ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

Academic Year 1988-89

C-Theorem and Four Fermion Interactions in Two Dimensions

Thesis submitted for the
degree of "Doctor Philosophiae"

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TRIESTE

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Acknowledgments

This thesis is the result of one year and a half spent at the Physics Department of Pisa University. Let me then acknowledge the warm hospitality of all the people I met there, and in particular I want to name Beppe Curci, Giampiero Paffuti and Paolo Rossi, with whom I cooperated more closely. A special mention is deserved by Andrea Viceré, with whom I shared many computer troubles and a lot of evenings at the Department.

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Chapter 1

Introduction

The main concern of this thesis is the study of an explicit realization of Zamolodchikov's C-theorem [1,2,3] as provided by the most general four fermion quantum field theory in two dimensions [32,34,47].

The interest in two dimensional four fermion models traces back to the by now classical works of Thirring [11], Kleiber [12,13] and Johnson [14], and, more recently, Gross and Neveu [21], and it was strongly motivated by the desire of gaining insights in the general properties of quantum field theory starting from the simple situation one faces in a two dimensional space. We may say that it was hoped to learn, in the "gymnasium" of two dimensions, things that could have been applied and developed in the real four dimensional world. If we consider for instance the case of the Thirring model [11] we can easily imagine that its exact solvability made it a favourite test bench for the new concepts arising in quantum field theory, as it is testified by the example of the study of operator products at short distances [15,16,17].

As the study of two dimensional quantum field theories was deepened, it became clear that two dimensional models would have a dignity of their own. The connections with statistical mechanics (the example of the Ising model is reviewed in refs.[56,57,58,59]) and their peculiar properties such as the equivalence of bosons and fermions [19,22,36,37,38,48], motivated an increasing interest culminated with the rediscovery of string theories [25,26,27] which put a strong emphasis on the conformal properties of two dimensional theories [2,3,4,27]. The use of conformal symmetry arguments (see ref.[18] for an example in the case of the Thirring model) has always provided many insights both in quantum field theory and in statistical mechanics, and Zamolodchikov's theorem [1,2,3] can be considered as a deep and interesting result in the study of conformal invariant quantum field theories. In particular it introduces a new point of view in the study of two dimensional lagrangian models, based on the interplay between the quantum properties of the theory and the geometrical characteristics of the coupling constants manifold [32,34]. Moreover it is the first attempt to extend in some way the consequences of conformal invariance also outside the renormaliztion group fixed point where it strictly holds [5,6].

This thesis is devoted to show the application of Zamolodchikov's C-theorem to four fermion models, and it presents some results [32,34] related to it. It is hoped that, besides a check of its validity and consistency, it will also provide to the reader an actual example of the new insights yielded by this theorem.

1.1 Outline

As already mentioned Zamolodchikov's C-theorem is one of the most interesting developements obtained in the last few years [1,2,3,4] and consists in the discovery of a close connection between two dimensional field theory and the structure of the abstract manifold defined by the parameters appearing in the interaction lagrangian:

$$\mathcal{L}_I = -\sum_{i=1}^{n} g^i O_i \tag{1.1.1}$$

where $\{O_i\}$ is a set of dimension two operators. The n-dimensional manifold $\mathcal{M}$ we want to analyze has the coupling constants g^i as coordinates and is naturally equipped with a Riemannian structure provided by a metric-like tensor G_{ij} defined in terms of the correlation functions of the operators O_i [1]:

$$G_{ij} \propto \langle O_i O_j \rangle \tag{1.1.2}$$

This definition suggests the possibility of a perturbative evaluation of this metric tensor, and its study in the case of four fermion models will be one of the aims of this work.

Zamolodchikov's theorem [1] yields a link between G_{ij} and the renormalization group β-functions defining in a standard way a set of curves on the manifold $\mathcal{M}$ describing the flow of renormalization group transformations

$$\mu \frac{dg^i}{d\mu} = \beta^i(g) \tag{1.1.3}$$

and introducing a vector field on $\mathcal{M}$. In fact it is possible to show that there exists a scalar function C defined on the space of parameters and related to the central charge of the model (that is to the response of the theory to two dimensional conformal transformations) which satisfies the following remarkable property

$$\mu\frac{dC}{d\mu} = -\beta^i\frac{\partial}{\partial g^i}C = -\frac{3}{4}G_{ij}\beta^i\beta^j \tag{1.1.4}$$

If we notice that in two dimensions the existence of a Hilbert space with positive definite norm insures that the metric G_{ij} is positive definite, we conclude that eq.(1.1.4) signals a general monotonicity character in the flow of renormalization group[1]. Relation (1.1.4) naturally leads to the conjecture

$$\frac{\partial}{\partial g^i}C = \frac{3}{4}G_{ij}\beta^j = \frac{3}{4}\beta_i \tag{1.1.5}$$

and, altough there is no clear proof of this relation to all orders of perturbation theory (see also ref. [40,41]), a next to leading computation supports this result, which in turn amounts to say that

$$\oint \beta_i dg^i = 0 \tag{1.1.6}$$

or, equivalently, that the Maxwell tensor constructed from β_i vanishes, and which gives very strong constraints on the theory.

The above discussion give the general framework of this work, which essentially tries to verify all these statements in a specific model. Before going on we must stress that a crucial condition for the validity of our considerations is the absence of hidden mass parameters in the theory, so that it must be free of infrared divergences, otherwise the structure of eq.(1.1.4) changes [29]. A simple non trivial infrared convergent model is the most general quartic interaction of N Dirac fermions [31,32,34,47], also called generalized Thirring model

$$\mathcal{L}_I = -g_S\frac{1}{2}(\bar\psi\psi)^2 - g_P\frac{1}{2}(\bar\psi\gamma^5\psi)^2 - g_V\frac{1}{2}(\bar\psi\gamma_\mu\psi\bar\psi\gamma^\mu\psi) \tag{1.1.7}$$

We will give a description of the renormalization structure of this model up to two loops using dimensional regularization, and, as the model (1.1.7) contains all the known special cases (SU(N) Thirring models, Gross-Neveu model etc.) this will have its own relevance [21,44,45].

We want to mention here the problem of "evanescent operators" [30,32,34,52,53, 54,55] which arises starting from the two loop level; its analysis, albeit a bit technical,

[1]This aspect is believed to have some consequences also in string models [2,3].

can be a useful tool in the study of other models, expecially in the supersymmetric case. The use of dimensional regularization implies that the theory has to be considered in $d = 2 + \varepsilon$ dimensions; in this case the Clifford algebra becomes infinite dimensional [42,43], the three operators appearing in eq.(1.1.7) do not longer form a complete set for the regularized model and an infinite series of "evanescent operators" living in $d-2$ dimensions, each generating its own β function, is needed, so that a renormalization group flow in an infinite dimensional space seems to appear. The problem can be overcome by a suitable projection technique [30,32,34,52,53,54,55], whose final outcome is given by the three relevant β functions. In this way the β functions can be computed in the minimal subtraction scheme.

As usual at this stage there is a freedom due to finite renormalizations and this can be used to preserve in an explicit way the symmetries of the model, without changing its physical content. In particular a large class of symmetries is related to the chiral nature of the classical theory and so it is formally obscured in the minimal subtraction scheme. In the text it is shown how to perform the finite renormalization needed to achieve an explicitly symmetric pattern for correlation functions. Being aware that a well defined procedure of fixing finite renormalizations is given by the study of Ward identities, it has been preferred to follow another approach which is more algebraic and problably more intuitive: the direct study of the implications of symmetries on the β functions and a check of the explicit computations against these requirements. A detailed study of Ward identities is left to a future work.

The final result will be a set of "symmetric" β functions which are listed here

$$\begin{aligned}\beta_S &= -\frac{1}{\pi}\left[(N-1)g_S^2 + g_S g_P - 2g_V(g_S - g_P)\right] \\ &+ \frac{1}{2\pi^2}\left[N g_S(g_S^2 + g_P^2) - g_S^2(g_S - g_P) + 2g_V(g_P^2 - g_S^2)\right. \\ &\qquad \left. + 2N g_V^2(g_S - g_P)\right] \qquad (1.1.8)\end{aligned}$$

$$\begin{aligned}\beta_P &= -\frac{1}{\pi}\left[(N-1)g_P^2 + g_P g_S - 2g_V(g_P - g_S)\right] \\ &+ \frac{1}{2\pi^2}\left[N g_P(g_P^2 + g_S^2) - g_P^2(g_P - g_S) + 2g_V(g_S^2 - g_P^2)\right. \\ &\qquad \left. + 2N g_V^2(g_P - g_S)\right] \qquad (1.1.9)\end{aligned}$$

$$\beta_V = -\frac{1}{\pi} g_S g_P$$

$$+ \quad \frac{1}{2\pi^2}\left[(N-1)g_V(g_S-g_P)^2 + g_S g_P(g_S+g_P)\right] \tag{1.1.10}$$

After the study of the renormalization structure of the model the problem of constructing the metric in the parameter space has been faced. While the leading three loop calculation is quite trivial the next order requires some work, and also in this computation a suitable choice of a "symmetric scheme" simplifies the result which finally turns out to be

$$\begin{aligned} G_{ij} &= \frac{N}{16\pi^4}\begin{pmatrix} 2N-1 & 1 & -2 \\ 1 & 2N-1 & -2 \\ -2 & -2 & 4N \end{pmatrix} + \\ &+ \left(1 + log(\pi\mu^2x^2) + \gamma_E\right)\frac{N(N-1)}{8\pi^5} \end{aligned}$$

$$\cdot \begin{pmatrix} (2N-1)g_S + g_P - 2g_V & g_S + g_P - 2g_V & -2(g_S-g_P) \\ g_S + g_P - 2g_V & (2N-1)g_P + g_S - 2g_V & 2(g_S-g_P) \\ -2(g_S-g_P) & 2(g_S-g_P) & 0 \end{pmatrix} \tag{1.1.11}$$

The choice of a finite renormalization amounts to a diffeomorphism of the metric and so it does not affect any physical conclusion. As one can see from eq.(1.1.11) the $\mathcal{O}(g)$ corrections to the metric disappear at a scale

$$x^2 = \frac{1}{\pi\mu^2}e^{-(\gamma_E+1)} \tag{1.1.12}$$

and moreover all the scale dependence can be expressed by a diffeomorphism driven by the β functions. This means that with the coordinate choice (1.1.12) the metric is given by the lowest order result, and, from a geometrical point of view, this amounts to say that our coordinates are locally euclidean in the origin. This is by no means trivial and it depends on the choice of the subtraction scheme; it signals a close connection between the metric structure on the parameter space and the fulfulliment of Ward identities in the two dimensional field theory. All these considerations are confirmed by a thorough algebraic analysis: the implementation of symmetries essentially fixes the form of the metric tensor G_{ij}. An higher order computation would allow to compute the curvature tensor of the manifold of parameters but this is beyond the scope of the present work. Finally eq.(1.1.5) permits the reconstruction of C by the formula

$$C = \frac{3}{4}\int_0^1 g^i\beta_i(tg)dt \tag{1.1.13}$$

and the final result is

$$\begin{aligned} C &= \frac{N}{8\pi^2} - \frac{3}{64\pi^5} N(N-1)\Big[(2N-1)(g_S^3+g_P^3) + 3g_S g_P(g_S+g_P) \\ &\qquad -6g_V(g_S-g_P)^2\Big] \\ &+ \frac{3}{512\pi^6} N(N-1)\Big[(2N-1)(g_S^2+g_P^2)^2 + 4g_S g_P(g_S^2+g_P^2+g_S g_P) \\ &\qquad -8g_V(g_S-g_P)(g_S^2-g_P^2) + 8g_V^2(g_S-g_P)^2\Big] \\ &- \frac{3}{8}\left[1+\gamma_E+log(\pi\mu^2x^2)\right] G_{ij}^{(0)}\beta^{(1)\,i}\beta^{(1)\,j} \end{aligned} \tag{1.1.14}$$

while a direct perturbative evaluation would have required to consider Feynman diagrams up to the fifth loop order.

1.2 Summary

We now give a brief summary of the content of the various chapters.

- In chapter 2 we define the generalized Thirring model and list its classical symmetries which can be studied with the help of the technique of abelian bosonization, which is also sketched.
- In chapter 3 we outline the renormalization of the model with a particular emphasis on the problem of evanescent operators, whose solution is also surveied. The computation of the one loop β functions is reported for completeness.
- In chapter 4 we give a detailed account of the calculation of the two loop β functions. The subject is quite technical and we give the explicit results for the full list of relevant Feynman diagrams. It is also shown in detail how evanescent operators disappear from the renormalization group equations. We give the results for the β functions both in the minimal subtraction scheme and in a "symmetric" scheme (these last results are summarized in table 6.3). Moreover we study in an algebraic way the constraints imposed on the β functions by the symmetries of the model and verify that the "symmetric" results satisfy all these constraints.
- In chapter 5 we give an account of Zamolodchikov's theorem. The proof is given in the framework of the generalized Thirring model, but is by no means model

dependent. Rather the aim is to state as clearly as possible all the hypotheses involved. We also discuss briefly the problem of infrared divergences and point out some relevant and interesting cosequences of the theorem both for the study of our model and in the context of statistical mechanics.

- In chapter 6 the computation of the three and four loops contributions to the metric G_{ij} of the manifold of coupling constants is presented. The algebraic implications of symmetry properties on G_{ij} are worked out and it is shown how this analysis, together with Zamolodchikov's theorem, puts severe constraints on the β functions. Finally the five loops contribution to C is reported.

- Chapter 7 contains the conclusions and some indication for future studies.

- Several appendices are added in order to deal with technical problems. Appendix (C) in particular is devoted to an explicit analysis of $SU(N)$ Thirring model. This study has its own interest and can provide a check for the results of chapter 4 and 6 .

Chapter 2

General definitions

In this chapter we will survey the characteristics of the generalized Thirring model [32,33,34,47] with a particular emphasis on its symmetry properties. Moreover we will outline the technique of abelian bosonization [19,22,36,37,38,48], which is useful in this kind of analysis.

2.1 General two-dimensional lagrangian

For the sake of clarity it is convenient to begin with the definition of the lagrangian we have to consider. As discussed in the introduction we will study the most general two dimensional four-fermion interaction involving N Dirac fermions and enjoying $U(N)$ symmetry. The problem has already been discussed in the literature and the following massless lagrangian has been introduced

$$\mathcal{L} = \bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}g_S(\bar{\psi}\psi)^2 - \frac{1}{2}g_V(\bar{\psi}\gamma_\mu\psi)^2 - \frac{1}{2}g_P(\bar{\psi}\gamma_5\psi)^2 \tag{2.1.1}$$

where euclidean metric is adopted and summation over $U(N)$ color indices is understood. Our conventions for γ matrices and Fierz identities are listed in appendix A. Equation (2.1.1) requires some comments: apparently we have rejected a parity-violating term

$$-g_5(\bar{\psi}\psi)(\bar{\psi}\gamma_5\psi) \tag{2.1.2}$$

However as long as chiral transformations can be performed without generation of anomalies, it is possible to eliminate the term (2.1.2) by such a transformation accompanied by a redefinition of the coupling. Let us indeed list the chiral transformation properties of fermion quadrilinear forms under the mapping

$$\begin{aligned} \psi &\rightarrow [\exp \frac{\alpha}{2}\gamma_5]\psi \\ \bar{\psi} &\rightarrow \bar{\psi}[\exp \frac{\alpha}{2}\gamma_5] \end{aligned} \tag{2.1.3}$$

We obtain

$$\begin{aligned} (\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\psi)^2 &\rightarrow (\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\psi)^2 \\ (\bar{\psi}\gamma_\mu\psi)^2 &\rightarrow (\bar{\psi}\gamma_\mu\psi)^2 \\ (\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2 &\rightarrow \cos 2\alpha[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2] + \sin 2\alpha(2\bar{\psi}\psi\bar{\psi}\gamma_5\psi) \\ 2\bar{\psi}\psi\bar{\psi}\gamma_5\psi &\rightarrow \cos 2\alpha(2\bar{\psi}\psi\bar{\psi}\gamma_5\psi) - \sin 2\alpha[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2] \end{aligned} \tag{2.1.4}$$

Therefore the interaction

$$-\frac{1}{2}\left\{\frac{g_S - g_P}{2}[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2] + 2g_5(\bar{\psi}\psi\bar{\psi}\gamma_5\psi)\right\} \tag{2.1.5}$$

can be chirally transformed to

$$-\frac{1}{2}\sqrt{(\frac{g_S - g_P}{2})^2 + {g_5}^2} \quad [(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2] \tag{2.1.6}$$

without affecting the other terms in the lagrangian. We refer to the appendix for the discussion of interaction terms explicitly depending on color generator matrices: suffice it to say that they can all be reabsorbed in the lagrangian (2.1.1) by a Fierz transformation.

2.2 Abelian bosonization

A very useful heuristic tool in the discussion of the symmetry properties of two-dimensional fermion models is the abelian bosonization technique [19,22,36,37,38,48]. This is one of the most interesting features of field theories in two dimensions and states the equivalence between theories constructed with fermions and theories constructed with bosons. A theory in which the basic field operator satisfies a manifest anticommutation rule may be intrinsically related to a theory in which the field operator satisfies only a commutation relation. A well known example of that [19] is the equivalence between the so called "quantum sine-Gordon" model, with lagrangian

$$\mathcal{L}_{SG} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{\alpha}{\beta^2}cos(\beta\phi) - \frac{\alpha}{\beta^2}$$

and the massive Thirring model,

$$\mathcal{L}_{Th} = \bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}g(\bar{\psi}\gamma_\mu\psi)^2 - m\bar{\psi}\psi.$$

This equivalence may be shown by comparing the perturbative expressions of the Green's functions of the two models; it is possible to identify them provided we adjust the parameters in the following way:

$$\frac{4\pi}{\beta^2} = 1 + \frac{g}{\pi}$$

$$m = \frac{\alpha}{\beta^2}$$

It should be mentioned that there is also another method to establish equivalence between a fermion and a boson model. It consists in the construction of an operator solution of the field equation in one theory in term of a nonlocal expression of operators in the other. This procedure, also called boson representation, transforms directly a theory of fermions into a theory of bosons, and their equivalence is a consequence. In the case of our model (2.1.1), the boson representation of it is realized on the basis of the following set of rules:

$$\begin{aligned} \bar{\psi}_a\gamma_\mu\psi_a &= -\frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial_\nu\phi_a \\ \bar{\psi}_a(1\pm\gamma_5)\psi_a &= \,:\exp \pm i\sqrt{4\pi}\phi_a: \\ \bar{\psi}_a\partial\!\!\!/\psi_a &= \frac{1}{2}\partial_\mu\phi_a\partial_\mu\phi_a \end{aligned} \tag{2.2.1}$$

and in this language the quadrilinear forms become:

$$\begin{aligned} (\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\psi)^2 &= -\frac{1}{\pi}\sum_a(\partial_\mu\phi_a)^2 + 2\sum_{a>b}\cos\sqrt{4\pi}(\phi_a - \phi_b) \\ (\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2 &= 2\sum_{a>b}\cos\sqrt{4\pi}(\phi_a + \phi_b) \\ 2\bar{\psi}\psi\bar{\psi}\gamma_5\psi &= 2\sum_{a>b}\sin\sqrt{4\pi}(\phi_a + \phi_b) \\ (\bar{\psi}\gamma_\mu\psi)^2 &= -\frac{1}{\pi}(\partial_\mu\sum_a\phi_a)^2 \end{aligned} \tag{2.2.2}$$

while the chiral transformations are simply shifts in the scalar fields

$$\phi_a \to \phi_a - \frac{\alpha}{\sqrt{4\pi}}. \tag{2.2.3}$$

When put in its bosonic form, a fermion theory can be investigated in an easier way for what concern its symmetry properties and connections with other fermion theories, since many unusual and hidden features can become apparent in the boson representation (see, e.g., [31]). In this respect, examples of particular interest are provided by the boson formulation of fermion theories with chiral and internal symmetries, such as the $SU(N)$ Thirring model [22,37,48]. Here we have the following problem: the model possesses a $U(1)$ chiral symmetry which should prevent fermions from acquiring a mass. However a $1/N$ expansion seems to produce a spontaneous breaking of chiral symmetry, while fermions become massive and a Goldstone boson appears contradicting Coleman's theorem on the absence of continuous symmetry breaking in two dimensions [20]. Indeed a detailed analysis [22], which can be based on the use of abelian bosonization [37,48], shows that the $1/N$ expansion, if treated carefully, is a good guide to the properties of the model. As a matter of fact it can be proven that the symmetry is not spontaneously broken and that the massless particle which appears is not a Goldstone boson.

In order to have a feeling of what happens and to give a simple example of how bosonization works we shall describe very briefly the case of a solvable model with chiral $U(1)$ symmetry [22,39], having the lagrangian

$$\mathcal{L} = \bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}\lambda\left[\bar{\psi}(1+\gamma_5)\psi e^{\frac{i\sigma}{a}} + \bar{\psi}(1-\gamma_5)\psi e^{\frac{-i\sigma}{a}}\right]. \tag{2.2.4}$$

Introducing the boson representation (2.2.1) involving a single scalar field ϕ, and considering the linear combinations

$$\tilde{\phi} = \frac{\sqrt{4\pi}\phi + \sigma/a}{\sqrt{4\pi + 1/a^2}}$$

$$\tilde{\sigma} = \frac{\sqrt{4\pi}\sigma - \phi/a}{\sqrt{4\pi + 1/a^2}}$$

the lagrangian (2.2.4) takes the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \tilde{\phi})^2 + \frac{1}{2}(\partial_\mu \tilde{\sigma})^2 - \lambda \cos\left(\sqrt{4\pi + \frac{1}{a^2}}\ \tilde{\phi}\right)$$

thus describing a massless scalar $\tilde{\sigma}$ and a sine-Gordon field $\tilde{\phi}$, which gives a spectrum containing a massive fermion and a massive antifermion. By writing the axial current A_μ in terms of the field $\tilde{\phi}$ and $\tilde{\sigma}$ we can understand why fermions can acquire a mass despite chiral symmetry: we have

$$A_\mu = -\sqrt{\frac{1}{\pi} + 4\pi a^2} \;\; \partial_\mu \tilde{\sigma}$$

which involves only the field $\tilde{\sigma}$. This means that the field $\tilde{\phi}$, and therefore the fermions associated with this field, are neutral under chirality; therefore they can have a mass without any apparent contradiction. Finally, again by using the boson representation, it can be shown that chirality violating Green functions vanish as required by Coleman's theorem, so that the massless scalar of this model is not a decoupled Goldstone boson: it is simply a non Goldstone massless boson.

Such kind of analysis can be generalized to more complicated cases, and indeed can be useful also for finding connections between different models; since lagrangian (2.1.1) pretends to be a general theory of fermions in two dimensions, abelian bosonization is profitable if we want to study its properties and symmetries. Therefore we conclude this section by enumerating some results concerning our model that can be established by making use of the boson representation and that will be emploied in the following [22,31,38,48].

1. The equivalence of the model $g_S = g_P = Ng_V = g/2$ with the $SU(N)$ non-abelian Thirring model.

$$\mathcal{L} = \bar{\psi}\partial\!\!\!/ \psi + \frac{1}{2} g (\bar{\psi} T^a \gamma_\mu \psi)^2 \tag{2.2.5}$$

   where T^a are the $SU(N)$ generators in the fundamental representation.

2. The decoupling of the $U(1)$ Thirring model [50]

$$\mathcal{L}_I = -\frac{1}{2} \delta_V (\bar{\psi} \gamma_\mu \psi)^2 \tag{2.2.6}$$

   from the above mentioned $SU(N)$ Thirring model.

3. The $O(2N)$ symmetry enjoyed by the model $g_P = g_V = 0$ (Gross-Neveu model) and by the related model $g_S = g_V = 0$ [46].

4. The quantum equivalence of all the $N = 1$ models, based on the Fierz identity for $U(1)$ fermions:
$$(\bar{\psi}\psi)^2 = (\bar{\psi}\gamma_5\psi)^2 = -\frac{1}{2}(\bar{\psi}\gamma_\mu\psi)^2 \tag{2.2.7}$$
and the vanishing of their β-functions [51].

5. The $SO(4)$ symmetry of general $SU(2)$ models with $g_V = g_P/2$ [31].

6. The quantum equivalence of $U(3)$ Gross-Neveu and $SU(4)$ Thirring models.

Chapter 3

Renormalization properties

The purpose of this chapter is to illustrate the renormalization properties of the generalized Thirring model (2.1.1). By power counting, lagrangian (2.1.1) is renormalizable and free of infrared divergencies; however the regularization procedure we are going to use, namely dimensional regularization, will pose some technical problems that have to be solved. It is therefore useful, before attacking any actual calculation, to settle the general framework in which they can be treated [32,34].

3.1 Regularization and renormalization in the dimensional scheme

The general form of the lagrangian we are dealing with is

$$\mathcal{L} = \bar{\psi}\partial\!\!\!/\psi - \sum_i g^i O_i \tag{3.1.1}$$

where O_i are the four-fermion operators of eq.(2.1.1). As usual the existence of the renormalization group implies that renormalized n-point Green's functions $\Gamma^{[n]}$ satisfy the equation

$$\left[\mu\frac{\partial}{\partial\mu} + \beta^k\frac{\partial}{\partial g^k} + \frac{1}{2}n\gamma\right]\Gamma^{[n]} = 0 \tag{3.1.2}$$

This means that a scale transformation can be seen as a flow governed by the vector β^k in a parameter space that can be formally seen as a manifold whose coordinates are the couplings g^i. Physics must be invariant under general coordinate transformations

$$g^i \to g'^i(g) \tag{3.1.3}$$

From equation (3.1.2) we expect β^k to be a (contravariant) vector in the parameter space, and therefore its transformation properties under (3.1.3) should be

$$\beta^k(g') = \frac{\partial g'^k}{\partial g^l}\beta^l[g(g')]. \tag{3.1.4}$$

Equations (3.1.3) and (3.1.4) have the following infinitesimal form

$$\begin{aligned} g'^k &= g^k + G^k(g) \\ \beta'^k(g) &= \beta^k(g) + \left[\frac{\partial G^k}{\partial g^l}\beta^l - \frac{\partial \beta^k}{\partial g^l}G^l\right] \end{aligned} \tag{3.1.5}$$

and this is the well known transformation law of the β-functions obtained by renormalization group arguments.

On the basis of the previous considerations one naively expects that a scale transformation of our models may be expressed as a three-dimensional flow. However this statements can hold without qualifications only as long as our renormalization procedure does not enlarge the parameter space, that is as long as the lagrangian is explicitly multiplicatively renormalizable. In practice, because of the presence of the axial interaction, this condition is not satisfied and the reduction of the renormalization group flow to the three-dimensional parameter space requires a detailed analysis. For definiteness we shall choose the standard dimensional regularization scheme, that insures the fulfillment of the Ward identities of vector currents. In this scheme the multiplicative renormalizability in the three-parameter space is immediately lost, since in d dimensions a complete basis of the Clifford algebra involves an infinite number of operators O_i [42,43]. An explicit form of this basis [42] is

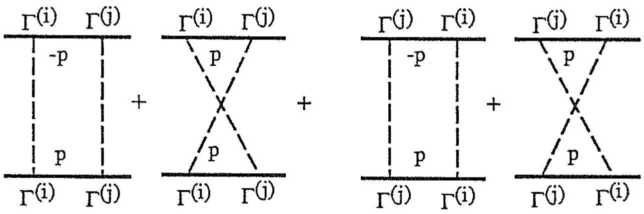
$$O_k = \frac{1}{2}\bar{\psi}\Gamma^{(k)}\psi\bar{\psi}\tilde{\Gamma}^{(k)}\psi \tag{3.1.6}$$

where

$$\Gamma^{(k)} = \gamma_{[\mu_1}\cdots\gamma_{\mu_k]} \tag{3.1.7}$$

Even if we started with a lagrangian containing only the operators O_0, O_1, O_2, radiative corrections would induce mixings with the "evanescent operators" $O_i, i > 2$ [52,53,54]. By the term "evanescent operators" we mean operators whose matrix elements are all vanishing at the tree level; they will however give a contribution to the Green's functions when matching a divergence in a higher loop diagram. What renders the renormalization of our model less awkward than it may seem is the fact that in the loop expansion only a finite number of evanescent operators will appear at each order.

Figure 1



For instance, the divergent part of the one loop diagrams in figure 1 can be shown to be proportional to the operator

$$\bar{\psi} X_{\mu}^{(ij)} \psi \bar{\psi} X_{\mu}^{(ij)} \psi \tag{3.1.8}$$

where the operator

$$X_{\mu}^{(ij)} = \Gamma^{(i)} \gamma^{\mu} \Gamma^{(j)} - \Gamma^{(j)} \gamma^{\mu} \Gamma^{(i)} \tag{3.1.9}$$

can be decomposed (see Appendix A) in the basis of the operators $O_i, i \leq 4$.

The strategy we have to use in order to solve the problem of "evanescent operators" [30,32,34,52,53,54,55] can be better understood if we consider, at first, a general quantum field theory in two dimensions specified by the following tree level lagrangian

$$\mathcal{L}_{Tree} = \sum_{i=1}^{n} g^i O_i \tag{3.1.10}$$

As we have seen the regularization procedure will yield the appearance of an infinite set of evanescent operators, which we mark with a bar, $\{\bar{O}_k\}_{k=1}^{\infty}$, thus giving the following general form of the renormalized action S

$$S = \int dz \mu^{\varepsilon} \{ \sum_{i=1}^{n} (g^i + P^i(\mathrm{g}, \varepsilon)) O_i + \sum_{k=1}^{\infty} \bar{P}^k(\mathrm{g}, \varepsilon) \bar{O}_k \} \tag{3.1.11}$$

where the counterterms $P^i, \bar{P}^k$ can be expanded in the number of loops L

$$P^i(\mathrm{g}, \varepsilon) \;=\; \sum_{L=1}^{\infty} \sum_{\nu=1}^{L} \frac{1}{\varepsilon^{\nu}} P^{i(L,\nu)}(\mathrm{g}) \tag{3.1.12}$$

and the scale of mass, μ, has been explicitly factored out in order to keep the dimensions of the fields and the couplings at their canonical value for $d = 2$. We will make use of the notion of normal products or normal ordered operators $N[O_i]$, that in the minimal subtraction scheme can be defined as follows

$$N[O_i] = O_i + \sum_{j=1}^{n} X_i^j(g,\varepsilon)O_j + \sum_{k=1}^{\infty} \bar{X}_i^k(g,\varepsilon)\bar{O}_k \tag{3.1.13}$$

where $\mathbf{X}, \bar{\mathbf{X}}$ are the mixing or anomalous dimensions matrices. Equivalently it can be seen that the zero momentum insertion of a normal product $N[O_i]$ into a Green's function of r fields ϕ_j can be written as

$$\langle \int dz \;\; \mu^{\varepsilon} N[O_i](z)\phi_1 \cdots \phi_r\rangle = \frac{\partial}{\partial g^i}\langle \phi_1 \cdots \phi_r\rangle \tag{3.1.14}$$

This implies for the renormalized action S, eq.(3.1.11), the relation

$$\begin{aligned} \int dz \mu^{\varepsilon} N[O_j](z) \;&=\; \frac{\partial S}{\partial g^j} = \\ = \int dz \mu^{\varepsilon}\Big[O_j \;&+\; \sum_{i=1}^{n} \frac{\partial P^i}{\partial g_j} O_i + \sum_{k=1}^{\infty} \frac{\partial \bar{P}^k}{\partial g_j}\bar{O}_k\Big] \end{aligned} \tag{3.1.15}$$

which shows that the quantity $\frac{\partial S}{\partial g^j}$ is finite. On the other hand, since in eq.(3.1.11) we have factored out the dependence on μ, one can easily obtain

$$\mu\frac{\partial}{\partial \mu}S \;=\; \int dz \mu^{\varepsilon}\varepsilon\Big[\sum_{i=1}^{n}(g^i + P^i)O_i + \sum_{k=1}^{\infty}\bar{P}^k\bar{O}_k\Big] \tag{3.1.16}$$

and hence the following relation holds

$$(\mu\frac{\partial}{\partial \mu} - \varepsilon \mathrm{g}\cdot\frac{\partial}{\partial \mathrm{g}})S = \int dz \;\; \mu^{\varepsilon}\varepsilon\Big[(1-\mathrm{g}\cdot\frac{\partial}{\partial \mathrm{g}})\sum_{i=1}^{n} P^i O_i + (1-\mathrm{g}\cdot\frac{\partial}{\partial \mathrm{g}})\sum_{k=1}^{\infty}\bar{P}^k\bar{O}_k\Big] \tag{3.1.17}$$

It is apparent that the l.h.s. of this equation is finite, and so must be the r.h.s., which can then be written in terms of normal operators; furthermore the explicit presence in it of the cut off ε assures that the only contributions we have to take into account are those coming from the residues of the simple poles $(r.s.p)$. Then we may write

$$\begin{aligned} (\mu\frac{\partial}{\partial \mu} - \varepsilon \mathrm{g}\cdot\frac{\partial}{\partial \mathrm{g}})S \;&=\; \int dz \;\; \mu^{\varepsilon}\varepsilon\Big[r.s.p\Big\{(1-\mathrm{g}\cdot\frac{\partial}{\partial \mathrm{g}})P^i\Big\}N[O_i] \\ &+ r.s.p\Big\{(1-\mathrm{g}\cdot\frac{\partial}{\partial \mathrm{g}})\bar{P}^k\Big\}N[\bar{O}_k]\Big] \end{aligned} \tag{3.1.18}$$

Let us now consider the differential operator $(1 - g \cdot \frac{\partial}{\partial g})$ which corresponds to the loop counting operator; when applied to the loop expansion of P^i it yields

$$(1 - g \cdot \frac{\partial}{\partial g})P^i = \sum_{L,\nu} L \frac{P^{i(L,\nu)}(g)}{\varepsilon^\nu} \qquad (3.1.19)$$

and therefore

$$r.s.p. \left\{(1 - g \cdot \frac{\partial}{\partial g})P^i\right\} = \sum_L L P^{i(L,1)}(g) \qquad (3.1.20)$$

with an analogous relation for $\bar{P}^k$. One is led to take eq.(3.1.20) as the usual definition of the β-function

$$\begin{aligned} \tilde{\beta}^i &= -\sum_L L P^{i(L,1)} \\ \bar{\beta}^k &= -\sum_L L \bar{P}^{k(L,1)} \end{aligned} \qquad (3.1.21)$$

and hence it would turn out that

$$(\mu \frac{\partial}{\partial \mu} - \varepsilon g \cdot \frac{\partial}{\partial g})S = -\int dz\ \mu^\varepsilon \Big[\sum_{i=1}^{n} \tilde{\beta}^i N[O_i] + \sum_{k=1}^{\infty} \bar{\beta} N[\bar{O}_k]\Big] \qquad (3.1.22)$$

However, this is not the whole story. In fact the identification (3.1.21) of the β-functions is not correct, since we have for the relevant couplings

$$\mu \frac{dg^i}{d\mu} \neq \tilde{\beta}^i \qquad (3.1.23)$$

because one does not take into account the effect of the evanescent operators[1].

The solution of this problem is based on the existence of a reduction formula [53,54] that allows to project the evanescent operators on the relevant ones and express,

[1] Indeed it can be easily seen that the scaling properties of a 1PI Green's function depend also on the evanescent operators $N[\bar{O}_k]_{k=1\cdots\infty}$

$$\mu \frac{\partial}{\partial \mu} \langle \phi_1 \cdots \phi_m \rangle_{1PI} = \int dz\ \mu^\varepsilon \left[\tilde{\beta}^i \langle N[O_i]\phi_1 \cdots \phi_m\rangle + \bar{\beta}^k \langle N[\bar{O}_k]\phi_1 \cdots \phi_m\rangle\right]$$

order by order in the loop expansion, the finite contribution of the former in terms of insertions of the latters; one obtains, as $\varepsilon \to 0$

$$\int dz\ N[\bar{O}_k] \ = \ \int dz\ \sum_{j=0}^{n} C_k^j(\mathrm{g}) N[O_j] \tag{3.1.24}$$

where the coefficients C_k^j start at the quantum level. Therefore the β-functions that give the correct renormalization group flow has to be defined as

$$\beta^i \ = \ \tilde{\beta}^i \ + \ \sum_{k=1}^{\infty} \bar{\beta}^k C_k^j(\mathrm{g}) \tag{3.1.25}$$

and eq.(3.1.22) yields

$$\left(\mu\frac{\partial}{\partial\mu} - \varepsilon \mathrm{g} \cdot \frac{\partial}{\partial \mathrm{g}}\right) S = \ - \ \int dz\ \mu^{\varepsilon} \ \sum_{i=1}^{n} \beta^i N[O_i] \tag{3.1.26}$$

which, recalling eq.(3.1.15), becomes finally the renormalization group equation for the action S

$$\left(\mu\frac{\partial}{\partial\mu} + \sum_{i=1}^{n} \beta^i \frac{\partial}{\partial g^i} - \varepsilon \mathrm{g} \cdot \frac{\partial}{\partial \mathrm{g}}\right) S \ = \ 0\ . \tag{3.1.27}$$

After having reviewed the general framework in which the problem of evanescent operators is set, let us now tackle the multiplicative renormalization of the model (3.1.1) [30,32,34], by introducing the bare lagrangian

$$\mathcal{L}_B = \bar{\psi}_B \partial\!\!\!/ \psi_B - \sum_{i=0}^{+\infty} {g_B}^i {O_i}^B \tag{3.1.28}$$

and the rescalings

$$\psi_B \ = \ \sqrt{Z_2}\ \psi \tag{3.1.29}$$

$${g_B}^i \ = \ \mu^{-\varepsilon}[g^i + P^i(g,\varepsilon)] \tag{3.1.30}$$

where, analogously to eq.(3.1.12) we have

$$Z_2 \ = \ 1 + \sum_{L=1}^{\infty} \sum_{\nu=1}^{L} \frac{1}{\varepsilon^{\nu}} Z_2^{(L,\nu)}(g) \tag{3.1.31}$$

$$P^i(g,\varepsilon) \ = \ \sum_{L=1}^{\infty} \sum_{\nu=1}^{L} \frac{1}{\varepsilon^{\nu}} P^{i(L,\nu)}(g) \tag{3.1.32}$$

The renormalization group β and γ functions can be found by means of a recursive relation, and are

$$\tilde{\beta}^i = -\sum_{L=1}^{\infty} L P^{i(L,1)}(g) \tag{3.1.33}$$

$$\tilde{\gamma} = -\sum_{L=1}^{\infty} L Z_2^{(L,1)}(g) \tag{3.1.34}$$

They appear in the standard renormalization group equation for an n-point 1PI Green's function $\Gamma^{[n]}$

$$\left[\mu\frac{\partial}{\partial\mu} + \tilde{\beta}^k\frac{\partial}{\partial g^k} + \frac{1}{2}n\tilde{\gamma}\right]\Gamma^{[n]} = 0 \tag{3.1.35}$$

which, in this form, corresponds to a flow in an infinite parameter space. The projection on the relevant physical parameters g_S, g_P, g_V can be obtained by defining first of all a "counting operator" $\mathcal{N}_\psi$ which counts the number of external fermion lines in a Green's function

$$\mathcal{N}_\psi \equiv \frac{1}{2}\int dz N\left[\bar{\psi}\frac{\partial S}{\partial\bar{\psi}} + \psi\frac{\partial S}{\partial\psi}\right](z) \tag{3.1.36}$$

Its insertion yields

$$\langle\mathcal{N}_\psi\ \bar{\psi}_1\psi_1\cdots\bar{\psi}_r\psi_r\rangle = r\langle\bar{\psi}_1\psi_1\cdots\bar{\psi}_r\psi_r\rangle \tag{3.1.37}$$

and, recalling (3.1.28), its expression is

$$\mathcal{N}_\psi = N\left[\bar{\psi}\partial\!\!\!/\psi + 2\sum_{k=0}^{\infty} g^k O_k\right] \tag{3.1.38}$$

Therefore we can rewrite the renormalization group equation (3.1.35) as an operator equation for the action S

$$\mu\frac{\partial S}{\partial\mu} = \int dz\sum_{k=0}^{\infty}\tilde{\beta}^k N[O_k] + \tilde{\gamma}\ N\left[\bar{\psi}\partial\!\!\!/\psi + 2\sum_{k=0}^{\infty} g^k O_k\right] \tag{3.1.39}$$

and the reduction formula which eliminates the evanescent operators in this case is

$$\begin{aligned}\int dz N[O_k](z) &\equiv \int dz\Bigg[\sum_{j=0}^{2} C_k{}^j(g)\ N[O_j](z) \\ &+ \rho^{(k)}(g)\ N\left[\bar{\psi}\partial\!\!\!/\psi + 2\sum_{j=0}^{2} g^j O_j\right](z)\Bigg]\Bigg|_{g_k=0,k>2}\end{aligned} \tag{3.1.40}$$

where $k > 2$. Our task is completed by defining the β-functions of the relevant physical parameters, which will be computed in the next chapter; analogously to eq.(3.1.25) this can be done by posing

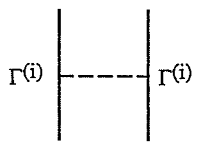
$$\beta^j(g_0, g_1, g_2) = \tilde{\beta}^j(g_0, g_1, g_2) + \sum_{k=3}^{\infty} \tilde{\beta}^k(g_0, g_1, g_2)\ C_k{}^j(g_0, g_1, g_2) \tag{3.1.41}$$

Moreover for the anomalous dimension we have

$$\gamma(g_0, g_1, g_2) = \tilde{\gamma}(g_0, g_1, g_2) + \sum_{k=3}^{\infty} \tilde{\beta}^k(g_0, g_1, g_2)\rho_{(k)}(g_0, g_1, g_2). \tag{3.1.42}$$

Before discussing two-loop calculations it is useful to repeat the computation of the one-loop contributions to the β-functions, in order to illustrate the previous discussion and fix the notation.

In our representation of Feynman diagrams we shall draw the four-fermion vertex by a dashed line joining two fermion lines

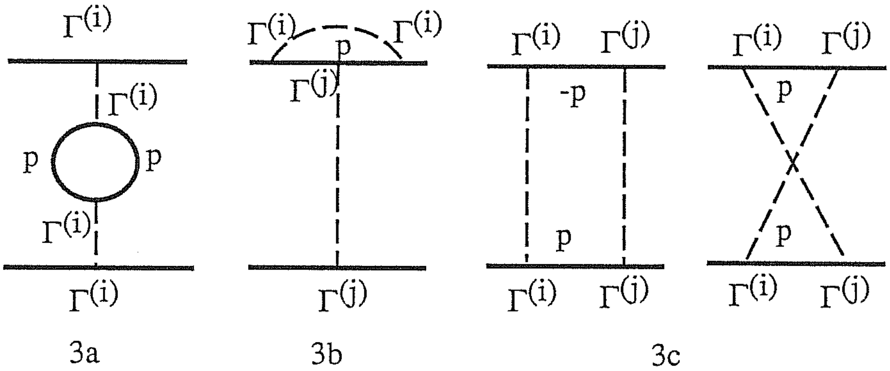


$g^i \Gamma^{(i)}_A \otimes \Gamma^{(i)}_A$

This allows for an automatic accounting of the different possible contractions between fermions. At the tree level the four fermion vertex is simply

$$g^i \Gamma^{(i)}_A \otimes \Gamma^{(i)}_A \tag{3.1.43}$$

where A is a set of antisymmetrized Lorentz indices and the sum over repeated A indices is understood. At the one loop level there are three essentially different contributions, drawn in diagrams (3a), (3b) and (3c), and the β-functions are determined by the coefficients of the $1/\varepsilon$ poles of these graphs. Since the theory is infrared finite, one can in principle compute the diagrams for arbitary external momenta and massless propagators and extract the singular part.



In the actual calculation it is simpler to introduce a mass in the propagator and set the external momenta to zero. Assuming the assignment of internal loop momenta as shown in the figure we obtain the following contributions

$$
\begin{aligned}
(3a) &= -Ng^ig^j \int \frac{d^dp}{(2\pi)^d}\, Tr\left[\Gamma_A^{(i)} \frac{1}{i\not p + m} \Gamma_B^{(j)} \frac{1}{i\not p + m}\right] \Gamma_A^{(i)} \otimes \Gamma_B^{(j)} \\
(3b) &= g^ig^j \int \frac{d^dp}{(2\pi)^d}\, \Gamma_A^{(i)} \frac{1}{i\not p + m} \Gamma_B^{(j)} \frac{1}{i\not p + m} \Gamma_A^{(i)} \otimes \Gamma_B^{(j)} \\
(3c) &= g^ig^j \int \frac{d^dp}{(2\pi)^d} \left[\Gamma_A^{(i)} \frac{1}{i\not p + m} \Gamma_B^{(j)} \otimes \Gamma_A^{(i)} \frac{1}{i\not p + m} \Gamma_B^{(j)}\right. \\
&+ \left. \Gamma_A^{(i)} \frac{1}{i\not p + m} \Gamma_B^{(j)} \otimes \Gamma_B^{(j)} \frac{1}{i\not p + m} \Gamma_A^{(i)}\right]
\end{aligned}
$$

Let us now establish a number of useful relationships and definitions (see also appendix A):

$$\frac{1}{2} Tr[\Gamma_A^{(i)}\Gamma_B^{(j)}] = \delta_{ij}\delta_{AB}\, \nu^{(i)}, \quad \nu^{(i)} \equiv (-1)^{\frac{i(i-1)}{2}} \frac{Tr[\mathbf{1}]}{2} \tag{3.1.44}$$

$$\sum_\mu \gamma_\mu \Gamma_A^{(i)} \gamma_\mu = c^{(i)}\Gamma_A^{(i)} \tag{3.1.45}$$

$$\sum_B \Gamma_B^{(j)}\Gamma_A^{(i)}\Gamma_B^{(j)} = d^{(ij)}\Gamma_A^{(i)} \tag{3.1.46}$$

The two following products appear very often in the evaluation of Feynman integrals

$$c^{(i)}\nu^{(i)} \equiv A^{(i)} \tag{3.1.47}$$

$$c^{(i)}d^{(ij)} \equiv B^{(ij)} \tag{3.1.48}$$

Moreover we need to introduce the following tensor combinations of $\Gamma^{(i)}$ matrices

$$\Gamma_A^{(i)}\gamma_\mu\Gamma_B^{(j)} - \Gamma_B^{(j)}\gamma_\mu\Gamma_A^{(i)} = X_{AB\mu}^{(ij)} \tag{3.1.49}$$

$$\Gamma_A^{(i)}\Gamma_B^{(j)} + \Gamma_B^{(j)}\Gamma_A^{(i)} = Y_{AB}^{(ij)} \tag{3.1.50}$$

These tensors have a decomposition in the basis of the $\Gamma^{(i)}$ matrices, and it is convenient to define the corresponding coefficients by

$$\sum_{A,B,\mu} X_{AB\mu}^{(ij)} \otimes X_{AB\mu}^{(ij)} = \sum_k E^{ijk} \sum_C \Gamma_C^{(k)} \otimes \Gamma_C^{(k)} \tag{3.1.51}$$

The only one-loop integrals we need are

$$\int \frac{d^dp}{(2\pi)^d}\frac{1}{p^2+m^2} = I \to -\frac{1}{2\pi\varepsilon} \text{ as } \varepsilon \to 0 \tag{3.1.52}$$

$$\int \frac{d^dp}{(2\pi)^d}\frac{m^2}{(p^2+m^2)^2} = I_2 = -\frac{\varepsilon}{2}I \tag{3.1.53}$$

In terms of the above defined quantities we obtain

$$(3a) = N(g^i)^2\Gamma_A^{(i)} \otimes \Gamma_A^{(i)}[A^{(i)}I + \nu^{(i)}I_2] \tag{3.1.54}$$

$$(3b) = -g^ig^j\Gamma_A^{(i)} \otimes \Gamma_A^{(i)}[B^{(ij)}I + d^{(ij)}I_2] \tag{3.1.55}$$

$$(3c) = \frac{1}{4}[g^ig^jX_{AB\mu}^{(ij)} \otimes X_{AB\mu}^{(ij)}I + g^ig^jY_{AB}^{(ij)} \otimes Y_{AB}^{(ij)}I_2] \tag{3.1.56}$$

Let us notice that all the terms proportional to I_2 are finite effects due to the infrared regularization and do not carry any contribution to the β-functions. The one-loop counterterms are obtained by taking the $\varepsilon = 0$ value of the functions A, B, E (which we denote by $\hat{A}, \hat{B}, \hat{E}$) times $1/2\pi\varepsilon$. The one-loop β-functions written in compact form are therefore

$$\beta^i = -\frac{1}{2\pi}[N\hat{A}^{(i)}(g^i)^2 - \sum_j \hat{B}^{(ij)}g^ig^j + \frac{1}{4}\sum_{jk}\hat{E}_{jki}g^jg^k] \tag{3.1.57}$$

and in standard notation the relevant components are [47]

$$\beta_S = -\frac{1}{\pi}[(N-1)g_S^2 + g_Sg_P - 2g_V(g_S - g_P)] \tag{3.1.58}$$

$$\beta_P = -\frac{1}{\pi}[(N-1)g_P^2 + g_Sg_P - 2g_V(g_P - g_S)] \tag{3.1.59}$$

$$\beta_V = -\frac{1}{\pi}g_Sg_P \tag{3.1.60}$$

3.2 The fate of evanescent operators

In this section we want to study the influence of evanescent operators on the renormalization group flow. Let us recall that the divergent terms of the one loop diagrams have been computed from the second order perturbation theory of the four point function at zero external momentum. Disregarding the irrelevant contribution of the mass term the result (3.1.54), (3.1.55), (3.1.56) can be recast in the compact form:

$$\begin{aligned} \frac{1}{2!} \quad & \langle\, g^j \int d^d z O_j(z) \;\; g^k \int d^d z O_k(z) \,\rangle \;= \\ = \;\; & g^j g^k \; C^i{}_{jk} \, \Gamma^{(i)}_A \otimes \Gamma^{(i)}_A I \equiv V^i \; \Gamma^{(i)}_A \otimes \Gamma^{(i)}_A \; I \end{aligned} \tag{3.2.1}$$

where the coefficients $C_{jk}{}^i$ are explicitly given by

$$C^i{}_{jk} \;=\; (N \; \delta_{ij}\delta_{ki} \; A^{(i)} - \delta_{ki} \; B^{(ij)} + \frac{1}{4} E_{jki}) \tag{3.2.2}$$

The counterterms and the β-functions are respectively given by

$$-\hat{I} \; g^j g^k \; \hat{C}^i{}_{jk} \; O_i \;\; \equiv \;\; -\hat{I}\hat{V}^i \; O_i \tag{3.2.3}$$

$$\beta^i = (\varepsilon \hat{I}) \; g^j g^k \hat{C}^i{}_{jk} \;\; \equiv \;\; (\varepsilon \hat{I}) \; \hat{V}^i \tag{3.2.4}$$

Clearly we have a contribution to β^i also for $i \geq 3$, which comes from evanescent operators. We notice that these operators, by definition, have vanishing matrix elements in $d = 2$, and therefore if $j, k \geq 3$,with $i \leq 2$ we must have

$$C^i{}_{jk} \sim \varepsilon, \tag{3.2.5}$$

as it can be explicitly checked. It is then convenient to distinguish the evanescent operators by labelling them with a greek index, O_α, so that their renormalization group flow terms are denoted by

$$\beta^\alpha \; \frac{\partial}{\partial g^\alpha} \tag{3.2.6}$$

As explained in section 3.1 the effect of the terms (3.2.6) can be studied by considering the insertions of the operators $\beta^\alpha O_\alpha$ in the flow of the theory with $g_\alpha = 0$: in order to express these insertions in terms of the usual three-parameter space we need a reduction formula

$$\beta^\alpha \; O_\alpha = \sum_{i \leq 2} \; \beta^\alpha \; C^i_\alpha O_i \tag{3.2.7}$$

The nature of the evanescent operators O_α is such that they can give matrix elements different from zero only when considered in divergent diagrams; since at the one loop level the only divergent graphs are those of the four point functions we will compute the insertion in them of the equation (3.2.7). As usual we put the external momenta to zero and make the computation in the theory with three couplings only: the result is given by

$$\langle \sum_\alpha \int dz\beta^\alpha O_\alpha(z) \;\; \sum_{i\leq 2} \int dy g^i O_i(y) \;\rangle_{4\psi} = \sum_{\alpha,i\leq 2} \beta^\alpha \; C^i_\alpha \; \Gamma^{(i)}_A \otimes \Gamma^{(i)}_A \tag{3.2.8}$$

By noticing that the l.h.s. of (3.2.8) is just proportional to the one loop term (3.2.1) we can write it as follows:

$$2\beta^\alpha \; g^k \; C^i{}_{\alpha k} \; \Gamma^{(i)}_A \otimes \Gamma^{(i)}_A I\Big|_{\varepsilon=0} = \beta^\alpha \; \frac{\partial}{\partial g^\alpha}\left(V^i \; \Gamma^{(i)}_A \otimes \Gamma^{(i)}_A\right) I\Big|_{\varepsilon=0} \tag{3.2.9}$$

As stated before $C^i{}_{\alpha k} \sim \varepsilon$, therefore setting $V^i = \hat{V}^i + \varepsilon\delta V^i$ we obtain at the one-loop level

$$C^i_\alpha \;\; = \beta^\alpha \frac{\partial}{\partial g^\alpha}(\delta V^i) \; (\varepsilon \hat{I}) \tag{3.2.10}$$

and hence we see that the evanescent operators induce a shift in the β-functions of the relevant operators, given by

$$\delta\beta^i = \beta^\alpha \frac{\partial}{\partial g^\alpha}(\delta V^i) \; (\varepsilon \hat{I}). \tag{3.2.11}$$

But the story does not finish here, since in the computation of the two-loopβ-function we must consider counterterms induced at one-loop by the evanescent operators. According to the usual argument their contribution is the following

$$\begin{aligned} & - \;\; \langle \hat{I} \; g^j g^k C^{\hat{\alpha}}{}_{jk} \int dz O_\alpha \; g^i \int dy O_i \; \rangle \\ & = \;\; -2\hat{I} g^j g^k \; \hat{C}^\alpha{}_{jk} g^i \; C^l{}_{\alpha i} \Gamma^{(l)}_A \otimes \Gamma^{(l)}_A, \end{aligned} \tag{3.2.12}$$

and, on the other hand, locality of the counterterms implies that the only combination which can appear in two-loop diagrams with a double pole in $1/\varepsilon$ is:

$$I^2 \; - \; 2I\hat{I}. \tag{3.2.13}$$

Therefore genuine two-loops diagrams must give a contribution of the form

$$I^2 \; g^j g^k \; C^\alpha{}_{jk} \; g^i C^l{}_{\alpha i} \Gamma^{(l)}_A \otimes \Gamma^{(l)}_A. \tag{3.2.14}$$

Since ${C^l}_{\alpha i}$ is proportional to ε, both (3.2.12) and (3.2.14) contain only a simple pole: collecting their contribution we see that the effect on β-function at two loops is

$$\Delta\beta^{(2)i} = -2g^j g^k g^s \; {\hat{C}^\alpha}_{jk} {C^i}_{\alpha s} \; \varepsilon \; I^2 \equiv \; -\beta^\alpha \; C^i_\alpha \tag{3.2.15}$$

so that

$$\Delta\beta^{(2)i} + \delta\beta^i = 0. \tag{3.2.16}$$

Thus we can conclude that the contribution of the evanescent operators disappears from the flow of the renormalization group; as a check of this claim we will have to find a term

$$\Delta\beta^{(2)i} = -\beta^\alpha \frac{\partial}{\partial g^\alpha}(\delta V^i) \; (\varepsilon \hat{I}) \tag{3.2.17}$$

when computing the β-functions at the two-loop level.

Chapter 4

Two loops β-functions

This chapter is devoted to illustrate the computation, up to two loops, of the β-functions of the model (2.1.1) [32,34]. While dealing with this technical subject, we will see how the contribution of the evanescent operators can be identified, thus allowing the definition of a "symmetric scheme" of subtraction in which the result satisfies the symmetry properties discussed in chapter 2. On the other hand exactly these symmetry requirements permit to envisage an alternative algebraic approach that leads to an almost complete determination of the two loops β-functions.

4.1 Perturbative computation

This section reports the results of the computation of two loops diagrams contributing to the β functions of the model described in eq.(2.1.1) [32,34].

The calculation of vertex diagrams has been performed at zero external momenta introducing an infrared cutoff in the form of a mass term $m\,\bar{\psi}\psi$ for the fermion field; the fermion propagator will be written as

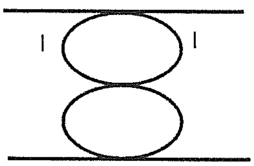
$$S(p) = \frac{1}{i\not{p} + m} = \frac{-i\not{p} + m}{p^2 + m^2} \tag{4.1.1}$$

In each diagram the mass term in the numerator will give a contribution to the pole term in $1/\varepsilon$, but on general grounds these terms do not contribute to β functions; moreover the infrared finiteness of the theory insures that these terms must cancel and it has been explicitely checked that this happens in the sum of each diagram and its counterterm. This amounts to say that all calculations can be performed with an effective propagator

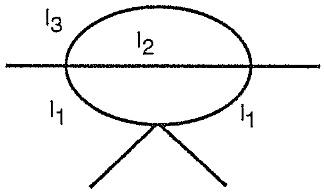
$$S(p) = \frac{-i\not{p}}{p^2 + m^2} \tag{4.1.2}$$

We will give the results in terms of the propagator (4.1.2) and collect in appendix B some examples of the cancellation of the mass terms.

The general topology of the four point two loops diagrams in momentum space is given by the following graphs:



(3a)



(3b)

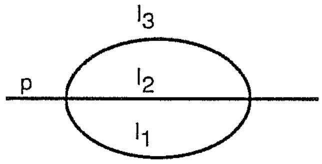
so that we have to consider only two types of integrals

$$(3a) \quad : \quad I_{\mu\nu} = \int_{l_1} \frac{l_{1\mu}\, l_{1\nu}}{(l_1^2 + m^2)^2} = \frac{1}{2}\delta_{\mu\nu} I \tag{4.1.3}$$

$$\begin{aligned} (3b) \quad : \quad I_{\mu\nu\alpha\beta} &= \int_{l_1 l_2} \frac{l_{1\mu}\, l_{1\nu}\, l_{2\alpha}\, l_{3\beta}}{(l_1^2 + m^2)^2 (l_2^2 + m^2)(l_3^2 + m^2)} = \\ & -\frac{I^2}{8}\delta_{\mu\nu}\delta_{\alpha\beta} - \frac{\varepsilon I^2}{32}\left[\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\mu\nu}\delta_{\alpha\beta}\right] \\ & \equiv -\frac{I^2}{8}\delta_{\mu\nu}\delta_{\alpha\beta} - \frac{\varepsilon I^2}{32} S_{\mu\nu\alpha\beta} \end{aligned} \tag{4.1.4}$$

where I is the scalar integral defined in (3.1.52), and we have written only the relevant divergent contributions.

To evaluate the β functions we have to compute also the field renormalization Z_2 which involves another type of integral, in this case at momentum $p \neq 0$, corresponding to the topology:

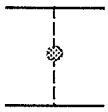


So we meet a new integral whose divergent part is

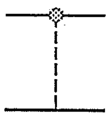
$$(3c): \quad \int_{l_1 l_2} \frac{l_{1\mu} l_{2\nu} l_{3\alpha}}{(l_1^2+m^2)^2(l_2^2+m^2)(l_3^2+m^2)} = \frac{\varepsilon I^2}{8}\left(\delta_{\mu\nu}p_\alpha + \delta_{\mu\alpha}p_\nu + \delta_{\nu\beta}p_\mu\right) \tag{4.1.5}$$

having put $p = l_1 + l_2 + l_3$.

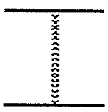
According to the notation defined in section 3.1 we denote the self energy, vertex and ladder counterterms as follows:



$$= -g_i^2 N \hat{c}_i \nu_i \hat{I} \Gamma_A^{(i)} \otimes \Gamma_A^{(i)}$$
$$\equiv -N g_i^2 \hat{A}^{(i)} \hat{I} \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} \tag{4.1.6}$$



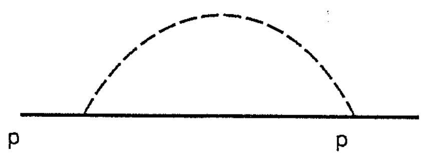
$$= \frac{1}{2} g_i g_j d^{(ij)} \hat{c}_i \hat{I} \Gamma_A^{(i)} \otimes \Gamma_A^{(i)}$$
$$\equiv \frac{1}{2} g_i g_j \hat{B}^{(ij)} \hat{I} \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} \tag{4.1.7}$$



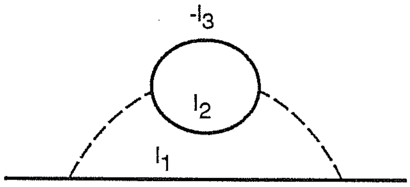
$$= -\frac{1}{4} g_i g_j \hat{I} \hat{X}_{AB}^{\mu(ij)} \otimes \hat{X}_{AB}^{\mu(ij)}$$
$$\equiv -\frac{1}{4} g_i g_j \hat{E}_{ijk} \hat{I} \Gamma_A^{(k)} \otimes \Gamma_A^{(k)} \tag{4.1.8}$$

The symbol $\hat{f}$ as usual define the leading term in the Laurent expansion in ε of the quantity f.

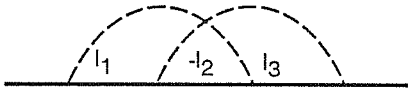
Let us start with the computation of Z_2.The relevant diagrams are:



(Z1)



(Z2A)



(Z2B)

The one loop diagram (Z1) gives trivially zero. At two loops we have again zero for the contribution of one loop vertex counterterm (its contribution has the same structure of the graph ($Z1$)) and we have only the genuine two loops diagrams (Z2A) and (Z2B)

$$\begin{aligned}(Z2A) &= -Ng_ig_j\int_{l_1l_2}\Gamma_A^{(i)}S(l_1)\Gamma_B^{(j)}\mathrm{Tr}\left[\Gamma_A^{(i)}S(l_2)\Gamma_B^{(j)}S(-l_3)\right] \\ &= i\frac{\varepsilon I^2}{16}\Gamma_A^{(i)}\gamma_\mu\Gamma_B^{(j)}\mathrm{Tr}\left[\Gamma_A^{(i)}\gamma_\alpha\Gamma_B^{(j)}\gamma_\beta\right]\left[\delta_{\mu\alpha}p_\beta+\delta_{\mu\beta}p_\alpha+\delta_{\alpha\beta}p_\mu\right]\end{aligned}$$

Let us note that εI^2 is a simple pole, so for the computation of the pole it is sufficient to perform the algebra of γ matrices in two dimensions. An easy computation gives

$$(Z2A) = i\not{p}\frac{\varepsilon I^2}{2}N\left(g_S^2+g_P^2+2g_V^2\right) \tag{4.1.9}$$

For the graph ($Z2B$) we have

$$\begin{aligned}(Z2B) &= g_ig_j\int_{l_1l_2}\Gamma_A^{(i)}S(l_1)\Gamma_B^{(j)}S(-l_2)\Gamma_A^{(i)}S(l_3)\Gamma_B^{(j)} \\ &= -i\frac{\varepsilon I^2}{16}g_ig_j\left[\delta_{\mu\alpha}p_\beta+\delta_{\mu\beta}p_\alpha+\delta_{\alpha\beta}p_\mu\right]\Gamma_A^{(i)}\gamma_\mu\Gamma_B^{(j)}\gamma_\alpha\Gamma_A^{(i)}\gamma_\beta\Gamma_B^{(j)} \\ &= -i\not{p}\frac{\varepsilon I^2}{4}\left[(g_S-g_P)^2+4g_V\left(g_S+g_P\right)\right]\end{aligned}$$

Summing up we obtain

$$\begin{aligned}Z_2^{(2)} = &- \frac{\varepsilon I^2}{2}N\left(g_S^2+g_P^2+2g_V^2\right) \\ &+ \frac{\varepsilon I^2}{4}\left[(g_S-g_P)^2+4g_V\left(g_S+g_P\right)\right]\end{aligned} \tag{4.1.10}$$

The vertex diagrams we have to consider come from the opening, in all the possible ways, of the four fermion vertex in graphs (3a) and (3b); they are displayed in the tables 4.1 and 4.2 at the end of this chapter, together with their counterterms. The final results can be summarized as follows

$$
\begin{aligned}
(A) &= N^2 g_i^3 \left[A^{(i)2} I^2 - 2\hat{A}^{(i)} A^{(i)} \hat{I} I\right] \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} \\
(B) &= N g_i^2 g_j \left[-\frac{1}{2} A^{(i)} B^{(ij)} I^2 + A^{(i)} \hat{B}^{(ij)}\right] \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} \\
(C) &= N g_i^2 g_j \left[-A^{(i)} B^{(ij)} I^2 + A^{(i)} \hat{B}^{(ij)} \hat{I} I + \hat{A}^{(i)} B^{(ij)} \hat{I} I\right] \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} \\
(D) &= \frac{N}{2} g_i g_j^2 \left[B^{(ij)} A^{(j)} I^2 + 2 B^{(ij)} \hat{A}^{(j)} \hat{I} I\right] \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} - 2N g_V^3 \varepsilon I^2 \gamma_\mu \otimes \gamma_\mu \\
(E) &= \frac{N}{4} g_i g_j g_k \left[A^{(i)} E_{jki} I^2 - 2 A^{(i)} \hat{E}_{jki} \hat{I} I\right] \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} \\
&\quad -2N g_S g_P g_V \varepsilon I^2 \gamma_\mu \otimes \gamma_\mu \\
(F) &= \frac{N}{4} g_j g_k^2 \left[E_{jki} A^{(k)} I^2 - 2 E_{jki} \hat{A}^{(k)} \hat{I} I\right] \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} \\
(G) &= g_i g_j g_k \Big[\left(\frac{1}{4} B^{(ij)} B^{(ik)} I^2 - \frac{1}{2} \hat{B}^{(ik)} B^{(ij)} \hat{I} I\right) \left(B^{(ik)} B^{(kj)} I^2 - B^{(ik)} \hat{B}^{(kj)} \hat{I} I\right) \\
&+ \left(-\frac{1}{8} B^{(is)} E_{jks} I^2 + \frac{1}{4} B^{(is)} \hat{E}_{jks} \hat{I} I\right)\Big] \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} \\
&+ \varepsilon I^2 \left[2 g_V^2 (g_S + g_P) - \frac{1}{2} g_V (g_S - g_P)^2\right] \gamma_\mu \otimes \gamma_\mu \\
(H) &= g_j g_k g_l \left[-\frac{1}{4} I^2 E_{jki} B^{(jl)} + \frac{1}{2} \hat{I} I E_{jki} \hat{B}^{(jl)}\right] \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} \\
&+ \varepsilon I^2 g_V (g_S + g_P) (g_P 1 \otimes 1 + g_S \gamma_5 \otimes \gamma_5) \\
(L) &= g_k g_j g_l \left[-\frac{I^2}{8} E_{kji} B^{(il)} + \frac{1}{4} \hat{I} I \hat{E}_{kji} B^{(il)}\right] \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} \\
&\quad +\varepsilon I^2 g_S g_P (g_S + g_P) \gamma_\mu \otimes \gamma_\mu \\
(M) &= g_j g_k g_l \left[\frac{I^2}{16} E_{kjs} E_{sli} - \frac{\hat{I} I}{16} \hat{E}_{kjs} E_{sli}\right] \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} \\
&- \frac{1}{2} \varepsilon I^2 \left(g_S^2 - g_P^2\right) (g_S 1 \otimes 1 - g_P \gamma_5 \otimes \gamma_5) \\
(N) &= g_i g_j g_k \left[\frac{I^2}{4} B^{(ij)} B^{(ik)} - \frac{1}{2} \hat{I} I B^{(ij)} \hat{B}^{(ik)}\right] \Gamma_A^{(i)} \otimes \Gamma_A^{(i)}
\end{aligned}
$$

Before collecting all terms together we can make a check on the structure of the double pole in $1/\varepsilon$. The β function at one loop is

$$\beta^{i(1)} = \varepsilon \hat{I} \left[N \hat{A}^{(i)} g_i^2 - \hat{B}^{(ij)} g_i g_j + \frac{1}{4} \hat{E}_{kji} g_k g_j \right] \tag{4.1.11}$$

and it is easily checked that the sum of the coefficients of double pole terms is given by

$$\frac{1}{2} \beta^{j(1)} \frac{\partial \beta^{i(1)}}{\partial g^j} \tag{4.1.12}$$

as prescribed by the renormalization group.

For what concern the proper calculation of the β-functions we notice first of all that diagrams (A), (C) and (N) do not give rise to simple poles, as expected from their factorized form. It is then convenient to set

$$A^{(i)} = \hat{A}^{(i)} + \varepsilon \delta A^{(i)} \tag{4.1.13}$$

$$B^{(ij)} = \hat{B}^{(ij)} + \varepsilon \delta B^{(ij)} \tag{4.1.14}$$

$$E_{ijk} = \hat{E}_{ijk} + \varepsilon \delta E_{ijk} \tag{4.1.15}$$

Therefore the simple pole terms have the following form:

$$\begin{aligned}
(B) &= N g_i^2 g_j \left(-\frac{1}{2} \hat{A}^{(i)} \delta B^{(ij)} + \frac{1}{2} \delta A^{(i)} \hat{B}^{(ij)} \right) \varepsilon \hat{I}^2 \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} \\
(D) &= N g_i g_j^2 \left(-\frac{1}{2} \delta A^{(j)} \hat{B}^{(ij)} + \frac{1}{2} \hat{A}^{(j)} \delta B^{(ij)} \right) \varepsilon \hat{I}^2 \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} \\
&\quad -2 N g_V^2 \varepsilon \hat{I}^2 \gamma_\mu \otimes \gamma_\mu \\
(E) &= N g_i g_j g_k \left(\frac{1}{4} \delta E_{jki} \hat{A}^{(i)} - \hat{E}_{jki} \delta A^{(i)} \right) \varepsilon \hat{I}^2 \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} \\
&\quad -2 N g_S g_P g_V \varepsilon \hat{I}^2 \gamma_\mu \otimes \gamma_\mu \\
(F) &= N g_j g_k^2 \left(\frac{1}{4} \hat{E}_{jki} \delta A^{(k)} - \delta E_{jki} \hat{A}^{(k)} \right) \varepsilon \hat{I}^2 \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} \\
(G) &= g_i g_j g_k \Big(\frac{1}{2} \hat{B}^{(ij)} \delta B^{(jk)} - \frac{1}{2} \delta B^{(ij)} \hat{B}^{(jk)} - \frac{1}{2} \delta B^{(ij)} \hat{B}^{(jk)} \\
&\qquad + \frac{1}{8} \delta B^{(is)} E_{jks} - \frac{1}{8} \hat{B}^{(is)} \delta E_{jks} \Big) \varepsilon \hat{I}^2 \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} \\
&+ \left[2 g_V^2 (g_S + g_P) - \frac{1}{2} g_V (g_S - g_P)^2 \right] \varepsilon \hat{I}^2 \gamma_\mu \otimes \gamma_\mu
\end{aligned}$$

$$
\begin{aligned}
(H) &= \frac{1}{4} g_j g_k g_l \left[\delta E_{kji} \hat{B}^{(jl)} - \hat{E}_{kji} \delta B^{(jl)}\right] \varepsilon \hat{I}^2 \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} \\
&\quad +2\varepsilon \hat{I}^2 g_V \left(g_S + g_P\right)\left(g_P 1 \otimes 1 + g_S \gamma_5 \otimes \gamma_5\right) \\
(L) &= \frac{1}{8} g_k g_j g_l \left[\hat{E}_{kji} \delta B^{(il)} - \delta E_{kji} \hat{B}^{(il)}\right] \varepsilon \hat{I}^2 \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} \\
&\quad +\varepsilon \hat{I}^2 g_S g_P \left(g_S + g_P\right) \gamma_\mu \otimes \gamma_\mu \\
(M) &= \frac{1}{16} g_j g_k g_l \left[\delta E_{kjs} \hat{E}_{sli} - \hat{E}_{kjs} \delta E_{sli}\right] \varepsilon \hat{I}^2 \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} \\
&\quad -\frac{1}{2} \varepsilon \hat{I}^2 \left(g_S^2 - g_P^2\right)\left(g_S 1 \otimes 1 - g_P \gamma_5 \otimes \gamma_5\right) \qquad (4.1.16)
\end{aligned}
$$

Let us now notice that all the contribution coming from the expansion of double poles can be written as

$$
\frac{1}{2}\left[G^k \frac{\partial \beta^i}{\partial g^k} - \beta^k \frac{\partial G^i}{\partial g^k}\right] \frac{1}{\varepsilon} \qquad (4.1.17)
$$

where

$$
G^i = (\varepsilon \hat{I}) \left(N \delta A^{(i)} g_i^2 - \delta B^{(ij)} g_i g_j + \frac{1}{4} \delta E_{kji} g_k g_j\right) \qquad (4.1.18)
$$

So including the contribution of Z_2, which is the same for all form factor and is given in eq.(4.1.10), the $\tilde{\beta}$ functions can be written as

$$
\begin{aligned}
\tilde{\beta}^i \Gamma_A^{(i)} \otimes \Gamma_A^{(i)} &= G_j \frac{\partial \beta^{i(1)}}{\partial g^j} - \beta^{j(1)} \frac{\partial G^i}{\partial g^j} + \qquad (4.1.19) \\
&+ \frac{1}{2\pi^2}\Big[N g_S \left(g_S^2 + g_P^2\right) - g_S^2 \left(g_S - g_P\right) \\
&\qquad +2 g_V \left(g_P^2 - g_S^2\right) + 2N g_V^2 \left(g_S - g_P\right)\Big] 1 \otimes 1 \\
&+ \frac{1}{2\pi^2}\Big[N g_P \left(g_P^2 + g_S^2\right) - g_P^2 \left(g_P - g_S\right) \\
&\qquad +2 g_V \left(g_S^2 - g_P^2\right) + 2N g_V^2 \left(g_P - g_S\right)\Big] \gamma_5 \otimes \gamma_5 \\
&+ \frac{1}{2\pi^2} \left[(N-1) g_V \left(g_S - g_P\right)^2 + g_S g_P \left(g_S + g_P\right)\right] \gamma_\mu \otimes \gamma_\mu
\end{aligned}
$$

where we have used $2(\varepsilon \hat{I})^2 = 1/2\pi^2$.

The contribution to the $\tilde{\beta}$ functions in eq.(4.1.19) that does not depend on G^i will be denoted by $\beta^{(sym)}$ [32,34]

$$\tilde{\beta}^{i(2)} = G^j \frac{\partial \beta^{i(1)}}{\partial g^j} - \beta^{i(1)} \frac{\partial G^i}{\partial g^j} + \beta^{i(sym)} \tag{4.1.20}$$

First of all let us analyze the contribution of evanescent couplings which we denote here by greek letters; their contribution to β functions of relevant operators is given by

$$\Delta\beta^i = G^\alpha \frac{\partial \beta^{i(1)}}{\partial g^\alpha} - \beta^{\alpha(1)} \frac{\partial G^i}{\partial g^\alpha} \tag{4.1.21}$$

but the insertion of evanescent operators in one loop graph cannot give pole terms and therefore $\beta^{i(1)}$ cannot depend on g^α. Hence

$$\Delta\beta^i = -\beta^{\alpha(1)} \frac{\partial G^i}{\partial g^\alpha} \tag{4.1.22}$$

This expression is exactly what we need in order to cancel the shift at two loops coming from $\beta^{\alpha(1)}$ as explained in section 3.2; actually if we note that the expressions for G^i and $\left(\delta V^i \varepsilon \hat{I}\right)$ are the same, eq.(4.1.22) is easily seen to coincide with eq.(3.2.11).

This means that the renormalization group flow is tridimensional and we can forget the contribution of evanescent operators in the expression (4.1.20). As it stands eq.(4.1.20) gives β functions in the minimal scheme for any given choice of relevant operators. We give here for completeness the G^i functions in the basis defined by eq.(3.1.6) [32,34]

$$G_S = -\frac{1}{2\pi}\left[(N-1)\,g_S^2 + 4g_Sg_P - 4g_Sg_V + 6g_Vg_P\right] \tag{4.1.23}$$

$$G_P = -\frac{1}{2\pi}\left[-(N-6)\,g_P^2 - g_Sg_P + 4g_Pg_V\right] \tag{4.1.24}$$

$$G_V = -\frac{1}{2\pi}\left[-Ng_V^2 + g_V\,(g_S+g_P) + 2g_Sg_P\right] \tag{4.1.25}$$

The form of the equation (4.1.20) suggests the possibility of performing a change of variables

$$g'^i = g^i + G^i \tag{4.1.26}$$

such that the resulting β-functions are simply

$$\beta' = \beta^{(1)} + \beta^{(sym)} \tag{4.1.27}$$

that in the usual components read

$$
\begin{aligned}
\beta_S &= -\frac{1}{\pi}\left[(N-1)g_S^2 + g_S g_P - 2g_V(g_S - g_P)\right] \\
&+ \frac{1}{2\pi^2}\Big[N g_S(g_S^2 + g_P^2) - g_S^2(g_S - g_P) + 2g_V(g_P^2 - g_S^2) \\
&\qquad +2N g_V^2(g_S - g_P)\Big] \\
\beta_P &= -\frac{1}{\pi}\left[(N-1)g_P^2 + g_P g_S - 2g_V(g_P - g_S)\right] \\
&+ \frac{1}{2\pi^2}\Big[N g_P(g_P^2 + g_S^2) - g_P^2(g_P - g_S) + 2g_V(g_S^2 - g_P^2) \\
&\qquad +2N g_V^2(g_P - g_S)\Big] \\
\beta_V &= -\frac{1}{\pi} g_S g_P \\
&+ \frac{1}{2\pi^2}\left[(N-1)g_V(g_S - g_P)^2 + g_S g_P(g_S + g_P)\right] \qquad (4.1.28)
\end{aligned}
$$

All special cases of equation (4.1.28) reproduce the known results for Gross-Neveu [28] and Thirring [32,34,35] models. In sections 4.2 and 6.2 we shall show that eq.(4.1.28) is the unique form of the β-functions compatible with an explicit realization of the classical symmetries.

4.2 Algebraic approach

In this section we shall take an alternative algebraic approach to the determination of the β-functions. Its strategy is based on the assumption that all the classical symmetries of the lagrangian (2.1.1) can be explicitly implemented in the quantum realization of the model, that is there always exists a change of variables such that these symmetries correspond to definite algebraic properties of the correlation functions. As we shall show these algebraic conditions will be sufficient for an almost complete determination of the symmetric two-loop β-functions. By comparison with our previous results we will be able to confirm explicitly, at the two-loop level, the change of variables suggested in eq.(4.1.26).

We have already listed in section 2.2 all the classical symmetries of our model. Let us now critically discuss them and present their consequences at the quantum level. In the following it will be useful to introduce a slight change in notation, by defining

$$g_{\pm} = \frac{1}{2}(g_S \pm g_P) \qquad (4.2.1)$$

and the corresponding components of the β vector

$$\beta_{\pm} = \frac{1}{2}(\beta_S \pm \beta_P). \qquad (4.2.2)$$

1. Chiral properties.

   Since we have restricted our operator space to only three terms, the only remnant of the covariance under global chiral transformations of the ψ fields are the symmetry properties resulting from the discrete chiral transformation ($\alpha = \pi/2$)

$$\begin{aligned} \psi &\rightarrow \frac{1+\gamma_5}{\sqrt{2}}\psi \\ \bar{\psi} &\rightarrow \bar{\psi}\frac{1+\gamma_5}{\sqrt{2}} \end{aligned} \qquad (4.2.3)$$

   such that

$$\begin{aligned} \bar{\psi}\psi &\rightarrow \bar{\psi}\gamma_5\psi \\ \bar{\psi}\gamma_5\psi &\rightarrow -\bar{\psi}\psi \\ \bar{\psi}\gamma_\mu\psi &\rightarrow \bar{\psi}\gamma_\mu\psi \end{aligned} \qquad (4.2.4)$$

   The operators $O_{\pm} = \frac{1}{2}[(\bar{\psi}\psi)^2 \pm (\bar{\psi}\gamma_5\psi)^2]$ are respectively even and odd under (4.2.3). Therefore all the correlation functions that are even or odd in O_- must be even or odd functions of the corresponding coupling g_-.

   A trivial consequence of this statement leads to the following property of the β-functions:

$$\begin{aligned} \beta_+(g_+, g_-, g_V) &= \beta_+(g_+, -g_-, g_V) \\ \beta_-(g_+, g_-, g_V) &= -\beta_-(g_+, -g_-, g_V) \\ \beta_V(g_+, g_-, g_V) &= \beta_V(g_+, -g_-, g_V) \end{aligned} \qquad (4.2.5)$$

   Along with the corresponding symmetry properties of the metric, these conditions put very strong constraints on the structure of the parameter space.

2. Thirring models.

The case of the Thirring model corresponds to the choice $g_- = 0$, implying from eq.(4.2.5)

$$\beta_-(g_+, 0, g_V) = 0 \tag{4.2.6}$$

Making use of the bosonization technique one can very easily argue that the $U(1)$ factor appearing in general in the Thirring model must be completely decoupled from the $SU(N)$ factor, and have vanishing β-function [50]. Moreover from the Fierz identities, which we assume to be explicitely realized in this formulation of the model, we can establish the equivalence

$$O_+ + \frac{1}{N} O_V = -\frac{1}{2l_R} (\bar{\psi}\gamma_\mu T^a \psi)(\bar{\psi}\gamma_\mu T^a \psi) \tag{4.2.7}$$

where T^a are the $SU(N)$ generators and

$$Tr[T^a T^b] = l_R \delta^{ab} \tag{4.2.8}$$

We can then reexpress the interaction lagrangian as

$$\begin{aligned} \mathcal{L}_I &= -g_+(O_+ + \frac{1}{N} O_V) - (g_V - \frac{1}{N} g_+) O_V \\ &\equiv -g_+ O_{SU(N)} - (g_V - \frac{1}{N} g_+) O_{U(1)} \end{aligned} \tag{4.2.9}$$

The consequence of the abovementioned properties on the β-functions can now be sinthetized as follows

$$\beta_V(g_+, 0, g_V) - \frac{1}{N}\beta_+(g_+, 0, g_V) = 0 \tag{4.2.10}$$

$$\frac{\partial \beta_+}{\partial g_V}(g_+, 0, g_V) = 0. \tag{4.2.11}$$

3. Gross-Neveu models.

By choosing $g_+ = g_- = g, g_V = 0$ we restrict our model to the so called Gross-Neveu model, which is known to be stable under renormalization. Therefore this choice of couplings must be preserved by the renormalization group transformations, and the β-functions must enjoy the properties

$$\beta_+(g, g, 0) = \beta_-(g, g, 0) \tag{4.2.12}$$

$$\beta_V(g, g, 0) = 0 \tag{4.2.13}$$

As a byproduct of our knowledge about the Gross-Neveu model we can also expect an overall multiplicative factor $(N-1)$ to appear in the nonvanishing β functions.

4. Special symmetries of $N=1$ and $N=2$ models.

   The quantum equivalence of all $N=1$ models, based on eq.(2.2.7) implies simply the relationship

$$\beta_V(N=1) = \beta_+(N=1). \tag{4.2.14}$$

   In order to discuss $N=2$ models, it is convenient to introduce a more natural parametrization of the interaction terms (see ref.[31])

$$\mathcal{L}_I = -g_+\ (O_+ + \frac{1}{2}O_V) - g_-\ (O_- - \frac{1}{2}O_V) - \delta_V O_V \tag{4.2.15}$$

   where

$$\delta_V = g_V - \frac{1}{2}g_+ + \frac{1}{2}g_- \tag{4.2.16}$$

   and the first two terms correspond to the decoupled antiself-dual and self-dual parts of the standard decomposition $O(4) = O(3) \otimes O(3)$. As already discussed the U(1) singlet term $\delta_V O_V$ couples only to the self-dual current interaction. As a consequence, we can deduce the following properties of the $N=2$ β-functions:

$$\beta_+ = \beta_2(g_+) \tag{4.2.17}$$

$$\beta_- = \beta_-(g_-, \delta_V), \qquad \beta_-(g_-, 0) = \beta_2(g_-), \qquad \beta_-(0, \delta_V) = 0 \tag{4.2.18}$$

$$\beta_V = \frac{1}{2}\beta_+ - \frac{1}{2}\beta_- + \beta_\delta(g_-, \delta_V) \qquad \beta_\delta(0, \delta_V) = 0 \tag{4.2.19}$$

   Obviously some of these conditions are just special cases of those previously discussed.

5. Group-theoretic factors and N dependence.

   Let us first observe that $U(N)$ Gross-Neveu models are nothing but $O(2N)$ Thirring models (ref.[31]). From an analysis of current-current interactions based on the formalism of non abelian bosonization [23,24] one is naturally led to conjecture that the Thirring model β-functions can only depend on the symmetry group through properly normalized Casimir invariants in the adjoint representation [34]. This statement can be explicitly shown to hold in one and two-loops computations. Now, recalling that

$$C_A^{SU(N)} = N \qquad\qquad C_A^{O(N)} = N-2 \tag{4.2.20}$$

we can reformulate eqs.(4.2.10) (4.2.11) (4.2.12) and (4.2.13) in terms of a single function of just one coupling $C_A\beta_T(g; C_A)$, such that:

$$\begin{aligned}
\beta_+(g_+, 0, g_V; N) &= N\beta_T(g_+; N) &(4.2.21)\\
\beta_V(g_+, 0, g_V; N) &= \beta_T(g_+; N) &(4.2.22)\\
\beta_+(g, g, 0; N) &= 2(N-1)\beta_T(g; 2(N-1)) &(4.2.23)\\
\beta_-(g, g, 0; N) &= 2(N-1)\beta_T(g; 2(N-1)) &(4.2.24)\\
\beta_V(g, g, 0; N) &= 0 &(4.2.25)
\end{aligned}$$

As a special case of eqs.(4.2.21) (4.2.23) we obtain

$$\beta_+(g, 0, 0; 4) = \beta_+(g, g, 0; 3) \tag{4.2.26}$$

reflecting the quantum equivalence of $U(3)$ Gross-Neveu and $SU(4)$ Thirring models at the level of β-functions.

Another useful hint comes from the diagrammatic analysis, performed in the $1/N$ expansion scheme, showing that the maximal power of N appearing in the coefficients of the β-functions at the L-loop order is $L-1$ for all $L > 1$. Therefore the power series expansion of the function β_T is

$$\beta_T(g; N) = b_1 g^2 + b_2 g^3 + (b_3 N + b_3')g^4 + \mathcal{O}(N^2 g^5) \tag{4.2.27}$$

where the b_i are N-independent numbers. The computation presented in the Appendix C allows us to fix the coefficients

$$b_1 = -\frac{1}{\pi}\,, \qquad b_2 = \frac{1}{\pi^2} \tag{4.2.28}$$

It is instructive to derive the most general allowable form of the one-loop β-functions based on the previous results. A straightforward application of eqs.(4.2.5) (4.2.21) (4.2.23) leads without any loss of generality to

$$\begin{aligned}
\beta_+^{(1)} &= b_1[Ng_+^2 + (N-2)g_-^2] &(4.2.29)\\
\beta_-^{(1)} &= b_1[2(N-1)g_+g_- + \lambda g_V g_-] &(4.2.30)\\
\beta_V^{(1)} &= b_1[g_+^2 - g_-^2] &(4.2.31)
\end{aligned}$$

with only one free N-independent parameter left, apart the normalization factor b_1.

Eqs.(4.2.14), (4.2.17) are automatically satisfied, but we can still make use of the special properties of $N = 2$ models embodied in eq.(4.2.18). The request that β_- be a function of g_- and $g_+ - 2g_V$ only implies immediately

$$\lambda = -4. \tag{4.2.32}$$

Therefore the one-loop β-functions are completely determined from symmetry considerations. Probably the most interesting consequence of this result does not reside in the actual computation of the β's (the one-loop analysis in section 3.1 did not prove so difficult after all) but, because of the uniqueness of the result, in the fact that the only allowable variable changes preserving all symmetries at the two-loop level are

$$g'^{\,i} = g^i + \beta^{i(1)} t \tag{4.2.33}$$

that is renormalization group transformations. Since these transformations, by definition, do not change the form of the β-functions, what we have just proven is the uniqueness of the symmetric form of the two loop β-functions.

The constraints implied by eqs.(4.2.5), (4.2.21-4.2.23), supplemented by eq.(4.2.27), can also be applied to the two loop β functions. The resulting general parametrization is

$$\begin{aligned}
\beta_+^{(2)} &= b_2[Ng_+^3 + (N-2)g_+g_-^2 + \overline{\mu} g_V g_-^2] && (4.2.34)\\
\beta_-^{(2)} &= b_2[(N-1)(g_+^2 + g_-^2) + \overline{\nu}(g_+^2 - g_-^2)\overline{\rho} g_V g_+ + \overline{\sigma} g_V^2]g_- && (4.2.35)\\
\beta_V^{(2)} &= b_2[g_+(g_+^2 - g_-^2) + \overline{\tau} g_V g_-^2] && (4.2.36)
\end{aligned}$$

where $\overline{\mu}, \overline{\nu}, \overline{\rho}, \overline{\sigma}, \overline{\tau}$ are linear functions of N. The special symmetries of $N = 1$ and $N = 2$ give further constraints whose implementation leads to the final result

$$\begin{aligned}
\beta_+^{(2)} &= b_2[Ng_+^3 + (N-2)g_+g_-^2 + \mu(N-2)g_V g_-^2] \qquad (4.2.37)\\
\beta_-^{(2)} &= b_2[(N-1)(g_+^2 + g_-^2)g_- - 4g_+g_Vg_- + 2Ng_V^2g_-\\
&\quad + \delta(g_+^2 - g_-^2 - 4g_+g_V + 4g_V^2)g_- + \nu(N-2)(g_+^2 - g_-^2)g_-\\
&\quad + \rho(N-2)g_Vg_+g_- + \sigma(N-2)g_V^2g_-]\\
\beta_V^{(2)} &= b_2[g_+(g_+^2 - g_-^2) + 2(N-1)g_Vg_-^2\\
&\quad + \mu(N-2)g_Vg_-^2]
\end{aligned}$$

where the five numerical parameters $\mu, \nu, \rho, \sigma, \delta$ are still undetermined. By comparison with eq(4.1.19), we know that the actual symmetric β's correspond to vanishing $\mu, \nu, \rho, \sigma, \delta$.

At this stage of our analysis we have not managed yet to identify the unique form of the two-loop β-function by purely algebraic arguments. However, as we shall show in section 6.1, more constraints can be obtained after having performed an analogous algebraic construction of the three and four loop metric and imposing Zamolodchikov's equation. The final conclusion will amount to an almost complete algebraic determination of the two loop β function.

Table 4.1: Two loops vertex graphs and their counterterms

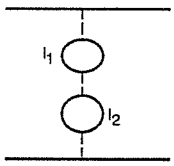
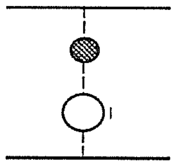
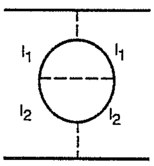
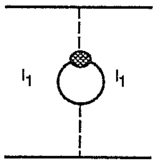
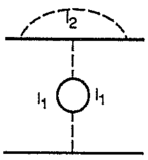
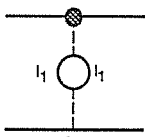
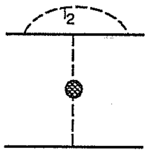
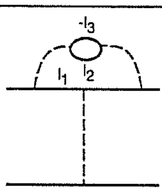
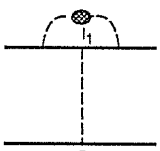
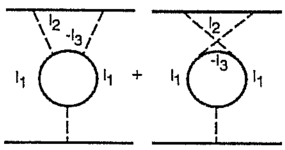
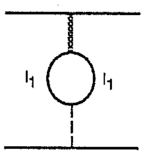
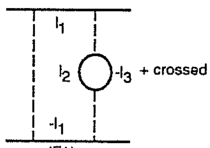
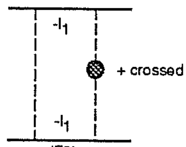
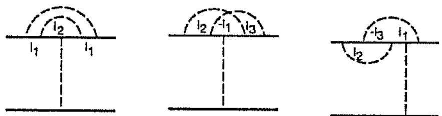
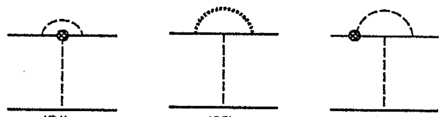
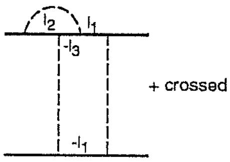
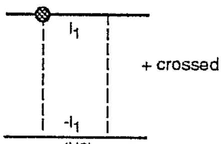
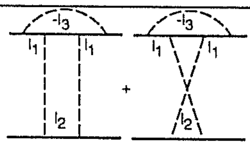
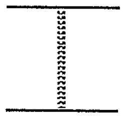
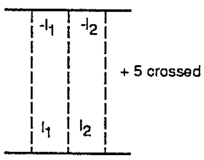
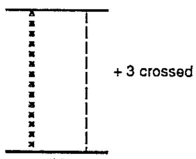
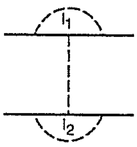
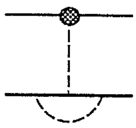
Label	*Graph*	*Counterterm*
A	 (A1)	 (A2)
B	 (B1)	 (B2)
C	 (C1)	 (C2)  (C3)
D	 (D1)	 (D2)
E	 (E1)	 (E2)

Table 4.2: Two loops vertex graphs and their counterterms (cont.d)

Label	*Graph*	*Counterterm*
F	 (F1)	 (F2)
G	 (G1) (G2) (G3)	 (G4) (G5) (G6)
H	 (H1)	 (H2)
L	 (L1)	 (L2)
M	 (M1)	 (M2)
N	 (N1)	 (N2)

Chapter 5

Zamolodchikov's C theorem

As anticipated in the introduction, one of the main motivations at the basis of this work is to provide, by means of the generalized Thirring model (2.1.1), a simple but non trivial explicit realization of Zamolodchikov's theorem [1,2,3], in order to show the close connection that it establishes between the two dimensional field theory and the structure of the manifold in which the coupling constants live.

From this standpoint it is certainly useful to review some general aspects of the theorem, and this chapter is intended as a brief survey of its proof, applications and validity.

5.1 The proof

Consistently with the above discussion, we begin by recalling few renormalization properties of the lagrangian (2.1.1). It is important to stress, however, that we will report a derivation of the theorem which is by no means model dependent; rather we wish to point out which are the general requirements fulfilled by the model (2.1.1) that allow us to proof Zamolodchikov's relation.

The generalized Thirring model is described by the lagrangian (2.1.1)

$$\mathcal{L} = \bar{\psi} \partial\!\!\!/ \psi - g^i O_i$$

and it is classically conformal invariant:

$$T^{(cl)\alpha}_{\alpha} = 0 \tag{5.1.1}$$

At the quantum level the scale invariance of the theory is broken by the renormalization procedure and the trace of energy momentum tensor is no longer vanishing:

$$T^{\alpha}_{\alpha} = \beta^i N[O_i] \tag{5.1.2}$$

The renormalized operators $N[O_i]$ were defined in chapter 3: their zero momentum insertion in each Green function is given by

$$\langle N[O_i]\prod_s \psi(x_s)\rangle \equiv \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\int d^2z N[O_i](z)\prod_s \psi(x_s)e^{-S} = \frac{\partial}{\partial g^i}\langle\prod_s \psi(x_s)\rangle \tag{5.1.3}$$

Thus the renormalized form of $\int d^2zN[O_i]$ in the minimal scheme can be directly seen from the definition of counterterms in the lagrangian. Power counting shows that the renormalized unintegrated operators can at most differ from their zero momentum insertion by a total derivative of dimension 1 operators; in the case of the model (2.1.1) the only operator we can conceive is the vector current, which is conserved and vanishes upon derivation. Therefore we conclude

$$N[O_i] = \mu^{-\varepsilon}O_k\left[\delta^k_i + \sum_{\substack{L=1\\ \nu}}\frac{\partial}{\partial g^i}\frac{P^{k(L,\nu)}}{\varepsilon^\nu}\right] \tag{5.1.4}$$

In eq.(5.1.4) and in the following we always understand possible additions proportional to the equations of motion; we are not interested in them as they do not give any contribution to the relations exploited below. From now on we omit for simplicity the symbol $N[\ ]$ in our formulas. Eq.(5.1.4) gives immediately the behaviour of O_i under renormalization group transformation [1,30] .

$$\mathcal{D}O_i \equiv \left(\mu\frac{\partial}{\partial\mu} + \beta^k\frac{\partial}{\partial g^k}\right)O_i = -\gamma^k_i O_k \tag{5.1.5}$$

where

$$\gamma^k_i = \varepsilon\delta^k_i + \frac{\partial\beta^k}{\partial g^i} \tag{5.1.6}$$

Let us now go back to the scale properties of the model. Following ref.[1] we consider the correlations function of energy momentum tensor; in the basis

$$x^+ = \frac{x+iy}{\sqrt{2}} \qquad x^- = \frac{x-iy}{\sqrt{2}} \tag{5.1.7}$$

they have the form

$$\langle T_{++}(x)\, T_{++}(0)\rangle = \frac{A_1(g_i)}{(x^+)^4} \tag{5.1.8}$$

$$\langle T_{-+}(x)\, T_{++}(0)\rangle = \langle T_{++}(x)\, T_{-+}(0)\rangle = \frac{A_2(g_i)}{(x^+)^3 x^-} \tag{5.1.9}$$

$$\langle T_{-+}(x)\, T_{-+}(0)\rangle = \frac{A_3(g_i)}{(x^+)^2 (x^-)^2} \tag{5.1.10}$$

Each form factor A_i is a function of the coupling constants which can be evaluated perturbatively, and Lorentz invariance implies that they are all scalars. The infrared finiteness of our model insures that they can only depend on $log(x^2\mu^2)$ at any order of perturbation theory. As a trivial consequence:

$$\frac{1}{2}\mu\frac{\partial}{\partial\mu}A_i = x^+\partial_+ A_i = x^-\partial_- A_i \tag{5.1.11}$$

Using the conservation of energy momentum tensor $\partial^\alpha T_{\alpha\beta} = 0$ we can write

$$\begin{aligned}\frac{1}{2}\mu\frac{\partial}{\partial\mu}(A_1 + A_2) &= x^-\partial_- A_1 + x^+\partial_+ A_2 = \\ &= 3A_2 + x^{+4}x^-\partial_-\langle T_{++}(x)\, T_{++}(0)\rangle + x^{+4}x^-\partial_+\langle T_{-+}(x)\, T_{++}(0)\rangle = 3A_2\end{aligned} \tag{5.1.12}$$

and

$$\begin{aligned}\frac{1}{2}\mu\frac{\partial}{\partial\mu}(A_2 + A_3) &= x^-\partial_- A_2 + x^+\partial_+ A_3 = \\ &= A_2 + 2A_3 + x^{+3}x^{-2}\partial_-\langle T_{++}(x)\, T_{-+}(0)\rangle + x^{+3}x^{-2}\partial_+\langle T_{-+}(x)\, T_{-+}(0)\rangle = \\ &= A_2 + 2A_3\end{aligned} \tag{5.1.13}$$

We now define the function C to be the following combination of A_i's

$$C = A_1 - 2A_2 - 3A_3 \tag{5.1.14}$$

and on the basis of the above considerations we can see that it satisfies the equation

$$\frac{1}{2}\mu\frac{\partial}{\partial\mu}C = -6A_3 \tag{5.1.15}$$

Using

$$T_{+-} = \frac{1}{2}T^\alpha_\alpha = \frac{1}{2}\beta^i O_i \tag{5.1.16}$$

we can write

$$A_3 = x^{+2} x^{-2} \frac{1}{4} \beta^i \beta^j \langle O_i(x) O_j(y) \rangle \tag{5.1.17}$$

and defining the metric-like tensor G_{ij} by

$$\langle O_i(x) O_j(0) \rangle = \frac{G_{ij}}{(x^2)^2} \tag{5.1.18}$$

eqs. (5.1.15,5.1.17) yield

$$\mu \frac{\partial}{\partial \mu} C = -\frac{3}{4} \beta^i \beta^j G_{ij} \tag{5.1.19}$$

which is Zamolodchikov's relation [1,2,3] expressing the effect of a scale transformation on the function C. Recalling the form of the operator $\mathcal{D}$ and the fact that the energy momentum tensor has vanishing anomalous dimension, we deduce from eq.(5.1.19) that

$$\beta^k \frac{\partial}{\partial g^k} C = \frac{3}{4} \beta^i \beta^j G_{ij} \tag{5.1.20}$$

Now, thanks to the absence of infrared divergences, we can notice that the metric of our Hilbert space is positive definite, and therefore we conclude that relation (5.1.19) implies a monotonic decrease of C under the action of a renormalization group transformation. Moreover we can also see from (5.1.19) that a stationarity point for C is given by the renormalization group fixed point, where all the β functions vanish,

$$\beta_i(g) = 0$$

At such a point the model is conformal invariant; from the definition of its central charge c_0, [4], and eqs. (5.1.8), (5.1.14), it can be shown that the stationary value of $C(g)$ coincides with c_0. In the case of our model where we have a fixed point at $g_i = 0$ a direct evaluation of the lowest order contribution to A_1 gives

$$C(g = 0) = c_0 = \frac{N}{8\pi^2} \,. \tag{5.1.21}$$

What we have just finished to proof can be summarized by stating the C-theorem: we have shown that there exists a function C of the coupling constants which is non increasing along renormalization group trajectories, which is stationary only at fixed points, and which, at a fixed point, is equal to the value of the central charge of the corresponding conformal invariant theory.

An intuitive and pictorial interpretation of this statement is to say that the renormalization group flow goes "downhill". From a more physical standpoint, the idea behind this theorem is that renormalization group transformations, with their coarse graining procedure, produce a loss of information about the degrees of freedom whose wavelenght is of the same order as the cut-off. This feature is at the heart of the irreversibility of the renormalization group flow, and there should exist some kind of entropy function, like $C(g)$, which measures this loss of information (remember that the central charge counts the degrees of freedom, and so does the C-function at any particular length scale).

This argument is general enough to be thought equally valid in any number of dimension, so that one may ask whether a "C-like" theorem can be stated also for $d \neq 2$ [5,9].Indeed it has been proposed [9] a generalization of the C-function for any even d, in the case of a theory defined on a sphere S^d, by posing, apart a normalization factor which depends on d

$$C_d \sim (-)^{\frac{d}{2}} \int_{S^d} \langle \Theta \rangle \sqrt{\gamma} d^2 x$$

where Θ is the trace of the stress energy tensor. Unfortunately it can only be proven that such a function is monotonically decreasing up to the first non trivial order in perturbation theory, while a general proof has not yet been worked out.

What we can say at least is that there are in fact some technical, if not conceptual, reasons that make the generalization of the C-theorem so hard to be established. In d dimensions rotational invariance and parity, which lie at the basis of Zamolodchikov's theorem for $d = 2$, fix the two points function of the stress energy tensor to have the form

$$\begin{aligned} \langle T_{\mu\nu}(x) T_{\lambda\sigma}(0) \rangle &= \frac{A}{x^{2d+4}} x_\mu x_\nu x_\lambda x_\sigma \\ &+ \frac{B}{x^{2d+2}} (x_\mu x_\nu \delta_{\lambda\sigma} + x_\lambda x_\sigma \delta_{\mu\nu}) \\ &+ \frac{K}{x^{2d+2}} (x_\mu x_\lambda \delta_{\nu\sigma} + x_\nu x_\lambda \delta_{\mu\sigma} + x_\nu x_\sigma \delta_{\mu\lambda} + x_\mu x_\sigma \delta_{\nu\lambda}) \\ &+ \frac{D}{x^{2d}} \delta_{\mu\nu} \delta_{\lambda\sigma} + \frac{E}{x^{2d}} (\delta_{\mu\lambda} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\lambda}) \end{aligned}$$

As we can see the invariant amplitudes $A, B, \ldots$ are more numerous than in the case of $d = 2$, and the best one can do [9] is to define a function $\tilde{C}$

$$\tilde{C} = -\frac{4}{d-1} [A + \frac{1}{2}(d^2 + d + 2) B + (d+3) K + \frac{1}{2} d(d+1) D + (d+1) E]$$

for which

$$\mu\frac{d}{d\mu}\tilde{C} = -4\frac{d+1}{d-1}\langle\Theta\Theta\rangle - 2(d-2)B.$$

This situation resembles in some sense the one we encounter even in $d = 2$ when the theory is affected by a bad infrared behavior; as we shall illustrate later on in the text, in the case of the generalized σ-model infrared divergences yield the appearance of an extra term at the r.h.s. of eq.(5.1.19) which spoils the monotonicity conclusions of the theorem [29].

Apart from these drawbacks, Zamolodchikov's theorem is an important qualitative result concerning two dimensional field theory, at least in two respects. First of all, as we already mentioned and as we hope to clarify further in the next sections, it provides a deep link between the quantum properties of a two dimensional model and the geometric structure of the space of the interaction parameters of the model [34]; eq.(5.1.19) indeed relates the renormalization group flow, represented by the β-function, to the metric in the space of coupling constants, defined as a correlation function of the interaction operators appearing in the lagrangian of a given theory.

On the other hand, while conformal invariance principles are valid only when we are at a critical point, Zamolodchikov's theorem has a significance that extends away from criticality, and goes in the direction of relating the behavior of a theory at a fixed point with its features in a neighborhood of it, or even at another fixed point [5,6,7,8,9,10]. In the context of statistical mechanics, where the critical conditions cannot be attained experimentally and only the scaling region around them can be investigated, this last feature could lead to interesting predictions [7,8] which could be, at least in principle, verified.

In the following, after reviewing briefly the problem of infrared divergences, which modify the theorem [29], we will try to explain, with slightly more details, these last two points.

5.2 The problem of infrared divergencies

One of the crucial properties which renders possible the proof of Zamolodchikov's theorem is the abscence of infrared divergencies in the theory; as we have seen in the preceding section this yields the positivity of the correlation function which defines the metric G_{ij}, and therefore the monotonicity of the C-function. It may be then natural to ask what happens to the theorem if the requirement of infrared finiteness is not fulfilled by the two dimensional field theory. The result, as we already mentioned in section 5.1, is that C is no longer decreasing since some extra terms appear at the r.h.s. of eq.(5.1.19) thus spoiling the theorem [29]. The aim of this short section is to provide a simple and qualitative discussion describing the situation we have in the case of σ-model [29], in order to give a feeling of the problems posed by infrared divergences. As such, the following lines do not pretend to clarify technical issues, nor to be a complete and thorough treatment of the subject. Rather, the reader who wishes to obtain more informations on that is suggested to consult the paper [29] which is the main reference throughout this section. Let us then consider the action [29]

$$S = \int dv \left[\frac{1}{2}\gamma^{\mu\nu} g^B_{ij} \partial_\mu X^i \partial_\nu X^j - \frac{\alpha'}{4} R^{(2)} \Phi^B\right] \frac{1}{2\pi\alpha'}$$

corresponding to the generalized σ-model, where X^i is a field with values in a target space M, $\gamma^{\mu\nu}$ and $R^{(2)}$ are the metric and the curvature in the world sheet, respectively, and Φ is the dilaton field.

We know that in order to proof a C-theorem one has to consider a linear combination of the stress energy correlations functions and show that its derivative is proportional to the trace-trace correlation. In the case at hand these correlations can be defined as follows

$$\begin{aligned} A_{\alpha\beta,\mu\nu}(x-y) &= \frac{2}{\sqrt{\gamma(x)}} \frac{\delta}{\delta\gamma^{\alpha\beta}(x)} \frac{2}{\sqrt{\gamma(y)}} \frac{\delta}{\delta\gamma^{\mu\nu}(y)} \mathcal{Z}\Big|_{\gamma_{\mu\nu}=\delta_{\mu\nu}} = \\ &= \langle T_{\alpha\beta}(x) T_{\mu\nu}(y)\rangle + \delta\text{-terms} \end{aligned}$$

where $\mathcal{Z}$ denotes the partition function. In momentum space it can be shown [49] that the structure of $A_{\alpha\beta,\mu\nu}$ is fixed by the energy momentum tensor conservation as follows

$$\begin{aligned} A_{\alpha\beta,\mu\nu}(p) &= -F_2(p^2)\frac{1}{p^2}(\delta_{\alpha\beta} - p_\alpha p_\beta)(\delta_{\mu\nu} - p_\mu p_\nu) \\ &\quad -F_3(p^2)\frac{1}{p^2}[(\delta_{\alpha\mu} - p_\alpha p_\mu)(\delta_{\beta\nu} - p_\beta p_\nu) + (\delta_{\alpha\nu} - p_\alpha p_\nu)(\delta_{\beta\mu} - p_\beta p_\mu)] \end{aligned} \tag{5.2.1}$$

Introducing light-cone coordinates in two dimensions

$$p_\mu = n_\mu p_- + m_\mu p_+ \tag{5.2.2}$$

where

$$n_\mu = \frac{1}{\sqrt{2}}(1, i) \qquad\qquad m_\mu = \frac{1}{\sqrt{2}}(1, -i) \tag{5.2.3}$$

and

$$p_- = p \cdot m \qquad\qquad p_+ = p \cdot n \tag{5.2.4}$$

it can be seen that $A_{\alpha\beta,\mu\nu}$ involves only one independent form factor, which we denote $F(p^2)$; we have

$$A_{\alpha\beta,\mu\nu} = \frac{F(p^2)}{p^2} S_\alpha(p) S_\beta(p) S_\mu(p) S_\nu(p) \tag{5.2.5}$$

where

$$F(p^2) = -F_2(p^2) - 2F_3(p^2) \tag{5.2.6}$$

and

$$S_\mu(p) = n_\mu p_- - m_\mu p_+ \tag{5.2.7}$$

If we had no infrared divergences we could conclude that at each perturbative order $F(p^2)$ depends only on $L(p^2) \equiv log(p^2/\mu^2)$; in the demonstration of the theorem this feature yields the relation

$$\mu \frac{\partial}{\partial \mu} = 2x^+ \partial_+ = 2x^- \partial_- \tag{5.2.8}$$

which in turn allows to proof eq.(5.1.19). The problem we have to face in the current situation is that the presence of infrared divergences involves, in general, the appearance of another mass scale m besides the ultraviolet one, μ.

Thus the functional dependence of $F(p^2)$ changes, and one has to take into account also the infrared cut-off m, appearing logarithmically as $log(m^2/\mu^2)$; therefore the naive scaling relation (5.2.8) does not hold any longer and the proof of Zamolodchikov's theorem fails. Indeed the careful analysis performed in [29] has shown that at the third loop level the form factor $F(p^2)$ contains a term of the form

$$L(m^2) = \frac{1}{2} \left[log \left(\frac{m^2}{4\pi\mu^2} \right) + \gamma_E \right] ,$$

and the whole result can be casted in terms of the various β-functions of the model as follows

$$F = \frac{1}{8\pi} \left\{ \left(1 - \frac{\alpha'}{2} L(m^2) \nabla^2 \right) (\beta_\Phi - \beta_k{}^k) + \beta_{ij} \beta^{ij} L(p^2) \right\} \tag{5.2.9}$$

If we now try to proof the theorem in the usual way by considering the correlation functions A_1, A_2, A_3 and by defining C as the combination (5.1.14), a direct consequence of the appearance of $L(m^2)$ is that the relation

$$\mu \frac{\partial}{\partial \mu} C = -12 A_3$$

is not verified, and actually implies a wrong relation between the Ricci tensor, the curvature scalar and the dilaton field [29]. Hence the theorem is modified by the presence of infrared divergences; relation (5.1.14) has to be corrected, and in fact it can be shown [29] that the following equation holds up to three loops

$$\mu\frac{\partial}{\partial\mu}C + 12A_3 = -\frac{3}{16\pi^2}\frac{\alpha'}{2}\nabla^2(\beta_\Phi - \beta_k{}^k)$$

This result contradicts both the monotonicity character of the C-function and the quadratic dependence upon the β-functions of its derivative, so that Zamolodchikov's theorem is not valid in its original form, apart from those particular cases in which $\nabla^2(\beta_\Phi - \beta_k{}^k)$ vanishes [29].

5.3 Properties of the metric G_{ij}.

As we have seen during the proof in section 5.1 an important ingredient of Zamolodchikov's theorem is the metric function G_{ij}, eq.(5.1.18), which will be the subject of the following chapter.

Before passing to the explicit calculation of G_{ij}, it may be interesting to dwell for a while upon its renormalization group properties; it is the aim of this section to point out the interplay between the geometric properties of the coupling constants manifold and renormalization aspects of two dimensional quantum field theory.

In the space of couplings in which the coordinates are g^i the operators O_i transform as the tangent vector $\partial/\partial g^i$ and the metric transforms correctly as a rank 2 covariant tensor under coordinate transformation $g \to g'$

$$G'_{ij}(g') = \frac{\partial g^k}{\partial g'^i}\frac{\partial g^l}{\partial g'^j}G_{kl}(g) \tag{5.3.1}$$

It is well known that β^i transforms like a contravariant vector

$$\beta'^i(g') = \frac{\partial g'^i}{\partial g^k}\beta^k(g) \tag{5.3.2}$$

due to the definition

$$\beta^i(g) = \mu\frac{dg^i}{d\mu} \tag{5.3.3}$$

Therefore eq.(5.1.19) defines a quantity which is a scalar in parameter space. The renormalization group properties of G_{ij} can be examined using eq.(5.1.5) in the $\varepsilon \to 0$ limit which yields

$$\begin{aligned} \mathcal{D}\frac{G_{ij}}{(x^2)^2} &= -\frac{1}{(x^2)^2}\left\{\gamma_i^k\langle O_k(x)O_j(0)\rangle + \gamma_j^k\langle O_i(x)O_k(0)\rangle\right\} \\ &= -\frac{1}{(x^2)^2}\left(\partial_i\beta^k G_{kj} + \partial_j\beta^k G_{ik}\right). \end{aligned} \quad (5.3.4)$$

After a little algebra we end up with

$$\mu\frac{\partial}{\partial\mu}G_{ij} = -\left(\partial_i\beta^k G_{kj} + \partial_j\beta^k G_{ik} + \beta^k\partial_k G_{ij}\right) \quad (5.3.5)$$

This means that a scale transformation acts on the metric as a Lie derivative along the vector β

$$\mu\frac{\partial}{\partial\mu}G_{ij} = -\mathcal{L}_\beta(G_{ij}) \quad (5.3.6)$$

and the same interpretation can be given to eq.(5.1.5) for O_i. Moreover we can note that all these relations are independent of the (possible) Riemannian connection of the manifold of parameters.

From a practical point of view eq.(5.3.5) can be used to fix all logarithms appearing in the function $G_{ij}(x;\mu)$. Setting

$$G_{ij}(x;\mu) = \sum_{s=0}^{\infty} G_{ij}^{(s)} log^s(x^2\mu^2) \quad (5.3.7)$$

we obtain from eq.(5.3.5)

$$G_{ij}^{(s)} = -\frac{1}{2s}\left[\partial_i\beta^k G_{kj} + \partial_j\beta^k G_{ik} + \beta^k\partial_k G_{ij}\right] \quad (5.3.8)$$

so that what really matters is the form of $G_{ij}^{(0)}$. Factorizing all possible numerical factors in a scale $M \propto \mu$ we can write

$$G_{ij} = G_{ij}^{(0)} + G_{ij}^{(1)} log(x^2M^2) + ... \quad (5.3.9)$$

choosing $x = 1/M$ we can say that $G_{ij}^{(0)}$ represents the real metric of the parameter space.

We want to stress that all relations we have written are covariant with respect to a redefinition of the couplings

$$g^i \rightarrow g^i + G^i \tag{5.3.10}$$

that in particular implies that any choice of renormalization procedure gives equivalent results and clearly does not mean that for every choice of couplings the physical content is transparent.

In the following chapter we will report both an analitical and an algebraic computation of the first nontrivial correction to the metric tensor G_{ij} which arises from a four loop correlation function and we extract the form of C using eq.(5.1.19) (a direct evaluation would require a five loop computation). Even this relatively simple step reveals some interesting aspects. For example the trivial fixed point of the theory $g^i = 0$ is locally euclidean (that is the Christoffel symbols are vanishing) only if the subtraction procedure preserves the symmetries of the lagrangian.

Let us then discuss some properties of the function C. Using the fact that the anomalous dimension of the energy momentum tensor is zero

$$\mathcal{D}T_{\alpha\beta} = 0 \tag{5.3.11}$$

eq.(5.1.19) can be recast in the form

$$\beta^k \frac{\partial}{\partial g^k} C = \frac{3}{4}\beta^i\beta^j G_{ij} \equiv \frac{3}{4}\beta^i\beta_i \tag{5.3.12}$$

As it stands eq.(5.3.12) must hold for every value of x and so we can choose the fixed scale $x = 1/M$ and consider only the metric $G_{ij}^{(0)}$ in the following relations.

It is tempting to argue from eq.(5.3.12) the stronger relation

$$\partial_i C = \frac{3}{4}\beta_i \tag{5.3.13}$$

Altough eq.(5.3.13) can be believed correct there is no clear proof of it (for a different but related approach see ref.[40,41]) so we just conjecture its validity. A necessary condition to be fulfilled is

$$\partial_i\beta_j - \partial_j\beta_i = 0 \tag{5.3.14}$$

that is the form

$$\beta_i dg^i \tag{5.3.15}$$

must be closed. If eq.(5.3.14) holds, Poincarè lemma allows us to construct C from eq.(5.3.13) in the perturbative expansion

$$C = C_0 + \frac{3}{4}\int_0^1 g^i \beta_i(tg)dt. \tag{5.3.16}$$

This is what will be done explicity in the following chapter; in particular the perturbative evaluation of the metric will confirm the analysis we have done here, showing that the contribution of the first non trivial order can be reabsorbed by a suitable choice of the scale. As a final remark we may add that a five loop computation G_{ij} would give some interesting informations about the space of parameters, expecially for what concers its curvature; unfortunalelly this calculation is technically quite difficult and therefore beyond the scope of this work.

5.4 Outside the fixed point

The last section of this chapter is devoted to a subject which, although lying slightly apart the main concern of this thesis, is very stimulating for its interdisciplinary character. As we already mentioned, an aspect of Zamolodchikov's theorem which is remarkably relevant, is that it relates the behaviour of a theory in a neighborhood of a fixed point with the conformal anomaly number c, which is a critical quantity [5,6,7]. Such issue may have greater importance for statistical mechanics systems [7,8], expecially for the study of the scaling region around a critical point, and this is what will be explained now.

In order to be as much general as possible in our argument, we consider the situation in which a fixed point action S^* is perturbed by some relevant operator ϕ so that the complete, perturbed action is

$$S = S^* - \lambda \int \phi(r)\, d^2r \tag{5.4.1}$$

Supposing that the operator ϕ has scaling dimensions $2h$, the coupling constant λ has dimensions $2(1-h)$, and therefore h must be less than one in order to have a relevant perturbation. One of the first thing one can imagine to do with the action (5.4.1) is to study its renormalization, for example around $\lambda = 0$. If the action is renormalizable, as we will assume, there will be introduced a certain, finite number of counterterms which are needed to define renormalized operators. In the case of the stress energy tensor, power counting shows that the renormalization procedure yields the appearance of at least one counterterm [5]; as a consequence quantum corrections affect the trace Θ of the energy-momentum tensor by shifting its value from zero to

$$\Theta(r) = -\, 4\pi\ \lambda\ (1-h)\ \phi(r) + \cdots \tag{5.4.2}$$

neglecting higher orders in λ. It is precisely at this point that the rôle of Zamolodchikov's theorem become apparent; in fact eq.(5.1.15) relates the derivative of the C-function to the correlation function of the trace Θ, and, if we recall the functional dependence on $log(\mu^2 r^2)$, implies

$$\frac{d}{dr}C = -\,\frac{3}{4}\, r^3\ \langle \Theta(r)\Theta(0)\rangle \tag{5.4.3}$$

This result, and eq.(5.4.2) allow to express the total change of C from short to large distances as

$$\Delta C = -12\pi^2\ \lambda^2\ (1-h)^2\ \int_{r_1}^{r_2} r^3 \langle \phi(r)\phi(0)\rangle\ dr \tag{5.4.4}$$

thus relating the variation of the quantity C, which characterizes a conformally invariant theory, to an integral of a correlation function computed away from the critical point.

What we have found may have some interesting physical consequences, for instance if we take the operator $\phi(r)$ to be the energy density operator $\epsilon(r)$, corresponding to a coupling constant λ which is the temperature difference

$$t = \beta_c - \beta$$

In such situation it usually happens that the renormalization group flows end at a trivial high- or low-temperature fixed point with $c = 0$ [5], and in the case of the Ising model eq.(5.4.4) gives the central charge c at the critical point as the second moment of the energy-energy correlations. Indeed the fixed point action of this model can be formulated [4,56,57,58,59] in terms of a pair of free Majorana fermion fields ψ and $\bar{\psi}$, as follows

$$S^* = \int\ \left(\psi\partial_-\psi + \bar{\psi}\partial_+\bar{\psi}\right) \tag{5.4.5}$$

while the energy density is given by

$$\epsilon(r)\ \sim\ \bar{\psi}\psi$$

so that moving away from the critical point corresponds to add a mass term

$$\sim\ t\ \int\ \bar{\psi}\psi\ d^2r$$

to S^*. The energy-energy correlation can be computed with the aid of Wick theorem and the form of the fermion propagators, and turns out to be [5]

$$\langle\, \epsilon(r)\epsilon(0)\, \rangle = \left(\frac{t}{2\pi}\right)^2 \left[K_1^2(tr) \;-\; K_0^2(tr)\right]$$

K_0 and K_1 being modified Bessel functions; therefore, after doing the integral, eq.(5.4.4) reproduces the known [4] value of the central charge of the Ising model

$$c = \frac{1}{2}$$

The kind of argument outlined above may provide a deeper insight in the universal properties of two dimensional critical system; in fact one can try to calculate universal combinations of amplitudes of quantities which become singular in the limit $T \to T_c$.

One of the most important [7,8] universal number is the singular part of the free energy per correlation volume, which we denote $f_s\xi^2$, where f_s is the singular part of the free energy per unite volume and ξ is the correlation length that can be defined as the second moment of the connected correlation function of the energy density $\epsilon(r)$

$$\xi^2 \;=\; \frac{\int r^2\langle\, \epsilon(r)\epsilon(0)\, \rangle_c\, d^2r}{\int \langle\, \epsilon(r)\epsilon(0)\, \rangle_c\, d^2r} \tag{5.4.6}$$

It is believed [7] that the product $f_s\xi^2$ tends to a constant as $T \to T_c$, while $\xi \sim B|t|^{-\nu}$ and $f_s \sim A|t|^{2-\alpha}$. We can now notice that the denominator in (5.4.6) is just the specific heat per unit volume whose singular part is

$$-\;\frac{\partial^2 f_s}{\partial t^2} \;\sim\; -\;[(2-\alpha)(1-\alpha)]t^{-2} f_s \tag{5.4.7}$$

Hence we can express $f_s\xi^2$ as follows

$$f_s\xi^2 \;\sim\; -\;[(2-\alpha)(1-\alpha)]^{-1} t^2 \int r^2\langle\, \epsilon(r)\epsilon(0)\, \rangle_c\, d^2r \tag{5.4.8}$$

where the integral is related to the C-function as in eq.(5.4.4). Indeed it can be shown [7] that the central charge at the critical fixed point can be computed as

$$c \;=\; 12\,\pi^2\, t^2\, \frac{1}{(2-\alpha)^2} \int_0^\infty r^3\langle\, \epsilon(r)\epsilon(0)\, \rangle_c\, dr \tag{5.4.9}$$

so that the universal amplitude we are studying can be related to c; we have, as $t \sim 0$

$$f_s \xi^2 = -\frac{c}{12\pi^2}(2-\alpha)(1-\alpha)^{-1} \tag{5.4.10}$$

and this equation is independent of t as expected.

The result we have obtained is a clear example of what was announced at the beginning of this section: an amplitude which refers to the scaling region around a critical point is expressed in terms of the critical number c by virtue of Zamolodchikov's theorem, and what is even more important is that a relation like (5.4.10) can be, at least in principle, verified experimentally, for example in absorbed systems [7,8].

As a final comment we may add that the approach to the perturbed action (5.4.1) can yield also other interesting consequences. In fact another possible application [5,6,10] is the study of a slightly relevant perturbation given by an operator ϕ with scaling dimension $h \sim 1$, as we have in the case of two very close fixed points. It is then possible to analyze the characteristics of one fixed point in terms of the other, and indeed relate the values of the central charge at the two points, in a perturbative way. The result, found in [10], is

$$c' = c - \frac{y^3}{b^2} + \cdots \tag{5.4.11}$$

where we see that the new central charge c' depends upon the renormalization group eigenvalue $y \equiv 2-2h$ of ϕ and on the operator product expansion coefficient b in the product $\phi\ \phi \sim -b\phi$.

These last remarks conclude the chapter devoted to the C-theorem. We hope to have provided a sufficiently complete survey of the subject, before attacking the computation of the metric G_{ij} and the C-function in the case of the generalized Thirring model (2.1.1), which will be performed in the next part of this work.

Chapter 6

The metric G_{ij} and the function C

As we have seen in the previous chapter the perturbative determination of Zamolodchikov's C-function requires the knowledge of the β-functions of the model as well as of the metric G_{ij} in the space of couplings. We will present [34] the computation of G_{ij} for the generalized Thirring model, eq.(5.1.18, by using two different approaches which are both in the framework of perturbation theory:

- an analytical approach, which amounts to the explicit calculation of the graphs which are relevant for the metric, up to four-loop order;
- an algebraic approach, which exploits the symmetry and group-theoretic properties of our models in a way very similar to what we have done in the case of β-functions.

6.1 Calculation of the metric G_{ij}

We shall keep the notation introduced in chapter 3, which turns out to be very convenient for the evaluation of Feynman graphs, and we will cast the final result both in the basis g_S, g_P, g_V and the one defined in eq.(4.2.1), where the algebraic considerations are more natural. The following transformation rules allow to relate the components of the metric in the two bases

$$
\begin{aligned}
G_{++} &= G_{SS} + 2G_{SP} + G_{PP} \\
G_{\pm V} &= G_{SV} \pm G_{SP} = G_{V\pm} \\
G_{+-} &= G_{SS} - G_{PP} = G_{-+}
\end{aligned}
$$

$$G_{--} = G_{SS} - 2G_{SP} + G_{PP} \tag{6.1.1}$$

and

$$\begin{aligned} G_{SS} &= \frac{1}{4}[G_{++} + 2G_{+-} + G_{--}] \\ G_{SP} &= \frac{1}{4}[G_{++} - G_{--}] \\ G_{SV} &= \frac{1}{4}[G_{+V} + G_{-V}] \\ G_{PV} &= \frac{1}{4}[G_{+V} - G_{-V}] \\ G_{PP} &= \frac{1}{4}[G_{++} - 2G_{+-} + G_{--}] \end{aligned} \tag{6.1.2}$$

Let us first describe the analytical approach, and start by recalling that the metric is given by the correlation functions of four-fermion operators

$$\langle O_i(x)O_j(0)\rangle = G_{ij}(g,x)\frac{1}{x^4} \tag{6.1.3}$$

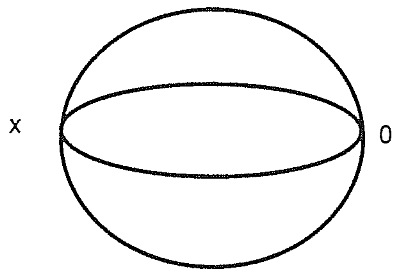
The perturbative computation of these Green's functions can be performed more easily in coordinate space where the d-dimensional zero mass fermion propagator is given by

$$\begin{aligned} S(x) &= \partial\!\!\!/ \int \frac{d^dp}{(2\pi)^d}\frac{e^{ip\cdot x}}{p^2} \\ &= -\frac{1}{2\pi^{\frac{d}{2}}}\Gamma(\frac{d}{2})\frac{\gamma\cdot x}{(x^2)^{\frac{d}{2}}} \end{aligned} \tag{6.1.4}$$

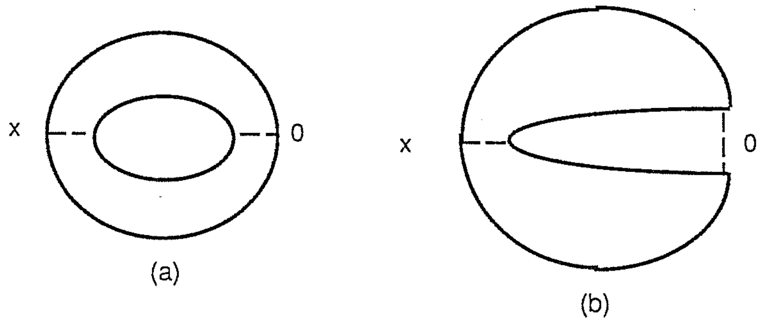
which reduces to the usual

$$S_2(x) = -\frac{1}{2\pi}\frac{\gamma\cdot x}{(x^2)} \tag{6.1.5}$$

for $d = 2$. It is apparent that the "tree" level calculation of the correlator in (6.1.3) actually involves diagrams with the following three loop topology:



We can read the corresponding graphs simply by opening the fermion vertices in all possible ways: this amounts to the Wick contractions of the product $O_i(x)O_j(0)$, and we are left with only two diagrams of order N^2 and N respectively.



Thus the first order computation of the metric is really a trivial matter, since there is no integration to perform and the result of (a) and (b) is certainly finite. Nevertheless we evaluate them keeping all the machinery introduced in section 3.1, in order to provide a warming up exercise about its use, which is very profitable at the next order. Recalling eq.(6.1.4) and paying attention to a combinatorial factor 2, the expressions for (a) and (b) are the following

$$\begin{aligned}(a) &= \frac{1}{2}N^2 Tr[\Gamma_A^{(i)}S(x)\Gamma_B^{(j)}S(x)]Tr[\Gamma_A^{(i)}S(x)\Gamma_B^{(j)}S(x)] \\ &= \frac{1}{2}N^2\left[\frac{\Gamma(1+\frac{\varepsilon}{2})}{2(\pi x^2)^{1+\frac{\varepsilon}{2}}}\right]^4 Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x] \\ &\quad Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x] \qquad (6.1.6)\\ (b) &= -\frac{1}{2}NTr[\Gamma_A^{(i)}S(x)\Gamma_B^{(j)}S(x)\Gamma_A^{(i)}S(x)\Gamma_B^{(j)}S(x)]\end{aligned}$$

$$= -\frac{1}{2}N\left[\frac{\Gamma(1+\frac{\varepsilon}{2})}{2(\pi x^2)^{1+\frac{\varepsilon}{2}}}\right]^4 Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x] \qquad (6.1.7)$$

Summing up the two contributions we obtain

$$\begin{aligned}\langle O_i(x)O_j(0)\rangle &= \\ = \frac{1}{2}\left[\frac{\Gamma(1+\frac{\varepsilon}{2})}{2(\pi x^2)^{1+\frac{\varepsilon}{2}}}\right]^4 &\Big[\ N^2Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x]Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x] \\ &- NTr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x]\Big] \qquad (6.1.8)\end{aligned}$$

This formula can be further reduced by making use of the property

$$\gamma \cdot x \Gamma_D^{(r)} \gamma \cdot x \otimes \gamma \cdot x \Gamma_D^{(r)} \gamma \cdot x = \Gamma_D^{(r)} \otimes \Gamma_D^{(r)} (x^2)^2 \tag{6.1.9}$$

which can be proven by symmetry arguments, and the traces can be evaluated with the aid of eq(3.1.44), (3.1.46) and defining a quantity $d^{(j)}$ as follows

$$\sum_A \Gamma_A^{(j)} \Gamma_A^{(j)} = d^{(j)} \mathbf{1} \tag{6.1.10}$$

The result we obtain is

$$\langle O_i(x) O_j(0) \rangle = \frac{1}{2} \frac{1}{(2\pi)^4} \left[4N^2 \nu^{(j)} d^{(j)} \delta_{ij} - 2N d^{(i)} d^{(ij)} \right] \tag{6.1.11}$$

and can be rewritten in a standard form by noticing that in two space time dimensions, and in the basis S, P, V we have

$$d^{(j)} = (1, -1, 2) \tag{6.1.12}$$

$$\nu^{(i)} = (1, -1, 1) \tag{6.1.13}$$

$$d^{(i)} d^{(ij)} = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix} \tag{6.1.14}$$

It is then very easy to see that the final outcome of the first order computation of the metric is

$$G_{ij}^{(0)} = \frac{N}{16\pi^4} \begin{pmatrix} 2N-1 & 1 & -2 \\ 1 & 2N-1 & -2 \\ -2 & -2 & 4N \end{pmatrix} \tag{6.1.15}$$

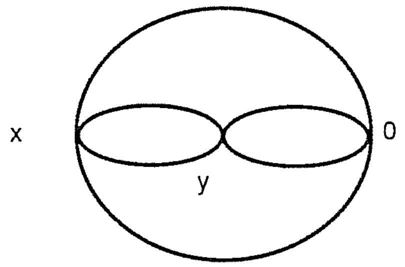
in the basis S, P, V; by using the transformation relations (6.1.1) we find

$$G_{ij}^{(0)} = \frac{N}{4\pi^4} \begin{pmatrix} N & 0 & -1 \\ 0 & N-1 & 0 \\ -1 & 0 & N \end{pmatrix} \tag{6.1.16}$$

in the basis $+, -, V$.

As a final comment we may say that all the symmetry properties, which have been used in section 4.2 and will be the starting point of the algebraic derivation of the metric, are trivially satisfied by this result.

We can now start to present the computation of the metric at the second perturbative order. As in the previous case we have only one possible topology, of the four-loop type, for the graphs we have to consider, and associated to that there is an integration over a coordinate y.



Even though the Wick contractions generate seven different kinds of diagrams, reported later on in the text, we have just one type of integral to evaluate, which is

$$\begin{aligned} J^{\alpha\beta\sigma\tau} &= \int d^d y \frac{y_\alpha y_\beta (x-y)_\sigma (x-y)_\tau}{(y^2)^d((x-y)^2)^d} \\ &= \frac{\pi}{4x^2}\frac{\Gamma(1+\frac{\varepsilon}{2})\Gamma^2(1-\frac{\varepsilon}{2})}{\Gamma(2-\varepsilon)\Gamma^2(2+\varepsilon)}\frac{\pi^{\frac{\varepsilon}{2}}}{(x^6)^{\frac{\varepsilon}{2}}}\Big[\delta_{\alpha\sigma}\delta_{\beta\tau}+\delta_{\alpha\beta}\delta_{\sigma\tau}+\delta_{\alpha\tau}\delta_{\beta\sigma} \\ &\quad -\frac{2+3\varepsilon}{x^2}(\delta_{\alpha\sigma}x_\beta x_\tau+\delta_{\alpha\tau}x_\sigma x_\beta+\delta_{\beta\sigma}x_\alpha x_\tau+\delta_{\beta\tau}x_\sigma x_\alpha) \\ &\quad +(4+3\varepsilon)(2+3\varepsilon)\frac{x_\alpha x_\beta x_\sigma x_\tau}{x^4} \\ &\quad -\frac{(2+3\varepsilon)(2-\varepsilon)}{\varepsilon}\frac{1}{x^2}(\delta_{\alpha\beta}x_\sigma x_\tau+\delta_{\sigma\tau}x_\alpha x_\beta)\Big] \end{aligned} \tag{6.1.17}$$

Different graphs have a different structure of traces of the Dirac γ matrices, with which the integral $J^{\alpha\beta\sigma\tau}$ has to be contracted. They can be seen in the table at the end of this section, where all the graphs are reported together with their expression. Employing our notation and taking into account combinatorial and other factors, such as the explicit N dependence of each diagram, we can write down the following list where $J^{\alpha\beta\sigma\tau}$ is factored out

$$\begin{aligned} J_7 &= N^3\frac{g_k}{64\pi^6 x^4}\left[\frac{\Gamma^6(1+\frac{\varepsilon}{2})}{(\pi^3x^2)^\varepsilon}\right] Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x] \\ &\quad Tr[\Gamma_A^{(i)}\gamma_\sigma\Gamma_C^{(k)}\gamma_\tau]Tr[\Gamma_C^{(k)}\gamma_\alpha\Gamma_B^{(j)}\gamma_\beta]J^{\alpha\beta\sigma\tau} \end{aligned}$$

$$J_6 = -N^2 \frac{g_k}{64\pi^6 x^4}\left[\frac{\Gamma^6(1+\frac{\varepsilon}{2})}{(\pi^3 x^2)^\varepsilon}\right] Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x]$$
$$Tr[\Gamma_A^{(i)}\gamma_\sigma\Gamma_C^{(k)}\gamma_\alpha\Gamma_B^{(j)}\gamma_\beta\Gamma_C^{(k)}\gamma_\tau]J^{\alpha\beta\sigma\tau}$$

$$J_5 = -N^2 \frac{g_k}{64\pi^6 x^4}\left[\frac{\Gamma^6(1+\frac{\varepsilon}{2})}{(\pi^3 x^2)^\varepsilon}\right]\Big\{Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma_\alpha\Gamma_C^{(k)}\gamma_\sigma]Tr[\Gamma_A^{(i)}\gamma_\tau\Gamma_C^{(k)}\gamma_\beta\Gamma_B^{(j)}\gamma\cdot x]$$
$$-Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma_\alpha\Gamma_C^{(k)}\gamma_\sigma]Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma_\beta\Gamma_C^{(k)}\gamma_\tau]\Big\}J^{\alpha\beta\sigma\tau}$$

$$J_4 = -N^2 \frac{g_k}{64\pi^6 x^4}\left[\frac{\Gamma^6(1+\frac{\varepsilon}{2})}{(\pi^3 x^2)^\varepsilon}\right]\Big\{Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x\Gamma_A^{(i)}\gamma_\sigma\Gamma_C^{(k)}\gamma_\tau]Tr[\Gamma_C^{(k)}\gamma_\alpha\Gamma_B^{(j)}\gamma_\beta]$$
$$+Tr[\Gamma_A^{(i)}\gamma_\sigma\Gamma_C^{(k)}\gamma_\tau]Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma_\alpha\Gamma_C^{(k)}\gamma_\beta\Gamma_B^{(j)}\gamma\cdot x]\Big\}J^{\alpha\beta\sigma\tau}$$

$$J_3 = -N\frac{g_k}{64\pi^6 x^4}\left[\frac{\Gamma^6(1+\frac{\varepsilon}{2})}{(\pi^3 x^2)^\varepsilon}\right] Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma_\alpha\Gamma_C^{(k)}\gamma_\sigma\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma_\beta\Gamma_C^{(k)}\gamma_\tau]J^{\alpha\beta\sigma\tau}$$

$$J_2 = N\frac{g_k}{64\pi^6 x^4}\left[\frac{\Gamma^6(1+\frac{\varepsilon}{2})}{(\pi^3 x^2)^\varepsilon}\right] Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma_\alpha\Gamma_C^{(k)}\gamma_\beta\Gamma_B^{(j)}\gamma\cdot x\Gamma_A^{(i)}\gamma_\sigma\Gamma_C^{(k)}\gamma_\tau]J^{\alpha\beta\sigma\tau}$$

$$J_1 = -N\frac{g_k}{64\pi^6 x^4}\left[\frac{\Gamma^6(1+\frac{\varepsilon}{2})}{(\pi^3 x^2)^\varepsilon}\right]\Big\{Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma_\alpha\Gamma_C^{(k)}\gamma_\sigma\Gamma_A^{(i)}\gamma_\tau\Gamma_C^{(k)}\gamma_\beta\Gamma_B^{(j)}\gamma\cdot x]$$
$$+Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x\Gamma_A^{(i)}\gamma_\sigma\Gamma_C^{(k)}\gamma_\alpha\Gamma_B^{(j)}\gamma_\beta\Gamma_C^{(k)}\gamma_\tau]\Big\}J^{\alpha\beta\sigma\tau}$$

At this point it is perhaps profitable to split the computation into two parts, one for the terms which carry the singularity in $1/\varepsilon$, and one for the others. In fact if we recall the form of the integral $J^{\alpha\beta\sigma\tau}$, eq(6.1.17), we immediately see that there is a set of form factors whose coefficients are regular as $\varepsilon \to 0$; let us denote this part by $F^{\alpha\beta\sigma\tau}$

$$F^{\alpha\beta\sigma\tau}\Big|_{\varepsilon=0} = \frac{\pi}{4x^2}\Big[\delta_{\alpha\sigma}\delta_{\beta\tau}+\delta_{\alpha\beta}\delta_{\sigma\tau}+\delta_{\alpha\tau}\delta_{\beta\sigma}$$
$$-\frac{2}{x^2}(\delta_{\alpha\sigma}x_\beta x_\tau+\delta_{\alpha\tau}x_\sigma x_\beta+\delta_{\beta\sigma}x_\alpha x_\tau+\delta_{\beta\tau}x_\sigma x_\alpha)$$
$$+8\frac{x_\alpha x_\beta x_\sigma x_\tau}{x^4}\Big] \qquad (6.1.18)$$

and proceed to contract it with the structure of each graph J_7 to J_1. Since we are dealing with finite quantities, this part of the computation can be performed in a way very similar to the evaluation of the first perturbative order, by considering everything in two space-time dimensions. The result can be conveniently expressed as the following matrix in the $+,-,V$ basis

$$G^{(1F)}_{ij} = \frac{N(N-1)}{2\pi^5}\begin{pmatrix} (N+1)g_+ & (N-1)g_- & 0 \\ (N-1)g_- & (N-1)g_+ - 2g_V & -2g_- \\ 0 & -2g_- & 0 \end{pmatrix} \tag{6.1.19}$$

or, in the S,P,V basis

$$\begin{aligned} G^{(1F)}_{ij} &= \frac{N(N-1)}{8\pi^5} \qquad (6.1.20)\\ &\begin{pmatrix} (2N-1)g_S + g_P - 2g_V & g_S + g_P - 2g_V & -2(g_S - g_P) \\ g_S + g_P - 2g_V & (2N-1)g_P + g_S - 2g_V & 2(g_S - g_P) \\ -2(g_S - g_P) & 2(g_S - g_P) & 0 \end{pmatrix}\end{aligned}$$

and can be seen to be equal to

$$G^{(1F)}_{ij} = -\left[G^{(0)}_{ik}\frac{\partial\beta^{(1)k}}{\partial g^j} + G^{(0)}_{jk}\frac{\partial\beta^{(1)k}}{\partial g^i}\right]. \tag{6.1.21}$$

Let us now consider the remaining part of $J^{\alpha\beta\sigma\tau}$ which is

$$\begin{aligned} D^{\alpha\beta\sigma\tau} &= -\frac{\pi}{4x^2}\frac{\Gamma(1+\frac{\varepsilon}{2})\Gamma^2(1-\frac{\varepsilon}{2})}{\Gamma(2-\varepsilon)\Gamma^2(2+\varepsilon)}\frac{\pi^{\frac{\varepsilon}{2}}}{(x^6)^{\frac{\varepsilon}{2}}} \\ &\left[\frac{(2+3\varepsilon)(2-\varepsilon)}{\varepsilon}\frac{1}{x^2}(\delta_{\alpha\beta}x_\sigma x_\tau + \delta_{\sigma\tau}x_\alpha x_\beta)\right]. \qquad (6.1.22)\end{aligned}$$

Performing the various contractions and recalling the definitions (3.1.47) (3.1.48) and (3.1.51) we obtain this table of results

$$\begin{aligned} J_7 &= -N^3(A^{(i)}g_i + A^{(j)}g_j)\frac{\Gamma^4(1+\frac{\varepsilon}{2})}{32\pi^5 x^8(\pi^4x^8)^{\frac{\varepsilon}{2}}}\left[\frac{1}{\varepsilon} - \frac{1}{2}\gamma_E - \frac{1}{2}log(\pi\mu^2x^2)\right] \\ &\quad Tr[\Gamma^{(i)}_A\gamma\cdot x\Gamma^{(j)}_B\gamma\cdot x]Tr[\Gamma^{(i)}_A\gamma\cdot x\Gamma^{(j)}_B\gamma\cdot x] \\ J_6 &= N^2(B^{(ik)}g_k + B^{(jk)}g_k)\frac{\Gamma^4(1+\frac{\varepsilon}{2})}{64\pi^5x^8(\pi^4x^8)^{\frac{\varepsilon}{2}}}\left[\frac{1}{\varepsilon} - \frac{1}{2}\gamma_E - \frac{1}{2}log(\pi\mu^2x^2)\right] \\ &\quad Tr[\Gamma^{(i)}_A\gamma\cdot x\Gamma^{(j)}_B\gamma\cdot x]Tr[\Gamma^{(i)}_A\gamma\cdot x\Gamma^{(j)}_B\gamma\cdot x]\end{aligned}$$

$$\begin{aligned}
J_5 &= -N^2\frac{\Gamma^4(1+\frac{\varepsilon}{2})}{128\pi^5x^8(\pi^4x^8)^{\frac{\varepsilon}{2}}}\left[\frac{1}{\varepsilon}-\frac{1}{2}\gamma_E-\frac{1}{2}log(\pi\mu^2x^2)\right] \\
&\Big\{E_{ikl}g_kTr[\Gamma_B^{(j)}\gamma\cdot x\Gamma_D^{(l)}\gamma\cdot x]Tr[\Gamma_B^{(j)}\gamma\cdot x\Gamma_D^{(l)}\gamma\cdot x] \\
&+E_{jkl}g_kTr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_D^{(l)}\gamma\cdot x]Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_D^{(l)}\gamma\cdot x]\Big\} \\
J_4 &= N^2\frac{\Gamma^4(1+\frac{\varepsilon}{2})}{64\pi^5x^8(\pi^4x^8)^{\frac{\varepsilon}{2}}}\left[\frac{1}{\varepsilon}-\frac{1}{2}\gamma_E-\frac{1}{2}log(\pi\mu^2x^2)\right] \\
&\Big\{2(A^{(i)}g_i+A^{(j)}g_j)Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x] \\
&+B^{(ki)}g_kTr[\Gamma_C^{(k)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x]Tr[\Gamma_C^{(k)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x] \\
&+B^{(kj)}g_kTr[\Gamma_C^{(k)}\gamma\cdot x\Gamma_A^{(i)}\gamma\cdot x]Tr[\Gamma_C^{(k)}\gamma\cdot x\Gamma_A^{(i)}\gamma\cdot x]\Big\} \\
J_2 &= -N\frac{\Gamma^4(1+\frac{\varepsilon}{2})}{64\pi^5x^8(\pi^4x^8)^{\frac{\varepsilon}{2}}}\left[\frac{1}{\varepsilon}-\frac{1}{2}\gamma_E-\frac{1}{2}log(\pi\mu^2x^2)\right] \\
&\Big\{B^{(ki)}g_kTr[\Gamma_B^{(j)}\gamma\cdot x\Gamma_C^{(k)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x\Gamma_C^{(k)}\gamma\cdot x] \\
&B^{(kj)}g_kTr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_C^{(k)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x\Gamma_C^{(k)}\gamma\cdot x]\Big\} \\
J_1+J_3 &= -N\frac{\Gamma^4(1+\frac{\varepsilon}{2})}{64\pi^5x^8(\pi^4x^8)^{\frac{\varepsilon}{2}}}\left[\frac{1}{\varepsilon}-\frac{1}{2}\gamma_E-\frac{1}{2}log(\pi\mu^2x^2)\right] \\
&\Big\{(B^{(ik)}g_k+B^{(jk)}g_k)Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x] \\
&-\frac{1}{2}E_{ikl}g_kTr[\Gamma_B^{(j)}\gamma\cdot x\Gamma_D^{(l)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x\Gamma_D^{(l)}\gamma\cdot x] \\
&-\frac{1}{2}E_{jkl}g_kTr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_D^{(l)}\gamma\cdot x\Gamma_A^{(i)}\gamma\cdot x\Gamma_D^{(l)}\gamma\cdot x]\Big\}
\end{aligned}$$

We are now ready to sum up all the various contributions, and if we do that it is easy to reconstruct and factor out a term which is the metric at the first perturbative order (see eq.(6.1.8))

$$\begin{aligned}
G_{ij}^{(0)} &= \frac{x^4}{2}\left[\frac{\Gamma(1+\frac{\varepsilon}{2})}{2(\pi x^2)^{1+\frac{\varepsilon}{2}}}\right]^4\Big[N^2Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x] \\
&Tr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x] \\
&-NTr[\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x\Gamma_A^{(i)}\gamma\cdot x\Gamma_B^{(j)}\gamma\cdot x]\Big] \qquad (6.1.23)
\end{aligned}$$

Therefore we can write the result of the computation of the graphs as follows

$$J_1+\ldots+J_7 = -\frac{1}{2\pi x^4}\left[\frac{1}{\varepsilon}-\frac{1}{2}log(\pi x^2\mu^2)-\frac{1}{2}\gamma_E\right]$$

$$\Big\{[2N(A^{(i)}g_i + A^{(j)}g_j) - (B^{(ik)}g_k + B^{(jk)}g_k)]G^{(0)}_{ij} - B^{(ki)}g_kG^{(0)}_{jk} - B^{(kj)}g_kG^{(0)}_{ik} + \frac{1}{2}(E_{ikl}g_kG^{(0)}_{jl} + E_{jkl}g_kG^{(0)}_{il})\Big\} \tag{6.1.24}$$

where we have introduced a scale μ. The corresponding result for the metric can be put in an even more compact form by using the definition of the one-loop vertex function W^j (cfr.(3.2.1))

$$\varepsilon\hat{I}V^j \equiv W^j = -\frac{1}{2\pi}[NA^{(j)}g_j^2 - \sum_k B^{(jk)}g_jg_k + \frac{1}{4}\sum_{kl} E_{klj}g_kg_l], \tag{6.1.25}$$

whose $\varepsilon = 0$ part gives exactly the β-functions eq.(3.1.57), and its derivative

$$\frac{\partial W^j}{\partial g^i} = -\frac{1}{2\pi}[2NA^{(i)}g_i\delta_{ij} - B^{(ik)}g_k\delta_{ij} - B^{(ji)}g_j + \frac{1}{2}E_{ikj}g_k] \tag{6.1.26}$$

We then obtain simply

$$G^{(1L)}_{ij} = \left[\frac{1}{\varepsilon} - \frac{1}{2}log(\pi x^2\mu^2) - \frac{1}{2}\gamma_E\right]\left[G^{(0)}_{jk}\frac{\partial W^k}{\partial g^i} + G^{(0)}_{ik}\frac{\partial W^k}{\partial g^j}\right] \tag{6.1.27}$$

The divergent part can be eliminated, as usual in the MS scheme, by subtracting the counterterms, which are given by

$$K_{ij} = \frac{1}{\varepsilon}\left[G^{(0)}_{jk}\frac{\partial\beta^k}{\partial g^i} + G^{(0)}_{ik}\frac{\partial\beta^k}{\partial g^j}\right]. \tag{6.1.28}$$

Recalling that, by definition from eq.(4.1.18), we have

$$W^k - \beta^k = \varepsilon G^k \tag{6.1.29}$$

after the subtraction of the counterterms we are left with

$$G^{(1L)}_{ij} = \left[G^{(0)}_{jk}\frac{\partial G^k}{\partial g^i} + G^{(0)}_{ik}\frac{\partial G^k}{\partial g^j}\right] - \frac{1}{2}\left[log(\pi x^2\mu^2) + \gamma_E\right]\left[G^{(0)}_{jk}\frac{\partial\beta^k}{\partial g^i} + G^{(0)}_{ik}\frac{\partial\beta^k}{\partial g^j}\right]. \tag{6.1.30}$$

Collecting also the result eq.(6.1.21), the final outcome of the computation of the second order metric is, in the MS scheme,

$$\begin{aligned} G^{(1)}_{ij} &= G^{(1L)}_{ij} + G^{(1F)}_{ij} \\ &= -\frac{1}{2}\left[log(\pi x^2\mu^2) + \gamma_E + 1\right]\left[G^{(0)}_{jk}\frac{\partial\beta^k}{\partial g^i} + G^{(0)}_{ik}\frac{\partial\beta^k}{\partial g^j}\right] \\ &+ \left[G^{(0)}_{jk}\frac{\partial G^k}{\partial g^i} + G^{(0)}_{ik}\frac{\partial G^k}{\partial g^j}\right]. \end{aligned} \qquad (6.1.31)$$

We can improve this result if we recall that the last two terms can be reabsorbed by a finite shift in the coupling constants: if we choose to change variables as in eq.(4.1.26)

$$g'^k = g^k + G^k \qquad (6.1.32)$$

it turns out that the metric changes by the amount

$$\delta G_{ij} = -G^{(0)}_{jk}\frac{\partial G^k}{\partial g^i} - G^{(0)}_{ik}\frac{\partial G^k}{\partial g^j} \qquad (6.1.33)$$

so that the first two terms disappear and:

$$G^{(1)}_{ij} = -\frac{1}{2}\left[log(\pi x^2\mu^2) + \gamma_E + 1\right]\left[G^{(0)}_{jk}\frac{\partial\beta^k}{\partial g^i} + G^{(0)}_{ik}\frac{\partial\beta^k}{\partial g^j}\right] \qquad (6.1.34)$$

As already noticed, this turns out to be

$$\begin{aligned} G^{(1)}_{ij} &= \left[log(\pi x^2\mu^2) + \gamma_E + 1\right]\frac{N(N-1)}{2\pi^5} \\ &\begin{pmatrix} (N+1)g_+ & (N-1)g_- & 0 \\ (N-1)g_- & (N-1)g_+ - 2g_V & -2g_- \\ 0 & -2g_- & 0 \end{pmatrix} \end{aligned} \qquad (6.1.35)$$

in the $+, -, V$ basis and

$$\begin{aligned} G^{(1)}_{ij} &= \left[log(\pi x^2\mu^2) + \gamma_E + 1\right]\frac{N(N-1)}{8\pi^5} \qquad (6.1.36) \\ &\begin{pmatrix} (2N-1)g_S + g_P - 2g_V & g_S + g_P - 2g_V & -2(g_S - g_P) \\ g_S + g_P - 2g_V & (2N-1)g_P + g_S - 2g_V & 2(g_S - g_P) \\ -2(g_S - g_P) & 2(g_S - g_P) & 0 \end{pmatrix} \end{aligned}$$

in the S, P, V basis. This concludes the computation of the first perturbative order. For the sake of completeness we will report in the table 6.4 at the end of this chapter the expressions of $G_{ij} = G^{(0)}_{ij} + G^{(1)}_{ij}$ in the two bases $+, -, V$ and S, P, V.

Table 6.1: Graphs contributing to the metric G_{ij}

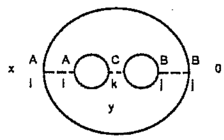
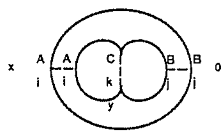
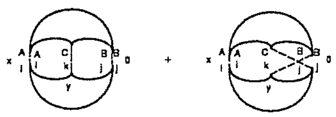
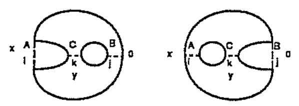
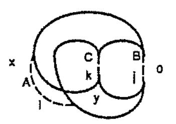
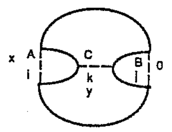
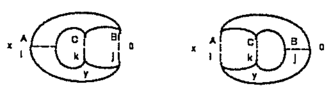
 J_7	$-N^3 g_k \int d^d y Tr[\Gamma_A^{(i)} S(x) \Gamma_B^{(j)} S(-x)]$ $\cdot Tr[\Gamma_A^{(i)} S(x-y) \Gamma_C^{(k)} S(y-x)]$ $\cdot Tr[\Gamma_C^{(k)} S(y) \Gamma_B^{(j)} S(-y)]$
 J_6	$N^2 g_k \int d^d y Tr[\Gamma_A^{(i)} S(x) \Gamma_B^{(j)} S(-x)]$ $\cdot Tr[\Gamma_A^{(i)} S(x-y) \Gamma_C^{(k)} S(y) \Gamma_B^{(j)} S(-y) \Gamma_C^{(k)} S(y-x)]$
 J_5	$N^2 g_k \int d^d y \Big\{ Tr[\Gamma_A^{(i)} S(x) \Gamma_B^{(j)} S(-y) \Gamma_C^{(k)} S(y-x)]\cdot$ $\cdot Tr[\Gamma_A^{(i)} S(x-y) \Gamma_C^{(k)} S(y) \Gamma_B^{(j)} S(-x)]$ $+ Tr[\Gamma_A^{(i)} S(x) \Gamma_B^{(j)} S(-y) \Gamma_C^{(k)} S(y-x)]\cdot$ $\cdot Tr[\Gamma_A^{(i)} S(x) \Gamma_B^{(j)} S(-y) \Gamma_C^{(k)} S(y-x)] \Big\}$

Table 6.2: Graphs contributing to the metric $G_{ij}(cont.d)$

 J4	$N^2 g_k \int d^d y \Big\{ Tr\Big[\Gamma_A^{(i)} S(x)\Gamma_B^{(j)} S(-x)\Gamma_A^{(i)} S(x-y)\cdot \Gamma_C^{(k)} S(y-x)\Big]\cdot Tr[\Gamma_C^{(k)} S(y)\Gamma_B^{(j)} S(-y)] + Tr[\Gamma_A^{(i)} S(x)\Gamma_B^{(j)} S(-y)\Gamma_C^{(k)} S(y)\Gamma_B^{(j)} S(-x)]\cdot Tr[\Gamma_A^{(i)} S(x-y)\Gamma_C^{(k)} S(y-x)]\Big\}$
 J3	$-N g_k \int d^d y Tr[\Gamma_A^{(i)} S(x)\Gamma_B^{(j)} S(-y)\Gamma_C^{(k)} S(y-x) \cdot \Gamma_A^{(i)} S(x)\Gamma_B^{(j)} S(-y)\Gamma_C^{(k)} S(y-x)]$
 J2	$-N g_k \int d^d y Tr[\Gamma_A^{(i)} S(x)\Gamma_B^{(j)} S(-y)\Gamma_C^{(k)} S(-y) \cdot \Gamma_B^{(j)} S(-x)\Gamma_A^{(i)} S(x-y)\Gamma_C^{(k)} S(y-x)]$
 J1	$-N g_k \int d^d y \Big\{ Tr[\Gamma_A^{(i)} S(x)\Gamma_B^{(j)} S(-y)\Gamma_C^{(k)} S(y-x)\cdot \Gamma_A^{(i)} S(x-y)\Gamma_C^{(k)} S(y)\Gamma_B^{(j)} S(-x)] + Tr[\Gamma_A^{(i)} S(x)\Gamma_B^{(j)} S(-x)\Gamma_A^{(i)} S(y-x)\cdot \Gamma_C^{(k)} S(y)\Gamma_B^{(j)} S(-y)\Gamma_C^{(k)} S(y-x)]\Big\}$

6.2 Metric and central charge in the symmetric scheme

The algebraic approach [34] presented in section 4.2 can equally be applied to the determination of the metric matrix elements.

Let us list the general properties of the metric in the symmetric scheme resulting purely from symmetry considerations.

- Chiral properties.

  The parity properties of O_- under the transformation (4.2.3) lead to the following properties of the metric:

$$G_{++}, G_{+V}, G_{VV}, G_{--} \qquad \text{are even functions of } g_- \tag{6.2.1}$$

$$G_{+-}, G_{V-} \qquad \text{are odd functions of } g_- \tag{6.2.2}$$

- Thirring and Gross-Neveu models.

  The group theoretic arguments that we have applied to the computation of the β functions can be adapted and extended to the metric. The matrix elements of the current-current interaction gO_T can be parametrized in terms of a universal function F_T:

$$G_{TT} = d_A\, F_T(g; C_A) \tag{6.2.3}$$

  where C_A and d_A are the Casimir invariant and the dimension of the adjoint representation of the symmetry group.

  A further constraint can be obtained by recalling that in the $g_- = 0$ theory there is a decoupled free field associated with the $U(1)$ subgroup. The decoupling is made apparent in the bosonized version of the theory.

  Therefore starting from eq.(4.2.9) we easily obtain the following constraints:

$$\left(G_{V+} + \frac{1}{N} G_{VV}\right)_{g_-=0} = 0 \tag{6.2.4}$$

$$\left(\frac{1}{N}\frac{\partial}{\partial g_V} + \frac{\partial}{\partial g_+}\right) G_{VV}\bigg|_{g_-=0} = 0 \tag{6.2.5}$$

Now by applying eq.(6.2.4) and (6.2.5) to $SU(N)$ Thirring models we obtain:

$$G_{++} + \frac{2}{N}G_{+V} + \frac{1}{N^2}G_{VV}\Big|_{g_-=0} = G_{++} + \frac{1}{N}G_{+V}\Big|_{g_-=0} = (N^2-1)F_T(g_+;N) \tag{6.2.6}$$

while in the case of Gross-Neveu models the corresponding condition is

$$G_{++} + 2G_{+-} + G_{--}\Big|_{\substack{g_+=g_-=g \\ g_V=0}} = N(2N-1)F_T(g;2(N-1)) \tag{6.2.7}$$

Let us notice that the request of free-field behavior with canonical commutation rules leads to the stronger costraint that $G_{VV}(g_-=0)$ be independent of the couplings and equal to its free field value.

- Special symmetries of N=1 and N=2 models.

When N=1 the following constraints hold:

$$O_- = 0 \qquad O_V = -O_+ \tag{6.2.8}$$

as a consequence

$$G_{+-}, G_{V-}, G_{--} \sim (N-1) \tag{6.2.9}$$

and

$$G_{++}(N=1) = G_{VV}(N=1) \tag{6.2.10}$$

Moreover the free field condition implies that G_{++} and G_{VV} must have their free theory value; this last condition can be expressed by the statement that all perturbative corrections to all matrix elements must be proportional to the factor (N-1).

In N=2 models we can make use of the special symmetries described in section 4.2 to obtain the following set of conditions

$$G_{+-} + \frac{1}{2}G_{V-}\Big|_{N=2} = 0 \tag{6.2.11}$$

$$G_{+V} + \frac{1}{2}G_{VV}\Big|_{N=2} = 0 \tag{6.2.12}$$

$$G_{++} + \frac{1}{2}G_{+V}\Big|_{N=2} = 3F_T(g_+;2) \tag{6.2.13}$$

$$G_{--} - G_{-V} + \frac{1}{4}G_{VV}\Big|_{\substack{N=2 \\ \delta_V=0}} = 3F_T(g_-;2) \tag{6.2.14}$$

Moreover the decoupling of the selfdual and antiselfdual components leads to the further constraint that when the metric is expressed as a function of g_+, g_-, δ_V only the matrix elements G_{++} can have a dependence on g_+.

- The last constraint on the metric concerns the dependence on N. A trivial diagrammatic analysis shows that there is an overall factor of N in every matrix element. Moreover from the $1/N$ expansion it is easily established that the leading behavior of the coefficients at the l-loop order is $g^{l-3}N^{l-1}$. Further constraints come from the fact that the metric is diagonal in the S,P,V basis in the leading order in the $1/N$ expansion [33].

As a trivial application of these results let's first compute (up to a normalization constant) the free field metric.

We parametrize the function F_T as

$$F_T(g; N) = f_0 + (f_1 N + f_1')g + \mathcal{O}(g^2) \tag{6.2.15}$$

where f_0 is independent of N and its actual value is

$$f_0 = \frac{1}{4\pi^4} \tag{6.2.16}$$

By simply applying the chiral properties and power counting in N we immediately obtain

$$G_{ij}^{(0)} = N f_0 \begin{bmatrix} N & 0 & -1 \\ 0 & N-1 & 0 \\ -1 & 0 & N \end{bmatrix} \tag{6.2.17}$$

It is easy to check that all other conditions are automatically satisfied.

Let us now consider the 4-loop contribution to the metric. On general grounds, by applying the chiral symmetry and the U(1) decoupling we can parametrize $G_{ij}^{(1)}$ by

$$G_{ij}^{(1)}(x) = \alpha N \begin{bmatrix} A g_+ + B g_V & C g_- & g_+ - N g_V \\ C g_- & D g_+ + E g_V & F g_- \\ g_+ - N g_V & F g_- & N^2 g_V - N g_+ \end{bmatrix} \tag{6.2.18}$$

where α is a numerical coefficient and A-F are functions of N (at most quadratic) and can depend linearly on $log(x)$: however this dependence is fixed by the renormalization group properties of G_{ij}. We can now make use of the information about the dependence on N coming from the 1/N expansion and eq.(6.2.18) to write:

$$G_{ij}^{(1)}(x) = \alpha N \begin{bmatrix} Ag_+ + bg_V & c(N-1)g_- & g_+ - Ng_V \\ c(N-1)g_- & (N-1)(dg_+ + eg_V) & f(N-1)g_- \\ g_+ - Ng_V & f(N-1)g_- & N^2 g_V - Ng_+ \end{bmatrix} \tag{6.2.19}$$

where

$$A = \mathcal{O}(N^2) \quad b, c, d = \mathcal{O}(N) \quad e, f = \mathcal{O}(1) \tag{6.2.20}$$

Enforcing eqs.(6.2.6) and (6.2.7) leads to the relationships

$$\alpha N \left[Ag_+ + bg_V + \frac{1}{N}(g_+ - Ng_V) \right] = (N^2-1)(f_1 N + f_1')g_+ \tag{6.2.21}$$

$$\alpha \left[A + 2c(N-1) + d(N-1) \right] = (2N-1)(2f_1(N-1) + f_1') \tag{6.2.22}$$

Some manipulations lead to the following representation

$$\begin{aligned} G_{ij}^{(1)}(x) &= -f_1' N \begin{bmatrix} g_V - Ng_+ & 0 & g_+ - Ng_V \\ 0 & (N-1)(2g_V - g_+) & 0 \\ g_+ - Ng_V & 0 & N^2 g_V - Ng_+ \end{bmatrix} \\ &+ f_1 \quad N(N-1) \begin{bmatrix} (N+1)g_+ & \tilde{c}g_- & 0 \\ \tilde{c}g_- & \tilde{d}g_+ - 2\tilde{e}g_V & \tilde{f}g_- \\ 0 & \tilde{f}g_- & 0 \end{bmatrix} \end{aligned} \tag{6.2.23}$$

with the condition

$$2\tilde{c} + \tilde{d} = 3(N-1) \qquad \tilde{e}, \tilde{f} = \mathcal{O}(1) \tag{6.2.24}$$

Making use of eq.(6.2.11) (6.2.12) (6.2.13) (6.2.14) we obtain

$$\tilde{e} = \tilde{d}(N=2) \qquad \tilde{f} = -2\tilde{c}(N=2) \tag{6.2.25}$$

It is convenient to solve eq.(6.2.24) in the form

$$\tilde{c} = (N-1) + \frac{\gamma}{f_1} \qquad \tilde{d} = (N-1) - 2\frac{\gamma}{f_1} \tag{6.2.26}$$

where from the 1/N expansion one can show that γ must be independent of N.

In conclusion the most general allowable form of the 4-loop contribution to the metric is

$$
\begin{aligned}
G^{(1)}_{ij}(x) &= f_1 N(N-1)\begin{bmatrix} (N+1)g_+ & (N-1)g_- & 0 \\ (N-1)g_- & (N-1)g_+ - 2g_V & -2g_- \\ 0 & -2g_- & 0 \end{bmatrix} + \\
&+ f_1' N \begin{bmatrix} Ng_+ - g_V & 0 & Ng_V - g_+ \\ 0 & (N-1)(g_+ - 2g_V) & 0 \\ Ng_V - g_+ & 0 & Ng_+ - N^2 g_V \end{bmatrix} \\
&+ \gamma N(N-1)\begin{bmatrix} 0 & g_- & 0 \\ g_- & 4g_V - 2g_+ & -2g_- \\ 0 & -2g_- & 0 \end{bmatrix}
\end{aligned}
\tag{6.2.27}
$$

Let us now apply our result eq.(6.2.27), together with eq.(4.2.37), to Zamolodchikov's equation in the form

$$
\frac{\partial C}{\partial g^i} = G_{ij}\beta^j \simeq \left(G^{(0)}_{ij} + G^{(1)}_{ij}\right)\left(\beta^{(1)j} + \beta^{(2)j}\right) \simeq G^{(0)}_{ij}\beta^{(1)j} + G^{(0)}_{ij}\beta^{(2)j} + G^{(1)}_{ij}\beta^{(1)j}
\tag{6.2.28}
$$

A trivial consequence of this equation is the order by order irrotationality of the covariant vector $G_{ij}\beta^j$. The lowest order term is[1]

$$
\beta^{(1)}_i = G^{(0)}_{ij}\beta^{(1)j} = f_0 b_1 N(N-1)\begin{pmatrix} (N+1)g_+^2 + (N-1)g_-^2 \\ 2(N-1)g_+g_- - 4g_Vg_- \\ -2g_-^2 \end{pmatrix}
\tag{6.2.29}
$$

and it is trivially seen to satisfy

$$
\beta^{(1)}_i = \partial_i\left[f_0 b_1 N(N-1)\left(\frac{N+1}{3}g_+^3 + (N-1)g_-^2 g_+ - 2g_V g_-^2\right)\right]
\tag{6.2.30}
$$

[1]It is convenient to recall eq.(4.2.28):

$$
b_1 = -\frac{1}{\pi}\,, \qquad b_2 = \frac{1}{\pi^2}
$$

implying

$$\frac{\partial \beta_i^{(1)}}{\partial g^j} = \frac{\partial \beta_j^{(1)}}{\partial g^i} \tag{6.2.31}$$

At the two loop level we obtain from eq.(6.2.27) and eq.(4.2.37)

$$\begin{aligned}
\beta_i^{(2)} &= G_{ij}^{(0)} \beta^{(2)\,j} + G_{ij}^{(1)} \beta^{(1)\,j} = \\
&= f_0 b_2 N(N-1) \begin{pmatrix} (N+1)g_+^2 + (N-1)g_+ g_-^2 - 2g_V g_-^2 \\ (N-1)(g_+^2 + g_-^2)g_- - 4g_+ g_- g_V + 2N g_V^2 g_- \\ -2g_+ g_-^2 + 2N g_V g_-^2 \end{pmatrix} \\
&+ f_1 b_1 N(N-1) \begin{pmatrix} N(N+1)g_+^3 + (3N^2-5N)g_+ g_-^2 - 4(N-1)g_V g_-^2 \\ g_- \left[(N^2-3N+4)g_-^2 + (3N^2-5N)g_+^2 - 8(N-1)g_+ g_V + 8g_V^2\right] \\ 8g_V g_-^2 - 4(N-1)g_+ g_-^2 \end{pmatrix} \\
&+ \quad f_0 b_2 N(N-1)\delta \begin{pmatrix} 0 \\ g_-(g_+^2 - g_-^2 - 4g_V g_+ + 4g_V^2) \\ 0 \end{pmatrix} \\
&+ \quad f_0 b_2 N(N-1)(N-2) \begin{pmatrix} \mu g_V g_-^2 \\ \nu(g_+^2 - g_-^2)g_- + \rho g_V g_+ g_- + \sigma g_V^2 g_- \\ \mu g_V g_-^2 \end{pmatrix} \\
&+ f_1' b_1 N(N-1) \begin{pmatrix} -2g_V g_-^2 + (N+1)g_+^3 + (N-1)g_+ g_-^2 \\ -4N g_V g_+ g_- + 2(N-1)g_+^2 g_- + 8g_V^2 g_- \\ -2g_+ g_-^2 + 2N g_V g_-^2 \end{pmatrix} \\
&+ \quad \gamma b_1 N(N-1) \begin{pmatrix} 2(N-1)g_+ g_-^2 - 4g_V g_-^2 \\ (2-3N)g_+^2 g_- + N g_+^3 + 8N g_V g_+ g_- - 16 g_V^2 g_- \\ 8g_V g_-^2 - 4(N-1)g_+ g_-^2 \end{pmatrix}
\end{aligned} \tag{6.2.32}$$

It is easily seen that the first two terms can be expressed as

$$\partial_i N(N-1)\Big\{ f_0 b_2 \left[\frac{N+1}{4} g_+^4 + \frac{N-1}{2} g_+^2 g_-^2 - 2g_V g_+ g_-^2 + N g_V^2 g_-^2 + \frac{N-1}{4} g_-^4\right] +$$
$$f_1 b_1 \Big[\frac{N(N+1)}{4} g_+^4 + (N^2-3N+4)g_-^4 +$$

$$\frac{3N^2-5N}{2}g_+^2g_-^2 - 4(N-1)g_Vg_+g_-^2 + 4g_V^2g_-^2\Big]\Big\} \qquad (6.2.33)$$

Therefore the irrotationality condition can be seen as a homogeneous set of equations in the parameters $\delta, \mu, \nu, \rho, \sigma$ and f_1', γ. This set of equations can be solved in the form

$$\begin{aligned} b_2\delta &= b_1[6\gamma - f_1'] \\ b_2\mu &= -4b_1\gamma \\ b_2\rho &= b_1[4f_1' - 16\gamma] \\ b_2\nu &= b_1[5\gamma - f_1'] \\ b_2\sigma &= b_1[2f_1' - 4\gamma] \end{aligned} \qquad (6.2.34)$$

As a consequence the most general two loop β functions are completely determined in terms of the most general admissible four loop metric:

$$\begin{aligned} \beta^{(2)} &= b_2 \begin{pmatrix} Ng_+^3 + (N-2)g_+g_-^2 \\ (N-1)(g_+^2+g_-^2)g_- - 4g_+g_-g_V + 2Ng_V^2g_- \\ g_+(g_+^2 - g_-^2) + 2(N-1)g_Vg_-^2 \end{pmatrix} \\ &+ b_1f_1' \begin{pmatrix} 0 \\ g_-\left[(N-1)(g_-^2 - g_+^2 + 4g_Vg_+) + 2(N-4)g_V^2\right] \\ 0 \end{pmatrix} \\ &+ b_1\gamma \begin{pmatrix} -4(N-2)g_Vg_-^2 \\ g_-\left[(5N-4)(g_+^2 - g_-^2) - 8(2N-1)g_+g_V - 4(N-8)g_V^2\right] \\ -4(N-2)g_Vg_-^2 \end{pmatrix} \end{aligned} \qquad (6.2.35)$$

Finally we can use irrotationality as an integrability condition and obtain

$$\begin{aligned} \beta_i^{(2)} &= \partial_i(f_0b_2 + f_1'b_1)N(N-1)\Big[\frac{N+1}{4}g_+^4 + \frac{N-1}{2}g_+^2g_-^2 \\ &- 2g_Vg_+g_-^2 + Ng_V^2g_-^2 + \frac{N-1}{4}g_-^4\Big] \\ &+ \partial_i\frac{f_1}{4b_1f_0}\left[G_{jl}^{(0)}\beta^{(1)j}\beta^{(1)l}\right] \\ &- \partial_i\gamma b_1N(N-1)g_-^2\left[(N-1)(g_-^2 - g_+^2 + 4g_Vg_+) + 2(N-4)g_V^2\right] \end{aligned} \qquad (6.2.36)$$

Going back to eq.(6.2.27), we can still make use of the free field normalization condition on $G_{VV}(g_- = 0)$, thus obtaining the constraint

$$f_1' = 0 \tag{6.2.37}$$

The second term in eq(6.2.36) is related to the renormalization group properties of the metric: it is easy to show that its coefficient must take the value

$$f_1 = -2b_1 f_0 \left[log(\pi\mu^2 x^2) + \gamma_E + 1\right] \tag{6.2.38}$$

In conclusion, our algebraic construction led us to an almost complete determination of the structure of the two loop β functions and the four loop metric. There is only one free parameter left, which does not correspond to an arbitrariness in reparametrization (there is no variable shift preserving the symmetry conditions to this order, apart from the trivial renormalization group transformation). From a direct evaluation of Feynman diagrams we found

$$\gamma = 0 \tag{6.2.39}$$

Assuming these results and the validity of eq.(5.3.13) we are now ready to compute the five loop central charge. From eq.(5.3.16) we obtain in the symmetric scheme the expression:

$$\begin{aligned} C &= c_0 + \frac{3}{4} b_1 f_0 N(N-1) \left[\left(\frac{N+1}{3}\right) g_+^3 + (N-1) g_+ g_-^2 - 2 g_V g_-^2\right] \\ &+ \frac{3}{4} b_2 f_0 N(N-1) \left[\frac{N+1}{4} g_+^4 + \frac{N-1}{2} g_+^2 g_-^2 + \frac{N-1}{4} g_-^4 - 2 g_V g_+ g_-^2 + N g_V^2 g_-^2\right] \\ &- \frac{3}{8}\left[1 + \gamma_E + log(\pi\mu^2 x^2)\right] G_{ij}^{(0)} \beta^{(1)\,i} \beta^{(1)\,j} \end{aligned} \tag{6.2.40}$$

All the scale dependence is contained in the last term, as expected from renormalization group invariance of C. The symmetries of the model that we already discussed, yield a set of constraints which are satisfied by the function C and which are reported below for completeness.

1. Chiral symmetry

$$C(g_+, g_-, g_V) = C(g_+, -g_-, g_V) \tag{6.2.41}$$

2. Group properties

$$C_T - c_0 = d_A K_T(g; C_A) \tag{6.2.42}$$

where K_T is a universal function related to β_T and F_T by

$$\begin{aligned} K_T(g; C_A) &= \frac{3}{4}\int_0^1 C_A g F_T(tg; C_A)\beta_T(tg; C_A) = \\ &= \frac{3}{4} C_A \left[\frac{1}{3} b_1 f_0 g^3 + \frac{1}{4}(b_2 f_0 + C_A b_1 f_1) g^4\right] \end{aligned} \tag{6.2.43}$$

3. Singlet decoupling

$$\frac{\partial C}{\partial g_V} \mid_{g_- = 0} = 0 \tag{6.2.44}$$

4. Conformal invariance of $U(1)$ models

$$C - c_0 \propto N(N-1) \tag{6.2.45}$$

5. Properties of N=2 models

$$C(g_+, g_-, g_V)_{N=2} = C_+(g_+) + C_-(g_-, \delta_V) \tag{6.2.46}$$

$$C_-(g_-, 0) = C_+(g_-) \tag{6.2.47}$$

At the end of this chapter it is useful to summarize all the results we found in a set of tables. Thus table 6.3 reports the explicit expression of the β-function up to two loops in the two bases S, P, V and $+, -, V$, table 6.4 contains the metric and table 6.5 the function C.

Table 6.3: Summary of β-functions

Basis	*Symmetric β-functions (upper indices)*
S	$\beta_S = -\frac{1}{\pi}\left[(N-1)g_S^2 + g_S g_P - 2g_V(g_S - g_P)\right] + \frac{1}{2\pi^2}\Big[N g_S(g_S^2 + g_P^2) - g_S^2(g_S - g_P) + 2g_V(g_P^2 - g_S^2) + 2N g_V^2(g_S - g_P)\Big]$
P	$\beta_P = -\frac{1}{\pi}\left[(N-1)g_P^2 + g_P g_S - 2g_V(g_P - g_S)\right] + \frac{1}{2\pi^2}\Big[N g_P(g_P^2 + g_S^2) - g_P^2(g_P - g_S) + 2g_V(g_S^2 - g_P^2) + 2N g_V^2(g_P - g_S)\Big]$
V	$\beta_V = -\frac{1}{\pi} g_S g_P + \frac{1}{2\pi^2}\left[(N-1)g_V(g_S - g_P)^2 + g_S g_P(g_S + g_P)\right]$
+	$\beta_+ = -\frac{1}{\pi}\left[N g_+^2 + (N-2)g_-^2\right] + \frac{1}{\pi^2}\left[N g_+^3 + (N-2)g_+ g_-^2\right]$
−	$\beta_- = -\frac{1}{\pi}\left[2(N-1)g_+ g_- - 4g_V g_-\right] + \frac{1}{\pi^2}\left[(N-1)(g_+^2 + g_-^2)g_- - 4g_+ g_V g_- + 2N g_V^2 g_-\right]$
V	$\beta_V = -\frac{1}{\pi}\left[g_+^2 - g_-^2\right] + \frac{1}{\pi^2}\left[g_+(g_+^2 - g_-^2) + 2(N-1)g_V g_-^2\right]$

Table 6.4: Metric

Basis	G_{ij}
S, P, V	$G_{ij} = \frac{N}{16\pi^4} \begin{pmatrix} 2N-1 & 1 & -2 \\ 1 & 2N-1 & -2 \\ -2 & -2 & 4N \end{pmatrix}$ $+ \frac{1}{8\pi^5} \left[log(\pi x^2 \mu^2) + \gamma_E + 1 \right] N(N-1) \cdot$ $\cdot \begin{pmatrix} (2N-1)g_S + g_P - 2g_V & g_S + g_P - 2g_V & -2(g_S - g_P) \\ g_S + g_P - 2g_V & (2N-1)g_P + g_S - 2g_V & 2(g_S - g_P) \\ -2(g_S - g_P) & 2(g_S - g_P) & 0 \end{pmatrix}$
$+, -, V$	$G_{ij} = \frac{N}{4\pi^4} \begin{pmatrix} N & 0 & -1 \\ 0 & N-1 & 0 \\ -1 & 0 & N \end{pmatrix}$ $+ \frac{[log(\pi x^2 \mu^2) + \gamma_E + 1]}{2\pi^5} N(N-1) \begin{pmatrix} (N+1)g_+ & (N-1)g_- & 0 \\ (N-1)g_- & (N-1)g_+ - 2g_V & -2g_- \\ 0 & -2g_- & 0 \end{pmatrix}$

Table 6.5: C-function

Basis	*C-function*
S,P,V	$C = \frac{N}{8\pi^2} - \frac{3}{64\pi^5}N(N-1)\Big[(2N-1)(g_S^3+g_P^3)$ $+3g_Sg_P(g_S+g_P) - 6g_V(g_S-g_P)^2\Big]$ $+ \frac{3}{512\pi^6}N(N-1)\Big[(2N-1)(g_S^2+g_P^2)^2 + 4g_Sg_P(g_S^2+g_P^2+g_Sg_P)$ $-8g_V(g_S-g_P)(g_S^2-g_P^2) + 8g_V^2(g_S-g_P)^2\Big]$ $- \frac{3}{8}\left[1+\gamma_E+log(\pi\mu^2x^2)\right] G^{(0)}_{ij}\beta^{(1)\,i}\beta^{(1)\,j}$
$+,-,V$	$C = \frac{N}{8\pi^2} - \frac{3}{16\pi^5}N(N-1)\Big[\left(\frac{N+1}{3}\right)g_+^3$ $+(N-1)g_+g_-^2 - 2g_Vg_-^2\Big]$ $+ \frac{3}{16\pi^6}N(N-1)\Big[\frac{N+1}{4}g_+^4 + \frac{N-1}{2}g_+^2g_-^2$ $+\frac{N-1}{4}g_-^4 - 2g_Vg_+g_-^2 + Ng_V^2g_-^2\Big]$ $- \frac{3}{8}\left[1+\gamma_E+log(\pi\mu^2x^2)\right] G^{(0)}_{ij}\beta^{(1)\,i}\beta^{(1)\,j}$

Chapter 7

Conclusions

This thesis was devoted to study an explicit realization of Zamolodchikov's C-theorem [1,2,3] by means of the generalized Thirring model (2.1.1) [32,34,47]. During this work we have been led to discuss several questions concerning the nature and the properties of the model at hand and the consistency and limits of validity of the theorem. Examples of that have been the study of the classical symmetries of the lagrangian, with the aid of abelian bosonization, or the analysis of the problems induced by the regularization procedure we have emploied to settle the renormalization of the quantum theory, and, on the other hand, the discussion of the influence of infrared divergences on the proof of the theorem and of its interesting consequences in relating quantum aspects of field theory and geometrical properties of the manifold of coupling constants. We hope to have provided as much details as possible in each part of the survey, compatibly with the limits and the scope of this thesis.

Beyond that the study of the application of Zamolodchikov's theorem led to consider some old and new quantities in quantum field theory, namely β-functions, the metric tensor G_{ij} in the parameter space and the function C. We have presented the computation of two of these functions in the framework of perturbation theory up to the highest perturbative order reachable without too much difficulty (second loop for the β-functions and fourth for G_{ij}), while the C-function has been worked out with the aid of Zamolodchikov's relation. Moreover we have discussed how to recover the symmetries of the model which are present at a classical level, and indeed how to exploit them as constraints for an algebraic approach to the determination of the above quantities.

At the end of this work we would like to point out some issues which deserve more work and which could be examined more closely in the future:

- A detailed study of the Ward identities of the model in order to understand better all the connections between the symmetries of the two dimensional model and the metric structure of the abstract mainifold of the coupling constants.

- A supersymmetric extension of the model. This would be interesting in two respects:

  1. The problem of γ_5 has been solved in a self consistent way in the dimensional scheme, and this could allow us to study the supersymmetric Ward identities in this scheme.
  2. The presence of scalar fields in this case would eventually give rise to infrared divergences, which spoil the original version of Zamolodchikov's theorem. From the study of this model we can learn how to answer several interesting questions about the approach to conformal symmetry in the supersymmetric case.

- As we have already stressed in the main text the computation of the next order (five loops) contribution to the metric would allow us to know additional informations on the geometric structure of the manifold of coupling constants, such as the curvature. This task involves a huge number of diagrams so that it is technically quite difficult.

- The four-fermion models can also be treated in a bosonized version (both abelian and non abelian): the first has been used in section 2.2 in order to exploit the symmetries of the model. In the second case it is already known that the conformal algebras of the bosonic and fermionic systems coincide: it would be interesting to study the perturbation theory of the bosonic model and check if this equivalence survives outside the critical point.

♣♣♣♣

Appendix A

Clifford algebras and Fierz identities in d-dimensions

Dimensional regularization requires an extension of our notion of a Clifford algebra in d spacetime dimensions, since we must analytically continue d to any (non-integer) value. We consider a representation of the d-dimensional Clifford algebra defined by

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\ \delta_{\mu\nu} I \tag{A.1}$$

The space of matrices acting on Dirac spinor indices is spanned by the completely antisymmetric products

$$\begin{aligned} \Gamma^{(k)} = \gamma_{[\mu_1} \cdots \gamma_{\mu_k]} &= \frac{1}{k!} \sum_{perm} (-1)^p\ \gamma_{\mu_1} \cdots \gamma_{\mu_i} \\ &\equiv \gamma_{\mu_1 \ldots \mu_k} \end{aligned} \tag{A.2}$$

When the number of dimensions is an integer, then $1 \leq d$ and the total number of these matrices is 2^d. However when d is noninteger the number of matrices must be considered as infinite. It is quite easy to prove the properties

$$\gamma_{\mu_1 \ldots \mu_k} = (-1)^{\frac{1}{2}k(k-1)} \gamma_{\mu_k \ldots \mu_1} \tag{A.3}$$

$$\gamma_\nu \gamma_{\mu_1 \ldots \mu_k} \gamma_\nu = (-1)^{k-1} \gamma_{\mu_1 \ldots \mu_k} \quad \nu \in \{\mu_k\} \tag{A.4}$$

$$\gamma_\nu \gamma_{\mu_1 \ldots \mu_k} \gamma_\nu = (-1)^{k} \gamma_{\mu_1 \ldots \mu_k} \quad \nu \notin \{\mu_k\} \tag{A.5}$$

where we have no summation on indices. As an immediate consequence

$$\Gamma^{(i)} \Gamma^{(j)} = \pm \Gamma^{(k)}\ k \leq i + j \tag{A.6}$$

$$\Gamma^{(j)} \Gamma^{(i)} \Gamma^{(j)} = \pm \Gamma^{(i)} \quad \text{no summation on } j \tag{A.7}$$

A few special consequences of the previous results are

$$\frac{1}{2}\left[\gamma_\nu \Gamma^{(i)}_{\mu_1\ldots\mu_i} + (-1)^i \Gamma^{(i)}_{\mu_1\ldots\mu_i}\gamma_\nu\right] = \Gamma^{(i+1)}_{\nu\mu_1\ldots\mu_i} \tag{A.8}$$

$$\frac{1}{2}\left[\gamma_\nu \Gamma^{(i)}_{\mu_1\ldots\mu_i} - (-1)^i \Gamma^{(i)}_{\mu_1\ldots\mu_i}\gamma_\nu\right] = \sum (-1)^p \delta_{\nu\mu_k}\, \Gamma^{(i-1)}_{\mu_1\ldots\mu_i[\mu_k]} \tag{A.9}$$

$$\sum_\nu \gamma_\nu \Gamma^{(i)} \gamma_\nu = (-1)^i (d-2i)\Gamma^{(i)} \tag{A.10}$$

and more generally

$$\sum_{\{\nu_j\}} \Gamma^{(j)}_{\nu_1\ldots\nu_j} \Gamma^{(i)} \Gamma^{(j)}_{\nu_1\ldots\nu_j} =$$

$$= (-1)^{\frac{1}{2}j(j-1)}(-1)^{ij} \sum \binom{i}{p}\binom{d-i}{j-p}\Gamma^{(i)} \tag{A.11}$$

We can therefore identify the functions defined in the text

$$c^{(i)} = (-1)^i (d-2i) \tag{A.12}$$

$$d^{(ij)} = (-1)^{\frac{1}{2}j(j-1)}(-1)^{ij} \sum \binom{i}{p}\binom{d-i}{j-p} \tag{A.13}$$

Having defined the matrices

$$X^{(ij)}_\mu = \Gamma^{(i)}\gamma_\mu\Gamma^{(j)} - \Gamma^{(j)}\gamma_\mu\Gamma^{(i)} \tag{A.14}$$

we can now explicitly compute their decomposition in the basis of matrices $\Gamma^{(k)}$. The relevant decompositions at the two-loop level are

$$X^{(0,0)}_\mu = 0 \tag{A.15}$$

$$X^{\mu(0,1)}_\alpha = 2\,\Gamma^{(2)}_{\mu\alpha} \tag{A.16}$$

$$X^{\mu(0,2)}_{\alpha\beta} = 2(\delta_{\mu\alpha}\gamma_\beta - \delta_{\mu\beta}\gamma_\alpha) \tag{A.17}$$

$$X^{\mu(1,1)}_{\alpha\beta} = 2\,\Gamma^{(3)}_{\alpha\mu\beta} \tag{A.18}$$

$$X^{\mu(1,2)}_{\alpha\beta\gamma} = 2(\delta_{\mu\beta}\delta_{\alpha\gamma} - \delta_{\alpha\beta}\delta_{\mu\gamma}) + 2\Gamma^{(4)}_{\beta\mu\gamma\alpha} \tag{A.19}$$

$$\begin{aligned} X^{\mu(2,2)}_{\alpha\beta\gamma\delta} = 2[\delta_{\beta\gamma}\Gamma^{(3)}_{\delta\mu\alpha} &- \delta_{\alpha\gamma}\Gamma^{(3)}_{\delta\mu\beta} + \delta_{\alpha\delta}\Gamma^{(3)}_{\gamma\mu\beta} - \delta_{\beta\delta}\Gamma^{(3)}_{\gamma\mu\alpha} \\ -\delta_{\mu\delta}\Gamma^{(3)}_{\alpha\beta\gamma} &+ \delta_{\mu\delta}\Gamma^{(3)}_{\alpha\beta\delta} + \delta_{\mu\beta}\Gamma^{(3)}_{\gamma\delta\alpha} - \delta_{\mu\alpha}\Gamma^{(3)}_{\gamma\delta\beta}] \end{aligned} \tag{A.20}$$

The Fierz identities are a consequence of the completeness in the matrix space spanned by the $\Gamma^{(k)}$. All the identities can easily be derived by the master formula

$$\sum_k (-1)^{\frac{1}{2}k(k-1)} \sum_{\mu_j} (\Gamma^{(k)}_{\mu_1 \ldots \mu_k})_{\alpha\beta} (\Gamma^{(k)}_{\mu_1 \ldots \mu_k})_{\gamma\delta} = 2^{d/2} \delta_{\alpha\delta} \delta_{\gamma\beta} \tag{A.21}$$

In the presence of an $SU(N)$ symmetry it may be useful to use the previous formula in conjunction with the relationship

$$\sum_a T^a_{ij} \, T^a_{lm} = l_R (\delta_{im} \delta_{lj} - \frac{1}{N} \delta_{ij} \delta_{lm}) \tag{A.22}$$

where T^a are the $SU(N)$ generators in some representation R and

$$Tr T^a T^b = l_R \delta_{ab} \tag{A.23}$$

Another important consequence of equation(A.21) is that it allows to decompose the operators in the basis $O^{(i)}$ as defined in the text. At the two-loop level the relevant non trivial relationship following from eq.(A.21) are

$$\sum_{\mu,\alpha,\beta} X^{\mu(0,2)}_{\alpha\beta} \otimes X^{\mu(0,2)}_{\alpha\beta} = 8(d-1) \gamma_\mu \otimes \gamma_\mu \tag{A.24}$$

$$\begin{aligned} \sum_{\mu,\alpha,\beta,\gamma} X^{\mu(1,2)}_{\alpha\beta\gamma} \otimes X^{\mu(1,2)}_{\alpha\beta\gamma} &= 8d(d-1) I \otimes I \\ &\quad +4 \sum_{\mu,\alpha,\beta,\gamma} \Gamma^{(4)}_{\beta,\mu,\gamma,\alpha} \otimes \Gamma^{(4)}_{\beta,\mu,\gamma,\alpha} \end{aligned} \tag{A.25}$$

$$\sum_{\mu,\alpha,\beta,\gamma,\delta} X^{\mu(2,2)}_{\alpha\beta\gamma\delta} \otimes X^{\mu(2,2)}_{\alpha\beta\gamma\delta} = 16(2d-5) \sum_{\alpha\beta\gamma} \Gamma^{(3)}_{\alpha\beta\gamma} \otimes \Gamma^{(3)}_{\alpha\beta\gamma} \tag{A.26}$$

and these allow an explicit computation of the functions E_{ijk} defined in the text.

Appendix B

Mass terms cancellation

In this appendix we show a simple example of the cancellation of mass terms in the computation of β functions.

The propagator of the fermion field will be

$$S(p) = \frac{-i\not{p} + m}{p^2 + m^2} \tag{B.1}$$

and we will be interested in the contribution to pole terms in two loop diagrams coming from the mass term in the numerator. Let us consider for example diagrams (E) in section 4.1.

$$\begin{aligned}(E1) &= -g_i g_j g_k \int_{l_1 l_2} \Gamma_A^{(i)} \otimes \{\Gamma_B^{(j)} S(l_2) \Gamma_C^{(k)} \mathrm{Tr}\left[S(l_1)\Gamma_A^{(i)} S(l_1) \Gamma_C^{(k)} S(-l_3) \Gamma_B^{(j)}\right] \\ &+ \Gamma_B^{(j)} S(-l_2) \Gamma_C^{(k)} \mathrm{Tr}\left[S(l_1)\Gamma_A^{(i)} S(l_1) \Gamma_B^{(j)} S(-l_3) \Gamma_C^{(k)}\right]\}\end{aligned} \tag{B.2}$$

For symmetry reasons only terms even in the variables l_i can appear in the computation, so we can meet only the following integrals

$$J_{\mu\nu}^{(1)} = \int_{l_1 l_2} \frac{m^2 l_{2\mu} l_{3\nu}}{(l_1^2 + m^2)^2 (l_2^2 + m^2)(l_3^2 + m^2)} \tag{B.3}$$

$$J_{\mu\nu}^{(2)} = \int_{l_1 l_2} \frac{m^2 l_{1\mu} l_{2\nu}}{(l_1^2 + m^2)^2 (l_2^2 + m^2)(l_3^2 + m^2)} \tag{B.4}$$

$$J_{\mu\nu}^{(3)} = \int_{l_1 l_2} \frac{m^2 l_{1\mu} l_{1\nu}}{(l_1^2 + m^2)^2 (l_2^2 + m^2)(l_3^2 + m^2)} \tag{B.5}$$

$$J^{(4)} = \int_{l_1 l_2} \frac{m^4}{(l_1^2+m^2)^2(l_2^2+m^2)(l_3^2+m^2)} \tag{B.6}$$

It is easily seen that only the first integral diverges and its pole is

$$J^{(1)}_{\mu\nu} = \frac{\varepsilon}{4}\hat{I}_2^2\delta_{\mu\nu} \tag{B.7}$$

This result implies that the pole contribution in eq.(B.2) coming from mass terms can be expressed as

$$\begin{aligned}(E1)_m &= -2N\frac{\varepsilon}{4}\hat{I}_2^2 g_i g_j g_k \Gamma_A^{(i)} \otimes \Gamma_B^{(j)}\gamma_\alpha\Gamma_C^{(k)}\Big[\mathrm{Tr}\left(\Gamma_A^{(i)}\Gamma_C^{(k)}\gamma_\alpha\Gamma_B^{(j)}\right) \\ &\quad -\mathrm{Tr}\left(\Gamma_A^{(i)}\Gamma_B^{(j)}\gamma_\alpha\Gamma_C^{(k)}\right)\Big] \\ &= \frac{\varepsilon}{4}\hat{I}_2^2 N g_i g_j g_k \Gamma_A^{(i)} \otimes X_{BC}^{\alpha(jk)}\mathrm{Tr}\left(\Gamma_A^{(i)}X_{BC}^{\alpha(jk)}\right)\end{aligned} \tag{B.8}$$

Let us now consider the counterterm (E2)

$$(E2) = \frac{N}{2}\hat{I}_2 g_i g_j g_k \hat{X}_{BC}^{\alpha(jk)}\int_{l_1}\mathrm{Tr}\left[\hat{X}_{BC}^{\alpha(jk)}S(l_1)\Gamma_A^{(i)}S(l_1)\right]\otimes\Gamma_A^{(i)} \tag{B.9}$$

The mass term can contribute only to the integral

$$J^{(0)} = \int_{l_1}\frac{m^2}{(l_1^2+m^2)^2} = -\frac{\varepsilon}{2}I_2 \tag{B.10}$$

So we have for the pole term

$$(E2)_m = -\frac{N}{4}g_i g_j g_k \varepsilon \hat{I}_2^2 \hat{X}_{BC}^{\alpha(jk)}\mathrm{Tr}\left[\hat{X}_{BC}^{\alpha(jk)}\Gamma_A^{(i)}\right]\otimes\Gamma_A^{(i)} \tag{B.11}$$

One can see that the contribution of eq.(B.11) exactly cancels the term appearing in eq.(B.9). The same mechanism works for all the graphs, as it can be easily checked.

Appendix C

$SU(N)$ Thirring model

In this appendix we give a brief summary of the explicit computation of the β functions for $SU(N)$ Thirring model [35], and this can provide a check of the results presented in this work.

The lagrangian of the model is:

$$\mathcal{L} = \bar{\psi} \partial\!\!\!/ \psi + \frac{1}{2} g J^a_\mu J^a_\mu \tag{C.1}$$

where

$$J^a_\mu = \bar{\psi} \gamma_\mu T^a \psi \tag{C.2}$$

The operators T^a are the generators of the Lie algebra of the group G in the representation R. Their normalization is

$$\begin{aligned} \mathrm{Tr}\left(T^a\, T^b\right) &= l_R \delta^{ab} && \text{(C.3)} \\ T^a T^a &= C_F = d_G \, \frac{l_R}{d_R} && \text{(C.4)} \\ \left[T^a, T^b\right] &= i\, f^{abc}\, T^c && \text{(C.5)} \end{aligned}$$

In eq.(C.4, C.5) d_G is the dimension of the group, d_R is the dimension of the representation R, f^{abc} are the structure constants.

We list some formulae which can help the reader in going through the computations:

$$f^{abc} f^{a'bc} = C_A \, \delta^{a\,a'} \tag{C.6}$$

$$S_{ab} \equiv \{T^a, T^b\} = \frac{2C_F}{d_G}\delta^{ab} + d_{abc}T^c \tag{C.7}$$

$$2\left(2C_F - \frac{1}{2}C_A\right) = \frac{1}{d_G}\left(4C_F + d_{abc}d_{abc}\right) \tag{C.8}$$

$$T^b T^a T^b = \left(C_F - \frac{1}{2}C_A\right)T^a \tag{C.9}$$

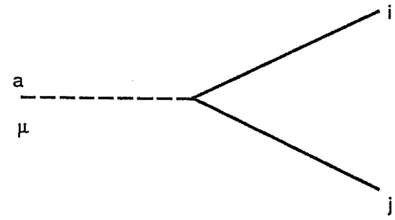
The values of the Casimir operators C_F and C_A for the fundamental and adjoint representations of $SU(N)$ are given by

$$C_F = \frac{N^2-1}{2N} \qquad C_A = N \tag{C.10}$$

It can be useful to rewrite the lagrangian (1) using a lagrangian multiplier

$$\mathcal{L}_A = \bar{\psi}\left(i\partial\!\!\!/ \; - \; iA\!\!\!/^{\,a}T^a\right) + \frac{1}{2g}A^a_\mu A^a_\mu \tag{C.11}$$

In this notation the vertices can be drawn as in figure



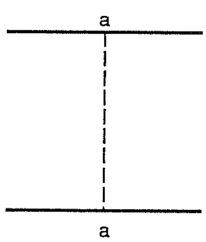
$$i\gamma_\mu T^a_{ij}$$

As we explained in the main text we use an effective propagator

$$S(p) = \frac{-i\not{p}}{p^2+m^2} \tag{C.12}$$

and as usual it can be verified that the inclusion of a mass term in the numerator of eq.(C.12) does not change the results for the β function.

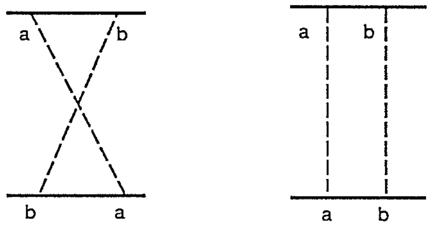
All diagrams are computed at zero external momenta and the results are given factorizing the group structure and the Lorentz factors; therefore the zero loop graph



takes the form

$$(A0) = -g^2 T^a \otimes T^a \gamma_\mu \otimes \gamma_\mu \tag{C.13}$$

At one loop order the only divergent graph is



$$\begin{aligned}(A1) &= g^2 T^a T^b \otimes T^a T^b \int_{l1} \gamma_\mu S(l_1)\gamma_\nu \otimes \gamma_\mu S(-l_1)\gamma_\nu \\ &\quad + g^2 T^a T^b \otimes T^b T^a \int_{l1} \gamma_\mu S(l_1)\gamma_\nu \otimes \gamma_\nu S(l_1)\gamma_\mu \\ &= -\frac{g^2}{4} C_A (3d-2)\, I_2 T^a \otimes T^a \gamma_\mu \otimes \gamma_\mu + \frac{1}{4} g^2 I_2 S_{ab} \otimes S_{ab} A_3 \otimes A_3\end{aligned} \tag{C.14}$$

where the results and the notations of appendix A have been used. It appears that at one loop we have a β function both for the relevant operator

$$O_{Th} = \frac{1}{2}\bar\psi\gamma_\mu T^a\psi\,\bar\psi\gamma_\mu T^a\psi \tag{C.15}$$

and for the evanescent operator

$$O_3 = \frac{1}{2}\bar\psi S_{ab}\Gamma^{(3)}\psi\,\bar\psi S_{ab}\Gamma^{(3)}\psi \tag{C.16}$$

$$\beta_{Th} = -\frac{1}{2\pi} g^2 C_A \tag{C.17}$$

$$\beta_3 = \frac{1}{8\pi} g^2 \tag{C.18}$$

The renormalization constant in the relevant operator is

$$Z_3^{(1)} g^2 = \frac{1}{2\pi} g^2 C_A \frac{1}{\varepsilon} \tag{C.19}$$

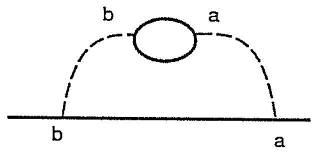
At two loops order, assuming

$$g_{bare} = g + Z_3^{(1)} g^2 + g^3 \left(Z_3^{(2)} - 2 Z_2^{(2)} \right) \tag{C.20}$$

where Z_2 is the wave function renormalization constant, we have

$$\beta_{Th}^{(2)} = -2\varepsilon\, g^3 \left(Z_3^{(2)} - 2 Z_2^{(2)} \right) \tag{C.21}$$

Let us start form the computation of $Z_2^{(2)}$. The only divergent graph is



and its computation leads to

$$Z_2^{(2)} g^2 = C_F (\varepsilon I_2) l_R g^2 = \frac{C_F l_R}{4\pi^2} g^2 \frac{1}{\varepsilon} \tag{C.22}$$

The vertex graphs are drawn in the table C.1 and a straightforward computation gives the following results for their relevant divergent part:

$$(G1) = g^3 T(R) C_A \varepsilon I_2^2 T^a \otimes T^a \gamma_\mu \otimes \gamma_\mu$$

$$(G2) = -g^3 T(R) C_A \varepsilon I_2^2 T^a \otimes T^a \gamma_\mu \otimes \gamma_\mu$$

$$(G3) = 2 g^3 T(R) \left(C_F - \tfrac{1}{2} C_A \right) \varepsilon I_2^2 T^a \otimes T^a \gamma_\mu \otimes \gamma_\mu$$

$$(G4) = -\tfrac{1}{4} g^3 I_2^2 \left(16 + d(d-2)\right) T^a T^b T^c \otimes \left(T^a T^b T^c + T^c T^b T^a \right) \gamma_\mu \otimes \gamma_\mu$$

$$(G5) = 2 g^3 I_2^2 \left(1 + \tfrac{9}{8}\varepsilon \right) T^a T^b T^c \otimes \left(T^a T^c T^b + T^b T^a T^c + T^c T^a T^b + T^b T^c T^a \right) \gamma_\mu \otimes \gamma_\mu$$

$$G(6) = 2 g^3 C_A \hat{I}_2 I_2 \left[\tfrac{1}{4} C_A (3d-2) T^a \otimes T^a \gamma_\mu \otimes \gamma_\mu \right] + \tfrac{3}{2} g^3 \varepsilon \hat{I}_2 I_2 \cdot$$

$$\cdot T^a T^b T^c \otimes \left(2 T^a T^b T^c + 2 T^c T^b T^a + T^a T^c T^b + T^b T^a T^c + T^c T^a T^b + T^b T^c T^a \right)$$

The second part of (G6) comes from the insertion of the operator O_3. If we denote the group factors by

$$\begin{aligned} F_1 &= T^aT^bT^c \otimes \left(T^aT^bT^c + T^cT^bT^a\right) \\ F_2 &= T^aT^bT^c \otimes \left(T^aT^cT^b + T^bT^aT^c + T^cT^aT^b + T^bT^cT^a\right) \\ 2F_1 - F_2 &= \frac{1}{2}C_A^2 T^a \otimes T^a \end{aligned} \tag{C.23}$$

the sum $(G4)+(G5)+(G6)$ takes the value

$$g^3\left[2\hat{I}_2 I_2 - I_2^2\right] C_A^2 T^a \otimes T^a \gamma_\mu \otimes \gamma_\mu + \frac{3}{4} g^3 \varepsilon I_2^2 \left(2F_1 + F_2\right) \gamma_\mu \otimes \gamma_\mu \tag{C.24}$$

We see that these graphs do not contribute to the β function of the relevant coupling O_{Th} but instead give a renormalization of the operator

$$\begin{aligned} \mathcal{F} &= \frac{1}{2}\bar{\psi} T^aT^bT^c \gamma_\mu \psi \bar{\psi}\left(2T^aT^bT^c + 2T^cT^bT^a\right. \\ &+ \left. T^aT^cT^b + T^bT^aT^c + T^cT^aT^b + T^bT^cT^a\right)\gamma_\mu \psi \\ \beta_{\mathcal{F}} &= -\frac{3}{4}\frac{1}{4\pi^2} g^3 \end{aligned} \tag{C.25}$$

Taking into account of the wave function renormalization, the graphs (G1-G3) give

$$\beta_{Th}^{(2)} = \frac{1}{2\pi^2} g^3 C_A l_R \tag{C.26}$$

Therefore, up to two loops, the renormalization group flow is driven by the operator

$$\left(\beta_{Th}^{(1)} + \beta_{Th}^{(2)}\right) O_{Th} + \beta_3^{(1)} O_3 + \beta_{\mathcal{F}}^{(1)} \mathcal{F} \tag{C.27}$$

We have to work out the reduction formula for the operator $\beta_3^{(1)} O_3$. The computation is essentially identical to the insertion of the counterterms proportional to O_3 in the graph $(G6)$ and gives

$$\beta_3^{(1)} O_3 = \frac{3}{4}\frac{1}{4\pi^2} g^3 \mathcal{F} \tag{C.28}$$

which exactly cancel the spurious contribution in eq.(C.27). Therefore the renormalization group is driven by the β function

$$\beta_{Th} = -\frac{1}{2\pi} g^2 C_A \left(1 - \frac{l_R}{\pi} g\right) \tag{C.29}$$

In order to compare with the results presented in chapter 4 we must recall that

$$O_{Th} = -l_R(O_+ + \frac{1}{N}O_V) \tag{C.30}$$

therefore the relationship between the coupling g and g_+ is

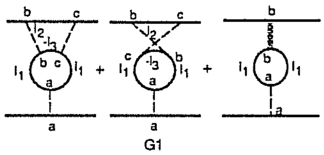
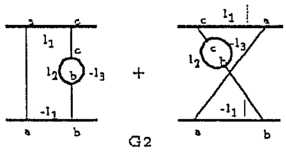
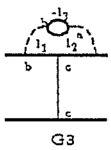
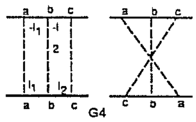
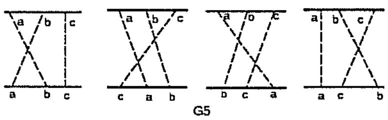
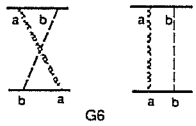
$$g_+ = l_R g \tag{C.31}$$

and consequently

$$\beta(g_+) = l_R \beta_{Th}(\frac{g_+}{l_R}) = -\frac{1}{2 l_R \pi} g_+^2 C_A (1 - \frac{g_+}{\pi}) \tag{C.32}$$

which leads to eq.(4.2.28) for $l_R = 1/2, C_A = N$.

Table C.1: Divergent vertex graph for the $SU(N)$ Thirring model

G1			
G2			
G3		G4	
G5		G6[1]	

1. G6 denotes the contribution of one loop counterterm.

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