \mathbb{R}. \tag{II.3.7}$$

In order to prove the Proposition II.3.1, we need the following lemma :

LEMMA II.3.2. *Assume that F satisfies (H1), (H2) and let F_ε as in the Definition II.3.1. There exists a function $\delta(\varepsilon) = \delta(\varepsilon, c_1, c_2)$ with*

$$\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0,$$

such that if x is a critical point of I_ε satisfying (II.3.6), then

$$\left(\int_0^1 E - F_\varepsilon(x)\right)\left(\int_0^1 |\dot{x}|^2\right) \geq \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)\left(\int_0^1 -\alpha_1 F_\varepsilon(x)\right) - \delta(\varepsilon). \tag{II.3.8}$$

The proof of Proposition II.3.2 is given in the Appendix; let us show how Proposition II.3.1 follows from Proposition II.3.2.

PROOF OF PROPOSITION II.3.1: Let x such a critical point of I_ε. First, from (II.3.3) we deduce that

$$\int_0^1 E - F_\varepsilon(x) \leq \frac{-\alpha_2 E}{2 - \alpha_2},$$

and therefore

$$\frac{1}{2}\int_0^1 |\dot{x}|^2 \geq I_\varepsilon(x)\left(\frac{2-\alpha_2}{-\alpha_2 E}\right) \geq c_1\left(\frac{2-\alpha_2}{-\alpha_2 E}\right) \tag{II.3.9}$$

It follows from (II.3.8) that

$$\int_0^1 E - F_\varepsilon(x) \geq \frac{-\alpha_1 E - \delta^*(\varepsilon)}{2 - \alpha_1},$$

where $\delta^*(\varepsilon) = \frac{\delta(\varepsilon)}{\int_0^1 |\dot{x}|^2}$ has $\lim_{\varepsilon \to 0} \delta^*(\varepsilon) = 0$. Therefore

$$\text{(II.3.10)} \qquad \frac{1}{2}\int_0^1 |\dot{x}|^2 \le I_\varepsilon(x)\Big(\frac{2-\alpha_1}{-\alpha_1 E - \delta^*(\varepsilon)}\Big) .$$

From (II.3.5) we deduce that, for every $\rho > 0$, we have

$$\text{(II.3.11)} \qquad \inf_{\substack{x\in S_\rho \\ |x(0)|=|x(1)|=\varepsilon}} \int_0^1 -F_\varepsilon(x) \ge \inf_{\substack{x\in S_\rho \\ |x(0)|=|x(1)|=\varepsilon}} \int_0^1 \frac{a}{|x|^\alpha} .$$

From (H4) and the fact that $\phi(\alpha) = \left(\sqrt{2}\pi\right)^{-\alpha} J(\alpha,1)$, we can fix γ and ξ small enough to have

$$\text{(II.3.12)} \qquad (\frac{2}{2+\xi})\,\frac{J(\alpha,1)-\gamma}{(\sqrt{2}\,\pi)^\alpha} > \frac{b}{a}\frac{\alpha_2}{\alpha_1}\left(\frac{\alpha}{\alpha_2}\right)^{\frac{\alpha}{2}}\left(\frac{2-\alpha}{2-\alpha_2}\right)^{\frac{2-\alpha}{2}} .$$

We apply Proposition II.2.3 with $\rho_1^2 = \frac{2-\alpha_1}{-\alpha_1 E}c_1$ and $\rho_2^2 = \frac{2-\alpha_1}{-\alpha_1 E-\delta^*(\varepsilon)}c_2$, finding an $\varepsilon(\gamma)$ such that (II.2.6) holds for every $\varepsilon \le \varepsilon(\gamma)$ and $\rho \in [\rho_1, \rho_2]$ (it is possible since $\lim_{\varepsilon\to 0}\delta^*(\varepsilon) = 0$). From Lemma II.3.2 we can take ε so small that

$$\delta(\varepsilon) \le \xi I_\varepsilon(x) ,$$

so that (II.3.8) becomes

$$\text{(II.3.13)} \qquad (2+\xi)I_\varepsilon(x) \ge \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)\left(\int_0^1 -\alpha_1 F_\varepsilon(x)\right) .$$

Assuming that $\min_{t\in[0,1]}|x(t)| \le \varepsilon$, from (II.3.11),(II.3.5), (II.2.6) and Proposition II.2.1.i) we obtain

$$\text{(II.3.14)} \qquad -\alpha_1\int_0^1 F_\varepsilon(x) \ge \alpha_1 a\,(J(\alpha,1)-\gamma)\frac{1}{\left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)^{\alpha/2}}$$

and therefore, from (II.3.9) and (II.3.13), (II.3.14) leads to

$$\begin{aligned}(2+\xi)\,I_\varepsilon(x) &\ge \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)^{\frac{2-\alpha}{2}} \alpha_1 a(J(\alpha,1)-\gamma) \\ &\ge \left[I_\varepsilon(x)(\frac{2-\alpha_2}{-\alpha_2 E})\right]^{\frac{2-\alpha}{2}} \alpha_1 a(J(\alpha,a)-\gamma) .\end{aligned}$$

The above inequality, together with (II.3.6) contradicts (II.3.12) (that is (H4)).◇

3.II.4. Proof of Theorem 3.II.1

In order to prove Theorem 1, first of all we are going to replace the "singular" F with a regular $F\varepsilon$, in accordance with Definition II.3.1. Then, because of the estimates of Section 3, we shall be in a position to apply the results of Section 2. Finally from the a priori estimate (Proposition II.3.1) we shall conclude that the critical point of I_ε found by the application of Theorem 2.1 does not interact with the truncation, that is, it is actually a critical point of I.

According with the notations of section II.1, let $0 < \varepsilon < 1$ and let

$$\begin{aligned}
\rho_b &= [\frac{(2-\alpha)b}{-2E}]^{\frac{1}{\alpha}} \\
\Sigma_1 &= \{x \in H \,/\, \frac{1}{2}\int_0^1 |\dot{x}|^2 = 2\pi^2\rho_b^2\} \\
\Sigma_2 &= \bigcup_{x\in S^{N-2}(e_1)}\bigcup_{\theta\in\{\varepsilon,\varepsilon^{-1}\}} \{\theta\rho_b((4x+(x-e_1)\cos 2\pi t + e_N\sin 2\pi t)\} \\
A^* &= \bigcup_{x\in S^{N-2}(e_1)}\bigcup_{\theta\in[\varepsilon,\varepsilon^{-1}]} \{\theta\rho_b((4x+(x-e_1)\cos 2\pi t + e_N\sin 2\pi t)\} \\
\mathcal{A}^* &= \mathcal{A}^*_{\Sigma_1,\Sigma_2}\,.
\end{aligned}$$

It follows from Theorem II.1.2 that $\mathcal{A}^* \neq \emptyset$, for every $0 < \varepsilon < 1$, so that (II.1.3) holds true.

We replace F with F_ε and I with I_ε as in Definition II.3.1. ε is taken so small in such a way that Proposition II.3.1 holds and (II.3.8) and (II.3.9) hold too.

Taking into account of (H4), let us fix $\gamma > 0$ so small that

$$\frac{J(\alpha,1)-\gamma}{(\sqrt{2}\pi)^\alpha} = \phi(\alpha) - \frac{\gamma}{(\sqrt{2}\pi)^\alpha} > \frac{b}{a}\,,$$

and let ε be given by Proposition II.2.3 so that (II.2.5) holds; then we have

$$\text{(II.4.1)}\qquad \begin{aligned}
\inf_{\Sigma_1\cap\partial A} I_\varepsilon \;\geq\; & \inf_{\substack{x\in\Sigma_1\\ |x(0)|=|x(1)|=\varepsilon}} 2\pi^2\rho_b^2\left(E + a\int_0^1 \frac{1}{|x|^\alpha}\right) \\
=\; & 2\pi^2\rho_b^2\left(E + \frac{a(J(\alpha,1)-\gamma)}{(\sqrt{2}\pi\rho_b)^\alpha}\right) > \pi^2 b^{\frac{2}{\alpha}}\alpha\left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}}.
\end{aligned}$$

From (H1), Definition II.3.1 and Propositions II.2.2 and II.2.3, we deduce that, for any small ε,

$$
\begin{aligned}
& \inf_{A\in\mathcal{A}^*_{\Sigma_1,\Sigma_2}} \sup_{A} I_\varepsilon \geq \inf_{A\in\mathcal{A}^*_{\Sigma_1,\Sigma_2}} \sup_{A\cap\Sigma_1} I_\varepsilon \\
& \geq \max\left(\inf_{A\in\mathcal{A}^*_{\Sigma_1,\Sigma_2}} \sup_{A\cap\Sigma_1} \,, \inf_{\substack{x\in\Sigma_1 \\ |x(0)|=|x(1)|=\varepsilon}} \right) 2\pi^2\rho_b^2 \left(E + a\int_0^1 \frac{1}{|x|^\alpha}\right) \qquad \text{(II.4.2)} \\
& \geq \pi^2\, a^{2/\alpha}\, \alpha \left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}}
\end{aligned}
$$

On the other hand, from (H1), Definition II.3.1 and Proposition II.2.2 we deduce

$$
\begin{aligned}
& \inf_{A\in\mathcal{A}^*_{\Sigma_1,\Sigma_2}} \sup_{A} I_\varepsilon \qquad \text{(II.4.3)} \\
& \leq \inf_{A\in\mathcal{A}^*_{\Sigma_1,\Sigma_2}} \sup_{A} \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)\left(\int_0^1 E + \frac{b}{|x|^\alpha}\right) = \pi^2\, b^{2/\alpha}\, \alpha \left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}}
\end{aligned}
$$

Moreover, we can fix ε small enough to have

$$
\sup_{\Sigma_2} I_\varepsilon \leq \sup_{\Sigma_2} \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)\left(\int_0^1 E + \frac{b}{|x|^\alpha}\right) < \pi^2\, a^{2/\alpha}\, \alpha \left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} . \qquad \text{(II.4.4)}
$$

Hence I_ε verifies assumptions (II.1.3), (II.1.4) and (II.1.5) of Theorem II.1.1. Finally we have to check the condition (II.1.6). To this aim we prove the following Proposition.

PROPOSITION II.4.1. *Assume that F satisfies (H1) and (H2), and let F_ε as in the Definition II.4.2 with $\varepsilon^\alpha < \frac{a}{-E}$. Then I_ε satisfies the condition (C) at any strictly positive level.*

PROOF: Let $(x_n)_n$ be a sequence such that

$$
I_\varepsilon(x_n) = c_n \to c > 0 \qquad \text{(II.4.5)}
$$

$$
-\left(\int_0^1 E - F_\varepsilon(x_n)\right)\ddot{x}_n = \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)\nabla F_\varepsilon(x_n) + h_n \quad \text{with} \quad h_n \to 0 \quad \text{in} \quad H^{-1} \qquad \text{(II.4.6)}
$$

$$
\left|\left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)\left(\int_0^1 E - F_\varepsilon(x_n)\right) - \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)\int_0^1 \nabla F_\varepsilon(x_n)\cdot x_n\right| \to 0 . \qquad \text{(II.4.7)}
$$

We claim that $\left(\|\dot{x}_n\|_{L^2}\right)_n$ is bounded. Assuming by the contrary that $\limsup_{n\to+\infty}\|\dot{x}\|_{L^2} = +\infty$, up to a subsequence, we get from (II.4.5) that

$$\text{(II.4.8)} \qquad \lim_{n\to+\infty} \int_0^1 E - F_\varepsilon(x_n) \;=\; 0\,.$$

Setting $A_n = \{t \in [0,1] \;/\; |x_n(t)| \geq \varepsilon\}$, we get from Tchebitchev inequality that

$$\text{(II.4.9)} \qquad \limsup_{n\to+\infty} \text{meas}(A_n) \;>\; 0$$

(indeed we have $\int_{[0,1]\setminus A_n} -F_\varepsilon(x_n) \geq \frac{a}{\varepsilon^\alpha}(1 - \text{meas}(A_n))$). From (II.3.2), (II.3.3) and (H2) we obtain

$$\text{(II.4.10)} \qquad 2c_n \;\geq\; \left(\frac{1}{2}\int_0^1 |\dot{x}_n|^2\right)\left(\int_{A_n} -\alpha_1 F_\varepsilon(x_n)\right) + \delta_n$$

with $\lim_{n\to+\infty}\delta_n = 0$.

Setting $M_n = \max_{t\in[0,1]}|x_n(t)|$, from (H1), (II.4.10)leads to

$$\text{(II.4.11)} \qquad 2c_n \;\geq\; \left(\frac{1}{2}\int_0^1 |\dot{x}_n|^2\right)\left(\alpha_1 a \;\text{meas}(A_n)\,\frac{1}{M_n^\alpha}\right) + \delta_n\,.$$

Since, for every n, $\int_0^1 -F_\varepsilon(x_n) \geq -E$, we have

$$\text{(II.4.12)} \qquad M_n^2 \;\leq\; d_1\frac{1}{2}\int_0^1 |\dot{x}_n|^2 + d_2\,,$$

for some $d_1, d_2 \in \mathbb{R}$ independent on n. Taking into account of (II.4.9), (II.4.11) and (II.4.12) imply the boundness of $\left(\|\dot{x}_n\|_{L^2}\right)_n$. This fact and (II.4.12) imply the existence of a subsequence $(x_{n_k})_k$ uniformly convergent, and by (II.4.6) we can conclude that (x_{n_k}) converges strongly in H^1_N. ◇

End of the proof of Theorem 1: For ε small enough, by the application of Theorem 2.1, we obtain the existence of a critical point x_ε of I_ε at the level $c^* \in [c_1, \pi^2 b^{\frac{2}{\alpha}}\alpha\left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}}]$. Since we can choose ε so small as we like, from Proposition II.3.1 we can conclude that x_ε is actually a critical point if I in Λ.◇

II.5. Proof of Corollary 3.II.2

We are going to prove Corollary II.2 as a consequence of Theorem II.1 and of the estimates here. Let x be the solution found by the application of Theorem 1: the main idea in proving Corollary II.2 consists in estimating some $L^{2\gamma}$ norm of the derivative of $|x(t)|^{\frac{2+\alpha}{2}}$.

Throughout this section, $x \in \Lambda$ is a critical point of I such that

$$0 < I(x) \leq \pi^2 b^{\frac{2}{\alpha}} \alpha \left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} . \tag{II.5.1}$$

$$-\ddot{x} = \lambda^2 \nabla F(x) \; ; \tag{II.5.2}$$

Hence we have

$$\frac{1}{2}|\dot{x}|^2 + \lambda^2 F(x) = \lambda^2 E \; ; \tag{II.5.3}$$

where

$$\lambda^2 \; = \; \frac{\frac{1}{2}\int_0^1 |\dot{x}|^2}{\int_0^1 E - F(x)} \; . \tag{II.5.4}$$

Taking the L^2 inner product of (II.5.2) by x, from (II.5.1) and (II.5.4) we obtain

$$\lambda^2 \leq I(x) \left(\frac{2-\alpha}{-\alpha E}\right)^2 \leq \frac{4\pi^2}{\alpha b} \left(\frac{(2-\alpha)b}{-2E}\right)^{\frac{2+\alpha}{\alpha}} \left(\frac{\alpha(2-\alpha_1)}{\alpha_1(2-\alpha)}\right)^2 . \tag{II.5.5}$$

Setting $y(t) = x(\lambda^{-1}t)$ and $\rho(t) = |y(t)|$, then ρ satisfies

$$\begin{cases} -\frac{1}{2}\ddot{\rho^2} = -2E - (2-\alpha_1)\frac{c(t)}{\rho^\alpha} \\ \rho(t+\lambda) = \rho(t) \; , \; \forall t \\ \rho(t) > 0 \; , \; \forall t \; , \end{cases} \tag{II.5.6}$$

where

$$c(t) = \frac{1}{2-\alpha_1}[-2F(y(t)) - \nabla F(y(t)) \cdot y(t)]|y(t)|^\alpha ;$$

so that (H2) imply

$$\frac{2-\alpha_2}{2-\alpha_1}a \leq c(t) \leq b \ , \ \forall t \in \mathbb{R} \ . \tag{II.5.7}$$

We set $c(t) = \frac{2-\alpha_2}{2-\alpha_1}a(1+c_0(t))$ with

$$0 \leq c_0(t) \leq \frac{(2-\alpha_1)b}{(2-\alpha_2)a} - 1 \ , \ \forall t \ . \tag{II.5.8}$$

By the change of variables $s(t) = \int_0^t \frac{1}{\rho^\alpha}$, and $\mu(s) = \rho^{2-\alpha}(s(t))$, (II.5.6) becomes equivalent to the system

$$\begin{cases} -\mu'' = -2(2-\alpha)E\mu^\gamma - (2-\alpha)(2-\alpha_2)a(1+c_0(s)) \\ \mu(s+\omega) = \mu(s) \ , \ \forall s \\ \mu(s) > 0 \ , \ \forall s \ , \end{cases} \tag{II.5.9}$$

where $(\cdot)'$ denotes the derivation with respect to the variable s and

$$\gamma = \frac{\alpha}{2-\alpha} \ . \tag{II.5.10}$$

Moreover, from (II.5.6) and (II.5.7) we have

$$\omega = \int_0^\lambda \frac{1}{\rho^\alpha} \leq \frac{-2E}{(2-\alpha_2)a}\lambda \ . \tag{II.5.11}$$

We denote $v = \mu'$ By derivating the equation in (II.5.6) and by taking the L^2 inner product with v and taking into account of (II.5.10), we obtain

$$\int_0^\omega (v')^2 ds \ = \ -2\alpha E \int_0^\omega \mu^{\gamma-1} v^2 ds + (2-\alpha)(2-\alpha_2)a \int_0^\omega c_0 v' ds \ . \tag{II.5.12}$$

From Holder inequality, taking into account of (II.58), (II.5.12) leads to

$$\begin{aligned} &\int_0^\omega (v')^2 ds \ \leq \\ &- 2\alpha E \left(\int_0^\omega \mu^\gamma ds\right)^{\frac{\gamma-1}{\gamma}} \left(\int_0^\omega v^{2\gamma} ds\right)^{\frac{1}{\gamma}} + c_1 a\sqrt{\omega} \left(\int_0^\omega (v')^2 ds\right)^{\frac{1}{2}} \ . \end{aligned} \tag{II.5.13}$$

where

$$c_1 = (2-\alpha)(2-\alpha_2)\left(\frac{(2-\alpha)b}{(2-\alpha_2)a} - 1\right) . \tag{II.5.14}$$

By integrating the equation in (II.5.9) and from (II.5.8) we deduce that $\int_0^\omega \mu^\gamma ds \leq \frac{(2-\alpha)b}{-2E}\omega$; thus, for any $0 < \delta < 1$ we have

$$\delta \int_0^\omega (v')^2 ds - \frac{c_1^2 a^2 \omega}{1-\delta} \leq -2\alpha E \left(\frac{(2-\alpha)b}{-2E}\omega\right)^{\frac{\gamma-1}{\gamma}} \left(\int_0^\omega v^{2\gamma} ds\right)^{\frac{1}{\gamma}} . \tag{II.5.14}$$

Assuming that μ is not constant, from Proposition II.2.3, Definition II.2.4 and from (II.5.14),(II.5.11) and (II.5.5) we obtain

$$\delta - \frac{c_2 a^2 \omega^{\frac{2\gamma+1}{\gamma}}}{(1-\delta)\Phi^*(\alpha)\left(\int_0^\omega (v)^{2\gamma} ds\right)^{\frac{1}{\gamma}}} \leq \left(\frac{(2-\alpha)b}{(2-\alpha_2)a}\right)^2 \frac{1}{\Phi^*(\alpha)} , \tag{II.5.16}$$

where

$$c_2 = \frac{2\pi^2 \alpha}{2-\alpha} c_1^2 , \tag{II.5.17}$$

where (according with the notations of Definition II.2.4),

$$\begin{aligned} \Omega^*(\alpha) = \Omega(\gamma) = \Omega\left(\frac{\alpha}{2-\alpha}\right) \\ = \frac{S^2\left(\frac{\alpha}{2-\alpha}, 1\right)}{4\pi^2(2-\alpha}. \end{aligned}$$

Now we fix

$$\delta = \frac{1}{2}\left(1 + \left(\frac{(2-\alpha)b}{(2-\alpha_2)a}\right)^2 \frac{1}{\Phi^*(\alpha)}\right) ,$$

(note that, by (H5), $0 > \delta > 1$), so that (II.5.16) becomes

$$\left(\int_0^\omega v^{2\gamma} ds\right)^{\frac{1}{\gamma}} \leq c_3 a^2 \omega^{\frac{2\gamma+1}{\gamma}} , \tag{II.5.18}$$

with

(II.5.19)
$$c_3 = \frac{4c_2}{\left(1-\left(\frac{(2-\alpha)b}{(2-\alpha_2)a}\right)^2 \frac{1}{\Phi^*(\alpha)}\right)^2 \Omega^*(\alpha)} .$$

By an easy computation one finds that

$$\left(\int_0^\lambda \left(\frac{d}{dt}\rho^{\frac{2+\alpha}{2}}\right)^{2\gamma} dt\right)^{\frac{1}{\gamma}} = \left(\frac{(2+\alpha)}{2(2-\alpha)}\right)^2 \left(\int_0^\omega (\mu')^{2\gamma} ds\right)^{\frac{1}{\gamma}} ,$$

so that,from (II.5.18), (II.5.19) and (II.5.11) we get

(II.5.20)
$$\left(\int_0^\lambda \left(\frac{d}{dt}\rho^{\frac{2+\alpha}{2}}\right)^{2\gamma} dt\right)^{\frac{1}{\gamma}} \leq c_4 a^2 \left(\frac{-2E}{(2-\alpha_2)a}\right)^{\frac{2+\alpha}{\alpha}} \lambda^{\frac{2+\alpha}{\alpha}} ,$$

where

(II.5.21)
$$c_4 = \left(\frac{2+\alpha}{2(2-\alpha)}\right)^2 c_3 .$$

Let us write $\rho_* = \left(\frac{(2-\alpha_2)a}{-2E}\right)^{\frac{1}{\alpha}}$. Remark that ρ_* is assumed by the function $\rho(t)$, since, from (II.5.6), every local minimum $\rho(t_0)$ has $\rho(t_0) \leq \rho_*$.From (II.5.20) and (II.5.5)we deduce

$$\begin{aligned} |\rho^{\frac{2+\alpha}{2}}(t) - \rho_*^{\frac{2+\alpha}{2}}| &\leq \int_0^\lambda |\frac{d}{dt}\rho^{\frac{2+\alpha}{\alpha}}|dt \\ &\leq \left(\int_0^\lambda \left(\frac{d}{dt}\rho^{\frac{2+\alpha}{2}}\right)^{2\gamma} dt\right)^{\frac{1}{2\gamma}} \lambda^{1-\frac{1}{2\gamma}} \\ &\leq \sqrt{c_4}a \left(\frac{-2E}{(2-\alpha_2)a}\right)^{\frac{2+\alpha}{2\alpha}} \lambda^2 \\ &\leq c_5 \rho_*^{\frac{2+\alpha}{2}} \end{aligned}$$

where

(II.5.22)
$$c_5 = \sqrt{c_4}\left(\frac{b}{a}\right)^{\frac{2}{\alpha}} \left(\frac{2-\alpha}{2-\alpha_2}\right)^{\frac{2+\alpha}{\alpha}} \left(\frac{\alpha(2-\alpha_1)}{\alpha_1(2-\alpha)}\right)^2 .$$

Finally we get

(II.5.23) $$\rho(t) \geq \sigma_1 \rho_* = \sigma \left(\frac{(2-\alpha)a}{-2E}\right)^{\frac{1}{\alpha}} \qquad \forall t \in \mathbb{R},$$

where

$$\sigma_1^{\frac{2+\alpha}{2}} = 1 - c_5 ,$$

and

(II.5.24) $$\sigma = \left(\frac{2-\alpha_2}{2-\alpha}\right)^{\frac{1}{\alpha}} \sigma_1$$

Finally, from (II.5.22),(II.5.21),(II.5.19),(II.5.17) and (II.5.14) we get

$$\sigma^{\frac{2+\alpha}{2}} = \{1 - \{\frac{2+\alpha}{2\pi}\sqrt{2-\alpha}\left(\frac{2-\alpha}{2-\alpha_2}\right)^{\frac{2+\alpha}{\alpha}} \left(\frac{b}{a}\right)^{\frac{2}{\alpha}} \left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a} - 1\right)$$

(II.5.25) $$\frac{\sqrt{\Omega^*(\alpha)}}{\Phi^*(\alpha) - \left(\frac{(2-\alpha)b}{(2-\alpha_2)a}\right)^2}\}\} \left(\frac{2-\alpha_2}{2-\alpha}\right)^{\frac{2+\alpha}{2\alpha}} .$$

The above equality and (II.5.23) prove Corollary 3.II.2, taking σas in (II.5.25).

APPENDIX : Proof of Lemma II.3.2.

Throughout this section, $x \in H$ is a critical point of I_ε satisfying

$(A.1)$ $$0 < c_1 \leq I_\varepsilon(x) \leq c_2 .$$

Therefore, setting

$$\lambda^2 = \frac{\frac{1}{2}\int_0^1 |\dot{x}|^2}{\int_0^1 E - F_\varepsilon(x)} ,$$

x solves

$(A.2)$ $$-\ddot{x} = \lambda^2 \nabla F_\varepsilon(x)$$

$$(A.3)\qquad \frac{1}{2}|\dot{x}|^2 + \lambda^2 F_\varepsilon(x) \;=\; \lambda^2 E\;.$$

Taking the L^2 product of (A.2) by x, and from (II.3.3), we get

$$(A.4)\qquad -\int_0^1 F_\varepsilon(x) \;\leq\; \frac{-2E}{2-\alpha_2}\;,$$

so that

$$(A.5)\qquad \lambda^2 \;\geq\; c_1\,\big(\frac{2-\alpha_2}{-\alpha_2 E}\big)^2\;.$$

STEP 1: For $\varepsilon > 0$ small enough, we have

$$(A.6)\qquad \frac{1}{2}\int_0^1 |\dot{x}|^2 \;\leq\; c_2\,\frac{2+\alpha_1}{-\alpha_1 E}$$

PROOF: Let us denote

$$\xi \;=\; \int_0^1 E - F_\varepsilon(x)\;,$$

and

$$A \;=\; \{t\in[0,1]\;/\; -F_\varepsilon(x(t)) \geq -2E\}\;.$$

It follows from Tchebicev inequality that

$$\text{meas}(A) \;\leq\; \frac{\xi}{-E}\;.$$

Moreover, for $\varepsilon > 0$ small enough we have $F_\varepsilon(x(t)) \;=\; F(x(t))$, $\forall\; t\in[0,1]\setminus A$; from (II.3.2),(II.3.3),(H2) and (A.2) we deduce that

$$\int_0^1 |\dot{x}|^2 = \frac{\frac{1}{2}\int_0^1|\dot{x}|^2}{\int_0^1 E - F_\varepsilon(x)}\int_0^1 \nabla F_\varepsilon(x)\cdot x \geq \frac{\frac{1}{2}\int_0^1|\dot{x}|^2}{\int_0^1 E - F_\varepsilon(x)}\int_{[0,1]\setminus A} -\alpha_1 F_\varepsilon(x)\;,$$

so that

$$2\xi \;\geq\; -\alpha_1\int_{[0,1]\setminus A} F_\varepsilon(x) \;\geq\; -\alpha_1 E\,[1-\text{meas}(A)]\;;$$

therefore

$$\xi \;\geq\; \frac{-\alpha_1 E}{(2+\alpha_1)}\;,$$

and (A.6) follows from this fact, since $\frac{1}{2}\int_0^1|\dot{x}|^2 = \frac{I_\varepsilon(x)}{\xi}$. Moreover we have

$$(A.7)\qquad \lambda^2 \;\leq\; c_2\,\big(\frac{2+\alpha}{-\alpha E}\big)^2\;.$$

◇

STEP 2: Assume that $\varepsilon^\alpha < \frac{(2-\alpha_2)a}{-2E}$, and

$$|x(0)| \;=\; \min_{t\in[0,1]} |x(t)| \leq \varepsilon \,.$$

Consider

$$\Omega \;=\; \big\{[t_1, t_{i+1}] \subseteq [0,1] \;/\; |x(t_i)| = |x(t_{i+1})| = \varepsilon \\ \text{and } |x(t)| > \varepsilon \,,\; \forall\, t \in (t_i, t_{i+1})\big\} \,.$$

Then

$$\sharp\,\Omega \;\leq\; \left[\left(\frac{(2-\alpha_2)a}{-2E}\right)^{1/\alpha} - \varepsilon\right]^{-2} \frac{c_2}{2}\left(\frac{2+\alpha_1}{-\alpha_1 E}\right) .$$

PROOF: From (II.3.2) and (II.3.3) we deduce that if $\bar{t} \in [t_1, t_{i+1}]$ is a local maximum, then

$$|x(\bar{t})|^\alpha \;\geq\; \frac{(2-\alpha_2)a}{-2E} \,.$$

Therefore, from Hölder inequality, we get

$$\begin{aligned}(t_{i+1}-t_i)^{1/2}\left(\int_0^1 |\dot{x}|^2\right)^{1/2} \;&\geq\; \int_{t_i}^{\bar{t}} |\dot{x}| + \int_{\bar{t}}^{t_{i+1}} |\dot{x}| \\ &\geq\; 2\left[\left(\frac{(2-\alpha_2)a}{-2E}\right)^{1/\alpha} - \varepsilon\right] ;\end{aligned}$$

the claim follows from this inequality and (A.6).◇

STEP 3: For $0 < \varepsilon \leq \bar{\varepsilon}$ ($\bar{\varepsilon}$ small) there is a function $\delta_1(\varepsilon)$ such that

$$\lim_{\varepsilon\to 0} \delta_1(\varepsilon) \;=\; 0$$

and, for any critical point of I_ε satisfying (A.1), if

$$B \;=\; \{t \in [0,1] \;/\; |x(t)| > \varepsilon\}$$

then

$$\int_{[0,1]\setminus B} -F_\varepsilon(x) \;\leq\; \delta_1(\varepsilon) \,. \tag{A.8}$$

PROOF: From (A.3), (II.3.2), (H1) and (A.7), if $|x(\bar{t})| = \varepsilon$, we deduce that

$$\begin{aligned}|\dot{x}|\,|x|^2 \;&=\; 2\lambda^2 E|x|^2 - 2\lambda^2 F_\varepsilon(x)|x|^2 \\ &\leq\; 2\lambda^2 E\varepsilon^2 + 2\lambda^2 b\varepsilon^{2-\alpha} \;=\; (\delta_2(\varepsilon))^2\end{aligned} \tag{A.9}$$

with $\lim_{\varepsilon\to 0}\delta_2(\varepsilon) = 0$.

Let Ω as in the step 2, and let $n = \sharp\Omega$. From (A.2), (A.7) and (II.3.3) we have that

$$\frac{d}{dt}x\cdot\dot{x}(t) \ \geq\ \lambda^2(2E-(2-\alpha_2)F_\varepsilon(x(t)))\ , \qquad ,\forall t\in[0,1]\ .$$

By integrating the above inequality and from (A.9) we deduce that

$$(A.10) \qquad (2-\alpha_2)\int_{[0,1]\setminus B} -F_\varepsilon(x)\ \leq\ -2E(1-\text{meas}(B))+\frac{1}{\lambda^2}\,n\,\delta_2(\varepsilon)$$

From (A.4), (II.3.5) and Tchebicev inequality we deduce

$$(A.11) \qquad 1-\text{meas}(B)\ \leq\ \frac{-2E}{2-\alpha_2}\,\frac{\varepsilon^\alpha}{a}$$

so the thesis holds from (A.10), (A.11) and (A.5), just by taking

$$\delta_1(\varepsilon)\ =\ \frac{1}{2-\alpha_2}\Big[\frac{(-2E)^2}{2-\alpha_2}\,\frac{\varepsilon^\alpha}{a}+\frac{1}{c_1}\,\Big(\frac{-\alpha_2 E}{2-\alpha_2}\Big)^2\,n\,\delta_2(\varepsilon)\Big]\ ,$$

since n has an upper bound independent on ε if $\varepsilon \leq \overline{\varepsilon} < \left(\frac{(2-\alpha_2)a}{-2E}\right)^{1/\alpha}$.◇

END OF THE PROOF: From (II.3.2), (II.3.3) (H2) and (A.2) we get

$$\left(\int_0^1 E-F_\varepsilon(x)\right)\left(\int_0^1|\dot{x}|^2\right) \geq \left(\frac{1}{2}\int_0^1|\dot{x}|^2\right)\left(\int_{[0,1]\setminus B}-\alpha_1 F_\varepsilon(x)\right)$$

Hence, from (A.6), (A.8) and the above inequality we have

$$\begin{aligned}&\left(\int_0^1 E-F_\varepsilon(x)\right)\left(\int_{[0,1]\setminus B}|\dot{x}|^2\right)\\ &\geq \left(\frac{1}{2}\int_0^1|\dot{x}|^2\right)\left(\int_0^1-\alpha_1F_\varepsilon(x)\right)\ -\ \left(\frac{1}{2}\int_0^1|\dot{x}|^2\right)\left(\int_{[0,1]\setminus B}-\alpha_1F_\varepsilon(x)\right)\\ &\geq \left(\frac{1}{2}\int_0^1|\dot{x}|^2\right)\left(\int_0^1-\alpha_1F_\varepsilon(x)\right)\ -\ c_2\left(\frac{2+\alpha_1}{-\alpha_1E}\right)\alpha_1\delta_1(\varepsilon)\ .\end{aligned}$$

Hence Lemma II.3.2 holds with $\delta(\varepsilon)=c_2\left(\frac{2+\alpha_1}{-E}\right)\delta_1(\varepsilon)$,by virtue of the Step 3.◇

3.III. The case $0 < \alpha < 1$

The discussion of 1.V shows that, when $0 < \alpha < 1$, Proposition II.2.1.iv) does not hold; therefore, the circular solutions to (P_E) do not correspond to the variational approach described in Theorem II.1.1. In this chapter we are going to show that the Linking Theorem can be applied to this situation. Moreover, an estimate similar to the one of Theorem 2.II.1 will allow us to avoid the collision solutions. As for the fixed period problem, the case $\alpha = 1$ is a limiting case, where both the two techniques are applicable; of course, in that case, no estimate is available in order to exclude the collision solutions (see also Part 2, chapter III).

We shall be concerned with the following assumptions:

$$\exists\, \alpha, \alpha_1, \alpha_2,\ , 0 < \alpha_1 \leq \alpha \leq \alpha_2 < 1,\ \exists b \geq a > 0 \text{ such that:}$$

(H1)
$$\frac{a}{|x|^\alpha} \leq -F(x) \leq \frac{b}{|x|^\alpha} \qquad \forall x \in \mathbb{R}^N \setminus \{0\}\,.$$

(H2)
$$-\alpha_1 F(x) \leq \nabla F(x) \cdot x \leq -\alpha_2 F(x) \qquad \forall x \in \mathbb{R}^N \setminus \{0\}\,.$$

The the main goal of this chapter is the following:

THEOREM 3.III.1. *Let $F \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfying (H1),(H2). Then there exists a function $\Phi : (0,1) \to \mathbb{R}$ such that, when*

(H3)
$$\frac{b}{a}\frac{\alpha_2}{\alpha_1} < \Phi(\alpha)\,,$$

then (P_E) has at least one solution if and only if $E < 0$. Moreover, Φ fulfils the following properties:

$$\Phi(1) > 1 \qquad \forall\, 0 < \alpha < 1$$
$$\lim_{\alpha \to 1} \Phi(\alpha) = 1$$
$$\lim_{\alpha \to 0} \Phi(\alpha) = 1\,.$$

As for the case $1 > \alpha > 2$ (Remark II.1), it easy to see that (H1) and (H2) imply that $E < 0$ is a necessary condition for the solvability of (P_E). Therefore, throughout this chapter,*E will denote a negative energy.*

III.1. The variational principle

Before recalling the Linking Theorem, let us give the following

DEFINITION III.1.1. *Let X be a Banach space, and let Σ_1 and Q be two closed subset of X. We say that Σ_1 and ∂Q link if*

$$\forall \text{ continuous } \gamma : Q \to X \text{ such that } \gamma_{|\partial Q} = id \quad \gamma(Q) \cap \Sigma_1 \neq \emptyset . \tag{III.1.1}$$

The following Theorem shows a relation between set intersection properties and the existence of critical point for (possibly unbounded) functionals. It is a generalized form of the Mountain Pass theorem and of the Saddle Point theorem ([8]).

THEOREM III.1.1 (LINKING THEOREM). *Assume that*

$$\Sigma_1 \text{ and } Q \text{ link }, \tag{III.1.2}$$

and let $I \in C^1(X; \mathbb{R})$ satisfy

$$\inf_{\Sigma_1} I > \sup_{\partial Q} I \tag{III.1.3}$$

Define

$$\Gamma = \{\gamma : Q \to X \text{ continuous } / \ \gamma_{|\partial Q} = id\} ,$$

and

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma(Q)} I ,$$

and assume moreover that I fulfils the Cerami condition at level c, (C_c). Then I has at least one critical point at level c.

We refer to [8] for the proof.

III.2. Some lemmas about the radial case

In this section we are going to define suitable Q and Σ_1 in such a way that, for the functional associated to the potential $F(x) = \frac{-a}{|x|^\alpha}$, the geometrical assumptions (III.1.2) and (III.1.3) of Theorem III.1.1. are satisfied. It will turn out that the corresponding critical level is the level of the circular solutions.

Let us consider the set of all the K-orbits of the problem

$$\text{(III.2.1)} \qquad \begin{cases} -\ddot{x} = (2\pi)^2 \left(\frac{(2-\alpha)a}{-2E}\right)^{\frac{3}{\alpha}} \frac{x}{|x|^3} \\ x(t+1) = x(t) \,. \end{cases}$$

Taking into account of the results of Part 1, chapter III, we know that the set of all the solutions (possibly crossing the origin) of (III.2.1) is (up to the S^1 symmetry) a 2-manifold homeomorphic to the ball B^2. Moreover, each K-orbit x verifies

$$\text{(III.2.2)} \qquad \int_0^1 |\dot{x}|^2 = (2\pi)^2 \left(\frac{(2-\alpha)a}{-2E}\right)^{\frac{2}{\alpha}}$$

$$\text{(III.2.3)} \qquad \int_0^1 \frac{1}{|x|} = \left(\frac{-2E}{(2-\alpha)a}\right)^{\frac{1}{\alpha}} \,.$$

Now, let us consider the continuous $g : B^2 \to H$ defined as

$$\text{(III.2.4)} \qquad g(z)(t) = \begin{cases} \left(\frac{(2-\alpha)a}{-2E}\right)^{\frac{1}{\alpha}} e^{2\pi i t} & \text{if } z = 0 \\ \text{elliptic 1-periodic Kepler orbit with symmetry axis } \dfrac{z}{|z|} & \text{if } 0 < |z| < \frac{1}{2} \\ \text{collision 1-periodic Kepler orbit} & \text{if } |z| = \frac{1}{2} \\ 2(1-|z|)g\left(\frac{z}{2|z|}\right)(t) + (2|z|-1)\frac{z}{|z|}\frac{a}{-E} & \text{if } \frac{1}{2} \le |z| \le 1 \,; \end{cases}$$

g admits a continuous extension $\overline{g} : B^2 \times B^{N-2}$ as

$$\text{(III.2.5)} \qquad \overline{g}(z,y) = \begin{cases} g(z) + 2\left(\frac{(2-\alpha)a}{-2E}\right)^{\frac{1}{\alpha}} y & \text{if } |y| \le \frac{1}{2} \\ 2(1-|z|)g(z, \frac{y}{2|y|}) + (2|z|-1)\left(\frac{(2-\alpha)a}{-2E}\right)^{\frac{1}{\alpha}} \frac{y}{|y|} & \text{if } \frac{1}{2} \le |y| \le 1 \,. \end{cases}$$

We observe that , for every $(z,y) \in B^2 \times B^{N-2}$,

$$
\text{(III.2.6)} \qquad \begin{aligned} &\int_0^1 |\frac{d}{dt}\overline{g}(z,y)(t)|^2 \le (2\pi)^2 \left(\frac{(2-\alpha)a}{-2E}\right)^{\frac{2}{\alpha}} \\ &\int_0^1 \frac{a}{|\overline{g}(z,y)(t)|^\alpha} \le a\left(\int_0^1 \frac{1}{|\overline{g}(z,y)(t)|}\right)^\alpha \le \frac{-2E}{2-\alpha} . \end{aligned}
$$

We define

$$
\text{(III.2.7)} \qquad Q_\varepsilon^R = [\varepsilon, R]\overline{g}(B^2 \times B^{N-2}) .
$$

Our first goal is the following result:

PROPOSITION III.2.1. *Let* $\Sigma_1 = \{x \in H \ / \ \int_0^1 |\dot{x}|^2 = (2\pi)^2 \left(\frac{(2-\alpha)a}{-2E}\right)^{\frac{2}{\alpha}} \ , \ \int_0^1 x = 0\}$. *Then , for every small* ε *and every large* $R > 0$, Σ_1 *and* ∂Q_ε^R *link.*

PROOF: First, the boundary $\partial Q_\varepsilon^R = \partial[\varepsilon, R]\overline{g}(B^2 \times B^{N-2}) = \varepsilon\overline{g}(B^2 \times B^{N-2}) \cup [\varepsilon, R]\partial\overline{g}(B^2 \times B^{N-2}) \cup R\overline{g}(B^2 \times B^{N-2})$. We observe that $\partial\overline{g}(B^2 \times B^{N-2}) = \overline{g}(\partial(B^2 \times B^{N-2}))$ is a subset of the space of all the constant functions, such that the identity has topological degree one, with respect to zero. Now let $\gamma \in \Gamma$, and let us consider P_0, the orthogonal projection of H into the space E_N of all the constant functions (remark that $P_0^{-1}(0) = \{x \in H \ / \ \int_0^1 x = 0\} = H_0$). We consider the continuous function $f : [\varepsilon, R] \times B^2 \times B^{N-2} \to E_N$ defined as $f(r, z) = P_0 \circ \gamma(r\overline{g}(z))$. Up to a small C^0 perturbation, we can assume that $f^{-1}(0)$ consists in a finite number of compact 1-manifolds. Moreover, since $\deg_0 f(r, \cdot) = 1$, for every $r \in [\varepsilon, R]$, and $f(\varepsilon, \cdot)^{-1}(0) = f(R, \cdot)^{-1}(0) = 0$, we can conclude that there is a unique connected component of $f^{-1}(0)$ connecting $\{\varepsilon\} \times (B^2 \times B^{N-2})$ with $\{R\} \times (B^2 \times B^{N-2})$. Let us call $(r(\lambda), z(\lambda))$ ($\lambda \in [0,1]$) a parametrization of such a 1-manifold; then we have $r(0) = \varepsilon$, $r(1) = R$, and $z(0) = z(1) = 0$ (since the circular function are the only functions having zero mean value of ∂Q_ε^R). Now, by (III.2.4) and (III.2.5), we have that $\int_0^1 |\frac{d}{dt}\gamma(r(0)\overline{g}(0))|^2 = (2\pi\varepsilon)^2 \left(\frac{(2-\alpha)a}{-2E}\right)^{\frac{2}{\alpha}}$ and $\int_0^1 |\frac{d}{dt}\gamma(r(1)\overline{g}(1))|^2 = (2\pi R)^2 \left(\frac{(2-\alpha)a}{-2E}\right)^{\frac{2}{\alpha}}$; therefore a $\lambda^* \in (0,1)$ exists such that $\int_0^1 |\frac{d}{dt}\gamma(r(\lambda^*)\overline{g}(z(\lambda^*)))|^2 = (2\pi)^2 \left(\frac{(2-\alpha)a}{-2E}\right)^{\frac{2}{\alpha}}$. Hence $\gamma(r(\lambda^*)\overline{g}(z(\lambda^*))) \in \Sigma_1$.$\diamond$

REMARK III.3.1: Of course, ∂Q_ε^R and $\theta\Sigma_1$ link for every $\theta \in (\varepsilon, R)$.$\diamond$

In order to estimate the critical level, we have the

PROPOSITION III.2.2. *Let $I(x) = \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)$ then, for each ε small and each $R > 0$ large enough, we have*

$$\text{(III.2.8)} \qquad \sup_{\partial Q_\varepsilon^R} I < \inf_{\Sigma_1} I = \pi^2\, a^{2/\alpha}\, \alpha \left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} ,$$

and moreover

$$\text{(III.2.9)} \qquad \inf_{\gamma\in\Gamma} \sup_{\gamma(Q)} I = \sup_{Q} I = \pi^2\, a^{2/\alpha}\, \alpha \left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} .$$

PROOF: We first observe that, by the convexity of the function $\frac{1}{t^\alpha}$, then

$$\inf_{\Sigma_1} \int_0^1 \frac{a}{|x|^\alpha} = a \inf_{\Sigma_1} \frac{1}{\left(\int_0^1 |x|^2\right)^{\frac{\alpha}{2}}} = \frac{-2E}{(2-\alpha)} ,$$

and therefore, since Σ_1 and ∂Q_ε^R link, we deduce that

$$\inf_{\gamma\in\Gamma} \sup_{\gamma Q_\varepsilon^R} I \geq \inf_{\Sigma_1} I = \pi^2\, a^{2/\alpha}\, \alpha \left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} .$$

On the other hand, for small values of ε and large values of R, we have

$$\sup_{\partial Q_\varepsilon^R} I \leq C_1 \varepsilon^{2-\alpha} ,$$

where C_1 depends only on the values of E and a.
Now let $x \in \Lambda$ be fixed: the computation of $\sup_{r>0} I(rx)$ leads to

$$\sup_{r>0} I(rx) = \frac{\alpha}{2}\left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{2}} \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)\left(\int_0^1 \frac{a}{|x|^\alpha}\right)^{\frac{2}{\alpha}} .$$

Therefore, from (III.2.6) and (III.2.7) we deduce that

$$\sup_{Q_\varepsilon^R} I = \sup_{[\varepsilon,R]\overline{g}(B^2\times B^{N-2})} I$$

$$= \sup_{\overline{g}(B^2\times B^{N-2})} \frac{\alpha}{2}\left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{2}} \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)\left(\int_0^1 \frac{a}{|x|^\alpha}\right)^{\frac{2}{\alpha}} = \pi^2\, a^{2/\alpha}\, \alpha \left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} .$$

III.3. Proof of Theorem III.1

As for the fixed period problem (cfr Part 2, chapter II), we are first going to show that (P_E) admits a generalized (i.e. possibly crossing the singularity) solutions. Then we shall exclude the collision solutions by means of some estimates on the level. To this aim, we consider a family of truncated potentials F_ε, as defined in Definition II.3.1, and we denote by I_ε the family of the associated functionals.

Our first step consists in proving the following result:

PROPOSITION III.3.1. *Assume (H1) and (H2) hold, and let I_ε be as in Definition II.3.1. Then, for every ε small, I_ε has a critical point x_ε satisfying*

$$\text{(III.3.1)} \qquad \pi^2\, a^{2/\alpha}\, \alpha \left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} \leq I_\varepsilon(x_\varepsilon) \leq \pi^2\, b^{2/\alpha}\, \alpha \left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} .$$

PROOF: We are going to apply Theorem III.1.1, together with the results of section III.2. Indeed, by Proposition III.2.1, assumption (III.1.2) is fulfilled (provided that ε is sufficiently small and R sufficiently large). Moreover,from (H1), (H2) and Definition II.3.1 and from Proposition III.2.2. we deduce that

$$\inf_{\gamma\in\Gamma} \sup_{\gamma(Q_\varepsilon^R)} I_\varepsilon \leq \pi^2\, b^{2/\alpha}\, \alpha \left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}}$$

and moreover, since $\inf_{\Sigma_1\cap\partial\Lambda} I_\varepsilon > \pi^2\, a^{2/\alpha}\, \alpha \left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}}$ (see also 1.VI), from Proposition III.2.2 (taking into account of Remark III.2.2), we obtain

$$\inf_{\gamma\in\Gamma} \sup_{\gamma(Q_\varepsilon^R)} I_\varepsilon \geq \pi^2\, a^{2/\alpha}\, \alpha \left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} > C_1\varepsilon^{2-\alpha} \geq \sup_{\partial Q_\varepsilon^R} I_\varepsilon$$

Therefore, since by Proposition II.4.1, the condition (C) of Cerami holds at any positive level, I_ε has at least one critical point x_ε satisfying (III.3.1).◇

PROPOSITION III.3.2. *Let x_ε be as in Proposition III.3.1. There exists a sequence $(x_{\varepsilon_n})_n$ converging in the strong topology of H to a limit x. Moreover x satisfies:*

$$\text{(III.3.2)} \qquad \pi^2\, a^{2/\alpha}\, \alpha \left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} \leq I(x) \leq \pi^2\, b^{2/\alpha}\, \alpha \left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}}$$

$$\text{(III.3.3)} \qquad \begin{aligned} &\left(\frac{2-\alpha_2}{-\alpha_2 E}\right)^2 \pi^2\, a^{2/\alpha}\, \alpha \left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} \leq \frac{\frac{1}{2}\int_0^1 |\dot{x}|^2}{\int_0^1 E - F(x)} \\ &\leq \left(\frac{2-\alpha_1}{-\alpha_1 E}\right)^2 \pi^2\, b^{2/\alpha}\, \alpha \left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} . \end{aligned}$$

PROOF: By the estimates of II.6, one first prove that the H-norm of the x_εs has a bound independent on ε if ε is sufficiently small. Therefore it admits a sequence uniformly converging (and H-weakly) converging to a limit $x \in H$. Of course, if $x \in \Lambda$, then x is a critical point of I and therefore (III.3.2), (III.3.3) obviously hold. If not, by the estimates of II.6, we can deduce that the convergence actually holds in the strong topology of H, and also that the terms $F_\varepsilon(x_\varepsilon$ converges to $F(x)$ in the L^1 topology. Then (III.3.2) and (III.3.3) easily follow from (III.3.1) and Proposition II.3.2 (?).◇

PROOF OF THEOREM III.1: Up to now we have found a strong limit x of a sequence of critical point of the truncated functionals satisfying moreover (III.3.2) and (III.3.3). Now, if $x \in Lambda$ the proof is complete, since x is a critical point of I. Thus we assume by the contrary that $x \in \partial\Lambda$ and we are going to apply Proposition II.2 of Part 2 to find a contradiction. To this end, we set

(III.3.4)
$$\lambda^2 = \frac{\frac{1}{2}\int_0^1 |\dot{x}|^2}{\int_0^1 E - F(x)} ;$$

then x solves

(III.3.5)
$$\begin{cases} -\ddot{x}(t) = \lambda^1 \nabla F(x(t)) & \forall\, t \notin x^{-1}(0) \\ \frac{1}{2}|\dot{x}(t)|^2 + \lambda^2 F(x(t)) = \lambda^2 E & \forall\, t \notin x^{-1}(0) \\ x(t+1) = x(t) & \forall\, t \in \mathsf{R} \\ x(t) \neq 0 & \text{a.e. in } [0,1] . \end{cases}$$

Setting $\rho(t) = |x(t)|$, from (III.3.5) we deduce that ρ satisfies

(III.3.6)
$$\begin{cases} \frac{1}{2}\ddot{\rho^2} = \lambda^2 \left(2E + (2-\alpha)a\frac{c(t)}{\rho^\alpha}\right) & \\ \rho(t+1) = \rho(t) & \forall t \in \mathsf{R} \\ \rho(t) \geq 0 & \forall t \in \mathsf{R} \\ \dot{\rho^2} & \text{absolutely continuous} , \end{cases}$$

where

(III.3.7)
$$c(t) = (-2F(x(t)) - \nabla F(x(t)) \cdot x(t)) \left(\frac{|x(t)|^\alpha}{a(2-\alpha)}\right) ;$$

of course (H1) and (H2) imply

(III.3.8)
$$\frac{2-\alpha_2}{2-\alpha} \leq c(t) \leq \frac{b(2-\alpha_1)}{a(2-\alpha)} \qquad \forall t \in \mathsf{R} .$$

Now let $\mu = \left(\frac{(2-\alpha)c_1^\alpha(1)}{-(2+\alpha)\lambda^2 E}\right)^{\frac{1}{2}}$: then μ satisfies

$$
\text{(III.3.9)} \qquad \begin{cases} \frac{1}{2}\ddot{\mu}^2 = -2\frac{2-\alpha}{2+\alpha}c_1^\alpha(1) + \frac{(2-\alpha)h(t)}{\mu^\alpha} \\ \mu(t+1) = \mu(t) \\ \mu(t) \geq 0 \\ \dot{\mu}^2 \qquad\qquad \text{absolutely continuous} , \end{cases}
$$

where

$$
h(t) = \left(\frac{(2-\alpha)c_1^\alpha(1)}{-(2+\alpha)\lambda^2 E}\right)^{\frac{2+\alpha}{2}} \lambda^2 ac(t)
$$

so that by the definition of c_1^α (???) and (III.3.4), (III.3.3) and (III.3.7) we deduce that

$$
\frac{a}{b}\left(\frac{\alpha_1}{\alpha_2}\right)^\alpha \left(\frac{(2-\alpha_2)}{(2-\alpha_1)}\right)^{\alpha+1} \leq \frac{a}{b}\left(\frac{(2-\alpha)\alpha_1}{(2-\alpha_1)\alpha}\right)^\alpha \frac{2-\alpha_2}{2-\alpha}
$$
$$
\leq h(t) \leq \frac{b}{a}\left(\frac{(2-\alpha)\alpha_2}{(2-\alpha_2)\alpha}\right)^\alpha \frac{2-\alpha_1}{2-\alpha} \leq \frac{b}{a}\left(\frac{\alpha_2}{\alpha_1}\right)^\alpha \left(\frac{(2-\alpha_1)}{(2-\alpha_2)}\right)^{\alpha+1} .
$$

Now let $h_0(\alpha)$ be as in Proposition II.2 of Part 2: we define

$$
\overline{\Phi}(\alpha) = 1 + h_0(\alpha) ,
$$

in such a way that, when (H3) holds, then both

$$
\left(\frac{b}{a}\left(\frac{\alpha_2}{\alpha_1}\right)^\alpha \left(\frac{(2-\alpha_1)}{(2-\alpha_2)}\right)^{\alpha+1} - 1\right) < h_0(\alpha) \text{, and } \left(1 - \frac{a}{b}\left(\frac{\alpha_1}{\alpha_2}\right)^\alpha \left(\frac{(2-\alpha_2)}{(2-\alpha_1)}\right)^{\alpha+1})\right) < h_0(\alpha) ,
$$

hold too. Hence, whenever $\frac{b\alpha_2}{a\alpha-1} < \Phi(\alpha)$, Proposition II.2. of Part 2 says that (III.3.9) does not admit any solution; hence we have a contradiction.◇

PART 4. MULTIPLICITY OF PERIODIC SOLUTIONS TO THE FIXED PERIOD PROBLEM

This part is devoted to two main purposes: the first one is to define a tool (homotopical index theory) which provides multiple critical points for functionals having a singularity and a lack of compactness. The second one is to apply the abstract results in order to find a multiplicity of periodic solutions to a class of dynamical systems with singular potential.

We shall deal with the problem of finding a multiplicity of solutions to the problem

$$(P_T) \qquad \begin{cases} -\ddot{x} = \nabla F(x) \\ x(t+T) = x(t) \\ x(t) \in \mathbb{R}^N \setminus \{0\}\,,\ N \geq 3 \end{cases}$$

where $T > 0$ is fixed and $F \in \mathcal{C}^1(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$ is a singular potential of an attractive type, that is, $F(x)$ behaves like $\frac{-a}{|x|^\alpha}$, for some $a > 0$ and $\alpha \geq 1$. If $F(x) = \frac{-a}{|x|^\alpha}$, we know that the problem (P_T) admits an infinity of solutions : indeed, for a fixed period T we have all the planar circular solutions $x(t) = Re^{i\omega t}$ (with $\omega = \frac{2\pi k}{T}$, and $R^{\alpha+2} = a\alpha\left(\frac{T}{2\pi k}\right)^2$) having $\frac{T}{k}$ as minimal period. For a more general potential F, we are interested in finding the set of the solutions of (P_T) corresponding in a variational sense to those of minimal period T.

4.I The homotopical index theory

In this chapter we are going to introduce a tool (homotopical index) which will turn out to be fit to treat the problem of the search of multiple critical point for singular functionals presenting a lack of coercivity at the level of the large constant functions.
Roughly speaking, we are going to work in the following situation:

As usual Λ denotes the subset of H of all the noncollision functions:

$$\Lambda = \{x \in H \ / \ x(t) \neq 0 \ \forall t \in \mathbb{R}\}$$

I is a positive functional of class C^1 in Λ, admitting a lower semicontinuous extension $\overline{I} : H \to \mathbb{R} \cup \{+\infty\}$. The features of I are such that

i) $\overline{I}(x) \geq 0$, for every x;
ii) $0 = \inf_{\Lambda} \overline{I}$ is not attained;
iii) 0 is the only level where the Palais-Smale condition fails;
iv) I is invariant under a group G of symmetries acting on H;
v) there is an $\varepsilon > 0$ such that the sublevel $\{x \in \Lambda \,/\, I(x) \leq \varepsilon\}$ can be contracted into E_N in a continuous and symmetry-preserving way, without crossing $\partial\Lambda = \{x \in H \,/\, \exists t \,, x(t) = 0\}$.

In this setting, it has been showed in Theorem I.2.1 of Part 2 that positive critical levels can be found as inf-sup levels of classes whose elements cannot be contracted in a continuous way into subsets of E_N without crossing $\partial\Lambda$.

We are going to make use of this kind arguments, together with the theory of the geometrical indices of [**12**].
Consider an index associated to the group G, that is a function $i : \mathcal{B} \to \mathrm{R} \cup \{+\infty\}$ (here $\mathcal{B}$ is the class of all the subsets of H which do not contain fixed points of G), satisfying

$$
\begin{array}{ll}
(i) & A \subseteq B \Longrightarrow i(A) \leq i(B) \\
(ii) & i(A \cup B) \leq i(A) + i(B) \\
(iii) & \text{if } h : A \to B \text{ is continuous and } G \text{ equivariant} \\
 & \quad \Longrightarrow i(A) \leq i(B) \\
(iv) & A \text{ compact } \Longrightarrow i(A) < +\infty \\
(v) & A \text{ compact } \Longrightarrow \exists\, \varepsilon > 0 \text{ such that } i(N_\varepsilon(A)) = i(A)\,.
\end{array}
$$

The homotopical index j of a set $A \subseteq \Lambda$ measures (in terms of the geometrical index i) how big is the set that one has to take off from A, in order to make it to be contractible into a subset of E_N, without crossing the boundary of Λ. The homotopical index will be shown to enjoy the following properties:

$$
\begin{array}{ll}
(i) & A \subseteq B \Longrightarrow j(A) \leq j(B) \\
(ii) & j(A \cup B) \leq j(A) + i(B) \\
(iii) & \text{if } h : A \times [0,1] \to H \text{ is a continuous and } G\text{-equivariant homotopy,} \\
 & \text{with } h(A,[0,1] \cap \partial\Lambda = \emptyset, \\
 & \qquad\qquad \Longrightarrow j(A) \leq j(h(A,1)) \\
(iv) & A \text{ compact } \Longrightarrow j(A) < +\infty \\
(v) & A \text{ compact } \Longrightarrow \exists\, \varepsilon > 0 \text{ such that } j(N_\varepsilon(A)) = j(A)\,,
\end{array}
$$

for any closed A, $B \in \mathcal{B} \cap 2^{\Lambda}$. Then multiple critical points will appear at the inf-sup levels of the classes $\Gamma_r^* = \{A \in \mathcal{B} \cap 2^{\Lambda} \text{ compact} \,/\, j(A) \geq r \,,\, \sup_A I \leq \inf_{\partial\Lambda} I\,\}$.

In order to obtain multiple solutions to the problem (P_T), we are going to exploit the invariance of the problem under the compact group of symmetries

$$G = \{P_s(x(\cdot)) = x(s - \cdot)\,,\, T_s(x(\cdot)) = x(s + \cdot)\}_{s \in [0,T]}\,.$$

The index of S^1 symmetry related to the subgroup of G, $\{T_s(x(\cdot)) = x(s+\cdot)\}_{s\in[0,T]}$, has been introduced by Benci in [**13**] where he proved a multiplicity result for problems without singularity (superquadratic and subquadratic potentials).

I.1. Some technical preliminaries

Let us recall, for the reader's convenience, some variants of classical results on minimax and deformations, which will be used in working with singular functionals. Although they have been implicitly used in several papers concerning singular potentials, we prefer to recall them here.

In this Section, $I \in C^1(\Lambda; \mathbb{R})$ admits a lower semicontinuous extension $\overline{I} : H \to \mathbb{R} \cup \{+\infty\}$, and we denote

$$c_0 = \inf_{\partial\Lambda} \overline{I}\ . \qquad \text{(I.1.1)}$$

As usual,

$$K_c = \{x \in \Lambda \ / \ I(x) = c\,,\ \nabla I(x) = 0\}.$$

Consider the gradient flow (in Λ) associated to I; when $I \in C^2(\Lambda; \mathbb{R})$then η is defined by

$$\begin{cases} \frac{d\eta}{d\sigma} = \frac{-\nabla I(\eta)}{1+\|\nabla I(\eta)\|} \\ \eta(x,0) = x\ , \end{cases} \qquad \text{(I.1.2)}$$

if not, one can define a locally Lipschitz continuous pseudogradient field, and η as the solution of associated Cauchy problem (see, for instance [**12**]). Since Λ is open, for each $x \in \Lambda$, $\eta(x,\sigma)$ is defined for all $\sigma \in [0, \varepsilon(x))$ for some $\varepsilon(x) \geq d(x, \partial\Lambda)$. Moreover, for every x such that $I(x) \leq c_0$, $\eta(x,\sigma)$ is defined for every $\sigma \in [0, +\infty)$.
Therefore, in the sublevel $\{x \in \Lambda \ / \ I(x) \leq c_0\}$ one can still make use of the deformation techniques. One can then prove the following theorem:

THEOREM I.1.1. (MIN-MAX PRINCIPLE).
Let $I \in C^2(\Lambda; \mathbb{R})$ admit a l.s.c. extension $\overline{I} : H \to \mathbb{R} \cup \{+\infty\}$, and let c_0 as in (I.1.1). Consider a class $\mathcal{A}$ of compact subsets of Λ satisfying :

$$\sup_A I \leq c_0\ , \qquad \forall\, A \in \mathcal{A}\ , \qquad \text{(I.1.3)}$$

$$\eta(A,\sigma) \in \mathcal{A}\ , \qquad \forall\, A \in \mathcal{A}\ ,\ \forall\, \sigma \in [0,+\infty)\ . \qquad \text{(I.1.4)}$$

Let

$$c = \inf_{A \in \mathcal{A}} \sup_{x \in A} I(x) \ ,$$

$(0 \leq c \leq c_0)$. *Then there exists a sequence* $(x_n)_n$ *such that :*

$$x_n \in \Lambda \quad , \quad \forall n \in \mathbb{N} \tag{I.1.5}$$

$$I(x_n) \to c \tag{I.1.6}$$

$$dI(x_n) \to 0 \ . \tag{I.1.7}$$

PROOF: Indeed, two cases are possible:
a) $c < c_0$. Then, from the lower semicontinuity of $\overline{I}$, and from the fact that I decreases along the flow η, one deduces that, for every $0 < \varepsilon < c_0 - c$, there exists a constant $M > 0$ such that the set $\{x \in \Lambda \ / \ I(x) \leq c + \varepsilon \ , \ d(x, \partial\Lambda) \geq M\}$ is invariant under the gradient flow. Hence the usual arguments can be applied to conclude the proof (see for instance [**37**]).
b) $c = c_0$. Then one has that $A \cap K_{c_0} \neq \emptyset$ for every $A \in \mathcal{A}$. Indeed, if not, a small deformation can be performed making the functional decrease (indeed $I(\eta(x, \cdot))$ is strictly decreasing unless $\nabla I(x) = 0$, and A is compact). $\diamond$

For any $c \in \mathbb{R}$, we call a Palais-Smale sequence in Λ at level c a sequence satisfying (I.1.5),(I.1.6),(I.1.7). We recall the compactness Palais-Smale condition:

(PS_c) every Palais-Smale sequence in Λ at level c possesses a subsequence converging to some limit in H

REMARK I.1.1: If $c < c_0$ then the condition $P.S_c$ is equivalent to

every Palais-Smale sequence in Λ at level c possesses a subsequence converging to some limit in Λ ,

because of the lower semicontinuity of $\overline{I}$.$\diamond$

A straightforward consequence of Theorem I.1.1 is the following corollary:

COROLLARY I.1.1. *Under the assumptions of Theorem I.1.1, assume moreover that PS_c holds. Then $K_c \neq \emptyset$.*

Finally, let us state the suitable version of the deformation lemma (see [**12**]):

LEMMA I.1.1 (DEFORMATION LEMMA). *Under the assumptions of Corollary I.1.1, for any $\varepsilon > 0$, let $N = N_\varepsilon(K_c)$; then there is an $\overline{\varepsilon}$ such that, for every $\varepsilon \leq \overline{\varepsilon}$ there are some $\delta > 0$ and $\sigma > 0$ such that:*

$$(if\ c < c_0) \qquad \sup_A I \leq c + \delta \implies \sup_{\eta((A\setminus N),\sigma)} I \leq c - \delta ,$$

$$(if\ c = c_0) \qquad \begin{array}{ll} \sup_{cl(\eta((A\setminus N),\sigma))} I < c , & \forall A \in \mathcal{A} , \\ A \cap K_c \neq \emptyset , & \forall A \in \mathcal{A} . \end{array}$$

I.2. The geometrical index of symmetry $\mathbb{Z}_2$

The purpose of this Section is to define a geometrical index (in the sense of [**12**]) related to the group

$$G = \{P_s, T_s\}_{s\in[0,T]} ,$$

where P_s and T_s are the unitary transformations of H respectively defined by :

$$\text{(I.2.1)} \qquad \begin{array}{lll} y = P_s(x) \iff y(t) = x(s-t) & \forall\, t \in [0,T] , \\ y = T_s(x) \iff y(t) = x(s+t) & \forall\, t \in [0,T] . \end{array}$$

Remark that the P_s and T_s are actually defined by periodicity for all $s \in \mathbb{R}$.
Consider the relations of H defined as

$$\text{(I.2.2)} \qquad \begin{array}{l} yPx \iff \exists\, s \in [0,T] \text{ such that } y = P_s(x) \\ yTx \iff \exists\, s \in [0,T] \text{ such that } y = T_s(x) . \end{array}$$

We point out that P and T will play different roles in the definition of our index; actually this index will be defined on the quotient space H/T.

At the end of this section, we shall prove a first "singular" multiplicity theorem, convenient to treat cases of coercive functionals. This result can be viewed as a slight generalization of the usual multiplicity theorem that one can obtain in the nonsingular case (see [**12**], Theorem 2.4).

We need to fix up some more notations. If $A \subseteq H$, we denote $P(A) = \{x \in H \ / \ \exists y \in A \ , \ xPy\}$ and $T(A) = \{x \in H \ / \ \exists y \in A \ , \ xTy\}$. A set A is P-invariant if $P(A) = A$, and it is T-invariant if $T(A) = A$.

A function $h : H \to H$ is said to be G-equivariant if $h \circ g = g \circ h \ , \ \forall \ g \in G$; for any set X, a function $h : H \to X$ is G -invariant if $h \circ g = h \ , \ \forall \ g \in G$. A set $A \subseteq H$ is G -invariant if $g(A) = A \ , \ \forall \ g \in G$. Two functions x, y in H are geometrically distinct if $y \neq P_s(x)$ and $y \neq T_s(x) \ , \ \forall \ s \in [0, T]$. F_0 denotes the set of all the fixed points of P :

$$F_0 = \{x \in H \ / \ xPx\} . \tag{I.2.3}$$

Remark that, by definition,

$$\begin{aligned} P_s &= T_s \circ P_0 \ , & \forall s \in [0,T] \\ T_s &= P_s \circ P_0 \ , & \forall s \in [0,T] \\ P_0^2 &= id \ . \end{aligned}$$

Let us first state the G -equivariant version of Dugundji's extension Theorem:

LEMMA I.2.1. *Let $A \subseteq H$ be closed and G -invariant. Let $f : A \to \mathrm{R}^k$ be any continuous function satisfying*

$$xPy \Longrightarrow f(x) = -f(y) \qquad \forall \ x, y \in A. \tag{I.2.4}$$

Then f admits a continuous extension $\overline{f} : H \to \mathrm{R}^k$ satisfying

$$xPy \Longrightarrow \overline{f}(x) = -\overline{f}(y) \qquad \forall \ x, y \in H \ . \tag{I.2.5}$$

PROOF: We remark that (I.2.4) is equivalent to

$$f(P_s(x)) = -f(x) \ , \qquad \forall \ x \in A \ , \forall \ s \in [0,T] \ ,$$

which also implies that

$$f(T_s(x)) = f(x)\ , \qquad \forall\, x \in A\ , \forall\, s \in [0,T]\ .$$

Since R^k is convex and A is closed, by virtue of Dugundji's extension theorem we can find a continuous $f_1 : H \to R^k$ such that $f_1|A = f$. Let $f_2 : H \to R^k$ be the continuous function defined by

$$f_2(x) \;=\; \frac{1}{T}\int_0^T f_1(T_s(x))\, ds \qquad \forall x \in H;$$

then one has $f_2(T_s(x)) = f_2(x)$, $\forall x \in H$,$\forall s \in [0,T]$, and moreover f_2 extends f too. Finally we define $\overline{f}$ as

$$\overline{f}(x) \;=\; \frac{1}{2}(f_2(x) - f_2(P_0(x)))\ , \qquad \forall\, x \in H\ .$$

Then obviously $\overline{f}$ extends f, and moreover, since $P_s = T_s \circ P_0$ ($\forall s \in [0,T]$), one has that

$$\overline{f}(P_s(x)) = -\overline{f}(x)\ , \qquad \forall\, x \in A\ , \forall\, s \in [0,T]\ ,$$

which is equivalent to (I.2.5). ◇

We are going to define the geometrical index as a function defined on a class of admissible sets and taking values in $\mathsf{N} \cup \{+\infty\}$. Let

(I.2.6) $$\mathcal{B} \;=\; \{A \subseteq H \text{ closed}/\ A \text{ is } G\text{-equivariant},\ A \cap F_0 = \emptyset\}\ ,$$

and, for a fixed integer k, let

(I.2.7) $$\mathcal{F}_k \;=\; \{f : H \to R^k \text{ continuous } /\ xPy \Longrightarrow f(x) = -f(y)\}\ .$$

DEFINITION I.2.1. *The index i is the function $i : \mathcal{B} \to \mathsf{N} \cup \{+\infty\}$ defined by :*

(I.2.8) $$i(\emptyset) \;=\; 0$$

(I.2.9) $$i(A) \;=\; +\infty \iff \forall\, k \in \mathsf{N}\ , \forall\, f \in \mathcal{F}_k\ ,\ 0 \in f(A)$$

(I.2.10) $$i(A) \;=\; k \iff k \text{ is the smallest integer such that } \exists\, f \in \mathcal{F}_k\ ,\ 0 \notin f(A)\ .$$

REMARK I.2.1: Because of Lemma I.2.1, it is enough to check (I.2.9) and (I.2.10) for every continuous $f : A \to \mathrm{R}^k$ such that $xPy \Longrightarrow f(x) = -f(y)$.◇

REMARK I.2.2: It follows from Borsuk's theorem that the definition of i is equivalent to

$$i(\emptyset) = 0$$

$$i(A) = +\infty \iff \text{for every closed finite } T\text{-invariant covering of } A\ ,\ A_1, \ldots, A_k\ ,\ \text{there exists } r \text{ such that } A_r \cap P(A_r) \neq \emptyset\ .$$

$$i(A) = k \iff k \text{ is the smallest integer such that there exist } k \text{ closed } T\text{-invariant sets } A_1, \ldots, A_k \text{ with } A_r \cap P(A_r) = \emptyset \quad (r = 1, \ldots, k) \text{ and } \{A_r \cup P(A_r)\}_r \text{ is a covering of } A.$$

The next proposition shows that an index i is an index in the sense [**BBF**], Definition 2.5.◇

PROPOSITION I.2.1. *Let $A, B \in \mathcal{B}$, then*

(*i*) $A \subseteq B \Longrightarrow i(A) \leq i(B)$

(*ii*) $i(A \cup B) \leq i(A) + i(B)$

(*iii*) *if $h : A \to B$ is continuous and G equivariant* $\Longrightarrow i(A) \leq i(B)$

(*iv*) *A compact* $\Longrightarrow i(A) < +\infty$

(*v*) *A compact* $\Longrightarrow \exists\, \varepsilon > 0$ *such that* $i(N_\varepsilon(A)) = i(A)$

(*vi*) *if A contains only a finite number of geometrically distinct orbits then $i(A) = 1$.*

PROOF: (i),(iii) and (vi) easily follow from the definition of i.

(ii) If either $i(A) = +\infty$ or $i(B) = +\infty$, the claim is obviously true; if not, let $f_A \in \mathcal{F}_{i(A)}$ and $f_B \in \mathcal{F}_{i(B)}$ such that $0 \notin f_A(A)$ and $0 \notin f_B(B)$ as in (I.2.10); then, taking $f : H \to \mathrm{R}^{i(A)+i(B)}$, $f = f_A \times f_B$, one has that $0 \notin f(A \cup B)$.

(iv) Indeed from the compactness of A, and from $A \cap F_0 = \emptyset$, for every $\varepsilon > 0$ small enough, we can find a finite set $\{x_1, \ldots, x_k\}$ such that $N_\varepsilon(T(\{x_r\}) \cap N_\varepsilon(P(\{x_r\})) = \emptyset$, and $(N_\varepsilon(T_r) \cup N_\varepsilon(P\{x_r\}))_{r=1,\dots,k}$ covers A. Therefore it follows from Remark I.2.2 that $i(A) \leq k$.

(v) Obviously $i(N_\varepsilon(A)) \geq i(A)$; on the other hand, one can cover A with k pairs of compact sets $A_1 \cup P(A_1), \dots, A_k \cup P(A_k)$, with $A_r \cap P(A_r) = \emptyset$, $(r = 1, \dots, k)$. For $\varepsilon > 0$ small enough we have $N_\varepsilon(A_r) \cap N_\varepsilon(P(A_r)) = \emptyset$ and the proof is complete. ◇

Now we are ready to prove our first multiplicity theorem. A nonsingular variant can be proved just by applying Theorem 2.4 of [**BBF**].

THEOREM I.2.1. *Let $E \subseteq H$ be a closed, G-invariant subspace, and let $I \in C^2(\Lambda \cap E; \mathbb{R})$ be a G invariant functional, admitting a lower semicontinuous extension $\overline{I} : E \to \mathbb{R} \cup \{+\infty\}$. We assume that*

$$\overline{I} \geq 0 \,, \qquad \forall x \in E \tag{I.2.11}$$

$$\begin{aligned} &\exists A \in \mathcal{B} \cap 2^{\Lambda \cap E} \,, \quad \exists k \geq 1 \text{ such that} \\ &i(A) \geq k \,,\ \sup_A I \leq c_0 = \inf_{x \in \partial\Lambda} \overline{I}. \end{aligned} \tag{I.2.12}$$

Hence, for $1 \leq r \leq k$, the classes

$$\Gamma_k = \{A \in \mathcal{B} \cap 2^{\Lambda \cap E} \text{ compact} \ / \ i(A) \geq k \,,\ \sup_A I \leq c_0\}$$

are nonempty and we define

$$c_r = \inf_{A \in \Gamma_r} \sup_A I \,, \qquad r = 1, \dots, k \,;$$

then one has that $0 \leq c_1 \leq \cdots \leq c_k \leq c_0$.

Assume moreover that

$$(P.S)^\Lambda_{c_r} \qquad (r = 1, \dots, k) \tag{I.2.13}$$

$$K_{c_r} \cap F_0 = \emptyset \,. \qquad (r = 1, \dots, k) \,. \tag{I.2.14}$$

hold. Then I has at least k geometrically distinct critical points in $\{x \in E \cap \Lambda \ / \ c_1 \leq I(x) \leq c_k\}$.

PROOF: First, by virtue of Corollary I.1.1, the c_r's are critical value of I in Λ, that is $K_{c_r} \neq \emptyset$, $(r = 1, \cdots, k)$. Thus, if $c_1 < c_2 < \cdots < c_k$, the proof is complete. We assume then that, for some $r, h \geq 1$, $c_r = \cdots = c_{r+h} = c$, and we consider the two cases:
a) $c < c_0$. Then the $(P.S)_c^\Lambda$ assumption implies that K_c is compact. We are going to show that $i(K_c) \geq h+1$.
Indeed, from Lemma I.1.1, for each fixed ε sufficiently small, we can find a σ and an $A \in \Gamma_{r+h}$ such that, setting $N = N_\varepsilon(K_c)$, then

$$\sup_{cl(\eta(A \setminus N, \sigma))} I < c\,.$$

Hence one necessarily has that $i(cl(\eta(A \setminus N, \sigma))) = i(cl(A \setminus N) \leq r - 1$; therefore one deduces from Proposition I.2.1.ii that $i(N) \geq h+1$. Moreover, is ε is small enough, then $i(N) = i(K_c)$ (Proposition I.2.1.v). This fact ends the proof by virtue of Proposition I.2.1.vi.
b) $c = c_0$ Then the $(P.S)_c^\Lambda$ assumption implies that $A \cap K_c$ is compact, for every $A \in \Gamma_{r+h}$. Arguing as in the case a) (but applying the second part of Lemma I.1.1, one finds that $i(A \cap K_c) \geq h+1$, $\forall A \in \Gamma_{r+h}$. This fact ends the proof by virtue of Proposition I.2.1.vi.◇

The following result provides example of subsets of Λ having nontrivial (i.e. different from one) geometrical indices. This result will be used in proving the Theorems of chapters II and III.

THEOREM I.2.2. *Let S^{N-1} be the unit sphere of $\mathbb{R}^N$ with $N \geq 3$ odd. We consider*

$$C_N = \{h \in E \;/\; h(t) = x\cos\omega t + y \sin\omega t\,,\quad \text{with } x\cdot y = 0\,,\; x, y \in S^{N-1}\,,\; \omega = \frac{2\pi}{T}.\}$$

Then $i(C_N) \geq N$.

PROOF: We are going to show that C_N contains a subset having an index larger or equal to N.

Let $(e_1, \ldots, e_N)$ be an orthonormal system of $\mathbb{R}^N$, and let $S^{N-2} = \{x \in S^{N-1} \;/\; x \cdot e_N = 0\}$. Define $\phi : S^{N-2} \times S^1 \to C_N$ as

$$\begin{aligned} &\phi(x_1, \ldots, x_{N-1}, \theta)(t) \\ &= (\cos\theta e_N + \sin\theta e_1)\; \cos\omega t + \\ &+ [x_1(-\sin\theta e_N + \cos\theta e_1) + x_2 e_2 + \ldots x_{N-1}e_{N-1}]\; \sin\omega t\,. \end{aligned} \tag{I.2.15}$$

Note that $\phi(-x_1,\cdots,-x_{N-1},\theta) = P_0(\phi(x_1,\cdots,x_{N-1},\theta))$, $\forall(x_1,\cdots,x_{N-1},\theta)$. We claim that $i(T(\phi(S^{N-2}\times S^1))) \geq N$. Assume by the contrary that there exists $f \in \mathcal{F}_{N-1}(C_N)$ such that

$$0 \notin f\circ\phi(S^{N-2}\times S^1)\,. \tag{I.2.16}$$

Consider $\varphi = f\circ\phi$. From (I.2.7), for every θ, $\varphi(\cdot,\theta)$ is a continuous odd function, so we deduce from Borsuk's theorem and from (I.2.16) that

$$\deg_0\ \varphi(\cdot,\theta) \text{ is odd and does not depend on } \theta. \tag{I.2.17}$$

From (I.2.15) we have

$$\phi(x_1,\ldots,x_{N-1},\theta+\pi)(t) = \phi(x_1,-x_2,\cdots-x_{N-1},\theta)(\pi+t)\,,$$

and therefore

$$\varphi(x_1,\ldots,x_{N-1},\theta+\pi) = \varphi(x_1,-x_2,\ldots,-x_{N-1},\theta)\,. \tag{I.2.18}$$

Let A be the linear isomorphism of R^{N-1} defined as $A(x_1,\ldots,x_{N-1}) = (x_1,-x_2,\ldots,-x_{N-1})$. Since N is odd, then $\deg A = -1$, so from (I.2.18) follows that $\deg_0\ \varphi(\cdot,\theta) = -\deg_0\ \varphi(\cdot,\theta+\pi)$, in contradiction with (I.2.17). ◇

I.3. The homotopical index related to the geometrical index

It should be clear by the setting of the problem that the main property one requires to the homotopical index is to be invariant under homotopies of Λ in Λ and to be not invariant under homotopies contracting subsets of Λ into sets of large constant functions. Although we are going to define the homotopical index related to the geometrical index introduced in section 2, it is clear that to each index (i.e a set function satisfying i),..,v) of Proposition I.2.1 for some group G) there corresponds an homotopical index.

We denote by $\mathcal{H}$ the class of all the G -equivariant homotopies homotopically equivalent to the identity:

$$\begin{aligned}\mathcal{H} = \{&h : H\times[0,1]\to H \text{ continuous such that}\\ &h(x,0)=x\,, h(g(x),\sigma) = g(h(x,\sigma))\,,\ \forall\, x\in H\,,\\ &\forall\,\sigma\in[0,1]\,,\ \forall\, g\in G\}\,.\end{aligned} \tag{I.3.1}$$

REMARK I.3.1: If the functional I is G-invariant, then the flow η as defined in (I.1.2) belongs to $\mathcal{H}$.

DEFINITION I.3.1. *Let $A \in 2^{\Lambda}$ be G-invariant. We say that A is G-contractible if there exists $h \in \mathcal{H}$ such that $h(x,\sigma) \in \Lambda$ and $h(x,1) \in E_N \setminus \{0\}$, $\forall\, x \in A$, $\forall\, \sigma \in [0,1]$.*

DEFINITION I.3.2. *For a given compact $A \in \mathcal{B} \cap 2^{\Lambda}$, let*

$$\mathcal{H}_0(A) \;=\; \{h \in \mathcal{H} \;/\; h(x,1) \in E_N \;,\; \forall\, x \in A\} \;. \tag{I.3.2}$$

We say that the homotopical index of A is k (j(A)=k) if

$$k \;=\; \min_{h \in \mathcal{H}_0(A)} i(\{x \in A \;/\; H(x,[0,1]) \cap \partial\Lambda \neq \emptyset\}) \;. \tag{I.3.3}$$

The following Lemma shows that $\mathcal{H}_0(A)$ doesn't actually depend on A.

LEMMA I.3.1. *Let $A \in \mathcal{B}$ be closed and let $h \in \mathcal{H}_0(A)$. Then $h_{|A\times[0,1]}$ admits a continuous extension $\overline{h}$ such that $\overline{h} \in \mathcal{H}_0(H)$.*

PROOF: By the application of Dugundji's Theorem one first extends $h_{|A\times\{1\}}$ to a continuous $h_1 : H \times \{1\} \to E_N$. Then the function $h_2 : (A \times [0,1]) \cup (H \times \{0\}) \cup (H \times \{1\}) \to H$ defined as

$$h_2(x,\sigma) = \begin{cases} h(x,\sigma) & \text{if } (x,\sigma) \in A \times [0,1] \\ x & \text{if } \sigma = 0 \\ h_1(x) & \text{if } \sigma = 1 \,, \end{cases}$$

is continuous and extends h too. Hence h_2 admits a continuous extension h_3 defined on the whole of $H \times [0,1]$. Remark that $h_3(H \times \{1\}) \subseteq E_N$.
Finally we define (see also the proof of Lemma I.2.1)

$$h_4(x,\sigma) = \frac{1}{T}\int_0^T T_{T-s} h_3(T_s(x),\sigma)\,ds \;, \qquad \forall\,(x,\sigma) \in H \times [0,1] \;,$$

and

$$\overline{h}(x,\sigma) = \frac{1}{2}\left(h_4(x,\sigma) + P_0\left(h_4(P_0(x),\sigma)\right)\right) \;.$$

One easily verifies that $\overline{h}$ is the desired extension of H.$\diamond$

REMARK I.3.2: If A is G -contractible obviously $j(A) = 0$. If A is not G -contractible, the homotopical index measures (in the sense of the geometrical index) how big is the set that one has to take out from A to make A be G-contractible. This can be used as an equivalent definition, as is shown in the following proposition.

PROPOSITION I.3.1. *For every compact $A \in \mathcal{B} \cap 2^\Lambda$ we have*

$$
\text{(I.3.4)} \qquad j(A) = \min\{i(B)/B \in \mathcal{B} \quad B \subseteq A \text{ and } A \setminus B \text{ is } G\text{-contractible}\}\,.
$$

PROOF: Let us call $\bar{j}(A)$ the right hand side of (I.3.4). For every $h \in \mathcal{H}_0(A)$, the set

$$
B = \{x \in A \,/\, h(x,[0,1]) \cap \partial\Lambda \neq \emptyset\}
$$

is closed and $A \setminus B$ is G-contractible, that is $j(A) \geq \bar{j}(A)$. On the other hand, for a given closed B such that $A \supseteq B$ and $A \setminus B$ is P -contractible, we can take $B' = N_\varepsilon(B)$ in such a way that $i(B') = i(B)$ (Prop. I.2.1.v)), and $cl(A \setminus B')$ is G-contractible ; hence there exists $h \in \mathcal{H}_0(cl(A \setminus B'))$ such that $h(cl(A \setminus B'),[0,1]) \cap \partial\Lambda = \emptyset$. From Lemma I.3.1 we can find an $\bar{h} \in \mathcal{H}_0(A)$ such that $\bar{h}(x,\sigma) = H(x,\sigma)$, for every $(x,\sigma) \in cl(A \setminus B') \times [0,1]$. Therefore $i(B) = i(B') \geq j(A)$, and hence $\bar{j}(A) \geq j(A)$. ◇

PROPOSITION I.3.2. *Let $A \in \mathcal{B} \cap 2^\Lambda$ be closed and let B be a closed subset of A. Then $j(A) \leq j(cl(A \setminus B)) + i(B)$.*

PROOF: If C is closed subset of $cl(A \setminus B)$ such that $(cl(A \setminus B)) \setminus C$ is P -contractible then $A \setminus (B \cup C)$ is P -contractible ; therefore, by Proposition I.3.1, $j(A) \leq i(B \cup C)$, and, from Proposition I.2.1.(iv), $j(A) \leq i(B) + i(C)$. By Proposition I.3.1, this fact proves that $j(A) \leq j(cl(A \setminus B)) + i(B)$. ◇

PROPOSITION I.3.3. *Let $A \in \mathcal{B} \cap 2^\Lambda$ be compact and let $h \in \mathcal{H}$ such that $h(A,[0,1]) \cap \partial\Lambda = \emptyset$, and $h(A,[0,1]) \cap F_0 = \emptyset$. Then $j(h(A,1)) \geq j(A)$.*

PROOF: Assuming on the contrary that $j(h(A,1)) < j(A)$, we can find a closed subset B of $h(A,1)$ such that $h(A,1) \setminus B$ is G-contractible, and $i(B) \leq j(A) - 1$. If $C = \{x \in A \,/\, h(x,1) \in B\}$, it follows from Proposition I.2.1.iii) that $i(C) \leq i(B) \leq j(A) - 1$. On the other hand, C is closed and obviously $A \setminus C$ is G-contractible, so $i(C) \geq j(A)$. ◇

Now we are in a position to prove the main result of this section:

THEOREM I.3.1. *Let $I \in C^2(\Lambda; \mathbb{R})$ be a G invariant functional, admitting a lower semi-continuous extension $\bar{I} : H \to \mathbb{R} \cup \{+\infty\}$. Assume that*

$$
\text{(I.3.5)} \qquad \bar{I} \geq 0\,, \qquad \forall x \in H
$$

$$\text{(I.3.6)} \qquad \begin{array}{l} \exists\, A \in \mathcal{B} \cap 2^{\Lambda}\,, \quad \exists k \geq 1 \textit{ such that} \\ j(A) \geq k\,,\ \sup_A I \leq c_0 = \inf_{\partial\Lambda} \overline{I}. \end{array}$$

Hence, for $1 \leq r \leq k$, the classes

$$\Gamma_k^* \;=\; \{A \in \mathcal{B} \cap 2^{\Lambda \cap E} \textit{ compact} \,/\, j(A) \geq k\,,\ \sup_A I \leq c_0\}$$

are nonempty and we define

$$c_r^* \;=\; \inf_{A \in \Gamma_r^*} \sup_A I\,, \qquad r = 1, \ldots, k\,;$$

then one has that $0 \leq c_1^* \leq \cdots \leq c_k^* \leq c_0$.

Assume moreover that

$$\text{(I.3.7)} \qquad (P.S)_{c_r^*}^{\Lambda} \qquad (r = 1, \ldots, k)$$

$$\text{(I.3.8)} \qquad K_{c_r^*} \cap F_0 \;=\; \emptyset\,. \qquad (r = 1, \ldots, k)\,.$$

hold. Then I has at least k geometrically distinct critical points in $\{x \in \Lambda \,/\, c_1^* \leq I(x) \leq c_k^*\}$.

PROOF: First c_r^* are critical levels for every $r = 1, \ldots, k$. Indeed the classes Γ_r^* are invariant under the gradient flow η as is defined in (I.1.2), and the Palais-Smale condition is fulfilled at every level c_r^* ; therefore it follows from Corollary I.1.1 that $K_{c_r^*} \neq \emptyset$.

Hence, if $c_1^* < \cdots < c_k^*$ the proof is complete. We assume that for some $r, h \geq 1$, $c_r^* = \cdots = c_{r+h}^* = c^*$, and we consider the two cases:
a) $c^* < c_0$. Then from (I.3.7) one deduces that K_{c^*} is compact.We are going to prove that $i(K_{c^*}) \geq h + 1$ (remark that (I.3.8) implies that $K_{c^*} \in \mathcal{B}$). Indeed it follows from Lemma I.1.1 that for every fixed $\varepsilon > 0$ small enough, we can find a $\sigma > 0$ and an $A \in \Gamma_{r+h}^*$ such that, if $N = N_\varepsilon(K_{c^*})$,then

$$\text{(I.3.9)} \qquad \sup_{cl(\eta(A \setminus N), \sigma))} I \;<\; c^*\,.$$

Let $A' = cl(\eta(A \setminus N, \sigma))$; then (I.3.9) implies that $j(A') \leq r - 1$. Moreover, it follows from Proposition I.3.2 and Proposition I.3.3, that $j(A) \leq j(A') + i(N) \leq r - 1 + i(N)$; therefore $i(N) \geq h + 1$. The proof is then complete by virtue of Proposition I.2.1.v and vi (indeed, for ε small, one has that $i(N) = i(K_{c^*})$).
b) $c^* = c_0$. Then, arguing as in the step a) (but applying the second part of Lemma I.1.1), one finds that $i(A \cap K_{c^*}) \geq h + 1$, $\forall a \in \Gamma_{r+h}^*$.$\diamond$

I.4. Computation of the homotopical index

In this Section we provide example of sets having non trivial homotopical index.

THEOREM I.4.1. *Let S^{N-1} be the unit ball of $\mathbb{R}^N$; we consider the set of all the great circles of S^{N-1},*

$$(\text{I.4.1})\quad C_N = \left\{z \in H \;/\; z(t) = x\sin\omega t + y\cos\omega t\,,\ \omega = \frac{2\pi}{T}\,,\ x,y \in S^{N-1}\,,\ ,x\cdot y = 0\right\}.$$

Then $j(C_N) \geq N-1$.

PROOF: The proof consists in two main parts. In the first one we show that proving Theorem I.4.1 is in fact equivalent to find zeroes of functions having some symmetry properties. In the second one we prove a convenient Borsuk Ulam type theorem.
Let e_1 be a unit vector of S^{N-1} and let $S^{N-2} = \{x \in S^{N-1} \;/$
$x \cdot e_1 = 0\}$: we consider the continuous function $F : S^{N-2} \times S^{N-2} \times S^1 \to \mathbb{R}^N$ defined as :

$$(\text{I.4.2})\qquad F(x,y,e^{i\omega t}) \;=\; x\cos\omega t + (x\cdot y e_1 + (y - (y\cdot x)x))\sin\omega t\,,$$

which gives a parametrization of C_N. We denote by F' the associated continuous $F' : S^{N-2} \times S^{N-2} \to C_N$.

In order to prove that $j(C_N) \geq N-1$, we have to prove that, for every $h \in \mathcal{H}_0(C_N)$,

$$(\text{I.4.3})\qquad i(\{z \in C_N \;/\; \exists\, \sigma \in [0,1]\,,\ h(z,\sigma) \in \partial\Lambda\}) \;\geq\; N-1\,.$$

To every $h \in \mathcal{H}_0(C_N)$ we can associate a continuous extension of F, $\Phi : S^{N-2} \times S^{N-2} \times B^2 \to \mathbb{R}^N$ of F in the following way :

$$(\text{I.4.4})\qquad \Phi(x,y,\rho e^{i\omega t}) \;=\; h(F'(x,y)\,,\,1-\rho)(t)\,,$$

and hence (I.4.3) can be written as

$$(\text{I.4.5})\qquad \begin{aligned} &i(F'(\{(x,y) \in S^{N-2} \times S^{N-2} \;/ \\ &\qquad \exists\, \rho e^{i\omega t} \in B^2\,,\ \Phi(x,y,\rho e^{i\omega t}) = 0\})) \geq N-1 \end{aligned}$$

From Definition I.2.1 we have to prove that for every $f \in \mathcal{F}_{N-2}$,

$$(I.4.7)\qquad 0 \in f(F'(\{(x,y) \in S^{N-2} \times S^{N-2} \, / \, \exists\, \rho e^{i\omega t} \in B^2 \ , \ \Phi(x,y,\rho e^{i\omega t}) = 0\})) \, .$$

So let $f \in \mathcal{F}_{N-2}$: we define $\varphi : S^{N-2} \times S^{N-2}(\times B^1) \to \mathbb{R}^{N-2}$ as

$$(I.4.8)\qquad \varphi(x,y,\rho e^{i\omega t}) \;=\; f(F'(x,y)) \; ;$$

Finally we have to prove that

$$0 \in \Phi \times \varphi(S^{N-2} \times S^{N-2} \times B^1).$$

Consider the equation

$$(I.4.9)\qquad \Phi \times \varphi(x,y,\rho e^{i\omega t}) = 0.$$

From the symmetry of the problem, if $(x,y,\rho e^{i\omega t})$ is a solution, then $(x,-y,\rho e^{-i\omega t})$, $(-x,-y+2(y-(y\cdot x)x), e^{i\omega(\pi-t)})$, and $(-x,y-2(y-(y\cdot x)x), e^{i\omega(\pi+t)})$ are solutions too. We are going to prove that, in a nondegenerate situation, the equation (I.4.9) has an odd number of 4-ples of solutions of the type $\{(x,y,\rho e^{i\omega t}),(x,-y,\rho e^{-i\omega t}),(-x,-y+2(y-(y\cdot x)x), e^{i\omega(\pi+t)}),(-x,y-2(y-(y\cdot x)x), e^{i\omega(\pi-t)})\}$. Moreover, we shall see that the nondegeneracy can be assumed without loss of generality, taking a special symmetry preserving perturbation of $\Phi \times \varphi$. To the aim of solving (I.4.9), we are going to introduce a special extension of F, say Φ_1, for which (I.4.9) has an odd number of 4-ples of solutions for almost every φ. Next we shall prove that, for a nondegenerate homotopy $((1-\lambda)\Phi_1 + \lambda\Phi) \times \varphi$, the oddness of the number of 4-ples of solutions also holds at $\lambda = 1$.
The proof is divided in several steps :

STEP 1: By definition, F enjoys the following symmetry properties :

$$\begin{aligned} F(x,-y,e^{i\omega t}) &= F(x,y,e^{-i\omega t}) \\ F(-x,y-2(y-(y\cdot x)x),e^{i\omega t}) &= F(x,y,e^{i\omega(\pi+t)}) \\ \forall\,(x,y,e^{i\omega t}) &\in S^{N-2} \times S^{N-2} \times B^2 \ , \end{aligned}$$

then the G -equivariance of h implies the symmetry properties of Φ:

$$(I.4.10)\qquad \begin{aligned} \Phi(x,-y,\rho e^{i\omega t}) &= \Phi(x,y,\rho e^{-i\omega t}) \\ \Phi(-x,y-2(y-(y\cdot x)x),\rho e^{i\omega t}) &= \Phi(x,y,\rho e^{i\omega(\pi+t)}) \\ \forall\,(x,y,\rho e^{i\omega t}) &\in S^{N-2} \times S^{N-2} \times B^2 \ . \end{aligned}$$

Moreover, from the symmetry properties of f (I.2.7) and from (I.4.8), we deduce

$$
\text{(I.4.11)} \qquad \begin{aligned}
&\varphi(x,-y,\rho e^{i\omega t}) = -\varphi(x,y,\rho e^{i\omega t}) \\
&\varphi(-x, y-2(y-(y\cdot x)x), \rho e^{i\omega t}) = \varphi(x,y,\rho e^{i\omega(\pi+t)}) \\
&\forall\,(x,y,\rho e^{i\omega t}) \in S^{N-2}\times S^{N-2}\times B^2 .
\end{aligned}
$$

STEP 2: We consider the continuous $\Phi_1 : S^{N-2}\times S^{N-2}\times B^2 \to \mathbb{R}^N$ defined by

$$
\text{(I.4.12)} \qquad \begin{aligned}
&\Phi_1(x,y,\rho e^{i\omega t}) \\
&\quad = \begin{cases} -8(1-\rho)e_1 + F(x,y,e^{i\omega t}) & \text{if } \frac{1}{2}\le\rho\le 1 \\ -4e_1 + 2\rho F(x,y,e^{i\omega t}) & \text{if } 0\le\rho\le\frac{1}{2} . \end{cases}
\end{aligned}
$$

For every $z \in B^N$ such that $z\cdot e_1 > 0$ the set

$$\{(x,y,\rho e^{i\omega t}) \in S^{N-2}\times S^{N-2}\times B^2 \;/\; \Phi_1(x,y,\rho e^{i\omega t}) = z\}$$

can always be parametrized by means of two continuous functions $g_1, g_2 : S^{N-2} \to S^{N-2}\times S^{N-2}\times B^2$, as

$$
\text{(I.4.13)} \qquad \begin{aligned}
g_1(x) &= (x, y_1(x), \rho(x)e^{i\omega t_1(x)}) \qquad 0 < t_1(x) < \pi \\
g_2(x) &= (x, y_2(x), \rho(x)e^{i\omega t_2(x)}) \qquad \pi < t_1(x) < 2\pi ,
\end{aligned}
$$

where $\rho(x)$ is the unique positive solution of the equation

$$|z|^2 + 16(1-\rho)z\cdot e_1 + 64(1-\rho)^2 = 1 ;$$

let us set

$$w = z + 8(1-\rho(x))e_1 ,$$

note that $|w\cdot x| \neq 1$; then (I.4.13) are defined by

$$
\begin{aligned}
&\cos t_1(x) = w\cdot x \quad , \quad 0 < t_1(x) < \pi \\
&\cos t_2(x) = w\cdot x \quad , \quad \pi < t_2(x) < 2\pi ; \\
&y_1(x) = \frac{1}{\sqrt{1-(w\cdot x)^2}}(w - x\cdot w\, x - w\cdot e_1\, e_1 + w\cdot e_1\, x) \\
&y_2(x) = \frac{-1}{\sqrt{1-(w\cdot x)^2}}(w - x\cdot w\, x - w\cdot e_1\, e_1 + w\cdot e_1\, x) .
\end{aligned}
$$

We remark that

$$y_i(-x) = -y_i(x) + 2(y_i(x) - x\cdot y_i(x)x) \quad i = 1,2 ,$$

so that, from (I.4.11), $\varphi \circ g_i(-x) = -\varphi \circ g_i(x), i = 1, 2$. Moreover remark that

$$t_2(x) = 2\pi - t_1(x) .$$

STEP 3: Even if Sard's Theorem doesn't apply directly without breaking the symmetries, we are going to prove a symmetry preserving modification of Sard's Theorem. More precisely we are going to show that, for every $\varepsilon > 0$, we can find a small perturbation φ_ε of φ satisfying (I.4.11), and we can find a small $z_\varepsilon \in B^N$ such that $z_\varepsilon \cdot e_1 > 0$, such that $(0,0)$ is a regular value of $(\Phi_1 - z_\varepsilon) \times \varphi_\varepsilon$.
To this aim, let us denote $\xi_1, \xi_2, \xi_3, \xi_4 : S^{N-2} \times S^{N-2} \to S^{N-2} \times S^{N-2}$ the functions: $\xi_1(x,y) = (x,y)$, $\xi_2(x,y) = (x,-y)$, $\xi_3(x,y) = (-x, y - 2(y \cdot x)x)$, $\xi_4 = (x,y) = (-x, -y + 2(y \cdot x)x)$; we can find an integer M and M closed sets $C_1, \dots, C_M$ such that $\xi_\ell(C_i) \cap \xi_m(C_i) = \emptyset$ for $\ell \neq m$, and $\{\xi_\ell(C_i)\ , 1 \leq \ell \leq 4\ , 1 \leq i \leq M\}$ covers $S^{N-2} \times S^{N-2}$.

We remark that by virtue of (I.4.10) and (I.4.11), if (z,c) is a regular value for $\Phi_1 \times \varphi$ then $(z, -c)$ is regular too.

For every $i = 1, \dots, M$, let $p_{1,i}$ a C^1 function satisfying

$$p_{1,i}(x) = \begin{cases} 1 & \text{if } x \in C_i \\ 0 & \text{if } x \in \xi_2(C_i) \cup \xi_3(C_i) \cup \xi_4(C_i) \end{cases}$$

and let $p_{e,i} = p_{1,i} \circ \xi_e^{-1}$.

From Sard's Theorem there exists a pair $(z_\varepsilon^1, c_\varepsilon^1)$, with $|z_\varepsilon^1| + |c_\varepsilon^1| < \varepsilon$ and $z_\varepsilon^1 \cdot e_1 > 0$,which is a regular value for $\Phi_1 \times \varphi$; therefore, setting

$$\varphi_\varepsilon^1 = \varphi - (p_{1,1} - p_{2,1} + p_{3,1} - p_{4,1}) c_\varepsilon^1 ,$$

$(0,0)$ is a regular value from $(\Phi_\varepsilon^1 - z_\varepsilon^1\ , \varphi_\varepsilon^1)$ on the domain $\bigcup_{\ell=1}^k \xi_\ell(C_1) \times B^2$.

Remark that φ_ε^1 still satisfies (I.4.11). Arguing by induction, we assume that we have found a very small z_ε^{k-1} (with $z_\varepsilon^{k-1} \cdot e_1 > 0$) and very small $c_\varepsilon^1, \dots, c_\varepsilon^{k-1}$ very small, such that, setting

$$D_{k-1} = \bigcup_{i=1}^{k-1} \bigcup_{\ell=1}^{4} \xi_l(C_i) \times B^2 ,$$

and

$$\varphi_\varepsilon^{k-1} = \varphi - \sum_{i=1}^{k-1} (p_{1,i} - p_{2,i} - p_{3,i} - p_{4,i}) c_\varepsilon^i ,$$

then $(0,0)$ is regular for $(\Phi - z_\varepsilon^{k-1}) \times \varphi_\varepsilon^1$ on D_{k-1}.

From Sard's Theorem, we can find a $(z_\varepsilon^k, c_\varepsilon^k)$ satisfying both the following conditions :

1) $(0,0)$ is regular for $(\Phi - z_\varepsilon^k\ ,\ \varphi_\varepsilon^{k-1} - c_\varepsilon^k)$ on $S^{N-2} \times S^{N-2} \times B^2$.

2) z_ε^k is so close to z_ε^{k-1} and c_ε^k is so small that $(0,0)$ is still regular for $(\Phi_1 - z_\varepsilon^k) \times \varphi_\varepsilon^k$ on D_k, where

$$\varphi_\varepsilon^k = \varphi_\varepsilon^{k-1} - (p_{1,k} - p_{2,k} + p_{3,k} - p_{4,k})c_k^\varepsilon ,$$

and

$$D_k = \left(\bigcup_{\ell=1}^{4} \xi_\ell(C_k) \times B^2\right) \cup D_{k-1} .$$

Thus $(0,0)$ is a regular value for $(\Phi_1 - z_\varepsilon^k) \times \varphi_\varepsilon^k$ on the whole of D_k. Arguing by recurrence, for $k = M$, one has found that $(0,0)$ is a regular value for $(\Phi_1 - z_\varepsilon^M) \times \varphi_\varepsilon^M$ on $S^{N-2} \times S^{N-2} \times B^2$.

STEP 4: First we remark that $[(\Phi_1 - z_\varepsilon) \times \varphi_\varepsilon]^{-1}(0,0) \neq \emptyset$, because $\varphi_\varepsilon \circ g_1$ and $\varphi_\varepsilon \circ g_2 : S^{N-2} \to \mathbb{R}^{N-2}$ are continuous and odd, by virtue of (I.4.11). Moreover, since $(0,0)$ is a regular value of $(\Phi_1 - z_\varepsilon) \times \varphi_\varepsilon$, it is the union of a finite number of symmetric 4 -ples $\{g_1(x), g_1(-x), g_2(x), g_2(-x)\}$. We can take another small symmetry preserving perturbation of φ_ε in such a way that 0 is a regular value for both $\varphi_\varepsilon \circ g_1$ and $\varphi_\varepsilon \circ g_2$. Hence, it follows from Borsuk's Theorem that $[\Phi_1 - z_\varepsilon) \times \varphi_\varepsilon]^{-1}(0,0)$ is the union of an odd number of 4 -ples $\{g_1(x), g_1(-x), g_2(x), g_2(-x)\}$: indeed there is an odd number of counterimages of 0 of $\varphi_\varepsilon \circ g_i$ for every half sphere $S^{N-2} \cap \{x \in \mathbb{R}^{N-1}\ /\ x \cdot e \geq 0\}$ such that $(\varphi_\varepsilon \circ g_i)^{-1}(0,0) \cap \{x \in \mathbb{R}^{N-1}\ /\ x \cdot e = 0\} = \emptyset$.

STEP 5: We consider the homotopy $\psi = [(1-\lambda)\Phi_1 + \lambda\Phi] \times \varphi_\varepsilon : [0,1] \times S^{N-2} \times S^{N-2} \times B^2 \to \mathbb{R}^N \times \mathbb{R}^{N-2}$, and, by the same technique introduced in the Step 3, we can assume without loss of generality that $(0,0)$ is a regular value of ψ. Then $\psi^{-1}(0,0)$ is the union of a finite number of compact 1 -manifolds. It is well known that a compact 1 -manifold imbedded in a compact manifold either is homeomorphic to S^1 or it starts and dies on the boundary. Since $\psi = \Phi_1 \times \varphi$ on the set $[0,1] \times S^{N-2} \times S^{N-2} \times S^1$, $\psi^{-1}(0,0)$ can intersect the boundary of the domain only in $[\{0\} \times S^{N-2} \times S^{N-2} \times B^2] \cup [\{1\} \times S^{N-2} \times S^{N-2} \times B^2]$. Remark that one can assume without loss of generality that each 1-manifold intersect this boundary in a transversal way. From the previous step we know that there is an odd number of symmetric 4 -ples of 1 -manifolds of $\psi^{-1}(0,0)$ starting at $\lambda = 0$. From the symmetry of the problem every 4 -ple is of the type $\{(x, y, \rho e^{i\omega t})\ ,\ (x, -y, \rho e^{-i\omega t})\ ,\ (-x, y - 2(y - (y \cdot x)x), \rho e^{i\omega(\pi + t)})\ ,\ (-x, -y + 2(y - (y \cdot x)x), \rho e^{i\omega(\pi - t)})\}$, so two symmetric (i.e. belonging to the same 4 -ple) 1 -manifolds can not intersect. Therefore we can conclude that at least one 4 -ple of 1 -manifolds of $\psi^{-1}(0,0)$ has to intersect $\{1\} \times S^{N-2} \times S^{N-2} \times B^2$.

◇

REMARK I.4.1: It should be clear that we have actually proved the following fact:

PROPOSITION I.4.1. *Let $\Phi : S^{N-2} \times S^{N-2} \times B^2 \to \mathbb{R}^N$ and $\varphi : S^{N-2} \times S^{N-2} \times B^2 \to \mathbb{R}^{N-2}$ be continuous satisfying (I.2.9), (I.2.11) and (I.2.12). Then*

(i) $$0 \in \Phi \times \varphi \left(S^{N-2} \times S^{N-2} \times B^2\right) ,$$

$\forall |z| < 1$, $\forall \delta > 0$ $\exists z_\delta$ and $\exists \varphi_\delta$ *satisfying (I.2.12)*

(ii) *such that* $|z_\delta - z$ *vert* $< \delta$, $|\varphi_\delta - \varphi|_{+\infty} < \delta$; and

$\left((\Phi - z_\delta) \times \varphi_\delta\right)^{-1}(0)$ *consists in an odd number of 4-ples of the type of (I.2.13)*.

4.II. THE STRONG FORCE CASE ($\alpha \geq 2$)

In this chapter we deal with the problem of finding a multiplicity of solutions to (P_T) when the potential F behaves in a similar way that one pf the form $F(x) = \frac{-a}{|x|^\alpha}$, with $\alpha \geq 2$. To this end, we shall apply the abstract multiplicity theorems of the last chapter.

Of course we are going to look for solutions of (P_T) as critical points of the action integral

$$I(x) = \int_0^T \frac{1}{2}|\dot{x}|^2 - F(x) ,$$

where

$$\Lambda = \{x \in H^1(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^N) \;/\; x(t) \neq 0 \;,\; \forall\, t \in [0,T]\} .$$

We consider the following assumptions on F:

$\exists\, a, b > 0$ and $\alpha \geq \alpha_1 \geq 2$, such that

$(H1)$ $$\frac{a}{|x|^\alpha} \leq -F(x) \leq \frac{b}{|x|^\alpha} , \qquad \forall\, x \in \mathbb{R}^N \setminus \{0\} ;$$

$(H2)$ $$\nabla F(x) \cdot x \geq -\alpha_1 F(x) , \qquad \forall\, x \in \mathbb{R}^N \setminus \{0\} ;$$

$(H3)$ $$\lim_{|x| \to +\infty} |\nabla F(x)| = 0 .$$

Remark that (H1) implies that

$$I_a^\alpha(x) \leq I(x) \leq I_b^\alpha(x) , \qquad \forall x \in \Lambda ,$$

where, for any constant $a > o$,

(II.1) $$I_a^\alpha(x) = \int_0^T \frac{1}{2}|\dot{x}|^2 + \frac{a}{|x|^\alpha} ,$$

For a potential of the form $F(x) = \frac{-a}{|x|^\alpha}$, $(a > 0, \alpha \geq 1)$, the planar circular solutions of (P_T) having minimal period T have the property of minimizing I_a^α on the set of all the solutions (see 1.Part 1, chapter V). According with the notations there, for every $a > 0$, we denote by $c_a^\alpha(T)$ the smallest critical level of I_a^α: we recall that their the values are given by

(II.2) $$c_a^\alpha(T) = T\Big(\frac{2\pi}{T}\Big)^{\frac{2\alpha}{\alpha+2}} \Big(\frac{\alpha+2}{2\alpha}\Big) a^{\frac{2}{\alpha+2}} \alpha^{\frac{2}{\alpha+2}} .$$

Indeed $c_a^\alpha(T) = I_a^\alpha(x_a)$, where $x_a = R_a e^{i\omega t}$ with $\omega = \frac{2\pi}{T}$, and $R_a^{\alpha+2} = \alpha a(\frac{T}{2\pi})^2$.

If $F(x) = \frac{-a}{|x|^\alpha}$, we know that the problem (P_T) admits an infinity of solutions : indeed, for a fixed period T we have all the planar circular solutions $x(t) = Re^{i\omega t}$ (with $\omega = \frac{2\pi k}{T}$, and $R^{\alpha+2} = a\alpha\big(\frac{T}{2\pi k}\big)^2$) having $\frac{T}{k}$ as minimal period. For a more general potential F, we are interested in finding the set of the solutions of (P_T) corresponding in a variational sense to those of minimal period T (that is to those at level $c_a^\alpha(T)$).

Our first goal is the following

THEOREM II.1. *Let $F \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ and assume that (H1), (H2) and (H3) hold. Then (P_T) has at least $N-1$ geometrically distinct solutions x_k such that $c_a^\alpha(T) \le I(x_k) \le c_b^\alpha(T)$.*

Under some more restrictive assumptions we can obtain informations on the minimal period of the solutions found by the application of Theorem II.1:

COROLLARY II.1. *Let $F \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ and assume that (H1), (H2) hold. Assume moreover that F satisfies*

$$(H4) \qquad \begin{array}{c} \exists\, \alpha_2 \ge \alpha \quad \textit{such that} \\ |\nabla F(x)| \le a\alpha_2 \dfrac{1}{|x|^{\alpha+1}}, \qquad \forall\, x \in \mathbb{R}^N \setminus \{0\}, \end{array}$$

with $\alpha > 2$ and $a, b, \alpha, \alpha_2, \gamma$ satisfying

$$(H5) \qquad \frac{\alpha_2^2\, b^{2/\alpha}}{\alpha_1^2\, a^{2/\alpha}} \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{\alpha+2}{\alpha}} < 4.$$

Then, for every fixed T, (P_T) has at least $N-1$ geometrically distinct solutions having T as their minimal period.

As an straightforward application of Corollary II.1 we have the

COROLLARY II.2. *Let $U \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfy*

$$\lim_{|x|\to 0} |x|^{\alpha+1} |\nabla U(x)| = 0$$

for some $\alpha > 2$, and let $F(x) = \frac{-a}{|x|^\alpha} + U(x)$. Then there exists $\overline{T} > 0$ such that for every $T \le \overline{T}$, (P_T) has at least $N-1$ geometrical distinct solutions having T as minimal period.

We are going to prove Theorem II.1 as a consequence of Theorem I.3.1. To this aim, we prove some preliminary propositions.

PROPOSITION II.1. *Let $F \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ and assume that (H1) holds. Then*

$$\lim_{x \to \partial\Lambda} I = +\infty \,.$$

Moreover, I has a natural l.s.c. extension $\overline{I}$, which is defined in the whole of H and takes values in $\mathbb{R} \cup \{+\infty\}$, such that $\inf_{\partial\Lambda} \overline{I} = +\infty$.

PROOF: As we have pointed out in I.1 of Part 2 , a potential of the form $F(x) = \frac{-a}{|x|^\alpha}$ fulfils the strong force condition (SF) if and only if $\alpha \geq 2$. As a natural extension of I we can take $\overline{I}(x) = +\infty$ for $x \in \partial\Lambda$.◇

PROPOSITION II.2. *Let $F \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfying (H1) and (H2). Then I fulfils the (PS_c) condition at any positive level.*

PROOF: Indeed, (H1) together with (H2) is stronger than the set of assumption (H) of 2.I. Therefore Proposition I.1.2 of Part 2 can be applied. ◇.

PROPOSITION II.3. *Assume that (H1) holds. Then there is ε such that the sublevel $I^\varepsilon = \{x \in \Lambda \;/\; I(x) < \varepsilon\}$ is G-contractible. Therefore each class of non G-contractible sets has a positive infsup level.*

PROOF: Let $I(x) < \varepsilon$ with a very small ε. Then one obviously has

$$\int_0^T \frac{1}{2}|\dot{x}|^2 \leq \varepsilon \,,$$
$$\int_0^T -F(x) \leq \varepsilon \,.$$

Therefore one deduces that

$$\left\| x - \frac{1}{T}\int_0^T x \right\|_\infty \leq \sqrt{T\varepsilon} \,,$$

and that $|\frac{1}{T}\int_0^T x|$ has to be very large. We define the homotopy h as $h(x,\sigma) = (1-\sigma)x + \sigma\frac{1}{T}\int_0^T x$ (of course this homotopy is G-equivariant) . ◇

PROPOSITION II.4. *Assume that (H1) and (H2) hold. Then if x solves (P_T), $x \notin F_0$.*

PROOF: Let x be a solution of (P_T) and assume that $x \in F_0$. Then there exists a t_0 such that $x(t_0 - t) = x(t)$, $\forall t \in \mathbb{R}$, and hence $\dot{x}(t_0) = 0$. Let $E = \frac{1}{2}|\dot{x}|^2 + F(x)$; then, since

$F(x) < 0$, we have $E < 0$.
On the other hand, it follows from (H2) that $E \geq 0$. Indeed let $\rho(t) = |x(t)|$, and let $\rho_m = \rho(t_m)$ be a point of minimum of ρ ; then $\frac{d^2\rho^2}{dt^2}(t_m) \geq 0$, that is $|\dot{x}(t_m)|^2 \geq \nabla F(x(t_m)) \cdot x(t_m) \geq -\alpha_1 F(x(t_m))$, and therefore $E = \frac{1}{2}|\dot{x}(t_m)|^2 + F(x(t_m)) \geq -(\frac{\alpha_1}{2} - 1)F(t_m) \geq 0$.◇

Proposition II.5. *Let $I = I_a^\alpha$ for some $a > 0$ and $\alpha \geq 1$. Then all the assumption of Theorem I.3.1 are satisfied. Moreover*

$$\inf_{A \in \Gamma_r^*} \sup_A I_a^\alpha = c_a^\alpha(T) \qquad r = 1, \cdots N-1 .$$

Proof: Indeed, let

$$A_0 = \left(\alpha a (\frac{T}{2\pi})^2\right)^{\frac{1}{\alpha+2}} C_N , \tag{II.3}$$

then from Theorem I.4.1, $A_0 \in \Gamma_{N-1}^*$. Moreover, from Propositions 1.V.2 of Part 1 and I.1.1 of Part 2we have that

$$\sup_{A_0} I_a^\alpha = c_a^\alpha(T) < \inf_{\partial\Lambda} I_a^\alpha = +\infty .$$

From this fact and the previous Propositions, we easily see that all the assumptions of Theorem I.3.1 are fulfilled. Thus

$$c_r^* = \inf_{A \in \Gamma_r^*} \sup_A I_a^\alpha \leq \sup_{A_0} I_a^\alpha = c_a^\alpha(T) \qquad r = 1, \cdots N-1 .$$

On the other hand, the c_a^*s are critical levels for I_a^α, so that, from Proposition I.?? we deduce that

$$c_r^* \geq c_a^\alpha(T) \qquad r = 1, \cdots N-1 ;$$

hence the equality holds.◇

Now we are in a position to prove Theorem II.1:
Proof of Theorem 1: We are going to prove Theorem 1 as an application of Theorem I.3.1. Indeed, by Proposition II.1, (I.3.5) is satisfied. From Proposition II.1 and from Theorem I.4.1, also (I.3.6) is fulfilled. Then the critical levels c_r^* $(r = 1, \cdots, N-1)$ are then well defined ad positive, because of Proposition II.3, so that (I.3.7) follows from Proposition II.1. Finally Proposition II.4 makes (I.3.8) be fulfilled. Remark that (H1), together with Proposition II.5 leads to $c_a^\alpha(T) \leq c_1^* \leq, \ldots, \leq c_{N-1}^* \leq c_b^\alpha(T)$.◇

Corollary II.1 follows from Theorem II.1 and from the following Proposition:

Proposition II.6. *Assume that for some $\alpha > 2$, (H1),(H2),(H4) and (H5) hold. If c is such that*

$$c \leq c_b^\alpha(T) ,$$

then every critical point of I in Λ at level c has minimal period T.

Proof: Let x be a critical point of I at level $c \leq c_b^\alpha(T)$, and let

$$E = \frac{1}{2}|\dot{x}|^2 + F(x) \tag{II.4}$$

its energy (remember that (P) is autonomous). Let $\rho(t) = |x(t)|^2$, and let $\rho_M = \rho(t_M)$ be a point of minimum of ρ : then $\frac{d^2\rho}{dt^2}(t_M) \geq 0$, that is $|\dot{x}(t_M)|^2 - \nabla F(x(t_M)) \cdot \dot{x}(t_M) \geq 0$. Therefore it follows from (H1) and (H2) that

$$\frac{1}{|x(t_M)|^\alpha} \leq \left(\frac{2}{\alpha_1 - 2}\right)\frac{E}{a} . \tag{II.5}$$

By multiplying both sides of $-\ddot{x} = \nabla F(x)$ by x and by integrating we get

$$\int_0^T |\dot{x}|^2 = \int_0^T \nabla F(x) \cdot x \leq -\alpha_2 \int_0^T F(x) ,$$

and therefore

$$c = \int_0^T \frac{1}{2}|\dot{x}|^2 - F(x) \leq -\left(\frac{\alpha_2}{2} + 1\right) \int_0^T F(x) . \tag{II.6}$$

On the other hand, by integrating (II.4) and from (II.6), we get

$$\begin{aligned} TE &= c + 2\int_0^T F(x) \leq \left(1 - 2\left(\frac{\alpha_2}{2} + 1\right)^{-1}\right)c \\ &= \frac{\alpha_2 - 2}{\alpha_2 + 2}c \leq \frac{\alpha_2 - 2}{\alpha_2 + 2}c_b^\alpha(T) . \end{aligned} \tag{II.7}$$

Now assuming that the minimal period of x if $\frac{T}{k}$, with $k \geq 1$, from Wirtinger inequality and (H2) we have

$$\int_0^T |\ddot{x}|^2 \geq \left(\frac{2\pi k}{T}\right)^2 \int_0^T |\dot{x}|^2 \geq -\alpha\left(\frac{2\pi k}{T}\right)^2 \int_0^T F(x) . \tag{II.8}$$

In the other hand, from (II.5) and (H4),

$$\int_0^T |\ddot{x}|^2 = \int_0^T |\nabla F(x)|^2$$

(II.9)
$$\begin{aligned}
&\leq \alpha_2^2 a^2 \int_0^T \frac{1}{|x|^{2\alpha+2}} \leq \frac{\alpha_2^2 a^2}{|x(t_M)|^{\alpha+2}} \int_0^T \frac{1}{|x|^\alpha} \\
&\leq \frac{-\alpha_2^2 a}{|x(t_M)|^{\alpha+2}} \int_0^T F(x) ,
\end{aligned}$$

and therefore, taking into account of (II.2) and (II.5), (II.8) together with (II.9) lead to

$$\begin{aligned}
(\frac{2\pi k}{T})^2 &\leq \frac{-\alpha_2^2 a}{\alpha_1} \frac{1}{|x(t_M)|^{\alpha+2}} \leq \frac{\alpha_2^2}{\alpha_1}\left[(\frac{2}{\alpha_1 - 2})\frac{E}{a}\right]^{\frac{\alpha+2}{\alpha}} \\
&\leq \frac{\alpha_2^2 a}{\alpha_1}\left[(\frac{2}{\alpha-2})\frac{1}{a}\frac{\alpha_2 - 2}{\alpha_2 + 2}\frac{c_b^\alpha(T)}{T}\right]^{\frac{\alpha+2}{\alpha}} \\
&\leq \frac{\alpha_2^2}{\alpha_1^2}\frac{b^{2/\alpha}}{a^{2/\alpha}}\left(\frac{\alpha_2 - 2}{\alpha_1 - 1}\right)^{\frac{\alpha+2}{\alpha}} (\frac{2\pi}{T})^2 .
\end{aligned}$$

Therefore (H5) implies $k < 2$, that is $k = 1$. ◇

PROOF OF COROLLARY II.2: Under the assumptions of Corollary II.1, the trajectory of a solution at level $c_a^\alpha(T) \leq I(x) \leq c_b^\alpha(T)$ satisfies

$$|x(t)|^\alpha \leq \frac{(\alpha_2)b}{2E} ,$$

where, by (II.1),

$$E \geq \frac{\alpha_1 - 2}{\alpha_1 + 2}\frac{c_a^\alpha(1)}{T^{\frac{2\alpha}{2+\alpha}}} .$$

Hence we obtain that

$$|x(t)| \leq C(\alpha_1, \alpha, \alpha_2) T^{\frac{2\alpha}{2+\alpha}} .$$

Therefore it is sufficient to assume that (H1), (H2) (H4) (with (H5)) hold in a small neighborhood of zero (and this is the case of the potential in Corollary II.2), provided that the period T is chosen sufficiently small.◇

4.III. The weak force case $(1 < \alpha < 2)$

As we have pointed out in I.2 of Part 2, the case $1 \leq \alpha < 2$ presents an additional difficulty due to the fact that $\liminf_{x \to \partial\Lambda} I < +\infty$. However, the meaning of the pinching condition introduced in Theorem I.2.2 of Part 2 is that the functional evaluated over a set of great circles is less than the minimum value over all the collision solutions. Therefore Theorem I.3.1 still apply to this situation.

In this chapter we shall use the notations of the previous one. Moreover, according with the discussion of Part 1, ch. V and Part 2, I.2,, let us recall the definition of Ψ:

$$\Psi(\alpha) = \frac{\min_{x \in \partial\Lambda} \int_0^T \frac{1}{2}|\dot{x}|^2 + \frac{a}{|x|^\alpha}}{c_a^\alpha(T)}, \tag{III.1}$$

where $c_a^\alpha(T)$ is the smallest critical level of the functional corresponding to the potential $F(x) = \frac{-a}{|x|^\alpha}$. let us recall that the value of $c_a^\alpha(T)$ corresponds to the circular solutions and

$$c_a^\alpha(T) = T\left(\frac{2\pi}{T}\right)^{\frac{2\alpha}{2+\alpha}} \left(\frac{2+\alpha}{2\alpha}\right) \alpha^{\frac{2}{2+\alpha}} a^{\frac{2}{2+\alpha}}. \tag{III.2}$$

We consider the following assumption on F:

$\exists\, a, b > 0$ and $1 \leq \alpha < 2$, such that

$$\frac{a}{|x|^\alpha} \leq -F(x) \leq \frac{b}{|x|^\alpha}, \qquad \forall\, x \in \mathbb{R}^N \setminus \{0\}; \tag{H1}$$

$$-2F(x) - \nabla F(x) \cdot x \leq (2-\alpha)\frac{a}{|x|^\alpha}, \qquad \forall\, x \in \mathbb{R}^N \setminus \{0\}; \tag{H2}$$

$$\lim_{|x| \to +\infty} \nabla F(x) = 0. \tag{H3}$$

Then the following Theorem holds:

THEOREM III.1. *Let $F \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ and assume that (H1),(H2), (H3) hold with*

$$\frac{b}{a} \leq \Psi(\alpha). \tag{III.3}$$

Then, for every $T > 0$, (P_T) has at least $N-1$ geometrically distinct solutions x_k such that $c_a^\alpha(T) \leq I(x_k) \leq c_b^\alpha(T)$.

The assumption (H2) is a technical one and it is introduced in order to avoid the fixed points of the symmetry.
In order to obtain an estimate on the minimal level of the solutions, an additional pinching condition has to be introduced:

COROLLARY III.1. *In addition to (H1) and (III.3), assume that*

$$\exists \alpha_2\ , \alpha_1 < 2\ , \alpha_2 \geq \alpha \geq \alpha_1 \text{ such that}$$

$$\text{(H4)} \qquad -\alpha_1 F(x) \leq \nabla F(x) \leq -\alpha_2 F(x)\ , \qquad \forall x \in \mathbb{R}^N \setminus \{0\}$$

$$\text{(H5)} \qquad |\nabla F(x)| \leq a\alpha_2 \frac{1}{|x|^{\alpha+1}}\ , \qquad \forall x \in \mathbb{R}^N \setminus \{0\}\ .$$

Then there are three functions $\Xi(\alpha)$, $\sigma_1(\frac{b}{a}, \alpha, \alpha_1, \alpha_2)$ *and* $\sigma_2(\frac{b}{a}, \alpha, \alpha_1, \alpha_2)$ *such that when*

$$\text{(III.4)} \qquad \left(\frac{b}{a}\right)^{\frac{2}{\alpha}} \frac{2-\alpha_2}{2-\alpha} \left(\frac{(2-\alpha_1)(2+\alpha)}{(2-\alpha)(2+\alpha_1)}\right)^{\frac{2+\alpha}{\alpha}} < \Xi(\alpha)\ ,$$

$$\text{(III.5)} \qquad \sigma_2(\frac{b}{a}, \alpha, \alpha_1, \alpha_2) \left(\frac{2-\alpha_1}{2}\right)^{\frac{1}{\alpha}} < 1\ ,$$

and

$$\text{(III.6)} \qquad \frac{b^{2/\alpha}}{a^{2/\alpha}} \frac{\alpha_2^2}{\alpha_1 \alpha} \left(\frac{(2-\alpha_1)(2+\alpha)}{(2-\alpha_2)(2+\alpha_1)}\right)^{\frac{2+\alpha}{\alpha}} < 4 \left(\sigma_1(\frac{b}{a}, \alpha, \alpha_1, \alpha_2)\right)^{2+\alpha}\ ,$$

then, for every $T > 0$, (P_T) *has at least* $N-1$ *geometrically distinct solutions having* T *has minimal period.*
Moreover, Ξ *and* σ *enjoy the following properties:*

$$\Xi(\alpha) \geq 2\ , \qquad \forall 1 < \alpha < 2$$

$$\lim_{\alpha \to 2} \Xi(\alpha) = +\infty\ ;$$

$$\sigma(\frac{b}{a}, \alpha, \alpha_1, \alpha_2) > 0$$

$$\lim_{\frac{\alpha_2 b}{\alpha_1 a} \to 1} \sigma(\frac{b}{a}, \alpha, \alpha_1, \alpha_2) = 1 \qquad \text{for every fixed } \alpha$$

$$\lim_{\alpha \to 2} \sigma(\frac{b}{a}, \alpha, \alpha_1, \alpha_2) = 1 \qquad \text{if } \frac{(2-\alpha_1)b}{(2-\alpha_2)a} \text{ remains bounded.}$$

The following result is just a streightforward consequence of the above Corollary:

COROLLARY III.2. *Let $U \in C^1(\mathbb{R}^n \setminus \{0\};\mathbb{R})$ satisfy*

$$\lim_{x \to 0} |x|^{\alpha+1} |\nabla U(x)| = 0 ,$$

and let $F(x) = \frac{-a}{|x|^\alpha}$, for some $a > 0$ and $1 < \alpha < 2$. Then there exists $\overline{T}$ such that, for every $T \leq \overline{T}$, then (P_T) has at least $N-1$ geometrically distinct solutions having T as minimal period.

Let us start by proving Theorem III.1:

PROOF OF THEOREM III.1: To carry out the proof, we are going to apply Theorem I.3.1, together with Theorem I.4.1. First of all, I has a natural (weakly) l.s.c. extension as

$$\overline{I}(x) = \begin{cases} \int_0^T \frac{1}{2}|\dot{x}|^2 - F(x) , & \text{if } \int_0^T -F(x) < +\infty ; \\ +\infty , & \text{if } \int_0^T -F(x) = +\infty . \end{cases}$$

Arguing as in the proof of Theorem II.1, one easily checks that assumptions (I.3.5) and (I.3.7) of Theorem I.3.1 are fulfilled. As far as condition (I.3.6) is concerned, one just consider that, by virtue of Theorem I.4.1, the set

$$A_0 = \left(b\alpha \left(\frac{T}{2\pi} \right)^2 \right)^{\frac{1}{2+\alpha}} C_N$$

belongs to Γ^*_{N-1}, and, from (H1) and (III.3), it enjoys the property that

$$\sup_{A_0} I \leq c_b^\alpha(T) \leq \inf_{\partial\Lambda} I_a^\alpha \leq \inf_{\partial\Lambda} I .$$

Indeed, A_0 is the set of all the circular critical points of I_b^α. Then the proof works as the one of Theorem II.1. The only difference consists in proving that (H2) implies (I.3.8). This fact can be proved as follows:
Let $E = \frac{1}{2}|\dot{x}|^2 + F(x)$; then, since $F(x) < 0$, we have $E < 0$. Moreover, since $-\ddot{x} = \nabla F(x)$, it follows from (H2) and the energy integral that

$$\text{(III.7)} \qquad \frac{d^2}{dt^2}\rho^2 = 4E - 4F(x) - 2\nabla F(x) \cdot x \leq 2\big(2E + (2-\alpha)\frac{a}{\rho^\alpha}\big) ,$$

where $\rho(t) = |x(t)|$. Therefore, if t_0 is such that $|\dot{x}(t_0)| = 0$, and then $E = F(x(t_0)) \leq \frac{-a}{|x(t_0)|^\alpha}$ it follows from (II.10) that $\rho(t_0)$ is a strict local maximum of ρ. Hence, $\dot{\rho}(t) \leq 0$ for every $t \in [t_0, t_1]$, for some $t_1 > t_0$, such that $\rho(t_1)$ is a local minimum. By multiplying (III.7) by $\dot{\rho}$ and by integrating, we obtain

$$\frac{1}{4}(\dot{\rho^2})^2 - 2E\rho^2 - 2a\rho^{2-\alpha} \geq -2E\rho(t_0)^2 - 2a\rho(t_0)^{2-\alpha} \geq 0 ;$$

hence $\rho(t_1) = 0$. Therefore $x \in \partial\Lambda$ while, condition (III.3) does not allow critical point at level less than $c_b^\alpha(T)$ in $\partial\Lambda$.
We are now in a position to apply Theorem I.3.1,finding at least N-1 critical points of the functional I in Λ such that $c_a^\alpha(T) \le I(x_k) \le c_b^\alpha(T)$. ◇

We remark that even a very small perturbation of the Keplerian potential ($\alpha = 1$) cannot be included in this discussion. Indeed if $a = b$, then $c_0 = c_a = c_b$, and no perturbation is allowed, as is shown in chapter III of Part 2.

PROOF OF COROLLARY III.1: Corollary III.1 follows from Theorem III.1 and the following proposition:

PROPOSITION III.1. *Let (H1) and (H4) holds. Then there are functions Ξ, σ_1 and σ_2, satisfying the properties of Corollary III.1, such that, when (III.4) and (III.5) hold then each solution x of (P_T) satisfying*

$$c_a^\alpha(T) \le I(x) \le c_b^\alpha(T)|; , \tag{III.8}$$

has T as minimal period and can not belong to F_0.

In order to prove Proposition III.1, some preliminaries are needed.
Let x be a solution of (P_T) such that (III.8) holds. Then x solves

$$-\ddot{x} = \nabla F(x) \tag{III.9}$$

$$\frac{1}{2}|\dot{x}|^2 + F(x) = E \tag{III.10}$$

where, from (III.8) and (H1),(H4), the energy E has the bounds:

$$\frac{2-\alpha_2}{2+\alpha_2}\frac{c_a^\alpha(T)}{T} \le -E \le \frac{2-\alpha_1}{2+\alpha_1}\frac{c_b^\alpha(T)}{T} . \tag{III.11}$$

Let us denote

$$\rho(t) = |y(t)| , \qquad \forall t \in \mathbb{R} \tag{III.12}$$

and

(III.13) $$c(t) = \frac{1}{a(2-\alpha_2)}|y(t)|^\alpha(-2F(y(t)) - \nabla F(y(t))\cdot y(t)) , \qquad \forall t \in \mathbb{R} .$$

Then from (H4) we deduce that

(III.14) $$1 \le c(t) \le \frac{(2-\alpha_1)b}{(2-\alpha_2)a} , \qquad \forall t \in \mathbb{R} .$$

Moreover, from (III.9) and (III.10) we have

(III.15) $$\begin{cases} -\frac{1}{2}\ddot{\rho^2} = -2E - (2-\alpha)\frac{c(t)}{\rho^\alpha} & \forall t \in \mathbb{R} \\ \rho(t+T) = \rho(t) & \forall t \in \mathbb{R} \\ \rho(t) > 0 & \forall t \in \mathbb{R} . \end{cases}$$

PROPOSITION III.2. *Assume that (H1) and (H4) hold, and assume moreover that* $\frac{b}{a} < \frac{2}{2-\alpha_1}$. *Assume that* $x \in F_0$. *Then there are* T_1 *and* T_2 *with* $T_2 - T_1 \le \frac{T}{2}$, *such that* $\rho(T_1) = \rho(T_2)$, $\dot{\rho}(T_1) = \dot{\rho}(T_2) = 0$, *and* $\min_{t\in[T_1,T_2]} |\dot{y}| = 0$.

PROOF: By definition, $x \in F_0$ implies that there exists $s \in [0,1]$ such that

$$x(s-t) = x(t) , \qquad \forall t \in \mathbb{R} ,$$

or, equivalently

$$x(\frac{s}{2}-t) = x(\frac{s}{2}+t) , \qquad \forall t \in \mathbb{R} .$$

One easily deduces that

$$x(\frac{s+T}{2}-t) = x(\frac{s+T}{2}+t) , \qquad \forall t \in \mathbb{R} ,$$

and therefore that

$$\dot{x}(\frac{s}{2}) = \dot{x}(\frac{s+T}{2}) = 0 .$$

Hence, it follows from (III.10) that

$$F(x(\frac{s}{2})) = F(x(\frac{s+T}{2})) = E ,$$

so that (H1) implies

$$|x(\frac{s}{2})|^\alpha \ge \frac{a}{-E} , \quad \text{and } |x(\frac{s+1}{2})|^\alpha \ge \frac{a}{-E} .$$

From (III.15), we can conclude that both $|x(\frac{s}{2})|$ and $|x(\frac{s+T}{2})|$ are strict local maxima for $|x(t)|$. One then deduces that there is at least one local minimum $|x(t^*)|$, with $t^* \in [\frac{s}{2}, \frac{s+T}{2}]$. Assuming for example that $t^* - \frac{s}{2} \le \frac{1}{4}$, one finds that $\rho(t^*) = \rho(s-t^*)$ and $\dot{\rho}(t^*) = \dot{\rho}(s-t^*) = 0.\diamond$

PROPOSITION III.3. *Assume that (H1) and (H4) hold. Assume that the minimal period of x is $\frac{1}{k}$, with $k \geq 2$. Then there are T_1 and T_2 with $T_2 - T_1 \leq \frac{T}{2}$, such that $\rho(T_1) = \rho(T_2)$, $\dot\rho(T_1) = \dot\rho(T_2) = 0$.*

PROOF: In an obvious consequence of the fact that $\rho(t + \frac{T}{k}) = \rho(t)$, $\forall t \in \mathbb{R}$.$\diamond$

PROPOSITION III.4. *Under the assumptions of Proposition III.2 (respectively Proposition III.3), there are three functions $\sigma_1, \sigma_2 : (1,2)\times[1,+\infty) \to (0,+\infty)$ and $\Xi : (1,2) \to (1,+\infty)$ such that, if (III.4) holds, then*

$$\rho(t) \geq \sigma_1(\frac{b}{a}, \alpha_2, \alpha, \alpha_1)\left(\frac{(2-\alpha_1)a}{-2E}\right)^{\frac{1}{\alpha}}, \quad \forall t \in [T_1, T_2],$$

and

$$\rho(t) \leq \sigma_2(\frac{b}{a}, \alpha_2, \alpha, \alpha_1)\left(\frac{(2-\alpha_2)b}{-2E}\right)^{\frac{1}{\alpha}}, \quad \forall t \in [T_1, T_2].$$

Moreover, the following properties hold:

$$\Xi(\alpha) \geq 2 \qquad \forall \alpha \in (1,2);$$
$$\lim_{\alpha \to 2} \Xi(\alpha) = +\infty;$$

$$\lim_{\frac{b_1}{a_1} \to 1} \sigma_i(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) = 1, \qquad \forall \alpha \in (1,2)$$

$$\lim_{\alpha \to 2} \sigma_i(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) = 1, \qquad \text{if } \frac{(2-\alpha_1)b}{(2-\alpha_2)a} \text{ remains bounded}.$$

PROOF. STEP 1: By the change of variables $s(t) = \int_0^t \frac{1}{\rho^\alpha}$, and $\mu(s) = \rho^{2-\alpha}(s(t))$, from Proposition III.2 (resp . Proposition III.3), (III.15) becomes equivalent to

$$\text{(III.16)} \qquad \begin{cases} -\mu'' = -2(2-\alpha)E\mu^\beta - (2-\alpha)^2 c(s) \\ \mu(0) = \mu(\omega), \\ \mu'(0) = \mu'(\omega) = 0 \\ \mu(s) > 0, \ \forall s. \end{cases}$$

Where $(\cdot)'$ denotes the derivation with respect to the variable s

$$\text{(III.17)} \qquad \beta = \frac{\alpha}{2-\alpha}.$$

and

(III.18)
$$\omega = \int_{T_1}^{T_2} \frac{1}{\rho^\alpha}.$$

Note that, by Proposition III.2 (resp. Proposition III.3), by integrating the equation in (III.15) one obtains

(III.19)
$$\omega = \int_{T_1}^{T_2} \frac{1}{\rho^\alpha} \le \frac{-2E}{(2-\alpha_2)a}(T_2 - T_1) \le \frac{-2E}{(2-\alpha_2)a}\frac{T}{2} .$$

We set $c(s) = (1 + c_0(s))$, so that, from (III.14) we have

(III.20)
$$0 \le c_0(s) \le \frac{(2-\alpha_1)b}{(2-\alpha_2)a}, \qquad \forall s \in [0,\omega] .$$

By derivating the equation in (III.16) and by taking the L^2 inner product with μ' we obtain

(III.21)
$$\int_0^\omega (\mu")^2 ds = -2\alpha E \int_0^\omega \mu^{\beta-1}(\mu')^2 ds + (2-\alpha)(2-\alpha_2)a \int_0^\omega c_0 \mu" ds ,$$

and, from Holder inequality, taking into account of (III.20),

(III.22)
$$\int_0^\omega (\mu")^2 ds$$
$$\le -2\alpha E \int_0^\omega \mu^{\beta-1}(\mu')^2 ds + (2-\alpha)(2-\alpha_2)a \left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a} - 1\right)\sqrt{\omega}\left(\int_0^\omega (\mu")^2 ds\right)^{\frac{1}{2}} .$$

Now, for any $0 < \delta < 1$, the inequality
$$\delta x^2 - \frac{d^2}{4(1-\delta)} \le x^2 - dx , \qquad \forall x, d \in \mathbb{R} ,$$
together with (III.22) imply

(III.23)
$$\delta \int_0^\omega (\mu")^2 ds - \frac{(2-\alpha)^2"2-\alpha_2)^2 a^2 \left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a} - 1\right)^2 \omega}{4(1-\delta)} \le -2\alpha E \int_0^\omega \mu^{\beta-1}(\mu')^2 ds .$$

STEP 2: Let us denote by H^2_ω the Sobolev space $\{u \in H^2(\mathbb{R};\mathbb{R}) \ / \ u(s+\omega) = u(s) \forall s \in \mathbb{R}\}$. Let us consider, for every $\beta > 1$

(III.24) $$\Xi^*(\beta,\omega) = \inf_{H^2_\omega} \frac{\left(\int_0^\omega |u|^\beta\right)^{\frac{\beta-1}{\beta}} \int_0^\omega |\ddot{u}|^2}{\int_0^\omega |\dot{u}|^2 |u|^{\beta-1}}$$

and

(III.25) $$\Omega^*(\beta,\omega) \ = \ \inf_{u\in H^2_\omega} \frac{\int_0^\omega (u")^2}{\left(\int_0^\omega (u')^{2\beta}\right)^{\frac{1}{\beta}}} \ .$$

The following properties hold

(III.26) $$\Xi^*(\beta,\omega) = \frac{1}{\omega^{\frac{\beta+1}{\beta}}}\Xi^*(\beta,1) \ , \qquad \forall \beta eq1 \ , \forall \omega > 0 \ ;$$

(III.27) $$\Omega^*(\beta,\omega) = \frac{1}{\omega^{\frac{\beta+1}{\beta}}}\Omega^*(\beta,1)) \ , \qquad \forall \beta \geq 1 \ , \forall \omega > 0 \ ;$$

(III.28) $$\Xi^*(\beta,1) \geq \Omega^*(\beta,1) \ , \qquad \forall \beta \geq 1 \ ;$$

(III.29) $$\Omega^*(\beta,1) = \frac{1}{\omega^{\frac{1+\beta}{\beta}}} \frac{(1+\beta)^{\frac{1+\beta}{\beta}}}{\beta} \{ \int_0^{2\pi} |\sin\theta|^{\frac{1+\beta}{\beta}} d\theta \}^2 \ .$$

Taking into account of (III.17), we set

(III.30) $$\begin{aligned} \Xi(\alpha) &= \frac{4}{(2\pi)^2(2-\alpha)}\Xi^*(\frac{\alpha}{2-\alpha},1) \ , \\ \Omega(\alpha) &= \frac{4}{(2\pi)^2(2-\alpha)}\Omega^*(\frac{\alpha}{2-\alpha},1) \ , \ . \end{aligned}$$

The proof of (III.26) and (III.27) obviously follows from the definitions. Formula (III.28) follow from the definitions and Holder's inequality. Finally (III.29) was proved in 3.II. Remark that Ξ fulfils all the properties of the claim of Corollary III.1

Now, from our change of variables we have

(III.31) $$\int_0^\omega \mu^\beta = T_2 - T_1 \ .$$

Hence the inequality (III.23) can be rewritten as

$$\text{(III.32)}\qquad \begin{aligned} &\delta\int_0^\omega (\mu")^2 ds \;-\; \frac{(2-\alpha)^2(2-\alpha_2)^2 a\left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a}-1\right)^2\omega}{4(1-\delta)} \\ &\leq\; -2\alpha E\left(\frac{T}{2}\right)^{\frac{\beta-1}{\beta}} \frac{\int_0^1(\mu')^2\mu^{\beta-1}}{\left(\int_0^\omega \mu^\beta\right)^{\frac{\beta-1}{\beta}}}\,. \end{aligned}$$

Assuming that μ is not constant, from (III.32), (III.23) (III.26), we deduce

$$\text{(III.33)}\qquad \begin{aligned} &\delta \;-\; \frac{(2-\alpha)^2(2-\alpha_2)^2 a\left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a}-1\right)^2\omega}{4(1-\delta)\int_0^\omega(\mu")^2} \\ &\leq\; -2\alpha E\left(\frac{T}{2}\right)^{\frac{\beta-1}{\beta}} \frac{1}{\Xi^*(\beta,1)}\omega^{\frac{1+\beta}{\beta}}\,, \end{aligned}$$

and therefore from (III.17), (III.18),(III.10), (III.2) and (III.30), we obtain

$$\text{(III.34)}\qquad \begin{aligned} &\delta \;-\; \frac{(2-\alpha)^2(2-\alpha_2)^2 a\left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a}-1\right)^2\omega}{4(1-\delta)\int_0^\omega(\mu")^2} \\ &\leq\; \left(\frac{b}{a}\right)^{\frac{2}{\alpha}}\frac{2-\alpha_2}{2-\alpha}\left(\frac{(2-\alpha_1)(2+\alpha)}{(2-\alpha)(2+\alpha_1)}\right)^{\frac{2+\alpha}{\alpha}}\frac{1}{\Xi(\alpha)}\,. \end{aligned}$$

Now, from (III.4) we have

$$\text{(III.35)}\qquad \left(\frac{b}{a}\right)^{\frac{2}{\alpha}}\frac{2-\alpha_2}{2-\alpha}\left(\frac{(2-\alpha_1)(2+\alpha)}{(2-\alpha)(2+\alpha_1)}\right)^{\frac{2+\alpha}{\alpha}} < \Xi(\alpha)\,.$$

Let us denote

$$\text{(III.36)}\qquad \tilde{\xi}_1\left(\frac{b}{a},\alpha_2,\alpha,\alpha_1\right) = \left(\frac{b}{a}\right)^{\frac{2}{\alpha}}\frac{2-\alpha_2}{2-\alpha}\left(\frac{(2-\alpha_1)(2+\alpha)}{(2-\alpha)(2+\alpha_1)}\right)^{\frac{2+\alpha}{\alpha}}\,,$$

We set

$$\delta = \frac{1}{2}\left(1+\frac{\tilde{\xi}_1(\frac{b}{a},\alpha_2,\alpha,\alpha_1)}{\Xi(\alpha)}\right)\,;$$

it follows from (III.4) that $0 < \delta < 1$. Now (III.34) becomes

$$\text{(III.37)}\qquad \begin{aligned}&\int_0^\omega (\mu")^2 \\ &\leq (2-\alpha)^2(2-\alpha_2)^2 a\omega \left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a}-1\right)^2 \left(1-\frac{\tilde{\xi}_1((\frac{b}{a},\alpha_2,\alpha,\alpha_1)}{\Xi(\alpha)}\right)^{-2},\end{aligned}$$

and hence, from (III.27),

$$\text{(III.38)}\qquad \begin{aligned}&\left(\int_0^\omega (\mu')^{2\beta}\right)^{\frac{1}{\beta}} \\ &\leq \frac{1}{\Omega(\alpha)}(2-\alpha)^2(2-\alpha_2)^2 a^2 \omega^{\frac{2\beta+1}{\beta}} \left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a}-1\right)^2 \left(1-\frac{\tilde{\xi}_1((\frac{b}{a},\alpha_2,\alpha,\alpha_1)}{\Xi(\alpha)}\right)^{-2}.\end{aligned}$$

Now, an easy computation shows that

$$\left(\int_{T_1}^{T_2}\left(\frac{d}{dt}\rho^{\frac{2+\alpha}{2}}\right)^{2\beta} dt\right)^{\frac{1}{\beta}} = \left(\frac{(2+\alpha)}{2(2-\alpha)}\right)^2 \left(\int_0^\omega (\mu')^{2\beta} ds\right)^{\frac{1}{\beta}},$$

so that,from (III.38), (III.17) we obtain that

$$\text{(III.39)}\qquad \left(\int_{T_1}^{T_2}\left(\frac{d}{dt}\rho^{\frac{2+\alpha}{2}}\right)^{2\beta} dt\right)^{\frac{1}{\beta}} \leq \tilde{\xi}_2(\frac{b}{a},\alpha_2,\alpha,\alpha_1)a^2 \left(\frac{-2E}{(2-\alpha_2)a}\right)^{\frac{2+\alpha}{\alpha}} \left(\frac{T}{2}\right)^{\frac{2+\alpha}{\alpha}},$$

where

$$\begin{aligned}&\tilde{\xi}_2(\frac{b}{a},\alpha_2,\alpha,\alpha_1) \\ &= \left(\frac{(2+\alpha)}{2(2-\alpha)}\right)^2 \frac{1}{\Omega(\alpha)}(2-\alpha)^2(2-\alpha_2)^2 \left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a}-1\right)^2 \left(1-\tilde{x}(\frac{b}{a},\alpha_2,\alpha,\alpha_1)\frac{1}{\Xi(\alpha)}\right)^{-2},\end{aligned}$$

has the following properties:

$$\text{(III.40)}\qquad \begin{aligned}&\lim_{\frac{(2-\alpha_1)b}{(2-\alpha_2)a}\to 1} \tilde{\xi}_2(\frac{b}{a},\alpha_2,\alpha,\alpha_1) = 0 \qquad && \forall\, 1<\alpha<2 \\ &\lim_{\alpha\to 2} \tilde{\xi}_2(\frac{b}{a},\alpha_2,\alpha,\alpha_1) = 0 && \text{if } \frac{(2-\alpha_1)b}{(2-\alpha_2)a} \text{ remains bounded}\end{aligned}$$

Let us write $\rho_* = \left(\frac{(2-\alpha_2)a}{-2E}\right)^{\frac{1}{\alpha}}$. Remark that, since, from (III.15), every local maximum $\rho(t_0)$ has $\rho(t_0) \geq \rho_*$, then either ρ_* is assumed by the function $\rho(t)$ of $\rho(t) \geq \rho_*$, $\forall t$.From (III.39) we deduce

$$|\rho^{\frac{2+\alpha}{2}}(t) - \rho_*^{\frac{2+\alpha}{2}}| \leq \int_{T_1}^{T_2} |\frac{d}{dt}\rho^{\frac{2+\alpha}{\alpha}}|dt$$

$$\leq \left(\int_{T_1}^{T_2} \left(\frac{d}{dt}\rho^{\frac{2+\alpha}{2}}\right)^{2\beta} dt\right)^{\frac{1}{2\beta}} T^{1-\frac{1}{2\beta}}$$

$$\leq \sqrt{\bar{\xi}_2}\left(\frac{-2E}{(2-\alpha_2)a}\right)^{\frac{2+\alpha}{2\alpha}} a \left(\frac{T}{2}\right)^2$$

Now, (III.11) implies that

$$\frac{-2E}{(2-\alpha_2)a} \leq \left(\frac{b}{a}\right)^{\frac{2}{2+\alpha}} \frac{2-\alpha_1}{2-\alpha_2}\,\frac{2c_1^{\alpha}(1)}{2+\alpha_2}\,\frac{1}{(aT^2)^{\frac{\alpha}{2+\alpha}}},$$

so that the above inequality leads to

$$\text{(III.41)} \qquad |\rho^{\frac{2+\alpha}{2}}(t) - \rho_*^{\frac{2+\alpha}{2}}| \leq \tilde{\xi}_3 \left(\frac{(2-\alpha_2)a}{-2E}\right)^{\frac{2+\alpha}{2\alpha}} = \tilde{\xi}_3 \rho_*^{\frac{2+\alpha}{2}},$$

where

$$\tilde{\xi}_3(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) = \sqrt{\bar{\xi}_2}\left(\frac{b}{a}\right)^{\frac{2}{\alpha}} \left(\frac{c_1^{\alpha}(1)(2-\alpha_1)}{(2+\alpha_1)(2-\alpha_2)}\right)^{\frac{2+\alpha}{\alpha}} \qquad III.42$$

still satisfies

$$\text{(III.40)} \qquad \begin{aligned} &\lim_{\frac{(2-\alpha_1)b}{(2-\alpha_2)a}\to 1} \tilde{\xi}_3(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) = 0 \qquad \forall\, 1<\alpha<2 \\ &\lim_{\alpha\to 2} \tilde{\xi}_3(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) = 0 \qquad \text{if } \frac{(2-\alpha_1)b}{(2-\alpha_2)a} \text{ remains bounded} \end{aligned}$$

Finally obtain

$$\text{(III.44)} \qquad \rho(t) \geq \sigma_1\rho_* = \sigma\left(\frac{(2-\alpha_2)a}{-2E}\right)^{\frac{1}{\alpha}} \qquad \forall t \in [T_1, T_2],$$

and

$$\rho(t) \leq \sigma_2 \rho_* = \sigma \left(\frac{(2-\alpha_2)a}{-2E} \right)^{\frac{1}{\alpha}} \qquad \forall t \in [T_1, T_2], \tag{III.45}$$

where

$$\left(\sigma_1(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) \right)^{\frac{2+\alpha}{2}} = 1 - \tilde{\xi}_3(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) \,,$$

and

$$\left(\sigma_2(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) \right)^{\frac{2+\alpha}{2}} = 1 - \tilde{\xi}_3(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) \,,$$

From (III.43), both σ_1 and σ_2 satisfy

$$\begin{aligned} &\lim_{\frac{(2-\alpha_1)b}{(2-\alpha_2)a} \to 1} \sigma_i(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) = 0 \qquad \forall\, 1 < \alpha < 2 \\ &\lim_{\alpha \to 2} \sigma_i(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) = 0 \qquad \text{if } \frac{(2-\alpha_1)b}{(2-\alpha_2)a} \text{ remains bounded} \end{aligned} \tag{III.46}$$

Hence Proposition III.4 is proved.

Proof of Proposition III.1: Assume first that $x \in F_0$. Then Proposition III.2 says that there is $\bar{t} \in [T_1, T_2]$ such that $\dot{x}(\bar{t}) = 0$. We then deduce from (III.10), (H1) and Proposition III.4 that $\sigma_2(\frac{b}{a}, \alpha_2, \alpha, \alpha_1)^{\alpha} < \frac{2}{2-\alpha_1}$, that is a contradiction.

Now we assume that the minimal period of x is $\frac{T}{k}$, for some integer $k \geq 2$. Then Wirtinger inequality leads to

$$\left(\frac{2\pi k}{T} \right)^2 \int_0^T |\dot{x}|^2 \leq \int_0^T |\ddot{x}|^2 \,. \tag{III.47}$$

On the other hand, from (H2) we have

$$\int_0^T |\dot{x}|^2 \geq -\alpha_1 \int_0^T F(x) \tag{III.48}$$

and, from (H1), (H5) and Proposition III.4 we obtain

(III.49)

$$\int_0^T |\ddot{x}|^2 = \int_0^T |\nabla F(x)|^2 \leq a^2 \alpha_2 \int_0^T \frac{1}{|x|^{2\alpha+2}} \leq a\alpha_2^2 \left(\frac{-2E}{(2-\alpha_1)a} \right)^{\frac{2+\alpha}{\alpha}} \frac{1}{\sigma_1^{2+\alpha}} \int_0^T -F(x) \,.$$

From (III.48) and (III.49) we deduce

$$\left(\frac{2\pi k}{T}\right)^2 \leq a\alpha_2^2 \left(\frac{-2E}{(2-\alpha_1)a}\right)^{\frac{2+\alpha}{\alpha}} \frac{1}{\sigma_1^{2+\alpha}} .$$

Therefore the estimate in the energy (III.11), together with (III.2) implies

$$k^2 \leq \frac{b^{2/\alpha}}{a^{2/\alpha}} \frac{\alpha_2^2}{\alpha_1\alpha} \left(\frac{(2-\alpha_1)(2+\alpha)}{(2-\alpha_2)(2+\alpha_1)}\right)^{\frac{2+\alpha}{\alpha}} \frac{1}{\sigma_1^{2+\alpha}} < 4$$

Therefore $k = 1.\diamond$

PROOF OF COROLLARY III.2: It is a matter of facts that, under the assumptions of Corollary III.1, each solution satisfying (III.??) is constrained in the set $\{x \in \mathbb{R}^N \;/\; |x|^\alpha \leq \frac{b}{-E}\}$. Therefore (III.11) implies that each solution satisfying (III.???) has the bound $|x(t)|^\alpha \leq \frac{(2+\alpha_2)b}{(2-\alpha_2)c_a^\alpha(1)} T^{\frac{2\alpha}{2+\alpha}}$. Thus, it is enough to assume that (H1), (H4) and (H5) hold in a small neighborhood of the origin, provided the period is chosen sufficiently small. Now a potential of the form of the one in Corollary III.2 obviously fulfils (H1), (H4) and (H5) with $\frac{b}{a}$ and $\frac{\alpha_2}{\alpha_1}$ as close to one as we want, in a sufficiently small neighborhood of zero. $\diamond$

4.IV. THE EVEN CASE

As is should be clear from the results contained in Part 2, chapter III, the approach of the last chapter apply strictly when $\alpha > 1$. In facts, the pinching condition of Theorem III.1 becomes $\frac{b}{a} = 1$, when $\alpha = 1$. However, as it has been shown by Degiovanni, Marino and Giannoni in [26], when F has some symmetrical properties, one can overcome the degeneracy occurring at $\alpha = 1$. We shall see in Part 6 other interesting properties of the symmetrical case. We are going to use the approach of [26] in order to exclude the collision solutions. As we shall see, the symmetry constraint allows to work with coercive functionals. In such a situation, the multiplicity result will be proved via the application of Theorem I.2.1.

Throughout this chapter, the potential F is even, that is it satisfies

$$\text{(S)} \qquad F(-x) = F(x) \quad , \quad \forall\, x \in \mathbb{R}^N \setminus \{0\}$$

Then it is a well known fact that the critical points of the associated functional I constrained on

$$\Lambda^S = \{x \in \Lambda \ / \ x(t + \frac{T}{2}) = -x(t) \ , \forall t \in \mathbb{R}\}$$

are actually critical point of I over Λ and hence they are solutions of (P_T). Moreover, the quadratic part of the functional is coercive.

The main goal of this section is the following:

THEOREM IV.1. *Let $F \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfy (S) in addition to*

$$\text{(H1)} \qquad \exists\ a, b > 0 \text{ and } \alpha > 0, \text{ such that} \quad \frac{a}{|x|^\alpha} \le -F(x) \le \frac{b}{|x|^\alpha}, \qquad \forall\, x \in \mathbb{R}^N \setminus \{0\}\,;$$

with

$$\text{(IV.1)} \qquad \frac{b}{a} < 3^\alpha\,.$$

Then there exists a function $\overline{\Psi} : (0, +\infty] \to \mathbb{R}$ such that when

$$\text{(IV.2)} \qquad \frac{b}{a} < \overline{\Psi}(\alpha)\,,$$

then (P_T) *has at least N-1 (N if N is odd) geometrically distinct solutions having* T *as minimal period. Moreover,* $\overline{\Psi}$ *enjoys the properties:*

$$\lim_{\alpha\to 0}\overline{\Psi}(\alpha)=1$$
$$\overline{\Psi} \text{ is increasing}$$
$$\lim_{\alpha\to 1}\overline{\Psi}(\alpha)=+\infty$$

In order to prove Theorem IV.1, some preliminary proposition are needed.

PROPOSITION IV.1. *Let* $F(x)=\frac{-a}{|x|^\alpha}$, *for some* $a>0$ *and* $\alpha>0$. *Then*

$$c_a^\alpha(T)=\min_{\Lambda^S}\left(\int_0^T\frac{1}{2}|\dot{x}|^2+\frac{a}{|x|^\alpha}\right)$$
$$<\min_{\partial\Lambda^S}\left(\int_0^T\frac{1}{2}|\dot{x}|^2+\frac{a}{|x|^\alpha}\right)$$

PROOF: First of all, the functional $I_a^\alpha(x)=\int_0^T\frac{1}{2}|\dot{x}|^2+\frac{a}{|x|^\alpha}$ admits a natural weakly lower semicontinuous extension to the whole of $H^S=\{x\in H\ /\ x(t+\frac{T}{2})=-x(t)\ ,\forall t\in \mathrm{R}\}$; we shall still denote by I_a^α this extension. As such, I_a^α is coercive, so that is minimum is attained. Moreover, by the convexity of each function $\frac{1}{s^\beta}$ we have

$$I_a^\alpha(x)\geq\int_0^T\frac{1}{2}|\dot{x}|^2+aT^{\frac{2+\alpha}{2}}\frac{1}{\left(\int_0^T|x|^2\right)^{\frac{\alpha}{2}}}, \tag{IV.3}$$

and the strict inequality holds, unless x has constant modulus. From (IV.3) we deduce that Proposition IV.1 holds and moreover that the function

$$\overline{\Psi}(\alpha)=\frac{c_a^\alpha(T)}{\min_{\partial\Lambda^S}\left(\int_0^T\frac{1}{2}|\dot{x}|^2+\frac{a}{|x|^\alpha}\right)} \tag{IV.4}$$

fulfils all the desired properties.◇

PROPOSITION IV.2. *Assume that (H1) holds. Then I satisfies the (PS) condition (in* H^S*) at every level.*

PROOF: It is an easy consequence of the fact that i is coercive .◇

PROPOSITION IV.3.

$$F_0 \cap H^S \subseteq \partial \Lambda^S$$

PROOF: Indeed, if $x \in F_0 \cap H^S$ there exists s such that$x(s-t) = x(t)$, $\forall t \in \mathbb{R}$; on the other hand, since $x(t+\frac{T}{2}) = -x(t)$, $\forall t \in \mathbb{R}$, we deduce that, when $s - t_0 = t_0 + \frac{T}{2}$, then $x(t_0) = 0$.◇

PROPOSITION IV.4. *Assume that (H1) and (IV.1) hold. Then every critical point of I in Λ^S such that $I(x) \leq c_b$ has minimal period T.*

PROOF: We assume on the contrary that the minimal period of x is $\frac{T}{k}$, for some integer $k \geq 2$. We first deduce that k is odd, and therefore that $k \geq 3$. Hence, from (H1), we have

$$I(x) \geq \min_{k\geq 3} \min_{x \in \Lambda^S_k} \left(\int_O^T \frac{1}{2}|\dot{x}|^2 + \frac{a}{|x|^\alpha} \right) . \qquad \text{(IV.5)}$$

Now, is is clear from Proposition IV.1, and from the value of $c_a^\alpha(T)$ that the right hand side of (IV.5) is equal to $3^{\frac{2\alpha}{2+\alpha}} c_a^\alpha(T)$. Condition (IV.1) then implies that $k = 1$.◇

Now we are in a position to prove Theorem IV.1:

PROOF OF THEOREM IV.1: We are going to prove Theorem II.3 as an application of Theorem I.2.1. It follows from Theorem I.2.2 that the set

$$A_0 = \left(b\alpha \left(\frac{T}{2\pi} \right)^2 \right)^{\frac{1}{2+\alpha}}$$

has index larger than N-1 (N if N is odd). Therefore, from (H1), (IV.2) and the definition of $\overline{\Psi}$ we have that A_0 belongs to Γ_{N-1} (Γ_N if N is odd), since

$$\sup_{A_0} I \leq \sup_{A_0} \int_0^T |\dot{x}|^2 + \frac{b}{|x|^\alpha} = c_b^\alpha .$$

Moreover, by Propositions IV.2 and IV.3, assumptions (??) and (???) of Theorem I.2.1 are fulfilled. Hence Theorem I.2.1 can be applied, and N-1 (N is N is odd) geometrically distinct critical points of I are founds.

The above inequality also implies that these critical points are at level less or equal to $c_b^\alpha(T)$; therefore from Proposition IV.4 we deduce that their minimal period is exactly T.◇

PART 5. MULTIPLICITY OF PERIODIC SOLUTION TO THE FIXED ENERGY PROBLEM.

For a given potential $F \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ and a fixed energy E, we look for multiple solutions to the problem

$$(P_E) \qquad \begin{cases} -\ddot{x} = \nabla F(x) \\ \frac{1}{2}|\dot{x}|^2 + F(x) = E \\ x(t+\lambda) = x(t) & \forall t \in \mathbb{R} \\ x(t) \neq 0 & \forall t \in \mathbb{R}\,, \end{cases}$$

where the unknowns are both the function x and its period λ.
As usual, the potential F presents a singularity at the origin and it behaves like a potential of the form $F(x) = \frac{-a}{|x|^\alpha}$ for some $a, \alpha > 0$, behavior that will be clear from the assumptions of the theorems in chapter II and III.

Taking into account of the necessary and sufficient conditions for the solvability of (P_E) contained in Part 3, we shall treat separately the two cases:

(1) $\alpha > 2$ and $E > 0$;
(2) $1 < \alpha < 2$ and $E < 0$.

Of course, we shall look for solutions of (P_E) as critical points of the functional

$$I(x) = \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)\left(\int_0^1 E - F(x)\right)$$

over its natural domain (taking into account of the singularity of F)

$$\Lambda = \{x \in H \;/\; x(t) \neq 0\,, \forall t \in \mathbb{R}\}\,.$$

As for the fixed period problem, in order to obtain a multiplicity of critical points, we shall exploit the invariance of the functional under the group of symmetries $G = \{T_s, P_s\}$ where $T_s(x)(t) = x(s+t)$ and $P_s(x)(t) = x(s-t)$. In this order, a convenient tool to treat the problem in the case (1) will turn out to be the homotopical index related to the group G, which has been defined in I.1 of Part 4. In facts,when $\alpha > 2$ and $E > 0$ the features of the functional associated to (P_E) are similar to the ones of functional associated to the fixed period problem; in other words, I is positive and it presents a lack of coercivity only at the level of the large constant functions. Therefore the results of I.3 of Part 4 are (almost) applicable.

As far as the case (2) is concerned, an additional difficulty arises from the fact that, when the energy is negative (and it is the natural choice when $\alpha < 2$), then the associated functional is unbounded (from below and from above). To overcome this, a new homotopical pseudo-index theory is introduced. Roughly speaking, the homotopical pseudo-index of a given set is the homotopical index of its intersection with a given closed set of the function space. This approach will turn out to be profitable to treat the case (2), since the restriction of the associated functional to each set of the type $\{x \ / \ \int_0^1 |\dot{x}|^2 - \rho^2\}$ is bounded from below and it presents a lack of compactness only at the level of the large constant functions.

5.I. The homotopical pseudo-index theory

5.I.1. Definition of the homotopical pseudo-index and abstract multiplicity results.

The first purpose of this section is to define the homotopical pseudo-index related to the homotopical index of I.2 of Part 4 and to state its basic properties. The notion of homotopical pseudo-index has been introduced in [**41**] in order to study critical points of *indefinite* singular functionals. Our second goal consists in use the homotopical pseudo-index as a tool in order to prove an abstract multiplicity result.

Altough we are going to defined the homotopical pseudo-index related to the group G defined in I.1 of Part 4, it will be clear from the definition that to any homotopical index one can associate an homotopical pseudo-index and that a multiplicity theorem analogous to Theorem I.3.1 of Part 4 holds.

In this chapter we shall make use of all the notations and results of the first chapter of Part 4. According with the definitions there, let us consider

$$\text{(I.1.1)} \qquad \begin{aligned} \mathcal{H} \ = \ \{h : H \times [0,1] \to H \text{ continuous such that} \\ h(x,0) = x \ , h(g(x),\sigma) = g(h(x,\sigma)) \ , \ \forall\, x \in H \ , \\ \forall\, \sigma \in [0,1] \ , \ \forall\, g \in G\} \ . \end{aligned}$$

and, for any $A \in \mathcal{B} \cap 2^{\Lambda}$

$$\text{(I.1.2)} \qquad \mathcal{H}_0(A) = \{h \in \mathcal{H} \ / \ h(A,1) \subseteq E_N\} \ .$$

We assume that two closed G-invariant subsets of H, say Σ_1 and Σ_2 are given such that

$$\Sigma_1 \cap \Sigma_2 \ = \ \emptyset \ ;$$

and we consider

$$(I.1.3)\qquad \begin{aligned}\mathcal{H}^*_{\Sigma_1,\Sigma_2} = \{h \in \mathcal{H} \ / \ & h(\cdot,s) \text{ is a } G\text{-equivariant homeomorphism},\\ & h(x,s_1+s_2) = h(h(x,s_1),s_2)\\ & h(x,s) = x \ \forall x \in (\Sigma_1 \cap \partial\Lambda) \cup \Sigma_2, \forall s \in [0,1]\},\end{aligned}$$

The homotopical pseudo-index related to $\mathcal{H}^*_{\Sigma_1,\Sigma_2}$ of a set A measures the homotopical index of $A \cap h(\Sigma_1)$; more precisely

DEFINITION I.1.1. *Let $A \in \mathcal{B}$, $A \cap (\Sigma_1 \cap \partial\Lambda = \emptyset$. We say that the homotopical pseudo-index of A is k (and we denote $j^*(A) = k$) if*

$$(I.1.4)\qquad k = \min_{h \in \mathcal{H}^*_{\Sigma_1,\Sigma_2}} j\left(h(A,1) \cap \Sigma_1\right) .$$

REMARK I.1.1: This definition makes sense;indeed, from (I.1.3) one has that $h(A,1) \cap (\Sigma_1 \cap \partial\Lambda) = \emptyset$, for each $h \in \mathcal{H}^*_{\Sigma_1,\Sigma_2}$.◇

PROPOSITION I.1.1. *Let $A, B \in \mathcal{B}$ be such that $(A \cup B) \cap (\Sigma_1 \cap \partial\Lambda) = \emptyset$, then*

(*i*) $A \subseteq B \Longrightarrow j^*(A) \leq j^*(B)$

(*ii*) $j^*(A \cup B) \leq j^*(A) + i(B)$

(*iii*) *if* $h \in \mathcal{H}^*_{\Sigma_1,\Sigma_2}$ $\Longrightarrow j^*(A) = j^*(h(A,1))$

(*iv*) *A compact* $\Longrightarrow j^*(A) < +\infty$

(*v*) *A compact* $\Longrightarrow \exists\, \varepsilon > 0$ *such that* $j^*(N_\varepsilon(A)) = j^*(A)$.

PROOF: i) As a matter of fact, for any $h \in \mathcal{H}^*_{\Sigma_1,\Sigma_2}$ one has that $h(A,1) \subseteq h(B,1)$, so that by proposition the monotonicity of j one deduces that

$$j(h(A,1) \cap \Sigma_1) \leq j(h(B,1) \cap \Sigma_1) .$$

One concludes just by taking the minimum of both sides for every $h \in \mathcal{H}^*_{\Sigma_1,\Sigma_2}$.

ii) Let $h \in \mathcal{H}^*_{\Sigma_1,\Sigma_2}$ be fixed. From Propositions I.3.2 and I.3.3 of Part 4 one has that

$$\begin{aligned}& j\left(h(A \cup B,1) \cap \Sigma_1\right) \leq j\left((h(A,1) \cap \Sigma_1) \cup h(B,1)\right)\\ & j\left(h(A,1) \cap \Sigma_1\right) + i(h(B,1)) = j\left(h(A,1) \cap \Sigma_1\right) + i(B) .\end{aligned}$$

As before, one concludes by minimizing for every $h \in \mathcal{H}^*_{\Sigma_1,\ Sigma_2}$.

iii) easily follows from the fact that $\mathcal{H}^*_{\Sigma_1,\Sigma_2}$ is a group.

iv) is a direct consequence of the following inequality:

$$j^*(A) \leq j(A \cap \Sigma_1) \leq i(A \cap \Sigma_1) .$$

v) easily follows from the properties of j of I.3 of Part 4.◇

Let us recall a weaker condition than the Palais-Smales's, but still sufficient to have a deformation lemma: the Cerami condition (at the level c):

(C_c) every sequence $(x_n)_n$ in H such that
$\lim_{n\to+\infty} I(x_n) = c$
$\lim_{n\to+\infty} (1 + ||x_n||)\, ||dI(x_n)|| = 0$
possesses a subsequence converging to some limit in H.

Condition (C_c) will turn out to be easier to verify in the setting of Theorem III.1.

In the following, we shall denote

$$K_c = \{x \in H \ / \ I(x) = c \ , \ dI(x) = 0\} .$$

Now we state the suitable version of the Deformation Lemma:

PROPOSITION I.1.2. *Let $I \in C^1(H;\mathbb{R})$ be a G-invariant functional. Let $c \in \mathbb{R}$ such that*

$$\text{(I.1.5)} \qquad \inf_{x\in\Sigma_1\cap\partial\Lambda} I(x) > c^* > \sup_{x\in\Sigma_2} I(x) \ .$$

Assume that, for some $\gamma > 0$, I satisfies the condition $(C)_c$, for all $c \in [c^*-\gamma, c^*+\gamma]$. Then there exists ε_0 such that, for every $\varepsilon \leq \varepsilon_0$, there are δ and $\eta \in \mathcal{H}^*_{\Sigma_1,\Sigma_2}$ such that, if $N = N_\varepsilon(K_{c^*})$, then:

$$\text{(I.1.6)} \qquad \sup_A I \leq c^* + \delta \quad \Longrightarrow \quad \sup_{\eta(A\setminus N,1)} I \leq c^* - \delta \ .$$

Moreover, if $K_{c^*} = \emptyset$, then

$$\text{(I.1.7)} \qquad \sup_A I \leq c^* + \delta \quad \Longrightarrow \quad \sup_{\eta(A,1)} I \leq c^* - \delta \ .$$

PROOF: The proof is standard (see for instance [**12**]).◇

Now we are in a position to prove the main goal of this section: for every integer r we denote

$$\Gamma_r^* = \{A \in \mathcal{B} \cap 2^\Lambda \text{ compact } / \ j(A)^* \geq r\}$$

THEOREM I.1.1. *Let $I \in C^1(H;\mathbb{R})$. Assume that there exist $\Sigma_1,\Sigma_2 \subseteq H$ closed, $k \in \mathbb{N}$, and $c > 0$ such that*

$$\Gamma_k^* \neq \emptyset. \tag{I.1.8}$$

$$\sup_A I \geq c > 0\ , \qquad \forall A \subseteq \Sigma_1\ , j(A) \geq 1. \tag{I.1.9}$$

Then the numbers

$$c_r^* = \inf_{A\in\Gamma_r^*} \sup_A I$$

are well defined. Assume moreover that

$$\inf_{\Sigma_1\cap\partial\Lambda} I > c_r^* > \sup_{\Sigma_2} I, \qquad (r = 1,\ldots,k) \tag{I.1.10}$$

and that

$$(C)_{c_r^*} \qquad (r = 1,\ldots,k) \tag{I.1.11}$$

$$K_{c_r^*} \cap F_0 = \emptyset\ . \qquad (r = 1,\ldots,k)\ . \tag{I.1.12}$$

hold. Then I has at least k geometrically distinct critical points in $\{x \in H \ / \ c_1^* \leq I(x) \leq c_k^*\}$.

PROOF: First of all, we claim that the c_r^*'s are critical levels for every $r = 1,\ldots,k$. Indeed, the classes Γ_r^* are invariant under all the homotopies of $\mathcal{H}^*_{\Sigma_1,\Sigma_2}$. Hence, assuming that $K_{c_r^*} = \emptyset$ for some r, one deduces from the second part of Proposition I.1.2 (which can be applied because of (I.1.10) and (I.1.11)) that there is $A \in \Gamma_r^*$ such that $\sup_A I < c_r^*$, in contradiction with the definition of c_r^*.

Hence, if $c_1^* < \cdots < c_k^*$ the proof is complete. Now we assume that, for some $r,h \geq 1$, $c_r^* = \cdots = c_{r+h}^* = c^*$ holds, and we claim that $i(K_{c^*}) \geq h+1$ (observe that, by (I.1.12), the index i of K_{c^*} is well defined). This fact will end the proof, taking into account of Proposition 1.1.vi).

Let us fix $\varepsilon > 0$ such that Lemma 2.1 holds, and let $A \in \Gamma^*_{r+h}$ be such that $\sup_A I \leq c^* + \delta$. Let $N = N_\varepsilon(K_{c^*})$ and η be as in Proposition I.1.2: then one deduces from (I.1.6) that $j^*(cl(\eta(A \setminus N, \sigma))) \leq r - 1$. Remark that, from Proposition 2.3.iii)?, one has that $j^*(cl(\eta(A \setminus N, 1))) = j^*(cl(A \setminus N))$, since $\eta \in \mathcal{H}^*_{\Sigma_1,\Sigma_2}$. On the other hand, it follows from Proposition 2.3.ii) that $j^*(A) \leq j^*(cl(A \setminus N)) + i(A \cap N)$, so that $i(N) \geq i(A \cap N) \geq h + 1$. The proof is then complete by virtue of Proposition 2.1.v ?(indeed, for ε small, one has that $i(N) = i(K_{c^*})$).$\diamond$

I.2. Computation of the indices

In this section we provide examples of sets having nontrivial (i.e. different from 1) homotopical indices. To this end we shall make use of some results of about the homotopical index contained in Part 4.

We denote by C_N the set of all the grat circles of S^{N-1} :

(I.2.1)
$$C_N = \{z \in H \;/\; z(t) = x \sin 2\pi t + y \cos 2\pi t \quad |x| = |y| = 1 \;,\; x \cdot y = 0\}$$

Let us fix up some more notations: let $\rho > 0$ and $0 < \varepsilon < 1$ be fixed and consider

(I.2.2)
$$\Sigma_1 = \{x \in H \;/\; \frac{1}{2}\int_0^1 |\dot{x}|^2 = 2\pi^2\rho^2\} \,,$$

(I.2.3)
$$\Sigma_2 = \{x \in H \;/\; \frac{1}{2}\int_0^1 |\dot{x}|^2 = 2\pi^2\rho^2\varepsilon^2\} \cup \{x \in H \;/\; \frac{1}{2}\int_0^1 |\dot{x}|^2 = 2\pi^2\rho^2\varepsilon^{-2}\} \,,$$

and let

(I.2.4)
$$C^*_N = [\varepsilon, \varepsilon^{-1}]\rho C_N \,.$$

The following theorem holds:

THEOREM I.2.2. *For every $\rho > 0$, $0 < \varepsilon < 1$ and $N \in \mathbb{N}$,*

$$j^*(C_N^*) = N - 1 \,. \tag{I.2.5}$$

PROOF: It is not difficult to see that $j(C_N^*) \leq j(C_N)$, and hence, from Theorem I.4.1 of the part 4, that $j^*(C_N^*) \leq N-1$. Thus we have to prove the reversed inequality.From the definitions of the homotopival pseudo-index (Definition I.1.1) and from Definitions I.2.1, I.3.1 (recalling Lemma I.3.1) of Part 4, proving it is equivalent to prove that

$$\begin{aligned} &\forall h_1 \in \mathcal{H}_{\Sigma_1,\Sigma_2} \,,\ \forall h_2 \in \mathcal{H}_0(H) \,,\ \forall f \in \mathcal{F}_{N-2} \,, \\ &0 \in f(\{z \in h_1(C_N^*, 1) \cap \Sigma_1 \;/\; h_2(z, [0,1]) \cap \partial\Lambda \neq \emptyset\}) \,. \end{aligned} \tag{I.2.6}$$

Let $h_1 \in \mathcal{H}_{\Sigma_1,\Sigma_2}$, $h_2 \in \mathcal{H}_0(H)$ and $f \in \mathcal{F}_{N-2}$ be fixed. Let $e_1 \in S^{N-1}$ and consider $S^{N-2} = \{x \in S^{N-1} \;/\; x \cdot e_1 = 0\}$, and consider the continuous $F^* : S^{N-2} \times S^{N-2} \times S^1 \times [\varepsilon\rho, \varepsilon^{-1}\rho] \to \mathbb{R}^N$, defined by

$$F^*(x, y, e^{2\pi i t}, \lambda) = \lambda\left(x \cos 2\pi t + (x \cdot y e_1 + (y - (y \cdot x)x)) \sin 2\pi t\right) \,, \tag{I.2.7}$$

F^* induces a parametrization of C_N^*, $\overline{F}^* : S^{N-2} \times S^{N-2} \times [\varepsilon\rho, \varepsilon^{-1}\rho] \to C_N^*$. Define the continuous $\Phi^* : S^{N-2} \times S^{N-2} \times B^2 \times [\varepsilon\rho, \varepsilon^{-1}\rho] \to \mathbb{R}^N$, say $\varphi^* : S^{N-2} \times S^{N-2} \times B^2 \times [\varepsilon\rho, \varepsilon^{-1}\rho] \to \mathbb{R}^{N-2}$, $\xi^* : S^{N-2} \times S^{N-2} \times B^2 \times [\varepsilon\rho, \varepsilon^{-1}\rho] \to \mathbb{R}$ as

$$\Phi^*(x, y, \tau e^{2\pi i t}, \lambda) \;=\; \begin{cases} h_1(\overline{F}^*(x, y, \lambda), 2 - 2\tau)(t) & \text{if } \frac{1}{2} \leq \rho \leq 1 \\ h_2(h_1(\overline{F}^*(x, y, \lambda), 1), 1 - 2\tau) & \text{if } 0 \leq \rho \leq \frac{1}{2} \,. \end{cases} \tag{I.2.8}$$

The continuity of Φ^* follows from (1.10) and (1.12).

$$\varphi^*(x, y, \tau e^{2\pi i t}, \lambda) \;=\; f(h_1(\overline{F}^*(x, y, \lambda), 1)) \,, \tag{I.2.9}$$

and

$$\xi^*(x, y, \tau e^{i 2\pi t}, \lambda) \;=\; \begin{cases} \int_0^1 \left| \frac{d}{dt}(h_1(\overline{F}^*(x, y, \lambda), 2 - 2\tau) \right|^2 dt & \text{if } \frac{1}{2} \leq \rho \leq 1 \,, \\ \int_0^1 \left| \frac{d}{dt}(h_1(\overline{F}^*(x, y, \lambda), 1) \right|^2 dt & \text{if } 0 \leq \rho \leq \frac{1}{2} \,. \end{cases} \tag{I.2.10}$$

We claim that if

$$(0, 0, 2\pi\rho^2) \in \Phi^* \times \varphi^* \times \xi^*(S^{N-2} \times S^{N-2} \times B^2 \times [\varepsilon\rho, \varepsilon^{-1}\rho]) \,, \tag{I.2.11}$$

then (I.2.2)0 is satisfied. Indeed, let $\Phi^* \times \varphi^* \times \xi^*(x_0, y_0, \tau_0 e^{2\pi i t_0}, \lambda_0) = (0, 0, 2\pi^2)$. Then, from (1.12) and (I.2.8) (remember that $h_1 \in \mathcal{H}_{\Sigma_1, \Sigma_2}$), we obtain that $0 \leq \tau_0 \geq \frac{1}{2}$. Setting

$$z_0 = h_1(\overline{F}^*(x_0, y_0, \lambda_0), 1)$$

, from (I.2.8), (I.2.9) and (I.2.10) we deduce

$$\begin{aligned} f(z_0) &= 0 \\ z_0 &\in h_1(C_N^*, [0,1]) \cap \Sigma_1 \\ h_2(z_0, 1 - 2\tau_0)(t_0) &= 0\,; \end{aligned}$$

that is (I.2.6) holds.

In order to prove (I.2.11), we first remark that, by the G -equivariance of h_1 and h_2, and from (1.5), the following symmetry properties hold:

(I.2.12)
$$\begin{aligned} &\Phi^*(x, -y, \tau e^{i\omega t}, \lambda) = \Phi(x, y, \tau e^{-i\omega t}, \lambda) \\ &\Phi^*(-x, y - 2(y - (y \cdot x)x), \tau e^{i\omega t}, \lambda) = \Phi^*(x, y, \tau e^{i\omega(\pi + t)}, \lambda) \\ &\forall\, (x, y, \tau e^{i\omega t}, \lambda) \in S^{N-2} \times S^{N-2} \times B^2 \times [\varepsilon\rho, \varepsilon^{-1}\rho]\,, \end{aligned}$$

and

(I.2.13)
$$\begin{aligned} &\varphi^*(x, -y, \tau e^{i\omega t}, \lambda) = -\varphi^*(x, y, \tau e^{i\omega t}, \lambda) \\ &\varphi^*(-x, y - 2(y - (y \cdot x)x), \tau e^{i\omega t}, \lambda) = \varphi^*(x, y, \tau e^{i\omega(\pi + t), \lambda}) \\ &\forall\, (x, y, \tau e^{i\omega t}, \lambda) \in S^{N-2} \times S^{N-2} \times B^2 \times [\varepsilon\rho, \varepsilon^{-1}\rho]\,. \end{aligned}$$

By the same perturbations arguments used in proving Theorem I.4.1 of Part 4 for every $\delta > 0$, one can find z_δ and φ_δ^* with $z_\delta < \delta$, $|\varphi_\delta^* - \varphi^*|_\infty < \delta$ and φ_δ^* still satisfying (I.2.13) such that:

a) $(0, 0)$ is a regular value for $(\Phi^* - z_\delta) \times \varphi_\delta^*$;

b) $(0, 0)$ is a regular value for both

$$((\Phi^* - z_\delta) \times \varphi_\delta^*)_{|S^{N-2} \times S^{N-2} \times B^2 \times \{\varepsilon\rho\}}$$

and

$$((\Phi^* - z_\delta) \times \varphi_\delta^*)_{|S^{N-2} \times S^{N-2} \times B^2 \times \{\varepsilon^{-1}\rho\}}$$

.

Indeed, one is in a position to apply Proposition I.4.1 of the part 4.

Hence $((\Phi^* - z_\delta) \times \varphi_\delta^*)^{-1}(0,0)$ consists in a finite number of 1-manifolds. From the symmetry of the problem, these 1-manifolds appear in 4-ples of the type

$$\begin{aligned}
&\{(x(\sigma), y(\sigma), \tau(\sigma)e^{i\omega t(\sigma)}), (x(\sigma), -y(\sigma), \tau(\sigma)e^{-i\omega t(\sigma)}), \\
&(-x(\sigma), -y(\sigma) + 2(y(\sigma) - (y(\sigma)\cdot x(\sigma))x(\sigma)), \tau(\sigma)e^{i\omega(\pi - t(\sigma))}), \\
&(-x(\sigma), y(\sigma) - 2(y(\sigma) - (y(\sigma)\cdot x(\sigma))x(\sigma)), e^{i\omega(\pi + t(\sigma))})\}_{\sigma\in[0,1]} \,.
\end{aligned} \tag{I.2.14}$$

It is a well-known fact that a compact 1-manifold imbedded in a compact manifold is either homeomorphic to S^1 or it start and die on the boundary. From (I.2.8), each 1-manifold can intersect the boundary only in $S^{N-2} \times S^{N-2} \times B^2 \times \{\varepsilon\rho, \varepsilon^{-1}\rho\}$.
Moreover both $((\Phi^* - z_\delta) \times \varphi_\delta^*)^{-1}_{|S^{N-2}\times S^{N-2}\times B^2\times\{\varepsilon\rho\}}$ and $((\Phi^* - z_\delta) \times \varphi_\delta^*)^{-1}_{|S^{N-2}\times S^{N-2}\times B^2\times\{\varepsilon^{-1}\rho\}}$ consist in an odd number of 4-ples of the type of (I.2.13). One can assume without loss of generality that the 1-manifolds starting at the boundary intersect it transversally.
Since two symmetric 1-manifolds (i.e; belonging to the same 4-ple) can never intersect, one can conclude that at least one 4-ple of solutions has to connect $S^{N-2} \times S^{N-2} \times B^2 \times \{\varepsilon\rho\}$ with $S^{N-2} \times S^{N-2} \times B^2 \times \{\varepsilon^{-1}\rho\}$.

Let us denote by $(x(\sigma), y(\sigma), \rho(\sigma)e^{2\pi i t(\sigma)}, \lambda(\sigma))_{\sigma\in[0,1]}$ one of such 1-manifolds: then one has that $\lambda(0) = \varepsilon\rho$ and $\lambda(1) = \varepsilon^{-1}\rho$, so that from (I.2.10) one deduces that

$$\xi^*(x(0), y(0), \rho(0)e^{2\pi i t(0)}, \lambda(0)) = 2\pi^2\varepsilon^2\rho^2 < 2\pi^2\rho^2 < 2\pi^2\varepsilon^{-2}\rho^2\xi^*(x(1), y(1), \rho(1)e^{2\pi i t(1)}, \lambda(1))$$

. Hence a $\sigma_0 \in [0,1]$ exists such that

$$\xi^*(x(\sigma_0), y(\sigma_0), \rho(\sigma_0)e^{2\pi i t(\sigma_0)}, \lambda(\sigma_0)) = 2\pi^2\rho^2$$

. Therefore

$$\Phi^* \times \varphi^* \times \xi^*(x(\sigma_0), y\sigma(0), \rho(\sigma_0)e^{2\pi i t(\sigma_0)}, \lambda(\sigma_0)) = (0, 0, 2\pi^2\rho^2)$$

, that is, (I.2.11) is satisfied.◇

5.II. The strong force case .

We have seen in Part 3, chapter 1, that a natural condition for the solvability of (P_E), when $\alpha > 2$, is that the energy is positive. For this reason, the associated functional will be positive, and the homotopical index theory introduced in 4.I will be the suitable tool for the search of multiple critical points.

We are going to prove the following results:

THEOREM II.1. *Let* $F \in C^1(\mathrm{R}^N \setminus \{0\}; \mathrm{R})$ *satisfy:*

$$\exists\, b \geq a > 0\ ,\ \exists\, \alpha_2 \geq \alpha \geq \alpha_1 > 2\ ,\ \textit{such that}$$

$$(H1) \qquad \frac{a}{|x|^\alpha} \leq -F(x) \leq \frac{b}{|x|^\alpha}\ , \qquad \forall x \in \mathrm{R}^N \setminus \{0\}$$

$$(H2) \qquad -\alpha_1 F(x) \leq \nabla F(x)\dot{x} \leq -\alpha_2 F(x) \qquad \forall x \in \mathrm{R}^N \setminus \{0\}$$

$$(H3) \qquad |\nabla F(x)| \leq \frac{a\alpha_2}{|x|^{\alpha+1}} \qquad \forall x \in \mathrm{R}^N \setminus \{0\}\ ,$$

and assume moreover that a, b, α_2, α, α_1 *satisfy*

$$(\mathrm{II}.1) \qquad \frac{b^{2/\alpha}}{a^{2/\alpha}}\, \frac{\alpha}{\alpha_1} \left(\frac{\alpha_2 - 2}{\alpha - 2}\right)^2 < 4\,.$$

Then, for every positive energy E, (P_E) *has at least N-1 geometrically distinct solutions having minimal period in the interval*

(II.2)
$$[\left(\frac{(\alpha_1 - 2)\alpha}{(\alpha - 2)\alpha_1}\right)^2 \frac{(2\pi)^2}{\alpha} a^{2/\alpha} \left(\frac{\alpha - 2}{2E}\right)^{\frac{\alpha+2}{\alpha}}\ ,\ \left(\frac{(\alpha_2 - 2)\alpha}{(\alpha - 2)\alpha_2}\right)^2 \frac{(2\pi)^2}{\alpha} b^{2/\alpha} \left(\frac{\alpha - 2}{2E}\right)^{\frac{\alpha+2}{\alpha}}]\,.$$

REMARK: It it not difficult to see that, when (H1) and (H2) hold, the solutions of (P_E) are constrained in the ball of radius $\left(\frac{(\alpha_2-2)b}{2E}\right)^{\frac{1}{\alpha}}$, so that the following Corollary easily follows from Theorem II.1:

COROLLARY II.1. *Let* $U \in C^1(\mathrm{R}^N \setminus \{0\}; \mathrm{R})$ *satisfy*

$$\exists \alpha > 2\ \textit{such that}$$

$$\lim_{x \to 0} |x|^{\alpha+1} |\nabla U(x)| = 0\ ,$$

and let $F(x) = \frac{-a}{|x|^\alpha}$ *for some* $a > 0$. *Then there exists* $\overline{E} > 0$ *such that, for every* $E > \overline{E}$, *then* (P_E) *has at least N-1 geometrically distinct solutions having minimal period in the interval (II.2).*

From now on, the energy E is fixed positive. To carry out the proof of Theorem II.1, we are first going to add to the functional I a term inducing the strong force. Then we shall prove some preliminary propositions, that will allow the application of Theorem I.3.1 of Part 4. By virtue of an a priori estimate (Proposition II.1) we shall be in a position to conclude that the critical points found by Theorem I.3.1 of Part 4 are actually (up to the rescaling of the period) solutions of (P_E).

To this end, we start with the following:

DEFINITION II.1. *For any* $\varepsilon > 0$, $V_\varepsilon \in C^2(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ *denotes a function such that*

$$V_\varepsilon(x) \geq \frac{1}{|x|^2} \qquad \text{if } 0 < |x| \leq \varepsilon \tag{II.3}$$

$$V_\varepsilon(x) > 0 \qquad \text{if } 0 < |x| < 2\varepsilon \tag{II.4}$$

$$V_\varepsilon(x) = 0 \qquad \text{if } |x| \geq 2\varepsilon \tag{II.5}$$

$$\nabla V_\varepsilon(x) \cdot x \leq -2V_\varepsilon(x) \qquad \forall\, x \in \mathbb{R}^N - \{0\}\,. \tag{II.6}$$

Define

$$I_\varepsilon(x) = I(x) + \int_0^1 V_\varepsilon(x)$$

In the following we shall make use of some results which have been proved in chapter I of Part 3. For the reader's convenience we are going to recall them here.

PROPOSITION II.1. *(A priori estimate). Assume (H1), (H2) hold and let* $x \in \Lambda$ *be a critical point of* I_ε, *such that* $I_\varepsilon(x) > 0$. *If*

$$(2\varepsilon)^\alpha \leq \frac{(\alpha_1 - 2)a}{2E} \tag{II.7}$$

then

$$|x(t)| \geq 2\varepsilon \qquad \forall\, t \in [0,1]\,. \tag{II.8}$$

Proposition II.1 is nothing else than Proposition I.1 of Part 3; also Proposition II.2, II.3 and II.4 were proved in Part 3, chapter I. By virtue of Proposition II.1, looking for critical points of I is equivalent to looking for critical points of I_ε, provided that (II.7) holds. We are going to show that I_ε satisfies the assumptions of Theorem 2.1.

PROPOSITION II.2. *Assume that (H1) holds. Then*

(II.9)
$$\inf_{x \to \partial\Lambda} I_\varepsilon(x) = +\infty .$$

PROPOSITION II.3. *Assume (H1) and (H2) hold. Then, for any $c > 0$, I_ε satisfies the (PS_c) condition.*

PROPOSITION II.4. *Assume (H1), (H2) and (II.7) hold. Let $x \in \Lambda$ be a critical point of I_ε at positive level. Then $x \notin F_0$.*

PROOF: Let x be such a critical point; by Proposition II.1, we know that x is actually a critical point of I, so that it satisfies

$$-\ddot{x} = \lambda^2 \nabla F(x) ,$$
$$|\dot{x}|^2 + \lambda^2 F(x) = \lambda^2 E ,$$

where

$$\lambda^2 = \frac{\frac{1}{2}\int_0^1 |\dot{x}|^2}{\int_0^1 E - F(x)} > 0$$

since $I(x) > 0$. Now assume that $x \in F_0$, that is that there exists s such that $x(s-t) = x(t)$, $\forall t \in \mathbb{R}$. Therefore $\dot{x}(s) = 0$. One then deduce from the conservation of the energy that $F(x(s)) = E$, and hence, from (H1), that $E < 0$.◇

PROPOSITION II.5. *Assume (H1),(H2),(H3) and (II.1) hold. Let x be a critical point of I such that*

(II.10)
$$0 < I(x) \leq \pi^2 b^{2/\alpha} \alpha \left(\frac{2E}{\alpha - 2}\right)^{\frac{\alpha-2}{\alpha}} .$$

Then x has 1 as its minimal period.

PROOF: First, arguing as in the proof of Proposition II.1, we deduce from (H1) and (H2) that

$$(\text{II}.11) \qquad |x(t)|^\alpha \ \geq \ \frac{a(\alpha-2)}{2E} \qquad \forall\, t \in \mathbb{R}\,.$$

Since x is a critical point of I at positive level, we have

$$(\text{II}.12) \qquad -\ddot{x} \ = \ \frac{\frac{1}{2}\int_0^1 |\dot{x}|^2}{\int_0^1 E - F(x)}\, \nabla F(x)$$

and taking the L^2 product of (II.12) by x, from (H1) and (H2) we obtain

$$\int_0^1 |\dot{x}|^2 \ \leq \ \frac{\frac{1}{2}\int_0^1 |\dot{x}|^2}{\int_0^1 E - F(x)} \int_0^1 -\alpha_2 F(x)\,,$$

and hence

$$(\text{II}.13) \qquad \int_0^1 E - F(x) \ \geq \ (\frac{\alpha_2}{\beta - 2})\, E\,.$$

Setting

$$(\text{II}.14) \qquad \lambda^2 \ = \ \frac{\frac{1}{2}\int_0^1 |\dot{x}|^2}{\int_0^1 E - F(x)} \ = \ \frac{I(x)}{\left(\int_0^1 E - F(x)\right)^2}\,,$$

from (II.13) we deduce

$$(\text{II}.24) \qquad \lambda^2 \ \leq \ (\frac{\alpha_2 - 2}{\alpha_2 E})^2\, I(x)\,.$$

(II.12) and (H3) lead to

$$\int_0^1 |\ddot{x}|^2 \ \leq \ \lambda^4 a^2 \alpha_2^2 \int_0^1 \frac{1}{|x|^{2(\alpha+1)}}\,;$$

hence, from (II.12) and (H1) we obtain

$$(\text{II}.15) \qquad \int_0^1 |\ddot{x}|^2 \ \geq \ \lambda^4 a \alpha_2^2 \left[\frac{2E}{(\alpha-2)a}\right]^{\frac{\alpha+2}{\alpha}} \int_0^1 -F(x)\,.$$

Since $\dot{x}$ has zero mean value, Wirtinger inequality says that

(II.16)
$$\int_0^1 |\ddot{x}|^2 \geq (2k\pi)^2 \int_0^1 |\dot{x}|^2 ,$$

where $\frac{1}{k}$ is the minimal period of x ; therefore, from (II.12) and (H2) we find

(II.16)
$$\int_0^1 |\ddot{x}|^2 \geq (2k\pi)^2 \lambda^2 \int_0^1 -\alpha_1 F(x) .$$

(II.15) together with (II.16) lead to

$$k^2 \leq \frac{1}{\alpha_1} \frac{1}{(2\pi)^2} \lambda^2 a \alpha_2^2 \left[\frac{2E}{(\alpha_1 - 2)a} \right]^{\frac{\alpha+2}{\alpha}} ,$$

and therefore, from (II.14) and (II.10),

$$k^2 \leq \frac{b^{2/\alpha}}{a^{2/\alpha}} \frac{\alpha}{\alpha_1} \left(\frac{\alpha_2 - 2}{\alpha - 2} \right)^2 .$$

Hence (II.1) implies that $k^2 < 4$, and therefore (k is an integer), $k = 1$.◇

Now we turn to the proof of Theorem 1.

PROOF OF THEOREM II.1: To carry out the proof, we first replace the functional I with I_ε as in Definition II.3, with ε so small that (II.7) holds. Then, by virtue of Proposition II.1, the critical points of I_ε at positive levels are solutions of (P_E), up to the rescaling of the period. In order to find multiple critical points of I_ε, we shall apply the results of sections 2 and 3.

We claim that I_ε satisfy all the assumptions of Theorem I.3.1 of Part 4. Indeed, from Proposition II.2, I_ε admits a lower semicontinuous extension to the whole of H as $\overline{I}_\varepsilon(x) = +\infty$, when $x \in \partial\Lambda$. Then (I.3.5) (Part 4) holds. Moreover, Proposition II.2 also implies that $\inf_{\partial\Lambda} \overline{I}_\varepsilon = +\infty$, so that, by Theorem 3.1, (I.3.6)(Part 4) is satisfied. By Proposition II.4, (I.3.8)(Part 4) is fulfilled too. In order to check (I.3.7)(Part 4), from Proposition II.3 we have to prove that the critical levels are positive.

To do this, we assume on the contrary that, for some r, $c_r = 0$. Then there is a sequence $(A_n)_n$ in Γ_R such that $\sup_{A_n} I_\varepsilon \to 0$. We are then going to find a contradiction proving that, for large values of n, the A_ns are G-contractible sets. Indeed, from (II.??) we deduce that $\sup_{A_n} \int_0^1 |\dot{x}|^2 \to 0$ and $\inf_{A_n} \min_t |x(t)| \to +\infty$. Therefore, for large values on n, the G-equivariant continuous homotopy $h(x, \sigma) = (1-\sigma)x + \sigma \int_0^1 x$ can be performed, contracting the A_Ns into a set of large constant functions, without crossing the boundary of Λ.

Hence Theorem I.3.1 of Part 4 can be applied, providing the existence of at least $N-1$ geometrically distincts critical points.

We remark that, taking $R_b = \left[\frac{(\alpha-2)b}{2E}\right]^{1/\alpha}$, from (H1) we have

$$\sup_{R_b C_N} I_\varepsilon = \sup_{R_b C_N} I \le \sup_{R_b C_N} \left\{ \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)\left(\int_0^1 E + \frac{b}{|x|^\alpha}\right)\right\}$$
$$= \pi^2\, b^{2/\alpha}\, \alpha\, \left(\frac{2E}{\alpha-2}\right)^{\frac{\alpha-2}{\alpha}} .$$

Therefore, from Theorem I.3.1 of Part 4, $c_{N-1} \le \pi^2\, b^{2/\alpha}\, \alpha\, \left(\frac{2E}{\alpha-2}\right)^{\frac{\alpha-2}{\alpha}}$, so that from Proposition II.5 we deduce that the minimal period of these critical points is exactly one.

Now, we point out that the same method apply to the functional corresponding to the potential $F(x) = -a|x|^\alpha$. It is not difficult to see that, in that case $c_1 = \cdots = c_{N-1} = \pi^2\, a^{2/\alpha}\, \alpha\, \left(\frac{2E}{\alpha-2}\right)^{\frac{\alpha-2}{\alpha}}$ (see also III.3 of Part 3). Hence, from (H1) we deduce that

$$\pi^2\, a^{2/\alpha}\, \alpha\, \left(\frac{2E}{\alpha-2}\right)^{\frac{\alpha-2}{\alpha}} \le c_r \le \pi^2\, b^{2/\alpha}\, \alpha\, \left(\frac{2E}{\alpha-2}\right)^{\frac{\alpha-2}{\alpha}} , \quad r = 1 \ldots N-1 .$$

Finally, arguing as in the proof of Proposition II.5, from (H2) the periods of the solutions satisfy

$$\left(\frac{\alpha_1-2}{\alpha_1 E}\right)^2 I(x_r) \le \lambda_r^2 \le \left(\frac{\alpha_2-2}{\alpha_2 E}\right)^2 I(x_r) ;$$

hence we deduce from the above discussion that the minimal period of these solutions belong to the interval

$$\left[\left(\frac{(\alpha_1-2)\alpha}{(\alpha-2)\alpha_1}\right)^2 \frac{(2\pi)^2}{\alpha} a^{2/\alpha} \left(\frac{\alpha-2}{2E}\right)^{\frac{\alpha+2}{\alpha}} , \left(\frac{(\alpha_2-2)\alpha}{(\alpha-2)\alpha_2}\right)^2 \frac{(2\pi)^2}{\alpha} b^{2/\alpha} \left(\frac{\alpha-2}{2E}\right)^{\frac{\alpha+2}{\alpha}} \right] .$$

◇

5.III. The weak force case

When the weak force case is examined from a variational point of view, the main difficulty arises from the fact that, when the energy is negative (and, from the results of Part 3, chapter II, it is the natural choice when $\alpha < 2$), then the associated functional is unbounded (from below and from above); moreover both $\liminf_{x \to \partial\Lambda} I(x) = -\infty$ and $\limsup_{x \to \partial\Lambda} I(x) = +\infty$. However, as we have seen in Part 3, chapter II, when $1 < \alpha < 2$ the features of the associated functional are such that its restriction to each set of the type $\{x \ / \ \int_0^1 |\dot{x}|^2 - \rho^2\}$ is bounded from below and it presents a lack of compactness only at the level of the large constant functions. The homotopical pseudo-index theory seems then to be a profitable tool in order to obtain a multiplicity of solutions.

We shall prove the following results:

THEOREM III.1. *Let $F \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfies:*

$$\exists b \geq a > 0 \ , \ , \exists 2 > \alpha_2 \geq \alpha \geq \alpha_1 > 1 \ , \ \textit{such that}$$

$$(H1) \qquad \frac{a}{|x|^\alpha} \leq -F(x) \leq \frac{b}{|x|^\alpha} \ , \qquad \forall x \in \mathbb{R}^N \setminus \{0\}$$

$$(H2) \qquad -\alpha_1 F(x) \leq \nabla F(x) \dot{x} \leq -\alpha_2 F(x) \qquad \forall x \in \mathbb{R}^N \setminus \{0\}$$

$$(H3) \qquad |\nabla F(x)| \leq \frac{a\alpha_2}{|x|^{\alpha+1}} \qquad \forall x \in \mathbb{R}^N \setminus \{0\} \ .$$

Then there exist three functions $\Psi^*(\alpha)$ and $\sigma_1(\frac{b}{a}, \alpha_2, \alpha, \alpha_1)$, $\sigma_2(\frac{b}{a}, \alpha_2, \alpha, \alpha_1)$ such that, when

$$\text{(III.1)} \qquad \frac{b}{a} \left(\frac{(2-\alpha_1)\alpha}{(2-\alpha)\alpha_1} \right)^\alpha \frac{2-\alpha}{2-\alpha_2} < \Psi^*(\alpha) \ ,$$

$$\text{(III.2)} \qquad \sigma_2(\frac{b}{a}, \alpha, \alpha_1, \alpha_2) \left(\frac{2-\alpha_1}{2} \right)^{\frac{1}{\alpha}} < 1 \ ,$$

and

$$\text{(III.3)} \qquad \frac{b^{2/\alpha}}{a^{2/\alpha}} \frac{\alpha_2^2}{\alpha_1^2} \left(\frac{(2-\alpha_1)\alpha}{(2-\alpha)\alpha_1} \right)^2 \left(\frac{2-\alpha}{2-\alpha_2} \right)^{\frac{2+\alpha}{\alpha}} \frac{1}{\sigma_1^{2+\alpha}} < 4 \ ,$$

then, for every negative energy E, (P_E) has at least N-1 geometrically distinct solutions having minimal period in the interval

$$\text{(III.4)} \qquad [\pi^2 a^{2/\alpha}\alpha\left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}}\left(\frac{2-\alpha_2}{-\alpha_2 E}\right)^2 \; , \; \pi^2 b^{2/\alpha}\alpha\left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}}\left(\frac{2-\alpha_1}{-\alpha_1 E}\right)^2] \; .$$

Moreover Ψ^*, σ_1 and σ_2 enjoy the following properties:

$$\begin{array}{ll}
\Psi^*(\alpha) > 1 & \forall 1 < \alpha < 1 \\
\Psi^*(1) = 1 & \\
\lim_{\alpha \to 2} \Psi^*(\alpha) = +\infty & \\
\sigma_i(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) > 0 & \\
\lim_{\frac{(2-\alpha_1)b}{(2-\alpha_2)a} \to 1} \sigma_i(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) = 1 & \forall \textit{fixed } \alpha \\
\lim_{\alpha \to 2} \sigma_i(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) = 1 & \textit{if } \frac{(2-\alpha_1)b}{(2-\alpha_2)a} \textit{ remains bounded.}
\end{array}$$

The properties of Φ^*, σ_1 and σ_2 simply mean that, for each fixed $1 < \alpha < 2$, the field of conditions (III.1), (III.2) and (III.3) is non empty, and that, when $\alpha \to 2$ they converge to the limit condition:

$$\frac{b^{2/\alpha}}{a^{2/\alpha}}\frac{\alpha_2^2}{\alpha_1^2}\left(\frac{(2-\alpha_1)\alpha}{(2-\alpha)\alpha_1}\right)^2\left(\frac{2-\alpha}{2-\alpha_2}\right)^{\frac{2+\alpha}{\alpha}} < 4$$

REMARK: It is immediate to check that, when (H1) holds, the motion of each possible solution of (P_E) is constrained in the ball of radius $\left(\frac{b}{-E}\right)^{\frac{1}{\alpha}}$. Therefore all the hypotheses of Theorem III.1 can be assumed to hold true just for every x in this ball. Therefore, the following result directly follows from Theorem III.1:

COROLLARY III.1. *Let $U \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfy*

$$\text{(III.5)} \qquad \begin{array}{l} \exists \alpha \,, 1 < \alpha < 2 \textit{ such that} \\ \lim_{x \to 0} |x|^{\alpha+1}|\nabla U(x)| = 0 \,, \end{array}$$

and let $F(x) = \frac{-a}{|x|^\alpha}$ for some $a > 0$. Then there exists $\overline{E} < 0$ such that, for every $E < \overline{E}$, then (P_E) has at least N-1 geometrically distinct solutions having minimal period in the interval (III.4).

III.1. Proof of Theorem III.1

To carry out the proof of Theorem III.1 we shall apply the results of chapter I about the homotopical pseudo-index. First of all, we are going to replace the "singular" F with a regular F_ε, which is defined and smooth even at the origin. Next we shall provide estimates which will allow us to apply Theorem I.1.1. Finally from the a priori estimate (Proposition III.1.1) we shall conclude that the critical points found in this way do not interact with the truncation.

Although some of the estimates below were proved in 3.II, we shall recall them here for the reader's convenience.

We start by defining the suitable truncation of F:

DEFINITION III.1.1. *Assume that F satisfies (H1),(H2). For any* $\varepsilon > 0$, $F_\varepsilon \in C^1(\mathbb{R}^N;\mathbb{R})$ *denotes a function such that*

$$\begin{array}{ll} F_\varepsilon(x) = F(x) & \text{if } |x| \geq \varepsilon \\ 0 \leq \nabla F_\varepsilon(x)\cdot x \leq -\alpha_2 F_\varepsilon(x) & \forall\, x \in \mathbb{R}^N \\ -F_\varepsilon(x) \leq \dfrac{b}{|x|^\alpha} & \forall\, x \in \mathbb{R}^N \end{array}$$

$$\begin{cases} \text{there is a non decreasing } f:\mathbb{R}\to\mathbb{R}\text{, with} & \\ f(|x|) \geq \frac{a}{\varepsilon^\alpha} & \text{if } |x| \leq \varepsilon \\ \text{such that } -F_\varepsilon(x) \geq f(|x|) & \forall x \in \mathbb{R}^N . \end{cases}$$

We set

$$I_\varepsilon(x) = \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)\left(\int_0^1 E - F_\varepsilon(x)\right) \qquad \forall\, x \in H .$$

The following proposition provides the above mentioned a priori estimate: (notice that Proposition III.1.1 is nothing else than Proposition II.3 of Part 3)

PROPOSITION III.1.1. *(A priori estimate) Let F satisfy (H1), (H2), and let* F_ε, I_ε *as in Definition III.1.1. Let* $c_1 > 0$ *be fixed. There exist* $\bar{\varepsilon} > 0$ *and a function* $\Psi_1 : [0,1] \to [1,+\infty)$ *such that if*

$$\text{(III.1.6)} \qquad \frac{b}{a}\left(\frac{(2-\alpha_1)\alpha}{(2-\alpha)\alpha_1}\right)^{\frac{\alpha}{2}} \frac{2-\alpha}{2-\alpha_2} < \Psi_1(\alpha) ;$$

then, for every $0 < \varepsilon \leq \bar{\varepsilon}$, *each critical point of* I_ε *at level*

$$\text{(III.1.7)} \qquad c_1 \leq I_\varepsilon(x) \leq c_2 \leq \pi^2\, b^{2/\alpha}\, \alpha\left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} ,$$

satisfies

$$|x(t)| \geq \varepsilon \qquad \forall\, t \in \mathbf{R}\,.$$

Moreover, Ψ_1 *fulfils the following properties:*

$$\Psi_1(1) = 1\,;$$
$$\Psi_1 \text{ is increasing;}$$
$$\lim_{\alpha\to 2} \Psi_1(\alpha) = +\infty\,.$$

It is clear from Proposition III.1.1 that finding critical points of I_ε such that (III.1.7) holds is in fact equivalent to finding critical points of I.

According with the definitions of I.1 and I.2, let us define

(III.1.8)
$$\rho_b = \left[\frac{(2-\alpha)a}{-2E}\right]^{\frac{1}{\alpha}}\,;$$

(III.1.9)
$$\Sigma_1 = \{x \in H \,/\, \frac{1}{2}\int_0^1 |\dot{x}|^2 = 2\pi^2\rho_b^2\}\,;$$

(III.1.10)
$$\Sigma_2 = \varepsilon\rho_a C_N \cup \varepsilon^{-1}\rho_b C_N\,;$$

(III.1.11)
$$C_N^* = [\varepsilon\rho_b, \varepsilon^{-1}\rho_b]C_N\,.$$

Let us recall some other results of Part 3, chapter II:

PROPOSITION III.1.2. *Let* Ψ_1 *be defined in Proposition III.1.1 and assume that (H1) holds with* $\frac{b}{a} < \Psi_1(\alpha)$. *Then there exists* $\overline{\varepsilon} > 0$ *such that, for every* $0 < \varepsilon \leq \overline{\varepsilon}$, *then*

(III.1.12)
$$\inf_{\Sigma_1\cap\partial\Lambda} I_\varepsilon > 2\pi^2\rho_b^2\left(E + \left(\frac{-2E}{2-\alpha}\right)\right) = \pi^2 b^{2/\alpha}\alpha\left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} \leq \sup_{C_N^*} I_\varepsilon\,.$$

PROPOSITION III.1.3. *Let* $F(x) = \frac{-a}{|x|^\alpha}$, *for some* $a > 0$ *and* $1 < \alpha < 2$. *Then the smallest positive critical level of the associated functional* I *in* Λ *is* $\pi^2 a^{2/\alpha}\alpha\left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}}$.

PROPOSITION III.1.4. *Let* $F(x) = \frac{-a}{|x|^\alpha}$, *and let* I_ε *be as in Definition 4.1.For any* $\rho > 0$, *consider the class*

$$\mathcal{A}^*_\rho = \{A \in \mathcal{B} \cap 2^\Lambda \; / \; j(A \cap S_\rho) \geq 1\} \; ;$$

and let us define

(III.1.13)
$$K(\alpha,\rho) = \inf_{A \in \mathcal{A}^*_\rho} \sup_A \int_0^1 F_\varepsilon(x) \; .$$

Then,

(III.1.14)
$$K(\alpha,\rho) = K(\alpha,1)\rho^{-\alpha} \; , \qquad \forall \rho > 0 \, , \forall 1 \leq \alpha < 2 \; ;$$

(III.1.15)
$$K(\alpha,1) = a(\sqrt{2}\pi)^\alpha \; , \qquad \forall 1 \leq \alpha < 2 \; .$$

PROPOSITION III.1.5. *Assume (H1),(H2) hold and let I_ε be defined in Definition III.1.1. If ε is sufficiently small, then, for every $0 > c_1 \leq c_2$ and every $c^* \in [c_1, c_2]$, I_ε satisfies condition $(C)_{c^*}$.*

The proof of next proposition is very similar to the one of Proposition III.1 of Part 3. For sack of completeness we shall give it in the next section.

PROPOSITION III.1.6. *Let (H1) (H2) and (H3) holds. Then there are functions Ξ, σ_1 and σ_2, such that, when*

(III.1.16)
$$\frac{b}{a}\left(\frac{(2-\alpha_1)\alpha}{(2-\alpha)\alpha_1}\right)^\alpha \frac{2-\alpha}{2-\alpha_2} \; < \; \Xi(\alpha) < \Xi(\alpha) \; ,$$

(III.1.17)
$$\sigma_2(\frac{b}{a}, \alpha, \alpha_1, \alpha_2)\left(\frac{2-\alpha_1}{2}\right)^{\frac{1}{\alpha}} < 1 \; ,$$

and

(III.1.18)
$$\frac{b^{2/\alpha}}{a^{2/\alpha}} \frac{\alpha\alpha_2^2}{\alpha_1^3}\left(\frac{2-\alpha_1}{2-\alpha}\right)^2\left(\frac{2-\alpha}{2-\alpha_2}\right)^{\frac{2+\alpha}{\alpha}} \frac{1}{\sigma_1^{2+\alpha}} < 4 \; ,$$

then, for every energy $E < 0$, each critical point of I at level

$$\text{(III.1.19)} \qquad \pi^2 a^{2/\alpha}\alpha\left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} \leq I(x) \leq \pi^2 b^{2/\alpha}\alpha\left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} ;,$$

has 1 as minimal period and can not belong to F_0.
Moreover, Ξ and σ enjoy the following properties:

$$\Xi(\alpha) \geq 2\,, \qquad \forall 1 < \alpha < 2$$
$$\lim_{\alpha\to 2} \Xi(\alpha) = +\infty\,;$$
$$\sigma(\frac{b}{a}, \alpha, \alpha_1, \alpha_2) > 0$$
$$\lim_{\frac{\alpha_2 b}{\alpha_1 a}\to 1} \sigma(\frac{b}{a}, \alpha, \alpha_1, \alpha_2) = 1 \qquad \textit{for every fixed } \alpha$$
$$\lim_{\alpha\to 2} \sigma(\frac{b}{a}, \alpha, \alpha_1, \alpha_2) = 1 \qquad \textit{if } \frac{(2-\alpha_1)b}{(2-\alpha_2)a} \textit{ remains bounded.}$$

Now we are in a position to prove Theorem III.1:

PROOF OF THEOREM III.1: To carry out the proof, we are going to apply the results of I.1 and I.2 about the homotopical pseudo-index.

We are going to define $\Psi^*(\alpha) = \min\left(\Psi_1(\alpha), \Xi(\alpha), \frac{2}{2-\alpha}\right)$, in such a way that the assumptions of all the above Propositions are satisfied.

Let us first replace F with F_ε as in Definition III.1.1. ε can be taken so small that all the previous Propositions hold true. Then we claim that the associated functional I_ε fulfils all the assumptions of Theorem 2.2. Indeed, let Σ_1, Σ_2 be defined in (III.1.9), (III.1.10). Then, from Theorem 3.2, we know that C_N^* as defined in (III.1..11) belongs to Γ_{N-1}^*. Moreover, we deduce from (H1) and Proposition III.1.4 that, for every $A \subseteq \Sigma_1$, $j(A) \geq 1$, we have

$$\sup_A I_\varepsilon \geq -2\pi^2\rho_b^2 E\left(\frac{2a}{(2-\alpha)b} - 1\right) > 0\,.$$

So that (I.1.8) and (I.1.9) hold true. Moreover, by Proposition III.1.2, when ε is small, then

$$\text{(III.1.20)} \qquad \inf_{\Sigma_1} I_\varepsilon > \pi^2 b^{2/\alpha}\alpha\left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} \geq c_r^* \qquad r = 1, \ldots N-1\,,$$

while, Proposition III.1.4 leads to

$$\text{(III.1.21)} \qquad c_r^* \geq \inf_{\substack{A\subseteq\Sigma_1 \\ j(A)\geq 1}} \geq -2\pi^2\rho_b^2 E\left(\frac{2a}{(2-\alpha)b-1}\right) > \sup_{\Sigma_2} I_\varepsilon\,; \qquad r = 1, \ldots N-1\,,$$

(indeed, $\sup_{\Sigma_2} I_\varepsilon \leq C_1\varepsilon^{2-\alpha}$, for some constant C_1 independent of ε). Therefore (I.1..10) is fulfilled and, by Proposition III.1.5, also (I.1.11) holds true. Finally, from the above inequality, Proposition III.1.6 allows us to exclude the existence of elements of F_O at the critical levels ((I.1.12)). Thus Theorem I.1.1 provides the existence of at least $N-1$ distinct critical points x_r $(r = 1, \dots N-1)$ of I_ε. Let us observe that the same method provides critical point for the potential $\frac{-a}{|x|^\alpha}$, so that from (III.1.20) and Proposition III.1.3 we deduce that the following estimate holds:

$$\pi^2 b^{2/\alpha}\alpha\left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} \geq I_\varepsilon(x_r) \geq \pi^2 a^{2/\alpha}\alpha\left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}} .$$

By virtue of Proposition III.1.1, we can say that these critical points do not interact with the truncation, that is, they are critical point of I. As such, up to the rescaling of the period, they are solutions of (P_E) having minimal (because of Proposition III.1.6) period in the interval

$$\left[\left(\frac{(\alpha_1-2)\alpha}{(\alpha-2)\alpha_1}\right)^2 \frac{(2\pi)^2}{\alpha} a^{2/\alpha}\left(\frac{\alpha-2}{2E}\right)^{\frac{\alpha+2}{\alpha}}, \left(\frac{(\alpha_2-2)\alpha}{(\alpha-2)\alpha_2}\right)^2 \frac{(2\pi)^2}{\alpha} b^{2/\alpha}\left(\frac{\alpha-2}{2E}\right)^{\frac{\alpha+2}{\alpha}}\right].$$

◇

III.2. Proof of Proposition III.1.6

In this section we turn to the proof of Proposition III.1.6. To do this, some preliminaries are needed.

Let x be a critical point of I such that (III.1.19) holds, and let $y(t) = x(\lambda^{-1}t)$ be the corresponding solution of (P_E). We shall actually prove that $y \notin F_0$ and that the minimal period of y is λ.

From (H2) and the estimate on the level of x, we deduce the following estimates on the period λ of y:

$$\text{(III.2.1)}\quad \pi^2 a^{2/\alpha}\alpha\left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}}\left(\frac{2-\alpha_2}{-\alpha_2 E}\right)^2 \leq \lambda^2 \leq \pi^2 b^{2/\alpha}\alpha\left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{\alpha}}\left(\frac{2-\alpha_1}{-\alpha_1 E}\right)^2 .$$

As a solution of (P_E), y satisfies

$$\text{(III.2.2)}\qquad -\ddot{y} = \nabla F(y)$$

$$\text{(III.2.3)}\qquad \frac{1}{2}|\dot{y}|^2 + F(y) = E$$

Let us denote

$$\text{(III.2.4)} \qquad \rho(t) = |y(t)| \ , \qquad \forall t \in \mathbb{R}$$

and

$$\text{(III.2.5)} \qquad c(t) = \frac{1}{a(2-\alpha_2)} |y(t)|^{\alpha} \left(-2F(y(t)) - \nabla F(y(t)) \cdot y(t)\right) \ , \qquad \forall t \in \mathbb{R} \ .$$

Then from (H2) we deduce that

$$\text{(III.2.6)} \qquad 1 \leq c(t) \leq \frac{(2-\alpha_1)b}{(2-\alpha_2)a} \ , \qquad \forall t \in \mathbb{R} \ .$$

Moreover, from (III.2.2) and (III.2.3) we have

$$\text{(III.2.7)} \qquad \begin{cases} -\frac{1}{2}\ddot{\rho^2} = -2E - (2-\alpha_2)a\frac{c(t)}{\rho^{\alpha}} & \forall t \in \mathbb{R} \\ \rho(t+\lambda) = \rho(t) & \forall t \in \mathbb{R} \\ \rho(t) > 0 & \forall t \in \mathbb{R} \ . \end{cases}$$

Proposition III.2.1. *Assume that (H1) and (H2) hold, and assume moreover that* $\frac{b}{a} < \frac{2}{2-\alpha_1}$. *Assume that* $y \in F_0$. *Then there are* λ_1 *and* λ_2 *with* $\lambda_2 - \lambda_1 \leq \frac{\lambda}{2}$, *such that* $\rho(\lambda_1) = \rho(\lambda_2)$, $\dot{\rho}(\lambda_1) = \dot{\rho}(\lambda_2) = 0$, *and* $\min_{t\in[\lambda_1,\lambda_2]} |\dot{y}| = 0$.

Proof: By definition, $y \in F_0$ implies that there exists $s \in [0,1]$ such that

$$y(s-t) = y(t) \ , \qquad \forall t \in \mathbb{R} \ ,$$

or, equivalently

$$y(\frac{s}{2} - t) = y(\frac{s}{2} + t) \ , \qquad \forall t \in \mathbb{R} \ .$$

One easily deduces that

$$y(\frac{s+\lambda}{2} - t) = y(\frac{s+\lambda}{2} + t) \ , \qquad \forall t \in \mathbb{R} \ ,$$

and therefore that

$$\dot{y}(\frac{s}{2}) = \dot{y}(\frac{s+\lambda}{2}) = 0 \ .$$

Hence, it follows from (III.2.3) that

$$F(y(\frac{s}{2})) = F(y(\frac{s+\lambda}{2})) = E ,$$

so that (H1) implies

$$|y(\frac{s}{2})|^\alpha \geq \frac{a}{-E} , \quad \text{and } |y(\frac{s+1}{2})|^\alpha \geq \frac{a}{-E} .$$

From (III.2.7), we can conclude that both $|y(\frac{s}{2})|$ and $|y(\frac{s+\lambda}{2})|$ are strict local maxima for $|y(t)|$. One then deduces that there is at least one local minimum $|y(t^*)|$, with $t^* \in [\frac{s}{2}, \frac{s+\lambda}{2}]$. Assuming for example that $t^* - \frac{s}{2} \leq \frac{1}{4}$, one finds that $\rho(t^*) = \rho(s - t^*)$ and $\dot\rho(t^*) = \dot\rho(s - t^*) = 0$.◇

Proposition III.2.2. *Assume that the minimal period of y is $\frac{\lambda}{k}$, with $k \geq 2$. Then there are λ_1 and λ_2 with $\lambda_2 - \lambda_1 \leq \frac{\lambda}{2}$, such that $\rho(\lambda_1) = \rho(\lambda_2)$, $\dot\rho(\lambda_1) = \dot\rho(\lambda_2) = 0$.*

Proof: In an obvious consequence of the fact that $\rho(t + \frac{\lambda}{k}) = \rho(t)$, $\forall t \in \mathbb{R}$.◇

Proposition III.2.3. *Under the assumptions of Proposition III.2.1 (respectively Proposition III.2.2), there are three functions $\sigma_1, \sigma_2 : (1,2) \times [1, +\infty) \to (0, +\infty)$ and $\Xi : (1,2) \to (1, +\infty)$ such that, if*

$$\frac{b}{a} \left(\frac{(2-\alpha_1)\alpha}{(2-\alpha)\alpha_1} \right)^\alpha \frac{2-\alpha}{2-\alpha_2} \leq \Xi(\alpha) ,$$

holds, then

$$\rho(t) \geq \sigma_1(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) \left(\frac{(2-\alpha_1)a}{-2E} \right)^{\frac{1}{\alpha}} , \quad \forall t \in [\lambda_1, \lambda_2] ,$$

and

$$\rho(t) \leq \sigma_2(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) \left(\frac{(2-\alpha_2)b}{-2E} \right)^{\frac{1}{\alpha}} , \quad \forall t \in [\lambda_1, \lambda_2] .$$

Moreover, the following properties hold:

$$\Xi(\alpha) \geq 1 \qquad \forall \alpha \in (1,2) ;$$

Ξ *is increasing*

$$\lim_{\alpha \to 2} \Xi(\alpha) = +\infty ;$$

$$\lim_{\frac{b_1}{a_1} \to 1} \sigma_i(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) = 1 , \qquad \forall \alpha \in (1,2)$$

$$\lim_{\alpha \to 2} \sigma_i(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) = 1 , \qquad \text{if } \frac{(2-\alpha_1)b}{(2-\alpha_2)a} \text{ remains bounded} .$$

PROOF. STEP 1: By the change of variables $s(t) = \int_0^t \frac{1}{\rho^\alpha}$, and $\mu(s) = \rho^{2-\alpha}(s(t))$, from Proposition III.2.1 (resp . Proposition III.2.2), (III.2.7) becomes equivalent to

$$\text{(III.2.8)} \qquad \begin{cases} -\mu'' = -2(2-\alpha)E\mu^\beta - (2-\alpha)^2 c(s) \\ \mu(0) = \mu(\omega)\ , \\ \mu'(0) = \mu'(\omega) = 0 \\ \mu(s) > 0\ ,\ \forall s\ . \end{cases}$$

Where $(\cdot)'$ denotes the derivation with respect to the variable s

$$\text{(III.2.9)} \qquad \beta = \frac{\alpha}{2-\alpha}\ .$$

and

$$\text{(III.2.10)} \qquad \omega = \int_{\lambda_1}^{\lambda_2} \frac{1}{\rho^\alpha}.$$

Note that, by Proposition III.2.1 (resp. Proposition III.2.2), by integrating the equation in (III.2.7) one obtains

$$\text{(III.2.11)} \qquad \omega = \int_{\lambda_1}^{\lambda_2} \frac{1}{\rho^\alpha} \le \frac{-2E}{(2-\alpha_2)a}(\lambda_2 - \lambda_1) \le \frac{-2E}{(2-\alpha_2)a}\frac{\lambda}{2}\ .$$

We set $c(s) = (1 + c_0(s))$, so that, from (III.2.6) we have

$$\text{(III.2.12)} \qquad 0 \le c_0(s) \le \frac{(2-\alpha_1)b}{(2-\alpha_2)a}\ , \qquad \forall s \in [0,\omega]\ .$$

By derivating the equation in (III.2.8) and by taking the L^2 inner product with μ' we obtain

$$\text{(III.2.13)} \qquad \int_0^\omega (\mu'')^2 ds \ = \ -2\alpha E \int_0^\omega \mu^{\beta-1}(\mu')^2 ds + (2-\alpha)(2-\alpha_2)a \int_0^\omega c_0 \mu'' ds\ ,$$

and, from Holder inequality, taking into account of (III.2.12),

$$\text{(III.2.14)} \qquad \begin{aligned} &\int_0^\omega (\mu'')^2 ds - (2-\alpha)(2-\alpha_2)a \left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a} - 1 \right) \sqrt{\omega} \left(\int_0^\omega (\mu'')^2 ds \right)^{\frac{1}{2}} \\ &\le -2\alpha E \int_0^\omega \mu^{\beta-1}(\mu')^2 ds\ . \end{aligned}$$

Now, for any $0 < \delta < 1$, the inequality

$$\delta x^2 - \frac{d^2}{4(1-\delta)} \le x^2 - dx \,, \qquad \forall x, d \in \mathbb{R} \,,$$

together with (III.2.14) imply

$$\text{(III.2.15)} \qquad \begin{aligned} &\delta \int_0^\omega (\mu")^2 ds \ - \ \frac{(2-\alpha)^2(2-\alpha_2)^2 a^2 \left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a} - 1\right)^2 \omega}{4(1-\delta)} \\ &\le -2\alpha E \int_0^\omega \mu^{\beta-1}(\mu')^2 ds \,. \end{aligned}$$

◇

STEP 2: Let us denote by H^2_ω the Sobolev space $\{u \in H^2(\mathbb{R};\mathbb{R}) \ / \ u(s+\omega) = u(s) \forall s \in \mathbb{R}\}$. Let us consider, for every $\beta > 1$

$$\text{(III.2.16)} \qquad \Xi^*(\beta,\omega) = \inf_{H^2_\omega} \frac{\left(\int_0^\omega |u|^\beta\right)^{\frac{\beta-1}{\beta}} \int_0^\omega |\ddot{u}|^2}{\int_0^\omega |\dot{u}|^2 |u|^{\beta-1}}$$

and

$$\text{(III.2.17)} \qquad \Omega^*(\beta,\omega) \ = \ \inf_{u \in H^2_\omega} \frac{\int_0^\omega (u")^2}{\left(\int_0^\omega (u')^{2\beta}\right)^{\frac{1}{\beta}}} \,.$$

The following properties hold

$$\text{(III.2.18)} \qquad \Xi^*(\beta,\omega) = \frac{1}{\omega^{\frac{\beta+1}{\beta}}} \Xi^*(\beta,1) \,, \qquad \forall \beta \le 1 \,, \forall \omega > 0 \,;$$

$$\text{(III.2.19)} \qquad \Omega^*(\beta,\omega) = \frac{1}{\omega^{\frac{\beta+1}{\beta}}} \Omega^*(\beta,1)) \,, \qquad \forall \beta \ge 1 \,, \forall \omega > 0 \,;$$

$$\text{(III.2.20)} \qquad \Xi^*(\beta,1) \ge \Omega^*(\beta,1) \,, \qquad \forall \beta \ge 1 \,;$$

$$\text{(III.2.21)} \qquad \Omega^*(\beta,1) = \frac{1}{\omega^{\frac{1+\beta}{\beta}}} \frac{(1+\beta)^{\frac{1+\beta}{\beta}}}{\beta} \left\{ \int_0^{2\pi} |\sin\theta|^{\frac{1+\beta}{\beta}} d\theta \right\}^2 \,.$$

Taking into account of (III.2.9), we set

$$(\Xi(\alpha))^{\frac{2}{\alpha}} = \frac{4}{(2\pi)^2(2-\alpha)}\Xi^*(\frac{\alpha}{2-\alpha},1)\,, \qquad \text{(III.2.22)}$$

$$\Omega(\alpha) = \frac{4}{(2\pi)^2(2-\alpha)}\Omega^*(\frac{\alpha}{2-\alpha},1)\,, \,.$$

The proof of (III.2.18) and (III.2.19) obviously follows from the definitions. Formula (III.2.20) follow from the definitions and Holder's inequality. Finally (III.2.21) was proved in 3.II.??

REMARK III.2.1: From (III.2.22), one deduces that

$$\Omega(\alpha) \geq \frac{4}{\pi^2}\left(\frac{\pi}{2-\alpha}\right)^{\frac{2}{\alpha}} ;$$

and then that

$$\Xi(\alpha) \geq \left(\frac{2}{\pi}\right)^{\alpha} \frac{\pi}{2-\alpha} .$$

since the right hand side of the above inequality is increasing, we deduce that Ξ fulfils all the properties of the claim of Corollary III.2.???1

Now, from our change of variables we have

$$\int_0^\omega \mu^\beta = \lambda_2 - \lambda_1 \,. \qquad \text{(III.2.23)}$$

Hence the inequality (III.2.15) can be rewritten as

$$\delta\int_0^\omega (\mu")^2 ds \; - \; \frac{(2-\alpha)^2(2-\alpha_2)^2 a\left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a} - 1\right)^2 \omega}{4(1-\delta)} \qquad \text{(III.2.24)}$$

$$\leq \; -2\alpha E\left(\frac{\lambda}{2}\right)^{\frac{\beta-1}{\beta}} \frac{\int_0^1 (\mu')^2\mu^{\beta-1}}{\left(\int_0^\omega \mu^\beta\right)^{\frac{\beta-1}{\beta}}} \,.$$

Assuming that μ is not constant, from (III.2.24), (III.2.15) (III.2.18), we deduce

$$\delta \; - \; \frac{(2-\alpha)^2(2-\alpha_2)^2 a\left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a} - 1\right)^2 \omega}{4(1-\delta)\int_0^\omega (\mu")^2} \qquad \text{(III.2.25)}$$

$$\leq \; -2\alpha E\left(\frac{\lambda}{2}\right)^{\frac{\beta-1}{\beta}} \frac{1}{\Xi^*(\beta,1)}\omega^{\frac{1+\beta}{\beta}} \,,$$

and therefore from (III.2.22), (III.2.9), (III.2.23) and (III.2.1), we obtain

(III.2.26)
$$\delta - \frac{(2-\alpha)^2(2-\alpha_2)^2 a\left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a}-1\right)^2 \omega}{4(1-\delta)\int_0^\omega (\mu")^2}$$
$$\leq \left(\frac{b}{a}\right)^{\frac{2}{\alpha}} \left(\frac{(2-\alpha_1)\alpha)}{(2-\alpha)\alpha_1}\right)^2 \left(\frac{2-\alpha}{2-\alpha_2}\right)^{\frac{2}{\alpha}} \frac{1}{(\Xi(\alpha))^{\frac{2}{\alpha}}}.$$

Now, according with the claim of Proposition III.2.3, we assume that

(III.2.27)
$$\frac{b}{a}\left(\frac{(2-\alpha_1)\alpha}{(2-\alpha)\alpha_1}\right)^\alpha \frac{2-\alpha}{2-\alpha_2} < \Xi(\alpha).$$

Let us denote

(III.2.28)
$$\xi_1(\frac{b}{a},\alpha_2,\alpha,\alpha_1) = \left(\frac{b}{a}\right)^{\frac{2}{\alpha}} \left(\frac{(2-\alpha_1)\alpha)}{(2-\alpha)\alpha_1}\right)^2 \left(\frac{(2-\alpha)}{(2-\alpha_2}\right)^{\frac{2}{\alpha}},$$

We set

$$\delta = \frac{1}{2}\left(1+\frac{\xi_1(\frac{b}{a},\alpha_2,\alpha,\alpha_1)}{(\Xi(\alpha))^{\frac{2}{\alpha}}}\right);$$

it follows from (III.2.4) that $0<\delta<1$. Now (III.2.26) becomes

(III.2.29)
$$\int_0^\omega (\mu")^2$$
$$\leq (2-\alpha)^2(2-\alpha_2)^2 a\omega \left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a}-1\right)^2 \left(1-\frac{\xi_1((\frac{b}{a},\alpha_2,\alpha,\alpha_1)}{(\Xi(\alpha))^{\frac{2}{\alpha}}}\right)^{-2},$$

and hence, from (III.2.19),

(III.2.30)
$$\left(\int_0^\omega (\mu')^{2\beta}\right)^{\frac{1}{\beta}} \leq$$
$$\leq \frac{1}{\pi^2(2-\alpha)\Omega(\alpha)}(2-\alpha)^2(2-\alpha_2)^2 a^2 \omega^{\frac{2\beta+1}{\beta}} \left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a}-1\right)^2 \left(1-\frac{\xi_1((\frac{b}{a},\alpha_2,\alpha,\alpha_1)}{(\Xi(\alpha))^{\frac{2}{\alpha}}}\right)^{-2}.$$

Now, an easy computation shows that

$$\left(\int_{\lambda_1}^{\lambda_2}\left(\frac{d}{dt}\rho^{\frac{2+\alpha}{2}}\right)^{2\beta}dt\right)^{\frac{1}{\beta}} = \left(\frac{(2+\alpha)}{2(2-\alpha)}\right)^2\left(\int_0^{\omega}(\mu')^{2\beta}ds\right)^{\frac{1}{\beta}} ,$$

so that,from (III.2.30), (III.2.9) we obtain that

$$\text{(III.2.31)}\qquad \left(\int_{\lambda_1}^{\lambda_2}\left(\frac{d}{dt}\rho^{\frac{2+\alpha}{2}}\right)^{2\beta}dt\right)^{\frac{1}{\beta}} \leq \xi_2(\frac{b}{a},\alpha_2,\alpha,\alpha_1)a^2\left(\frac{-2E}{(2-\alpha_2)a}\right)^{\frac{2+\alpha}{\alpha}}\left(\frac{\lambda}{2}\right)^{\frac{2+\alpha}{\alpha}} ,$$

where

$$\begin{aligned}
&\xi_2(\frac{b}{a},\alpha_2,\alpha,\alpha_1) = \\
&= \left(\frac{(2+\alpha)}{2(2-\alpha)}\right)^2\frac{1}{\pi^2\Omega(\alpha)}(2-\alpha)(2-\alpha_2)^2\cdot \\
&\cdot\left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a}-1\right)^2\left(1-\frac{\xi_1(\frac{b}{a},\alpha_2,\alpha,\alpha_1)}{(\Xi(\alpha))^{\frac{2}{\alpha}}}\right)^{-2} ,
\end{aligned}$$

has the following properties (see Remark III.2.1):

$$\text{(III.2.32)}\qquad \begin{aligned}
&\lim_{\frac{(2-\alpha_1)b}{(2-\alpha_2)a}\to 1}\xi_2(\frac{b}{a},\alpha_2,\alpha,\alpha_1) = 0 && \forall\, 1<\alpha<2 \\
&\lim_{\alpha\to 2}\xi_2(\frac{b}{a},\alpha_2,\alpha,\alpha_1) = 0 && \text{if } \frac{(2-\alpha_1)b}{(2-\alpha_2)a} \text{ remains bounded}
\end{aligned}$$

Let us write $\rho_* = \left(\frac{(2-\alpha_2)a}{-2E}\right)^{\frac{1}{\alpha}}$. Remark that, since, from (III.2.7), every local maximum $\rho(t_0)$ has $\rho(t_0)\geq\rho_*$, then either ρ_* is assumed by the function $\rho(t)$ of $\rho(t)\geq\rho_*$, $\forall t$.From (III.2.31) we deduce

$$\begin{aligned}
|\rho^{\frac{2+\alpha}{2}}(t)-\rho_*^{\frac{2+\alpha}{2}}| &\leq \int_{\lambda_1}^{\lambda_2}|\frac{d}{dt}\rho^{\frac{2+\alpha}{\alpha}}|dt \\
&\leq \left(\int_{\lambda_1}^{\lambda_2}\left(\frac{d}{dt}\rho^{\frac{2+\alpha}{2}}\right)^{2\beta}dt\right)^{\frac{1}{2\beta}}\lambda^{1-\frac{1}{2\beta}} \\
&\leq \sqrt{\xi_2}\left(\frac{-2E}{(2-\alpha_2)a}\right)^{\frac{2+\alpha}{2\alpha}}a\left(\frac{\lambda}{2}\right)^2
\end{aligned}$$

and hence, from (III.2.1),

$$\text{(III.2.33)}\qquad |\rho^{\frac{2+\alpha}{2}}(t)-\rho_*^{\frac{2+\alpha}{2}}| \leq \xi_3\left(\frac{(2-\alpha_2)a}{-2E}\right)^{\frac{2+\alpha}{2\alpha}} = \xi_3\rho_*^{\frac{2+\alpha}{2}} ,$$

where

$$\text{(III.2.34)} \qquad \xi_3(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) = \sqrt{\xi_2}\frac{\pi^2}{\alpha}\left(\frac{b}{a}\right)^{\frac{2}{\alpha}}\left(\frac{(2-\alpha_1)\alpha}{(2-\alpha)\alpha_1}\right)^2\left(\frac{2-\alpha}{2-\alpha_2}\right)^{\frac{2+\alpha}{\alpha}},$$

still satisfies

$$\text{(III.2.35)} \qquad \begin{aligned} &\lim_{\frac{(2-\alpha_1)b}{(2-\alpha_2)a}\to 1} \xi_3(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) = 0 \qquad \forall\, 1 < \alpha < 2 \\ &\lim_{\alpha\to 2} \xi_3(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) = 0 \qquad \text{if } \frac{(2-\alpha_1)b}{(2-\alpha_2)a} \text{ remains bounded} \end{aligned}$$

Finally obtain

$$\text{(III.2.36)} \qquad \rho(t) \geq \sigma_1\rho_* = \sigma_1\left(\frac{(2-\alpha_2)a}{-2E}\right)^{\frac{1}{\alpha}} \qquad \forall t \in [\lambda_1, \lambda_2],$$

and

$$\text{(III.2.37)} \qquad \rho(t) \leq \sigma_2\rho_* = \sigma_1\left(\frac{(2-\alpha_2)a}{-2E}\right)^{\frac{1}{\alpha}} \qquad \forall t \in [\lambda_1, \lambda_2],$$

where

$$\left(\sigma_1(\frac{b}{a}, \alpha_2, \alpha, \alpha_1)\right)^{\frac{2+\alpha}{2}} = 1 - \xi_3(\frac{b}{a}, \alpha_2, \alpha, \alpha_1),$$

and

$$\left(\sigma_2(\frac{b}{a}, \alpha_2, \alpha, \alpha_1)\right)^{\frac{2+\alpha}{2}} = 1 + \xi_3(\frac{b}{a}, \alpha_2, \alpha, \alpha_1),$$

From (III.2.35), both σ_1 and σ_2 satisfy

$$\text{(III.2.38)} \qquad \begin{aligned} &\lim_{\frac{(2-\alpha_1)b}{(2-\alpha_2)a}\to 1} \sigma_i(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) = 1 \qquad \forall\, 1 < \alpha < 2 \\ &\lim_{\alpha\to 2} \sigma_i(\frac{b}{a}, \alpha_2, \alpha, \alpha_1) = 1 \qquad \text{if } \frac{(2-\alpha_1)b}{(2-\alpha_2)a} \text{ remains bounded} \end{aligned}$$

Hence Proposition III.2.3 is proved.

PROOF OF PROPOSITION III.1.6: We assume that (III.1.16) (III.1.17) and (III.1.18) hold with Ξ, σ_1 and σ_2 respectively defined in (III.2.22), (III.2.36), (III.2.37). Assume first that $y \in F_0$. Then Proposition III.2.1 says that there is $\bar{t} \in [\lambda_1, \lambda_2]$ such that $\dot{y}(\bar{t}) = 0$. We then deduce from (III.2.3), (H1) and Proposition III.2.3 that $\sigma_2(\frac{b}{a}, \alpha_2, \alpha, \alpha_1)^\alpha < \frac{2}{2-\alpha_1}$, that is a contradiction.

Now we assume that the minimal period of y is $\frac{T}{k}$, for some integer $k \geq 2$. Then Wirtinger inequality leads to

$$\left(\frac{2\pi k}{\lambda}\right)^2 \int_0^T |\dot{y}|^2 \leq \int_0^\lambda |\ddot{y}|^2 . \tag{III.2.39}$$

On the other hand, from (H2) we have

$$\int_0^\lambda |\dot{y}|^2 \geq -\alpha_1 \int_0^\lambda F(y) \tag{III.2.40}$$

and, from (H1), (H3) and Proposition III.2.3 we obtain

$$\begin{aligned} \int_0^\lambda |\ddot{y}|^2 &= \int_0^\lambda |\nabla F(y)|^2 \leq a^2 \alpha_2^2 \int_0^\lambda \frac{1}{|y|^{2\alpha+2}} \\ &\leq a\alpha_2^2 \left(\frac{-2E}{(2-\alpha_2)a}\right)^{\frac{2+\alpha}{\alpha}} \frac{1}{\sigma_1^{2+\alpha}} \int_0^\lambda -F(y) . \end{aligned} \tag{III.2.41}$$

From (III.2.40) and (III.2.41) we deduce

$$\left(\frac{2\pi k}{\lambda}\right)^2 \leq a\alpha_2^2 \left(\frac{-2E}{(2-\alpha_2)a}\right)^{\frac{2+\alpha}{\alpha}} \frac{1}{\sigma_1^{2+\alpha}} .$$

Therefore (III.2.1) implies

$$k^2 \leq \frac{b^{2/\alpha}}{a^{2/\alpha}} \frac{\alpha\alpha_2^2}{\alpha_1^3} \left(\frac{2-\alpha_1}{2-\alpha}\right)^2 \left(\frac{2-\alpha}{2-\alpha_2}\right)^{\frac{2+\alpha}{\alpha}} \frac{1}{\sigma_1^{2+\alpha}} < 4 ,$$

that is (k is an integer) $k = 1$.◇

PART6 SYMMETRIC CASES
A THREE BODY PROBLEM

As we have already pointed out in Part 4, chapter IV, one can overcome the strong degeneracy occurring at $\alpha = 1$, provided that the potential F possesses some symmetry properties. As a matter of facts, the symmetry of the potential allow the introduction of some symmetry constraints in the function space. The first effect of this constraint is that the functional becomes coercive on the space of the symmetric functions; for example, we have seen in IV of Part 4 that, when F is even, the natural domain of the functional is the space of all the functions in H such that $x(t+\frac{T}{2}) = -x(t)$, $\forall t \in \mathbb{R}$. The main problem in solving (P_T) then becomes to avoid the collision solutions. In all this part, our abstract setting will be the following. Λ is an open and dense subset of a function space H, where I is defined, coercive and (weakly) lower semicontinuous; moreover I is regular (of class C^2) in Λ. Of course Λ is the set of all the noncollision functions, and the problem of avoiding collision can be faced by wondering whether

(1) $$\inf_{\Lambda} I < \inf_{\partial\Lambda} \,.$$

When $F(x) = \frac{-a}{|x|^\alpha}$, we have shown in IV of Part 4 that the symmetry constraint $x(t+\frac{T}{2}) = -x(t)$ implies that (1) holds, for every $a, \alpha > 0$ This fact has been showed in [**26**], where a pinching condition is introduced in order to treat potentials similar to the ones of the above form (see also Part 4, chapter IV).

We are going to show that the minimization approach provides noncollision solution under *local* assumptions on the potential near the singularity.

In this part we shall be concerned with the following problems: in chapter I we shall treat the problem

$$\begin{cases} -\ddot{x} = \nabla F(x,t) & \\ x(t+T) = x(t)\ qquad & \forall t \\ x(t+\frac{T}{2}) = -x(t) & \forall t \\ x(t) \neq 0 & \forall t\,, \end{cases}$$

when F is T-periodic in t and satisfies some symmetry properties, as well as the the problem

$$\begin{cases} -\ddot{x} = \nabla F(x,t) \\ x(0) = x_0 \ , \ x(T) = x_1 \\ x(t) \neq 0 & \forall t \ , \end{cases}$$

We shall show that, under suitable local assumption at zero, the associated functionals fulfils (1).

The method introduced there will turn out to be profitable also in treating a restricted two body problem; more precisely, we assume that the trajectory of two bodies x_1 and x_2 is fixed T-periodic and such that $x_i(t + \frac{T}{2}) = -x_i(t)$, $\forall t \in \mathbb{R}$. We assume moreover that $x_i \in \mathcal{C}^2(\mathbb{R} \setminus (x_1 - x_2)^{-1}(0); \mathbb{R}^3)$, and we consider the problem

$$\begin{cases} -\ddot{x} = \nabla F_1(x - x_1) + \nabla F_2(x - x_2) \\ x(t+T) = x(t) & \forall t \\ x(t + \frac{T}{2}) = -x(t) & \forall t \\ x(t) - x_1(t) \neq 0 & \\ & \forall t \notin (x_1 - x_2)^{-1}(0) \ , \\ x(t) - x_2(t) \neq 0 & \end{cases}$$

Finally we shall consider the three body problem

$$\begin{cases} -\ddot{x}_i = \sum_{\substack{j \neq i \\ j=1}}^{3} \nabla F_{i,j}(x_i - x_j) & i = 1,2,3 \\ x_i(t+T) = x_i(t) & \forall t \ , i = 1,2,3 \\ x(t + \frac{T}{2}) = -x(t) & \forall t \ , i = 1,2,3 \\ x_i(t) - x_j(t) \neq 0 & \forall t \ , i = 1,2,3 \ . \end{cases}$$

Also this problem will be treated under local assumptions on the potentials $F_{i,j}$ near the origin.

I The symmetric case

In this chapter we are concerned with some classes of conservative differential systems with boundary conditions and singular potentials of the type $F(x) = \frac{-a}{|x|^\alpha}$, $a > 0$ and $0 < \alpha < 2$. The existence of solutions to this kind of problems can be derived from the minimization of a suitable functional, the main difficulty being to avoid the "collisions", that is, the functions which pass through the singularity of the potential.

As we have already pointed out, the main purpose of this chapter is to show how one can make use of *local* hypotheses on the potential (in some neighborhood of its singularity) in order to get existence of at least one noncollision solution. We wish to point out that our argument will provide orbits which remain only for a very short time in the neighborhood of zero where we make our assumptions; therefore our method cannot be viewed simply as a perturbation technique.

We begin by fixing some notations: for a fixed $F \in C^2(\{\mathbb{R}^N \setminus \{0\}\} \times \mathbb{R}; \mathbb{R})$ we write

$$F(x,t) = \frac{-a}{|x|^\alpha} + U(x,t) ,$$

for some $a > 0$ and $0 < \alpha < 2$.
Concerning U we shall make the following assumptions:

(H1) $$\limsup_{|x|\to 0} |x|^\alpha \left|\frac{\partial U}{\partial t}(x,t)\right| < +\infty \qquad \text{uniformly in t;}$$

(H2) $$\lim_{|x|\to 0} |x|^{\alpha+2} |\nabla^2 U(x,t)| = 0 \qquad \text{uniformly in t;}$$

(H3) $$U(x,t) = \frac{\lambda}{2}|x|^2 + V(x,t) \qquad \text{for some } \lambda \in \mathbb{R} \text{ , and}$$
$$\lim_{|x|\to\infty} \frac{|\nabla V(x,t)|}{|x|} = 0 \qquad \text{uniformly in t.}$$

REMARK I.1: It is easy to see that (H2) implies the following growth conditions for U at zero:

$$\lim_{|x|\to 0} |x|^{\alpha+1} |\nabla U(x,t)| = 0 \qquad \text{uniformly in t;}$$

$$\lim_{|x|\to 0} |x|^{\alpha} |U(x,t)| = 0 \qquad \text{uniformly in t.}$$

◇

Suitable conditions on λ will be imposed later in order to ensure the coercivity of the associated functional.
Our main results are summarized in the following theorems. Although these are formulated for $N \geq 3$, we will show in Section I.2 that the case $N = 2$ can be treated with similar but more restrictive hypotheses. We remark that the theorems contained in this paper are still true when $\alpha \geq 2$, but, as is well known, this case permits a standard approach to avoid collisions ([28]), and therefore we shall not consider it here.

THEOREM I.1. *Let F be defined as above, for some $N \geq 3$, $a > 0$ and $0 < \alpha < 2$. Suppose that F is T-periodic in t and satisfies the symmetry condition*

$$\text{(S)} \qquad F(-x, t + \frac{T}{2}) = F(x,t) \qquad \forall x \in \mathbb{R}^N \setminus \{0\}, \forall t \in \mathbb{R}.$$

Assume moreover that (H1), (H2), (H3) hold with $\lambda < \left(\frac{\pi}{T}\right)^2$. Then the problem

$$\text{(1)} \qquad \begin{cases} -\ddot{x} = \nabla F(x,t) & \\ x(t+T) = x(t) & \forall t \in \mathbb{R} \\ x(t + \frac{T}{2}) = -x(t) & \forall t \in \mathbb{R} \\ x(t) \neq 0 & \forall t \in \mathbb{R} \end{cases}$$

has at least one solution.

COROLLARY I.1. *Under the hypotheses of Theorem I.1 assume that F does not depend on t. Then, $\forall T > 0$, problem (1) has at least one solution having T as minimal period.*

THEOREM I.2. *Let F be defined as above, for some $N \geq 3$, $a > 0$ and $0 < \alpha < 2$. Suppose that F is T-periodic in t and that (H1), (H2), (H3) hold with $\lambda < 0$. Then the problem*

$$\text{(2)} \qquad \begin{cases} -\ddot{x} = \nabla F(x,t) & \\ x(t+T) = x(t) & \forall t \in \mathbb{R} \\ x(t) \neq 0 & \forall t \in \mathbb{R} \end{cases}$$

has at least one solution.

THEOREM I.3. *Let F be defined as above, for some $N \geq 3$, $a > 0$ and $0 < \alpha < 2$. Let $T > 0$ and $x_1, x_2 \in \mathbb{R}^N \setminus \{0\}$ be fixed. Assume that (H1), (H2), (H3) hold with $T^2\lambda < \lambda(x_1, x_2)$, the best Sobolev constant of the injection of $\{x \in H^1([0,1]; \mathbb{R}^N) \ / \ x(0) = x_1 \ , \ x(1) = x_2\}$ into L^2. Then the problem*

$$\text{(3)} \qquad \begin{cases} -\ddot{x} = \nabla F(x,t) \\ x(0) = x_1 \ , \ x(T) = x_2 \\ x(t) \neq 0 \qquad \forall t \in [0,T] \end{cases}$$

has at least one solution.

REMARK 2: Problem (3) becomes nontrivial when the angle between x_1 and x_2 is not too small.$\diamond$

REMARK 3: Hypotheses (H1) and (H2) are obviously fulfilled if U is regular (of class C^2) in the whole of $\mathbb{R}^N$; therefore we can solve (1) whenever $F(x,t) = \frac{\lambda}{2}|x|^2 - \frac{a}{|x|^\alpha} + V(x,t)$ is such that $V \in C^2(\mathbb{R}^N \times \mathbb{R}; \mathbb{R})$, $\lambda < \left(\frac{\pi}{T}\right)^2$ and $\nabla V(x,t)$ is sublinear at infinity.◇

Solving problems (1), (2) and (3) is equivalent to finding critical points of the functional:

$$I(x) = \int_0^T \{\frac{1}{2}|\dot{x}|^2 + \frac{a}{|x|^\alpha} - U(x,t)\}dt$$

respectively on the sets

$$\Lambda_1 = \{x \in H^1(\mathbb{R}; \mathbb{R}^N) \ / \ x(t+T) = x(t),\ x(t+\frac{T}{2}) = -x(t),\ x(t) \neq 0\ \forall t \in \mathbb{R}\},$$

$$\Lambda_2 = \{x \in H^1(\mathbb{R}; \mathbb{R}^N) \ / \ x(t+T) = x(t),\ x(t) \neq 0\ \forall t \in \mathbb{R}\},$$

$$\Lambda_3 = \{x \in H^1([0,T]; \mathbb{R}^N) \ / \ x(0) = x_1,\ x(T) = x_2,\ x(t) \neq 0\ \forall t \in [0,T]\}.$$

In the setting of Theorems 1, 2 and 3, the functional I is weakly lower semicontinuous and coercive, so that its infimum is always attained in $\overline{\Lambda}_1$, $\overline{\Lambda}_2$ and $\overline{\Lambda}_3$ respectively.

The main problem being to avoid collisions, we shall use the following argument: first we perturb I with a suitable "strong force" term in a neighborhood of zero, thereby obtaining a noncollision minimum for the corresponding functional; next we prove that any such minimum cannot interact with the perturbation, and therefore is a critical point of I in Λ_i, $i = 1,2,3$. To this end the main estimate consists in showing that if the minimum approaches too much the singularity, then a small variation can be performed making the functional decrease.

As a by-product of our approach we shall show in Section 4 that in each case one actually has

$$\inf_{\partial \Lambda_i} I > \inf_{\Lambda_i} I \ , \ i = 1,2,3.$$

N.B.: the three theorems can be proved with nearly the same argument and the same technical lemmas. An additional difficulty arises in the proof of Theorem I.1 because the variation mentioned above has to be symmetric. This is the reason why we perform only the proof of Theorem I.1.

NOTATIONS: Throughout this chapter we shall denote by $B_r = \{x \in \mathbb{R}^N \ / \ |x| < r\}$, the ball of radius r in $\mathbb{R}^N$ (centered at zero), and by S^{N-1} the unit sphere: $S^{N-1} = \{x \in$

$\mathbb{R}^N \ / \ |x| = 1\}$.
The sets Λ_i, $i = 1, 2, 3$, defined above are open and dense in

$$H_1 = \{x \in H^1(\mathbb{R}; \mathbb{R}^N) \ / \ x(t+T) = x(t), \ x(t + \frac{T}{2}) = -x(t), \ \forall t \in \mathbb{R}\},$$

$$H_2 = \{x \in H^1(\mathbb{R}; \mathbb{R}^N) \ / \ x(t+T) = x(t), \ \forall t \in \mathbb{R}\},$$

$$H_3 = \{x \in H^1([0,T]; \mathbb{R}^N) \ / \ x(0) = x_1, \ x(T) = x_2, \ \},$$

respectively. Therefore $\overline{\Lambda}_i$ and $\partial\Lambda_i$, are to be considered in H_i, $i = 1, 2, 3$.

I.1. Locally radial potentials

In this section we prove Theorem I.1 under the additional assumption that the potential is radially symmetric in a neighborhood of the singularity. This result will be used in the proof of the general case of section I.2. We wish to point out that in presence of local radial symmetries, the assumption (U2) considered here is weaker that (H2), so that Theorem I.1.1 is not just a particular case of Theorem I.1.

According to the notations of section 1, we write

$$F(x,t) = \frac{-a}{|x|^\alpha} + U(x,t), \quad \forall\, (x,t) \in (\mathrm{R}^N \setminus \{0\}) \times \mathrm{R}\,, \tag{I.1.1}$$

and we suppose that the function U satisfies the additional assumptions:

(U1)
$$\begin{aligned} &\exists \varepsilon > 0\ \exists \phi : (0,\varepsilon] \to \mathrm{R} \text{ of class } C^2 \text{ such that} \\ &U(x,t) = \phi(|x|) \qquad \forall x\,,\ 0 < |x| \le \varepsilon\,,\ \forall t \in \mathrm{R} \end{aligned}$$

(U2)
$$\lim_{s\to 0} s^{\alpha+1}|\phi'(s)| = 0\,.$$

Remark that (U2) implies that

$$\lim_{s\to 0} s^{\alpha}|\phi(s)| = 0\,.$$

Then we have the following

THEOREM I.1.1. *Let F be as in (I.1.1) with $a > 0$, $N \ge 2$ and $0 < \alpha < 2$. Suppose that F is T-periodic in t and satisfies the symmetry condition (S). Assume that (U1), (U2) and (H3) hold with $\lambda < \left(\frac{\pi}{T}\right)^2$; then problem (1) has at least one solution which minimizes I in Λ_1.*

PROOF: Consider the functional defined in Λ_1

$$I(x) = \int_0^T \{\frac{1}{2}|\dot{x}|^2 + \frac{a}{|x|^\alpha} - U(x,t)\}dt\,. \tag{I.1.2}$$

Because of the symmetry properties of F, the critical points of I in Λ_1 are solutions of (1). Assumption (H3) together with the fact that $\lambda < \left(\frac{\pi}{T}\right)^2$ imply that I is coercive, that is there are constants μ, $\nu > 0$ such that:

$$(I.1.2) \qquad I(x) \geq \mu||\dot{x}||_2^2 - \nu\,, \qquad \forall x \in \Lambda_1\,.$$

For each $\delta > 0$ we take a function $f_\delta \in C^2((0,+\infty);\mathrm{R})$ such that

$$(i) \qquad f_\delta(s) = \begin{cases} 0\,, & \text{if } s \geq \delta \\ -\frac{1}{s^2}\,, & \text{if } 0 < s \leq \frac{\delta}{2} \end{cases}$$

$$(ii) \qquad f'_\delta(s) \geq 0 \qquad s \in (0,\delta]\,.$$

It is immediate to check that such a function exists for every $\delta > 0$. Now we define $F_\delta \in C^2(\mathrm{R}^N \setminus \{0\};\mathrm{R})$ as $F_\delta(x) = f_\delta(|x|)$ and we consider the functional

$$(I.1.3) \qquad J_\delta(x) = I(x) - \int_0^T F_\delta(x)dt\,.$$

The term F_δ induces the strong force and therefore, as is well-known, $\inf_{x\in\partial\Lambda_1} J_\delta = +\infty$, so that no critical point of J_δ at a finite level can cross the singularity. Since (I.1.2) holds for J_δ as well we can conclude that there exists $x_\delta \in \Lambda_1$ such that $\inf_{u\in\Lambda_1} J_\delta(u) = J_\delta(x_\delta)$. In order to prove that x_δ is a solution of problem (1) we are going to show that $\min_{t\in[0,T]} |x_\delta(t)| \geq \delta$ if δ is sufficiently small.

To this aim we suppose that for every $\delta > 0$, the minimum x_δ found above satisfies the property $\min_{t\in[0,T]} |x_\delta(t)| = |x_\delta(t_\delta)| < \delta$ and we show that this leads to a contradiction. In what follows the C_j 's, j=1,2,... will denote real constants *independent* of δ.

Now, since x_δ is an absolute minimum of J_δ on Λ_1, there exists C_1 such that

$$(I.1.4) \qquad I(x_\delta) \leq J_\delta(x_\delta) \leq C_1\,;$$

then from (I.1.2) it follows that there exist C_2 and C_3 such that

$$(I.1.5) \qquad ||\dot{x}_\delta||_2^2 \leq C_2$$

and

$$(I.1.6) \qquad \int_0^T \frac{a}{|x_\delta|^\alpha} dt \leq C_3\,.$$

Therefore there exists an $\varepsilon' \leq \varepsilon$ (independent of δ) such that, if $A_{\varepsilon'} = \{t \;/\; |x_\delta(t)| \leq \varepsilon'\}$, then $meas(A_{\varepsilon'}) < \frac{T}{2}$.

At this point we can take an interval $[t_0, t_1] \subset [0, T]$ such that

i) $|x_\delta(t_0)| = |x_\delta(t_1)| = \varepsilon'$, $|t_1 - t_0| < \frac{T}{2}$;

ii) $|x_\delta(t)| < \varepsilon'$, $\forall t \in (t_0, t_1)$;

iii) $\min_{t\in[t_0,t_1]} |x_\delta(t)| < \delta$.

Moreover, from (I.1.5) it follows that there exists C_4 such that

$$C_4 \leq |t_1 - t_0| < \frac{T}{2} . \tag{I.1.7}$$

Repeating the above argument for $\frac{\varepsilon'}{2}$ we can find another interval $[s_0, s_1] \subset (t_0, t_1)$ such that $|x_\delta(s_0)| = |x_\delta(s_1)| = \frac{\varepsilon'}{2}$, $\forall t \in (s_0, s_1)$ and $\min_{t\in[s_0,s_1]} |x_\delta(t)| < \delta$. Exactly as above,then, there exist C_5, C_6, C_7 such that $|s_1 - s_0| \geq C_5$, $|t_0 - s_0| \geq C_6$, $|t_1 - s_1| \geq C_7$. ◇

STEP 1 ($N \geq 3$): Now, since the potential is radial in $\overline{B}_\varepsilon \setminus \{0\}$, x_δ is planar in the same set, and precisely it lies in the plane spanned by $x(t_0)$ and $\dot{x}(t_0)$. Assuming that $N \geq 3$, then there exists a vector $v_\delta \in S^{N-1}$ orthogonal to x_δ in $[t_0, t_1]$.
We are going to use v_δ to show that there exists at least one direction along which the second derivative of I_δ is negative.
To this end, let $\xi_\delta : [0, T] \to [0, 1]$ be a continuous, piecewise linear function which satisfies

$$\xi_\delta(s) = \begin{cases} 1 & \text{if } s \in [s_0, s_1] \\ 0 & \text{if } s \notin [t_0, t_1] \, ; \end{cases} \tag{I.1.8}$$

we extend ξ_δ by periodicity to R and we let $w_\delta = \left(\xi_\delta(t) - \xi_\delta(t + \frac{T}{2})\right) v_\delta$: it is clear that $w_\delta + x_\delta \in \Lambda_1$; moreover, we claim that $\nabla^2 J_\delta(x_\delta)(w_\delta, w_\delta) < 0$, contradicting the fact that x_δ is a minimum point in Λ_1. To carry over this estimate we need to analyze the form of $\nabla^2 J_\delta(x_\delta)(w_\delta, w_\delta)$. An easy computation shows that

$$\begin{aligned} \nabla^2 J_\delta(x_\delta)(w_\delta, w_\delta) = & \int_0^T |\dot{w}_\delta|^2 dt + a\alpha(\alpha+2) \int_0^T \frac{(x_\delta \cdot w_\delta)^2}{|x_\delta|^{\alpha+4}} dt - a\alpha \int_0^T \frac{|w_\delta|^2}{|x_\delta|^{\alpha+2}} dt \\ & - \int_0^T \{\nabla^2 U(x_\delta, t)(w_\delta, w_\delta) + \nabla^2 F_\delta(x_\delta)(w_\delta, w_\delta)\} dt . \end{aligned} \tag{I.1.9}$$

Notice that whenever $w_\delta(t) \neq 0$, we have $|x_\delta(t)| \leq \varepsilon'$, so that we can substitute U with ϕ. Now, it is easy to see that in $\overline{B}_\varepsilon \setminus \{0\}$,

$$\nabla^2 \phi(|x|)(\zeta, \zeta) = K(x)(x \cdot \zeta)^2 + \frac{\phi'(|x|)}{|x|} |\zeta|^2 , \tag{I.1.10}$$

for some suitable $K \in C^0(\overline{B}_\varepsilon \setminus \{0\}; \mathbb{R})$, and for all $\zeta \in \mathbb{R}^N$; moreover, a similar expression holds for $\nabla^2 F_\delta$. Thus, since by definition $f'_\delta(|x|) \geq 0$ for all $x \neq 0$ and $x_\delta(t) \cdot w_\delta(t) = 0\ \forall t$, we get, also taking into account the symmetry of x_δ and w_δ,

$$(I.1.11) \qquad \begin{aligned} \nabla^2 J_\delta(x_\delta)(w_\delta, w_\delta) &\leq \frac{2}{C_6} + \frac{2}{C_7} - a\alpha \int_0^T \frac{|w_\delta|^2}{|x_\delta|^{\alpha+2}} dt - \int_0^T \frac{\phi'(|x_\delta|)}{|x_\delta|} |w_\delta|^2 dt \leq \\ &\frac{2}{C_6} + \frac{2}{C_7} - 2a\alpha \int_{t_0}^{t_1} \frac{|w_\delta|^2}{|x_\delta|} dt - \int_{t_0}^{t_1} \frac{\phi'(|x_\delta|)}{|x_\delta|} |w_\delta|^2 dt\ . \end{aligned}$$

Without loss of generality we can suppose ε' so small that by (U2) $-\frac{a\alpha}{|x_\delta|^{\alpha+2}} - \frac{\phi'(|x_\delta|)}{|x_\delta|} \leq -\frac{S}{|x_\delta|^{\alpha+2}}$ for some $S > 0$ and all $x \in \overline{B}_{\varepsilon'} \setminus \{0\}$, so that

$$(I.1.12) \qquad \nabla^2 J_\delta(x_\delta)(w_\delta, w_\delta) \leq \frac{2}{C_6} + \frac{2}{C_7} - 2S \int_{s_0}^{s_1} \frac{1}{|x_\delta|^{\alpha+2}} dt\ .$$

This completes the proof since, as is well-known, if $\min_{t\in[t_0,t_1]} |x_\delta(t)| < \delta \to 0$, and $\int_{t_0}^{t_1} |\dot{x}_\delta|^2$ is bounded, then

$$(I.1.13) \qquad \int_{s_0}^{s_1} \frac{1}{|x|^{\alpha+2}} dt \to +\infty\ ,$$

which means that for δ small enough $\nabla^2 J_\delta(x_\delta)(w_\delta, w_\delta) < 0$, that is, x_δ is not a minimum point for J_δ in Λ_1.◇

STEP 2 ($N = 2$): The case $N = 2$ can be proved as follows: by the estimates until the step 1, one has that

$$(I.1.14) \qquad \begin{aligned} &\int_{t_0}^{t_1} \{\frac{1}{2}|\dot{x}_\delta|^2 + \frac{a}{|x_\delta|^\alpha} - \phi(|x_\delta|) - F_\delta(|x_\delta|)\} \\ &= \min_{y\in\Sigma_2} \int_{t_0}^{t_1} \{\frac{1}{2}|\dot{y}|^2 + \frac{a}{|y|^\alpha} - \phi(|y|) - F_\delta(|y|)\}\ , \end{aligned}$$

where

$$\begin{aligned} \Sigma_2 = \{y \in H^1([t_0, t_1]; \mathbb{R}^2)\ /\ y(t_0) = x_\delta(t_0)\ ,\ y(t_1) = x(t_1) \\ |y(t)| \leq \varepsilon'\ , \forall t \in [t_0, t_1]\}\ . \end{aligned}$$

Now, since the potential in (I.1.4) is radial, we can immerge the minimization problem of (I.1.14) in a higher dimension, that is we consider a minimizer $\overline{x}$ of the integral in (I.1.14) on the set

$$\Sigma_N = \{y \in H^1([t_0,t_1];\mathbb{R}^N) \; / \; y(t_0) = x_\delta(t_0) \; , \; y(t_1) = x(t_1)$$
$$|y(t)| \leq \varepsilon' \; , \forall t \in [t_0,t_1]\} \; ,$$

for some $N \geq 3$. Again, because of the radial symmetry of the potential and of the constraint, we can assume that $\overline{x}$ is planar, and precisely it lies on the same plane that the one of x_δ, so that $\overline{x}$ minimizes the integral of the right hand side of (I.1.14) on Σ_2 as well as x_δ. Therefore one has $x_\delta = \overline{x}$ in $[t_0,t_1]$. Finally we know from step 1 that (for δ small) $\overline{x}$ cannot interact with the strong force perturbation.

We have thus proved that there exists a solution of Problem (1) which is a local minimum for I in Λ_1. We shall show in section 4 that this solution is actually a global minimum for I in Λ_1.◇

We remark that the argument used in the above proof can be repeated with some straightforward modifications (actually simplifications) to obtain the following results:

THEOREM I.1.2. *Let F be as in (I.1.1) with $a > 0$, $N \geq 2$ and $0 < \alpha < 2$. Suppose F is T-periodic in t and (U1), (U2), (H3) hold with $\lambda < 0$. Then problem (2) has at least one solution which minimizes I in Λ_2.*

THEOREM I.1.3. *Let F be as in (I.1.1) with $a > 0$, $N \geq 2$ and $0 < \alpha < 2$. Suppose F is T-periodic in t and (U1), (U2), (H3) hold. Let x_1, x_2 be fixed points in $\mathbb{R}^N \setminus \{0\}$ and let $T > 0$. Assume $T^2\lambda < \lambda(x_1,x_2)$ as in Theorem I.3; then problem (3) has at least one solution which minimizes I in Λ_3.*

I.2. The general case (Proof of Theorem I.1)

In this section we turn to the proof of Theorem I.1. To this end we shall make use of the results of section 2, since we shall first work with a truncated potential which is radially symmetric in a neighborhood of the singularity. Then we shall obtain a more precise estimate on the behavior of the minimum found by the application of Theorem I.1.1. With the aid of these estimates we shall show that any such minimum cannot interact with the truncation by the construction of a suitable variation along which the functional decreases.

Throughout this section δ is the radius of the neighborhood where the truncation is located, and therefore δ can always be taken less than one.

DEFINITION I.2.1. *Let $\varphi \in C^2([0,+\infty);[0,1])$ be a fixed function satisfying*

$$\varphi(s) = \begin{cases} 0 & \text{if } s \leq \frac{1}{2} \\ 1 & \text{if } s \geq 1 . \end{cases} \tag{I.2.1}$$

Define

$$\varphi_\delta(s) = \varphi\left(\frac{s}{\delta}\right) , \qquad \forall s \in [0,+\infty) \tag{I.2.2}$$

$$U_\delta(x,t) = \varphi_\delta(|x|)U(x,t) , \tag{I.2.3}$$

$$F_\delta(x,t) = \frac{-a}{|x|^\alpha} + U_\delta(x,t) , \tag{I.2.4}$$

and

$$I_\delta(x) = \int_0^T \{\frac{1}{2}|\dot{x}|^2 - F_\delta(x,t)\}dt \qquad \forall x \in \Lambda_1 . \tag{I.2.5}$$

PROPOSITION I.2.1. *Let U satisfy (H1) and (H2). Then U_δ satisfies (H1) and (H2) too, uniformly in δ.*

PROOF: (H1) is obviously fulfilled. Concerning (H2), remark that since φ is of class C^2 and its derivatives have compact support in $[0,1]$, then there exist constants d,$d' > 0$ such that $|\nabla\varphi_\delta(|x|)| \leq \frac{d}{\delta}$ and $|\nabla^2\varphi_\delta(|x|)| \leq \frac{d'}{\delta^2}$. To complete the proof, just consider that

$$|\nabla^2 U_\delta(x,t)| \le |U(x,t)||\nabla^2\varphi_\delta(|x|)| + 2|\nabla U(x,t)||\nabla\varphi_\delta(|x|)| + |\nabla^2 U(x,t)||\varphi_\delta(|x|)| \,.$$

From now on the C_j's will denote constants *independent* of δ.

By the application of Theorem I.1.1 and by Proposition I.2.1 we can find an $x_\delta \in \Lambda_1$ such that:

(I.2.6)
$$x_\delta \text{ is a local minimum for } I_\delta \text{ and it minimizes } I_\delta \text{ on a set } \{x \in \Lambda_1 \;/\; |x(t)| \ge \rho\} \text{ for some } \rho < \frac{\delta}{2}\,.$$

(I.2.7)
$$\exists C_1 \text{ such that } I_\delta(x_\delta) \le C_1\,.$$

The following proposition contains the main estimate which will be used in the proof of Theorem I.1.

PROPOSITION I.2.2. *Assume (H1), (H2), (H3) hold. Then there exists $\sigma > 0$ such that $\forall\delta > 0$, every x satisfying (I.2.6) and (I.2.7) verifies*

(I.2.8)
$$\min_{t\in[0,T]} |x(t)| \ge \sigma\delta.$$

PROOF: Let x be such a local minimum, assume that

(I.2.9)
$$|x(\bar t)| = \min_{t\in[0,T]} |x(t)| \le \frac{\delta}{4}\,,$$

and take an interval $[t_0, t_1] \subset [0,T]$, containing $\bar t$ and such that $|x(t_0)| = |x(t_1)| = \frac{\delta}{2}$, $|x(t)| < \frac{\delta}{2}$ for $t \in (t_0, t_1)$. Then we have

(I.2.10)
$$-\ddot x = \frac{a\alpha}{|x|^{\alpha+2}}x \qquad \text{in}[t_0,t_1]\,,$$

(I.2.11)
$$\frac{1}{2}|\dot x|^2 - \frac{a}{|x|^\alpha} = E \qquad \text{in}[t_0,t_1]\,,$$

(I.2.12)
$$-\ddot x = \nabla F_\delta(x,t) \qquad \text{in}[0,T]\,,$$

(I.2.13)
$$\frac{1}{2}|\dot x|^2 + F_\delta(x,t) = E + \int_{t_0}^{t} \frac{\partial U_\delta}{\partial t}(x,s)ds \quad \text{in}[0,T]\,.$$

Let us set $\omega = t_1 - t_0$.

STEP 1. *There exists C_2 such that*

$$|E| \leq C_2 . \tag{I.2.14}$$

PROOF OF STEP 1: Because of the growth assumptions on U_δ at infinity and at the origin and Proposition I.2.1, the boundedness of $I_\delta(x)$ implies the boundedness of each term $\int_0^T |\dot{x}|^2 dt$, $\int_0^T \frac{a}{|x|^\alpha} dt$, $\int_0^T U_\delta(x,t) dt$ with constants independent of δ. (I.2.14) then follows from (I.2.13).◇

STEP 2. *There is a constant C_3 such that*

$$\omega \leq C_3 \delta^{\frac{\alpha+2}{2}} . \tag{I.2.15}$$

PROOF OF STEP 2: First remark that by (I.2.10),(I.2.11)

$$\frac{1}{2}\frac{d^2}{dt^2}|x|^2 = 2E + (2-\alpha)\frac{a}{|x|^\alpha} \qquad \text{in } [t_0,t_1] , \tag{I.2.16}$$

so that $|x|^2$ is convex in $[t_0,t_1]$ if δ is small, and that, by construction, one has

$$x(t_0)\cdot\dot{x}(t_0) \leq 0 \text{ and } x(t_1)\cdot\dot{x}(t_1) \geq 0 . \tag{I.2.17}$$

Taking the L^2 product of (I.2.10) by x one finds that

$$\begin{aligned} \int_{t_0}^{t_1} |\dot{x}|^2 &= \alpha \int_{t_0}^{t_1} \frac{a}{|x|^\alpha} - x(t_0)\cdot\dot{x}(t_0) + x(t_1)\cdot\dot{x}(t_1) \\ &\geq \alpha \int_{t_0}^{t_1} \frac{a}{|x|^\alpha} \geq \alpha\omega 2^\alpha \frac{a}{\delta^\alpha} . \end{aligned} \tag{I.2.18}$$

Moreover from (I.2.6) we deduce that

$$c(t_0,t_1) := \int_{t_0}^{t_1} \{\frac{1}{2}|\dot{x}|^2 + \frac{a}{|x|^\alpha}\} = \min_{y\in\Sigma} \int_{t_0}^{t_1} \{\frac{1}{2}|\dot{y}|^2 + \frac{a}{|y|^\alpha}\} , \tag{I.2.19}$$

where

$$\Sigma = \{y \in H^1([t_0,t_1];\mathbb{R}^N) \;/\; y(t_0) = x(t_0) ,\; y(t_1) = x(t_1) ,\rho \leq |y(t)| \leq \frac{\delta}{2}\} ,$$

so that

$$c(t_0, t_1) \le \frac{\delta^2 \pi^2}{8\omega} + a 2^\alpha \frac{\omega}{\delta^\alpha} \; ; \tag{I.2.20}$$

(just take the circular path joining $x(t_0)$ and $x(t_1)$).

Finally from (I.2.18) and (I.2.20) we get

$$\omega^2 \le 2^{-\alpha} \frac{\pi^2 \delta^{2+\alpha}}{8a\alpha} \; ,$$

which implies (I.2.15).◇

STEP 3. *There exists a constant C_4 such that*

$$\int_{t_0}^{t_1} |\dot{x}|^2 \le C_4 \delta^{\frac{2-\alpha}{2}} \; . \tag{I.2.21}$$

PROOF OF STEP 3: First remark that, from (I.2.11) and(I.2.14), one has

$$|x(t_i) \cdot \dot{x}(t_i)| \le |x(t_i)||\dot{x}(t_i)| =$$
$$= \sqrt{2E\frac{\delta^2}{4} + 2a\left(\frac{\delta}{2}\right)^{2-\alpha}} \le C_4' \delta^{\frac{2-\alpha}{2}} \; , \quad i = 1, I.1.$$

By integrating (I.2.16), and by Steps 1 and 2, one finds

$$\int_{t_0}^{t_1} \frac{a}{|x|^\alpha} = \frac{1}{2-\alpha}[-2E\omega - x(t_0) \cdot \dot{x}(t_0) + x(t_1) \cdot \dot{x}(t_1)] \le C_4'' \delta^{\frac{2-\alpha}{2}} \; ;$$

hence

$$\int_{t_0}^{t_1} |\dot{x}|^2 = 2E\omega + 2 \int_{t_0}^{t_1} \frac{a}{|x|^\alpha} \le C_4 \delta^{\frac{2-\alpha}{2}} \; .$$

◇

STEP 4. *There exists a constant C_5 such that*

$$\omega \ge C_5 \delta^{\frac{2+\alpha}{2}} \; . \tag{I.2.22}$$

PROOF OF STEP 4: Indeed by Hölder's inequality and (I.2.21) one has

$$\omega \ge \frac{\delta^2}{16} \frac{1}{\int_{t_0}^{t_1} |\dot{x}|^2 dt} \ge C_5 \delta^{\frac{2+\alpha}{2}} \; .$$

◇

END OF THE PROOF OF PROPOSITION I.2.2: Since the potential F_δ is radial in $B_{\frac{\delta}{2}}$ and since x satisfies (I.2.11), then $x(t)$ is planar in $[t_0, t_1]$. Let $v \in S^{N-1}$ be a vector orthogonal to the plane spanned by $x(t_0), \dot{x}(t_0)$. Following the argument used in the proof of Theorem I.1.1 (and taking a smaller δ if it is necessary) one finds a function w such that $w(t) = \psi(t)v$, $x + w \in \Lambda_1$ and

$$w(t) = \begin{cases} 0 & \text{if } t \notin [t_0, t_1] \cup [t_0 + \frac{T}{2}, t_1 + \frac{T}{2}] \\ v & \text{if } t \in [t_0 + \frac{\omega}{4}, t_0 + \frac{3\omega}{4}] \,, \end{cases}$$

$$\int_{t_0}^{t_1} |\dot{w}|^2 dt \leq \frac{C_6}{\omega} \,.$$

Let $\sigma = \frac{2}{\delta} \min_{t \in [0,T]} |x(t)| = \frac{2}{\delta} |x(\bar{t})|$, that is,

$$\min_{t \in [0,T]} |x(t)| = \sigma \frac{\delta}{2} \,.$$

The computation of $\nabla^2 I_\delta(x)(w,w)$ leads to

$$\nabla^2 I_\delta(x)(w,w) \leq \frac{C_6}{\omega} - \frac{C_7}{\delta^\alpha \int_{t_0}^{t_1} |\dot{x}|^2} \left(\frac{1}{\sigma^{\frac{\alpha}{2}}} - 1 \right)^2 .$$

Indeed one has

$$\frac{2}{\alpha} \left(\frac{1}{|x(\bar{t})|^{\frac{\alpha}{2}}} - \frac{1}{|x(t_0)|^{\frac{\alpha}{2}}} \right) \leq \int_{t_0}^{t_1} \left| \frac{2}{\alpha} \frac{d}{dt} \frac{1}{|x|^{\frac{\alpha}{2}}} \right|$$
$$\leq \left(\int_{t_0}^{t_1} |\dot{x}|^2 \right)^{\frac{1}{2}} \left(\int_{t_0}^{t_1} \frac{1}{|x|^{\alpha+2}} \right)^{\frac{1}{2}} .$$

Since x is a local minimum and by the estimates (I.2.21) and (I.2.22), one finds

$$0 \leq \frac{1}{\delta^{\frac{2+\alpha}{2}}} \left(\frac{C_6}{C_5} - \frac{C_7}{C_4} \left(\frac{1}{\sigma^{\frac{\alpha}{2}}} - 1 \right)^2 \right) ;$$

hence

$$\frac{C_6}{C_5} - \frac{C_7}{C_4} \left(\frac{1}{\sigma^{\frac{\alpha}{2}}} - 1 \right)^2 \geq 0 \,,$$

and therefore, since that constant C_j's do not depend on δ, σ has a lower bound independent of δ.◇

We are now ready to complete the proof of the main Theorem.

PROOF OF THEOREM I.1: To carry over the proof of Theorem I.1 we begin by making a truncation of the potential according with Definition I.2.1. By means of Theorem I.1.1 the corresponding truncated functional can be shown to posses a local minimum satisfying moreover (I.2.6) and (I.2.7), and therefore one gets in this way a noncollision solution x_δ for the corresponding differential problem. As before, the problem is to show that x_δ does not interact with the truncation, that is to say $\min_{t\in[0,T]} |x_\delta(t)| \geq \delta$.

First of all we remark that the bound on the levels $I_\delta(x_\delta)$ implies the following estimates independent of δ:

$$|E_\delta| \leq C_1$$

$$|\int_{t_0}^{t} \frac{\partial U_\delta}{\partial t}(x_\delta, t)| \leq C_2 \ .$$

Consider now $B_\delta(t)$, the angular momentum of x_δ: then we have

$$B_\delta(t) \cdot x_\delta(t) = 0 \ , \qquad \forall t \in \mathrm{R} \ , \tag{I.2.23}$$

$$|B_\delta(t)| = |x_\delta(t)|^2 |\frac{d}{dt}\frac{x_\delta(t)}{|x_\delta(t)|}| \ , \tag{I.2.24}$$

$$|\frac{d}{dt} B_\delta(t)| \leq |\nabla U_\delta(x_\delta, t)||x_\delta(t)| \ . \tag{I.2.25}$$

Define $\rho_\delta(t) = |x_\delta(t)|$. The energy integral can be written as

$$\frac{1}{2}\dot\rho_\delta^2 + \frac{1}{2}\frac{B_\delta^2(t)}{\rho_\delta^2} - \frac{a}{\rho_\delta^\alpha} + U_\delta(x_\delta, t) = E_\delta + \int_{t_0}^{t} \frac{\partial U_\delta}{\partial t} \ . \tag{I.2.26}$$

◇

PROPOSITION I.2.3. *Assume (H1) and (H2) hold. Then there are constants δ_0 and μ (independent of δ) such that*

$$|x_\delta(t)| \leq \delta_0 \quad \Rightarrow \quad B_\delta^2(t) \geq \mu |x_\delta(t)|^{2-\alpha} \ .$$

PROOF: Let σ be as in Proposition I.2.2 and let $a' > a'' \geq a > a''' > 0$ be fixed constants such that

$$a'''\sigma - a' + a'' \geq \frac{a}{2}\sigma \ . \tag{I.2.28}$$

Thanks to Proposition I.2.1 and (H.2), there exists δ_0 such that, for every $\delta \leq 1$ and every $|E| \leq C_1$,

$$2E + (2-\alpha)\frac{a}{|x|^\alpha} - 2U_\delta(x,t) - \nabla U_\delta(x,t)\cdot x + 2C_2$$
$$\le 2E + (2-\alpha)\frac{a'}{|x|^\alpha}\,, \quad |x| \le \delta_0\,, \tag{I.2.29}$$

$$2E|x|^2 + 2a'|x|^{2-\alpha} \ge 2a''|x|^{2-\alpha}\,, \quad |x| \le \delta_0\,, \tag{I.2.30}$$

$$2E|x|^2 + 2a|x|^{2-\alpha} - 2U_\delta(x,t) - 2C_2|x|^2 \ge 2E|x|^2 + 2a'''|x|^{2-\alpha}\,, \quad |x| \le \delta_0\,, \tag{I.2.31}$$

$$2E + (2-\alpha)\frac{a}{|x|^\alpha} - 2U_\delta(x,t) - \nabla U_\delta(x,t)\cdot x - 2C_2 \ge 0\,, \quad |x| \le \delta_0\,. \tag{I.2.32}$$

Since $\frac{1}{2}\ddot{\rho^2} = |\dot{x}|^2 + x\cdot\ddot{x}$, then (I.2.28) and (I.2.29) imply

$$0 \le \frac{1}{2}\ddot{\rho^2} \le 2E + (2-\alpha)\frac{a'}{\rho^\alpha} \quad \text{if } \rho \le \delta_0\,. \tag{I.2.33}$$

Now let $\bar{t}$ be such that $\rho(\bar{t}) < \delta_0$ is a local minimum for ρ; then we can assume that $\dot{\rho}(t) \ge 0$ for $t \in [\bar{t}, t^*]$, with $\rho(t^*) \ge \delta_0$. By multiplying (I.2.33) by $\dot{\rho}^2$ one finds

$$\frac{d}{dt}\left(\frac{1}{4}\dot{\rho^2}^2 - 2E\rho^2 - 2a'\rho^{2-\alpha}\right) \le 0\,. \tag{I.2.34}$$

Hence we have

$$\frac{1}{4}\dot{\rho^2}^2 - 2E\rho^2 - 2a'\rho^{2-\alpha} \le -2E\rho(\bar{t})^2 - 2a'\rho(\bar{t})^{2-\alpha}\,. \tag{I.2.35}$$

Notice that if δ_0 is small enough, then the right-hand-side of (I.2.35) is strictly negative. If we denote it by $-B_0^2 = -2E\rho$, then from (I.2.30) we have

$$B_0^2 \ge 2a''\rho(\bar{t})^{2-\alpha}\,, \tag{I.2.36}$$

and from (I.2.26) and (I.2.31) we deduce that

$$B^2(t) \ge -\frac{1}{4}\dot{\rho^2}^2 + 2E\rho^2 + 2a'''\rho^{2-\alpha}\,;$$

therefore, from the above inequality, (I.2.35) and (I.2.28) we find

$$B^2(t) \geq B_0^2 - 2(a' - a''')\rho^{2-\alpha} \geq a\sigma\rho^{2-\alpha} \,, \tag{I.2.37}$$

if $t < t^*$. It is obvious that, by reversing the time, the same inequality holds for $t \leq \bar{t}$, and for any local minimum $\rho(\bar{t}) \leq \delta_0$.◇

Let now δ_0 be fixed and assume that

$$\min_{t\in[0,T]} |x(t)| \leq \frac{\delta_0}{2} \,.$$

Take an interval $[t_0, t_1]$ such that $|x(t_0)| = |x(t_1)| = \delta_0$, $|x(t)| < \delta_0$ in (t_0, t_1) and $\min_{t\in[t_0,t_1]} |x(t)| \leq \frac{\delta_0}{2}$. Define

$$\omega = t_1 - t_0 \,.$$

Then we have the following estimate:

PROPOSITION I.2.4. *Assume (H1) and (H2) hold. There are constants (independent of δ) $\bar{\delta}$, ν_1, $\nu_2 > 0$ such that, for any $\delta_0 \leq \bar{\delta}$ one has*

$$\nu_1 \delta_0^{\frac{2+\alpha}{2}} \leq \omega \leq \nu_2 \delta_0^{\frac{2+\alpha}{2}} \,.$$

PROOF: The proof works essentially in the same way as that of steps 2,3,4 of Proposition I.2.3. Just remark that by Proposition I.2.1 the constants ν_1 and ν_2 can be taken as close as we please to the constants C_3 and C_5 of that proof, provided we work in a sufficiently small neighborhood of zero.◇

END OF THE PROOF OF THEOREM I.1: Let us begin by fixing δ_0 in accordance with Propositions I.2.3 and I.2.4, and assume that x_δ ($\delta \leq \frac{\delta_0}{2}$) is a local minimum found via the above arguments such that

$$\min_t < |x_\delta(t)| \leq \delta \,.$$

Working as in the proof of Theorem I.1.1, one considers an interval $[t_0, t_1]$ such that

$$|x_\delta(t_1)| = |x_\delta(t_0)| = \frac{\delta_0}{2} \,,$$

$$|x_\delta(t)| < \frac{\delta_0}{2} \,, \qquad \forall t \in (t_0, t_1) \,.$$

Next we consider $\bar{t} \in (t_0, t_1)$ such that $|x_\delta(\bar{t})| = \min_{t\in[t_0,t_1]} |x_\delta(t)|$, as we have done before. Moreover, let us take the interval $[s_0, s_1] \subseteq [t_0, t_1]$ such that $|x_\delta(s_0)| = |x_\delta(s_1)| = \frac{\delta_0}{4}$

and $|x_\delta(t)| < \frac{\delta_0}{4}$, $\forall t \in (s_0, s_1)$.

Now we take a piecewise linear function $\psi_\delta : [0,T] \to [0,1]$ which behaves like ξ_δ of Theorem I.1.1 on $[t_0, t_1]$ and we consider

$$P_\delta(t) = \left(\psi_\delta(t) - \psi_\delta(t + \frac{T}{2})\right) \frac{B_\delta(t)}{|B_\delta(t)|}$$

It is clear from (I.2.23) that $P_\delta(t) + x_\delta(t) \in \Lambda_1$ and that $P_\delta(t) \cdot x_\delta(t) = 0$,$\forall t \in [0,T]$. Moreover, from (I.2.24), (I.2.25) and (H2) one has that

$$|\dot{P}_\delta(t)|^2 \leq C_3 \left(|\dot{\psi}_\delta|^2 + \frac{|\dot{B}_\delta(t)|^2}{|B_\delta(t)|} \right) ,$$

so that

$$\int_0^T |\dot{P}_\delta(t)|^2 \leq C_4 \left(\frac{1}{\delta_0} + \int_{t_0}^{t_1} \frac{C(\delta_0)}{|x_\delta|^{\alpha+2}} \right) ,$$

with $C(\delta_0) \to 0$ as $\delta_0 \to 0$ and C_4 independent of δ_0.

From the estimate of proposition I.2.4, one deduces that the fractions $\frac{s_0 - t_0}{s_1 - s_0}$ and $\frac{t_1 - s_1}{s_1 - s_0}$ are bounded by a constant independent of δ_0, so that one deduces that

$$\int_0^T |\dot{P}_\delta(t)|^2 \leq C_5 \left(\frac{1}{\delta_0} + \int_{s_0}^{s_1} \frac{C(\delta_0)}{|x_\delta|^{\alpha+2}} \right) .$$

with C_5 independent of δ_0. Now we fix δ_0 such that $C_5\delta_0 \leq \frac{\alpha a}{2}$. Then the computation of $\nabla^2 I_\delta(x_\delta)(P_\delta, P_\delta)$, together with the fact that x_δ is a local minimum of I_δ, leads to the following estimate:

$$0 \leq \nabla^2 I_\delta(x_\delta)(P_\delta, P_\delta) \leq C_6 - \frac{\alpha a}{2} \int_{s_0}^{s_1} \frac{1}{|x_\delta|^{\alpha+2}} , \tag{I.2.38}$$

with C_6 depending on δ_0 but not on δ (if $\delta \leq \frac{\delta_0}{4}$). From Proposition I.2.4, $s_1 - s_0$ has a lower bound independent of δ, so that the integral in (I.2.38) diverges whenever $\delta \to 0$.◇

As it should be clear by now, the main idea in proving Theorems 1,2,3 can be summarized in the following Lemma:

LEMMA I.3.1. *Assume the hypotheses of Theorems 1 (resp. 2 or 3) be fulfilled. Then for any fixed $C \in \mathbb{R}$ there exists a $\delta_0 > 0$ such that, if $x \in \Lambda_1$ (resp. in Λ_2 or Λ_3) is a local minimum of I with $I(x) \leq C$, then $|x(t)| \geq \delta_0$, $\forall\, t \in \mathbb{R}$.*
The same result holds (with the same δ_0) for the perturbed functional I_δ, where

$$I_\delta(x) = \int_0^T \{\frac{1}{2}|\dot{x}|^2 + \frac{a}{|x|^\alpha} - U_\delta(x,t) - F_\delta(|x|)\}dt$$

(F_δ is the strong force term as in (I.1.3), and U_δ is as in Definition I.2.1).

REMARK I.3.1: This result is false for $N = 2$. Indeed, assume that $F(x) = \frac{-a}{|x|}$, let $T > 0$ be fixed, and let $C = \frac{3}{2}(2\pi a)^{\frac{2}{3}} T^{\frac{1}{3}}$. Let $x_0 \in \mathbb{R}^2$, $|x_0| \geq a^{\frac{1}{3}} \left(\frac{T}{2\pi}\right)^{\frac{2}{3}}$ be fixed. Then for any value of the eccentricity $0 \leq e < 1$, there is at least one elliptic function x_e such that $x_e(0) = x_e(1) = x_0$ with $I(x_e) = C$. Moreover x_e is a local minimum of the functional

$$I(x) = \int_0^T \frac{1}{2}|\dot{x}|^2 + \frac{a}{|x|} ,$$

on the set

$$\Lambda = \{x \in H^1([0,T];\mathbb{R}^2) \;/\; x(0) = x(T) = x_0\}$$

(see also [27]). It is clear that $\min_{t\in[0,T]} |x_e| \to 0$ as $e \to 1$.
This fact does not contradict Theorem I.1.3, since the absolute minimum of I is strictly less than C, and it is attained on a path having topological degree zero, with respect to the origin.◇

Another consequence of Lemma I.3.1 is the following estimate:

THEOREM I.3.1. *Under the hypotheses of Theorem 1,2 and 3 respectively, one has*

$$\inf_{x\in\partial\Lambda_i} I(x) > \inf_{x\in\Lambda_i} I(x) \qquad i = 1,2,3\,.$$

PROOF: Assume by contradiction that

$$c_i = \inf_{x\in\partial\Lambda_i} I(x) = \inf_{x\in\Lambda_i} I(x)\,.$$

Let us denote by K_{c_i} the (compact) set of all the minima of I in $\Lambda_i \cup \partial\Lambda_i$. Then it follows from Lemma I.3.1 that

$$dist(\Lambda_i \cap K_{c_i}, \partial\Lambda_i) = d_i > 0 \,, \qquad i = 1,2,3 \,.$$

Modify I to I^* so that

$$I^*(x) = \begin{cases} I(x) \,, & \text{if } dist(x, \Lambda_i \cap K_{c_i}) \geq \frac{d_i}{2} \\ I(x) + \varepsilon \,, & \text{if } dist(x, \Lambda_i \cap K_{c_i}) \leq \frac{d_i}{4} \,, \end{cases}$$

$$I^*(x) > I(x) \qquad \text{if } dist(x, \Lambda_i \cap K_{c_i}) < \frac{d_i}{2} \,.$$

Then $c_i = \inf_{x \in \partial\Lambda_i} I^*(x) = \inf_{x \in \Lambda_i} I^*(x)$. With a slight modification of the arguments used in proving Theorem I.1 (resp. 2 or 3) one easily proves that I^* admits a minimum $x^* \in \Lambda_i$, at level $I^*(x^*) = c_i$, hence $dist(x^*, \Lambda_i \cap K_{c_i}) > \frac{d_i}{2}$, so that x^* minimizes I as well and this is a contradiction .◇

II. A restricted three body problem in $\mathbb{R}^3$

In this chapter we deal with a restricted symmetric three bodies problem; more precisely, let $x_1, x_2 \in H$ such that $x_1, x_2 \in C^2(\mathbb{R} \setminus (x_1 - x_2)^{-1}(0); \mathbb{R}^3)$. We look for solutions to the problem

$$(P) \qquad \begin{cases} -\ddot{x} = \nabla F_1(x - x_1) + \nabla F_2(x - x_2) & \\ x(t+T) = x(t) & \forall t \\ x(t + \frac{T}{2}) = -x(t) & \forall t \\ x(t) - x_1(t) \neq 0 & \\ x(t) - x_2(t) \neq 0 & \forall t \notin (x_1 - x_2)^{-1}(0) \,. \end{cases}$$

A situation of this type occurs, for example, when two bodies, say the Sun and the Eart, are moving in a symmetrical way under the effect of a gravitational-type low, and a third body, much more smaller that the first two, say the Moon, moves in the force field generated by the Sun and the Eart.
This problem has a variationa structure, since each solution of (P) is a critical point of the associated functional

$$\text{(II.1)} \qquad I(x) = \int_0^T \frac{1}{2}|\dot{x}|^2 - F_1(x - x_1) - F_2(x - x_2) \,,$$

in the subset of H,

$$\text{(II.2)} \qquad \Lambda = \{x \in H \;/\; x(t) - x_1(t) \neq 0 \,, x(t) - x_2(t) \neq 0 \,, \forall t \notin (x_1 - x_2)^{-1}(t)\} \,.$$

Throughout this chapter H is the space of all the T-periodic functions $H^1_{loc}(\mathbb{R}; \mathbb{R}^3)$ satisfying the symmetry constraint $x(t + \frac{T}{2}) = -x(t)$.
We consider the following assumptions on x_1 and x_2:

$$\text{(III.3)} \qquad x_1 \,,\; x_1 \in H \cap C^2(\mathbb{R} \setminus (x_1 - x_2)^{-1}(0); \mathbb{R}^3)$$

and, on F_1 F_2:

$$(S) \qquad F_i(x) = F_i(-x) \qquad i = 1,2 \,, \forall x \in \mathbb{R}^3 \setminus \{0\}$$

$\exists a_1, a_2 \,,\; \alpha > 0 \; \exists U_1, U_2$, such that

$$(H1) \qquad F_i(x) = \frac{-a_i}{|x|^\alpha} + U_i(x) \qquad i = 1,2 \,, \forall x \in \mathbb{R}^3 \setminus \{0\}$$

$$(H2) \qquad \lim_{x \to 0} |x|^{\alpha+2} |\nabla^2 U_i(x)| = 0 \qquad i = 1,2 \,.$$

$$(H3) \qquad \lim_{|x| \to +\infty} \frac{|\nabla U_i(x)|}{|x|} = 0 \qquad i = 1,2 \,.$$

Our main goal is the following Theorem:

THEOREM II.1. *Let x_1, x_2 be given such that (II.3) holds, and let F_1, F_2 satisfy (S), (H1), (H2) and (H3). Then, for every period T, (P) has at least one solution having minimal period T. Moreover*

$$\inf_{\Lambda} I \textit{ is attained in } \Lambda \,. \tag{II.4}$$

PROOF: We first remark that, under our hypotheses, the functional I admits a natural extension to the whole of H; as such, I is coercive (because of the symmetry constraint on the function space) and it is weakly lower semicontinuous. It is clear that, when (II.4) holds, the proof is done. Therefore we assume by tha contrary that (II.4) is false, that is that I admits a minimizer in $H \setminus \Lambda$. Let x be such a minimizer: then x verifies

$$I(x) = \inf_{H} I \,, \qquad x \in H \setminus \Lambda \,. \tag{II.5}$$

We shall make use of the results of the last chapter to find a contradiction. From (II.2) saying that $x \in \partial\Lambda$ is equivalent to

$$\exists \bar{t} \in [0,T] \setminus (x_1 - x_2)^{-1}(0) \,,\ \exists i \in \{1,2\} \,/\, x(\bar{t}) = x_i(\bar{t}) \,. \tag{II.6}$$

Of course we can assume without loss of generality that $i = 1$, so that at the time $\bar{t} \in [0,T] \setminus (x_1 - x_2)^{-1}(0)$ x collides with x_1. Since x_1 and x_2 are continuous functions, $[0,T] \setminus (x_1 - x_2)^{-1}(0)$ is open in $[0,T]$. Moreover, since the measure of the set $\{t \in [0,T] \,/\, x(t) = x_i(t)\}$ $(i = 1,2)$ is zero, we can find an interval $[t_1, t_2] \subset [0,T]$ such that

$$\begin{aligned} &x(t_0) \neq x_1(t_0) \,,\ x(t_1) \neq x_1(t_1) \\ &x(t_0) \neq x_2(t_0) \,,\ x(t_1) \neq x_2(t_1) \\ &\bar{t} \in (t_0, t_1) \\ &|x_1(t) - x_2(t)| \geq C_1 > 0 \,, \qquad ,\forall t \in [t_1, t_2] \,. \end{aligned} \tag{II.7}$$

Of course, x solves the minimization problem

$$\inf_{\substack{y \in H^1([t_1,t_2];\mathbf{R}^3) \\ y(t_0)=x(t_0)\,,\ y(t_1)=x(t_1)}} \left(\int_{t_0}^{t_1} \frac{1}{2}|\dot{y}|^2 - F_1(y - x_1) - F_2(y - x_2) \right) \,. \tag{II.8}$$

Let us denote

$$c(t_0,t_1) = \int_{t_0}^{t_1} \frac{1}{2}|\dot{x}|^2 - F_1(x-x_1) - F_2(x-x_2)$$

then we have

$$\int_{t_0}^{t_1} |\dot{x}|^2 \leq 2c(t_1,t_2)$$

Just by taking a smaller interval (if necessary), we can assume without loss of generality that (II.7) holds in addition to

$$\text{(II.9)} \qquad \left(\sqrt{2c(t_0,t_1)} + \left(\int_0^T |\dot{x}_1|^2\right)^{\frac{1}{2}}\right)(t_1-t_2)^{\frac{1}{2}} \leq \frac{C_1}{2}\,,$$

so that x satisfies

$$\text{(II.10)} \qquad |x(t)-x_2(t)| \geq \frac{C_1}{2}\,, \qquad \forall t \in [t_0,t_1]\,,$$

and the same estimate holds for every minimizer of (II.8).

Now, by the change ov variable $z = y - x_1$, the minimizing (II.8) becomes equivalent to minimize

$$\text{(II.11)} \qquad \begin{aligned} &\inf_{\substack{z\in H^1([t_1,t_2];\mathbb{R}^3)\\ z(t_0)=x(t_0)-x_1(t_0)\\ z(t_1)=x(t_1)-x_1(t_1)}} \Big(\int_{t_0}^{t_1}\left(\frac{1}{2}|\dot{z}|^2 + \frac{1}{2}|\dot{x}_1|^2 - z\cdot\ddot{x}_1\right) \\ &+ \int_{t_0}^{t_1} (-F_1(z) - F_2(z+x_1-x_2)) + x(t_1)\cdot\dot{x}_1(t_1) - x(t_0)\cdot\dot{x}_1(t_0)\Big)\,. \end{aligned}$$

Although the term F_2 is not regular, thanks to (II.9) and (II.10) we can treat it as regular, since every minimizer (or minimizing sequence) can not interact with the set of the singularity of F_2. Thus, from (H1), (H2) and (H3), the potential of the above integral are exactly of the form of the ones of Theorem I.?. Indeed, up to the constant terms, it has the form $\frac{-a_1}{|z|^\alpha} + U_1(z) + z\cdot\ddot{x}_1 + F_2(z+x_1-x_2)$. Now Theorem I.3 actually says that the infimum of (II.11) is attained in the set of the $z \in H^1([t_1,t_2];\mathbb{R}^3)$, such that $z(t_0) = x(t_0) - x_1(t_0)$, $z(t_1) = x(t_1) - x_1(t_1)$ and $z(t) \neq 0$, $\forall t \in [t_0,t_1]$. Now, if z minimizes (II.11), then $y = z + x_1$ minimizes (II.8); therefore $y(t) = x(t)$, $\forall t \in [t_0,t_1]$, and $x(t) \neq x_1(t)$, $\forall t \in [t_0,t_1]$, in contradiction with our assumption that $x(\bar{t}) = x_1(\bar{t})$. Of course the same argument apply for every collision time $\bar{t}$ between x and x_1, or x and x_2. The proof is then complete. ◇

III. A three body problem in $\mathbb{R}^3$

As a three body problem we consider

$$(P_T) \qquad \begin{cases} -m_i\ddot{x}_i = \sum_{\substack{j=1\\ j\neq i}}^{3} \nabla F_{i,j}(x_i - x_j) & i = 1,2,3 \\ x_i(t+T) = x(t) & i = 1,2,3\ ,\ \forall t \\ x_i(t+\frac{T}{2}) = -x_i(t) & i = 1,2,3\ ,\ \forall t \\ x_i(t) \neq x_j(t) & i \neq j\ ,\ i,j = 1,2,3\ ,\ \forall t \in \mathbb{R} \end{cases}$$

where the potentials $F_{i,j} \in C^1(\mathbb{R}^3 \setminus \{0\}; \mathbb{R})$ have a singularity at zero of actractive type. In order to fix your mind, think of the situations of three bodies subjcet to the mutual effect of the universal low of gravitation:

$$-m_i\ddot{x}_i = \sum_{\substack{j=1\\ j\neq i}}^{3} \frac{m_i m_j}{|x_i - x_j|} \qquad i = 1,2,3\ .$$

This problem has a variational structure provided that

$$(H1) \qquad F_{i,j}(x) = F_{j,i}(-x)\ , \qquad \forall x \in \mathbb{R}^3 \setminus \{O\}\ , i,j = 1,2,3\ .$$

In that case, the functional naturally associated to (P) is

$$\text{(III.1)} \qquad I(x) = \frac{1}{2}\int_0^T \left(\sum_{i=1}^{3} m_i|\dot{x}_i|^2 - \sum_{\substack{i\neq j\\ i,j=1}}^{3} F_{i,j}(x_i - x_j) \right)$$

as for the restricted two bodies problem we shall deal with even potential, that is we shall assume that

$$(S) \qquad F_{i,j}(x) = F_{i,j}(-x)\ , \qquad \forall x \in \mathbb{R}^3 \setminus \{0\}\ , i,j = 1,2,3\ ;$$

then the natural domain of I is

$$\text{(III.2)} \quad \Lambda = \{X \in H\ /\ X = (x_1, x_2, x_3)\ ,\ , x_i(t) \neq x_j(t)\ , \forall i \neq j\ , i,j = 1,2,3\ , \forall t \in \mathbb{R}\}$$

Where $H = \{X \in H^1loc(\mathrm{R},\mathrm{R}^9) \ / \ X(t+T) = X(t) \ , X(t+\frac{T}{2}) = -X(t) \ , \forall t \in \mathrm{R}^9\}$. We shall write

$$F(X) = \sum_{\substack{i \neq j \\ i,j=1}}^{3} F_{i,j}(x_i - x_j) \ .$$

Our first result is concerned with locally radial symmetric potentials:

THEOREM III.1. *Let $F_{i,j} \in C^2(\mathrm{R}^3 \setminus \{0\};\mathrm{R})$ satisfy (H1), (S) in addition to*

$$\exists \alpha > 0 \ , \exists a_{i,j} > 0 \ , \text{such that}$$

$$F_{i,j} = \frac{-a_{i,j}}{|x|^\alpha} + U_{i,j}(x) \qquad i,j = 1,2,3 \ , i \neq j$$

$$\exists \varepsilon > 0 \ \exists \phi_{i,j} : (0,\varepsilon] \to \mathrm{R} \text{ of class } C^2 \text{ such that}$$

(H2) $$U_{i,j}(x) = \phi_{i,j}(|x|) \qquad \forall x \ , \ 0 < |x| \leq \varepsilon \ ,$$

(H3) $$\lim_{s \to 0} s^{\alpha+1} |\phi'_{i,j}(s)| = 0 \ .$$

(H4) $$\lim_{|x| \to +\infty} \frac{|\nabla F_{i,j}|}{|x|} = 0 \ .$$

Then, for every period $T > 0$, (P_T) has at least one solution having minimal period T. Moreover

(III.3) $$\inf_{\Lambda} I < \inf_{x \in \partial\Lambda} I \ .$$

We notice that, when $F_{i,j}(x) = F_{j,i}(x) = \frac{-a_{i,j}}{|x|^\alpha}$ then (III.3) holds. Therefore, pinching conditions can be introduced in treating potentials without radial symmetry:

THEOREM III.2. *Let $F_{i,j} \in C^2(\mathrm{R}^3 \setminus \{0\};\mathrm{R})$ satisfy (H1), (S) in addition to*

$$\exists \alpha > 0 \exists a_{i,j} > 0 \ , \exists C > 1 \ , \text{such that}$$

$$\frac{a_{i,j}}{|x|^\alpha} \leq -F_{i,j}(x) \leq \frac{C a_{i,j}}{|x|^\alpha} \ , \qquad \forall x \in \mathrm{R}^3 \setminus \{0\} \ ,$$

Then there exists a function $\Psi(\alpha, \frac{m_i}{m_j}, \frac{a_{i,j}}{a_{i,k}})$ such that, when

(III.4) $$C \leq \Psi(\alpha, \frac{m_i}{m_j}, \frac{a_{i,j}}{a_{i,k}}) \ , \qquad \forall i,j,k = 1,2,3 \ .$$

Then, for every period $T > 0$, (P_T) has at least one solution having T as minimal period. Moreover Ψ enjoys the following properties:

$$\Psi(\alpha, \frac{m_i}{m_j}, \frac{a_{i,j}}{a_{i,k}}) > 1 \;, \forall \alpha > 0 \;,$$
$$\lim_{\alpha \to 2} \Psi(\alpha, \frac{m_i}{m_j}, \frac{a_{i,j}}{a_{i,k}}) = +\infty$$
$$\lim_{\alpha \to 0} \Psi(\alpha, \frac{m_i}{m_j}, \frac{a_{i,j}}{a_{i,k}}) = 1 \;.$$

REMARK: Thinking of Newton's low of gravitation, a remarkable application of the above theorem holds when $F_{i,j}(x) \simeq \frac{a m_i m_j}{|x|^\alpha}$. In that case, the pinching condition reduces to $C \leq \Psi(\alpha, \frac{m_i}{m_j})$.$\diamond$

As direct consequences of Theorem III.2 we obtain the following Corollaries:

COROLLARY III.1. *Assume that $m_1 = m_2 = m_3 = m$, and let $F_{i,j} \in C^2(\mathbb{R}^3 \setminus \{0\}; \mathbb{R})$ satisfy (H1), (S) in addition to*

$$\exists \alpha > 0 \exists a, b > 0 \;, \textit{such that}$$
$$\frac{a}{|x|^\alpha} \leq -F_{i,j}(x) \leq \frac{b}{|x|^\alpha} \;, \qquad \forall x \in \mathbb{R}^3 \setminus \{0\} \;,$$

Then there exists a function $\Psi(\alpha)$ such that, when

$$\frac{b}{a} \leq \Psi(\alpha) \tag{III.5}$$

Then, for every period $T > 0$, (P_T) has at least one solution having T as minimal period. Moreover Ψ enjoys the following properties:

$$\Psi(\alpha) > 1 \;, \forall \alpha > 0 \;,$$
$$\lim_{\alpha \to 2} \Psi(\alpha) = +\infty$$
$$\lim_{\alpha \to 0} \Psi(\alpha) = 1 \;.$$

The following result is just an immediate consequence of Corollary III.1:

COROLLARY III.2. *Let $F_{i,j} \in C^2(\mathbb{R}^3 \setminus \{0\};\mathbb{R})$ satisfy (H1), (S) in addition to*

$$\begin{aligned}
&\exists\alpha > 0\,, \exists a_{i,j} > 0\,, \textit{such that} \\
&F_{i,j} = \frac{-a_{i,j}}{|x|^\alpha} + U_{i,j}(x) \qquad i,j = 1,2,3\,, i \neq j \\
&\lim_{x\to 0} |x|^\alpha |U_{i,j}(x)| = 0 \qquad i,j = 1,2,3\,, i \neq j
\end{aligned}$$

Then, for every $T > 0$, (P_T) has infinitely many solutions.

III.1 Proof of the results

PROOF OF THEOREM III.1: We first observe that the associated functional I admits a natural extension to the whole of H, just by taking the expression in (III.1) if it is finite, and $+\infty$ otherwise. As such, I is weakly lower semicontinuous and, from (H4), it is coercive. Now we assume by sack of absurdity that $\inf_\Lambda I = \inf_{\partial\Lambda} I = \inf_H I$. Then I admits a point of minimum $X = (x_1, x_2, x_3) \in \partial\Lambda$ such that

$$I(X) = \inf_\Lambda I = \inf_{\partial\Lambda} I = \inf_H I$$

From (III.2), $X \in \partial\Lambda$ just means that

$$\exists\bar{t} \in [0,T]\,, \exists i,j \in \{1,2,3\}\,, i \neq j\,, x_i(\bar{t}) = x_j(\bar{t}) \tag{III.1.1}$$

We say that $\bar{t}$ is a time of *double collision* if $x_k(\bar{t}) \neq x_i(\bar{t}) = x_j(\bar{t})$, for $k \neq i$, $k \neq j$, while $\bar{t}$ is a time of *triple collision* if $x_k(\bar{t}) = x_i(\bar{t}) = x_j(\bar{t})$, for $k \neq i$, $k \neq j$. We are going to treat separately the two cases.

We first prove the following result:

PROPOSITION III.1. *Assume that (H2) (H3) hold, and let X be such that $I(x) = \inf_H I$. Then X can not have double collisions.*

PROOF: We assume by the contrary that there exist $i \neq j$ such that $x_i(\bar{t}) = x_j(\bar{t})$ and $x_k(\bar{t}) \neq x_i(\bar{t})$ for $k \neq i$, $k \neq j$, say $x_2(\bar{t}) = x_3(\bar{t})$ and $x_1(\bar{t}) \neq x_2(\bar{t})$.

We observe that x_3 is a minimizer of the functional

$$I_{x_1,x_2}(x_3) = \int_0^T \frac{1}{2}|\dot{x}_3|^2 - F_{1,3}(x_3 - x_1) - F_{2,3}(x_3 - x_2)$$
$$+ \int_0^T \frac{1}{2}\left(|\dot{x}_1|^2 + |\dot{x}_2|^2\right) - F_{1,2}(x_1 - x_2)$$

over $\tilde{H} = \{x_3 \in H^1_{loc}(\mathrm{R};\mathrm{R}^3) \ / \ x_3(t+T) = x_3(t) \ , \ x_3(t+\frac{T}{2}) = -x_3(t) \ , \forall t\}$. We can then apply the results of Theorem II.1 (see also Theorem I.3) to prove that $\min_{\tilde{H}} I_{x_1,x_2}(x_3)$ is attained by a function x_3 which do not cross neither x_1, or x_2 for all $\bar{t}$ such that $x_1(\bar{t}) \neq x_2(\bar{t})$. Since this argument works for every instant of double collision, we can conclude that the minimizer X can just have triple collisions.◇

Next proposition will end the proof of Theorem III.1:

PROPOSITION III.2. *Assume that (H2) (H3) hold, and let X be such that $I(X) = \inf_X I$. Then X is free of triple collision. Moreover*

$$\inf_\Lambda I < \inf_{x \in \partial\Lambda} I$$

The proof of Proposition III.2 will be given in the next section. Now we turn to the proof of Theorem III.2 and the related results.◇

PROOF OF THEOREM III.2: By the change of variables $x_i(t) = \left(T^2 \frac{a_{i_0,k_0}}{m_{j_0}}\right)^{\frac{1}{2+\alpha}} v_i(\frac{t}{T})$ we easily prove that

$$\inf_\Lambda \frac{1}{2}\int_0^T \left(\sum_{i=1}^{3} m_i|\dot{x}_i|^2 + \sum_{\substack{i\neq j\\ i,j=1}}^{3} \frac{a_ij}{|x_i - x_j|^\alpha}\right)$$
$$= \left(T^{2-\alpha} a^2_{i_0,k_0} m^\alpha_{j_0}\right)^{\frac{1}{2+\alpha}} \inf_\Lambda \frac{1}{2}\int_0^1 \left(\sum_{i=1}^{3} \frac{m_i}{m_{j_0}}|\dot{x}_i|^2 + \sum_{\substack{i\neq j\\ i,j=1}}^{3} \frac{a_ij}{a_{i_0,k_0}|x_i - x_j|^\alpha}\right)$$

and

$$\inf_{x\in\partial\Lambda} \frac{1}{2}\int_0^T \left(\sum_{i=1}^{3} m_i|\dot{x}_i|^2 + \sum_{\substack{i\neq j\\ i,j=1}}^{3} \frac{a_ij}{|x_i - x_j|^\alpha} \right)$$

$$= \left(T^{2-\alpha} a_{i_0,k_0}^2 m_{j_0}^\alpha\right)^{\frac{1}{2+\alpha}} \inf_{x\in\partial\Lambda} \frac{1}{2}\int_0^1 \left(\sum_{i=1}^{3} \frac{m_i}{m_{j_0}}|\dot{x}_i|^2 + \sum_{\substack{i\neq j\\ i,j=1}}^{3} \frac{a_ij}{a_{i_0,k_0}|x_i - x_j|^\alpha} \right) .$$

Therefore, the right condition on C is that

$$C^{\frac{2}{2+\alpha}} \leq \Psi(\alpha, \frac{m_i}{m_j}, \frac{a_{i,j}}{a_{i,k}})$$

where Ψ is defined as

$$\Psi(\alpha, \frac{m_i}{m_j}, \frac{a_{i,j}}{a_{i,k}}) =$$

$$\max_{i_0,j_0,k_0} \left(\frac{\inf_{x\in\partial\Lambda} \frac{1}{2}\int_0^1 \left(\sum_{i=1}^3 \frac{m_i}{m_{j_0}}|\dot{x}_i|^2 + \sum_{\substack{i\neq j\\ i,j=1}}^3 \frac{a_ij}{a_{i_0,k_0}|x_i-x_j|^\alpha} \right)}{\inf_{x\in\Lambda} \frac{1}{2}\int_0^1 \left(\sum_{i=1}^3 \frac{m_i}{m_{j_0}}|\dot{x}_i|^2 + \sum_{\substack{i\neq j\\ i,j=1}}^3 \frac{a_ij}{a_{i_0,k_0}|x_i-x_j|^\alpha} \right)} \right)$$

$\diamond$

PROOF OF COROLLARY III.2: From the above discussion one easily sees that, under the assumptions of Theorem III.2, a constant C_1 (depending on the masses and the values of the $a_{i,j}$'s) exists such that the minimizer X found by the application of Theorem III.1 satisfies $\|\dot{X}\|_{L^2} \leq C_1 T^{\frac{2-\alpha}{2+\alpha}}$. One then concludes just by taking a small T.$\diamond$

II.2. Proof of Proposition III.1.2

The purpose of this section is to show that a minimizer of I can not have any triple collision. This fact implies that $\inf_\Lambda I < \inf_{\partial\Lambda} I$, since we have seen in Proposition III.1.1 that all minimizers are free of double collisions.

We are going to accomplish our goal by means of a main estimate based on a series of preliminary lemmas. Some of these lemmas are almost immediate extensions of what is classically known as Sundman's Theorem. This theorem states that in the classical three-body problem a triple collision orbit has zero angular momentum. The proof of Sundman's Theorem can be found in [**33**]. Since our arguments start by following the line proposed there, we shall prove only the results which are not trivial extensions of those in [**33**], otherwise we shall refer to it for proofs.

Since we are interested in the minima of the action integral, of course there is an instant of triple collision which is isolated in $[0, T]$ and therefore the motion can be assumed to be regular in a neighborhood of that instant. Although we are going to prove that each minimum is free of isolated triple collision, by a density argument, one easily concludes that all minimizers are free of triple collisions.

Definition III.2.1. *We set, for a generic orbit* $X = (x_1, x_2, x_3)$

$$T = T(\dot{X}) = \sum_{i=1}^{3} m_i |\dot{x}_i|^2 , \qquad \text{and} \qquad G = G(X) = \sum_{i=1}^{3} m_i |x_i|^2 .$$

The following Proposition makes precise the kind of orbits we are dealing with.

Proposition III.2.1. *Suppose* $X \in H$ *is such that* $I(X) = \inf_H I$*; then*
i) X *is free of double collisions.*
ii) X *has at most two triple collisions, both at zero.*
iii) $\exists\ E \in \mathrm{R}$ *such that* $T - F = E$ *(conservation of energy).*
iv)$\sum_{i=1}^{3} m_i x_i = \sum_{i=1}^{3} m_i \dot{x}_i = 0, \quad \forall t \in [0, T]$ *(conservation of center of mass and momentum).*
v) Suppose that $\forall t \in (a, b)$, $\forall i \neq j$, $|x_i(t) - x_j(t)| \leq \varepsilon$.
Then $\exists B_0 \in \mathrm{R}^3$ *such that* $\forall t \in (a, b)$,

$$B := \sum_{i=1}^{3} m_i \dot{x}_i(t) \times x_i(t) = B_0$$

(conservation of angular momentum for small distances).

Proof: We set $I(X) = \int_0^T L(X, \dot{X}) dt$, and $I_{[a,b]}(X) = \int_a^b L(X, \dot{X}) dt$.
We recall that the equations of motion associated to I are

$$-m_i\ddot{x}_i = \sum_{\substack{j=1\\j\neq i}}^{3} \nabla_{x_i} F_{i,j}(x_i - x_j).$$

i) See Proposition III.1.1

ii)Suppose that X has more than two triple collisions. It is not restrictive to assume that $t = 0$ and $t = T_1$ are instants of triple collision such that X solves the equations of motion in $(0, T_1)$ and moreover

$$I_{[0,T_1]}(X) \leq \frac{2T_1}{T} I(X)$$

Setting $V(t) = X(t) - \frac{\sum m_i x_i}{\sum m_i}$ we have that V has only triple collisions at zero and $I(V) \leq I(X)$ (strictly if the center of mass of X is not zero).

Let $U(t) = V(\frac{2T_1}{T}t)$ for $t \in [0, \frac{T}{2}]$ (and symmetrically in $[\frac{T}{2}, T]$); then $U \in H$ and

$$I(U) = \frac{T}{2T_1} I_{[0,T_1]}(V) + (\frac{2T_1}{T} - \frac{T}{2T_1}) \sum_{i=1}^{3} m_i \int_0^{T_1} |\dot{v}_i|^2 dt \leq \frac{T}{2T_1} I_{[0,T_1]}(V) \leq I(X),$$

(and the strict inequality holds when X has more than two triple collisions). This is a contradiction.

iii)Since F is a function of the x_i's only, it follows immediately that if X is a solution of the equations of motion, then

$$\exists E \in \mathbb{R}\text{ , such that}$$

(III.2.1) $$T - F = E\,.$$

iv) By summing the three equations of motion one obtains

$$\sum_{i=1}^{3} m_i \ddot{x}_i = \sum_{\substack{i,j=1\\i\neq j}}^{3} \nabla F_{i,j}(x_i - x_j);$$

but since, by (H1)

$$\nabla F_{i,j}(x_i - x_j) = -\nabla F_{j,i}(x_j - x_i),$$

we have $\sum_{i=1}^{3} m_i \ddot{x}_i = 0$, which implies $\sum_{i=1}^{3} m_i x_i = ct + d$, for some constants $c, d \in \mathbb{R}^3$. By periodicity, $c = 0$, and by the symmetry condition (S), $d = 0$, which means that both the center of mass $\sum_{i=1}^{3} m_i x_i$ and the linear momentum $\sum_{i=1}^{3} m_i \dot{x}_i$ are identically zero throughout the motion.

v)Indeed, $\forall t \in (a, b)$, by virtue of (H3), we have

$$m_i\ddot{x}_i \times x_i = \sum_{\substack{j=1\\ j\neq i}}^{3} \alpha m_i m_j \frac{x_j \times x_i}{|x_i - x_j|^{\alpha+2}} + \phi'_{i,j}(|x_i - x_j|)\frac{x_i \times x_j}{|x_i - x_j|} \, .$$

Summing over i one sees that the j-th term of the i-th equation cancels the i-th term of the j-th equation. Therefore

$$\dot{B} = \frac{d}{dt}\sum_{i=1}^{3} m_i \dot{x}_i \times x_i = \sum_{i=1}^{3} m_i \ddot{x}_i \times x_i = 0 \, , \qquad \forall t \in (a,b)$$

and B is constant in (a,b).

Now we state some preliminary lemmas.

LEMMA III.2.1. *$\forall \gamma > 0$, $\exists \sigma_\gamma > 0$ such that, if $|x_i| \leq \sigma_\gamma$ $i = 1,2,3$, then*

$$\sum_{i=1}^{3} x_i \frac{\partial F}{\partial x_i} \geq -(\alpha + \gamma) F \, . \tag{III.2.2}$$

PROOF: We remark that equality holds in (III.2.2), with $\gamma = 0$, if F is homogeneous of degree $-\alpha$; this is the case, for example, of the classical three body problem ($\alpha = 1$).

In our case it is enough to compute (recall (H3))

$$\sum_{i=1}^{3} x_i \frac{\partial F}{\partial x_i} = \sum_{\substack{i,j=1\\ i<j}}^{3} -\alpha \frac{m_i m_j}{|x_i - x_j|^\alpha} + \phi'_{i,j}(|x_i - x_j|)|x_i - x_j|$$

and to see that, for σ_γ small, one has

$$\begin{aligned} &\sum_{i=1}^{3} x_i \frac{\partial F}{\partial x_i} + (\alpha + \gamma) F \\ &= \sum_{\substack{i,j=1\\ i<j}}^{3} \frac{1}{|x_i - x_j|^\alpha} \{\gamma m_i m_j + \phi'_{i,j}(|x_i - x_j|)|x_i - x_j|^{\alpha+1} + \\ &+ (\alpha + \gamma)\phi_{i,j}(|x_i - x_j|)|x_i - x_j|^\alpha\} \;\geq\; 0 \end{aligned}$$

thanks to the growth assumption (H3).

where μ will be conveniently chosen later; extend $\overline{w}$ to $\mathbb{R}$ by periodicity and define

$$\overline{\overline{w}}(t) = \overline{w}(t) - \overline{w}(t + \frac{T}{2}) .$$

Finally let $V \in H$ be the function

$$V(t) = (v_1(t), v_2(t), v_3(t)) = (\overline{\overline{w}}(t), 0, -\overline{\overline{w}}(t)) \ :$$

it is clear that $v_j(t) + x_j(t) \neq v_k(t) + x_k(t)$, $\forall j \neq k$, $\forall t \in [-\delta, \delta]$ and moreover that $v_k(t) \cdot x_j(t) = 0$, $\forall j, k$, $\forall t \in \mathbb{R}$.

We are going to show that $I(X + V) < I(X)$, contradicting the fact that $I(X)$ is the infimum of I over Λ.

We start by remarking that by symmetry and by the choice of V,

$$I(X+V) - I(X) = 2\{\sum_{i=1}^{3} \frac{m_i}{2} \int_{-\delta}^{\delta} |\dot{x}_i + \dot{v}_i|^2 - |\dot{x}_i|^2 dt +$$
$$+ \frac{1}{2} \sum_{\substack{i,j=1 \\ j \neq i}}^{3} \int_{-\delta}^{\delta} F_{i,j}(x_i + v_i - x_j - v_j) - F_{i,j}(x_i - x_j) dt\} .$$

The estimate of the kinetic part leads to

$$\sum_{i=1}^{3} m_i \int_{-\delta}^{\delta} |\dot{x}_i + \dot{v}_i|^2 - |\dot{x}_i|^2 dt = \sum_{i=1}^{3} m_i \int_{-\delta}^{\delta} |\dot{v}_i|^2 dt \leq C_1 \mu^2 , \tag{III.2.4}$$

where $C_1 > 0$ is independent of μ.
Therefore

$$I(X+V) - I(X) \leq$$
$$\leq 2\{\frac{1}{2} \sum_{\substack{i,j=1 \\ j \neq i}}^{3} \int_{-\delta}^{\delta} F_{i,j}(x_i + v_i - x_j - v_j) - F_{i,j}(x_i - x_j) dt\} + C_1 \mu^2 . \tag{III.2.5}$$

We want to show that if μ is taken sufficiently small, then this quantity is strictly negative. To do this, we can certainly assume that δ and μ are so small that the potential takes the form given by (H3) throughout the interval $[-\delta, \delta]$. In other words, if we set

$$T_{i,j} = \int_{-\delta}^{\delta} (\frac{m_i m_j}{|x_i + v_i - x_j - v_j|^\alpha} + \phi_{i,j}(|x_i + v_i - x_j - v_j|) -$$
$$- \frac{m_i m_j}{|x_i - x - j|^\alpha} - \phi_{i,j}(|x_i - x_j|)) dt,$$

(III.2.5) becomes

$$\text{(III.2.6)} \qquad I(X+V)-I(X) \le \sum_{\substack{i,j=1\\ j\neq I}}^{3} T_{i,j} + C_1\mu^2 \, .$$

Now we estimate the generic term $T_{i,j}$ of (III.2.5):

$$\begin{aligned} T_{i,j} &= \int_{-\delta}^{\delta}\int_0^1 \frac{d}{d\lambda}\Big(\frac{m_i m_j}{|x_i - x_j + \lambda v_i - \lambda v_j|^\alpha} + \phi_{i,j}(|x_i - x_j + \lambda v_i - \lambda v_j|)\Big) d\lambda dt \\ &= \int_{-\delta}^{\delta}\int_0^1 -\frac{\lambda |v_i - v_j|^2}{|x_i - x_j + \lambda v_i - \lambda v_j|^{\alpha+2}} \\ &\cdot\big(\alpha m_i m_j - \phi'_{i,j}(|x_i - x_j + \lambda v_i - \lambda v_j|)|x_i - x_j + \lambda v_i - \lambda v_j|^{\alpha+1}\big) d\lambda dt \, . \end{aligned}$$

Now for μ and δsmall enough, the quantity in the round brackets is larger or equal than $\frac{1}{2}\alpha m_i m_j$, because of (H3). Hence

$$\begin{aligned} T_{i,j} &\le \frac{1}{2}\int_{-\delta}^{\delta}\int_0^1 -\frac{\lambda |v_i - v_j|^2}{|x_i - x_j + \lambda v_i - \lambda v_j|^{\alpha+2}} d\lambda dt \le \\ &\le \frac{\alpha m_i m_j}{2}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}}\int_0^1 -\frac{\lambda \mu^2}{|x_i - x_j + \lambda v_i - \lambda v_j|^{\alpha+2}} d\lambda dt \le \\ &\le \frac{\alpha m_i m_j}{4}\mu^2 \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}}\int_{\frac{1}{2}}^1 \frac{-1}{|x_i - x_j + \lambda v_i - \lambda v_j|^{\alpha+2}} d\lambda dt \, . \end{aligned}$$

Now for δsmall enough, by Lemma III.2.4, there exists $C_2 > 0$ such that

$$|x_i| \le C_2 t^{\frac{2}{\alpha+\gamma+2}} \, , \qquad i = 1, 2, 3 \, ,$$

for some $\gamma > 0$, $\alpha+\gamma < 2$; therefore we have $|x_i - x_j| \le \mu$, for all t such that $2C_2 t^{\frac{2}{\alpha+\gamma+2}} \le \mu$.

Restricting the interval of integration to

$$[-\sigma, \sigma] := [-\left(\frac{\mu}{2C_2}\right)^{\frac{\alpha+\gamma+2}{2}}, \left(\frac{\mu}{2C_2}\right)^{\frac{\alpha+\gamma+2}{2}}],$$

we obtain

$$T_{i,j} \le \frac{\alpha m_i m_j}{4}\mu^2 \int_{-\sigma}^{\sigma}\int_{\frac{1}{2}}^1 \frac{-1}{3^{\alpha+2}\mu^{\alpha+2}} d\lambda dt \le -C_3 \mu^{\frac{2-\alpha+\gamma}{2}} \, ,$$

where C_3 is a positive constant independent of μ.
Summing the $T_{i,j}$'s and recalling (III.2.4) we obtain

LEMMA III.2.2. *Let $t = 0$ be an isolated triple collision instant for X. Then there exists $\tilde{t} > 0$ such that*

$$\forall t \in (-\tilde{t}, \tilde{t}) \setminus 0, \qquad B(t) = 0.$$

PROOF: By the conservation of center of mass, the triple collision can occur only at zero. Moreover we can assume that X is regular of class C^2 in $(-\delta, \delta) \setminus \{0\}$, for some $\delta > 0$. From now on the proof works exactly like that of the classical Sundman's Theorem reported in [**33**], pag 26. The only difference is that in our case one has to use the inequality

$$\frac{1}{2}G'' \geq (2 - \alpha - \gamma)T + (\alpha + \gamma)E \,. \tag{III.2.3}$$

This inequality (which is obtained by differentiating twice G and by using $T - F = E$) holds $\forall \gamma > 0$ such that $\alpha + \gamma < 2$ if $|t|$ is small enough. We omit further details.

LEMMA III.2.3. *Let X solve the equations of motion in (a, b) and suppose $B(t) = 0$,$\forall t \in (a, b)$.*
Then the components of X, (x_1, x_2, x_3), lie on the same fixed plane of $\mathbb{R}^3$ for all $t \in (a, b)$.

PROOF: See [**33**], pag 28.

Adding the fact that X is a minimum of the action integral, we can prove the following

PROPOSITION III.2.2. *Suppose (H1)-(H3) hold and let $X \in H^1$ be such that $I(X) = \inf_\Lambda I$. Assume that an isolated triple collision takes place at zero at time $t = 0$.*
Then there exists $\tilde{X} \in H^1$ such that $I(\tilde{X}) = I(X)$ and $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ lie on the same fixed plane of $\mathbb{R}^3$ for all $|t| < \delta$, with $\delta > 0$ small enough.

PROOF: By Lemma III.2.2 we can assume $B(t) = 0$ in $(-\delta, 0) \cup (0, \delta)$ for some $\delta > 0$. By Lemma III.2.3 X is planar both in $(-\delta, 0)$ and in $(0, \delta)$, but the two planes need not coincide.
Set $I(X) = \int_0^T L(X, \dot{X})dt$; it is not restrictive to assume that

$$\int_0^{\frac{T}{4}} L(X, \dot{X})dt \leq \int_{\frac{T}{4}}^{\frac{T}{2}} L(X, \dot{X})dt,$$

so that by the symmetry of X it follows

$$\int_{\frac{T}{2}}^{\frac{3T}{4}} L(X, \dot{X})dt \leq \int_{\frac{3T}{4}}^{T} L(X, \dot{X})dt.$$

Now consider the orbit defined by

$$\tilde{X}(t) = \begin{cases} X(t) & \text{if } t \in [0, \frac{T}{4}] \cup [\frac{T}{2}, \frac{3T}{4}] \\ -X(-t) & \text{otherwise in } [0, T] \, ; \end{cases}$$

it is immediate to check that $\tilde{X} \in H^1, I(\tilde{X}) \leq I(X)$, and $\tilde{X}(-s) = -\tilde{X}(s)$ for all s small enough, which means that $\tilde{X}$ minimizes I and enters and leaves the collision on the same plane.

Next lemma provides an important estimate of the behavior of the bodies when approaching or leaving a triple collision. The fundamental ideas to prove this lemma are all contained in [**33**], where it is actually proved for the keplerian case.

LEMMA III.2.4. *Let (H1)-(H3) hold and consider an isolated triple collision for X at time $t = 0$.*
Then $\forall \gamma > 0$ such that $\alpha + \gamma < 2$, $\exists C > 0$ such that

$$|x_i(t)| \leq t^{\frac{2}{2+\alpha+\gamma}}, \qquad \forall i = 1,2,3 \text{ and } \forall t \text{ small enough.}$$

PROOF: This proof too works like that of [**33**], pag.69, for the keplerian case. We just add that one has to estimate $\frac{d}{dt} G^{\frac{4}{2+\alpha+\gamma}}$ instead of $\frac{d}{dt} G^{\frac{4}{3}}$ and to use (III.2.3) instead of $\frac{1}{2} F'' = T + E$. Otherwise no change is necessary.

Having gathered the estimates we need, we can now prove the main result of this section. We recall that if X minimizes I, then it is free of double collision; we now are in a position to show that X can not have any triple collision either.

PROPOSITION III.2.3. *Assume that (S), (H1), (H2), (H3) hold and let X such that $I(X) = \min_\Lambda I$. Then X can not have any triple collision.*

PROOF: Let X be such a minimizer and suppose that X has an isolated triple collision at time $t = 0$. By Proposition 1 we can replace X by another minimizer (still denoted by X) with the further property that X is planar in $[-\delta, \delta]$, for some $\delta > 0$. Therefore, there exists $w \in S^2$ such that

$$x_j(t) \cdot w = 0 \, , \qquad \forall j = 1,2,3 \, , \forall t \in [-\delta, \delta] \, .$$

Let $\overline{w}$ be the piecewise liner function defined by

$$\overline{w}(t) = \begin{cases} 0 & \text{if } t \notin [-\delta, \delta] \\ \mu w & \text{if } t \in [-\frac{\delta}{2}, \frac{\delta}{2}] \, , \end{cases}$$

$$I(X+V) - I(X) \leq C_1\mu^2 - C_4\mu^{\frac{2-\alpha+\gamma}{2}}$$

and, since $0 < \frac{2-\alpha+\gamma}{2} < 2$, we have that $I(X+V) - I(X) < 0$, when μ is small enough, contradicting the fact that $I(X) = \inf_\Lambda I$.

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