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OTHER RESEARCH LINES

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A chi dovesse leggere

Se le pagine di questo libro ammettono qualche verso felice, voglia perdonarmi il lettore la sgarbería di averlo usurpato io, anticipatamente. Le nostre quisquilie differiscono poco; ordinaria e fortuita è la circostanza che tu sia il lettore di questi esercizi, e che io ne sia l'estensore.

(From *Elogio de la sombra,* J.L. Borges.)

ABSTRACT

We investigate the recuperation of expected invariant behaviours in a pregeometric gravity theory in which the fundamental degrees of freedom are fermions ψ_i and the spin connection ω_μ^{ab}, and the full general relativistic invariance is broken spontaneously. We show how dangerous increasing energy behaviours of physical amplitudes cancel in a highly nontrivial way. This evidences the expected loss of the vacuum generated scale in the UV regime and gives support for the consistency of spontaneously broken gravity theories.

This class of theories are characterized by a generic "potential" which depends on $\bar{\psi}\cdot\psi$. We find that the requirement of the additional local symmetry: $\psi(x) \to \lambda(x)\psi(x)$, uniquely determines it. The resulting action is

$$S_{\rm TG} = A \int d^4x \; \frac{\det \frac{i}{2}(\bar{\psi}\gamma^a \overrightarrow{D}_\mu \psi - \bar{\psi}\overleftarrow{D}_\mu \gamma^a \psi)}{(\bar{\psi}\cdot\psi)^4} \quad , \quad A = \text{const.}$$

We argue that this action defines a consistent (UV well-behaved (plausibly finite), unitary, etc.) quantum field theory which describes Einstein gravity as a spontaneous symmetry breaking of conformal invariance.

Then we study the analogous theories in two-dimensional space-time. We find that in 2d the connection with gravity theories can be established exactly. Within this framework, we show that the Thirring model coupled to gravity can be derived from a theory in which the underlying Lagrangian is zero (modulo topological invariants). This result is also interesting for our sake since the Thirring model coupled to gravity is the precise 2d analog of our (spontaneously broken) conformal invariant 4d theory.

I. INTRODUCTION

Probably inspired by a previous idea of Zel'dovich [1], long ago Sakharov [2] suggested that the Einstein-Hilbert action may be not a fundamental action, but rather an effective action induced by quantum structure. This idea influenced a number of authors, who attempted to generate the graviton as a composite [3-8]. Akama *et al* [5] constructed a theory in which no fundamental vierbein is assumed, and the only fundamental degrees of freedom are fermions. Amati and Veneziano [7] extended this idea by providing, through the inclusion of a potential, a mechanism for spontaneous symmetry breaking of diffeomorphisms and local Lorentz symmetries. In a somewhat different context, Tomboulis [9] realized that the (massless-) fermion quantum fluctuations induce an R^2-type gravity action and hence [10] lead to a power counting renormalizable and, through the implementation of the Lee-Wick mechanism [11], unitary (ghost-free) theory.

The theory discussed here is very similar to that of ref. [7], but not exactly the same. Instead of introducing the spin connection as a composite field of fermions, it is defined as an independent field, perhaps for the sake of simplicity, or (and this reason may be implied by the former) just because the theories of spin 1-particles -unlike spin 2- are perfectly consistent and therefore it makes sense to speak about fundamental spin-1 particles. In any case, as far as the essential conclusions are concerned, we do not expect any change for a theory with a composite spin connection.

Among the vast number of possible models of this kind, we will find that the requirement of conformal invariance selects only one. This model is interesting because also Weyl invariance is spontaneously broken (there is higgs mechanism, etc.) and the low energy theory still reproduces Einstein theory of gravity. This is a surprise since the spontaneous generation of a Planck mass is unexpected in a Weyl invariant theory (it might arise through Weyl anomaly, but in $D = 4$ it starts at 2-loop level [12]). On the other hand, there had been hitherto no way to eliminate the ghosts from conformal gravity. In the present case, the original action is *linear* in time derivatives, so the troubles characteristic of higher derivative theories should be absent in physical quantities. Indeed, the R^2 terms induced by integrating out fermions do not introduce ghosts, as remarked by Tomboulis [9].

Pregeometric theories of gravity might provide a suitable framework to describe the unbroken phase of quantum gravity, which in the usual treatment corresponds to an untractable $g_{\mu\nu} = 0$ classical value [13,14]. The general idea is that at short distances the theory possesses full gauge symmetry implemented by a non metric action which is broken spontaneously by a vacuum induced length beyond which a space-time metric emerges, with the dynamics dictated by the Einstein-Hilbert action. The gauge symmetry includes at least diffeomorphisms and local Lorentz invariance required by general relativity, and may be larger, as proposed in specific approaches (see, e.g. [15]) or as it is conjectured for superstring theories [12,16].

There is a general consensus that the symmetric theory should have a good UV behaviour and that this should not be spoiled by the spontaneous breaking. But while this is proven correct for YM gauge theories where both symmetric and spontaneously broken realizations are renormalizable, the expectation is much less firmly grounded for theories including gravity.

As well known, the Einstein-Hilbert action, when treated as a fundamental quantum action, leads to a nonrenormalizable quantum field theory and to scattering amplitudes with unitary-violating increasing energy behaviour. Here we will show in that the present theory, for energies much greater than the vacuum generated scale (associated to the Planck mass), scale invariance is restored; highly non-trivial cancellations occur confirming the expected good behaviour of the theory and the renormalizability by naive power counting. This is much the same of what happens in spontaneously broken YM theories where scalar particles, with couplings exactly as in the Higgs model, are required to cancel the unwanted increasing energy behaviour of certain tree-level 4-point amplitudes [17,18]. Indeed, we shall show how "tree" 4-point amplitudes satisfy this unitary bound in accordance with the "good" scaling behaviour.

Masses, couplings and propagators of all states are explicitly predicted by the theory and consistently computable in a loop ($1/N$) expansion. The correct high energy behaviour found at the leading order is thus a clear sign of the expected consistency of the theory in which Einstein gravity arises as a low energy aproximation.

Finally, we will move on two dimensions, where it is possible to analyse the theory in the unbroken phase and, as we will see, the connection between these induced gravity models and gravity can be established exactly. Some unexpected issues will be met.

II. SPONTANEOUS SYMMETRY BREAKING OF GENERAL COVARIANCE

II.1 DEFINITION OF THE CLASSICAL THEORY

Let us take as fundamental degree of freedom 1 Dirac fermion and the $SO(3,1)$ spin connection $\omega_{\mu,ab}$. The invariances are postulated to be diffeomorphisms and local Lorentz transformations [a]. The starting action is [5,7]

$$S = \int d^4x \;\; \det W \;\; V(\bar{\psi}\psi) \; , \tag{II.1}$$

$$W^a_\mu \equiv \frac{i}{2}(\bar{\psi}\gamma^a \overrightarrow{D}_\mu \psi - \bar{\psi}\overleftarrow{D}_\mu \gamma^a \psi) \; , \quad D_\mu = \partial_\mu + \sigma^{ab}\omega_{\mu,ab} \; .$$

Under $SO(3,1)$ infinitesimal local transformations of parameter τ^{ab}, the fields change according to

$$\psi(x) \longrightarrow (1 - \frac{i}{4}\sigma_{ab}\tau^{ab}(x))\psi(x)$$

$$\bar{\psi}(x) \longrightarrow \bar{\psi}(x)(1 + \frac{i}{4}\sigma_{ab}\tau^{ab}(x)) \tag{II.2}$$

$$\omega_{\mu,ab} \longrightarrow \omega_{\mu,ab} + \frac{i}{4}\partial_\mu \tau_{ab} + \frac{i}{2}(\tau^{ac}\omega_{\mu,cb} - \tau^{bc}\omega_{\mu,ca})$$

Under diffeomorphisms, ψ transforms as a scalar and the spin connection transforms as a vector.

Remark: It is possible to add other terms like

$$I = \int d^4x \; g(\bar{\psi}\psi)\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}W^a_\mu W^b_\nu R^{cd}_{\rho\sigma}(\omega) + \alpha(\bar{\psi}\psi)\epsilon^{\mu\nu\rho\sigma}R^{ab}_{\mu\nu}R_{\rho\sigma,ab} + \text{etc.} \; , \tag{II.3}$$

[a] As described in [7], further gauge symmetries can also be incorporated. They are inessential to the present research and thus we will not consider them here.

while preserving diffeomorphisms and local Lorentz invariance. We will not attempt the introduction of these additional terms •

It is straightforward to extend the single spinor case by introducing N fermion replicas in order to define a $1/N$ expansion. To explore the spectrum of the theory and the low energy effective theory we introduce composite operators[b] and corresponding Lagrange multipliers in such a way that fermions can be integrated out (we follow the approach used in ref.[7]):

$$S' = \int d^4x \det W[V(\rho) - \xi^{\mu a} W_{\mu a} + \phi\rho + \sum_{i=1}^{N} (\frac{i}{2}\xi^{\mu a}\bar{\psi}_i(\gamma_a \overrightarrow{D}_\mu - \overleftarrow{D}_\mu \gamma_a)\psi_i - a_i \phi \bar{\psi}_i \cdot \psi_i] \ . \quad (II.4)$$

Note that ψ_i, $i = 1, ...N$ are N adimensional spinor fields and the relative weights a_i with which each of them contributes to the composite ρ and W represent N adimensional parameters.

II.2 $(\bar{\psi}\psi)^{-4}$ MODEL

The potential V can be an arbitrary function of $\bar{\psi}\psi$; the requirement of diffeomorphism and local $SO(3,1)$ invariances gives no restriction on V. However, if we require an additional gauge invariance

$$\psi \to e^{-\sigma/2}\psi \quad , \quad \bar{\psi} \to e^{-\sigma/2}\bar{\psi} \quad , \quad \omega_\mu^{ab} \to \omega_\mu^{ab} \ , \qquad (II.5)$$

the potential V is uniquely fixed and the resulting theory is defined by the action

$$S_{\rm TG} = A \int d^4x \ \frac{\det W}{(\bar{\psi} \cdot \psi)^4} \quad , \quad A = {\rm const.} \qquad (II.6)$$

As we shall see later, the transformation (II.5) exactly corresponds to Weyl invariance in the induced gravity theory. In sect. V we will see that this theory is the precise 4D analog of the Thirring model coupled to gravity (hence the notation $S_{\rm TG}$). We will refer to this model as the $(\bar{\psi}\psi)^{-4}$ model or simply 4D TG theory.

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[b] The formal definition of such operators as the product of Dirac fields is ambiguous, just as in standard 4d theories. Usually these ambiguities are eliminated by demanding that the composite operators satisfy the proper Ward identities.

The gauge invariance (II.5) also constrains the σ-model (in N-dimensional space) couplings in (II.3) to be given by $g = B/(\bar{\psi}\psi)^2$ and $\alpha =$const., etc.

For $V(\bar{\psi}\psi) = A/(\bar{\psi}\cdot\psi)^4$, S' has correspondingly the additional local symmetry

$$\psi_i \to e^{-\sigma/2}\psi_i \ , \quad \bar{\psi}_i \to e^{-\sigma/2}\bar{\psi}_i \ , \quad W_{\mu a} \to e^{-\sigma}W_{\mu a} \ , \quad \omega_\mu^{ab} \to \omega_\mu^{ab} \ ,$$
$$\xi^{\mu a} \to e^{5\sigma}\xi^{\mu a} \ , \quad \rho \to e^{-\sigma}\rho \ , \quad \phi \to e^{5\sigma}\phi \ . \qquad (II.7)$$

II.3 VACUUM PROPERTIES

Let V be a generic potential. The theory can now be studied within the $1/N$ expansion provided by the fermion loops. Expanding the bosonic fields around their VEVs, the fermion loops provide linear terms in the fluctuations whose vanishing conditions determine the vacuum. Likewise, fermion loops provide bilinear and higher order terms in the bosonic fluctuations all proportional to N. Bilinears give the induced kinetic and mass terms; their diagonalization leads to boson propagators proportional to $1/N$, which implies that bosonic loops will generate non leading $1/N$ contributions.

Fermion loops give rise to adimensional integrals that have to be regularized. We will compute them by dimensional regularization.

Consider the bosonic effective theory which remains after integrating out fermions. The vacuum equations essentially admit two solutions, both of vanishing action: the trivial, $\langle W_{\mu a}\rangle = 0$; the non trivial or non symmetric solution given by

$$\langle \xi^{\mu\alpha}\rangle = \Lambda^{-1}\eta^{\mu\alpha}, \ \langle W_{\mu\alpha}\rangle = b\Lambda\eta_{\mu\alpha}, \ \langle\phi\rangle = v, \ \langle\rho\rangle = Db/v \ , \ \langle\omega_\mu^{ab}\rangle = 0 \ , \qquad (II.8)$$

with

$$\langle W_{\mu a}\rangle\langle\xi^{\mu a}\rangle = \langle\phi\rangle\langle\rho\rangle = DV(\langle\rho\rangle) = Db \ , \ \langle\phi\rangle = -V'(\langle\rho\rangle) \ , \qquad (II.9)$$

$$b^{D+1} = -\sum_{i=1}^{N}\int \frac{d^D r}{(2\pi)^D}\frac{r^2}{r^2 + \langle\phi\rangle^2 a_i^2} \ . \qquad (II.10)$$

Note that Λ has dimension of mass. We will assume that the dynamics selects the non symmetric vacuum. The symmetric phase will only be considered in sect. V, in the case of two-dimensional space-time.

From eqs.(II.4,8) we see that we may identify in $\Psi_i = b^2\Lambda^{3/2}\psi_i$ the usual 3/2 dimensional spinor fields and in $m_i = va_i\Lambda$ their lagrangean masses. We also see that small fermion masses are protected ($a_i << 1$).

It is convenient to separate the fields $W_{\mu a}$ and ρ into classical and quantum fluctuation parts:

$$W_{\mu a} = W^0_{\mu a} + \tilde{W}_{\mu a} \ , \quad \rho = \rho_0 + \tilde{\rho} \tag{II.11}$$

where $W^0_{\mu a}$ and ρ^0 are the solutions of the classical equations of motion, i.e.

$$V'(\rho^0) = -\phi \ , \quad W^0_{\mu a} = \frac{1}{5}\xi^{-1}_{\mu a}(V(\rho) + \phi\rho) \ . \tag{II.12}$$

Inserting into the bosonic effective action (cf. (II.4)) one obtains

$$S_{\text{eff}} = \int d^4x(\det\xi^{-1})\Big(\frac{1}{5}(V[\rho_0(\phi)] + \phi\rho_0(\phi))\Big)^5 + O(\tilde{\rho}^2) + O(\tilde{W}^2)$$
$$+ \sum_{i=1}^{N} \operatorname{tr}\log[\frac{i}{2}\xi^{\mu a}(\gamma_a \overrightarrow{D}_\mu - \overleftarrow{D}_\mu\gamma_a) - a_i\phi] \ . \tag{II.13}$$

The quantum fluctuations $\tilde{\rho}$ and $\tilde{W}_{\mu a}$ will be relevant in loop calculations. Indeed, $\tilde{W}_{\mu a}$ and $\tilde{\rho}$ are not coupled to fermions and thus acquire kinetic terms only at higher order in $1/N$. This means that their masses will be of the order $N\Lambda$ so they will not contribute to the order we shall explore here.

In the Weyl invariant $(\bar{\psi}\psi)^{-4}$ case (II.6) the vacuum values obey the relations

$$\langle\phi\rangle\langle\rho\rangle^5 = 4A \neq 0, \quad b^5 = \frac{A\langle\phi\rangle^4}{256} \ , \tag{II.14}$$

and the bosonic effective action (II.13) reduces to

$$S^{\text{eff}}_{\text{TG}} = \frac{A}{256}\int d^4x(\det\xi^{-1})\phi^4 + O(\tilde{\rho}^2) + O(\tilde{W}^2) + \sum_{i=1}^{N} \operatorname{tr}\log(\frac{i}{2}\xi^{\mu a}(\gamma_a \overrightarrow{D}_\mu - \overleftarrow{D}_\mu\gamma_a) - a_i\phi) \tag{II.15}$$

Now let us expand action (II.13) around the vacuum values:

$$\phi = v(1 + \tilde{\phi}) \ , \quad \xi^{\mu a} = \Lambda^{-1}(\eta^{\mu a} + \tilde{\xi}^{\mu a}) \ . \tag{II.16}$$

We obtain

$$S_{\text{eff}} = \int d^4x \; b^5 \Lambda^4 [1 - 4\tilde{\phi}\tilde{\xi}^a_a + 2\tilde{\phi}^2 (\frac{16}{5} + \frac{V'(\langle\rho\rangle)}{\langle\rho\rangle V''(\langle\rho\rangle)}) + \frac{1}{2}(\eta_{\mu a}\eta_{\nu b} + \eta_{\mu b}\eta_{\nu a})\tilde{\xi}^{\mu a}\tilde{\xi}^{\nu b} + ...]$$

$$+ \sum_{i=1}^{N} \operatorname{tr}\log \Lambda^{-1} [\frac{i}{2}\gamma^a \overleftrightarrow{\partial}_a - a_i v\Lambda + \omega_{\mu,ab}\gamma_5\gamma_c\epsilon^{\mu abc} + \tilde{\xi}^{\mu a}(\gamma_a \overrightarrow{D}_\mu - \overleftarrow{D}_\mu \gamma_a) - a_i v \Lambda \tilde{\phi}] \quad (II.17)$$

Expanding the logarithm in powers of the quantum fluctuations $\tilde{\xi}, \tilde{\phi}, \ \omega$ one can obtain propagators and n-point interactions which characterize the effective theory.

III. EFFECTIVE LAGRANGIAN AND HIGGS PHENOMENON

III.1 COMPUTATION OF QUADRATIC TERMS

Let us now discuss the quadratic terms in the field fluctuations $\tilde{\phi}$, $\tilde{\xi}^{\mu a}$ and $\omega_{\mu,ab}$ around their VEVs. They have been discussed in [7] as a low momentum expansion. Since we want to investigate the high energy regime, it is useful to have the exact expressions for the bilinear terms.

The integrals that we have to compute are the following ones:

$$\langle\tilde{\xi}_{\mu a}(q)\tilde{\xi}_{\nu b}(-q)\rangle_{\log} = \sum_{i=1}^{N}\int\frac{d^4p}{(2\pi)^4}\frac{p_\mu p_\nu}{D_q D_{-q}}\mathrm{tr}[\gamma_a(\not{p}+\frac{1}{2}\not{q}+m_i)\gamma_b(\not{p}-\frac{1}{2}\not{q}+m_i)]\ , \quad (III.1)$$

$$\langle\tilde{\xi}_{\mu a}\tilde{\phi}\rangle_{\log} = -\sum_{i=1}^{N}\int\frac{d^4p}{(2\pi)^4}\frac{p_\mu m_i}{D_q D_{-q}}\mathrm{tr}[(\not{p}+\frac{1}{2}\not{q}+m_i)\gamma_a(\not{p}-\frac{1}{2}\not{q}+m_i)]\ , \quad (III.2)$$

$$\langle\tilde{\phi}\tilde{\phi}\rangle_{\log} = \sum_{i=1}^{N}\int\frac{d^4p}{(2\pi)^4}\frac{m_i^2}{D_q D_{-q}}\mathrm{tr}[(\not{p}+\frac{1}{2}\not{q}+m_i)(\not{p}-\frac{1}{2}\not{q}+m_i)]\ , \quad (III.3)$$

$$\langle\omega_{\mu,ab}\omega_{\nu,cd}\rangle_{\log} = -\epsilon_{\mu abe}\epsilon_{\nu cdf}\sum_{i=1}^{N}\int\frac{d^4p}{(2\pi)^4}\frac{1}{D_q D_{-q}}\mathrm{tr}[\gamma_5\gamma^e(\not{p}+\frac{1}{2}\not{q}+m_i)\gamma_5\gamma^f(\not{p}-\frac{1}{2}\not{q}+m_i)]\ , \quad (III.4)$$

$$\langle\omega_{\mu,ab}\tilde{\xi}_{\nu c}\rangle_{\log} = i\epsilon_{\mu abd}\sum_{i=1}^{N}\int\frac{d^4p}{(2\pi)^4}\frac{p_\mu}{D_q D_{-q}}\mathrm{tr}[\gamma_c(\not{p}+\frac{1}{2}\not{q}+m_i)\gamma_5\gamma^d(\not{p}-\frac{1}{2}\not{q}+m_i)]\ ; \quad (III.5)$$

$$\langle\omega_{\mu,ab}\tilde{\phi}\rangle_{\log} = -i\epsilon_{\mu abd}\sum_{i=1}^{N}\int\frac{d^4p}{(2\pi)^4}\frac{m_i}{D_q D_{-q}}\mathrm{tr}[(\not{p}+\frac{1}{2}\not{q}+m_i)\gamma_5\gamma^d(\not{p}-\frac{1}{2}\not{q}+m_i)]\ , \quad (III.6)$$

where

$$D_{\pm q} = (p \pm \frac{1}{2}q)^2 - m_i^2$$

It is convenient to decompose $\tilde{\xi}^{\mu a}$ into symmetric and antisymmetric parts, $\tilde{\xi}^{\mu a} = \tilde{\xi}^{\mu a}_{\mathrm{s}} + \tilde{\xi}^{\mu a}_{\mathrm{a}}$. A somewhat long calculation leads to the following results:

$$\langle \tilde{\xi}^{\mathrm{s}}_{\mu a}(q)\tilde{\xi}^{\mathrm{s}}_{\nu b}(-q)\rangle = \frac{1}{5}q^2(3f(q^2)+2g(q^2))(\theta_{\mu\nu}\theta_{ab}+\theta_{\mu b}\theta_{a\nu}-\frac{2}{3}\theta_{\mu a}\theta_{\nu b}) + \frac{2}{3}q^2 g(q^2)\theta_{\mu a}\theta_{\nu b} \ , \tag{III.7}$$

$$\langle \tilde{\xi}^{\mathrm{s}}_{\mu a}\tilde{\phi}\rangle = -2q^2 g(q^2)\theta_{\mu a} \ , \tag{III.8}$$

$$\langle \tilde{\phi}\tilde{\phi}\rangle = 6q^2 g(q^2) + c \ , \tag{III.9}$$

$$\langle \tilde{\xi}^{\mathrm{a}}_{\mu a}(q)\tilde{\xi}^{\mathrm{s}}_{\nu b}(-q)\rangle = 0 \ , \quad \langle \tilde{\xi}^{\mathrm{a}}_{\mu a}(q)\tilde{\xi}^{\mathrm{a}}_{\nu b}(-q)\rangle = f(q^2)q^\rho q^\sigma \epsilon_{\mu a\rho c}\epsilon_{\nu b\sigma}{}^{c} \ , \tag{III.10}$$

$$\langle \omega_{\mu,ab}\tilde{\phi}\rangle = 0 \ , \quad \langle \tilde{\xi}_{\mu a}\omega_{\nu,bc}\rangle = 4f(q^2)q^\rho \epsilon_{\mu a\rho\sigma}\epsilon_{\nu bc}{}^{\sigma} \ , \tag{III.11}$$

$$\langle \omega_{\mu,ab}\omega_{\nu,cd}\rangle = \epsilon_{\mu ab\rho}\epsilon_{\nu cd\sigma}[16f(q^2)(\eta^{\rho\sigma} - \frac{q^\rho q^\sigma}{q^2}) - 24g(q^2)\frac{q^\rho q^\sigma}{q^2}] \ , \tag{III.12}$$

where $\theta_{\mu a} = \eta_{\mu a} - q_\mu q_a/q^2$ and

$$(4\pi)^2 f(q^2) = \sum_{i=1}^{N} \frac{q^2}{12}\left[\log(a_i^2 v^2/4\pi) + J(q^2/m_i^2)\right] + \frac{m_i^2}{2}\log(a_i^2 v^2/4\pi) + \frac{m_i^2}{3}J(q^2/m_i^2) \ , \tag{III.13}$$

$$(4\pi)^2 g(q^2) = -\frac{1}{3}\sum_{i=1}^{N} m_i^2\left[\log(a_i^2 v^2/4\pi) + J(q^2/m_i^2)\right] \ , \tag{III.14}$$

$$(4\pi)^2 c = \sum_{i=1}^{N} 4m_i^4 \log\frac{a_i^2\langle\phi\rangle^2}{4\pi}\left(\frac{V'(\langle\rho\rangle)}{\langle\rho\rangle V''(\langle\rho\rangle)} + \frac{1}{5}\right) \ , \tag{III.15}$$

with

$$J(x) \equiv 2\int_0^{1/2} dy \log(1+\frac{x}{4} - xy^2) = -2 - A^{1/2}\log B \quad , \quad . \tag{III.16}$$

$$A \equiv 1 + \frac{4}{x} , \quad B \equiv 1 + \frac{x}{2}(1 - A^{1/2}) .$$

These lengthy expressions when expanded to second order in q reproduce the small momentum regime analysed in [7] (see subsect.3).

We have absorbed the $1/(D-4)$ infinity into a renormalization of

$$\phi \to Z^{-1}\phi, \; \xi^{\mu a} \to Z^{-1}\xi^{\mu a} , \quad \log Z = 1/(D-4) .$$

Thus ($Z' \equiv Z^{1/5}$)

$$\rho \to Z'\rho , \quad W_{\mu a} \to Z' W_{\mu a} , \quad \omega_\mu^{ab} \to \omega_\mu^{ab} \tag{III.17}$$

which in the original action just amounts to a wave function renormalization: $\bar{\psi}_i \to Z'^{1/2}\bar{\psi}_i$, $\psi_i \to Z'^{1/2}\psi_i$. Generically, this implies a renormalization of the constants appearing in the potential V. In the 4D TG model, instead, the renormalization constant Z actually cancels out because of Weyl invariance (cf. eq. (II.7)). This will be interpreted in sect. VI as a sign of finiteness of the $(\bar{\psi}\psi)^{-4}$ theory.

III.2 HIGGS MECHANISM

The $\tilde{\xi}\omega$ sector is diagonalized by

$$B_{\mu,ab} = \omega_{\mu,ab} + \frac{i}{4}\partial_\mu \tilde{\xi}^{\mathrm{a}}_{ab} + \dots , \tag{III.18}$$

where dots represent the non abelian part which is irrelevant for the bilinears we are discussing. Eq.(III.18) shows the emergence of the Higgs mechanism found in [7], caused by the spontaneous breaking of the local Lorentz symmetry. Indeed, $\tilde{\xi}^{\mu a}_{\mathrm{a}}$ can be completely eliminated from the theory by a $SO(3,1)$ gauge transformation with parameter $\tau^{\mu a} = -\frac{i}{4}\tilde{\xi}^{\mu a}_{\mathrm{a}}$. The Lorentz connection becomes massive having eaten the six Goldstone bosons $\tilde{\xi}^{\mu a}_{\mathrm{a}}$. This shows that the breaking is really of spontaneous nature.

In the $(\bar{\psi}\psi)^{-4}$ model (II.6), also the symmetry (II.5) or (II.7) is broken by the vacuum values. We would like to see whether this breaking is of spontaneous nature as well. Inserting the 4D TG potential $V = A/(\bar{\psi}\psi)^4$ into eq. (III.15), we find that c is identically zero. In this case $\tilde{\phi}$ can be completely eliminated by fixing the additional symmetry,

$$\bar{\xi}^{s\prime}_{\mu a} = \bar{\xi}^{s}_{\mu a} + \alpha\theta_{\mu a} \ , \quad \bar{\phi}' = \bar{\phi} + \alpha \ , \tag{III.19}$$

in the "unitary" gauge $\alpha = -\bar{\phi}$. The symmetry (III.19) is nothing but the infinitesimal version of (II.7) [c]. The low energy analysis (see subsect.3) shows that the mode $\Lambda\xi^{\mu a}$ is nothing but the usual vierbein. This means that the symmetry (II.5) indeed corresponds to Weyl invariance in the usual sense. The mode ϕ plays the role of Goldstone boson. We thus arrive at the desired point: the 4D TG model is a conformal invariant theory in which also Weyl invariance is spontaneously broken.

Let us introduce the notation $\bar{e}^{s}_{\mu a} = \bar{\xi}^{s}_{\mu a} - \bar{\phi}\theta_{\mu a}$. What remains for the quadratic terms of the Lagrangian in the $\bar{\xi}_{\mu a}\bar{\phi}$ sector is just

$$\frac{1}{2}\bar{e}^{\mu a}_{s}(q)\bar{e}^{\nu b}_{s}(-q)[\frac{1}{5}q^2(3f(q^2) + 2g(q^2))(\theta_{\mu\nu}\theta_{ab} + \theta_{\mu b}\theta_{a\nu} - \frac{2}{3}\theta_{\mu a}\theta_{\nu b}) + \frac{2}{3}q^2 g(q^2)\theta_{\mu a}\theta_{\nu b}] \ . \tag{III.20}$$

For any other potential V, c is non vanishing and therefore the theory is no longer symmetric under the transformations (III.19). Diagonalization of the quadratic terms leads to an additional term $\frac{1}{2}c\phi^2$. The mode ϕ remains, but it does not propagate at this order.

III.3 LOW ENERGY EFFECTIVE ACTION

Now we would like to learn what is the low energy physics predicted by this theory, where by "low energy" we mean energies much less than the Planck mass, which in this theory can only be proportional to Λ, the scale that has appeared upon the symmetry breaking. So the strategy is to expand the quadratic lagrangian in powers of q^2/Λ^2. In doing so we get

$$\mathcal{L}_{\text{quad}} = \frac{1}{12}M_P^2\bar{e}^{s}_{\mu a}(q)\bar{e}^{s}_{\nu b}(-q)[q^2(\theta_{\mu\nu}\theta_{ab} + \theta_{\mu b}\theta_{a\nu} - 2\theta_{\mu a}\theta_{\nu b}) + O(q^4/\Lambda^2)]$$
$$+\frac{1}{2}B_\rho B_\sigma[8M_P^2\eta^{\rho\sigma} + O(q^2)] + \frac{1}{2}c\phi^2 \tag{III.21}$$

[c] Because of the residual invariance under linearized diffeomorphisms, one can also write the transformation (III.19) as $\bar{\xi}^{s\prime}_{\mu a} = \bar{\xi}^{s}_{\mu a} + \alpha\eta_{\mu a}$, $\bar{\phi}' = \bar{\phi} + \alpha$, which is the direct linearization of (II.7). We have preferred the form (III.19) since it is naturally read from eqs. (III.7-9).

$$(4\pi)^2 M_P^2 \equiv \Lambda^2 \sum_{i=1}^{N} a_i^2 \langle\phi\rangle^2 \log \frac{a_i^2 \langle\phi\rangle^2}{4\pi} \ ,$$

where we have defined $B_\mu = \epsilon_\mu{}^{\nu\alpha\beta} B_{\nu,\alpha\beta}$ which is the only component of the spin connection that propagates to this order in $1/N$. We see from eq. (III.21) that B_μ has a mass of order M_P. The mode ϕ (absent in the $(\bar\psi\psi)^{-4}$ theory) will get a kinetic term at higher order in $1/N$. This means that its mass will be of order $N\Lambda$.

In coordinate space the $\bar{e}$ quadratic term reads

$$\frac{1}{6} M_P^2 \left[\Box(\bar{e}^{\mu a}\bar{e}_{\mu a} - \bar{e}^\mu_\mu \bar{e}^\mu_\mu) - 2\partial_a\partial_b \bar{e}^{\mu a}\bar{e}^b_\mu + 2\partial_a\partial_\mu \bar{e}^{\mu a}\bar{e}^b_b\right] \qquad (III.22)$$

By associating M_P to the Planck mass and identifying $\bar{e}$ with the fluctuation of the usual vierbein, eq.(III.22) coincides with the bilinear term in the expansion of the Einstein-Hilbert action. Therefore, the low energy (LE) theory will have the following form

$$S_{\rm LE} = \frac{1}{16\pi G} \int d^4x \det e R(e) + O(\Box^2) \ , \qquad (III.23)$$

$$G^{-1} = \Lambda^2 \left(\frac{1}{12\pi} \sum_{i=1}^{N} (a_i v)^2 \log(a_i^2 v^2/4\pi)\right) \ ,$$

which implies that the physics described by these models is in agreement with the observed phenomenology.

IV. HIGH ENERGY REGIME AND UNITARITY BOUNDS

IV.1 PROPAGATORS AND "GHOST" FREEDOM

It is convenient to separate $\tilde{e}_s^{\mu a}$ in the following way:

$$\tilde{e}_s^{\mu a} = \tilde{e}_0^{\mu a} + \theta^{\mu a}\sigma \qquad (IV.1)$$

where $\tilde{e}_0^{\mu a}\theta_{\mu a} = 0$ and $\tilde{e}_0^{\mu a} = \tilde{e}_0^{a\mu}$ (hence $\sigma = \frac{1}{3}\tilde{e}_s^{\mu a}\theta_{\mu a}$).

The inversion for the $\tilde{e}_0$ quadratic term needs the elimination of the zero mode implied by the residual invariance under linearized diffeomorphisms $\delta\tilde{e}_s^{\mu a} = \partial_\mu\epsilon_a + \partial_a\epsilon_\mu + O(\epsilon^2)$. This is fixed in the standard way by requiring $\partial_\mu\tilde{e}_s^{\mu a} = 0$ through the addition of the usual gauge fixing term. The quadratic Lagrangian becomes

$$\begin{aligned}\mathcal{L}_{\rm quad} = &\frac{1}{2}\tilde{e}_0^{\mu a}(q)\tilde{e}_0^{\nu b}(-q)\big[\frac{1}{5}q^2(3f(q^2) + 2g(q^2))(\theta_{\mu\nu}\theta_{ab} + \theta_{\mu b}\theta_{a\nu} - \frac{2}{3}\theta_{\mu a}\theta_{\nu b})\\ &+\lambda(\theta_{\mu\nu}q_a q_b + \theta_{ab}q_\mu q_\nu + \theta_{\mu b}q_\nu q_a + \theta_{\nu a}q_\mu q_b + 4q_\mu q_a q_\nu q_b/q^2)\big]\\ &+3q^2 g(q^2)\sigma^2 + \frac{1}{2}B_\rho B_\sigma[16f(q^2)(\eta^{\rho\sigma} - \frac{q^\rho q^\sigma}{q^2}) - 24g(q^2)\frac{q^\rho q^\sigma}{q^2}] \,.\end{aligned} \qquad (IV.2)$$

The inversion of these quadratic terms give rise to the following propagators:

$$\pi_{\mu\nu} \equiv \langle B_\mu B_\nu\rangle^{-1} = \frac{\eta_{\mu\nu}}{16f(q^2)} - \frac{q_\mu q_\nu}{q^2}\frac{2f(q^2) + 3g(q^2)}{48f(q^2)g(q^2)} \,, \qquad (IV.3)$$

$$\pi \equiv \langle\sigma\sigma\rangle^{-1} = \frac{1}{6q^2 g(q^2)} \,, \qquad (IV.4)$$

$$\pi_{\mu a,\nu b}(q) = \frac{1}{2}A(q^2)(\theta_{\mu\nu}\theta_{ab} + \theta_{\mu b}\theta_{a\nu} - \frac{2}{3}\theta_{\mu a}\theta_{\nu b} - q_\mu q_\nu q_a q_b/q^4) \,, \qquad (IV.5)$$

where (we choose the Landau gauge, $\lambda \to \infty$)

$$A(q^2) = \frac{5}{q^2(3f(q^2) + 2g(q^2))} \,. \qquad (IV.6)$$

In the high-energy regime we have (see eq. (III.16))

$$J(q^2/m_i^2) \sim -2 + \log q^2/m_i^2,\ q^2/m_i^2 >> 1\ , \tag{IV.7}$$

hence we obtain (see eqs. (III.13,14))

$$(4\pi)^2 f(q^2) = \frac{q^2}{12}(-2 + \log\frac{q^2}{4\pi\Lambda^2}) + O(\Lambda^2)\ ,\quad (4\pi)^2 g(q^2) = O(\Lambda^2)\ . \tag{IV.8}$$

Inserting into eq. (IV.2) we get a term of the form

$$\tilde{e}_0^{\mu a}(q)\tilde{e}_0^{\nu b}(-q)q^4(1 - \frac{1}{2}\log\frac{q^2}{4\pi\Lambda^2})(\theta_{\mu\nu}\theta_{ab} + \theta_{\mu b}\theta_{a\nu} - \frac{2}{3}\theta_{\mu a}\theta_{\nu b}) \tag{IV.9}$$

This resembles the bilinear term in the expansion of the conformal invariant action around $e_0^{\mu a} = \delta^{\mu a} + \tilde{e}_0^{\mu a}$:

$$\int \sqrt{g}(3R^{\mu\nu}R_{\mu\nu} - R^2)\ ,\quad g^{\mu\nu} = e^{\mu a}e_a^{\nu}\ .$$

This is a theory which classically has indefinite energy and quantum mechanically has an indefinite metric Hilbert space with ghosts. None of these troubles are expected in the present theory. The original action (II.1) is *linear* in time derivatives. This means that, unlike R^2 theories, the initial value problem is well defined. On the other hand, eq. (IV.9) contains logarithms which pick an imaginary part as soon as q^2 is greater than four times the square mass of the lightest fermion (cf eqs.(III.13,14)). Hence there are no real bound state poles, but a complex conjugate pair of unstable, unphysical particles [9]. Therefore such pair does not contribute to the absorptive part. It is well known [11] that in this case a unitary S-matrix defined between physical particle asymptotic state exists. The price that one has to pay is that the usual global analytical properties of the S matrix are modified by the extra poles and cuts. This is the Lee-Wick mechanism [11] and it was implemented in the context of induced "R^2" gravity terms by Tomboulis [9] (further discussions can be found in refs. [12,19]).

The "non analyticity" of the S-matrix will lead to the occurrence of non causal effects. This point was also discussed in ref. [9]. The author concludes that no logical paradox could arise in a scattering experiment (one has well-defined S-matrix between in and out physical states). As the energy increases, the complex ghosts produce more important effects but, at the same time, the interaction decreases due to the asymptotic freedom of the theory. The fact that the theory is asymptotically free (which can be seen from eq.(IV.9)) is consistent with the $1/N$ approach and the introduction of composite fields.

IV.2 CALCULATION OF EFFECTIVE INTERACIONS

Unitarity also requires that an n-point tree amplitude, with the ratio of all invariant fixed, should be bounded by E^{n-4} in the limit $E \to \infty$. A violation of such a bound would lead to incontrollable divergences in loop diagrams. One of the celebrated issues of spontaneously broken YM theories was in fact to realize that the precise, predicted couplings involving Higgs are exactly what is needed in order to "magically" cancel unitarity-violating behaviours in 4-point amplitudes [17,18]. In this section IV we will show that the 4-point amplitudes of the present theory satisfies such unitarity bounds thanks to even more striking cancellations.

Much as happens for YM broken theories, it is possible to identify physical scattering amplitudes such that when all kinematic invariants (s, t and u for $2 \to 2$ processes) become large, i.e. much larger than the vacuum generated scale Λ, single exchanged particles give rise to unwanted increasing energy behaviour. A similar mechanism should work here if the spontaneously broken theory is UV correct as commonly expected. The leading amplitudes contains fermion loops not only in the bilinear already discussed but also in trilinear and higher order terms appearing as effective interactions of the B_μ, σ and $\tilde{e}_0^{\mu a}$ fields. It does not contain however loops of those fields; they will appear to next order in $1/N$.

As in the YM case, the potentially dangerous amplitudes are those involving longitudinal massive bosons B_μ, i.e., with polarization $\epsilon_\mu = \frac{1}{m_B} p_\mu$, $p^2 = m_B^2$, where m_B is the mass of the boson B_μ which, to this leading order in $1/N$, is just given by the zero of the function $f(q^2)$ (cf. eq. (IV.3)). It is convenient to introduce the notation $B_{long.} = \frac{1}{m_B} p_\mu B^\mu$. Let us now compute the $B_\mu B_{long.} B_{long.}$ effective interaction (see fig. 1), which will be used in the analysis of the processes considered in the subsects. IV.3 and IV.4. It is given by

$$\langle B_\mu B_{long.} B_{long.} \rangle =$$

$$-\frac{2i}{m_B^2} \sum_{i=1}^{N} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{D_0 D_1 D_2} \mathrm{tr}[\gamma_5 \not{p}_1 (\not{k} + m_i) \gamma_5 \not{p}_2 (\not{k} - \not{p}_2 + m_i) \gamma_5 \gamma_\mu (\not{k} + \not{p}_1 + m_i)] \ , \quad (IV.10)$$

where

$$D_0 \equiv k^2 - m_i^2 \ , \quad D_1 \equiv (k+p_1)^2 - m_i^2 \ , \quad D_2 \equiv (k-p_2)^2 - m_i^2 \ . \tag{IV.11}$$

Explicit computation of the trace shows that

$$\langle B_\mu B_{long.} B_{long.} \rangle = 0 \tag{IV.12}$$

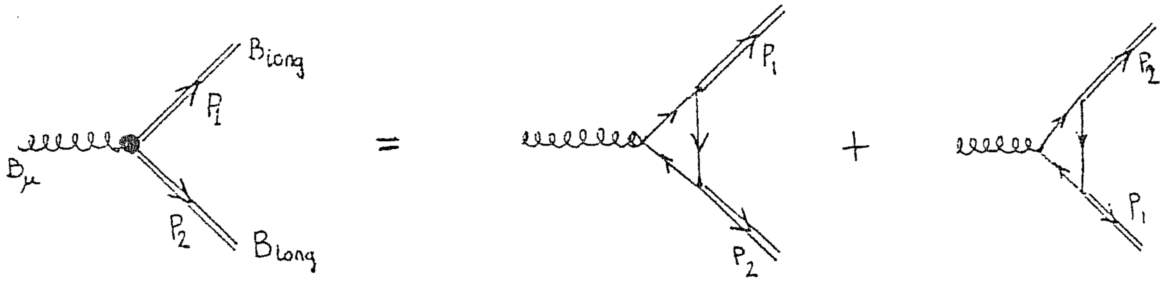

Fig.1: Diagrams contributing to the $B_\mu B_{long}.B_{long}.$ effective interaction.

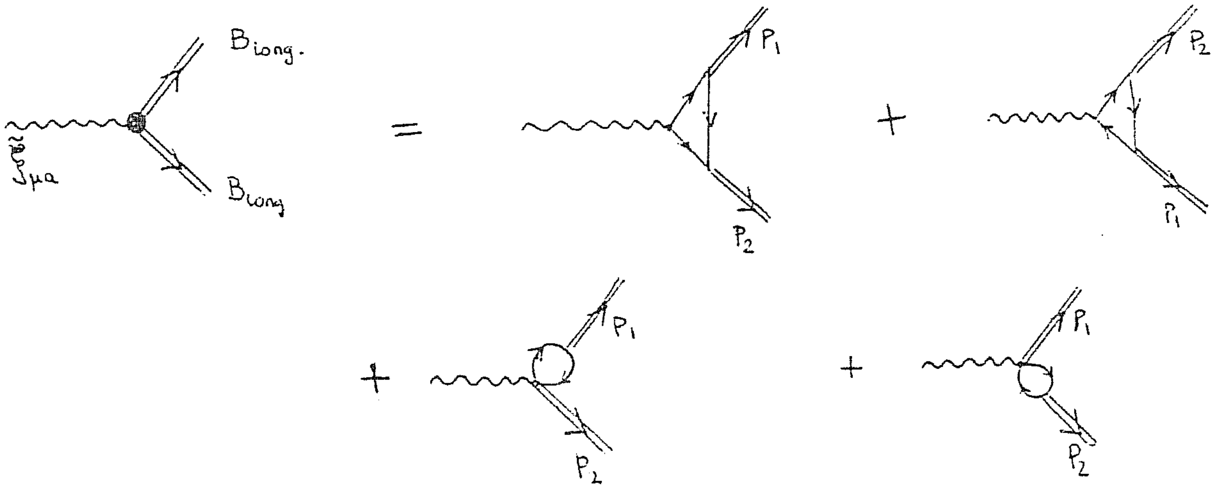

Fig.2: Diagrams contributing to the $\tilde{\xi} B_{long}.B_{long}.$ effective interaction.

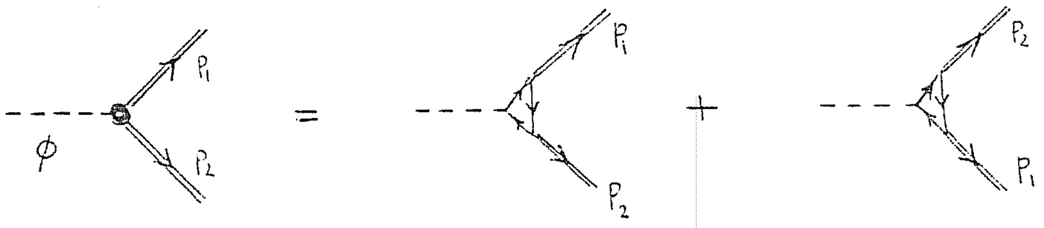

Fig.3: Diagrams contributing to the $\tilde{\phi} B_{long}.B_{long}.$ effective interaction.

Now consider the $\tilde{\xi} B_{long.}B_{long.}$ effective interaction. The diagrams which contribute are depicted in fig.2. This interaction can be decomposed as: $V_{\mu a} \equiv \langle \tilde{\xi}_{\mu a} B_{long.}B_{long.}\rangle = V_{\mu a}^{(1)} + V_{\mu a}^{(2)}$, where

$$V_{\mu a}^{(1)} = -\frac{1}{m_B^2}\sum_{i=1}^{N}\int \frac{d^4k}{(2\pi)^4}\frac{(2k+p_1-p_2)_\mu}{D_0D_1D_2}\mathrm{tr}[\gamma_5 \not{p}_1(\not{k}+m_i)\gamma_5\not{p}_2(\not{k}-\not{p}_2+m_i)\gamma_a(\not{k}+\not{p}_1+m_i)] \;, \qquad (IV.13)$$

$$V_{\mu a}^{(2)} = -\frac{1}{3m_B^2}(\eta_{\mu a}p_{1\rho} - \eta_{\rho a}p_{1\mu})\sum_{i=1}^{N}\int \frac{d^4k}{(2\pi)^4}\frac{1}{D_{p_2}D_{-p_2}}\mathrm{tr}[\gamma_5\not{p}_2(\not{k}+\frac{1}{2}\not{p}_2+m_i)$$

$$\gamma_5\gamma^\rho(\not{k}-\frac{1}{2}\not{p}_2+m_i)] + (p_1 \leftrightarrow p_2) \;, \qquad (IV.14)$$

where $D_{\pm q}$ was defined in sect. III. A somewhat long computation gives

$$V_{\mu a}^{(1)} = 12\frac{(p_1+p_2)^2}{m_B^2}g[(p_1+p_2)^2]\eta_{\mu a} + \frac{24}{m_B^2}\big(g(m_B^2) - g[(p_1+p_2)^2]\big)(p_{1\mu}p_{2a}+p_{2\mu}p_{1a})$$

$$-\frac{16}{m_B^2}g(m_B^2)(p_{1\mu}p_{1a}+p_{2\mu}p_{2a}) + O(\Lambda^2) \;, \qquad (IV.15)$$

$$V_{\mu a}^{(2)} = -\frac{8}{m_B^2}g(m_B^2)(2\eta_{\mu a}p_1\cdot p_2 - p_{1\mu}p_{2a} - p_{2\mu}p_{1a}) \;. \qquad (IV.16)$$

The function $g(q^2)$ was defined in eq. (III.14).

Another vertex we need is the $\tilde{\phi} B_{long.}B_{long.}$ vertex (see fig.3), which we will denote by V_ϕ. It is given by the following integral

$$V_\phi = -\frac{2}{m_B^2}\sum_{i=1}^{N}\int \frac{d^4k}{(2\pi)^4}\frac{m_i}{D_0D_1D_2}\mathrm{tr}[\gamma_5\not{p}_1(\not{k}+m_i)\gamma_5\not{p}_2(\not{k}-\not{p}_2+m_i)(\not{k}+\not{p}_1+m_i)] \;. \qquad (IV.17)$$

We obtain

$$V_\phi = -24\frac{(p_1+p_2)^2}{m_B^2}g[(p_1+p_2)^2] + O(\Lambda^2) \;. \qquad (IV.18)$$

The physical modes which diagonalize quadratic terms in the $\bar{\xi}\bar{\phi}$ sector were found in sect. III to be $\bar{e}^{s}_{\mu a} = \bar{\xi}^{s}_{\mu a} - \bar{\phi}\theta_{\mu a}$ and $\bar{\phi}' = \bar{\phi}$. In the Weyl invariant $(\bar{\psi}\psi)^{-4}$ case the mode $\bar{\phi}$ was completely eliminated from the bilinear terms upon this transformation. If the Weyl symmetry is to really be a symmetry in the effective 4D TG theory, the mode $\bar{\phi}$ should automatically disappear also from all n-point interactions[d]; in particular, from the (three-point interaction) terms we are analysing, namely

$$\bar{\phi}B_{long.}B_{long.}V_{\phi} + \bar{\xi}^{s}_{\mu a}B_{long.}B_{long.}V^{\mu a} = \bar{\phi}B_{long.}B_{long.}(V_{\phi} + \theta_{\mu a}V^{\mu a}) + \bar{e}^{s}_{\mu a}B_{long.}B_{long.}V^{\mu a} \tag{IV.19}$$

Moreover, since at the order we are considering these three-point interactions do not depend at all on the potential, the mode $\bar{\phi}$ should actually disappear from eq. (IV.19) for an arbitrary potential. To check this, we use eqs. (IV.15,16) to compute

$$\theta_{\mu a}V^{\mu a} = 24\frac{(p_1+p_2)^2}{m_B^2}g[(p_1+p_2)^2] \tag{IV.20}$$

Therefore $V_{\phi} + \theta_{\mu a}V^{\mu a} = 0$. Q.E.D. (cf. eq.(IV.19)).

Writing, as in eq. (IV.1), $\bar{e}^{\mu a}_{s} = \bar{e}^{\mu a}_{0} + \theta^{\mu a}\sigma$, we thus have

$$\langle \bar{e}^{\mu a}_{0} B_{long.}B_{long.}\rangle = V_{\mu a} \tag{IV.21}$$

$$\langle \sigma B_{long.}B_{long.}\rangle = -V_{\phi} = 24\frac{(p_1+p_2)^2}{m_B^2}g[(p_1+p_2)^2] + O(\Lambda^2) \tag{IV.22}$$

IV.3 ANALYSIS OF THE PROCESS $\psi_j\bar{\psi}_j \to B_{long.}B_{long.}$

Let us first discuss the physical process $\psi_j\bar{\psi}_j \to B_{long.}B_{long.}$ in a kinematical configuration in which s, t and u ($s+t+u = 2m_j^2 + 2m_B^2$) are proportional to $E^2 >> \Lambda^2$ (i.e. $s,t,u >> m_j^2, m_B^2$). The diagrams contributing to the leading $1/N$ amplitudes are depicted in fig.4. We could expect, in principle, leading E^2 and non leading E behaviours which should be absent in the whole amplitude.

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[d] Provided we add the proper $O(\bar{\phi}^2)$ corrections in the definition of $\bar{e}^{s}_{\mu a}$ which make $\bar{\phi}$ exponentiate (cf. eq.(II.7)). In the present case we look at a vertex which is linear in $\bar{\phi}$ so such corrections are irrelevant.

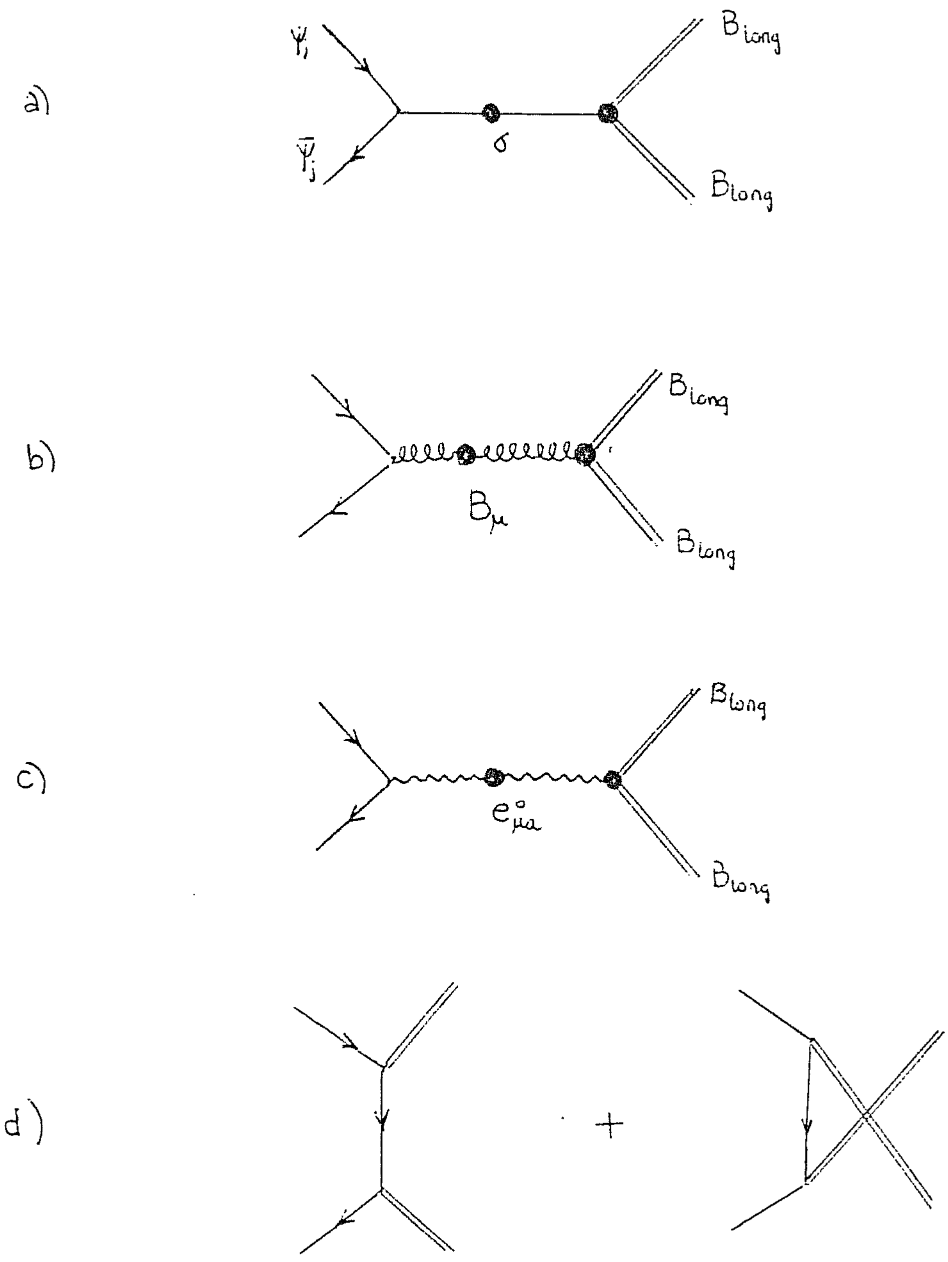

Fig.4: Tree-level diagrams contributing to the process $\psi_j\bar{\psi}_j \to B_{long}.B_{long}.$.

Using eqs. (IV.4,5,6,8,12,21,22), it is now a simple matter to evaluate the four contributions of fig.4. The results are

$$T_a = \frac{4m_j}{m_B^2}\bar{v}(q_2)u(q_1) + O(1) \ , \qquad (IV.23)$$

$$T_b = 0 \ , \qquad (IV.24)$$

$$T_c = O(1) \ , \qquad (IV.25)$$

$$T_d = -\frac{4m_j}{m_B^2}\bar{v}(q_2)u(q_1) + O(1) \ . \qquad (IV.26)$$

In the present high-energy regime $\bar{v}(q_2)u(q_1) \propto E$ for all spinor polarizations. The leading E^2 behaviour has cancelled between the two diagrams of fig. 4d. $\tilde{e}_0$-exchange gives regular contribution due to the $1/q^4$ high energy behaviour of its propagator. The remaining subleading contributions finally cancel in $T = T_a + T_b + T_c + T_d$.

The whole amplitude thus satisfy the required unitary bound.

IV.4 ANALYSIS OF THE PROCESS $B_{long.}B_{long.} \to B_{long.}B_{long.}$

A more striking cancellation takes place in the analysis of the process $B_{long.}B_{long.} \to B_{long.}B_{long.}$. Besides the tree diagrams of figs.5a-c we find a contribution from an effective $4B$ interaction (fig.5d) that arises from the box diagrams of fig.6.

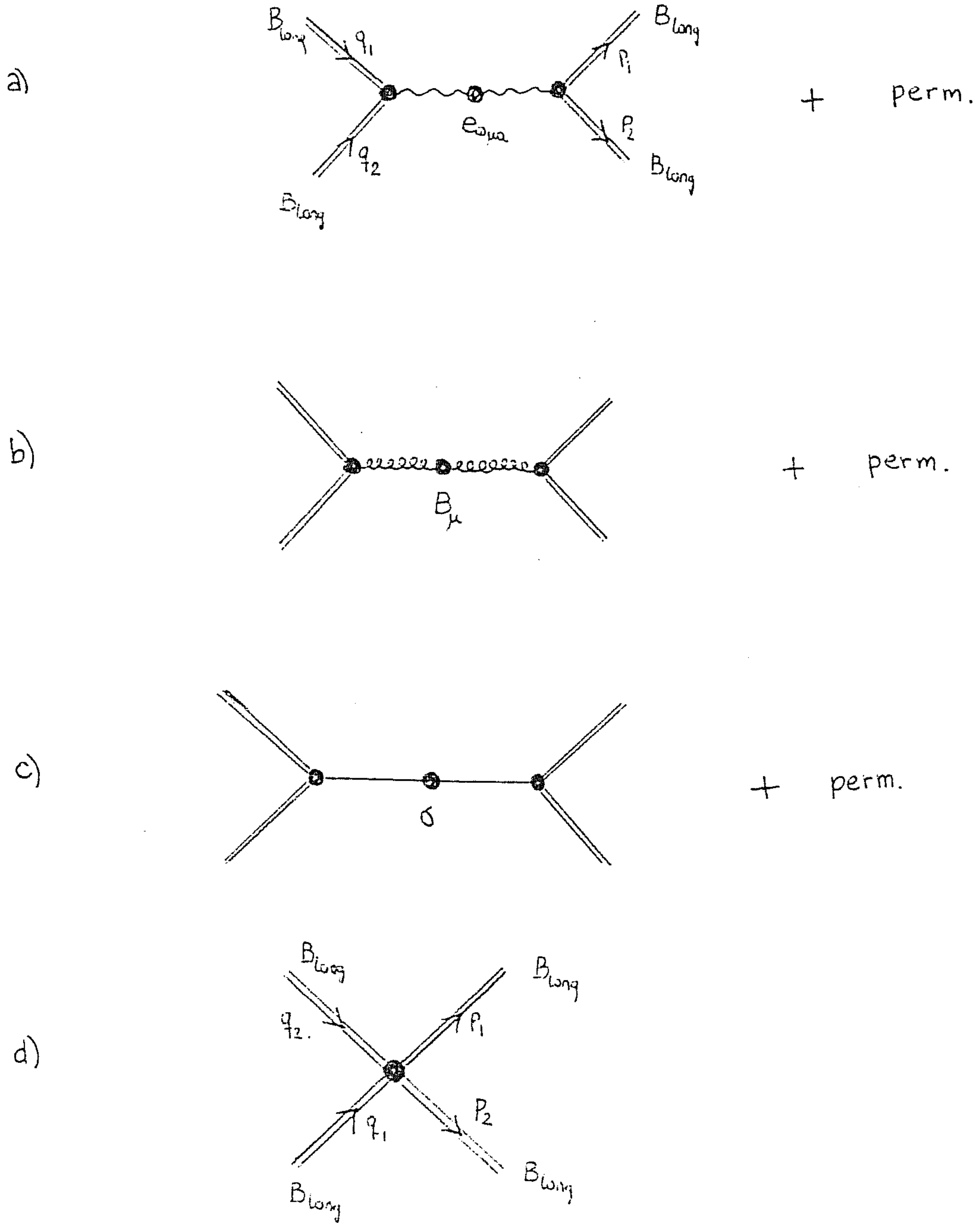

Fig.5: Tree-level diagrams contributing to the process $B_{long.}B_{long.} \rightarrow B_{long.}B_{long.}$.

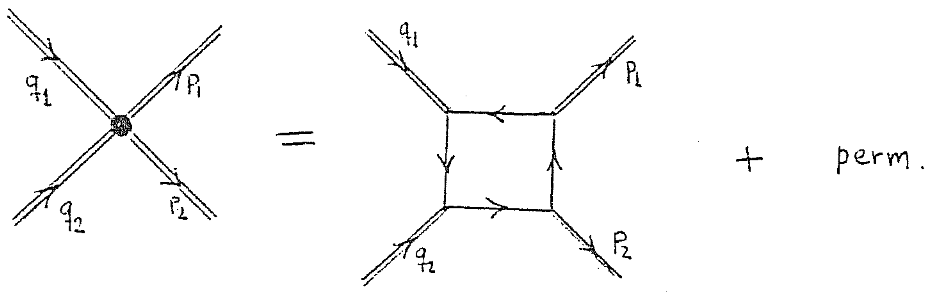

Fig.6: Effective 4-point interaction for $B_{long.}$. The permutations of external lines give the six non equivalent box diagrams.

Again, the $\tilde{e}_0$ exchange gives regular contribution T_a and the B exchange contribution T_b vanishes. Using eqs. (IV.4,22), we find that the σ exchange of fig.5c gives instead

$$T_c = \frac{24}{m_B^4}(sg(s) + tg(t) + ug(u)) + O(1) \ . \tag{IV.27}$$

We see that T_c increases with the energy as E^2/Λ^2 (cf. eq.(IV.9)). The box diagram of fig. 6 is given by

$$T_d = \langle B_{long.} B_{long.} B_{long.} B_{long.} \rangle = \frac{1}{m_B^4} \sum_{i=1}^{N} \int \frac{d^4k}{(2\pi)^4} \frac{1}{D D_0 D_1 D_2}$$

$$\mathrm{tr}[\gamma_5 \not{p}_1(\not{k}+m_i)\gamma_5 \not{p}_2(\not{k}-\not{p}_2+m_i)\gamma_5 \not{q}_2(\not{k}+\not{p}_1-\not{q}_1+m_i)\gamma_5 \not{q}_1(\not{k}+\not{p}_1+m_i)] + \text{perm.} \, , \tag{IV.28}$$

where $D = (k + p_1 - q_1)^2 - m_i^2$.

The computations are quite long and most of them have been made by computer. The result is (we give only one channel; $V_i : T_d = \sum_{i=1}^{6} V_i$)

$$V_1 = V_{E^4}^{(1)} + V_{E^2}^{(1)} + O(1) \tag{IV.29}$$

where

$$V_{E^4}^{(1)} = -\frac{1}{12}\frac{1}{m_B^4}(h(s)s^2 + 2h(s)st + 2h(t)st + h(t)t^2) \tag{IV.30}$$

$$V_{E^2}^{(1)} = \frac{1}{3m_B^2}(h(s)s+h(u)u) + \frac{1}{3m_B^4}\left[-12g(m_B^2)(s+t) - \frac{1}{2}(g(t)t+g(s)s) + 11(g(s)t+g(t)s)\right] \tag{IV.31}$$

where

$$(4\pi)^2 h(s) = -\sum_{i=1}^{N} \left(\log \frac{a_i^2 v^2}{4\pi} + J(s/m_i^2)\right) \tag{IV.32}$$

Each box diagram gives a leading E^4 contribution. Summing over all channels we find

$$\sum_{i=1}^{6} V_{E^4}^{(i)} = -\frac{4}{3}\frac{1}{m_B^2}(h(s)s + h(u)u + h(t)t) \tag{IV.33}$$

$$\sum_{i=1}^{6} V_{E^2}^{(i)} = \frac{4}{3}\frac{1}{m_B^2}(h(s)s + h(u)u + h(t)t) - \frac{24}{m_B^4}(sg(s) + tg(t) + ug(u)) \ . \tag{IV.34}$$

We see that the leading E^4 contribution has cancelled in the sum. It has remained a subleading E^2 contribution. From eqs. (IV.33,34) we finally find that the total $4B$ effective interaction is

$$T_d = \sum_{i=1}^{6} V_i = \sum_{i=1}^{6} V_{E^4}^{(i)} + \sum_{i=1}^{6} V_{E^2}^{(i)} + O(1) = -\frac{24}{m_B^4}(sg(s) + tg(t) + ug(u)) + O(1) \ . \quad (IV.35)$$

Now, using eqs. (IV.27) and (IV.35), we see that the total amplitude $T = T_a + T_b + T_c + T_d$ is again regular, i.e. $T = O(1)$, due to a cancellation of terms growing as E^2 coming from $4B$ effective interaction and (graviton's trace) σ exchange.

We thus find a correct high energy behaviour of 4-point tree amplitudes. This is a strong evidence of the consistency of this spontaneously broken theory and a non trivial sign for renormalizability.

V. 2d INDUCED QUANTUM GRAVITY AND TOPOLOGICAL PHASE OF THE THIRRING MODEL

V.1 PRELIMINARIES

The usual way to avoid the enormous mathematical complication of a quantum field theory is to study it in two space-time dimensions. In 2d we will be able to establish an exact relation betwen the fermion and the gravity theory [20].

A very familiar example of induced gravity in two dimensions is the Polyakov ansatz for string theory [21] (see also ref.[22,23]), where integration of scalars (and fermions, for the fermionic string) gives rise to the Liouville lagrangian (or its supersymmetric extension) which in the critical dimension cancels against the contribution coming from the Weyl anomaly.

Another possible approach for two-dimensional gravity is that of "topological gravity", which arises by starting from the classical "trivial" Lagrangian $\mathcal{L}_{\rm grav} = 0$ and fixing an enlarged gauge invariance by introducing appropriate ghosts and second generation ghosts [24]. A somewhat similar case we will meet here: the Thirring model [25] coupled to 2d gravity can be described in terms of a theory whose underlying Lagrangian is a topological invariant.

Here we will consider the 2d analog of the theory considered in previous sections. The only degrees of freedom are Dirac fermions and the action is invariant under local Lorentz transformations and diffeomorphisms:

$$S = \int d^2x \; \epsilon^{\mu\nu}\epsilon_{ab} \; \bar{\psi}\gamma^a \overleftrightarrow{\partial}_\mu\psi \; \bar{\psi}\gamma^b \overleftrightarrow{\partial}_\nu\psi \; V(\bar{\psi}\psi) \; . \qquad (V.1)$$

In two dimensions the spin connection does not contribute. The Dirac matrices obey $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$, where η^{ab} has signature $= \{1, -1\}$. Our notation is as follows:

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; , \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \; , \quad \psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \; . \qquad (V.2)$$

V.2 $1/N$ APPROACH

It is amusing to first study the two dimensional theory with the same tools that we have investigated the four dimensional case. In passing, we will check the consistency of the method. Extending the action (V.1) to describe N Dirac fermions (so as to define a $1/N$ expansion) we have (cf. eq.(II.4))

$$S = \int d^2x \det W[V(\rho) - \xi^{\mu a}W_{\mu a} + \phi\rho + \sum_{i=1}^{N}(\frac{i}{2}\xi^{\mu a}\bar{\psi}_i\gamma_a \overleftrightarrow{\partial}_\mu\psi_i - a_i\phi\bar{\psi}_i\cdot\psi_i)] \tag{V.3}$$

In four dimensions, we have seen that the low energy effective theory reduces to Einstein theory. In the high energy limit, the expectation is that scale invariance should be restored. In fact, we have seen that in this UV regime the 4d effective theory in the gravity sector is described by the conformal invariant action $\int\sqrt{g}(3R^{\mu\nu}R_{\mu\nu} - R^2)$. In two dimensions the situation is slightly different: it is the Einstein-Hilbert action that is conformal invariant; so it should emerge as the *high* energy limit of the effective theory[e]. Let us see that this is indeed the case. The 1-loop computation of the graviton two-point function from action (V.3) yields (the details of the derivation are omitted):

$$\langle\tilde{\xi}^s_{\mu a}(q)\tilde{\xi}^s_{\nu b}(-q)\rangle = A(q^2)(\theta_{\mu\nu}\theta_{ab} + \theta_{\mu b}\theta_{a\nu} - 2\theta_{\mu a}\theta_{\nu b}) + B(q^2)\theta_{\mu a}\theta_{\nu b}\ , \tag{V.4}$$

where

$$(4\pi)A = \sum_{i=1}^{N}\frac{1}{6}q^2\log\frac{a_i^2\langle\phi\rangle^2}{4\pi} + \frac{1}{6}(q^2 + 8m_i^2)(J(q^2/m_i^2) + 2)\ ,$$

$$(4\pi)B = -2\sum_{i=1}^{N}m_i^2(J(q^2/m_i^2) + 2)\ ,\quad m_i = a_i\langle\phi\rangle\Lambda\ . \tag{V.5}$$

The function $J(x)$ has been defined in eq. (III.16). In the limit $q^2/\Lambda^2 \to \infty$, the first term in (V.4) dominates ($A(q^2) \sim q^2\log(q^2/\Lambda^2)$, $B(q^2) \sim \Lambda^2\log(q^2/\Lambda^2)$) and it can be identified with the bilinear term in the expansion of the Einstein-Hilbert lagrangian. Q.E.D.

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[e] As well known, in two dimensions the Einstein-Hilbert lagrangian is a total derivative; dynamics comes from the induced Liouville action $\int R\Delta^{-1}R$. What is intended to prove here is that, in the high energy limit, the quadratic part in the lagrangian reduces to that coming from Einstein-Hilbert lagrangian, as required if scale invariance is to be restored.

V.3 CONNECTION WITH GEOMETRY

The quantum theory which follows from the action (V.1) is equivalent to the quantum theory which follows from

$$S' = \int d^2x \; e \; (U(\bar{\psi}\psi) + \frac{i}{2} e^{\mu a} \bar{\psi} \gamma_a \overleftrightarrow{\partial}_\mu \psi) \; , \qquad (V.6)$$

where $U = \frac{1}{8} V^{-1}$ and an auxiliar zweibein field $e_{\mu a}$ has been introduced. To prove this, we write $ee^{\mu a} = e_{\nu b} \epsilon^{\mu\nu} \epsilon^{ab}$ and observe that the dependence of S' on each component of the zweibein is linear. It is a straightforward exercise to integrate out $e_{\mu a}$ exactly, component by component, since two of them can be treated as Lagrange multipliers. One easily obtains

$$\exp(-S) = \int De \; \exp(-S') \; , \quad De \equiv De_{00} De_{10} De_{01} De_{11} \; . \qquad (V.7)$$

All symmetries have been preserved. Eq. (V.7) is an exact result and it holds true also for $N > 1$ Dirac fermions. But it does not hold in higher dimensions and it does not hold either as soon as one incorporates scalar fields.

In the conformal gauge [23], $e_{\mu a} = e^{\sigma} \eta_{\mu a}$, the connection with the quantum theory defined by action (V.1) is not evident. It is plausible -but not obvious- that the corresponding gauge choice (which picks the same integrand, Liouville action, etc.) in the functional integral of theory (V.1) should be of the form $\bar{\psi} \gamma_a \overleftrightarrow{\partial}_\mu \psi \sim e^{\sigma'} \eta_{\mu a}$, with the identification $\sigma' \sim \sigma$.

Thus we see that in two space-time dimensions theories defined by action (V.1) can be expressed in more familiar terms, where geometry appears in a standard way. For example, in the simplest case, $U = A$ =const., eqs. (V.6) and (V.7) imply that the theory (V.1) describes a "free" Dirac fermion ($c = 1$ matter) coupled 2d gravity.

V.4 TOPOLOGICAL PHASE OF THE THIRRING MODEL

To quantize the theory one can start either from action (V.1) or (V.6). It is somewhat similar to the operator quantization of Nambu-Goto or Polyakov string. There is no time derivative of $e_{\mu a}$ in eq.(V.6) so its equation of motion is a constraint:

$$e_{\mu a} = -i4V\bar{\psi}\gamma_a \overleftrightarrow{\partial}_\mu \psi \ . \qquad (V.8)$$

Local Lorentz symmetry can be fixed by eliminating the antisymmetric part of the zweibein. Diffeomorphisms are then fixed by demanding that $e_{\mu a} = e^\sigma \eta_{\mu a}$. The resulting equations of motion are

$$\partial_-\psi_+ + \frac{1}{2}\partial_-\sigma\psi_+ = -e^\sigma U'\psi_- \ , \quad \partial_+\psi_- + \frac{1}{2}\partial_+\sigma\psi_- = e^\sigma U'\psi_+ \ , \qquad (V.9)$$

$$\partial_-\psi_+^* + \frac{1}{2}\partial_-\sigma\psi_+^* = -e^\sigma U'\psi_-^* \ , \quad \partial_+\psi_-^* + \frac{1}{2}\partial_+\sigma\psi_-^* = e^\sigma U'\psi_+^* \ . \qquad (V.10)$$

Eq. (V.8) implies:

i) "Lorentz" condition:

$$\psi_-^* \overleftrightarrow{\partial}_+ \psi_- = \psi_+^* \overleftrightarrow{\partial}_- \psi_+ \ , \qquad (V.11)$$

ii) "Virasoro" conditions:

$$\psi_-^* \overleftrightarrow{\partial}_- \psi_- = 0 \ , \quad \psi_+^* \overleftrightarrow{\partial}_+ \psi_+ = 0 \ , \qquad (V.12)$$

iii) "Weyl" condition:

$$\psi_-^* \overleftrightarrow{\partial}_+ \psi_- + \psi_+^* \overleftrightarrow{\partial}_- \psi_+ = 4ie^\sigma U \ . \qquad (V.13)$$

The constraints (V.11-13) are (normal ordered) operators and they are imposed by declaring that a "physical state" is a state of the Fock space of solutions which is annihilated by the constraints. Note that the Lorentz condition (V.11) is identically satisfied for any state belonging to the Fock space. Inserting the equations of motion (V.9,10), the "Weyl" condition becomes

$$\bar{\psi}\cdot\psi \ U' = 2U \ . \qquad (V.14)$$

For a generic U, this constrains the value of $\bar{\psi}\psi$. However, there is a special case of potential U which deserves a separate treatment. If we regard (V.14) as a differential equation for U (in other words, we are requiring that (V.14) holds for arbitrary $\bar{\psi}\psi$ -a new gauge symmetry) one finds

$$U(\bar{\psi}\psi) = C(\bar{\psi}\cdot\psi)^2 = -2C\psi_+^*\psi_+\psi_-^*\psi_- \ , \quad C = \text{const.} \qquad (V.15)$$

This interaction corresponds to the Thirring model [25]. The Thirring model was extensively studied and we will not reproduce the usual treatment here (see, e.g. ref. [26]). For this particular value of the potential the action (V.6) is invariant under Weyl transformations $e_{\mu a} \to \lambda(x) e_{\mu a}$, $\psi \to \lambda(x)^{-1/2}\psi$, $\bar{\psi} \to \lambda(x)^{-1/2}\bar{\psi}$. The fermion contribution to the central charge is 1 [27].

We have seen that $e_{\mu a}$ can be exactly integrated out to give rise to the fundamental action (V.1) which, when U is given by (12), is still Weyl invariant under $\psi \to \lambda(x)^{-1/2}\psi$, $\bar{\psi} \to \lambda(x)^{-1/2}\bar{\psi}$. It is this new invariance which renders the theory "topological". A short way to see that the classical theory has only topological degrees of freedom is by writing the action (V.1) in components:

$$S = 4 \int d^2x \; \psi_+^* \overleftrightarrow{\partial}_\mu \psi_+ \; \psi_-^* \overleftrightarrow{\partial}_\nu \psi_- \epsilon^{\mu\nu} V \; . \qquad (V.16)$$

Inserting eq. (V.15) we obtain

$$S_{\rm TH} = \frac{2}{C} \int d^2x \; \partial_\mu \log \frac{\psi_+}{\psi_+^*} \partial_\nu \log \frac{\psi_-^*}{\psi_-} \epsilon^{\mu\nu} \; . \qquad (V.17)$$

In terms of the closed form $id\theta_\pm \equiv d\log(\psi_\pm/\psi_\pm^*)$ eq.(V.17) reads

$$S_{\rm TH} = \frac{2}{C} \int_M d\theta_+ \wedge d\theta_- \; , \qquad (V.18)$$

This is a topological invariant, the degree (or "winding number") of the map $\Theta : M \to S^1 \times S^1$, $\Theta \equiv (i\theta_+, i\theta_-)$. As a concrete example, let M be a Riemann surface of genus g with p punctures. Using Cauchy theorem on the "cut" Riemann surface [28] one gets

$$S_{\rm TH} = \frac{2}{C} \sum_{i=1}^{g} \left(\int_{a_i} d\theta_+ \int_{b_i} d\theta_- - \int_{a_i} d\theta_- \int_{b_i} d\theta_+ \right) + \frac{2}{C} \sum_{k=1}^{p} \oint_{C_k} \theta_+ d\theta_- = \frac{8\pi^2}{C} n \; , \quad (V.19)$$

where $n \in \mathbb{Z}$. In particular, if M is topologically a two-sphere, then $S_{TH} = 0$.

One possible approach to define a meaningful quantum field theory starting from $S_{\rm TH}$ is to derive a "cohomological" theory (see, e.g., refs. [29,30]) by an appropriate BRST gauge fixing of (V.18). This idea was explored for pure 2d gravity in ref. [24] (see also [31]). Eq. (V.18) suggests that the present theory (for M compact and oriented) is a particular case of the topological sigma model considered in [29]. This would imply a relation between

$c = 1$ conformal theories coupled to 2d gravity and the topological sigma models of [29]. An argument in favor of this unexpected issue is the following: the Thirring model can be thought of as representing a *free* Dirac fermion with twisted boundary conditions. This in turn is equivalent to a scalar field compactified on a circle. On the other hand, the action of 2d gravity basically contains the Liouville mode (contributing with $c = 1$) plus ghost terms. The whole action is thus similar to the topological action which results from the BRST quantization of (V.18). This is just an argument, as the one given by Distler [31] to establish a possible connection between the $c = -2$ conformal theory coupled to gravity and topological gravity. But here we have performed a further step towards a rigorous proof: we have shown that the classical action is a topological invariant and coincides with the corresponding action of the topological sigma model. We know what is the convenient gauge to choose -$\partial_\mu \theta_a$ self dual- in order to get an identical topological sigma model action through BRST quantization of (V.18). This is under current investigation.

V.5 REMARK ON CLASSICAL THEORIES

A curious result comes out in the classical analysis of the more general case $U = A + B\bar{\psi} \cdot \psi + C(\bar{\psi} \cdot \psi)^2$. This is a massive Thirring model with cosmological term (cf. eq.(V.6)) (for a study of what is usually called the massive Thirring model see, e.g., ref. [32]). Rescaling the fermion fields as $\varphi = e^{\sigma/2}\psi$ the equations of motion (V.9), (V.10) take the form

$$\partial_- \varphi_+ = -Be^\sigma \varphi_- + 2iC\varphi_-^* \varphi_+ \varphi_- \ , \quad \partial_+ \varphi_- = Be^\sigma \varphi_+ + 2iC\varphi_+^* \varphi_- \varphi_+ \ , \tag{V.20}$$

$$\partial_- \varphi_+^* = -Be^\sigma \varphi_-^* - 2iC\varphi_+^* \varphi_- \varphi_-^* \ , \quad \partial_+ \varphi_-^* = Be^\sigma \varphi_+^* - 2iC\varphi_-^* \varphi_+ \varphi_+ \ . \tag{V.21}$$

By inserting eqs. (V.20), (V.21) into eq. (V.13) one obtains

$$e^\sigma = -i\frac{B}{2A}(\varphi_+^* \varphi_- - \varphi_-^* \varphi_+) \ . \tag{V.22}$$

Substituting e^σ into the equations of motion (V.20), (V.21) we learn that this model, which is not manifestly Weyl invariant, is classically equivalent to a massless Thirring model with coupling $C' = C - B^2/4A$. Moreover, when the rate between the square mass and four times the cosmological constant, $B^2/4A$, equals the coupling of the four-fermion interaction, the classical theory transmutes into a "free" fermion theory coupled to gravity. This is readily extended to the case of N Dirac fermions (Gross-Neveu model [33]).

In the quantum theory these relations are no longer true. These are the consequence of the classical equation of motion for λ ($\equiv e^{\sigma}$). But if we intend to perform an integration over λ, we pick up a Jacobian with extra λ-dependent terms -coming from the Liouville action- which do not permit the naive "gaussian" integration.

VI. OUTLOOK

The underlying ideas behind induced gravity theories are, like the Burkean Foundations or the Christian Democratic tradition, extremely conservative. Gravity is not postulated as a fundamental force, but as a *derived* phenomenon. These theories are somewhat conceited: they pretend to explain the bizarre phenomena underlying general relativity (space-time deformation, black holes, etc.) just with the tools of gauge theories, without adding *anything* new. But, truly speaking, gauge theories are perfectly consistent and provide the most accurate description of natural phenomena ever known. On the other hand, nature is full of examples which resemble "long-distance" or "derived" or "induced" effects. These facts render the idea of induced gravity enormously appealing. A familiar example of long-distance effect can be found in history: for thousands of years people believed that matter was continuous. An example of derived phenomenon is provided by the Van der Waals force between atoms. It is not a *fundamental* force but a residue (or "dipole") of the electromagnetic interaction. Another example is Fermi theory of weak interactions. As expected for a theory with a dimensional coupling constant, the Fermi theory is non-renormalizable, and repeated attempts to quantize the weak interactions starting from the Fermi theory as the fundamental quantum action have met with frustration. It is now known that the Fermi theory is only a long-wavelength effective theory. The fundamental quantum theory for the weak interactions is the renormalizable gauge theory of Glashow, Salam and Weinberg. What about gravity? That symmetries of gravity are spontaneously broken is an experimental fact: the vacuum expectation value of the vierbein, $\langle e_{\mu a}\rangle$, is the minkowski metric $\eta_{\mu a}$, breaking local Lorentz transformations and diffeomorphisms down to linearized diffeomorphisms. The standard lore of spontaneously broken theories is that at short distances the theory should look as symmetric, i.e. invariances should be restored. But in Einstein theory such restoration is definitely not possible, for the appearence of the inverse metric $g^{\mu\nu}$ which makes the unbroken phase unavalaible [13,14]. The phase $\det\langle e_{\mu a}\rangle = 0$ may be explored in the first-order form. This formalism of gravity is equivalent to quantized Einstein theory only in the phase $\det\langle e_{\mu a}\rangle \neq 0$ [13]. One may wonder why solutions with $\det\langle e_{\mu a}\rangle = 0$ are to be considered physical at all. We have not an answer to such a question, but it would be unnatural to exclude such configurations from the path integral (Hawkings pointed out that changes of space-time topology occur through these degenerate metrics [34]).

The theories investigated here were shown to reduce to Einstein theory for energies much less than the Planck mass and they are known [7] to have room to naturally incorporate the other interactions, e.g., by including a Grand Unification gauge group like $SO(10)$, etc. The next and most important question is whether a consistent quantum field theory can be defined from action (II.1) and this is precisely the point on which we have focused our attention throughout this work. We have seen that all physical quantities can be computed within the frame of $1/N$ expansion and that one has well-defined S-matrix between in and out physical states. The cancellations we revealed are a strong indication that the theory is consistent and renormalizable in the $1/N$ expansion. Unlike Einstein theory, scattering amplitudes grow at most as a logarithm as the energy increases, which proves tree-level unitarity and guarantees "good" loop behaviour (power counting renormalizability). In particular, this implies that when all energies overcome the breaking scale the scale is lost, thus meaning the absence of E/M_{Planck} singularities for $M_{\text{Planck}} \to 0$ or, in other words, a restoration of scale invariance at short distances.

It is interesting to recognize that the mechanism of these cancellations involve the massive connection B_μ and the scalar σ (graviton's trace), much as in spontaneously broken YM theory. The e_0-gravity sector has the same power counting and invariances as conformal invariant gravity theory, which is known to be renormalizable [10].

We have seen that, among the class of theories parametrized by V, there is a special case in which the theory is also Weyl invariant. Furthermore, we have identified a Higgs mechanism corresponding to the spontaneous breaking of Weyl invariance. The Goldstone boson is the ("dilaton") mode ϕ which completely disappears in the unitary gauge. Einstein gravity still arises in the low energy regime of this theory and -since the computations of sect. IV were made for an arbitrary potential- this theory possesses good ultraviolet behaviour as well. There is some indication that this theory might even be finite [e]. This is strictly true at the 1-fermion loop order we have analysed. In fact, the renormalization $\phi \to Z\phi$, etc. is unnecessary in this case since, as we have remarked in sect. III, the

[e] Provided potential two-loop conformal anomaly somehow cancels (see, e.g., [35])

renormalization constant Z cancels out because of Weyl invariance. Consider the case $N = 1$. In this Weyl invariant theory, renormalizability, if true, implies finiteness. Indeed, renormalizability of this theory means that at any loop order all infinities can be absorbed into a renormalization of $\psi \to Z\psi$ in action (II.6). But since the Z factor cancels out, again, no counterterm is induced implying finiteness.

In the non conformally invariant case, Weyl invariance is broken explicitly through the terms of the potential V (see eqs.(II.4,7)). In this case, one has to renormalize the parameters defining the potential, which are therefore expected to run. For example, if $V = A_0 + B_0\bar{\psi}\psi$, then $A_R = Z'^4 A_0$, $B_R = Z'^5 B_0$ (see eqs. (II.4) and (III.17)).

Although a rigorous proof of renormalizability might seem straightforward [f], such a proof coud not be accomplished without performing 2-loop computations for the kinetic terms of the W, ρ- sector, which we have not explored in this work. Such a computation is extremely involved (it would be comparable with a three loop calculation in Einstein gravity) and probably insufficient to prove renormalizability.

We believe that the next step is to give up the $1/N$ expansion and try an understanding of the theory in the unbroken phase. We have partly succeeded in doing this in two space-time dimensions, where it was shown that it is possible to describe the Thirring model coupled to gravity in terms of a theory in which the underlying lagrangian is a topological invariant. Unfortunately, in four dimensions things are not so straightforward (there is no direct connection with geometry, it is not possible to choose a conformally flat metric, etc.). But it is a fact of life that, if we really intend to understand the short-distance structure of induced gravity, we have to learn how to deal with the seemingly unseizeble unbroken phase, where there is no metric, and the merely fermionic nature of the theory emerges.

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[f] One starts from action (II.4) and define the quantum action by adding gauge-fixing and corresponding ghosts terms, e.g.

$$S_Q = S' + \int d^4x \left[\frac{\alpha}{2}(\partial \cdot \omega)^2 + \frac{\lambda}{2}(\partial \cdot W)^2 + \partial \bar{c} \cdot Dc + \partial_\mu \bar{\eta}_a (\partial_\nu W^a_\mu + W^a_\nu \partial_\mu)\eta^\nu \right] ,$$

write down BRST transformations associated to diffeomorphisms and local $SO(3,1)$ invariances, derive corresponding Slavnov identities for the generating functional of proper vertices which restrict the possible counterterms, etc.

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OTHER RESEARCH LINES

Here we describe other research lines performed during the period of studies at SISSA.

Sect. A contains an example of the construction of a global operator formalism for conformal field theories on Riemann surfaces, subject dealt in a Master Thesis [A1-A5]. In refs. [A6-A8], which are consecutive of that Thesis, some applications of this formalism have been implemented. The operator formalism is illustrated by showing a direct application in a simple example, namely the scalar field which takes values on a circle. This is then used to construct vertex operators and to derive bosonization formulas.

The sect. B outlines the investigations performed in collaboration with Prof. A. Tseytlin. We have studied the renormalization of tadpole divergences in string theory and the correspondence with the effective action, and made several non trivial checks of the renormalizability by explicit calculations at genus 1, 2 and 3. This is the first nontrivial sign that there exists RG in string theories. The fact that the renormalization group acts within the first quantized string theory, by relating different loop orders, is crucial to understand the structure of string perturbation theory. This seems to be highly nontrivial and may provide a hint to understand second quantized string field theory [B4]. Besides, the renormalization group implies a differential equation for the partition function which expresses its divergent part in terms of the finite one (the dilaton potential). This is a linear "evolution" equation and hence can be solved explicitly. In particular, the resummation of the leading tadpole divergences is found to be $d_1 \exp(-\frac{1}{4} D d_1 g^2 \log \epsilon)$, where d_1 is the genus one vacuum amplitude and g is the string coupling.

A. GLOBAL OPERATOR FORMALISM FOR CONFORMAL FIELD THEORIES ON RIEMANN SURFACES

A.1 INTRODUCTION

In this section we illustrate the global operator formalism (developed in collaboration with the people listed in the references) by applying it to the study of Fermi-Bose equivalence (we follow sects. 2-4 of ref.[A8]). We show by explicit computation that the central terms , as well as correlation functions, corresponding to the Bose and Fermi models agree at arbitrary genus.

The construction of this operator formalism follows the standard procedure (the equations of motion are linear): a) obtain the classical equations of motion from a variational principle or from a hamiltonian; b) express the most general solution in terms of a basis; c) promote fields to operators acting on a Fock space and obeying canonical commutation relations. We use the bases introduced in refs. [A12,A13]. In ref.[A1] we have generalized the algebras and bases to treat the supersymmetric case. In ref. [A2] we apply this formalism to perform the Sugawara construction on arbitrary Riemann surface. In ref. [A3] we obtain explicit expressions (in terms of theta functions, prime forms, etc.) of the bases and construct the operator formalism for bc systems of arbitrary weight $\lambda \in \mathbb{Z},\ \mathbb{Z}+1/2$, and compute correlation functions. All results are expressed in terms of globally well-defined tensors. In [A4] we construct the operator formalism for string theory (scalar matter). A hamiltonian dictating time evolution is introduced; the standard two-point functions naturally arise. The formalism for superstring theory is constructed in [A6], where it is also obtained an interesting expression for the world-sheet supersymmetry generator at higher genus. In ref. [A7] it is shown that the operator formalism naturally carries an Arakelov metric (with respect to the Bergmann metric in the two-punctured Riemann surface). Tachyon amplitudes are computed in detail; a discussion about the residual $U(1)\times U(1)$ of the torus is given. In [A8] these techniques are applied to the study of bosonization. Some interesting issues emerge. Refs. [A9-A11] are reviews. Refs. [A14-A15] are some works of other people on this subject.

A.2 GLOBAL QUANTIZATION OF THE SCALAR FIELD ON A CIRCLE

We parametrize the Riemann surface Σ by using a "time" defined as follows

$$\tau(P) = Re \int_{P_0}^{P} dk \quad ; \quad P_0, P \in \Sigma \tag{A.1}$$

where dk is the third-kind abelian differential with simple poles at two points P_+ and P_- in general position with residues +1 and -1 respectively , and such that its periods over all cycles are imaginary *. Under these conditions the one-form dk is uniquely determined, and in terms of prime-forms and θ-functions reads

$$dk = d\left(\log \frac{E(P,P_+)}{E(P,P_-)}\right) - \pi\eta_a(\Omega_2^{-1})_{ab} \int_{P_-}^{P_+} (\eta_b - \bar{\eta}_b) \tag{A.2}$$

where η_a are the g abelian differentials normalized according to $\oint_{\alpha_a} \eta_b = \delta_{ab}$; $\oint_{\beta_a} \eta_b = \Omega_{ab}$. It follows from eq.(A.1) that $\tau(P)$ goes to $\mp\infty$ for $P \to P_\pm$. The level curves of the function $\tau(P)$ are conventionally called C_τ.

Our starting point is the equation of motion for a free scalar field which takes values on a circle of radius one:

$$\partial\bar{\partial}\varphi(P) = 0 \quad ; \quad P \in \Sigma\ , \quad P \neq P_\pm \tag{A.3}$$

This implies that $\partial\varphi(P)$ ($\bar{\partial}\varphi(P)$) is an everywhere holomorphic (antiholomorphic) one form excepting the points $P_\pm$. Let $\{\omega_n\}$ denote the basis for meromorphic 1-differential, holomorphic outside $P_\pm$, obtained in [A12]. Then we can expand

$$\partial\varphi(P) = \sum_n \alpha_n\omega_n \equiv \sum_{i=1}^{g} \epsilon_i\eta_i + \sum_{n\notin I} \alpha_n\omega_n \tag{A.4}$$

$$\bar{\partial}\varphi(P) = \sum_n \bar{\alpha}_n\bar{\omega}_n \equiv \sum_{i=1}^{g} \bar{\epsilon}_i\bar{\eta}_i + \sum_{n\notin I} \bar{\alpha}_n\bar{\omega}_n \tag{A.5}$$

* These points are the analog of $z = 0, \infty$ in the sphere. The time parametrization given by $\tau(P)$ was already used by S.Mandelstam (Phys. Rep.C13,(1974)259) for the light cone interacting string. dk is the 1-form implicit in this work and studied in greater detail by Krichever and Novikov [A12] and S.Giddings and S.Wolpert (Commun. Math. Phys.109, (1987)177).

where $I \equiv [-g/2, g/2)$ and $\omega_{g/2} \equiv dk$. Since $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, the following conditions must be imposed

$$\oint_{C_\tau} d\varphi = 2\pi l \;\; ; \;\; \oint_{\alpha_a} d\varphi = 2\pi n_a \quad \oint_{\beta_a} d\varphi = 2\pi m_a \;\; ; \;\; l, n_a, m_a \in \mathbb{Z} \tag{A.6}$$

Insertion of the expansions (A.4), (A.5) into the above conditions gives

$$\begin{aligned} \alpha_{g/2} - \bar{\alpha}_{g/2} = -il \;\; ; \;\; \alpha_{g/2} + \bar{\alpha}_{g/2} \equiv -ip \\ \epsilon_a + \bar{\epsilon}_a = 2\pi n_a - \sum_{n \notin I} (\alpha_n a_n^a + \bar{\alpha}_n \bar{a}_n^a) \\ \epsilon_b \Omega_{ba} + \bar{\epsilon}_b \bar{\Omega}_{ba} = 2\pi m_a - \sum_{n \notin I} (\alpha_n b_n^a + \bar{\alpha}_n \bar{b}_n^a) \end{aligned} \tag{A.7}$$

where $a_n^a = \oint_{\alpha_a} \omega_n$, $b_n^a = \oint_{\beta_a} \omega_n$. This can be regarded as a system of $2g+2$ equations with $2g+2$ unknowns $\epsilon_a, \bar{\epsilon}_a, \alpha_{g/2}, \bar{\alpha}_{g/2}$. Inserting the solution into $d\varphi$ and then integrating we obtain

$$\begin{aligned} \varphi(P) =& \phi - ip\tau(P) + l\sigma(P) + \pi i [\Omega_2^{-1}(\bar{\Omega} n - m) \int_{P_0}^{P} \eta - c.c] + \\ & \sum_{n > g/2} (\alpha_n \phi_n(P, P_-) + \bar{\alpha}_n \bar{\phi}_n(P, P_-)) + \sum_{n < -g/2} (\alpha_n \phi_n(P, P_+) + \bar{\alpha}_n \bar{\phi}_n(P, P_+)) \end{aligned} \tag{A.8}$$

where $\phi_n(P, Q_0)$ are harmonic (single-valued) functions given by

$$\phi_n(P, Q_0) = \int_{Q_0}^{P} (\omega_n - (F_{na}\eta_a + \bar{G}_{na}\bar{\eta}_a)) \tag{A.9}$$

with

$$F_{na} = \frac{i}{2} (\Omega_2^{-1})_{ab} (\bar{\Omega}_{bc} a_n^c - b_n^b) \; , \quad G_{na} = \frac{i}{2} (\Omega_2^{-1})_{ab} (\bar{\Omega}_{bc} \bar{a}_n^c - \bar{b}_n^b) \tag{A.10}$$

Using the results of ref [A3] for the ω_n and η_a, one can obtain explicit expressions for the present $\phi_n(P, Q_0)$ in terms of prime-forms and θ-funtions. $\sigma(P)$ is an angular variable given by $\sigma(P) = Im \int_{P_0}^{P} (dk - 2F_{ag/2}\eta_a)$. This new variable is a harmonic single-valued function on the Riemann surface without $P_\pm$ and a slit from P_+ to P_-.

In the quantized theory the α_n, $\bar{\alpha}_n$ are operators acting on a Fock space. The vacuum state $|0; p = 0>$ is defined to be the state annihilated by α_n, $\bar{\alpha}_n$ with $n > g/2$; the dual vacuum state $< 0; p = 0|$ is annihilated by α_n, $\bar{\alpha}_n$ with $n < -g/2$. From the canonical commutation relations for $\varphi(P)$ it follows

$$[\alpha_n, \alpha_m] = 2\gamma_{nm} \; , \quad [\bar{\alpha}_n, \bar{\alpha}_m] = 2\bar{\gamma}_{nm} \; , \quad [\alpha_n, \; \bar{\alpha}_m] = 0 \; , \quad [\phi, p] = i \; . \tag{A.11}$$

One may similarly treat the case of compactification on orbifolds by suitably modifying the monodromy conditions (A.6) and by constructing the appropriate basis $\{\omega'_n\}$, according to the number of punctures or branch cuts of the relevant Riemann surface. Further restrictions (this time on the eigenvalues of p) may arise if one requires the "wave function" $e^{ip.X}$ to be single-valued.

A.3 BOSONIZATION OF FERMIONS OF SPIN λ

It has been argued in ref. [A.16] that a fermion bc-system of weights λ, $1-\lambda$ is equivalent to a bosonic scalar field coupled to a background charge $Q = 2\lambda - 1$ described by the action

$$S[\varphi] = \frac{1}{2\pi}\int(\partial\varphi\bar{\partial}\varphi - \frac{i}{4}Q\sqrt{g}R\varphi d^2z) \tag{A.12}$$

The equation of motion is

$$\partial\bar{\partial}\varphi = \frac{i}{8}Q\sqrt{g}Rd^2z \tag{A.13}$$

The quantization is performed as in the previous section; the additional background term in the equation of motion is regarded as a perturbation. The energy-momentum tensor corresponding to the action (A.12) is *

$$T = -\frac{1}{2}\partial\varphi\partial\varphi - \frac{i}{2}Q\nabla_z\partial_z\varphi(dz)^2 \tag{A.14}$$

The generalized Virasoro operators (or KN operators) will be therefore given by

$$L_r = -\frac{1}{2}\sum_{n,m} l^r_{nm} : \alpha_n\alpha_m : -\frac{i}{2}Q\sum_m S^r_m\alpha_m \tag{A.15}$$

where $l^r_{nm} \equiv \frac{1}{2\pi i}\oint_{C_\tau} e_r\omega_n\omega_m$ and $S^r_m \equiv \frac{1}{2\pi i}\oint_{C_\tau} e_r\nabla_z\omega_m$. Now we would like to compare the algebra (1.8) corresponding to a bc-system of weight λ, $1-\lambda$ with the algebra of L_n, j_n ($j(P) \equiv \partial\varphi = \sum_n \alpha_n\omega_n$ and thus $j_n = \alpha_n$) corresponding to the soliton field φ. Using the algebra (A.11) of the mode operators one readily finds the desired result: the algebras are identical, up to trivial cocycles.

Primary fields of conformal dimension $(\frac{q(q+Q)}{2}, \frac{q(q+Q)}{2})$ are represented by a vertex operator

* As usual, the conjugate counterparts are not written explicitely.

$$V_q(P) = h^{\frac{q(q+Q)}{2}} : e^{iq\varphi(P)} : \qquad (A.16)$$

where $h \equiv |\omega|^2$ is the Arakelov-type metric induced by the operator formalism [A7] and is given by

$$|\omega|^2 = \left| \frac{E(P_+, P_-)}{E(P, P_+)E(P, P_-)} \right|^2 exp(-\pi \int_{P_-}^{P} (\eta - \bar{\eta})_a (\Omega_2^{-1})_{ab} \int_{P_+}^{P} (\eta - \bar{\eta})_b) \qquad (A.17)$$

Indeed, by following the same lines as [A14], it is an easy exercise to prove that

$$[L_n, V_q] = L_{e_n} V_q \ , \quad [\bar{L}_n, V_q] = L_{\bar{e}_n} V_q \qquad (A.18)$$

Remarkably, the form to be used in the definition of the vertices is uniquely determined (up to a multiplicative constant) by the conditions (A.18) which, as explained in [A4], are needed for unitarity. That $|\omega|^2$, which has been found by looking at scattering amplitudes, is precisely the form which follows from conditions (A.18), has been shown in [A14].

In terms of the metric (A.17), the perturbation term in (A.13) reads

$$\sqrt{g} R d^2 z = -4\partial\bar{\partial} \log h(P) = -8\pi\eta_a (\Omega_2^{-1})_{ab} \bar{\eta}_b \quad P \neq P_+, \ P_- \qquad (A.19)$$

We see from eq.(A.13) that, in this case, $\partial\bar{\partial}\varphi$ is proportional to the Bergmann metric on the Riemann surface without the points $P_\pm$.

It is convenient to separate the soliton, center of mass, and quantum fluctuation parts by writing $\varphi(P) = \phi(\tau) + \varphi_{nm} + \hat{\varphi}$. Now, consider for example the case $\lambda = 1/2$. Using the algebra (A.11) and the definition of the vacuum Fock space one can show that the zero soliton part of $< V_{q_1} V_{q_2} >$ is

$$\begin{aligned} &< 0|T\{V_{q_1}(P_1)V_{q_2}(P_2)\}|0 >_{00} = \\ &= |E(P_1, P_2)|^{-2q^2} exp[\frac{-q^2\pi}{2} \int_{P_1}^{P_2} (\eta - \bar{\eta})_a (\Omega_2^{-1})_{ab} \int_{P_1}^{P_2} (\eta - \bar{\eta})_b] 2\pi\delta(q_1 + q_2) \\ &\equiv (F(P_1, P_2))^{-q^2} 2\pi\delta(q_1 + q_2) \qquad (A.20) \end{aligned}$$

where we have used the non trivial identity [A7] (see also [A4])

$$\frac{1}{2} \sum_{n>g/2} \sum_{m<-g/2} \gamma_{mn} \phi_m(P_j, P_+) \phi_n(P_i, P_-) + c.c. =$$

$$= \frac{1}{2}\log\left|\frac{E(P_i,P_j)E(P_-,P_+)}{E(P_-,P_j)E(P_i,P_+)}\right|^2 - \frac{\pi}{2}\sum_{a,b=1}^{g}\int_{P_-}^{P_i}(\eta-\bar\eta)_a(\Omega_2^{-1})_{ab}\int_{P_+}^{P_j}(\eta-\bar\eta)_b. \qquad (A.21)$$

This kind of expansion generalizes the notion of Laurent series of the complex plane.

Eq. (A.20) is in agreement with well known results (see, e.g., [A17]).

Vertex operators with $q = \pm 1$ describe the fermi bilinears $b\bar{b}$ and $c\bar{c}$ respectively. To show that this is in fact the case, let us consider the following correlation function

$$A(P_1, ..., P_I, P, Q) \equiv$$

$$\frac{< 0|T\{(\prod^I h_i^\lambda : e^{i\varphi(P_i)} :)(h^\lambda : e^{i\varphi(P)} :)(h^{1-\lambda} : e^{-i\varphi(Q)} :) : e^{-\frac{iQ}{8\pi}\int\sqrt{g}R\varphi} :\}|0 >}{< 0|T\{(\prod^I h_i^\lambda : e^{i\varphi(P_i)} :) : e^{-\frac{iQ}{8\pi}\int\sqrt{g}R\varphi} :\}|0 >} \qquad (A.22)$$

The quantum fluctuation part is computed in a similar way as we computed (A.20), namely using the algebra of modes operators and the identity (A.21). The soliton part coincides with the *ad-hoc* expression given in [A17]. The average over all soliton sectors leads to a sum over spin structures of the Fermi theory [A17]. Taking a particular even spin structure one finds that the anholomorphic parts exactly cancel leaving a correlation function which is the modulus squared of a meromorphic function in P_i, Q_j. By using standard mathematical identities [A18], the chiral projection can be written as

$$\frac{1}{E(P,Q)}\left(\frac{E(P,P_-)}{E(Q,P_-)}\right)^{(2\lambda-1)(g-1)}\left(\frac{\sigma(P)}{\sigma(Q)}\right)^{(2\lambda-1)}\frac{\theta(Q-P+u(\lambda))}{\theta(u(\lambda))} \qquad (A.23)$$

which is the well-known bc system correlation function $< 0|T\{b(P)c(Q)\}|0 >$, found in this framework in ref. [A3].

Bosonization has been a useful tool in the study of string theory and conformal field theories in general. It has been used, for instance, in proving the equivalence of the GS and NSR superstring or in constructing the covariant fermion emission vertex, and it played an important role in understanding the symmetries of the heterotic string.

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B. RENORMALIZATION GROUP AND STRUCTURE OF STRING PERTURBATION THEORY

B.1 INTRODUCTION

A remarkable property of string theory is that the tree level string dynamics is dictated by the fixed points of the 2d renormalization group (RG) [B1]. The extension of this equivalence to the string loop level would be important since it would imply the existence of "non-perturbative conformal theories", which move out of the tree level fixed point to determine the true string vacuum structure. The mechanism to implement this idea was suggested by Fischler and Susskind [B2], who observed that in general renormalization of string loop tadpole infinities can be reinterpreted as renormalization of string σ-model couplings (for previous suggestions, see ref.[B3]; for a recent review and more references see, e.g., refs. [B4]). However, it is by no means trivial that string combinatorics is such that tadpole loop infinities can be consistently cancelled out by renormalizing σ-model couplings. In order for the renormalization group (RG) to be realized in a consistent way, the coefficients of $\log^n \epsilon$-infinities should satisfy certain relations, namely they should be expressed in terms of the $\log \epsilon$-coefficients (β-function coefficients). In particular, one should check that $\log \epsilon$ infinities properly "exponentiate" (this was partly checked for external leg infinities in ref. [B5]). There had been no discussion of multiple *tadpole* logarithmic infinities and their "exponentiation". In ref. [B6], we have checked the realization of RG in string loops taking as an example the $\log^2 \epsilon$ massless tadpole infinities in two-loop tachyonic ampiltudes and in the three-loop partition function in the closed Bose string theory.

B.2 GLUING TOPOLOGIES BY THE RENORMALIZATION GROUP [B9]

The generating functional for the massless closed string amplitudes has the following structure (see for details ref.[B4])

$$\hat{Z} = \hat{Z}_0 + \hat{Z}_{\text{loop}} \ , \ \hat{Z}_0 = S_0 + O(\log \epsilon) \ , \qquad (B.1)$$

$$S_0 = (\alpha')^{-D/2} \int d^D x \sqrt{G} e^{-2\Phi} [-\frac{2}{3}(D-26) + \alpha'(R + 4(\partial\Phi)^2) + ...] \ , \tag{B.2}$$

$$\hat{Z}_{\rm loop} = \bar{Z} + \text{“momentum dependent” contributions} \ , \tag{B.3}$$

$$\bar{Z} = (\alpha')^{-D/2} \int d^D x \sqrt{G} W \ , \quad W = \sum_{n=1}^{\infty} \hat{d}_n e^{2(n-1)\Phi} \ . \tag{B.4}$$

$\hat{d}_n$ contain logarithmic divergences corresponding to massless tadpole factorizations

$$\hat{d}_n = d_n + \sum_{k=1}^{n-1} d_n^{(k)} t^k \ , \quad t \equiv \log \epsilon \ , \quad d_n = \text{finite} \ , \tag{B.5}$$

i.e. $W(\Phi, 0) = \sum_{n=1}^{\infty} d_n e^{2(n-1)\Phi} \equiv V(\Phi)$.

The basic assumption of renormalizability of $\hat{Z}$, $\hat{Z}(G(\epsilon), \Phi(\epsilon); \epsilon) = \hat{Z}_R(G^R, \Phi^R)$, implies that

$$\frac{\partial \hat{Z}}{\partial \log \epsilon} - \beta^G_{\mu\nu} \frac{\partial \hat{Z}}{\partial G_{\mu\nu}} - \beta^\Phi \frac{\partial \hat{Z}}{\partial \Phi} = 0; \tag{B.6}$$

The tadpole contributions to the β-functions can be found from the single tadpole factorization rule [B7,B4]*

$$\beta^G_{\mu\nu} = -\frac{1}{4} e^{2\Phi} (2V + \frac{\partial V}{\partial \Phi}) G_{\mu\nu} \ , \tag{B.7}$$

$$\beta^\Phi = -\frac{1}{16} e^{2\Phi} (2DV + (D-2) \frac{\partial V}{\partial \Phi}) \ , \tag{B.8}$$

As a result, the β-functions (B.7), (B.8) are related to the corresponding “potential” term in the effective action

$$\beta^i = \kappa^{ij} \frac{\partial S_{\rm pot.}}{\partial \varphi^j} \ , \tag{B.9}$$

$$\kappa^{ij} = k \begin{pmatrix} G_{\mu\alpha} G_{\nu\beta} & \frac{1}{4} G_{\alpha\beta} \\ \frac{1}{4} G_{\mu\nu} & \frac{1}{16}(D-2) \end{pmatrix} , \ k = -e^{2\Phi} \alpha'^{D/2} G^{-1/2} \ ,$$

$$S_{\rm pot.}(\varphi) = \bar{Z}_R(\varphi) = (\alpha')^{-D/2} \int d^D x \sqrt{G} V(\varphi) \ . \tag{B.10}$$

κ^{ij} is the matrix which appears in the graviton-dilaton propagator corresponding to the tree effective action (B.2). Substituting (B.9) into (B.8) we find the following equation for the partition function (B.3)

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* An analysis of β functions accounting for both “local” and “modular” contributions can be found in ref.[B10].

$$\frac{\partial \bar{Z}}{\partial t} = \frac{\partial \bar{Z}_R}{\partial \varphi^i} \kappa^{ij} \frac{\partial \bar{Z}}{\partial \varphi^j} , \quad \bar{Z} = \bar{Z}(\varphi, t) , \quad \bar{Z}_R(\varphi) = \bar{Z}(\varphi, 0) . \tag{B.11}$$

Rewriting (B.11) explicitly in terms of $W(\Phi, t)$ using eq.(B.4) we get

$$\frac{\partial W}{\partial t} = -\frac{1}{16} e^{2\Phi} [(D-2) \frac{\partial V}{\partial \Phi} \frac{\partial W}{\partial \Phi} + 2D(\frac{\partial V}{\partial \Phi} W + V \frac{\partial W}{\partial \Phi}) + 4DVW] , \tag{B.12}$$

The explicit solution is

$$\frac{\partial W}{\partial t} = A(\Phi, \frac{\partial}{\partial \Phi}) W , \quad W(\Phi, t) = e^{tA} V(\Phi) , \tag{B.13}$$

i.e.

$$W(\Phi, t) = \exp \left(-\frac{1}{16} t e^{2\Phi} \left[2D(2V + \frac{\partial V}{\partial \Phi}) + (2DV + (D-2) \frac{\partial V}{\partial \Phi}) \frac{\partial}{\partial \Phi} \right] \right) V(\Phi) . \tag{B.14}$$

Eq.(B.14) gives explicit expression for the divergent parts of the partition function in terms of its finite part V. In particular, we obtain for the $\log \epsilon$-part $W^{(1)}(\Phi)$ of W (cf. (B.4), (B.5))

$$W^{(1)}(\Phi) = -\frac{1}{16} e^{2\Phi} [(D-2)(\frac{\partial V}{\partial \Phi})^2 + 4DV \frac{\partial V}{\partial \Phi} + 4DV^2] . \tag{B.15}$$

One can also find the resummation of the most divergent terms at each genus ($d_1^{(0)} = d_1$)

$$\tilde{W}(\Phi, t) \equiv \sum_{n=1}^{\infty} d_n^{(n-1)} e^{2(n-1)\Phi} t^{n-1} . \tag{B.16}$$

Since the most divergent contributions correspond to the "maximal" factorizations on genus-one parts the result for $\tilde{W}$ is found from (B.14) by replacing V by its one loop value d_1 *

$$\tilde{W} = d_1 \exp(-\frac{1}{4} t e^{2\Phi} D d_1) . \tag{B.17}$$

The RG thus implies that the coefficients of the leading divergences at genus n should be given by

$$d_n^{(n-1)} = \frac{1}{(n-1)!} (-\frac{1}{4})^{n-1} d_1^n D^{n-1} . \tag{B.18}$$

* To find a similar expression for the sum of subleading ($O(\log^{n-2} \epsilon)$) divergences one is to replace V by $d_1 + e^{2\Phi} d_2$ and linearize (20) with respect to d_2.

This is a prediction from the renormalizability of the massless divergences. In ref. [B9] it has been then analysed the leading logarithmic divergence in the closed string partition function at arbitrary genus n and showed (using Schottky parametrization) that its coefficient is proportional to D^{n-1} (D is the space-time dimension). This conclusion is consistent with (B.18).

B.3 RENORMALIZATION OF MULTIPLE INFINITIES

Including a graviton and dilaton background we get the σ-model action

$$I(G,\Phi) = \frac{1}{4\pi\alpha'}\int d^2z\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + \frac{1}{4\pi}\int d^2z\sqrt{h}R^{(2)}\Phi(X). \qquad (B.19)$$

Computing the amplitudes with the action (B.19) and regularizing the massless scalar tadpole divergences with the help of a cutoff ϵ we should be able to check that all the ϵ dependence cancels out. In terms of the renormalized fields, the action (B.19) reads

$$I(G(\epsilon),\Phi(\epsilon)) = I(G^R,\Phi^R) + \delta I(G^R,\Phi^R;\epsilon). \qquad (B.20)$$

The counterterm action δI will be given as an expansion in the renormalized string coupling $g = e^{\Phi^R_{(const)}}$ ($\Phi^R_{(const)}$ is the constant part of the renormalized dilaton field) and α'. The "tadpole" part of the counterterm will have the form

$$\delta I = \frac{1}{4\pi\alpha'}\int d^2z\sqrt{h}\partial_a X^\mu \partial^a X^\nu \delta G^R_{\mu\nu} + \frac{1}{4\pi}\int d^2z\sqrt{h}R^{(2)}\delta\Phi^R. \qquad (B.21)$$

Taking into account the (tadpole) counterterm, the scattering amplitude can be represented as

$$A_M = < V_1 ... V_M e^{-\delta I} >_0 + < V_1 ... V_M e^{-\delta I} >_1 + < V_1 ... V_M e^{-\delta I} >_2 + ... \, . \qquad (B.22)$$

and should be finite. A_M can be rewritten as (we shall restrict consideration to the g^2-order. Note that the tree level contribution is $\sim g^{-2}$)

$$A_M = (A_M)_{\text{finite}} + (A_M)_{\log\epsilon} + (A_M)_{g^2\log\epsilon} + (A_M)_{g^2\log^2\epsilon} \qquad (B.23)$$

One should be able to prove that all divergent parts are automatically cancelled upon the insertion of the counterterms. Indeed, in [B6] we have proved that the $\log^2\epsilon$ counterterms are universal (e.g., the same counterterms provide finiteness both of two-loop scattering amplitudes and of the three-loop partition function) and are related to the $\log\epsilon$-counterterms (beta-functions) in the standard way dictated by the renormalization group. We illustrate this by taking as an example the $(A_M)_{g^2 \log^2 \epsilon}$ part. From the relations for the counterterms implied by the RG and the correspondence with effective action, it turns out [B6] that the $\log^2 \epsilon$ part of the two loop amplitude should obey the following relation:

$$< V_1 ... V_M >_2 \Big|_{\log^2 \epsilon} = \frac{d_1}{4 d_0} g^2 \log \epsilon < V_1 ... V_M V_g >_1 \Big|_{\log \epsilon} . \qquad (B.24)$$

We would like to know whether string perturbation theory satisfies this requirement. This two-loop check would imply that scattering amplitudes are finite even to the non trivial order $g^2 \log^2 \epsilon$ and it would confirm that the RG consistently acts on string loops. This is important since, as we have seen in the previous section, the RG provides loop information and may give a hint to resum string perturbation theory [B4].

It is convenient to use the "period matrix" representation for the two loop measure. The two loop contribution to the M-tachyon scattering amplitude can be written as

$$< V_1 ... V_M >_2 = c_2 g^2 \int_{F_2} d\mu_2 \int \prod_{i=1}^{M} d^2 z_i \prod_{i<j} F(z_i, z_j)^{\alpha' k_i . k_j} \qquad (B.25)$$

where

$$F(z_i, z_j)^2 = |E(z_i, z_j)|^2 exp[\frac{\pi}{2} \int_{z_i}^{z_j} (\omega_a - \bar{\omega}_a) \int_{z_i}^{z_j} (\omega_b - \bar{\omega}_b)(\Omega_2^{-1})_{ab}] \ ; \quad \Omega_2 \equiv Im\Omega \ . \ (B.26)$$

Consider the limit $\Omega_{12} \to 0$. The genus-2 Riemann surface degenerates to two tori Σ_1, Σ_2. The behaviour of the measure is displayed in eq.(A.6). As concerns the integrand, we are going to use the following formulas, derived in appendix C of ref.[B6]:

$$E(z, w) \to E_1(z, w) + O(\Omega_{12}^2), \qquad \text{if} \ \ z, w \in \Sigma_1, \qquad (B.27)$$

$$\omega_1(z) \to \nu(z) + O(\Omega_{12}^2), \qquad \text{if} \ \ z \in \Sigma_1, \qquad (B.28)$$

$$\omega_2(z) \to \frac{\Omega_{12}}{2\pi i} \frac{\partial}{\partial z} \left(\frac{\theta_1'(z - p_1|\tau)}{\theta_1(z - p_1|\tau)} \right) dz + O(\Omega_{12}^2), \quad \text{if} \ \ z \in \Sigma_1, \qquad (B.29)$$

where $E_1(z, w) = \frac{\theta_1(z-w|\tau)}{\theta_1'(0|\tau)} (dz)^{-1/2} (dw)^{-1/2}$, $\nu(z) = dz$ are respectively the prime-form and abelian differential on the torus Σ_1 and $\tau = \Omega_{11} + O(\Omega_{12}^2)$.

In particular, using eq. (B.27), we obtain

$$\prod_{i<j}^{M} |E(z_i, z_j)|^{\alpha' k_i . k_j} \to \prod_{i<j}^{M} |E_1(z_i, z_j)|^{\alpha' k_i . k_j} + O(\Omega_{12}^2) \; ; \quad z_i, z_j \in \Sigma_1 \tag{B.30}$$

Using eqs.(B.28) and (B.29), it is a straightforward exercise to obtain

$$\sum_{i,j=1}^{M} \pi\alpha' k_i.k_j \int_{z_i}^{z_j} (\omega_a - \bar{\omega}_a) \int_{z_i}^{z_j} (\omega_b - \bar{\omega}_b)(\Omega_2^{-1})_{ab} =$$

$$\sum_{i,j=1}^{M} \pi\alpha' k_i.k_j \left[\frac{1}{Im\Omega_{11}}(z_{ji} - \bar{z}_{ji})^2 - \frac{|\Omega_{12}|^2}{2\pi^2 Im\Omega_{22}} K(z_j|\tau)\bar{K}(z_i|\tau) + ...\right] \tag{B.31}$$

where we have also used energy-momentum conservation, $\sum_{i=1}^{M} k_i = 0$. From eqs.(B.30) and (B.31), one can see that the amplitude (B.25) has the form

$$< V_1 ... V_M >_2 \sim \int_\epsilon \frac{d^2\Omega_{12}}{|\Omega_{12}|^4} + \int_\epsilon \frac{d^2\Omega_{12}}{|\Omega_{12}|^2} + \text{finite}$$

The first term corresponding to a tachyon exchange * , the second term to a massless scalar exchange, and the remaining to higher-level contributions. Explicitly, the massless exchange contribution is

$$< V_1 ... V_M >_2 \Big|_{\log\epsilon} = \frac{c_2}{2} g^2 \frac{4\pi}{(2\pi^2)^D} \left(\frac{2}{\pi}\right)^4 \int_{F_1} \frac{d^2\Omega_{22}}{(Im\Omega_{22})^{14} |\Delta_{(1)}|^{16}} \int_\epsilon \frac{d^2\Omega_{12}}{|\Omega_{12}|^2}$$
$$\times \left(\int_{F_1} \frac{d^2\Omega_{11}}{(Im\Omega_{11})^{14} |\Delta_{(1)}|^{16}} \int \prod_{i=1}^{M} d^2 z_i \prod_{i<j}^{M} \chi(z_{i,j}|\tau)^{\alpha' k_i . k_j} \right.$$
$$\left. \left[\frac{D}{4} - \sum_{i,j} \frac{\alpha' k_i.k_j}{16\pi} K(z_j|\tau)\bar{K}(z_i|\tau)\right] \right) \tag{B.32}$$

We recognize the scattering amplitude of M tachyons and 1 massless scalar particle on the torus. Eq. (B.32) thus gives

$$< V_1 ... V_M >_2 \Big|_{\log\epsilon} = \frac{d_1}{4 d_0} g^2 \log\epsilon < V_1 ... V_M V_g >_1 \; . \tag{B.33}$$

which proves (B.24) and thus establishes $(A_M)_{g^2 \log^2 \epsilon} = 0$.

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* We assume the analytic continuation prescription of ref. [B11] to eliminate the divergences associated with the tachyon mode.

B.4 CONCLUDING REMARKS

The present results give strong evidence -which was absent earlier- that the combinatorics of string diagrams, with the particular choice of weights for string loop contributions implied by the correspondence with the effective action, is precisely what we need in order to have RG.

RG "glues" together different orders of the string coupling constant. This suggests a reformulation of 1st-quantized string theory in which the RG is incorporated in a manifest way (as in field theory) and perturbation theory is improved (i.e., we have automatic "exponentiation").

We have also seen that the very assumption of renormalizability determines all divergent parts in terms of the finite part, and written down explicit expressions in some cases. It is interesting to note that under some assumptions one can also get differential equations for the finite part which, as in matrix models, might provide a nonperturbative definition for V. For example (see ref. [B9]), if we assume that $V \in \mathrm{Ker}(A)$ in eq. (B.13) we obtain a differential equation with a "non-perturbative" (for $D > 2$) solution $V(\Phi) = b\,\exp(a\Phi)$, where $a = \frac{-2\sqrt{D}}{\sqrt{D}+\sqrt{2}}$.

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