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Strada Costiera 11

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CANDIDATE

MARCO MATONE

SUPERVISOR

PROF. LORIANO BONORA

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Shut up'n play yer guitar

Frank Zappa

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CONTENTS

0. INTRODUCTION

One of the more interesting aspects in conformal field theories concerns their formulation on Riemann surfaces. The determination of correlators on surfaces of higher topology is an important problem of mathematical physics with many implications in string theory. Indeed it is a well – known result that conformal field theories correspond to vacuum solutions of strings. Another motivation in studying conformal field theory in higher genus is that correlation functions on the torus (whose relevance in statistical mechanics is obvious) can be recovered from the higher genus correlators by pinching cycles. However the basic remark that motivates the study of this subject is that primary fields (the relevant objects of CFT) have tensorial properties under conformal transfomations. Then it is natural to formulate CFT on the moduli space of Riemann surfaces since, by definition, they have analytic transition functions between coordinate patches. Let us recall how it is possible to get informations by formulating conformal field theory in higher genus. Consider a primary field correlator G and let us denote with $m \equiv (m_1, ..., m_{3g-3})$ the moduli parameter. It turns out that G can be expressed in terms of analytic and antianalytic building blocks:

$$G(z, m; \bar{z}, \bar{m}) = \sum_{\bar{I}, J} \overline{\mathcal{F}}_{\bar{I}}(\bar{z}, \bar{m}) h_{\bar{I}, J} \mathcal{F}_J(z, m),$$

where $h_{\bar{I},J}$ is an hermitean metric. The requirement for G to be single valued and modular invariant, imposes restrictions on $h_{\bar{I},J}$. In particular these requirements give constraints on the operator content. Similar ideas have been used in classifying the modular invariant partition functions. Another relation between CFT and Riemann surfaces is due to the Verlinde's conjecture stating that modular transformations diagonalizes the fusion rules.

A first systematic approach to the formulation of conformal field theories in higher genus was given in the fundamental paper by Friedan and Shenker. In this thesis we will describe some related arguments. In particular we will introduce basis for differentials on a Riemann surface (Σ) that allows us to formulate an operator formalism that closely resembles the euclidean radial quantization approach to CFT formulated on the sphere. In order to do this we will generalize the basis for differentials introduced by Krichever and Novikov to holomorphic differentials on Riemann surfaces with more than two - punctures.

Obviously this may be relevant for studying the explicit formulation of CFT on the space of punctured surfaces introduced by Vafa.

Another interesting aspect in the CFT is its relation with integrable systems. In chapter 4 we will show how it is possible to relate the KdV equation to the Riemann surfaces in a new way. The idea is based on a covariantization procedure that allows us to recover the KN algebra starting from the KdV equation. The covariantization procedure consists in using a vector field e (i.e. a -1 – form) to define the derivative on Σ. For example the (non covariant) third – derivative of the KdV field (a 2 – form) becomes

$$u''' \longrightarrow e^{-1}(e(e(e^2u)')')' .$$

This covariantization procedure turns to be related to the equations for the null states in CFT.

In the last chapter we will give several examples about the formulation of CFT in terms of real – weights $b-c$ systems. They play the analog rôle of the compactified scalar field. The difference lies in the particular features of the $b-c$ system. In particular in formulating CFT in terms of the $b-c$ system it turns out that a "symmetry requirenment" for the relevant differentials gives in a natural way the Kac's formula (see eq.(8.106)).

1. GENERALITIES ON 2D CONFORMAL FIELD THEORIES

The aim of this chapter is to provide a short introduction to two - dimensional conformal field theory emphasizing some facts that will be used in following chapters. The basic reference is the original paper by Belavin, Polyakov and Zamolodchikov[1]. For an updated collection of articles on the subject see [2]. An excellent review on the argument is the recent paper by P. Ginsparg [3]. In [4] may be found applications to statistical mechanical models. For applications to string theories see for example [5,6,7]. Other good reviews are listed in [8].

1.1 The Bootstrap Program

In statistical physics a given physical system at the second order phase transition point can be described by a massless euclidean quantum field theory. In this case the theory is classified by a set of operators $\{A_j(x)\}$ that under scaling transformations $x \to \lambda x$ transform as $A_j(x) \to \lambda^{-\Delta_j} A_j(\lambda x)$. The non - negative parameter Δ_j is the anomalous scaling dimension of $A_j(x)$. Since in critical phenomena the fields are in general interacting, Δ_j does not correspond to the canonical dimensions of free fields. The computation of the spectrum $\{\Delta_j\}$ is of fundamental importance because it determines the critical exponents of the theory.

In [9] Polyakov formulated the hypothesis that at the critical point the theory should also be invariant under the local scaling $x \to \lambda(x)x$. This is a conformal symmetry of critical phenomena. In two - dimensions this local rescaling can be extended to the infinite dimensional group of analytic transformations.

In the set $\{A_j\}$ there are some fields with the same transformation properties of tensors under analytic coordinate variations. These fields, called primary fields, are the basic objects of the theory. In particular any correlation function can be computed, in principle, in terms of correlators involving primary fields only.

In formulating the bootstrap program Polyakov proposed to solve the theory using conformal invariance and operator algebra[10]. Conformal invariance serves to classify the theory in terms of representations of the conformal group. Assuming the existence of a set of operators $\{A_j(0)\}$ containing the identity operator and any derivative of each field involved, the operator algebra hypothesis consists in the assumption of the validity of the following expansion

$$A_i(x)A_j(0) \sim \sum_k C_{ijk}(x)A_k(0)\,. \tag{1.1}$$

This expansion is understood in a weak sense, i.e. as an exact expansion inside a correlation function. The requirement of locality imposes on the c - number function $C_{ijk}(x)$ to be single - valued. The conformal symmetry fixes the functional dependence of $C_{ijk}(x)$ while its dependence on anomalous dimensions and numerical factors is determined by an infinite set of equations coming from the requirement of associativity which is the main dynamical principle of the operator algebra. This idea was suggested also for conformal field theories in arbitrary dimensions; however this conformal bootstrap program proves to be too difficult to be implemented. Nevertheless, owing to its infinite - dimensional symmetry in 2d - conformal field theories, these equations are more manageable.

1.2 The Conformal Group

Let us begin with the description of the conformal group in an n - dimensional space with flat metric $g_{ab} = \delta_{ab}$. It is defined to be the subgroup of coordinate transformations

$$\xi^a \to \eta^a(\xi)\,, \qquad a = 0, ..., n-1\,, \tag{1.2}$$

leaving the metric invariant up to a multiplicative factor

$$g_{ab} \to \frac{\partial \xi^{a'}}{\partial \eta^a}\frac{\partial \xi^{b'}}{\partial \eta^b} g_{ab} = \rho(\xi) g_{ab}\,. \tag{1.3}$$

Then the coordinate transformations must satisfy the following equation

$$\partial_a \eta_b + \partial_b \eta_a = \frac{2}{n}\delta_{ab}\partial_c \eta_c\,. \tag{1.4}$$

In more than two - dimensions the conformal group is finite dimensional and consists of translations, rotations, dilatations and inversions. In two - dimensions the conditions (1.4) are the Cauchy - Riemann equations, hence the two - dimensional conformal group $\mathcal{G}$ is the infinite dimensional group of analytic coordinate transformations $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$, where $z = \xi^1 + i\xi^2$ and $\bar{z} = \xi^1 - i\xi^2$. The analytic transformations $z \to f(z) = z - z^{n+1}$ and $\bar{z} \to f(\bar{z}) = \bar{z} - \bar{z}^{n+1}$ are generated by $\mathcal{L}_n = -z^{n+1}\partial_z$ and $\bar{\mathcal{L}}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$ respectively. Their algebra is

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}\,, \qquad [\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] = (m-n)\bar{\mathcal{L}}_{m+n}\,, \qquad [\mathcal{L}_m, \bar{\mathcal{L}}_n] = 0\,. \tag{1.5}$$

Since $[\mathcal{L}_m, \bar{\mathcal{L}}_n] = 0$ we can regard z and $\bar{z}$ as independent coordinates: $(z, \bar{z}) \in \mathbb{C}^2$. Then the conformal group decomposes in the direct product of "chiral" groups

$$\mathcal{G} = \Gamma \otimes \bar{\Gamma}\,. \tag{1.6}$$

For this reason in the following we shall frequently consider only the z dependent part.

Unlike the $n \geq 3$ case where the algebra is globally defined, in two - dimensions we have to distinguish the local algebra from the global one. The unique global generators (generating the global conformal group) are $\{\mathcal{L}_{-1}, \mathcal{L}_0, \mathcal{L}_1\} \cup \{\bar{\mathcal{L}}_{-1}, \bar{\mathcal{L}}_0, \bar{\mathcal{L}}_1\}$, which are the generators of $sl(2, \mathbb{C})$. Indeed they are the unique non singular generators on the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$, both in $z = 0$ and $w = 0$ ($w = -1/z$). In other words by Riemann - Roch theorem dim $H^0(K^{-1}, S^2) = 3$ where K is the canonical line bundle. The elements of the global conformal group $SL(2, \mathbb{C})/\mathbb{Z}_2 \sim SO(3,1)$ are the global automorphisms of S^2 acting as the group of fractional linear transformations

$$z \to f(z) = \frac{az + b}{cz + d}\,, \qquad a, b, c, d \in \mathbb{C}\,, \quad ad - bc = 1\,. \tag{1.7}$$

Notice that $\mathcal{L}_0 + \bar{\mathcal{L}}_0$ generates dilatations and $i(\mathcal{L}_0 - \bar{\mathcal{L}}_0)$ rotations. As we will show, in the quantum case the algebra (1.5) has a central extension. This algebra is called Virasoro algebra and it is generated by the modes L_n and $\bar{L}_n$ of the Laurent expansion of the energy momentum tensor (see below). Given an eigenstate $|\phi >$ of L_0 ($\bar{L}_0$) with eigenvalues $h \in \mathbb{R}$ ($\bar{h} \in \mathbb{R}$), its anomalous scaling dimension and spin are $\Delta = h + \bar{h}$ and $s = h - \bar{h}$ respectively.

The $\mathcal{L}_n$'s with $|n| \geq 2$ generate the so - called local conformal group. Since under the change of variables (1.2) the variation of the action of a physical system is

$$\delta S = \int d^n\xi T^{ab}(\partial_a \eta_b + \partial_b \eta_a)\,, \tag{1.8}$$

the invariance of the action under conformal transformations implies that the energy momentum tensor is traceless

$$T^a{}_a = 0\,. \tag{1.9}$$

In a two - dimensional conformal field theory the only non - vanishing components of T_{ab} are (in complex notation)

$$T_{zz} = T_{00} - T_{11} - 2iT_{10}\,, \qquad T_{\bar{z}\bar{z}} = T_{00} - T_{11} + 2iT_{10}\,. \tag{1.10}$$

Due to the equations of motion T_{ab} is conserved

$$g^{ab}\partial_a T_{bc} = 0\,, \tag{1.11}$$

then, since $T^a{}_a = 0$, we have $\partial_{\bar{z}} T_{zz} = 0$ and $\partial_z T_{\bar{z}\bar{z}} = 0$. In the following we use the notation: $T(z) \equiv T_{zz}(z)$, $\overline{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z})$.

1.3 Euclidean Radial Quantization

As mentioned above, in two - dimensional conformal field theories a fundamental role is played by the primary fields. They transform as tensors, that is under the conformal reparametrization $z \to f(z)$, $\bar{z} \to \bar{f}(\bar{z})$ the primary field $\phi_j(z,\bar{z})$ with conformal weights $(h_j, \bar{h}_j)$ transforms according to

$$\phi_j(z,\bar{z}) \to \tilde{\phi}_j(z,\bar{z}) = \left(\frac{df}{dz}\right)^{h_j} \left(\frac{d\bar{f}}{d\bar{z}}\right)^{\bar{h}_j} \phi_j(f,\bar{f})\,. \tag{1.12}$$

Fields without this transformation property are called secondary fields. Among the secondary fields there are the quasi - primary fields, that is fields transforming as (1.12) under $SL(2,\mathbb{C})$. (The group associated to a chiral part of the theory is $SL(2,\mathbb{R})$). Under the infinitesimal transformation $f(z) = z + \epsilon(z)$, $\bar{f}(\bar{z}) = \bar{z} + \bar{\epsilon}(\bar{z})$ we have

$$\phi_j(z,\bar{z}) \to \delta_{\epsilon\bar{\epsilon}}\phi_j(z,\bar{z}) \equiv \tilde{\phi}_j(z,\bar{z}) - \phi_j(z,\bar{z}) \sim \left(\epsilon(z)\partial_z + h_j\epsilon'(z) + \bar{\epsilon}(\bar{z})\partial_{\bar{z}} + \bar{h}_j\bar{\epsilon}'(\bar{z})\right)\phi_j(z,\bar{z})\,. \tag{1.13}$$

In order to define correlators we first introduce the operator formalism in the euclidean variables $(\tau,\sigma) \in \mathbb{R}$ related to z and $\bar{z}$ by the conformal transformation

$$z = e^w\,, \qquad \bar{z} = e^{\bar{w}}\,, \tag{1.14}$$

where $w = \tau + i\sigma$ and $\bar{w} = \tau - i\sigma$. It is clear that the massless character of field theories near the critical point is at the origin of severe infrared singularities in the correlation functions of the fields (infrared problem). The absence of positivity caused by these singularities is the reason for the non standard features of these theories. In particular they appear naturally states and fields invariant under translations; these "infrared" fields have been yet introduced long time ago in the literature on conformally invariant models [11,12,13] but their strict connection with the infrared behaviour of the theory has been explained only recently [14,15,16]. In the last works it has also been shown the great utility of these objects when discussing the symmetries of these infrared singular quantum field theories. We however shall be concerned on other features of the conformal invariant models and avoid infrared problems by compactifying the space coordinate on a cylinder: $0 < \sigma \leq 2\pi$. The correlation functions can be expressed in the operator formalism

$$< \phi_{j_1} ... \phi_{j_n} > = < 0| T \{ \phi_{j_1}(\sigma_1, \tau_1) ... \phi_{j_n}(\sigma_n, \tau_n) \} |0 > , \tag{1.15}$$

where T is the euclidean chronological ordering. The equal time curves on the cylinder correspond to constant radius circles on S^2 and the T ordering is replaced by the radial ordering operator R. Its definition is

$$R \{ \phi_j(z) \phi_k(w) \} = \theta(|z| - |w|) \phi_j(z) \phi_k(w) - \epsilon \theta(|w| - |z|) \phi_k(w) \phi_j(z) , \tag{1.16}$$

where $\epsilon = 1\ (-1)$ for fermionic (bosonic) operators. In this euclidean radial quantization procedure[17] the generator of dilatations on the sphere is, up to a shift proportional to the central charge (see below for its definition), the hamiltonian of the system on the cylinder: $H_{cyl} = (L_0 + \bar{L}_0)_{cyl} = L_0 + \bar{L}_0 - \frac{c}{12}$. The asymptotic times $\tau \to -\infty$ and $\tau \to \infty$ (where the in and out vacua $|0>$ and $<0|$ live) correspond to the points $z = 0$ and $z = \infty$. Then the computations can be performed on S^2 where we can use the power of the complex analysis calculus.

In order to make more transparent the relation with the operator formalism let us recall that under a coordinate transformation the variation of an operator ϕ_j is

$$\delta_\epsilon \phi_j(x) = [T_\epsilon, \phi_j(x)] , \qquad T_\epsilon = \int dy^1 \epsilon(y^1) T(y) , \tag{1.17}$$

where the integral is over a spacelike slice. In a euclidean theory the operator products are defined only if time ordered. Then on the complex plane we have

$$\delta_\epsilon \phi_{j_k}(z_k) = \frac{1}{2\pi i}\left(\oint\limits_{|z|>|z_k|} dz\epsilon(z)T(z)\phi_{j_k}(z_k) - \oint\limits_{|z_k|>|z|} dz\epsilon(z)\phi_{j_k}(z_k)T(z) \right) =$$

$$= \frac{1}{2\pi i}\left(\oint\limits_{|z|>|z_k|} - \oint\limits_{|z_k|>|z|} \right) dz\epsilon(z)R\left\{T(z)\phi_{j_k}(z_k)\right\} = \frac{1}{2\pi i}\oint \epsilon(z)R\left\{T(z)\phi_{j_k}(z_k)\right\} , \quad (1.18)$$

where the last contour integral is a closed curve around z_k separating it from the origin.

A delicate aspect of radial quantization is how to define the adjoint operation for euclidean fields corresponding to hermitian fields in Minkowski space. Here we consider the case of a primary field ϕ_j; the general case is discussed in ref.[18]. The first remark is that since $t = i\tau$, the complex conjugation is achieved by reversing the cylinder time. On the complex plane this corresponds to the substitution $z \to \bar{z}^{-1}$. Next we define the *in* - and *out* - states:

$$|\phi_j > \equiv \lim_{z,\bar{z}\to 0} \phi_j(z,\bar{z})|0> , \qquad <\phi_j| \equiv \lim_{w,\bar{w}\to 0} <0|\tilde{\phi}_j(w,\bar{w}) , \quad (1.19)$$

where $w = z^{-1}$ is a local coordinate in a neighborhood of $z = \infty$ and

$$\tilde{\phi}_j(w,\bar{w}) = \left(\frac{dw}{dz}\right)^{-h_j}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}_j}\phi_j(z,\bar{z}) . \quad (1.20)$$

On the other hand

$$<\phi_j| = |\phi_j>^\dagger = \lim_{z,\bar{z}\to 0} <0|\phi_j(z,\bar{z})^\dagger , \quad (1.21)$$

so that after the substitution $z \to \bar{z}^{-1}$ and using the transformation property of $\phi_j(z,\bar{z})$ we get (up to a phase)

$$\phi_j(z,\bar{z})^\dagger = z^{-2\bar{h}_j}\bar{z}^{-2h_j}\phi_j(\bar{z}^{-1},z^{-1}) . \quad (1.22)$$

1.4 Ward Identities

Now we show how the correlation functions with energy momentum tensor insertions can be expressed in terms of primary field correlation functions. This is a consequence of the conformal Ward identity. In particular correlation functions involving only the energy momentum tensor are determined explicitly. This property is due to the fact that $T(z)$ generates conformal transformations and, owing to the conformal invariance of the theory, they can be expressed in terms of primary fields. Let us consider the Ward identity involving a singular $T(z)$. It is the quantum analogue of the classical equation $\partial_{\bar{z}} T(z) = 0$. In order to derive it we perform a conformal transformation in the functional integral and look at it as a change of variable, then we use: $\delta_\epsilon S = \int d^2 z T(z) \partial_{\bar{z}} \epsilon(z)$ and get

$$\partial_{\bar{z}} < T(z)\phi_{j_1}(z_1)...\phi_{j_n}(z_n) >= \sum_{k=1}^{n} < \phi_{j_1}(z_1)...\delta_\epsilon \phi_{j_k}(z_k)...\phi_{j_n}(z_n) > . \tag{1.23}$$

In two - dimensions this equation can be solved explicitly, indeed integrating by part and exploiting the relation $\partial_{\bar{z}} \frac{1}{z-w} = 2\pi\delta^{(2)}(z-w)$, we have

$$< T(z)\phi_{j_1}(z_1)...\phi_{j_n}(z_n) >= \sum_{k=1}^{n} \left(\frac{h_{j_k}}{(z-z_k)^2} + \frac{1}{(z-z_k)} \partial_{z_k} \right) < \phi_{j_1}(z_1)...\phi_{j_n}(z_n) > . \tag{1.24}$$

From this equation we can derive the operator product expansion (OPE) of $T(z)$ with ϕ_j. Alternatively it follows from comparing (1.18) with the transformation of ϕ_j given in (1.13) with $\bar{\epsilon}(\bar{z}) = 0$. The result is (because of the radial ordering, we assume $|w| > |z|$ in the OPE)

$$T(w)\phi_j(z) \sim \frac{h_j}{(w-z)^2}\phi_j(z) + \frac{1}{w-z}\partial_z \phi_j(z) + \sum_{n \geq 2} (w-z)^{n-2} \widehat{L}_n \phi_j(z) , \tag{1.25}$$

where

$$\widehat{L}_{-n}\phi_j(z) = \frac{1}{2\pi i} \oint dw (w-z)^{-n+1} T(w) \phi_j(z) . \tag{1.26}$$

Observe that $\widehat{L}_{n>0}\phi_j(z) = 0$, $\widehat{L}_0\phi_j(z) = h_j\phi_j(z)$ and $\widehat{L}_{-1}\phi_j(z) = \partial_z\phi_j(z)$. The fields $\widehat{L}_n\phi_j \equiv \phi_j^{(-n)}$ with $n > 1$, are called descendants of ϕ_j. $T(z)$ is an example of a descendant field, indeed it is just: $1^{(-2)}(z) = (\widehat{L}_2 1)(z) = \frac{1}{2\pi i} \oint dw (z-w)^{-1} T(w) 1 = T(z)$, where 1 is the identity operator. Besides the descendant fields there are also the descendants of the descendants:

$$\phi_j^{(-n,-m)}(z) = \frac{1}{2\pi i}\oint dw(w-z)^{-n+1}T(w)\phi_j^{(-m)}(w)\,. \tag{1.27}$$

The set of all descendant fields, each one denoted by an arbitrary number of $\widehat{L}_n$ acting on ϕ_j, is called a conformal family

$$[\phi_j] = \{\phi_j, \widehat{L}_{-n}\phi_j, \widehat{L}_{-m}\widehat{L}_{-n}\phi_j, ..., \widehat{L}_{-n_1}...\widehat{L}_{-n_k}\phi_j\}\,, \quad k \geq 1\,. \tag{1.28}$$

The set of all fields $\{A_k\}$ in a given two - dimensional conformal field theory decomposes in a finite or infinite sum of conformal families

$$\{A_k\} = \oplus_j[\phi_j]\,. \tag{1.29}$$

The correlators with descendant field insertions can be obtained from those of the primary fields. It is easy to show that

$$< \phi_{j_1}(w_1,\bar{w}_1)...\phi_{j_n}(w_n,\bar{w}_n)\phi_j^{(-k_1,...,-k_m)}(z,\bar{z}) >=$$

$$= \Lambda_{-k_1}...\Lambda_{-k_m} < \phi_{j_1}(w_1,\bar{w}_1)...\phi_{j_n}(w_n,\bar{w}_n) >\,, \tag{1.30}$$

where for $k \geq 2$

$$\Lambda_{-k} = -\sum_{l=1}^{n}\left(h_{j_l}(1-k)(w_l-z)^{-k} + (w_l-z)^{-k+1}\partial_{w_j}\right)\,. \tag{1.31}$$

The Λ's provide a realization of the $\mathcal{L}$'s algebra.

Now we introduce the central charge c. This is twice the coefficient of the $(z-w)^{-4}$ term (allowed by scale invariance) in the operator product expansion of $T(z)$ with itself

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w)\,. \tag{1.32}$$

The central charge (or Schwinger term) is a dynamical parameter of the theory. From the OPE it follows that $< T(z)T(w) >= \frac{c/2}{(z-w)^4}$, then the requirement of positive norm of the space of the states gives $c \geq 0$. It turns out that for $c = 0$ the Virasoro algebra (defined below) has only a unitary representation. This representation is the trivial one since the highest weight state is just the vacuum while the states $L_{-n}|0>$ are set to zero because of zero norm [19].

A characteristic of $T(z)$ is that it is not a primary field. Indeed from the OPE we can see that its variation under conformal transformations is

$$\delta_\epsilon T(z) = \epsilon(z)\partial_z T(z) + 2\partial_z\epsilon(z)T(z) + \frac{c}{12}\partial_z^3\epsilon(z)\,. \tag{1.33}$$

Under the finite transformation ($z \to f(z)$) we have

$$T(z) \to (\partial f)^2 T(f(z)) + \frac{c}{12}S(f,z)\,, \tag{1.34}$$

where $S(f,z)$ is the Schwartzian derivative:

$$S(f,z) = \frac{\partial_z f \partial_z^3 f - (3/2)(\partial_z^2 f)^2}{(\partial_z f)^2}\,. \tag{1.35}$$

Since $S(f,z)$ vanishes under $SL(2,\mathbb{R})$ transformations, the energy momentum tensor is a $SL(2,\mathbb{R})$ primary field. Analogous formulae hold for the right - chiral sector.

The central charge corresponds to an anomaly of the theory. Indeed from translation invariance it follows that all one - point functions (and therefore $< T(z) >$) vanish. Nevertheless for transformations not in $SL(2,\mathbb{R})$ the central charge breaks the symmetry under diffeomorphisms, in particular $< T(z) > \neq 0$. This effect has an interesting interpretation in terms of the Casimir effect[20]. Consider the conformal map from S^2 to the cylinder. From (1.14) and (1.34) it follows that

$$T_{cyl}(w) = z^2 T(z) - \frac{c}{24} = \sum_{n\in\mathbb{Z}} \left(L_{-n} - \frac{c}{24}\delta_{n,0}\right) e^{-nw}\,, \tag{1.36}$$

$$(L_0)_{cyl} = L_0 - \frac{c}{24}\,. \tag{1.37}$$

Computing the partition function

$$Z = \mathrm{Tr} e^{-\beta(L_0+\bar{L}_0)_{cyl}} = \mathrm{Tr} e^{-\beta(L_0+\bar{L}_0-\frac{c}{12})}\,, \tag{1.38}$$

it follows that the central charge can be seen as a Casimir effect connected with the finite geometry of the cylinder (when the radius of the cylinder goes to infinity $< T(z)_{cyl} > \to 0$).

1.5 Virasoro Algebra

Now we derive the algebra generated from $T(z)$. First of all observe that identifying $\phi(z)$ with the identity operator in (1.26), the $\widehat{L}_n$'s coincide with the modes in the expansion of the energy momentum tensor: $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n$ (the choice of the z's exponent corresponds to assigning to L_n scaling dimension $-n$). The L_n's are the generators of conformal transformations acting on the space of correlation functions; indeed multiplying (1.24) by z^{n+1} and integrating around a contour containing all the points $\{z_i\}$ we have

$$< L_n\phi_{j_1}(z_1)...\phi_{j_m}(z_m) >= \delta_{\epsilon_n} < \phi_{j_1}(z_1)...\phi_{j_m}(z_m) > , \qquad \epsilon_n(z) = z^{n+1} . \tag{1.39}$$

This formula is equivalent to the commutation relation

$$[L_n, \phi_j(z)] = \frac{1}{2\pi i}\oint dw w^{n+1}T(w)\phi_j(z) = h_j(n+1)z^n\phi_j(z) + z^{n+1}\partial_z\phi_j(z) . \tag{1.40}$$

It is easy to show that the algebra generated by L_n's, called Virasoro algebra[21], is

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} , \qquad [L_m, c] = 0 . \tag{1.41}$$

This algebra is isomorphic to the universal central extension of the complexified algebra $\mathcal{D}iff(S^1)$ of infinitesimal diffeormophisms of the circle. This central extension has been discovered by Gelfand and Fuks [22]. The direct sum $\mathcal{V}ir \oplus \overline{\mathcal{V}ir}$ can be obtained by viewing $\mathcal{V}ir$ as a real Lie algebra and by tensoring it with $\mathbb{C}$ (over $\mathbb{R}$). The cocycle that multiplies the central charge is the only possible (up to trivial cocycles) compatible with the Jacobi identity. The $\bar{L}_n$'s algebra is identical to the L_n's algebra with c replaced with $\bar{c}$; moreover, since the operator product expansion of T with $\overline{T}$ has no singularity, we have $[L_n, \bar{L}_m] = 0$. Notice that $sl(2,\mathbb{C})$ is a subalgebra of the Virasoro algebra (for a detailed analysis of infinite dimensional algebras see refs.[23,24]).

To reproduce the $SL(2,\mathbb{C})$ symmetry on the space of the physical states we assume that there exists an $SL(2,\mathbb{C})$ - invariant vacuum $|0> \equiv |0>_L \otimes |0>_R$ (the left and right "chiral" vacua are $SL(2,\mathbb{R})$ invariant)

$$L_{0,\pm1}|0> = \bar{L}_{0,\pm1}|0> = 0 . \tag{1.42}$$

This requirement is equivalent to the condition that $T(z)|0>$ be regular at $z=0$.

As shown above, associated with $\phi_j(z)$ there is the *in* state $|\phi_j>$ (this is a particular feature of two - dimensional conformal field theories, see for example ref.[3]). In order to analyze the modes content of this state we expand the primary field in Laurent - series (as we will show a primary field of weight $(h,0)$ depends only on the z - variable)

$$\phi_j(z) = \sum_{n\in\mathbb{Z}-h_j} \phi_{j,n} z^{-n-h_j}\,, \qquad \phi_{j,n} = \frac{1}{2\pi i}\oint dz \phi_j(z) z^{n+h_j-1}\,. \tag{1.43}$$

Then the primary state associated to ϕ_j is

$$|\phi_j>=\phi_{j,-h_j}|0>\ . \tag{1.44}$$

From eq.(1.26) it follows that

$$L_0|\phi_j>=h_j|\phi_j>\,, \qquad L_n|\phi_j>=0\,, \quad n>0\,. \tag{1.45}$$

A distinguished feature of a conformal families is that under conformal transformations every field in a given family is mapped into an element of the same family. Then it follows that each $[\phi_j]$ is a representation of the conformal algebra. In particular eq.(1.45) defines an infinite dimensional highest - weight representation of $\mathcal{V}ir$ called Verma module. The set of descendant states of level N of $|\phi_j>$ is

$$|N,\phi_j>\equiv\{L_{-n_1}...L_{-n_k}|\phi_j>\}\,, \qquad \sum_i n_i = N\,, \quad 0<n_1\le n_2...\le n_k\,. \tag{1.46}$$

They have conformal dimension h_j+N. The number of states of level N coincides with the number of partitions of N into positive integer parts; it is denoted by $P(N)$ and can be expressed in terms of the generating function

$$\prod_{n\ge 1}(1-q^n)^{-1} = \sum_{N\ge 0} P(N) q^N\,, \qquad P(0)\equiv 1\,. \tag{1.47}$$

The representation generated by $|\phi_j>$ is characterized by h_j and c. Since the Cartan subalgebra of $\mathcal{V}ir$ contains only L_0 and the identity operator, the space of descendant states is L_0 graded, that is they are labelled by the eigenvalues of L_0. Among the representations generated from primary fields that which is generated by the identity operator plays a particular role: its conformal dimension is zero and the associated highest - weight is just the vacuum $|0>$. From eqs.(1.42) and (1.45) it follows that

$$L_n|0>=0\,,\qquad n\geq 0\,. \tag{1.48}$$

Notice that the requirement of $SL(2,\mathbb{C})$ invariance of the vacuum sets to zero the zero norm (see eq.(1.66)) highest - weight state $L_{-1}|0>$.

From eq.(1.22) it follows that the physical requirement for the energy momentum operator in Minkowski space to be hermitian is equivalent, in euclidean radial quantization, to the condition

$$L_n^\dagger = L_{-n}\,. \tag{1.49}$$

From this it follows that $<0|L_n = (L_{-n}|0>)^\dagger$. From the above discussion it follows that the defining equations for the *in* and *out* vacua are

$$L_n|0>=0\,,\quad n\geq -1 \qquad <0|L_n=0\,,\quad n\leq 1\,, \tag{1.50}$$

$$\bar{L}_n|0>=0\,,\quad n\geq -1 \qquad <0|\bar{L}_n=0\,,\quad n\leq 1\,, \tag{1.51}$$

This implies that L_0, L_1+L_{-1} and $i(L_1-L_{-1})$ together with $\bar{L}_0$, $\bar{L}_1+\bar{L}_{-1}$ and $i(\bar{L}_1-\bar{L}_{-1})$ generate the (globally well - defined) unitary representation of $SL(2,\mathbb{C})$. The $SL(2,\mathbb{C})$ invariance of the space of states allows us to fix the coordinate dependence of two - and three - point functions (the constant appearing in the three - point function is determined by the underlying physical system). Indeed let U represent a $SL(2,\mathbb{C})$ transformation in the Hilbert space, then under $z\to w=\dfrac{az+b}{cz+d}$ we have

$$\phi(z,\bar{z}) = (cz+d)^{-2h_j}(\bar{c}\bar{z}+\bar{d})^{-2\bar{h}_j}\,\tilde{\phi}(z,\bar{z})\,, \tag{1.52}$$

$$\tilde{\phi}(w,\bar{w}) = U\phi(z,\bar{z})U^{-1}\,. \tag{1.53}$$

Hence since $U|0>=|0>$ and $<0|U^\dagger=<0|$ we get (in the following the radial ordering is understood)

$$<0|\phi_{j_1}(z_1,\bar{z}_1)\ldots\phi_{j_n}(z_n,\bar{z}_n)|0>=$$

$$= \left(\prod_{k=1}^{n} (cz_i + d)^{-2h_{j_k}} (\bar{c}\bar{z}_i + \bar{d})^{-2\bar{h}_{j_k}} \right) < 0|\phi_{j_1}(w_1, \bar{w}_1)...\phi_{j_n}(w_n, \bar{w}_n)|0> . \qquad (1.54)$$

From this equation we can find the coordinate dependence of two - and three - point functions. The normalization constant in the two - point function is arbitrary. Choosing a normalization for a basis of the primary fields such that

$$< 0|\phi_{j_1}(z_1, \bar{z}_1)\phi_{j_2}(z_2, \bar{z}_2)|0 >= (z_1 - z_2)^{-2h_{j_1}} (\bar{z}_1 - \bar{z}_2)^{-2\bar{h}_{j_1}} \delta_{h_{j_1} h_{j_2}} \delta_{\bar{h}_{j_1} \bar{h}_{j_2}} , \qquad (1.55)$$

it follows that the coefficient $C_{j_1 j_2 j_k}$ in the OPE

$$\phi_{j_1}(z_1, \bar{z}_1)\phi_{j_2}(z_2, \bar{z}_2) \sim \sum_k C_{j_1 j_2 j_k} (z_1 - z_2)^{h_{j_k} - h_{j_1} - h_{j_2}} (\bar{z}_1 - \bar{z}_2)^{\bar{h}_{j_k} - \bar{h}_{j_1} - \bar{h}_{j_2}} \phi_{j_k}(z_2, \bar{z}_2) , \qquad (1.56)$$

is completely symmetric in its indices. By a suitable limit procedure and using (1.55) it is easy to see that $C_{j_1 j_2 j_3}$ coincides with the coefficient appearing in the three - point function ($z_{ij} \equiv z_i - z_j$, $\bar{z}_{ij} \equiv \bar{z}_i - \bar{z}_j$)

$$< 0|\phi_{j_1}(z_1, \bar{z}_1)\phi_{j_2}(z_2, \bar{z}_2)\phi_{j_3}(z_3, \bar{z}_3)|0 >=$$
$$= C_{j_1 j_2 j_3} z_{12}^{h_{j_3} - h_{j_2} - h_{j_1}} z_{23}^{h_{j_1} - h_{j_2} - h_{j_3}} z_{13}^{h_{j_2} - h_{j_3} - h_{j_1}} \bar{z}_{12}^{\bar{h}_{j_3} - \bar{h}_{j_2} - \bar{h}_{j_1}} \bar{z}_{23}^{\bar{h}_{j_1} - \bar{h}_{j_2} - \bar{h}_{j_3}} \bar{z}_{13}^{\bar{h}_{j_2} - \bar{h}_{j_3} - \bar{h}_{j_1}} . \qquad (1.57)$$

Observe that, after identification of $\bar{z}$ with the complex conjugate of z, the requirement of singlevaluedness restricts the spin to being integral.

Now we discuss the role of the coefficient C_{ijk}. First of all we rewrite the OPE of ϕ_{j_1} with ϕ_{j_2} by grouping together all secondary fields in their conformal family

$$\phi_{j_1}(z, \bar{z})\phi_{j_2}(0,0) \sim \sum_{k\{l\bar{l}\}} C_{j_1 j_2 j_k}^{\{l\bar{l}\}} z^{h_{j_k} - h_{j_1} - h_{j_2}} \bar{z}^{\bar{h}_{j_k} - \bar{h}_{j_1} - \bar{h}_{j_2}} \phi_{j_k}^{\{l\bar{l}\}}(0,0) , \qquad (1.58)$$

where $\phi_{j_k}^{\{l\bar{l}\}}$ is the secondary field belonging to the conformal family $[\phi_{j_k}]$:

$$\phi_{j_k}^{\{l\bar{l}\}} \equiv \widehat{L}_{-l_1} ... \widehat{L}_{-l_n} \widehat{\bar{L}}_{-l_1} ... \widehat{\bar{L}}_{-l_m} \phi_{j_k} . \qquad (1.59)$$

Performing a conformal transformation on both sides of (1.58), and comparing terms, leads to the following relations

$$C_{ijk}^{\{l\bar{l}\}} = C_{ijk} \beta_{ij}^{k\{l\}} \bar{\beta}_{ij}^{k\{\bar{l}\}} , \qquad (1.60)$$

where C_{ijk} is the operator product coefficient between primary fields. The function $\beta_{ij}^{k\{l\}}$ (and analogously $\bar{\beta}_{ij}^{k\{\bar{l}\}}$) depends on the central charge and on the weights h_i, h_j, h_k and, in principle, can be computed by conformal symmetry. The factorized form of the eq.(1.60) is a consequence of the fact that the representation $[\phi_k]$ decomposes in the direct product of the representations of Γ and $\overline{\Gamma}$: $[\phi_k] = \mathcal{V}_k \otimes \overline{\mathcal{V}}_k$.

From the previous discussion it follows that the theory is completely characterized by c, the conformal weights of the highest - weight states and by the coefficients C_{ijk}. These coefficients can be determined by the set of equations coming from the the requirement of associativity of the operator algebra. This is the main dynamical principle stated in the bootstrap program. Now we briefly show how this requirement can be implemented. Let us consider the primary field four point function:

$$A(1,2,3,4) \equiv < \phi_{j_1}(z_1,\bar{z}_1)\phi_{j_2}(z_2,\bar{z}_2)\phi_{j_3}(z_3,\bar{z}_3)\phi_{j_4}(z_4,\bar{z}_4) > \ . \tag{1.61}$$

Then by associativity hypothesis it follows that

$$\lim_{z_1 \to z_2, z_3 \to z_4} A(1,2,3,4) = \lim_{z_1 \to z_3, z_2 \to z_4} A(1,2,3,4)\,. \tag{1.62}$$

This equation can be rewritten in terms of "conformal blocks"[1],

$$\sum_p C_{j_1 j_2 p} C_{j_3 j_4 p} \mathcal{F}_{j_1 j_2}^{j_3 j_4}(p|x) \overline{\mathcal{F}}_{j_1 j_2}^{j_3 j_4}(p|\bar{x}) = \sum_q C_{j_1 j_3 q} C_{j_2 j_4 q} \mathcal{F}_{j_1 j_3}^{j_2 j_4}(q|1-x) \overline{\mathcal{F}}_{j_1 j_3}^{j_2 j_4}(q|1-\bar{x})\,, \tag{1.63}$$

where x and $\bar{x}$ are the anharmonic quotients[1]. By $SL(2,\mathbb{C})$ invariance we can choose $z_1 = \bar{z}_1 = \infty$, $z_2 = \bar{z}_2 = 1$, $z_3 = x$, $\bar{z}_3 = \bar{x}$, $z_4 = \bar{z}_4 = 0$. The expression for the conformal blocks is

$$\mathcal{F}_{j_1 j_2}^{j_3 j_4}(p|x) = x^{h_p - h_{j_3} - h_{j_4}} \sum_{\{k\}} \beta_{j_1 j_2}^{p\{k\}} x^{\sum k_i} \frac{< k|\phi_{j_3}(1,1) L_{-k_1} \ldots L_{-k_N}|p >}{< k|\phi_{j_3}(1,1)|p >}\,. \tag{1.64}$$

These functions are the building blocks of the theory because any correlator can be expressed in terms of them. For further details see [1].

1.6 Kac Formula

As a consequence of our definition of adjoint and from eq.(1.55) we have that $|\phi_j>$ has norm one. Moreover since L_0 and $\bar{L}_0$ are hermitian, their eigenvectors corresponding to different eigenvalues are orthonormal

$$< \phi_j|\phi_k >= \delta_{h_j h_k} \,. \tag{1.65}$$

One of the consequences of the definition of adjoint is that all nonvanishing correlators of ϕ_j must fall as $z^{-2h_j}\bar{z}^{-2\bar{h}_j}$ as $z,\ \bar{z} \to \infty$. In order to have a physically suitable theory the Hilbert space must have positive norm. This requirement imposes some restrictions on the values of c and on the conformal weights. For example from

$$< \phi_j|L^\dagger_{-n}L_{-n}|\phi_j >= 2nh_j + \frac{c}{12}(n^3 - n)\,, \tag{1.66}$$

it follows, besides the already known condition $c \geq 0$ (large n), the constraint $h_j \geq 0$ (set $n = 1$), where the equal holds iff $|\phi_j >\equiv |0>$. Notice that from (1.66) it follows that a primary field ϕ_j with $\bar{h} = 0$ depends only on the z - variable. Indeed from the commutation relation: $[\bar{L}_{-1}, \phi_j] = \partial_{\bar{z}}\phi_j$ (see eq.(1.40)), and observing that $\bar{L}_{-1}\phi_j|0>$ has zero norm, it follows that $\partial_{\bar{z}}\phi_j = 0$.

Now we briefly show how it is possible to find constraints for a highest weight representation to be unitary. As shown, the states with different dimensions are orthogonal:

$$< N, \phi_j|M, \phi_j >= 0\,, \quad if \quad N \neq M\,, \tag{1.67}$$

then the unitarity can be examined level by level in L_0. Let $M_N(c,h)$ be the $P(N) \times P(N)$ matrix of inner products of the kind (1.67) with $N = M$. Then a necessary condition for unitary is that $\det M_N(c,h)$ be non - negative. In [25] Kac found (proven in [26]) the determinant of $M_N(c,h)$ at each level in an arbitrary Verma module, it reads

$$\det M_N(c,h) = \alpha_N \prod_{pq \leq N} (h - h_{p,q}(c))^{P(N-pq)}\,, \tag{1.68}$$

where α_N is a positive constant independent of c and h. The expression for the weights is (in the following we denote with $\phi_{p,q}$ the primary state of weight $h_{p,q}$)

$$h_{p,q} = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)} = h_{m-p,m+1-q}\,, \tag{1.69}$$

where the dependence of m from the central charge is

$$m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}}\,. \tag{1.70}$$

Observe that in general $m \in \mathbb{C}$. From (1.69) it follows that if $c > 1$ then $h_{p,q} \notin \mathbb{R}^+$, therefore Kac's determinant has no zeroes. Moreover it turns out that $\det M_N(c,h)$ is positive for all values of h, so that positivity of Kac's determinant imposes no restrictions in the plane $c > 1\,, h > 0$. The requirement of positivity of Kac's formula imposes no restriction on c and h. In ref.[27] Friedan, Qiu and Shenker have shown that for $c \leq 1$, $\det M_N(c,h)$ is non - negative only if $m = 3, 4, \ldots$ and $1 \leq p \leq m-1$, $1 \leq q \leq p$. In ref.[28] it was shown that this condition turns out to be sufficient for the highest weight representation generated by $|\phi_{p,q}>$ to be unitary.

The expression of $\det M_N(c,h)$ indicates that at level pq there is a null vector, that is a descendant state $|\chi>$ which has zero norm and satisfying the conditions

$$L_0|\chi> = (h+pq)|\chi>\,, \qquad L_n|\chi> = 0\,, \qquad n > 0\,, \tag{1.71}$$

which are characteristic of primary states. A primary field $\phi_{p,q}$ generating a null vector is called degenerate field. Because of the null vector the representation is reducible; however since any correlation function with a null field vanishes, it can be set to zero. As shown above, correlation functions with secondary fields can be expressed in terms of primary field correlation functions. Then setting the null vector to zero allows us to obtain a differential equation of order pq for correlation functions with the insertion of $\phi_{p,q}$. From these equations we get the following OPE or fusion rules

$$[\phi_{p_1,q_1}][\phi_{p_2,q_2}] \sim \sum_{k_1=|p_1-q_1|+1}^{p_1+q_1-1} \; \sum_{k_2=|p_2-q_2|+1}^{p_2+q_2-1} [\phi_{k_1,k_2}]\,. \tag{1.72}$$

For theories composed entirely of degenerate representations one can in principle construct all the correlation functions. Since minimal models have this property their dynamics is completely determined by conformal symmetry. Nevertheless in the general case these differential equations are not sufficient to completely determine the correlation functions.

2. BASIS FOR MEROMORPHIC DIFFERENTIALS

In the study of two - dimensional conformal field theories on higher genus Riemann surfaces an important rôle is played by the bases for meromorphic differentials introduced by Krichever and Novikov(*) [30]. These can be seen as the generalization to higher genus of the monomials (or λ - differentials) $z^n(dz)^\lambda$ on the sphere.

In this chapter we present a detailed analysis of KN differentials and of its generalization to real weight differentials. In particular we write these bases in terms of theta functions and prime form.

In the following chapters we will use them to define a global operator formalism for conformal field theory on Riemann surfaces that closely resembles the operator formalism on the sphere [31,32,33].

Here we consider only holomorphic differentials on Riemann surfaces with two punctures; the case with more than two punctures will be discussed in chapter 7.

2.1 Real Weight Differentials: Definitions

Real weight differentials in general are well defined only on a covering (in general with infinite sheets) of a Riemann surface Σ [32] (for related arguments see [34]). However for brevity we will refer to them as λ - differentials on Σ. Let $P_+, P_- \in \Sigma$ be two distinguished points in general position (see below for its precise meaning) that for $g = 0$ can be identified with $0, \infty$. In the following $z_\pm$ will denote local coordinates vanishing at $P_\pm$. For $\lambda \in \mathbb{R}$ we define a λ - differential holomorphic outside $P_\pm$ with the following behaviour in a neighborhood of these points:

$$f_j^{(\lambda,l)}(z_\pm) = a_j^{(\lambda)\pm}(l) z_\pm^{\pm j - s(\lambda)}(1 + \mathcal{O}(z_\pm))(dz_\pm)^\lambda = h_j^{(\lambda,l)}(z_\pm)(dz_\pm)^\lambda \,, \tag{2.1}$$

$$s(\lambda) \equiv \frac{g}{2} - \lambda(g-1) \,,$$

where $j \in \mathbb{Z} + s(\lambda) + \lambda(l-1)$, $l \in \mathbb{Z}$ and $a_j^{(\lambda)\pm}(l)$ are constants. When $\lambda = m/n$, with m and n relatively prime numbers, there are only n distinct sectors, i.e. l =1,...,n, (notice that for irrational λ inequivalent sectors are labelled by both negative and positive l).

(*) Global expansion on a Riemann surface were first discussed in [29].

When we go once around P_+ along a trivial homological cycle separating P_+ and P_-, $h_j^{(\lambda,l)}$ picks up a phase (observe that a contour oriented positively with respect to the z_+ coordinate is oriented negatively with respect to the z_- coordinate):

$$h_j^{(\lambda,l)} \to e^{2\pi i\lambda(l\mp 1-g\pm g)} h_j^{(\lambda,l)} . \tag{2.2}$$

For instance when $\lambda \in \mathbb{Z}+1/2$ the Neveu - Schwarz ($l=1$) and the Ramond ($l=2$) sectors are recovered[31].

We remark that on the sphere there is the following relation ($U_+ \cup U_- = S^2$, $z_\pm \in U_\pm$, $z_+ = z_-^{-1}$ on $U_- \cap U_+$)

$$\tilde{h}(z_-) = h(z_+)(-z_-)^{-2\lambda} , \qquad \text{on } U_- \cap U_+ , \tag{2.3}$$

where the tilde emphasizes the different functional dependence. For $\lambda \in \mathbb{Z}/2$ the phases in the RHS of eq.(2.2) coincide (this is true for any g). For $\lambda \in \mathbb{R}$ the difference between these phases is just equal to the phase coming from the term $z_-^{-2\lambda}$. An analogue result is true for any genus, the only delicate aspect concerns the z_- functional dependence of z_+ (for details see ref.[35]). Notice that $f_j^{(\lambda,l)}(z)$ is a well - defined λ - differential on Σ only if $l=1$ (singlevaluedness of $h_j^{(\lambda,1)}$ in P_+), and

$$Q(\lambda) \equiv -2s(\lambda)+1 = (2\lambda-1)(g-1) , \tag{2.4}$$

is an integer (singlevaluedness in P_-). The same information can be gotten from the Riemann - Roch theorem. Indeed when $\lambda > 1$, $g > 1$, the λ - form $f_j^{(\lambda,1)}(z)$ is holomorphic for $s(\lambda) \le j \le -s(\lambda)$, therefore eq.(2.1) gives all the zero modes of the Cauchy - Riemann operator $\bar{\partial}$ coupled to λ - differentials and therefore $Q(\lambda)$ must be an integer.

Due to the Riemann - Roch theorem eq.(2.1) must be modified in a few specific cases which are listed and treated in detail later.

For $\lambda \in \frac{\mathbb{Z}}{2}$ the Riemann - Roch theorem guarantees the existence and uniqueness of $f_j^{(\lambda,l)}(z)$ up to the multiplicative constants $a_j^{(\lambda)+}(l)$ ($a_j^{(\lambda)-}(l)$ is completely determined once the $a_j^{(\lambda)+}(l)$ is chosen). For arbitrary λ we will show the existence of $f_j^{(\lambda,l)}(z)$ by explicit construction. The uniqueness follows from the fact that given two λ - differentials satisfying eq.(2.1), their quotient is a meromorphic function with g poles in general position; therefore, by the Weierstrass (Noether) gap theorem[36], this function is a constant.

On Σ it is possible to introduce the global euclidean time. It is defined to be the harmonic function[37]

$$\tau(Q) = Re \int_{Q_0}^{Q} f^{(1)}_{-\frac{g}{2}}(z) , \tag{2.5}$$

where $f^{(1)}_{-\frac{g}{2}}(z)$ is the third kind differential with simple poles in $P_\pm$ with residues ± 1, normalized in such a way that the periods along the homology cycles be purely imaginary; its explicit expression is given below. For a given value of τ we define the level line

$$C_\tau = \left\{ Q \in \Sigma \middle| Re \int_{Q_0}^{Q} f^{(1)}_{-\frac{g}{2}}(z) = \tau \right\} . \tag{2.6}$$

For $\tau \to \pm\infty$ the level lines become small circles around $P_\mp$.

The dual $f^j_{(1-\lambda,l)}(z)$ of $f^{(\lambda,l)}_j(z)$ is defined by means of

$$\frac{1}{2\pi i} \oint_{C_\tau} f^{(\lambda,l)}_i(z) f^j_{(1-\lambda,l)}(z) = \delta^j_i . \tag{2.7}$$

The analogue of the value of $|z|$ on S^2 is given by the function $r(Q) = e^{\tau(Q)}$. Notice that r takes the following "critical" values: $r(P_+) = 0$, $r(Q_0) = 1$, $r(P_-) = \infty$. By analogy with the case of euclidean radial quantization discussed in chapter 1, where the equal - time curves on the cylinder correspond to circles on S^2, we can introduce the equal - time curves on Σ

$$\Gamma_r = \left\{ Q \in \Sigma \middle| \exp Re \int_{Q_0}^{Q} f^{(1)}_{-\frac{g}{2}}(z) = r \right\} . \tag{2.8}$$

r is called the radius of the curve Γ_r. We define the open disc of radius r to be the set

$$D(r) = \{ Q \in \Sigma | r(Q) < r \} . \tag{2.9}$$

Associated to two discs of radii r_1 and r_2 there is the open annulus

$$A(r_1, r_2) = D_{r_2} \setminus \overline{D}_{r_1} , \qquad r_2 > r_1 > 0 , \tag{2.10}$$

where the bar denotes the closure symbol.

We conclude this section showing how it is possible to define pointwise convergence on the space of meromorphic λ - forms. We denote this space by $\Gamma(\Sigma, \mathcal{O}^{\lambda})$ where $\mathcal{O} \equiv \mathcal{O}^0$ is the sheaf of meromorphic functions on Σ and $\mathcal{O}^{\lambda} \equiv (\mathcal{O}^1)^{\otimes\lambda}$. $\mathcal{O}^1$ is the sheaf of meromorphic 1 - forms. First of all we choose a metric $\hat{g}$ on Σ. We take the pull - back of the canonical flat metric on the Jacobian $J(\Sigma)$ via Jacobi map $I : \mathrm{Div}(\Sigma) \to J(\Sigma)$

$$\hat{g} = \sum_{k=1}^{g} (\overline{\omega}_k \otimes \omega_k + \omega_k \otimes \overline{\omega}_k) . \tag{2.11}$$

Then we define the absolute value of a section $h \in \Gamma(\Sigma, \mathcal{O}^{\lambda})$ at $z \in \Sigma$ to be

$$|h(z)| = \left(\bar{h}(z)h(z)\hat{g}(z)^{-\lambda}\right)^{\frac{1}{2}} . \tag{2.12}$$

This absolute value defines the notion of pointwise convergence of series of meromorphic differentials on Σ and was used in [38] to compute generalized Cauchy kernels.

2.2 Real Weight Differentials: Explicit Costruction (General Case)

Let us start now with the construction of the λ - differential $f_j^{(\lambda,l)}(z)$ in terms of theta functions and prime forms. The expression

$$\frac{E(z,P_+)^{j+s(\lambda)-2\lambda}}{E(z,P_-)^{j+s(\lambda)}} , \tag{2.13}$$

is a (multivalued) λ - differential with degree in P_+ exceeding that of $f_j^{(\lambda,l)}(z)$ by $2s(\lambda) - 2\lambda = g(1-2\lambda)$. Since by the Riemann vanishing theorem[39] the divisor of $\theta(P - gP_+ + \Delta)$ is gP_+, we put

$$f_j^{(\lambda,l)}(z) = \frac{E(z,P_+)^{j+s(\lambda)-2\lambda}}{E(z,P_-)^{j+s(\lambda)}} \frac{\theta(z + (j - s(\lambda))P_+ - (j + s(\lambda))P_- + (1-2\lambda)\Delta)}{\theta(z - gP_+ + \Delta)^{1-2\lambda}} . \tag{2.14}$$

where Δ is the Riemann class. The θ - function in the denominator insures the correct singularity in P_+, while the θ - function in the numerator guarantees the singlevaluedness of $f_j^{(\lambda,l)}(z)$. Since the latter has g zeroes, it follows that the degree of $f_j^{(\lambda,l)}(z)$ is precisely $2\lambda(g-1)$.

Using the definition of the σ - differential (see Appendix A) and inserting the theta characteristics, eq.(2.14) is generalized to the λ - differential with $[^\delta_\epsilon]$ - structure, i.e.

$$f_j^{(\lambda,l)}(z) = \frac{\theta[^\delta_\epsilon](z + (j - s(\lambda))P_+ - (j + s(\lambda))P_- + (1 - 2\lambda)\Delta)\sigma(z)^{2\lambda-1}}{E(z,P_+)^{-j+s(\lambda)}E(z,P_-)^{j+s(\lambda)}}, \tag{2.15}$$

where $j \in \mathbb{Z} + s(\lambda) + \lambda(l-1)$, $l \in \mathbb{Z}$. For $\delta, \epsilon = 0$ eq.(2.15) differs from eq.(2.14) by the constant term $s(P_+, ..., P_+)$ defined in Appendix A. When $\lambda = m/n$ then

$$\delta_i, \epsilon_i \in \{0, \frac{1}{n}, ..., \frac{n-1}{n}\}, \tag{2.16}$$

so that there are n^{2g} "$[^\delta_\epsilon]$ - *structures*", n for each one of the a_i and b_i homology cycles. However, for the time being, we will leave the "$[^\delta_\epsilon]$ - *structures*" undetermined. The dual of $f_j^{(\lambda,l)}(z)$ is

$$f^j_{(1-\lambda,l)}(z) = \frac{N(\lambda,l,j)\theta[^{-\delta}_{-\epsilon}](z - (j - s(\lambda) + 1)P_+ + (j + s(\lambda) - 1)P_- + (2\lambda - 1)\Delta)}{E(z,P_+)^{j-s(\lambda)+1}E(z,P_-)^{-j-s(\lambda)+1}\sigma(z)^{2\lambda-1}}, \tag{2.17}$$

where $N(\lambda, l, j)$ is a suitable normalization constant fixed by the condition (2.7), that, for the sake of brevity, we do not write down explicitly.

Now we are able to give the definition of "general position". It means that there is a countable set of points of Σ which have to be avoided. More precisely P_+ and P_- must not coincide with Weierstrass points. Moreover the same condition should be satisfied by the g zeroes implicitly fixed by (2.1) (and then from the choice of the points $P_\pm$). Indeed in this particular case by the Riemann vanishing theorem the theta function is identically zero. It would be interesting to investigate the explicit form of differentials with particular choices of the points $P_\pm$.

The multivaluedness of $f^j_{(1-\lambda,l)}(z)$ is given by (2.2) with λ replaced by $1 - \lambda$, so that the integrand in eq.(2.7) is a well - defined 1 - differential.

To conclude this section we give the expression of the covariant delta function for λ - differentials in the sector l:

$$\Delta^{(l)}(z, w) = \sum_j f_j^{(\lambda,l)}(z) f^j_{(1-\lambda,l)}(w), \tag{2.18}$$

that is, if $g(z)$ is a smooth λ - differential with the multivaluedness given in eq.(2.2), we have

$$g(z) = \oint_{C_\tau} \Delta^{(l)}(z,w)g(w)\,. \tag{2.19}$$

2.3 Real Weight Differentials: Explicit Costruction (Particular Cases)

Now we discuss some particular cases in which bases defined above need to be modified. Due to the Riemann - Roch theorem, eq.(2.1) does not work in the following cases (this follows also from the application of the Riemann vanishing theorem to the explicit form of eq.(2.1) given in eq.(2.15))

$$\forall g \; for \; |j| \leq \frac{g}{2} \; and \; \lambda = 0,1\,;$$

$$\forall g \; for \; |j| = \frac{1}{2} \; and \; \lambda = \frac{1}{2}, \quad l = 1 \; with \; odd \; spin \; structure\,;$$

$$g = 1 \; for \; |j| = \frac{1}{2} \; and \; \lambda \in \mathbb{Z}\,;$$

$$g = 1 \; for \; |j| = \frac{1}{2} \; and \; \lambda \in \mathbb{Z} + \frac{1}{2}, \;\; l = 1\,.$$

Then we define

$$f_j^{(0)}(z_\pm) = a_j^{(0)\pm} z_\pm^{\pm j - \frac{g}{2} - \frac{1\mp 1}{2}}(1 + \mathcal{O}(z_\pm)), \qquad -\frac{g}{2} \leq j \leq \frac{g}{2} - 1\,, \tag{2.20}$$

and $f_{\frac{g}{2}}^{(0)}(z) = 1$. For $\lambda = 1$ we put

$$f_j^{(1)}(z_\pm) = a_j^{(1)\pm} z_\pm^{\pm j + \frac{g}{2} - \frac{1\pm 1}{2}}(1 + \mathcal{O}(z_\pm))dz_\pm, \qquad -\frac{g}{2} + 1 \leq j \leq \frac{g}{2}\,, \tag{2.21}$$

and choose $f_{-\frac{g}{2}}^{(1)}(z)$ to be the third kind abelian differential with simple poles at $P_\pm$ with residue ± 1, normalized in such a way that its periods be purely imaginary, i.e.

$$Re \oint_{a_i} f_{-\frac{g}{2}}^{(1)}(z) = Re \oint_{b_i} f_{-\frac{g}{2}}^{(1)}(z) = 0\,. \tag{2.22}$$

For $g=1$ we define

$$f_j^{(\lambda)}(z) = f_j^{(0)}(z)(f_{\frac{1}{2}}^{(1)}(z))^\lambda, \quad \lambda \in Z\,, \tag{2.23}$$

$$f_j^{(\lambda,1)}(z) = f_j^{(0)}(z)(f_{\frac{1}{2}}^{(1)}(z))^\lambda, \quad \lambda \in Z + \frac{1}{2}\,, \tag{2.24}$$

where the spin structure of $(f_{\frac{1}{2}}^{(1)}(z))^{\frac{1}{2}}$ is chosen to be odd. Finally, for $\lambda = 1/2$, $l = 1$ and odd spin structure:

$$f_{-\frac{1}{2}}^{(\frac{1}{2},1)}(z_\pm) = a_{-\frac{1}{2}}^{(\frac{1}{2})\pm}(1) z_\pm^{-1}(1 + \mathcal{O}(z_\pm))(dz_\pm)^{\frac{1}{2}}\,, \tag{2.25}$$

$$f_{\frac{1}{2}}^{(\frac{1}{2},1)}(z_\pm) = a_{\frac{1}{2}}^{(\frac{1}{2})\pm}(1)(1 + \mathcal{O}(z_\pm))(dz_\pm)^{\frac{1}{2}}\,. \tag{2.26}$$

The multivaluedness of $f^j_{(1-\lambda,1)}(z)$ is given by (2.2) with λ replaced by $1-\lambda$, so that the integrand in eq.(2.7) is a well defined 1 - differential.

The explicit form of the differentials in eqs.(2.20 - 2.21) is

$$f_j^{(0)}(z) = \frac{\theta(z + (j - \frac{g}{2})P_+ + R - (j + \frac{g}{2} + 1)P_+ + \Delta)}{E(z,R)^{-1} E(z,P_-)^{j+\frac{g}{2}+1} E(z,P_+)^{-j+\frac{g}{2}} \sigma(z)}\,, \qquad -\frac{g}{2} \le j \le \frac{g}{2} - 1\,, \tag{2.27}$$

$$f_j^{(1)}(z) = \frac{\theta(z + (j + \frac{g}{2} - 1)P_+ - R - (j - \frac{g}{2})P_- - \Delta)}{E(z,R) E(z,P_-)^{j-\frac{g}{2}} E(z,P_+)^{-j-\frac{g}{2}+1} \sigma(z)^{-1}}\,, \qquad -\frac{g}{2} + 1 \le j \le \frac{g}{2}\,, \tag{2.28}$$

where the point $R \in \Sigma$ is arbitrary; its presence in (2.27) reflects the fact that the meromorphic function defined in (2.27) is unique up to addition of a constant. In eq.(2.28) the pole in R is cancelled by the zero of the θ - function; indeed, being $\theta(-z) = \theta(z)$, from the Riemann vanishing theorem we have for $1 - g/2 \le j \le g/2$

$$\theta((j + \frac{g}{2} - 1)P_+ - (j - \frac{g}{2})P_- - \Delta) = 0\,. \tag{2.29}$$

Note that this result allows us to identify the differential $f_{\frac{1}{2}}^{(1)}(z)$ in eq.(2.23) with $\sigma(z)^2$.

For $j = -g/2$ the 1 - differential is

$$f^{(1)}_{-\frac{g}{2}}(z) = d\ln\frac{E(z,P_+)}{E(z,P_-)} - 2\pi i \sum_{j,k=1}^{g} \left(\mathrm{Im} \int_{P_-}^{P_+} \omega_j \right) (\mathrm{Im}\Omega)^{-1}_{jk}\omega_k(z) . \tag{2.30}$$

Finally the conditions (2.25) and (2.26) give

$$f^{(\frac{1}{2},1)}_{-\frac{1}{2}}(z) = \frac{E(z,S)}{E(z,P_+)E(z,P_-)}\theta[^{\delta}_{\epsilon}](z+S-P_+-P_-) , \tag{2.31}$$

$$f^{(\frac{1}{2},1)}_{\frac{1}{2}}(z) = \frac{\theta[^{\delta}_{\epsilon}](z-P_-)}{E(z,P_-)} , \tag{2.32}$$

where the θ - characteristics are odd. The presence in eq.(2.31) of the arbitrary point $S \in \Sigma$ is due to the fact that eq.(2.25) does not fix uniquely $f^{(\frac{1}{2},1)}_{-\frac{1}{2}}(z)$. Note that $f^{(\frac{1}{2},1)}_{\frac{1}{2}}(z)$, as defined in (2.32), is equal to the expression (2.15) with $j = 1/2$, $\lambda = 1/2$, $l = 1$. This means that in the framework of θ - functions theory the modification (2.26) to eq.(2.1) is automatically taken into account. Moreover since $z = P_-$ is a zero of $\theta[^{\delta}_{\epsilon}](z-P_-)$, $f^{(\frac{1}{2},1)}_{\frac{1}{2}}(z)$ does not have poles, it is the zero - mode associated to the odd spin structure $[^{\delta}_{\epsilon}]$. In particular using the definition of the prime form, we have

$$f^{(\frac{1}{2},1)}_{\frac{1}{2}}(z) = h(P_-)h(z) , \tag{2.33}$$

so that we can identify $f^{(\frac{1}{2},1)}_{\frac{1}{2}}(z)$ with the 1/2 - differential which appears in eq.(2.24).

3. VIRASORO AND HEISENBERG ALGEBRAS IN HIGHER GENUS

3.1 Krichever Novikov Algebra

In this section we describe the KN algebra[30]. This is the higher genus analogue of the Virasoro algebra. Before discussing it we give some definitions.

An algebra $G = \sum_{i\in Z} G_i$ is said to be k - graded if for all G_i, G_j we have

$$G_i G_j \subset \sum_{s=i+j-k}^{i+j+k} G_s \,. \tag{3.1}$$

0 - graded algebras are called $\mathbb{Z}$ - graded. An l - graded module M over a k - graded algebra has the property

$$G_i M_j = \sum_{s=-k-l}^{k+l} M_{i+j-s} \,. \tag{3.2}$$

Next, in order to find the central extension of the KN algebra, we recall some facts about the cohomology for a Lie algebra $\mathcal{A}$ (see for example [40]). A k - cochain is an antisymmetric map $c : \mathcal{A} \to \mathbb{C}$. A k - cocycle is a k - cochain closed under the coubondary operation

$$\delta c(a_1, ..., a_{k+1}) = \sum_{i\geq j} (-1)^{i+j} c([a_i, a_j], a_1, ..., \check{a}_i, ..., \check{a}_j, ..., a_k) = 0 \,, \tag{3.3}$$

where the symbol $\check{}$ denotes omission. Notice that $\delta^2 = 0$. By means of a central element t we extend the algebra $\mathcal{A}$ to the algebra $\widehat{\mathcal{A}}$ defined by the new commutator

$$[a_i, a_j]_{c.e.} = [a_i, a_j] + tc(a_i, a_j) \,, \qquad [a_i, t] = 0 \,. \tag{3.4}$$

Observe that the new commutator $[a_i, a_j]_{c.e.}$ is still a 2 - cocycle. In other words, it satisfies the Jacobi identity. A k - cocycle of the kind $\delta c(a_1, ..., a_k)$ is said to be trivial.

Now we introduce the g_0 - graded algebra $\mathcal{V}_\Sigma$ of meromorphic vector fields $e_j(z) \equiv f_j^{(-1)}(z)$. Their commutator is

$$[e_i, e_j] = \sum_{s=-g_0}^{g_0} C_{ij}^s e_{i+j-s} \,, \qquad g_0 \equiv \frac{3}{2} g \,, \tag{3.5}$$

where

$$C_{ij}^{s} = \frac{1}{2\pi i} \oint_{C_\tau} f_{(2)}^{i+j-s}(z)[e_i(z), e_j(z)] \,. \tag{3.6}$$

Eq.(3.5) follows from an analysis of the singularities in $P_\pm$. For $s = -g_0$ and $s = g_0$ ($g > 1$) we have

$$C_{ij}^{g_0} = j - i \,, \qquad C_{ij}^{-g_0} = (i-j)\frac{a_i^{(-1)-} a_j^{(-1)-}}{a_{i+j+g_0}^{(-1)-}} \,, \tag{3.7}$$

where

$$a_k^{(-1)-} = (-1)^{i+g_0-1} E(P_-, P_+)^{2k} \frac{\theta(P_- + (k - s(-1))P_+ - (k + s(-1))P_- + 3\Delta)}{\theta(P_+ + (k - s(-1))P_+ - (k + s(-1))P_- + 3\Delta)} \,. \tag{3.8}$$

The KN algebra $\widehat{\mathcal{V}}_\Sigma$ is the central extension of $\mathcal{V}_\Sigma$ with commutator

$$[e_i, e_j] = \sum_{s=-g_0}^{g_0} C_{ij}^{s} e_{i+j-s} + t\chi(e_i, e_j), \qquad [e_i, t] = 0 \,, \tag{3.9}$$

where the cocycle is defined by

$$\chi(e_i, e_j) = \frac{1}{24\pi i} \oint_{C_\tau} \left(\frac{1}{2}(e_i''' e_j - e_j''' e_i) - \mathcal{R}(e_i' e_j - e_j' e_i)\right) . \tag{3.10}$$

The projective connection $\mathcal{R}$ assures that the integrand is a well - defined 1 - form. If $\mathcal{R}$ has polar degree $mP_+ + nP_-$ with $m, n \leq 2$, then the cocycle satisfies the "locality" condition:

$$\chi(e_i, e_j) = 0 \qquad \text{for } |i + j| > 3g \,. \tag{3.11}$$

It turns out that this cocycle is unique up to trivial cocycles[30,41]. In particular when $g = 0$ this algebra reduces to the Virasoro algebra. The Neveu - Schwarz and Ramond superalgebras in higher genus have been constructed in ref.[42].

3.2 A Covariantization Method

An explicit expression for the projective connection in eq.(3.10) can be obtained by means of an arbitrary vector field $\tilde{e}(z)$[32,43]. The derivative of a λ - differential is covariantly well defined only for $\lambda = 0$, therefore ($' \equiv dz\partial_z$)

$$\left(\left(\left(\frac{e_i}{\tilde{e}}\right)'\tilde{e}\right)'\tilde{e}\right)'\frac{e_j}{\tilde{e}} - \left(\left(\left(\frac{e_j}{\tilde{e}}\right)'\tilde{e}\right)'\tilde{e}\right)'\frac{e_i}{\tilde{e}}, \tag{3.12}$$

is a well - defined one - form, equal to twice the integrand in eq.(3.10) with

$$\mathcal{R} = \frac{\tilde{e}''}{\tilde{e}} - \frac{1}{2}\left(\frac{\tilde{e}'}{\tilde{e}}\right)^2 . \tag{3.13}$$

This method of covariantization plays a central rôle in defining the KdV equation on Riemann surfaces[32,43]. If, in a neighborhood of a point $P_i \in \Sigma$, $\tilde{e}(z) = z^a g(z)$, with $z(P_i) = 0$, then the polar behaviour of the projective connection is

$$\mathcal{R} \sim \frac{a^2 - 2a}{2z^2} + a\frac{g'}{g}\frac{1}{z} . \tag{3.14}$$

Due to the poles in $P_i \neq P_\pm$, the cocycle $\chi(e_i, e_j)$ is τ - dependent; to get a τ - independent cocycle we define the "Baker - Akhiezer vector field"

$$\tilde{e}_H(z) = \frac{\sigma(P_+)\theta(P_+ - gP_- + \Delta)E(z, P_-)}{E(P_+, P_-)E(z, P_+)^{g-1}\sigma(z)\theta(z - gP_- + \Delta)} e^{-2\pi i \sum_1^\infty c_k \int_{P_0}^z \eta_k} , \tag{3.15}$$

where η_k is the normalized (i.e. $\oint_{a_i} \eta_k = 0,\ i = 1, ..., g$) differential of second kind with poles of order $k+1$ in P_+, given in a local coordinate by

$$\eta_k(z) = \frac{1}{k!}\partial_w^k \partial_z \ln E(z, w) dz_{|_{w=P_+}} . \tag{3.16}$$

In order for $\tilde{e}_H(z)$ to be single - valued, the constants c_k's must satisfy the equation

$$\sum_{k=1}^\infty c_k \oint_b \eta_k = 2\Delta - (g-1)(P_+ + P_-), \tag{3.17}$$

where $b \equiv (b_1, ..., b_g)$.

3.3 Heisenberg Algebra On Σ

Another interesting generalization to higher genus of an algebra defined on S^2 is the Heisenberg algebra[30,41]. By analogy with the previous case we look at the central extension of the commutative algebra $\mathcal{H}_\Sigma$ of meromorphic functions $A_j \equiv f_j^{(0)}$ on Σ. The central extension is described in terms of the 2 -cocycle

$$\gamma_{ij} = \frac{1}{2\pi i} \oint_{C_\tau} A_i dA_j \,. \tag{3.18}$$

The centrally extended algebra $\widehat{\mathcal{H}}_\Sigma$ is generated by A_i and by the central element t, together to the commutation relations

$$[A_i, A_j] = \gamma_{ij} t \,, \qquad [A_i, t] = 0 \,. \tag{3.19}$$

This cocycle is local with respect to the sum of its indices, i.e.

$$\gamma_{ij} = 0 \,, \qquad \text{for } |i+j| > g \,. \tag{3.20}$$

On S^2 the centrally extended algebra coincides with the Heisenberg algebra. Indeed, setting $p_k = A_k$ and $q_k = -A_k$, we have

$$[p_k, p_l] = [q_k, q_l] = 0 \,, \quad [p_k, q_l] = k\delta_{k,l} t \,, \quad [p_k, t] = [q_k, t] = 0 \,. \tag{3.21}$$

Global representations of Heisenberg algebra on Schottky doubles of an open Riemann surface was given by A. Jaffe, S. Klimek and A. Lesniewski in ref.[44].

4. KN BASES: SOME APPLICATIONS

In this chapter we will review some research done using the KN bases[30,41,42,45,46]. In particular we show that one can explicitly construct in the bosonic string case[45] a BRST charge over any Riemann surface and show that it is nilpotent in the critical (D=26) dimension. Then we show that one can extend the construction of Krichever and Novikov so as to generalize the Neveu - Schwarz and Ramond algebras to Riemann surfaces of arbitrary genus, construct a corresponding BRST charge and recover the expected (D=10) critical dimension in the superstring case. Next we show that the Sugawara construction can be extended over an arbitrary Riemann surface[46].

4.1 Ramond - KN And Neveu - Schwarz - KN Superalgebras

Let us consider the KN differential:

$$g_{(\alpha,l)}(z_\pm) \equiv f_\alpha^{(-\frac{1}{2},l)}(z_\pm) = a_\alpha^\pm z_\pm^{\pm\alpha-g+\frac{1}{2}}(1+\mathcal{O}(z_\pm))(dz_\pm)^{-\frac{1}{2}}\,, \tag{4.1}$$

where $\alpha \in \mathbb{Z} + \frac{l}{2}$. As shown in chapter 2, $l = 1(2)$ correspond to the Neveu - Schwarz (Ramond) sector.

Let us come now to the binary operations which will allow us to define a superalgebra. Let us concentrate on the e_i's and the g_α's. As for the first we take the Lie bracket[30,41] $[e_i, e_j]$, for the latter we have the tensor product of sections and set $\{g_\alpha, g_\beta\} \equiv g_\alpha g_\beta + g_\beta g_\alpha$. Finally we set $[e_i, g_\alpha] \equiv \mathcal{L}_{e_i} g_\alpha$, where $\mathcal{L}_e g = \big(\epsilon(z)\partial\gamma(z) + \lambda\gamma(z)\partial\epsilon(z)\big)(dz)^\lambda$ in a local patch where $e = \epsilon(z)\frac{\partial}{\partial z}$ and $g = \gamma(z)(dz)^\lambda$). For integer λ, $\mathcal{L}_e$ reduces to the Lie derivative along the vector field e. Then from an analysis of the singularities in $P_\pm$ we obtain

$$[e_i, e_j] = \sum_{s=-g_0}^{g_0} C_{ij}^s e_{i+j-s}\,, \qquad g_0 \equiv \frac{3}{2}g\,, \tag{4.2}$$

$$[e_i, g_\alpha] = \sum_{s=-g_0}^{g_0} H_{i\alpha}^s g_{i+\alpha-s}\,, \tag{4.3}$$

$$\{g_\alpha, g_\beta\} = \sum_{p=-g}^{g} B_{\alpha\beta}^p e_{\alpha+\beta-\frac{p}{2}}\,. \tag{4.4}$$

The coefficients C^s_{ij}, $H^s_{i\alpha}$,$B^p_{\alpha\beta}$ can be calculated from the constants appearing in the expansion of e_i and g_α near $P_\pm$. For example, in the simplest case, we have (see also eq.(3.7)) $C^{g_0}_{ij} = j - i$, $H^{g_0}_{i\alpha} = \alpha - \frac{i}{2} - g + \frac{g_0}{2}$, $B^g_{\alpha\beta} = 2$.

Eq.(4.2) defines the (not centrally extended) KN algebra, while eqs.(4.2 - 4.4) together define the R - KN superalgebra or the NS - KN superalgebra for $\alpha, \beta, \gamma, \ldots$ integer or half integer, respectively. We will denote by $\mathcal{V}_\Sigma$ the algebra generated by the e_i's through eq.(4.2), and by $\mathcal{A}_\Sigma$ the superalgebra generated by the e_i's and the g_α's through eqs.(4.2 - 4.4). The algebra $\mathcal{V}_\Sigma$ splits according to

$$\mathcal{V}_\Sigma = \mathcal{V}^+_\Sigma \oplus \mathcal{V}^0_\Sigma \oplus \mathcal{V}^-_\Sigma \, , \tag{4.5}$$

where $\mathcal{V}^\pm_\Sigma$ are the subalgebras generated by the e_i's with $\pm i \geq g_0 - 1$, and generate diffeomorphisms. The complement $\mathcal{V}^0_\Sigma$ generated by the e_i with $|i| \leq g_0 - 2$ corresponds to deformations that change the conformal structure, its complex dimension is $3g - 3$ and it is naturally identified with the tangent space to the moduli space. Indeed the e_i with $|i| \leq g_0 - 2$ are vector fields with poles both in P_+ and P_-. The rôle of these meromorphic vector fields in changing the complex structure will be discussed in more details later on.

Similarly the superalgebra $\mathcal{A}_\Sigma$ splits according to

$$\mathcal{A}_\Sigma = \mathcal{A}^+_\Sigma \oplus \mathcal{A}^0_\Sigma \oplus \mathcal{A}^-_\Sigma \, , \tag{4.6}$$

where $\mathcal{A}^\pm_\Sigma$ are the superalgebras generated by e_i with $\pm i \geq g_0 - 1$ and g_α with $\pm\alpha \geq g - \frac{1}{2}$. These generate superconformal transformations. The complement $\mathcal{A}^0_\Sigma$ generated by e_i with $|i| \leq g_0 - 2$ and g_α with $|\alpha| < g - \frac{1}{2}$ correspond to deformations that change the superconformal structure. $\mathcal{A}^0_\Sigma$ is naturally identified with the tangent space to the supermoduli space. One can easily see that the complex dimension of $\mathcal{A}^0_\Sigma$ is $3g-3+2g-2$, the dimension of the supermoduli space.

4.2 The Central Extensions

In order to define the central extension of the NS - KN and R - KN superalgebras, let us introduce the following cocycle

$$\phi(g_\alpha, g_\beta) = \frac{1}{6\pi i} \oint_{C_\tau} \tilde{\phi}(g_\alpha, g_\beta)\,, \tag{4.7}$$

where $\tilde{\phi}$ is defined as follows.

Let ρ and σ have weight $-\frac{1}{2}$ and be holomorphic on Σ except possibly for poles or branch points in $P_\pm$ (with associated branch cut), and let $\rho = \rho(z_+)(dz_+)^{-\frac{1}{2}}$, $\sigma = \sigma(z_+)(dz_+)^{-\frac{1}{2}}$. Then

$$\tilde{\phi}(\rho, \sigma) = \rho' \sigma' dz_+ + \frac{\mathcal{R}}{2} \rho \sigma dz_+\,, \tag{4.8}$$

where $\mathcal{R}$ is a Schwarzian connection. It is immediate to see that it verifies the following properties:

(i) $\phi(g_\alpha, g_\beta) = \phi(g_\beta, g_\alpha)$;

(ii) it is independent of the coordinate system;

(iii) it satisfies the cocycle condition:

$$\phi(\rho, [\sigma, f]) - \phi(\sigma, [f, \rho]) + \chi(f, \{\rho, \sigma\}) = 0\,; \tag{4.9}$$

(iv) it is "local", in the sense that

$$\phi(g_\alpha, g_\beta) = 0 \quad for \quad |\alpha + \beta| > 2g\,, \tag{4.10}$$

as follows from an elementary computation of the zeroes and poles in $P_\pm$.

So finally we can centrally extend both NS - KN and R - KN superalgebras as follows (the cocycle $\chi(e_i, e_j)$ was defined in chapter 3)

$$[e_i, e_j] = \sum_{s=-g_0}^{g_0} C_{ij}^s e_{i+j-s} + t\chi(e_i, e_j)\,, \tag{4.11}$$

$$[e_i, g_\alpha] = \sum_{s=-g_0}^{g_0} H_{i\alpha}^s g_{i+\alpha-s}\,, \tag{4.12}$$

$$\{g_\alpha, g_\beta\} = \sum_{p=-g}^{g} B^p_{\alpha\beta} e_{\alpha+\beta-\frac{p}{2}} + t\phi(g_\alpha, g_\beta), \tag{4.13}$$

$$[e_i, t] = [g_\alpha, t] = 0\,. \tag{4.14}$$

A few final remarks:

- The cocycles χ and ϕ are easily calculated in a few cases. For example, for $\mathcal{R} = 0$,

$$\chi(e_i, e_{3g-i}) = \frac{1}{12}((i - g_0)^3 - (i - g_0)), \tag{4.15}$$

$$\phi(g_\alpha, g_{2g-\alpha}) = -\frac{1}{3}(\alpha - g)^2 + \frac{1}{12}\,. \tag{4.16}$$

- It has been shown by Krichever and Novikov[30] that up to trivial cocycles there is only one cocycle satisfying the "locality" condition (3.11).
- the above superalgebras reduce to the usual Virasoro, Neveu - Schwarz and Ramond superalgebras in the genus 0 case.

4.3 A String Realization

Now we want to realize the above algebras as intrinsic algebras of (super)string theories, first from a classical and then from a quantum point of view. In the following the superstring case will be treated explicitly. The purely bosonic case can be easily recovered by setting to zero the fermionic variables and the corresponding ghosts. The strategy consists as usual in defining the relevant energy - momentum tensor and supercurrent, and then analyzing it over the KN basis of quadratic and 3/2 - differentials, respectively. The algebra we are looking for will appear as algebra of the corresponding "momenta" ensuing from the classical Poisson brackets. Finally we will consider the expansion coefficients as operators acting on suitable Fock spaces. Such realizations gives rise, via normal ordering, to central extensions. In order to define a nilpotent BRST charge we introduce suitable ghosts. Their contributions exactly matches the "matter" contribution in the critical (D=10) dimension.

We start from the energy - momentum tensor that, in local coordinates, is given by $T = T^{X\psi} + T^{gh}$

$$T^{X\psi} \equiv -\partial X^\mu \partial X_\mu - \frac{1}{2}\partial\psi^\mu\psi_\mu\,, \qquad T^{gh} \equiv c\partial b + 2\partial c b - \frac{1}{2}\gamma\partial\beta - \frac{3}{2}\partial\gamma\beta\,, \tag{4.17}$$

and the supersymmetric current $J = J^{X\psi} + J^{gh}$

$$J^{X\psi} = \psi_\mu \partial X^\mu, \quad J^{gh} = 2c\partial\beta + 3\partial c\beta - \gamma b, \tag{4.18}$$

where $X^\mu(Q)$ and $\psi^\mu(Q)$ are fields of weight 0 and 1/2 respectively. $b(Q)$ and $c(Q)$ ($\beta(Q)$ and $\gamma(Q)$) are anticommuting (commuting) ghost fields of weight 2 and -1 (3/2 and -1/2) respectively. T and J have weight 2 and 3/2.

We can use the bases $\{e_i\}, \{\omega^i\}$, etc., $\{g_\alpha\}, \{h_\alpha\}$, etc., introduced above in order to expand these fields (the coefficients will be later interpreted as creation and annihilation operators in suitable Fock spaces)

$$\lambda = -1: \quad c(Q) = \sum_i c^i e_i(Q), \quad c^i = \frac{1}{2\pi i} \oint_{C_\tau} c(Q)\Omega^i(Q), \tag{4.19}$$

$$\lambda = 2: \quad b(Q) = \sum_i b_i \Omega^i(Q), \quad b_i = \frac{1}{2\pi i} \oint_{C_\tau} b(Q) e_i(Q), \tag{4.20}$$

$$\lambda = -\frac{1}{2}: \quad \gamma(Q) = \sum_\alpha \gamma^\alpha g_\alpha(Q), \quad \gamma^\alpha = \frac{1}{2\pi i} \oint_{C_\tau} \gamma(Q) k^\alpha(Q), \tag{4.21}$$

$$\lambda = \frac{1}{2}: \quad \psi^\mu(Q) = \sum_\alpha d^\mu_\alpha h_\alpha(Q), \quad d^\mu_\alpha = \frac{1}{2\pi i} \oint_{C_\tau} \psi^\mu(Q) h^+_\alpha(Q) \tag{4.22}$$

$$\lambda = \frac{3}{2}: \quad \beta(Q) = \sum_\alpha \beta_\alpha k^\alpha(Q), \quad \beta_\alpha = \frac{1}{2\pi i} \oint_{C_\tau} \beta(Q) g_\alpha(Q), \tag{4.23}$$

$$\lambda = 1: \quad dX^\mu(Q) + P^\mu(Q) = \sqrt{2} \sum_i \alpha^\mu_i \omega^i(Q), \tag{4.24}$$

$$\sqrt{2}\alpha^\mu_i = \frac{1}{2\pi i} \oint_{C_\tau} (dX^\mu(Q) + P^\mu(Q)) A_i(Q), \tag{4.25}$$

where P^μ is the conjugate momentum of X^μ and C_τ are the level curves of the univalent function $\tau(Q) = Re \int_{Q_0}^{Q} \omega_{\frac{g}{2}}$ over Σ (see chapter 2). These level curves can be interpreted as representing closed string configurations on the Riemann surface and τ as a proper time. Recall that as τ tends to $\pm\infty$, C_τ tends to a circle around $P_\pm$. Now we introduce the Poisson brackets

$$[X^\mu(Q), P^\nu(Q')] = 2\pi\eta^{\mu\nu}\Delta_\tau(Q, Q'), \qquad Q, Q' \in C_\tau, \tag{4.26}$$

$$\{\psi^\mu(Q), \psi^\nu(Q')\} = 2\pi\eta^{\mu\nu}\delta_\tau(Q,Q'), \tag{4.27}$$

$$\{c(Q), b(Q')\} = 2\pi D_\tau(Q,Q'), \tag{4.28}$$

$$[\gamma(Q), \beta(Q')] = 2\pi d_\tau(Q,Q'), \tag{4.29}$$

The symbols in the r.h.s. play the rôle of δ - functions over C_τ for smooth tensors of weight 0, 1/2, -1, $-1/2$, respectively. For example for a generic smooth function $f(Q)$ over C_τ we have (see chapter 2)

$$f(Q) = \oint_{C_\tau} \Delta_\tau(Q,Q')f(Q'), \qquad Q, Q' \in C_\tau. \tag{4.30}$$

As a consequence of eqs.(4.26 - 4.29) we have the following Poisson brackets for the coefficients of the expansions (4.19 - 4.25):

$$[\alpha_i^\mu, \alpha_j^\nu] = -i\gamma_{ij}\eta^{\mu\nu}, \tag{4.31}$$

$$\{d_\alpha^\mu, d_\beta^\nu\} = -i\eta^{\mu\nu}\delta_{\alpha\beta}, \tag{4.32}$$

$$\{b_i, c^j\} = -i\delta_i^j, \tag{4.33}$$

$$[\gamma^\alpha, \beta_\beta] = -i\delta_\beta^\alpha, \tag{4.34}$$

where γ_{ij} is given in eq.(3.18).

Now let us consider $L_i = L_i^{X\psi} + L_i^{gh}$ and $G_\alpha = G_\alpha^{X\psi} + G_\alpha^{gh}$ defined by:

$$T(Q) = \sum_i L_i\Omega^i(Q), \quad J(Q) = \sum_\alpha G_\alpha k^\alpha(Q). \tag{4.35}$$

So

$$L_i^{X\psi} = -\frac{1}{2}\sum_{jk} l_i^{jk}\alpha_i \cdot \alpha_k + \frac{1}{4}\sum_{\alpha\beta} d_\alpha \cdot d_\beta F_i^{\alpha\beta}, \tag{4.36}$$

$$L_i^{gh} = \sum_j \sum_{s=-g_0}^{g_0} C_{ij}^s c^j b_{i+j-s} - \sum_\alpha \sum_{s=-g_0}^{g_0} H_{i\alpha}^s \gamma^\alpha \beta_{i+\alpha-s}, \tag{4.37}$$

and

$$G_\alpha^{X\psi} = \sum_{\beta j} d_\beta \cdot \alpha_j D_\beta^{\alpha j} \,, \tag{4.38}$$

$$G_\alpha^{gh} = -2 \sum_j \sum_{s=-g_0}^{g_0} c^j \beta_{j+\alpha-s} H_{j\alpha}^s - \frac{1}{2} \sum_\beta \sum_{p=-g}^{g} B_{\alpha\beta}^p \gamma^\beta b_{\alpha+\beta-\frac{p}{2}} \,, \tag{4.39}$$

where

$$l_i^{jk} = \frac{1}{2\pi i} \oint_{C_\tau} \omega^i \omega^k e_i, \quad F_i^{\alpha\beta} = \frac{1}{2\pi i} \oint_{C_\tau} (h_\beta \partial h_\alpha - h_\alpha \partial h_\beta) e_i \,, \tag{4.40}$$

$$D_\beta^{\alpha j} = \frac{1}{2\pi i} \oint_{C_\tau} h_\beta \omega^j g_\alpha \,. \tag{4.41}$$

Then the Poisson brackets for L_i and G_α are:

$$[L_i, L_j] = -i \sum_{s=-g_0}^{g_0} C_{ji}^s L_{i+j-s} \,, \tag{4.42}$$

$$[L_i, G_\alpha] = -i \sum_{s=-g_0}^{g_0} H_{i\alpha}^s G_{i+\alpha-s} \,, \tag{4.43}$$

$$\{G_\alpha, G_\beta\} = -i \sum_{p=-g}^{g} B_{\alpha\beta}^p L_{\alpha+\beta-\frac{p}{2}} \,. \tag{4.44}$$

These are a representation of eqs.(4.2 - 4.4), apart from the opposite sign in the first equation and the $-i$ factor. Of course eq.(4.42) alone defines a representation of the KN algebra.

4.4 Quantization

All the classical quantities considered so far are promoted to operators acting in a Fock space. The Poisson brackets are replaced by quantum commutators according to the recipe: $[\,,\,]_{P.B.} \to -i[\,,\,]_{quantum}$. In order to avoid ambiguities we have to define normal ordering. As for the α_i's, the normal ordering prescription is any one given in [41]. For the other relevant operators it is defined by considering as annihilation operators b_i for $i > 0$ and c_i for $i \leq 0$, d_α and γ_α for $\alpha \leq 0$ and β_α for $\alpha > 0$, and as creation operators the complementary ones (choosing another discriminating value for the normal ordering instead of zero, would amount to modifying the central charges by trivial cocycles). With this prescription we have calculated the algebra of $:L_i:$ and $:G_\alpha:$ and we have obtained:

$$[:L_i:,:L_j:] = \sum_{s=-g_0}^{g_0} C^s_{ji} :L_{i+j-s}: + \hat{\chi}_{ij}\,, \tag{4.45}$$

$$[:G_\alpha:,:L_i:] = \sum_{s=-g_0}^{g_0} H^s_{i\alpha} :G_{i+\alpha-s}:\,, \tag{4.46}$$

$$\{:G_\alpha:,:G_\beta:\} = \sum_{p=-g}^{g} B^p_{\alpha\beta} :L_{\alpha+\beta-p}: + \hat{\phi}_{\alpha\beta}\,. \tag{4.47}$$

This again is a replica of eqs.(4.11 - 4.14). Of course the crucial quantities are the central charges $\hat{\chi}_{ij}$ and $\hat{\phi}_{\alpha\beta}$. In order to give an idea of the problems involved without introducing too many technicalities, from now on we will limit ourselves to the purely bosonic string case. The relevant central charge can be written as:

$$\hat{\chi}_{ij} = D\chi^\Lambda_{ij} + \tilde{\chi}_{ij}\,, \tag{4.48}$$

where D is the target space dimension, χ^Λ_{ij} is given in ref.[41] and $\tilde{\chi}_{ij}$ is eq.(21) of [45]. That this central charge is a cocycle is a rather nontrivial fact. One can easily prove that it is antisymmetric and satisfies the locality condition (3.11). But the Jacobi identity is more complicated to deal with. By using an explicit construction of the KN algebra by means of semi - infinite forms we have been able to prove that both χ^Λ_{ij} and $\tilde{\chi}_{ij}$ are indeed cocycles and are proportional to $\chi(e_i, e_j)$. Therefore it is enough to calculate them for a particular value of the indices in order to know the proportionality constant. We have calculated $\hat{\chi}_{ij}$ for $i + j = 3g$ and found

$$\chi^\Lambda_{i,3g-i} = \frac{1}{12}(i-g_0)^3 + (i-g_0)A(\Lambda)\,, \tag{4.49}$$

$$\tilde{\chi}_{i,3g-i} = -\frac{13}{6}(i-g_0)^3 + (i-g_0)(\frac{1}{6} + g_0^2 - g_0)\,, \tag{4.50}$$

where $A(\Lambda)$ is a number depending on the normal ordering prescription chosen for the α operators. Eqs.(4.49 - 4.50) should be compared with eq.(4.15). The trivial parts which depends on the normal ordering or on the Schwarzian connection, can be taken care of by a suitable redefinition of the generators. The non trivial parts allow us to calculate the proportionality constant. Up to trivial cocycles we have

$$\hat{\chi}_{ij} = (D - 26)\chi(e_i, e_j)\,. \tag{4.51}$$

Following an analogous procedure, in the superstring case we find

$$\hat{\chi}_{ij} = (\frac{3}{2}D - 15)\chi(e_i, e_j)\,, \qquad \hat{\phi}_{\alpha\beta} = -(\frac{3}{2}D - 15)\phi(g_\alpha, g_\beta)\,. \tag{4.52}$$

4.5 The BRST Operator

It is now easy to define a BRST operator on Σ corresponding to the NS - KN and R - KN superalgebras. We define

$$Q = \frac{1}{2\pi i}\oint_{C_\tau}(T^{X,\psi}(Q)c(Q) + J^{X,\psi}(Q)\gamma(Q) + \frac{1}{2}B(Q)[C(Q), C(Q)]+ \tag{4.53}$$

$$-\beta(Q)[C(Q), \gamma(Q)] - \frac{1}{2}\{\gamma(Q), \gamma(Q)\}b(Q))\,.$$

The integrand in eq.(4.53) is a global expression and the commutators are geometrical commutators (in the sense of eq.(4.2 - 4.4)).

After quantization we have to consider $\hat{Q} =: Q :$. We obtain

$$\hat{Q}^2 = \{\hat{Q}, \hat{Q}\} = \sum_{i,j} \hat{\chi}_{i,j} : c^i c^j : + \sum_{\alpha,\beta} \hat{\phi}_{\alpha\beta} : \gamma^\alpha \gamma^\beta : \,. \tag{4.54}$$

From eq.(4.52) we have that up to trivial cocycles $\hat{Q}^2 = 0$ for $D = 10$. The BRST operator for the purely bosonic case is obtained from eq.(4.53) by setting to zero the fermionic fields and relevant ghosts. Due to eq.(4.53) nilpotence holds for $D = 26$.

4.6 The Sugawara Construction

A question which naturally arises is to what extent properties of the Virasoro algebra extend to KN algebras. One important topic concerns the Sugawara construction. The Virasoro algebra can be realized by constructing the generators as (suitably normal ordered) bilinears in the generators of a Kac - Moody algebra. Is it possible to extend this construction to a KN generalization of a (non Abelian) Kac - Moody algebra? This question was answered in the affirmative in [46]. Here we give a brief summary. As outlined in [30], a generalization of a Kac - Moody algebra for a genus g Riemann surface is obtained by considering the tensor product of a Lie algebra $\mathcal{G}$ and the algebra of meromorphic functions over Σ with respect to multiplication. Let us call T^a a basis of $\mathcal{G}$ and set $J_i^a \equiv A_i \otimes T^a$. We have immediately the KN - Kac - Moody algebra $\mathcal{K}_\Sigma$

$$[J_i^a, J_j^b] = f^{abc} \alpha_{ij}^s J_s^c + t\gamma_{ij}\delta^{ab}\,, \qquad [J_i^a, t] = 0\,, \tag{4.55}$$

where

$$\alpha_{ij}^s = \frac{1}{2\pi i} \oint A_i A_j \omega^s\,. \tag{4.56}$$

In eq.(4.55) we understand the summation over s which is limited to a finite range (of width $g+1$). From now on, for simplicity, two repeated lower and upper indices are understood to be summed from $-\infty$ to $+\infty$.

Let us construct the 1 - differential "fields"

$$J^a(Q) = J_i^a \omega^i(Q)\,, \qquad Q \in \Sigma\,, \tag{4.57}$$

where now J_i^a are to be considered as expansion coefficients satisfying (4.55). Then

$$[J^a(Q), J^b(Q')] = f^{abc}\Delta(Q,Q')J^c(Q) + td\Delta(Q,Q')\,. \tag{4.58}$$

Let us now introduce the Sugawara energy momentum tensor

$$T(Q) = -\frac{1}{c_v + k} : J^a(Q) J^a(Q) :\,, \tag{4.59}$$

and its "momenta"

$$L_i = \frac{1}{2\pi i} \oint T(Q) e_i(Q) = -\frac{1}{c_v + k} l_i^{pq} : J_p^a J_q^a :\,. \tag{4.60}$$

Here c_v is the second Casimir of the adjoint representation $c_v \delta_{ab} = f^{acd} f^{bcd}$, l_i^{pq} was defined above, whereas the normal ordering is defined by

$$: J^a_p J^b_q := \begin{cases} J^a_p J^b_q & p < N\,; \\ J^b_q J^a_q & p \geq N\,, \end{cases} \tag{4.61}$$

where N is a fixed integer. Computing $[L_i, J^b_k]$ one obtains:

$$[L_i, J^b_k] = \frac{c_v}{c_v + k} \Theta^l_{ik} J^b_l - \frac{k}{c_v + k} \mathcal{S}^l_{ik} J^b_l\,, \tag{4.62}$$

where

$$\mathcal{S}^l_{ik} = \frac{1}{2\pi i} \oint \omega^l e_i dA_k, \quad \Theta^l_{ik} = \Bigg(\sum_{\substack{p \geq N \\ q < N}} - \sum_{\substack{p < N \\ q \geq N}} \Bigg) l^{pr}_i \alpha^q_{pk} \alpha^l_{rq}\,. \tag{4.63}$$

In [46] it was proved that

$$\Bigg(\sum_{\substack{p \geq N \\ q < N}} - \sum_{\substack{p < N \\ q \geq N}} \Bigg) \omega^p(Q) \omega^q(Q') A_p(Q') A_q(Q) = d' \Delta(Q', Q)\,. \tag{4.64}$$

As it turns out this equation is independent of N. As a consequence of eq.(4.64) $\Theta^l_{ik} = -\mathcal{S}^l_{ik}$, and

$$[L_i, J^b_k] = -\mathcal{S}^l_{ik} J^b_l\,. \tag{4.65}$$

In the $g = 0$ case this equation reduces to the well known one

$$[L_i, J^k_b] = -k J^b_{i+k}\,. \tag{4.66}$$

Using eq.(4.65) it is now not hard to find

$$[L_i, L_j] = \frac{1}{c_v + k} \bigg(\big(l^{pq}_i \mathcal{S}^k_{jq} - l^{pq}_j \mathcal{S}^k_{iq} \big) : J^a_p J^a_k : + k \dim \mathcal{G} \chi_{ij} \bigg)\,, \tag{4.67}$$

where

$$\chi_{ij} = \frac{1}{2} \Bigg(\sum_{\substack{p \geq N \\ q < N}} - \sum_{\substack{p < N \\ q \geq N}} \Bigg) \mathcal{S}^q_{ip} \mathcal{S}^p_{jq}\,. \tag{4.68}$$

But one easily verifies that

$$l^{pq}_i \mathcal{S}^k_{jq} - l^{pq}_j \mathcal{S}^k_{iq} = -C^s_{ij} l^{pk}_{i+j-s}$$

So eq.(4.67) becomes

$$[L_i, L_j] = C^s_{ij} L_{i+j-s} + \frac{k \dim \mathcal{G}}{(c_v + k)} \chi_{ij}\,. \tag{4.69}$$

Finally it was demonstrated in [46] that χ_{ij} coincides up to trivial cocycles with $\chi(e_i, e_j)$. Eq.(4.69) completes the construction of a representation of a KN algebra over a Riemann surface of genus g by means of the Sugawara ansatz.

5. REPRESENTATIONS OF KN AND VIRASORO ALGEBRAS

It is well known that the representations of the Virasoro algebra form a natural classifying grid for conformal models in genus 0 (the Riemann sphere, the complex plane)[1]. The question is whether they are still a good classifying tool for conformal models in higher genus Riemann surfaces. The main point advocated in this chapter is that they must be substituted in this function by the representations of the appropriate KN algebra[47].

It is clear that the main point in this context is the understanding of the relation between the Virasoro algebra representations and the KN algebra representations. By this we mean the following: we can localize a copy of the Virasoro algebra in a small circle around any point of an arbitrary Riemann surface[48,49]; this algebra has a definite relation with the restriction to the circle of the relevant (globally defined) KN algebra; the problem is then to study the relation between the corresponding representations.

Here we set out to analyze this problem on very simple examples (anticommuting b - c systems, in which Clifford algebra representations play a central rôle). If on one side these examples show the complexity of the problem, on the other hand they allow us to draw a few conclusions. First of all, the representations of $\mathcal{V}ir$ and KN constructed starting from the same b - c system are, in general, not equivalent. A KN representation turns out to decompose, in general, into an infinite sum of $\mathcal{V}ir$ representations. Because of this and other reasons a suitable classifying grid for conformal theories over a higher genus Riemann surface appears to be provided by the representations of the appropriate KN algebra.

Conformal field theories in genus 0 are characterized by a high degeneracy as compared with the situation in higher genus. The above splitting of the KN representations in higher genus can be understood in this way. At the end of the paper we give another example of the genus 0 degeneracy.

5.1 Some Notations

Let $\Omega^j(z_\pm) \equiv f^{(2)}_{-j}$ where $f^{(2)}_{-j}$ is the quadratic differential defined in eq.(2.1). This differential is the dual of the vector field e_i:

$$\frac{1}{2\pi i}\oint e_i \Omega^j = \delta_i^j \,. \tag{5.1}$$

Now we write the Lie brackets of the basis elements e_i redefining the upper index of the structure constants:

$$[e_i, e_j] = C_{ij}^k e_k \,, \tag{5.2}$$

where here and henceforth repeated lower and upper indices are understood to be summed from $-\infty$ to $+\infty$. The structure constants

$$C_{ij}^k = \frac{1}{2\pi i}\oint \Omega^k [e_i, e_j] \,, \tag{5.3}$$

vanish for $\mid i + j - k \mid > g_0$. The centrally extended version of this algebra is the KN algebra described in chapter 3.

Let us now consider the circle $S^1 = \{z_+ : |z_+| = 1\}$ (for the sake of simplicity z_+ will be denoted z from now on) as well as the bases of vector fields $\{\tilde{e}_n\}$ and quadratic differentials $\{\tilde{\Omega}^m\}$

$$\begin{aligned} \tilde{e}_n &= z^{n+1}\frac{d}{dz} \,, \\ \tilde{\Omega}^m &= z^{-m-2}(dz)^2 \,, \end{aligned} \tag{5.4}$$

over S^1, which of course extend to an annulus A around S^1. Moreover

$$\frac{1}{2\pi i}\oint_{S^1} \tilde{e}_n \tilde{\Omega}^m = \delta_n^m \,. \tag{5.5}$$

The indices n, m, p, q are understood to be integers (of $\mathcal{V}ir$ - type) throughout this chapter. These vector fields satisfy the commutation relations

$$[\tilde{e}_n, \tilde{e}_m] = (m - n)\tilde{e}_{n+m} \,. \tag{5.6}$$

As discussed in chapter 1 the centrally extended version of this algebra is the Virasoro algebra

$$[\tilde{e}_n, \tilde{e}_m] = (m - n)\tilde{e}_{n+m} + c\tilde{\chi}(\tilde{e}_n, \tilde{e}_m), \qquad [\tilde{e}_n, c] = 0 \,, \tag{5.7}$$

where $\tilde{\chi}(\tilde{e}_n, \tilde{e}_m) = \frac{1}{12}(n^3 - n)\delta_{n+m,0}$ up to trivial cocycles.

The representations we are interested in are representations of the KN algebra. Before coming to representations let us study the relation between $\mathcal{V}ir$ and KN.

The bases $\{e_i\}$ and $\{\Omega^j\}$ restricted to S^1 are dense[30] in the linear span of $\{\tilde{e}_n\}$ and $\{\tilde{\Omega}^m\}$ respectively. So we can express the restrictions of the former (which will be denoted with the same symbols as the unrestricted quantities) in terms of the latter, and viceversa:

$$e_i = A_i^n \tilde{e}_n\,, \qquad \tilde{e}_n = (A^{-1})_n^i e_i\,, \tag{5.8}$$

$$\Omega^i = B_n^i \tilde{\Omega}^n\,, \qquad \tilde{\Omega}^n = (B^{-1})_i^n \Omega^i\,, \tag{5.9}$$

As a consequence of the duality relations (5.1) and (5.5), we have

$$A_i^n B_n^j = \delta_i^j\,, \qquad B_n^i A_i^m = \delta_n^m\,. \tag{5.10}$$

The matrix elements A_i^n and B_j^m can be easily calculated:

$$A_i^n = \frac{1}{2\pi i}\oint_S e_i \tilde{\Omega}^n, \qquad B_m^j = \frac{1}{2\pi i}\oint_S \tilde{e}_m \Omega^j\,. \tag{5.11}$$

One sees that

$$A_i^n = \begin{cases} 0 & for\ n < i - g_0\,; \\ 1 & for\ n = i - g_0\,, \end{cases} \qquad B_m^j = \begin{cases} 0 & for\ m > j - g_0\,; \\ 1 & for\ m = i - g_0\,, \end{cases} \tag{5.12}$$

while they are in general nonvanishing in the complementary ranges. The matrix elements A_i^n and B_j^m are the coefficients appearing in the Laurent expansion near P_+ of e_i and Ω^j, respectively. Actually, instead of eq.(2.1) we can write near P_+ ($z_+ \equiv z$):

$$\begin{aligned} e_i &= \sum_{n\geq 0} A_i^{i+n-g_0}\ z^{i+n-g_0+1}\frac{\partial}{\partial z}\,, \\ \Omega^j &= \sum_{n\geq 0} B_{j-n-g_0}^j\ z^{-j+n-g_0-2}(dz)^2\,. \end{aligned} \tag{5.13}$$

By comparing now (5.2) and (5.6) and using eqs.(5.8 - 5.9) we obtain the important relation

$$B_i^m B_j^n C_{ij}^k A_k^p = \tilde{C}_{nm}^p\,, \tag{5.14}$$

with $\tilde{C}^p_{nm} = (m-n)\delta^p_{n+m}$, between the KN and $\mathcal{V}ir$ structure constants, together with the inverse one. Similarly we can write $\chi(e_i, e_j)$ by means of eqs.(5.8 - 5.9). We find that

$$\chi(e_i, e_j) = A^n_i A^m_j \tilde{\phi}_{nm}, \qquad |i+j| \leq 3g\,, \tag{5.15}$$

and vanishes otherwise. The cocycle $\tilde{\phi}_{nm}$ has the general structure

$$\tilde{\phi}_{nm} = \tilde{\chi}(\tilde{e}_n, \tilde{e}_m) + \sum_{N=0}^{4g_0} \rho_N (m-n)\delta_{N+n+m,0}\,, \tag{5.16}$$

and ρ_N are numbers characterizing the Schwarzian connection $\mathcal{R}$.

Summarizing, by restricting to the circle S^1, we obtain two maps

$$\phi:\ KN \longrightarrow \overline{Vir}, \qquad \psi:\ Vir \longrightarrow \overline{KN}\,, \tag{5.17}$$

defined by

$$\phi(e_i) = A^n_i \tilde{e}_n, \qquad \psi(\tilde{e}_n) = B^i_n e_i\,, \tag{5.18}$$

$$\phi(t) = c, \qquad \psi(c) = t\,, \tag{5.19}$$

which are linear and preserve the Lie brackets through the identity (5.14) and (5.15), provided that we replace $\tilde{\chi}(\tilde{e}_n, \tilde{e}_m)$ in eq.(5.7) by $\tilde{\phi}_{nm}$. The bar in eq.(5.17) denote formal completion, that is the linear span of the basis elements in which infinite combinations are allowed. This is in order to take into account that the combinations appearing in eqs.(5.17) or (5.8 - 5.9) are (in general$^{(*)}$) infinite. For this reason we cannot speak about isomorphism concerning the maps ϕ and ψ.

5.2 Representations

Now we come to the representations of $\mathcal{V}ir$ and KN. We will discuss in detail a simple example based on a b - c system characterized by two anticommuting fields b and

$^{(*)}$ See the comment after eq.(5.55)

c of weight 2 and -1 respectively (the ghost system of the bosonic string), with energy momentum tensor

$$T(Q) = b(Q)\partial c(Q) + 2(\partial c(Q))b(Q)\,, \tag{5.20}$$

where $Q \in \Sigma$. We expand b, c and T in the appropriate bases on Σ

$$c(Q) = c^i e_i(Q)\,, \quad b(Q) = b_i \Omega^i(Q)\,, \quad T(Q) = L_i \Omega^i(Q)\,. \tag{5.21}$$

If we restrict these fields to S^1 we can expand them on the bases $\{\tilde{e}_n\}$ and $\{\tilde{\Omega}^n\}$

$$c = \tilde{c}^n \tilde{e}_n\,, \qquad b = \tilde{b}_n \tilde{\Omega}^n\,, \qquad T = \tilde{L}_n \tilde{\Omega}^n\,. \tag{5.22}$$

The canonical commutation relations are

$$\{b_i, c^j\} = \delta_i^j\,, \tag{5.23}$$

$$\{\tilde{b}_n, \tilde{c}^m\} = \delta_n^m\,, \tag{5.24}$$

respectively. We represent the two vacua on which these operators act by means of two semiinfinite forms[30,31,41]:

$$|0)_\Sigma = \Omega^{g_0-2} \wedge \Omega^{g_0-3} \wedge \Omega^{g_0-4} \wedge \ldots\,, \tag{5.25}$$

$$|0)_0 = \tilde{\Omega}^{-2} \wedge \tilde{\Omega}^{-3} \wedge \tilde{\Omega}^{-4} \wedge \ldots\,. \tag{5.26}$$

In general we will call $\mathcal{F}_\Sigma$ and $\mathcal{F}_0$ the linear space of semiinfinite forms of type (5.25) and (5.26) respectively. The action of the operators b_i, c^j and $\tilde{b}_n$, $\tilde{c}^m$ is specified by the rules

$$c^i = \Omega^i \wedge\,, \qquad\qquad b_i = i_{e_i}\,, \tag{5.27}$$

and

$$\tilde{c}^n = \tilde{\Omega}^n \wedge\,, \qquad\qquad \tilde{b}_n = i_{\tilde{e}_n}\,, \tag{5.28}$$

where

$$i_{e_j} \Omega^k = \frac{1}{2\pi i} \oint e_j \Omega^k = \delta_j^k\,, \qquad etc.. \tag{5.29}$$

In this way we obtain a representation of the Clifford algebras (5.23) and (5.24). Moreover we have

$$c^i|0)_\Sigma = 0 \qquad i < g_0 - 1\,, \qquad\qquad b_i|0)_\Sigma = 0 \qquad i \geq g_0 - 1\,, \tag{5.30}$$

and

$$\tilde{c}^n|0)_0 \qquad n < -1\,, \qquad\qquad \tilde{b}_n|0)_0 = 0 \qquad n \geq -1\,. \tag{5.31}$$

Consequently we define the normal ordering for monomials of b_i, c^j and $\tilde{b}_n$, $\tilde{c}^m$ by

$$:b_i c^j: = \begin{cases} b_i c^j & j < g_0 - 1\,; \\ -c^j b_i & j \geq g_0 - 1\,, \end{cases} \tag{5.32}$$

and

$$:\tilde{b}_n \tilde{c}^m: = \begin{cases} \tilde{b}_n \tilde{c}^m & m < -1\,; \\ -\tilde{c}^m \tilde{b}_n & m \geq -1\,. \end{cases} \tag{5.33}$$

The generators of KN and $\mathcal{V}ir$ extracted from eqs.(5.21) and (5.22) are

$$L_i = C_{ij}^k : b_k c^j :\,, \tag{5.34}$$

$$\tilde{L}_n = \tilde{C}_{nm}^p : \tilde{b}_p \tilde{c}^m : \tag{5.35}$$

Their commutation relations are

$$[L_i, L_j] = -C_{ij}^k L_k + \chi_{ij}\,, \tag{5.36}$$

and

$$\chi_{ij} = \Bigg(\sum_{\substack{k < g_0 - 1 \\ l \geq g_0 - 1}} - \sum_{\substack{k \geq g_0 - 1 \\ l < g_0 - 1}} \Bigg) C_{ik}^l C_{jl}^k\,. \tag{5.37}$$

Similarly

$$[\tilde{L}_n, \tilde{L}_m] = (n - m)\tilde{L}_{n+m} + \tilde{\chi}_{nm}\,, \tag{5.38}$$

where, as is well known,

$$\tilde{\chi}_{nm} = -\frac{13}{6}(n^3 - n)\delta_{n+m,0} = -26\tilde{\chi}(\tilde{e}_n, \tilde{e}_m)\,. \tag{5.39}$$

Using eqs.(5.8 - 5.9) and (5.14) it is not difficult to prove that

$$\chi_{ij} = A_i^n A_j^m \hat{\phi}_{nm} = -26\chi(e_i, e_j)\,, \tag{5.40}$$

where $\hat{\phi}_{nm}$ is of the type (5.16) with a suitable choice of the Schwarzian connection $\mathcal{R}$, as can be inferred from the fact that χ_{ij} is coordinate independent. So eqs.(5.36) and (5.38) are realizations (up to a minus sign) of commutation relations of the KN and Virasoro algebras respectively. Moreover we have

$$L_i|0)_\Sigma = 0 \qquad i > g_0, \qquad\qquad L_{g_0}|0)_\Sigma = 0\,, \tag{5.41}$$

and

$$\tilde{L}_n|0)_0 = 0 \qquad n > 0, \qquad\qquad \tilde{L}_0|0)_0 = 0\,. \tag{5.42}$$

The Verma modules $\mathcal{V}_\Sigma$ constructed over $|0)_\Sigma$ and $\mathcal{V}_0$ constructed over $|0)_0$ are therefore a representation space for a representation λ of KN and ρ of $\mathcal{V}ir$, respectively:

$$\begin{aligned} \lambda :\ KN &\longrightarrow End\,\mathcal{V}_\Sigma\,; \\ \rho :\ Vir &\longrightarrow End\,\mathcal{V}_0\,. \end{aligned} \tag{5.43}$$

Both are lowest weight representations characterized by the same eigenvalues $(13/6, 0)$ of (t, L_{g_0}) and $(c, \tilde{L}_0)$.

Now an important remark, which will be essential below. It is well known that the eigenvalue of $\tilde{L}_0$ can be changed by an arbitrary amount $\tilde{a}_0$ at the price of modifying the cocycle $\tilde{\chi}_{nm}$ by a trivial part. This can be accomplished by the redefinition

$$\tilde{L}_0 \rightarrow \tilde{L}_0' = \tilde{L}_0 + \tilde{a}_0 c\,. \tag{5.44}$$

In order to fix such a choice (which we call normalization), an independent input is needed. An analogous problem exists for the KN algebra representations. In this case we can change not only the eigenvalue of L_{g_0}, but modify any L_i for $-g_0 \le i \le g_0$ without changing the relevant representation. This can be done by redefining

$$L_i \rightarrow L_i' = L_i + a_i t\,, \tag{5.45}$$

provided we modify the cocycle χ_{ij} which defines the central extension by a term $\sim C_{ij}^k a_k$. We will see below an example of how the normalization can be fixed in a specific problem.

5.3 From Virasoro To KN Representations

Since the two representations $\mathcal{V}_\Sigma$ and $\mathcal{V}_0$ stem from the same b - c system, it is natural to ask whether they are equivalent. In such a case there must exist an invertible linear map $K : \mathcal{V}_0 \to \mathcal{V}_\Sigma$ such that for any $X \in \mathcal{V}ir$ we have(*)

$$K \cdot \rho(X) = \lambda(\psi(X)) \cdot K \,. \tag{5.46}$$

Looking at the semiinfinite forms (5.25) and (5.26) and at the relations (5.8), (5.9) one realizes that $|0)_\Sigma$ lies in the space of infinite linear combinations of semiinfinite forms belonging to $\mathcal{F}_0$. Therefore one may hope to find an explicit expression of it in terms of the operators $\tilde{b}_n$ and $\tilde{c}^m$. This is what happens. It is not hard to prove that

$$K = \,{}^\circ_\circ exp\left(\sum_{n<p} F^p_n \tilde{c}^n \tilde{b}_p\right){}^\circ_\circ\,, \tag{5.47}$$

where $F^p_n = B^{p+g_0}_n$ for $n < p$ and $= 0$ otherwise, satisfies

$$|0)_\Sigma = K|0)_0 \,. \tag{5.48}$$

In eq.5.47 the ${}^\circ_\circ \cdot {}^\circ_\circ$ ordering is defined by carrying in any monomial all the b_j operators to the right (with the appropriate sign). The operator K is invertible and

$$K^{-1} = \,{}^\circ_\circ exp\left(\sum_{p>n} F^p_n \tilde{b}_p \tilde{c}^n\right){}^\circ_\circ\,, \tag{5.49}$$

(*) This definition of equivalence between two representations is more general than the usual one for representations of isomorphic algebras. However one could envisage an even more general equivalence problem by considering a map

$$\psi(\tilde{e}_n) = \Gamma^p_n B^i_p e_i,$$

where Γ is an invertible matrix. One can easily verify that Γ must preserve the Virasoro structure constants.

We do not pursue further this subject here. In this context see also the considerations developed in ref.[50].

where now the c^k operators have to be carried to the right in all monomials. K and K^{-1} can also be written as

$$K = \,{}^{\circ}_{\circ} exp\left(\sum_{p>n} E^p_n \tilde{b}_p \tilde{c}^n\right) {}^{\circ}_{\circ} \,, \tag{5.50}$$

and

$$K^{-1} = \,{}^{\circ}_{\circ} exp\left(\sum_{p>n} E^p_n \tilde{c}^n \tilde{b}_p\right) {}^{\circ}_{\circ} \,, \tag{5.51}$$

where $E^p_n = A^p_{n+g_0}$ for $p > n$ and $= 0$ otherwise. All these equations can be proved after a lengthy but straightforward algebra. One can now prove that

$$\begin{aligned} K\tilde{c}^n K^{-1} &= c^{n+g_0}\,, \qquad \forall n\,, \\ K\tilde{b}_n K^{-1} &= b_{n+g_0}\,, \qquad \forall n\,. \end{aligned} \tag{5.52}$$

That is, K preserves the Clifford algebra anticommutation relations (see eqs.(5.23 - 5.24)). The map K is analogous, but not exactly equal, to the Bogoliubov transformations which one meets in the literature[51].

Now let us return to the equivalence problem, that is to eq.(5.46). Choosing $X = \tilde{e}_n$, so that $\rho(\tilde{e}_n) = -\tilde{L}_n$, $\lambda\left(\psi(\tilde{e}_n)\right) = -B^i_n L_i$, and using eq.(5.52) together with (5.14) (notice that K preserves the ordering in (5.34) and (5.35)), one finds that the following equations must be satisfied for any j, k and n

$$(j - g_0 - n)\delta^k_{j+n} = \sum_m (m - n) A^m_j B^k_{n+m}\,, \tag{5.53}$$

and

$$\tilde{a}_0 = B^i_0 a_i = B^{g_0}_0 a_{g_0} = a_{g_0}\,. \tag{5.54}$$

Notice that the summation over m in eq.(5.53) is in fact limited to the range $j - g_0 \leq m \leq k - g_0 - n$ due to eqs.(5.8 - 5.9)). It is easy to see that eqs.(5.53 - 5.54) are satisfied for $j \geq k - n$. For $j < k - n$ this is not guaranteed. For example for $j = k - n - 1$

$$0 = (k - 2n - g_0 - 1)B^k_{k-g_0-1} + (k - 2n - g_0)A^{k-n-g_0}_{k-n-1}\,, \qquad \forall n, k\,, \tag{5.55}$$

and so on. The matrix elements A^n_i, B^j_m depend, through the period matrix which characterizes the basis $\{e_i\}$ and $\{\Omega^j\}$[31], on the relevant point of the moduli space corresponding to the given Riemann surface Σ. In general eq.(5.55) is not satisfied for a generic point of the moduli space (for the case of the torus see [30,52]). We cannot exclude that eq.(5.55) be satisfied in particular points of the moduli space (see the remark after eq.(5.66) for an example of a sort of vanishing which could be related to this problem). But, at least in general, we can conclude that the two representations of KN and $\mathcal{V}ir$ are not equivalent.

This result is not surprising. Looking at the defining eq.(5.25) and at eqs.(5.8 - 5.9) we see that $|0)_\Sigma$ corresponds to an infinite linear combination of semiinfinite forms ($\in \mathcal{F}_0$) of the type

$$|m)_0 = \tilde{\Omega}^{-2-m} \wedge \tilde{\Omega}^{-3-m} \wedge \ldots, \qquad m \geq 0 .$$

These are "vacua" on which new representations $\mathcal{V}_0^{(m)}$ of $\mathcal{V}ir$[24] can be constructed, characterized by the same value of c but with weight $\frac{1}{2}m(m+3)$ units less than the weight of $\mathcal{V}_0$. Thus it is clear that the KN representation based on $|0)_\Sigma$ contains an infinite superposition of $\mathcal{V}ir$ representations, unless a miraculous cancellation occurs. The appearance of these new vacua can be interpreted as an interaction of the b - c system. The latter looks like a free theory as long as we consider expansions in the KN bases. However let us consider as basic modes the "in" and "out" modes, that is the modes with respect to bases on a small circle around P_+ and P_-. Then the interaction of the system with the geometry manifest itself, for instance, through the presence the above higher weight representations.

We want now to see the same problem on another example. To this end we introduce the spaces $\mathcal{F}'_\Sigma$ of dual semiinfinite forms and, in particular, the dual vacuum

$$\ldots \wedge e_{g_0-4} \wedge e_{g_0-3} \wedge e_{g_0-2} = \ _\Sigma(0| , \tag{5.56}$$

on which the creation and annihilation operators act by right action (the one defined by eqs.(5.27) and (5.28) is the left action)

$$i_{\Omega^j} = c^j , \qquad \wedge e_i = b_i . \tag{5.57}$$

The interior product acts first on the rightmost element and then from right to left with the usual rule: $i_{\Omega^j} e_k = \delta^j_k$. Therefore we find

$$_\Sigma(0|c^j = 0 \qquad j \geq g_0 - 1 , \qquad _\Sigma(0|b_j = 0 \qquad j < g_0 - 1 . \tag{5.58}$$

L_i are defined as in eq.(5.34), but on dual semiinfinite forms the leftmost elements act first. We find

$$_{\Sigma}(0|L_i = 0 \qquad i < -g_0\,, \qquad _{\Sigma}(0|L_{-g_0} = 0\,. \tag{5.59}$$

We will call the representation constructed over $_{\Sigma}(0|$ with the right action the dual representation.

Of course we can do the same for the corresponding $\mathcal{V}ir$ representation, starting from the dual space of semiinfinite forms $\mathcal{F}_0'$ and the dual vacuum $_0(0|$, defined in the obvious way. Then one can verify that

$$_0(0|K^{-1} = \ _{\Sigma}(0|\,, \tag{5.60}$$

with the definition (5.49) or (5.51) and in the sense of the right action.

We have an obvious pairing[30] between the space of semiinfinite forms and the space of the dual ones, which gives in particular

$$_{\Sigma}(0|0)_{\Sigma} = 1\,, \qquad\qquad _0(0|0)_0 = 1\,. \tag{5.61}$$

One can verify that

$$\begin{aligned} _{\Sigma}(0|L_i|0)_{\Sigma} &= 0 \qquad \forall i\,; \\ _0(0|\tilde{L}_n|0)_0 &= 0 \qquad \forall n\,. \end{aligned} \tag{5.62}$$

Now let us consider the correlation functions[31]

$$_{\Sigma}(0|R\left(b(Q_1)\ldots b(Q_n)c(P_1)\ldots c(P_n)\right)|0)_{\Sigma}\,, \tag{5.63}$$

where R represent the equivalent of the radial ordering in Σ (see below). Let us suppose that all the points $Q_1,\ldots,Q_n,P_1,\ldots,P_n$ are contained in an annulus A around S^1. We may wonder whether calculating the correlation functions on Σ and then restricting to the annulus A we obtain the same result as calculating the correlation functions directly on the annulus (which is equivalent to computing them on the complex plane). Since only the Clifford algebra of the creation and annihilation operators is involved in this calculation (see [31]) and this is preserved by the transformation induced by K, it is easy to see that the answer is yes.

The next question concerns the calculus of correlation functions involving condensates of the fundamental fields, such as the energy momentum tensor T. It is evident that the above question reformulated for this kind of correlation functions is no, in general. Let us see a simple example. The vacuum expectation value of T as calculated by Eguchi and Ooguri[48] by means of the relevant Ward identities, is

$$< T >= \sum_{i=-g_0+2}^{g_0-2} \Omega^i \frac{\partial}{\partial y^i} ln \mathcal{Z} \, , \tag{5.64}$$

where $\mathcal{Z}$ is the partition function of the relevant system, y^i are the modular parameters. We recall that Ω^i for $-g_0 + 2 \leq i \leq g_0 - 2$ are holomorphic quadratic differentials. It has been remarked that the RHS of eq.(5.64) cannot be set to zero but at the price of violating Ward identities[53]. The KN realization of the b - c system provides a natural interpretation of eq.(5.64). It is enough to set

$$\begin{aligned} {}_\Sigma(0|L'_i|0)_\Sigma &= 0 \, , \qquad |i| \geq g_0 - 1 \, , \\ {}_\Sigma(0|L'_i|0)_\Sigma &= \frac{\partial}{\partial y^i} \ln \mathcal{Z} \, , \qquad |i| \leq g_0 - 2 \, . \end{aligned} \tag{5.65}$$

This provides a normalization for L'_i. If we restrict the RHS of eq.(54) to S^1 and try to interpret it as the VEV of the same T restricted to S^1, we find that a condition must be satisfied. As a consequence of eqs.(5.44) and 5.62 we have

$$\begin{aligned} {}_0(0|\tilde{L}_n|0)_0 = \; 0 \; &= B^i_n \frac{\partial}{\partial y^i} \ln \mathcal{Z} \, , \qquad n \neq 0 \, , \\ {}_0(0|\tilde{L}'_0|0)_0 = \; \tilde{a}_0 \; &= B^i_0 \frac{\partial}{\partial y^i} \ln \mathcal{Z} = 0 \, , \end{aligned} \tag{5.66}$$

which, of course, cannot be satisfied in general. In ref.[53] it was shown that, in genus 1, the VEV of T for the free bosonic field vanishes at the orbifold points of the moduli space. Whenever this occurs, the preceding discussion is pointless: the two procedures give the same result. So one is led to speculate that, corresponding to the orbifold points of the moduli space, the relation between $\mathcal{V}ir$ and KN representations becomes particularly simple.

As a result of the preceding examples, it is clear that the representations of the Virasoro algebra are not the best tool in order to classify conformal field theories over Riemann surfaces, while the representations of the KN algebra seem to lend themselves for such a purpose.

The relation illustrated above between KN and $\mathcal{V}ir$ representations are but an example of the degeneracy which characterizes a conformal field theory in genus zero and disappears in higher genus Riemann surfaces. Another example is the following. In genus zero $\tilde{L}_0$ defines the Cartan subalgebra of $\mathcal{V}ir$, and, on the other hand, the real and imaginary part of $\tilde{e}_0$ generate the radial motions and the rotations in the complex plane, respectively. From this we can easily identify the Hamiltonian of the system.

On higher genus Riemann surfaces L_{g_0} identifies the Cartan subalgebra of KN, but e_{g_0} generates radial motions and rotations only in a neighbourhood of P_+. To better explain the point note that the time $\tau(Q)$ (see chapter 2), turns out to be a Morse function. Its critical points are the points where the level curves C_τ split and rejoin. The generators of the motion along τ and along C_τ are constructed as follows. Consider the vector field e_H such that $e_H \otimes \omega = 1$, in the sense of the product of sections of the relevant line bundles. The vector field e_H has $2g$ simple poles which correspond to the critical points of $\tau(Q)$. The generator of the motion along τ is $e_H + \bar{e}_H$ (the bar meaning complex conjugate), while $e_H - \bar{e}_H$ "rotates" C_τ. The vector field e_H behaves like e_{g_0} only near P_+. More details will be given in a forthcoming paper where we intend to study the consequences of this fact in conformal field theory.

To conclude this chapter let us mention that what we have done above can be repeated for any b - c system with arbitrary λ. Of course we find different A and B matrices, different constants in L_i, etc.. But the method is the same and we find analogs, in particular, of eqs.(5.53).

6. KdV EQUATION IN HIGHER GENUS

Not many equations have proved as successful as the KdV equation. Invented almost a century ago [54] to explain the behaviour of waves in shallow water, it turned out to have the right amount of nonlinearity as to allow for soliton solutions. More recently [55] an infinite set of conserved charges were discovered, together with an infinite hierarchy of equations, built starting from the KdV equation. Another reason of interest stems from the connection [56] between this equation and the Virasoro algebra with a non trivial central extension. The above explains sufficiently the interest stirred lately by the KdV equation. We can add to this that a modified version of the KdV equation is connected with the Sine - Gordon equation and that generalizations of the modified KdV equation lead to the Toda field theory [54,57]. Moreover higher order generalizations of the KdV hamiltonian system have been identified recently [58] with classical analogs of the W - algebras [59] There is enough here to wonder whether this prototype of integrable systems, formulated up to now only on the cylinder, can be generalized to less trivial topologies, in particular to Riemann surfaces.

There are probably several ways of relating the KdV equation to Riemann surfaces (see, for example, ref.[54,60]). In this chapter we want to generalize it in such a way that it represent a dynamics on a given Riemann surface Σ [43,32]. To this end we covariantize the original KdV equation so that the new one, hereafter referred to as RKdV equation, is globally defined on Σ. Once this is done, we find a generalized infinite set of conserved charges and the corresponding equations of motion. Moreover we find that the hamiltonian system connected with the RKdV equation is related to a realization of the Krichever and Novikov algebra [30,41].

6.1 KdV Equation On The Cylinder

Let us first recall a few basic properties of the classical KdV equation on the cylinder:

$$-\frac{c}{12}u_\tau = u''' + 3uu', \tag{6.1}$$

where $-\infty < \tau < +\infty$ and $0 < \sigma \leq 2\pi$, and u is periodic in σ, $u' \equiv \dfrac{\partial u}{\partial \sigma}$, $u_\tau \equiv \dfrac{\partial u}{\partial \tau}$. The factor $-\frac{c}{12}$ in the LHS is introduced for later purposes. Eq.(6.1) is a bi - hamiltonian system: it can be regarded as the result of two distinct dynamics

$$u_\tau = \{u, H_{n-k+1}\}_{(k)} = \mathcal{D}^{(k)} \frac{\delta H_{n-k+1}}{\delta u}, \quad n = 3, \quad k = 1, 2, \tag{6.2}$$

where

$$H_2 = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \mathcal{H}_2 = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \frac{u^2}{2}, \tag{6.3}$$

$$H_3 = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \mathcal{H}_3 = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \frac{1}{2}(u^3 - (u')^2), \tag{6.4}$$

and

$$\mathcal{D}^{(1)} = -\frac{12}{c}\partial_\sigma, \qquad \mathcal{D}^{(2)} = -\frac{12}{c}(\partial_\sigma^3 + 2u\partial_\sigma + u'), \tag{6.5}$$

$$\{F, G\}_{(k)} = \int_0^{2\pi} d\sigma \frac{\delta F}{\delta u} \mathcal{D}^{(k)} \frac{\delta G}{\delta u}, \qquad k = 1, 2. \tag{6.6}$$

The other equations of the KdV hierarchy are written down in terms of eq.(6.2) with generic n. The hamiltonians H_n are easily constructed recursively from the formula

$$\mathcal{D}^{(1)} \frac{\delta H_n}{\delta u} = \mathcal{D}^{(2)} \frac{\delta H_{n-1}}{\delta u}, \tag{6.7}$$

implied by eq.(6.2). For example

$$H_1 = \int_0^{2\pi} d\sigma u, \qquad H_4 = \frac{1}{2} \int_0^{2\pi} d\sigma \left(\frac{5}{4}u^4 + \frac{5}{2}u^2 u'' + (u'')^2 \right). \tag{6.8}$$

All these hamiltonians are in involution

$$\{H_n, H_m\}_{(k)} = 0, \qquad \forall n, m \geq 1, \; k = 1, 2. \tag{6.9}$$

6.2 KdV Equation On The Complex Plane: Covariantization

Let us now transfer eq.(6.1) from the cylinder to the complex plane $\mathbb{C}$. To do this we interpret u as a 2 - form and consider the conformal map defined by $z = e^{\tau + i\sigma}$. For the sake of simplicity we assume u to be a (meromorphic) quadratic differential in $\mathbb{C}$. Precisely we write

$$U \equiv u(\sigma,\tau)\,(d\sigma)^2 + v(\sigma,\tau)d\sigma d\tau + w(\sigma,\tau)\,(d\tau)^2 = \tilde{u}(z)\,(dz)^2 \equiv \mathcal{U}\,, \tag{6.10}$$

so that locally:

$$u = -\frac{i}{2}v = -w = -z^2\tilde{u}\,, \tag{6.11}$$

and assume that u satisfy eq.(6.1). We write the LHS of eq.(6.1) multiplied by $(d\tau)^2 + 2id\tau d\sigma - (d\sigma)^2$ as

$$\mathcal{L}_{\partial_\tau} U = \mathcal{L}_{e_0+\bar{e}_0}\mathcal{U}\,, \tag{6.12}$$

where $\mathcal{L}$ denotes the Lie derivative and $e_0 = z\partial_z$ with reference to the Witt basis $e_n = z^{n+1}\partial_z$ and its conjugate. The RHS of eq.(6.1) can be written as

$$i\left(z^{-1}\partial_z\left(z\partial_z\left(z\partial_z\left(\tilde{u}z^2\right)\right)\right) + 3\tilde{u}z\partial_z\left(\tilde{u}z^2\right)\right)(dz)^2 =$$

$$= i\left(e_0^{-1}\partial\left(e_0\partial\left(e_0\partial\left(\mathcal{U}e_0^2\right)\right)\right) + 3\mathcal{U}e_0\partial\left(e_0^2\mathcal{U}\right)\right)\,, \tag{6.13}$$

where $\partial = dz\partial_z$ and, in general, for two tensors ϕ and ω of weight λ and μ we write

$$\mathcal{L}_\phi\omega = -\lambda\phi\partial\omega + \mu\partial\phi\omega\,. \tag{6.14}$$

Now putting together eqs.(6.12) and (6.13) it is easy to see that

$$i\mathcal{L}_{\partial_\tau}\mathcal{U} = \{\mathcal{U}, H_{n-k+1}\}_{(k)}\,, \qquad for\ n = 3,\ k = 1,2\,, \tag{6.15}$$

where

$$H_2 = \frac{1}{2\pi i}\oint \mathcal{H}_2 = \frac{1}{2\pi i}\oint \frac{1}{2}e_0^3\mathcal{U}^2\,, \tag{6.16}$$

$$H_3 = \frac{1}{2\pi i}\oint \mathcal{H}_3 = \frac{1}{2\pi i}\oint \frac{1}{2}\left(e_0^5\mathcal{U}^3 - e_0\left(\partial\left(e_0^2\mathcal{U}\right)\right)^2\right)\,, \tag{6.17}$$

where the integration contour encircles the origin once and

$$\mathcal{D}_c^{(1)} = -\frac{12}{c}e_0^{-1}\partial e_0^{-1}\,, \tag{6.18}$$

$$\mathcal{D}_c^{(2)} = -\frac{12}{c}\left(e_0^{-1}\partial e_0\partial e_0\partial e_0^{-1} + 2\mathcal{U}e_0\partial e_0^{-1} + e_0^{-2}\left(\partial\left(e_0^2\mathcal{U}\right)\right)\right)\,, \tag{6.19}$$

where ∂ acts on anything is on the right, and $\mathcal{D}_c^{(k)}$ are to be applied to vector fields (the label c stands for covariant). In particular we have

$$\{\mathcal{U}(z), \mathcal{U}(w)\}_{(2)} = \mathcal{D}_c^{(2)}(z)\delta(z,w) . \tag{6.20}$$

If we analyze now the $\mathcal{U}$ on the Witt basis

$$L_n = -\frac{c}{24\pi i} \oint \mathcal{U} e_n , \tag{6.21}$$

we find a realization of the Virasoro algebra

$$\{L_n, L_m\} = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} + \frac{cn}{12}\delta_{n+m,0} , \tag{6.22}$$

up to a redefinition of L_0.

This is enough to suggest the generalization to the higher genus Riemann surfaces. The field e_0 was inserted not for academic reasons: for example $e_0^2\mathcal{U}$ is a meromorphic function, $\partial\left(e_0^2\mathcal{U}\right)$ is a 1 - differential, etc.

6.3 KdV Equation On Riemann Surfaces

Let Σ be any genus g compact Riemann surface. As shown in chapter 5, the vector fields

$$X_{\pm} = e_H \pm \bar{e}_H , \tag{6.23}$$

generate on Σ the τ - flow and the flow tangent to C_τ respectively. It is now easy to generalize the KdV equation to higher genus Riemann surfaces. The LHS, following eq.(6.1), writes $\mathcal{L}_X\mathcal{U} = \mathcal{L}_e\mathcal{U}$, $\mathcal{U}$ being a meromorphic 2 - differential. The RHS will be modelled over eq.(6.13), with e_0 replaced by the meromorphic vector field e_H simply denoted e. So finally the KdV equation takes the form

$$i\mathcal{L}_e\mathcal{U} = e^{-1}\partial\left(e\partial\left(e\partial\left(e^2\mathcal{U}\right)\right)\right) + 3\mathcal{U}e\partial\left(e^2\mathcal{U}\right) . \tag{6.24}$$

This equation is globally defined on Σ. We will refer to it as the RKdV equation. We remark that the RKdV equation depends on the field e since the RHS and LHS of the equation do not have the same degree of homogeneity in e. We remark also that we could have used any other meromorphic vector field in order to covariantize the KdV equation. However if we use a field different from e_H we loose the nice geometrical picture connected with the coordinatization described above.

The hamiltonian character of eq.(6.24) is evident

$$\mathcal{L}_e\mathcal{U} = \{\mathcal{U}, H_{n-k+1}\}_{(k)} \,, \qquad \text{for } n = 3,4,\ldots \text{ and } k = 1,2\,, \tag{6.25}$$

$$\mathcal{D}_c^{(1)}\frac{\delta H_n}{\delta\mathcal{U}} = \mathcal{D}_c^{(2)}\frac{\delta H_{n-1}}{\delta\mathcal{U}}\,, \tag{6.26}$$

where we have

$$H_1 = \frac{1}{2\pi i}\oint_{C_\tau} e\mathcal{U}\,, \tag{6.27}$$

$$H_2 = \frac{1}{2\pi i}\oint_{C_\tau} \frac{1}{2}e^3\mathcal{U}^2\,, \tag{6.28}$$

$$H_3 = \frac{1}{2\pi i}\oint_{C_\tau} \frac{1}{2}\left(e^5\mathcal{U}^3 - e\left(\partial\left(e^2\mathcal{U}\right)\right)^2\right)\,, \quad \text{etc.}. \tag{6.29}$$

When we choose, for example, e to be a suitable "Baker - Akhiezer" vector field, H_n do not depend on τ (see below).

The integration contour is a simple curve separating P_+ and P_- and

$$\mathcal{D}_c^{(1)} = -\frac{12}{c}e^{-1}\partial e\,, \tag{6.30}$$

$$\mathcal{D}_c^{(2)} = -\frac{12}{c}\left(e^{-1}\partial e\partial e\partial e^{-1} + 2\mathcal{U}e\partial e^{-1} + e^{-2}\partial\left(e^2\mathcal{U}\right)\right)\,, \tag{6.31}$$

with the same meaning as in eqs.(6.18 - 6.19). In particular

$$\{\mathcal{U}(Q),\mathcal{U}(Q')\}_{(2)} = \mathcal{D}_c^{(2)}(Q)\Delta(Q,Q')\,, \tag{6.32}$$

where

$$\frac{1}{2\pi i}\oint_{C_\tau} \Delta^{(2)}(Q,Q')v(Q') = v(Q)\,, \tag{6.33}$$

and

$$\Delta^{(2)}(Q,Q') = \frac{\delta v(Q)}{\delta v(Q')}\,, \tag{6.34}$$

for any vector field $v(Q)$. Eq.(6.32) can be rewritten in terms of components in any local coordinate patch as

$$\{\tilde{u}(z), \tilde{u}(z')\}_{(2)} = \frac{12}{c}\left(-2\tilde{u}(z)\partial_z - \tilde{u}'(z) - \partial_z^3 + 2\mathcal{R}(z)\partial_z + \mathcal{R}'(z)\right)\Delta(z,z'), \qquad (6.35)$$

where $\Delta(z,z')$ is the component form of $\Delta(Q,Q')$, and

$$\mathcal{R} = \frac{\epsilon''}{\epsilon} - \frac{1}{2}\left(\frac{\epsilon'}{\epsilon}\right)^2, \qquad e = \epsilon(z)\partial_z\,. \qquad (6.36)$$

It is easy to prove that $\mathcal{R}$ is a projective connection. The fact that the charges (6.27 - 6.29) commute can be proved [32] along the same lines as in ref.[55]. Another procedure is the one used for example in ref.[56] (see also [54,55,61])(*).

6.4 KdV Equation and KN Algebra

We want now to exhibit the relation between the Poisson brackets (6.35) and the Krichever - Novikov algebra discussed in chapter 3. First of all we expand $\mathcal{U}$ with respect to the quadratic differentials $\Omega^k \equiv f_{-k}^{(2)}$ ($f_{-k}^{(2)}$ was defined in (2.1)):

$$\mathcal{U} = \sum_k L_k \Omega^k\,, \qquad (6.37)$$

where

$$L_k = -\frac{c}{24\pi i}\oint_{C_\tau} \mathcal{U} e_k\,. \qquad (6.38)$$

(*) One can consider a sort of "square root" of the hamiltonian structure (6.32) by introducing a "Miura" transformation. Consider a one form ω which is related to $\tilde{u}$ by the Riccati - type equation

$$a) \qquad \tilde{u} = \omega^2 + e^{-1}\partial(e\omega)\,. \qquad (a)$$

Suppose ω satisfies the Poisson bracket

$$b) \qquad \{\omega(Q), \omega(Q')\} = \partial\Delta^{(1)}(Q,Q')\,. \qquad (b)$$

For $\Delta^{(1)}(Q,Q')$, see the definition (2.18). One can prove that if (a) and (b) are assumed then eq.(6.32) follows. Inserting (a) in (6.24) we obtain the so - called modified KdV equation.

Then eq.(4.53) implies (notice that $\Delta^{(2)}(Q,Q') = \sum_k e_k(Q)\Omega^k(Q')$)

$$\{L_i, L_j\}_{(2)} = \sum_{k=i+j-g_0}^{i+j+g_0} C_{ij}^k L_k + \frac{c}{12}\chi(e_i, e_j)\,, \tag{6.39}$$

where $\chi(e_i, e_j)$ is given by ($e_i = e_i(z)\partial_z$):

$$\chi(e_i, e_j) = \frac{1}{2\pi i}\oint_{C_\tau} dz\,\left(\left(e_i'''(z)e_j(z) - e_j'''(z)e_i(z)\right) - \mathcal{R}(z)\left(e_i'(z)e_j(z) - e_j'(z)e_i(z)\right)\right)\,. \tag{6.40}$$

Let us concentrate now on this bilinear object. If it

1) is antisymmetric

2) satisfies the cocycle identity

$$\chi([e_i, e_j], e_k) + \text{ cyclic permutations} = 0\,, \tag{6.41}$$

3) is coordinate independent

4) satisfies the "locality condition"

$$\chi(e_i, e_j) = 0\,, \text{ for } |i+j| > 3g\,, \tag{6.42}$$

then (6.39) is a centrally extended version of the KN algebra with the central charge c. The first three properties can be easily verified. The fourth is the critical one: its validity depends on the explicit form of $\mathcal{R}$, i.e., by eq.(6.36), on the field e we have used to covariantize the KdV equation.

If, following the analogy with the $g = 0$ case, we use the field e_H defined by (see chapter 5)

$$e_H \otimes \omega_{\frac{g}{2}}\,, \tag{6.43}$$

where $\omega_{\frac{g}{2}} \equiv f_{\frac{g}{2}}^{(1)}$ ($f_{\frac{g}{2}}^{(1)}$ is defined in eq.(2.30)), $\mathcal{R}$ will have quadratic poles at the critical points of the "global time" $\tau(Q)$ and, in particular, eq.(6.42) is not satisfied. However, we can proceed as follows. A projective connection is defined up to meromorphic quadratic differential Ω_H which has exactly the poles of $\mathcal{R}$ at the critical points. Then the projective connection ($\Omega_H = \Omega_H(z)(dz)^2$)

$$\tilde{\mathcal{R}} = \mathcal{R} - \Omega_H(z)\,, \tag{6.44}$$

is everywhere holomorphic except at P_+ and P_-. Now, suppose we replace $\mathcal{R}$ with $\tilde{\mathcal{R}}$ in eq.(6.40). This correspond to the redefinition

$$L_i \longrightarrow \tilde{L}_i = L_i - S_i\,, \tag{6.45}$$

where

$$S_i = \frac{c}{24\pi i}\oint_{C_\tau} \Omega_H e_i\,. \tag{6.46}$$

This of course corresponds to the substitution

$$\mathcal{U} \longrightarrow \tilde{\mathcal{U}} = \mathcal{U} - \Omega_H\,. \tag{6.47}$$

So if $\mathcal{U}$ is a solution of the RKdV equation, or anyhow a quadratic differential satisfying (6.32), then $\tilde{\mathcal{U}}$ generates an analog of the KN algebra (i.e. with "local" central charge).

It is perhaps instructive to examine other vector fields than e_H to covariantize the KdV equation. Another possible choice for a meromorphic vector field in the place of e in eq.(6.24) is the field e_{g_0} of the KN basis (introduced in chapters 2 and 3), since, like e_H, it behaves as $z_+\partial_{z_+}$ near P_+ (but they behave differently in P_-). Unfortunately it has g zeroes outside $P_\pm$ so that $\mathcal{R}$ has correspondingly quadratic poles here. We must therefore proceed to a redefinition as above.

A more appealing possibility is the following one. We can construct a field which behaves like e_H and e_{g_0} near P_+ and is holomorphic outside P_-, so that the corresponding cocycle, eq.(6.40) does not depend on τ. We call it "Baker - Akhiezer" vector field

$$e_B(Q) = \frac{E(Q,P_+)E(Q,P_-)\theta(P_+ - gP_- + \Delta)^2}{\theta(Q - gP_- + \Delta)^2 E(P_+,P_-)} e^{-2\pi i \sum\limits_{k\geq 0} c_k \int\limits_{P_0}^{Q} \eta_k}\,. \tag{6.48}$$

In this formula η_k are normalized second kind differentials with poles of order $k+1$ in P_-, i.e.

$$\eta_k(z) = \frac{1}{k!}\partial_w^k \partial_z \ln E(z,w) dz_{|_{w=P_-}}\,, \tag{6.49}$$

normalized in such a way that

$$\oint\limits_{a_i} \eta_k = 0\,, \qquad i = 1,...,g\,. \tag{6.50}$$

The constant c_k are normalized in such a way as to render $e_B(Q)$ single - valued, i.e. they must satisfy

$$\sum_{k\geq 0} c_k \oint_{b_i} \eta_k = 2\Delta_i + I_i(P_+) - (2g-1)I_i(P_-)\,, \qquad i = 1, ..., g\,. \tag{6.51}$$

In these equations Δ is the vector of Riemann constants and $I(P)$ is the jacobi map. It is now clear that the corresponding $\mathcal{R}$ will have a quadratic pole in P_+ and a pole in P_- of order N double to the order to the pole of the highest η_k appearing in the exponent of eq.(6.48). So the corresponding cocycle $\chi(e_i, e_j)$ satisfies the "locality" condition in the form

$$\chi(e_i, e_j) = 0\,, \qquad for\; i+j > 2g_0\,, \quad i+j < -2g_0 - N + 2\,. \tag{6.52}$$

If we want to recover eq.(6.42) we have to subtract a suitable quadratic differential with a pole of order N in P_-. Another interesting possibility is the "Baker - Akhiezer" vector field introduced in chapter 3

$$\tilde{e}_B(Q) = \frac{\sigma(P_+)\theta(P_+ - gP_- + \Delta)E(Q,P_-)}{E(P_+,P_-)E(Q,P_+)^{g-1}\sigma(Q)\theta(Q - gP_-\Delta)} e^{-2\pi i \sum_{k=1}^{\infty} c_k \int_{P_0}^{Q} \eta_k}\,. \tag{6.53}$$

The discussion is parallel to the previous one and will not be repeated. We only note that on the torus $e_B(Q)$ is a costant and that the RHS of eq.(6.51) is now equal to $2\Delta - (g-1)(P_- + P_-)$.

6.5 Generalizations

Let us now consider some generalizations of the KN algebra (6.39) or, which is equivalent, (6.35). The way we arrived at eq.(6.35) has been described above, but there is also another procedure: covariantize ∂^3 by means of the vector field e, and obtain $e^{-1}\partial e \partial e \partial e^{-1}$, i.e. in local coordinates

$$\partial_z^3 + 2\mathcal{R}\partial_z + \mathcal{R}'\,, \tag{6.54}$$

where $\mathcal{R}$ is given by (6.36); finally simply replace $\mathcal{R}$ by $\mathcal{R} + \tilde{u}$ where $\mathcal{U} = \tilde{u}(z)(dz)^2$ (since $\tilde{u}$ is a quadratic differential this does not change the covariance properties of the operator). The generalization is as follows. Define

$$\overline{\mathcal{D}}_c^{(\lambda)} = e^{1-\lambda}\partial e \partial e ... \partial e^{1-\lambda}\,. \tag{6.55}$$

This maps tensors of weight $1-\lambda$ into tensors of weight λ. We require it to be homogeneus of degree 0 in e (we want the possible dependence on e to appear only trough th $\mathcal{R}$, and its derivatives). Then if n is the number of ∂'s in (6.55) the following condition must be satisfied

$$n = 2\lambda - 1 . \tag{6.56}$$

This equation can be satisfied only for λ integer or half - integer. Let us consider first the integral case. In particular for $\lambda = 1$, $\overline{\mathcal{D}}_c^{(1)} = \partial$; for $\lambda = 2$, we reobtain - up to a constant factor - the third order covariant differential operator discussed above. Let us discuss the case $\lambda = 3$, in local coordinates

$$\overline{\mathcal{D}}_c^{(3)}(\mathcal{R}) = \left(\partial_z^5 - 10\mathcal{R}\partial_z^3 - 15\mathcal{R}'\partial_z^2 - 9\mathcal{R}''\partial_z + 16\mathcal{R}^2\partial_z + 16\mathcal{R}\mathcal{R}' - 2\mathcal{R}'''\right)(dz)^5 , \tag{6.57}$$

where $\mathcal{R}$ is again given by eq.(6.36), etc. Notice that

$$\chi\left(f_i^{(1-\lambda)}, f_j^{(1-\lambda)}\right) = \frac{1}{2\pi i}\oint_{C_\tau} f_i^{(1-\lambda)}\overline{\mathcal{D}}_c^{(\lambda)} f_j^{(1-\lambda)} , \tag{6.58}$$

has the property 1) and 3) above. If the $\mathcal{R}$ appearing in $\overline{\mathcal{D}}_c^{(\lambda)}$ is suitably chosen it can also satisfy the locality condition in the form

$$\chi\left(f_i^{(1-\lambda)}, f_j^{(1-\lambda)}\right) = 0 , \qquad for\ |i+j| > (2\lambda - 1)g = ng . \tag{6.59}$$

Therefore they can give rise to central extensions of suitable algebras. The case $\lambda = 2$ has already been described. Let us consider $\lambda = 3$, in this case $\overline{\mathcal{D}}_c^{(3)}(\mathcal{R})$ maps quadratic vector fields into weight 3 tensors; it is therefore appropriate for the Poisson brackets of a tensor $\mathcal{V}$ of weight 3. However if we want a non trivial algebra we have to replace $\mathcal{R}$ by $\mathcal{R} - \tilde{u}$, where $\mathcal{U} = \tilde{u}(dz)^2$ is a quadratic differential

$$\{\mathcal{V}(Q), \mathcal{V}(Q')\}_{(3)} = \mathcal{D}_c^{(3)}(Q)\Delta^{(3)}(Q, Q') , \tag{6.60}$$

where

$$\mathcal{D}_c^{(3)}(Q) \equiv \overline{\mathcal{D}}_c^{(3)}(\mathcal{R} - \tilde{u}) . \tag{6.61}$$

Now a closed algebra will be defined by eqs.(6.35,6.60), and

$$\{\mathcal{U}(Q),\mathcal{V}(Q')\} = \left(\mathcal{V}(Q)\partial + \frac{2}{3}\partial(\mathcal{V})\right)\Delta^{(3)}(Q,Q') \,. \tag{6.62}$$

It can be proved that

$$-\{\mathcal{U}(Q'),\mathcal{V}(Q')\} = \{\mathcal{V}(Q),\mathcal{U}(Q')\} = \left(\mathcal{V}(Q)\partial + \frac{1}{3}\partial(\mathcal{V})\right)\Delta^{(2)}(Q,Q') \,. \tag{6.63}$$

Eq.(6.62) expresses the fact that $\mathcal{V}$ has weight 3. Using eqs.(6.35), (6.60) and (6.62) we can define a generalization of Zamolodchikov's spin 3 algebra [59] to higher genus, in the same way as the KN algebra represents a generalization of the Virasoro algebra.

Let us see next the half integer case. It is convenient in this case to introduce a $-\frac{1}{2}$ weight tensor $\rho = \rho(z)(dz)^{-\frac{1}{2}}$. In terms of $\rho(z)$, $\mathcal{R}$ takes a very simple form:

$$\mathcal{R} = 2\rho''(z)\rho(z) \,. \tag{6.64}$$

Now

$$\overline{\mathcal{D}}_c^{(\lambda)} = \rho^{2-2\lambda}\partial\rho^2\partial\rho^2 ... \partial\rho^{2-2\lambda} \,. \tag{6.65}$$

Some examples:

$$\overline{\mathcal{D}}_c^{(\frac{3}{2})} = \left(\partial_z^2 - \frac{\mathcal{R}}{2}\right)(dz)^2 \,, \tag{6.66}$$

$$\overline{\mathcal{D}}_c^{(\frac{5}{2})} = \left(\partial_z^4 - 10\frac{\mathcal{R}}{2}\partial_z^2 - 10\frac{\mathcal{R}'}{2}\partial_z + 9\left(\frac{\mathcal{R}}{2}\right)^2 - 3\frac{\mathcal{R}''}{2}\right)(dz)^4 \tag{6.67}$$

$$\overline{\mathcal{D}}_c^{(\frac{7}{2})} = \left(\partial_z^6 - 35\frac{\mathcal{R}}{2}\partial_z^4 - 70\frac{\mathcal{R}'}{2}\partial_z^3 + 259\left(\frac{\mathcal{R}}{2}\right)^2\partial_z^2 - 63\frac{\mathcal{R}''}{2} + \right.$$

$$\left. + \frac{\mathcal{R}}{2}\frac{\mathcal{R}'}{2}\partial_z - 28\frac{\mathcal{R}'''}{2}\partial_z - 225\left(\frac{\mathcal{R}}{2}\right)^3 + 130\left(\frac{\mathcal{R}'}{2}\right)^2 + 155\frac{\mathcal{R}}{2}\frac{\mathcal{R}'}{2} - 5\frac{\mathcal{R}'''}{2}\right)(dz)^6 \,. \tag{6.68}$$

In the appropriate KN basis we have the cocycle

$$\varphi(\psi_\alpha^{(1-\lambda)},\psi_\beta^{(1-\lambda)}) \equiv \frac{1}{2\pi i}\oint_{C_\tau} \psi_\alpha^{(1-\lambda)}\overline{\mathcal{D}}_c^{(\lambda)}\psi_\beta^{(1-\lambda)} \,, \tag{6.69}$$

is symmetric in $\alpha \leftrightarrow \beta$ and is coordinate independent. Moreover by suitably choosing $\mathcal{R}$, we find that they vanish for $|\alpha+\beta| > ng$. As a consequence they provide cocycles for central extensions of spin one - half algebras. Indeed, proceeding as above we find that from (6.66) we obtain the NS or R extension of the Virasoro algebra and from (6.67) the generalization of Zamolodchikov's spin - 5/2 algebra. Given the above hamiltonian structures, we can define generalizations of the KdV equation and corresponding hierarchies (see, for example, [62]). As one can see from eq.(6.68), in $\mathcal{D}_c^{(7/2)}$ there appear cubic terms in $\tilde{u}$ after the shift $\mathcal{R} \rightarrow \mathcal{R} - \tilde{u}$. Terms of order higher than quadratic appear in any $\mathcal{D}_c^{(\lambda)}$ with $\lambda \geq 7/2$. So these operators are not relevant to W - algebras, which close over quadratic terms [59]. A more refined construction is necessary in order to recover higher order W - algebras. Nevertheless the operators $\mathcal{D}_c^{(\lambda)}$ are of interest in connection with another problem: on the cylinder these operators intervene in the equations that determine the null vectors of the Virasoro algebra in the limit of c tending to infinity. One is therefore motivated to conjecture that the covariant operators we have introduced here are connected with the null vectors of the KN algebra in the same limit.

7. CONFORMAL TECHNIQUES IN HIGHER GENUS

Many informations on conformal field theories come from their formulation on Riemann surfaces[6,63,64,65]. This feature is based on the observation[6] that primary fields have tensor properties under conformal transformations and then it is natural to formulate CFT on the moduli space of Riemann surfaces that, by definition, have analytic transition functions between coordinate patches.

Another peculiarity of CFT is that factorization property of primary field correlators holds in any genus. In particular a correlator can be written in terms of analytic and antianalytic building blocks[6]:

$$G(z,m;\bar{z},\bar{m}) = \sum_{\bar{I},J} \overline{\mathcal{F}}_{\bar{I}}(\bar{z},\bar{m}) h_{\bar{I},J} \mathcal{F}_J(z,m)\,, \tag{7.1}$$

where $z \equiv (z_1, ..., z_n)$ are the primary field coordinates, $m \equiv (m_1, ..., m_{3g-3})$ are the moduli of Σ and $h_{\bar{I},J}$ is an hermitean metric. The functions $\overline{\mathcal{F}}_{\bar{I}}(\bar{z},\bar{m})$ and $\mathcal{F}_J(z,m)$ are multivalued and not modular invariant. Then the requirement for G to be both singlevalued and modular invariant gives constraints on $h_{\bar{I},J}$. In genus one this requirement implies restrictions on the operator content[63] of the conformal theory. Similar arguments play a fundamental rôle in classifying the modular invariant partition functions[66]. Another relevant result coming from the analysis of CFT on Riemann surfaces is due to the Verlinde's conjecture[64], proved in [67,68,69], stating that the modular transformation $S : \tau \longrightarrow -1/\tau$ diagonalizes the fusion rules (τ is the torus period matrix).

In this chapter, after reviewing basic facts on the Riemannn surfaces theory, we discuss some properties of CFT in higher genus. In particular we show how the KN algebra acts on the moduli space and analyze its relation with the standard formulation of the operator formalism. In section 5 we introduce a basis for holomorphic differentials on punctured Riemann surfaces. Finally and in section 6 we review the light - cone formulation of the bosonic string.

7.1 Generalities On Riemann Surfaces

In this section we will recall some results from Riemann surfaces theory. A standard reference on this subject is the book by H. M. Farkas and I. Kra [36]. Other books on this argument are listed in [70 - 74]. For applications to string theories see for example [75 - 83].

Given a Riemann surface Σ it is always possible to introduce in any patch isothermal coordinates $(\eta^1_\alpha, \eta^2_\alpha)$ so that the metric is euclidean (up to a conformal rescaling $\rho(\eta^1_\alpha, \eta^2_\alpha))^{(*)}$

$$\hat{g}_\alpha = g_{ab}(\eta) d\eta^a_\alpha d\eta^b_\alpha, \qquad g_{ab}(\eta) = \rho(\eta^1_\alpha, \eta^2_\alpha)\delta_{ab} \,. \tag{7.2}$$

Replacing the real coordinates with the complex ones

$$z_\alpha = \eta^1_\alpha + i\eta^2_\alpha, \qquad \bar{z}_\alpha = \eta^1_\alpha - i\eta^2_\alpha \,, \tag{7.3}$$

the metric takes the form

$$\hat{g}_\alpha = \rho dz_\alpha d\bar{z}_\alpha \,; \tag{7.4}$$

and the only nonvanishing components of the metric and its inverse are

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}\rho, \qquad g^{z\bar{z}} = g^{\bar{z}z} = 2\rho^{-1} \,. \tag{7.5}$$

Since isothermal coordinates on two intersecting patches are related by conformal transformation it is obvious that the transition functions must be analytic (antianalytic) with respect to z_α, $(\bar{z}_\alpha)$:

$$z_\beta = f_{\beta\alpha}(z_\alpha), \qquad \bar{z}_\beta = \bar{f}_{\beta\alpha}(\bar{z}_\alpha) \,. \tag{7.6}$$

This means that the complex coordinates are well - defined on any patch, so Riemann surfaces are complex manifolds. Associated to $\hat{g}$ there is a two - form on Σ

$$V = \frac{i}{2}\sqrt{g} dz \wedge d\bar{z} \,, \tag{7.7}$$

where $g \equiv \det g_{ab} = g_{z\bar{z}}$. If the Riemann surface was equipped with a non - conformally flat metric, $g_{ab}(\xi) d\xi^a_\alpha d\xi^b_\alpha$, the choice (7.4) could be recovered locally by a reparametrization

(*) The correct expression for the metric would be the symmetrized product but for conciseness we omit it.

$\xi \to \eta(\xi)$ and a Weyl rescaling $g_{ab}(\xi) \to \phi(\xi)g_{ab}(\xi)$ (these transformations form the "gauge" group: $Conf(\Sigma) \otimes Diff(\Sigma)$). In general, global obstructions do not allow for such a solution in every patch. This aspect is connected with the moduli problem that we discuss below. To introduce complex coordinates starting from (in general) non - isothermal coordinates we introduce the complex structure

$$J_a{}^b = \sqrt{g}\epsilon_{ac}g^{cb}\,, \tag{7.8}$$

where ϵ_{ab} is the completely antisymmetric tensor with $\epsilon_{12} = 1$, whose existence follows from the fact that a Riemann surface is oriented. Notice that $J_a{}^b$ is Weyl invariant and reparametrization covariant, moreover

$$J_a{}^c J_c{}^b = -\delta_a^b, \qquad \nabla_b J_a{}^c = 0\,, \tag{7.9}$$

where ∇_a is the covariant derivative. Associated with $J_a{}^b$ there are the complex coordinates z, $\bar{z}$:

$$J_a{}^b\partial_b z_\alpha = i\partial_a z_\alpha, \qquad J_a{}^b\partial_b \bar{z}_\alpha = -i\partial_a \bar{z}_\alpha, \qquad \partial_a \equiv \frac{\partial}{\partial\xi_\alpha^a}\,. \tag{7.10}$$

In these harmonic coordinates the metric takes the conformal form

$$\hat{g}_\alpha = g_{ab}d\xi_\alpha^a d\xi_\alpha^b = \rho dz_\alpha d\bar{z}_\alpha\,. \tag{7.11}$$

A metric is useful for contracting z and $\bar{z}$ indices in pairs, so that a $(p+q,p)$ - tensor $t = t_{z\ldots\bar{z}\ldots}(dz)^{p+q}(d\bar{z})^p$, can be written in terms of q - differentials

$$t = t(z\bar{z})(dz)^q\,, \qquad t(z,\bar{z}) \equiv (g^{z\bar{z}})^p t_{z\ldots\bar{z}\ldots}\,. \tag{7.12}$$

These q - differentials are sections of the line bundle K^q, where K is the holomorphic cotangent bundle (or canonical line bundle), i.e. $K = T^*(\Sigma)$. $T^*(\Sigma)$ is the holomorphic part of the complexified version of the cotangent space $T^*_{\mathbb{R}}(\Sigma)$ to Σ with local basis $(d\xi_\alpha^1, d\xi_\alpha^2)$. Recall that the complexified version of the tangent $T_{\mathbb{R}}(\Sigma)$ and of the its dual $T^*_{\mathbb{R}}(\Sigma)$ are defined by permitting the coefficients of the frames $(\partial/\partial\xi_\alpha^1, \partial/\partial\xi_\alpha^2)$ and $(d\xi_\alpha^1, d\xi_\alpha^2)$ to be complex

$$T_{\mathbb{C}}(\Sigma) = T_{\mathbb{R}}(\Sigma) \otimes \mathbb{C} = \left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial\bar{z}_\alpha}\right) \equiv T(\Sigma) \oplus \overline{T}(\Sigma)\,, \tag{7.13}$$

$$T^*_{\mathbb{C}}(\Sigma) = T^*_{\mathbb{R}}(\Sigma) \otimes \mathbb{C} = (dz_\alpha, d\bar{z}_\alpha) \equiv T^*(\Sigma) \oplus \overline{T}^*(\Sigma) . \tag{7.14}$$

Let us now introduce the covariant derivative and its adjoint:

$$\nabla^q_z : K^q \to K^{q+1}$$

$$\nabla^q_{\bar{z}} t(z,\bar{z}) dz^{q+1} = (g_{z\bar{z}})^q \, \partial_{\bar{z}} \left((g_{z\bar{z}})^{-q} \, t(z,\bar{z}) \right) dz^{q+1} . \tag{7.15}$$

The two - form V allows us to define the adjoint of ∇^q_z. Indeed using the scalar product

$$< t_1 | t_2 > = \int_\Sigma \frac{i}{2} dz \wedge d\bar{z} \, (g_{z\bar{z}})^{1-q} \, \bar{t}_1(z,\bar{z}) t_2(z,\bar{z}) , \quad t_1, t_2 \in K^q , \tag{7.16}$$

we have $\nabla^{q\dagger} = \nabla_{q+1}$, where

$$\nabla^z_q : K^q \to K^{q-1} ,$$

$$\nabla^{\bar{z}}_q t(z,\bar{z}) dz^{q-1} = -g^{z\bar{z}} \partial_{\bar{z}} t(z,\bar{z}) dz^{q-1} . \tag{7.17}$$

By means of the scalar product it is possible to define two - different laplacians

$$\Delta^{(+)}_q = \nabla^z_{q+1} \nabla^q_z , \qquad \Delta^{(-)}_q = \nabla^{q-1}_z \nabla^z_q . \tag{7.18}$$

It is easy to show that

$$\Delta^{(+)}_q - \Delta^{(-)}_q = -q \frac{R}{2} , \tag{7.19}$$

where R is the scalar curvature:

$$R = -2 g^{z\bar{z}} \partial_z \partial_{\bar{z}} \log g_{z\bar{z}} . \tag{7.20}$$

The curvature can be written in terms of the Christoffel symbol (called also hermitean connection):

$$\Gamma^z_{zz} = (g_{z\bar{z}})^{-1} \partial_z g_{z\bar{z}} . \tag{7.21}$$

7.2 Moduli Deformations

To discuss the moduli problem we start with the analysis of the variation of the metric under conformal and conformal transformations. In particular under the joint action of infinitesimal conformal transformations $\delta\rho = \rho\delta\phi$ ($\rho \equiv e^{\phi}$) and reparametrizations $z \rightarrow w = z + v^z$ we have

$$\delta g_{z\bar{z}} = \rho\delta\phi + \partial_z(\rho v^z) + \partial_{\bar{z}}(\rho\bar{v}^z), \qquad \delta g_{\bar{z}\bar{z}} = \rho\partial_{\bar{z}}v^z, \qquad \delta g_{zz} = \rho\partial_z\bar{v}^z\,. \tag{7.22}$$

Next we introduce the inner product in the space of the all metrics on genus g Riemann surfaces

$$(\delta\hat{g},\delta\hat{g}) = \int d^2\xi\sqrt{g}\left(cg^{ac}(\xi)g^{bd}(\xi) + g^{ab}(\xi)g^{cd}(\xi)\right)\delta g_{ac}(\xi)\delta g_{bc}(\xi)\,. \tag{7.23}$$

In the following we use this equation setting the arbitrary constant c equal to zero without affecting any physical result. This inner product determines the volume elements on the space of the metrics in the same way that the finite dimensional volume $d^2\xi\sqrt{g}$ is determined by the inner product $g_{ab}(\xi)\delta\xi^a\delta\xi^b$ on variations of ξ. Let us introduce the metric deformations

$$\delta\hat{g}' = \delta\phi'\rho dzd\bar{z} + \Omega dzdz + \overline{\Omega}d\bar{z}d\bar{z}\,. \tag{7.24}$$

The variations that cannot be described by reparametrizations and/or Weyl transformations are given by the orthogonality condition

$$(\delta\hat{g},\delta\hat{g}') = \frac{i}{2}\int dz\wedge d\bar{z}\left(\delta g_{z\bar{z}}\delta\phi'\rho + \partial_{\bar{z}}v^z\Omega + \partial_z\bar{v}^z\overline{\Omega}\right) = 0\,, \tag{7.25}$$

where

$$\delta\hat{g} = \delta g_{z\bar{z}}dzd\bar{z} + \delta g_{\bar{z}\bar{z}}d\bar{z}d\bar{z} + \delta g_{zz}dzdz\,, \tag{7.26}$$

The solution of eq.(7.25) is $\delta\phi' = \partial_{\bar{z}}\Omega = \partial_z\overline{\Omega} = 0$. This means that the dimension of the space of holomorphic quadratic differentials Ω is the same as the dimension of the moduli space of genus g Riemann surfaces

$$\mathrm{M}_g = \frac{space\ of\ all\ metrics}{Conf(\Sigma)\otimes Diff(\Sigma)}\,. \tag{7.27}$$

Let us now introduce the Beltrami differentials. Suppose that we integrated eq.(7.10). This means that a set of coordinate patches $\{U_\alpha\}$ is given on Σ such that the conformal structure is everywhere $\hat{g}_\alpha = \rho_0(z_\alpha,\bar{z}_\alpha)|dz_\alpha|^2$ (the positivity of the metric requires that

$\rho_0(z_\alpha)$ be real and positive). Any other metric can be parametrized by means of Beltrami differentials $\mu = \mu_{\bar{z}}{}^{z} d\bar{z}(dz)^{-1}$:

$$\hat{g}_\alpha(\mu) = \rho(z_\alpha, \bar{z}_\alpha)|dz_\alpha + \mu d\bar{z}_\alpha|^2 \,. \tag{7.28}$$

Globally it is possible to put the new metric in the form (in any patch)

$$\hat{g}_\alpha(\mu) = \rho(z_\alpha, \bar{z}_\alpha)|\partial_{z_\alpha} w_\alpha|^{-2}|dw_\alpha|^2 = \rho(z_\alpha, \bar{z}_\alpha)|\partial_{z_\alpha} w_\alpha|^{-2}|\partial_{z_\alpha} w_\alpha dz_\alpha + \partial_{\bar{z}_\alpha} w_\alpha d\bar{z}_\alpha|^2 \,, \tag{7.29}$$

only if the Beltrami equation (in the following we suppress the subscript α)

$$\partial_{\bar{z}} w = \mu_{\bar{z}}{}^{z} \partial_z w \,, \tag{7.30}$$

admits a global solution. In this case, since everywhere $\hat{g}_\alpha(\mu) = \rho(z_\alpha, \bar{z}_\alpha)|\partial_{z_\alpha} w_\alpha|^{-2}|dw_\alpha|^2$ we are changing the conformal (=complex) structure $J_a{}^b$ only by a diffeomorphism. A local solution can be found looking at the eq.(7.30) with

$$w(z, \bar{z}) = z + v^z \,, \tag{7.31}$$

then

$$\mu_{\bar{z}}{}^{z} = \frac{\partial_{\bar{z}} v^z}{1 + \partial_z v^z} \,. \tag{7.32}$$

For infinitesimal μ the Beltrami equation is: $\mu_{\bar{z}}{}^{z} = \partial_{\bar{z}} v^z$ that as a consequence of Poincaré's lemma always admits a local solution. So locally the conformal structures are "gauge" equivalent. However the local solutions on any patch on Σ may not match. To parametrize different metrics we consider solutions of eq.(7.32) with discontinuities along a closed curve. Consider Σ with the puncture P_+ and a local coordinate z such that $z(P_+) = 0$. Let us denote with Σ^+ the disc defined by $z \leq 1$ (we suppose that $P_- \notin \Sigma^+$) and with A an annulus whose center is P_+. Let Σ^- be defined by the conditions: $\Sigma^+ \cup \Sigma^- = \Sigma$ and $\Sigma^+ \cap \Sigma^- = A$. To obtain a new Riemann surface we deform the annulus:

$$z \to w = z + \epsilon e_i(z), \qquad \epsilon \in \mathbb{C}, \quad z \in A \,, \tag{7.33}$$

where $e_i = e_i(z)(dz)^{-1}$ is a KN vector field (recall that it is holomorphic everywhere except possibly for poles in P_+ and/or P_-). The data $\{\Sigma, P_+, z\}$ define a punctured Riemann surface with a choice of a local coordinate. The new surface Σ' is obtained by identifying the new annulus with the previous collar on Σ^+. The new metric is

$$\hat{g}(\mu) = \rho(z)|dz + \epsilon\mu_i d\bar{z}|^2 , \tag{7.34}$$

where the Beltrami differential is

$$\mu_i(P) = \begin{cases} \bar{\partial} e_i & \text{if } P \in \Sigma^+; \\ 0 & \text{if } P \in \Sigma^- \backslash A. \end{cases} \tag{7.35}$$

Notice that $\delta g_{\bar{z}\bar{z}} = 2\eta_{\bar{z}\bar{z}} g_{z\bar{z}}$ and (since $\delta g^{zz} = -\left(g^{z\bar{z}}\right)^2 \delta g_{\bar{z}\bar{z}}$) $\delta g^{zz} = -2g^{z\bar{z}}\mu_{\bar{z}}{}^z$.

The KN holomorphic differentials Ω^i form a dual basis with respect to μ_i, integrating by part we have

$$\frac{1}{2\pi i}\int_\Sigma \mu_i \Omega^j = \delta_i^j . \tag{7.36}$$

If $e_i(z)$ is vanishing at P_+ then we are changing only the coordinate z, while $e_{g_0-1}(z)$ ($g_0 \equiv \frac{3}{2}g$) further moves the puncture P_+. For $i \leq -g_0 + 1$, e_i is holomorphic on $\Sigma\backslash\{P_+\}$, so Σ' is identical to Σ because the variation induced in the annulus can be reabsorbed in a holomorphic coordinate transformation on $\Sigma\backslash\Sigma^+$. For $|i| \leq g_0 - 2$ the vector field $\tilde{e}_i$ has poles both in P_+ and P_-. This change in Σ corresponds to an infinitesimal moduli deformation. Notice that the dimension of the space of these vector fields is just $3g - 3$.

To summarize we can write $\mathcal{V}_\Sigma$, the space of KN vector fields, as a direct sum of three subspaces

$$\mathcal{V}_\Sigma = \mathcal{V}_\Sigma^+ \oplus \mathcal{V}_\Sigma^0 \oplus \mathcal{V}_\Sigma^- , \tag{7.37}$$

where $\mathcal{V}_\Sigma^\pm = \{e_i | \mp i \geq g_0 - 1\}$ is the space of vector fields extending holomorphically on $\Sigma\backslash\{P_\pm\}$ and $\mathcal{V}_\Sigma^0 = \{e_i | -g_0 + 2 \leq i \leq g_0 - 2\}$ is the space of vector fields generating the infinitesimal moduli deformations (for further details see , refs.[49,30,85,86,84,86]).

We consider now the variation of the period matrix under a metric deformation. First we look at the variation of the holomorphic differentials $\omega_i = \omega_i(z)dz$. In any local patch arbitrary metric deformations are equivalent to an infinitesimal reparametrization combined with Weyl rescaling. Then we can express the local variation of the (Weyl invariant) holomorphic differentials by means of the following OPE

$$\delta_{ww}\omega_j(z) \sim \left(\frac{1}{(z-w)^2} + \frac{1}{z-w}\partial_z\right)\omega_j(z), \qquad \delta_{ww} \equiv \frac{4\pi}{\sqrt{g}}\frac{\delta}{\delta g^{ww}}. \tag{7.38}$$

This equation gives only the local behaviour. The requirement of right normalization for the new holomorphic differentials $\omega_i' \sim \omega_i + \delta\omega_i$ gives the global constraint:

$$\oint_{a_i} dz \delta_{ww}\omega_j(z) = 0. \tag{7.39}$$

The joint action of eqs.(7.38 - 7.39) gives

$$\delta_{ww}\omega_i(z) = \omega_i(w)\partial_w\partial_z \log E(z,w), \tag{7.40}$$

where $E(z,w)$ is the prime form. The variation of the period matrix is

$$\delta_{ww}\Omega_{ij} = \oint_{b_i} dz\delta_{ww}\omega_j(z) = 2\pi i\omega_i(w)\omega_j(w), \tag{7.41}$$

then $(d^2z \equiv \frac{i}{2}dz \wedge d\bar{z})$

$$\delta_k\Omega_{ij} \equiv \frac{1}{2\pi}\int_\Sigma d^2z\sqrt{g}\delta_{zz}\Omega_{ij}\delta g^{zz} = -\int_\Sigma \omega_i\omega_j\mu_k = -\oint \omega_i\omega_j e_k, \tag{7.42}$$

where the integration contour separates P_+ from P_- and can be identified with C_τ. The last equality follows from the fact that μ_k vanishes outside the disc and $\omega_i(z)\omega_j(z)\partial_{\bar{z}}e_k(z) = \partial_{\bar{z}}(\omega_i(z)\omega_j(z)e_k(z))$. Notice that (as expected)

$$\delta_k\Omega_{ij} = 0, \qquad \text{for} |k| \geq g_0 - 1. \tag{7.43}$$

Observe that the expression of $\delta_k\Omega_{ij}$ is explicit, indeed it the vector field in the LHS is given in chapter 2 in terms of theta functions and prime forms. This may be useful in studying the Schottky problem which is one of the difficulties in the Polyakov approach to string theory. This problem consists in identifying the period matrices in the Siegel upper - half plane. This space is the $1/2g(g+1)$ dimensional space of the symmetric gxg - matrices with positive imaginary part and it is equal to the dimension of the Moduli space only for $g \leq 3$. The answer to this problem has been recently given by Shiota and Mulase [8788]. On the other hand this solution is given in a very implicit way and we need further insights to investigate the moduli dependence of the theta functions appearing in the string path integral. Indeed it turns out that a matrix in the Siegel upper - half plane is a period matrix if and only if the corresponding τ - function satisfies the Hirota's bilinear relation that does not look manageable for our purpose. For details on the grassmannian approach to string theory see for example refs.[93 - 91]. Standard references for the mathematical aspects of τ - function and infinite dimensional grassmannians theory are [97 - 100].

7.3 The Standard Operator Formalism

In this section we describe (with some modifications) the operator formalism developed by Vafa [101] (see also ref.[84]) who has given an interpretation of the Ishibashi, Matsuo and Ooguri grassmannian approach[93] in terms of "infinite conserved charges".

Let $\Sigma_0 = \Sigma \backslash \Sigma^+$, associated with it there is a vector $|\partial\Sigma_0>$ corresponding to the state coming from the path integral evaluated on Σ_0. Then the operators' action on the Hilbert space "attached" to the boundary $\partial\Sigma_0$ can be obtained from the action of these operators on the boundary value of the field which is fixed in the path integral. Owing to the peculiarity of $2d$ conformal field theory, the ket state $|\partial\Sigma_0>$ is fixed (up to a moduli dependent constant) by the infinitely many conditions

$$Q_i|\partial\Sigma_0>= 0\,, \tag{7.44}$$

where the Q_i charges correspond to the infinite symmetries of the action under a shift of the field by a holomorphic section on Σ_0.

To be specific we consider the anticommuting $b-c$ system defined by the first - order action

$$S = \int b\bar{\partial}c + c\bar{\partial}b + c.c.\,, \tag{7.45}$$

where the fields b and c are λ - and $(1-\lambda)$ - differentials respectively. Here we consider the case $\lambda \in \mathbb{Z}/2$. The anticommutation relations are

$$\{b^i, c_j\} = \delta^i_j\,, \qquad \{b^i, b^j\} = \{c_i, c_j\} = 0\,. \tag{7.46}$$

On $\Sigma\backslash\Sigma_0$ the action S is invariant both under the shift

$$b \to b + \epsilon f^{(\lambda)}_j, \qquad j \geq s(\lambda)\,, \tag{7.47}$$

and

$$c \to c + \epsilon f^j_{(1-\lambda)}, \qquad j \leq s(\lambda) - 1\,, \tag{7.48}$$

where $f^{(\lambda)}_j$ and $f^j_{(1-\lambda)}$ are KN - differentials. The corresponding charges are

$$Q_j = \frac{1}{2\pi i} \oint\limits_{\partial\Sigma_0} c f^{(\lambda)}_j\,, \tag{7.49}$$

and

$$Q^j = \frac{1}{2\pi i} \oint\limits_{\partial\Sigma_0} b f^j_{(1-\lambda)}\,. \tag{7.50}$$

Next we develop the operator formalism on the semi - infinite cylinder (with P_+) at infinity resulting from a conformal transformation of the disc Σ^+. To the point P_+ we associate the standard vacuum $< 0|$ and the state $|\partial\Sigma_0 >$ with the boundary $\partial\Sigma_0$(*). The state $|\partial\Sigma_0 >$ is the Bogoliubov transformation of $|0 >$ (for the definition of Bogoliubov transformations and several applications see the excellent book by F. Strocchi [102]).

Let us start with the expansion of the b and c fields on the disc

$$b(z) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} b^k z^{k-\lambda} (dz)^\lambda\,, \tag{7.51}$$

$$c(z) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} c_k z^{-k-1+\lambda} (dz)^{1-\lambda}\,. \tag{7.52}$$

(*) According with notations of [31], in the chapter devoted to the global operator formalism we will associate to the points P_+ and P_- the "global" *in* and *out* vacua respectively.

Let us now write the KN - differentials considering only their expansion with respect to the point $z \equiv z_+$ (here we consider for $\lambda \in \mathbb{Z} + 1/2$ only the NS - sector and use the notation $f_j^{(\lambda)} \equiv f_j^{(\lambda,1)}$)

$$f_j^{(\lambda)}(z) = \left(z^{j-s(\lambda)} + \sum_{k=1}^{\infty} B_{jk} z^{j-s(\lambda)+k} \right) (dz)^\lambda = h_j^{(\lambda)} (dz)^\lambda , \tag{7.53}$$

$$f_{(1-\lambda)}^j = \left(z^{-j+s(\lambda)-1} + \sum_{k=1}^{\infty} \tilde{B}_{jk} z^{-j+s(\lambda)+k} \right) (dz)^{1-\lambda} = h_{(1-\lambda)}^j (dz)^{1-\lambda} , \tag{7.54}$$

where $j \in \mathbb{Z} + s(\lambda)$. The coefficients B_{jk} and $\tilde{B}_{jk}$ can be written in the following form

$$B_{jk} = \frac{1}{2\pi i} \oint_{C_\tau} dz h_j^{(\lambda)}(z) z^{-j+s(\lambda)-k-1} , \tag{7.55}$$

$$\tilde{B}_{jk} = \frac{1}{2\pi i} \oint_{C_\tau} dz h_{(1-\lambda)}^j (z) z^{j-s(\lambda)-k} . \tag{7.56}$$

In these integrals the contour C_τ is equivalent to $\partial\Sigma_0$. A straightforward computation gives

$$Q_j = c_a + B_{jl} c_{a+l} , \tag{7.57}$$

$$Q^j = b_a + \tilde{B}_{jl} b_{a-l} , \tag{7.58}$$

where $a = j + g(\lambda - 1/2)$. The next step consists in finding the expression of the Bogoliubov state $|\partial\Sigma_0>$ in terms of the b_k's and c_k's modes acting on the vacuum $|0>$. In order to find the explicit expression a fundamental relation is

$$\sum_{k,l=1}^{\infty} B_{ik} \tilde{B}_{jl} \oint_{\partial\Sigma_0} dz z^{i-j+k+l} = 0 , \tag{7.59}$$

which is just the duality condition for the KN - differentials. We omit the details and give the formal expression for correlators involving an arbitrary number of b's and c's fields

$$< X(1,...,N)\prod_{j=1}^{L}(b(z_j)c(w_j)) >= \frac{< 0|R\{X(1,...,N)\prod_{j=1}^{L}(b(z_j)c(w_j))\}|\partial\Sigma_0 >}{< 0|X(1,...,N)|\partial\Sigma_0 >}, \qquad (7.60)$$

where $X(1,...,N)$ denotes the zero - mode insertions and R is the radial ordering.

7.4 CFT AND MODULI DEFORMATIONS

In ref.[84] was shown that the ket state $|\partial\Sigma_0' >$ corresponding to the new Riemann surface Σ' (related to Σ by the transformations (7.33 - 7.35) is

$$|\partial\Sigma_0' >= T(e_i)|\partial\Sigma_0 > +|\partial\Sigma_0 >, \qquad (7.61)$$

where

$$T(e_i) = \frac{1}{2\pi i}\oint_{\partial\Sigma_0} T(z)e_i(z) = \sum_n a_n L_n , \qquad (7.62)$$

and

$$T(z) = \sum_n L_n z^{n-2}(dz)^2 .$$

To demonstrate eq.(7.61) first of all note that since

$$(Q_j + \delta Q_j)|\partial\Sigma_0' >= (Q^j + \delta Q^j)|\partial\Sigma_0' >= 0 , \qquad (7.63)$$

eq.(7.61) is equivalent to

$$\delta Q_j = \epsilon[T(e_i), Q_j] , \qquad \delta Q^j = \epsilon[T(e_i), Q^j] . \qquad (7.64)$$

On the other hand it is a standard result that

$$[L_n, b(z)] = -\left(z^{-n+1}\partial_z + \lambda(1-n)z^{-n}\right) b(z) , \qquad (7.65)$$

$$[L_n, c(z)] = -\left(z^{-n+1}\partial_z + (1-\lambda)(1-n)z^{-n}\right) c(z) , \qquad (7.66)$$

so that

$$[T(e_i), b(z)] = \mathcal{L}_{e_i} b(z) , \qquad (7.67)$$

$$[T(e_i), c(z)] = \mathcal{L}_{e_i} c(z)\,, \tag{7.68}$$

Therefore we have

$$[T(e_i), Q_j] = -\oint_{\partial\Sigma_0} f_j^{(\lambda)} \mathcal{L}_{e_i} c(z)\,, \tag{7.69}$$

$$[T(e_i), Q^j] = -\oint_{\partial\Sigma_0} f^j_{(1-\lambda)} \mathcal{L}_{e_i} b(z)\,, \tag{7.70}$$

where $\mathcal{L}$ is the Lie derivative. Let us conclude this section by summaraizing the action of the Virasoro generators on the state $|\partial\Sigma_0>$. From the previous discussion and from eq.(7.37) it follows that

$$L_n|\partial\Sigma_0>\,, \tag{7.71}$$

changes the coordinates for $n \leq 0$, moves the puncture for $n = 1$ while changes the moduli for $n = 2, ..., 3g-3$.

REMARK. It is obvious that these variations can be expressed in terms of KN - generators, in particular the relation between the Virasoro and KN algebras was given in ref.[31]. These relations involve sum with infinite terms. Nevertheless it is possible to introduce "global" and "symmetric" *in* and *out* vacua (denoted in ref.[31] by $|0)_\Sigma$, ${}_\Sigma(0|$), whit respect to which the action of the KN generators play the rôle analog of the Virasoro generators acting on the state $|\partial\Sigma_0>$.

7.5 Holomorphic Differentials On Punctured Riemann Surfaces

Now, generalizing the KN - differentials, we introduce a basis for λ - differentials holomorphic on the N punctured Riemann surface $\Sigma\backslash\{P_1, ..., P_N\}$. By Riemann - Roch theorem

$$h^0\left(K^\lambda, \Sigma\right) - h^0\left(K^{1-\lambda}, \Sigma\right) = (2\lambda - 1)(g-1)\,, \tag{7.72}$$

it follows that in order to obtain existence and uniqueness (up to a multiplicative constant) their divisor (restricted to P_i) $\sum_{i=1}^N n_i P_i$ must have degree $\sum_{i=1}^N n_i = -2s(\lambda)$. It is easy to see that this basis can be written in terms of KN basis. First of all any holomorphic (everywhere on Σ) differential can be written in terms of KN holomorphic differentials. Moreover any differential with poles only in $\{P_1, ..., P_N\}$ can be written as a (finite) linear combination of KN differentials after appropriate identifications of P_+, P_- with pairs in the set $\{P_1, ..., P_N\}$. So we introduce, following ref.[103], (see also ref.[104])(*), the differentials

$$f_{j,1}^{(\lambda)}(z) = f^{(\lambda)}(j, 0, ..., 0, -2s(\lambda) - j), \qquad j \in \mathbb{Z}, \tag{7.73}$$

where $f^{(\lambda)}(j_1, ..., j_N)$, with $j_1, ..., j_{N-1} \in \mathbb{Z}$, $j_N = -2s(\lambda) - \sum_{k=1}^{N} j_k$ is the unique (up to a multiplicative constant) holomorphic λ - differential on $\Sigma \backslash \{P_1, ..., P_N\}$ with the following expansion around each puncture P_k, $k = 1, ..., N$:

$$f^{(\lambda)}(j_1, ..., j_N) = a_k z_k^{j_k}(1 + O(z_k))(dz_k)^\lambda, \qquad z_k(P_k) = 0. \tag{7.74}$$

Unicity up to a multiplicative constant means that the only degree of freedom is the value to assign to one of the a_k's constants. After this choice all a_k's are fixed. We fix the normalization by setting $a_N = 1$. Notice that for $0 \leq j \leq -2s(\lambda)$, $f_{j,1}^{(\lambda)}(z)$ is a zero modes basis. Moreover any differential $f^{(\lambda)}(0, ..., j, ..., 0, -2s(\lambda) - j)$, $j \geq 0$, is a (finite) linear combination of $f_{j,1}^{(\lambda)}(z))$, $j \geq 0$. It is obvious that a set of linearly independent differentials for the vector space

$$F^{(\lambda)}(P_1, ..., P_N) = \{f^{(\lambda)}(j_1, ..., j_N) | j_1, ..., j_{N-1} \in \mathbb{Z},\ j_N = -2s(\lambda) - \sum_{k=1}^{N} j_k\}, \tag{7.75}$$

is given by $f_{j,1}^{(\lambda)}(z)$, $j \in \mathbb{Z}$ and

$$f_{j,n}^{(\lambda)}(z) = f^{(\lambda)}(0, .., j, .., 0, -2s(\lambda) - j), \qquad j < 0, \qquad n = 2, ..., N-1, \tag{7.76}$$

where the index j is in the $n - th$ slot. Any basis element is a KN differential:

$$f_{j,k}^{(\lambda)}(z) = f_{j-s(\lambda)}^{(\lambda)}(z|P_k, P_N), \qquad k = 1, ..., N-1, \tag{7.77}$$

where $f_{j-s(\lambda)}^{(\lambda)}(z|P_k, P_l)$ corresponds to the KN differential $f_{j-s(\lambda)}^{(\lambda)}(z)$ with the substitutions $P_- \to P_k$ and $P_+ \to P_l$.

(*) An example of holomorphic differentials on punctured Riemann surface was given in ref.[105] where the term depending from the punctures ($\neq P_\pm$) was considered as a normalization constant.

In order to define the generalized $C_\tau^{(N)}$ contours we introduce the third kind differential (we set $a_{kk} = 0$)

$$\omega^{(N)}(z) = \sum_{k\neq l} a_{kl} f^{(1)}_{-\frac{g}{2}}(z|P_k, P_l)\,. \tag{7.78}$$

It has N poles and $2g+N-2$ zeroes. On Σ we define the euclidean time to be the harmonic function

$$\tau^{(N)}(Q) = Re\int_{Q_0}^{Q} \omega^{(N)}(z)\,, \qquad \omega^{(N)}(Q_0) = 0\,. \tag{7.79}$$

We require that the residue in P_k, $k = 1, ..., N$, be non - vanishing

$$Res_{P_k}\left(\omega^{(N)}\right) = \sum_{l=1}^{N}(a_{kl} - a_{lk}) \neq 0\,, \qquad k = 1, ..., N\,. \tag{7.80}$$

Notice that the residue sum is identically zero. For $Q \to P_k$, the euclidean time has the following limit

$$\lim_{Q\to P_k} s\tau^{(N)}(Q) \to -\infty\,, \tag{7.81}$$

where s is the sign of $Res_{P_k}\left(\omega^{(N)}\right)$. It is a well - known result that a third kind differential ω_{PQ} with polar divisor $(-P - Q)$ normalized in such a way that

$$Re\oint_{a_i} \omega_{PQ} = Re\oint_{b_i} \omega_{PQ} = 0\,, \tag{7.82}$$

(we fix the multiplicative constant by setting $Res_P\left(\omega_{PQ}\right) = -Res_Q\left(\omega_{PQ}\right) = 1$) satisfies the relation

$$Re\int_{S}^{R} \omega_{PQ} = Re\int_{Q}^{P} \omega_{RS}\,. \tag{7.83}$$

Then the euclidean time can be written in the equivalent form

$$\tau^{(N)}(Q) = Re\sum_{k\neq l}\int_{P_l}^{P_k} f^{(1)}_{-\frac{g}{2}}(z|Q, Q_0)\,. \tag{7.84}$$

The level line

$$C_{\tau}^{(N)} = \{Q \in \Sigma | Re \int_{Q_0}^{Q} \omega^{(N)}(z) = \tau^{(N)}\}, \tag{7.85}$$

is a union of disjoint real closed curves. The zeroes $\{Q_1, ..., Q_{2g+N-2}\}$ of $\omega^{(N)}(z)$ correspond to string interaction points where some curves split or join. In the limit $\tau^{(N)}(Q) \to +\infty$ $(-\infty)$ the level line $C_{\tau}^{(N)}$ is a collection of p $(N-p)$ curves diffeomorphic to S^1 where p $(N-p)$ is the number of poles with negative (positive) residue. $\tau^{(N)}(Q)$ is a height Morse function. Its critical values are $C_i = \tau^{(N)}(Q_i)$. A possible choice for the residues $c_k \equiv Res_{P_k}\left(\omega^{(N)}\right)$ is[103]:

$$c_k = \frac{1}{2p}, \qquad k = 1, ..., p, \qquad c_k = -\frac{1}{2(N-p)}, \qquad k = p+1, ..., N. \tag{7.86}$$

Let us now introduce the third kind differential

$$\rho = -\frac{1}{N-1} \sum_{j=1}^{N-1} \tilde{\omega}_{P_j P_N}. \tag{7.87}$$

The associated euclidean time

$$\tilde{\tau}^{(N)} = Re \int_{Q_0}^{Q} \rho(z), \tag{7.88}$$

defines the level line

$$C_{\tilde{\tau}}^{(N)} = \{Q \in \Sigma | Re \int_{Q_0}^{Q} \rho(z) = \tilde{\tau}^{(N)}\}. \tag{7.89}$$

For $\tilde{\tau}^{(N)} \to -\infty\,(+\infty)$, $C_{\tilde{\tau}}^{(N)}$ is a (collection of) circle C^N (circles C^k) with center P_N (P_k, $k = 1, ..., N-1$). Let us now consider a $(1-\lambda)$ - differential h holomorphic on $\Sigma \backslash \{P_1, ..., P_N\}$. We can expand h in a finite linear combination of $f_{j,k}^{(1-\lambda)}$ (recall that $j \in \mathbb{Z}$ for $k = 1$ and $j < 0$ for $k = 2, ..., N-1$)

$$h = \sum_{j<0} \sum_{k=1}^{N-1} c^{jk} f_{j,k}^{(1-\lambda)} + \sum_{j \geq 0} c^{j1} f_{j,1}^{(1-\lambda)}. \tag{7.90}$$

Before computing c^{jk} notice that for a one form g belonging to $F^{(1)}(P_1, ..., P_N)$ we have

$$\sum_{k=1}^{N-1} \frac{1}{2\pi i} \oint\limits_{C^k} g = \frac{1}{2\pi i} \oint\limits_{C_\tau^{(N)}} g = \frac{1}{2\pi i} \oint\limits_{C^N} g \,. \tag{7.91}$$

The coefficients are

$$c^{j1} = \frac{1}{2\pi i} \oint\limits_{C^N} h f^{(\lambda)}_{-j-1,1} = \frac{1}{2\pi i} \oint\limits_{C_\tau^{(N)}} h f^{(\lambda)}_{-j-1,1}, \qquad j \geq 0 \,, \tag{7.92}$$

$$c^{j1} = \frac{1}{2\pi i} \oint\limits_{C^1} h f^{(\lambda)}_{-j-1,1}, \qquad j < 0 \,, \tag{7.93}$$

$$c^{-1k} = \frac{1}{2\pi i} \oint\limits_{C^k} h f^{(\lambda)}_{0,1} \,, \tag{7.94}$$

$$c^{-2k} = \frac{1}{2\pi i} \oint\limits_{C^k} h f^{(\lambda)}_{1,1} - c^{-1k} \alpha_{1,-1,k}(\lambda) \,, \tag{7.95}$$

......

$$c^{-rk} = \frac{1}{2\pi i} \oint\limits_{C^k} h f^{(\lambda)}_{r-1,1} - \sum_{l=1}^{r-1} c^{-lk} \alpha_{r-1,-l,k}(\lambda) \,. \tag{7.96}$$

With respect to the $N = 2$ case there is the new term

$$\alpha_{r,l,k}(\lambda) = \frac{1}{2\pi i} \oint\limits_{C^k} f^{(\lambda)}_{r,1} f^{(1-\lambda)}_{l,k}, \qquad r \geq 0, \quad l < 0, \quad k = 2, ..., N-1 \,, \tag{7.97}$$

a straightforward computation gives $\alpha_{j,-j-1,k}(\lambda) = 1$ and $\alpha_{j,l,k}(\lambda) = 0$ for $l < -j-1$.

7.6 Holomorphic Differentials On Punctured Riemann Surfaces And Light - Cone Diagrams

The differentials with more than two - punctures play a fundamental rôle in formulating both string (see for example [75,106,107,108]) and conformal field theories[109−115] in higher genus.

In this section we discuss a physical application for these differentials. In particular we show how the global time $\tau^{(N)}$ can be used in formulating the light - cone diagram for the closed bosonic string in D - dimension.

Let us introduce the light - cone coordinates

$$X^{\pm} = \left(X^0 \pm X^{D-1}\right)/\sqrt{2}, \tag{7.98}$$

The light - cone gauge corresponds to the choice

$$X^{+}(\sigma,\tau) = P^{+}\tau + \ constant\,. \tag{7.99}$$

In the Mandelstam formulation of the interacting string picture one retain only the transverse, physical, positive norm excitations of the string. Because one can write a self - adjoint hamiltonian, the corresponding quantized theory turns out to be unitary. Moreover in the critical dimension it is also Poincaré covariant. In [116] D'Hoker and Giddings have shown the unitarity of the scattering amplitudes of the Polyakov closed bosonic string (integrated over a single copy of moduli space) demonstrating the equivalence with the light - cone interacting string. Now we show how the third kind differentials play first fiddle in the light - cone formulation. A light - cone diagram consists of flat tubes (each one representing the bosonic propagator) except for curvature singularities at the string interaction points $\{Q_1, ..., Q_{2g+N-2}\}$. In the light - cone gauge the external (internal) string circumference $2\pi i\alpha_i$ $(2\pi i\beta_i)$ is proportional to the P_i^+ (Q_i^+) component of the string momentum. The moduli characterizing a given diagram are

$$\alpha_i = P_i^+ = \frac{1}{2\pi i}\oint\limits_{C^i} \omega^{(N)}, \qquad i = 1, ..., N\,, \tag{7.100}$$

$$\beta_i = Q_i^+ = \frac{1}{2\pi i}\oint\limits_{a_i} \omega^{(N)}, \qquad i = 1, ..., g\,, \tag{7.101}$$

$$\gamma_i \in [0, 2\pi) \qquad i = 1, ..., 3g+N-3, \qquad internal\ string\ twist\ angles\,, \tag{7.102}$$

$$\tau_i^{(N)} \equiv \tau^{(N)}(Q_i) \qquad i = 1, ..., 2g+N-2\,, \qquad interaction\ times\,. \tag{7.103}$$

Notice that the condition of vanishing of the residue summation for one differentials corresponds to the conservation of the total external momentum. Really the number of interactions is equal to the number of zeroes of $\tau_i^{(N)}$ i.e. $2g+N-2$; on the other hand the interaction times $\tau_i^{(N)}$ are defined up to a overall shift that can be used to set $\tau_1^{(N)}=0$. By a conformal transformation, the tubes corresponding to the external states are identified with the punctures $P_1,...,P_N$. Fixing the external momenta P_i^+ corresponds to fix the $\omega^{(N)}$ residues. In this case the number of parameters is just $6g+2N-6$ which is the real dimension of the moduli space of N punctured Riemann surfaces. In [117] Giddings and Wolpert showed that the light - cone diagram provides a single copy of moduli space. Moreover any fixed light - cone diagram always defines an abelian differential and viceversa. In order to illustrate how it is possible to define a flat structure starting from $\omega^{(N)}$, let us introduce the coordinate z defined outside the divisor of $\omega^{(N)}$:

$$z=\int_{Q_0}^{Q}\omega^{(N)}\,. \tag{7.104}$$

This coordinate can be used to define on $\Sigma\backslash\{P_1,...,P_N,Q_1,...,Q_{2g+N-2}\}$ the flat metric

$$\hat{g}_{flat}=dzd\bar{z}=|\omega^{(N)}|^2\,. \tag{7.105}$$

At the zeroes Q_i we have $dz\sim(Q-Q_i)dQ$ that is $(z-z_0)\sim(Q-Q_i)^2$. At the poles $z\sim ln(Q-Q_0)$ so the mapping (7.104) corresponds to take a small disc whose center is P_i to a tube propagating off to infinity in the z coordinate. The choice of all imaginary periods for $\omega^{(N)}$ is crucial for invariance under the mapping class group. Unlike the abelian differentials with a real part contribution of their periods, differentials with purely imaginary periods depend only on the conformal structure (see [36]). In the case in which a differential with purely imaginary periods has no poles we find that the corresponding Riemann surface is degenerate i.e.

$$\det(Im\Omega_{ij})=0\,. \tag{7.106}$$

This case corresponds to a Riemann surface with one handle pinched off to form two punctures.

8. b - c SYSTEM APPROACH TO MINIMAL MODELS

Here we discuss a new method based on the use of b - c systems for the study of minimal models[118,105]. The reason the method works lies in the fact that the bosonized version of suitable anticommuting b - c systems coincides with the (chiral) bosonized versions of the minimal models (Coulomb gas[119]). Duality between bosons and fermions in two dimensions is a well established property. Free fermion theories can be bosonized on any Riemann surface[120−123]. The natural generalization of this is to express free chiral anticommuting b - c systems and commuting $\beta - \gamma$ systems in terms of bosonic ones. While this bosonization has been carried out for b - c systems (or β - γ systems) with integral or half integral weight, little is known so far for systems of generic weight. The problem is not merely academic. Take for example, the rational conformal field theories corresponding to the series of minimal models. It is well - known from the work of Feigin - Fuks[124] and in a more explicit way from Dotsenko - Fateev's[119] that such models can be formulated as bosonic field theories with a (in general irrational) charge at infinity. This fact brought about an important improvement in the study of minimal models as it allowed to calculate correlation functions through an integral procedure instead of as solutions of complicated differential equations.

It is evident from the work of Feigin and Fuks that there is another formulation beside the bosonic one. In ref.[124] it appears in terms of semiinfinite forms constructed starting from bases of real (or complex) weight tensors. Now these bases can be envisaged as bases of b - c systems. Therefore one expects that there exists also a second field theory formulation (beside the Coulomb gas one). This was indeed proved, for the genus zero case, in ref.[118], where we introduced a b - c system of suitable (in general irrational) weight, to describe any given minimal model, and we eventually arrived at the same results as with the Coulomb gas method.

We notice that, in the light of recent works on Wess - Zumino - Novikov - Witten[125] and coset models in which the latter are formulated in terms of suitable bosonic fields, the latter models too can probably be formulated in terms of suitable b - c and β - γ systems.

The aim of the present chapter is to review the formalism of refs.[118,105] We deal first of all the genus zero case. In particular, inspired by the calculations of the spin field correlation functions, we apply these results to calculate correlation functions of field insertions whose interpretation depends on the particular b - c system we are considering. To this end we will introduce suitable b - c systems, essentially characterized by the bases over which we expand them. The bases contain the information concerning the field insertions we want to describe.

Then we generalize this approach to higher genus using the real - weight differentials defined in chapter 2. We will test our formalism by rederiving known results of spin field correlation functions and generalize them, by calculating correlation functions of spin fields for rational λ. In this case field insertions represent exactly spin fields. Finally we will apply our method to bases of b - c systems ("fat" bases) such that the corresponding field insertions represent chiral vertex operators in minimal models.

A few distinctive features of our approach deserve to be pointed out. Our bases allow us to define bra and ket vacua both depending on the Riemann surface. This is why we can generalize to higher genus the approach on the sphere discussed in section 1. Furthermore we obtain the conservation of the total charge in the correlation functions as a consequence of geometrical consistency on the basis (the total charge corresponds to the higher genus "charge at infinity" of the Coulomb gas approach). This is an example of the connection between conformal field theory and geometry exposed by the b - c system approach.

8.1 The Genus Zero Case

In this section we treat the genus 0 case. The higher genus case will be discussed later on. In fact one of the virtues of our approach is that it appears as the natural framework for the generalization to higher genus Riemann surfaces, as it automatically takes into account the geometrical factors such as curvature and holonomy.

Let us introduce first a real weight chiral b - c system and the corresponding bases. This system is described in terms of two anti - commuting fields b and c of weight λ and $(1-\lambda)$ respectively. Unless otherwise specified it will be understood that λ is a real number. Here we consider the anticommuting case. The action is

$$S = \int_{\Sigma} b\bar{\partial}c, \tag{8.1}$$

In the operator formalism these fields are expanded in a basis of λ - differentials in such a way that the equations of motion $\bar{\partial}b = 0$ and $\bar{\partial}c = 0$ are satisfied everywhere except possibly at $P_{\pm}$.

The energy momentum tensor of the system is

$$T = (1-\lambda)\partial bc - \lambda b\partial c\,. \tag{8.2}$$

Different bases for the b fields on the sphere are given by:

$$f_j^{(\lambda,l)}(z) = (P_+ - P_-)^{j+\lambda}(z-P_+)^{j-\lambda}(z-P_-)^{-j-\lambda}(dz)^\lambda h_j^{(\lambda,1)}(z)(dz)^\lambda\,, \tag{8.3}$$

where $j \in \mathbb{Z} + \lambda l$ and $l \in \mathbb{Z}$ is a sector index. When going once along a closed curve C around P_+ (separating P_+ from P_-) we have

$$h_j^{(\lambda,1)}(z) \longrightarrow e^{2\pi i\lambda(l\mp 1)}h_j^{(\lambda,1)}(z)\,. \tag{8.4}$$

When λ is rational, i.e. $\lambda = m/n$, with m, n relatively prime integers, then there are only n distinct sectors $l = 1, 2, ..., n$. When λ is half - integer we have the Neveu - Schwarz ($l = 1$) and the Ramond ($l = 2$) sectors. For the c fields we have similarly the bases

$$f^j_{(1-\lambda,l)}(z) = (P_+ - P_-)^{-j-\lambda+1}(z-P_+)^{-j+\lambda-1}(z-P_-)^{j+\lambda-1}(dz)^{1-\lambda}\,, \tag{8.5}$$

with the same conventions as above, so that the following duality relation holds:

$$\frac{1}{2\pi i}\oint_C f^i_{(1-\lambda,l)}(z)f_j^{(\lambda,l)}(z) = \delta^i_j\,, \tag{8.6}$$

where $i, j \in \mathbb{Z} + \lambda l$. Notice that for integer λ eq.(8.5) gives the $2\lambda - 1$ zero - modes of the c fields.

We expand now, in each sector l, the fields b and c :

$$b^l(z) = \sum_j b^{j,l} f_j^{(\lambda,l)}(z)\,, \qquad c^l(z) = \sum_j c^l_j f^j_{(1-\lambda,l)}(z)\,, \tag{8.7}$$

and assume the following anticommutation relations

$$\{b^{i,l}, c^l_j\} = \delta^i_j\,, \qquad \{b^{i,l}, b^{j,l}\} = \{c^l_i, c^l_j\} = 0\,. \tag{8.8}$$

For each sector the vacuum is defined as follows [32]:

$$\begin{aligned} b^{j,l}|0>_l = {}_l<0|c_j{}^l = 0\,, &\qquad for\; j \leq \lambda + \overline{\lambda(l-1)} - 1\,; \\ c^l_j|0>_l = {}_l<0|b^{j,l} = 0, &\qquad for\; j \geq \lambda + \overline{\lambda(l-1)}\,. \end{aligned} \tag{8.9}$$

The bar denotes the non - integer part of $\lambda(l-1)$. When λ is integer or half - integer the bra vacuum takes into account the zero - modes insertions [31], moreover ${}_l<0|0>_l = 1$. This choice of the vacuum comes from the requirement that $b^l(z)|0>_l$ be non singular in $z = P_+$; on the other hand the first value of j corresponding to non - negative frequency in the expansion of $b(z)$ is just

$$j = \lambda + \overline{\lambda(l-1)}\,. \tag{8.10}$$

The propagator is defined as follows:

$$\begin{aligned} S^{(\lambda,l)}(z,w) &\equiv {}_l<0|R\left(b^l(z)c^l(w)\right)|0>_l = \\ &= \begin{cases} {}_l<0|b^l(z)c^l(w)|0>_l\,, & \text{if } |\zeta(z)| > |\zeta(w)|\,; \\ -{}_l<0|c^l(w)b^l(z)|0>_l\,, & \text{if } |\zeta(w)| > |\zeta(z)|\,. \end{cases} \end{aligned} \tag{8.11}$$

where R denotes the radial ordered product and $\zeta(x) = \dfrac{x-P_+}{x-P_-}$. Inserting the expansion (8.7) in eq.(8.11) we get

$$S^{(\lambda,l)}(z,w) = \begin{cases} \sum_{j\leq\lambda+\overline{\lambda(l-1)}-1} f_j^{(\lambda,l)}(z) f^j_{(1-\lambda,l)}(w)\,, & \text{if } |\zeta(z)| > |\zeta(w)|\,; \\ -\sum_{j\geq\lambda+\overline{\lambda(l-1)}} f_j^{(\lambda,l)}(z) f^j_{(1-\lambda,l)}(w)\,, & \text{if } |\zeta(w)| > |\zeta(z)|\,. \end{cases} \tag{8.12}$$

Then the expression for the propagator $S^{(\lambda,l)}(z,w)$ is:

$$S^{(\lambda,l)}(z,w) = \frac{1}{(z-w)}\left(\frac{z-P_+}{w-P_+}\right)^{\overline{\lambda(l-1)}}\left(\frac{z-P_-}{w-P_-}\right)^{-2\lambda+1-\overline{\lambda(l-1)}}(dz)^{\lambda}(dw)^{1-\lambda}\,. \tag{8.13}$$

It is easy to see that there is a direct connection of $S^{(\lambda,l)}(z,w)$ with the covariant delta function for λ differentials in the sector l

$$\Delta^l(z,w) = \sum_j f_j^{(\lambda,l)}(z) f^j_{(1-\lambda,l)}(w)\,,\,. \tag{8.14}$$

i.e.

$$g(z) = \oint_C \Delta^l(z,w) g(w)\,, \tag{8.15}$$

where $g(z)$ is any smooth λ - differential with the multivaluedness given in (8.4).

Our aim now is to generalize the bases (8.7) in such a way as to record in the bases themselves the insertions of suitable fields which will be assimilated to the vertex operators of the bosonized formulation. To this end we notice that the propagator $S^{(\lambda,l)}(z,w)$ can be seen as the propagator in the sector $l = 1$ with the insertion of a "λ - spin fields" at the points $P_\pm$:

$$S^{(\lambda,l)}(z,w) = \frac{{}_1<0|R\left(S^{-,l}(P_-)b^1(z)c^1(w)S^{+,l}(P_+)\right)|0>_1}{{}_1<0|R\left(S^{-,l}(P_-)S^{+,l}(P_+)\right)|0>_1} \tag{8.16}$$

where $S^{\pm,1}$ are the identity operators. The factors $\left(\frac{z-P_+}{w-P_+}\right)^{\overline{\lambda(l-1)}}$ and $\left(\frac{z-P_-}{w-P_-}\right)^{-\overline{\lambda(l-1)}}$ in eq.(8.13) are the effects of the λ-spin field insertions $S^{+,l}(P_+)$ and $S^{-,l}(P_-)$. The weights of the λ - spin fields can be calculated and are equal to

$$\pm\frac{1}{2}\overline{\lambda(l-1)}\left[\pm\overline{\lambda(l-1)}+2\lambda-1\right]. \tag{8.17}$$

The factor $\left(\frac{z-P_-}{w-P_-}\right)^{1-2\lambda}$ is due to the vacuum charge (see below). In order to reproduce (8.16) with the insertion of the λ-spin fields and of the charge in arbitrary points on the sphere we have to modify the bases in the following multiplicative way:

$$\begin{aligned}\tilde{f}_j^{(\lambda,l)}(z) = {} & (P_+-P_-)^{j-\lambda+1-\overline{\lambda(l-1)}}(z-P_+)^{j-\lambda-\overline{\lambda(l-1)}}(z-P_-)^{-j+\lambda-1+\overline{\lambda(l-1)}}\cdot \\ & \cdot(z-P_1)^{\overline{\lambda(l-1)}}(z-P_2)^{-\overline{\lambda(l-1)}}(z-P_3)^{1-2\lambda}(dz)^\lambda\,,\end{aligned} \tag{8.18}$$

$$\begin{aligned}\tilde{f}^j_{(1-\lambda,l)}(z) = {} & (P_+-P_-)^{-j+\lambda+\overline{\lambda(l-1)}}(z-P_+)^{-j-1+\lambda+\overline{\lambda(l-1)}}(z-P_-)^{j-\lambda-\overline{\lambda(l-1)}}\cdot \\ & \cdot(z-P_1)^{-\overline{\lambda(l-1)}}(z-P_2)^{\overline{\lambda(l-1)}}(z-P_3)^{2\lambda-1}(dz)^{1-\lambda}\,.\end{aligned} \tag{8.19}$$

The exponents of $(z-P_\pm)$ in eqs.(8.18 - 8.19) is integer and l - independent. In particular

$$\begin{aligned}(z-P_+)^{j-\lambda-\overline{\lambda(l-1)}}(z-P_-)^{-j+\lambda-1+\overline{\lambda(l-1)}} &= (z-P_+)^k(z-P_-)^{-k-1}\,, \\ k = (j-\lambda-\overline{\lambda(l-1)}) &\in \mathbb{Z}\,.\end{aligned} \tag{8.20}$$

In this formalism we have only the vacuum $|0>_1$(*).

Using these modified bases we obtain the propagator

$$\tilde{S}^{(\lambda,l)}(z,w)=\frac{1}{(z-w)}\left(\frac{z-P_1}{w-P_1}\right)^{\overline{\lambda(l-1)}}\left(\frac{z-P_2}{w-P_2}\right)^{-\overline{\lambda(l-1)}}\left(\frac{z-P_3}{w-P_3}\right)^{1-2\lambda}(dz)^{\lambda}(dw)^{1-\lambda} \tag{8.21}$$

Notice that with the use of the bases in eqs.(8.18 - 8.19) the propagator is independent of the points $P_\pm$ where the bra and ket vacua are defined. The sum $(1-2\lambda)$ of the exponents in eq.(8.21) plays the role of the total charge. When λ is an integer there are not λ-spin fields and the factor $\left(\frac{z-P_3}{w-P_3}\right)^{-2\lambda+1}$ is just the effect of the zero - mode insertions, the number $1-2\lambda$ being the Riemann - Roch index on the sphere. The c zero - modes can be inserted at arbitrary points on the sphere. In our formalism this is achieved by a further modification of the bases:

$$\hat{f}_j^{(\lambda,1)}(z)=(P_+-P_-)^{j-\lambda+1}(z-P_+)^{j-\lambda}(z-P_-)^{-j+\lambda-1}\prod_{i=1}^{2\lambda-1}(z-P_i)^{-1}(dz)^{\lambda}, \tag{8.22}$$

and

$$\hat{f}^j_{(1-\lambda,1)}(z)=(P_+-P_-)^{-j+\lambda}(z-P_+)^{-j+\lambda-1}(z-P_-)^{j-\lambda}\prod_{i=1}^{2\lambda-1}(z-P_i)(dz)^{1-\lambda}. \tag{8.23}$$

The propagator becomes

$$\begin{aligned}\hat{S}^{(\lambda,1)}(z,w)&=\frac{1}{(z-w)}\prod_{i=1}^{2\lambda-1}\left(\frac{w-P_i}{z-P_i}\right)(dz)^{\lambda}(dw)^{1-\lambda}=\\&={}_1<0|R\left(b^1(z)c^1(w)\right)|0>_1=\frac{<\prod_i c(P_i)b(z)c(w)>}{<\prod_i c(P_i)>}.\end{aligned} \tag{8.24}$$

The RHS refers to the standard formalism. From eq.(8.21), for $\lambda\in\mathbb{R}$, the charge $1-2\lambda$ can be seen as a sort of generalized Riemann-Roch index. As we will show later this is true in any genus. This result plays an essential role in our approach; in particular the background charge for the bosonized version of the minimal models in higher genus is just

(*) With this basis the first non-negative frequency in P_+ in the expansion of $b(z)|0>$ is just $(z-P_+)^0=1$. This reflects the fact that we have inserted the λ - spin fields away from the points $P_\pm$ where the bra and ket vacua are defined.

equal to the generalized Riemann - Roch index $(2\lambda - 1)(g-1)$, $\lambda \in \mathbb{R}$[32] As a natural extension of the above procedure we introduce a "fat" b - c system (we denote it as B - C system). To do this we reshuffle the location of the λ - spin fields and of the zero modes and represent them as insertions of V - fields at points P_i. As we will show these fields $V^i(P_i)$ are to be assimilated to (the chiral part of) insertions of vertex operators $: e^{i\alpha_i \phi(P_i)} :$; namely

$$S(z,w) = {}_1 < 0|R(B(z)C(w))\,|0>_1 = \frac{<\prod_k V^k(P_k) b(z) c(w)>}{<\prod_k V^k(P_k)>} \,. \tag{8.25}$$

These insertions are obtained by a further modification of the b - c system bases.

For the B field we have the expansion:

$$B(z) = \sum_j B^j g_j^{(\lambda)}(z)\,, \tag{8.26}$$

$$g_j^{(\lambda)}(z) = (P_+ - P_-)^{j-\lambda+1} \frac{(z-P_+)^{j-\lambda}}{(z-P_-)^{j-\lambda+1}} \prod_{i=1}^{n} (z-P_i)^{\tilde{\alpha}_i} (dz)^{\lambda}\,. \tag{8.27}$$

The corresponding expansion for the dual C fields is

$$C(z) = \sum_j C_j g^j_{(1-\lambda)}(z)\,, \tag{8.28}$$

$$g^j_{(1-\lambda)}(z) = (P_+ - P_-)^{-j+\lambda} \frac{(z-P_+)^{-j+\lambda-1}}{(z-P_-)^{-j+\lambda}} \prod_{i=1}^{n} (z-P_i)^{-\tilde{\alpha}_i} (dz)^{1-\lambda}\,, \tag{8.29}$$

where $j \in \mathbb{Z} + \lambda$. As we have seen the total charge conservation gives the constraint

$$\sum_i \tilde{\alpha}_i = 1 - 2\lambda\,. \tag{8.30}$$

The propagator $S(z,w)$ is given by:

$$S(z,w) = \frac{1}{(z-w)} \prod_i \left(\frac{z-P_i}{w-P_i}\right)^{\tilde{\alpha}_i} (dz)^{\lambda} (dw)^{1-\lambda}\,. \tag{8.31}$$

We now compute, with a procedure similar to the one introduced by Dixon *et al.* in [126], the correlation functions $<\prod_i V^i(P_i)>$. The stress-energy tensor satisfies the following OPE with any weight h primary field $V(P)$:

$$T(z)V(P) = \frac{hV(P)}{(z-P)^2} + \frac{1}{(z-P)}\partial_P V(P) \,. \tag{8.32}$$

Therefore

$$< T(z)V^1(P_1)...V^n(P_n) >= \sum_i [\frac{h_i}{(z-P_i)^2} + \frac{\partial_{P_i}}{(z-P_i)}] < V^1(P_1)...V^n(P_n) > \,. \tag{8.33}$$

(8.33) gives us a set of first order differential equations which can be solved to obtain the correlator $< V^1(P_1)...V^n(P_n) >$. In our case we have

$$\frac{< T(z)V^1(P_1)...V^n(P_n) >}{< V^1(P_1)...V^n(P_n) >} = \lim_{z\to w} [(1-\lambda)\partial_z - \lambda\partial_w][S(z,w) - \frac{1}{z-w}] \,. \tag{8.34}$$

By solving the equations we get

$$< \prod_i V^i(P_i) >= const \prod_{i<j} (z_{P_i} - z_{P_j})^{\tilde{\alpha}_i\tilde{\alpha}_j} \,. \tag{8.35}$$

The conformal weight of the vertex operator in P_i is given by(*):

$$h_{\tilde{\alpha}_i} = \frac{1}{2}\tilde{\alpha}_i(\tilde{\alpha}_i + 2\lambda - 1) \,. \tag{8.36}$$

The same conformal weight is obtained both from $\tilde{\alpha}_i$ and from $2\tilde{\alpha}_0 - \tilde{\alpha}_i$ (here $2\tilde{\alpha}_0 \equiv 1-2\lambda$). The central charge of the fermionic system is:

$$c(\lambda) = -12\lambda^2 + 12\lambda - 2 = 1 - 12\tilde{\alpha}_0^2 \,. \tag{8.37}$$

At this point it is evident that we have made contact with the Coulomb gas formalism. We recall a few basic facts about it. The stress energy tensor is

$$T_{zz} = -\frac{1}{4}\partial_z\phi\partial_z\phi + i\alpha_0\partial^2{}_z\phi \,, \tag{8.38}$$

where $-2\alpha_0$ is the charge placed at ∞. The central charge is

$$c = 1 - 24\alpha^2{}_0 \,. \tag{8.39}$$

The primary fields are represented by vertex operators $V_\alpha =: e^{i\alpha\phi}$: of conformal weight

$$h_{\alpha_i} = \alpha_i{}^2 - 2\alpha_i\alpha_0 = h_{2\alpha_0 - \alpha_i} \,. \tag{8.40}$$

(*) For $\lambda \in \frac{\mathbb{Z}}{2}$ a similar formula can be found in ref.[127].

Therefore the relation between the two formalisms is established by setting:

$$\tilde{\alpha}_i = \sqrt{2}\alpha_i\,, \qquad \tilde{\alpha}_0 = \sqrt{2}\alpha_0\,. \tag{8.41}$$

In particular for the chiral part we have:

$$< \prod_i V^i(P_i) >=< \prod_i V_{\alpha_i}(P_i) > \,. \tag{8.42}$$

We notice however that the condition

$$\sum\nolimits_i \alpha_i = 2\alpha_0\,, \tag{8.43}$$

for the non - vanishing of the correlator in the Coulomb gas case, is obtained in the b - c formalism as a topological prescription (generalized Riemann - Roch index). From now on the two formalisms proceed in a parallel way. In order to fulfill the condition (8.30) we have in general to introduce screening charges in the correlators. A screening charge is determined by the condition $h_{\tilde{\alpha}} = 1$, i.e.

$$\tilde{\alpha} = \tilde{\alpha}_\pm = \tilde{\alpha}_0 \pm \sqrt{\tilde{\alpha}_0^2 + 2}\,. \tag{8.44}$$

As a consequence the allowed charges $\tilde{\alpha}$ are quantized:

$$\tilde{\alpha}_{r,s} = \frac{1}{2}(1-r)\tilde{\alpha}_+ + \frac{1}{2}(1-s)\tilde{\alpha}_-\,. \tag{8.45}$$

In particular for the unitary series we have

$$c = 1 - \frac{6}{p(p+1)} \qquad \tilde{\alpha}_0 = \pm\frac{1}{\sqrt{2p(p+1)}}\,. \tag{8.46}$$

Therefore the relevant b - c system will have

$$\lambda = \frac{1}{2} \pm \frac{1}{\sqrt{2p(p+1)}}\,. \tag{8.47}$$

So, in general λ will be irrational, apart from special values ($p = 8, 49, 288, \ldots$).

In conclusion we have reproduced the results of the Coulomb gas approach by means of anticommuting b - c systems. We remark that our approach is strictly related to the

Feigin - Fuchs analysis[124]. The bases (2.2) are of the type considered by Feigin and Fuchs to construct modules of the Virasoro algebra. The condition (8.45), as is well - known, denotes the existence of singular vectors. The similarity of the field theory language and the formalism of semi - infinite forms deserves further investigations.

8.2.1 The Higher Genus Case

As in the genus zero case we start with the chiral anticommuting b - c system. The anticommutation relations are similar to eq.(8.8)

$$\{b^{i,l}, c_j^l\} = \delta_j^i, \qquad \{b^{i,l}, b^{j,l}\} = 0 = \{c_i^l, c_j^l\}, \tag{8.48}$$

the unique difference is that now: $j \in \mathbb{Z} + s(\lambda) + \lambda(l-1)$, $l \in \mathbb{Z}$, $\lambda \in \mathbb{R}$. In the operator formalism these fields are expanded in a basis of λ - differentials in such a way that the equations of motion $\bar{\partial} b = 0$ and $\bar{\partial} c = 0$ are satisfied everywhere except possibly at $P_\pm$. We use the differentials defined in chapter 2 as bases to expand the b and c fields

$$b^{(l)}(z) = \sum_j b^{j,l} f_j^{(\lambda,l)}(z), \qquad c^{(l)}(z) = \sum_j c_j^l f^j_{(1-\lambda,l)}(z). \tag{8.49}$$

For any $\lambda \in \mathbb{R}$ we define the vacuum $|0>_l$ associated to Σ, following ref.[32] (for $\lambda = 0, 1$ or $g = 1$ there are some modifications that for brevity we do not consider)

$$\begin{aligned} b^{j,l}|0>_l = {}_l<0|c_j^l = 0, &\qquad for \;\; j \le s(\lambda) + \mu(l) - 1; \\ c_j^l|0>_l = {}_l<0|b^{j,l} = 0, &\qquad for \;\; j \ge s(\lambda) + \mu(l), \end{aligned} \tag{8.50}$$

$${}_l<0|0>_l = 1,$$

$$\mu(l) = \overline{\lambda(l-1)},$$

where the bar denotes the non integral part of $\lambda(l-1)$. Notice that the requirement ${}_l<0|0>_l = 1$ is consistent with the algebra (8.48).

Since they are going to play a very important role in the following, let us digress a bit on the properties of the vacua we have just defined. They are different from the vacua one usually meets in the literature. First of all both $|0>_l$ and ${}_l<0|$ depend on the moduli of Σ. The second remark is that the vacuum ${}_1<0|$ accounts for the insertion of the total charge $Q(\lambda)$, which is entirely concentrated in P_-. The subscript l means a further insertion of "λ-spin fields" $S^{-,l}(P_-)$ and $S^{+,l}(P_+)$ in ${}_1<0|$ and $|0>_1$, respectively. In particular as we will show below we have

$$|0>_l = S^{+,l}(P_+)|0>_1\,, \qquad {}_l<0| = \frac{{}_1<0|S^{-,l}(P_-)}{{}_1<0|S^{-,l}(P_-)S^{+,l}(P_+)|0>_1}\,, \tag{8.51}$$

where $S^{-,1}(P_-)$ and $S^{+,1}(P_+)$ are the identity operators. We recall that for $Q(\lambda)\in\mathbb{Z}$ the higher genus generalization of the standard (i.e. without zero mode insertion) vacuum are defined by the requirement that $b^{(1)}(z)|0>_\Sigma$, $c^{(1)}(z)|0>_\Sigma$ and ${}_\Sigma<0|b^{(1)}(z)$, ${}_\Sigma<0|c^{(1)}(z)$ be holomorphic in P_+ and P_- respectively. This condition gives $|0>_\Sigma = |0>_1$ and

$$\begin{aligned} {}_\Sigma<0|c_j^1 &= 0\,, \qquad for\;\; j\le -s(\lambda)\,; \\ {}_\Sigma<0|b^{j,1} &= 0\,, \qquad for\;\; j\ge 1-s(\lambda)\,. \end{aligned} \tag{8.52}$$

If $Q(\lambda)\notin\mathbb{Z}$, it is not possible to define a bra vacuum such that the exponents of z_- in the expansion of ${}_\Sigma<0|b^{(1)}(z)$, ${}_\Sigma<0|c^{(1)}(z)$ be integral. The solution of this problem is to use a modified basis, where an "amount $Q(\lambda)$ of singularity" in P_- of the differential $f_j^{(\lambda,1)}(z_\pm)$ $(f^j_{(1-\lambda,1)}(z_\pm))$ is shifted to other points. As we will show, this corresponds to a shift in the location of the charge $Q(\lambda)$. Moreover, using this modified basis in the expansion of the fields $b^{(1)}(z)$ and $c^{(1)}(w)$, the vacuum ${}_1<0|$ is defined by the holomorphicity condition on ${}_1<0|b^{(1)}(z)$, ${}_1<0|c^{(1)}(z)$ in P_-. An analogous argument holds for the vacua in an arbitrary sector l. The bases representing λ-spin field insertions at arbitrary points will be discussed later.

After this digression concerning the vacua (8.50), let us compute the following propagator

$$S^{(l)}(z,w)\equiv {}_l<0|\mathcal{R}(b^{(l)}(z)c^{(l)}(w))|0>_l = \begin{cases} {}_l<0|b^{(l)}(z)c^{(l)}(w)|0>_l\,, & \text{if } \tau_z>\tau_w\,; \\ -{}_l<0|c^{(l)}(w)b^{(l)}(z)|0>_l\,, & \text{if } \tau_w>\tau_z\,. \end{cases} \tag{8.53}$$

Inserting the expansions (8.49) into eq.(8.53) we obtain

$$S^{(l)}(z,w) = \begin{cases} \sum_{j\le s(\lambda)+\mu(l)-1} f_j^{(\lambda,l)}(z) f^j_{(1-\lambda,l)}(w)\,, & \text{if } \tau_z > \tau_w\,; \\ -\sum_{j\ge s(\lambda)+\mu(l)} f_j^{(\lambda,l)}(z) f^j_{(1-\lambda,l)}(w)\,, & \text{if } \tau_w > \tau_z\,. \end{cases} \tag{8.54}$$

To evaluate $S^{(l)}(z,w)$ one looks at the behaviour of the right hand side of eq.(8.54) in a neighborhood of $P_\pm$ and in the limit $z \to w$. A careful analysis similar to the one carried out in [31] shows that

$$S^{(l)}(z,w) = \frac{1}{E(z,w)} \left(\frac{E(z,P_-)}{E(w,P_-)}\right)^{Q(\lambda)-\mu(l)} \left(\frac{E(z,P_+)}{E(w,P_+)}\right)^{\mu(l)} \cdot$$

$$\cdot \left(\frac{\sigma(z)}{\sigma(w)}\right)^{2\lambda-1} \frac{\theta[^\delta_\epsilon](z-w+(Q(\lambda)-\mu(l))P_- + \mu(l)P_+ - (2\lambda-1)\Delta)}{\theta[^\delta_\epsilon]((Q(\lambda)-\mu(l))P_- + \mu(l)P_+ - (2\lambda-1)\Delta)}\,. \tag{8.55}$$

That equations (8.54) and (8.55) coincide, can be seen also in another way: one considers the propagator $S^{(l)}(z,w)$ in (8.55) as λ - differential in z and expands it in the basis $f_j^{(\lambda)}(z)$

$$S^{(l)}(z,w) = \sum_j a^j(w) f_j^{(\lambda)}(z)\,, \tag{8.56}$$

where

$$a^j(w) = \frac{1}{2\pi i} \oint_{C_\tau} S^{(l)}(z,w) f^j_{(1-\lambda)}(z)\,, \tag{8.57}$$

It is easy to verify that

$$a^j(w) = \begin{cases} f^j_{(1-\lambda)}(w)\,, & \text{if } j \le s(\lambda)+\mu(l)\,, \\ 0\,, & \text{if } j \ge s(\lambda)+\mu(l)+1\,, \end{cases} \qquad \tau_z > \tau_w\,; \tag{8.58}$$

$$a^j(w) = \begin{cases} 0\,, & \text{if } j \le s(\lambda)+\mu(l)\,, \\ -f^j_{(1-\lambda)}(w)\,, & \text{if } j \ge s(\lambda)+\mu(l)+1\,, \end{cases} \qquad \tau_w > \tau_z\,. \tag{8.59}$$

Notice also that $S^{(l)}(z,w)$ can be seen as the generalization of the Szegö kernel to λ - and $(1-\lambda)$ - differentials in the w and z variables respectively.

Let us come now to the main point of this section (and a crucial point of our construction), i.e. generalizing the bases (2.15 - 2.17) in such a way as to record in the bases themselves the insertion of suitable fields (V - fields) which, as in the genus zero case, will be eventually assimilated to vertex operators of the bosonized formulation. In the following we would like to motivate the introduction of these bases by discussing several intermediate steps that lead to these equations, bearing in mind that the basic idea is simply to shift (inside the bases) the singularities corresponding to the charge $Q(\lambda)$ and the spin - fields, from P_+ and P_- to generic points on Σ. The first step in this direction consists, following the procedure for the genus zero case, in locating the zero modes (whose total number is $Q(\lambda)$) outside the point P_- (zero modes exist only when $Q(\lambda) \in \mathbb{Z}^+$ in the sector $l = 1$; however their presence is reflected in any sector l). This corresponds to expanding the b field in terms of[32]

$$\tilde{f}_j^{(\lambda,l)}(z) = \frac{\prod_{i=1}^{Q(\lambda)} E(z,P_i)\theta[^{\delta}_{\epsilon}](z+u)}{E(z,P_-)^{j-s(\lambda)+1}E(z,P_+)^{-j+s(\lambda)}\sigma(z)^{1-2\lambda}}, \tag{8.60}$$
$$u = (j-s(\lambda))P_+ - (j-s(\lambda)+1)P_- + \Sigma_{i=1}^{Q(\lambda)}P_i + (1-2\lambda)\Delta\,,$$

where $j \in \mathbb{Z} + s(\lambda) + \lambda(l-1)$, $l \in \mathbb{Z}^{(*)}$ and $Q(\lambda) \in \mathbb{Z}^+$. The dual basis (up to a normalization) is

$$\tilde{f}^j_{(1-\lambda,l)}(z) = \frac{\theta[^{-\delta}_{-\epsilon}](z-(j-s(\lambda)+1)P_+ + (j-s(\lambda))P_- - \Sigma_{i=1}^{Q(\lambda)}P_i + (2\lambda-1)\Delta)}{E(z,P_+)^{j-s(\lambda)+1}E(z,P_-)^{-j+s(\lambda)}\sigma(z)^{2\lambda-1}\prod_{i=1}^{Q(\lambda)}E(z,P_i)}\,. \tag{8.61}$$

The propagator with the insertion of zero-modes at the points $P_1, ..., P_{Q(\lambda)}$ is

$$S^{(l)}(z,w) = \frac{1}{E(z,w)}\left(\frac{E(z,P_+)E(w,P_-)}{E(w,P_+)E(z,P_-)}\right)^{\mu(l)}\left(\prod_{i=1}^{Q(\lambda)}\frac{E(z,P_i)}{E(w,P_i)}\right)\cdot$$
$$\cdot\left(\frac{\sigma(z)}{\sigma(w)}\right)^{2\lambda-1}\frac{\theta[^{\delta}_{\epsilon}](z-w-\mu(l)P_- + \mu(l)P_+ + \Sigma_{i=1}^{Q(\lambda)}P_i - (2\lambda-1)\Delta)}{\theta[^{\delta}_{\epsilon}](-\mu(l)P_- + \mu(l)P_+ + \Sigma_{i=1}^{Q(\lambda)}P_i - (2\lambda-1)\Delta)}\,. \tag{8.62}$$

When $\lambda \in \mathbb{R}$ is generic we cannot do the same as above, i.e. insert vertex fields with charge ± 1. However we can, for example, insert a vertex field at the point $P \in \Sigma$ that

(*) The range of distinct values of l is determined by the value of λ (see comments after eq.(2.1)).

absorbs the entire charge $Q(\lambda)$. This can be done substituting in eqs.(8.60 - 8.61) the terms $\prod_{i=1}^{Q(\lambda)} E(z,P_i)$ and $\Sigma_{i=1}^{Q(\lambda)} P_i$ with $E(z,P)^{Q(\lambda)}$ and $Q(\lambda)P$, respectively. The propagator becomes

$$S^{(l)}(z,w) = \frac{1}{E(z,w)} \left(\frac{E(z,P_+)E(w,P_-)}{E(w,P_+)E(z,P_-)} \right)^{\mu(l)} \left(\frac{E(z,P)}{E(w,P)} \right)^{Q(\lambda)} \cdot$$

$$\cdot \left(\frac{\sigma(z)}{\sigma(w)} \right)^{2\lambda-1} \frac{\theta[^\delta_\epsilon](z-w-\mu(l)P_- + \mu(l)P_+ + Q(\lambda)P - (2\lambda-1)\Delta)}{\theta[^\delta_\epsilon](-\mu(l)P_- + \mu(l)P_+ + Q(\lambda)P - (2\lambda-1)\Delta)} . \tag{8.63}$$

Notice that $S^{(l)}(z,w)$ can be seen as the propagator in the sector $l=1$ with the insertion of λ-spin fields at the points $P_\pm$:

$$S^{(l)}(z,w) = \frac{{}_1 < 0|R\left(S^{-,l}(P_-)b^1(z)c^1(w)S^{+,l}(P_+)\right)|0>_1}{{}_1 < 0|R\left(S^{-,l}(P_-)S^{+,l}(P_+)\right)|0>_1} , \tag{8.64}$$

where $S^{\pm,1}$ are identity operators. The factors $\left(\frac{E(z,P_+)}{E(w,P_+)} \right)^{\mu(l)}$ and $\left(\frac{E(z,P_-)}{E(w,P_-)} \right)^{-\mu(l)}$ in eqs.(8.62 - 8.63) are the effects of the λ - spin field insertions $S^{+,l}(P_+)$ and $S^{-,l}(P_-)$.

The remarks just made, concerning eq.(8.64), suggests that also the spin-fields can be moved away from P_+ and P_-. Let us consider a simple example. We want to find the analog of eqs.(8.60 - 8.61) with the insertion of the λ - spin fields at arbitrary points Q_1 and Q_2. The bases fit for that are for $Q(\lambda) \in \mathbb{Z}^+$ (for $\lambda \in \mathbb{R}$ the modification is analogous)

$$\hat{f}_j^{(\lambda,l)}(z) = \left(\frac{E(z,Q_1)}{E(z,Q_2)} \right)^{\mu(l)} \frac{\prod_{i=1}^{Q(\lambda)} E(z,P_i)\sigma(z)^{2\lambda-1}\theta[^\delta_\epsilon](z+u)}{E(z,P_+)^{-j+s(\lambda)+\mu(l)}E(z,P_-)^{j-s(\lambda)+1-\mu(l)}} , \tag{8.65}$$

where $u = (j-s(\lambda)-\mu(l))(P_+-P_-)-P_-+\mu(l)(Q_1-Q_2)+(1-2\lambda)\Delta, \quad j \in \mathbb{Z}+s(\lambda)+\lambda(l-1)$. For the sake of conciseness we do not write down explicitly the dual basis. We remark that the numbers $j-s(\lambda)-\mu(l)$ are l-independent and integral. So the conditions (8.50) define a unique (l - independent) vacuum.

Using such modified bases we find a propagator $\hat{S}^{(\lambda,l)}$, which is equal to (8.62) with $P_+(P_-)$ replaced by $Q_1(Q_2)$. As a consequence this propagator is independent of the points $P_\pm$ where the vacua are defined.

As a final natural extension of the procedure outlined above, we introduce now "fat" b - c systems, referred to as b - c systems. The idea is to reshuffle the location of the λ - spin fields and of the total charge, and to represent them as insertions of V-fields at the points P_i. As we have shown in the $g = 0$ case the fields $V^i(P_i)$ can be regarded as the fermionized counterparts of (chiral) insertions of vertex operators : $e^{i\alpha_i\phi(P_i)}$: (for suitable α_i). In other words

$$S(z,w) \ = < 0|\mathcal{R}((B(z)C(w))|0 > = \frac{< \prod_k V^k(P_k)b(z)c(w) >}{< \prod_k V^k(P_k) >}. \tag{8.66}$$

The last expression refers to the standard formalism.

In order to implement the idea just outlined, we have to use modified bases

$$g_j^{(\lambda)}(z) = \frac{\prod_{i=1}^n E(z,P_i)^{\tilde{\alpha}_i}\theta[^\delta_\epsilon](z+u)}{E(z,P_-)^{j-s(\lambda)+1}E(z,P_+)^{-j+s(\lambda)}\sigma(z)^{1-2\lambda}}, \tag{8.67}$$

where $u = (j - s(\lambda))P_+ - (j - s(\lambda) + 1)P_- + \sum_{i=1}^n \tilde{\alpha}_i P_i + (1-2\lambda)\Delta, \quad j \in \mathbb{Z} + s(\lambda)$. We expand

$$B(z) \ = \ \sum_j B^j g_j^{(\lambda)}(z). \tag{8.68}$$

The dual bases are

$$g^j_{(1-\lambda)}(z) = \frac{\theta[^{-\delta}_{-\epsilon}](z + (-j + s(\lambda) - 1)P_+ - (-j + s(\lambda))P_- - \sum_{i=1}^n \tilde{\alpha}_i P_i + (2\lambda - 1)\Delta)}{E(z,P_+)^{j-s(\lambda)+1}E(z,P_-)^{-j+s(\lambda)}\prod_{i=1}^n E(z,P_i)^{\tilde{\alpha}_i}\sigma(z)^{2\lambda-1}}, \tag{8.69}$$

and

$$C(z) \ = \ \sum_j C_j g^j_{(1-\lambda)}(z). \tag{8.70}$$

The vacuum $|0>$ in eq.(8.66) is defined by the analog of eq.(8.50) and since in eqs.(8.67 - 8.69) $j \in \mathbb{Z} + s(\lambda)$ it corresponds to $|0>_1$. The requirement that $g_j^{(\lambda)}$ be of weight λ in z gives the constraint

$$\sum_{i=1}^n \tilde{\alpha}_i = (2\lambda - 1)(g-1). \tag{8.71}$$

From this equation we can see the topological origin of the constraint over the total charge of the V - fields.

The propagator of the b - c system is

$$S(z,w) \equiv < B(z)C(w) >=$$

$$\frac{1}{E(z,w)} \prod_{i=1}^{n} \left(\frac{E(z,P_i)}{E(w,P_i)} \right)^{\tilde{\alpha}_i} \left(\frac{\sigma(z)}{\sigma(w)} \right)^{2\lambda-1} \frac{\theta[^\delta_\epsilon](z-w+\sum_{i=1}^{n} \tilde{\alpha}_i P_i - (2\lambda-1)\Delta)}{\theta[^\delta_\epsilon](\sum_{i=1}^{n} \tilde{\alpha}_i P_i - (2\lambda-1)\Delta)} . \quad (8.72)$$

8.2.2 Spin - Field Correlation Functions

As an introduction to the calculation of V - field correlation functions in the next section, here we show how one can compute correlation functions of spin fields in a straightforward way. In particular we test the formalism introduced in the previous section by recovering the 2 - point spin - field insertions in the $\lambda = 1/2$ and in the commuting $\lambda = 3/2$ cases[128]. We then generalize these calculations to rational λ. We recall that λ - spin fields are a particular case of V - fields. We start with the $\lambda = 1/2$ case. Let $S^{\pm}(P_{\pm})$ be the spin-field connecting the Neveu-Schwarz with the Ramond vacuum(*):

$$S^{(2)}(z,w) =_2 < 0|R(b(z)c(w))|0 >_2 = \frac{< S^-(P_-)b(z)c(w)S^+(P_+) >}{< S^-(P_-)S^+(P_+) >} . \quad (8.73)$$

Once the propagator $S^{(2)}(z,w)$ is known, the correlation function $< S^-(P_-)S^+(P_+) >$ can be computed with a procedure similar to the one introduced by Dixon et al. in [126].

By means of the operator product expansion

$$< T(z)V^1(P_1)...V^n(P_n) >= \sum_i \left(\frac{h_i}{(z-P_i)^2} + \frac{\partial_{P_i}}{(z-P_i)} \right) < V^1(P_1)...V^n(P_n) >, \quad (8.74)$$

we obtain a set of first order differential equations which can be solved to get the correlator $< V^1(P_1)...V^n(P_n) >$.

For $\lambda = 1/2$ we have the following correlation function for the stress - energy tensor in presence of spin - field insertions:

$$\frac{< S^-(P_-)T(z)S^+(P_+) >}{< S^-(P_-)S^+(P_+) >} = \lim_{z\to w} \frac{1}{2}(\partial_z - \partial_w) \left(S^{(2)}(z,w) - \frac{1}{(z-w)} \right), \quad (8.75)$$

with

(*) For brevity here we do not consider odd spin structures.

$$S^{(2)}(z,w)=\frac{1}{E(z,w)}\left(\frac{E(z,P_+)E(w,P_-)}{E(w,P_+)E(z,P_-)}\right)^{\frac{1}{2}}\frac{\theta[^\delta_\epsilon](z-w+\frac{1}{2}(P_+-P_-))}{\theta[^\delta_\epsilon](\frac{1}{2}(P_+-P_-))}. \qquad (8.76)$$

We obtain

$$< S^-(P_-)S^+(P_+) >= K_{\delta,\epsilon}E(P_-,P_+)^{-\frac{1}{4}}\theta[^\delta_\epsilon](\frac{1}{2}(P_+-P_-)), \qquad (8.77)$$

where $K_{\delta,\epsilon}$ is an integration constant which carries a dependence on the spin structure. The conformal weight of the spin-fields $S^{\pm}(P_{\pm})$ is $1/8$ as can be seen in two independent ways: either by looking at the residue in the leading singularity in (8.75) or directly from the geometrical weight in eq.(8.77).

Before addressing the commuting $\lambda = 3/2$ case, we consider the anticommuting $\lambda = 3/2$ system. In order for the propagator to be non vanishing we must put $2g-2$ insertions of b zero - modes at the points P_i. Therefore the propagator for the $\lambda = 3/2$ case is

$$\tilde{S}^{(2)}(z,w) = {}_2< 0|R(b(z)c(w))|0 >_2 = \frac{< S^-(P_-)b(z)c(w)\prod_{i=1}^{2g-2}b(z_i)S^+(P_+) >}{< S^-(P_-)\prod_{i=1}^{2g-2}b(z_i)S^+(P_+) >}, \qquad (8.78)$$

where

$$\tilde{S}^{(2)}(z,w)=\frac{1}{E(z,w)}\left(\frac{E(z,P_+)E(w,P_-)}{E(w,P_+)E(z,P_-)}\right)^{\frac{1}{2}}\prod_{i=1}^{2g-2}\frac{E(z,P_i)}{E(w,P_i)}\left(\frac{\sigma(z)}{\sigma(w)}\right)^2\frac{\theta[^\delta_\epsilon](z-w+u)}{\theta[^\delta_\epsilon](u)}, \qquad (8.79)$$

with $u = \frac{1}{2}(P_+ - P_-) + \sum_{i=1}^{2g-2} P_i - 2\Delta$.

The same procedure outlined above allows us to compute the correlation function with zero-modes and spin-field insertions:

$$\begin{aligned}< S^-(P_-)b(P_1)...b(P_{2g-2})S^+(P_+) >&= K_{\delta,\epsilon}\frac{\sigma(P_+)}{\sigma(P_-)}\prod_i\sigma(P_i)^2E(P_-,P_+)^{-\frac{1}{4}}\cdot\\ \cdot\prod_{j,k}E(P_-,P_j)^{-\frac{1}{2}}E(P_k,P_+)^{\frac{1}{2}}&\prod_{l<m}E(P_l,P_m)\theta[^\delta_\epsilon](\frac{1}{2}(P_+-P_-)+\sum_i P_i-2\Delta).\end{aligned} \qquad (8.80)$$

In P_-, P_+ the conformal weight is respectively $-3/8$ and $5/8$. The $2g-2$ points P_i represent the zero-mode insertions, as can be seen by noting that the conformal weight at P_i is precisely $3/2$.

Let us now discuss the commuting case. The quantization of a generic commuting β - γ system of weight λ and $1-\lambda$ respectively is achieved by imposing the commutation relations

$$[\beta^{i,l},\gamma^l_j]=\delta^i_j\,,\qquad [\beta^{i,l},\beta^{j,l}]=0=[\gamma^l_i,\gamma^l_j]\,. \tag{8.81}$$

The bra and ket vacua of the bosonic system are assumed to satisfy the relation (8.50) as for the anticommuting case.

Once a specific choice of the zero-mode insertions is made, the bosonic propagator

$$\bar{S}^{(l)}(z,w)\equiv_l<0|R(\beta^{(l)}(z)\gamma^{(l)}(w))|0>_l\,,$$

coincides with the fermionic one

$$S^{(l)}(z,w)\equiv_l<0|R(b^{(l)}(z)c^{(l)}(w))|0>_l\;.$$

The regularized bosonic stress energy tensor is

$$T(z)=-\lim_{z\to w}\left((1-\lambda)\partial_z-\lambda\partial_w\right)\left(\beta(z)\gamma(w)-\frac{1}{z-w}\right)\,. \tag{8.82}$$

For $\lambda=\frac{3}{2}$ a possible expression for the bosonic propagator is the one given in eq.(8.79). The points P_i represent the insertion of vertex operators $V_1(P_i)$ which carry a charge 1. These vertex operators can no longer (as in the fermionic case) be considered as zero-mode insertions because their conformal weight, as shown below, is $-3/2$ (*). Starting from the correlator of the stress-energy tensor with vertex operators V_1 and spin-field insertions

$$\frac{<S^-(P_-)\left(\prod_{i=1}^{2g-2}V_1(P_i)\right)T(z)S^+(P_+)>}{<S^-(P_-)\left(\prod_{i=1}^{2g-2}V_1(P_i)\right)S^+(P_+)>}\,,$$

we get the correlation function:

(*) In the path integral approach these vertex operators are represented by a delta function $\delta(\beta)$.

$$< S^-(P_-)V_1(P_1)...V_1(P_{2g-2})S^+(P_+) >= K_{\delta,\epsilon}\frac{\sigma(P_-)}{\sigma(P_+)}\prod_i \sigma(P_i)^{-2}E(P_-,P_+)^{\frac{1}{4}}\cdot$$
$$\cdot\prod_{j,k} E(P_-,P_j)^{\frac{1}{2}}E(P_k,P_+)^{-\frac{1}{2}}\prod_{l<m} E(P_l,P_m)^{-1}\frac{1}{\theta[{}^\delta_\epsilon](\frac{1}{2}(P_+-P_-)+\sum_i P_i - 2\Delta)}. \tag{8.83}$$

In P_-, P_+ the conformal weight is respectively $3/8$ and $-5/8$. In order to compare our result with Atick-Sen's[128] we notice that they absorbed the extra charge by inserting, instead of $2g-2$ vertex operators with charge 1, $g-1$ vertex operators V_2 with charge 2 (which turn out to have conformal weight -4)(*). In our formalism their result can be recovered starting from the propagator

$$\tilde{S}^{(2)}(z,w) = \frac{1}{E(z,w)}\left(\frac{E(z,P_+)E(w,P_-)}{E(w,P_+)E(z,P_-)}\right)^{\frac{1}{2}}\prod_{i=1}^{g-1}\left(\frac{E(z,P_i)}{E(w,P_i)}\right)^2\left(\frac{\sigma(z)}{\sigma(w)}\right)^2\frac{\theta[{}^\delta_\epsilon](z-w+u)}{\theta[{}^\delta_\epsilon](u)}, \tag{8.84}$$

with $u = \frac{1}{2}(P_+-P_-)+2\sum_{i=1}^{g-1} P_i - 2\Delta$. With this choice the correlator is

$$< S^-(P_-)V_2(P_1)...V_2(P_{g-1})S^+(P_+) >= K_{\delta,\epsilon}\frac{\sigma(P_-)}{\sigma(P_+)}\prod_i \sigma(P_i)^{-4}E(P_-,P_+)^{\frac{1}{4}}\cdot$$
$$\cdot\prod_{j,k} E(P_-,P_j)E(P_k,P_+)^{-1}\prod_{l<m} E(P_l,P_m)^{-4}\frac{1}{\theta[{}^\delta_\epsilon](\frac{1}{2}(P_+-P_-)+2\sum_i P_i - 2\Delta)}. \tag{8.85}$$

This result completely agrees with Atick and Sen's. From the discussion carried out in section 8.2.1, it is clear that our approach allows us to compute the correlation function for an arbitrary number of spin - field insertions and, moreover, that the vertex operators, which must be inserted in order to fulfill the condition (8.71) on the total charge, can be placed in general position and assumed to have the most general charge. The corresponding

(*) The relation between the conformal weight and the charge in the case of a fermionic b - c system of weight λ is shown in the next section. We remark here that since the stress - energy tensor for the bosonic system differs by a minus sign from the fermionic one, in the bosonic case the relation between conformal weight and charge is just the opposite: $h_{\tilde{\alpha}_i} = -\frac{1}{2}\tilde{\alpha}_i^2+\frac{1}{2}(1-2\lambda)\tilde{\alpha}_i$. An immediate consequence of this fact is that only in a certain range of values for λ is it possible to introduce real charge operators which reproduce the $(\lambda, 1-\lambda)$ $\beta-\gamma$ system.

formulas can be obtained by specializing the expressions given in the next section for real λ.

In this section however we limit ourselves to give some more examples with rational λ. The general expression for the spin - field correlation functions in the case of anticommuting b - c systems of half - integer weight $N/2$, N odd, is:

$$\begin{aligned}
&< S^+(P_1)...S^+(P_q)b(Q_1)...b(Q_r)c(R_1)...c(R_s)S^-(T_1)...S^-(T_t) >= \\
&= K_{\delta,\epsilon}\prod_i \sigma(P_i)^{\frac{1}{2}(N-1)}\prod_j \sigma(Q_j)^{N-1}\prod_k \sigma(R_k)^{1-N}\prod_l \sigma(T_l)^{\frac{1}{2}(1-N)}\cdot \\
&\cdot\prod_{i<j} E(P_i,P_j)^{\frac{1}{4}}\prod_{k<l} E(Q_k,Q_l)\prod_{m<n} E(R_m,R_n)\prod_{p<q} E(T_p,T_q)^{\frac{1}{4}}\cdot \\
&\cdot\prod_{i,j,k,l} E(P_i,Q_j)^{\frac{1}{2}}E(P_i,R_k)^{-\frac{1}{2}}E(P_i,T_l)^{-\frac{1}{4}}E(Q_j,R_k)^{-1}E(Q_j,T_l)^{-\frac{1}{2}}E(R_k,T_l)^{\frac{1}{2}}\cdot \\
&\cdot\theta[^\delta_\epsilon](\frac{1}{2}(\sum_i P_i-\sum_l T_l)+\sum_j Q_j-\sum_k R_k-(N-1)\Delta)\,.
\end{aligned} \tag{8.86}$$

The charge conservation condition requires the constraint

$$\frac{1}{2}(q-t)+r-s=(N-1)(g-1)\,, \tag{8.87}$$

to be satisfied, otherwise the correlation function vanishes. The spin-fields $S^\pm$ have charge $\pm 1/2$ and their conformal weight is $(2N-1)/8$ for S^+ and $(3-2N)/8$ for S^-.

Likewise the general spin-field correlation function for a commuting $\beta-\gamma$ system with $\lambda = N/2$ is:

$$\begin{aligned}
&< S^+(P_1)...S^+(P_q)V(Q)S^-(T_1)...S^-(T_s) >= \\
&= K_{\delta,\epsilon}\prod_i \sigma(P_i)^{-\frac{1}{2}(N-1)}\sigma(Q)^{\gamma(1-N)}\prod_j \sigma(T_j)^{\frac{1}{2}(N-1)}\cdot \\
&\cdot\prod_{i<j} E(P_i,P_j)^{-\frac{1}{4}}\prod_{k<l} E(T_k,T_l)^{-\frac{1}{4}}\prod_m E(P_m,Q)^{-\frac{1}{2}\gamma}\prod_n E(T_n,Q)^{\frac{1}{2}\gamma}\prod_{t,u} E(P_t,T_u)^{\frac{1}{4}}\cdot \\
&\cdot\left[\theta[^\delta_\epsilon](\frac{1}{2}(\sum_i P_i-\sum_l T_l)+\gamma Q-(N-1)\Delta)\right]^{-1}\,.
\end{aligned} \tag{8.88}$$

The spin - fields $S^\pm$ have charge $\pm 1/2$ and conformal weight $(1-2N)/8$ for S^+ and $(2N-3)/8$ for S^-. The extra charge has been absorbed by inserting at the point Q a single operator V, whose charge γ is given by

$$\gamma = \frac{2(N-1)(g-1)+s-q}{2}\,. \tag{8.89}$$

Finally we consider the case of a fermionic b - c system of rational weight $\lambda = N/M$, with N, M relatively prime integers. The λ - spin fields with charge $\pm k, k = 1/M, ..., (M-1)/M$ are denoted by $S_k^{\pm}$. These fields, of course, live on a suitable covering of the Riemann surface Σ. The correlation function with the insertion of $N_{\pm k}$ λ-spin fields $S_k^{\pm}$ at the points $P_{\pm k,i}$, where $(\pm k; i) = (\pm k1; 1, ..., N_{\pm k})$, is given by:

$$\begin{aligned}
&< S_1^+(P_{1,1})..S_1^+(P_{1,N_1})S_2^+(P_{2,1})..S_{M-1}^+(P_{M-1,N_{M-1}})b(Q_1)..b(Q_r)\cdot \\
&\cdot c(R_1)..c(R_s)S_{M-1}^-(P_{M-1,1})..S_1^-(P_{-1,N_{-1}}) >= \\
&= K_{\delta,\epsilon}[\prod_k \prod_{l=1,...,N_{\pm k}} \sigma(P_{\pm k,l})^{\pm(2N-M)\frac{k}{M^2}}] \prod_i \sigma(Q_i)^{\frac{2N-M}{M}} \prod_j \sigma(R_j)^{\frac{M-2N}{M}}\cdot \\
&\cdot [\prod_{k,i}\prod_{k',j} E(P_{\pm k,i}, P_{\mp k',j})^{-\frac{kk'}{2M^2}}] \prod_{k,l,m} E(P_{\pm k,l}, Q_m)^{\pm\frac{k}{M}} \prod_{k,n,q} E(P_{\pm k,n}, R_q)^{\mp\frac{k}{M}}\cdot \\
&\cdot \prod_{(k,i)\neq(k',j)} E(P_{\pm k,i}, P_{\pm k',j})^{\frac{kk'}{2M^2}} \prod_{l<m} E(Q_l, Q_m) \prod_{n<p} E(R_n, R_p)\cdot \\
&\cdot \theta[^\delta_\epsilon]\left(\pm\sum_{k,i}\frac{k}{M}P_{\pm k,i} + \sum_j Q_j - \sum_l R_l - \frac{(2N-M)}{M}\Delta\right).
\end{aligned} \tag{8.90}$$

The correlator is non - vanishing only if the constraint

$$\frac{2N-M}{M}(g-1) = r - s + \sum_{k=1}^{M-1}(\frac{k}{M}N_k - \frac{k}{M}N_{-k})\,, \tag{8.91}$$

is satisfied. The conformal weight of $S_k^{\pm}$ is given by

$$\frac{k^2 \mp (M-2N)k}{2M^2}\,. \tag{8.92}$$

The correlation functions calculated so far are in general non - single - valued as the points, where the insertions occur, are shifted by a homology cycle. So the above formulas are to be considered as starting points where singlevaluedness and modular invariance are still to be implemented.

8.2.3 Vertex Insertion Correlation Functions for Minimal Models

In this section we return to the general case. We consider the "fat" $b-c$ systems defined in eqs.(8.66 - 8.72) and apply to them the method applied in the last section to a few simple cases. We will use

$$\frac{< T(z)V^1(P_1)\dots V^n(P_n) >}{< V^1(P_1)\dots V^n(P_n) >} = \lim_{z\to w}((1-\lambda)\partial_z - \lambda\partial_w)\left(S(z,w) - \frac{1}{z-w}\right), \quad (8.93)$$

where $S(z,w)$ is given by eq.(8.72).

Analyzing the residues of the leading and subleading singularities at P_i, and comparing them with the OPE of $T(z)V^i(P_i)$ we are able to identify the weights h_i of the fields V^i and to extract differential equations which allow us to determine the form of $< \prod_i V^i(P_i) >$. Specifically we obtain the relation which determines the conformal weight in terms of the charge $\tilde{\alpha}_i$:

$$h_i \equiv h_{\tilde{\alpha}_i} = \frac{1}{2}\tilde{\alpha}_i^2 - \frac{1}{2}(1-2\lambda)\tilde{\alpha}_i\,, \quad (8.94)$$

and

$$\partial_{P_i} ln < \prod_k V^k(P_k) > = (2\lambda-1)\tilde{\alpha}_i\frac{\sigma'(P_i)}{\sigma(P_i)} + \tilde{\alpha}_i\frac{\theta'[^{\delta}_{\epsilon}](v)}{\theta[^{\delta}_{\epsilon}](v)} + \sum_{i\neq j}\tilde{\alpha}_i\tilde{\alpha}_j\frac{E'(P_i,P_j)}{E(P_i,P_j)}, \quad (8.95)$$

where

$$v = \sum_i \tilde{\alpha}_i P_i - (2\lambda-1)\Delta\,. \quad (8.96)$$

Integrating (8.95) we obtain

$$< \prod_k V^k(P_k) >= K_{\delta,\epsilon}\prod_k \sigma(P_k)^{\tilde{\alpha}_k(2\lambda-1)}\prod_{i<j}E(P_i,P_j)^{\tilde{\alpha}_i\tilde{\alpha}_j}\theta[^{\delta}_{\epsilon}](v)\,, \quad (8.97)$$

where $K_{\delta,\epsilon}$ is an integration constant.

As for eq.(8.94) we remark that the weight of V^i can be obtained directly from eq.(8.97) by calculating the conformal weight of the RHS at the points P_i and applying the constraint (8.71).

In general the correlation functions given by eq.(8.97) are not single-valued on Σ. They have cuts with endpoints P_i as expected. But they, in general, pick up a phase and the theta characteristics are shifted when P_i winds around a homology cycle. Precisely, when $P_i \to P_i + na + mb$, eq.(8.97) goes over to

$$K_{\delta,\epsilon} \prod_k \sigma(P_k)^{\tilde{\alpha}_k(2\lambda-1)} \prod_{i<j} E(P_i,P_j)^{\tilde{\alpha}_i\tilde{\alpha}_j} e^{-2\pi i\tilde{\alpha}_i^2 mn - 2\pi i\tilde{\alpha}_i m\epsilon} \theta[^{\delta+\tilde{\alpha}_i m}_{\epsilon+\tilde{\alpha}_i n}](v) \,. \tag{8.98}$$

The phase and the shifts can be rational or irrational depending on the values taken by $\tilde{\alpha}_i, \epsilon$ and δ. In order to carry on the discussion of this very important point we need to be more specific, so let us refer as in the genus zero case to the minimal models.

Let us add the information that B - C systems represent minimal models. This is was shown in the genus 0 case where "fat" b - c systems constitute a dual (fermionized) version of the Coulomb gas approach, so that by the use of b - c systems we can represent primary fields and reconstruct their correlation functions. The correspondence between the b - c system formalism and Dotsenko - Fateev's Coulomb gas approach is established by means of the identifications

$$\tilde{\alpha}_i = \sqrt{2}\alpha_i, \qquad \tilde{\alpha}_0 \equiv \frac{1}{2}(1-2\lambda) = \sqrt{2}\alpha_0 \,, \tag{8.99}$$

and the value of λ suitable for a given minimal model is obtained by equating the central charges

$$c(\lambda) = -12\lambda^2 + 12\lambda - 2 = 1 - 24\alpha_0^2 \equiv 1 - \frac{6(p-q)^2}{pq} \,. \tag{8.100}$$

Here α_0 and α_i are the same symbols appearing in [119]. So for the minimal model identified by the integers p and q we have

$$\lambda = \frac{1}{2} + \frac{p-q}{\sqrt{2pq}} \,. \tag{8.101}$$

Therefore in general λ is irrational. Incidentally we remark that exchanging p and q turns λ into $1-\lambda$. This is the way the electromagnetic duality of the bosonic formulation is recovered in our formalism. We remember that the constraint (8.71) on the bases implies that the V-fields correlation functions are non - vanishing only if the total charge $\sum_i \tilde{\alpha}_i$ satisfies the relation:

$$\sum_i \tilde{\alpha}_i = -2\tilde{\alpha}_0(g-1) \,. \tag{8.102}$$

We can now introduce screening charges, i.e. weight one V - fields that by eq.(8.94) have charge

$$\tilde{\alpha}_{\pm} = \tilde{\alpha}_0 \pm \sqrt{\tilde{\alpha}_0^2 + 2}\,. \tag{8.103}$$

Eqs.(8.102) and (8.103) imply charge quantization, the same as in the case $g = 0$:

$$\tilde{\alpha}_{r,s} = \frac{1}{2}(1-r)\tilde{\alpha}_+ + \frac{1}{2}(1-s)\tilde{\alpha}_- = (1-r)\sqrt{\frac{q}{2p}} - (1-s)\sqrt{\frac{p}{2q}}\,, \tag{8.104}$$

where $1 \le r \le p-1, 1 \le s \le q-1$. We remark that in higher genus, w.r.t. the genus 0 case, an extra charge $2\tilde{\alpha}_0 g$ must be reabsorbed (we can think of $2\tilde{\alpha}_0(g-1)$ as a "bra vacuum charge"). In general this can easily be done by inserting in the correlation function an appropriate number of screening charges and relative contour integrations; this is possible thanks to the fact that for the $\tilde{\alpha}_i's$ given by eqs.(8.99 - 8.102) the following relation holds

$$(p-1)\tilde{\alpha}_+ + (q-1)\tilde{\alpha}_- = -2\tilde{\alpha}_0 = \sqrt{2}\frac{p-q}{\sqrt{pq}}\,, \tag{8.105}$$

with the parametrization of $\tilde{\alpha}_i$ and λ given by eqs.(8.98 - 8.101).

For the sake of simplicity, from now on, we will be dealing only with unitary minimal models. The relevant formulas are obtained from the previous ones by the substitutions $q \to p$ and $p \to p+1$.

Let us now return to eq.(8.97) with the parametrization of $\tilde{\alpha}_i$ and λ given in eqs.(8.104) and (8.101), respectively, and the above substitutions. Looking at eq.(8.98) we see that the shifts of the theta - characteristics are in general irrational. This fact is not surprising since it is inherited from the bases (8.67) we started from: they are well-behaved as far as the z-dependence is concerned, but not with respect to the P_i's (well-behaved meaning that the relevant shifts along homology cycles involve rational phases and shifts). Now, if phases and shifts in (8.98) were rational (as they are in some particular cases), we could take the attitude of the previous section and consider eq.(8.97) as our basic result, from which single-valued quantities can be calculated by taking linear combinations of the RHS of (8.97) with suitable coefficients $K_{\delta,\epsilon}$. Since this is not the case, we take another attitude (see however the remark at the end of this section): we change the bases (8.67). In order to construct the new bases we argue as follows.

When a P_i, i.e. a point where a vertex is inserted, winds around a homology cycle, in general the theta - characteristics change and a phase appears. The phase and the theta - characteristic shifts must correspond to the behaviour of the vertex insertions at the points P_i. We describe the latter in the following way: we say that there exists a finite covering Σ' of the Riemann surface Σ, such that on Σ' the correlation functions $< \prod_k V^k(P_k) >$ are single-valued (up to a possible phase depending only on the θ - characteristics). We will see that this simple statement will allow us not only to characterize the covering, but also to determine the θ - characteristics δ and ϵ.

To see this let us modify the bases (8.67) as follows

$$g_j^{(\lambda)}[{}^{\delta}_{\epsilon}](z, P_1, ..., P_n) = \frac{E(z,P_+)^{j-s(\lambda)}}{E(z,P_-)^{j-s(\lambda)+1}} \prod_{i=1}^{n} \left(\frac{E(P_i,P_+)^{j-s(\lambda)}}{E(P_i,P_-)^{j-s(\lambda)+1}} \right)^{\tilde{\alpha}_i} \prod_{i=1}^{n} E(z,P_i)^{\tilde{\alpha}_i} \cdot$$

$$\cdot \prod_{i\leq j}^{n} E(P_i,P_j)^{\tilde{\alpha}_i\tilde{\alpha}_j} \sigma(z)^{2\lambda-1} \prod_{i=1}^{n} \sigma(P_i)^{\tilde{\alpha}_i(2\lambda-1)} \theta[{}^{\delta}_{\epsilon}](cz + cv|d\Omega), \qquad c, d \in \mathbb{R}\,, \qquad (8.106)$$

where $u = (j - s(\lambda))P_+ - (j - s(\lambda) + 1)P_- + \sum_{i=1}^{n} \tilde{\alpha}_i P_i + (1 - 2\lambda)\Delta), \quad j \in \mathbb{Z} + s(\lambda)$, and c, d are numerical constants to be determined. We recover the bases (8.67) by setting $c = d = 1$, up to a normalization. This normalization is actually very important [35] since it allows us to put the conformal weights $h_{\tilde{\alpha}_i} = \frac{1}{2}\tilde{\alpha}_i^2 - \frac{1}{2}(1 - 2\lambda)\tilde{\alpha}_i$ at the points P_i. We point out that this normalization is not *ad hoc*: it is possible only if condition (8.71) is satisfied, being thus another indication of the geometrical consistency of our method.

Our aim now is to determine c and d in such a way that the bases be single-valued (up to theta-characteristic dependent phases) when one of the P_i winds around a homology cycle of the covering Σ'. This will allow us to determine also δ and ϵ. The detailed derivation is given in the following section. Here we write down the result.

Using the freedom in the choice of theta - function basis, pointed out in the next section, we can write the new bases (8.106) in the form:

$$g_j^{(\lambda)}[{}^{\delta}_{\epsilon}](z, P_1, ..., P_n) = \frac{E(z,P_+)^{j-s(\lambda)}}{E(z,P_-)^{j-s(\lambda)+1}} \prod_{i=1}^{n} \left(\frac{E(P_i,P_+)^{j-s(\lambda)}}{E(P_i,P_-)^{j-s(\lambda)+1}} \right)^{\tilde{\alpha}_i} \prod_{i=1}^{n} E(z,P_i)^{\tilde{\alpha}_i} \cdot$$

$$\cdot \prod_{i\leq j}^{n} E(P_i,P_j)^{\tilde{\alpha}_i\tilde{\alpha}_j} \sigma(z)^{2\lambda-1} \prod_{i=1}^{n} \sigma(P_i)^{\tilde{\alpha}_i(2\lambda-1)} \theta[{}^{\frac{u}{2p(p+1)}}_{0}](\sqrt{2p(p+1)}(z+v)|2p(p+1)\Omega)\,.$$

(8.107)

where $u = 0, 1, ..., 2p(p+1) - 1$. These bases are characterized by the fact that when any P_i winds $2p(p+1)$ times around any homology cycle of Σ, they remain unchanged. This is not true for the z - dependence: the effect of a z - shift along the homology cycles implies an irrational shift both for the δ and ϵ characteristics. Evidently the z-dependence can be understood only in terms of an infinite covering of Σ. The propagator of the new b - c system is given by

$$\begin{aligned} \mathcal{S}(z,w) &= \frac{1}{E(z,w)} \prod_{i=1}^{n} \left(\frac{E(z,P_i)}{E(w,P_i)} \right)^{\tilde{\alpha}_i} \left(\frac{\sigma(z)}{\sigma(w)} \right)^{2\lambda-1} \cdot \\ &\cdot \frac{\theta[^{\frac{u}{2p(p+1)}}_{0}](\sqrt{2p(p+1)}(z - w + \sum_{i=1}^{n} \tilde{\alpha}_i P_i - (2\lambda - 1)\Delta)|2p(p+1)\Omega)}{\theta[^{\frac{u}{2p(p+1)}}_{0}](\sqrt{2p(p+1)}(\sum_{i=1}^{n} \tilde{\alpha}_i P_i - (2\lambda - 1))\Delta)|2p(p+1)\Omega)}. \end{aligned} \tag{8.108}$$

Redoing now the calculations from (8.93) to (8.97) with these new bases we find

$$< \prod_k V^k(P_k) >= K_\delta \prod_k \sigma(P_k)^{\tilde{\alpha}_k(2\lambda-1)} \prod_{i<j} E(P_i,P_j)^{\tilde{\alpha}_i\tilde{\alpha}_j} \theta[^\delta_0](v|2p(p+1)\Omega), \tag{8.109}$$

where v is given by:

$$v = \sqrt{2p(p+1)}(\Sigma_j \tilde{\alpha}_j P_j + (1 - 2\lambda)\Delta),$$

and the characteristic δ is of the form $u/2p(p+1)$ where $u = 0, ..., 2p(p+1) - 1$.

A few comments are in order. The result (8.109) is now single-valued when any P_i winds $n = 2p(p+1)$ times around any homology cycle. We have already noticed that this in general cannot be true for any integer n for the formula (8.97). However it is likely that one can recover eq.(8.109) starting from the formula (8.97), by means of an averaging procedure over the theta characteristics similar to the one outlined in ref.[129].

Eq.(8.109) is our final result in this section. A similar result was obtained via the bosonic approach in ref.[130,131] (see also [132]). It can be taken as a starting point for computing conformal blocks of minimal models in a generic Riemann surface.

8.2.4 Compactification Radius

Our aim in this section is to determine the values of c, d, δ and ϵ in eq.(8.106). This is equivalent to determine the compactification radius of the corresponding scalar field in the bosonized version of the real - weight b - c system. To this end we will use the formula

$$\theta[{}^{\delta}_{\epsilon}](cz+cn+cm\Omega|d\Omega) = e^{-(1/d)(\pi icm\Omega cm+2\pi icm(cz+\epsilon+cn))}\theta[{}^{\delta+cm/d}_{\epsilon+cn}](cz|d\Omega)\,, \qquad (8.110)$$

which holds for any b, c, δ and ϵ. As a consequence of this the change of the basis under a shift along the homology cycles is

$$g_j^{(\lambda)}[{}^{\delta}_{\epsilon}](z,P_1,.,P_k+na+mb,.,P_n) = e^{-2\pi im(\tilde{\alpha}_k^2 n+\tilde{\alpha}_k\epsilon/c)}g_j^{(\lambda)}[{}^{\delta+\tilde{\alpha}_k m/c}_{\epsilon+c\tilde{\alpha}_k n}](z,P_1,.,P_k,.,P_n)\,, \qquad (8.111)$$

where we have put $d=c^2$ in order to eliminate the dependence on z, P_+, P_-, P_i, Ω and Δ. We notice that, had we started from an even more general expression of the bases, in which z, P_+, P_-, P_i, Ω and Δ all had a different coefficient in front of them in the argument of the θ function, the requirement of the phase being independent on z, P_+, P_-, P_i, Ω and Δ in eq.(8.106), would have implied the equality of the coefficients in front of z, P_+, P_-, P_i and Δ, and therefore would have entailed the same conclusion.

Now we impose the new bases (8.106) to live on the covering Σ', as far as the points P_i are concerned (while we ignore the z - dependence). In other words we assume that there exists an integer N such that when any P_i winds N times around an a or b cycle, the bases return to the initial form (up to a phase). From eq.(8.111) we have

$$g_j^{(\lambda)}[{}^{\delta}_{\epsilon}](z,P_1,.,P_k+Na,.,P_n) = g_j^{(\lambda)}[{}^{\delta}_{\epsilon+c\tilde{\alpha}_k N}](z,P_1,.,P_k,.,P_n)\,, \qquad (8.112)$$

and

$$g_j^{(\lambda)}[{}^{\delta}_{\epsilon}](z,P_1,.,P_k+Nb,.,P_n) = e^{-2\pi iN(\tilde{\alpha}_k\epsilon/c)}g_j^{(\lambda)}[{}^{\delta+\tilde{\alpha}_k N/c}_{\epsilon}](z,P_1,.,P_k,.,P_n)\,. \qquad (8.113)$$

Therefore, first of all, $c\tilde{\alpha}_k N$ and $\tilde{\alpha}_k N/c$ must be integers. Now due to eq.(5.11) with $q\to p$ and $p\to p+1$, we can write

$$\tilde{\alpha}_k = \frac{n_k}{\sqrt{2p(p+1)}}\,,$$

where n_k is an integer. Then we have

$$\frac{cNn_k}{\sqrt{2p(p+1)}} = m_k \in \mathbb{Z}\,, \qquad \frac{Nn_k}{c\sqrt{2p(p+1)}} = l_k \in \mathbb{Z}\,. \qquad (8.114)$$

It follows that

$$m_k = an_k, \qquad l_k = bn_k, \tag{8.115}$$

and since in any given minimal unitary model n_k always takes on the value 1, a and b must be integers. Therefore we can rewrite eq.(8.114) as follows

$$cN = a\sqrt{2p(p+1)}, \qquad \frac{N}{c} = b\sqrt{2p(p+1)}, \tag{8.116}$$

from which we have

$$N^2 = ab\ 2p(p+1), \qquad c^2 = \frac{a}{b}. \tag{8.117}$$

Now we choose $N = 2p(p+1)$ (we will discuss later all the other possible choices). Then

$$ab = 2p(p+1), \qquad c = \frac{\sqrt{2p(p+1)}}{b}. \tag{8.118}$$

The theta - characteristics δ and ϵ can be deduced from the same formulas (8.112 - 8.113). Under the shift $P_k \to P_k + a$ the characteristic ϵ becomes $\epsilon + c\tilde{\alpha}_k$. So ϵ must be an integer times $c\tilde{\alpha}_k = n_k/b$. Similarly, as a consequence of a shift of a b - cycle, we deduce that δ must be an integer times $\tilde{\alpha}_k/c = bn_k/2p(p+1)$. Finally we arrive at the bases (8.106) where the theta function is

$$\theta[^{bu/2p(p+1)}_{t/b}]\left(\frac{\sqrt{2p(p+1)}}{b}(z+v)|\frac{2p(p+1)}{b^2}\Omega\right), \tag{8.119}$$

where $t, u \in \mathbb{Z}$. Recalling now that b is an integer we remark that an equivalent basis of b - th order theta - functions[39,133] is given by eq.(8.119) with $b = 1$. We will use this freedom in the choice of a theta-function basis to write our new bases (8.106) in the form (8.107).

Let us recall now the choice we did after eq.(8.117) for N. We could have chosen also $N = 2p(p+1)s$ where s is an integer. In this case $ab = 2p(p+1)s$ and $c = \sqrt{2p(p+1)s}/b$; but this would have entailed a simple redefinition of the basis of theta functions of the type considered above. Similarly we could have chosen $N = p(p+1)$, so that $ab = p(p+1)/2$ etc.. But again this corresponds to a redefinition of the basis of the theta functions, as above. Other choices for N are possible for some particular values of p, but these cases too can be treated in the same way. So, in conclusion, there is no loss of generality in choosing the bases (8.107).

APPENDIX: THETA FUNCTIONS

In this appendix we recall some facts about theta functions theory[39,133]. The θ - function with characteristic $[{\alpha \atop \beta}]$ is defined by

$$\theta[{\alpha \atop \beta}](z) = \sum_{n\in\mathbb{Z}^g} e^{\pi i(n+\alpha)\Omega(n+\alpha)+2\pi i(n+\alpha)(z+\beta)} =$$

$$= e^{\pi i\alpha\Omega\alpha+2\pi i\alpha(z+\beta)}\theta(z+\beta+\Omega\alpha), \qquad (A.1)$$

$$\theta(z) = \theta[{0 \atop 0}](z), \qquad z\in\mathbb{C}^g, \quad \alpha,\beta\in\mathbb{R}^g,$$

where $\Omega_{i,j} \equiv \oint_{b_i}\omega_j$, $\Omega_{i,j} = \Omega_{j,i}$, $Im(\Omega) > 0$. The holomorphic differentials ω_i, $i = 1,...,g$ are normalized in such a way that $\oint_{a_i}\omega_j = \delta^i_j$, a_i, b_i being the homology cycles basis.

When $\alpha_i, \beta_i \in \{0, \frac{1}{2}\}$, the θ - function is even or odd depending on the parity of $4\alpha\beta$. The θ - function is multivalued under a lattice shift in the z - variable:

$$\theta[{\alpha \atop \beta}](z+n+\Omega m) = e^{-\pi im\Omega m-2\pi imz+2\pi i(\alpha n-\beta m)}\theta[{\alpha \atop \beta}](z). \qquad (A.2)$$

Riemann vanishing theorem.

The function

$$f(z) = \theta(I(z) - \Sigma_{i=1}^g I(P_i) + I(\Delta)), \qquad z, P_i\in\Sigma, \qquad (A.3)$$

either vanishes identically or it has g simple zeroes in $z = P_1, ..., P_g$.

Δ is the Riemann divisor class defined by

$$I_k(\Delta) = \frac{1-\Omega_{k,k}}{2} + \sum_{j\neq k}\oint_{a_j}\omega_j(z)I_k(z), \qquad (A.4)$$

where

$$I_k(z) = \int_{P_0}^z \omega_k, \qquad P_0, z\in\Sigma, \qquad (A.5)$$

is the Jacobi map (P_0 is an arbitrary reference point) and

$$I(D = \Sigma_{i=1}^n m_i P_i) \equiv \Sigma_{i=1}^n m_i I(P_i), \qquad m_i\in\mathbb{R}.$$

Another useful theorem, due to Riemann, states that

$$2\Delta = [K]\,, \tag{A.6}$$

where K is the canonical line bundle and $[K]$ denotes the associated divisor class. We recall that two divisors D_1, D_2 belong to the same divisor class $[D]$ if $D_1 - D_2$ is equal to the divisor of a meromorphic function.

Abel theorem.

Let D be a divisor on Σ. Then

$$I(D) = I([D]) \ mod.\Gamma \equiv \{v \in \mathbb{C}^g | \ v = n + \Omega m, \ \ n,m \in \mathbb{Z}^g\}\,. \tag{A.7}$$

The prime form is defined by

$$E(z,w) = \frac{\theta[^\alpha_\beta](I(z) - I(w))}{h(z)h(w)} = -E(w,z)\,, \qquad z,w \in \Sigma\,, \tag{A.8}$$

it is a holomorphic (multivalued) $(-\frac{1}{2}, -\frac{1}{2})$ - differential with a simple zero in $z = w$:

$$E(z,w) \sim z - w\,, \text{ as } z \to w\,. \tag{A.9}$$

$h(z)$ is the square root of $\Sigma_{i=1}^{g} \omega_i(z) \partial_{u_i} \theta[^\alpha_\beta](u)|_{u_i=0}$, it is the holomorphic $\frac{1}{2}$ - differential with non singular (i.e. $\partial_{u_i}\theta[^\alpha_\beta](u)|_{u_i=0} \neq 0$) odd spin structure $[^\alpha_\beta]$. Notice that $E(z,w)$ does not depend on the particular choice of $[^\alpha_\beta]$. The prime form has the following multivaluedness around the b's homology cycles:

$$E(z + na + mb, w) = e^{-\pi i m\Omega m - 2\pi i m(I(z) - I(w))} E(z,w)\,. \tag{A.10}$$

The σ - differential is defined by

$$\sigma(z) = \mathrm{e}^{-\sum\limits_{j=1}^{g} \oint\limits_{a_j} \omega_j(w) \ln E(w,z)}\,. \tag{A.11}$$

It is a (multivalued) $\frac{g}{2}$ - differential without zeroes and poles defined on a covering of Σ. It has the following multivaluedness

$$\sigma(z + na + mb) = \mathrm{e}^{\pi i (g-1) m\Omega m - 2\pi i m(I(\Delta) - (g-1)I(z))} \sigma(z)\,.$$

A useful relation involving $\sigma(z)$ is

$$\theta(z - I(P_1 + ... + P_g) + \Delta) = s(P_1, ..., P_g)\sigma(z)E(z, P_1)...E(z, P_g)\,, \qquad (A.12)$$

where $s(P_1, ..., P_g)$ is a holomorphic section of a line bundle of degree $g-1$ in each variable.

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