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of

"PHILOSOPHIAE DOCTOR"

GLOBAL OPERATOR FORMULATION OF QUANTUM CONFORMAL FIELD THEORIES ON RIEMANN SURFACES

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To E. and V.

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CHAPTER I:

MATHEMATICAL BACKGROUND

PART A: Some topics on Riemann surfaces[1,2,3]

1. General information

Let Σ be a two dimensional topological space. A holomorphic atlas on Σ is an open covering $\{U_\alpha\}$, $\cup U_\alpha=\Sigma$, with associated charts or patch coordinates (homeomorphisms)

$$\phi_\alpha:\ \Sigma\supset U_\alpha \longrightarrow V_\alpha\subset C$$

in such a way that for any overlap $U_\alpha\cap U_\beta\neq\phi$, the *transition functions*

$$f_{\alpha\beta}=\phi_\alpha\circ\phi_\beta^{-1}:\ V_\beta\supset\phi_\beta(U_\alpha\cap U_\beta) \longrightarrow \phi_\alpha(U_\alpha\cap U_\beta)\subset V_\alpha$$
$$z_\beta \longrightarrow z_\alpha=f_{\alpha\beta}(z_\beta)$$

are holomorphic functions. Two atlas are said to be equivalent if the charts of one of them are holomorphic in the sense explained with respect to the charts of the other and viceversa. An equivalence class of atlas defines a complex structure on Σ.

A 1-dimensional complex manifold, called algebraic curve or Riemann surface (RS), is a two dimensional connected manifold which admits a complex structure.

Let us consider a RS Σ. Locally, a *line bundle* on Σ is given by the cartesian product $U_\alpha x C$. The local cartesian products are put together by giving a *transition function* $g_{\alpha\beta}$ on each non-empty overlap $U_\alpha\cap U_\beta$. The transition function is a complex valued non-vanishing function on the overlap. The object that one constructs by putting together this collection of cartesian products is called a complex line bundle. If the transition functions are analytic, it is said to be a *holomorphic line bundle*. The set of holomorphic line bundles over Σ is a group under the tensor product $\otimes$ known as the Picard group Pic(Σ).

A *section* of a line bundle ζ is given by a collection $\{\Psi_\alpha\}$ of locally defined complex valued functions such that on the overlaps they are related by the appropiate transition function

$$\Psi_\alpha = g_{\alpha\beta}\Psi_\beta$$

On a non-empty triple overlap $U_\alpha\cap U_\beta\cap U_\gamma$ the transition functions must satisfy the consistency "cocycle" condition $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}=1$.

In all this work we will restrict ourselves to compact oriented RS without boundary. Any such a surface Σ is completely characterized topologically after we give its Euler number $\kappa(g)=2(1-g)$, where g is the number of handles or "holes" known as the genus of Σ.

Let us introduce on Σ a basis for the first homology group $H_1(\Sigma,Z)=Z^{2g}$, showed in Fig.I.1. This basis has the property that the intersection pairing of cycles (closed curves) satisfies

$$(a_i,a_j) = (b_i,b_j) = 0$$
$$(a_i,b_j) = -(b_i,a_j) = \delta_{ij} \quad , \quad i,j=1,\ldots,g \tag{I.1}$$

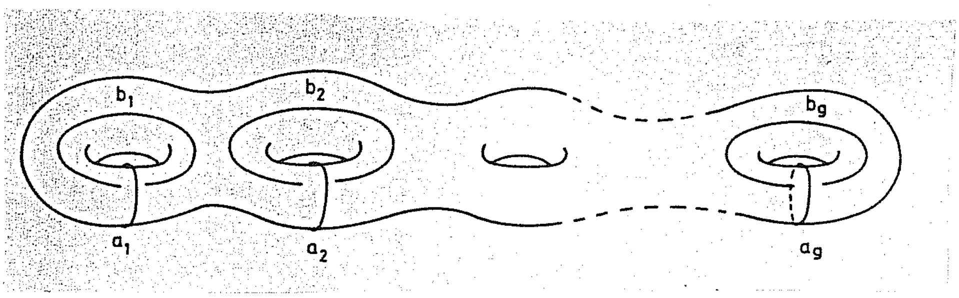

Fig.I.1: A canonical homology basis for a genus g Riemann surface.

Any basis satisfying (I.1) is called "canonical". In terms of it any cycle can be written as $\gamma=n^t a+m^t b$ with $(n,m)\in Z^{2g}$, and therefore

$$(\gamma,\gamma') = (n,m)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} n' \\ m' \end{pmatrix} \tag{I.2}$$

This quadratic form counts the number of intersections (including orientation) between γ and γ'.

Now, by the Hodge-De Rham theorem one can also find a set of harmonic 1-forms α_i, β_i, i=1,...g, dual to the homology basis

$$\oint_{a_j} \alpha_i = \oint_{b_j} \beta_i = \delta_{ij} \quad , \quad \oint_{a_j} \beta_i = \oint_{b_j} \alpha_i = 0 \tag{I.3}$$

But, by making use of the complex structure on Σ, we can combine the closed forms $\{\alpha_i,\beta_i\}$ into g holomorphic and antiholomorphic differentials $\{\eta^i,\bar{\eta}^i\}$; like $\{\alpha_i,\beta_i\}$ they are determined once a canonical basis or "marking" has been chosen. The standard way of normalizing the η^i's is to require

$$\oint_{a_j} \eta^i = \delta_{ij} \tag{I.4a}$$

Then the periods over the b-cycles are completely determined

$$\oint_{b_j} \eta^i = \tau_{ij} \tag{I.4b}$$

$\tau=\tau_1+i\tau_2$ is called the "period matrix" of Σ. It is possible to prove that is symmetric with $\tau_2=\mathrm{Im}\tau>0$, i.e., $\tau \in H^g$, the (left) Siegel half-plane. It can also be shown that no two inequivalent RS have the same τ (Torelli's theorem), so we can use H^g to parametrize surfaces. However this is a highly redundant description, since the same surface with two different markings will in general have two different matrices τ. Suppose the two canonical bases are related by

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{I.5}$$

where A,B,C and D $\in Z^{g\times g}$. To preserve the convention (I.1) the matrix in (I.5) must leave the symplectic form (I.2) invariant, i.e., it must be a symplectic matrix with integer coefficients, an element of Sp(2g,Z). This is called the Siegel modular group.

It is easy to compute the new η^i's (normalized according to (I.4)) in the new basis (I.5)

$$\eta' = [(C\tau+D)^{-1}]^t\eta \tag{I.6a}$$

The corresponding period matrix will be

$$\tau' = (A\tau+B)(C\tau+D)^{-1} \tag{I.6b}$$

One can verify by using the defining properties of a symplectic matrix that τ' is again an element of H^g. Since period matrices related by (I.6b) refer to the same RS, we can define $A^g=H^g/Sp(2g,Z)$, with every point on A^g representing an orbit of Sp(2g,Z) in H^g. To summarize, using the map (I.4b) and the equivalence (I.6b) we see that every RS is represented by a point in A^g; thus M^g, the *moduli space* of genus g RS, sits inside A^g. On the other hand, no every matrix in H^g corresponds in general to some surface Σ. Indeed the moduli space M^g is in general much smaller than A^g. For g=0,1,2,3 however it is true; the (complex) dimension of A^g is g(g+1)/2, equal to 0,1,3,6 in these cases, and coincides with the dimension of the moduli space. For g>2, $\dim M^g=3g-3$ and then $\dim M^g<\dim A^g$ for g>3. The problem of finding the subset $T^g \subset H^g$ called Teichmüller space ($M^g=T^g/Sp(2g,Z)$) corresponding to period matrices of RS is known as the Schottky problem, and was recently solved in Ref.[4].

We can now describe the group of disconnected diffeomorphisms of Σ. Let Diff(Σ) be the full group of orientation preserving diffeomorphisms on Σ, and $\mathrm{Diff}_0(\Sigma)$ the normal subgroup of diffeomorphisms connected to the identity. The quotient group $\Omega(\Sigma)=\mathrm{Diff}(\Sigma)/\mathrm{Diff}_0(\Sigma)$ is known as the *mapping class group* (MCG). Any non trivial element in $\Omega(\Sigma)$ is called a modular transformation and all are generated by the "Dehn twists". A Dehn twist around a non contractible loop $\gamma\in\Sigma$ is constructed as follows: given γ, choose a neighborhood of γ that is topologically equivalent to a cylinder. We can now cut Σ along γ, and keeping one of the edges of the cut fixed, we can twist the other by 2π and glue them together again. By this construction we can associate to every point of the original torus a new point, in a way which is smooth and yet clearly not continuously related to the identity map. There is a deep theorem that states that for any class in $\Omega(\Sigma)$ (modular transformation) we can always choose a representative given by a Dehn twist.

We can represent the Dehn twists in terms of matrices, writing their action on the homology basis. Let D_γ be the diffeomorphism defined by twisting around γ. The intersection matrix (I.1,2) is manifestly invariant under diffeomorphisms, so the matrix $M(D_\gamma)$ representing D_γ must be an element of Sp(2g,Z). In fact the set of matrices $M(D_\gamma)$ generate all of Sp(2g,Z). However, a Dehn twist along a homologically trivial cycle, while non trivial, does not affect the homology class of any curve and so maps to the unit matrix. Such twists generate a subgroup $T(\Sigma)$ of $\Omega(\Sigma)$, the "Torelli group". The quotient group $\Omega(\Sigma)/T(\Sigma)$ is precisely Sp(2g,Z).

2. Differential geometry on Riemann surfaces

The local differential geometry on Σ is quite simple. It is a well-known fact that we can choose on any Σ isothermal coordinates where a given metric h has the form

$$h = \rho(x)(dx^1 \otimes dx^1 + dx^2 \otimes dx^2) = h_{z\bar{z}}(z,\bar{z})(dz\otimes d\bar{z} + d\bar{z}\otimes dz)$$

$$z = x^1 + ix^2 \quad , \quad \rho(x) = 2\, h_{z\bar{z}}(z,\bar{z}) \tag{I.7}$$

Metrics related by multiplication of a scalar function define a conformal structure on Σ. It is easy to prove that complex structures and conformal structures on Σ are in 1-1 correspondence.

A metric h defines an integrable two-form H on Σ, locally given by

$$H = i dz\wedge d\bar{z}\; h_{z\bar{z}}(z,\bar{z}) = dx^1\wedge dx^2\; \rho(x) \tag{I.8}$$

From (I.7) it follows that we can always decompose an arbitrary tensor into its irreducible components with respect to O(2), the frame group of the surface. These can all be written in terms of tensors of the form

$$t = t(z,\bar{z})\; dz^p \quad , \quad p=0,\pm1,\pm2,\ldots \tag{I.9}$$

$t(z,\bar{z})$ transforms across the coordinate patches in such a way that t is invariant, that is, with transition functions $g_{\alpha\beta}=(\partial z_\alpha/\partial z_\beta)^{-p}$. More generally, an object of the form $t=t(z,\bar{z})dz^p d\bar{z}^q \in A_{(p,q)}(\Sigma)$ is said to be a tensor of rank (p,q), d=p+q and s=p-q being its "scaling dimension" and "spin" respectively.

In isothermal coordinates there are two non-vanishing components of the Christoffel symbols, the "hermitian connection" of h

$$\omega_z \equiv \Gamma^z_{zz} = \partial_z \log h_{z\bar{z}} \tag{I.10}$$

and its "complex conjugate" $\omega_{\bar{z}}=\partial_{\bar{z}}\log h_{z\bar{z}}$ (we will consistently omit "antiholomorphic" parts in what follows not to be reiterative). The tensors (I.9) are sections of the line bundle $A_{(p,0)}(\Sigma)=K^p$, where K is the holomorphic cotangent bundle $K\equiv A_{(1,0)}(\Sigma)$, the set of 1-forms with no $\bar{z}$ indices. K is called the "canonical bundle" of Σ. The covariant derivative acting on p-differentials is then the operator ($h^{z\bar{z}}=(h_{z\bar{z}})^{-1}$)

$$\nabla^p : K^p \longrightarrow K^{p+1}$$

$$t \longrightarrow \nabla^p t = \nabla^p_z t(z,\bar{z})\ dz^{p+1} = (h_{z\bar{z}})^p \partial_z ((h^{z\bar{z}})^p t(z,\bar{z}))dz^{p+1} \tag{I.11a}$$

and its adjoint (the "Cauchy-Riemann" operator $\bar{\partial}$ acting on p-differentials) with respect to the usual scalar product

$$<t|t'> = \int_\Sigma idz\wedge d\bar{z}\ (h_{z\bar{z}})^{1-d}\ t(z,\bar{z})\ t'(z,\bar{z})^* \tag{I.11b}$$

is $\nabla^{p+} = -\nabla_p$, with

$$\nabla_p : K^p \longrightarrow K^{p-1}$$

$$t \longrightarrow \nabla_p t = \nabla^z_p t(z,\bar{z})\ dz^{p-1} = h^{z\bar{z}}\ \partial_{\bar{z}} t(z,\bar{z})dz^{p-1} \tag{I.11c}$$

Using (I.11) we can construct two laplacians acting on p-differentials

$$\Delta^{(+)}_p = \nabla^z_{p+1}\nabla^p_z \quad , \quad \Delta^{(-)}_p = \nabla^{p-1}_z \nabla^z_p \tag{I.12a}$$

satisfying

$$\Delta^{(+)}_p - \Delta^{(-)}_p = [\nabla^z, \nabla_z]\big|_{p\text{-diff.}} = s/2\ R^h \tag{I.12b}$$

where R^h is the scalar curvature of (I.7)

$$R^h = -2\ h^{z\bar{z}} \partial_z \partial_{\bar{z}} \log h_{z\bar{z}} \tag{I.13}$$

3.Divisors, line bundles and the Riemann-Roch theorem

A *divisor* D on Σ is a set of points $P_i \in \Sigma$ with associated multiplicities $n_i \in Z$. It is usually denoted as a formal sum $D=\sum n_i P_i$. A divisor is said to be *positive* if $n_i > 0\ \forall i$. The *degree* of D is by definition the number $\deg D = \sum n_i$.

Given a section s of a holomorphic line bundle ζ, we can associate to it a divisor given by the set of zeros $\{P_i\}$ and poles $\{Q_i\}$ with multiplicities positive and negative respectively

$$D(s) = \sum n_i P_i - \sum m_i Q_i \quad , \quad n_i, m_i \in N$$

Let us pick up a canonical homology basis (I.1) on Σ with the associated period matrix τ. The *Jacobian torus* is defined by

$$J(\Sigma) = C^g/\Gamma(\Sigma) = \{\ [z]\ /\ [z]=\{z\in C^g\ /\ z\simeq z+v,\ v\in\Gamma(\Sigma)\}\ \}$$

where the *period lattice* $\Gamma(\Sigma)$ is

$$\Gamma(\Sigma) = \{v \in C^g: v=n+\tau m, \ (n,m) \in Z^{2g}\}$$

The Jacobi map $I:\Sigma \to J(\Sigma)$ is defined by

$$I(P) = \int_{P_0}^{P} \eta \quad , P_0, \ P \in \Sigma, \tag{I.14}$$

where P_0 is an arbitrary reference point on Σ. It is extended to arbitrary divisors by linearity. An important property of the Jacobi map is given by the Abel's theorem; it states that D is the divisor of a meromorphic function (section of K^0, the trivial tensor bundle) if and only if I(D)=0 and degD=0. Therefore (I.14) maps *divisor equivalence classes* [D] *to* $J(\Sigma)$, where two divisors D_1, D_2 are equivalent if (D_1-D_2) is the divisor of a meromorphic function. Often we will denote for compactness I(D) or [D] by D itself.

Thus, since two different sections s_1, $s_2 \in \zeta$ can be obtained by multiplication by a meromorphic function ($s_1/s_2 \in K^0$), we can associate a holomorphic line bundle with an equivalence class of divisors. Conversely, given a divisor D we can construct a line bundle as follows: if we restrict D to U_α, we can find a meromorphic function f_α on U_α whose divisor coicides with the restriction D_α. For two overlapping patches U_α, U_β, we have functions f_α, f_β. In the overlap $U_\alpha \cap U_\beta$, f_α and f_β have the same divisors. Thus we can define the transition function for a line bundle to be f_α/f_β. It is trivial to check that the cocycle conditions are satisfied. Hence we can think of holomorphic line bundles on Σ either in terms of transition functions, or in terms of divisors.

Another very useful result is the Jacobi inversion theorem: let Σ^g represent the set of positive divisors of degree g; then outside of a set of complex codimension 1 in $J(\Sigma)$, there is a one-to-one correspondence between positive divisors of degree g and points of the jacobian, that is

$$I(D) = \sum_{i=1}^{g} I(P_i) \equiv z \in J(\Sigma)$$

is invertible for almost all z.

Now, let us call $H^0(\Sigma,\zeta)$ the kernel of $\bar\partial$ acting on the space of sections of ζ; its dimension will therefore give the number of independent holomorphic sections of ζ, called usually "zero modes". The Riemann-Roch theorem can be formulated as follows

$$\dim H^0(\Sigma,\zeta) - \dim H^0(\Sigma,K\otimes\zeta^{-1}) = \deg\zeta - g + 1 \qquad (I.15)$$

Let us consider same simple applications of (I.15). By taking $\zeta\equiv K$, being $K\otimes K^{-1}$ the trivial bundle and therefore $\dim H^0(\Sigma,K\otimes K^{-1})=1$ (the constant) and $\dim H^0(\Sigma,K)=g$ (the g abelian differentials introduced in (I.4)), we learn: $\deg K=2g-2$ and then $\deg K^p=2p(g-1)$ (the degree is additive with respect to the tensor product). Now let us take $\zeta\equiv K^p$ with $p>1$, and $g>2$. In this case $\dim H^0(\Sigma,K^{1-p})=0$ because $\deg K^{1-p}=-(p-1)(2g-2)<0$ and therefore K^{1-p} can not have holomorphic sections, so (I.15) gives $\dim H^0(\Sigma,K^p)=(2p-1)(g-1)$ as the number of holomorphic p-differentials. We will return to (I.15) later, when we will discuss the Krichever-Novikov bases.

4. Spin bundles and spin structures

Geometrically, a spin bundle S is defined as a square root of the canonical bundle $S\simeq K^{1/2}$, i.e. as a bundle such that $S\otimes S=K$. In terms of transition functions, it corresponds to take the square roots of the transition functions of K $h_{\alpha\beta}=(\partial z_\alpha/\partial z_\beta)^{1/2}$ consistently with the cocyle condition $h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha}=1$. The question is whether such square roots exist and if they do one would like to know how many there are. We will show that there exist exactly 2^{2g} possibilities of choosing the signs for $h_{\alpha\beta}$, corresponding to 2^{2g} spin structures.

Let us consider the group of flat holomorphic line bundles (those with constant transition functions) $Pic_0(\Sigma)\subset Pic(\Sigma)$, and construct the "cut" surface $\tilde{\Sigma}$ as follows: choose a point A on Σ, and draw curves starting at A and homotopic to the canonical homology basis of Fig.I.1 (Fig.I.2a); then cut the surface along these curves and unfold it, obtaining a 4g-sided polygon (Fig.I.2b). If we assign transition functions across the cuts, then every flat bundle on Σ is equivalent to one whose transition functions are all constant phases, one for each homology generator. For the choosen basis, if s is a section of a general flat line bundle L, we identify the section s along a_i with $\exp(-i2\pi\phi_i)$ times s along a_i^{-1}, and s along b_i with $\exp(i2\pi\theta_i)$ times s along β_i^{-1}. Since $0\le\phi_i,\theta_i<1$, flat holomorphic line bundles are parametrized by a torus called the "Picard torus" of the surface $Pic_0(\Sigma)=R^{2g}/Z^{2g}$. Now let us consider $L=S_\alpha\otimes S_\beta^{-1}\subset Pic_0(\Sigma)$ where S_α and S_β are two spin bundles. Because $S_\alpha^2=S_\beta^2=K$, L^2 is trivial $(2I(D(L))=0)$, and therefore the difference between two spin structures is parametrized by a point of order two in $Pic_0(\Sigma)$. Since R^{2g}/Z^{2g} has 2^{2g} points of order two, we conclude that this is also the number of spin structures.

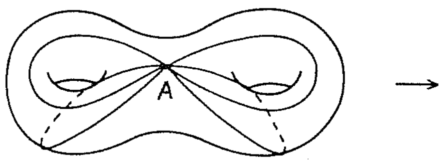

Fig.I.2a: Cutting a Riemann surface along a homology basis

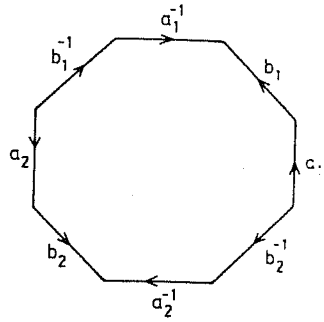

Fig.I.2b: The "cut" surface for g=2

PART B: Theta functions on Riemann surfaces[5,6,7]

5. Theta functions: definition and properties

Let us consider a gxg complex matrix $\tau \in H^g$. Any point in $e \in C^g$ can be written uniquely as $e=(\beta,\alpha)\begin{pmatrix}1\\ \tau\end{pmatrix}$ where 1 stands for the identity gxg matrix; $(\beta,\alpha)\in R^{2g}$ are the characteristics of e.

The first order θ-function with characteristics $[^{\alpha}_{\beta}]$ is defined by its Fourier series of the form

$$\theta[^{\alpha}_{\beta}](z|\tau) = \sum_{N\in Z^g} \exp(i\pi(N+\alpha)^t\tau(N+\alpha) + i2\pi(N+\alpha)^t(z+\beta))$$
$$= \exp(i\pi\alpha^t\tau\alpha + i2\pi\alpha^t(z+\beta))\ \theta(z+e|\tau) \quad ,$$
$$\theta(z|\tau) = \theta[^0_0](z|\tau) \quad , z \in C^g,\ \alpha,\beta \in R^g,\ \tau\in H^g \qquad (I.16)$$

We will use the notation $\theta(z|\tau)\equiv\theta(z)$ for compactness.

It follows from the definition (I.16) the transformation law

$$\theta[^{\alpha}_{\beta}](z+n+\tau m) = \exp(-i\pi m^t\tau m - i2\pi m^t z + i2\pi(\alpha^t n - \beta^t m))\ \theta[^{\alpha}_{\beta}](z) \qquad (I.17)$$

Indeed we can define the θ-function as the (unique up to multiplicative constant) section of a holomorphic line bundle (θ line bundle) over $J(\Sigma)$ with transition functions given by (I.17). It also follows that $\theta[^{\alpha+N}_{\beta+M}](z)=\exp(i2\pi\alpha^t M)\theta[^{\alpha}_{\beta}](z)$, $(N,M)\in Z^{2g}$, so it suffices to consider the characteristics (α,β) such that $0\le\alpha_i,\beta_i<1$, $i=1,\dots g$.

The characteristics (α,β) such that α_i,β_i are 0 or 1/2 are called half-periods; the corresponding 2^{2g} θ-functions are even or odd depending on the parity of $4\alpha^t\beta$ as it can be easily checked from (I.16). They are closely related to the spin structures of Σ (see Section 7).

6. Multivalued functions on Riemann surfaces

A fundamental theorem in θ-function theory is the *Riemann vanishing theorem*. It states that the function

$$F(P) = \theta(I(P) - \sum_{i=1}^{g} I(P_i) + \Delta) \tag{I.18}$$

either vanishes identically or it has exactly g simple zeros in $P=P_i$, $i=1,\dots,g$. $\Delta=(\Delta_k)$ in (I.18) is the Riemann constant vector defined by

$$\Delta_k = (1+\tau_{kk})/2 + \sum_{\substack{l=1\\ l\neq k}}^{g} \oint_{a_l} \eta_l(P) \int_{P_0}^{P} \eta_k \tag{I.19a}$$

and depends on P_0 and the marking. It will be shown in the next section that there is a particular spin bundle $S_{(0,0)}$ such that its divisor class $D_{(0,0)}$, called the Riemann class, satisfies

$$I(D_{(0,0)}) = \Delta + (g-1)I(P_0) \tag{I.19b}$$

and is independent of P_0. The Riemann class, as the divisor class of any spin bundle, is related to the canonical line bundle by $K=2D_{(0,0)}$. As a useful corollary of this theorem, we have

$$\theta(-\sum_{i=1}^{g-1} I(P_i) + \Delta) = 0 \quad , \ \forall\ (P_i) \in \Sigma^{g-1} \tag{I.20}$$

When one studies complex function theory on the sphere, the basic building blocks are monomials of the form $(z-z_i)$. The analogous object for an arbitrary RS is the "prime form" E(P,Q). It is a multivalued -1/2-differential without poles in both variables P and Q with a unique simple zero for Q=P. Let us sketch its construction. Consider the function

$$\theta[^{\alpha_o}_{\beta_o}](I(P-Q)) \tag{I.21a}$$

with $[^{\alpha_o}_{\beta_o}]$ a non-singular ($\partial z_i \theta[^{\alpha_o}_{\beta_o}](0)\neq 0$ for some i) odd characteristic. Keeping Q fixed it will vanish as a function of P in (g-1) points P_i, $i=1,\dots,g-1$, according to (I.18) and Jacobi inversion theorem. Similarly, as a function of Q, keeping P fixed, it will also vanish at the same points. But being the characteristic odd, it also vanishes for P=Q. If we now take P and Q very close to each other and to one of the P_i's, then (I.21a) behaves like

$$(P-Q)(P-P_i)(Q-P_i) \tag{I.21b}$$

Thus, if we differentiate (I.21a) with respect to P and then set Q=P, we obtain a holomorphic 1-form

$$h_0(P)^2 = d_P\theta[^{\alpha_o}_{\beta_o}](I(P-Q))\Big|_{Q=P} = \sum_{i=1} \partial z_i \theta[^{\alpha_o}_{\beta_o}](0)\ \eta^i(P) \tag{I.22}$$

But from (I.21b) we know that $h_0(P)^2$ has *double* zeros at the P_i's and does not vanish or blows up anywhere else. Consequently we can take its square root $h_0(P)$ without any fear of introducing cuts on Σ. Then $h_0(P)$ will be a holomorphic section of a spin bundle, and the prime form is given by

$$E(P,Q) = \theta[{}^{\alpha_o}_{\beta_o}](I(P-Q))/(h_0(P)h_0(Q)) = -E(Q,P) \quad ,$$
$$E(P,Q) \approx P-Q, \text{ as } Q \approx P, \qquad (I.23)$$

It can be shown that $E(P,Q)$ is independent of the particular choice of $[{}^{\alpha_o}_{\beta_o}]$. For cycles winding around P, it transforms as (see (I.17))

$$E(P,Q) \xrightarrow{P+na+mb} \exp(-i\pi m^t\tau\, m - i2\pi\, m^t I(P-Q))\, E(P,Q) \qquad (I.24)$$

Finally, we introduce the σ-differential

$$\sigma(P) = \exp(\, - \sum_{j=1}^{g} \oint_{a_j} \eta_j(Q) \ln E(Q,P)\,) \qquad (I.25a)$$

It is a (multivalued) g/2-differential without zeroes and poles, a section of a trivial line bundle. Its transformation property is

$$\sigma(P) \xrightarrow{P+na+mb} \exp(i\pi(g-1)m^t\tau\, m - i2\pi\, m^t(\Delta-(g-1)I(P)))\, \sigma(P) \qquad (I.25b)$$

7.Theta divisor and spin structures

The set $\Theta = \{z:\theta(z)=0\}$ is a variety of complex codimension one in $J(\Sigma)$ called Θ-*divisor*.

Let ζ be a degree (g-1) line bundle with a holomorphic section; then in its corresponding divisor class $[\zeta]$ there is a positive divisor $\sum_{i=1}^{g-1} P_i$. By eq.(I.15) $\dim H^0(\Sigma,\zeta)=\dim H^0(\Sigma,K\otimes\zeta^{-1})$ and therefore $K\otimes\zeta^{-1}$ has a divisor of the form $\sum_{i=1}^{g-1} Q_i$, verifying $I(\sum_{i=1}^{g-1} P_i)+I(\sum_{i=1}^{g-1} Q_i)=I(K)$. Let $D_{(\alpha\beta)}$ be a spin bundle, $2D_{(\alpha\beta)}=K$; then the set

$$S_{(\alpha\beta)} = \{I(\sum_{i=1}^{g-1} P_i - D_{(\alpha\beta)}) \;/\; P_1,\ldots, P_{g-1} \in \Sigma\} \subset J(\Sigma) \qquad (I.26)$$

is a symmetric subset with respect to the origin of $J(\Sigma)$ because

$$-I(\sum_{i=1}^{g-1} P_i - D_{(\alpha\beta)}) = I(\sum_{i=1}^{g-1} Q_i - D_{(\alpha\beta)}) \in S_{(\alpha\beta)}$$

From the corollary (I.20) as the points P_i sweep Σ we recover Θ

$$\Delta - \{ I(\sum_{i=1}^{g-1} P_i) \ / \ P_1, \ldots, P_{g-1} \in \Sigma \} = \Theta$$

Therefore

$$S_{(\alpha\beta)} = \gamma_{(\alpha\beta)} - \Theta \quad , \quad \gamma_{(\alpha\beta)} = \Delta - I(D_{(\alpha\beta)}) \tag{I.27}$$

Since Θ and $S_{(\alpha\beta)}$ are both symmetric subsets with respect to the origin of $J(\Sigma)$, we have

$$\Theta + 2\gamma_{(\alpha\beta)} = \Theta \tag{I.28}$$

This means that $\theta(z+2\gamma_{(\alpha\beta)})/\theta(z)$ is a constant on the compact space $J(\Sigma)$, and then (I.17) implies that $2\gamma_{(\alpha\beta)} \in \Gamma(\Sigma)$, that is, each $\gamma_{(\alpha\beta)}$ is one of the 2^{2g} points of order two. Being 2^{2g} also the number of spin structures, it follows from Abel's theorem that for each half-point of $J(\Sigma)$ there is a different $D_{(\alpha\beta)}$ and viceversa. Since we can write $\gamma_{(\alpha\beta)}=\beta+\tau\alpha$, then

$$\Theta_{(\alpha\beta)} \equiv \{z:\theta[{}^{\alpha}_{\beta}](z)=0\} = \Theta + \gamma_{(\alpha\beta)} \quad , \quad \alpha_i, \ \beta_i \in \{0,1/2\}$$

$$I(D_{(\alpha\beta)}) = \Delta - \beta - \tau\alpha \ . \tag{I.29}$$

Noting that

$$S_{(0,0)} = \{ I(\sum_{i=1}^{g-1} P_i - D_{(00)}) \ / \ (P_i) \in \Sigma^{g-1} \} = \Theta \tag{I.30}$$

and that θ is P_0-independent we see that $D_{(00)}$ depends only on the homology basis chosen. This particular spin structure is precisely the Riemann class introduced in (I.19b). From (I.27) it follows that there is a one-to-one correspondence between degree (g-1) line bundles for which $\bar{\partial}_\zeta$ has a zero mode (that is, the line bundles $\{I(\sum_{i=1}^{g-1} P_i)/ \ (P_i) \in \Sigma^{g-1}\}$) and points in Θ. Moreover, it turns out that $\dim H^0(\Sigma,\zeta)$ equals the multiplicity of the zero of $\theta(z)$ at $z=I(\Delta)-I(\zeta)$.

For odd characteristics $[{}^{\alpha}_{\beta}]$, $0 \in \Theta_{(\alpha\beta)}$ so that there is at least a set of points $P_1, \ldots, P_{g-1}$ such that

$$I(\sum_{i=1}^{g-1} P_i - D_{(\alpha\beta)}) = 0 \cdot \tag{I.31}$$

that is, $D_{(\alpha\beta)}$ has at least a holomorphic section with zeroes at $z=P_i$, $i=1,\ldots,g-1$, the $h_0(P)$ constructed in Section 6 and given explicitely by (I.22). In the case of even theta functions there are certain values of τ for which $0 \in \Theta_{(\alpha\beta)}$, for example for g=2 this happens when the period matrix is diagonal.

PART C: The Krichever-Novikov bases

8. Definition and explicit construction

In Ref.[8] Krichever and Novikov (KN) introduced bases for meromorphic λ-differentials on genus g Riemann surfaces which are holomorphic outside two distinguished points P_+ and P_-. These bases are uniquely determined up to numerical constants as a consecuence of the Riemann-Roch theorem. They are the generalization of the bases $\{z^{n-\lambda}dz^{\lambda}\}$ over the sphere, the points $P_+(P_-)$ playing the role of $z=0$ $(=\infty)$. Thus the KN bases provide a mean to globally Laurent-like expand any tensor field of the type described before on genus g RS. This fact will lend itself to operatorially formulate conformal field theories on RS, as we will see in the next chapter.

Case of integer λ

For integer $\lambda>1$ and $g>1$, the Riemann-Roch theorem guarantees the existence and uniqueness (up to a multiplicative constant) of tensors of rank $(\lambda,0)$ which in a neighborhood of P_+ and P_- have the following behaviour

$$f_j^{(\lambda)}(z_\pm) = \varphi_j^{(\lambda)\pm} \; z_\pm^{\pm j-s(\lambda)} \; (1+o(z_\pm)) \; (dz_\pm)^{\lambda}$$

$$s(\lambda) = g/2 - \lambda\,(g-1) <0 \qquad \text{(I.32)}$$

where $z_\pm(P_\pm)=0$, $z_\pm$ being local coordinates in $P_\pm$. The multiplicative constant may be fixed by requiring for example $\varphi_j^{(\lambda)+}=1$. The index j in eq.(I.32) takes either integer or half-integer values depending on whether g is even or odd, respectively. When $j=s(\lambda),\ldots,-s(\lambda)$ we have a basis for zero modes of the $\bar{\partial}$-operator acting on λ-differentials. These are in number $-2s(\lambda)+1=(2\lambda-1)(g-1)$, reproducing the result of Section 3.

The existence and uniqueness of the sections (I.32) can be proved as follows: let us rewrite (I.15) as

$$\dim H^0(\Sigma,K^{\lambda}\otimes\mathcal{O}(-D)) = \dim H^0(\Sigma,K^{1-\lambda}\otimes\mathcal{O}(D)) - \deg D - 2s(\lambda) + 1 \tag{I.33}$$

where for our purposes $\mathcal{O}(D)$ is the line bundle corresponding to the divisor D. An useful tool is given by the following lemma (see Ref.[9] for an easy proof): given two divisors Y and X, if $0\le\deg X\le\dim H^0(\Sigma,K^{\lambda}\otimes\mathcal{O}(-Y))$

$$\dim H^0(\Sigma,K^{\lambda}\otimes\mathcal{O}(-Y-X)) = \dim H^0(\Sigma,K^{\lambda}\otimes\mathcal{O}(-Y)) - \deg X \tag{I.34}$$

being zero if $\deg X\ge\dim H^0(\Sigma,K^{\lambda}\mathcal{O}(-Y))$. Let us take $\deg Y=-2s(\lambda)-g+1$; then $\deg K^{1-\lambda}\otimes\mathcal{O}(Y)=-1<0$ and from (I.33) we learn $\dim H^0(\Sigma,K^{\lambda}\otimes\mathcal{O}(-Y))=g$. By taking now $\deg X=g-1$ we obtain from (I.34)

$$\dim H^0(\Sigma,K^{\lambda}\otimes\mathcal{O}(-D)) = 1 \tag{I.35}$$

for any divisor D such that $\deg D=\deg Y+\deg X=-2s(\lambda)$. If $D=(-j+s(\lambda))P_+ +(j+s(\lambda))P_-$, then $\dim H^0(\Sigma,K^{\lambda}\otimes\mathcal{O}(-D))$ counts precisely the number of sections of the type (I.32). Analogous steps considering $(1-\lambda)<0$ in (I.33) prove the existence and uniqueness of the sections $f_j^{(1-\lambda)}$.

Note also that once one proves the existence of (I.32) by constructing it (as we will do later) then uniqueness follows from the Noether "gap" theorem. In fact, let us assume there are two sections $f_j^{(\lambda)}$ and $f'^{(\lambda)}_j$ of K^{λ} satisfying (I.32) and we define a function h as the quotient of them. Since $f_j^{(\lambda)}$ and $f'^{(\lambda)}_j$ have g zeroes outside $P_{\pm}$, it follows that h is a meromorphic function with a number of poles between zero and g. But the Noether "gap" theorem states that there exist no meromorphic functions with a number of poles in general position between 1 and g[1]. This implies that h is necessarily holomorphic and therefore a constant.

For $\lambda=0$ the behaviour is modified with respect to eq.(I.32) due to the Weirstrass gap theorem[2]. Let A_j, $|j|\ge g/2+1$, be the

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[1]The restricted version: "there is for general $P\in\Sigma$ no meromorphic function with a pole of order $n\le g$ in P" is known as Weierstrass gap theorem; the particular points where it does are called Weierstrass points.

[2]The proof of the assertions about existence and uniqueness of the sections introduced in the remainder of this section will not be presented here (see [9]).

unique function which in a neighborhood of $P_\pm$ has the Laurent expansion

$$A_j(z_\pm) = \alpha_j^\pm z_\pm^{\pm j-g/2}(1+o(z_\pm)) \qquad (I.36)$$

(As before, j is integer or half-integer depending on the parity of g). For j=-g/2,...,g/2-1 we take the functions with the following behaviour

$$A_j(z_\pm) = \alpha_j^\pm z_\pm^{\pm j-g/2 {+0 \atop -1}}(1+o(z_\pm)) \qquad (I.37)$$

These conditions define A_j uniquely up to addition of a constant, because (I.37) fixes poles in both P_+ and P_-. For j=g/2 we choose $A_{g/2}$ to be the holomorphic section (the constant); this completes the basis of meromorphic functions.

For λ=1, we take the basis of one-forms as follows: in the range $|j|\geq g/2+1$, $\omega^j=f^{(1)}_{-j}$, with $f^{(1)}_{-j}$ given by (I.32); for j=-g/2,...,g/2-1, those specified by the local series

$$\omega^j(z_\pm) = \beta_j^\pm z_\pm^{\mp j+g/2 {-1 \atop +0}} (1+o(z_\pm))dz_\pm \qquad (I.38)$$

Finally, we take $\omega^{g/2}$ as the Abelian differential of the third kind with simple poles in $P_\pm$ and residues ± 1, in such way that its periods over all cycles be purely imaginary.

In the g=1 case, $\deg K^\lambda=0$ and therefore the number of zeros is equal to the number of poles for any section of K^λ for any λ; the existence of a holomorphic (and therefore without zeroes and poles) one-form η enables us to write the KN bases as follows

$$f_j^{(\lambda)} = \text{constant } A_j \ \eta^\lambda \qquad (I.39)$$

where the A_j's are defined as before.

Let us now move to the explicit construction of these bases[10]. Looking at (I.32), we observe that this behaviour is correctely reproduced by using prime forms as follows .

$$f_j^{(\lambda)} \approx E(P,P_+)^{j-s(\lambda)} / E(P,P_-)^{j+s(\lambda)}$$

The correct weight in the P-variable is obtained by mean of the σ-differential

$$f_j^{(\lambda)} \approx (E(P,P_+)^{j-s(\lambda)} / E(P,P_-)^{j+s(\lambda)}) \ \sigma(P)^{2\lambda-1}$$

Finally, we require $f_j^{(\lambda)}$ to be single-valued. To this purpose we introduce a θ-function

$$f_j^{(\lambda)}(P) = N_j^{(\lambda)}(P_+,P_-) \frac{E(P,P+)^{j-s(\lambda)}}{E(P,P_-)^{j+s(\lambda)}} \sigma(P)^{2\lambda-1} \theta(P+e(\lambda,j)) \tag{I.40}$$

where

$$e(\lambda,j) = (j-s(\lambda))P_+ - (j+s(\lambda))P_- + (1-2\lambda)\Delta \quad ,$$

and

$$N_j^{(\lambda)}(P_+,P_-) = \frac{E(P+,P-)^{j+s(\lambda)} \ \sigma(P+)^{1-2\lambda} \ h_0(P+)^{2(j-s(\lambda)+\lambda)}}{\theta(P_+ + e(\lambda,j))}$$

Note that the θ-function gives the g zeroes of $f_j^{(\lambda)}$ outside $P_\pm$; the constant $N_j^{(\lambda)}(P_+,P_-)$ is choosen in order to satisfy a duality condition to be defined below in (I.46)[3].

For g=1 or λ=0,1, eq.(I.40) does not work in the interval $-g/2\le j\le g/2$. In fact, in these cases the θ-function has zeroes in P_+ and P_- which cancel the poles of the prime forms, as it is easy to verify by using the Riemann vanishing theorem.

For λ=0 the expressions for any g are defined by

$|j|>g/2$:

$A_j(P)$ = as (I.40) with λ=0,

$-g/2\le j\le g/2-1$:

$$A_j(P) = N_j^{(0)}(P_\pm,P_{g+1}) \frac{E(P,P+)^{j-g/2}E(P,P_{g+1})}{E(P,P_-)^{j+g/2+1}} \sigma(P)^{2\lambda-1}\theta(P+e(j))$$
$$- a_j(P_\pm,P_{g+1}) \tag{I.41}$$

where

$$e(j) = (j-g/2)P_+ - (j+g/2+1)P_- + P_{g+1} + \Delta \quad ,$$

$$a_j(P_\pm,P_{g+1}) = \frac{1}{2\pi i} \oint_C (A_j(P) + a_j(P_\pm,P_{g+1}))\ \omega^{g/2} \quad ,$$

and

$$N_j^{(0)}(P_\pm,P_{g+1}) = \frac{E(P+,P-)^{j+g/2+1} \ \sigma(P+) \ h_0(P+)^{2(j-g/2)}}{\theta(P_+ + e(j)) \ E(P_+,P_{g+1})}$$

Here P_{g+1} is an arbitrary point different from $P_\pm$ (as we have

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[3] One can check that this choice of the coefficient $N_j^{(\lambda)}$ makes $f_j^{(\lambda)}$ single-valued and with the right weight in both P_+ and P_- variables.

already said, the A_j's defined by (I.41) are fixed up to addition of a constant; this arbitrariness is reflected in (I.41) through the point P_{g+1}, that can be fixed by requiring $a_j(P_\pm, P_{g+1})=0$), $\omega^{g/2}$ is defined below in eq.(I.43), and C is any countour which separates P_+ and P_-. Finally we take $A_{g/2}=1$.

For $\lambda=1$ the elements of the basis of 1-forms take the following form

$|j|>g/2$

$$\omega^j(P) = f^{(1)}_{-j} \quad , \text{ according to (I.40)}$$

$-g/2\le j\le g/2-1$[4]

$$\omega^j(P) = N^{(1)}_{-j}(P_\pm, P_{g+1}) \frac{E(P,P_-)^{j+g/2}\ \sigma(P)}{E(P,P_+)^{j-g/2+1} E(P,P_{g+1})}\ \theta(P+e(j)) \tag{I.42}$$

where

$$e(j) = (j+g/2)P_- - (j-g/2+1)P_+ - P_{g+1} - \Delta \ ,$$

and

$$N^{(1)}_{-j}(P_\pm, P_{g+1}) = \frac{E(P_+,P_-)^{-j-g/2}\ \ E(P_+,P_{g+1})\ h(P_+)^{-2(j-g/2)}}{\theta(P_+ + e(j))\ \sigma(P_+)}$$

$j=g/2$

$$\omega^{g/2}(P) = d[\ln(E(P,P_+)/E(P,P_-))] - $$
$$- 2\pi i \sum_{i,j=1}^{g} \mathrm{Im}\int_{P_-}^{P_+} \eta^i\ (\tau_2^{-1})_{ij}\ \eta^j(P) \tag{I.43}$$

In the genus one case, considering eq.(I.39) we can define the following expressions:

for $|j|>1/2$

$$f^{(\lambda)}_j(P) = N^{(\lambda)}_j(P_+,P_-)\ \frac{E(P,P_+)^{j-1/2}}{E(P,P_-)^{j+1/2}}\ \sigma(P)^{2\lambda-1}\ \theta(P+e(j,\lambda)) \tag{I.44}$$

where

$$e(j,\lambda) = (j-1/2)P_+ - (j+1/2)P_- + (1-2\lambda)\Delta \quad ,$$

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[4] Note that in eq.(I.42) we have no pole in P_{g+1} because the θ-function has a zero there which cancels the zero of $E(P,P_{g+1})$ for any $j=-g/2,\ldots,g/2-1$; in rigor from the Weierstrass gap theorem it follows that (I.42) does not depend on P_{g+1}.

and

$$N_j^{(\lambda)}(P_+,P_-) = \frac{E(P+,P-)^{j+1/2}\ \sigma(P+)^{2j}}{\theta(P_+ + e(j))} \qquad ;$$

for $|j|=1/2$ (the zero mode)

$f_{1/2}^{(\lambda)}(P) = \omega^{-1/2}(P)^{\lambda}$ according to (I.42),

and

$$f_{-1/2}^{(\lambda)}(P) = N_{-1/2}^{(\lambda)}(P_\pm,P_2)\ \frac{E(P,P_2)\ \sigma(P)^{2\lambda-1}}{E(P,P_+)\ E(P,P_-)}\ \theta(P-e) - c(P_\pm,P_2)\ f_{1/2}^{(\lambda)}(P) \qquad \text{(I.45)}$$

where

$$e = P_+ + P_- - P_2 - \Delta ,$$

$$c(P_\pm,P_2) = \left(\frac{E(P+,P-)\ \sigma(P+)^{-1}}{\theta(P_+-e)\ E(P_+,P_2)}\right)^2 \frac{1}{4\pi i}\oint_C \left(\frac{E(P,P_2)\ \theta(P-e)}{E(P,P_+)\ E(P,P_-)}\right)^2$$

and

$$N_{-1/2}^{(\lambda)}(P_+,P_-) = \frac{E(P+,P-)}{\theta(P_+-e)\ E(P_+,P_2)}$$

Duality relation

The dual section of $f_j^{(\lambda)}$, $f^j_{(1-\lambda)}$, is defined by the following duality relation

$$\frac{1}{2\pi i}\oint_C\ f_i^{(\lambda)}\ f^j_{(1-\lambda)} = \delta_i^{\ j} \quad , \qquad \text{(I.46)}$$

where C is a contour separating P_+ and P_- (due to the holomorphicity outside $P_\pm$ of the f_j's, the contour integration does not depend on the particular C choosen). The bases defined before satisfy this relation with $f^j_{(1-\lambda)}=f_{-j}^{(1-\lambda)}$ (in the particular case: $\lambda=0$, for $g\geq 2$ this relation is also verified by the A_j's and ω^j's given in (I.41-43)). The constants were choosed in order for (I.46) to hold.

Case of half-integer λ

Let us now consider sections of K^{λ} with a given spin structure $[\alpha,\beta]$. We are interested in two kind of bases:

i) Basis for the space of tensors of weight λ with the spin structure $[\alpha,\beta]$ which are holomorphic outside P_+ and P_- and a slit from P_+ to P_- ("Ramond (R) -type" bases);

ii) Basis for the space of tensors of weight λ with the spin structure $[\alpha,\beta]$ which are holomorphic outside P_+ and P_- ("Neveu-Schwarz (NS) -type" bases).

By Riemann-Roch theorem , there exists a unique (up to a normalization constant) section $f_n^{(\lambda)}$ which in neighborhoods of $P_\pm$ have the form (when the spin structure is odd, the following expression is slightly modified in the NS sector in the cases $\lambda=1/2$ with $|n|=1/2$ or $g=1$, see below)

$$f_n^{(\lambda)}(z_\pm) = a_n^{\pm} z_\pm^{\pm n-s(\lambda)}(1+o(z_\pm))(dz_\pm)^{\lambda} \tag{I.47}$$

where n takes integer values in the R case i), and half-integer values in the NS case ii).

Even though (I.47) looks like (I.32), there is however a difference due to the fact that the indices j or n, run in general over distinct values.

Let us now consider the NS sector with odd spin structure, and $\lambda=1/2$. If $n\neq\pm1/2$, then eq.(I.47) still holds. For $n=\pm1/2$ we take the sections as follows

$$f_{-1/2}^{(1/2)}(z_\pm) = a_{-1/2}^{\pm} z_\pm^{-1} (1+o(z_\pm))(dz_\pm)^{1/2}$$

$$f_{1/2}^{(1/2)}(z_\pm) = a_{1/2}^{\pm} (1+o(z_\pm))(dz_\pm)^{1/2} \tag{I.48}$$

Considering as before the NS sector, odd spin structure, but $g=1$, we can define for any half integer λ

$$f_n^{(\lambda)} = \text{constant } A_n \eta^{\lambda} , \tag{I.49}$$

where we take the spin structure of $\eta^{1/2}$ to be odd ($\eta^{1/2}\propto h_0$).

The explicit construction of theses bases is made in the same way as for integer λ, but now we have to take into account the spin structure. This is accomplished by introducing θ-functions with characteristics $[{\alpha \atop \beta}]$. Then, we write

$$f_n^{(\lambda)}(P) = N_n^{(\lambda)}(P_+,P_-) \frac{E(P,P+)^{n-s(\lambda)}}{E(P,P_-)^{n+s(\lambda)}} \sigma(P)^{2\lambda-1} \theta[^\alpha_\beta](P+e(\lambda,n))$$

(I.50)

where

$$e(\lambda,n) = (n-s(\lambda))P_+ - (n+s(\lambda))P_- + (1-2\lambda)\Delta \quad ,$$

$$N_n^{(\lambda)}(P_+,P_-) = \frac{E(P+,P-)^{n+s(\lambda)} \quad \sigma(P+)^{2n}}{\theta[^\alpha_\beta](P_+ + e(\lambda,n))}$$

For the particular case of $[^\alpha_\beta]$ corresponding to an odd spin structure, NS sector and $\lambda=1/2$ we have for $|n|=1/2$

$$f_{1/2}^{(1/2)}(P) = N_{1/2}^{(1/2)}(P_+,P_-) \; \theta[^\alpha_\beta](P-P_-)/E(P,P_-) \qquad \text{(I.51)}$$

where

$$N_{1/2}^{(1/2)} \; (P_+,P_-) = \frac{E(P+,P-) \; \sigma(P+)}{\theta[^\alpha_\beta](P_+-P_-)} \quad ,$$

and

$$f_{-1/2}^{(1/2)}(P) = N_{-1/2}^{(1/2)}(P_\pm,R) \; \frac{E(P,R)}{E(P,P_+) \; E(P,P_-)} \; \theta[^\alpha_\beta](P+e)$$

(I.52)

where $\quad e = R - P_+ - P_- \quad$, and

$$N_{-1/2}^{(1/2)} \; (P_\pm,R) = \frac{E(P+,P-)}{E(P_+,R) \; \sigma(P_+)} \; \theta[^\alpha_\beta](P_+ +e)$$

Here R is a generic fixed point. The dual bases are defined as in the λ integer case. This completes the explicit construction of the KN bases.

9. The Krichever-Novikov algebra

Among the KN bases introduced in the precedent section the basis of vector fields ($\lambda=-1$) is a particular one. It can be used to define via Lie brackets a generalization at arbitrary genus of the algebra of complex-valued vector fields on the circle. The central extension of this algebra is the so-called KN algebra[8].

Let us consider a meromorphic vector field on Σ $u=u(z)\partial_z \in K^{-1}$. The Lie derivative of a tensor of rank (p,q) $t=t(z,\bar{z})dz^p d\bar{z}^q$

with respect to u is then defined by

$$L_u t = (u(z)\partial_z t(z,\bar{z}) + p\ \partial_z u(z)\ t(z,\bar{z}))dz^p d\bar{z}^q \tag{I.53}$$

If we take $v\in K^{-1}$, then the Lie derivative defines an antisymmetric bilinear form on K^{-1}, the Lie bracket

$$[u,v] \equiv L_v\ u = (v(z)\partial_z u(z) - u(z)\partial_z v(z))\partial_z = -L_u v \tag{I.54}$$

Now, if we denote by $e_j \equiv f_j^{(-1)}$ the basis of meromorphic vector fields given by (I.40) ($g>1$, for $g=1$ there are slight modifications in the following expressions)

$$e_j(P) = N_j^{(-1)}(P_+,P_-)\ \frac{E(P,P+)^{j-g_o+1}}{E(P,P_-)^{j+g_o-1}}\ \sigma(P)^{-3}\ \theta(P+e(-1,j)) \tag{I.55}$$

where $g_0=3g/2$, then the Lie bracket (I.54) define the algebra

$$[e_i,e_j] = \sum_{s=-g_0}^{g_0} C^s_{ij}\ e_{i+j-s} \tag{I.56}$$

The C^s_{ij} are the structure constants of the algebra. By using the basis of 2-differentials dual to (I.55) $\Omega^j=f^{(2)}_{-j}$ they can be written as

$$C^s_{ij} = 1/(2\pi i) \oint_C\ [e_i,e_j]\ \Omega^{i+j-s} \tag{I.57}$$

The locality condition in (I.56) then follows from an straightforward counting of zeroes and poles. We can obtain from (I.57) the C^S_{ij}'s in terms of θ-functions and their derivatives. For example ($\varphi_i^{(-1)+}=1$)

$$C^{g_o}_{ij} = (i-j)\quad ,\qquad C^{-g_o}_{ij} = (i-j)\ \frac{\varphi_i^{(-1)-}\ \ \varphi_j^{(-1)-}}{\varphi^{(-1)-}_{i+j+g}}$$

$$\varphi_i^{(-1)-} = (-1)^{i+g_o-1}\ E(P_-,P_+)^{2i}\ \frac{\theta(P- + e(-1,i))}{\theta(P_+ + e(-1,i))}\ \left(\frac{\sigma(P+)}{\sigma(P_+)}\right)^3 \tag{I.58}$$

For $s=-g_o+1,\ldots,\ g_o-1$, the structure constants will depend on the succesive coefficients of the expansion of e_i around P_+ or P_- (cf.(I.32)), which can also be written in terms of θ-functions and their derivatives using the formulas given above.

At $g=0$, $e_n(z)=z^{n+1}$ and (I.56) takes the form

$$[e_n,e_m]=(n-m)e_{n+m}$$

coinciding with the algebra of the generators of diffeomorphisms of the circle.

If we add to the generators of the algebra (I.56) (considered now as an abstract mathematical object) another one t commuting with all the e_i's, then the *central extended* version of (I.56) looks like

$$[e_i, e_j] = \sum_{s=-g_0}^{g_0} C^s_{ij}\ e_{i+j-s} + c/12\ t\ \kappa_{ij}$$

$$[e_i, t] = 0 \tag{I.59a}$$

where c is the so-called central charge of the algebra and κ_{ij} is a cocycle. However the condition for (I.59a) to be an algebra (and then it satisfies the Jacobi identity) imposes strong restrictions on κ_{ij}. Indeed KN showed that if we require the locality condition

$$\kappa_{ij} = 0 \quad \text{if} \quad |i+j|>3g$$

then the cocycle must be of the form

$$\kappa_{ij} = 1/(2\pi i) \oint_C dz\ (\ 1/2\ (\partial^3_z e_i(z)\ e_j(z) - (i\leftrightarrow j)\ ) -$$

$$- R(z)\ (\partial_z e_i(z)\ e_j(z) - (i\leftrightarrow j))\ ) \tag{I.59b}$$

where R(z) is a schwartzian connection transforming under $z \longrightarrow w=w(z)$ as

$$R'(w)dw^2 = R(z)dz^2 + \{w,z\}dz^2 \tag{I.60a}$$

$$\{w,z\} = \partial^3_z w(z)/\partial_z w(z) - 3/2\ (\partial^2_z w(z)/\partial_z w(z))^2 \tag{I.60b}$$

in order to make the integrand of (I.59b) a well-defined 1-form; $\{w,z\}$ is called the schwartzian derivative.

Eqs.(I.59) define the KN algebra. We note that two KN algebras with the same central charge and differents schwarztian connections R(z) and R'(z) are isomorphic. It is easy to verify that the corresponding generators e_i and e'_i are related by $e'_i=e_i+S_i$, where

$$S_i = c/(24\pi i) \oint_C dz\ e_i(z)\ (R'(z) - R(z)) \tag{I.61}$$

is a well-defined c-number because the difference of two schwarztian connections is a 2-differential[11]. It is said in this case that the two algebras differ by "trivial cocycles".

We finally recall that at g=0 eq.(I.59a) reduces to the famous Virasoro algebra (R(z)=0)

$$[L^V_n, L^V_m] = (n-m)\ L^V_{n+m} + c/12\ \delta_{n+m,0}\ (n^3-n) \tag{I.62}$$

CHAPTER II:

OPERATOR FORMULATION OF CONFORMAL FIELD THEORIES ON RIEMANN SURFACES

PART A: Hamiltonian formalism for generic CFT on RS[12]

1. The Krichever-Novikov parametrization

Let us recall some elementary facts of genus zero. When a "string" propagates in space-time it sweeps out a world-sheet which is topologically a cylinder (Fig.II.1a). This is conventionally parametrized by an angular coordinate σ and a time evolution parameter t. By going to euclidean time τ (=it), this cylinder can be mapped to the complex plane without the points z=0 and z=∞ by simply defining $z=\exp(\tau+i\sigma)$ as the coordinate of the complex plane (Fig.II.1b). This is conformal to a sphere without two points (Fig.II.1c). The inverse map can be defined by

$$\tau(z,\bar{z}) = \mathrm{Re}\int_1^z dz/z \qquad \text{(II.1a)}$$

$$\sigma(z,\bar{z}) = \mathrm{Im}\int_1^z dz/z \quad , \ \sigma \simeq \sigma + 2\pi n, \ n \in N \qquad \text{(II.1b)}$$

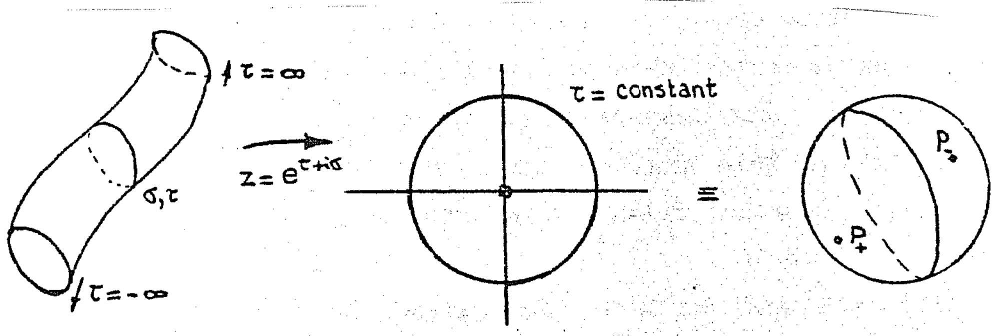

Fig.II.1a: Cylinder

Fig.II.1b: Complex plane without z=0,∞

Fig.II.1c: Sphere without two points

We see that the level curves of equal τ (which represent the string) are concentric circles around z=0 (Fig.II.1b).

At higher genus we have something similar: the string propagates but this time it splits and then joins giving rise to "holes" which topologically characterize the two dimensional surface (Fig.II.2a). Again, by going to euclidean time, this can be conformally mapped to a Riemann surface Σ without two points P_+ and P_- (Fig.II.2b). By noting that the integrand in eqs.(II.1) is no other thing that the 1-form with simple poles in $z=0,\infty$ and residues +1 and -1 respectively, we are led to define[5]

$$\tau(P) = \mathrm{Re}\int_{P_o}^{P} dk \quad ; \; P,\ P_o \in \Sigma \tag{II.2a}$$

where dk is a differential of the third kind with simple poles at P_+ and P_- with residues +1 and -1 respectively. This defines dk up to addition of holomorphic differentials. If we require τ to be unambiguously defined (as in (II.1a)), that is

$$\mathrm{Re}\oint_{\gamma_i} dk = 0 \tag{II.3}$$

with γ_i any homology cycle, then dk is fixed unambiguously being precisely the $\omega^{g/2}$ introduced in the Section 8 of the precedent chapter. From its explicit expression in (I.43) we obtain

$$\tau(P) = \mathrm{Re}\Big(\log\Big(\frac{E(P,P_+)E(P_o,P_-)}{E(P,P_-)E(P_o,P_+)}\Big) - i2\pi \sum_{i,j=1}^{g} \mathrm{Im}\int_{P_-}^{P_+} \eta^i \, (\tau_2^{-1})_{ij} \int_{P_o}^{P} \eta^j \Big) \tag{II.4}$$

By analogy with the g=0 case one could define

$$\sigma(P) = \mathrm{Im}\int_{P_o}^{P} dk \tag{II.2b}$$

but now we have to specify the path, otherwise σ is not well defined. It follows that $d\sigma=\mathrm{Im}(dk)$ is a well defined 1-form.

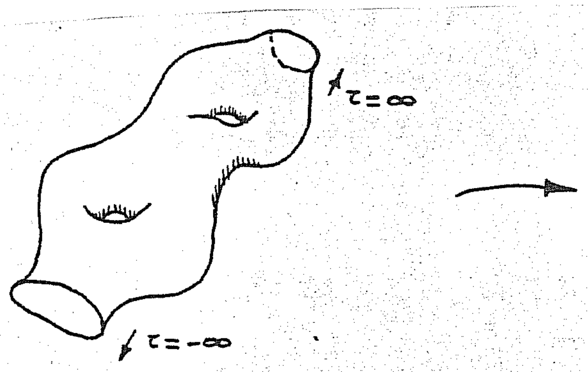

Fig.II.2a: Cylinder with holes

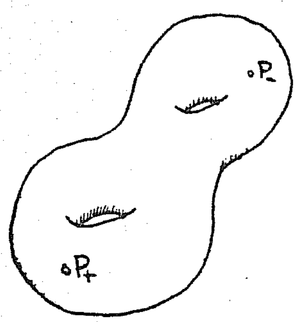

Fig.II.2b: Riemann surface with two punctures.

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[5]This parametrization was first considered time ago by Mandelstam in formulating the "interacting string picture" to define multiloop scattering amplitudes[13].

The string propagating along Σ will be represented by a one parameter family of contours C_τ defined as follows

$$C_\tau = \{ P \in \Sigma : \tau(P)=\tau \}$$

For $\tau\to\mp\infty$ the C_τ are small circles around $P_\pm$. As τ grows up, the string evolves with splitting and joinings until it reaches the point P_- (Fig.II.3).

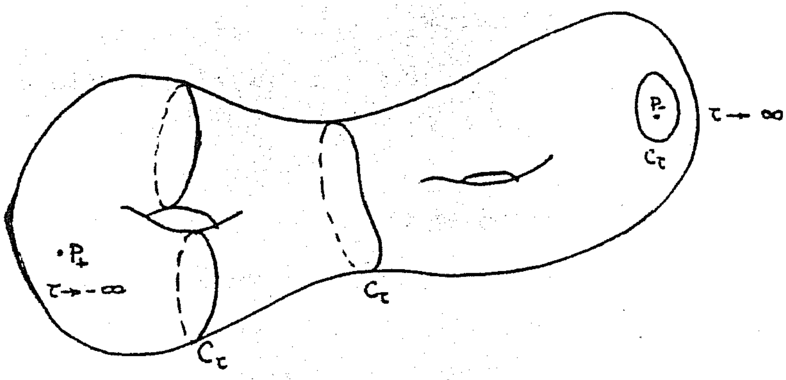

Figure II.3: String propagating and interacting

In Ref.[8] KN showed that the basis $\{f_j^{(\lambda)}\}$ is a complete set on C_τ in the sense that *any* tensor $F^{(\lambda)}(P)$ $(P \in C_\tau)$ of weight λ which is smooth (or piecewise smooth) over C_τ can be expanded in the basis $\{f_k^{(\lambda)}\}$

$$F^{(\lambda)}(P) = \sum_k a_k \; f_k^{(\lambda)}(P)$$

Therefore, if we have a tensor over Σ which is smooth except possibly in $P_\pm$, this expansion will hold for any C_τ, $-\infty<\tau<\infty$, with coefficients generally depending on τ. But if we consider a meromorphic $F^{(\lambda)}$ which is holomorphic outside $P_\pm$, the coefficients a_k will be τ-independent.

From this completeness property and the duality relation (I.46) it follows that the delta-function on C_τ takes the general form

$$\Delta_\tau(P,Q) = 1/(2\pi i) \sum_i f_i^{(\lambda)}(P) \; f^i_{(1-\lambda)}(Q) \qquad \text{(II.5)}$$

2.Hamiltonian, momentum and KN operators

Let us consider a field theory on a Riemann surface Σ defined by some action $S=S[\Phi;h]$ where Φ stands for the generic fields and

$h=h_{ab}(x)dx^a\otimes dx^b$ is a metric. The energy-momentum tensor (EMT) is by definition ($|h|=\det h_{ab}$)

$$t = \frac{4\pi}{|h|^{1/2}} \frac{\delta S[\Phi;h]}{\delta h^{ab}} dx^a\otimes dx^b \qquad \text{(II.6)}$$

The theory is classically Weyl invariant if S is invariant under Weyl rescalings of h, i.e., $S[\Phi;\rho h]=S[\Phi;h]$ for any function ρ. After "gauge fixing" to isothermal coordinates (I.7), all the dependence in h is missing and we remain with a gauge fixed theory which locally presents invariance under holomorphic change of coordinates $z \rightarrow w=w(z)$ that preserve the conformal form of the metric (I.7). This kind of theories are called "Conformal Field Theories" (CFT)[14,15,16].

Operator formulations of CFT over a generic RS, as opposed to the path integral formulation, have been object of intensive research in the last years[17]. The common feature of these approaches is that they privilege the local description of CFT over a disk cut out from the RS. The globalization is essentially obtained via Bogoliubov transformations relating states over the disk to states over the RS without disk. It is the aim of this chapter to show that is possible to operatorially formulate CFT on RS in a manifestly global way by using the machinery developed in the precedent sections.

Now let us take for any chart on Σ $x^1\equiv\tau$ and $\sigma^2\equiv\sigma$ according to the KN parametrization eqs.(II.2). We define the Hamiltonian in the standard way as the integral of t_{11} at fixed time

$$H(\tau) = 1/2\pi \oint_{C_\tau} d\sigma\, t_{11} \qquad \text{(II.7a)}$$

The momentum is similarly defined as

$$P(\tau) = 1/2\pi i \oint_{C_\tau} d\sigma\, t_{12} \qquad \text{(II.7b)}$$

H and P will become the generators of translation in τ and σ respectively.

Now from (II.6) it follows that t is symmetric and traceless, a well-known fact in CFT; then eqs.(II.7) can also be written as follows

$$H(\tau) = -1/2\pi \oint_{C_\tau} (t|e_\sigma) \qquad \text{(II.8a)}$$

$$P(\tau) = 1/2\pi i \oint_{C_\tau} (t|e_\tau) \qquad \text{(II.8b)}$$

where e_σ, e_τ are the following vector fields[6]

$$e_\tau = e_k + e_{\bar{k}} \quad , \qquad e_\sigma = i(e_k - e_{\bar{k}}) \qquad \text{(II.9)}$$

with e_k, $e_{\bar{k}}$, the meromorphic vector fields dual to dk and $d\bar{k}$ respectively in the sense

$$\begin{aligned} dk(e_k) &= (e_k|dk) = 1 = (e_{\bar{k}}|d\bar{k}) \\ d\bar{k}(e_k) &= (e_k|d\bar{k}) = 0 = (e_{\bar{k}}|dk) \end{aligned} \qquad \text{(II.10)}$$

Whenever dk has a zero (pole), e_k must have a pole (zero) in order for (II.10) to hold. Since by Riemann-Roch theorem dk has 2g zeroes outside $P_\pm$ we conclude that e_k has simple zeroes at $P_\pm$ and 2g poles outside them. Of course, the same occurs for its complex conjugate $e_{\bar{k}}$. At these 2g points one has $d\tau=0=d\sigma$. These points correspond to the critical points where the C_τ's split or join.

Now, from the tracelessness and symmetry of t it also follows that $t=T+\bar{T}$ where $T\in A_{(2,0)}$ and $\bar{T}\in A_{(0,2)}$ (the $T_{z\bar{z}}=T_{\bar{z}z}$ component is zero). This fact allows us to express $H(\tau)$ and $P(\tau)$ in the following way

$$H(\tau) = 1/(2\pi i) \oint_{C_\tau} (T|e_k) - (1/2\pi i) \oint_{C_\tau} (T|e_{\bar{k}}) \qquad \text{(II.11)}$$

$$P(\tau) = 1/(2\pi i) \oint_{C_\tau} (T|e_k) + 1/(2\pi i) \oint_{C_\tau} (\bar{T}|e_{\bar{k}}) \qquad \text{(II.12)}$$

It is another well-known fact in CFT that on-shell T and $\bar{T}$ are holomorphic and antiholomorphic respectively; then the following expansions will hold

$$T(Q) \equiv \sum_j L_j \, \Omega^j(Q) \quad , \quad L_j = 1(2\pi i) \oint_{C_\tau} e_j \, T \qquad \text{(II.13a)}$$

$$\bar{T}(Q) \equiv \sum_j \bar{L}_j \, \bar{\Omega}^j(Q) \qquad \bar{L}_j = -1(2\pi i) \oint_{C_\tau} \bar{e}_j \, \bar{T} \qquad \text{(II.13b)}$$

[6] The vector fields e_k, $e_{\bar{k}}$ introduced in eqs.(II.9,10) should not be confused with anyone belonging to the KN basis $\{e_j\}$.

The coefficients of these expansions are the KN operators. It will be showed in the next chapter that they provide a realization of the KN algebra (I.59).

Let us stres that $H(\tau)$ and $P(\tau)$ in (II.11,12) depend on time. This is due to the 2g poles of the vector fields e_k, $e_{\bar{k}}$. The variation with time is however very simple. It is like a step function in the sense that it remains constant until it reaches a splitting or joining of the C_τ (because the integrand picks a pole from e_k), where it changes value by a discrete quantity.

Finally let us note that at g=0 H and P reduce to the known expressions

$$H = L_0 + \bar{L}_0 \quad , \qquad P = L_0 - \bar{L}_0$$

PART B: b-c systems[10]

3. Equations of motion and mode expansions

In this part we start developing systematically the operator formalism over a generic Σ by means of the KN bases, by considering the simplest cases of conformal field theories: the b-c chiral systems, also called "ghost" systems.

Let b, c be a conjugate pair of fields belonging to $A_{(\lambda,0)}$ and $A_{(1-\lambda,0)}$ respectively, and $\bar{b}$, $\bar{c}$ their "antiholomorphic counterparts" with action[7]

$$S[b,c] = -1/2\pi \int_{\Sigma} (b\wedge\bar{\partial}c - \bar{b}\wedge\partial\bar{c}) \qquad \text{(II.14)}$$

The components of the EMT are given by

$$T(Q) = \lambda\ \partial c(Q)\ b(Q) - (1-\lambda)\ c(Q)\ \partial b(Q) \qquad \text{(II.15a)}$$

$$\bar{T}(Q) = \lambda\ \bar{\partial}\bar{c}(Q)\ \bar{b}(Q) - (1-\lambda)\ \bar{c}(Q)\ \bar{\partial}\bar{b}(Q) \qquad \text{(II.15b)}$$

So, the ghost Hamiltonian (II.8a) is

$$H^{gh}(\tau) = -1/2\pi \oint_{C_\tau} (e_\sigma|\lambda\partial cb - (1-\lambda)c\partial b + \lambda\bar{\partial}\bar{c}\bar{b} - (1-\lambda)\bar{c}\bar{\partial}\bar{b}) \qquad \text{(II.16)}$$

It gives by definition the time evolution of the fields of the theory via commutation. From now on, we restrict ourselves to the study of the "chiral" part b,c. By imposing the canonical anticommutation relation between b and c

$$\{b(P),c(Q)\} = i2\pi\Delta_\tau(P,Q) \quad , \quad P,Q \in C_\tau \qquad \text{(II.17)}$$

we obtain the equations of motion

$$L_{e_\tau} c(Q) \equiv [H,c(Q)] = -iL_{e_\sigma} c(Q) \qquad \text{(II.18a)}$$

$$L_{e_\tau} b(Q) \equiv [H,b(Q)] = -iL_{e_\sigma} b(Q) \qquad \text{(II.18b)}$$

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[7] This action can be derived from gauge-fixing a manifestly reparametrization invariant one; the energy-momentum tensor components (II.15) are obtained from it via (II.6), see ref.[18] for details.

These equations imply

$$\bar{\partial} c(Q) = 0 = \bar{\partial} b(Q) \quad , \quad Q \neq P_+ ; P_- \tag{II.19}$$

whose most general solution is

$$c(Q) = \sum_i c^i f_i^{(1-\lambda)}(Q) \tag{II.20a}$$

$$b(Q) = \sum_i b_i f^i_{(\lambda)}(Q) \tag{II.20b}$$

The Hamiltonian operator written in terms of the coefficients c_i, b^i takes the form

$$H^{gh}(\tau) = \sum_{i,j} S_i{}^j(\tau) \, c^i \, b_j \tag{II.21a}$$

$$S_i{}^j(\tau) = 1/(2\pi i) \oint_{C_\tau} (e_k | \lambda \partial f_i^{(1-\lambda)} f^j_{(\lambda)} - (1-\lambda) f_i^{(1-\lambda)} \partial f^j_{(\lambda)}) \tag{II.21b}$$

The anticommutation relation (II.17) implies that the coefficients b_k's and c^k's satisfy the standard anticommutation rules

$$\{b_i, c^j\} = \delta_i{}^j \quad , \quad \{c^i, c^j\} = 0 = \{b_i, b_j\} \tag{II.22}$$

4.The Fock space

In this section we will consider the general case $\lambda>1$, $g>1$. Upon quantization, $b(Q)$ and $c(Q)$ become operators acting on a Hilbert space, so it does the coefficients of the expansions (II.20). They are operators acting on a type-Fock space with vacuum state $|0)_\Sigma$ satisfying the conditions

$$c^k |0)_\Sigma = 0 \quad , \quad k < s(1-\lambda) \tag{II.23a}$$

$$b_k |0)_\Sigma = 0 \quad , \quad k \geq s(1-\lambda) \tag{II.23b}$$

We can represent the vacuum as the semi-infinite form

$$|0)_\Sigma = f^{s(1-\lambda)}_{(\lambda)} \wedge f^{s(1-\lambda)+1}_{(\lambda)} \wedge \ldots\ldots \tag{II.24}$$

(from now on we drop the label (λ) when it is not strictly necessary). Then the action of c^k and b_k on the vacuum admits the explicit representation (right action)[8]

$$b_k = f^k \wedge \quad ,$$

$$c^k |0)_\Sigma = i_{f_k} |0)_\Sigma = \sum_{j=s(1-\lambda)}^{\infty} (-1)^j \, i_{f_k} (f^j) \, f^{s(1-\lambda)} \wedge \ldots \wedge \check{f}^j \wedge \ldots \tag{II.25}$$

where i_{f_k} denotes the usual antiderivation defined by

$$i_{f_k} (f^j) = \frac{1}{2\pi i} \oint_{C_\tau} f_k \, f^j = \delta_k{}^j$$

and $^\vee$ denotes omission. Analogously we can define the dual vacuum $_\Sigma(0|$ by means of

$$_\Sigma(0| c^k = 0 \quad , \quad k \geq s(1-\lambda) \tag{II.26a}$$

$$_\Sigma(0| b_k = 0 \quad , \quad k < s(1-\lambda) \tag{II.26b}$$

It can be usefully given the representation[8]

$$_\Sigma(0| = f_{s(1-\lambda)} \wedge f_{s(1-\lambda)+1} \wedge \ldots\ldots \quad , \tag{II.27}$$

c^k and b_k act on this semi-infinite form (left action) as

$$c^k = f_k \wedge \quad , \qquad b_k = i_{f^k} \tag{II.28}$$

Moreover there exists a pairing such that

$$_\Sigma(0|0)_\Sigma = 1 \tag{II.29}$$

It is now time to comment on the meaning of this vacuum and of the excitations which are created out of it or destroyed by the operators c^k and b_k. First of all, we have to enphasize that both these operators and the vacuum state are globally defined over Σ, due to the fact that the KN bases are globally defined. In this sense our approach is different from the previous ones[17] where two vacua are generally used, one related to a disk singled out of the surface and the other related to the rest of Σ.

It is evident that the c^k and b_k modes are related in a complicated (g-dependent) way to the usual g=0 modes, which are the string modes with associated particle interpretation. Let us find the relation between the genus g modes and the genus zero modes. We pick a coordinate z near P_+, $z(P_+)=0$, and the circle $C=\{|z|=1\}$. A basis for tensors of weight λ over this circle is given by

$$\tilde{f}_n^{(1-\lambda)}(z) = z^{n-1+\lambda} \, dz^{1-\lambda} \tag{II.30a}$$

$$\tilde{f}^n_{(\lambda)}(z) = z^{-n-\lambda} \, dz^\lambda \tag{II.30b}$$

where n is integer. The restrictions $\{\bar{f}_j^{(1-\lambda)}\}$ and $\{\bar{f}_{(\lambda)}^{j}\}$ of the bases $\{f_j^{(1-\lambda)}\}$ and $\{f_{(\lambda)}^{j}\}$ on C are dense in the space of the corresponding tensors on C[8]. Therefore, we can expand

$$\bar{f}_j^{(1-\lambda)} = \sum_n A_j^{\ n}(1-\lambda)\ \tilde{f}_n^{(1-\lambda)} \tag{II.31a}$$

$$\tilde{f}_n^{(1-\lambda)} = \sum_j B_n^{\ j}(1-\lambda)\ \bar{f}_j^{(1-\lambda)} \tag{II.31b}$$

$$\bar{f}^j_{(\lambda)} = \sum_n C_n^{\ j}(\lambda)\ \tilde{f}^n_{(\lambda)} \tag{II.31c}$$

$$\tilde{f}^n_{(\lambda)} = \sum_j D_j^{\ n}(\lambda)\ \bar{f}^j_{(\lambda)} \tag{II.31d}$$

where

$$A_j^{\ n}(1-\lambda) = \frac{1}{2\pi i}\oint_C \bar{f}_j^{(1-\lambda)}\ \tilde{f}^n_{(\lambda)} \quad ,$$

$$B_n^{\ j}(1-\lambda) = \frac{1}{2\pi i}\oint_C \bar{f}^j_{(\lambda)}\ \tilde{f}_n^{(1-\lambda)} \quad , \text{etc.} \tag{II.31e}$$

It is easy to see that $A(1-\lambda)=B(1-\lambda)^{-1}$, $C(\lambda)=D(\lambda)^{-1}$ and $C(\lambda)=B(1-\lambda)$. The entries of the $A(1-\lambda)$ matrix vanish for $n<j+s(\lambda)-\lambda$, and are given by the coefficients of the Laurent tail in eq.(I.32) otherwise. Similarly $B_n^{\ j}(1-\lambda)$ vanish if $n>j+s(\lambda)-\lambda$, and is otherwise given by the coefficients of the Laurent expansion of $\bar{f}^{\ j}_{(1-\lambda)}$ near P_+. We remark that in general the matrices $A(1-\lambda)$ and $B(1-\lambda)$ have an infinite number of non-vanishing entries and stress that they can be explicitly calculated.

Now we are able to calculate the relation between creation and annihilation operators on the sphere and on genus g Riemann surfaces. Let $\bar{b}$ and $\bar{c}$ be the restrictions of b and c to the circle C, then we can consider both expansions

$$\bar{b}(P) = \sum_j b_j\ \bar{f}^j_{(\lambda)}(P) = \sum_n \tilde{b}_n\ \tilde{f}^n_{(\lambda)}(P) \tag{II.32a}$$

$$\bar{c}(P) = \sum_j c^j\ \bar{f}_j^{(1-\lambda)}(P) = \sum_n \tilde{c}^n\ \tilde{f}_n^{(1-\lambda)}(P) \tag{II.32b}$$

It is easy to find the relation between the $\tilde{b}_i$'s, b_i's, $\tilde{c}^i$'s and c^i's

$$\tilde{c}^n = \sum_j A_j^{\ n}(1-\lambda)\ c^j \quad , \quad \tilde{b}_n = \sum_j B_n^{\ j}(1-\lambda)\ b_j \tag{II.33}$$

These are infinite combinations, however one can realize that when applied to $|0)_\Sigma$ only a finite combination survives due to (II.23) and the properties of the A and B matrices listed after eq.(II.31). As a consequence this is true for any state constructed from $|0)_\Sigma$ by applying a finite number of $\tilde{b}_n$ and $\tilde{c}^n$ operators. In particular we have

$$\tilde{b}_n \ |0)_\Sigma = 0 \quad \text{for } n+\lambda-1\geq 0$$

$$\tilde{c}^n \ |0)_\Sigma = 0 \quad \text{for } n+\lambda-1<0$$

Of course, we could have started from the g=0 vacuum $|0)_0$ defined by eq.(II.24) where g=0, and tried to find the action of b_k and c^k over it. We would have found eq.(II.23) with $|0)_0$ instead of $|0)_\Sigma$. The relation between $|0)_0$ and $|0)_\Sigma$ for a genus $g\neq 0$ Σ can be reconstructed by means of the semi-infinite form representation of $|0)_\Sigma$ and the transformation of the basis elements. For example, for $\lambda=-1$

$$\Omega^j = \tilde{\Omega}^{j-g_0} + \sum_{n<j-g_0}^{\infty} B_n^{\ j}(2)\ \tilde{\Omega}^n$$

It is now worth spending a few words about the connection between the vacuum defined here and a vacuum frequently used in the literature[15,17]. A definition of the vacuum $|0>_\Sigma$ implicit in the path integral approach (see Section 7) is specified by means of (II.23) and

$$_\Sigma<0|c^k = 0 \quad , \ k > -s(1-\lambda) \qquad \text{(II.34a)}$$

$$_\Sigma<0|b_k = 0 \quad , \ k \leq -s(1-\lambda) \qquad \text{(II.34b)}$$

instead of (II.26). $|0>_\Sigma$ and $_\Sigma<0|$ can be represented by two semi-infinite forms, given respectively by eq.(II.24) and by

$$_\Sigma<0| = f_{-s(1-\lambda)+1}\wedge f_{-s(1-\lambda)+2}\wedge\ldots \qquad \text{(II.35)}$$

c^k and b_k act as in eqs.(II.25,28). It is evident that

$$_\Sigma<0|0>_\Sigma = 0$$

and that $_\Sigma<0|$ and $_\Sigma(0|$ differ exactly by the dual bases elements of the b zero modes for $\lambda>1$ and for $g\geq 2$.

For g=0, $\lambda=2$, let us define

$$|0>_0 = \tilde{\Omega}^{-1}\wedge \tilde{\Omega}^0\wedge\ldots \qquad \text{(II.36a)}$$

$$_0<0| = \tilde{e}_2\wedge \tilde{e}_3\wedge\ldots \qquad \text{(II.36b)}$$

and assume

$$|0\rangle_0^+ = {}_0\langle 0| \quad , \quad (c^k)^+ = c^{-k} \quad , \quad (b_k)^+ = b_{-k} \tag{II.37}$$

These definitions exactly realize the prescription

$$(c^{-1}|0\rangle_0)^+ = {}_0\langle 0|c^1 c^0$$

introduced in ref.[15] (remember that w.r.t. this reference $c^k \leftrightarrow c^{-k}$). It is therefore possible to interpret the pairing defined by the brackets ${}_\Sigma\langle \; | \; \rangle_\Sigma$ as a scalar product.

5.Correlation functions

Propagators for integer λ

Let us first consider the $g>1$, $\lambda>1$ case. The propagator $S(P,Q)$ is defined as

$$S(P,Q) = {}_\Sigma(0|T\{b(P)\ c(Q)\}|0)_\Sigma$$
$$= \begin{cases} {}_\Sigma(0|b(P)\ c(Q)|0)_\Sigma \quad , \quad \tau_P > \tau_Q \\ -{}_\Sigma(0|c(Q)\ b(P)|0)_\Sigma \quad , \quad \tau_Q > \tau_P \end{cases} \tag{II.38}$$

where $\tau_P(\tau_Q)$ means the value of τ at the point $P(Q)$. By inserting (II.20) into (II.38), and using (II.22,23,26) we easily obtain

$$S(P,Q) = \begin{cases} \sum_{k=s(1-\lambda)}^{\infty} f^k_{(\lambda)}(P)\ f_k^{(1-\lambda)}(Q) & \tau_P > \tau_Q \\ -\sum_{k=-\infty}^{s(1-\lambda)-1} f^k_{(\lambda)}(P)\ f_k^{(1-\lambda)}(Q) & \tau_Q > \tau_P \end{cases} \tag{II.39}$$

Now we would like to give a more compact expression for this Szegö kernel. Eventually we will find that $S(P,Q)$ reduces to the known result (see [19,20])

$$S(P,Q) = \frac{1}{E(P,Q)} \left(\frac{E(P,P_-)}{E(Q,P_-)}\right)^{(2\lambda-1)(g-1)} \left(\frac{\sigma(P)}{\sigma(Q)}\right)^{(2\lambda-1)} \times$$
$$\times \frac{\theta(Q-P+u(\lambda))}{\theta(u(\lambda))} \tag{II.40}$$

where

$$u(\lambda) = (2\lambda-1)(g-1)P_- + (2\lambda-1)\Delta$$

A check that eq.(II.40) and (II.39) coincide is the following. Consider the propagator to be a tensor $F^{(1-\lambda)}(Q)$ of weight $1-\lambda$

depending on Q, at fixed P. Then it can be expanded in terms of the basis $\{f_k^{(1-\lambda)}\}$

$$F^{(1-\lambda)}(Q) = \sum_k a_k \; f_k^{(1-\lambda)}(Q) \tag{II.41}$$

Multiplying by $f^k_{(\lambda)}$ and integrating over C_τ we obtain

$$a_k = \frac{1}{2\pi i} \oint_{C_\tau} F^{(1-\lambda)} \; f^k_{(\lambda)} \tag{II.42}$$

Now we can use equation (II.40) and the explicit expressions for $f^k_{(\lambda)}$ in order to arrive at eq.(II.39). In fact, solving the integral (II.42) we obtain

$$a_k = \begin{cases} f^k_{(\lambda)} & , \; k \geq s(1-\lambda) \\ 0 & , \; k < s(1-\lambda) \end{cases} \quad , \quad \tau_Q < \tau_P \; ,$$
$$a_k = \begin{cases} 0 & , \; k \geq s(1-\lambda) \\ -f^k_{(\lambda)} & , \; k < s(1-\lambda) \end{cases} \quad , \quad \tau_P < \tau_Q \; , \tag{II.43}$$

in agreement with eq.(II.39).

It is instructive to see how one can use a heuristic argument to pass from (II.39) to the compact form (II.40). This consists in looking at the behaviour of the sums (II.39), and then identifying the zeroes and poles of S(P,Q). After that, one uses the Riemann-Roch theorem to prove the existence and uniqueness of sections with such behaviours, and the explicit expression (II.40) easily follows.

Let us first consider the point P fixed. From (II.39) we get

$$F^{(1-\lambda)}(Q) = \begin{cases} \displaystyle\sum_{k=s(1-\lambda)}^{\infty} a_k \; f_k^{(1-\lambda)}(Q) & , \; \tau_P > \tau_Q \\ \displaystyle -\sum_{k=-\infty}^{s(1-\lambda)-1} a_k \; f_k^{(1-\lambda)}(Q) & , \; \tau_Q > \tau_P \end{cases} \tag{II.44}$$

In order to see the behaviour of $F^{(1-\lambda)}(Q)$ on the Riemann surface, we use eq.(I.32). It follows that

$$\begin{aligned} Q \to P_+ \quad &\Rightarrow \quad F^{(1-\lambda)}(Q) \text{ is holomorphic non-zero;} \\ Q \to P_- \quad &\Rightarrow \quad F^{(1-\lambda)}(Q) \text{ has a pole of order } (2\lambda-1)(g-1); \\ Q \to P \quad &\Rightarrow \quad F^{(1-\lambda)}(Q) \text{ has a simple pole.} \end{aligned} \tag{II.45}$$

Proceeding likewise, but at fixed Q, the propagator can be considered as a tensor of weigth λ which we denote by $F^+_{(\lambda)}$

$$F^{+}_{(\lambda)}(P)=\begin{cases}\sum\limits_{k=S(1-\lambda)}^{\infty} b_k\, f^k_{(\lambda)}(P) & ,\ \tau_P > \tau_Q\\ -\sum\limits_{k=-\infty}^{S(1-\lambda)-1} b_k\, f^k_{(\lambda)}(P) & ,\ \tau_Q > \tau_P\end{cases} \tag{II.46}$$

Similarly, from eq.(II.41) we extract the behaviour of $F^{+}_{(\lambda)}(P)$

$P \to P_+ \quad \Rightarrow \quad F^{+}_{(\lambda)}(P)$ is holomorphic non-zero;

$P \to P_- \quad \Rightarrow \quad F^{+}_{(\lambda)}(P)$ is holomorphic with a zero of order $(2\lambda-1)(g-1)$;

$P \to Q \quad \Rightarrow \quad F^{+}_{(\lambda)}(P)$ has a simple pole.

(II.47)

Riemann-Roch theorem tells us that $F^{(1-\lambda)}(Q)$ and $F^{+}_{(\lambda)}(P)$ are uniquely determined up to a constant $C_1(P,P_\pm)$ and $C_2(Q,P_\pm)$ respectively. So, there remains an indetermination by a constant $C(P_\pm)$. The propagator $S(P,Q)$ is now easily found from (II.45) and (II.47), and from the requirement of correct dimensionality and single-valuedness

$$S(P,Q) = \frac{C(P\pm)}{E(P,Q)}\left(\frac{E(P,P-)}{E(Q,P_-)}\right)^{(2\lambda-1)(g-1)}\left(\frac{\sigma(P)}{\sigma(Q)}\right)^{(2\lambda-1)} \times \frac{\theta(Q-P+u(\lambda))}{\theta(u(\lambda))} \tag{II.48}$$

where

$$u(\lambda) = (2\lambda-1)(g-1)P- + (2\lambda-1)\Delta.$$

The constant $C(P_\pm)$ can be determined by making $P\to P_-$ and $Q\to P_+$, where the propagator has a zero of order $(2\lambda-1)(g-1)$

$$S(P,Q) = \sum_{k=S(1-\lambda)}^{\infty} f^k_{(\lambda)}(P)\, f^{(1-\lambda)}_k(Q) \underset{Q\to P_+}{\overset{P\to P_-}{\approx}} f^{S(1-\lambda)}_{(\lambda)}(P)\, f^{(1-\lambda)}_{S(1-\lambda)}(Q) \tag{II.49}$$

Inserting the explicit expressions for $f^k_{(\lambda)}$ and $f^{(1-\lambda)}_k$ we obtain

$$S(P,Q) \underset{Q\to P_+}{\overset{P\to P_-}{\approx}} \frac{E(P,P-)^{(2\lambda-1)(g-1)}}{E(P_+,P_-)^{(2\lambda-1)(g-1)+1}}\left[\frac{\sigma(P-)}{\sigma(P_+)}\right]^{(2\lambda-1)} \times \frac{\theta(P- - P+ + u)}{\theta(u)} \tag{II.50}$$

Now, comparing (II.50) with (II.48) we conclude that $C(P_\pm)=1$.

Finally a remark. Had we chosen the vacuum state to be the state which is annihilated by the negative frequency modes of c(P) and b(P) with P near to P_-, then we would have found a similar result for S(P,Q) with P_+ instead of P_-. A useful relation to prove this statement is

$$f_k^{(\lambda)}(P;P_+,P_-) = c\, f_{-k}^{(\lambda)}(P;P_-,P_+)$$

where c is a constant independent of P.

Let us come now to the $\lambda=1$ case. We have already seen that the bases $\{A_i\},\{\omega^i\}$ are slightly modified with respect to the generic $f_k^{(\lambda)}$, eq.(I.40). The vacuum state in this case is defined by the conditions

$$c^k|0)_\Sigma = 0 \quad , \quad k \le g/2 \tag{II.51a}$$

$$b_k|0)_\Sigma = 0 \quad , \quad k > g/2 \tag{II.51b}$$

So, the propagator is

$$S(P,Q) = \begin{cases} \sum\limits_{k=g/2+1}^{\infty} \omega^k(P)\ A_k(Q) & , \tau_P > \tau_Q \\ -\sum\limits_{k=-\infty}^{g/2} \omega^k(P)\ A_k(Q) & , \tau_Q > \tau_P \end{cases} \tag{II.52}$$

The summations in (II.52) can be performed by the two methods explained before. We will not repeat the computation, since it follows the same lines as above. We just quote here the result which agrees with the well-known Szegö kernel for $\lambda=1$[25,26]

$$S(P,Q) = \frac{E(Q,P+)\ \theta(Q - P - u)\ \theta(P - P+ - u)}{E(P,Q)\ E(P,P_+)\ \theta(u)\ \theta(Q - P+ - u)} \tag{II.53}$$

where $u = g\,P_- - P_+ - \Delta$.

Finally, let us consider the genus one case. The vacuum state is defined by the conditions

$$c^k|0)_\Sigma = 0 \quad , \quad k \le 1/2 \tag{II.54a}$$

$$b_k|0) = 0 \quad , \quad k > 1/2 \tag{II.54b}$$

The propagator can be computed in much the same way as for the previous cases, and we obtain

$$S(P,Q)= \frac{1}{E(P,Q)}\ \frac{E(P,P-)\ E(Q,P+)}{E(P,P_+)\ E(Q,P_-)} \left(\frac{\sigma(P)}{\sigma(Q)}\right)^{(2\lambda-1)} \frac{\theta(Q - P + u)}{\theta(u)} \tag{II.55}$$

where $u = P_- - P_+ - \Delta$.

Propagators for half-integer λ

These kind of propagators are of interest in the study of CFT on RS, and in particular, in superstring theory, where we have matter with $\lambda=1/2$ and reparametrization ghosts with $\lambda=2$ as anticommuting system, as well as superconformal ghost (β,γ) as commuting system with $\lambda=3/2$ (see next chapter)

All of the results we have presented are easily extended to the commuting case and it is immediate to show that the propagator

$$S_{\beta\gamma}(P,Q) = {}_{\Sigma}(0|T\{\beta(P)\ \gamma(Q)\}|0)_{\Sigma} = {}_{\Sigma}(0|\beta(P)\gamma(Q)|0)_{\Sigma}\ \theta(\tau_P-\tau_Q) + + {}_{\Sigma}(0|\gamma(Q)\beta(P)|0)_{\Sigma}\ \theta(\tau_Q-\tau_P) \tag{II.56}$$

gives the same S(P,Q) as in the fermion case.

For half-integer λ the only subtleties which arise in the computation of S(P,Q) come from the presence of branch points in the R sector, though these branch points are absent in integrals of the type (II.42) (this is due to the fact that in the NS sector both $F^{(1-\lambda)}$ and $f^{j}_{(\lambda)}$ in eq.(II.42) have branch points in $P_{\pm}$). So, we limit ourselves to give the results:

$g \geq 2$, NS sector, $\lambda \neq 1/2$

$$S(P,Q) = \frac{1}{E(P,Q)} \left(\frac{E(P,P-)}{E(Q,P_-)}\right)^{(2\lambda-1)(g-1)} \left(\frac{\sigma(P)}{\sigma(Q)}\right)^{(2\lambda-1)} \times \times \frac{\theta[^{\alpha}_{\beta}](Q-P+u(\lambda))}{\theta[^{\alpha}_{\beta}](u(\lambda))} \tag{II.57}$$

where

$$u(\lambda) = -(2\lambda-1)(g-1)P_- + (2\lambda-1)\Delta \quad ,$$

$g \geq 2$, Ramond sector

$$S(P,Q) = \frac{1}{E(P,Q)} \left(\frac{E(P,P-)}{E(Q,P-)}\right)^{(2\lambda-1)(g-1)+1/2} \left(\frac{\sigma(P)}{\sigma(Q)}\right)^{(2\lambda-1)} \times \times \left(\frac{E(P,P+)}{E(Q,P_+)}\right)^{1/2} \frac{\theta[^{\alpha}_{\beta}](Q-P+u(\lambda))}{\theta[^{\alpha}_{\beta}](u(\lambda))} \tag{II.58}$$

where

$$u(\lambda) = 1/2(P_+-P_-) - (2\lambda-1)(g-1)P_- + (2\lambda-1)\Delta \quad .$$

In the $\lambda=1/2$ case, formulas (II.57,58) still hold, except when the spin structure is odd and the sector is NS, for which we have

$$S(P,Q) = \frac{E(P,P-)\ E(Q,P+)\ \theta[^{\alpha}_{\beta}](Q - P + P+ - P-)}{E(P,Q)\ E(P,P_+)\ E(Q,P_-)\ \theta[^{\alpha}_{\beta}](P_+ - P_-)} \qquad (II.59)$$

If g=1, and the spin structure is odd, the propagator in the NS sector for any half-integer λ is given by

$$S(P,Q) = \frac{1}{E(P,Q)}\ \frac{E(P,P-)}{E(P,P_+)}\ \frac{E(Q,P+)}{E(Q,P_-)}\ \left(\frac{\sigma(P)}{\sigma(Q)}\right)^{(2\lambda-1)} \times$$

$$\times \frac{\theta[^{\alpha}_{\beta}](Q-P+u(\lambda))}{\theta[^{\alpha}_{\beta}](u(\lambda))} \qquad (II.60)$$

where

$$u(\lambda) = P_+ - P_- + (2\lambda-1)\Delta\ .$$

For any other case with g=1, the propagator is given by eq.(III.57,58).

We finally make some remarks concerning to the N-points correlation functions. They can be calculated by using Wick's theorem. The only non-vanishing correlation functions are of the form ${}_{\Sigma}(0|T\{\prod_{i=1}^{N}(b(P_i)c(Q_i))\}|0)_{\Sigma}$. The rule to calculate them is

$${}_{\Sigma}(0|T\{\prod_{i=1}^{N}(b(P_i)c(Q_i))\}|0)_{\Sigma} =$$

$$= \begin{cases} \sum_{\sigma} (-)^{\sigma} \prod_{i=1}^{N} {}_{\Sigma}(0|T\{b(P_i)c(Q_i)\}|0)_{\Sigma} \\ \sum_{\sigma} \prod_{i=1}^{N} {}_{\Sigma}(0|T\{b(P_i)c(Q_i)\}|0)_{\Sigma} \end{cases} \qquad (II.61)$$

corresponding to an anticommuting (up) or commuting (down) b-c system; σ runs over all permutations.

6.Zero modes and Teichmüller deformations

It is interesting to note that the KN bases have among their elements the zero modes for λ-differentials.

For example, we observe that from the explicit expressions given in Chapter I, Section 8, the basis of meromorphic vector fields has three zero modes for i=±1,0 when g=0, one zero mode when g=1 (corresponding to i=1/2), and no zero mode if g≥2.

It is a well-known result that the number of zero modes of two-differentials sometimes called quadratic differentials coincides with the dimension of the moduli space. In fact, for λ=2 eq.(I.32) becomes

$$\Omega^j(z_\pm) = \varphi_j^{(2)\pm} \; z_\pm^{\pm j-2+g_o} \; (1+o(z_\pm)) \; (dz_\pm)^2 \tag{II.62}$$

This is a zero mode provided that $|j|\le g_o-2$; therefore there are 3g-3 quadratic differentials for $g\ge 2$ and no zero modes for g=0. If g=1 we saw that there is only one holomorphic section of K^λ for any $\lambda \in Z$, that labeled by i=1/2 in eq.(I.45).

On the other hand, we know that the number of zero modes of the 3/2-differentials plus the number of the quadratic differentials gives the dimension of the supermoduli space. In fact, for λ=3/2 we obtain (cf. eq.(I.47))

$$f_n^{(3/2)}(z_\pm) = a_n^{(3/2)\pm} \; z_\pm^{\pm n+g-3/2} \; (1+o(z_\pm))(dz_\pm)^{3/2} \tag{II.63}$$

from where we observe the existence of 2g-2 zero modes.

The explicit global expressions of the zero modes can be obtained from the formulas of Chapter I. As an example, let us write the basis of quadratic differentials (λ=2)

$$\Omega^j(P) = N_j^{(2)}(P_+,P_-) \frac{E(P,P+)^{j+g_o-2}}{E(P,P_-)^{j-g_o+2}} \sigma(P)^3 \; \theta(P+e(2,j)) \; .$$

where

$$e(2,j) = (j+g_o-2)P_+ - (j-g_o+2)P_- + 3\lambda\Delta$$

and

$$N_j^{(2)}(P_+,P_-) = \frac{E(P+,P-)^{j-g_o+2} \; \sigma(P+)^{-3} \; h(P+)^{3g}}{\theta((j+g_o-1)P_+ - (j-g_o+2)P_- - 3\lambda\Delta)}$$

for $j = -g_o+2,\dots,g_o-2$.

The dual to the Ω^j's form a basis for the Beltrami differentials μ^i. They obey the duality relation

$$\frac{1}{2\pi i}\int_\Sigma \mu_i \; \Omega^j = \delta_i^{\;j} \tag{II.64}$$

In order to discuss the deformations of the complex structure of Σ we will closely follow reference [3]. The vector fields e_i with $|i|\le g_o-2$ can be used to generate Teichmüller deformations of the Riemann surface in the following way. Divide the Riemann surface in two parts Σ^+ and Σ^- containing P_+ and P_- respectively such that Σ^+ be a small disk whose center is $P_+$and $\Sigma^+\cap\Sigma^-=A$, where A is an annulus. Take a local coordinate z on the disk. We can use the vector field e_i to obtain a new Riemann surface as follows. We deform $A \to A'$ by

$$z \to z + \epsilon e_i \quad , \; z \in A, \; \epsilon \in C \tag{II.65}$$

where $e_i = e_i(z)\partial_z$. Now Σ^- is glued to the disk Σ^+ by identifying the new annulus with the previous collar on Σ^+. This new Riemann surface is not analytically equivalent to the old one when e_i has poles both in P_+ and P_- which corresponds to $|i| \le g_o - 2$.

Under the infinitesimal deformation (II.65) the metric transforms to

$$\gamma(\mu) \propto |dz + \epsilon \, \mu_i \, d\bar{z}|^2 \tag{II.66}$$

where

$$\mu_i(P) = \begin{cases} \bar{\partial} \, \tilde{e}_i & \text{if } P \in \Sigma^+ \\ 0 & \text{if } P \in \Sigma^- - A \end{cases} \tag{II.67}$$

Integrating by parts we observe that the Beltrami differentials defined by (II.67) satisfy (II.64).

Now we are able to give the explicit expression for the variation of the period matrix τ under Teichmüller deformations[20]. Under the deformation of the complex structure given by (II.66,67) we have

$$\delta_k \tau_{ij} \equiv -\int_\Sigma \eta^i \, \eta^j \, \mu_k = -\oint_C \eta^i \, \eta^j \, e_k \tag{II.68}$$

where the integration countour C separates P_+ and P_-. Then it is easy to see that the variation $\delta_k \tau_{ij}$ vanishes if $|k| \ge g_o - 1$. Now suppose v is a linear combination of meromorphic vector fields e_k of the KN basis. Then the most general infinitesimal variation of the period matrix is given by eq.(II.68) with e_k replaced by (for $g>1$)

$$v = \sum_{k=-g_o+2}^{g_o-2} y_k \, e_k$$

where the e_k's are given in (I.55). The y_k's are the *moduli* of the surface, or better, form a local coordinate system in M^g around the particular surface with period matrix τ deformed according to (II.68).

7. Connection with the path integral approach

Zero modes enter in an essential way also in a definition of the vacuum state which is mutuated from the path integral approach. It would be natural to ask that $b(P)|0\rangle_\Sigma$, $c(P)|0\rangle_\Sigma$ be finite in P_+, and ${}_\Sigma\langle 0|b(P)$, ${}_\Sigma\langle 0|c(P)$ be finite in P_-. This leads for $\lambda>1$ and $g>1$ to

$$c^k|0\rangle_\Sigma = 0 \quad , \; k < s(1-\lambda) \tag{II.69a}$$

$$b_k|0\rangle_\Sigma = 0 \quad , \; k \geq s(1-\lambda) \tag{II.69b}$$

and

$${}_\Sigma\langle 0|c^k = 0 \quad , \; k > -s(1-\lambda) \tag{II.70a}$$

$${}_\Sigma\langle 0|b_k = 0 \quad , \; k \leq -s(1-\lambda) \tag{II.70b}$$

as in Section 4, eqs.(II.23,34). If we define the propagator by $\langle b(P)c(Q)\rangle = {}_\Sigma\langle 0|T\{b(P)c(Q)\}|0\rangle_\Sigma$, then we would obtain a vanishing result. This is because the vacuum defined above gives ${}_\Sigma\langle 0|0\rangle_\Sigma=0$, as it can be seen by using the algebra. For example, for even g and anticommuting c and b we have

$$0 = {}_\Sigma\langle 0|b_0c_0|0\rangle_\Sigma = {}_\Sigma\langle 0|0\rangle_\Sigma - {}_\Sigma\langle 0|c_0b_0|0\rangle_\Sigma = {}_\Sigma\langle 0|0\rangle_\Sigma \; .$$

This fact is similar to what takes place in the path integral approach

$$Z = \int [db\; dc]\; \exp(-S[b,c]) = 0$$

because of the presence of the zero modes.

In order to obtain meaningful results the partition function Z is redefined by inserting as many b and c fields as the number N and M of the corresponding zero modes, that is (for any λ and g)

$$Z(z_1,\ldots,z_N;w_1,\ldots,w_M) =$$

$$= \int [db\; dc]\; b(z_1)..b(z_N)\; c(w_1)..c(w_M)\; \exp(-S[b,c]) \tag{II.71}$$

A correlation function is then defined with respect to this measure

$$\langle b(P_1)\ldots b(P_r)\; c(Q_1)\ldots c(Q_s)\rangle = Z(z_1,\ldots,z_N;w_1,\ldots,w_M)^{-1} \times$$

$$\times \int [db\; dc]\; b(z_1)\ldots c(w_M)\ldots b(P_1)\ldots c(Q_s)\; \exp(-S[b,c]) \tag{II.72}$$

In our operator formalism we proceed likewise and define a correlation function in the following way

$$<b(P_1)\dots b(P_r)\ c(Q_1)\dots c(Q_s)> =$$

$$= \frac{{}_{\Sigma}<0|T\{b(z_1)\dots c(w_M)\dots b(P_1)\dots c(Q_s)\}|0>_{\Sigma}}{{}_{\Sigma}<0|T\{b(z_1)\dots c(w_M)\}|0>_{\Sigma}} \qquad \text{(II.73)}$$

We will show that (II.72,73) lead to the same results and also are in agreement with those of Section 5, provided we identify at the end of the computations the points z_i with P_- and w_i with P_+.

Let us first consider the case $\lambda>1$ for $g\geq 2$. The propagator is then defined according to (II.73)

$$<b(P)c(Q)> = \frac{{}_{\Sigma}<0|T\{b(z_1)\dots b(z_N)\ b(P)\ c(Q)\}|0>_{\Sigma}}{{}_{\Sigma}<0|T\{b(z_1)\dots b(z_N)\}|0>_{\Sigma}} \qquad \text{(II.74)}$$

where $N=-2s(\lambda)+1$.

Let us choose $z_1,\dots, z_N$ such that $\tau_1>\dots>\tau_N$; then we have

$${}_{\Sigma}<0|T\{b(z_1)\dots b(z_N)\}|0>_{\Sigma} = {}_{\Sigma}<0|b(z_1)\dots b(z_N)|0>_{\Sigma} =$$

$$= \det|f^i(z_j)|\ {}_{\Sigma}<0|b_{+s(\lambda)}\dots b_{-s(\lambda)}|0>_{\Sigma} \quad , \quad i=s(\lambda),\dots,-s(\lambda) \qquad \text{(II.75)}$$

Let us take now z_i near P_- and consider first $\tau_P>\tau_Q$ so that

$${}_{\Sigma}<0|T\{b(z_1)\dots b(P)c(Q)\}|0>_{\Sigma} = {}_{\Sigma}<0|b(z_1)\dots b(P)c(Q)|0>_{\Sigma} =$$

$$= \sum_{\{i_k\},j,k} {}_{\Sigma}<0|b_{i_1}\dots b_{i_N} b_j c^k|0>_{\Sigma}\ f^{i_1}(z_1)\dots f^{i_N}(z_N)\ f^j(P)\ f_k(Q)$$

$$= \sum_{\text{perm.}} (-)^P\ {}_{\Sigma}<0|b(z_1)\dots \check{b}(z_i)\dots b(z_N)\ b(P)|0>_{\Sigma} \sum_{j=s(1-\lambda)}^{\infty} f^j(z_i)\ f_j(Q) \qquad \text{(II.76)}$$

where z_i takes the values $z_1,\dots, z_N$, P. In a similar way one calculates the contribution for $\tau_P<\tau_Q$. The final result is given by

$$<b(P)c(Q)> = \det|g_i(z_j)|^{-1}\ \det\begin{pmatrix} S(P,Q) & S(z_1,Q) & \dots & S(z_N,Q) \\ g_1(P) & g_1(z_1) & \dots & g_1(z_N) \\ \dots & \dots & \dots & \dots \\ g_N(P) & g_N(z_1) & \dots & g_N(z_N) \end{pmatrix} \qquad \text{(II.77)}$$

where $g_1,\dots, g_N$ stands for $f^{+s(\lambda)},\dots, f^{-s(\lambda)}$.

By taking the limit $z_i\to P_-$, for which $S(z_i,Q)=0$ we get

$$\lim_{z_i\to P_-} <b(P)\ c(Q)> = S(P,Q) \qquad \text{(II.78)}$$

as it was asserted.

Let us consider as another example the case $\lambda=1$. We obtain

$$<b(P)\ c(Q)> = \det|\omega_i(z_j)|^{-1} \times$$

$$\times \det\begin{pmatrix} S(P,Q)-S(P,w) & S(z_1,Q)-S(z_1,w)\dots\dots S(z_g,Q)-S(z_g,w) \\ \omega_1(P) & \omega_1(z_1)\dots\dots\dots\dots\omega_1(z_g) \\ \dots & \dots\dots\dots\dots \\ \omega_g(P) & \omega_g(z_1)\dots\dots\dots\dots\omega_g(z_g) \end{pmatrix} \tag{II.79}$$

where the ω_i's, $i=1,\dots,g$ stands for the ω^j's, $j\in I$. In the limit $z_i\to P_-$, $w\to P_+$, we recover the expected result

$$\lim_{z_i\to P_-,\ w\to P_+} <b(P)\ c(Q)> = S(P,Q) \tag{II.80}$$

PART C: The scalar field[12]

8.Hamiltonian and equation of motion

Let us consider a free scalar field X(Q) on a Riemann surface Σ with the action

$$S[X,h] = -\frac{1}{4\pi}\int_{\Sigma} d^2x|h(x)|^{1/2}\, h^{ab}(x)\, \partial_a X(x)\, \partial_b X(x) \tag{II.81}$$

The phase space for this theory will be the space of functions X(Q) and differentials P(Q), with the Poisson bracket

$$\{P(Q),X(Q')\} = -\Delta_\tau(Q,Q') \quad ; \quad Q,Q' \in C_\tau \tag{II.82a}$$

where $\Delta_\tau(Q,Q')$ is the delta-function (cf. eq.(II.5))

$$\Delta_\tau(Q,Q') = 1/(2\pi i) \sum_j \omega^j(Q)\, A_j(Q') \tag{II.82b}$$

The components of the EMT are defined by

$$T(Q) = -1/4\, (dX(Q) + 2\pi P(Q))^2 \tag{II.83a}$$

$$\overline{T}(Q) = -1/4\, (dX(Q) - 2\pi P(Q))^2 \tag{II.83b}$$

T and $\overline{T}$ are holomorphic and antiholomorphic respectively outside $P_\pm$ provided X and P obey their equations of motion (see below).

Inserting (II.83) into (II.8a) we obtain the Hamiltonian, which takes the form

$$H(\tau) = 1/8\pi \oint_{C_\tau} (e_\sigma|(dX+2\pi P)^2 + (dX-2\pi P)^2) =$$

$$= 1/4\pi \oint_{C_\tau} (e_\sigma|dX^2 + 4\pi^2 P^2) \tag{II.84}$$

Now we impose the equations of motion which follow from this Hamiltonian. These are easily obtained by using (II.82)

$$L_{e_\tau} X(Q) \equiv i\{H,X(Q)\} = -2\pi i\, (P|e_\sigma)(Q) \tag{II.85a}$$

$$L_{e_\tau} P(Q) \equiv i\{H,P(Q)\} = 1/(2\pi i)\, d(dX(Q)|e_\sigma) \tag{II.85b}$$

The first one tells us that $P(Q)=1/(2\pi)(\partial-\overline{\partial})X(Q)$, so eqs.(II.83) become

$$T(Q) = -\partial X(Q)\, \partial X(Q) \quad , \quad \overline{T}(Q) = -\overline{\partial} X(Q)\, \overline{\partial} X(Q) \tag{II.86}$$

By contracting with e_σ, eq.(II.85b) becomes

$$L^2_{e_\tau} X(Q) = -L^2_{e_\sigma} X(Q) \qquad \text{(II.87)}$$

which is the analogue to the g=0 equation of motion $\ddot{X}+X''=0$. Equation (II.87) can also be written as

$$\partial\bar{\partial}X(Q)=0 \qquad \text{(II.88)}$$

It must hold everywhere except the points P_+ and P_- because they correspond to times $-\infty$ and $+\infty$ respectively. This already happens at genus zero where the points $z=0$ and $z=\infty$ are excluded. This implies that $\partial X(Q)$ $(\bar{\partial}X(Q))$ is holomorphic (antiholomorphic) everywhere except $P_\pm$. Therefore they can be expanded as follows[8]

$$\partial X(Q) = i/\sqrt{2} \sum_n \alpha_n \, \omega^n(Q) \qquad \text{(II.89a)}$$

$$\bar{\partial} X(Q) = i/\sqrt{2} \sum_n \bar{\alpha}_n \, \bar{\omega}^n(Q) \qquad \text{(II.89b)}$$

Now the requirement of single-valuedness for X implies the following relations $(d=\partial+\bar{\partial})$

$$\oint_{C_\tau} dX = 0 \qquad \text{(II.90a)}$$

$$\oint_{a_i} dX = 0 = \oint_{b_i} dX \qquad \text{(II.90b)}$$

where (a_i, b_i), $i=1,\ldots,g$, is a canonical basis of homology. By inserting the expansions (II.89), one finds that eq.(II.90a) implies

$$\alpha_{g/2} = \bar{\alpha}_{g/2} \equiv -p/\sqrt{2} \qquad \text{(II.91)}$$

just as in the genus zero case, whereas eqs.(II.90b) imply

$$\sum_j (\alpha_j \, a^j_i + \bar{\alpha}_j \, \bar{a}^j_i) = 0 \qquad \text{(II.92a)}$$

$$\sum_j (\alpha_j \, b^j_i + \bar{\alpha}_j \, \bar{b}^j_i) = 0 \qquad \text{(II.92b)}$$

where

$$a^j_i = \oint_{a_i} \omega^j \quad ; \quad b^j_i = \oint_{b_i} \omega^j \qquad \text{(II.92c)}$$

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[8] These expansions were first written down in Ref.[21]; however the nature of the α_n's, $\bar{\alpha}_n$'s was not clarified there.

Eqs.(II.92a,b) can be written as

$$\sum_{j\in I} (\alpha_j \, a_i^j + \bar{\alpha}_j \, \bar{a}_i^j) = A_i \tag{II.93a}$$

$$\sum_{j\in I} (\alpha_j \, b_i^j + \bar{\alpha}_j \, \bar{b}_i^j) = B_i \tag{II.93b}$$

where

$$A_i = - \sum_{|n|>g/2} (\alpha_n \, a_i^n + \bar{\alpha}_n \, \bar{a}_i^n) \tag{II.93c}$$

$$B_i = - \sum_{|n|>g/2} (\alpha_n \, b_i^n + \bar{\alpha}_n \, \bar{b}_i^n) \tag{II.93d}$$

Eqs.(II.93) can be seen as a system of 2g equations with 2g unknowns. Recall that for $n \in I$, the ω^j's in eq.(I.42) form a basis of holomorphic differentials related with the standard (I.4) by

$$\omega^j = \sum_{k=1}^{g} a_k^j \, \eta^k \tag{II.94}$$

Then eqs.(II.93a,b) become

$$\epsilon_i + \bar{\epsilon}_i = A_i \tag{II.95a}$$

$$\sum_{j=1}^{g} (\epsilon_j \tau_{ji} + \bar{\epsilon}_j \bar{\tau}_{ji}) = B_i \quad , \quad i=1,\dots,g \tag{II.95b}$$

where

$$\epsilon_i = \sum_{j\in I} \alpha_j \, a_i^j \quad ; \quad \bar{\epsilon}_i = \sum_{j\in I} \bar{\alpha}_j \, \bar{a}_i^j \tag{II.96}$$

In matrix notation

$$\begin{pmatrix} 1 & 1 \\ \tau & \bar{\tau} \end{pmatrix} \begin{pmatrix} \epsilon \\ \bar{\epsilon} \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \tag{II.97}$$

It follows

$$\begin{pmatrix} \epsilon \\ \bar{\epsilon} \end{pmatrix} = i/2 \begin{pmatrix} \tau_2^{-1}\bar{\tau} & -\tau_2^{-1} \\ -\tau_2^{-1}\tau & \tau_2^{-1} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \tag{II.98}$$

X(Q) is obtained by integration

$$X(Q) = \int_{Q_o}^{Q} dX = x - ip\, \tau(Q) + i/\sqrt{2} \sum_{j\neq g/2} (\alpha_j \, B^j(Q) + \bar{\alpha}_j \, \bar{B}^j(Q)) \tag{II.99}$$

where

$$B^j(Q) = \int_{Q_o}^{Q} \omega^j \quad , \quad \tau(Q) = \mathrm{Re}\int_{Q_o}^{Q} \omega^{g/2}$$

By using eqs.(II.98) and (II.93c,d) the ϵ_i, $\bar{\epsilon}_i$, $i=1,\dots,g$, are expressed in terms of α_j , $\bar{\alpha}_j$ with $|j|>g/2$. One obtains

$$\epsilon^{\mu}_{i} = - \sum_{|j|>g/2} (F^{j}_{i}\, \alpha_{j} + G^{j}_{i}\, \bar{\alpha}_{j}) \qquad \text{(II.100a)}$$

$$\bar{\epsilon}_{i} = - \sum_{|j|>g/2} (\bar{G}^{j}_{i}\, \alpha_{j} + \bar{F}^{j}_{i}\, \bar{\alpha}_{j}) \qquad \text{(II.100b)}$$

with

$$F^{j}_{i} = i/2 \sum_{k=1}^{g} ((\tau_2^{-1}\bar{\tau})_{jk} a^{k}_{i} - (\tau_2^{-1})_{jk} b^{k}_{i}) \qquad \text{(II.100c)}$$

$$G^{j}_{i} = i/2 \sum_{i=1}^{g} ((\tau_2^{-1}\bar{\tau})_{jk} \bar{a}^{k}_{i} - (\tau_2^{-1})_{jk} \bar{b}^{k}_{i}) \qquad \text{(II.100d)}$$

Inserting this into eq.(II.99), it follows

$$X(Q) = x - ip\, \tau(Q) + i/\sqrt{2} \sum_{|n|>g/2} (\alpha_n\, \phi_n(Q) + \bar{\alpha}_n\, \bar{\phi}_n(Q)) \qquad \text{(II.101)}$$

where $\phi_n(Q)$ are harmonic (single-valued) functions given by

$$\phi_n(Q) = \int_{Q_o}^{Q} (\omega^n - \sum_{j=1}^{g} (F^{n}_{j}\, \eta^{j} + \bar{G}^{n}_{j}\, \bar{\eta}^{j})) \qquad \text{(II.102)}$$

with $Q_0 \equiv P_-$ (P_+) for $n>g/2$ $(<-g/2)$ (this particular choice is dictated from the genus zero case, and will become very useful). As in the tree level case, p represents the center of mass momentum, i.e., $p = \oint P$.

The next step is the quantization of the theory: the coefficients of these expansions become second quantized operators acting on a Fock space, whose commutation rules are to be derived from the canonical commutation relation

$$[P(Q),X(Q')] = -i\, \Delta_{\tau}(Q,Q') \quad ; \quad Q,Q' \in C_{\tau} \qquad \text{(II.103)}$$

This leads to the following commutation rules for $\alpha_n, \bar{\alpha}_n, p$, x[21]:

$$[\alpha_n,\alpha_m] = \gamma_{nm} \quad , \quad [\bar{\alpha}_n,\bar{\alpha}_m] = \bar{\gamma}_{nm}$$

$$[\alpha_n,\bar{\alpha}_m] = 0 \quad , \quad [x,p] = i \qquad \text{(II.104a)}$$

where

$$\gamma_{nm} = 1/(2\pi i) \oint_{C_{\tau}} dA_n\, A_m \qquad \text{(II.104b)}$$

In terms of α_n, $\bar{\alpha}_n$ the Hamiltonian and momentum operators take the following form

$$H(\tau) = 1/2 \sum_{n,m} (l^{nm}(\tau)\, :\alpha_n\, \alpha_m: + \bar{l}^{nm}(\tau)\, :\bar{\alpha}_n\, \bar{\alpha}_m:) \qquad \text{(II.105a)}$$

$$P(\tau) = i/2 \sum_{n,m} (l^{nm}(\tau)\, :\alpha_n\, \alpha_m: - \bar{l}^{nm}(\tau)\, :\bar{\alpha}_n\, \bar{\alpha}_m:) \qquad \text{(II.105b)}$$

where

$$1^{nm}(\tau) = 1/(2\pi i) \oint_{C_\tau} (e_k|\omega^n)\ \omega^m = 1^{mn}(\tau) \qquad \text{(II.105c)}$$

9.Fock space and computation of propagators

The Fock space is defined as usual to be the space generated by the α_n, $\bar{\alpha}_n$, with the vacuum state defined by

$$\alpha_n|0\rangle = \bar{\alpha}_n|0\rangle = 0 \quad , \quad n\geq g/2 \qquad \text{(II.106a)}$$

$$\langle 0|\alpha_n = \langle 0|\bar{\alpha}_n = 0 \quad , \quad n<-g/2 \text{ or } n=g/2 \qquad \text{(II.106b)}$$

The normal ordered product introduced in eqs.(II.105a,b) is that induced by eqs.(II.106)

$$:\alpha_n\alpha_m: = \begin{cases} \alpha_n\alpha_m & \text{if } n<-g/2 \text{ or } m>g/2 \\ \alpha_m\alpha_n & \text{if } n>g/2 \text{ or } m<-g/2 \end{cases} \qquad \text{(II.107)}$$

Consider first the correlation function $\langle\partial X(Q)\bar{\partial}X(Q')\rangle$. By definition it is given by

$$\langle 0|T\{\partial X(Q)\bar{\partial}X(Q')\}|0\rangle = f(Q,Q')\theta(\tau_Q-\tau_{Q'}) + g(Q,Q')\theta(\tau_{Q'}-\tau_Q)$$

(II.108a)

where

$$f(Q,Q') = \langle 0|\partial X(Q)\partial X(Q')|0\rangle \qquad \text{(II.108b)}$$

$$g(Q,Q') = \langle 0|\bar{\partial}X(Q')\partial X(Q)|0\rangle \qquad \text{(II.108c)}$$

By inserting the expansions (II.89) and using eqs.(II.106) one obtains

$$f(Q,Q') = \sum_{i,j=1}^{g} C_{ij}\ \eta^i(Q)\ \bar{\eta}^j(Q') = g(Q,Q') \qquad \text{(II.109)}$$

where

$$\begin{aligned} C_{ij} &= -1/2\ \langle 0|\epsilon_i\ \bar{\epsilon}_j\ |0\rangle \\ &= -1/2 \sum_{m<-g/2}\ \sum_{n>g/2} (\gamma_{nm}F^n_i\ \bar{G}^m_j + \bar{\gamma}_{nm}\bar{F}^m_j\ G^n_i) \end{aligned} \qquad \text{(II.110)}$$

In order to calculate C_{ij} we use the relations

$$\sum_m \gamma_{nm}\ F^m_i = \sum_m \gamma_{nm}\ \bar{G}^m_i = 0 \qquad \text{(II.111)}$$

where one has

$$F^n_i = a^n_i \quad , \quad G^n_i = 0 \quad ; \qquad n \in I \qquad \text{(II.112)}$$

Therefore C_{ij} can be rewritten as

$$C_{ij} = - \sum_{n\in I} a_i^n D_{nj} \tag{II.113}$$

with

$$D_{nj} = 1/2 \sum_{m>g/2} \gamma_{nm} \bar{G}_j^m \tag{II.114}$$

By using eq.(II.100d) it follows

$$D_{nj} = i/4 \sum_{i=1}^{g} (\tau_2^{-1})_{ji} K_n^i \tag{II.115}$$

where

$$K_n^i = \sum_{m>g/2} \gamma_{nm} (b_i^m - \sum_{j=1}^{g} \tau_{ik} a_k^m) \tag{II.116}$$

Now we use the following identity (see appendix at the end of this section, eq.(II.135))

$$b_i^m - \sum_{j=1}^{g} \tau_{ij} a_i^m = \oint_{C_+} I_i \omega^m \quad , \quad m>g/2 \tag{II.117}$$

where C_+ is a small contour around P_+. It follows

$$K_n^i = \oint_{C_+} I_i \sum_{m>g/2} \gamma_{nm} \omega^m = \oint_{C_+} I_i dA_m = -2\pi i \ (a^{-1})_n^i \tag{II.118}$$

where we have used the fact that ω^n, $n<g/2$ are holomorphic in P_+. Now from (II.115) we have

$$D_{nj} = \pi/2 \sum_{l=1}^{g} (\tau_2^{-1})_{jl} (a^{-1})_n^l \tag{II.119}$$

Inserting this result into (II.113) we get

$$C_{ij} = -\pi/2 \ (\tau_2^{-1})_{ij} \tag{II.120}$$

Thus we finally obtain

$$<0|T\{\partial X(Q)\bar{\partial} X(Q')\}|0> = -\pi/2 \sum_{i,j=1}^{g} (\tau_2^{-1})_{ij} \eta^i(Q) \bar{\eta}^j(Q') \tag{II.121}$$

which coincides with the well-known result quoted in the literature, computed by other methods [20,22].

Let us now consider the correlation function $<\partial X(P)\partial X(Q)>$. This is

$$<0|T\{\partial X(P)\partial X(Q)\}|0> = A(P,Q)\theta(\tau_P-\tau_Q) + A(Q,P)\theta(\tau_Q-\tau_P)$$

$$A(P,Q) \equiv <0|\partial X(P)\partial X(Q)|0> \tag{II.122}$$

From eq.(II.89a) we have

$$A(P,Q) = -1/2 \sum_{n,m} \langle 0|\alpha_n \alpha_m|0\rangle\, \omega^n(P)\, \omega^m(Q) \qquad \text{(II.123)}$$

By using the vacuum definition and the commutation rules for the α_n's one finds

$$A(P,Q) = -1/2(\, A_1\,(P,Q) + A_2\,(P,Q) + A_3\,(P,Q)\,) \qquad \text{(II.124a)}$$

where

$$A_1(P,Q) = \sum_{\substack{n>g/2\\ m}} \gamma_{nm}\, \omega^n(P)\, \omega^m(Q) \qquad \text{(II.124b)}$$

$$A_2(P,Q) = \sum_{\substack{n\in I\\ m<-g/2}} \gamma_{nm}\, \omega^n(P)\, \omega^m(Q) \qquad \text{(II.124c)}$$

$$A_3(P,Q) = \sum_{i,j=1}^{g} \langle 0|\epsilon_i \epsilon_j|0\rangle\, \eta^i(P)\, \eta^j(Q) \qquad \text{(II.124d)}$$

Since $\partial A_n(Q)=\sum_m \gamma_{nm}\omega^m(Q)$, $A_1(P,Q)$ can be written as

$$A_1(P,Q) = \sum_{n>g/2} \partial A_n(Q)\, \omega^n(P) \equiv \partial_Q S(P,Q) \qquad \text{(II.125)}$$

where S(P,Q) is given by eq.(II.53).

In order to compute $\langle\epsilon_i\epsilon_j\rangle$ we use eqs.(II.100). These leads to

$$\langle\epsilon_i\epsilon_j\rangle = \sum_{m<-g/2}\ \sum_{n>g/2} (\, \gamma_{nm}\, F^n_i\, F^m_j + \bar{\gamma}_{nm}\, G^m_j\, G^n_i) \qquad \text{(II.126)}$$

The second sum on the r.h.s. is easily shown to vanish because of eqs.(II.111,112). Into the first term we replace

$$F^m_j = -\bar{G}_j^{\ m} + a^m_j \qquad \text{(II.127)}$$

Then we use the result found above (eq.(II.119)). We obtain

$$\langle\epsilon_i\epsilon_j\rangle = -\pi\, (\tau_2^{-1})_{ij} - \sum_{n\in I} a^n_i \sum_{m<-g/2} \gamma_{nm} a^n_i) \qquad \text{(II.128)}$$

Now eq.(II.124a) can be written as

$$A(P,Q) = -1/2\ (\, \sum_{n\in I} \omega^n(P) \sum_{m<-g/2} \gamma_{nm}\ (\omega^m(Q) - \sum_{j=1}^{g} a^m_j\, \eta^j(Q))\ -$$

$$- \pi \sum_{i,j=1}^{g} (\tau_2^{-1})_{ij}\, \eta^i(P)\, \eta^j(Q) + \partial_Q S(P,Q) \qquad \text{(II.129)}$$

By inserting this expression into (II.122) one arrives to the conclusion that the correlation function (II.122) has only a (double) pole at P=Q coming from $\partial_Q S(P,Q)$. By Riemann-Roch theorem, up to the addition of holomorphic terms and a multiplicative constant, there exists only one 1-form which has a double pole and which is holomorphic everywhere else. Therefore (II.122) can be written in the following form

$$<0|T\{\partial X(P)\partial X(Q)\}|0> = -1/2\ \partial_P\partial_Q \log E(P,Q) + \sum_{i,j=1}^{g} A_{ij}\ \eta^i(P)\eta^j(Q) \tag{II.130}$$

where A_{ij} is a g×g matrix which can be determined by integrating (II.129) along the a_i-cycles in both variables Q and P. After integration in the variable Q, the term $\partial_Q S(P,Q)$ disappears, and so does the first term on the r.h.s of (II.130). One finally obtains

$$A_{ij} = \pi/2\ (\tau_2^{-1})_{ij} \tag{II.131}$$

Thus we get[22,23]

$$<0|T\{\partial X(P)\partial X(Q)\}|0> = -1/2\ (\partial_P\partial_Q \log E(P,Q) - \pi \sum_{i,j=1}^{g} (\tau_2^{-1})_{ij}\ \eta^i(P)\ \eta^j(Q)) \tag{II.132}$$

Appendix

Let us consider a Riemann surface Σ of genus g with a given marking (a_i,b_i), i=1,...,g, As we saw in Chapter I, Section 4, this basis has the property that under cutting along these cycles, Σ becomes a 4g-gon $\tilde{\Sigma}$. Each cycle goes to a pair of sides (a_i,a_i^{-1}), (b_i,b_i^{-1}) of $\tilde{\Sigma}$, which are identified on Σ (see Fig.II.4 for g=2). Let $\{\eta^i\}$ be the standard basis of holomorphic differentials, and $P \in \Sigma$. Then the Jacobi map (I.14)

$$I(P) = \int_{P_0}^{P} \eta$$

defines functions which are single-valued *on the cut surface* $\tilde{\Sigma}$. It is straightforward to verify that the following relations hold[7]

$$I_i(P_j) - I_i(P'_j) = -\int_{P_j}^{P'_j} \eta_i \equiv -\oint_{b_j} \eta_i = -\tau_{ij} \quad ; P_j \in a_j,\ P'_j \in a_j^{-1} \tag{II.133a}$$

$$I_i(P_j) - I_i(P'_j) = -\int_{P_j}^{P'_j} \eta_i \equiv \oint_{a_j} \eta_i = \delta_{ij} \quad ; P_j \in b_j,\ P'_j \in b_j^{-1} \tag{II.133b}$$

where P_j and P'_j are identified on Σ (see Fig.II.4).

Now let us take a meromorphic one-form ω, holomorphic outside P_0. Let D be a domain containing P_0 whose boundary is a contour C_0. Since ω is closed ($d\omega=0$) on $(\tilde{\Sigma}-D)$, a simple application of the Cauchy's theorem yields

$$0 = \int_{\tilde{\Sigma}-D} \eta^i \wedge \omega = \int_{\tilde{\Sigma}-D} d(I_i \omega) = -\oint_{C_0} I_i \omega + \\ + \sum_{j=1}^{g} \left((\oint_{a_j} + \oint_{a_j^{-1}}) I_i \omega + (\oint_{b_j} + \oint_{b_j^{-1}}) I_i \omega \right) \qquad \text{(II.134)}$$

Then, by using the relations (II.133) we finally get

$$\oint_{b_i} \omega - \sum_{j=1}^{g} \tau_{ij} \oint_{a_j} \omega = \oint_{C_0} I_i \, \omega \qquad \text{(II.135)}$$

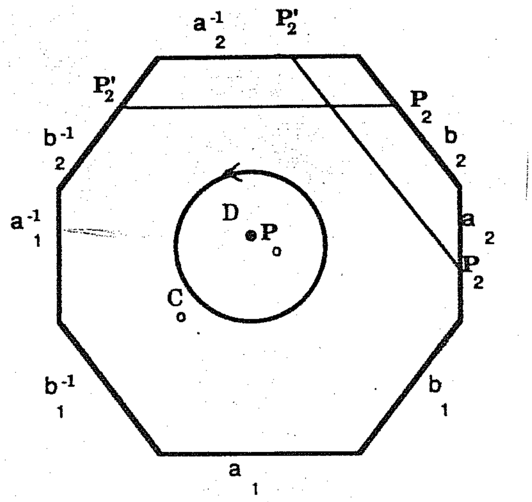

Fig.II.4: The "cut" Riemann surface for g=2

CHAPTER III:

APPLICATION TO STRING THEORIES

PART A: The role of the KN algebra in CFT[24]

1.The expectation value of the energy-momentum tensor

In ref.[21] KN introduced the energy-momentum tensor presented in Chapter II, Section 2, within the framework of an operator formalism for string theory, and showed that its components should realize the algebra (I.59). However the nature of the KN EMT and in particular its relation with the standard EMT defined in the path integral approach (PIA) to CFT has not been clarified there. In this section we will do it.

As we saw in Chapter II, the KN EMT can be decomposed as

$$t = T + \overline{T}$$

where T $(\overline{T})$ is (on-shell) holomorphic (antiholomorphic) outside $P_{\pm}$, obeying the expansions (II.13). In what follows we will put special attention in the T-component of the EMT. The results concerning the antiholomorphic part will become obvious.

T may be consider as the generator of infinitesimal conformal transformations

$$z \rightarrow z + \epsilon(z) \quad , \epsilon = \epsilon(z)\partial_z = \sum_i \epsilon^i e_i \quad , \epsilon^i \in C$$

on the punctured surface; in particular the operator L_i is the generator of $z \rightarrow z + e_i(z)$. Thus the KN operators become a generalization of the Virasoro operators on the sphere.

Coming back to the standard EMT considered in the path integral formalism, this is an object T^{PI} which presents the following operator product expansions[15]

$$T^{PI}(z)T^{PI}(w) = c/2\ (z-w)^{-4} + 2\ T^{PI}(w)\ (z-w)^{-2} + \\ + \partial_w T^{PI}(w)\ (z-w)^{-1} + \text{reg.terms}$$

$$T^{PI}(z)\phi(w) = \sum_n (z-w)^{-n-2}\ L^V_n\ \phi(w) \tag{III.1}$$

where ϕ is an arbitrary field. These statements are equivalent to saying that the L^V_n-operators in (III.1) satisfy the Virasoro algebra (I.62).

The KN EMT presents however two obvious differences with respect to (III.1):

i) It is a well-defined meromorphic two differential, at difference of T^{PI} that transforms with a schwartzian derivative[14,15];

ii) Due to the definition via normal ordering of the L_i operators the vacuum expectation value (VEV) of T (and $\bar{T}$) should be zero.

In the present operator formalism, we guess that the analoguous object T^S should be one that, in neighborhoods of P_+ where the following expansion holds ($z_+(P_+)=0$)

$$T^S(z_+)dz_+^2 = \sum_n L_n^S z_+^{-n-2} dz_+^2 \tag{III.2}$$

be so that the L_n^S-operators verify (I.62). It is not difficult to show by using the expansions of the KN bases near P_+ as in eqs.(II.30,31) that the desired object is given by

$$T^S(z) dz^2 = T(z) dz^2 - c/12 R(z) dz^2 \tag{III.3}$$

From the remarks exposed above it follows

$$\langle T^s(z)\rangle = \langle 0|T^s(z)|0\rangle = - c/12 R(z) \tag{III.4}$$

This equation tells us that the VEV of the standard EMT of an arbitrary QCFT should be essentially given by the schwartzian connection of the KN algebra associated with the theory. In the following sections we shall verify eq.(III.4) in the simple cases of the free field theories studied in the precedent chapter.

2.Computation of the KN albebra for the ghost system

Let us consider an anticommuting chiral b-c system of weights λ and $(1-\lambda)$ respectively. For simplicity we will consider the general case $g>1$, integer $\lambda>1$ or half integer $\lambda\geq1/2$ in the Neveau-Schwarz sector with some defined spin structure (α,β), avoiding the special case $\lambda=1/2$, odd spin structure. The corresponding results for the other cases are straightforwardly obtained by following the same steps presented here and taking into account the results of Section II, Part B.

We will denote in this chapter by $I(q)=q(g-1)=-2s(\lambda)+1$ the number of independent zero modes in K^{λ}, $q=2\lambda-1$.

With respect to the vacuum definition (II.23,26) the normal ordering is

$$:b_n c^m: = \begin{cases} b_n c^m & \text{if } n\le -s(\lambda) \\ -c^m b_n & \text{if } n> -s(\lambda) \end{cases} \tag{III.5}$$

The EMT is given by eq.(II.15a). By using the expansions (II.20) and eq.(II.13a) we get the KN operators

$$L_i = \sum_{n,m} S^n_{im} :c_n b^m: \tag{III.6}$$

$$S^n_{im} = 1/(2\pi i) \oint_{C_\tau} e_i \, (\lambda\, \partial f_n^{(1-\lambda)} f^m_{(\lambda)} - (1-\lambda)\, f_n^{(1-\lambda)} \partial f^m_{(\lambda)})$$

From eqs.(II.22) and (III.5,6) we learn its commutator[9]

$$[L_i, L_j] = \sum_{s=-g_0}^{g_0} C^s_{ij}\, L_{i+j-s} + R_{ij} \tag{III.7}$$

where

$$R_{ij} = \sum_{n\le -s(\lambda)} \sum_{m>-s(\lambda)} S^n_{im}\, S^m_{jn} - (i\leftrightarrow j) \tag{III.8}$$

where the antisymmetry of R_{ij} has been used.

Now let us consider the propagators of these systems given by the kernels (II.40)

$$\begin{aligned} K_\lambda(P,Q) &= \langle 0|T\{b(P)c(Q)\}|0\rangle = \\ &= E(P,Q)^{-1}\, (E(P,P_-)/E(Q,P_-))^{I(q)}\, (\sigma(P)/\sigma(Q))^q \times \\ &\times \theta[^\alpha_\beta](Q-P+v(q))/\theta[^\alpha_\beta](v(q)) \end{aligned}$$

$$v(q) = q\Delta - I(q)P_- \tag{III.9}$$

Taking into account the definition of S^n_{im} eq.(III.6) we can rewrite R_{ij} as follows

$$\begin{aligned} R_{ij} = &- \oint_{C_{\tau_P}} \oint_{C_{\tau_Q<\tau_P}} e_i(P)\, e_j(Q) \times [\, \lambda^2\, \partial_P K_\lambda(Q,P)\, \partial_Q K_\lambda(P,Q) + \\ &+ (1-\lambda)^2\, \partial_P K_\lambda(P,Q)\, \partial_Q K_\lambda(Q,P) - \lambda(1-\lambda)\, (\partial_P\partial_Q K_\lambda(P,Q)\, K(Q,P) + \\ &+ \partial_P\partial_Q K_\lambda(Q,P)\, K_\lambda(P,Q))\,] - (i\leftrightarrow j) = \\ &= - \oint_{C_{\tau_P}} e_j(P) \oint_{C_P} e_i(Q)\, [\, \lambda^2\, \partial_P K_\lambda(Q,P)\, \partial_Q K_\lambda(P,Q) + \\ &+ (1-\lambda)^2\, \partial_P K_\lambda(P,Q)\, \partial_Q K_\lambda(Q,P) - \lambda(1-\lambda)\, (\partial_P\partial_Q K_\lambda(P,Q)\, K_\lambda(Q,P) + \\ &+ \partial_P\partial_Q K_\lambda(Q,P)\, K_\lambda(P,Q))\,] \end{aligned} \tag{III.10}$$

[9] The computation of the first term in the RHS of (III.7) was carried out in Refs.[25,26].

where C_P is a contour enclosing P. Now let us consider the expansion of the kernels (III.9) $K_\lambda(P,Q)=K_\lambda(z,w)dz^\lambda dw^{(1-\lambda)}$ for w near z

$$K_\lambda(z,w) = (z-w)^{-1}(1 + c_1(z)(z-w) + (a_2(z)+c_2(z))(z-w)^2 + \dots) \qquad \text{(III.11)}$$

where $a_2(z)$ is the schwartzian connection defined by the expansion of the prime form $E(P,Q)=E(z,w)dz^{-1/2}dw^{-1/2}$ for w near z[5]

$$E(z,w)^{-1} = (z-w)^{-1} (1 + a_2(z) (z-w)^2 + \dots) \qquad \text{(III.12a)}$$

explicitely given by

$$a_2(z) = -1/8 (\partial_z \log(h_0(z)^2))^2 + 1/12\, h_0(z)^{-2} \partial_z^2 h_0(z)^2 -$$
$$- 1/6\, h_0(z)^{-2} \sum_{i,j,k=1}^{g} \partial_i \partial_j \partial_k \theta[^{\alpha_0}_{\beta_0}](0)\, \eta^i(z)\, \eta^j(z)\, \eta^k(z) , \qquad \text{(III.12b)}$$

and

$$c_1(z) = \partial_z \log(E(z,P_-)^{I(q)} \sigma(z)^q\, \theta[^\alpha_\beta](w-z+v(q)))\Big|_{w=z} \qquad \text{(III.13a)}$$

$$c_2(z) = 1/2\, (c_1(z)^2 + \partial_w^2 \log\theta[^\alpha_\beta](w-z+v(q))\Big|_{w=z} -$$
$$- \partial_z^2 \log(E(z,P_-)^{I(q)} \sigma(z)^q)) \qquad \text{(III.13b)}$$

By using (III.11) is straightforward to compute (III.10), the final result being

$$R_{ij} = c_\lambda/12\, \kappa_{ij} \quad , \quad c_\lambda = -2\, (6\lambda^2 - 6\lambda + 1) \qquad \text{(III.14)}$$

where κ_{ij} is an expression of the form (I.59b) with the schwartzian connection given by

$$R_\lambda(z) = -12/c_\lambda\, (\, a_2(z) + 1/2\, c_1(z)^2 - q/2\, \partial_z c_1(z) +$$
$$+ 1/2 \sum_{i,j=1}^{g} \partial_i \partial_j \log\theta[^\alpha_\beta](v(q))\, \eta^i(z)\, \eta^j(z)\,) \qquad \text{(III.15)}$$

Now from the vacuum definition (II.23,26) and (III.6) it follows that <T(Q)>=0, so (III.4) applies giving the result

$$\langle T^s(z)\rangle = a_2(z) + 1/2\, c_1(z)^2 - q/2\, \partial_z c_1(z) +$$
$$+ 1/2 \sum_{i,j=1}^{g} \partial_i \partial_j \log\theta[^\alpha_\beta](v(q))\, \eta^i(z)\, \eta^j(z) \qquad \text{(III.16)}$$

which coincides with that obtained by path integral methods (see [19,20,27]) by defining the composite operator T^{PI} by substracting the leading singularity (this is the reason because it transform as a schwartzian connection[11])

$$T^{PI}(z) = \lim_{w\to z} \left((\lambda\, \partial_z c(z)b(w) - (1-\lambda)\, c(z)\partial_w b(w)) + (z-w)^{-2} \right)$$

An important remark is in order. As we saw in Chapter II, Section 7, in path integral formulation of b-c systems the presence of zero modes constrains us to insert I(q) b-fields to make sense of correlation functions (cf. (II.72))

$$\langle O\rangle \equiv \int [dbdc]\ b(Q_1)\ldots b(Q_{I(q)})\ O \Big/ \int [dbdc]\ b(Q_1)\ldots b(Q_{I(q)})$$

for any operator O. We emphasized there that the propagators obtained in this approach coincide with those given in (III.9) provided that we take $Q_i=P_-$, $i=1,\ldots,I(q)$. It is also in this sense that the afirmation made below eq.(III.16) must be understood[10].

3.Computation of the KN algebra for the scalar field

For computations in free scalar field theory is useful to introduce the coefficients $\tilde{\gamma}_{nm}$ and $\tilde{s}_{nm}$ as follows

$$\alpha_n\alpha_m = :\alpha_n\alpha_m: + \tilde{\gamma}_{nm} \qquad \text{(III.17a)}$$

$$\alpha_n\bar{\alpha}_m = :\alpha_n\bar{\alpha}_m: + \tilde{s}_{nm} \qquad \text{(III.18a)}$$

They can be read directly from eqs.(II.100,104,107)

$$\tilde{\gamma}_{nm} = \begin{cases} 0 & \text{if } n<-g/2 \text{ or } m>g/2 \\ \gamma_{nm} & \text{if } n>g/2 \text{ or } m<-g/2 \end{cases} \qquad \text{(III.17b)}$$

and if $(n,m) \in I\times I$ ($\tilde{s}_{nm}=0$ otherwise)

$$\tilde{\gamma}_{nm} = \sum_{i,j=1}^{g} (a^{-1})^i_n\, (a^{-1})^j_m \sum_{\substack{l>g/2\\ k<-g/2}} \gamma_{lk} F^l_i\, F^k_j \qquad \text{(III.17c)}$$

$$\tilde{s}_{nm} = \pi \sum_{i,j=1}^{g} (a^{-1})^i_n\, (\bar{a}^{-1})^j_m\, (\tau_2^{-1})_{ij} \qquad \text{(III.18b)}$$

---

[10] In particular the coefficient $c_1(z)$ in eqs.(III.11,13a) is precisely minus the VEV in the sense explained of the ghost current j(z) defined by[20]

$$j(z) = \lim_{w\to z} (c(z)b(w) - (z-w)^{-1})$$

The $\tilde{\gamma}_{nm}$'s satisfy the relation

$$\tilde{\gamma}_{nm} - \tilde{\gamma}_{mn} = \gamma_{nm} \qquad \text{(III.17d)}$$

The components of the KN-EMT are given by (II.86)

$$T(Q) = -:\partial X(Q)\partial X(Q): \qquad \text{(III.19)}$$

By using eqs.(II.13,89) we get the KN operators in the form

$$L_i = 1/2 \sum_{n,m} l_i^{nm} :\alpha_n \alpha_m: \qquad \text{(III.20a)}$$

where

$$l_i^{nm} = 1/(2\pi i) \oint_{C_\tau} (e_i|\omega^n)\omega^m = l_i^{mn} \qquad \text{(III.20b)}$$

Now from the commutation rules (II.104) and the definition (III.17a) it follows that they satisfy the KN algebra (I.59) with c=1 and the cocycle given by

$$\kappa_{ij} = 6 \sum_{n,m,k,s} \gamma_{mk} \tilde{\gamma}_{ns} l_i^{mn} l_j^{ks} - (i\leftrightarrow j) \qquad \text{(III.21)}$$

By using the relation (III.17d) and the antisymmetry of κ_{ij} we can write (III.21) as

$$\kappa_{ij} = 6 \sum_{n,m,k,s} \tilde{\gamma}_{mk} \tilde{\gamma}_{ns} l_i^{mn} l_j^{ks} - (i\leftrightarrow j) \qquad \text{(III.22)}$$

In order to get an explicit expression for κ_{ij} we consider the following Laurent-like series[11]

$$G(P,Q) = \partial_P\partial_Q \log E(P,Q) - \pi \sum_{i,j=1}^{g} \eta^i(P) (\tau_2^{-1})_{ij} \eta^j(Q) =$$

$$= \begin{cases} \sum_{n,m} \tilde{\gamma}_{nm} \omega^n(P) \omega^m(Q) & , \tau_P > \tau_Q \\ \sum_{n,m} \tilde{\gamma}_{nm} \omega^n(Q) \omega^m(P) & , \tau_Q > \tau_P \end{cases} \qquad \text{(III.23)}$$

By using this expansion and the definition of the coefficients l_i^{nm} eq.(III.20b), we can express (III.22) as follows

[11] This expansion is no other thing that the corresponding one to the ∂X-propagator $\langle 0|T(\partial X(P)\partial X(Q))|0\rangle = -1/2G(P,Q)$, computed in Chapter II, Section 9.

$$\kappa_{ij} = 6 \oint_{C_{\tau_P}} e_j(P) \oint_{C_{\tau_Q > \tau_P}} e_i(Q)\ G(P,Q)^2 - (i \leftrightarrow j)$$

$$= 6 \oint_{C_\tau} e_j(P) \oint_{C_P} e_i(Q)\ G(P,Q)^2 \qquad \text{(III.24)}$$

By inserting (III.23) into (III.24) we arrive at an expression of the form (I.59b), with the schwartzian connection given by

$$R^X(z) = -12\ (a_2(z) - \pi/2 \sum_{i,j=1}^{g} \eta^i(z)\ (\tau_2^{-1})_{ij}\ \eta^j(z)) \qquad \text{(III.25)}$$

As in the ghost case we get $<0|T(Q)|0>=0$, so (III.4) reads

$$<T^s(z)> = a_2(z) - \pi/2 \sum_{i,j=1}^{g} \eta^i(z)\ (\tau_2^{-1})_{ij}\ \eta^j(z) \qquad \text{(III.26)}$$

which coincides with the well-known result coming in PIA from the definition (see [27] for example)

$$T^{PI}(z) = - \lim_{w \to z} \left(\partial_z X(z) \partial_w X(w) + (z-w)^{-2} \right)$$

of the composite operator $T^{PI}(z)$.

PART B: Physical states and unitarity

4. The Feynmann-Polyakov quantization

Let us consider the path integral formulation of the free scalar quantum field theory defined by the action (II.81)[28]. The partition function at genus g is defined by

$$Z_g = \int_{\Sigma_g} [dh][dX] \exp(-S[X;h]) \qquad \text{(III.27)}$$

where [dh] stands for the measure in the space of metrics on Σ. Because of the local invariance of (II.81) under arbitrary reparametrizations, we must fix a "gauge" to compute (III.27). A covariant choice is the so-called "conformal gauge"

$$h = \rho \hat{h}(y) \qquad \text{(III.28)}$$

where $\hat{h}$ is a reference metric depending on the the moduli $\{y_i\}$ of the surface and ρ an arbitrary function. As usual, the gauge fixing involves the introduction of a Fadeev-Popov determinant, while the path integral on X just gives the determinant of the scalar Laplacian, so (III.27) takes the form[12] (we omit infinite factors coming from the volumes of gauge groups)

$$Z_g = \int_{M^g} [dy] \int [d\rho] \det(FP)(\rho\hat{h}) \det(\Delta)(\rho\hat{h}) \qquad \text{(III.29)}$$

where the FP determinant can be expressed as a path integral over an anticommuting ghost system of weight $\lambda=2$

$$\det(FP) = \int [dbdcd\bar{b}d\bar{c}] \exp(-S[b,c]) \qquad \text{(III.30)}$$

Since the actions (II.14,81) are invariant under Weyl rescalings, we could wait for a decoupling of the conformal factor ρ in (III.27). However, due to intrinsic ρ-dependent definitions of the measures and problems with the regularization of the determinants this is not the case in general. Polyakov[28] showed that all the dependence in ρ is encoded in the Liouville action, i.e.

[12] The discussion presented here intends to be a "schematic" one, for rigorous derivations see [29-31].

$$\det(\ldots)(\rho\hat{h}) = \det(\ldots)(\hat{h}) \exp(-c\, S_{Liouv.}[\rho;\hat{h}])$$

$$S_{Liouv.}[\rho;\hat{h}]) \propto \int dx^2 |\hat{h}(x)|^{1/2} \, (1/2\, \hat{h}^{ab}(x)\partial_a \log\rho(x)\partial_b \log\rho(x) + \\ + R^{\hat{h}} \log\rho(x) + \mu^2\rho(x)) \qquad (III.31)$$

where the factor c is equal to 1 and -26 for the scalar field and λ=2 ghost system respectively. Then, if we take D=26 scalar fields the decoupling of ρ indeed occurs and (III.29) reads

$$Z_g = \int_{M^g} [dy] \det(FP)(\hat{h}(y)) \det(\Delta)(\hat{h}(y)) = \int_{M^g} d\mu(y) \qquad (III.32)$$

reducing to an integration in the moduli space. We quote for completeness the measure on M^g $d\mu(y)$ (see [20] and references therein)

$$d\mu(y) = W(y)\wedge\overline{W}(\bar{y}) \, (\det\tau_2)^{-13}$$
$$W(y) = dy^1\wedge\ldots\wedge dy^{3g-3} \, Z_2(P_1,\ldots P_{3g-3}) \, Z_1^{-13} \, / \, \det\Omega^i(P_j) \qquad (III.33)$$

where the y^i's are analytic coordinates on M^g corresponding to the variation (II.66) of the metric $\hat{h}(y)$, Z_2 is the partition function of a b-c system of weight 2 with P_i arbitrary points (cf. (II.71)) and Z_1 is the corresponding to λ=1 with the zero modes projected out.

If we consider now an arbitrary multilocal operator O (functional in general of X and h) then from (III.27,32) we have

$$\langle O\rangle_g = \int_{M^g} d\mu(y) \, \langle O\rangle \qquad (III.34a)$$

$$\langle O\rangle = \int [dX] \exp(-S[X;\hat{h}]) \, O \, / \int [dX] \exp(-S[X;\hat{h}]) \qquad (III.34b)$$

However it is not said that the decoupling of ρ occurs for any O. The operators for which it does are said to be Weyl invariant or conformal operators, and are the only ones which make sense in the theory. The operator formalism developed in the precedent chapter allows us to compute $\langle\ldots\rangle$; the total correlation function will be given by (III.34a).

5. Equivalence with the operator formalism

Let us consider the bosonic closed string theory defined by a set of D scalar fields $\{X^\mu,\ \mu=0,\dots,D-1\}$ with action (II.81) $(\alpha'=1)$ and Minkowski metric $(\eta_{\mu\nu})=\mathrm{diag}(-1,1,\dots,1)$, that represent the space-time coordinates of the string. From (III.29-31) we saw that the decoupling of the conformal factor ρ comes from the cancellation of the "anomalies" (III.31) between a set of 26 scalar fields and a $\lambda=2$ ghost system. This may be visualized from another point of view. As we said in Chapter II, Section 2, after gauge fixing the scalar theory to the conformal gauge we remain with a theory invariant under local holomorphic change of coordinates, the generators being the KN operators. But a quantum level its algebra does not close, giving rise to the central extension (I.59) with central charge c=D. So we may hope that the addition of the $\lambda=2$ ghost system makes the work of closing the algebra, leaving a theory fully conformal invariant. Indeed it is the case. The total KN EMT will be

$$T^t(Q) = \sum_i L_i^t\, \Omega^i(Q) \quad , \qquad L_i^t = L_i^X + L_i^{gh} \tag{III.35}$$

From (III.7,14,25) we have $(c_2=-26)$

$$[L_i^t, L_j^t] = \sum_{s=-g_0}^{g_0} C_{ij}^s\, L_{i+j-s}^t + (13/6)\ 1/(2\pi i) \oint_{C_\tau} \Omega\, [e_i, e_j]$$

$$\Omega(Q) = (R_2(z) - R^X(z))dz^2 \tag{III.36}$$

$\Omega(Q)$, being the difference between two schwartzian connections, is a well defined tensor of weight $(2,0)$[11], and from (III.15,25) is holomorphic outside P_- where has a pole of second order. By redefining L_i^t as follows

$$L_i^{st} = L_i^t + (13/6)\ 1/(2\pi i) \oint_{C_\tau} e_i\, \Omega \tag{III.37}$$

we get

$$[L_i^{st}, L_j^{st}] = \sum_{s=-g_0}^{g_0} C_{ij}^{\ s}\, L_{i+j-s}^{st} \tag{III.38}$$

giving a KN algebra without anomaly. In Ref.[25] a related result was predicted by considering the nilpotency in D=26 of the BRST charge associated.

Finally let us note that the EMT corresponding to (III.37)

$$T^{st}(Q) = \sum_i L_i^{st}\, \Omega^i(Q) = T^X(Q) + T^{gh}(Q) + 13/6\ \Omega(Q) \qquad \text{(III.39)}$$

has a VEV given by

$$<T^{st}(Q)> = 13/6\ \Omega(Q) \qquad \text{(III.40)}$$

The existence of this anomaly-free KN EMT on Σ with a well defined VEV could give a hint for defining the measure on the moduli space in this operator framework, pursuing lines of reasoning in somewhat similar to those of ref.[17].

Also the concept of a Weyl invariant operator has its counterpart in our framework. Being the KN operators the generators of conformal transformations in the gauge fixed theory, a *conformal operator* of weights (p,q) W(Q) should satisfy the covariance conditions

$$[L_n, W(Q)] = L_{e_n} W(Q) \qquad \text{(III.41a)}$$

$$[\bar{L}_n, W(Q)] = L_{\bar{e}_n} W(Q) \qquad \text{(III.41b)}$$

Examples are $\partial X(Q)$ ($\bar{\partial} X(Q)$) with weight (1,0) ((0,1)), b ((λ,0)), c ((1-λ,0)), $\bar{b}$ ((0,λ)) and $\bar{c}$ ((0,1-λ)), as can be easily checked. Other kind of conformal operators, the vertex operators, will be defined and extensively studied in the next sections.

6.Vertex operators and scattering amplitudes

Qualitatively a vertex operator (VO) is a local operator W(Q) which creates from the vacuum the excitations or "particles" of the string. When a string is propagating, the insertion of a VO is interpreted as the scattering of an off-shell particle (the string) and an on-shell particle (that represented by the VO). The origin of this interpretation is greatly explained in Ref.[18].

A scattering amplitude A^M of M external particles is understood as the VEV of a string of VO $\{V_i,\ i=1,...M\}$ that represent the particles. Let A^M_g the genus g contribution (g loop-order in the perturbative expansion). Then, according to the Polyakov recipe[28], we have

$$A^M = \sum_{g=0}^{\infty} k^{2g+M-2} A^M_g \qquad \text{(III.42)}$$

where k is a coupling constant. In order to obtain A^M_g, we have first to calculate $A^M[\Sigma_g]$, i.e., the amplitude of a process performed on an equivalence class of Riemann surfaces $[\Sigma_g]$. So, according with (III.34)

$$A^M_g = \int_{M^g} d\mu(y)\ A^M[\Sigma_g] \tag{III.43}$$

Then the definition of $A^M[\Sigma_g]$ in our context will be

$$A^M[\Sigma_g] = \langle 0|\ T(V_1 \ldots V_M)|0\rangle \tag{III.44}$$

Now we would like to find the right definition of the vertex operators within the present formalism. Let us start remembering that at genus zero a VO is an object of the form

$$V = \int_{S^2} i(dz/z)\wedge(d\bar{z}/\bar{z})\ w(z,\bar{z}) \tag{III.45}$$

where $z=\exp(\tau+i\sigma)$, τ being the euclidean time parameter and σ the angular coordinate of the string. It is required to have conformal dimensions (1,1) in the sense[18]

$$[L^v_n, w(z,\bar{z})] = (z^{n+1}\partial_z + n\ z^n)\ w(z,\bar{z}) \tag{III.46a}$$

$$[\bar{L}^v_n, w(z,\bar{z})] = (\bar{z}^{n+1}\partial_z + n\ \bar{z}^n)\ w(z,\bar{z}) \tag{III.46b}$$

where L^v_n, $\bar{L}^v_n$ are the Virasoro operators. This is exactly equivalent to impose that the two-form

$$W(z,\bar{z}) = i(dz/z)\wedge(d\bar{z}/\bar{z})\ w(z,\bar{z})$$

transforms according (III.41), which are equivalent to impose that V be conformal invariant. It must represent an on-shell string state of momentum p^μ with the correct Lorentz numbers of the represented particle. Moreover, it must change the momentum of any state by p^μ, so it must carry a factor $\exp(ip.X(Q))$. Then a VO at arbitrary genus will be of the form

$$V = \int_\Sigma W(Q;p) \tag{III.47}$$

with $W(Q;p)$ a (1,1) conformal operator valued 2-form.

The requirements (III.41) can be understood from another point of view. At g=0 the Fock space of the string theory presents negative norm states due to the time-like oscillators α^0_n, $\bar{\alpha}^0_n$. However there are spurious degrees of freedom coming from the gauge fixing. Classically there are constraint equations which follows from the equation of motion $\delta S[X;h]/\delta h^{ab}=0$ that looks like $T(Q)=\bar{T}(Q)=0$. At quantum level the quantization goes *a la* Gupta-Bleuler, imposing constraints that select a physical subspace of the whole Hilbert space. They are given by the Virasoro conditions[18], whose generalization at arbitrary genus is[21]

$$L_n\ |\phi> = 0 \quad ; \quad \bar{L}_n\ |\phi> = 0 \quad , \quad n>g_0 \tag{III.48a}$$

$$L_{g_0}\ |\phi> = |\phi> \quad ; \quad \bar{L}_{g_0}\ |\phi> = |\phi> \tag{III.48b}$$

and

$$<\phi|\ L_n = 0 \quad ; \quad <\phi|\ \bar{L}_n = 0 \quad , \quad n<-g_0 \tag{III.48c}$$

$$<\phi|L_{-g_0} = \epsilon_-\ <\phi| \quad ; <\phi|\ \bar{L}_{-g_0} = \bar{\epsilon}_-\ <\phi| \tag{III.48d}$$

for any physical state ϕ, where ϵ_- is the coefficient of the leading term in the expansion of the vector field e_{-g_0} around P_-.

A consequence of the commutation relations (III.41) is that vertex operators can be used to map physical states to physical states. In fact, let us see what happens at g=0. If $|\phi>$ is a physical state, $(L^v_n-\delta_{n,0})|\phi>=0$, $n\geq0$, then $|\phi'>=V|\phi>$ is also a physical state, as it can be seen from the fact that W commutes with L_n up to a time derivative, which creates a spurious state. Indeed, although the commutator $[L^v_n,V]=\int_\Sigma[L^v_n,W]$ is not zero, it decouples from any amplitude. This is seen by writing $z\partial/\partial z=\partial/\partial\tau+i\partial/\partial\sigma$ $(z=e^{\tau+i\sigma})$. The integral of the part with a derivative in σ is zero because the integrand is single-valued. What remains is something which is a derivative in τ. Even though this is not zero, it does not yield any contribution to an amplitude. This is usually showed by the "canceled propagator argument"[18]: the time derivative $dA/d\tau$ can be written as $dA/d\tau=[L^v_0+\bar{L}^v_0,A]$. Inserting this into the amplitude, the $L^v_0+\bar{L}^v_0$ factors next to A either cancel against an adjacent propagator or else annihilate against the states at the ends of the expression. Now a term with a canceled propagator is a holomorphic function of the Mandelstam variable s of the corresponding channel. Then it follows that there are regions of the invariant energy variables in which the term in question is analytic and vanishes as $|s|\to\infty$. Thus by a standard theorem of complex analysis, the amplitude must be identically zero in the region described, and zero everywhere else as well by analyticity. Essentially, what happens is that whenever one has a time derivative this leads, after integration in the corresponding time variable, to correlation functions with two vertex operators valued at the same time. This is either singular or zero depending on the region of s that one is considering, so again it follows that there are regions of the invariant energy variables in which the correlation function in question is vanishing, and by analyticity it must be zero everywhere else.

Now let us go to the interesting case of $g>1$. A physical state is a state which obeys the Virasoro-like conditions (III.48). Unitarity requires that only physical states (which are positive norm states in the $g=0$ language) contribute as poles in the amplitude. This is equivalent to the requirement that VO map physical states to physical states. This implies that $\delta V\equiv[L_n,V]$, $\delta V\equiv[\bar{L}_n,V]$, if not zero, must create a spurious or "ghost" state which decouples from the amplitudes (III.44) at least for $|n|\geq g_0$ (cf. (III.48)). More precisely, the state

$$\int_{\tau_i>\ldots>\tau_M} W_i(Q_i)\ldots W_M(Q_M)|0>$$

when it is on shell, should be annihilated by the L_n's for $n>g_0$. This is indeed the case provided that eqs.(III.41) holds. In fact, by acting with L_n on this state and commuting it with the vertex operators on the left one arrives to the following state

$$\int_{\tau_i>\ldots>\tau_M} W_i(Q_i)\ldots L_n W_M(Q_M)|0> \qquad \text{(III.49)}$$

plus states containing a commutator between two vertex operators, say, at times τ_{j-1} and τ_{j+1}. The integral corresponding to this commutator involves the region between $C\tau_{j-1}$ and $C\tau_{j+1}$. From (III.41) it follows

$$\int d(e_n|\,W_j) = \oint_{C_{\tau_{j-1}}} (e_n|\,W_j) + \oint_{C_{\tau_{j+1}}} (e_n|\,W_j)$$

and this gives no contribution because of the "canceled propagator argument" explained above. Now consider the state (III.49). When one commutes L_n with V_M obtains a commutator whose integral involves the region between $-\infty$ and τ_M. For $Q_M=P_+$, $[L_n,W_M(Q_M)]$ is not given just by (III.41), but there is a further contribution of the form

$$(e_n|dk)(z_M)\ \delta^2(z_M) \quad , \quad z_M(P_+) = 0$$

which after integration gives zero if $n>g_0$ and 1 if $n=g_0$. An analogous reasoning can be done for a state of the form

$$\int_{\tau_1>\ldots>\tau_j} \langle 0|W_1(Q_1)\ldots W_j(Q_j),$$

this time giving zero for $n<-g_0$ and ϵ_- for $n=-g_0$ (cf. (III.48c,d)).

Part C: Scattering amplitudes in String Theory

7. The tachyon vertex operator

Vertex operators have been extensively studied in the literature[15,32-35] within the Feynmann-Polyakov formulation of string theory. Also an operator study at g=0 in the case of the open string theory can be found in ref.[36]. In this part we will performe a systematic construction of VO for the bosonic closed string theory at arbitrary genus in our operator context.

Let us start rewriting the expansions of the scalar fields

$$X^{\mu}(Q) = x^{\mu}(\tau) + X^{\mu}_{osc}(Q) \tag{III.50a}$$

$$x^{\mu}(\tau) = x^{\mu} - ip^{\mu}\,\tau(Q) \tag{III.50b}$$

$$X^{\mu}_{osc}(Q) = i/\sqrt{2} \sum_{|n|>g/2} (\alpha^{\mu}_{n}\,\phi_{n}(Q) + \bar{\alpha}^{\mu}_{n}\,\bar{\phi}_{n}(Q)) \tag{III.50c}$$

The tachyon VO is defined by[13]

$$:W_{T}(Q): = \Omega(Q)\ :\exp(ip.X(Q)): \quad ,\ p^{2} = 4 \tag{III.51a}$$

$$:\exp(ip.X(Q)): \equiv \exp(ip.x(\tau))\ :\exp(ip.X_{osc}(Q)): \tag{III.51b}$$

where $\Omega(Q)=idz\wedge d\bar{z}\Omega_{z\bar{z}}$ is some integrable 2-form on Σ. We would like that (III.41) to hold for $:W_{T}:$; this requirement will fix (up to a multiplicative constant, of course) $\Omega(Q)$. So, let us compute the commutators in (III.41). From the formulas given in Chapter II and Part A of this chapter we get

$$[L_{n},:\exp(ip.X(Q)):] = -1/(2\sqrt{2}) \sum_{k,l} l^{kl}_{n}(A_{l}(Q)\ p.\alpha_{k}:\exp(ip.X(Q)): + :\exp(ip.X(Q)):\ p.\alpha_{l}\ A_{k}(Q)) \tag{III.52}$$

In the following calculations the formulas

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14 The value $p^{2}=4$ is indeed neccesary for $:W_{T}:$ to be a (1,1) conformal operator (see (III.61)); we may however think (III.51) in general as an object of conformal weights $(p^{2}/4,\ p^{2}/4)$.

$$\alpha^\mu_k :f(\alpha,\bar\alpha): = :\alpha^\mu_k f(\alpha,\bar\alpha): + \sum_s (\tilde\gamma_{ks} :\frac{\partial f(\alpha,\bar\alpha)}{\partial \alpha^\mu_s}: + \tilde s_{ks} :\frac{\partial f(\alpha,\bar\alpha)}{\partial \bar\alpha^\mu_s}:) \tag{III.53a}$$

$$:f(\alpha,\bar\alpha): \alpha^\mu_k = :\alpha^\mu_k f(\alpha,\bar\alpha): + \sum_s (\tilde\gamma_{sk} :\frac{\partial f(\alpha,\bar\alpha)}{\partial \alpha^\mu_s}: + \tilde s_{ks} :\frac{\partial f(\alpha,\bar\alpha)}{\partial \bar\alpha^\mu_s}:) \tag{III.53b}$$

valid for any smooth function f of the oscillators α_n ,$\bar\alpha_n$ will be extremely useful. By using them we obtain

$$[L_r, :\exp(ip.X(Q)):] = (L_{e_r} + p^2/4\ \kappa_r(Q)):\exp(ip.X(Q)):$$

$$[\bar L_r, :\exp(ip.X(Q)):] = (L_{\bar e_r} + p^2/4\ \bar\kappa_r(Q)):\exp(ip.X(Q)): \tag{III.54}$$

where the "anomalous" functions $\kappa_r(Q)$ are given by

$$\kappa_r(Q) = \sum_{n,m} 1^{nm}_r \sum_{|k|>g/2} (\tilde\gamma_{km}\ A_n(Q) + \tilde\gamma_{nk}\ A_m(Q))\ \phi_k(Q) \tag{III.55}$$

Let us compute these functions. From eqs.(II.100c,d) and (II.102,104b) follows the relation

$$A_n(Q) = \sum_{|m|>g/2} \gamma_{nm}\ \phi_m(Q) + \delta_{n,g/2} \tag{III.56}$$

and then we can write (III.55) as

$$\kappa_r(Q) = \sum_{n,m} 1^{nm}_r \sum_{|k|,|l|>g/2} (\tilde\gamma_{nk}\ \tilde\gamma_{ml} - \tilde\gamma_{kn}\ \tilde\gamma_{lm})\ \phi_k(Q)\ \phi_l(Q) +$$

$$+ \sum_n 1^{n,g/2}_r \sum_{|k|>g/2} (\tilde\gamma_{nk} + \tilde\gamma_{kn})\ \phi_k(Q) \tag{III.57}$$

where we have used (III.17d). Now, let us introduce the following Laurent-like expansions obtained by using the methods of Chapters I and II[15]

$$\omega(P,Q;P_+) = \sum_n \sum_{|k|>g/2} \tilde\gamma_{nk}\ \omega^n(P)\ \phi_k(Q) \quad , \tau_P>\tau_Q \tag{III.58a}$$

$$\omega(P,Q;P_-) = \sum_n \sum_{|k|>g/2} \tilde\gamma_{kn}\ \omega^n(P)\ \phi_k(Q) \quad , \tau_Q>\tau_P \tag{III.58b}$$

where

15 They essentially are those corresponding to the propagator <X∂X> of a scalar field X.

$$\omega(P,Q;P_0) = \partial_P \log(E(P,Q)/E(P,P_0)) - \pi \int_{P_0}^{Q} (\eta-\bar{\eta})^t \, \tau_2^{-1} \, \eta(P) \tag{III.59a}$$

is a meromorphic 1-form in P. It satisfies the relation (see (I.43))

$$\omega(P,Q;P_-) = \omega(P,Q;P_+) + \omega^{g/2}(P) \tag{III.59b}$$

With the help of (III.58,59) and the definition of the coefficients l_r^{nm}, eq.(III.20b), we can rewrite (III.57) as

$$\kappa_r(Q) = 1/(2\pi i) \oint_{C_Q} e_r(P) \, \omega(P,Q;P_+) \, \omega(P,Q;P_-) \tag{III.60}$$

The computation of the RHS by using (III.59a) is straightforward and we obtain

$$\kappa_r(z,\bar{z}) = \partial_z e_r(z) + \omega_z(z,\bar{z}) \, e_r(z) \tag{III.61a}$$

$$\bar{\kappa}_r(z,\bar{z}) = \partial_{\bar{z}} \bar{e}_r(z) + \omega_{\bar{z}}(z,\bar{z}) \, \bar{e}_r(z) \tag{III.61b}$$

where

$$\omega_z(z,\bar{z}) = -\partial_z \log(E(z,P_+)E(z,P_-)) - \pi \left(\int_{P_+}^{z} + \int_{P_-}^{z}\right)(\eta-\bar{\eta})^t \, \tau_2^{-1} \, \eta(z) \tag{III.62a}$$

$$\omega_{\bar{z}}(z,\bar{z}) = -\partial_{\bar{z}} \log(\bar{E}(\bar{z},P_+)\bar{E}(\bar{z},P_-)) + \pi \left(\int_{P_+}^{z} + \int_{P_-}^{z}\right)(\eta-\bar{\eta})^t \, \tau_2^{-1} \, \bar{\eta}(\bar{z}) \tag{III.62b}$$

Now we impose the conditions (III.41) on $:W_T(Q):$; from eqs.(III.51,54) these are equivalent to

$$\kappa_r(z,\bar{z}) = \nabla_z e_r(z) \quad , \quad \bar{\kappa}_r(z,\bar{z}) = \nabla_{\bar{z}} \bar{e}_r(z) \tag{III.63}$$

where ∇_z, $\nabla_{\bar{z}}$ are the covariant derivatives with respect to the connections

$$\omega_z(z,\bar{z}) = \partial_z \log\Omega_{z\bar{z}}(z,\bar{z}) \quad , \quad \omega_{\bar{z}}(z,\bar{z}) = \partial_{\bar{z}} \log\Omega_{z\bar{z}}(z,\bar{z}) \tag{III.64}$$

From (III.62,64) we read $\Omega(Q)$ [37,38]

$$\Omega(Q) = i \, \nu(Q)\wedge\bar{\nu}(Q)$$

$$\nu(Q) = E(P_+,P_-)/(E(Q,P_+)E(Q,P_-)) \exp\left(-\pi/2\int_{P_-}^{Q}(\eta-\bar{\eta})^t \tau_2^{-1}\int_{P_+}^{Q}(\eta-\bar{\eta})\right) \tag{III.65}$$

We remark $\Omega(Q)$ is a well-defined (single-valued) form, though this is not the case for $\nu(Q)$.

8. An Arakelov-type metric

We can give a particular interpretation of the form $\Omega(Q)$. As it is well known the propagator for the scalar fields $\langle X(P)X(Q)\rangle$ is not uniquely defined due to the zero mode. In the operator formalism this fact is manifested in the indetermination of the scalar products $\langle 0|x\, x|0\rangle$ and $\langle 0|x|0\rangle$, the first one giving rise to an indetermined constant and the second one to an indetermination in a function of P plus a function of Q. It is straightforward to verify

$$\langle 0|T\{(X(P)-x)(X(Q)-x)\}|0\rangle = -1/2\ g'(P,Q)$$

$$g'(P,Q) = \vartheta(\tau_P-\tau_Q)\ g(P,Q;P_\pm) + \vartheta(\tau_Q-\tau_P)\ g(Q,P;P_\pm) \qquad \text{(III.66a)}$$

where $\vartheta(\tau_P-\tau_Q)$ stands for the step function and[37] $(\tau_P>\tau_Q)$

$$g(P,Q;P_\pm) = \sum_{|k|,|l|>g/2} \tilde{\gamma}_{kl}\ \phi_k(P)\ \phi_l(Q) + \text{h.c.} =$$

$$= \log|E(P,Q)E(P_+,P_-)/(E(P,P_+)E(Q,P_-))|^2 - \pi\int_{P_-}^{P}(\eta-\bar{\eta})^t\tau_2^{-1}\int_{P_+}^{Q}(\eta-\bar{\eta})$$

(III.66b)

Because of the ambiguity mentioned above, we can *define* in our context a propagator $\Delta^\Omega(P,Q)$ for the scalar field by adding to (III.66a) a suitable sum of functions of P and Q in such a way that

$$\Delta^\Omega(P,Q) = 1/2\ (\ g(P,Q;P_\pm) + (Q\leftrightarrow P)\ ) =$$

$$= \log(|E(P,Q)E(P_+,P_-)|^2/|(E(P,P_+)E(P,P_-)E(Q,P_+)E(Q,P_-)|) -$$

$$- \pi/2\ (\int_{P_-}^{P}(\eta-\bar{\eta})^t\tau_2^{-1}\int_{P_+}^{Q}(\eta-\bar{\eta}) + (Q\leftrightarrow P)) \qquad \text{(III.67)}$$

More generically, given a metric h on Σ we can introduce an associated propagator by[39]

$$\Delta^h(P,Q) = \log(|E(P,Q)|^2 h(P)^{1/2} h(Q)^{1/2}) +$$

$$+ \pi/2 \int_Q^P(\eta-\bar{\eta})^t\ \tau_2^{-1} \int_Q^P(\eta-\bar{\eta}) \qquad \text{(III.68)}$$

It is easy to prove that, if $h=h_{z\bar{z}}(dz\otimes d\bar{z}+d\bar{z}\otimes dz)$ in some isothermal coordinates, then the following identity holds

$$h_{z\bar{z}}(z,\bar{z}) = \exp(\lim_{w\to z} (\Delta^h(z,\bar{z};w,\bar{w}) - \log|z-w|^2) \qquad \text{(III.69)}$$

Now, the two-form (III.65) $\Omega = idz\wedge d\bar{z}\Omega_{z\bar{z}}$ with $\Omega_{z\bar{z}}$ real given by

$$\Omega_{z\bar{z}}(z,\bar{z}) = |E(P_+,P_-)/(E(z,P_+)E(z,P_-))|^2 \times$$
$$\times \exp(-\pi\int_{P_-}^{Q}(\eta-\bar{\eta})^t\tau_2^{-1}\int_{P_+}^{Q}(\eta-\bar{\eta})) \qquad \text{(III.70)}$$

defines the singular metric $h^{\Omega}=\Omega_{z\bar{z}}(dz\otimes d\bar{z}+d\bar{z}\otimes dz)$ whose connections are given by (III.62). With a little bit of algebra is showed that this metric corresponds to eq.(III.69) with the propagator given by (III.67). The definition (III.69) is reminiscent to that of the Arakelov metric (for exactness, the Arakelov metric is a non-singular metric defined with respect to the propagator determined by the Bergmann metric

$$\mu_{z\bar{z}}(z,\bar{z}) = 1/(2g) \sum_{i,j=1}^{g} \eta^i(z)(\tau_2^{-1})_{ij}\bar{\eta}^j(\bar{z}) \ ;$$

see for example refs.[27,39]). Let us note also that the curvature of h^{Ω}

$$R^{\Omega}(z,z) = -2\ \Omega^{z\bar{z}}(z,\bar{z})\ \partial_z\partial_{\bar{z}}\log\Omega_{z\bar{z}}(z,\bar{z}) =$$
$$= 4\pi\ (-\Omega^{z\bar{z}}\ \eta(z)^t\tau_2^{-1}\bar{\eta}(\bar{z}) + 1(z,P_+) + 1(z,P_-)) \qquad \text{(III.71)}$$

where $1(z,w)=1/(2\pi)\Omega^{z\bar{z}}(z,\bar{z})\partial_z\partial_{\bar{z}}\log|z-w|^2$ is the covariant δ-function, verifies

$$1/(4\pi)\int_{\Sigma}\Omega\ R^{\Omega} = \kappa(g) = 2(1-g) \qquad \text{(III.72)}$$

as it should. In terms of R^{Ω}, the propagator (III.67) obeys the equation ($\Box^{\Omega}=-2\Omega^{z\bar{z}}\partial_z\partial_{\bar{z}}$)

$$-1/(4\pi)\ \Box^{\Omega}\Delta^{\Omega}(z,\bar{z};w,\bar{w}) = 1(z,w) - 1/2\ (1(z,P_+) + 1(z,P_-)) =$$
$$= 1(z,w) - 1/(8\pi)\ R^{\Omega}(z,\bar{z}) - g\ \Omega^{z\bar{z}}\ \mu_{z\bar{z}} \qquad \text{(III.73)}$$

This equation indeed defines a class of propagators[39] associated with a metric Ω, whose general solution is given by (III.68).

9. The M-tachyon multiloop amplitude[37]

Insertion of M VO (III.51) into $A^M[\Sigma_g]$ leads us to the calculation of correlation functions of the form

$$<0|:e^{ip_1\cdot X(Q_1)}:\ldots:e^{ip_M\cdot X(Q_M)}:|0> \qquad \text{(III.74)}$$

Writing $:e^{ip\cdot X(Q)}:=e^{ip.x(\tau)}W_0(Q)\bar{W}_0(Q)$, with

$$W_0(Q) = \exp(-1/2 \sum_{n<-g/2} p.\alpha_n\ \phi_n(Q))\ \exp(-1/2 \sum_{n>g/2} p.\alpha_n\ \phi_n(Q)) \qquad \text{(III.75)}$$

one obtains

$$<0|e^{ip_1\cdot x(\tau_1)}\ldots e^{ip_M\cdot x(\tau_M)}|0> =$$
$$= (2\pi)^{26}\ \delta^{26}(\sum_i p_i)\ \prod_{i=1}^{M} e^{p_i^2/2\ \tau_i}\ \prod_{i<j} e^{p_i\cdot p_j\tau_i} \qquad \text{(III.76)}$$

$$<0|W_0(Q_1)\ldots W_0(Q_M)|0> = \prod_{i<j} \exp\ p_i\cdot p_j\ \tilde{G}(Q_i,Q_j) \qquad \text{(III.77)}$$

where $\tilde{G}(P,Q)$ + h.c = $g(P,Q;P_\pm)$ (see (III.66b)). Defining $R(P_i,P_j)\equiv \tau(P_i)+g(P_i,P_j;P_\pm)$, the amplitude becomes

$$A^M_{tach.}[\Sigma_g] = \int_\Sigma (\prod_{i=1}^{M} \Omega(Q_i)\ e^{2\tau_i})\ \prod_{i\neq j} e^{p_i\cdot p_j/2\ R(Q_i,Q_j)}\ \times$$
$$\times\ (2\pi)^{26}\ \delta^{26}(\sum p_i) \qquad \text{(III.78)}$$

(We have used $p_i^2=4$). Now we insert the explicit expressions for $\Omega(Q)$, $\tau(Q)$, and $R(P,Q)$ and arrive to the following result

$$A^M_{tach.}[\Sigma_g] = (2\pi)^{26}\ \delta^{26}(\sum p_i) \int_\Sigma \prod_{i<j} |E(P_i,P_j)|^{p_i\cdot p_j}\ \times$$
$$\times\ \exp(\pi/4\ p_i\cdot p_j \int_{P_i}^{P_j} (\eta^k-\bar{\eta}^k)\ (\tau_2^{-1})_{kl} \int_{P_i}^{P_j} (\eta^l-\bar{\eta}^l)] \qquad \text{(III.79)}$$

in agreement with well-known results[40].

We would like to remark the following fact. Let us consider the following expression

$$A_N(\{p\}) = \prod_{i=1}^{N} (\int_\Sigma H(Q_i))\ \prod_{i<j=1}^{N} \exp(1/2p_i\cdot p_j\Delta^h(Q_i,Q_j)) \qquad \text{(III.80)}$$

where Δ^h is given by (III.68) and $H(Q)=idz\wedge d\bar{z}h_{z\bar{z}}$. It is easy to show (taking into account momentum conservation) that A_N does not depend on $h_{z\bar{z}}$. It indeed gives (III.79); the expression considered in refs.[40,35] corresponds to the metric $|h_0(Q)^2|^2$ while that appears in ref.[37] corresponds to the metric (III.70).

10. Higher massive levels vertex operators

Having a "natural" metric on Σ in our operator context, we define covariant derivatives acting on tensors of rank (n,m) by (cf. eqs.(I.11))

$$\nabla_z^{(n)} = \partial_z - n\,\omega_z \quad , \quad \nabla_{\bar z}^{(m)} = \partial_{\bar z} - m\,\omega_{\bar z} \qquad \text{(III.81)}$$

with the connections given by (III.62).

A vertex operator that represents a particle belonging to the n^{th} level will be in general a linear combination of terms of the form $\epsilon_{(\mu\bar\mu)}(p){:}W^{(\mu,\bar\mu)}(Q){:}$ with the polarization tensors $\epsilon_{(\mu\bar\mu)}(p)$ obeying suitable conditions in order for (III.41) to hold. The operators $W^{(\mu,\bar\mu)}(Q)$ are local monomial expressions constructed from the covariant derivatives of the fields X^μ and the exponential momentum factor. So it will contain terms like $\nabla_z^n X^\mu$, $\nabla_{\bar z}^m X^\mu$, but also $\nabla_{\bar z}^1\nabla_z^n\nabla_{\bar z}^m X^\mu$, $\nabla_z^1\nabla_{\bar z}^n\nabla_z^m X^\mu$, etc. However by using repetitively the relation (I.12b)

$$[\nabla_{\bar z},\nabla_z]\ t(Q) = s/2\ \Omega_{z\bar z}\ R^{\Omega}(Q)\ t(Q)$$

it is easy to see that terms like $\nabla_{\bar z}^1\nabla_z^n\nabla_{\bar z}^m X^\mu$, $\nabla_z^1\nabla_{\bar z}^n\nabla_z^m X^\mu$, etc, generate all the couplings of $\nabla_z^n X^\mu \nabla_{\bar z}^m X^\nu$ to the covariant derivatives of the curvature. So we are lead to consider the following most general expression[16]

$$:W^{(\mu,\bar\mu)}(Q): \ = \Omega(Q)\ (\Omega^{z\bar z})^{r(n)}\ f(Q)\ \exp(i/2p.x(\tau))\ \times$$

$$\times\ :W_{osc}^{(\mu,\bar\mu)}(Q):\ \exp(i/2p.x(\tau)) \qquad \text{(III.82a)}$$

where

$$f(Q) = \prod_{l=1}^{N} \nabla_z^{n_l} R^{\Omega}(Q)\ \prod_{l=1}^{\bar N} \nabla_{\bar z}^{\bar n_l} R^{\Omega}(Q)\ \prod_{l=1}^{P} \nabla_z^{p_l}\nabla_{\bar z}^{\bar p_l} R^{\Omega}(Q) \qquad \text{(III.82b)}$$

$$W_{osc}^{(\mu,\bar\mu)}(Q) = \prod_{l=1}^{M} \nabla_z^{m_l} X^{\mu_l}(Q)\ \prod_{l=1}^{\bar M} \nabla_{\bar z}^{\bar m_l} X^{\bar\mu_l}(Q)\ \exp(ip.X_{osc}(Q))$$

(III.82c)

[16] We find convenient to treat the zero mode part in this symmetric way.

$$r(n) = \sum_{l=1}^{M} m_l + \sum_{l=1}^{N} n_l + \sum_{l=1}^{P} p_l = \sum_{l=1}^{\bar{M}} \bar{m}_l + \sum_{l=1}^{\bar{N}} \bar{n}_l + \sum_{l=1}^{P} \bar{p}_l =$$

$$= n - (N + \bar{N} + P) \geq 0 \qquad \text{(III.82d)}$$

Here $(m_l, \bar{m}_l) \in N$ and $(n_l, \bar{n}_l, p_l, \bar{p}_l) \in N\cup\{0\}$.

Computation of a commutator

In order to impose the conditions (III.41) we must compute the commutators of $:W^{(\mu,\bar{\mu})}(Q):$ with the KN generators. The calculations are very tedious but straightforward, the full ingredients needed being contained in the previous sections. We quote here the final result

$$[L_r, :W^{(\mu,\bar{\mu})}(Q):] - L_{e_r} :W^{(\mu,\bar{\mu})}(Q): = -L_{e_r} f(Q) :W^{(\mu,\bar{\mu})}(Q):/f(Q)$$

$$+ \sum_{s=1}^{M} :W^{(\mu,\bar{\mu})}(Q)/\nabla_z^{m_s} X^{\mu}{}_s(Q) \; [\nabla_z^{m_s}, L_{e_r}] X^{\mu}{}_s(Q): +$$

$$+ (p^2/4 + r(n) - 1) \; \kappa_r(Q) \; :W^{(\mu,\bar{\mu})}(Q):$$

$$- i/2 \sum_{s=1}^{M} p^{\mu}{}_s \; A_r^{(m_p)}(Q) \; :W^{(\mu,\bar{\mu})}(Q)/\nabla_z^{m_s} X^{\mu}{}_s(Q): -$$

$$- 1/4 \sum_{\substack{s,t=1 \\ s\neq t}}^{M} \eta^{\mu_s \mu_t} B_r^{(m_s, m_t)}(Q) \; :W^{(\mu,\bar{\mu})}(Q)/(\nabla_z^{m_s} X^{\mu}{}_s(Q) \; \nabla_z^{m_t} X^{\mu}{}_t(Q)):$$

$$- i/4 \sum_{s=1}^{\bar{M}} \bar{p}^{\mu}{}_s \; \nabla_{\bar{z}}^{\bar{m}_s} \kappa_r(Q) \; :W^{(\mu,\bar{\mu})}(Q)/\nabla_{\bar{z}}^{\bar{m}_s} X^{\bar{\mu}}{}_s(Q): -$$

$$- 1/2 \sum_{s=1}^{M} \sum_{t=1}^{\bar{M}} \eta^{\mu_s \bar{\mu}_t} C_r^{(m_s, \bar{m}_t)}(Q) \; :W^{(\mu,\bar{\mu})}(Q)/(\nabla_z^{m_s} X^{\mu}{}_s(Q) \nabla_{\bar{z}}^{\bar{m}_t} X^{\bar{\mu}}{}_t(Q)):$$

(III.83a)

where the commutator between Lie and covariant derivative is

$$[\nabla_z^{m_s}, L_{e_r}] X^{\mu}{}_s(Q) = \sum_{k=2}^{m_s} \binom{m_s}{k} \nabla_z^{k-1} \kappa_r(Q) \; \nabla_z^{m_s - k + 1} X^{\mu}{}_s(Q) \qquad \text{(III.83b)}$$

if $m_s > 1$, being zero otherwise, and the tensors $A_r^{(p)}$, $B_r^{(p,q)}$, $C_r^{(p,q)}$ are given by

$$A_r^{(p)}(Q) = 1/(2\pi i) \oint_{C_Q} e_r(P) \; \omega(P,Q) \; (\nabla_Q)^p \omega(P,Q) \qquad \text{(III.84a)}$$

$$B_r^{(p,q)}(Q) = 1/(2\pi i) \oint_{C_Q} e_r(P)\ (\nabla_Q)^p \omega(P,Q)\ (\nabla_Q)^q \omega(P,Q)$$

$$\omega(P,Q) = 1/2\ (\omega(P,Q;P_+) + \omega(P,Q;P_-)) \qquad \text{(III.84b)}$$

$$C_r^{(p,q)}(Q) = \pi\ (\nabla_z)^p (e_r(Q)\eta(Q)^t)\ \tau_2^{-1}\ (\nabla_{\bar z})^{q-1}\ \bar\eta(Q) \qquad \text{(III.84c)}$$

The first few terms useful in what follows are[24]

$$A_r^{(1)}(Q) = 1/2\ \nabla_z \kappa_r(Q) \quad , \quad A_r^{(2)}(Q) = 1/2\ \nabla_z^2 \kappa_r(Q) - B_r^{(1,1)}(Q)$$

$$C_r^{(1,1)}(Q) = 1/2\ \nabla_z \nabla_{\bar z} \kappa_r(Q)$$

$$C_r^{(1,2)}(Q) = \nabla_{\bar z} C_r^{(1,1)}(Q) \quad , \quad C_r^{(2,1)}(Q) = \nabla_z C_r^{(1,1)}(Q)$$

$$C_r^{(2,2)}(Q) = \nabla_z C_r^{(1,2)}(Q) - 1/2\ R^{\Omega}(Q)\ C_r^{(1,1)}(Q)$$

(III.85)

Similar formulas to (III.83-85) are obtained for the commutator with $\bar{L}_r$.

Construction of vertex operators for physical states

We will consider the lowest massive levels; the extension to higher levels is made following the same lines described here.

1) Massless level: $n=1$, $p^2=0$.

The most general operator is given by

$$:W(Q): = \epsilon_{\mu\bar\mu}(p)\ :W^{\mu\bar\mu}(Q): + \beta(p)\ :W'(Q): \qquad \text{(III.86a)}$$

where

$$W^{\mu\bar\mu}(Q) = \Omega(Q)\ \Omega^{z\bar z}\ \nabla_z X^{\mu}(Q)\ \nabla_{\bar z} X^{\bar\mu}(Q)\ \exp(ip.X(Q)) \qquad \text{(III.86b)}$$

$$W'(Q) = \Omega(Q)\ R^{\Omega}(Q)\ \exp(ip.X(Q)) \qquad \text{(III.86c)}$$

[24] It can be see that the only "independent" anomalous tensors are κ_r and $B_r^{(1,1)}$; all the others can be expressed as linear combinations of the covariant derivatives of these two ones and the curvature R^{Ω}.

From (III.83) we get[25]

$$[L_r,:W(Q):] - L_{e_r} :W(Q): = \kappa_r(Q)\ A_{\mu\bar{\mu}}(p)\ :W^{(\mu,\bar{\mu})}(Q): \tag{III.87a}$$

with

$$A_{\mu\bar{\mu}}(p) = -1/4\ (p_\mu\ p^\rho \epsilon_{\rho\bar{\mu}}(p) + p_{\bar{\mu}}\ p^\rho \epsilon_{\mu\rho}(p)) + $$
$$+ 1/4\ (\epsilon_\rho{}^\rho - 8\beta(p))\ p_\mu p_{\bar{\mu}} \tag{III.87b}$$

If $\epsilon_{\mu\bar{\mu}}(p)$ is traceless, the cancellation of the RHS in (III.87a) implies $\beta(p)=0$ and the transversality conditions

$$p^\mu \epsilon_{\mu\bar{\mu}}(p) = p^{\bar{\mu}} \epsilon_{\mu\bar{\mu}}(p) = 0 \tag{III.88}$$

The symmetric and antisymmetric parts of $\epsilon_{\mu\bar{\mu}}(p)$ represent the graviton and the antisymmetric particles respectively. If instead we consider the trace part $\epsilon_{\mu\bar{\mu}}(p)=\eta_{\mu\bar{\mu}}$ then (III.87b) gives $\beta(p)=(D-2)/8$, leading to the the dilaton vertex operator

$$:W_D(Q): = \eta_{\mu\bar{\mu}}:W^{\mu\bar{\mu}}(Q): + (D-2)/8\ \Omega(Q)\ R^\Omega(Q)\ :\exp(ip.X(Q)): \tag{III.89}$$

The coupling of the dilaton wave function :exp(ip.X(Q)): to the curvature in (III.89) was originally proposed by Fradkin and Tseytlin on dimensional grounds, in the context of defining an effective action for the modes of the string[41], and used by Callan et al. to formulate the conformal invariance σ-model approach to the vacuum string problem[42]. It was however in ref.[34] that its origin was discovered in searching for a Weyl invariant vertex operator for the dilaton in the path integral approach to string perturbation theory. Here we recover it from the conformal invariance requirement in the gauge fixed theory.

[25] Total derivative terms will be neglected in what follows due to the "cancelled propagator" argument explained in Section 6, as it is naturally done in the path integral formulation.

2) First massive level n=2, $p^2=-4$.

The most general object to be considered is given by (we omit the $\Omega(Q)\exp(ip.X(Q))$ factors)[26]

$$W(Q) = \epsilon_{\mu\nu\bar{\mu}\bar{\nu}}(p)\, W^{\mu\nu\bar{\mu}\bar{\nu}}(Q) + \epsilon_{\mu\bar{\mu}}(p)\, W_1^{\mu\bar{\mu}}(Q) + \epsilon'_{\mu\bar{\mu}}(p)\, W_2^{\mu\bar{\mu}}(Q) +$$

$$+ \epsilon_{\mu\nu\bar{\mu}}(p)\, W_1^{\mu\nu\bar{\mu}}(Q) + \bar{\epsilon}_{\bar{\mu}\bar{\nu}\mu}(p)\, W_2^{\bar{\mu}\bar{\nu}\mu}(Q) + \epsilon(p)\, W'(Q) \tag{III.90a}$$

where

$$W^{\mu\nu\bar{\mu}\bar{\nu}}(Q)=\nabla_z X^{\mu}(Q)\nabla_z X^{\nu}(Q)\nabla^{z}X^{\bar{\mu}}(Q)\nabla^{z}X^{\bar{\nu}}(Q) \quad , \quad W'(Q)=R^{\Omega}(Q)^2$$

$$W_1^{\mu\bar{\mu}}(Q)=\nabla_z^2 X^{\mu}(Q)\nabla^{z2}X^{\bar{\mu}}(Q) \quad , \quad W_1^{\mu\nu\bar{\mu}}(Q)=\nabla_z X^{\mu}(Q)\nabla_z X^{\nu}(Q)\nabla^{z2}X^{\bar{\mu}}(Q)$$

$$W_2^{\mu\bar{\mu}}(Q)=R^{\Omega}(Q)\nabla_z X^{\mu}(Q)\nabla^{z}X^{\bar{\mu}}(Q) \quad , \quad W_2^{\bar{\mu}\bar{\nu}\mu}(Q)=\nabla_z^2 X^{\mu}(Q)\nabla^{z}X^{\bar{\mu}}(Q)\nabla^{z}X^{\bar{\nu}}(Q) \tag{III.90b}$$

The general formula (III.83a) gives the following result for the commutator

$$[L_r, :W(Q):] - L_{e_r} :W(Q): =$$

$$= \kappa_r(Q)\ \big(A_{\mu\nu\bar{\mu}\bar{\nu}}(p)W^{\mu\nu\bar{\mu}\bar{\nu}}(Q) + B_{\mu\bar{\mu}}(p)W_1^{\mu\bar{\mu}}(Q) + B'_{\mu\bar{\mu}}(p)W_2^{\mu\bar{\mu}}(Q) +$$

$$+ C^1_{\mu\nu\bar{\mu}}(p)W_1^{\mu\nu\bar{\mu}}(Q) + C^2_{\bar{\mu}\bar{\nu}\mu}(p)W_2^{\bar{\mu}\bar{\nu}\mu}(Q) + D(p)W'(Q) + E^1_{\mu}(p)W_1^{\mu}(Q) +$$

$$+ E^2_{\bar{\mu}}(p)W_2^{\bar{\mu}}(Q) + D'(p)W''(Q)\ \big) + B_r^{(1,1)}(Q)\ H_{\bar{\mu}}(p)\nabla^{z2}X^{\bar{\mu}}(Q) +$$

$$+ B_r^{(1,1)}(Q)\ G_{\bar{\mu}\bar{\nu}}(p)\nabla^{z}X^{\bar{\mu}}(Q)\nabla^{z}X^{\bar{\nu}}(Q) \tag{III.91}$$

where the tensors $A_{\mu\nu\bar{\mu}\bar{\nu}}(p)$, etc. are, as in (III.87b), linear combinations of the polarization tensors. The cancellation of these tensors (not all of them independent) gives a set of conditions to be imposed on the physical polarization tensors. They can be recast in the following form (we omit the "conjugate" ones)

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[26] There exist another three possible terms of the form

$$\epsilon_{\mu}\, W_1^{\mu} = \epsilon_{\mu}\ \nabla^{z}R^{\Omega}\nabla_z X^{\mu} \quad , \quad \epsilon_{\bar{\mu}}\ W_2^{\bar{\mu}} = \epsilon_{\bar{\mu}}\ \nabla_z R^{\Omega}\nabla^{z}X^{\bar{\mu}} \quad , \quad \epsilon'\ W'' = \epsilon'\ \nabla^{z}\nabla_z R^{\Omega},$$

that under integration on Σ are absorbed in $\epsilon'_{\mu\bar{\mu}}W_2^{\mu\bar{\mu}}$ via the gauge transformation

$$\epsilon'_{\mu\bar{\mu}}(p) \to \epsilon'_{\mu\bar{\mu}}(p) - \epsilon_{\mu}(p)\ p_{\bar{\mu}} - \epsilon_{\bar{\mu}}(p)\ p_{\mu} - \epsilon'(p)\ p_{\mu}\ p_{\bar{\mu}}$$

$$0 = \epsilon^{\rho}{}_{\rho\bar{\mu}}(p) - ip^{\rho}\epsilon_{\rho\bar{\mu}}(p) = \bar{\epsilon}^{\bar{\rho}}{}_{\bar{\rho}\mu}(p) - ip^{\bar{\rho}}\epsilon_{\mu\bar{\rho}}(p) \qquad \text{(III.92a)}$$

$$0 = \epsilon''(p) - 1/32\ (2\epsilon'{}_{\rho}{}^{\rho}(p) + \epsilon_{\rho}{}^{\rho}(p)) \qquad \text{(III.92b)}$$

$$0 = p^{\rho}(2\epsilon'_{\rho\bar{\mu}}(p) - \epsilon_{\rho\bar{\mu}}(p)) \qquad \text{(III.92c)}$$

$$0 = \epsilon'_{\mu\bar{\mu}}(p) - 1/2\epsilon_{\mu\rho\bar{\mu}}{}^{\rho}(p) + i/4\ (\epsilon_{\mu\rho}{}^{\rho}(p)p_{\bar{\mu}} + \bar{\epsilon}_{\bar{\mu}\bar{\rho}}{}^{\bar{\rho}}(p)p_{\mu}) + $$
$$+ 1/8\ \epsilon_{\rho}{}^{\rho}(p)p_{\mu}p_{\bar{\mu}} \qquad \text{(III.92d)}$$

$$0 = \epsilon_{\mu\nu\bar{\mu}}(p) - i/2p^{\bar{\rho}}\epsilon_{\mu\nu\bar{\mu}\bar{\rho}}(p) + i/4\ p_{\bar{\mu}}(\epsilon_{\mu\nu\bar{\rho}}{}^{\bar{\rho}}(p) + i\ p^{\bar{\rho}}\epsilon_{\mu\nu\bar{\rho}}(p)) \qquad \text{(III.92e)}$$

$$0 = \epsilon_{\mu\bar{\mu}}(p) - i/2p^{\rho}\epsilon_{\rho\mu\bar{\mu}}(p) + i/8\ p_{\mu}(\epsilon^{\rho}{}_{\rho\bar{\mu}}(p) + ip^{\rho}\epsilon_{\rho\bar{\mu}}(p)) \qquad \text{(III.92f)}$$

$$0 = ip^{\bar{\rho}}\epsilon_{\mu\nu\bar{\rho}}(p) - (\eta^{\bar{\rho}\bar{\sigma}} + p^{\bar{\rho}}p^{\bar{\sigma}})\epsilon_{\mu\nu\bar{\rho}\bar{\sigma}}(p) \qquad \text{(III.92g)}$$

$$0 = ip^{\bar{\rho}}\epsilon_{\mu\bar{\rho}}(p) - (\eta^{\bar{\rho}\bar{\sigma}} + p^{\bar{\rho}}p^{\bar{\sigma}})\bar{\epsilon}_{\bar{\rho}\bar{\sigma}\mu}(p) \qquad \text{(III.92h)}$$

In ref.[35] vertex operators were read from the residues of the poles for the intermediate states in the multitachyon amplitude (III.79), a method dictated by unitarity arguments (the need of doing it so was first remarked in ref.[15]). The vertices obtained here are in perfect agreement with them, a fact consistent with the statement that eqs.(III.41) are required by unitarity of scattering amplitudes (the equivalence of vertices for n=0,1 is obvious, to see the equivalence for n=2, we must identify our polarization tensors in (III.90a) with those of the "factorized" form of the vertex in ref.[35]

$$(e_{\mu\nu}(p)\nabla_z X^{\mu}(Q)\nabla_z X^{\nu}(Q) + e_{\mu}(p)\nabla_z^2 X^{\mu}(Q))\times$$
$$\times(\bar{e}_{\bar{\mu}\bar{\nu}}(p)\nabla^z X^{\bar{\mu}}(Q)\nabla^z X^{\bar{\nu}}(Q) + \bar{e}_{\bar{\mu}}(p)\nabla^{z\,2} X^{\bar{\mu}}(Q))\ \exp(ip.X(Q))$$

in the way: $\epsilon_{\mu\nu\bar{\mu}\bar{\nu}} \equiv e_{\mu\nu}\bar{e}_{\bar{\mu}\bar{\nu}}$, $\epsilon_{\mu\nu\bar{\mu}} \equiv e_{\mu\nu}\bar{e}_{\bar{\mu}}$, $\bar{\epsilon}_{\bar{\mu}\bar{\nu}\mu} \equiv \bar{e}_{\bar{\mu}\bar{\nu}}e_{\mu}$, $\epsilon_{\mu\bar{\mu}} \equiv e_{\mu}\bar{e}_{\bar{\mu}}$).

11. Some comments about Superstring Theories

The Virasoro algebra (I.62) has N=1 supersymmetric generalizations known as Ramond and Neveau-Schwarz algebras[27]. They display together with the T component of the EMT an anticommuting current of conformal weight 3/2

$$G(z)\, dz^{3/2} = \sum_r G_r\, z^{r-3/2}\, dz^{3/2} \tag{III.93}$$

where the index r runs over the integers (half-integers) in the R (NS) case. The components G_r obey the commutation (anti) relations[18]

$$[L_m, G_r] = (m/2 - r)\, G_{m+r} \tag{III.94a}$$

$$\{G_r, G_s\} = 2\, L_{r+s} + c/3\, (r^2-1/4)\, \delta_{r+s,0} \tag{III.94b}$$

It is natural to guess that, as the KN algebra is the higher genus generalization of the Virasoro algebra, there should be also a generalization of the superconformal algebra (I.62), (III.94). Its construction was worked out in Ref.[26]. Let us denote by $g_r \equiv f_r^{(-1/2)}$ the KN basis corresponding to $K^{-1/2}$ (a spin structure and R or NS boundary conditions are understood). Then the binary operations

$$[e_i, g_r] \equiv L_{e_i} g_r = \sum_{s=-g_0}^{g_0} H^s_{ir}\, g_{i+r-s} \tag{III.95a}$$

$$\{g_r, g_s\} \equiv g_r g_s + g_s g_r = \sum_{p=-g}^{g} B^p_{ir}\, e_{r+s-p/2} \tag{III.95b}$$

define un extension of eq.(I.56). The structure constants are obtained as usual by means of the dual bases and the corresponding duality relations (cf. Chapter I, Part C). The central extension is made by introducing the following cocycle[26]

$$\varphi_{rs} = 1/(2\pi i) \oint_{C_\tau} dz\, (\partial_z g_r(z)\, \partial_z g_s(z) + 1/2\, R(z)\, g_r(z)\, g_s(z)) \tag{III.96}$$

[27] The N=2 superconformal algebras have became very popular in the last time due to its close relation with the vacuum of the superstring, see [43,44] and references therein.

Then

$$[L_i, G_r] = \sum_{s=-g_0}^{g_0} H^s_{ir} G_{i+r-s} \tag{III.97a}$$

$$\{G_r, G_s\} = \sum_{p=-g}^{g} B^p_{ir} L_{r+s-p/2} + c/3\ t\ \varphi_{rs} \tag{III.97b}$$

$$[G_r, t] = 0 \tag{III.97c}$$

together with (I.59) defines the N=1 KN superconformal algebra.

Now the action of the superstring in the Neveau-Schwarz-Ramond (NSR) formulation is the sum of D copies of (II.81) plus

$$S_f[\psi,\bar\psi] = 1/(2\pi) \int i dz\wedge d\bar z\ \eta_{\mu\nu}(\psi^\mu\bar\partial\psi^\nu + \bar\psi^\mu\partial\bar\psi^\nu) \tag{III.98}$$

where $(\psi^\mu,\bar\psi^\mu)$ are the components of a Majorana spinor. The free field theory (III.98) can be worked out as we made in Chapter II with the scalar and ghost fields, starting from the components of the EMT (for a detailed discussion see Ref.[45])

$$T_f(Q) = i/2\ \psi(Q).\partial\psi(Q) \tag{III.99a}$$

$$\bar T_f(Q) = i/2\ \bar\psi(Q).\bar\partial\bar\psi(Q) \tag{III.99b}$$

The equations of motion for ψ^μ and $\bar\psi^\mu$ tell us that they are holomorphic and antiholomorphic respectively outside P_+ and P_- (and a possible cut if we are considering the R sector), and thus the following expansions hold

$$\psi^\mu(Q) = \sum_r d^\mu_r f^{(1/2)}_r(Q) \tag{III.100a}$$

$$\bar\psi^\mu(Q) = \sum_r \bar d^\mu_r \bar f^{(1/2)}_r(Q) \tag{III.100b}$$

where r runs as in (III.93). The operators d^μ_r, d^μ_r satisfy the canonical anticommutation relations

$$\{d^\mu_r, d^\nu_s\} = \eta^{\mu\nu}\ \delta_{r+s,0} \quad , \qquad \{\bar d^\mu_r, \bar d^\nu_s\} = \eta^{\mu\nu}\ \delta_{r+s,0} \tag{III.101}$$

The Virasoro components of (III.99) satisfy a KN algebra with c=1/2D and the schwartzian connection depending on the propagator (that is, on the kind of spinor we are considering) of the ψ's.

The action (III.98) is a gauge-fixed version of a complicated reparametrization invariant one, which presents also a local supersymmetric invariance together with a superWeyl invariance[18]. After superconformal gauge-fixing we remain with (III.98) (plus the scalars). The superconformal transformations are generated by the components of the supercurrent

$$G(Q) = \psi(Q).\partial X(Q) = \sum_r G_r f_r^{(3/2)}(Q) \qquad \text{(III.102a)}$$

$$\bar{G}(Q) = \bar{\psi}(Q).\bar{\partial}\bar{X}(Q) = \sum_r \bar{G}_r \bar{f}_r^{(3/2)}(Q) \qquad \text{(III.102b)}$$

and correspond to the transformations which leave the superWeyl gauge unmodified. The super FP determinant arising from the super gauge-fixing may be writen in this case as a path integral over a pair of *commuting* (bosonic) ghosts (β,γ) of weights 3/2 and -1/2 respectively, together with its complex conjugate $(\bar{\beta},\bar{\gamma})$, the so-called superghost system. The central charge of the KN algebra for this kind of systems of weight λ is given by minus that in (III.14) (the change of sign is due to the change in the normal ordering definition for a commuting system). In our case $\lambda=3/2$, so c=11. The whole system of matter plus ghost realize the superconformal algebra (I.57,III.99) ($G^{gh}=-c\partial\beta+1/2\gamma b-3/2\partial c\beta$) with total central charge

$$c_T = 1\ D + 1/2\ D + (-26) + 11 = 3/2\ (D-10) \qquad \text{(III.103)}$$

displaying the well-known critical dimension D=10 of the superstring. In Ref.[46] Polyakov showed that it is neccesary for the decoupling of the Weyl and superWeyl factors from the path integral (c_T multiplies in this case the superLiouville action, cf. (III.31)). The construction of an anomaly-free super-KN algebra for the superstring in the critical dimension is made by following similar steps to those in Section 5.

The superconformal invariance conditions for a superconformal field of weight (p,q) are given by (III.41) and[45]

$$[G_r, W(Q)] = L_{g_r} U(Q) \qquad \text{(III.104a)}$$

for some operator $U(Q) \in K^{p-1/2}\otimes\bar{K}^q$, and similarly for $\bar{G}_r$. The "Lie derivative" with respect to the -1/2-differential g_r acting on p-differentials is given by

$$L_{g_r} \omega^p(Q) = g_r(Q)\ \partial\omega^p(Q) + 2\ p\ \partial g_r(Q)\ \omega^p(Q) \qquad \text{(III.104b)}$$

At this point, in what respects to the definition of superscattering amplitudes, we should search for vertex operators V(Q) of conformal weight (1,1) satisfying (III.41,104). By carefully repeating the arguments given in Section 6 one concludes that V(Q) maps physical states to physical states up to spurious states, which are shown to decouple from the amplitudes provided that eqs.(III.41,104) hold[45].

The construction of VO for the superstring goes in a very elegant way in the superspace language[15]. This amounts to look for (1/2,1/2) local operators $W(Q,\theta,\bar{\theta})$ so that

$$V = \int_{\Sigma^s} d^2\theta \; W(Q,\theta,\bar{\theta}) \tag{III.105}$$

be superconformal invariant. Here θ, $\bar{\theta}$ are the holomorphic Grassman coordinates of the super RS Σ^s. For this end we introduce the matter superfields

$$Y^\mu(Q,\theta,\bar{\theta}) = X^\mu(Q) + i\,\bar{\theta}\,\psi^\mu(Q) + i\,\bar{\psi}^\mu(Q)\,\theta \tag{III.106}$$

and the super Cauchy operators

$$D = \bar{\theta}\,\nabla + \partial_{\bar{\theta}} \quad , \quad \bar{D} = \theta\,\bar{\nabla} + \partial_\theta \tag{III.107}$$

where ∇ is the covariant derivative (III.81).

The monomials to consider now consist of supercovariant derivatives of the superfields Y^μ (see [33] for details). As examples, let us write down the tachyon and graviton vertices[45]

$$V_{tach} = \int_{\Sigma^s} d^2\theta\; \Omega^{1/2}\; e^{ip.Y} = \int_{\Sigma} \Omega^{1/2}\; p.\psi\; p.\bar{\psi}\;\; e^{ip.X} \tag{III.108a}$$

$$\begin{aligned} V_{grav} &= \xi_{\mu\nu}(p) \int_{\Sigma^s} d^2\theta\; DY^\mu \wedge \bar{D}Y^\nu\; e^{ip.Y} = \\ &= \xi_{\mu\nu}(p) \int_{\Sigma} (\partial X^\mu - i\psi^\mu p.\psi) \wedge (\bar{\partial}X^\nu - i\bar{\psi}^\nu p.\bar{\psi}) e^{ip.X} \end{aligned} \tag{III.108b}$$

They obey the commutation relations (III.41,104) with $p^2=4$ and $p^2=0$, $\xi_{\mu\nu}(p)p^\nu=0$ respectively. By following the same lines as for the tachyon amplitude in the bosonic string computed in Section 9, we find that the M tachyon-scattering amplitude in the RNS superstring is[47]

$$A^M_{tach}[\Sigma] = \int_{\Sigma^s} \prod_{i=1}^{M} d^2\theta_i \prod_{i<j} |E(P_i,P_j)-2\bar{\theta}_i\bar{\theta}_j \frac{\theta[^\alpha_\beta](P_i-P_j)}{\theta[^\alpha_\beta](0)}|^{p_i\cdot p_j} \times$$

$$\times \exp(\pi p_i\cdot p_j/4 \int_{P_i}^{P_j} (\eta^k-\bar{\eta}^1)(\tau_2^{-1})_{kl} \int_{P_i}^{P_j} (\eta^k-\bar{\eta}^1)) \qquad \text{(III.109)}$$

Since the integrand contains an even number of θ_i, the amplitude vanishes unless M is even, consistent with the notion of a multiplicative conserved "G-parity" quantum number.

Similarly one can compute the M-graviton scattering amplitude. We find[47]

$$A^M_{grav}[\Sigma] = \xi^1_{\mu\nu_1}(p_1)\ldots\xi^M_{\mu\nu_M}(p_M)\int_\Sigma \prod_{i=1}^{M} d^2\theta_i$$

$$\times \exp(\pi p_i\cdot p_j/4 \int_{P_i}^{P_j} (\eta^k-\bar{\eta}^1)\ (\tau_2^{-1})_{kl} \int_{P_i}^{P_j} (\eta^k-\bar{\eta}^1)]\ |F_M|^2 \times$$

$$\times \prod_{i<j} |E(P_i,P_j)-2\bar{\theta}_i\bar{\theta}_j \frac{\theta[^\alpha_\beta](P_i-P_j)}{\theta[^\alpha_\beta](0)}|^{p_i\cdot p_j} \qquad \text{(III.110)}$$

where

$$F_M = \{ \prod_{i<j} \exp(\theta_i\xi_i\cdot p_j\langle\partial X_i X_j\rangle+\theta_j\xi_j\cdot p_i\langle X_i\partial X_j\rangle-\theta_i p_i\cdot\xi_j\langle\psi_i\psi_j\rangle-$$

$$- \theta_j p_j\cdot\xi_i\langle\psi_i\psi_j\rangle+\theta_i\theta_j\xi_i\cdot\xi_j\langle\partial X_i\partial X_j\rangle+\xi_i\cdot\xi_j\langle\psi_i\psi_j\rangle)\}\Big|_{\text{linear in } \xi^\mu_\iota}$$

and "<...>" denotes the propagators (a sum over spin structures and an integration over the supermoduli space is left over in (III.109,110)).

However this can not be the whole story. The Hilbert space of the superstring is a tensor product of left and right spaces. Due to the existence of two sectors R and NS for the spinor fields, we have four sectors in the Hilbert space, denoted commonly by R-R, NS-NS, R-NS, NS-R. The first two ones give rise to space-time bosons, while the last two to fermions[28].

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[28] This is due to the fact that in the R case the vacuum (and in general any state) must furnish a representation of the "zero mode" algebras $\{d^\mu_0, d^\nu_0\}=\eta^{\mu\nu}$, $\{\bar{d}^\mu_0, \bar{d}^\nu_0\}=\eta^{\mu\nu}$ (cf. (III.100), which are spinors under SO(1,D-1) (or spin(1,D-1))[18].

The vertex construction sketched before corresponds to emission or absorption of bosons. A qualitative way of seeing this is to recall that both the elementary fields X^μ and ψ^μ are space-time Lorentz vectors, while a fermion vertex should be a Lorentz spinor. Clearly this kind of VO can not be simply exp(ip.X) times a polynomial in the elementary fields. A related way to express this difficulty is to note that emission of a fermion from the string must turn a fermionic (bosonic) state into a bosonic (fermionic) one. But in order to pass from the R-R or NS-NS sectors to R-NS or NS-R the fermionic vertex must change the boundary conditions of the ψ^μ (or $\bar{\psi}^\mu$), that is, it must create a "cut" in the ψ^μ field. Equivalently, there should exist an operator which relates the NS and R vacua. Such operators are called "spin fields"[15,48]. They are the fundamental key to construct fermionic vertices (and also to write the fermionic charges of the space-time superPoincare algebra[15]).

We will not go into the construction of fermionic VO, but only remark some problems found in its definition and use within our operator context[49]. First of all, the g=0 construction of Refs.[15,48] is based in a "holo + antiholo" decomposition of the bosonic fields X^μ, thing that globally does not happen at higher genus (cf. eq.(II.101)). Also our master metric (III.70) has not a holomorphic square root. Secondly, fermionic VO include the spin fields of the β-γ system of weigth 3/2. Correlation functions of spin fields for fermionic and anticommuting ghost systems are easily computed by means of the bosonization prescription[50,20], and this framework can be straightforwardly adapted to our context in these cases[51]. But unfortunately the bosonization of commuting ghost systems seems not to be possible on arbitrary RS[49], and we need a field representation to compute correlation functions. Within the PIA there are however several approaches to do it (CFT techniques[52], fermionization[53]).

We finally remark that the no existence of a "chiral" vertex seems to be also a hard obstacle to carry out the Frenkel-Kac construction[54] and to fix a light cone gauge in string theories at arbitrary genus[55].

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