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IMPULSIVE CONTROL SYSTEMS

Dynamics, optimization problems, and applications to mechanics

Thesis submitted for the degree of Doctor Philosophiae

Academic Year 1988/1989

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Introduction

Impulsive control systems. In the classical theory of ordinary differential equations, solutions are sought within some family of absolutely continuous maps. In particular, if a *control differential equation* of the form

$$\text{(E)} \qquad \dot{x}(t) = f(t, x(t), u(t))$$

is considered; with f sufficiently regular, then the control maps $u(\cdot)$ have to be at least measurable.

On the other hand, control problems deriving from applied sciences might be modelled by a more complex dynamics than (E). In particular, one could encounter control differential equations whose right-hand sides contain the first derivative of the control .Then, more regularity must be required to the control mappings in order to remain within the classical theory of O.D.E.

Alternatively, one can supply some new concept of *solution* . For example, if discontinuous controls are to be used, one is forced to drop the assumption of continuity for the solutions, since one expects that a first kind discontinuity of the control generates a first kind discontinuity of the solution. That is the reason why we call *impulsive* the control systems containing the first derivative of the control on their right-hand sides. And it is just this kind of control systems we are going to treat in this thesis.

More precisely, we shall further require that the dependence on the derivative of the control is affine, i.e., to say, we shall deal with systems of the form

$$\text{(I.E.)} \qquad \dot{x} = f(t, x(t), u(t)) + \sum_{\alpha=1}^{m} g_\alpha(t, x(t), u(t))\, \dot{u}^\alpha(t) .$$

Applications. Before describing the topics of analytical character contained in this thesis, we wish to discuss some motivations for studying such a control system. Actually, these can be found in the Appendix, which is devoted to expose some contributions given by the author (see [31], [33]) to a mechanical theory recently introduced by Aldo Bressan (see [10-13]). Within this theory, a Lagrangian system Σ is considered, and an additional moving holonomic constraint on Σ is thought as a control.
If $(q_i, \gamma_\alpha)_{\alpha=1,\dots,M}^{i=1,\dots,N}$ denotes a set of Lagrangian coordinates for Σ, this additional constraint has a kinematic representation of the form $\gamma^\alpha = \hat{\gamma}^\alpha(t)$.

This implies that the dynamical equations for the system $\Sigma_{\hat{\gamma}}$ obtained from Σ by adding the constraint $\gamma^\alpha = \hat{\gamma}^\alpha(t)$, contain the derivative of the control $u^\alpha(\cdot) = \hat{\gamma}^\alpha(\cdot)$ on their right-hand sides. Moreover, under suitable assumptions on the coordinates (q^i, γ^α) these derivatives appear linearly. Hence, a control system of form (I.E.) governs the evolution of $\Sigma_{\hat{\gamma}}$. In particular, the control $u^\alpha(\cdot) = \hat{\gamma}^\alpha(\cdot)$ may be used in order to minimize some cost function, e.g. the kinetic energy of $\Sigma_{\hat{\gamma}}$ at a certain instant T .

These mechanical applications are our primary motivation for investigating control problems governed by a dynamics of form (I.E.). Also, some results (see [9]) obtained by the author in collaboration with Alberto Bressan on the robustness of solutions to (I.E.) has been very recently applied by G.Ferreyra (see [18]) in order to prove an approximation rule for the optimal solution of a minimizing problem in advertising.

Generalized solutions corresponding to scalar controls. After some geometrical preliminaries, in the first chapter of the thesis we present a number of results which mainly concern the continuous dependence of the (generalized) solutions to (I.E.) with respect to the controls $u(\cdot)$.

Actually, unless $u(\cdot)$ is differentiable (or, at least, absolutely continuous), it is not clear what one should intend by solution of (I.E.). A first satisfactory answer to this question was given by Sussmann [36] (see Chapter 1) in the case of scalar continuous inputs. Indeed he considered the input-output functional Φ which maps a smooth control $u(\cdot)$ into the corresponding trajectory $x(u,\cdot)$ of (1.4) on the time interval $[0,T]$. By using a method of successive approximations he showed that Φ admits a unique continuous (w.r. to the C^0 norms) extension $\hat{\Phi}$ defined for controls u which are merely continuous, possibly with unbounded variation. Finally he defined the solution relative to a continuous control u to be the image $\hat{\Phi}(u)$ of u under the map $\hat{\Phi}$.

In [7] Alberto Bressan pushed Sussmann's approach further, in order to include discontinuous (scalar) inputs as well(see Chapter 1). In fact , he proved that the input-output map Φ is Lipschitz continuous with respect to L^1 norms on the spaces of controls and trajectories, hence it admits a unique extension to a functional $\tilde{\Phi}$ that maps L^1-equivalence classes of controls into L^1-equivalence classes of trajectories. Moreover this correspondence was further refined by constructing a version of $\tilde{\Phi}$ which is Lipschitz continuous with respect to the norm of uniform convergence in $[0,T]$. Thanks to these results, a definition of (generalized) solution to (I.E.) corresponding to a scalar summable control was presented in [7] .

Generalized solutions corresponding to vector-valued controls. Sussmann's and Bressan's results cannot be extended to the case of vector valued controls, unless all Lie brackets $[g_i, g_j]$, $i,j = 1,\dots,m$, vanish identically. Indeed, in the general case the input-output map Φ is not continuous with respect to the C^0 norms on the spaces of trajectories and controls. This is due to the interactions which occur among the components of the inputs, as consequence of the non commutativity of the Lie-brackets $[g_i, g_j]$. Examples of this discontinuous behaviour of Φ are provided e.g. by Sussmann [36] and Fliess [20].

Yet, Bressan-Rampazzo [9] proved that the input-output map remains (Lipschitz) continuous (w.r. to C^0 norms) in the multidimensional case whenever one restricts the class of controls to a set of maps having equibounded Lipschitz constants. In other words, if a bound is imposed to the velocities of the input, then the above mentioned interactions cannot occur. This fact allowed to treat also the case of (possibly discontinuous) vector-valued controls with bounded total variations. For this prupose, in [9], a free-parameter distance δ between two paths and a concept of Lipschitzean *graph completion* φ for the graph of a control u are preliminarily introduced. Then, one shows that the (suitably defined) trajectory $x(\varphi,\cdot)$ corresponding to φ can be approximated by the solutions $x(u_n,\cdot)$ corresponding to smooth controls u_n, provided the graphs of the u_n tend (in the metric δ) to φ, when $n \to +\infty$. These result are reported in Chapter 2.

In [16], Dal Maso-Rampazzo study the vector-valued case from a measure theoretical point of view (see Chapter 3). For this purpose, suitable extensions of (I.E.) are proposed in the case of a control u with bounded total variation. For each of these extensions a result of continuous dependence of the (generalized) solutions with respect to the control is proved, provided certain hypotheses on the variation of the approximating (regular) controls are assumed. Furthermore, only conditions on the original controls $u(\cdot)$ are considered, without mentioning their graph completions.

Minimization problems for impulsive control systems. In the second part of this thesis (Chapters 4 and 5), an optimal control problem for systems of the form (I.E.) is treated. Precisely, one seeks the minimum value of

$$\{ \gamma(x(u,T)),\ u \in \mathcal{U} \},$$

where γ is a continuous function of the state, $\mathcal{U}$ is a family of *admissible controls* taking values in a compact subset $U \subseteq \mathbb{R}^n$, and $x(u,T)$ denotes the final value of the solution $x(u,\cdot)$ of (I.E.) corresponding to the control $u \in \mathcal{U}$.

We have already mentioned a possible application of this problem to mechanics. Furthermore, two concrete examples, concerning the ski and the swing, can be found in [12]. Yet, those examples are treatable by means of the ordinary theory of optimal control, for, thanks to a certain choice of the coordinates, the derivatives of the control do not appear in the differential constraint. Of course, this is not the general case, since , as long as the control is vector valued, such coordinates exist only under special conditions on the Riemannian structure of the constraint manifols. Hence mechanics demand an extension of the ordinary theory of optimal control to problems having an equation of the form (I.E.) as differential constraint.

In Chapter 4 the optimization problem with an a priori bound on the total variation of the controls is investigated. In view of the results contained in the previous chapters, one expects that the optimal controls contain *instantaneous* arcs. Actually, a result on the existence of such an optimal control is proved. Also, it is shown that the minimun value of the cost function may be approximated by means of Lipschitz continuous controls. These results have been obtained by the author of this thesis and have been recently proposed for publication.

Chapter 5 is formed by the first part of a joint work of the author and Alberto Bressan (see [8]).In this work, which is now in an advanced state of preparation, the authors tackle the optimization problem without any constraint on the total variations of the controls.
In the part presented in this thesis, we investigate the case in which all the Lie brackets $[g_\alpha, g_\beta]$ vanish identically. By means of a suitable diffeomorphism of the space of the couples state-control, we are able to refer the existence problem to the non-impulsive case. Also, a necessary condition for optimality is proved in the form of a maximum principle.

The investigation of the general case, in which the brackets $[g_\alpha, g_\beta]$ are not all equal to zero, is now going to be completed and will appear in [8].

Chapter 0

Some differential geometric preliminaries

In this Chapter, which is principally based on [3] and [23], some basic facts on vector fields, flows and Lie differentiations are rapidly recalled. These notions are important in order to understand why certain results which hold for scalar inputs (Chapter 1) are no longer valid in the case of multidimensional controls (Chapter 2).

For the sake of completeness, definitions and theorems are given for a differentiable manifold, althought in the next chapters the state variable merely belongs to an open subset of R^n.

1 Flows and Lie derivatives

Throughout this chapter the qualification *smooth* always means "having a suitable degree of differentiability so that all the required differentiations can be actually performed".

Let M be an n-dimensional smooth manifold, and let f be a smooth vector field on M . With f we associate two objects:

1)The *one parameter group * of diffeomorhisms* or *flow* $\Phi_f^t : M \to M$, where $\forall x^0 \in M$, $\Phi_g^t(x^0)$ is the value at time t of the solution to the Cauchy Problem

$$(1.1) \qquad \begin{cases} \dfrac{dx}{dt} = f(x) \\ x(0) = x^0 \end{cases}$$

2) The first order differential operator L_f , called the *Lie derivative in the direction of* f : for any differentiable function $\varphi : M \to R$, $L_f\varphi$ is a new function from M into R, whose value at a point $x \in M$ is

---

*Φ_f^t is really a group if and only if f is *complete* , i.e. if it is defined for all $t \in R$. In general Φ_f^t is defined only for sufficiently small t.

(1.2) $$L_f\varphi(x) = \frac{d}{dt}\Big|_{t=0} \varphi(\Phi_f^t(x))$$

It is trivial to verify that L_f is a linear operator from C^p functions $(p \geq 1)$ into C^{p-1} functions, which satisfies Liebniz formula

(1.3) $$L_f(\varphi_1 \varphi_2) = \varphi_1 L_f\varphi_2 + \varphi_2 L_f\varphi_1$$

In terms of local coordinates $(x^1,\ldots,x^n)$ the differential system $(1.1)_1$ is represented by

(1.4) $$dx^1/dt = f^1(x),\ldots, dx^n/dt = f^n(x) \quad ,$$

where $f^1(x),\ldots,f^n(x)$ are the components of $f(x)$ in the coordinate system $(x^1,\ldots,x^n)$.

Therefore

(1.5) $$L_f\varphi = f^1 \frac{\partial\varphi}{\partial x^1} + \ldots + f^n \frac{\partial\varphi}{\partial x^n}$$

We could say that in the coordinates $(x^1,\ldots,x^n)$ the operator L_f has the form

(1.6) $$L_f = f^1 \frac{\partial}{\partial x^1} + \ldots + f^n \frac{\partial}{\partial x^n};$$

this is the general form of a first-order linear differential operator on coordinate space. incidentally let us notice that the right-hand side of (1.6) may be interpreted as the expression of the vector field f in terms of the coordinate base $(\partial/\partial x^1,\ldots,\partial/\partial x^n)$ of the tangent bundle TU, where $U \subseteq M$ is the domain of the coordinates $(x^1,\ldots,x^n)$.

One can easily prove that the correspondences between vector fields f, flows Φ_f^t and differentiations L_f are one to one.

2 Lie brackets

Suppose that two smooth vector fields f and g are given on a manifold M . The corresponding flows do not, in general, commute:

$$\phi_f^t \circ \phi_g^s \neq \phi_g^s \circ \phi_f^t$$

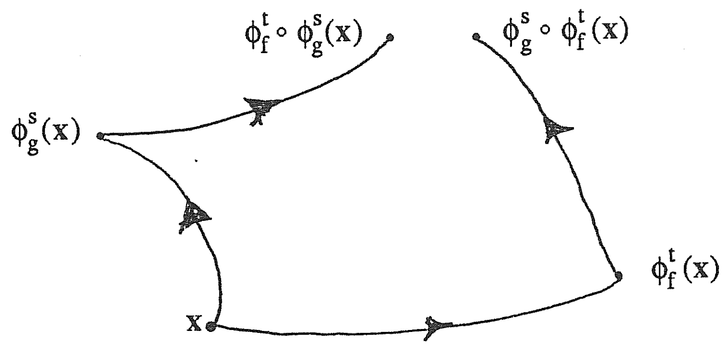

fig. 1

Example 2.1. Take the fields $f = \partial/\partial x^1$, $g = x^1\partial/\partial x^2$ on the (x^1, x^2) plane:

$$\phi_f^t \circ \phi_g^s(\bar{x}^1, \bar{x}^2) = (\bar{x}^1 + t, \bar{x}^2 + \bar{x}^1 s)$$

$$\phi_g^s \circ \phi_f^t(\bar{x}^1, \bar{x}^2) = (\bar{x}^1 + t, \bar{x}^2 + \bar{x}^1 s + ts)$$

To measure the degree of non-commutativity of the two flows ϕ_f^t and ϕ_g^s , let us fix x and consider the points $\phi_g^s \circ \phi_f^t(x)$ and $\phi_f^t \circ \phi_g^s(x)$. In order to estimate the difference between these points, let us compare the value at them of some smooth function φ from M into **R** . The difference

$$A(t,s,x) = \varphi(\phi_g^s \circ \phi_f^t(x)) - \varphi(\phi_f^t \circ \phi_g^s(x))$$

is clearly a differentiable function which is zero for s = 0 and for t = 0 . Therefore, the first term different from 0 in the Taylor expansion in s and t of A at (0,0) contains st , and the other terms of second order vanish. The coefficient of st is given by the value of $\partial^2 A/\partial s\partial t$ at (0,0) .

Lemma 2.1. The mixed partial derivative $\partial^2 A/\partial s\partial t$ at $(0,0)$ is equal to the commutator of differentiation in the directions f and g :

$$\text{(2.1)} \qquad \left.\frac{\partial^2}{\partial s\partial t}\right|_{s=t=0} \{\varphi(\phi_g^s \circ \phi_f^t(x)) - \varphi(\phi_f^t \circ \phi_g^s(x))\} = (L_f L_g\, \varphi - L_g\, L_f\, \varphi)(x)$$

Proof. By the definition of L_g ,

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \varphi(\phi_g^s \circ \phi_f^t(x)) = (L_g\, \varphi)\, (\phi_f^t(x)) \,.$$

Setting $\psi = L_g\varphi$, by the definition of L_f one has

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \psi(\phi_f^t(x)) = (L_f\, \psi) \,.$$

Thus,

$$\left.\frac{\partial}{\partial t\partial s}\right|_{s=t=0} \varphi(\phi_g^s \circ \phi_f^t(x)) = (L_f L_g\, \varphi)(x)) \,,$$

which proves (2.1) .

At first glance $L_f L_g - L_g L_f$ looks like a second-order differential operator. On the contrary:

Lemma 2.2. *The operator* $L_f L_g - L_g L_f$ *is a first-order linear differential operator.*

Proof. Let $(f^1,\ldots,f^n)$ and $(g^1,\ldots,g^n)$ be the components of f and g in a local coordinate system $(x^1,\ldots,x^n)$. Then

$$L_fL_g\varphi = \sum_{i=1}^{n} f^i\frac{\partial}{\partial x^i}\sum_{j=1}^{n} g^j\frac{\partial\varphi}{\partial x^j} = \sum_{i,j=1}^{n} f^i\frac{\partial g^j}{\partial x^i}\frac{\partial\varphi}{\partial x^j} + \sum_{i,j=1}^{n} f^ig^j\frac{\partial^2\varphi}{\partial x^i\partial x^j} \ .$$

If one subtracts $L_gL_f\varphi$, the terms with the second derivatives of φ vanish. Hence

(2.2) $$(L_fL_g - L_gL_f)\varphi = \sum_{i,j=1}^{n} (f^i\frac{\partial g^j}{\partial x^i} - g^i\frac{\partial f^j}{\partial x^i})\frac{\partial\varphi}{\partial x^j} \, .$$

QED

Since every first-order linear differential operator is given by a vector field, the operator $L_fL_g - L_gL_f$ also corresponds to some vector field :

Definition 2.1 The *Lie brackets* or *commutator* of two vector fields f and g are the vector field [f, g] for which

(2.3) $$L_{[f,\,g]} = L_fL_g - L_gL_f \ .$$

Setting $\varphi(x) = x^r(x)$ $(r= 1,\ldots,n)$ in Lemma 2.2, one proves that the r-th component of [f, g] in a coordinate system $(x^1,\ldots,x^n)$ is given by

(2.4) $$[f, g]^r = \sum_{i=1}^{n} (f^i\frac{\partial g^r}{\partial x^i} - g^i\frac{\partial f^r}{\partial x^i}).$$

3. The Lie Algebra of C^∞ vector fields

Definition 3.1 A Lie Algebra is a vector space L , together with a bilinear skewsymmetric operation $[\cdot,\cdot] : L\times L \to L$ which satisfy the *Jacoby identity* :

(3.1) $$[[A.B],C] + [[B,C],A] + [[C,A],B] = 0$$

for all triples A,B,C of elements of L . The operation $[\cdot,\cdot]$ is usually called the *commutator.*

Proposition 3.1. The Lie brackets make the vector space of C^∞ vector fields on a manifold M into a Lie algebra.

Proof. Bilinearity and skew-simmetry of the Lie brackets are clear. Let us prove the Jacobi identity. By definition of Lie brackets one has, for all triples f,g,r, of smooth vector fields

$$L_{[[f,g],r]} = L_{[f,g]}L_r - L_rL_{[f,g]} = L_fL_gL_r - L_gL_fL_r - L_rL_fL_g + L_rL_gL_f.$$

There will be 12 terms in all the sum $L_{[[f,g],r]} + L_{[[r,f],g]} + L_{[[g,r],f]}$. Each term appears twice, with opposite sign. By the linearity of L_f with respect to f , this proves the theorem.

4. A condition for the commutativity of flows

By Lemma 2.1. vanishing of [f,g] is necessary in order that the flows ϕ_f^t and ϕ_g^s commute. The following theorem proves that it is also sufficient.

Theorem 4.1. Let f and g be smooth vector fields on a manifold M , and assume [f, g] = 0. Then

(4.1) $$\phi_f^t \circ \phi_g^s(x) = \phi_g^s \circ \phi_f^t(x),$$

for all $x \in M$ and for all t and s for which the expressions on both sides are meaningful.

Proof. Let $x \in M$. For fixed value of t , the two curves in s given by the right- and left-hand side of (4.1.) have the same initial value, namely $\phi_f^t(x)$. The curve on the right

$$s \longmapsto \phi_g^s \circ \phi_f^t(x)$$

is by definition the integral curve of g. The curve on the left

$$s \longmapsto \phi_f^t \circ \phi_g^s(x)$$

is the image under ϕ_f^t of the integral curve of g having initial condition x.

What we must show is that these two curves satisfy the same differential equation. Let us compute the tangent vectors to the curve on the left:

(4.2) $$\frac{d}{ds}(\phi_f^t \circ \phi_g^s(x)) = D\phi_f^t(\phi_g^s(x)) \cdot \frac{d}{ds}\phi_g^s(x) = D\phi_f^t(\phi_g^s(x)) \cdot g(\phi_g^s(x)) .$$

where $D\,\phi_f^t(\cdot)$ is used to denote the tangent map to $x \longmapsto \phi_f^t(x)$.

Now let us fix s and let us denote by $F(t)$ the last expression in (4.2). One must show that if

$$G(t) = g(\phi_f^t(\phi_g^s(x))),$$

then

$$F(t) = G(t).$$

One has trivially $F(0) = G(0)$, since $D\phi_f^0(y) = y \ \ \forall y \in M$. On the other hand,

$$\frac{d}{dt}F(t) = Df(\phi_f^t \circ \phi_g^s(x)) \circ g(\phi_g^s(x))$$

and

$$\frac{d}{dt}G(t) = Dg(\phi_f^t \circ \phi_g^s(x)) \circ f(\phi_f^t \circ \phi_g^s(x)) = Df(\phi_f^t \circ \phi_g^s(x)) \quad g(\phi_f^t \circ \phi_g^s(x)) ,$$

where Df $[Dg]$ denotes the tangent map to the map $x \longmapsto f(x)$ $[g(x)]$ and the last equality is a consequence of the hypothesis $0 = [f, g] = Dg \cdot f - Df \cdot g$.

Hence the two curves F and G satisfy the same differential equation and start from the same point, whence they are equal. This proves the theorem.

PART I

Impulsive control systems

In this part we present a number of results which mainly concern the continuous dependence on controls of some kinds of generalized solutions to

$$\text{(I.E.)} \qquad \dot{x} = f(t, x(t), u(t)) + \sum_{\alpha=1}^{m} g_\alpha(t, x(t), u(t))\, \dot{u}^\alpha(t) .$$

We refer to the Introduction for a brief description of the contents of this part and for some mentions to the applications in classical mechanics. As for the latter,one can also refer to the Appendix of the present thesis.

Chapter 1

Impulsive systems with scalar controls

In this chapter one deals with an impulsive system with scalar control, i.e., to say, one consider a Cauchy problem of the form

$$(0.1) \qquad \begin{cases} \dot{x} = f(x) + g(x)\,\dot{u} \\ x(0) = \bar{x}\ . \end{cases}$$

For this kind of problem one obtains much stronger results than in the case of multidimensional controls, principally due to the absence of those phenomena of interaction between the components of the input which occur when $m > 1$.

As long as u is continuosly differentiable, the classical theory of ordinary differential equations applies to (0.1), provided f and g have a sufficient degree of regularity. However, if u is not so regular, its derivative has only a distributional meaning and the classical (Caratheodory) definition of solution cannot be used.

In [36] Sussmann tackled the problem for (scalar) continuous inputs. First, he proved the continuity of the input-output map with respect to the C^0 norms; then he defined the solution $x(u,\cdot)$ corresponding to a continuous u as the uniform limit of the solutions $x(u_n,\cdot)$ corresponding to a sequence of smooth controls u_n which converge uniformely to u when n tends to infinity. He also extented to this kind of solutions some classical theorems of uniqueness, existence and continuous dependence on the control and the initial data .

Bressan[7] pushed Sussmann's approach further, in order to include (summable) discontinuous controls as well. Unlike Sussmann, which had adopted a method of successive approximations, Bressan made use of a fixed point argument,.

It is important to point out that all the results of this Chapter can be extended to the case where f and g depend on t and u as well, simply by adding the new variables $x^{n+1} = t$, $x^{n+2} = u$.

An alternative way to face the problem was followed by Pandit-Deo [29], that defined solutions in the distributional sense. Yet, this work will be not reported within the present paper because of some contradictory results containedin it [see Hajek [21]].

The main tools of Sussmann's paper [36] together with some sketches of the proofs form the first Section of this chapter. The subsequent Section provides a detailed exposition of the work by Bressan [7].

1 A survey of Sussmann's results on continuous scalar inputs

BASIC DEFINITIONS

If $x=(x^1,\dots,x^n)\in \mathbf{R}^n$ and M is a square matrix, then $|x|$ and $|M|$ denote the Euclidean norm of x and the matrix norm of M, respectively, i.e.,

$$|x| = \Big(\sum_{k=1}^{n}(x^k)^2\Big)^{1/2}, \qquad |M| = \sup\{|Mx| : x\in \mathbf{R}^n,\ |x|=1\}.$$

If ϕ is a scalar-, or vector-, or matrix-valued function defined on an open subset Ω of $\mathbf{R}^n$, one says that ϕ is *Lipschitz continuous* on a set $S\subseteq\Omega$ if there is a constant C such that $|\phi(x) - \phi(y)| \le C|x - y|$ for all x, y in S. One calls ϕ *locally Lipschitz continuous* if ϕ is Lipschitz continuous on every compact subset of Ω. One says that ϕ satisfies a linear growth condition if there is a constant C such that $|\phi(x)| \le C(1+|x|)$ for all $x\in\Omega$.

If f is a vector field in Ω which is of class C^1, then Df denotes the matrix

$$Df = (\partial f^i/\partial x^j)_{i,j}$$

of the partial derivatives of the components of f. It is clear that, if Df is uniformely bounded, then f satisfies a linear growth condition, but the converse is not true.

Let a<b and let $C^r([a, b])$ $\big[C^r([a, b], \mathbf{R}^n)\big]$ denote the space of real-valued $\big[\mathbf{R}^n$-valued$\big]$ r times continuously differentiable functions on]a, b[which can be extended to continuous funtions on [a, b]. Let us consider vector fields f and g on a open subset Ω of $\mathbf{R}^n$ and let $t_0\in [a, b]$, $x_0\in\Omega$ be given. Denote by Φ the input-output functional which maps a control $u\in C^1([a, b])$ into the corresponding solution (in the ordinary sense,if it exists) $x(u\ \cdot)$ on [a, b] of the Cauchy Problem

$$(1.1)\qquad \begin{cases} \dot{x} = f(x) + g(x)\,\dot{u} \\ x(t_0) = x_0\,. \end{cases}$$

Definition 1.1. Let $u \in C^0([a, b])$. A curve $\gamma : t \mapsto x(t)$, $a \le t \le b$, $x(t) \in \Omega$, is said to be a solution of the initial value problem (1.1) if there exist a neighborhood N of u in $C^0([a, b])$ and a continuous map $\hat{\Phi} : N \mapsto C^0([a, b], R^n)$ such that:

(i) for each $v \in N \cap C^1([a, b])$, $\hat{\Phi}(v) = \Phi(v)$ $[\equiv x(v,\cdot)]$ and

(ii) $\hat{\Phi}(u) = \gamma$

The concept of solution for arbitrary intervals I of the real line is given in an obvious way:

Definition 1.2. A curve $\gamma : I \to \Omega$ is said to be a solution of the initial value problem (1.1) if, for every closed bounded interval $I' \subseteq I$ such that $t_0 \in I'$, the restriction of γ to I' is a solution of (1.1).

THE MAIN THEOREMS

Theorem 1.1. *Assume that*

(i) Ω *is an open subset of* R^n,

(ii) f,g *are vector fields on* Ω,

(iii) f *is locally Lipschitz continuous,*

(iv) g *is of class* C^1 *and its partial derivatives are locally Lipschitz continuous.*

Let I *be an interval of the real line, and let* t_0 *belong to the interior of* I, $x_0 \in \Omega$. *Let* u *be a real-valued continuous function on* I. *Then*

(1) *There is an interval* I', *conteining* t_0 *in its interior, and a curve* $t \mapsto x(t)$, $t \in I' \subseteq I$, *which is a solution of* (1.1).

(2) *If* I' *is any such interval, then the solution of* (1.1) *which is defined on* $I' \subseteq I$ *is unique.*

Theorem 1.2. *With the same hypotheses as in* Theorem 1.1 *assume, in addition, that* $\Omega = \mathbf{R}^n$, *that* f *satisfies a linear growth condition, and that* Dg *is uniformely bounded.*

Then, for every choice of I, t_0, x_0, u, *there is a solution of* (1.1) *defined on the whole interval* I.

Moreover, the solution depends continuosly on t_0, x_0 *and* u.

Remark. The assumption that Dg is uniformely bounded implies, in particular, that g satisfies a linear growth condition. Sussmann shows (with an example) that Theorem 2 is not true if the requirement that Dg be bounded is eliminated, even if g is required to grow linearly.

SKETCH OF THE PROOFS.

Here we want to quote only the main ideas of the proofs, refering to the original paper [36] for all calculations and estimates.

First, it is clear that both the existence and uniqueness results follow for arbitrary I if they are true for all compact I. Moreover, it clearly suffices to assume that $t_0=0$.

The uniqueness is trivial. Indeed, if γ_1: $t \mapsto x_1(t)$ and γ_2: $t \mapsto x_2(t)$ are solutions of (1.1) on I=[a, b], for a continuous u:[a, b]$\to\mathbf{R}$, then there are neighborhoods N_1, N_2 of u in $C^0(I, \mathbf{R})$, and continuous maps $\hat{\Phi}_1$, $\hat{\Phi}_2$ from N_1, N_2 into $C^0(I, \mathbf{R}^n)$ such that, for i=1,2, $\hat{\Phi}_i(u)=\gamma_i$ and that, whenever $v\in N_i$ is of class C^1, then $\hat{\Phi}^i(v)=\Phi(v)$ $[\equiv x(v,\cdot)]$. Since the C^1 functions are dense in $N_1 \cap N_2$ it follows that $\hat{\Phi}^1(u)=\hat{\Phi}^2(u)$, i.e. $\gamma^1=\gamma^2$.

Concerning existence, first the result of Theorem 1.1 is proved and then it is used to deduce the result of Theorem 1.2. An important role is played in the proof by a certain change of ccordinates, which reduces the initial value problem (1.1) to the special case in which

$$g = \underline{e}_1 = (1,0,0,\ldots,0).$$

Then one shows that proving the existence result for this special case is equivalent to prove it in general. Since this same transomation of coordinates is performed by Bressan in [7] , one can find it in the next Section of this chapter, where [7] is entirely reported and commented on.

Sussmann uses a method of successive approximations in order to prove existence of solutions in the special case

(1.2)
$$\begin{cases} \dot{x} = f(x) + \underline{e}_1 \dot{u} \\ x(t_0) = x_0 . \end{cases}$$

More precisely, he lets, for $t \in [a, b]$,

(1.3)
$$x_0(t) = x_0$$

$$x_{k+1}(t) = x_0 + \int_0^t f(x_k(s))\, ds + [u(t) - u(0)]\, \underline{e}_1$$

and shows that:

(T) the convergence of $(x_k(\cdot))_{k \in N}$ is uniform in t, x_0, u, as long as x_0 remains within a compact subset of $\mathbf{R}^n$ and u belongs to a bounded subset of $C^0[a, b]$.

Then, for each k, one lets $L_k : \mathbf{R}^n \times C^0[a, b] \to C^0[a, b]$ to be the functional which, to every $x_0 \in \mathbf{R}^n$ and every $u \in C^0[a, b]$, assigns the curve $t \longmapsto x_k(t)$. The L_k are continuous and, by (T), have a limit L as $k \to +\infty$, this convergence being uniform on the bounded subsets of $\mathbf{R}^n \times C^0[a, b]$. Therefore L is continuous.

Now let $u \in C^0[a, b]$ and $x_0 \in \mathbf{R}^n$. One proves that $L(x_0, u)$ is a solution to (1.1) in the sense of Definition 1.1. Indeed, if v is a C^1 function on [a, b], then

(1.4)
$$v(t) - v(0) = \int_0^t \dot{v}(\tau)\, d\tau$$

Therefore the successive approximations $t \longmapsto L_k(x_0, v)(t)$ satisfy

(1.5)
$$L_{k+1}(x_0, v) = x_0 + \int_0^t [f(x_k(\tau)) + \dot{v}(\tau)\, \underline{e}_1\, d\tau] .$$

So the $L_k(x_0, v)$ are the ordinary successive approximations that are used to construct the solutions of

(1.6)
$$\begin{cases} \dot{x} = f(x) + \underline{e}_1 \dot{v} \\ x(t_0) = x_0 . \end{cases}$$

It follows that, for every x_0 and every v of class C^1 the function $L(x_0,v)$ is the solution of (1.6), i.e. $L(x_0,v) = \Phi(x_0,v)$ [$\equiv x(v,\cdot)$]. Since $u \mapsto L(x_0,u)$ is continuous, by setting $\hat{\Phi}(u)=:L(x_0,u)$, it follows that, for each $x_0 \in \mathbf{R}^n$, $u \in C^0[a, b]$, the curve $t \mapsto L(x_0,u)(t)$ is a solution of (1.2), according with definition (1.1). Since L also depends continuously on x_0, it is clear that the last assertion of Theorem 2 follows. This concludes the proof of Theorem 2 in the special case (1.2). The proof of the general case follows thanks to the above mentioned transformation of coordinates.

Lastly, Theorem 1 is easily deduced by Theorem 2 by considering new vector fields $\mathcal{f}$ and $\mathcal{g}$ defined by

$$\begin{cases} \mathcal{f}(x) = r(x)\, f(x) \\ \mathcal{g}(x) = r(x)\, g(x) \end{cases} \quad \text{for } x \in \Omega \;, \quad \begin{cases} \mathcal{f}(x) = f(x) \\ \mathcal{g}(x) = g(x) \end{cases} \quad \text{for } x \notin \Omega$$

where $r: \Omega \to \mathbf{R}$ is a C^0 function which vanishes on the complement of a compact subset of Ω and is equal to one in a neighborhood of x_0.

2. Summable inputs

In this Section Bressan' s paper is reported and commented on. For the sake of readability, the transformation of coordinates which is used in the proof of the main theorem is made here more explicit than in the original paper. As it has been mentioned in the previous Section, this same transformation is used by Sussmann in [36].

BASIC ESTIMATES

Let Ω be an open set in $\mathbf{R}^n$ and let f, g be C^1 and C^2 vector fields on W. Given a scalar control $u(\bullet)$ belonging to the space $\mathcal{L}^1[0, T]$ of summable real maps on [0, T], let us denote (as in Section 1) by $x(u,\bullet)$ the solution (if it exists) of the Cauchy Problem

$$(2.1) \qquad \begin{cases} \dot{x} = f(x) + g(x)\, \dot{u} \\ x(0) = x_0 \in \Omega \end{cases}$$

on the time interval [0, T]. Using the coordinates $x = (x^1,\dots,x^n)$, (2.1) becomes

$$
(2.2) \qquad \begin{cases} \dot{x}^1 = f^1(x) + g^1(x)\,\dot{u} & x^1(0) = x_0^1 \\ \dots\dots & \dots\dots \\ \dot{x}^n = f^n(x) = g^n(x)\,\dot{u} & x^n(0) = x_0^n \end{cases}
$$

In order to extend the input-output map $\Phi : u(\cdot) \longmapsto x(u,\cdot)$ from $C^1[0, T]$ to a broader class of controls , it is necessary to investigate the continuity of Φ with respect to weaker norms on the spaces of controls and trajectories.

Theorem 2.1. *Let* $U \subset C^1[0, T]$ *and let* $K \subset \Omega$, $K' \subset R$ *be compact sets such that*

i) *all controls take values inside* K',

ii) *for every* $u \in U$, *the solution* $x(u,\bullet)$ *of* (2.1) *exists on* [0, T] *and takes values inside* K.

Then there exists a constant M *such that*

$$
(2.3) \qquad |x(u,\tau) - x(v,\tau)| + \int_0^T |x(u,t) - x(v,t)|\,dt \le
$$

$$
\le M[\,|u(0) - v(0)| + |u(\tau) - v(\tau)| + \int_0^T |u(t) - v(t)|\,dt\,]
$$

for all $u, v \in U$, $\tau \in [0, T]$.

Proof. The Theorem will be proven first for control systems of the form

$$
(2.4) \qquad \begin{cases} \dot{x} = f(x) + \underline{e}\,\dot{u} \\ x(0) = x_0 \in \mathbf{R}^n , \end{cases}
$$

where f is a C^1 vector field with compact support in $\mathbf{R}^n$ and $\underline{e}$ is a unit vector, then in the general case. For any $u \in C^1[0, T]$, (2.4) is equivalent to the integral equation

(2.5) $$x(t) = x_0 + \int_0^t f(x(s))\, ds + \underline{e}\,[u(t) - u(0)]\,,$$

which can be written in the more compact form

(2.6) $$x(u) = \psi(u,x(u))$$

with

(2.7) $$\psi(u, x)(t) = x_0 + \int_0^t f(x(s))\, ds + \underline{e}\,[u(t) - u(0)]\,.$$

In order to show that the functional $\Phi : u \longmapsto x(u,\cdot)$, implicitly defined by (2.6), is Lipschitz continuous (with respect to suitable norms) one relies upon the following corollary of the Contraction Mapping Theorem[17].

Lemma 2.1. Let E, F be Banach spaces, $\psi : E\times F \to F$ be a map such that $\forall\ u,v\in E$, $\forall\ x,y\in F$ one has

(2.8) $$\| \psi(u,x) - \psi(u,y) \|_F \le 1/2\ \| x - y \|_F$$ $$\| \psi(u,x) - \psi(v,x) \|_F \le L\ \| u - v \|_E$$

for some constant L. Then for each $u\in E$ there exists a unique $x = x(u)\in F$ such that $x(u) = \psi(u,x(u))$. Moreover

(2.9) $$\| x(u) - x(v) \|_F \le 2L\ \| u - v \|_E.$$

Proof of the lemma. For each $u\in E$, $x(u)$ exists and is unique, being the fixed point of the strict contraction $x \longmapsto \psi(u,x)$ in F. Moreover

$$\| x(u) - x(v) \|_F \le \| \psi(u,x(u)) - \psi(u,x(v)) \|_F + \| \psi(u,x(v)) - \psi(v,x(v))\|_F \le$$
$$\le 1/2\ \| x(u) - x(v) \|_F + L\| u - v \|_E,$$

from which (2.9) follows.

To prove (2.3) for the special system (2.4), choose a constant N≥1 such that the operator norm of the derivative $Df = (\partial fi/\partial xj)_{i,j}$ of f satisfies

$$\| Df(x) \| \leq N \qquad \forall x \in \mathbb{R}^n. \tag{2.10}$$

Lemma (2.1) will be applied to the functional ψ defined by (2.7) on the spaces $E = \left\{ u;\ u \in \mathcal{L}^1[0, T] \right\}$ with norm

$$\|u\|_E = |u(0)| + |u(\tau)| + \int_0^T |u(t)|\, dt$$

and $F = \left\{ u;\ u \in \mathcal{L}^1[0, T] \right\}$ with norm

$$\|x\|_F = \frac{e^{-4NT}}{4N} |x(\tau)| + \int_0^T e^{-4Nt} |x(t)|\, dt .$$

The assumption (2.8) are both satisfied. Indeed, if $u \in E$, $x,y \in F$, recalling (2.10) one has

$$\|\psi(u,x) - \psi(u,y)\|_F = \frac{e^{-4NT}}{4N} \left|\int_0^\tau [f(x(t)) - f(y(t))]dt\right| + \int_0^T e^{-4Nt} \left|\int_0^t [f(x(s)) - f(y(s))]ds\right| \leq$$

$$\leq \frac{e^{-4NT}}{4N} \int_0^\tau N\, |x(t) - y(t)|\, dt + \int_0^T e^{-4Nt} \int_0^t N\, |x(s) - y(s)|\, ds\, dt \leq$$

$$\leq \frac{1}{4} \int_0^\tau e^{-4Nt} |x(t) - y(t)|\, dt + \int_0^T |x(s) - y(s)| \int_s^T N e^{-4Nt}\, dt\, ds \leq$$

$$\leq \frac{1}{4} \|x - y\|_F + \int_0^T |x(s) - y(s)| \circ \frac{1}{4} [e^{-4Ns} - e^{-4NT}]\, ds \leq \frac{1}{2} \|x - y\|_F ,$$

hence $(2.8)_1$ holds. As to $(2.8)_2$ one has, for $u,v \in E$, $x \in F$:

$$\|\psi(u, x) - \psi(v, x)\|_F \leq |u(\tau) - v(\tau)| + \int_0^T e^{-4Nt} |(u(t) - u(0)) - (v(t) - v(0))|\, dt \leq \|u - v\|_E .$$

This yields $(8.2)_2$ with $L = 1$.

By Lemma 2.1, the map $u \mapsto x(u,\cdot)$, implicitly defined by (2.6), (2.7), is Lipschitz continuous with constant 2. This means that, for all $u,v \in C^1[0, T]$,

$$\frac{|x(u,\tau) - x(v,\tau)|}{4Ne^{4NT}} + \int_0^T e^{-4Nt} |x(u,t) - x(v,t)|\, dt \le$$

$$2\,[|u(0) - v(0)| + |u(\tau) - v(\tau)| + \int_0^T |u(t) - v(t)|\, dt] \;,$$

and yields (2.3) with $M = 8Ne^{4NT}$.

To achieve the proof in the general case, notice first that if f,g in (2.1) are replaced by vector fields f^*, g^* with compact support such that

$$f^*(x) = f(x), \qquad g^*(x) = g(x) \qquad\qquad \forall x \in K,$$

then the input-output map $\Phi : u \mapsto x(u,\cdot)$ does not change on U. One can thus assume that f and g have already compact support.

Consider the system on $\mathbf{R}^{n+1}$ obtained by adjoining to (2.2) the trivial equation $\dot{x}^0 = \dot{u}$, $x^0(0) = 0$, which yields $x^0(t) = u(t) - u(0)$. This can ce written in the form

$$(2.11) \qquad \begin{cases} \dot{x} = f(x) + g(x)\,\dot{u} \\ x(0) = (0,x^0) \end{cases}$$

with $x = (x^0,\ldots,x^n) = (x^0,x) \in \mathbf{R}^{n+1}$, $f(x) = f(x^0,x) = (0,f^1(x),\ldots,f^n(x))$, $g(x) = g(x^0,x) = (1,g^1(x),\ldots,g^n(x))$.

Construct on $\mathbf{R}^{n+1}$ a new set of coordinates $y = (y^0,\ldots,y^n)$ as follows. Given the n+1-tuple $(y^0,\ldots,y^n)$, let $s \mapsto (x^0(s),\ldots,x^n(s))$ be the solution of the Cauchy problem

$$(2.12) \qquad \begin{cases} \dot{x}^0(s) = 1 & x^0(0) = 0 \\ \dot{x}^1(s) = g^1(x(s)) & x^1(0) = y^1 \\ \quad\cdots\cdots & \quad\cdots\cdots \\ \dot{x}^n = g^n(x(s)) & x^n(0) = y^n \;. \end{cases}$$

Define $y = (y^0,y) = (y^0,\ldots,y^n) = T(x^0,\ldots,x^n)$ as the new coordinates of the point $P = (x^0,\ldots,x^n)$ in $\mathbf{R}^{n+1}$ reached by the solution of (2.12) at time $s = y^0$. Using the notations introduced in Chapter 2, this transformation is given by

(2.13) $$(y^0,\ldots,y^n) = T(x^0,\ldots,x^n) = (x^0,\phi_g^{-x^0}(x^1,\ldots,x^n)).$$

By the properties of continuous differentiability of the solutions of O.D.E. with respect to initial data, it is easy to verify that the coordinate transformation T is a C^2 homeomorphism of $\mathbf{R}^{n+1}$ into itself, and that in the new coordinates the vector field **g** has the constant expression

$$\mathbf{g}(y) = \underline{e}_1 = (1,0,\ldots,0),$$

while the components of **f** are still C^1 functions with compact support:

$$\mathbf{f}(y) = \mathbf{f}(y^0,y) = \partial T/\partial x \circ \mathbf{f}(x) = (0,\ D(\phi_g^{-y^0}) \circ (0,\ \mathbf{f}(\phi_g^{-y^0}(y))$$

Hence, for each smooth u, the initial value problem (2.11) is equivalent to

(2.14) $$\begin{cases} \dot{y} = f(y) + \underline{e}_1\ \dot{u} \\ y(0) = T(0,x^0)\ , \end{cases}$$

in the sense that if $\mathbf{x}(u,\cdot)$, $\mathbf{y}(u,\cdot)$ are the solutions of (2.11), (2.14), respectively, then they are related by

$$y(u,t) = T(\mathbf{x}(u,t)), \qquad \forall\ t \in [0,\ T].$$

By the first part of the proof, Theorem (2.1) holds for the Cauchy problem (2.14), whence it holds for (2.11). Therefore it holds for (2.1) as well. **QED**

Remark. As in the case where the controls are vector-valued (see § 4.4), one can easily prove a result, similar to the above theorem, in which no a-priori boundness assumptions on controls and trajectories are required.

A CLASS OF GENERALIZED SOLUTIONS

In analogy with [36] (see Section 1), a notion of generalized solution for (2.1) can now be introduced.

Definition 2.1. Given an equivalence class of bounded controls $u \in \mathcal{L}^1[0, T]$ and an initial value $u(0)$, a trajectory $t \longmapsto x(u,t)$ is a *generalized solution* of (2.1) if there exists a sequence of uniformely bounded controls $v_k \in C^1[0, T]$ such that $v_k(0)=u(0)$, $v_k \to u$ in $\mathcal{L}^1[0, T]$, and the corresponding trajectories $x(v_k,\cdot)$ have uniformely bounded values and tend to $x(u,\cdot)$ in the $\mathcal{L}^1$ norm.

Thanks to the estimate (2.3), any uniform a-priori bound on $x(v_k,t)$, $t \in [0, T]$, for some sequence $v_k \to u$ will provide the existence of a generalized solution to (2.1). Such solution is unique up to $\mathcal{L}^1$-equivalence and depends continuously on the control. In the case where u is defined pointwise on [0, T], the trajectory $x(u,\cdot)$ can also be pointwise determined. Indeed, for any fixed $\tau \in [0, T]$ one can construct a sequence of C^1 controls w_k^τ such that $w_k^\tau(0)=u(0)$, $w_k^\tau(\tau)=u(\tau)$ and $w_k^\tau \to u$ in $\mathcal{L}^1[0, T]$. The estimate (2.3) then implies that, as $k \to \cdot$, $x(w_k^\tau,\cdot)$ tends to $x(u,\cdot)$ in $\mathcal{L}^1[0, T]$ and $x(w_k^\tau,\tau)$ has a limit, say $x(\tau)$. Repeating this construction for all τ, one obtains a function $t \longmapsto x(t)$ defined pointwise on [0, T]. Notice that from any sequence v_k converging to u in $\mathcal{L}^1[0, T]$ one can extract a subsequence v'_k which converges pointwise to u on the complement $[0, T]\backslash N$ of a set N of measure zero. The estimate (2.3) implies that $x(v'_k,\tau)$ converges to $x(\tau)$ for all $\tau \notin N$, hence $x(\cdot)$ is a generalized solution of (2.1). More generally, if the control u is pointwise determined at $t = 0$ and on some subset Iå $\subset$ [0 T], the same is true for the corresponding trajectory.

The Lipschitz continuity of the trajectory with respect to changes in the initial condition x^0 can also be proven.

JUMPS

It is interesting to study the behavior of the trajectory at points t where the control has a jump.

Proposition 2.1. Assume that there exists the limits

$$\lim_{t \to \tau^-} u(t) = u^-, \quad \lim_{t \to \tau^+} u(t) = u^+.$$

Then the limits

$$\lim_{t \to \tau^-} x(u,t) = x^-, \quad \lim_{t \to \tau^+} x(u,t) = x^+$$

exists and

$$x^+ = \phi_g^{(u^+-u^-)}(x^-), \tag{2.15}$$

where, as in Chapter 2, the right-hand side of (2.15) denotes the value at time $t = u^+-u^-$ of the solution to the Cauchy Problem $dz/dt = g(z)$, $z(0)=x^-$.

Proof. By the same change of variable used in the proof of Theorem 2.1, it suffices to prove the result for the system (2.4), in which case (2.15) becomes simply

$$x^+ = x^- + (u^+ - u^-)\underline{e},$$

and the Proposition follows from (2.5).

Chapter 2

Impulsive systems with vector-valued controls

1. Introduction.

The results of the previous chapter do not hold when in the system

(1.1)
$$\begin{cases} \dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} g_i(x(t))\, \dot{u}_i(t) \\ x(0) = \bar{x} \in \Omega \ , \end{cases}$$

the control is vector-valued, i.e., to say, m>1, unless all Lie brackets $[g_i,g_j]$ vanish identically. Indeed, let us consider the following counterexample, due to Sussmann ([36]).

Example 1.1. Let A,B, be two n x n matrices such that $[B,A] = AB - BA = C \neq 0$. For $x \in R^n$, put $g_1(x) = Ax$, $g_2(x) = Bx$.

Let the interval [0, T] be partitioned into n equal n intervals $I_j^n = [(j-1)/n, j/n]$, $j=1,\dots,n$. Partition each I_j^n into four equal intervals $I_{j,i}^n$, i=1,2,3,4. Define w_n^1 (t) to be equal to $4n^{1/2}$ for $t \in I_{j,1}^n$, to $-4n^{1/2}$ for $t \in I_{j,3}^n$, and to zero for all other t. Similarly, let w_n^2 (t) be equal to $4n^{1/2}$ for $t \in I_{j,2}^n$, to $-4n^{1/2}$ for $t \in I_{j,4}^n$ and to zero for all other t. Let u_n^1 , u_n^2 be the indefinite integrals of w_n^1 , w_n^2 chosen so that $w_n^1(0) = w_n^2(0) = 0$). It is easy to see that u_n^1 and u_n^2 converge to zero uniformely, as $n \to \infty$. On the other hand, the solutions $x(u_n,\cdot)$ of

(1.1)
$$\begin{cases} \dot{x}(t) = \sum_{i=1,2}^{1} g_i(x(t))\, \dot{u}_n^i(t) \\ x(0) = \bar{x} \end{cases}$$

have the limit

$$x(t) = \varphi^t_{[g_2,g_1]}(\overline{x}) \equiv e^{t[A,B]}\overline{x} \ , \qquad t \in [0, T] \ ,$$

as $n \to \infty$.

The previous example shows that, if $x(u,\cdot)$ denotes the Carathèodory solution of the Cauchy Problem (1.1), the input-output map $\Phi : u(\cdot) \longmapsto x(u,\cdot)$ defined on the space of Lipschitz continuous controls is not continuous with respect to C^0 norms.

In the following Section we report the text of Bressan-Rampazzo's paper [8], in which one proves that Φ is continuous with respect to the C^0 topologies when restricted to a subset of controls having a uniform Lipschitz constant. When the control u is discontinuous but has bounded variation, one can parametrize the (*completion* of the) graph of u in a Lipschitz continuous way, and apply the previous result to a suitable augmented system, with $(m+1)$-dimensional controls. This permits to give a notion of generalized solution for controls with bounded total variation. Of course, it is important to understand the relations between generalized and classical solutions of (1.1). This is the object of some Theorems on continuous dependence and approximations given in Sections 4-5.

Moreover, like in the case of scalar inputs all the results of this Chapter can be extended to the case where f and g depend on t and u as well, simply by adding the new variables $x^0 = t$, $x^{n+\alpha} = u^\alpha$, $a=1,\dots,m$.

2. Graph-completions.

Let $u : [0,T] \to \mathbb{R}^m$ be continuously differentiable and let $\varphi(s) = (\varphi_0,\dots,\varphi_m)(s) = (t(s),u(t(s)))$, $s \in [0,S]$, be a Lipschitz continuous parametrization of the graph of u . Together with (1.1) consider the (n+1)-dimensional Cauchy problem

$$\text{(2.1)} \qquad \begin{cases} \dot{y}(s) = \tilde{f}(y(s))\,\dot{\varphi}_0 + \sum_{i=1}^{m} \tilde{g}_i(y(s))\,\dot{\varphi}_i(s) \\[2ex] y(0) = (0, \overline{x}) \end{cases}$$

with $f(x_0,x) = (1,f(x))$, $g_i(x_0,x) = (0,g_i(x))$. If $y(s) = (y_0,y_1,\dots,y_n)(s)$ solves (2.1) , then $y_0(s)= t(s)$, moreover $x(t) = (y_1,\dots,y_n)(s(t))$ yields the solution of (1.1) . In the case where u is not absolutely continuous, one may still be able to construct a Lipschitz parametrization j of the graph of u , solve the corresponding Cauchy problem (2.1) and use this solution $y(\cdot)$ to recover a generalized solution of (1.1) . To implement this program, we introduce the following definition.

Definition 2.1. Let $u : [0.T] \rightarrow \mathbf{R}^m$ have finite total variation . A *graph-completion* of u is a Lipschitz continuous map $\varphi:[0,S]\rightarrow[0,T]\times\mathbf{R}^m$, $\varphi(s) = (t(s) , v(s))$ such that

i) $\varphi(0) = (0, u(0))$, $\varphi(S) = (T, u(T))$

ii) $0 \leq r < s \leq T \Rightarrow t(r) \leq t(s)$

iii) $\forall t\in [0, T]$, $\exists s\in [0, S]$ such that $\varphi(s) = (t, u(t))$.

Notice that by iii) the range of j is a compact connected set containing the graph of u. Among all possible graph-completions of a control u , a natural one is obtained by "bridging" the discontinuities of u , on the graph, by straight segments as shown in the following

Example 2.1. Let u be right-continuous with bounded variation. For $t\in [0,T]$, let V(t) be the total variation of u on [0,t] . Since the function $t \rightarrow t+V(t)$ is right-continuous and strictly increasing, for every $s\in [0,T+V(T)]$ there exists a unique time $t = t(s)$ such that

(2.2) $$s \leq t + V(t) , t' + V(t') < s \qquad \forall t' < t .$$

For $t\in [0,T]$, define

(2.3) $$u^-(t) = \lim_{t'\rightarrow t^-} u(t') , \quad V^-(t) = \lim_{t'\rightarrow t^-} V(t') ,$$

(2.4) $$W(t) = t + V(t) , \quad W^-(t) = t + V(t') .$$

If u has a jump at t , we thus have

(2.5) $$\|u(t) - u^-(t)\| = V(t) - V^-(t) = W(t) - W^-(t)$$

A map $\varphi_u : [0,W(T)] \to [0,T]\times\mathbf{R}^m$ can now be defined by setting $\varphi_u(s) = (t(s), v(s))$, with $t(s)$ defined by (2.2) and $v(s) = u(t(s))$ if u is continuous at $t(s)$, while

$$v(s) = \frac{s - W^-(t)}{W(t) - W^-(t)} u(t) + \frac{W(t) - s}{W(t) - W^-(t)} \tag{2.6}$$

if u has a jump at time $t = t(s)$.

Using (2.5) one checks that φ_u is continuous with Lipschitz constant 1. Moreover, conditions i)÷iii) in Definition 2.1 hold. In particular, if $t\in[0,T]$, $(t,u(t)) = \varphi_u(s)$ for $s=t+V(t)$. The function φ_u, uniquely determined by the above construction, will be called the *canonical graph-completion* of u.

Definition 2.2. Let $u:[0,T] \to \mathbf{R}^m$ be a control with bounded variation, let $\varphi=(\varphi_0,\ldots,\varphi_m):[0,S]\to[0,T]\times\mathbf{R}^m$ be a graph-completion of u, and let $y = (y_0,\ldots,y_n)$ be the corresponding solution of (2.1). The (possibly multivalued) map $x(\varphi,\cdot)$ defined by

$$x(\varphi, t) = \{(y_1, \ldots, y_n)(s)\ ;\ t = y_0(s)\} \tag{2.7}$$

is called the *generalized solution of (1.1) relative to* φ.

Remark 2.1. By choosing $s^+(t) = \max\{s;\ t = y_0(s)\}$ and setting $x^+(\varphi,t) = (y_1,\ldots,y_n)(s^+(t))$ one obtains a right-continuous selection of the multivalued function $x(\varphi,\cdot)$ defined at (2.7).

Since the graph of $x(\varphi,\cdot)$ by definition coincides with the image $y([0,S])$, which is compact, it follows that the map $t\to x(\varphi,\cdot)$ is Hausdorff upper semicontinuous [1,p.41]. In general it is clear that these generalized solutions are not unique, since they depend on the choice of a particular graph-completion φ of u. Yet, one can obtain uniqueness by prescribing a canonical way to construct the map φ, as in Example 2.1. In physical applications, it can also happen that the graph-completion is naturally suggested by the problem.

3. Equivalent paths.

Our goal is to investigate the dependence of the generalized solution $x(\varphi,\cdot)$ on the particular graph-completion φ , and the relation between classical and generalized solutions of (1.1) . As a preliminary, observe that every Lipschitz continuous map $\varphi:[0,S]\to\mathbf{R}^n$ can be reparametrized by means of its total variation. More precisely, for $s\in[0,S]$, let $V(s)$ be the total variation of φ on $[0,S]$. V is thus a non-decreasing, Lipschitz continuous map from $[0,S]$ onto $[0,V(S)]$. For $\tau\in[0,V(S)]$, set $\varphi^{©}(\tau)=\varphi(s)$ if $\tau=V(s)$. It is readily checked that $\varphi^{©}$ is a well defined map with Lipschitz constant 1.

Definition 3.1. The map $\varphi^{©}:[0,V(S)]\to\mathbf{R}^n$ constructed above is the *canonical parametrization of* φ . We say that two continuous maps $\varphi_i:[0,S_i]\to\mathbf{R}^n$ $(i=1,2)$ are *equivalent*, and write $\varphi_1\sim\varphi_2$, if their canonical parametrizations coincide.

Proposition 3.1 . *If* φ *is a graph completion of a control* u *, such is also its canonical parametrization* $\varphi^{©}$ *. Moreover, the generalized solutions* $x(\varphi,\cdot)$ *and* $x(\varphi^{©},\cdot)$ *of* (1.1) *coincide.*

Proof. For the map $\varphi^{©}$, conditions i) and ii) in Def. 1 are obvious. Moreover, iii) holds because $\mathrm{graph}(u)\subseteq\mathrm{range}(\varphi)=\mathrm{range}(\varphi^{©})$. This yields the first assertion. If y , $y^{©}$ are the solutions of (2.1) corresponding to φ, $\varphi^{©}$ respectively, then $y(s)=y^{©}(V(s))$ for all $s\in[0,S]$, hence

$$x(\varphi,t)=\left\{(y_1,\dots,y_n)(s)\ ;\ t=y_0(s)\right\}=$$

$$=\left\{(y_1^{©},\dots,y_n^{©})(V(s))\ ;\ t=y_0^{©}(V(s))\right\}=x(\varphi^{©},t)\,.$$

In view of the above result, equivalent graph-completions of u yield the same generalized solution of (1.1). In the study of the dependence of $x(\varphi,\cdot)$ on φ , it is therefore natural to look for estimates which do not depend on the particular parametrization of φ . This is achieved by using a metric which is parameter-free.

Definition 3.2 . Given two continuous paths $\varphi_i:[0,S_i]\to\mathbf{R}^n$, $i=1,2$, their distance $\delta(\varphi_1,\varphi_2)$ is defined as

(3.1)
$$\delta(\varphi_1, \varphi_2) = \inf_{\gamma_1, \gamma_2} \max_{s \in [0,1]} \|\varphi_1(\gamma_1(s)) - \varphi_2(\gamma_2(s))\| ,$$

the inf being taken over all couples of continuous, non-decreasing, surjective maps $\gamma_i:[0,1] \to [0,S_i]$.

It is interesting to observe that the inf in (3.1) is actually a minimum. To prove this, we need

Lemma 3.1. *Given two maps* $\gamma_i:[0,1] \to [0,S_i]$, $i = 1,2$, *as in* Definition 3.2, *there exist nondecreasing surjective maps* $\gamma'_i:[0,1] \to [0,S_i]$, *continuous with Lipschitz constant* $L=1+S_1+S_2$, *such that*

(3.2)
$$\max_{s \in [0,1]} \|\varphi_1(\gamma_1(s)) - \varphi_2(\gamma_2(s))\| = \max_{s \in [0,1]} \|\varphi_1(\gamma'_1(s)) - \varphi_2(\gamma'_2(s))\|$$

Proof. For $t \in [0,1]$, set $\beta(t) = [t+\gamma_1(t)+\gamma_2(t)] / [1+S_1+S_2]$. β is then a strictly increasing, continuous surjective map from $[0,1]$ onto $[0,1]$. Since $d\beta(t)/dt \geq [1+S_1+S_2]^{-1}$ a.e., the inverse map β^{-1} is also increasing with Lipschitz constant $L = 1 + S_1 + S_2$. Define γ'_i , $i = 1,2$, by setting $\gamma'_i(s) = \gamma_i(\beta^{-1}(s))$. It is now clear that (3.2) holds. Concerning the Lipschitz condition, if $r < s$ we have

$$|\gamma'_i(r) - \gamma'_i(s)| = \gamma_i(\beta^{-1}(s)) - \gamma_i(\beta^{-1}(r)) \leq [\beta^{-1}(s) + \gamma_1(\beta^{-1}(s)) + \gamma_2(\beta^{-1}(s))] -$$
$$- \left[\beta^{-1}(r) + \gamma_1(\beta^{-1}(r)) + \gamma_2(\beta^{-1}(r))\right] = [1 + S_1 + S_2] \cdot \left[\beta(\beta^{-1}(s) - \beta(\beta^{-1}(r)\right] =$$
$$\left[1 + S_1 + S_2\right] \cdot [s - r] . \qquad (i = 1, 2)$$

Proposition 3.2 . *Given two continuous paths* $\varphi_i:[0,S_i] \to \mathbf{R}^n$, $i = 1,2$, *there exist two continuous nondecreasing surjective maps* $\gamma_i:[0,1] \to [0,S_i]$ *such that*

(3.3)
$$\delta(\varphi_1, \varphi_2) = \max_{s \in [0,1]} \|\varphi_1(\gamma_1(s)) - \varphi_2(\gamma_2(s))\| .$$

Proof. For every $n \geq 1$, choose γ_1^n, γ_2^n such that

(3.4)
$$\max_{s\in[0,1]} \|\varphi_1(\gamma_1^n(s)) - \varphi_2(\gamma_2^n(s))\| \leq \delta(\varphi_1, \varphi_2) + \frac{1}{n} .$$

By Lemma 3.1, it is not restrictive to assume that γ_1^n, γ_2^n are both Lipschitz continuous with constant $1+S_1+S_2$. Ascoli's Theorem now implies the existence of two maps γ_1, γ_2 such that $\varphi_i(\gamma_i(\cdot))$ (i=1,2) satisfies the conditions in Definition 2.1 and

$$\gamma_i(s) = \lim_{n'\to\infty} \gamma_i^{n'}(s)$$

for some subsequence $n'\to\infty$, uniformly on $[0,1]$. The continuity of φ_1, φ_2 together with (3.4) now imply (3.3).

QED

We conclude this Section by showing that the distance δ is in fact a pseudometric.

Proposition 3.3. *The distance δ satisfies the following:*

i) $\delta(\varphi_1, \varphi_2) = \delta(\varphi_2, \varphi_1) \geq 0$,

ii) $\delta(\varphi_1, \varphi_2) = 0$ if and only if $\varphi_1 \sim \varphi_2$,

iii) $\delta(\varphi_1, \varphi_2) + \delta(\varphi_2, \varphi_3) \geq \delta(\varphi_1, \varphi_3)$.

The proofs of i) and ii) are straightforward. To prove iii), let $\varphi_i:[0,S_i]\to\mathbf{R}^n$ (i=1,2,3) be continuous, and use Proposition 3.2 to construct functions α_1, α_2, β_2, β_3 such that

$$\delta(\varphi_1, \varphi_2) = \max_{s\in[0,1]} \|\varphi_1(\alpha_1(s)) - \varphi_2(\alpha_2(s))\| ,$$

$$\delta(\varphi_2, \varphi_3) = \max_{s\in[0,1]} \|\varphi_2(\beta_2(s)) - \varphi_3(\beta_3(s))\| .$$

For every $\varepsilon > 0$, the map $\alpha_{2\varepsilon}:[0,1]\to[0,S_2]$, $\alpha_{2\varepsilon}(s) = (1-\varepsilon)\alpha_2(s) + \varepsilon s S_2$ is strictly increasing, hence it has a continuous inverse $\alpha_{2\varepsilon}^{-1}$. Define $\alpha_{1\varepsilon} = \alpha_1 \circ \alpha_{2\varepsilon}^{-1} \circ \beta_2 : [0,1] \to [0,S_1]$. Then, for every $\varepsilon>0$,

$$\delta(\varphi_1,\varphi_2) \leq \max_{s\in[0,1]} \|\varphi_1 \circ \alpha_{1\varepsilon}(s) - \varphi_3 \circ \beta_3(s)\| \leq$$

$$\leq \max_{s\in[0,1]} \|\varphi_1 \circ \alpha_1 \circ \alpha_{2\varepsilon}^{-1} \circ \beta_2(s) - \varphi_2 \circ \alpha_2 \circ \alpha_{2\varepsilon}^{-1} \circ \beta_2(s)\| +$$

$$+ \max_{s\in[0,1]} \|\varphi_2 \circ \alpha_2 \circ \alpha_{2\varepsilon}^{-1} \circ \beta_2(s) - \varphi_2 \circ \beta_2(s)\| + \max_{s\in[0,1]} \|\varphi_2 \circ \beta_2(s) - \varphi_3 \circ \beta_3(s)\| =$$

$$= \delta(\varphi_1, \varphi_2) + \sigma(\varepsilon) + \delta(\varphi_2, \varphi_3) ,$$

with $\sigma(\varepsilon) = \max_{s\in[0,1]} \|\varphi_2 \circ \alpha_2 \circ \alpha_{2\varepsilon}^{-1} \circ \beta_2(s) - \varphi_2 \circ \beta_2(s)\|$.

Since $\sigma(\varepsilon)\to 0$ as $\varepsilon \to 0$ because of Lemma 3.2 below, this yields iii) .

Lemma 3.2. *Let* $\alpha : [a,b]\to[0,S]$ *be continuous, surjective and non-decreasing. For* $0<\varepsilon\leq 1$ *the map* α_ε *defined by*

(3.5) $$\alpha_\varepsilon(r) = [1 - \varepsilon]\, \alpha(r) + \varepsilon S[r - a] / [b-a]$$

is strictly increasing, and its inverse $\alpha_\varepsilon^{-1} : [0,S]\to[a,b]$ *satisfies*

(3.6) $$|\alpha(\alpha_\varepsilon^{-1}(t)) - t| \leq \varepsilon S \qquad \forall t\in [0, S] .$$

Indeed

$$|\alpha \circ \alpha_\varepsilon^{-1}(t) - t| = |\alpha \circ \alpha_\varepsilon^{-1}(t) - \alpha_\varepsilon \circ \alpha_\varepsilon^{-1}(t)| = |\varepsilon\alpha \circ \alpha_\varepsilon^{-1}(t) - \varepsilon S[\alpha_\varepsilon^{-1}(t) - a] / [b - a]| \leq \varepsilon S .$$

4. Continuous dependence.

Consider again a control system of the form

$$(4.1)\qquad \begin{cases} \dot{y}(t)=\sum_{i=1}^{m} g_i(y(t))\,\dot{u}_i(t) \\ \\ y(0) = (0,\ \bar{y})\in \mathbf{R}^n \end{cases} \qquad t\in [0, T]$$

where the vector fields g_i are defined and continuously differentiable in some open set $\Omega \subseteq \mathbf{R}^n$. We denote by $y(u,\cdot)$ the solution of (4.1), if it exists, corresponding to the control u. The next results extablish the continuity of the input-output map, when restricted to suitable sets of controls.

Theorem 4.1 . *Fix* $L > 0$ *, let* $\mathcal{U} \subset C^0([0,T];\mathbf{R}^m)$ *be the set of all controls with Lipschitz constant* L *, and call* $\mathcal{U}^*$ *the set of all controls* $u\in \mathcal{U}$ *for which the corresponding solution* $y(u,\cdot)$ *of* (4.1)) *exists in* Ω.

Then

i) $\mathcal{U}^*$ *is a relatively open subset of* $\mathcal{U}$.

ii) *The restriction to* $\mathcal{U}^*$ *of the map* $\Phi : u(\bullet)\to y(u,\bullet)$ *is locally Lipschitz continuous with respect to the* C^0 *norms on the spaces of controls and trajectories.*

The proof will rely on the following corollary of the Contraction Mapping Theorem [17] . A quite similar result appears in [2a] .

Lemma 4.1. *Let* A , B *be closed subsets of the Banach spaces* E,F *respectively. Let* $\Psi:A\times B\to B$ *be a map such that* $\forall u,v\in A$, $\forall x,y\in B$ *one has*

$$(4.2)\qquad \|\Psi(u, x) - \Psi(u, y)\|_F \le \frac{1}{2}\|x - y\|_F$$

$$(4.3)\qquad \|\Psi(u, x) - \Psi(v, x)\|_F \le C\,\|u - v\|_E$$

for some constant C . *Then for each* $u \in A$ *there exists a unique* $x = x(u) \in B$ *such that* $x(u) = \Psi(u, x(u))$. *Moreover*

(4.4) $$\|x(u) - x(v)\|_F \le 2C\, \|u - v\|_E .$$

Proof of Theorem 4.1. Let $u^* \in \mathcal{U}^*$ and let $y^* = y(u^*, \cdot)$ be the corresponding solution of (4.1) . Since the image $y^*([0,T])$ is a compact subset of Ω , there exists $\delta > 0$ so small that

(4.5) $$\Omega' = \bigcup_{t \in [0,T]} B(y(u^*, t), \delta) \subseteq \Omega .$$

Here and in the sequel $B(y,\delta)$ denotes the open ball centered at y with radius δ . For $i = 1,\dots,m$, construct a $\mathcal{C}^1$ vector field h_i with compact support in $\mathbf{R}^n$, which coincides with g_i on Ω' . Let N be an upper bound for both h_i and the operator norms of the Jacobian matrices of h_i , i.e.

(4.6) $$\|h_i(x)\| \le N \ , \ \|\frac{\partial h_i}{\partial x}(x)\| \le N \qquad \forall x \in \mathbf{R}^n , i = 1, \dots, m .$$

Consider the spaces $E = \mathcal{C}^0([0,T];\mathbf{R}^n)$ with the usual norm and $F = \mathcal{C}^0([0,T];\mathbf{R}^n)$ with the equivalent norm

(4.7) $$\|x\|_F = \sup\{e^{-\lambda t}\, \|x(t)\|\ t \in [0, T]\} \ , \lambda = 2N\, mL .$$

The new Cauchy problem

(4.8) $$\begin{cases} \dot{z}(t) = \sum_{i=1}^{m} h_i(z(t))\, \dot{u}_i(t) \\ z(0) = (0, \bar{y}) \end{cases} \qquad t \in [0, T]$$

now has a unique solution $z(u,\cdot)$ for every $u \in \mathcal{U}$.
Observe that $z(u,\cdot)$ is the solution of the implicit equation

(4.9) $$z = \Psi\, (u,z)$$

with

$$(4.10) \qquad \Psi(u, z)(t) = \bar{y} + \int_0^t \sum_{i=1}^m h_i(z(s))\ \dot{u}_i(s))\ ds\ .$$

We now apply Lemma 4.1 , taking $A = \mathcal{U}$ and letting $B \subset F$ be the set of all trajectories with Lipschitz constant mNL . This choice guarantees that Ψ in (4.10) maps $A \times B$ into B . To check (4.2), let $\|x-y\|_F = m$. Since $\|x(t) - y(t)\| \le m e^{\lambda t}$ we have

$$e^{-\lambda t} \|\Psi(u, x)(t) - \Psi(u, y)(t)\| \le$$

$$\le e^{-\lambda t} \int_0^t \sum_{i=1}^m \|g_i(x(s)) - g_i(y(s))\| \circ \|\dot{u}_i(s)\|\ ds)$$

$$(4.11)$$

$$\le e^{-\lambda t} \int_0^t mN\ \|x(s) - y(s)\| \cdot L\ ds \le e^{\lambda t} \int_0^t mN\mu e^{\lambda s}\ L\ ds$$

$$\le \mu m N L \lambda^{-1} = 1/2\ \|x - y\|_F$$

To obtain (4.3), an integration by parts yields

$$\|\Psi(u, x)(t) - \Psi(v, x)(t)\| \le \int_0^t \sum_{i=1}^m \left|\frac{d}{ds} g_i(x(s))\right\| \cdot \|u_i(s) - v_i(s)\| ds)$$

$$(4.12) \qquad + \sum_{i=1}^m \|g_i(x(0))\| \cdot \|u_i(0) - v_i(0)\| + \sum_{i=1}^m \|g_i(x(t))\| \cdot \|u_i(t) - v_i(t)\|$$

$$\le T m^2 N^n L\ \|u - v\|_E + 2m\ N\ \|u - v\|_E\ ,$$

because

$$\left\|\frac{d}{ds} g_i(x(s))\right\| \le \|(g_i)_x\ (x(s))\| \cdot \|\dot{x}(s)\| \ \le\ N \circ m\,NL\ .$$

This yields (4.3) with $C = Tm^2N^2L + 2mN$. By Lemma 4.1, the input-output map $u \to z(u,\cdot)$ is thus Lipschitz continuous on $\mathcal{U}$. Statements i) and ii) are now clear, because whenever the trajectory $z(u,\cdot)$ of (4.8) is entirely contained inside W' , it coincides with the solution $y(u,\cdot)$ of (4.1).

QED

The next result is an analog of Theorem 4.1 for the pseudometric δ defined at (3.1).

Theorem 4.2. *Fix* $K > 0$ *, let* $\mathcal{V}$ *be the set of all Lipschitz continuous maps* $v:[0,S_v] \to R^m$ *with total variation* $\leq K$ *, and call* $\mathcal{V}^*$ *the set of maps* $v \in V$ *for which the corresponding solution* $y(v,\cdot)$ *of* (4.1) *exists. Then*

i) $\mathcal{V}^*$ *is relatively open in* $\mathcal{V}$,

ii) *the restriction of the input-output map* $v(\cdot) \to y(v,\cdot)$ *to* $\mathcal{V}^*$ *is continuous, provided that the pseudometric defined at (3.1) is used on the spaces of controls and trajectories.*

The Theorem will be proved by showing that, given any sequence $(v_n)_{n\geq 0}$ of elements of $\mathcal{V}$ such that $v_0 \in \mathcal{V}^*$ and

$$\lim_{n\to\infty} \delta(v_n, v_0) = 0 \ , \tag{4.13}$$

there exists a subsequence $v_{n'}$ such that $v_{n'} \in \mathcal{V}^*$ for all n' suitably large and

$$\lim_{n'\to\infty} \delta(y(v_{n'}, \cdot) , y(v_0, \cdot)) = 0 \ . \tag{4.14}$$

By the results in Section 3, it is not restrictive to assume that each map v_n coincides with its canonical parametrization. In particular, we can assume that every v_n has Lipschitz constant 1 and is defined on some interval $[0,S_n] \subseteq [0,K]$. If (4.13) holds, by Definition 4 there exist maps β_n , γ_n $(n \geq 1)$ such that

$$\lim_{n\to\infty} \max_{t\in[0,1]} \|v_n(\beta(t)) - v_0(\gamma_n(t))\| = 0 \ . \tag{4.15}$$

By Lemma 3.1 we can also assume that β_n and γ_n are all Lipschitz continuous with constant $2K + 1$. Hence, for a suitable subsequence $n' \to \infty$, there exists a map γ such that $\gamma_{n'}(t) \to \gamma(t)$ uniformly on [0,1] . Moreover, (4.15) implies that the sequence of controls $u_{n'} = v_{n'} \circ \beta_{n'}$

converges to $u_0 = v_0 \circ \gamma$ uniformly on [0,1]. Since the solution of (4.1) corresponding to u_0 exists by assumption, Theorem 4.1 implies that the trajectories $y(u_{n'},\cdot)$ also exist for all n' suitably large, and tend to $y(u_0,\cdot)$ uniformly on [0,1]. This proves (4.14) because

$$\lim_{n\to\infty} \delta(y(v_{n'}, \cdot), y(v_0, \cdot)) \le \lim_{n\to\infty} \max_{t\in[0,1]} \|y(v_{n'}, \beta_{n'}(t)) - y(v_0, \gamma(t))\|$$

$$= \lim_{n'\to\infty} \max_{t\in[0,1]} \|y(v_n \circ \beta_{n'}, t) - y(v_0 \circ \gamma, t)\| = 0 \ .$$

We now specialize the above result in the case where the v_n represent the graph-completions of a sequence of controls u_n.

Corollary 4.1. *For every* $n \ge 0$, *let* $u_n:[0,T]\to R^m$ *be a control with bounded variation, let* φ_n *be a graph-completion of* u_n, *and let* $x(\varphi_n,\cdot)$ *be the corresponding generalized solution of* (1.1). *If* $\delta(\varphi_n, \varphi_0)\to 0$ *as* $n\to\infty$ *and if the total variation of the maps* φ_n *are uniformly bounded, then the graphs of* $x(\varphi_n,\cdot)$ *tend to the graph of* $x(\varphi_0, \cdot)$ *in the Hausdorff metric* [1.p.65] .

Corollary 4.2 . *Let* $\varphi_0:[0,S]\to R^{m+1}$ *be a graph-completion of a control* $u:[0,T]\to R^m$, *and let* $(u_n)_{n\ge 1}$ *be a sequence of Lipschitz continuous controls with uniformly bounded variation which approximate* φ_0 *in the sense that, setting* $\varphi_n(t) = (t,u_n(t))$, *one has*

$$\lim_{n'\to\infty} \delta(\varphi_0, \varphi_n) = 0 \ .$$

Then the generalized solution $x(\varphi_0,\cdot)$ of (1.1) *relative to* φ_0 *satisfies*

$$x(\varphi_0, t) = \lim_{n\to\infty} x(u_n, t)$$

at every $t\in[0,T]$ *where* $x(\varphi_0,t)$ *is single-valued, hence almost everywhere.*

Proof.. Set $x_0 = x(\varphi_0,\cdot)$, $x_n = x(u_n,\cdot)$ for $n \ge 1$. Fix $\varepsilon > 0$ and $t\in[0,T]$ such that $x(\varphi_0,t)$ is single-valued. Since x_0 is upper semicontinuous at t , there exists $\delta\in(0,\varepsilon/2)$ such that $x_0(s) \subseteq B(x_0(t) , \varepsilon/2)$ whenever $|s-t| \le \delta$. Choose N so large that the Hausdorff distance between the graphs of x_n and x_0 is smaller than δ, for all $n \geq N$. This implies the existence of $s_n \in [t-\delta , t+\delta]$ such that

$$x_n(t) \in B(x_0(s_n), \delta) \subseteq B(x_0(t), \varepsilon) , \qquad \forall n \geq N .$$

Since ε was arbitrary , Corollary 4.2 is proved.

Corollary 4.3 . *In addition to the assumption of* Corollary 4.2, *suppose that* u_n *converges to* u *uniformly on some interval* $[\tau,\tau+\sigma]$. *Then*

$$\lim_{n\to\infty} x(u_n, \tau) = x^+(\varphi_0, \tau)$$

where $x^+(\varphi_0, t)$ *was defined in the* Remark 2.1 .

Proof. Assume that there exists a subsequence $u_{n'}$ and a point x^* such that

$$\lim_{n'\to\infty} x(u_{n'}, \tau) = x^* \neq x^+(\varphi_0, \tau) .$$

Then on the interval $[\tau,\tau+\sigma]$ the sequence of solutions $x(u_{n'},\cdot)$ converges uniformly to the solution $x(t)$ of the Cauchy Problem

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{m} g_i(x)\, \dot{u}_i(t) \\ x(\tau) = x^* . \end{cases}$$

Since $x^+(\varphi_0,\bullet)$ is a right continuous function of t and $x(\bullet)$ is continuous on $[\tau,\tau+\sigma]$, the distance $|x^+(\varphi_0,t)-x(t)|$ remains strictly positive on some interval of the form $[\tau,\tau+\varepsilon]$. This is impossible because Corollary 4.2 implies $x^+(\varphi_0,t) = x(t)$ almost everywhere.

5. Mollifications.

Given a (possibly discontinuous) control $u:[0,T]\to\mathbf{R}^m$, approximate solutions of (1.1) can be constructed by means of a mollification. More precisely, let ψ be a $\mathcal{C}^1$ function with compact support for which

$$\int_{-\infty}^{\infty}\psi(t)dt=1, \quad \psi(t)\geq 0 \qquad \forall t\in\mathbf{R}. \tag{5.1}$$

For $\eta>0$, set $\psi_\eta(t)=\eta^{-1}\psi(\eta^{-1}t)$, and define the convolution $u_h=u*\psi_\eta$:

$$u_\eta(t)=\int_{-\infty}^{\infty}u(s)\,\psi_\eta(t-s)\,ds \tag{5.2}$$

with the convention that in (5.2) the function u has been extended outside $[0,T]$ by setting $u(t)=u(0)$ if $t<0$, $u(t)=u(T)$ if $t>T$. The mollified control u_η is then $\mathcal{C}^1$, and one can now look for a classical solution of (1.1) corresponding to u_η. It is interesting to determine the limit of this solution as $\eta\to 0$.

Theorem 5.1. *Assume that* $u:[0,T]\to\mathbf{R}^m$ *is right continuous with bounded variation, and that* $\lim_{t\to T-} u(t)=u(T)$. *Fix a* $\mathcal{C}^1$ *map* ψ *with compact support, satisfying* (5.1) *and define* $u_\eta=u*\psi_\eta$ *as in* (5.2). *Then, as* $\eta\to 0$, *the graph of the corresponding solution* $x(u_\eta,\cdot)$ *of (1.1) tends to the graph of* $x(\varphi_u,\cdot)$ *in the Hausdorff metric,* φ_u *being the canonical graph-completion of* u. *In particular,*

$$\lim_{\eta\to 0} x(u_\eta,t)=x(\varphi_u,t)$$

at every time t *where* u *is continuous*.

Proof. Define $\varphi_\eta:[0,T]\to\mathbf{R}^{m+1}$ by setting $\varphi_\eta(t)=(t,u_h(t))$ and let $\varphi_u:[0,T+V(T)]\to\mathbf{R}^{m+1}$ be the canonical graph-completion of u. All notations introduced in Example 2.1, Section 2 will be again used here. Since φ_u and φ_η $(\eta>0)$ have uniformly bounded variation, by Corollaries 1 and 2 it suffices to prove that $\delta(\varphi_u,\varphi_\eta)\to 0$ as $\eta\to 0$. For simplicity, we assume that the support of ψ is contained inside $[-1,1]$, which is not restrictive. Fix $\varepsilon>0$. Since u has bounded variation, there exist at most finitely many times t_i , $0<t_i\ldots<t_k<T$ where the jump $\|u(t_i)-u^-(t_i)\|$ is larger than $\varepsilon/4$. Choose $\varepsilon'\in(0,\varepsilon/4)$ such that

i) the intervals $(t_i-2\varepsilon' , t_i+2\varepsilon')$ are all disjoint and contained inside $[0,T]$.

ii) $V^-(t_i+2\varepsilon') - V(t_i) \le \varepsilon/4$, $V^-(t_i) - V(t_i-2\varepsilon') \le \varepsilon/4$ for all $i=1,\ldots,k$.

Now choose $\sigma\in(0,\varepsilon')$ such that, for every t in the compact set $J = [0,T]\setminus\bigcup_i B(t_i,\varepsilon')$,

$$V^-(t+\sigma) - V(t-\sigma) \le \varepsilon/2. \tag{5.3}$$

The Theorem will be proved by showing that

$$\delta(\varphi_u,\varphi_\eta) \le \varepsilon \qquad \forall\,\eta\in(0,\sigma). \tag{5.4}$$

Indeed, for any $\eta\in(0,s)$, we will construct a non-decreasing, continuous, surjective map $\beta : [0,W(T)] \to [0,T]$ such that

$$\|\varphi_u(s) - \varphi_\eta(\beta(s))\| \le \varepsilon \qquad \forall s\in[0,\ W(T)] \tag{5.5}$$

Let the canonical graph-completion $\varphi_u(s)=(t(s),v(s))$ be defined as in Example 2.1. In the construction of $\beta(s)$ we consider three cases.

I) If $t(s)\in J=[0,T]\setminus\bigcup_i B(t_i,\varepsilon')$, set $\beta(s) = t(s)$

II) If $s\in(W(t_i-\varepsilon'),\ W^-(t_i))$, set $b(s) = t_i - \varepsilon'$,

If $s\in(W(t_i),\ W^-(t_i+\varepsilon'))$, set $\beta(s) = t_i + \varepsilon'$.

III) If $t(s) = t_i$, i.e. $s\in[W^-(t_i),\ W(t_i)]$, define first the map α_μ: $[-\varepsilon', \varepsilon']\to[0,1]$,

$$\alpha_\mu(r) = \mu\,\frac{r+\varepsilon'}{2\varepsilon'} + (1-\mu)\cdot\int_{-\infty}^{\infty}\psi_\eta(s)ds\ . \tag{5.6}$$

For every $\mu \in (0,1]$, α_μ is strictly increasing, continuous and surjective. Call α_μ^{-1} its inverse. Define

(5.7) $$\beta(s) = t_i + \alpha_\mu^{-1}(\lambda(s))$$

with

(5.8) $$\lambda(s) = \frac{s - W^-(t_i)}{W(t_i) - W^-(t_i)}\,, \quad \mu = \min\left\{ \frac{\varepsilon/2}{W(t_i) - W^-(t_i)}, 1 \right\}.$$

Relying on the fact that $\text{supp}(\psi_\eta) \subseteq [-\eta,\eta]$, with $\eta < \sigma < \varepsilon'$, we will prove that, in all three cases, (5.5) holds.

I) If $t(s) \in J$, then $\|\varphi_u(s) - \varphi_\eta(\beta(s))\| = \|u(t(s)) - u_\eta(t(s))\| \leq \varepsilon/2$ because both $u(t(s))$ and $u_\eta(t(s))$ lie in the convex closure of the values of u on $(t(s)-\sigma, t(s)+\sigma)$ and, by (5.3), the oscillation of u on this interval is bounded by $\varepsilon/2$.

II) If $s \in (W(t_i-\varepsilon'), W^-(t_i))$, then $\|\varphi_u(s) - \varphi_\eta(\beta(s))\| \leq \|t(s) - (t_i - \varepsilon')\| + \|u(t(s)) - u_\eta(t_i - \varepsilon)\| \leq \varepsilon' + \varepsilon/4 \leq \varepsilon/2$.
Indeed, $t(s) \in (t_i - \varepsilon', t_i)$, both $u(t(s))$ and $u_\eta(t_i - \varepsilon')$ lie in the convex closure of the values of u on $(t_i - 2\varepsilon', t_i)$, and the oscillation of u on such interval is bounded by $\varepsilon/4$. The case $s \in (W(t_i), W^-(t_i + \varepsilon'))$ is entirely similar.

III) If $t(s) = t_i$, then

$$\|\varphi(s) - \varphi_\eta(\beta(s))\| \leq \|t_i - \beta(s)\| + \|v(s) - u_\eta(\beta(s))\|.$$

By construction, $\|t_i - \beta(s))\| \leq \varepsilon' < \varepsilon/4$. To estimate the second term, we use Lemma 3.2 in Section 3 and obtain

$$|\alpha_0(\alpha_\mu^{-1}(\lambda)) - \lambda| \leq \mu \leq \frac{\varepsilon/2}{W(t_i) - W^-(t_i)} \qquad \forall \lambda \in [0,1].$$

Moreover, (5.6) and (5.7) imply

$$(5.10) \qquad \int_{t_i}^{\infty} \Psi_\eta(\beta(s)-\xi)d\xi = \int_{-\infty}^{\beta(s)-t_i} \Psi_\eta(\xi')d\xi' = \alpha_0\,(\beta(s)-t_i) = \alpha_0(\alpha_\mu^{-1}(\lambda(s)))\ .$$

Recalling that the oscillation of u on $(t_i-2\varepsilon', t_i)$ and on $(t_i, t_i+2\varepsilon')$ is bounded by $\varepsilon/4$ and using (5.9), (5.10) with $\lambda = \lambda(s)$ we now obtain

$$\|v(s) - u_\eta(\beta(s))\| \le \|\,[\lambda\, u(t_i) + (1-\lambda)u^-(t_i)] - [\alpha_0 \circ \alpha_\eta^{-1}(\lambda)\, u(t_i) + (1-\alpha_0 \circ \alpha_\eta^{-1}(\lambda))\, u^-(t_i)]\,\| +$$

$$+ \left\| \int_{t_i-2\varepsilon'}^{t_i} u^-(t_i)\psi_\eta(\beta(s)-\xi)d\xi + \int_{t_i}^{t_i+2\varepsilon'} u(t_i)\psi_\eta(\beta(s)-\xi)d\xi - \int_{t_i-2\varepsilon'}^{t_i+2\varepsilon'} u(\xi)\psi_\eta(\beta(s)-\xi)d\xi \right\| \le$$

$$\le \|u(t_i) - u^-(t_i)\|\,\mu + \varepsilon/4 \ \le\ 3\varepsilon/4\ .$$

This completes the proof of (5.5), which in turn yields (5.4). **QED**

Chapter 3

Impulsive control systems from the measure-theoretical point of view

In this Chapter we report the results of a paper (see [16]) written by the author of this thesis in collaboration with Gianni Dal Maso. In this work suitable extensions of

$$\text{(E)} \qquad \begin{cases} \dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} g_i(x(t))\dot{u}^i(t) \\ x(0) = \bar{x} \ , \end{cases}$$

are proposed in the case of a control u with bounded total variation. For each of these extensions a result of continuous dependence of the (generalized) solutions with respect to the control is proved, provided certain hypothesis on the variation of the approximating (regular) controls are assumed. Furthermore, only conditions on the original controls $u(\cdot)$ are considered, without mentioning their graph completions.

1. Introduction

The function u with bounded variation on $[0,T]$ will be assumed to be left continuous at each point of $]0,T]$ and right continuous at 0. We shall denote by $BV^-([0,T],\mathbf{R}^m)$ the space of all these functions. We have chosen this space in order to simplify the exposition. However it will be clear from the statements of the theorems that that the results presented in this chapter can be extended to the class of all functions with bounded total variation, up to obvious changes.

If $u \in BV^-([0,T],\mathbf{R}^m)$, by $\dot{u}$ we denote the distributional derivative of u, which is an $\mathbf{R}^m$–valued Radon measure on $]0,T[$. We expect that the solution x of (E) is a function of bounded variation too, so that its distributional derivative $\dot{x}$ is an $\mathbf{R}^n$–valued Radon measure.

At a discontinuity instant t of u the evolution process x is subject to the joint effects of the jumps of $f(x(\cdot))$, $g_i(x(\cdot))$, and the concentrated measure generated by the differentiation of u at t. The problem is tipically nonlinear, since a product between the measure $\dot{u}^i$ and the function $g_i(x(\cdot))$ has to be defined. As Hajyek pointed out in [21], the interpretation of the products $g_i(x(\cdot))\,\dot{u}^i$ which is directly supplied by measure theory is not satisfactory for treating the evolution problem at issue. For example, the continuous dependence of solutions on controls gets lost, also in very elementary cases (see [21]).

In order to extend (E) to the case $u \in BV^-([0, T], \mathbf{R}^m)$ preserving all nice properties of well posed problems, we introduce the functions $G_i : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$, $i = 1, 2, \ldots, m$, defined by

$$G_i(z,p) = \int_0^1 g_i(\exp(\sigma \sum_{j=1}^m p^j g_j)z)d\sigma \,,$$

where the symbol $\exp(\sigma \sum_{j=1}^m p^j g_j)z$ denotes the value at the time $s = \sigma$ of the solution of the Cauchy problem

$$\begin{cases} \dfrac{dw}{ds} = \sum\limits_{j=1}^m p^j g_j(w(s)) \\ w(0) = z. \end{cases}$$

Then, in the case $u \in BV^-([0, T], \mathbf{R}^m)$, we consider the problem

$$(E)_{\mathcal{S}} \qquad \begin{cases} \dot{x} = f(x) + \sum\limits_{i=1}^m G_i(x(t^-),\dot{u}(\{t\}))\dot{u}^i \,, \\ x(0^+) = \bar{x} \,, \end{cases}$$

where the first line is interpreted as an equality between measures (the function $f(x)$ is identified, as usual, with the measure $f(x(t))\,dt$), and

$$x(t\pm) = \lim_{s \to t\pm} x(s)\,.$$

Since $G_i(z, 0) = g_i(z)$ for every $z \in \mathbf{R}^n$ and $\dot{u}(\{t\}) = u(t^+) - u(t^-)$ for every $t \in]0, T[$, problem $(E)_{\mathcal{S}}$ reduces to (E) when u is continuous. This shows that $(E)_{\mathcal{S}}$ can be considered as an extension of the classical problem (E).

We shall prove that, for every $u \in BV^-([0, T], \mathbf{R}^m)$, the extended Cauchy problem $(E)_S$ has one and only one solution $x \in BV^-([0, T], \mathbf{R}^n)$. As for the continuous dependence of the solutions x on the controls u, we prove the following result(Theorem 4.3): if (u_h) is a sequence in $BV^-([0, T], \mathbf{R}^m)$ which converges pointwise a.e. in $[0, T]$ to a function $u \in BV^-([0, T], \mathbf{R}^m)$, and the total variations of the u_h in $[0, T]$ converge to the total variation of u in $[0, T]$, then the solutions x_h of $(E)_S$ corresponding to the controls u_h converge pointwise a.e. in $[0, T]$ to the solution x of $(E)_S$ corresponding to u.
In particular, if the u_h are differentiable, then the x_h are nothing but the corresponding classical solutions of (E). Hence, $(E)_S$ can be viewed as a limit problem of a sequence of classical problems.

Following an idea developed in [9], all results concerning the solutions of $(E)_S$ are obtained by considering the $n + 1$-dimensional Cauchy problem

$$(E)_{n+1} \qquad \begin{cases} \dfrac{dy^0}{ds} = \dfrac{d\varphi^0}{ds} \\[2ex] \dfrac{dy}{ds} = f(y(s))\dfrac{d\varphi^0}{ds} + \displaystyle\sum_{i=1}^{m} g_i(y(s))\dfrac{d\varphi^i}{ds} \\[2ex] y^0(0) = 0 \\[1ex] y(0) = \bar{x} \ , \end{cases}$$

where the solution $\tilde{y} = (y^0, y)$ is a function from $[0,1]$ into $[0,T]\times\mathbf{R}^n$, and the control $\varphi = (\varphi^0, \ldots, \varphi^m)$ maps $[0,1]$ into $[0,T]\times\mathbf{R}^m$.

If u is absolutely continuous and $\varphi = (\varphi^0, \ldots, \varphi^m)$ is any absolutely continuous reparametrization of the graph of u, it is straightforward to check that x is the solution of (E) if and only if

$$(1.1) \qquad x(t) = y\big((\varphi^0)^{-1}(t)\big) \quad \forall t \in [0, T] ,$$

where $\tilde{y} = (y^0, y)$ is the solution of $(E)_{n+1}$.

If u is just a function with bounded variation and φ is the *canonical graph completion* of u (see Definition 2.4) introduced in [9], then x is a solution of $(E)_S$ if and only if (1.1) holds for the solution $\tilde{y} = (y^0, y)$ of $(E)_{n+1}$. The proof of this fact is based on a general chain rule for distributional derivatives of functions with bounded variation, recently proved in [9] and [2].

In the proof of the continuous dependence for $(E)_{\mathcal{S}}$, the convergence of the total variation of the controls u_h is crucial. In the last section we drop this hypothesis and study the limits x of sequences of solutions (x_h) of (E) corresponding to regular controls u_h, with uniformly bounded total variation, which converge pointwise a.e. in $[0, T]$ to a function $u \in BV^-([0, T], \mathbf{R}^m)$.

We prove that these limit functions x can be characterized as the solutions of a family of differential equations in the sense of measures, which generalize $(E)_{\mathcal{S}}$ and depend not only on the limit control u, but also on an arbitrary (countable) family $\mathcal{T}$ of time instants, which includes all discontinuity points of u. The jump of x at each point t of $\mathcal{T}$ is determined by solving an auxiliary Cauchy problem, with initial value $x(t-)$, which takes into account the behaviour of the sequence (u_h) in an arbitrarily small neighborhood of t.

2. A generalization of problem (E)

By $x = (x^1, x^2, \ldots, x^n)$ we denote a vector of the Euclidean space $\mathbf{R}^n$. Given a control $u = (u^1, \ldots, u^m)$ from the time interval $[0, T]$ into $\mathbf{R}^m$, let us consider the Cauchy problem

$$(E) \qquad \begin{cases} \dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} g_i(x(t))\dot{u}^i(t) \\ x(0) = \bar{x} \end{cases} \qquad t \in [0,T],$$

where the vector fields $f, g_1, \ldots, g_m$ from $\mathbf{R}^n$ into $\mathbf{R}^n$ are assumed to be continuously differentiable and globally bounded, and dots denote differentiation with respect to time.

One can naturally associate with (E) the $n + 1$-dimensional Cauchy problem

$$(E)_{n+1} \qquad \begin{cases} \dfrac{dy^0(s)}{ds} = \dfrac{d\varphi^0(s)}{ds} \\ \dfrac{dy(s)}{ds} = f(y(s))\dfrac{d\varphi^0(s)}{ds} + \sum_{i=1}^{m} g_i(y(s))\dfrac{d\varphi^i(s)}{ds} \\ y^0(0) = 0 \\ y(0) = \bar{x} \end{cases} \qquad s \in [0,1],$$

where: $\tilde{y} = (y^0, y)$: $[0,1] \to [0,T] \times \mathbf{R}^n$, φ^0 is a non decreasing map from $[0,1]$ onto $[0,T]$, and $\varphi = (\varphi^0, \ldots, \varphi^m)$ is an $m+1$-dimensional control from $[0,1]$ into $[0,T] \times \mathbf{R}^m$.

In fact, if $u \in C^1([0,T], \mathbf{R}^m)$, i.e. u is continuously differentiable, then (E) has a solution x, uniquely defined on $[0,T]$. Let $\varphi : [0,1] \to [0,T] \times \mathbf{R}^m$ be any C^1 reparametrization of the graph of u, i.e. $\varphi^0 \in C^1([0,1])$, $\varphi^0(0)=0$, $\varphi^0(1)=T$, $\frac{d\varphi^0(s)}{ds} \geq 0$, and $u^i(\varphi^0(s)) = \varphi^i(s)$ for every $s \in [0,1]$. Then it is trivial to check that the solution $\tilde{y} = (y^0, y)$ of $(E)_{n+1}$ corresponding to φ satisfies $x(\varphi^0(s)) = y(s)$ for every $s \in [0,1]$. Hence

$$x(t) = y\big((\varphi^0)^{-1}(t)\big) \qquad \forall\, t \in [0, T],$$

provided φ^0 is strictly increasing.

If u is not so regular one cannot apply the above argument. In particular (E) has not a classical meaning. Still, by using the concept of graph completion, which has been introduced in [9], we are setting down a relationship between the solutions of an extension (in the sense of measures) of Cauchy problem (E) and the (classical) solutions of $(E)_{n+1}$.

We begin by fixing some notation and recalling some definitions concerning functions with bounded total variations.

Definition 2.1. *A function* $f : [0,T] \to \mathbf{R}^q$ *is said to have bounded variation on the subinterval* $[a, b] \subseteq [0,T]$ *if there exists a constant* $C \geq 0$ *such that, for each finite set of points* $\{t_0, \ldots, t_p\}$ *satisfying*

$$a = t_0 < t_1 < \ldots < t_p = b,$$

the inequality

$$\sum_{k=1}^{p} |f(t_k) - f(t_{k-1})| \leq C$$

holds, where $|\cdot|$ *denotes the Euclidean norm. The least* C *which satisfies the above condition is called the variation of* f *on* $[a, b]$, *and it is denoted by* $V_a^b(f)$. *The number* $V_0^T(f)$ *will be called the total variation of* f.

The symbol $BV([0,T], \mathbf{R}^q)$ will denote the class of functions from $[0,T]$ into $\mathbf{R}^q$ with bounded total variation, whereas $BV^-([0,T], \mathbf{R}^q)$ will indicate the set of left continuous functions of $BV([0,T], \mathbf{R}^q)$ which are right continuous at 0.

For every $u \in BV([0, T], \mathbf{R}^m)$ the distributional derivative $\dot{u}$ is an $\mathbf{R}^m$-valued Radon measure on $]0, T[$. If $u \in BV^-([0, T], \mathbf{R}^m)$, then $\dot{u}$ is characterized by the equality

$$\dot{u}([t_1, t_2[) = u(t_2) - u(t_1)$$

for every subinterval $[t_1, t_2[\subseteq]0, T[$. In particular, for every $t \in]0, T[$,

$$\dot{u}(\{t\}) = \Delta u(t) := u(t^+) - u(t^-),$$

where $u(t^+)$, $u(t^-)$ are the right and the left limits of u at t, respectively. The integral of a function $f:[0, T] \to \mathbf{R}$ with respect to the measure $\dot{u}$ on a Borel subset A of $]0, T[$ will be denoted by

$$\int_A f\dot{u}.$$

If $u \in BV^-([0, T], \mathbf{R}^m)$, then, for every subinterval $[t_1,t_2[\subseteq]0,T[$, we have

$$|\dot{u}|([t_1,t_2[) = V_{t_1}^{t_2}(u),$$

where $|\dot{u}|$ is the total variation of the measure $\dot{u}$. Moreover

$$|\dot{u}|(]0,T[) = V_0^T(u),$$

for every $u \in BV^-([0, T], \mathbf{R}^m)$.

Remark 2.1. With each $u \in BV([0, T], \mathbf{R}^m)$ one can associate the left continuous map $u^- \in BV^-([0, T], \mathbf{R}^m)$ defined by

$$u^-(t) := u(t^-) = \lim_{s \to t^-} u(s), \qquad t \in]0, T],$$
$$u^-(0) := u(0^+) = \lim_{s \to 0^+} u(s).$$

It is well known that u^- coincides with u almost everywhere, which implies $\dot{u}^- = \dot{u}$ as measures on $]0, T[$.
We shall give our definitions and state our results for the class $BV^-([0,T],\mathbf{R}^m)$. It will be clear that analogous results hold true for the functions belonging to $BV([0, T], \mathbf{R}^m)$, up to obvious changes.

When $u \in BV^-([0, T], \mathbf{R}^m)$, problem (E) has not a classical meaning. In principle, one could interpret (E) as an equality between measures. Yet, in our opinion, this choice has to be refused since, on one hand, one cannot guarantee the existence of a solution. On the other hand, such a solution would not be robust, i.e. it would not depend continuously on controls, as it follows from the results in Section 4 (see also [21]).

In order to overcome the above objections, we propose to extend (E) to a new Cauchy Problem $(E)_S$ in the sense of meausres, which reduces to (E) when u is regular. In this section and in the following one, existence and uniqueness of the solution of the extended Cauchy Problem $(E)_S$ will be proved. It will be seen (Theorem 2.2) that this solution coincides with the solution corresponding to the *canonical graph completion* of the control u, which has been introduced in [9]. Moreover, the introduction of $(E)_S$ will be motivated by a robustness argument, which will be treated Section 4.

Let $(z, p) \in \mathbf{R}^n \times \mathbf{R}^m$. For every $i = 1, \ldots, m$, let the function $G_i : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ be defined by

$$G_i(z,p) = \int_0^1 g_i\Big(\exp(\sigma \sum_{j=1}^m p^j g_j)z\Big)d\sigma ,$$

where the symbol $\exp(\sigma \sum_{j=1}^m p^j g_j)z$ denotes the value at the time $s = \sigma$ of the solution of the Cauchy Problem

$$\begin{cases} \dfrac{dw}{ds} = \displaystyle\sum_{j=1}^m p^j g_j(w(s)) \\ w(0) = z . \end{cases}$$

Remark 2.2. It is trivial to check that, for every $(z,p) \in \mathbf{R}^n \times \mathbf{R}^m$, the identity

$$\sum_{i=1}^{m} p^i G_i(z,p) = \exp\left(\sum_{i=1}^{m} p^i g_i\right)z - z$$

holds true.

Let us consider the Cauchy Problem

$$(E)_S \qquad \begin{cases} \dot{x} = f(x) + \sum_{i=1}^{m} G_i\big(x(t^-), \dot{u}(\{t\})\big)\, \dot{u}^i & \text{on }]0,T[\\ x(0^+) = \bar{x}\ . \end{cases}$$

Definition 2.2. *Let* $u \in BV^-([0,T], R^m)$ *. A solution of* $(E)_S$ *is a map* $x \in BV([0,T],R^n)$ *which satisfies* $(E)_S$ *in the sense of measures on* $]0,T[$ *. In other words,* x *satisfies*

$$\int_B \dot{x} = \int_B f(x(t))\, dt + \sum_{i=1}^{m} \int_B G_i\big(x(t^-), \dot{u}(\{t\})\big)\dot{u}^i$$

for every Borel subset B of $]0,T[$.

We are now in a position to state the following theorem.

Theorem 2.1. *For each* $u \in BV^-([0,T], R^m)$ *there exists a unique solution of* $(E)_S$.

Remark 2.3. Since the measures involved in the Definition 2.2 do not change if we modify x on a set of Lebesque measure zero, by uniqueness we mean uniqueness up to sets of Lebesgue measure zero .

Remark 2.4. A solution x of $(E)_S$ is determined up to a set of Lebesgue measure zero. However, at each $t \in]0,T[$, the left limit $x(t^-)$ and the right limit $x(t^+)$ are uniquely determined. Hence, it is meaningful to speak of the *jump* $\Delta x(t) = x(t^+) - x(t^-)$ *of* x *at* t. This is given by the formula

$$\Delta x(t) = \exp\left(\sum_{i=1}^{m} \Delta u^i(t) g_i\right) x(t^-) - x(t^-) . \tag{2.1}$$

Indeed, $\Delta x(t) = \dot{x}(\{t\})$, $\Delta u^i(t) = \dot{u}^i(\{t\})$, and (2.1) follows directly from $(E)_S$ and Remark 2.2. Note that equality (2.1) generalizes the formula (3.1) in [7], which gives the jumps in the case of scalar controls.

Theorem 2.1 is a straightforward consequence of Theorem 2.2 below and of the uniqueness of the solution of the Cauchy Problem $(E)_{n+1}$. To state Theorem 2.2 we need the definition of canonical graph completion (see [9]).

Definition 2.3. (Graph completion) *Let* u *belong to* $BV^-([0,T], \mathbf{R}^m)$. *A graph completion of* u *is a Lipschitz continuous map* $\varphi = (\varphi^0, \ldots, \varphi^m)$: $[0,1] \to [0,T] \times \mathbf{R}^m$ *such that*

i) $$0 \leq r < s \leq T \Rightarrow \varphi^0(r) \leq \varphi^0(s) ;$$

ii) $$\forall t \in [0,T] \ , \ \exists s \in [0,1] \text{ such that } \varphi(s) = (t, u(t)) .$$

Note that this definition is slightly more general than the one in [9], because we do not require that $\varphi(0) = (0, u(0))$ and $\varphi(1) = (T, u(T))$. Anyway, condition ii) implies that there exist s_0 and s_1 such that $\varphi(s_0) = (0, u(0))$ and $\varphi(s_1) = (T, u(T))$, so that the restriction of φ to $[s_0,s_1]$ is a graph completion of u in the sense of [9].

Definition 2.4. (Canonical graph completion) *Let* u *belong to* $BV^-([0,T], \mathbf{R}^m)$ *and set*

$$W(t) := \frac{t + V_0^t(u)}{T + V_0^T(u)} , \qquad t \in [0,T] .$$

The canonical graph completion φ *of* u *is defined by*

$$\varphi(s) := (t, u(t)) \qquad \text{if } s = W(t),$$

$$\varphi(s) := (t, u(t) + \frac{s - W(t)}{\Delta W(t)} \Delta u(t)) \qquad \text{if } s \in]W(t), W(t^+)[.$$

Remark 2.5. If φ is the canonical graph completion of u , it is straightforward to verify that

i) $$\Delta W(t) = \frac{|\Delta u(t)|}{T + V_0^T(u)};$$

ii) $$\frac{d\varphi}{ds}(s) = \left(0 \ , \ \frac{\Delta u(t)}{\Delta W(t)}\right) \qquad \forall s \in]W(t), W(t^+)[;$$

iii) φ is Lipschitz continuous with constant $T + V_0^T(u)$.

Theorem 2.2. *Let* u *belong to* $BV^-([0,T], \mathbf{R}^m)$. *Then a map* $x \in BV([0,T], \mathbf{R}^n)$ *is a solution of* $(E)_S$ *if and only if there exists a solution* $\tilde{y} = (y^0, y)$ *of* $(E)_{n+1}$ *corresponding to the canonical graph completion* φ *such that*

(2.2) $$x(t) = y(W(t))$$

for almost every $t \in]0,T[$.

Theorem 2.2 will be proved in the next section.

Remark 2.6. If $x \in BV^-([0,T], \mathbf{R}^n)$, then (2.2) holds for every $t \in [0,T]$. In fact, $x(\cdot)$ and $y(W(\cdot))$ are functions of $BV^-([0,T], \mathbf{R}^n)$ which coincide almost everywhere on]0,T[, thus they are equal everywhere on [0,T].

3. Proof of Theorem 2.2.

In the proof of Theorem 2.2 we shall use Volpert's averaged superposition (see[37]) and a related theorem recently proved in [9] and, by a different approach, in [2].

Definition 3.1. (Volpert's averaged superposition). *Let* $A : \mathbf{R}^p \to \mathbf{R}^q$ *be a bounded Borel function, and let* v *belong to* $BV([0,T], \mathbf{R}^p)$. *The function* $\hat{A}(v) : [0,T] \to \mathbf{R}^q$ *defined by*

$$\hat{A}(v)(t) := \int_0^1 A\big(v(t^-) + \sigma(v(t^+) - v(t^-))\big)d\sigma$$

is called the averaged superposition of A *and* v.

Remark 3.1. $\hat{A}(v)(t) = A(v(t))$ at each point t where v is continuous.

Theorem 3.1. ([9] and [2]). *Let* $\psi : [0,1] \to \mathbf{R}^n$ *be a Lipschitz continuous function and let* $z \in BV([0,T], [0,1])$. *If the map* α *is defined by*

$$\alpha(t) := \psi(z(t)) \quad \forall t \in [0,T],$$

then:

i) $\alpha \in BV([0,T], \mathbf{R}^n)$;

ii) *the identity of measures*

$$\dot{\alpha} = \hat{\psi}_*(z)\dot{z}$$

holds, where ψ_* *denotes any Borel function coinciding with the derivative* $\frac{d\psi}{ds}$ *almost everywhere with respect to the Lebesgue measure.*

The following proposition, proved in [9], Proposition 5.3, is an easy consequence of the chain rule given by Theorem 3.1.

Proposition 3.1. *Let* v *be a strictly increasing map from* $[0,T]$ *into* $[0,1]$ *and let* $k : [0,1] \to \mathbf{R}$ *be a bounded Borel map. Then,*

$$\int_{]0,t[} \hat{k}(v)\dot{v} = \int_{v(0^+)}^{v(t^-)} k(s)ds .$$

for every $t \in]0,T]$.

We shall make use of the following change of variable formula.

Proposition 3.2. *Let* v *and* k *be as in Proposition 3.1. Then, for any Borel subset* B *of*]0,T[*formed by continuity points of* v *, the identity*

$$\int_B k(v)\dot{v} = \int_{v(B)} k(s)\, ds \tag{3.1}$$

holds true.

Proof. Denoting by χ_A the characteristic function of a set A and by k_B the map $k \cdot \chi_{v(B)}$, we have

$$\hat{k}_B(v)(t) = k(v(t))\chi_B(t) \tag{3.2}$$

for every $t \in]0,T[$. In fact, if v is continuous at t, then by definition we have

$$\hat{k}_B(v)(t) = k_B(v(t)) = k(v(t))\chi_{v(B)}(v(t)) = k(v(t))\chi_B(t) .$$

If v is discontinuous at t, then $\chi_B(t) = 0$ and

$$v(B) \cap]v(t^-), v(t^+)[= \emptyset ,$$

hence $\chi_{v(B)}\big(v(t^-) + \sigma(v(t^+) - v(t^-))\big) = 0$ for every $\sigma \in]0,1[$. This implies that

$$\hat{k}_B(v)(t) = \int_0^1 k_B\big(v(t^-)+\sigma(v(t^+)-v(t^-))\big)d\sigma = 0 = k(v(t))\chi_B(t)$$

and concludes the proof of (3.2).

By Proposition 3.1 and by (3.2) we have

$$\int_{v(B)} k(s)ds = \int_{v(0^+)}^{v(T^-)} k_B(s)ds = \int_{]0,T[} \hat{k}(v)\dot{v} = \int_{]0,T[} k(v)\chi_B \dot{v} = \int_B k(v)\dot{v},$$

which proves (3.1).

QED

Proof of Theorem 2.2. Let $\tilde{y} = (y^0, y)$ be a solution of $(E)_{n+1}$ corresponding to the canonical graph completion φ of the control u. Let us consider the map x defined by

(3.3) $$x(\cdot) := y(W(\cdot)) ,$$

and let us show that it is a solution of $(E)_S$. To this aim, we shall apply Theorem 3.1. By definition, y is a solution of the integral equation

$$y(s) = \int_0^s \left(f(y(\xi))\varphi_*^0(\xi) + \sum_{i=1}^m g_i(y(\xi))\varphi_*^i(\xi) \right) d\xi ,$$

where φ_*^0 and φ_*^i are Borel functions coinciding with the derivatives $\frac{d\varphi^0}{ds}$ and $\frac{d\varphi^i}{ds}$ almost everywhere with respect to the Lebesgue measure. Since f and the g_i are bounded, and the canonical graph completion φ is Lipschitz continuous, y turns out to be Lipschitz continuous. Furthermore, W belongs to $BV^-([0,T], [0,1])$. Then, Theorem 3.1 yields

(3.4) $$\dot{x} = \hat{y}_*(W)\, \dot{W} ,$$

where

$$y_*(s) = f(y(s))\, \varphi_*^0(s) + \sum_{i=1}^m g_i(y(s))\, \varphi_*^i(s) .$$

Let E_c denote the set of points at which u (hence W) is continuous, and let A be a subset of E_c. One has

$$\dot{x}(A) = \int_A \hat{y}_*(W)\, \dot{W} ,$$

and, by the continuity of W at every $t \in A$,

$$\hat{y}_*(W)(t) = y_*(W(t)) = f(y(W(t))\ \varphi_*^0(W(t)) + \sum_{i=1}^{m} g_i(y(W(t)))\ \varphi_*^i(W(t))\ .$$

Then,

$$\dot{x}(A) = \int_A f(y(W))\varphi_*^0(W)\dot{W} + \sum_{i=1}^{m} \int_A g_i(y(W))\ \varphi_*^i(W)\ \dot{W}\ .$$

Since $\varphi^0(W(t)) = t$ for every $t \in A$, by Theorem 3.1 $\varphi_*^0(W)\ \dot{W}$ coincides with the Lebesgue measure dt on A. Analogously, since $\varphi^i(W(t)) = u^i(t)$, we have $\varphi_*^i(W)\ \dot{W} = \dot{u}^i$ as measures on A, for each $i = 1, \ldots, m$. Hence,

$$\int_A \dot{x} = \int_A f(x(t))dt + \sum_{i=1}^{m} \int_A g_i(x)\dot{u}^i\ .$$

Since for every $t \in E_c$ and for every $i = 1, \ldots, m$ it is $g_i(x(t)) \equiv G_i(x(t^-), 0) = G_i(x(t),\dot{u}(\{t\}))$, the above equality can be written in the form

$$\int_A \dot{x} = \int_A f(x(t))dt + \sum_{i=1}^{m} \int_A G_i(x(t^-),\dot{u}(\{t\}))\dot{u}^i \tag{3.5}$$

Now, let $t \in]0,T[\setminus E_c$. Then (3.4) implies:

$$\dot{x}(\{t\}) = \hat{y}_*(W)(t)\ \dot{W}(\{t\})\ . \tag{3.6}$$

By ii) in Remark 2.5 and by the equation $(E)_{n+1}$, for every $\sigma \in]0, \dot{W}(\{t\})[$, it is

$$y(W(t) + \sigma) = \exp\left(\sigma \sum_{i=1}^{m} \frac{\dot{u}^i(\{t\})}{\dot{W}(\{t\})} g_i\right) y(W(t))\ .$$

Hence

(3.7)
$$\hat{y}_*(W)(t) = \int_0^1 \frac{dy}{ds}(W(t)+\sigma\dot{W}(\{t\}))\, d\sigma = \frac{1}{\dot{W}(\{t\})} \int_{W(t)}^{W(t)+\dot{W}(\{t\})} \frac{d\,y}{ds}(s)ds =$$
$$= \frac{1}{\dot{W}(\{t\})}\left[\exp\left(\sum_{i=1}^{m} \dot{u}^i(\{t\})g_i\right)y(W(t)) - y(W(t))\right].$$

Since $y(W(t)) = x(t)$, by (3.6) and (3.7) one obtains

$$\int_{\{t\}} \dot{x} = \exp\left(\sum_{i=1}^{m} \dot{u}^i(\{t\})g_i\right)x(t) - x(t).$$

By Remark 2.2, the right-hand side coincides with the vector $\sum_{i=1}^{m} G_i\big(x(t),\dot{u}(\{t\})\big)\dot{u}^i(\{t\})$.

Moreover, $\int_{\{t\}} f(x(t))dt = 0$. Hence,

(3.8)
$$\int_{\{t\}} \dot{x} = \int_{\{t\}} f(x(t))dt + \sum_{i=1}^{m} \int_{\{t\}} G_i\big(x(t),\dot{u}(\{t\})\big)\dot{u}^i .$$

Since identities (3.5) and (3.8) hold for every $A \subseteq E_c$ and for every $t \in]0,T[\setminus E_c$, respectively, the map x defined in (3.3) is a solution of $(E)_S$, according to Definition 2.2 .

Conversely, let us show that, if x is any solution of $(E)_S$, then there exists a solution $\tilde{y} = (y^0, y)$ of $(E)_{n+1}$ corresponding to the canonical graph completion φ of u, such that (3.1) holds almost everywhere in $[0,T]$. Without loss of generality we can assume that x is left continuous on $]0,T]$ and right continuous at $t = 0$. Let us consider the map $\tilde{y} = (y^0, y) : [0,1] \to [0,T]\times\mathbf{R}^m$ defined by

$$\tilde{y}(s) = (y^0, y)(s) = \big(t\, , x(t)\big), \quad \text{if } s = W(t)\, , t \in E_c,$$

$$\tilde{y}(s) = (y^0, y)(s) = \Big(t\, , \exp\Big[(s - W(t)) \sum_{i=1}^{m} \frac{\dot{u}^i(\{t\})}{\dot{W}(\{t\})} g_i\Big]x(t)\Big), \quad \text{if } s \in [W(t), W(t^+)]\, , t \in [0,1]\setminus E_c.$$

We shall prove that $\tilde{y}$ is a Lipschitz continuous map which solves $(E)_{n+1}$ with φ equal to the canonical graph completion of u.

Let us start by proving that $\tilde{y}$ is Lipschitz continuous on $W([0,T])$. First of all, we note that $\tilde{y}(W(t)) = (t, x(t))$ for every $t \in [0,T]$. If $s_1 = W(t_1)$, $s_2 = W(t_2)$, and $t_1 \le t_2$, then

$$|\tilde{y}(s_1) - \tilde{y}(s_2)| \le |t_1 - t_2| + |x(t_1) - x(t_2)| \le$$

$$\le (1 + \|f\|)\, |t_1 - t_2| + \|(g_1, \ldots, g_m)\|\, V_{t_1}^{t_2}(u) \le \tag{3.9}$$

$$\le C\big(W(t_2) - W(t_1)\big) = C(s_2 - s_1),$$

where $\|f\|$ denotes the sup norm of f, $\|(g_1, \ldots, g_m)\| = \max\{\|g_i\| : i = 1, \ldots, m\}$, and C is a suitable positive constant. This proves that $\tilde{y}$ is Lipschitz continuous on $W([0,T])$: in particular,

$$\lim_{\substack{s \to W(t^+)^+ \\ s \in W([0,T])}} \tilde{y}(s) = \lim_{\tau \to t^+} \tilde{y}(W(\tau)) = (t, x(t^+)) \tag{3.10}$$

for every $t \in [0,T] \setminus E_c$. Let us prove that

$$\tilde{y}(W(t^+)) = (t, x(t^+)) \tag{3.11}$$

for every $t \in [0,T] \setminus E_c$. Since $W(t^+) - W(t) = \dot{W}(\{t\})$, by the definition of $\tilde{y}$ we obtain

$$\tilde{y}(W(t^+)) = \Big(t\,,\, \exp[\dot{W}(\{t\}) \sum_{i=1}^{m} \frac{\dot{u}^i(\{t\})}{\dot{W}(\{t\})} g_i]\; x(t) \Big) =$$

$$= \Big(t\,,\, \exp[\sum_{i=1}^{m} \dot{u}^i(\{t\})\, g_i]\; x(t) \Big).$$

By Remark 2.2 we have

$$\tilde{y}(W(t^+)) = \Big(t,\, x(t) + \sum_{i=1}^{m} G_i\big(x(t)\,,\, \dot{u}(\{t\})\big)\, \dot{u}^i(\{t\})\Big).$$

Therefore, since the map x solves $(E)_S$, we obtain

$$\tilde{y}(W(t^+)) = \big(t, x(t) + \dot{x}(\{t\})\big) = \big(t, x(t^+)\big),$$

which concludes the proof of (3.11).

Furthermore, if $t \in [0,T] \setminus E_c$ and $s \in]W(t), W(t^+)[$ we have

$$\left|\frac{d\tilde{y}}{ds}(s)\right| = \left|\frac{dy}{ds}(s)\right| = \left|\sum_{i=1}^{m} \frac{\dot{u}^i(\{t\})}{\dot{W}(\{t\})} g_i(y(s))\right| \leq (T + V_0^T(u)) \, \|(g_1, \ldots, g_m)\|,$$

hence

(3.12)
$$|\tilde{y}(s_1) - \tilde{y}(s_2)| \leq (T + V_0^T(u)) \, \|(g_1, \ldots, g_m)\| \, |s_1 - s_2|$$

for every $s_1, s_2 \in [W(t), W(t^+)]$. The Lipschitz continuity of $\tilde{y}$ follows on (3.9), (3.10), (3.11), and (3.12).

To conclude the proof of Theorem 2.2, let us show that $\tilde{y}$ solves $(E)_{n+1}$, when φ coincides with the canonical graph completion of u .
It is sufficient to check that

(3.13)
$$\int_A \frac{dy}{ds}(s)\, ds = \int_A \left[f(y(s))\frac{d\varphi^0}{ds}(s) + \sum_{i=1}^{m} g_i(y(s))\, \frac{d\varphi^i}{ds}(s)\right] ds,$$

for each Borel subset A of [0,1]. Let us consider the partition of A formed by the subsets

$$A_1 = A \cap W(E_c), \quad A_2 = A \cap ([0,1] \setminus W(E_c)).$$

Then A_1 is a Borel set, and $A_1 = W(B)$, where B is a Borel subset of E_c .

Applying Proposition 3.2 and Theorem 3.1, and using the fact that $G_i(x(t), \dot{u}(\{t\})) = g_i(x(t))$ for every $t \in B$ and for every $i = 1, \ldots, m$, we obtain the following chain of identities

$$\int_{A_1} \frac{dy}{ds}(s)\, ds = \int_B \frac{dy}{ds}(W)\, \dot{W} = \int_B \dot{x} = \int_B f(x(t))dt + \int_B \sum_{i=1}^{m} g_i(x)\dot{u}^i =$$

(3.14)

$$\int_B \Big[f(y(W))\frac{d\varphi^0}{ds}(W) + \sum_{i=1}^{m} g_i(y(W))\frac{d\varphi^i}{ds}(W)\Big]\dot{W} = \int_{A_1} \Big[f(y(s))\frac{d\varphi^0}{ds}(s)) + \sum_{i=1}^{m} g_i(y(s))\frac{d\varphi^i}{ds}(s)\Big]ds.$$

In particular, the second identity in (3.14) follows on the definition of y on E_c.

On the other hand, for every $s \in A_2$, there exists a $t \in]0,T[$ such that $s \in]W(t), W(t^+)[$. Hence, by the definition of y,

$$\frac{dy}{ds}(s) = \sum_{i=1}^{m} \frac{\dot{u}^i(\{t\})}{\dot{W}(\{t\})}\, g_i(y(s)) .$$

Since for every $s \in [W(t), W(t^+)]$ we have $\frac{d\varphi}{ds}(s) = (0, \frac{\dot{u}(\{t\})}{\dot{W}(\{t\})})$, one obtains

(3.15) $$\frac{dy}{ds}(s) = f(y(s))\frac{d\varphi^0}{ds}(s) + \sum_{i=1}^{m} g_i(y(s))\frac{d\varphi^i}{ds}(s) \qquad \text{a.e. in } A_2 ,$$

which, together with (3.14), yields (3.13). QED

4. Continuos dependence on the controls for solutions of the Cauchy problem $(E)_s$

In Section 2 we have introduced the Cauchy problem $(E)_S$, which is meaningful whenever u merely belongs to $BV([0,T], R^m)$. On the other hand, $(E)_S$ can be considered as a generalization of the Cauchy problem (E), since $G_i(x(t^-), \dot{u}(\{t\})) = g_i(x(t))$, $i = 1, \ldots, m$, for each t where the control u is continuous. Now, we shall show (see Theorem 4.2) that this generalization agrees with a robustness argument.

First of all, we state Theorem 4.1, which concerns the continuous dependence of the Carathéodory solutions of (E) on controls, when the latter are uniformly Lipschitz continuous, and the spaces of controls u and solutions x are endowed with the C^0 norms. This result is a straightforward consequence of Theorem 1, Section 4, in [9], where the Lipschitz continuity of the input-output map $\Phi : u \to x$ has been proved. Theorem 4.1 follows also from the results on continuous dependence on a parameter in [30] and [5]. Still, we provide a direct proof, since this turns out to be very elementary.

Theorem 4.1. *Let* L *be a positive constant and let* $\mathcal{U} \subseteq C^0([0,T], \mathbf{R}^m)$ *be the subset of Lipschitz continuous controls with Lipschitz constant* L *. Let* Φ *be the input-output map which, with any control* u, *associates the corresponding solution* x *of* (E) .

Then the restriction of Φ *to* $\mathcal{U}$ *is continuous with respect to the* C^0*-norms on the space of controls and solutions.*

Proof. Let $(u_h)_{h\in N}$ be a sequence of controls in $\mathcal{U}$, and let $(x_h)_{h\in N}$ be the corresponding (Carathéodory) solutions of (E). Let the controls u_h converge uniformly to a control $u \in \mathcal{U}$. Let us show that the functions x_h converge uniformly to the solution x corresponding to u . By hypothesis, for every $h \in N$, it is

$$|\dot{u}_h(t)| \leq L , \tag{4.1}$$

for each t where $\dot{u}_h(t)$ is defined, i.e. almost everywhere in [0,T]. Since u_h converges to u uniformly, one obtains

$$\lim_{h\to\infty} \dot{u}_n = \dot{u} \qquad \text{in } L^\infty\text{-}w^* , \tag{4.2}$$

i.e., in the space of bounded functions endowed with the weak*- topology. By (4.1) it follows, for each $h \in N$ and for almost every $t \in [0,T]$,

$$|\dot{x}_h(t)| \leq \| f \| + \|(g_1, \ldots, g_m)\|_\infty \cdot L,$$

where $\| \cdot \|$ denotes the sup norm. Hence, the maps $(x_h)_{h\in N}$ are equi-Lipschitzian .

Then, by Ascoli-Arzelà theorem, there exists a subsequence $(x_{h'})$ of (x_h) which converges uniformly to a (Lipschitz continuous) map z. In particular, this yields

$$\lim_{h'\to\infty} \dot{x}_{h'} = \dot{z} , \qquad \text{in } L^\infty\text{-}w^* .$$

Moreover,

$$\lim_{h'\to\infty} \|f(x_{h'}(\cdot)) - f(z(\cdot))\| = 0 ,$$

and, for each $i = 1, \ldots, m$,

(4.3)
$$\lim_{h'\to\infty} \|g_i(x_{h'}(\cdot)) - g_i(z(\cdot))\| = 0 .$$

From (4.2) and (4.3) we get

$$\lim_{h'\to\infty} g_i(x_{h'})\dot{u}_{h'} = g_i(z)\,\dot{u} , \qquad \text{in } L^\infty\text{-}w^* .$$

Therefore

$$\begin{cases} \dot{z} = f(z) + \sum_{i=1}^{m} g_i(z)\dot{u}^i \\ z(0) = \bar{x} \end{cases}$$

i.e., z is a solution of (E) corresponding to u . By the uniqueness of solution of (E) , we obtain $z = x$. Hence every converging subsequence $(x_{h'})$ of (x_h) converges to x , which implies that (x_h) converges to x .

QED

The following theorem concerns the robustness of the solution of the Cauchy problem $(E)_S$, which has been mentioned at the beginning of this section. Although the thesis is the same as in Corollary 2 in [9], the hypothesis differs in that it concerns only the controls u_h, without any reference to their (canonical) graph completions.

Theorem 4.2. *Let* (u_h) *be a sequence in* $BV^-([0,T], R^m)$ *and let* $u \in BV^-([0,T], R^m)$. *Assume that*

i) $\lim_{h\to\infty} u_h(t) = u(t)$ *for* a.e. $t \in [0,T]$, *and*

ii) $$\lim_{h\to\infty} V_0^T(u_h) = V_0^T(u) .$$

Let $x_h \in BV^-([0,T], \mathbf{R}^n)$ *and* $x \in BV^-([0,T], \mathbf{R}^n)$ *be the solutions of* $(E)_S$ *corresponding to the controls* u_h *and* u *, respectively. Then*

$$\lim_{h\to\infty} x_h(t) = x(t)$$

for each $t \in [0,T]$ *where* u *is continuous, i.e. almost everywhere in* $[0,T]$.

Remark 4.2. By Theorem 4.1 , hypothesis ii) is not necessary, since it is trivial to find an equi-Lipschitzian sequence of controls $(u_h)_{h\in \mathbf{N}}$ which converges uniformly to a function u, whereas the total variations $V_0^T(u_h)$ do not converge to $V_0^T(u)$. On the other hand, if we drop hypothesis ii) we cannot guarantee the convergence of (x_h) to x, as it will be clear in the next section.

Remark 4.3. By Remark 2.1, in order to reformulate Theorem 4.2 for controls u_h belonging merely to $BV([0,T], \mathbf{R}^m)$, one has to replace condition ii) with

$$\lim_{h\to\infty} V_0^T(u_h^-) = V_0^T(u^-)$$

or, equivalently,

$$\lim_{h\to\infty} |\dot{u}_h|(]0,T[) = |\dot{u}|(]0,T[).$$

To prove Theorem 4.2 we need the following lemma.

Lemma 4.1. *Let* u *and* u_h, $h \in \mathbf{N}$, *be maps belonging to* $BV^-([0,T], \mathbf{R}^m)$ *and satisfying hypotheses* i) *and* ii) *in Theorem 4.2. Then*

$$\lim_{h\to 0} V_0^t(u_h) = V_0^t(u) \quad , \quad \forall t\in E_c ,$$

where E_c *denotes the set of continuity points of* u .

Proof. For every $\alpha, \beta\in [0,T]$, $\alpha < \beta$, it is

$$\liminf_{h\to\infty} V_{\alpha^+}^{\beta}(u_h) \geq V_{\alpha^+}^{\beta}(u) , \tag{4.4}$$

where, for any $u\in BV^-([0,T], R^m)$, we set

$$V_{\alpha^+}^{\beta}(u) = \lim_{s\to\alpha+} V_s^{\beta}(u).$$

Indeed, let $\varepsilon > 0$ and let $\mathcal{N}\subseteq[0,T]$ be a subset of measure zero such that $u_h(t) \to u(t)$ for every $t\in [0,T] \setminus \mathcal{N}$. Then, there exists a natural number $N > 0$ and N instants $t_1 , \ldots, t_N$, with $t_i\in [0,T] \setminus \mathcal{N}$ and $\alpha < t_1 \ldots < t_N \leq \beta$, such that

$$V_{\alpha^+}^{\beta}(u) - \varepsilon \leq \sum_{i=1}^{N} |u(t_i) - u(t_{i-1})| =$$

$$= \lim_{h\to\infty} \sum_{i=1}^{N} |u_h(t_i) - u_h(t_{i-1})| \leq \liminf_{h\to\infty} V_{\alpha^+}^{\beta}(u_h) ,$$

which yileds (4.4), by the arbitrariness of ε . To prove the Lemma, assume, by contradiction, that there is a $t\in E_c$ such that the sequence $(V_0^t(u_h))$ does not converge to $V_0^t(u)$. Then, by (4.4),

$$\limsup_{h\to\infty} V_0^t(u_h) > V_0^t(u)$$

(recall that $V_{0^+}^t(u) = V_0^t(u)$, because u is right continuous at 0). Furthermore, by (4.4) and by the continuity of u at t, we have

$$\liminf_{h\to\infty} V_{t^+}^{T}(u_h) \geq V_{t^+}^{T}(u) = V_t^T(u) .$$

Hence, there exists $\delta > 0$ and a subsequence $(u_{h'})$ of (u_h) such that, for every $h' \in \mathbf{N}$,

$$V_0^t(u_{h'}) > V_0^t(u) + 2\,\delta\,, \qquad V_{t+}^T(u_{h'}) \geq V_t^T(u) - \delta\,.$$

Then, for every h' we have

$$V_0^T(u_{h'}) \geq V_0^t(u_{h'}) + V_{t+}^T(u_{h'}) > V_0^T(u) + \delta\,,$$

which contradicts the hypothesis.

QED

Proof of Theorem 4.2. Let us denote by φ and φ_h the canonical graph completions of u and u_h, respectively. Furthermore, let W and W_h be the functions introduced in Definition 2.4 corresponding to u and u_h, respectively. Hypothesis ii) implies that the φ_h are equi-Lipschitzean with a constant converging to $T + V_0^T(u)$ (see Remark 2.4).

We claim that

(4.5) $$\lim_{h\to\infty} \|\varphi_h - \varphi\| = 0\,,$$

where $\|\cdot\|$ denotes the C^0 norm.

In fact, by Ascoli-Arzelà theorem, there exists a subsequence $(\varphi_{h'})$ of (φ_h) which converges uniformly to a Lipschitz continuous function ψ having a constant less or equal to $T + V_0^T(u)$. Let us show that $\psi = \varphi$.

Let E_c and $\mathcal{N}$ have the same meaning as in the previous Lemma. By assumption

(4.6) $$\lim_{h\to\infty} u_h(t) = u(t) \quad \forall t \in [0,T] \setminus \mathcal{N}.$$

Let us prove that

(4.7) $$\lim_{h\to\infty} \varphi_{h'}(W(t)) = \varphi(W(t)) \quad , \forall t \in E_c \setminus \mathcal{N}.$$

Indeed,

$$|\varphi_{h'}(W(t)) - \varphi(W(t))| \leq$$

$$\leq |\varphi_{h'}(W(t)) - \varphi_{h'}(W_{h'}(t))| + |\varphi_{h'}(W_{h'}(t)) - \varphi(W(t))| .$$

The second term on the right-hand side converges to zero, since (4.6) can be written

$$\lim_{h\to\infty} \varphi_h(W_h(t)) = \varphi(W(t)) , \forall t\in [0,T] \setminus \mathcal{N}.$$

Furthermore, for a suitable $L > 0$ and for every h', we have

$$|\varphi_{h'}(W(t)) - \varphi_{h'}(W_{h'}(t))| \leq L\, |W(t) - W_{h'}(t)| .$$

By Lemma 4.1, $|W(t) - W_{h'}(t)|$ converges to zero as h' tends to infinity, for every $t\in E_c$. Hence (4.7) is proved.

Actually, (4.7) holds true for every $t\in E_c$. Indeed, for every $t, \tau\in [0,T]$ we have

$$|\varphi_{h'}(W(t)) - \varphi(W(t))| \leq$$

$$\leq |\varphi_{h'}(W(t)) - \varphi_{h'}(W(\tau))| + |\varphi_{h'}(W(\tau)) - \varphi(W(\tau))| + |\varphi(W(\tau) - \varphi(W(t))| \leq$$

$$\leq |\varphi_{h'}(W(\tau)) - \varphi(W(\tau))| + 2\, L\, |W(\tau) - W(t)| .$$

The limit (4.7) implies that, if $\tau\in E_c \setminus \mathcal{N}$ and $t\in E_c$, then the last expression is smaller than any prescribed $\varepsilon > 0$ as soon as τ is sufficiently close to t and h' is sufficiently large. Therefore we can conlude that

$$\lim_{h'\to\infty} \varphi_{h'}(W(t)) = \varphi(W(t)) \quad \forall t\in E_c ,$$

that is, to say,

$$(4.8) \qquad \psi(s) = \varphi(s) \qquad \forall s\in W(E_c) .$$

Furthermore, it is

(4.9) $$\psi(s) = \varphi(s)\,, \quad \text{if } s \in [W(t^-), W(t^+)], \text{ for some } t \in [0,T]\,.$$

Indeed, if $s = W(t^-)$,

$$\psi(s) = \lim_{\xi \to s^-} \psi(\xi) = \lim_{\substack{\tau \to t^- \\ \tau \in E_c}} \psi(W(\tau)) =$$

$$= \lim_{\substack{\tau \to t^- \\ \tau \in E_c}} \varphi(W(\tau)) = \lim_{\xi \to s^-} \varphi(\xi) \equiv \varphi(s)\,.$$

Similarly, (4.9) can be proved for $s = W(t^+)$. It remains to check that, for every $t \in [0,T] \setminus E_c$,

(4.10) $$\psi(s) = \varphi(s)\,, \quad \text{if } s \in]W(t^-), W(t^+)[\,.$$

At such s, φ is affine and

$$\left|\frac{d\varphi}{ds}\right| = T + V_0^T(u)\,.$$

Then (4.10) is a straightforward consequence of the fact that ψ is Lipschitz continuous with a constant less or equal to $T + V_0^T(u)$ and that $\psi(s) = \varphi(s)$ at the endpoints $s = W(t^-)$ and $s = W(t^+)$. From (4.8) and (4.10) we obtain $\psi = \varphi$. Since this holds true for every converging subsequence $\varphi_{h'}$ of φ_h, (4.5) is proved.

To conclude the proof, let us denote by $\tilde{y} = (y^0, y)$ and $\tilde{y}_h = (y_h^0, y_h)$, $h \in \mathbf{N}$, the solutions of $(E)_{n+1}$ corresponding to the paths φ and φ_h, respectively. If we apply Theorem 4.1 to the Cauchy problem $(E)_{n+1}$, then by (4.5) we obtain

(4.11) $$\lim_{h \to \infty} \|\tilde{y}_h - \tilde{y}\| = 0\,.$$

Since, by Theorem 2.2 and Remark 2.6,

$$|x_h(t) - x(t)| = |y_h(W_h(t)) - y(W(t))| \leq$$

$$\leq |y_h(W_h(t)) - y(W_h(t))| + |y(W_h(t)) - y(W(t))| \leq$$

$$\leq |y_h(W_h(t)) - y(W_h(t))| + L\, |W_h(t) - W(t)|$$

for a suitable constant L, (4.11) and Lemma 4.1 imply that

$$\lim_{h\to\infty} x_h(t) = x(t), \qquad \forall t\in E_c .$$

QED

5. A limiting family of measure differential equations.

In the previous sections the emphasis has been put on the canonical graph completion of a control $u\in BV^{-}([0,T], \mathbf{R}^m)$. In particular, in Section 4 it has been shown that this is the appropriate graph completion whenever u is thought as the limiting element of a sequence of regular maps (u_h) which converge to u in the sense that

$$\lim_{h\to\infty} u_h = u \text{ a.e. and } \lim_{h\to\infty} V_0^T(u_h) = V_0^T(u) .$$

The latter is the condition which forces the graphs of the u_h to approximate (uniformly) the canonical graph completion of u .

If the hypothesis $\lim_{h\to\infty} V_0^T(u_h) = V_0^T(u)$ is dropped, one can no longer expect that the rectilinear bridging which is used in the construction of the canonical graph completion still plays a privileged role. Then, the problem arises of looking for the a.e. limits of solutions x_h corresponding to regular controls u_h which satisfy the weaker condition:

$$\lim_{h\to\infty} u_h = u \text{ a.e. and } \limsup_{h\to\infty} V_0^T(u_h) < +\infty .$$

It will be proved that there exists a family of measure differential equations, each of which has a unique solution coinciding with one of the a.e. limits of the sequence (x_h) considered above.

We shall need the following auxiliary Cauchy problem, which governs the *instantaneous evolution* of the state at the jump points of the control:

$$(J) \qquad \begin{cases} \dfrac{dz}{d\sigma} = \displaystyle\sum_{i=1}^{m} g_i(z(\sigma)) \frac{d\psi^i}{d\sigma}, & \sigma \in [0,1] , \\ z(0) = \bar{z} \end{cases}$$

where $\psi = (\psi^1, \ldots, \psi^m)$ is a Lipschitz continuous map from $[0,1]$ into $\mathbf{R}^m$. If z denotes the solution of (J), let us set

$$z(\bar{z}, \psi) := z(1) - \bar{z} .$$

Remark 5.1. The choice of $[0,1]$ as interval of parametrization is not restrictive, since (J) has a geometrical character, i.e. the trajectory described by a solution of (J) is independent of the parametrization of the path ψ.

Remark 5.2. On the basis of the previous remark, one can easily check that, for each $\bar{z}$ and each ψ, there exists a map χ from $[0,1]$ into $\mathbf{R}^m$ with Lipschitz constant $V_0^1(\psi)$, and such that $z(\bar{z}, \psi) = z(\bar{z}, \chi)$.

Now we can construct the family of measure differential equations whose solutions correspond to a.e. limits of solutions of (E).

Let $u \in BV^-([0,T], \mathbf{R}^m)$, and let $\mathcal{T}$ be a countable subset of $[0,T[$ which contains 0 and all discontinuity points of u. Furthermore, let $(\psi_t)_{t\in\mathcal{T}}$ be a family of Lipschitz continuous maps from $[0,1]$ into $\mathbf{R}^m$ such that

$$(H) \qquad \sum_{t\in\mathcal{T}} V_0^1(\psi_t) < +\infty, \text{ and } \psi_t(0) = u(t^-) \ , \ \psi_t(1) = u(t^+) \text{ for every } t \in \mathcal{T}.$$

In the case $t = 0$, the condition on $\psi_0(0)$ is meaningless, and we require only $\psi_0(1) = u(0^+)$.

Let us consider the following generalization in the sense of measures of the Cauchy problem (E):

$$(E)_{\mathcal{M}} \qquad \begin{cases} \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)\dot{u}^i|_{E_c} + \sum_{t\in\mathcal{T}} z(x(t^-),\psi_t)\delta_t & \text{on }]0,T[\\ x(0^+) = z(\bar{x},\psi_0)\ , \end{cases}$$

where δ_t denotes the Dirac measure concentrated in t, E_c is the set of all continuity points of u, and $\dot{u}|_{E_c}$ is defined by $\dot{u}|_{E_c}(B) = \dot{u}(B\cap E_c)$, for each Borel subset B of $]0,T[$.

The definition of solution of $(E)_{\mathcal{M}}$ is completly analogous to the definition of solution of $(E)_{\mathcal{S}}$.

Definition 5.1. *A solution of* $(E)_{\mathcal{M}}$ *is a map* $x\in BV([0,T], \mathbf{R}^n)$ *which satisfies* $(E)_{\mathcal{M}}$ *in the sense of measures on* $]0,T[$ *. In other words*

$$\int_B \dot{x} = \int_B f(x(t))dt + \sum_{i=1}^{m} \int_{B\cap E_c} g_i(x)\dot{u}^i + \sum_{t\in\mathcal{T}\cap B} z(x(t^-),\psi_t)$$

for every Borel subset B *of* $]0,T[$.

Remark 5.3. The condition $\sum_{t\in\mathcal{T}} V_0^1(\psi_t) < +\infty$ in (H) implies that $\sum_{t\in\mathcal{T}} |z(x(t^-),\psi_t)| < +\infty$, which guarantees that the right-hand side of $(E)_{\mathcal{M}}$ is a measure.

Remark 5.4. If $u\in BV^-([0,T], \mathbf{R}^m)$ and $(\psi_t)_{t\in\mathcal{T}}$ is defined for every $t\in\mathcal{T}$ by

$$\psi_t(s) = u(t) + s\,(u(t+) - u(t))\quad ,\ s\in[0,1]\ ,$$

then problem $(E)_{\mathcal{M}}$ coincides with $(E)_{\mathcal{S}}$ (see Remark 2.2).

Theorem 5.1. *For each* $u \in BV^{-}([0,T], \mathbf{R}^m)$ *and each family* $(\psi_t)_{t\in \mathcal{T}}$ *satisfying* (H), *the Cauchy problem* $(E)_{\mathcal{M}}$ *has a unique solution (up to a set of Lebesgue measure zero).*

Theorem 5.1 is a straightforward consequence of Theorem 5.2 below and the uniqueness of the solutions of the Cauchy problem $(E)_{n+1}$. To state the latter, we need a canonical way to construct a graph completion of u corresponding to the family $(\psi_t)_{t\in \mathcal{T}}$. On the basis of Remark 5.2, we can assume that

for every $t \in \mathcal{T}$, *the Lipschitz constant of* ψ_t *equals* $V_0^1(\psi_t)$.

Since $\mathcal{T}$ is a countable set, we can write $\mathcal{T} = \{t_i \mid i \in \mathbf{N}\}$. For every $i \in \mathbf{N}$, let us set

$$a_i := V_0^1(\psi_{t_i}) \ , \ a := \sum_{i=1}^{\infty} a_i \ .$$

Let us define the map $\mathcal{W}$: $[0,T] \rightarrow [0,1]$ by

$$\mathcal{W}(t) := \frac{1}{1+a}\left[W(t) + \sum_{t_i<t} a_i\right] , \tag{5.1}$$

where W is the map used in the definition of canonical graph completion of u (Definition 2.4). Then $\mathcal{W}$ is an increasing, left continuous map, and the only discontinuity points of $\mathcal{W}$ are contained in $\mathcal{T}$. At each $t_i \in \mathcal{T}$, the jump of $\mathcal{W}$ is given by

$$\Delta \mathcal{W}(t_i) = \frac{\Delta W(t_i) + a_i}{1 + a} \ .$$

Let us define the graph completion φ of u corresponding to the family $(\psi_t)_{t\in \mathcal{T}}$ by

$$\varphi(s) = (t , u(t)) \qquad \text{if } s = \mathcal{W}(t) , \ t \in [0,T] \setminus \mathcal{T},$$

(5.2)

$$\varphi(s) = \left(t_i, \psi_{t_i}\left(\frac{s - \mathcal{W}(t_i)}{\Delta \mathcal{W}(t_i)}\right)\right) \qquad \text{if } s \in [\mathcal{W}(t_i) , \mathcal{W}(t_i^+)] , \ t_i \in \mathcal{T}.$$

It is easy to check that φ is Lipschitz continuous with constant $L = (1 + a)[T + V_0^T(u)]$.

Theorem 5.2. *Let* u *and the family* $(\psi_t)_{t\in \mathcal{T}}$ *be as in Theorem 5.1. Then a map* $x\in BV([0,T], \mathbf{R}^n)$ *is a solution of* $(E)_{\mathcal{M}}$ *if and only if there exists a solution* $\tilde{y} = (y^0, y)$ *of* $(E)_{n+1}$ *, corresponding to the graph completion* φ *defined by* (5.2), *such that*

$$x(t) = y(\mathcal{W}(t))$$

for almost every $t\in [0,T]$.

Remark 5.5. If $x\in BV^-([0,T], \mathbf{R}^n)$, then the equality $x(t) = y(\mathcal{W}(t))$ holds for every $t\in]0,T]$ as in the case of Theorem 2.2, but now

$$x(0) = x(0^+) = y(\mathcal{W}(0^+)) ,$$

where $\mathcal{W}(0^+) = V_0^1(\psi_0)/(1 + a)$. Therefore, in general, we do not have $x(0) = y(\mathcal{W}(0)) = y(0)$.

Proof of Theorem 5.2. The proof is very similar to the proof of Theorem 2.2. Hence, we shall limit ourselves to indicate the points where they are different.

Let $\tilde{y} = (y^0, y)$ be the solution of $(E)_{n+1}$ corresponding to the graph completion φ defined by (5.2). Let us show that the map x defined by

$$x(t) := y(\mathcal{W}(t)) \quad , \quad t\in [0,T] ,$$

is a solution of $(E)_{\mathcal{M}}$. In this part of the proof we do not use the fact that $\mathcal{W}$ is given by (5.1). It is enough to assume that $\mathcal{W}(0) = 0$ and that $\mathcal{W}$ is increasing, left continuous on $]0,T]$, and continuous on $]0,T[\setminus \mathcal{T}$. Moreover, we assume that the function φ defined by (5.2) is Lipschitz continuous.

By definition, y is a solution of the integral equation

$$y(s) = \int_0^s \Big(f(y(\xi))\varphi_*^0(\xi) + \sum_{i=1}^m g_i(y(\xi))\varphi_*^i(\xi) \Big) d\xi ,$$

where φ_*^0 and φ_*^i are Borel functions coinciding with the derivatives $\frac{d\varphi^0}{ds}$ and $\frac{d\varphi^i}{ds}$ almost everywhere with respect to the Lebesgue measure. Since f and the g_i are bounded, and φ is Lipschitz continuous, y turns out to be Lipschitz continuous. Furthermore, $\mathcal{W}$ belongs to $BV([0,T], [0,1])$. Then, Theorem 3.1 yields

$$\dot{x} = \hat{y}_*(\mathcal{W})\, \dot{\mathcal{W}} , \tag{5.3}$$

where

$$y_*(s) = f(y(s))\, \varphi_*^0(s) + \sum_{i=1}^m g_i(y(s))\, \varphi_*^i(s) .$$

By an argument similar to the one used in the proof of Theorem 2.2, one can show that (5.3) implies

$$\int_A \dot{x} = \int_A f(x(t))dt + \sum_{i=1}^m \int_{A\cap E_c} g_i(x)\dot{u}^i + \sum_{t\in \mathcal{T}\cap A} z(x(t^-),\psi_t) , \tag{5.4}$$

for every Borel subset A of $]0,T[\setminus \mathcal{T}$. Note that the last term on the right-hand side of (5.4) is zero, since $\mathcal{T}\cap A = \emptyset$. We have written it just to render (5.4) formally similar to the integral representation of a solution of $(E)_{\mathcal{M}}$ given in Definition 5.1.

By (5.3), for every $t_j \in \mathcal{T}\cap]0,T[$, one has

$$\dot{x}(\{t_j\}) = \hat{y}_*(\mathcal{W})(t_j) \cdot \dot{\mathcal{W}}(\{t_j\}) . \tag{5.5}$$

By the definition of Volpert's averaged superposition (see Definition 3.1), and by the equality $\mathcal{W}(t^+) - \mathcal{W}(t^-) = \dot{\mathcal{W}}(\{t\})$, one has

$$\hat{y}_*(\mathcal{W})(t_j) = \int_0^1 \frac{dy}{ds}\big(\mathcal{W}(t_j) + \sigma \dot{\mathcal{W}}(\{t_j\})\big) d\sigma =$$

(5.6)

$$= \frac{1}{\dot{\mathcal{W}}(\{t_j\})} \int_0^1 \frac{dz}{d\sigma}(\sigma)\, d\sigma \;= \frac{1}{\dot{\mathcal{W}}(\{t_j\})}[z(1) - y(\mathcal{W}(t_j))] ,$$

where $z(\sigma) = y(\mathcal{W}(t_j) + \sigma\dot{\mathcal{W}}(\{t_j\}))$. By the definition (5.2) of φ on $[\mathcal{W}(t_j) , \mathcal{W}(t_j^+)]$ and by the equation $(E)_{n+1}$ satisfied by $\tilde{y}$, the function z satisfies the auxiliary Cauchy problem (J) with $\psi = \psi_{t_j}$ and $\bar{z} = y(\mathcal{W}(t_j))$. Therefore (5.5) and (5.6) imply, for every $t_j \in \mathcal{T}$,

$$\dot{x}(\{t_j\}) \;=\; z\big(y(\mathcal{W}(t_j))\, ,\, \psi_{t_j}\big) \;=\; z\big(x(t_j^-)\, ,\, \psi_{t_j}\big)\, .$$

Hence, we can write, for each $t_j \in \mathcal{T} \cap]0,T[$,

$$\int_{\{t_j\}} \dot{x} \;=\; \int_{\{t_j\}} f(x(t))dt \;+ \sum_{i=1}^{m} \int_{\{t_j\}\cap E_c} g_i(x)\,\dot{u}^i \;+\; z(x(t_j^-)\, ,\, \psi_{t_j})\, . \tag{5.7}$$

Note that only the last term on the right-hand side is non-vanishing. As in (5.4), the remaining terms have been written to obtain the form of the integral relation in Definition 5.1.

By (5.4) and (5.7) we can conclude that the map $x(\cdot) = y(\mathcal{W}(\cdot))$ satisfies the equation in the definition of $(E)_{\mathcal{M}}$. To prove that the Cauchy condition $x(0^+) = z(\bar{x},\psi_0)$ is fulfilled, it is enough to observe that $x(0^+) = y(\mathcal{W}(0^+))$ and that , by (5.2), the equation satisfied by y on $[0,\mathcal{W}(0^+)]$ yields $y(\mathcal{W}(0^+)) = z(\bar{x},\psi_0)$.

To conclude the proof, we have to show that if x is a solution on $(E)_{\mathcal{M}}$, then it coincides with $y(\mathcal{W})$ almost everywhere, where $\tilde{y} = (y^0, y)$ is a solution of $(E)_{n+1}$ corresponding to the graph completion φ defined by (5.1) and (5.2).

For this purpose, given any solution x of $(E)_{\mathcal{M}}$, let

$$\tilde{y}(s) = (y^0, y)(s) = (t, x(t^-)) \qquad \text{if } s = \mathcal{W}(t)\, ,\; t \notin \mathcal{T}$$

$$\tilde{y}(s) = (y^0, y)(s) = \left(t_j,\, z_j\!\left(\frac{s - \mathcal{W}(t_j)}{\dot{\mathcal{W}}(\{t_j\})}\right)\right) \qquad \text{if } s \in [\mathcal{W}(t_j), \mathcal{W}(t_j^+)]\, ,\; t_j \in \mathcal{T},$$

where $\mathcal{W}$ is defined by (5.1), and z_j denotes the solution of the auxiliary Cauchy problem (J) when $\psi = \psi_{t_j}$ and $\bar{z} = x(t_j^-)$.

One can check that

i) $\tilde{y}$ is Lipschitz continuous;

ii) $\tilde{y}$ is the solution of $(E)_{n+1}$ corresponding to the graph completion φ defined in 5.1.

The proof of i) and ii) are quite similar to the corresponding proofs in Theorem 2.2 and therefore are omitted. QED

The following theorem characterizes the solutions of Cauchy problems $(E)_{\mathcal{M}}$ as the almost everywhere limits of solutions x_h of (E) corresponding to sequences of regular controls u_h with equi-bounded variations which converge to u almost everywhere on [0,T].

Theorem 5.3. *Let* $(u_h)_{h\in N}$ *be a sequence of Lipschitz continuous controls with equi-bounded total variations converging a.e. to a function* $u\in BV^{-}([0,T],R^m)$. *Then there exists a subsequence* $(u_{h'})$ *of* (u_h) *such that the corresponding solutions* $x_{h'}$ *of the original Cauchy problem* (E) *converge almost everywhere to the solution* x *of the Cauchy problem* $(E)_{\mathcal{M}}$ *corresponding to* u *and to a suitable family* $(\psi_t)_{t\in \mathcal{T}}$ *satisfying* **(H)**.

Conversely, let u *be a map belonging to* $BV^{-}([0,T], R^m)$, *let* $(\psi_t)_{t\in \mathcal{T}}$ *be a family of Lipschitz continuous functions satisfying* **(H)**, *and let* x *be the corresponding solution of* $(E)_{\mathcal{M}}$. *Then, there exists a sequence* $(u_h)_{h\in N}$ *of Lipschitz continuous controls converging a.e. to* u *with equi-bounded total variations, such that the corresponding solutions* x_h *of the original Cauchy problem* (E) *converge to* x *almost everywhere.*

Theorem 5.3 will be proved in the next section.

6. Proof of Theorem 5.3.

The proof of Theorem 5.3 is based on Theorem 4.1 and on the following lemma.

Lemma 6.1. *Let* $u\in BV^{-}([0,T], R^m)$, *and let* $(u_h)_{h\in N}$ *be a sequence of continuous maps from* [0,T] *into* R^m *with equi-bounded total variations such that* (u_h) *converges to* u a.e. *in* [0,T] . *Then the graphs of the functions* u_h *can be reparametrized by means of maps* $\varphi_h : [0,1] \to [0,T]\times R^m$ *such that:*

i) *the family* $(\varphi_h)_{h\in N}$ *is equi-Lipschitzian;*

ii) *there exists a subsequence of* $(\varphi_h)_{h\in N}$ *which converges to a graph completion of* u .

Proof. Let φ_h be the canonical graph completion of u_h introduced in Definition 2.4. By (iii) of Remark 2.4 the Lipschitz constants of the functions φ_h are bounded uniformly with respect to h .

By applying Ascoli-Arzelà theorem to the family of maps $\varphi_h : [0,1] \to [0,T]\times\mathbf{R}^m$, we obtain the existence of a subsequence $(\varphi_{h'})$ of (φ_h) which converges uniformly on $[0,1]$ to a Lipschitz continuous map φ . We shall show that φ is a graph completion of u .

By contradiction, let us suppose that φ is not a graph completion of u . Since φ^0 is non decreasing, this means that there exists $t\in[0,T]$ such that

$$(6.1) \qquad (t, u(t)) \neq \varphi(s) \qquad \forall s\in[0,1] .$$

Since $K := \varphi([0,1])$ is a compact set, (6.1) implies that there is a $\delta > 0$ such that

$$(6.2) \qquad d(K, (t, u(t))) > \delta ,$$

where d denotes the usual distance between a point and a subset of $\mathbf{R}^{m+1}$. Since $(\varphi_{h'})$ converges to φ uniformly, setting $K_{h'} = \varphi_{h'}([0,1])$, by (6.2) there exists a natural number N such that

$$(6.3) \qquad d(K_{h'}, (t, u(t))) > \frac{\delta}{2} \qquad \forall h' > N .$$

Since u is left continuous on $]0,T]$, if $t > 0$ there exists a left neighborhood U_t of t such that

$$(6.4) \qquad |(\tau, u(\tau)) - (t, u(t))| < \frac{\delta}{4}, \qquad \forall \tau\in U_t .$$

Since u is right continuous at 0 , if $t = 0$ there exists a right neighborhood U_t of t such that (6.4) holds.

Let $\mathcal{N}$ be a subset of measure zero of $[0,T]$, such that

$$\lim_{h\to\infty} u_h(\tau) = u(\tau) \qquad \forall \tau\in [0,T] \setminus \mathcal{N}.$$

If $\tau\in U_t \setminus \mathcal{N}$, then there exists $h' > N$ such that

(6.5) $$|(\tau, u_{h'}(\tau)) - (\tau, u(\tau))| < \frac{\delta}{4}.$$

Since for every h' there exists an $s_{h'}\in [0,1]$ such that

$$\varphi_{h'}(s_{h'}) = (\tau, u_{h'}(\tau)) ,$$

from (6.4) and (6.5) we obtain that

$$d(K_{h'}, (t, u(t))) < \frac{\delta}{2},$$

which contradicts (6.3). This concludes the proof of the lemma.

QED

Proof of Theorem 5.3. Let φ_h be the canonical graph completion of u_h introduced in Definition 2.4. By Lemma 6.1 there exists a subsequence $(\varphi_{h'})$ of (φ_h) which converges uniformly on $[0,1]$ to a graph completion φ of u. Let $\mathcal{R}: [0,T] \to [0,1]$ be defined by

$$\mathcal{R}(t) = \inf \{ s\in [0,1] : \varphi^0(s) = t \}.$$

Then $\mathcal{R}$ is increasing, left continuous, and

(6.6) $$\varphi^0(s) = t \quad \Leftrightarrow \quad \mathcal{R}(t) \le s \le \mathcal{R}(t^+) ,$$

where we set $\mathcal{R}(T^+) = 1$.
Using the fact that φ is a graph completion of u and that u is left continuous on $]0,T]$, it is easy to see that

(6.7) $$(t, u(t)) = \varphi(\mathcal{R}(t)) \qquad \forall t\in]0,T] .$$

Let W_h be the functions introduced in Definition 2.4 corresponding to u_h. Let us prove that

$$(6.8) \qquad \mathcal{R}(t) \le \liminf_{h'\to\infty} W_{h'}(t) \le \limsup_{h'\to\infty} W_{h'}(t) \le \mathcal{R}(t^+)$$

for every $t\in[0,T[$. By the definition of φ_h we have $\varphi_h^0(W_h(t)) = t$. Since $(\varphi_{h'})$ converges to φ uniformly on $[0,1]$, we obtain

$$\varphi^0(\liminf_{h'\to\infty} W_{h'}(t)) = \varphi^0(\limsup_{h'\to\infty} W_{h'}(t)) = t\,,$$

which gives immediately (6.8).

Let $\tilde{y} = (y^0, y)$ and $\tilde{y}_h = (y_h^0, y_h)$ be the solutions of $(E)_{n+1}$ corresponding to φ and φ_h, respectively. Since the Lipschitz constants of the functions φ_h are uniformly bounded (Remark 2.4) and $(\varphi_{h'})$ converges to φ uniformly on $[0,1]$, if we apply Theorem 4.1 to the Cauchy problem $(E)_{n+1}$ we obtain that $(\tilde{y}_h)$ converges to $\tilde{y}$ uniformly on $[0,1]$. Since (6.8) gives

$$\mathcal{R}(t) = \lim_{h'\to\infty} W_{h'}(t)$$

at every continuity point t of $\mathcal{R}$ on $[0,T[$, from the uniform convergence of $(y_{h'})$ to y we get

$$(6.9) \qquad y(\mathcal{R}(t)) = \lim_{h'\to\infty} y_{h'}(W_{h'}(t))$$

for almost every $t\in[0,T]$.

Let $\mathcal{T}$ be the union of $\{0\}$ and of the set of all discontinuity points of $\mathcal{R}$. For every $t\in\mathcal{T}$ we define ψ_t by

$$\psi_t^i(s) = \varphi^i(\mathcal{R}(t) + s\,\Delta\mathcal{R}(t)) \;,\quad s\in[0,1] \;,\quad i = 1, \dots, m\,.$$

Using (6.6) and (6.7) we obtain that φ satisfies (5.2) with $\mathcal{W} = \mathcal{R}$. Therefore the first part of the proof of Theorem 5.2 shows that $x(\cdot) := y(\mathcal{R}(\cdot))$ is the solution of $(E)_{\mathcal{M}}$ corresponding to the control u and the family $(\psi_t)_{t\in\mathcal{T}}$. On the other hand, by Theorem 2.2 we have $x_{h'}(\cdot) = y_{h'}(W_{h'}(\cdot))$ a.e. on $[0,T]$. Therefore (6.9) yields

$$x(t) = \lim_{h'\to\infty} x_{h'}(t) \qquad \text{for a.e. } t\in[0,T] ,$$

which concludes the proof of the first part of Theorem 5.3 .

Let us prove the second part. Let u and the family $(\psi_t)_{t\in\mathcal{T}}$ be as in the hypothesis and let x be the corresponding solution of $(E)_{\mathcal{M}}$. Let $\varphi = (\varphi^0, \varphi^1, \ldots, \varphi^m)$ be the graph completion of u defined by (5.1) and (5.2), and let $(\varphi_h^0)_{h\in\mathbf{N}}$ be the sequence of maps from $[0,1]$ into $[0,T]$ defined by

$$\varphi_h^0(s) := \frac{h\varphi^0(s) + s}{h + 1} , \quad s\in[0,1] .$$

Since φ is Lipschitz continuous with constant $L = (1 + a)[T + V_0^T(u)]$ (see Section 5), the derivatives of the functions φ_h^0 satisfy, for almost every $s\in[0,1]$, the inequalities

$$\text{(6.10)} \qquad \frac{1}{h+1} \leq \frac{d\,\varphi_h^0(s)}{ds} \leq \frac{h\,L+1}{h+1} .$$

For every $h\in\mathbf{N}$, let us define the map $\varphi_h : [0,1] \to [0,T]\times\mathbf{R}^m$ by

$$\varphi_h(s) := (\varphi_h^0, \varphi^1, \ldots, \varphi^m)(s) , \quad s\in[0,1] .$$

By (6.10), the function φ_h^0 is invertible and, for every $h\in\mathbf{N}$, its inverse s_h is Lipschitz continuous with constant $h + 1$. Then the controls $u_h : [0,T] \to \mathbf{R}^m$ defined by

$$u_h(t) := (\varphi^1, \ldots, \varphi^m)(s_h(t))$$

are Lipschitz continuous and their total variations are uniformly bounded by L . Furthermore, for every $h\in\mathbf{N}$, φ_h is a reparametrization of the graph of u_h . Then, denoting by x_h the solution of (E) corresponding to the control u_h , and $\tilde{y}_h = (y_h^0, y_h)$ the solution of $(E)_{n+1}$ corresponding to φ_h , it is

$$\text{(6.11)} \qquad x_h(t) = y_h(s_h(t)) \quad \forall t\in[0,T] .$$

Note that (φ_h) converges to φ uniformly on $[0,1]$ and, by (6.10), the Lipschitz constants of the functions φ_h are uniformly bounded. Then, applying Theorem 4.1 to the

Cauchy problem $(E)_{n+1}$ we obtain that $(\tilde{y}_h)$ converges uniformly on $[0,1]$ to the solution $\tilde{y} = (y^0, y)$ of $(E)_{n+1}$ corresponding to φ .

Let $\mathcal{W}$ be the function defined in (5.1). Since $\mathcal{W}(t) = \inf\{ s \in [0,1] : \varphi^0(s) = t \}$ and $\varphi_h^0(s_h(t)) = t$ for every $t \in [0,T]$, arguing as in the first part of the proof we obtain that

$$\mathcal{W}(t) = \lim_{h \to \infty} s_h(t)$$

at every continuity point t of $\mathcal{W}$ on $[0,T[$. Therefore the uniform convergence of (y_h) to y gives

$$y(\mathcal{W}(t)) = \lim_{h \to \infty} y_h(s_h(t)) \quad \text{for a.e. } t \in [0,T] .$$

By (6.11) and Theorem 5.2 we conclude that

$$\lim_{h \to \infty} x_h(t) = x(t)$$

for almost every $t \in [0,T]$. QED

PART II

Minimization problems for impulsive control systems

In the next two chapters , optimal control problems for systems of the form

$$\text{(I.E.)} \qquad \dot{x} = f(t, x(t), u(t)) + \sum_{\alpha=1}^{m} g_\alpha(t, x(t), u(t))\, \dot{u}^\alpha(t)$$

are treated. Precisely, one seeks the minimum value of

$$\{ \gamma(x(u,T)),\ u \in \mathcal{U} \},$$

where γ is a continuous function of the state, $\mathcal{U}$ is a family of *admissible controls* taking values on a compact subset $U \subseteq \mathbf{R}^n$, and $x(u,T)$ denotes the final value of the solution $x(u,\cdot)$ of (I.E.) corresponding to the control $u \in \mathcal{U}$.

We refer to the Introduction for the mechanical applications which demand the study of such non standard optimal control problems.

In Chapter 4 an optimization problem with an a priori bound on the total variation of the controls is investigated. In view of the results contained in the previous chapters, one expects that the optimal controls contain instantaneous arcs. Actually, a result on the existence of such an optimal control is proved. Also, it is shown that the minimun value of the cost function may be approximated by means of Lipschitz continuous controls. These results have been obtained by the author of this thesis and have been recently proposed for publication.

Chapter 5 is formed by the the first part of a paper which is going to be written by the author of this thesis and Alberto Bressan (see [8]). In this work, which is now in an advanced state of preparation, we tackle the optimization problem without any constraint on the total variations of the controls.
In the part presented in this thesis, we investigate the case in which all the Lie brackets $[g_\alpha, g_\beta]$ vanish identically. By means of a suitable diffeomorphism of the space of the couples state-control, we are able to refer the existence problem to the non impulsive case. Also, a necessary condition for optimality is proved in the form of a maximum principle.

The investigation of the general case, in which the brackets $[g_\alpha, g_\beta]$ are not all equal to zero, is now going to be completed and will appear in [8].

Chapter 4

An optimization problem with a constraint on the total variations of the controls

1. Introduction.

Let U be a compact, connected subset of $\mathbf{R}^m$, and for $K > 0$ and $\bar{u} \in U$, let $\mathcal{U}_K$ the family of Lipschitz continuous maps $u : [0, T] \rightarrow U$, $u(0) = \bar{u}$, with total variation $V_0^T(u)$ less or equal than K.
Let $\Omega \subset \mathbf{R}^n$ be an open subset and let x denote an element of Ω. Given the vector fields $\tilde{f}$ and $\tilde{g}_\alpha$, $\alpha=1,\dots,m$, we consider the Cauchy problem

$$(1.1) \qquad \begin{cases} \dot{x} = \tilde{f}(t, x, u(t)) + \sum_{\alpha=1}^{m} \tilde{g}_\alpha(t, x, u(t))\dot{u}^\alpha \\ \\ x(0) = \bar{x} , \end{cases}$$

the dot denoting differentiation with respect to time, together with the minimization problem

$$(\mathcal{P}) \qquad \min\{\gamma(x(u, T)) \quad u(\cdot) \in \mathcal{U}_K\},$$

where γ is a continuous real function of the state, and $x(u, \cdot)$ denotes the solution (if it exists) of (2.1) corresponding to the control $u \in \mathcal{U}_K$.
It can be easily seen that an optimal control $\hat{u} \in \mathcal{U}_K$ for $\mathcal{P}$ does not exist in general, as it is shown by the following example.

Example. Consider the Cauchy problem in [0, T]

(C.P.)
$$\begin{cases} \dot{x}^1 = (\operatorname{tg} x^2+\operatorname{tg} x^3)\cdot\left(\frac{1}{2}-t\right) \\ \dot{x}^2 = \dfrac{1}{1+\operatorname{tg}^2 x^2}\,\dot{u}^2 \\ \dot{x}^3 = \dfrac{1}{1+\operatorname{tg}^2 x^3}\,\dot{u}^1 \\ (x^1,x^2,x^3)(0) = (0,0,0)\ , \end{cases}$$

where $(x^1, x^2, x^3)\in \mathbf{R}\times\,]-\frac{\pi}{2},\frac{\pi}{2}[\,\times\,]-\frac{\pi}{2},\frac{\pi}{2}[$, and $u(\cdot)$ is allowed to be a Lipschitz continuous map from $[0, T]$ into the compact set U defined as

$$U = \{(u^1, u^2) \mid (u^1, u^2)\in \mathbf{R}^2\ , 0 \le u^i \le 1\ , i = 1, 2\ , u^2 - (u^1)^2 \le 0\}\ .$$

Moreover we assume that $u(0) = (0, 0)$ and that the variation of $u(\cdot)$ is less or equal than $\frac{3}{2}$.

Let us denote by $\mathcal{U}_{\frac{3}{2}}$ the family of such maps and let us consider the optimization problem

(P)
$$\min\{x^1(u, T)\ , u\in \mathcal{U}_{\frac{3}{2}}\}\ ,$$

where $x^1(u, \cdot)$ denotes the first component of the solution of (C.P.) corresponding to the control $u\in \mathcal{U}_{\frac{3}{2}}$.

By means of the diffeomorphism $\phi : \mathbf{R}\times\,]-\frac{\pi}{2},\frac{\pi}{2}[\,\times\,]-\frac{\pi}{2},\frac{\pi}{2}[\ \to \mathbf{R}^3$ defined by

$$\phi(x^1, x^2, x^3) = (x^1, \operatorname{arctg} x^3, \operatorname{arctg} x^2)\ ,$$

(C.P.) is transformed into the new Cauchy Problem

(C.P.)'
$$\begin{cases} \dot{y}^1 = (y^2 + y^3)\cdot\left(\frac{1}{2}-t\right) \\ \dot{y}^2 = \dot{u}^1 \\ \dot{y}^3 = \dot{u}^2 \end{cases}$$

and the optimization problem (**P**) is replaced by

$$(\mathbf{P}_y) \qquad \min\{y^1(u, T)\ ,\ u\in \mathcal{U}_{\frac{3}{2}}\}\ .$$

Clearly the infimum value of $y^1(u, T)$ is achieved if and only if

$$(u^1 + u^2)(t) = (y^2 + y^3)(t) = 0 \quad \forall t\in [\,0\,, \tfrac{1}{2}\,[\,,$$

and

$$(u^1 + u^2)(t) = (y^2 + y^3)(t) = 2 \quad \forall t\in [\,\tfrac{1}{2}\,, 0\,[\,,$$

and this minimum is equal to $\frac{3}{4}$.
Obviously such a map $u(\cdot)$ cannot be extended to a Lipschitz continuous functions on $[0, T]$.
On the other hand, the maps u_k defined, for every $k\in \mathbf{N}$, by

$$u_k(t) = (0, 0) \quad \forall t\in [0, \tfrac{1}{2} - 2^{-k}]$$

$$u_k(t) = \left(\frac{t - \frac{1}{2} + 2^{-k}}{2^{-k+1}}\ ,\ \left(\frac{t - \frac{1}{2} + 2^{-k}}{2^{-k+1}}\right)^2 \right) \quad \forall t\in [\tfrac{1}{2} - 2^{-k}\,, \tfrac{1}{2} + 2^{-k}]$$

$$u_k(t) = (1, 1) \quad \forall t\in [\tfrac{1}{2} + 2^{-k}\,, 1]$$

belong to $\mathcal{U}_{\frac{3}{2}}$ and

$$\lim_{k\to\infty} y^1(u_k, T) = \inf\{y^1(u, T)\ ,\ u\in \mathcal{U}_{\frac{3}{2}}\} = \frac{3}{4}\,.$$

Notice that the graphs $(t, u_k(t))$ *converge* (in a sense to be precised in Section 2) to a *space-time control* containing an instantaneous arc at $t = 0$.

On the basis of the results in [9] (see Chapter 2) it seems natural to embed the optimization problem $(\mathcal{P})$ into a new problem $(\mathcal{P}')$ in which $\mathcal{U}_K$ is identified with the set Φ_K^+ of the reparametrizations of the graphs of $u \in \mathcal{U}_K$. The problem $(\mathcal{P}')$ is obtained from $(\mathcal{P})$ by adding one dimension both to the controls and the states. As a family of admissible controls for the extended problem $(\mathcal{P}')$ we shall consider the set Φ_K formed by all the continuous paths $(\varphi^0, \tilde{\varphi}) = (\varphi^0, \dots, \varphi^m) : [0, 1] \to [0,T] \times U$ such that $\varphi^0 : [0, 1] \to [0, T]$ is non decreasing and surjective, and $\tilde{\varphi}$ has total variation $V_0^1(\tilde{\varphi})$ less or equal than K. Notice that a path $\varphi \in \Phi_K$

may contain istantaneous arcs, i.e., to say, $\frac{d\varphi^0}{ds}$ may be zero on some subinterval of $[0, 1]$. Actually, if $\mathbf{\Phi}_K^+$ and $\mathbf{\Phi}_K$ denote the quotients obtained by reparametrization from Φ_K^+ and Φ_K, respectively, we shall show that the closure $\bar{\mathbf{\Phi}}_K^+$ of $\mathbf{\Phi}_K^+$ with respect to the parameter-free metric $\boldsymbol{\delta}$ introduced in [9] *coincides* with $\mathbf{\Phi}_K$.
Moreover, by the compactness of the latter one obtains the existence of an optimal control $\varphi \in \mathbf{\Phi}_K$ for the extended problem. Then, the identity $\bar{\mathbf{\Phi}}_K^+ = \mathbf{\Phi}_K$ and the results of continuous dependence of solutions on controls proved in [9] imply that the minimum of the extended problem coincides with the infimum of the original problem. Moreover,we construct explicitly (Theorem 4.2) a sequence of controls $u_n \in \mathcal{U}$ such that the corresponding values of the cost funtion tend to the minimum.

2. Space-time controls.

Let $\mathcal{U}_K$ be the family of controls introduced in Section 1. Let us assume that the fields $\tilde{f}$, $\tilde{g}_\alpha$, $\alpha = 1, \dots, m$, are continuously differentiable in all arguments and that the solution of (1.1) exists for every $u \in \mathcal{U}_K$.
By adding the state variables $x^0 = t$, $x^\alpha = u^\alpha$, $\alpha = 1, \dots, m$, and the equations $\dot{x}^0 = 1$, $\dot{x}^\alpha = \dot{u}^\alpha$, we can assume that (1.1) has the form

$$(2.1) \qquad \begin{cases} \dot{y} = f(y) + \sum g_\alpha(y)\dot{u}^\alpha \\ y(0) = \bar{y} = (0, \bar{x}, \bar{u}), \end{cases}$$

where $y = (x^0, \dots, x^{n+m})$, and the fields f, g_α, $\alpha = 1, \dots, m$, are defined as

(2.2) $$f = (1, \tilde{f}, 0, \ldots, 0)^t, \qquad g_\alpha = (0, \tilde{g}, 0, 1, 0)^t.$$

In (2.2) the symbol ^{t} means transposition, and the last m components of g_α are all zero but the $(n+1+\alpha)$-th, which is 1 .

Let $u \in \mathcal{U}_K$ and let $\varphi = (\varphi^0, \tilde{\varphi}) : [0, 1] \to [0, T] \times U$ be any *Lipschitzean reparametrization* of the graph $(t, u(t))$, i.e. φ^0 is Lipschitzean, $\varphi^0(0) = 0$ $\varphi^0(1) = T$, $\frac{d\varphi^0}{ds} \geq 0$ a.e. in $[0, 1]$, and

$$u(\varphi^0(s)) = \tilde{\varphi}(s) = (\varphi^1, \ldots, \varphi^m)(s),$$

for every $s \in [0, 1]$. Then, if $\check{y}(\cdot)$ denotes the solution of (2.1), if is trivial to check that the solution $y(s)$ of the Cauchy problem

(2.3) $$\begin{cases} \dfrac{dy}{ds} = f(y)\dfrac{d\varphi^0}{ds} + \displaystyle\sum_{\alpha=1}^{m} g_\alpha(y)\dfrac{d\varphi^\alpha}{ds} \\ y(0) = \bar{y} \end{cases}$$

satisfies

$$y(s) = \check{y}\big(\varphi^0(s)\big), \quad \forall s \in [0, 1].$$

Hence

$$\check{y}(t) = y((\varphi^0)^{-1}(t)) = \big(t, y^{\mathcal{R}}((\varphi^0)^{-1}(t))\big)_{\mathcal{R}=1,\ldots,n+m}, \quad \forall t \in [0, T],$$

provided φ^0 is strictly increasing.

Remark 2.1. As in [9], we notice that (2.3) has a free-parameter character (see also [9]). Indeed let φ_1 be a Lipschitz continuous map from $[0, 1]$ into $\mathbf{R} \times \mathbf{R}^m$ and $k : [0, 1] \to [0, 1]$ is a non decreasing Lipschitzean map such that $k(0) = 0$, $k(1) = 1$. Consider the reparametrization φ_2 of φ_1 defined by $\varphi_2(\cdot) := \varphi_1(k(\cdot))$. If y_1 and y_2 denote the solutions of (2.3) corresponding to φ_1 and φ_2 , respectively, then it is straightforward to verify that

$$y_2(s) = y_1(k(s)) \ .$$

We now introduce the set Φ_k^+ formed by Lipschitzean reparametrizations of graphs of maps belonging to $\mathcal{U}_K$.
Throughout this chapter we shall denote by $W^{1,\infty}(X, Y)$ the family of Lipschitz continuous maps from a metric space X into a metric space Y . Moreover, if $f : [a, b] \to \mathbf{R}^q$ is a functions with bounded variation, the symbol $V_\alpha^\beta(f)$ will stand for the *variation* of f in the interval $[\alpha, \beta] \subseteq [a, b]$ (see Chapter 3).

Definition 2.1. We set

$$\Phi_K^+ = \Big\{ \varphi \mid \varphi = (\varphi^0, \tilde{\varphi}) = (\varphi^0, \dots, \varphi^m) \in W^{1,\infty}\big([0,1]\ , [0,T]\times U\big) \ ,$$
$$\varphi^0(0) = 0 \quad \varphi^0(1) = T \ \frac{d\varphi^0}{ds} > 0 \ \text{ for a.e. } s \in [0, 1] \ ,$$
$$V_0^1(\tilde{\varphi}) \leq K \ , \text{ and } (\varphi^1, \dots, \varphi^m) \circ (\varphi^0)^{-1}(\cdot) \in \mathcal{U}_K \Big\} \ .$$

Φ_K^+ will be said the set of *graph-reparametrizations of* $\mathcal{U}_K$.
Motivated by Remark 2.1, now we recall the notion of equivalence of Lipschitz continuous maps introduced Definition 3.1 of [9].
As a preliminary, observe that every Lipschitz continuous map $f : [0, 1] \to \mathbf{R}^q$ can be reparametrized by means of its total variation. More precisely, let us define the map $f^{©}$ by

$$f^{©}\Big(\frac{V_0^s(f)}{V_0^1(f)}\Big) = f(s) \quad \forall s \in [0, 1] \ .$$

By the elementary properties of the map $s \longmapsto V_0^s(f)$, it is readily checked that $f^{©}$ is a well defined map with Lipschitz constant $V_0^1(f)$.

Definition 2.2. The map $f^{©} : [0, 1] \to \mathbf{R}^q$ constructed above is the *canonical parametrization of* f . We say that two continuous maps $f_i : [0, 1] \to \mathbf{R}^q$, i = 1, 2 , are *equivalent* , and write $f_1 \sim f_2$, if their canonical parametrizations coincide.

Definition 2.3. The quotient

$$\Phi_K^+ := \frac{\Phi_K^+}{\sim}$$

will be called the *set of graphs of* $\mathcal{U}_K$.

In analogy with [9] we are going to embed the set Φ_K^+ into a larger family of paths Φ_K which are allowed to be instantaneous, i.e. to say $\frac{d\varphi^0}{ds} = 0$, on some subinterval of their domain.

Definition 2.4. Let us set

$$\Phi_K = \Big\{ \varphi \mid \varphi = (\varphi^0, \tilde{\varphi}) = (\varphi^0, \ldots, \varphi^m) \in W^{1,\infty}\big([0,1]\,,\,[0,T]\times U\big)\,,$$
$$\varphi^0(0) = 0 \quad \varphi^0(1) = T\,, \ \frac{d\varphi^0(1)}{ds} \geq 0 \ \text{ for a.e. } s \in [0, 1]\ ,$$
$$V_0^1(\tilde{\varphi}) \leq K\ , (\varphi^0) = (0, \bar{u}) \Big\}\ .$$

The quotient

$$\Phi_K^+ := \frac{\Phi_K}{\sim}$$

will be called *the set of space-time controls with spatial variation less or equal than* K .

The set Φ_K is a metric space as long as it is endowed with the following distance δ, which has been introduced in [B.R.] .

Definition 2.5. Given two continuous paths $\varphi_i : [0, 1] \to \mathbf{R}^q$, $i = 1, 2$, their *distance* $\delta(\varphi_1, \varphi_2)$ defined as

$$\delta(\varphi_1, \varphi_2) = \inf_{k_1,k_2} \max_{s\in[0,1]} |\varphi_1(k_1(s)) - \varphi_2(k_2(s))| ,$$

the inf being taken over all couplets of continuous, non-decreasing, surjective maps $k_i : [0, 1] \to [0, 1]$, $i = 1,2.$.

In [9] it has been proved that the inf in the definition of δ is actually a minimum. Moreover δ is in fact a pseudometric on Φ_K, and $\delta(\varphi_1, \varphi_2) = 0$ if and only if $\varphi_1 \sim \varphi_2$. Then δ induces a metric on the quotient space $\mathbf{\Phi}_K$, which will be denoted by $\boldsymbol{\delta}$. Hereafter we shall think $\mathbf{\Phi}_K$ as a metric space endowed with the metric $\boldsymbol{\delta}$.

3. An extended minimization problem which admits existence of optimal controls.

We now extend the optimal control problem $(\mathcal{P})$ to a new problem $(\mathcal{P}')$ having $\mathbf{\Phi}_K$ as space of controls.

As a preliminary, we notice that, on the basis of Remark 2.1, *we can look at the Cauchy problem (2.3) as an input-output operator which maps an equivalence class* $\boldsymbol{\varphi} \in \mathbf{\Phi}_K$ *into the equivalence class* $\mathbf{y}(\boldsymbol{\varphi})$ defined as

$$y(\boldsymbol{\varphi}) := \{y \mid y \in C^0([0, 1], \mathbf{R}^{1+n+m}) \ , \ y \sim y(\varphi^{©}, \cdot)\} ,$$

where $y(\varphi^{©},\cdot)$ denotes the solution of (2.4) corresponding to the canonical representative $\varphi^{©}$ of $\boldsymbol{\varphi}$.

Since $y(1) = y(\varphi^{©}, 1)$ for every $y \in \mathbf{y}(\boldsymbol{\varphi})$ it is also meaningful to speak of *the final value*

$$\mathbf{y}(\boldsymbol{\varphi}) := y(\varphi^{©}, 1)$$

of the equivalence class $\mathfrak{y}(\varphi)$.

We are now in position of introducing the extended optimal control problem $(\mathcal{P}')$ defined as

$$(\mathcal{P}') \qquad \min\{\nu(\mathfrak{y}(\varphi, 1)) \quad \varphi\in\Phi_K\} \ ,$$

where $\nu(\cdot)$ is the (continuous) function defined on $\mathbf{R}\times\Omega\times\mathbf{R}^m$ as

$$(3.1) \qquad \nu(y^0, \ldots, y^{1+n+m}) := \gamma(y^1, \ldots, y^n) \ .$$

Theorem 3.1. *There exists an optimal control* $\hat{\varphi}\in\Phi_K$ *for the extended problem* $(\mathcal{P}')$, *that is, to say,*

$$(3.2) \qquad \nu(\mathfrak{y}(\hat{\varphi}, 1)) \leq \nu(\mathfrak{y}(\varphi, 1))$$

for every $\varphi\in\Phi_K$.

This theorem is a corollary of Lemma 3.1 and 3.2 below.

Lemma 3.1. *The imap* $c : \Phi_K \to \mathbf{R}$ *defined as*

$$(3.3) \qquad c(\varphi) := \nu(\mathfrak{y}(\varphi, 1))$$

is continuous.

Proof. Let

$$i : \varphi \longmapsto \mathfrak{y}(\varphi)$$

be the input-output map which, to a space-time control φ associates the corresponding solution $\mathfrak{y}(\varphi)$ of 2.4. By Theorem 2 in [9] (see Theorem 4.2 in Chapter 2) the map i is continuous with respect to the topologies induced by the metric δ both on the space of controls φ and the space of trajectories $\mathfrak{y}$.

Moreover, the function $m : \mathcal{Y} \to \mathcal{Y}(1)$ which maps an equivalence class of rectifiable paths into its final value is trivially continuous. Hence, the functional $c = v \circ n \circ i$ is continuous. QED

Lemma 3.2. *For each* $K > 0$, *the metric space* Φ_K *is compact.*

Proof. Let $(\varphi_n)_{n\in N}$ be a sequence of space-time controls $\varphi_n \in \Phi_K$. Consider the sequence $(\varphi_n^{©})_{n\in N}$ of the corresponding canonical representative. For each $n \circ N$, $\varphi_n^{©}$ has Lipschitz constant less or equal than $K + T$. Then, by Ascoli-Arzelà theorem, there exists a subsequence $\psi_n = (\tilde{\psi}_n^0, \tilde{\psi}_n)$ which converges uniformly to a Lipschitz continuous function $\psi = (\psi^0, \tilde{\psi})$. Moreover, $\psi \in \Phi_K$, since one can trivially verify that ψ^0 is a non decreasing surjective map from $[0, 1]$ into $[0, T]$, and

$$V_0^1(\tilde{\psi}) \leq \lim_{n\to+\infty} V_0^1(\tilde{\psi}_n) \leq K.$$

Since the uniform convergence implies the convergence in the metric δ, the subsequence $(\psi_n)_{n\in N}$ formed by the equivalence classes of the ψ_n converges to the equivalence class $\psi \in \Phi_K$ of the function ψ. QED

4. Approximation of the minimum by means of Lipschitz controls.

Theorem 3.1 states the existence of an optimal control $\hat{\varphi}$ for the extended problem $(\mathcal{P}')$. Obviously this does not guarantee the existence of an optimal control for the original problem $(\mathcal{P})$.

Yet the minimum value $v(\mathcal{Y}(\hat{\varphi}, 1))$ turns out to be equal to the infimum of the values of the cost functional in problem $(\mathcal{P})$. Precisely, we have

Theorem 4.1. *Let* $\hat{\varphi}$ *be an optimal (space-time) control for problem* $(\mathcal{P}')$. *Then there exists a sequence of controls* $u_n \in \mathcal{U}$ *such that*

$$\lim_{n\to+\infty} \gamma\big(x(u_n, T)\big) = v\big(\mathcal{Y}(\hat{\varphi}, 1)\big). \tag{4.1}$$

Since we have

$$\{v(y(\varphi, 1)) \ , \ \varphi\in \Phi_K^+\} = \{\gamma(x(u, T)) \ , u\in \mathcal{U}_K\},$$

Theorem 4.1 is a corollary of the following density result, whose proof supplies also an explicit construction of the approximating sequence (u_n).

Theorem 4.2. *Let* $\overline{\Phi_K^+}$ *denote the closure of* Φ_K^+ *in the space* $\frac{C^0([0,1],\mathbf{R}^{1+m})}{\sim}$ *endowed with the metric* δ. *Then*

$$\overline{\Phi_K^+} = \Phi_K . \tag{4.2}$$

Proof. Since the set Φ_K is closed and $\Phi_K^+ \subset \Phi_K$, the inclusion

$$\overline{\Phi_K^+} \subseteq \Phi_K \tag{4.3}$$

is trivial.
Let us prove the inclusion

$$\Phi_K \subseteq \overline{\Phi_K^+} . \tag{4.4}$$

Let $\varphi\in\Phi_K$ and let $\varphi^{©} = (\varphi^0, \tilde{\varphi})$ be the canonical representative of φ. For every $n\in\mathbf{N}$ and every $s\in[0, 1]$, we set

$$\psi_n^0(s) := \frac{T}{T+\frac{1}{n}} (\varphi^0(s) + \frac{1}{n} s)$$

and

$$\psi_n(\cdot) := (\psi_n^0, \tilde{\varphi})(\cdot) \ .$$

For every $n \in N$, the map $u_n : [0, T] \to U$ defined by

$$u_n(\cdot) = \tilde{\varphi} \circ (\psi_n^0)^{-1}(\cdot)$$

is Lipschitz continuous. Indeed, for every $s_1, s_2 \in [0, 1]$, $s_1 < s_2$, one has

$$\psi_n^0(s_2) - \psi^0(s_1) \geq \frac{T}{nT+1}(s_2 - s_1) \ .$$

Since $\tilde{\varphi}$ is Lipschitz continuous with constant $K + 1$, setting $t_1 = (\psi_n^0)^{-1}(s_1)$ and $t_2 = (\psi_n^0)^{-1}(s_2)$, we obtain

$$|u_n(t_2) - u_n(t_1)| \leq (K+1)\big((\psi_n^0)^{-1}(s_2) - (\psi_n^0)^{-1}(s_1)\big) \leq \frac{(K+1)(nT+1)}{T}(t_2 - t_1) \ .$$

Hence, each space-time control ψ_n belongs to the metric space Φ_K^+.
For each $n \in N$, let $\boldsymbol{\psi}_n \in \boldsymbol{\Phi}_K^+$ denote the equivalence class of ψ_n. Since the sequence $(\psi_n)_{n\in N}$ converges to $\varphi^{©}$ uniformly, we conclude that the sequence $(\boldsymbol{\psi}_n)_{n\in N}$ converges to $\boldsymbol{\varphi}$ in the metric $\boldsymbol{\delta}$. **QED**

Chapter 5

An optimal control problem for impulsive systems with commutative vector fields

1. Introduction.

In the present chapter we present the results contained in the first part of a paper which is going to be written by the author of this thesis and Alberto Bressan (see [8]). Since the second part of this paper, which concerns the non-commutative case, has not yet been put into a definitive version, herewe prefer to omit it and refer to the original paper.

Let us consider again the control system

$$(1.1) \qquad \begin{cases} \dot{x} = \tilde{f}(x, u(t)) + \sum_{\alpha=1}^{m} \tilde{g}_\alpha(x, u(t))\, \dot{u}^\alpha \\ x(0) = \bar{x} \end{cases}$$

where $x \in \mathbf{R}^n$ and the dots denote differentiation with respect to time.
We have seen that the case $m > 1$ differs from the case $m = 1$ in that:

i) only controls with bounded variation are allowed in order to avoid evolutions in the directions of the Lie brackets generated by the g_α;
ii) at the discontinuity points of $u(\cdot)$ the *trajectory of the jump* of u must be specified in order to achieve a robust definition of solution for (1.1).

In this chapter we shall see that if the fields g_α commute on $\mathbf{R}^{n+m}$, that is to say, $[g_\alpha, g_\beta] = 0$, for each α, $\beta = 1, \ldots, M$, then we can disregard the restrictions specified in i) and ii), and we can treat the vector-valued case in full analogy with the scalar case (see Chapter 3). In particular, thanks to a change of coordinates which transforms (1.1) into a non-impulsive control system, we will be able to deal with merely integrable controls and to speak simply of jumps, without mentioning the instantaneous trajectories. Furthermore, we shall see that it is possible to extend the results concerning (1.1) to the corresponding (variational and) adjoint variational systems. In particular, the latter will turn out to be well defined also in the case of integrable controls. This is useful to deal with optimization problems having (1.1) as differential constraint.

An optimization problem is introduced in section 3, where the question of the existence of an optimal control is treated, too. In section 4, a necessary condition for optimal controls is proved in the form of a maximum principle.

2. Commutative systems.

Let $x \in \mathbf{R}^n$ and let $u(\cdot)$ be a control function from $[0, T]$ into a compact, path-connected subset U of $\mathbf{R}^m$. Let us consider the differential system

$$(2.1) \qquad \begin{cases} \dot{x} = \tilde{f}(x, u(t)) + \sum_{\alpha=1}^{m} \tilde{g}_\alpha(x, u(t))\, \dot{u}^\alpha \\ \\ x(0) = \bar{x} \quad , \end{cases}$$

where the fields $\tilde{f}$ and $\tilde{g}_\alpha$, $\alpha = 1, \ldots, m$, belong to $C^1(\mathbf{R}^n \times \mathbf{R}^m, \mathbf{R}^m)$. The case in which f and g^α depend explicitly on time may be recovered by adding a variable $x^0 = t$ and the differential equation $\dot{x}^0 = 1$. Moreover, by adding the variables $x^{n+\alpha}$ and the differential equations $\dot{x}^{n+\alpha} = \dot{u}^\alpha$, $\alpha = 1, \ldots, m$ we can assume (2.1) has the form

$$(2.2) \qquad \begin{cases} \dot{y} = f(y) + \sum g_\alpha(y)\, \dot{u}^\alpha \\ \\ y(0) = \bar{y} = (x(0), u(0)) \quad , \end{cases}$$

where the symbol y denotes an element of $\mathbf{R}^n \times \mathbf{R}^m$. The vector fields f and g_α are defined as

$$(\tilde{f}^1, \ldots, \tilde{f}^n, 0, \ldots, 0)^t \quad (\tilde{g}_\alpha^1, \ldots, \tilde{g}_\alpha^n, 0, \ldots, 1, \ldots, 0)^t ,$$

respectively, where, for each $\alpha = 1, \ldots, m$, in the expression of $\tilde{g}_\alpha$ the last m entries are all zero but the α-th, which is 1, and t denotes transposition. Throughout this chapter we shall always assume that the vector fields $g_1, \ldots, g_m$ commute, i.e.,

$$(2.3) \qquad [g_\alpha, g_\beta] = 0,$$

for every $\alpha, \beta = 1, \ldots, M$, where the $[\cdot, \cdot]$ denotes Lie bracketing (see Chapter 0).
In this case the system (2.2) will be said to be *commutative*. We recall that, under assumption (2.3), one has

$$(2.4) \qquad \exp(a\, g_\alpha) \circ \exp(b\, g_\beta)\, y = \exp(b\, g_\beta) \circ \exp(a\, g_\alpha)\, y = \exp(a\, g_\alpha + b\, g_\beta)\, y$$

for every $\alpha, \beta = 1, \ldots, m$, and every $(a, b) \in \mathbf{R}^2$ and $y \in \mathbf{R}^{n+m}$ for which the above expressions can be defined. This symbol exp has the same meaning as in Chapter 2.
Let us suppose that there exists an element $\bar{u} \in U$ and an open ball $B \subset \mathbf{R}^m$ centered in $\bar{u}$ and containing U such that the expression

$$(2.5) \qquad \exp\Big(\sum_{\alpha=1}^{m} (u^\alpha - \bar{u}^\alpha) g_\alpha\Big) y =$$

$$\left[\exp(u^m - \bar{u}^m) g_m) \circ \exp((u^{m-1} - \bar{u}^{m-1}) g_{m-1}) \circ \ldots \circ (\exp((u^1 - \bar{u}^1) g_1) y\right]$$

may be defined for each $u \in B$ and every $y \in \mathbf{R}^{n+m}$. Up to a translation in $\mathbf{R}^m$ we can assume that $\bar{u} = (0, \ldots, 0) \in \mathbf{R}^m$, so that (2.5) becomes

$$(2.6) \qquad \exp\Big(\sum_{\alpha=1}^{m} u^\alpha\, g_\alpha\Big) y\ .$$

For each $u \in B$ and for each $x \in \mathbf{R}^n$ let us set

$$(2.7) \qquad k(x, u) := p_1 \circ \exp\Big(-\sum_{\alpha=1}^{m} u^\alpha\, g_\alpha\Big)(x, u)\ ,$$

where p_1 stands for the projection of $\mathbf{R}^n \times \mathbf{R}^m$ onto $\mathbf{R}^n$.
The function k satisfies the group action properties

$$k(k(x, u_1), u_2) = k(k(x, u_2), u_1) = k(x, u_1 + u_2)$$

(2.8)

$$k(x, 0) = x$$

for every $x \in \mathbf{R}^n$ and every $u_1, u_2 \in B$ such that $u_1 + u_2 \in B$. Moreover, for each $u \in B$, $k(\cdot, u)$ is a diffeomorphism of $\mathbf{R}^n$, and, by (2.8),

$$k^{-1}(\cdot, u) = k(\cdot, -u) \quad . \tag{2.9}$$

Let us consider the diffeomorphism ϕ of $\mathbf{R}^n \times B$ defined by

$$\phi(x, u) = (k(x, u), u) \ . \tag{2.10}$$

Let $\phi_* g_\alpha$ denote the push-forward of the vector field g_α , that is to say, the vector field which associates the tangent vector $\phi_*(x, u)\ g_\alpha(x, u) \in T_{(\xi,u)}(\mathbf{R}^n \times B)$ to the point $(\xi, u) = \phi(x, u)$, where $\phi_*(x, u)$ denotes the differential of ϕ at (x, u) .

Lemma 2.1. *For each* $\alpha = 1, \ldots, M$, *the vector field* $\phi_* g_\alpha$ *coincides with* $\dfrac{\partial}{\partial u^\alpha}$, *that is, to say,*

$$\phi_* g_\alpha(\xi, u) = (0, \ldots, 1, \ldots, 0)^t , \qquad \forall (\xi, u) \in \mathbf{R}^n \times B \ , \tag{2.11}$$

where, on the right-hand side, all the entries but the $(n+\alpha)$*-th are zero.*

Proof. (2.1) is satisfied if and only if $\exp(t\phi_* g_\alpha)(\bar{\xi}, \bar{u}) = (\bar{\xi}, \bar{u} + (0, \ldots, t, \ldots, 0))$, with t at the α-th position . Since $g_\alpha(x, u) = (g^1_\alpha, \ldots, g^n_\alpha, 0, \ldots 0, 1, \ldots, 0)^t$, and $\phi^{n+\alpha}(x, u) = u^\alpha$, it is sufficient to show that

$$0 = \frac{d}{dt}\big(k(\exp t\, g_\alpha(x, u))\big) = p_{1*} \circ \phi_*\, g_\alpha\big(\exp(t\, \phi_* g_\alpha)(\phi(x, u))\big) \qquad \forall (x, u) \in \mathbf{R}^n \times B \ .$$

Actually, one has

$$\frac{d}{dt}\big(k(\exp t\, g_\alpha(x, u))\big) = \frac{d}{dt}\Big[p_1 \circ \exp\big(-\sum_{\beta=1}^{m}(u^\beta + \delta^{\alpha,\beta}\, t)\, g_\beta\big) \circ \exp\,(t\, g_\alpha)(x, u)\Big]$$

$$= \frac{d}{dt}\Big[p_1 \circ \exp(-\sum_{\beta=1}^{m} u^\beta g_\beta)(x, u)\Big] = 0$$

where the symbol $\delta^{\alpha,\beta}$ equals zero if $\alpha \neq \beta$, otherwise it equals one.

QED

More generally, one has

Lemma 2.2. *If* $(v_1, v_2) \in T_x R^n \times T_u B$ *then*

$$\phi_*(x, u)(v_1, v_2) = \Big(v_1(1) - \sum_{\alpha=1}^{m} g_\alpha(\phi(x,u))v_2^\alpha \ , v_2(1)\Big) \in T_{\phi(x,u)} R^n \times B \ ,$$

where, by setting $(x(s), u(s)) = \exp(-s \sum_{\alpha=1}^{m} u^\alpha g_\alpha)(x, u)$, *we have written* $(v_1(1), v_2(1))$ *for the value at time* 1 *of the solution to the variational equation*

$$\text{(2.12)} \qquad \begin{cases} \dfrac{dv}{ds} = \displaystyle\sum_{\alpha=1}^{m} u^\alpha \nabla g_\alpha(x(s), u(s))\, v \\ v(0) = (v_1, v_2) \ . \end{cases}$$

In particular, $v_2(1) = v_2$. *In* (2.12) *the symbol* ∇g_α *denotes the Jacobian matrix of* g_α.

Corollary 2.1. *Let* $u : [0, T] \to U$ *be a Lipschitz continuous map, and let us suppose that the corresponding solution* $x(\cdot)$ *of* (2.1) *exists. Then, the map* $\xi(t) := p_1 \circ \phi(x(t), u(t)) = k(x(\cdot), u(\cdot))$ *solves the Cauchy problem*

$$\text{(2.13)} \qquad \begin{cases} \dot{\xi} = F(\xi, u(t)) \\ \xi(0) = k(\bar{x}, u(0)) \ , \end{cases}$$

where $(F(\xi, u), 0)$ *is the solution at time* 1 *of* (2.12), *with* $(v_1, v_2) = (f(\phi^{-1}(\xi, u))) = (\tilde{f}(\phi^{-1}(\xi, u)), 0)$.
Conversely, if $\xi(\cdot)$ *is the solution of* (2.13), *t hen* $x(\cdot) := p_1 \circ \phi^{-1}(\xi(t), u(t)) = k(\xi(t), -u(t))$ *solves* (2.1).

Proof of Lemma 2.2. By

$$\phi(x, u) = (p_1 \circ \exp(-\sum_{\alpha=1}^{m} u^\alpha g_\alpha)(x,u)\ ,\ u)\ ,$$

if p_2 stands for the projection of $R^n \times B$ onto B , we have

$$\sum_{\alpha=1}^{m} \frac{\partial(p_1 \circ \phi)}{\partial u^\alpha} v_2^\alpha = -\sum_{\alpha=1}^{m} g_\alpha(x, u) v_2^\alpha\ ,$$

$$\sum_{i=1}^{n} \frac{\partial(p_1 \circ \phi)}{\partial x^i} v_1^i = \sum_{i=1}^{n} \frac{\partial}{\partial x^i}\Big(p_1 \circ \exp(-\sum_{\alpha=1}^{m} u^\alpha g_\alpha)(x, u)\Big) v_1^i\ ,$$

$$\frac{\partial(p_2 \circ \phi)}{\partial x^i} = 0\ , \text{ and}$$

$$\sum_{\alpha=1}^{m} \frac{\partial(p_2 \circ \phi)}{\partial u^\alpha} v_2^\alpha = v_2 = \sum_{\alpha=1}^{m} \frac{\partial}{\partial u^\alpha}\Big(p_2 \circ \exp(-\sum_{\alpha=1}^{m} u^\alpha g_\alpha)(x, u)\Big)\ .$$

Since the function of s defined by

$$\Big(\sum_{i=1}^{n} \frac{\partial}{\partial x^i}(p_1 \circ \exp(-\sum_{\alpha=1}^{m} s u^\alpha g_\alpha)(x, u)) v_1^i\ ,\ \sum_{\alpha=1}^{m} \frac{\partial}{\partial u^\alpha}(p_2 \circ \exp(-\sum_{\alpha=1}^{m} s u^\alpha g_\alpha)(x, u)) v_2^\alpha\Big)$$

coincides with the function $v(s)$ solving (2.12), by the above identities and by

$$\phi_*(x,u)(v_1,v_2)=\Big(\sum_{i=1}^{n} \frac{\partial(p_1\circ\phi)}{\partial x^i} v_1^i + \sum_{\alpha=1}^{m} \frac{\partial(p_1\circ\phi)}{\partial u^\alpha} v_2^\alpha\ ,\ \sum_{i=1}^{n} \frac{\partial(p_2\circ\phi)}{\partial x^i} v_1^i + \sum_{\alpha=1}^{m} \frac{\partial(p_2\circ\phi)}{\partial u^\alpha} v_2^\alpha\Big)$$

we get the thesis. **QED**

Proof of Corollary 2.1. By Lemmas 2.1 and 2.2 the vector fields $\phi_* g_\alpha$ and $\phi_* f$ coincide with $\frac{\partial}{\partial u^\alpha}$ and $(F, 0)$, respectively. Hence $(x(\cdot), u(\cdot))$ is the solution of (2.2) corresponding to the control $u(\cdot)$ if and only if $(\xi(\cdot), u(\cdot)) = \phi(x(\cdot), u(\cdot))$ is the solution of

$$(2.14) \qquad \begin{cases} \dot{\xi} = F(\xi, \eta(t)) \\ \dot{\eta} = \dot{u}(t) \\ \xi(0) = p_1 \circ \phi(\bar{x}, u(0)) = k(\bar{x}, u(0)) \\ \eta(0) = u(0) . \end{cases}$$

Since $(\xi(\cdot), u(\cdot))$ [resp. $(x(\cdot), u(\cdot))$] is the solution of (2.14), [resp. (2.2)] if and only if $\xi(\cdot)$ [resp. $x(\cdot)$] is the solution of (2.13) [resp. (2.1)], the corollary is proved. **QED**

The following theorem will allow us to give a robust definition of solution to (2.1) corresponding to an integrable control $u(\cdot)$, as well as in the case m=1 (see Chapter 3) . Actually, the following theorem holds.

Theorem 2.1. *Let* $B \subseteq R^m$ *be the ball used in (2.7), and let* $C^1([0, T], B)$ *be the set of maps* $u \in C^1([0, T], R^m)$ *which take values in* B. *Moreover, let* $K \subset R^n$ *be a compact set such that a compact set such that, for every* $u \in C^1([0, T], B)$, *the corresponding solution* $x(u, \cdot)$ of (2.1) *exists and takes values inside* K .
Then, there exists a constant M *such that*

$$(2.15) \qquad |x(u, \tau) - x(v, \tau)| + \int_0^T |x(u, t) - x(v, t)|dt \le \\ \le M\Big[|u(0) - v(0)| + |u(\tau) - v(\tau)| + \int_0^T |u(t) - v(t)|dt\Big]$$

for all $u, v \in C^1([0, T], B), \tau \in [0, T]$.

Thanks to the diffeomorphism ϕ , the proof of this theorem can be obtained from the proof of Theorem 1 in [7], by substituting the functional $\psi(y, u)$ involved in that work with the functional

$$\chi\big((\xi,\eta),u\big)(t) = \int_0^t \big(F(\xi(s),\eta(s))\,,0\big)\,ds \;+\; \sum_{\alpha=1}^{m} [u^\alpha(t) - u^\alpha(0)]\,\frac{\partial}{\partial u^\alpha}\,,$$

where $\frac{\partial}{\partial u^\alpha}$ denotes the $n+\alpha$-th element of the canonical basis of $\mathbf{R}^{n+m}$. Then, the theorem can be proved for the control system (2.14), and hence for the system (2.2). This implies that it holds true for the control system (2.1) as well.

We are now in position to give a notion of generalized solution of (2.1) analogous to the one given in [7] for the scalar case.

Definition 2.1 [Generalized solution to 2.1] Let u be a measurable control such that $u(t)\in U$, for every $t\in[0, T]$, and let $u(0)\in U$ be an initial value.
A trajectory $t\to x(u,t)$ is a *generalized solution of* (2.1) if there exists a sequence of controls $v_k\in C^1([0, T], \mathbf{R}^n)$, $v_k(t)\in U$ $\forall$ $t\in[0, T]$, such that $v_k(0)=u(0)$, $v_k\to u$ in the L^1 norm, and the corresponding trajectories $x(v_k,\cdot)$ have uniformly bounded values and tend to $x(u,\cdot)$ in the L^1 norm.

Thanks to the estimate (2.15), any uniform a-priori bound on $x(v_k,t)$, $t\in[0, T]$, for some sequence v_k converging to u in L^1, will provide the existence of a generalized solution to (2.1). Such solution is unique up to L^1-equivalence and depends continuously on the control.
In the case where u is defined pointwise on $[0, T]$, the trajectory $x(u,\cdot)$ can also be pointwise determined. Indeed, for any fixed $\tau\in[0, T]$ one can construct a sequence of C^1 controls w_k^τ such that $w_k^\tau(0) = u(0)$, $w_k^\tau(\tau) = u(\tau)$ and $w_k^\tau\to u$ in $L^1[0, T]$. The estimate (2.15) then implies that, as $k\to\infty$, $x(w_k^\tau,\cdot)$ tends to $x(u,\cdot)$ in $L^1[0, T]$ and $x(w_k^\tau,\tau)$ has a limit, say $x(\tau)$. Repeating this construction for all τ, one obtains a function $t\mapsto x(t)$ defined pointwise on $[0, T]$.
Notice that , for any $t\in[0, T]$ one can extract a subsequence $(w_{k'}^t)$ from (w_k^t) which converges pointwise to u on the complement $[0, T]\setminus N$ of a set N of measure zero. Moreover, for any $\tau\in[0, T]$, by the estimate (2.15) one obtains

$$|x(\tau) - x(w_{k'}^t,\tau)| \le |x(\tau) - x(w_{k'}^\tau,\tau)| + |x(w_{k'}^\tau,\tau) - x(w_{k'}^t,\tau)|$$

$$\le |x(\tau) - x(w_{k'}^\tau,\tau)| + M\Big[\,|w_k^\tau(\tau) - w_{k'}^t(\tau)| + \int_0^T |w_k^\tau(\sigma) - w_{k'}^t(\sigma)|d\sigma\,\Big]$$

Now, $(w_{k'}^\tau - w_{k'}^t)$ converges to zero in the L^1 norm, and, for each $\tau\notin N$, $x(w_{k'}^\tau,\tau)$ and $w_{k'}^t(\tau)$ converge to $x(\tau)$ and $u(\tau)$, respectively. Furthermore, $w_k^\tau(\tau) = u(\tau)$, for every k'..

Hence the sequence $x\,(w^t_{k'},)$ converges to $x\,(\cdot)$ almost everywhere in [0,T], which implies that $x\,(\cdot)$ is a generalized solution of (2.1). More generally, if the control u is pointwise determined at $t = 0$ and on some subset $I \subset [0\ T]$, the same is true for the corresponding trajectory.

Definition 2.2. Let u be a measurable control as in Definition 2.1, pointwise determined on a subset $I \subset [0, T]$. A generalized solution to (2.1) determined on I as described above will be called a *(generalized) solution to* (2.1) *defined on* I.

In the following Proposition we prove that a result analogous to Corollary 2.1 holds for measurable controls.

Proposition 2.1. *Let* $u \in L^1([0, T], \mathbf{R}^m)$ *be a control with values in* U *such that the solution* ξ *to* (2.13) *exists on* [0, T] *. Then the map*

$$x(t) := k(\xi(t), u(t)) \tag{2.16}$$

is the generalized solution of (2.1) *corresponding to* u *. Conversely, if the generalized solution* $\tilde{x}$ of (2.1) *corresponding to the control* u *exists on* [0, T] *, then the equality*

$$\xi(t) = k(\tilde{x}(t), u(t)) \tag{2.17}$$

holds almost everywhere.

Proof. By definition (ξ, u) is a generalized solution to (2.14). Let $u_k \in C^1([0, T], \mathbf{R}^n)$ be a sequence of controls with values in U and such that the corresponding solutions (ξ_k, u_k) converge to (ξ, u) in L^1. Since ϕ^{-1} is Lipschitz continuous on compact sets, we have

$$\lim_{n \to +\infty} \phi^{-1}(\xi_k, u_k) = \phi^{-1}(\xi, u) \text{ in } L^1.$$

Since by Corollary 2.1 $(x_k, u_k) := \phi^{-1}(\xi_k, u_k)$ is the solution to (2.2) corresponding to u_k, it follows that

$$(x, u) := \phi^{-1}(\xi, u)$$

is a generalized solution of (2.1).

Conversely, if $\tilde{x}$ is a generalized solution of (2.1) corresponding to u , then $\tilde{x}$ coincides almost everywhere with the map $x(\cdot)$ defined in (2.16) .
Since by (2.16) and (2.9) we have

$$\xi(t) = k(x(t), u(t)) \ ,$$

the equality (2.17) holds almost everywhere in [0, T]. **QED**

As for generalized solutions defined on some subset $I \subseteq [0, T]$, it turns out that the equality (2.17) holds for every $t \in I$. Precisely we have:

Proposition 2.2. *Let* $u \in L^1([0, T], R^m)$ *be a control pointwise determined on a subset* $I \subseteq [0, T]$ *, with values in* U, *and suppose that there exists the corresponding solution* x *of* (2.1) *determined on* I.
Then, if ξ *is the Carathéodory solution of* (2.13) *corresponding to* u *, the equality*

$$x(t) = k(\xi(t), -u(t)) \tag{2.18}$$

holds for each $t \in I$.

Proof. Let $t \in I$. By the definition of generalized solution determined on I we have

$$x(t) = \lim_{n \to \infty} x_k(t) \ ,$$

where the x_k are the solutions of (2.1) corresponding to a sequence (u_k) of continuously differentiable controls such that $u_k(0) = u(0)$, $u_k(t) = u(t)$ and $u_k \to u$ in L^1 .
By Corollary 2.1, it is

$$x_k(t) = k(\xi_k(t), -u_k(t)) = k(\xi_k(t), -u(t))$$

where ξ_k denotes the solution of (2.13) corresponding to u_k . Since $\xi_k(t)$ tends to $\xi(t)$ and the map $k(\cdot, -u(t))$ is continuous, we obtain

$$x(t) = \lim_{n \to \infty} x_k(t) = \lim_{n \to \infty} k(\xi_k(t), -u(t)) = k(\xi(t), -u(t)) \ .$$

QED

3. An optimal control problem of Mayer type. Statement and existence results.

Let us consider again the control system

$$(2.1)\qquad \begin{cases} \dot{x} = \tilde{f}(x, u(t)) + \sum_{\alpha=1}^{m} \tilde{g}_\alpha(x, u(t))\, \dot{u}^\alpha \\ x(0) = \bar{x} \quad , \end{cases}$$

We have seen in the previous section that a solution corresponding to a measurable control u can be pointwise determined on a given subset $I\subset[0, T]$ (see Definition 2.2), provided the control u is defined on I. In particular let I be equal to the singleton $\{T\}$. Therefore we can speak of the *terminal point* $x_T(T, u)$ *of the solution* $x_T(\cdot, u)$ *corresponding to a control* u *which is pointwise determined in* T. Let $U\subseteq\mathbf{R}^m$ be a compact set, as in the previous section, and let $0\in U$. We shall consider the class $\mathcal{U}$ of controls defined by

$$\mathcal{U} = \{u : [0, T] \to U ,\ u \text{ measurable}, u(0) = 0\} .$$

Given a continuously differentiable cost function $\gamma : \mathbf{R}^n \to \mathbf{R}$, we are going to consider the optimization problem

$$(\mathcal{P}_T) \qquad \min\{\gamma(x_T(u, T)) ,\ u\in \mathcal{U}\},$$

i.e.we wish to minimize the value of γ at the terminal point $x_T(u, T)$ of the solution $x_T(u, \cdot)$ determined in T of (2.1) corresponding to the control $u\in \mathcal{U}$.
Actually this is equivalent to consider the problem of minimizing γ at the terminal point $x(u, T)$ of the *pointwise determined solution* $x(u, \cdot)$ of 2.1 corresponding to the control u.
Here and hereafter, by *pointwise determined solution* we mean the generalized solution determined on the whole $[0, T]$, according to Definition 2.2.
Indeed, by Proposition 2.2, we obtain the following corollary.

Corollary 3.1. *Let us denote by* $\mathcal{R}_T(T)$ *the reachable set at time* T *corresponding to the solutions of* (2.1) *determined in* T, *i.e.,*

$$\mathcal{R}_T(T) = \{x_T(u, T), u \in \mathcal{U}\} .$$

Moreover, let $\mathcal{R}(T)$ *be the reachable set at time* T *corresponding to the pointwise determined solutions of* (2.1), *i.e.,*

$$\mathcal{R}(T) = \{x(u, T) , u \in \mathcal{U}\} .$$

Lastly, let us denote by $\mathcal{R}'(T)$ *the reachable set at time* T *of the control system*

$$(2.13) \qquad \begin{cases} \dot{\xi} = F(\xi, u(t)) \\ \xi(0) = k(\bar{x}, u(0)), \end{cases}$$

i.e., to say, let us set

$$\mathcal{R}'(T) = \{\xi(u, T), u \in \mathcal{U}\} ,$$

where $\xi(u, \cdot)$ *denotes the solution of* (2.13) *corresponding to the control* u .
Then

$$\mathcal{R}_T(T) = \mathcal{R}(T) = \{x : x = k(\xi, -v), \quad (\xi, v) \in \mathcal{R}'(T) \times U\} .$$

On the basis of the above corollary, rather then studying problem $(\mathcal{P}_T)$, we shall consider the optimization problem

$$(\mathcal{P}) \qquad \min\{\gamma(x(u, T)) , u \in \mathcal{U}\}$$

Definition 3.1. We shall say that a control $\hat{u} \in \mathcal{U}$ is *optimal for* $(\mathcal{P})$ if the corresponding pointwise determined solution $x(\hat{u}, \cdot)$ of (2.1) satisfies

$$(3.1) \qquad \gamma(x(\hat{u}, T)) \leq \gamma(x) \qquad \forall x \in \mathcal{R}(T) .$$

We shall put problem $(\mathcal{P})$ in relation with the following auxiliary optimization problem $(\mathcal{P}')$:

$(\mathcal{P}')$ $$\min\{\psi(\xi(u, T)) \ , \ u \in \mathcal{U}\}$$

with ψ defined by

(3.2) $$\psi(\xi) = \min_{v \in U} \gamma(k(\xi(, -v)) \ .$$

Definition 3.2. A control $\hat{u} \in \mathcal{U}$ is said to be *optimal for* $(\mathcal{P}')$ if the corresponding Carathéodory solution $\xi(\hat{u}, \cdot)$ of (2.13) satisfies

(3.3) $$\psi(\xi(\hat{u}, T)) \leq \psi(\xi) \qquad \forall \xi \in \mathcal{R}'(T) \ .$$

Theorem 3.1. *A control* $\hat{u} \in \mathcal{U}$ *is optimal for* $(\mathcal{P})$ *if and only if it is optimal for* $(\mathcal{P}')$ *and satisfies*

(3.4) $$\psi(\xi(\hat{u}, T)) = \gamma\big(k(\xi(\hat{u}, T), \hat{u}(T)\big) \ .$$

Proof. Let $\hat{u} \in \mathcal{U}$ be optimal for $(\mathcal{P})$ and let $x(\hat{u}, \cdot)$ be the corresponding pointwise determined solution of (2.1). If $\xi(\hat{u}, \cdot)$ is the solution of (2.13) corresponding to $\hat{u}$, then Proposition 2.2 yields

$$x(\hat{u}, T) = k(\xi(\hat{u}, T), -\hat{u}(T)) \ .$$

Hence, by Corollary 3.1 it follows that

$$\gamma(x(\hat{u}, T)) = \gamma\big(k(\xi(\hat{u}, T), -\hat{u}(T))\big) \leq \gamma\big(k(\xi(\hat{u}, T), -v)\big) \qquad \forall v \in U \ ,$$

which, by the definition of ψ, implies (3.4).
Corollary 3.1 implies also that

$$\gamma(x(\hat{u}, T)) \leq \gamma\big(k(\xi, -v)\big) \qquad \forall (\xi, v) \in \mathcal{R}'(T) \times U \ .$$

In particular, for every $\xi \in \mathcal{R}'(T)$ we obtain

$$\psi(\xi(\hat{u}, T)) = \gamma(x(\hat{u}, T)) \leq \min_{v \in U} \gamma(k(\xi, -v)) = \psi(\xi) \ ,$$

and hence $\hat{u}$ is optimal for problem $(\mathcal{P}')$.
Conversely, let $\hat{u} \in \mathcal{U}$ be optimal for $(\mathcal{P}')$ and such that the value u(T) satisfies (3.4).
Then

$$\gamma\big(k(\xi(\hat{u}, T), -\hat{u}(T))\big) = \psi\big(\xi(\hat{u}, T)\big) \leq \psi(\xi) \qquad \forall \xi \in \mathcal{R}'(T) . \tag{3.5}$$

Since for each $v \in U$ and each $\xi \in \mathcal{R}'(T)$ we have

$$\psi(\xi) \leq \gamma(k(\xi, -v)),$$

by (3.5) and Corollary 3.1 we obtain

$$\gamma\big(k(\xi(u, T), -u(T))\big) \leq \gamma(x) \qquad \forall x \in \mathcal{R}(T) .$$

Since Proposition 2.2 implies that $x(u, T) = k(\xi(u, T)), -u(T))$, the lemma is proved.

QED

Thanks to Theorem 3.1 the existence problem for $(\mathcal{P})$ is reduced to the existence problem for $(\mathcal{P}')$, to which the classical results on the existence of an optimal control can be applied.
For instance, by Filippov theorem, the convexity of the values of the multifunctions

$$\mathcal{F}(\xi) = \bigcup_{u \in U} F(\xi, u) \tag{3.4}$$

and some suitable compactness and growth-conditions guarantee the existence of an optimal control for $(\mathcal{P}')$ (see e.g.[19],pag. 63), and hence for $(\mathcal{P})$. Notice that the expression of $\mathcal{F}(\xi)$ is provided by Corollary 2.1.
If the sets $\mathcal{F}(\xi)$ are not convex, then only the existence of optimal chattering controls can be proved (see e.g [24], pag. 266) under the same compactness and growth-conditions of Filippov theorem. We recall that a *chattering control* is a map which assigns to each t not merely a point $u(t) \in U$ but rather a probability measure $\nu(t)$ on U . It turns out, since $U \subset \mathbf{R}^m$, that it suffices to consider atomic probability measures $\nu(t)$ concentrated on no more than $m + 1$ points of U .

Remark 3.1. If the g_α do not depend on x, i.e.,if $g_\alpha = g_\alpha(u)$, $\forall\alpha = 1, \ldots, M$, then the expression of $\mathcal{F}(\xi)$ is quite simple. Indeed, if this is the case, one has

$$\xi = k(x, u) = x - \int_0^1 \sum_{\alpha=1}^{m} g_\alpha((1-t)u)dt .$$

Hence the Jacobian matrix $\left(\frac{\partial k^i(x,u)}{\partial x^j}\right)$ coincides with the unit matrix (δ^i_j) . It follows that

$$F(\xi, u) = f(\phi^{-1}(\xi, u)) = f(\xi + \int_0^1 \sum_{\alpha=1}^{m} u^\alpha g_\alpha((1-t)u)dt , u) ,$$

and hence

$$\mathcal{F}(\xi) = \bigcup_{u\in U} f(\xi + \int_0^1 \sum_{\alpha=1}^{m} u^\alpha g_\alpha((1-t)u)dt , u) .$$

4. A maximum principle for optimal controls of commutative impulsive systems.

We seek necessary conditions in the form of a maximum principle for a control which is optimal for $(\mathcal{P})$.

We begin by showing that many properties of the system (2.2) are extendible to the corresponding variational and adjoint systems. By *variational system* associated to a differential equation $\dot{z} = f(t, z)$, $z \in \mathbf{R}^q$, we mean the differential system on the tangent bundle $T\,\mathbf{R}^q$ defined by

(4.1)
$$\begin{cases} \dot{z} = f(t, z) \\ \dot{v} = \nabla f\,(t, z) \bullet v \end{cases}$$

where ∇f denotes the Jacobian matrix of f with respect to z , and $(z, v) \in T\,\mathbf{R}^q$. By *adjoint system* associated to $\dot{z} = f(t, z)$ we mean the differential system of the cotangent bundle $T^*\mathbf{R}^q$ defined by

(4.2)
$$\begin{cases} \dot{z} = f(t, z) \\ \dot{\lambda} = -\,\lambda \cdot \nabla f(t, z), \end{cases}$$

where $(z, \lambda) \in T^*\mathbf{R}^q$.

Let $u : [0, T] \to \mathbf{R}^m$ be a continuously differentiable map.

Then

(4.3)
$$\begin{cases} \dot{y} = f(y) + \sum_{\alpha=1}^{m} g_\alpha(y)\dot{u}^\alpha \\ \dot{v} = \left(\nabla f(y) + \sum_{\alpha=1}^{m} \nabla g_\alpha(y)\dot{u}^\alpha\right) \cdot v \end{cases}$$

and

$$(4.4) \qquad \begin{cases} \dot{y} = f(y) + \sum_{\alpha=1}^{m} g_\alpha(y)\dot{u}^\alpha \\ \\ \dot{\lambda} = -\lambda \cdot \left(\nabla f(y) + \sum_{\alpha=1}^{m} \nabla g_\alpha(y)\dot{u}^\alpha\right) \end{cases}$$

are the variational system and adjoint system associated to (2.2), respectively.
Note that they are of the same form of (2.2), i.e. their right-hand sides are affine in $\dot{u}$. Moreover, the following proposition states that they inherit the commutativity property from the system (2.2).

Proposition 4.1. *Let us assume the fields* g_α *are of class* C^2, *and let the control system* (2.2) *be commutative. Then the variational system* (4.3) *and the adjoint system* (4.4) *associated to* (2.2) *are commutative too.*

Proof. Let h and k be C^2 vector fields on $\mathbf{R}^q$ such that

$$[h, k] = 0 .$$

Then, the vector fields $\hat{h}$, $\hat{k}$ on the tangent bundle $T\,\mathbf{R}^q$

$$\hat{h}(x, v) = (h(y), \nabla f(y) \cdot v) , \qquad \hat{k}(y, v) = (k(y), \nabla k(y) \cdot v)$$

and the vector fields $\check{h}$, $\check{k}$ on the cotangent bundle $T^*\mathbf{R}^q$

$$\check{h}(y, \lambda) = (h(y) , -\lambda \bullet \nabla h(y)) , \qquad \check{k}(y, \lambda) = (k(y) , -\lambda \bullet \nabla k(y))$$

commute as well, that is, to say,

$$[\hat{h} , \hat{k}] = 0, \qquad [\check{h} , \check{k}] = 0.$$

Indeed, denoting the l-th component of $[\hat{h} , \hat{k}]$ by $[\hat{h} , \hat{k}]^l$, it is straightforward to check that

$$[\hat{h}, \hat{k}]^r(x, v) = [h, k]^r(x) = 0 ,$$

and

$$[\hat{h}, \hat{k}]^{q+r}(x, v) = \sum_{\ell=1}^{q} \frac{\partial}{\partial x^{\ell}} [h, k]^{r}(x)\, v^{\ell} = 0 \ ,$$

for every $(x, v) \in T\,\mathbf{R}^q$ and every $r = 1, \ldots, q$.
Similarly, one has

$$[\check{h}, \check{k}]^{r}(x, \lambda) = [h, k]^{r}(x) = 0 \ ,$$

and

$$[\check{h}, \check{k}]^{q+r}(x, \lambda) = -\sum_{\ell=1}^{q} \frac{\partial}{\partial x^{r}} [h, k]^{\ell}\, \lambda_{\ell} = 0 \ ,$$

for every $(x, \lambda) \in T^*\mathbf{R}^q$ and every $r = 1, \ldots, q$. By setting $h = g_\alpha$ and $k = g_\beta$, $\alpha, \beta = 1,\ldots,m$, we obtain the thesis, since $\hat{g}_1, \ldots, \hat{g}_m$, and $\check{g}_1, \ldots, \check{g}_m$ are just the vector fields which are multiplied by the $\dot{u}^\alpha$ on the right-hand sides of (4.3) and (4.4), respectively.

QED

Since the adjoint system (4.4) is commutative we can apply the results of the previous sections. Note that the control system

$$(4.5) \qquad \begin{cases} \dot{x} = \tilde{f}(x,u) + \displaystyle\sum_{\alpha=1}^{m} \tilde{g}_\alpha(x,u)\dot{u}^\alpha \\ \dot{\pi} = -\pi \left(\nabla_x \tilde{f}(x,u) + \displaystyle\sum_{\alpha=1}^{m} \nabla_x \tilde{g}_\alpha(x,u) \right) \dot{u}^\alpha \ , \end{cases}$$

where $(x, \pi) \in T^*\mathbf{R}_n$ and the symbol ∇_x denotes differentiation with respect to the variable x, is the adjoint system associated to (2.1). Given a differentiable control u, it is straightforward to check that $(x, \pi)(\cdot)$ is a solution to (4.5) if and only if for every $\ell \in \mathbf{R}^m$, $(y, \lambda)(\cdot) = (x, u, \pi, \ell)(\cdot)$ is a solution of (4.4).

In particular, as long as an initial value has been given, it is meaningful to speak of the pointwise determined solution (x, π) to (4.5).
Before stating a necessary condition for an optimal control for $(\mathcal{P})$ we need the following definition.

Definition 4.1. For every $v \in U$, let $k_*(\cdot, v)$ denote the differential of the application $k(\cdot, v)$. Let $(\bar{x}, \bar{u}) \in R^n \times U$ and $u \in U$. The tangent vector

$$k_*(k(\bar{x}, \bar{u}-u), u-\bar{u}) \cdot \tilde{f}(k(\bar{x}, \bar{u}-u), u) \in T_{\bar{x}}R^n$$

will be said the u-*transported of* $\tilde{f}$ *at* $(\bar{x}, \bar{u})$, and it will be denoted by $\mathcal{T}_u\tilde{f}(\bar{x}, \bar{u})$.

Theorem 4.1. (maximum principle for commutative impulsive systems). *Let* $\hat{u} \in \mathcal{U}$ *be an optimal control for the problem*

$$(\mathcal{P}) \qquad \min\{\gamma(x(u, T), u \in \mathcal{U}\},$$

where $x(u, \cdot)$ *denotes the pointwise determined solution of*

$$(2.1) \qquad \begin{cases} \dot{x} = \tilde{f}(x, u) + \sum_{\alpha=1}^{m} \tilde{g}_\alpha(x, u)\, \dot{u}^\alpha \\ \\ x(0) = \bar{x}\ . \end{cases}$$

Let us consider the solution $(\hat{x}, \hat{\pi}) : [0, T] \to T^*R^n$ *to the adjoint Cauchy Problem*

$$
\text{(A)} \qquad \begin{cases}
\dot{x} = \tilde{f}(x, \hat{u}) + \sum\limits_{\alpha=1} \tilde{g}_{\alpha}(x, \hat{u})\, \dot{\hat{u}}^{\alpha} \\[2ex]
\dot{\pi} = -\pi \cdot \left(\nabla_x \tilde{f}(x,\hat{u}) + \sum\limits_{\alpha=1}^{m} \nabla_x \tilde{g}(x,\hat{u})\dot{\hat{u}}^{\alpha} \right) \\[2ex]
x(T) = x(\hat{u},T) \\[2ex]
\pi(T) = \nabla\gamma(x(\hat{u}, T)) \quad ,
\end{cases}
$$

where the symbol $\nabla\gamma$ *denotes the gradient of* γ. *Then, the triple* $(\hat{u}, \hat{x}, \hat{\pi})$ *satisfies the inequality*

$$
\text{(M)} \qquad < \hat{\pi}(t)\, ,\; \mathcal{T}_u \tilde{f}(\hat{x}(t), \hat{u}(t)) - \tilde{f}(\hat{x}(t), \hat{u}(t)) > \; \leq 0
$$

for almost every $t \in [0, T]$ *and for each* $u \in U$, *where, for each* $x \in R^n$, $<\cdot, \cdot>$ *denotes the duality between* $T^*_x R^n$ *and* $T_x R^n$.
Moreover, the inequality

$$
\text{(J)} \qquad \gamma(\hat{x}(T)) \leq \gamma\big(k(\hat{x}(T), \hat{u}(T) - u)\big)
$$

holds, for every $u \in U$.

In order to prove Theorem 4.1, let us introduce a class $\{\mathcal{P}_{\bar{u}}, \bar{u} \in U\}$ of auxiliary optimization problems, to which the classical Pontryagin's maximum principle is applicable. For each $\bar{u} \in U$, we consider the function $\psi_{\bar{u}}$ from R^n into R defined by

$$
(4.8) \qquad \psi_{\bar{u}}(\xi) = \gamma\big(k(\xi, -\bar{u})\big) .
$$

The map $\psi_{\bar{u}}$ is of class C^1 since it is the superposition of two C^1 functions. Moreover,

$$
\psi(\xi) = \max_{\bar{u} \in U} \psi_{\bar{u}}(\xi)
$$

holds for every $\xi \in R^n$.

For every $\bar{u} \in U$, the optimization problem $(\mathcal{P}_{\bar{u}})$ is defined by

$$(\mathcal{P}_{\bar{u}}) \qquad \min\{\psi_{\bar{u}}(\xi(u, T)) \quad u \in \mathcal{U}\} ,$$

where $\xi(u, T)$ denotes the solution to the Cauchy Problem

$$(2.13) \qquad \begin{cases} \dot{\xi} = F(\xi, u(t)) \\ \xi(0) = k(\bar{x}, u(0)) \ . \end{cases}$$

Lemma 4.1. *If a control* $\hat{u} \in \mathcal{U}$ *is optimal for the problem* $(\mathcal{P}')$, *then* $\hat{u}$ *is optimal also for the problem* $(\mathcal{P}_{\hat{u}(T)})$.

Proof. By the definition of $\psi_{\hat{u}(T)}$ and by the equality (3.4) we have

$$\psi_{\hat{u}(T)}\big(\xi(\hat{u}, T)\big) = \psi\big(\xi(\hat{u}, T)\big) .$$

Since for each $\xi \in \mathcal{R}'(T)$ one has

$$\psi\big(\xi(\hat{u}, T)\big) \le \psi(\xi)$$

by

$$\psi(\xi) = \min_{v \in U} \gamma(k(\xi, -v)) \le \psi_{\hat{u}(T)}(\xi)$$

we obtain the thesis. **QED**

By means of of Lemma 4.1 we obtain the following result which is a corollary of the ordinary Pontryagin's maximum principle

Lemma 4.2. *Let* $\hat{u}$ *be an optimal control for problem* $(\mathcal{P})$ *and let* $\hat{\xi}$ *be defined by* $\hat{\xi}(t) := \xi(\hat{u}, t)$. *Let* $\hat{\omega} : [0, T] \to \mathbf{R}^n$ *be the solution of the adjoint Cauchy problem*

(4.9)
$$\begin{cases} \dot{\omega} = -\omega \cdot \nabla_\xi F(\hat{\xi}(t), \hat{u}(t)) \\ \omega(T) = \nabla\psi_{\hat{u}(T)}(\hat{\xi}(T)) \ . \end{cases}$$

Then the triple $(\hat{u}, \hat{\xi}, \hat{\omega})$ *satisfies the inequality*

(4.10)
$$<\hat{\omega}(t), F(\hat{\xi}(t), u) - F(\hat{\xi}(t), \hat{u}(t))> \ \leq 0$$

for almost every $t \in [0, T]$ *and for every* $u \in U$.

Indeed, by Lemma 4.1, $\hat{u}$ is an optimal control for the problem $(\mathcal{P}_{\hat{u}(T)})$, and Lemma 4.2 is nothing but the Pontryagin's principle applied to the problem $(\mathcal{P}_{\hat{u}(T)})$.

The inequality (M) in Theorem 4.1 will be obtained from (4.10), by means of a suitable C^1 automorphism of $T^*(R^n \times B)$, where B is the ball containing U, introduced in Section 2. Actually, let $\check{\phi}$ denote the cotangent map of ϕ, i.e. the inverse of the pullback ϕ^*. It is easily seen that $\check{\phi}$ transforms the adjoint system (4.4) into

(4.11)
$$\begin{cases} \dot{\xi} = F(\xi, \eta) \\ \dot{\eta} = \dot{u} \\ \dot{\omega} = -\omega \cdot \nabla_\xi F(\xi,\eta) - \nu \bullet \nabla_\eta F(\xi, \eta) \\ \dot{\nu} = 0 \quad , \end{cases}$$

which is the adjoint system associated to the equation in (2.14). In (4.6) the symbol $((\xi, \eta), (\omega, \nu))$ denotes an element of the cotangent bundle $T^*(R^n \times B)$.
It is straightforward to check that $\check{\phi}$ is a C^1 automorphism of $T^*(R^n \times B)$ as soon as the fields $g_1, \ldots, g_m$ are of class C^2 . It is also easy to verify that, for each $((x, u), (\pi, \eta)) \in T^*(R^n \times B)$,

(4.12)
$$\check{\phi}(x, u, \pi, \eta) = (k(x, u), u, k^*(x, -u)\pi, \eta)$$

holds, where, for every $v \in B$ and $x \in \mathbf{R}^n$, the symbol $k^*(x, v)$ indicates the transposed map of the differential $k_*(x, v)$ of $k(\cdot, v)$ at x.
In full analogy with Proposition 2.2, one can prove the following result.

Proposition 4.1. *Let* $(\bar{x}, \bar{\pi}) \in T^*\mathbf{R}^n$ *and let* $u \in \mathcal{U}$ *be a control such that both the pointwise determined solution* (x, π) *to*

$$\text{(4.13)} \qquad \begin{cases} \dot{x} = \tilde{f}(x, u) + \sum_{\alpha=1}^{m} \tilde{g}_\alpha(x, u)\dot{u}^\alpha \\ \dot{\pi} = -\pi \left(\nabla_x \tilde{f}(x, u) + \sum_{\alpha=1}^{m} \nabla_x \tilde{g}_\alpha(x, u)\dot{u}^\alpha\right) \\ x(T) = \bar{x} \\ \pi(T) = \bar{\pi} \end{cases}$$

and the Carathéodory solution (ξ, ω) *to*

$$\text{(4.14)} \qquad \begin{cases} \dot{\xi} = F(x, u) \\ \dot{\omega} = -\omega \cdot \nabla_x F(x,u) \\ \xi(T) = k(\bar{x}, u(T)) \\ \omega(T) = k^*(\bar{x}, -u(T))\, \bar{\pi} \end{cases}$$

exist.
Then, the equality

$$\text{(4.15)} \qquad (\xi(t), \omega(t)) = \big(k(x(t), u(t)) \ , \ k^*(x(t), -u(t))\, \pi(t)\big)$$

holds, for every $t \in [0, T]$.
Moreover, if $t \in [0, T]$ and $v \in T_{\xi(t)}\mathbf{R}^n$, *then*

(4.16) $$<\omega(t), v> \;=\; <\pi(t), k_*(x(t), -u(t))\, v>.$$

Indeed, as soon as the system (4.4), with $\pi(0) = (\bar{\pi}, 0)$, plays the role of the system (2.2) in Proposition 2.2, by 4.12 the latter yields the equality (4.15). And (4.13) is a straightforward consequence of 4.16.

Proof of Theorem.1. First, let us observe that, by the uniqueness of the pointwise determined solution to Cauchy problem (2.1), one has

$$\hat{x}(t) = x(\hat{u}, t)$$

for every $t \in [0, T]$.

Since $\hat{u}$ is optimal for $(\mathcal{P})$, by Theorem 3.1 it is also optimal for $(\mathcal{P}')$. Then Lemma 4.2 applies to $\hat{u}$, and by Proposition 4.1 we obtain

(4.17) $$<\hat{\pi}(t)\;,\; k_*(\hat{x}(t), -\hat{u}(t))(F(\hat{\xi}(t), u) - F(\hat{\xi}(t), \hat{u}(t))> \leq 0$$

for almost every $t \in [0, T]$, and for every $u \in U$, where, by Proposition 2.2, $\hat{\xi}(t) = k(\hat{x}(t), \hat{u}(t))$.

In order to show that the inequality (4.17) coincides with the inequality (M), we first observe that the equality

$$k(k(x, u_1), u_2) = k(x, u_1 + u_2)$$

implies

(4.18) $$k_*(k(x, u_1), u_2) \circ k_*(x, u_1) = k_*(x, u_1 + u_2),$$

for every $x \in \mathbf{R}^n$ and every $u_1, u_2 \in B$ such that $u_1 + u_2 \in B$.

Moreover, Corollary 2.1 implies that for every $(\xi, v) \in \mathbf{R}^n \times B$ it is

$$F(\xi, v) = k_*\big(k(\xi, -v), v\big)\, f(k(\xi, -v), v).$$

In particular, since $\hat{\xi}(t) = k(\hat{x}(t), \hat{u}(t))$, we have

$$F(\hat{\xi}(t), u) = k_*\big(k(\hat{x}(t), \hat{u}(t) - u), u\big)\ f\big(k(\hat{x}(t), \hat{u}(t) - u), u\big) ,$$

(4.19)

$$F(\hat{\xi}(t), \hat{u}(t)) = k_*\big(\hat{x}(t), \hat{u}(t)\big)\ f\big(\hat{x}(t), \hat{u}(t)\big) ,$$

for every $t \in [0, T]$.

Then, accordingly to Definition 4.1,by (4.18)-(4.19) we obtain

$$k_*\big(\hat{x}(t), -\hat{u}(t)\big)(F(\hat{\xi}(t), u) - F(\hat{\xi}(t), \hat{u}(t))) = \mathcal{T}_u f(\hat{x}(t), \hat{u}(t)) - f(\hat{x}(t), \hat{u}(t)) \ .$$

Hence, by (4.17), the inequality (**M**) holds for almost every $t \in [0, T]$ and for every $u \in U$. Finally, the inequality (**J**) in Theorem 4.1 follows from Proposition (2.2) and Theorem 3.1. Indeed, for every $u \in U$, one has

$$\gamma(\hat{x}(T)) = \gamma\big(k(\hat{\xi}(T), -\hat{u}(T)\big) = \psi(\hat{\xi}(T)) \ \leq \gamma\big(k(\hat{\xi}(T), -u)\big) = \gamma\big(k(\hat{x}(T), \hat{u}(T) - u)\big) \ .$$

QED

Appendix

Moving holonomic constraints as controls in classical mechanics

1.Introduction

Recently, Aldo Bressan has developed a theory concerning the control of Lagrangian systems by means of additional moving constraints.

A very simple example is supplied by a pendulum with variable length γ (see [12]). In this case, the constraint $\gamma = \gamma(\cdot)$ is thought as a control $u(\cdot)$, which may be implemented in order to minimize a given cost functional, e.g. the kinetic energy of the pendulum at a certain instant.

In its general setting, the mechanical model considered by Bressan consists of a Lagrangian system Σ subjected to holonomic, bilateral, time-independent, ideal constraints, and locally described by Lagrangian coordinates (q^i,γ^α), $i = 1,\dots, N$, $\alpha = 1, \dots, M$.
If p_i denotes the momentum conjugate to q^i, $i = 1, \dots, N$, the subsystem $\Sigma_{\hat{\gamma}}$ obtained from Σ by implementing the additional, time-dependent, ideal constraints $\gamma^\alpha = \hat{\gamma}^\alpha(t)$ is governed by equations of the form

$$(1.1)\qquad \begin{cases} \dot{q}^i = F^i(q,p,\hat{\gamma}(t),\dot{\hat{\gamma}}(t)) \\ \dot{p}_i = G_i(q,p,\hat{\gamma}(t),\dot{\hat{\gamma}}(t)) \end{cases} \qquad i = 1, \dots, N,$$

where the dot indicates differentiation with respect to time.

If the trajectories $\tilde{\gamma}^\alpha(\cdot)$ of the last M coordinates γ^α are fixed a priori, a scalar control $u(t)$ for (1.1) may be considered by setting $\hat{\gamma}^\alpha(t) = \tilde{\gamma}^\alpha(u(t))$. More generally, on the basis of the results in [9] and[16], one can drop the assumption of fixed trajectories for the coordinates γ^α and consider a vector-valued control $(u^1,\dots, u^M)(\cdot)$, to be identified directly with $(\hat{\gamma}^1,\dots,\hat{\gamma}^M)(\cdot)$ (see [31]-[33]).
In both cases the right-hand side of (1.1) contains the control $u(\cdot)$ and its derivative $\dot{u}(\cdot)$. Typically, the latter appears quadraticly, provided the forces depend linearly on the velocities of Σ. This is a strict consequence of the fact that some terms of the right-hand side of the system (1.1) are obtained by derivation w.r.to the state of the kinetic energy of Σ.

On the other hand, (F^i,G_i) is linear in $\dot{\hat{\gamma}}$ if and only if (F^i,G_i) is linear as a function of $\dot{u}$.

In this Appendix, we begin by briefly recalling some tools of the theory proposed by Aldo Bressan, with an approach slightly different from the original one. Secondly, some results obtained by the author of this thesis are reported by the two articles [33] and [31].

These results concern the study of the so-called *M-fit coordinate systems* (q^i,γ^α), which are defined as those coordinate systems in which the vector field (F^i,G_i) becomes linear in the argument $\dot{\hat{\gamma}}$. Moreover, they concerns also the *strongly M-fit coordinate systems*, in which (F^i,G_i) is independent of the argument $\dot{\hat{\gamma}}$.

As it has been shown in the first part of the present thesis, such linear dependence is essential in order to characterize the continuity of the input-output map $\phi : u(\cdot) \to (q,p)(\cdot)$ associated with a Cauchy problem for (1.1).

For example (see [9],[10],[36]), this linearity is equivalent to the continuity of ϕ with respect to the C^0-norms, which means that (sufficiently) small changes in the values of $u(\cdot)$ have the effect of producing (arbitrarily preassigned) small changes in the values of the corresponding $(q,p)(\cdot)$. Such a behaviour could be unexpected, since the presence of $\dot{\hat{\gamma}}(\cdot)$ –and hence of $\dot{u}$– on the right-hand side of (1.1) makes it impossible to apply the classical results on the continuous dependence of solutions on controls, unless e.g. the C^1-norm is used on the space of controls.

The above linearity of (F^i,G_i) in $\dot{\hat{\gamma}}$ is also essential in the characterization of the continuity of ϕ with respect to topologies which are weaker than the C^0 topology. (see [7],[9],[13]) This allows e.g. the study of the effects produced on the evolution of $\Sigma_{\hat{\gamma}}$ by a first kind discontinuity of $u(\cdot)$ (i.e., an *hyperimpulse* –see [10],[11]–) .

This Appendix contains six sections. In Section 2 some basic tools from [11] are briefly presented, by an approach similar to the one used in [33].

In Section 3 we recall some results on M-fit coordinates from [33], and prove a characterization of strongly M-fit coordinates for the case of positional forces.

In Section 4 we report a characterization for 1-fit ccordinate systems from [33].

Afterwards, we observe that the notion of an M-fit coordinate system does not have a geometrical meaning. On the other hand, it is trivial to check that M-fitness is conserved by changes of coordinates of the form

$$\tilde{q} = \tilde{q}(q,\gamma) \qquad \tilde{\gamma} = \tilde{\gamma}(\gamma).$$

An equivalence class of coordinate systems that are related by such transformations is generally thought as a *foliation* formed by the submanifolds of the form $\gamma = c \in \mathbf{R}^M$.

Then, the concept of an $\mathcal{U}$-M-*fit foliation* for a coordinate neighborhood $\mathcal{U}$ of the constraint manifold $\mathcal{M}$ is introduced in Section 5 . In particular, an M-*fit foliation*, i.e., a foliation of $\mathcal{M}$ which is $\mathcal{U}$-M-fit for each coordinate neighborhood $\mathcal{U}$, may be considered as the natural, global extension of the idea of an M-fit coordinate system.

In Theorem 5.2 a foliation $\mathcal{F}$ is shown to be $\mathcal{U}$-M-fit if and only if the (restriction to $\mathcal{U}$ of the) kinetic metric is *bundle-like* with respect to $\mathcal{F}$ (see Definition 5.3). Here, we use a basic result on these metrics –which were introduced by Reinhart in 1959 (see [34])– to characterize a $\mathcal{U}$-M-fit foliation in terms of the *spontaneous motions* of Σ (i.e., the geodesics of $\mathcal{M}$).

As for strong fitness, we know from [33] that each equivalence class of 1-fit coordinate systems, i.e., each 1-fit foliation, contains a strongly 1-fit element (q^i,γ) , $i = 1, \ldots, N$. The analogue for a M-fit foliation , with $M > 1$, is false, unless the orthogonal bundle is integrable (see Theorem 5.3).

In Section 6, Theorems 5.2 and 5.3 are proved.

The seventh section is mainly concerned with some examples of M-fit foliations . In particular, the configuration space of a rigid body is seen to be partitioned by foliations of this kind.

2. Additional moving constraints as controls

Let Σ be a mechanical system subjected to holonomic, bilateral, time-independent, ideal constraints. These define a differentiable (*) manifold $\mathcal{M}$, called *configuration manifold* or *constraint manifold*, which we assume to be D-dimensional, $D < \infty$. Let $T_m\mathcal{M}$ [resp. $T\mathcal{M}$] be the tangent space of $\mathcal{M}$ at m [resp. the tangent bundle of $\mathcal{M}$]. A *motion* m of Σ is a curve $m : I \to \mathcal{M}$, where I is an interval of $\mathbb{R}$. If $m(\cdot)$ is differentiable at t, and $m_*(t)$ denotes its differential, the *velocity* $\dot{m}(t)$ at the time t is defined by $\dot{m}(t) = m_*(t)\frac{d}{dt}(t) \in T_{m(t)}\mathcal{M}$, and the *kinetic energy* $\mathcal{T} = \mathcal{T}(m(t), \dot{m}(t))$ of $m(\cdot)$ is given by

$$(2.1) \qquad \mathcal{T}(m(t)), \dot{m}(t)) := \frac{1}{2} A(m(t)) (\dot{m}(t), \dot{m}(t)) ,$$

where A is a (suitable) metric tensor field on $\mathcal{M}$, and, $\forall m \in \mathcal{M}$, $A(m)(\cdot,\cdot)$ denotes the scalar product on $T_m\mathcal{M}$ induced by A.

Let us consider local coordinates $(\chi^{\mathcal{R}})$, $\mathcal{R} = 1, ..., D$, and, given a natural number $N < D$, let us set $q^i := \chi^i$, $i = 1, ..., N$, and $\gamma^\alpha := \chi^{N+\alpha}$, $\alpha = 1, \dots, M := D - N$ (**). Then the tensor field A has the following (local) expression

$$(2.2) \qquad A(m(q,\gamma)) = \sum_{i,j=1}^{N} A_{ij}(q,\gamma) dq^i \otimes dq^j + 2\sum_{i=1}^{N} \sum_{\alpha=1}^{M} A_{i,N+\alpha}(q,\gamma) dq^i \otimes d\gamma^\alpha$$

$$+ \sum_{\alpha,\beta=1}^{M} A_{N+\alpha,N+\beta}(q,\gamma) d\gamma^\alpha \otimes d\gamma^\beta ,$$

where: i) q and γ mean $(q^1, ..., q^N)$ and $(\gamma^1, ..., \gamma^M)$, respectively; ii) $m(q,\gamma)$ denotes the element of $\mathcal{M}$ having (q,γ) as coordinates; iii) for each (q,γ), the *kinetic matrix* $(A_{\mathcal{R}\mathcal{S}})_{\mathcal{R},\mathcal{S}=1,\dots,D}(q,\gamma)$ is positive definite.

Hence, if $(q,\gamma)(\cdot)$ [resp. $(\dot{q},\dot{\gamma})(\cdot)$] locally represents the motion $m(\cdot)$ [resp. the velocity $\dot{m}(\cdot)$], (2.1) can be written in the form

(*) Throughout this chapter we tacitly assume that the constraints and the functions under consideration have a sufficient degree of smoothness, so that the required differentiations can be actually performed.

(**) Throughout this chapterlower case Latin indexes run from 1 to N, lower case Greek indexes run from 1 to M, upper case Latin indexes run from 1 to D, and N+M=D.

(2.3) $$\mathcal{T}(m(t)), \dot{m}(t)) := \frac{1}{2} \sum_{i,j=1}^{N} A_{ij}(q(t),\gamma(t))\, \dot{q}^i(t)\, \dot{q}^j(t) + $$
$$+ \sum_{i=1}^{N} \sum_{\alpha=1}^{M} A_{i,N+\alpha}(q(t),\gamma(t))\dot{q}^i(t)\, \dot{\gamma}^\alpha(t) + \frac{1}{2} \sum_{\alpha,\beta=1}^{M} A_{N+\alpha,N+\beta}(q(t),\gamma(t))\, \dot{\gamma}^\alpha(t)\, \dot{\gamma}^\beta(t) .$$

At each point $m \in \mathcal{M}$, the metric A establishes an isomorphism ι_m between the tangent space $T_m\mathcal{M}$ and the cotangent space $T^*_m\mathcal{M}$, by means of the implicit relation

(2.4) $$(w, \iota_m(v))_m = A(m)(v,w),$$

where $v, w \in T_m\mathcal{M}$, and $(\cdot,\cdot)_m$ denotes the duality between $T_m\mathcal{M}$ and $T^*_m\mathcal{M}$.

$\iota_m(v)$ is called the *conjugate momentum* of v.

If $\iota_m(v)_\mathcal{R}$ [resp. $v^\mathcal{R}$], $\mathcal{R} = 1, \ldots, D$, denotes the $\mathcal{R}$-th components of $\iota_m(v)$ [resp. v] with respect to the standard basis $(d\chi^1, \ldots, d\chi^D)$ [resp. $(\frac{\partial}{\partial\chi^1}, \ldots, \frac{\partial}{\partial\chi^D})$] of $T^*_m\mathcal{M}$ [resp. $T_m\mathcal{M}$], the isomorphism ι_m is explicitly given by

(2.5) $$\iota_m(v)_\mathcal{R} = \sum_{\mathcal{S}=1}^{D} A_{\mathcal{R}\mathcal{S}}(q,\gamma)\, v^\mathcal{S} \left(\equiv \frac{\partial \mathcal{T}}{\partial v^\mathcal{R}} \right), \qquad \mathcal{R} = 1, \ldots, D .$$

Besides A, let us consider the metric A^{-1} which acts on the fibers of $T^*\mathcal{M}$ and, for each $m \in \mathcal{M}$, induces an inner product $A^{-1}(m)(\cdot,\cdot)$ on $T^*_m\mathcal{M}$ defined by

(2.6) $$A^{-1}(m)\,(\xi,\eta) := A(m)\,(\iota_m^{-1}(\xi), \iota_m^{-1}(\eta)) ,$$

where ξ, η are arbitrary elements of $T^*_m\mathcal{M}$.

For each $m \in \mathcal{M}$ and each $\xi \in T^*_m\mathcal{M}$, the Hamiltonian $\mathcal{H}$ associated with the Lagrangian function $\mathcal{T}$ is given by

(2.7) $$\mathcal{H}(m,\xi) := \frac{1}{2} A^{-1}(m)(\xi,\xi) \ .$$

In coordinates $(\chi^{\mathcal{R}}) \equiv (q^i,\gamma^\alpha)$, $A^{-1}(m)$ has a local representation of the form

(2.8) $$A^{-1}(m(q,\gamma)) = \sum_{i,j=1}^{N} A^{ij}(q,\gamma) \frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial q^j} + \\ + 2\sum_{i=1}^{N} \sum_{\alpha=1}^{M} A^{i,N+\alpha}(q,\gamma) \frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial \gamma^\alpha} + \sum_{\alpha,\beta=1}^{M} A^{N+\alpha,N+\beta}(q,\gamma) \frac{\partial}{\partial \gamma^\alpha} \otimes \frac{\partial}{\partial \gamma^\beta} \ ,$$

where $\left(A^{\mathcal{R},\mathcal{S}}(q,\gamma)\right)$ is the inverse of the matrix $\left(A_{\mathcal{R},\mathcal{S}}(q,\gamma)\right)$. Hence, if $\xi(p_i,p_\alpha) := \sum_{i=1}^{N} p_i dq^i + \sum_{\alpha=1}^{M} p_\alpha d\gamma^\alpha$, (2.7) may be locally expressed by

(2.9) $$\mathcal{H}(m(q,\gamma), \xi(p_i,p_\alpha)) = \frac{1}{2}\sum_{i,j=1}^{N} A^{ij}(q,\gamma)\, p_i p_j + \sum_{i=1}^{N} \sum_{\alpha=1}^{M} A^{i,N+\alpha}(q,\gamma)\, p_i p_\alpha + \\ + \frac{1}{2} \sum_{\alpha,\beta=1}^{M} A^{N+\alpha,N+\beta}(q,\gamma)\, p_\alpha p_\beta \ ,$$

From the definition of A^{-1} , it follows that, for each $m \in \mathcal{M}$, ι_m is an isometry between $T_m\mathcal{M}$ and $T^*_m\mathcal{M}$, when these are endowed with the scalar products $A(m)(\cdot,\cdot)$ and $A^{-1}(m)(\cdot,\cdot)$, respectively. Then,

(2.10) $$\mathcal{T}(m,v) = \mathcal{H}(m,\iota_m(v)) \qquad \forall v \in T_m\mathcal{M} \ .$$

In order to write the dynamical equations for Σ one needs the concept of applied force. For this purpose, one uses the notion of vertical section of a fiber bundle (see [1]).

Definition 2.1. *Let* $\pi : F \to X$ *be a fiber bundle and let* Γ (F,TF) *denote the set of sections of the tangent bundle* TF *. An element* $v \in \Gamma$ (F,TF) *is said to be vertical if* $\pi_*(f)v(f) = 0$, $\forall f \in F$, *where* $\pi_*(f)$ *denotes the differential (at* f*) of the projection map* π *.*

The physical notion of "forces applied to Σ" is represented by *a vertical section* $\mathcal{Q}$ *of the tangent bundle* $TT^*\mathcal{M}$, and the equations of motion of Σ form the differential system

$$\dot{z} = X_{\mathcal{H}}(z) + \mathcal{Q}, \tag{2.11}$$

where z denotes a point of $T^*\mathcal{M}$, and $X_{\mathcal{H}}$ is the *Hamiltonian vector field* associated with the *energy function* $\mathcal{H}$.

Locally, the verticality of the section $\mathcal{Q}$ is expressed by the fact that its first $D(\equiv N+M)$ components –with respect to the basis $\left(\frac{\partial}{\partial q^1},\dots,\frac{\partial}{\partial q^N},\frac{\partial}{\partial \gamma^1},\dots,\frac{\partial}{\partial \gamma^M},\frac{\partial}{\partial p_1},\dots,\frac{\partial}{\partial p_N},\frac{\partial}{\partial \mathrm{p}_1},\dots,\frac{\partial}{\partial \mathrm{p}_M}\right)$ – vanish identically.

Hence, in the coordinates $(q^i,\gamma^\alpha,p_i,\mathrm{p}_\alpha)$, (2.11) is expressed by

$$\begin{cases} \dot{q}^i = \dfrac{\partial \mathcal{H}}{\partial p_i} \\ \dot{\gamma}^\alpha = \dfrac{\partial \mathcal{H}}{\partial \mathrm{p}_\alpha} \\ \dot{p}_i = -\dfrac{\partial \mathcal{H}}{\partial q^i} + \mathcal{Q}_i \\ \dot{\mathrm{p}}_\alpha = -\dfrac{\partial \mathcal{H}}{\partial \gamma^\alpha} + \mathcal{Q}_{N+\alpha} \end{cases} \qquad \begin{array}{l} i = 1, \dots, N \\ \alpha = 1, \dots, M, \end{array} \tag{2.12}$$

where $(0, \dots, 0, \mathcal{Q}_1, \dots, \mathcal{Q}_N, \mathcal{Q}_{N+1}, \dots, \mathcal{Q}_{N+M})$ are the components of $\mathcal{Q}$ with respect to the basis $\left(\frac{\partial}{\partial q^1},\dots,\frac{\partial}{\partial q^N},\frac{\partial}{\partial \gamma^1},\dots,\frac{\partial}{\partial \gamma^M},\frac{\partial}{\partial p_1},\dots,\frac{\partial}{\partial p_N},\frac{\partial}{\partial \mathrm{p}_1},\dots,\frac{\partial}{\partial \mathrm{p}_M}\right)$. Sometimes the $\mathcal{Q}_{\mathcal{R}}$ are called the *Lagrangian components of the active sollecitation..*

In [11], Bressan considers the problem of adding some time-dependent constraints, kinematically expressed by $\gamma^\alpha = \hat{\gamma}^\alpha(t) := \bar{\gamma}^\alpha(u(t))$, where the $\bar{\gamma}^\alpha(\cdot)$ are fixed trajectories, and u(t) is a scalar function to be assumed as control. More generally, in [33] a vector-valued control $u(\cdot) = (u^1,\dots, u^M)(\cdot)$ is considered by identifying $(\hat{\gamma}^1, \dots,\hat{\gamma}^M)(\cdot)$ directly with

$(u^1,\ldots,u^M)(\cdot)$. The addition of such new constraints reduces Σ to a subsystem $\Sigma_{\hat{\gamma}}$, whose configuration manifold is locally parametrized by the q^i . Moreover, these constraints are assumed to be ideal with respect to $\Sigma_{\hat{\gamma}}$, which means that their Lagrangian components $\mathcal{Q}_1,\ldots,\mathcal{Q}_N,\mathcal{Q}_{N+1},\ldots,\mathcal{Q}_{N+M}$ satisfy

(2.13) $$\mathcal{Q}_h = 0 \qquad h = 1, \ldots, N.$$

Of course, the $\mathcal{Q}_{N+\alpha}$ must be consistent with the the prescribed evolutions $\hat{\gamma}^1(t),\ldots,\hat{\gamma}^M(t)$ of the coordinates $\gamma^1,\ldots,\gamma^M$. This is easily achieved by the following procedure.
First of all, by partially inverting (2.5), express the p_α as functions of q,γ,p, and $\dot{\gamma}$, where p stands for $(p_1, \ldots, p_N)$. One obtains

(2.13') $$p_\alpha = p_\alpha(q,\gamma,p,\dot{\gamma}) = \sum_{\rho=1}^{M} C_{\alpha\rho}\dot{\gamma}^\rho - \sum_{\rho=1}^{M}\sum_{s=1}^{N} C_{\alpha\rho}\, A^{N+\rho,s}\, p_s ,$$

where $(C_{\alpha\rho})_{\alpha,\rho=1,\ldots,M}$ is the inverse of the matrix $(A^{N+\alpha,N+\rho})_{\alpha,\rho=1,\ldots,M}$ (see [33]).

Secondly, consider the solutions $\hat{q}(\cdot)$, $\hat{p}(\cdot)$ of the differential system

(2.14) $$\begin{cases} \dot{q}^h = \dfrac{\partial \hat{\mathcal{H}}}{\partial p_h} \\ \dot{p}_h = -\dfrac{\partial \hat{\mathcal{H}}}{\partial q^h} + \hat{\mathcal{Q}}_h \end{cases} , \qquad h = 1, \ldots, N ,$$

where $\hat{\mathcal{H}} = \hat{\mathcal{H}}(t,q,p)$ and $\hat{\mathcal{Q}}_h = \hat{\mathcal{Q}}_h(t,q,p)$ are defined by

$$\hat{\mathcal{H}}(t,q,p) = \mathcal{H}\big(q,\hat{\gamma}(t),p,p(q,\hat{\gamma}(t),p,\dot{\hat{\gamma}}(t))\big)$$

$$\hat{\mathcal{Q}}_h(t,q,p) = \mathcal{Q}_h\big(t,q,\hat{\gamma}(t),p,p(q,\hat{\gamma}(t),p,\dot{\hat{\gamma}}(t))\big) .$$

Finally, denoting the function $p_\alpha(\hat{q}(\cdot),\hat{\gamma}(\cdot),\hat{p}(\cdot),\dot{\hat{\gamma}}(\cdot))$ by $\hat{p}_\alpha(\cdot)$, one finds that the $\mathcal{Q}_{N+\alpha}(\cdot)$ are implicitly defined by

$$\dot{\hat{p}}_\alpha(t) = -\frac{\partial \mathcal{H}}{\partial \gamma^\alpha}(\hat{q}(t),\hat{\gamma}(t),\hat{p}(t),\hat{p}(t)) + \mathcal{Q}_{N+\alpha}(t,\hat{q}(t),\hat{\gamma}(t),\hat{p}(t),\hat{p}(t)) + \mathcal{Q}_{N+\alpha}(t) \qquad \alpha = 1, \ldots, M.$$

By hypothesis (2.13) on the additional constraints, the motion of $\Sigma_{\hat{\gamma}}$ will be governed by the differential system (2.14). Both in the scalar case – i.e., $(\hat{\gamma}^1,\ldots,\hat{\gamma}^M)(\cdot)=(\tilde{\gamma}^1(u(\cdot)),\ldots,\tilde{\gamma}^M(u(\cdot)))$– and in the vector-valued case – i.e., $(\hat{\gamma}^1,\ldots,\hat{\gamma}^M)(\cdot) = (u^1,\ldots,u^M)(\cdot)$ –, the right-hand side of this system contains (throught $\dot{\hat{\gamma}}$) the derivative $\dot{u}$ of the control.

In the next section we will continue the investigation, begun in [33], on the existence problem of coordinate systems (q^i,γ^α) in which $\dot{\hat{\gamma}}$ – and hence $\dot{u}$ – appears at the most linearly on the right-hand side of (2.14).

The physical and mathematical motivations which lead to the study of these coordinates have been touched on in the introduction, where the differential system (1.1) has to be identified with (2.14). For a more detailed discussion about these motivations we refer directly to the original papers (see [7], [9],[10],[11],[13],[33],[36]) .

3. M-fit and strongly M-fit coordinate systems

Definition 3.1.[11] *A coordinate system* (q^i,γ^α) *, will be called* M-*fit [resp. strongly* M-*fit] if, for every differentiable map* $\hat{\gamma}(\cdot) = (\hat{\gamma}^1,\ldots, \hat{\gamma}^M)(\cdot)$*, the right-hand side of (2.14) is affine in* $\dot{\hat{\gamma}}$ *[resp. is independent of* $\dot{\hat{\gamma}}$ *] .*

Let us observe that the right-hand sides of the former N equations in (2.14) are already affine in $\dot{\hat{\gamma}}$. This is due to the fact that (2.14) is formed by some equations belonging to (2.12), which is a first order reduction of a second order differential system.
As for the right-hand sides of the latter N equations in (2.14), the terms $\frac{\partial \mathcal{H}}{\partial q^h}$ represent the constraint-reactions and, in general, are quadratic in $\dot{\hat{\gamma}}$. On the contrary, the Q_h are independent of [resp. affine in] $\dot{\hat{\gamma}}$ whenever the forces applied to Σ are independent of [resp. affine in] the velocity $\dot{m}(t)$.
Then, it seems quite natural to assume one of the following hypotheses.

H.1. *The forces applied to* Σ *are affine functions in the velocity* $\dot{m}(t)$
H.2 *The forces applied to* Σ *depend only on time and the configuration of* Σ .

Theorem 3.1.*Let us assume hypothesis H.1 on the applied forces. Then a system of coordinates* (q^i,γ^α) , $i = 1, \ldots, N$, $\alpha = 1, \ldots, M$, *is* M-*fit if and only if the* $NM(M + 1)/2$ *identities (in* q *and* γ *)*

$$(3.1)\qquad \frac{\partial A^{N+\alpha,N+\beta}}{\partial q^h} \equiv 0 \qquad \begin{matrix} \alpha, \beta = 1,\ldots,M \\ h = 1,\ldots,N \end{matrix}$$

hold on the range of (q^i,γ^α) *, where* $(A^{\mathcal{R}\mathcal{S}})$, $\mathcal{R}, \mathcal{S} = 1, \ldots, N + M$ *, is the inverse of the kinetic matrix* $(A_{\mathcal{R}\mathcal{S}})$.

Proof. By equality (2.13'), the term

$$(3.2)\qquad \frac{1}{2}\sum_{\alpha,\beta=1}^{M} A^{N+\alpha,n+\beta}\, \mathbb{p}_\alpha\, \mathbb{p}_\beta$$

depends linearly and quadraticly on the $\dot{\gamma}^\alpha$, whereas the remaining part of $\mathcal{H}$,

$$-\frac{1}{2}\sum_{i,l=1}^{N} A^{i,l}\, p_i\, p_l + \sum_{i=1}^{N}\ \sum_{\alpha=1}^{M} A^{i,N+\alpha}\, p_i\, \mathbb{p}_\alpha\ ,$$

is affine in the $\dot{\gamma}^\alpha$.

By substituting the $\mathbb{p}$'s in (3.2) with their expression (2.13'), one trivially checks that the quadratic dependence of $\frac{-\partial\mathcal{H}}{\partial q^h}$ on the velocities $\dot{\gamma}^\alpha$ is given by the term

$$\frac{1}{2}\sum_{\alpha,\beta,\rho,\delta}^{M}\left(\frac{\partial}{\partial q^h} A^{N+\alpha,N+\beta}\right) C_{\alpha,\rho}\, C_{\beta,\delta}\, \dot{\gamma}^\rho\, \dot{\gamma}^\delta$$

where $(C_{\alpha,\beta})_{\alpha,\beta=1,\dots,M}$ denotes the inverse of the matrix $(A^{N+\alpha,N+\beta})_{\alpha,\beta=1,\dots,M}$.

Since, by

$$\sum_{\alpha=1}^{M} A^{N+\alpha,N+\beta}\, C_{\alpha\rho} = \delta^\beta_\rho \ (3)$$

one has

$$\sum_{\alpha=1}^{M}\left(\frac{\partial}{\partial q^h} A^{N+\alpha,N+\beta}\right) C_{\alpha\rho} = -\sum_{\alpha=1}^{M} A^{N+\alpha,N+\beta}\, \frac{\partial C_{\alpha\rho}}{\partial q^h}\ , \tag{3.3}$$

the expression in (3.3) coincides with

$$\frac{1}{2}\sum_{\alpha,\beta=1}^{M} \frac{\partial C_{\alpha,\beta}}{\partial q^h}\, \dot{\gamma}^\alpha\, \dot{\gamma}^\beta\ . \tag{3.4}$$

By (3.4), the coefficient of $\dot{\gamma}^\alpha \cdot \dot{\gamma}^\beta$ in $\frac{-\partial\mathcal{H}}{\partial q^h}$ is given by

$$\frac{1}{2}\frac{\partial C_{\alpha,\beta}}{\partial q^h}\ . \tag{3.5}$$

Then, for every $h=1,\dots,N$, the term $\frac{-\partial\mathcal{H}}{\partial q^h}$ is linear in the $\dot{\gamma}^\alpha$ if and only if the identities

(3.6) $$\frac{\partial C_{\alpha,\beta}}{\partial q^h} = 0$$

hold, $\alpha,\beta = 1,\ldots,M$.
By the equality

$$\sum_{\alpha=1}^{M} A^{N+\rho,N+\alpha}\, C_{\alpha\beta} = \delta^{\rho}_{\beta} \qquad \rho, \beta = 1, \ldots, M ,$$

the identities (3.6) are equivalent to the identities (3.1). **QED**

Theorem 3.2. *Let us assume hypothesis H.2 on the forces. Then a coordinate system* (q^i,γ^α), $i = 1, \ldots, N$, $\alpha = 1, \ldots, M$, *is strongly* M*-fit if and only if the identities*

(3.7) $$\frac{\partial A_{N+\alpha,N+\beta}}{\partial q^h} \equiv 0, \quad A_{N+\rho,h} \equiv 0 \qquad \begin{array}{l}\alpha, \beta = 1,\ldots,M \\ h = 1,\ldots,N\end{array}$$

hold on the range of (q^i,γ^α) .

This Theorem can be proved by imposing that the coefficients of the $\dot{\hat{\gamma}}^\alpha$ and $\dot{\hat{\gamma}}^\beta\dot{\hat{\gamma}}^\delta$ in (2.14) vanish identically, for all $\alpha, \beta, \delta = 1, \ldots, M$. By a different approach, it has been proved in [11] (Theorem 4.1)

Thanks to Theorem 3.1, we can introduce an equivalence relation on the set of M-fit coordinate systems. Indeed, let us define an equivalence relation E on the family of coordinate systems for $\mathcal{M}$.

Definition 3.2. *We say that the coordinate systems* (q,γ) , $(\tilde{q},\tilde{\gamma})$ *are* E*-equivalent, and we write* $(q,\gamma)E(\tilde{q},\tilde{\gamma})$, *if*

i) (q,γ) , $(\tilde{q},\tilde{\gamma})$ *have the same domain, and*

ii) $\dfrac{\partial \tilde{\gamma}^\alpha}{\partial q^i} \equiv 0$, $\forall i = 1, \ldots, N$, $\forall \alpha = 1, \ldots, M$.

Then, one has the following proposition.

Proposition 3.1. E *induces an equivalence relation on the set of* M-*fit coordinate systems.*

Proof. The inverse $(\tilde{A}^{\mathcal{RS}})$ of the kinetic matrix in the coordinates $(\tilde{q},\tilde{\gamma})$ is obtained from the corresponding matrix $(A^{\mathcal{RS}})$ in the coordinates (q,γ) by the following transformation rule: $\tilde{A}^{\mathcal{RS}} = \sum_{\mathcal{L},\mathcal{M}=1}^{D} \frac{\partial \tilde{\chi}^{\mathcal{R}}}{\partial \chi^{\mathcal{L}}} \frac{\partial \tilde{\chi}^{\mathcal{S}}}{\partial \chi^{\mathcal{M}}} A^{\mathcal{LM}}$, $\mathcal{R},\mathcal{S}=1,\ldots,D(\equiv N+M)$. Hence, by theorem 3.1, $(\tilde{q},\tilde{\gamma})$ is M-fit whenever $(\tilde{q},\tilde{\gamma})E(q,\gamma)$ and (q,γ) is M-fit. This yields the thesis, since E is an equivalence relation on the family of coordinate systems for $\mathcal{M}$.

Q.E.D.

Putting off to Section 5 the investigation on the existence problem of M-fit coordinate systems for $M \geq 1$, in the following section we are examining the case $M = 1$.

4. Scalar controls

Let us investigate the case M=1, i.e., to say, let us assume that the variable γ is scalar.
One can consider *locally geodesic coordinates* (q^i,γ), $i = 1, \ldots, N$, which are characterized by the fact that the local expression $(A_{\mathcal{R}\mathcal{S}})$ of the metric tensor satisfies the conditions

$$A_{N+1,\mathcal{R}}(q^i,\gamma) = \delta_{N+1,\mathcal{R}}{}^{(*)}, \qquad \mathcal{R} = 1,\ldots,N+1,$$

identically on the range of (q^i,γ). Then, theorems 3.1-3.2 imply that,

under the assumption **H.1** *[resp.* **H.2***] on the forces, each system of locally geodesic coordinates is* 1-fit *[resp. strongly* 1-fit].

From the above remark one obtains that *there exist infinitely many 1-fit systems of coordinates.*

Indeed, at every point $m \in \mathcal{M}$ and for every vector $v \in T_m\mathcal{M}$, one can find a neighborhood $\mathcal{U}$ of m and a system of locally geodesic coordinates (q^i,γ) on $\mathcal{U}$ such that $\frac{\partial}{\partial\gamma}(m) = v$. These coordinate systems do not fill the family of 1-fit coordinate systems; still, any element of this family may be characterized as the image of a (suitable) system of locally geodesic coordinates (q^i,γ), under a diffeomorphism which sends hypersurfaces $\gamma = k \in \mathbf{R}$ into hypersurfaces of the same kind. More precisely the following result holds.

Theorem 4.1 *Let us assume hypothesis H.1 on the forces, and let* $(\tilde{q},\tilde{\gamma})$ *be a 1-fit system of coordinates with range* $\tilde{U}\times\tilde{I}$, $\tilde{U}\subseteq\mathbb{R}^N$, $\tilde{I}\subseteq\mathbb{R}$.
Then, for each $c \in \tilde{I}$, *there exist an open subset* $\tilde{\Omega}_c \subseteq \tilde{U}\times\tilde{I}$, *intersecting the hypersurface* $H_c = \{ (\tilde{q},c) \mid q \in \tilde{U}\}$, *and a diffeomorphism*

$$f = (f^1,\ldots,f^{N+1}) : \tilde{\Omega}_c \longrightarrow \Omega_c \quad (\equiv f(\tilde{\Omega}_c))$$

$$(\tilde{q}^i,\tilde{\gamma}) \longmapsto (q^i,\gamma) \quad (\equiv f(\tilde{q},\tilde{\gamma}))$$

such that

(*) Here and henceforth $\delta_{a,b} = \delta^{a,b} = 1$ if a=b, and $\delta_{a,b} = \delta^{a,b} = 0$ if $a \neq b$.

i) f *induces the identity on* H_c ,

ii) the equalities

$$\frac{\partial \gamma}{\partial \tilde{q}^i} = \frac{\partial f^{N+1}}{\partial \tilde{q}^i} = 0 \tag{4.1}$$

hold identically on $\tilde{\Omega}_c$, $\forall i = 1, \ldots, N$, *and*

iii) (q,γ) *is a system of locally geodesic coordinates .*

Conversely, if (q,γ) *is a system of locally geodesic coordinates with range* $\Omega \subseteq \mathbb{R}^N \times \mathbb{R}$ *and*

$$g = (g^1, \ldots, g^N, g^{N+1}) : \Omega \longrightarrow \tilde{\Omega}\ [=g(\Omega)]$$
$$(q,\gamma) \longmapsto (\tilde{q},\tilde{\gamma})[=g(q,\gamma)]$$

is a diffeomorphism such that

$$\frac{\partial \tilde{\gamma}}{\partial q^i} = \frac{\partial g^{N+1}}{\partial q^i} = 0 \quad \forall i = 1, \ldots, N(= D - 1) , \tag{4.2}$$

identically on Ω, *then* $(\tilde{q}, \tilde{\gamma})$ *is a system of 1-fit coordinates.*

Proof. The second part of the thesis is trivial. Indeed, let $g:\Omega \rightarrow \Omega'$ be a diffeomorphism satisfying (4.2), and let $A^{-1}=[A^{\mathcal{RS}}(q^i,\gamma)]_{\mathcal{RS}=1,\ldots,D}$ be the inverse of the matrix $A=[A_{\mathcal{RS}}(q^i,\gamma)]_{\mathcal{RS}=1,\ldots,D}$, representing the metric tensor in the coordinates $(q^i,\gamma)[\equiv(\chi^{\mathcal{R}})]$. Then,g transforms A^{-1} into the matrix $(A')^{-1}$, of components

$$A'^{\mathcal{RS}}(q'^i, \gamma') = \sum_{\mathcal{P},\mathcal{Q}=1}^{D} \frac{\partial \chi'^{\mathcal{R}}}{\partial \chi^{\mathcal{P}}} \frac{\partial \chi'^{\mathcal{S}}}{\partial \chi^{\mathcal{Q}}} A^{\mathcal{PQ}}(g^{-1}(q'^i, \gamma')) . \tag{4.3}$$

Since the chart $(W,(q^i,\gamma))$ is locally geodesic, one has $A^{D\mathcal{Q}}= \delta^{D\mathcal{Q}}$ identically on Ω, for $\mathcal{Q}= 1,\ldots,D$. Hence,by (4.2) and (4.3) one obtains

$$A'^{DD}(\gamma') = \left(\frac{\partial\gamma'}{\partial\gamma}\right)^2, \quad \frac{\partial A'^{DD}}{\partial q'^h} = 0 .$$

Then Corollary 3.1 allows to conclude that $(W,(q'^i,\gamma'))$ is 1-fit.

Now, let us prove the first part of the thesis. If $h : \Omega_C' \to \Omega_C$ is any diffeomorphism, $[\,(q^i,\gamma) \equiv\,](\chi^{\mathcal{R}}) := h(\chi'^{\mathcal{R}})$, then the matrix $(A')^{-1} = [A'^{\mathcal{RS}}(q^{i\prime},\gamma')]_{\mathcal{R},\mathcal{S}=1,\dots,D}$ is transformed by h into the matrix $(A)^{-1} = [A^{\mathcal{RS}}(q^i,\gamma)]_{\mathcal{R},\mathcal{S}=1,\dots,D}$, whose elements are given by

$$A^{\mathcal{RS}}(q^i,\gamma) = \sum_{P,Q=1}^{D} \frac{\partial\chi^{\mathcal{R}}}{\partial\chi'^{\mathcal{P}}}\,\frac{\partial\chi^{\mathcal{S}}}{\partial\chi'^{\mathcal{Q}}}\,A'^{\mathcal{PQ}}(h^{-1}(q^i,\gamma)) \; . \tag{4.4}$$

By using (4.4), one can rapidly check that

A) *if there exists a D-ple of maps* $(f^1,\dots,f^N,f^D)$ *such that:*

a) $f^1,\dots,f^N$, *and* f^D *are defined on an open subset* W'_C, *w hich intersects the hypersurface* H_c;

b) *for each* $i=1,\dots,N(=D-1)$ f^i *solves the Dirichlet problem*

$$(4.5)_i \qquad \begin{cases} \displaystyle\sum_{j=1}^{N} \frac{\partial\phi}{\partial q'^j} A'^{Dj}(q'^h,\gamma') + \frac{\partial\phi}{\partial\gamma'} A'^{DD}(\gamma') = 0 \\ \phi(q'^h,\gamma') = q'^i \quad \text{on} \quad H_c\;; \end{cases}$$

c) f^D *solves the Cauchy problem*

$$\begin{cases} \dfrac{d\zeta}{d\gamma'} = |A'^{DD}(\gamma')|^{-\frac{1}{2}} \\ \zeta(c) = c\;. \end{cases} \tag{4.6}$$

d) *the Jacobian matrix* $\partial(f^1,\dots,f^N,f^D)/\partial(q'^i,\gamma')$ *has full rank on* Ω'_C *; then, setting* $f=:(f^1,\dots,f^N,f^D):\Omega'_C \longrightarrow \Omega_C$ $(=: f(\Omega'_C))$, *the diffeomorphism* f *satisfies* i) *to* iii).

Indeed,if $(\chi^{\mathcal{R}})=(q^i,\gamma)=:(f^1,\ldots,f^N,f^D)(q'^i,\gamma')$ satisfy a), b), c), d) one has, for i=1,...,N,

$$(4.7)\qquad \frac{\partial\gamma}{\partial q'^i}=0\ ,\ \frac{\partial\gamma}{\partial\gamma'}\left(\sum_{j=1}^{N} A'^{Nj}\frac{\partial q^i}{\partial\gamma'}+A^{NN}\frac{\partial q^i}{\partial\gamma'}\right)=0\ ,\left(\frac{\partial\gamma}{\partial\gamma}\right)^2 A^{NN}=1\ ,$$

and $f(q'^1,\ldots,q'^N,c)=(q'^1,\ldots,q'^N,c),\quad \forall(q'^1,\ldots,q'^N,c)\in\Omega'_c$.

Since $(4.7)_1$,(4.4) imply that, $\forall i\in\{1,\ldots,N\}$, the right-hand side of $(4.7)_2$, $[(4.7)_3]$ coincide with A^{Di} $[A^{DD}]$, the (q^i,γ) turn out to be locally geodesic.
Let us observe that the possibility of restricting the choice of the transformations to those satisfying $(4.7)_1$ is equivalent to the condition $A'^{DD}=A'^{DD}(\gamma')$, i.e., to the 1-fitness of the coordinates (q'^1,γ') (see Theorem 3.1).
Then it remains to verify that a D-ple as in A) actually exists.
Since $(A')^{-1}$ is positive definite, the condition $A'^{DD}\neq 0$ is satisfied at any point of Ω'_c. This implies that the vector field of components $(A'^{D\mathcal{R}})_{\mathcal{R}=1,\ldots,D}$(in the coordinates (q'^1,γ')) is transversal to H_c. Therefore (see e.g. [4]), for each i=1,...,N, the problem $(4.5)_i$ has a local solution f^i. Moreover the (assumed) uniqueness for the solution of the differential system for the characteristic lines

$$(4.8)\qquad \frac{dy^{\mathcal{R}}}{ds}=A'^{D\mathcal{R}}(y(s))\quad \mathcal{R}=1,\ldots,D$$

implies that the f^i's are independent, i.e. the rank of the NxD Jacobian matrix $\partial(f^i)/\partial(\chi'^{\mathcal{R}})$ equals N at each point in which, $\forall i=1\ldots N$, f^i is defined.
Since A'^{DD} is (supposed) continuous, by $A'^{DD}\neq 0$ it follows that (4.6) has a local solution f^D and the functions $f^1,\ldots,f^N,f^D$ satisfy the required rank condition d).This concludes the proof.

QED

When the strong fitness of $(\tilde{q},\tilde{\gamma})$ is assumed, a sharper result can be obtained. In fact, in the following theorem we are proving that $(\tilde{q},\tilde{\gamma})$ may be characterized as the image of a system of locally geodesic coordinates (q,γ), under a diffeomorphism of the form $\tilde{q}=\tilde{q}(q)$, $\tilde{\gamma}=\tilde{\gamma}(\gamma)$.

Theorem 4.2. *Let us assume hypothesis H.2 on the forces acting on* Σ, *and let* $(\tilde{q},\tilde{\gamma})$ *be a system of strongly 1-fit coordinates . Let the* $(\tilde{q},\tilde{\gamma})$ *take values in* $\tilde{U}\times\tilde{I}$ *, where* $\tilde{U}\subseteq\mathbb{R}^N$ *, and* $\tilde{I}\subseteq\mathbb{R}$ *is an interval of* $\mathbb{R}$ *.*

Then, for each $c \in \tilde{I}$ *, there exists a diffeomorphism*

$$f^{N+1} : \tilde{I} \longrightarrow I \;\; (\equiv f^{N+1}(\tilde{I}))$$

$$\tilde{\gamma} \longmapsto \gamma \;\; (\equiv f^{N+1}(\tilde{\gamma}))$$

such that

i) $f^{N+1}(c) = c$ *, and*

ii) $(q,\gamma) := (\tilde{q},\gamma(\tilde{\gamma}))$ *is a system of locally geodesic coordinates.*

Conversely, if (q^i,γ) *is a system of locally geodesic coordinates taking values in* $U \times I \subseteq \mathbb{R}^N \times \mathbb{R}$ *, and* g^{N+1} *is a diffeomorphism from* I *onto* $\tilde{I}$ $(\subseteq \mathbb{R})$ *, then, for every diffeomorphism* $(g^1, ..., g^N)$ *of* U *onto itself, the system* $(\tilde{q},\tilde{\gamma})$ *of coordinates defined by* $(\tilde{q},\tilde{\gamma}) = (g^1(q),...,g^N(q),g^{N+1}(\gamma))$ *is strongly 1-fit.*

Proof. The second part of the thesis is trivial. Indeed, let g^{N+1} and $(g^1, ..., g^N)$ be as in the hypothesis and let $(A^{\mathcal{RS}}(q,\gamma))_{\mathcal{R},\mathcal{S}=1,...,N+1}$ be the local representation of the tensor field $\mathbf{A}^{-1}$ (see sect. 2) in the coordinates $(q^i,\gamma)_{i=1,...,N}$. Then, in the coordinates $(\tilde{\chi}^{\mathcal{R}})_{\mathcal{R}=1,...,N+1} \equiv (\tilde{q}^i,\tilde{\gamma})_{i=1,...,N} := g(q,\gamma)$ $(\equiv (g^1, ..., g^{N+1})\,(q,\gamma))$, $\mathbf{A}^{-1}$ is locally expressed by

$$\tilde{A}^{\mathcal{RS}}(\tilde{q},\tilde{\gamma}) = \sum_{\mathcal{L},\mathcal{M}=1}^{D} \frac{\partial \tilde{\chi}^{\mathcal{R}}}{\partial \chi^{\mathcal{L}}} \frac{\partial \tilde{\chi}^{\mathcal{S}}}{\partial \chi^{\mathcal{M}}} A^{\mathcal{LM}}(g^{-1}(\tilde{q},\tilde{\gamma})) \;, \qquad \mathcal{R}, \mathcal{S} = 1,\ldots,N+1 \;.$$

Because of the hypotheses on the coordinates (q,γ) and the diffeomorphism g , it follows that

$$\tilde{A}^{N+1,i} = \sum_{j=1}^{N} \frac{\partial \tilde{\gamma}}{\partial \gamma} \frac{\partial \tilde{q}^i}{\partial q^j} A^{N+1,j} \qquad \forall i = 1, ..., N \text{, and}$$

$$\tilde{A}^{N+1,N+1} = \left(\frac{\partial \tilde{\gamma}}{\partial \gamma}\right)^2 .$$

Then,

$$\frac{\partial \tilde{A}^{N+1,N+1}}{\partial \tilde{q}^i} \left(\equiv \frac{\partial}{\partial \tilde{q}^i} \left(\frac{\partial \tilde{\gamma}}{\partial \gamma}\right)^2 \right) = 0 \qquad \forall i = 1, ..., N \;,$$

so that

$$\tilde{A}_{N+1,i} = 0 = \frac{\partial \tilde{A}_{N+1,N+1}}{\partial \tilde{q}^i}, \ \forall i = 1, ..., N .$$

Hence, by theorem 3.2, the system of coordinates $(\tilde{q}^i,\tilde{\gamma})_{i=1,...,N}$ is 1-fit.

Now, let us prove the first part of the thesis.
If $f : (\tilde{q}^1, ..., \tilde{q}^N, \tilde{\gamma}) \longmapsto f(\tilde{q}^1, ..., \tilde{q}^N, \tilde{\gamma}) := (q^1, ..., q^N, \gamma) \ (\equiv(\chi^1, ..., \chi^{N+1}))$ is any diffeomorphism, in the coordinates (q,γ) the components of the tensor field A^{-1} are given by the entries of the matrix

(4.9) $$(A^{\mathcal{R}\mathcal{S}}(q,\gamma))_{\mathcal{R},\mathcal{S}=1,...,N+1} = \left(\frac{\partial\chi^{\mathcal{R}}}{\partial\tilde{\chi}^{\mathcal{L}}}\frac{\partial\chi^{\mathcal{S}}}{\partial\tilde{\chi}^{\mathcal{M}}}\tilde{A}^{\mathcal{L}\mathcal{M}}(f^{-1}(q,\gamma))\right)_{\mathcal{R},\mathcal{S}=1,...,N+1}.$$

Let $f^{N+1} : \tilde{\gamma} \longmapsto f^{N+1}(\tilde{\gamma})$ be a diffeomorphism defined on I, and let us specify f by setting $f(\tilde{q},\tilde{\gamma}) := (\tilde{q}, f^{N+1}(\tilde{\gamma}))$. Then (3.6) yields

(4.10)
$$A^{N+1,N+1}(\gamma) = \left(\frac{\partial f^{N+1}}{\partial\tilde{\gamma}}\right)^2 \cdot \tilde{A}^{N+1,N+1}$$
$$A^{N+1,i} \equiv 0 .$$

Since the matrix $(\tilde{A}_{i,j})_{i,j=1,...,N}$ [resp. $(\tilde{A}_{\mathcal{R},\mathcal{S}})_{\mathcal{R},\mathcal{S}=1,...,N+1}$] is positive definite , its determinant a [resp. $\mathcal{A}$] is positive. It follows that the function $\tilde{A}^{N+1,N+1}(\tilde{q},\tilde{\gamma}) = \frac{a}{\mathcal{A}}$ is (differentiable and) nowhere vanishing, so that the map $(\tilde{A}^{N+1,N+1})^{-1/2}$ is regular. Hence, the primitive $f^{N+1}(\tilde{\gamma}) = c + \int_c^{\tilde{\gamma}} (\tilde{A}^{N+1,N+1})^{-1/2} ds$ can be defined on $\tilde{I}$, and, by (4.10), one obtains

(4.11) $$A^{N+1,\mathcal{R}}(q,\gamma) \equiv \delta^{N+1,\mathcal{R}} , \qquad \forall\, \mathcal{R} = 1, ..., N + 1.$$

Since (4.11) implies $A_{N+1,\mathcal{R}} = \delta_{N+1,\mathcal{R}}$, $\forall\ \mathcal{R} = 1, \ldots, N+1$, the coordinates $(q^1,\ldots,q^N,\gamma) := (\tilde{q}^1,\ldots, \tilde{q}^N, f^{N+1}(\tilde{\gamma}))$ turn out to be geodesic.

QED.

5. $\mathcal{U}$-M-fit foliations

Let Σ be a mechanical system as in Section 2, and let us assume the hypothesis H.1 on the forces applied to Σ. Aim of this section is to provide a geometrical and kinematical interpretation to condition (3.1), which characterizes an M-fit coordinate system. The results concerning the case $M = 1$ have been presented in the previous section. In particular, we have seen that there exist infinitely many 1-fit coordinate systems for Σ and, by theorem 4.1, each of them is suitably related to a system of locally geodesic coordinates. On the contrary, when $M > 1$, M-fit coordinate systems for Σ could fail to exist, since, in principle, (3.1) might be false, for each choice of the local parametrization (q,γ).

As a first step to attack this existence problem, we shall replace the notion of an M-fit coordinate system with the notion of an $\mathcal{U}$-M-fit foliation, which, in particular, has a chart-independent character. Afterwards, we shall show (see theorem 5.2) that condition (3.1) on the coordinates (q,γ) means that the kinetic metric A is *bundle-like* (see definition 5.4) with respect to the foliation formed by the submanifolds $\gamma = c \in \mathbb{R}^M$.

As a consequence, the existence of $\mathcal{U}$-M-fit foliations – and hence of M-fit coordinate systems– will be related to the behavior of the geodesics of $\mathcal{M}$ (see corollary 5.1). Since these geodesics coincide with the so called *spontaneous motions* of Σ, the above relationship can be read from a kinematical point of view. Lastly, theorem 5.3 shows that further integrability hypotheses are needed in order to guarantee the existence of strongly M-fit coordinate systems.

To begin with, let us briefly recall some elementary facts from differential geometry of foliations (see e.g. [22],[35]).

Here, by a *distribution* on a (D-dimensional) manifold $\mathcal{M}$, we mean a map $\Delta : m \longmapsto \Delta_m$ which associates a subspace $\Delta_m < T_m\mathcal{M}$ to each point $m \in \mathcal{M}$. We shall consider only *nonsingular* distributions, i.e., distributions Δ such that Δ_m has constant dimension $N \leq D$, when m varies on $\mathcal{M}$. Moreover, we shall tacitly assume that Δ is differentiable, which means that every point $m \in \mathcal{M}$ has a neighborhood $\mathcal{V}$ and N differentiable vector fields on $\mathcal{V}$, which form a basis of Δ_n at every $n \in \mathcal{V}$. The number N [resp. $M = D - N$] , is called the *dimension* [resp. *codimension*] of the distribution Δ. A nonsingular distribution Δ of dimension N is said to be integrable if, for each $m \in \mathcal{M}$, one can find a neighborhood $\mathcal{U}$ of m with coordinates functions $(q^1, ..., q^N, \gamma^1,..., \gamma^M)$ such that

$$\Delta_n = \text{span}\left\{ \frac{\partial}{\partial q^1}(n),\ \dots,\ \frac{\partial}{\partial q^N}(n)\right\},$$

for every $n \in \mathcal{U}$. Such systems of coordinates are called Δ-*adapted* or *adapted to* Δ; the subsets $\mathcal{S}_c = \{n \in \mathcal{M} \mid \gamma(n) = c\}$, $c \in \gamma(\mathcal{U})$, are the *slices* of Δ , and the family $\{\mathcal{S}_c\}_{c \in \gamma(\mathcal{U})}$ is said to be the *foliation of* $\mathcal{U}$ *defined by the distribution* Δ .
A global version of this concept is given by means of the notion of an integral manifold. A connected submanifold $\mathcal{N}$ of $\mathcal{M}$ is an *integral manifold* of Δ if $f_*(T_n\mathcal{N}) = \Delta_n$ for all $n \in \mathcal{N}$, where f is the imbedding of $\mathcal{N}$ in $\mathcal{M}$, and f_* is its differential. If there is no other integral manifold $\mathcal{P}$ which contains $\mathcal{N}$, $\mathcal{N}$ is called a *maximal integral manifold of* Δ . A distribution Δ on $\mathcal{M}$ is said to have the *maximal integral manifolds property* if through every point $m \in \mathcal{M}$ passes a (unique) maximal integral manifold.
The partition of $\mathcal{M}$ formed by the maximal integral manifolds of Δ is called the *foliation of* $\mathcal{M}$ *defined by* Δ. Given any $m \in \mathcal{M}$, the (unique) maximal integral manifold containing m will be called the *leaf through* m .

A necessary condition for the integrability of a distribution is involutivity: a distribution Δ is *involutive* if, whenever two vector fileds $\mathbf{X}$, $\mathbf{Y}$ belong to Δ (i.e., $\mathbf{X}(m),\mathbf{Y}(m) \in \Delta_m$ $\forall m \in \mathcal{M}$), then $[\mathbf{X},\mathbf{Y}] \in \Delta$, where $[\mathbf{X},\mathbf{Y}]$ denotes the *Lie bracket* of $\mathbf{X}$ and $\mathbf{Y}$. Involutivity is easily seen to be also sufficient for the integrability of a distribution, as it is stated by the classical theorem of Frobenius.

Frobenius theorem. *Let* Δ *be an involutive nonsingular distribution on* $\mathcal{M}$ *. Then* Δ *is integrable. Furthermore, it has the maximal integral manifolds property.*

Remark. If a system of coordinates (q,γ) is Δ-adapted , and the functions $\tilde{q} = \tilde{q}(q,\gamma)$, $\tilde{\gamma} = \tilde{\gamma}(q,\gamma)$ satisfy

$$(*) \qquad \frac{\partial \tilde{\gamma}^\alpha}{\partial q^i} = 0, \ \ \alpha = 1, \ldots, M\ , \ \ i = 1, \ldots, N,$$

then the system $(\tilde{q},\tilde{\gamma})$ is easily seen to be Δ-adapted. Provided that the domains of the $(\tilde{q},\tilde{\gamma})$ and (q,γ) coincide, the identities (*) imply that (q,γ), $(\tilde{q},\tilde{\gamma})$ are E-equivalent.(see definition 3.2). In other words, given an M-fit system of coordinates (q,γ) on an open subset $\mathcal{U}$ of $\mathcal{M}$, the E-equivalence class $\{(\tilde{q},\tilde{\gamma}) \mid (\tilde{q},\tilde{\gamma})\, \mathbf{E}(q,\gamma)\}$ coincides with the family of coordinate systems which are defined on $\mathcal{U}$ and are adapted to the distribution $\Delta : m \to \text{span}\left\{\frac{\partial}{\partial q^1}(m), \ldots, \frac{\partial}{\partial q^N}(m)\right\}$.

Definition 5.1. *Let $\mathcal{M}$ be a* D*-dimensional Riemannian manifold and let $\mathcal{U}$ be an open subset of $\mathcal{M}$. A foliation of $\mathcal{U}$ defined by a (involutive) distribution Δ of codimension* M *is said to be $\mathcal{U}$ - M -fit if there exists a system of coordinates on $\mathcal{U}$ which is Δ-adapted and* M*-fit.*

The previous remark implies immediately the following proposition.

Proposition 5.1. *Let $\mathcal{M}$, $\mathcal{U}$,* and Δ *be as in Definition 5.1. The foliation defined by Δ is $\mathcal{U}$-M-fit if and only if every Δ-adapted system of coordinates on $\mathcal{U}$ is* M*-fit.*

Definition 5.2. *Let $\mathcal{M}$ and Δ be as in Definition 5.1. We say that the foliation of $\mathcal{M}$ defined by Δ is* M*-fit if it is $\mathcal{U}$-M-fit for every $\mathcal{U} \subseteq \mathcal{M}$ on which a system of Δ-adapted coordinates may be defined.*

In order to study the geometry of $\mathcal{U}$ -M-fit foliations, we shall use the notion of a *bundle-like metric* (see [34]). For this purpose, one needs to define the concept of an *orthogonal vector* and an *orthogonal distribution.*

Definition 5.3. *Let Δ be an involutive (nonsingular) distribution of codimension* $M = D - N$, $M \geq 1$, *defined on a* D*-dimensional manifold $\mathcal{M}$, and let* A *be a metric on $\mathcal{M}$. If $m \in \mathcal{M}$, a vector $X \in T_m\mathcal{M}$ will be called orthogonal to Δ if it is orthogonal to every vector of Δ_m, i.e.,* $A(m)(X,Y) = 0$, $\forall Y \in \Delta_m$.
The distribution which, at every $m \in \mathcal{M}$, associates the subspace $\Delta_m^{\perp}$ of vectors orthogonal to Δ will be called orthogonal to Δ. This distributionwill be denoted by $\Delta^{\perp}$.

Definition 5.4 (Reinhart, see [34]). *Let $\mathcal{M}$, Δ, and* A *be as in definition 5.3. The metric* A *is said to be bundle-like with respect to the foliation defined by Δ if, for every system of Δ-adapted coordinates (q,γ), it has a local representation of the form*

$$(5.1) \qquad \sum_{i,j=1}^{N} A_{ij}(q,\gamma)\, \omega^i \otimes \omega^j + \sum_{\alpha,\beta=1}^{M} A_{\alpha\beta}(\gamma)\, d\gamma^\alpha \otimes d\gamma^\beta \ ,$$

where, $\omega^1, \ldots, \omega^N$ are (independent) covector fields such that, for each $m \in \mathcal{M}$:

i) $(\omega^1(m), \ldots, \omega^N(m), d\gamma^1(m)\ldots, d\gamma^M(m))$ *is a basis for the cotangent space* $T^*_m\mathcal{M}$;

ii) $(\omega^i(m), X)_m = 0$, *for every* $X \in \Delta^{\perp}_m$, *and for every* $i = 1, \ldots, N$.

Henceforth a curve on $\mathcal{M}$ will be called *orthogonal* to a foliation $\mathcal{F}$ if all its tangent vectors are orthogonal to the distribution defining $\mathcal{F}$.
An intuitive characterization of bundle-like metrics is supplied by the following theorem, which is due to Reinhart (see e.g.[35] pp. 155-156).

Theorem 5.1. *Let* $\mathcal{F}$ *be a foliation on* $\mathcal{M}$. *The following conditions are equivalent*

i) A *is bundle-like with respect to a foliation* $\mathcal{F}$;

ii) *If one of the tangent vectors of a geodesic* c *is orthogonal (to the distribution defining* $\mathcal{F}$*), then* c *is orthogonal to* $\mathcal{F}$.

Sometimes ii) is shortly referred to by saying that the orthogonal bundle is *totally geodesic.*

The following result connects the notion of a bundle-like metric with the notion of an $\mathcal{U}$-M-fit foliation.

Theorem 5.2. *Let* $\mathcal{M}$ *be the D-dimensional configuration manifold of a mechanical system* Σ *subjected to holonomic, bilateral, time-independent, ideal constraints, and let* A *be the metric on* $\mathcal{M}$ *which defines the kinetic energy of* Σ. *Moreover, let the forces applied to* Σ *satisfy hypothesis* H.1 *(see sect. 3). Let* Δ *be an* M-*codimensional* ($M \geq 1$), *integrable distribution on* $\mathcal{M}$ *which defines a foliation* $\mathcal{F}$.
If $\mathcal{U}$ *is an open subset of* $\mathcal{M}$ *on which a* Δ*-adapted system of coordinates can be defined, then the following are equivalent:*

i) $\mathcal{F}$ *is* $\mathcal{U}$-M-*fit:*

ii) *The restriction of the metric* A *to the subset* $\mathcal{U}$ *is bundle-like with respect to* $\mathcal{F}$.

The proof of this theorem will be given in the next section. Theorems 5.1-5.2 yield:

Corollary 5.1. *Let $\mathcal{F}$ and Δ be as in theorem 5.2. Then $\mathcal{F}$ is $\mathcal{U}$-M-fit if and only if each geodesic having one tangent vector orthogonal to Δ is orthogonal to $\mathcal{F}$.*

This result has a direct physical interpretation, since geodesics relative to the kinetic metric A coincide with the so called *spontaneous motions* of Σ, i.e., those motions which occur when Σ is acted on only by the constraints.

Let us conclude this section by considering the problem of the existence of strongly M-fit coordinate systems.
If one adds H.2 to the assumptions in theorem 4.1, then one obtains that, within each E-equivalence class of 1-fit coordinate systems defined on an open subset $\mathcal{U}\subseteq\mathcal{M}$, there exists a strongly 1-fit element, e.g. a locally geodesic system of coordinates. In other words every $\mathcal{U}$-1-fit foliation is strongly $\mathcal{U}$-1-fit. This is no longer true if one replaces 1 with $M > 1$.
More precisely, the theorem below provides a necessary and sufficient condition for the existence of strongly M-fit coordinate systems.
Before stating this result, let us recall that a submanifold $\mathcal{S}$ of a Riemannian manifold $\mathcal{R}$ is said to be *totally geodesic* if, for every $s\in\mathcal{S}$ and every $v\in T_s\mathcal{S}$, the geodesic passing through s and having v as tangent vector at s lies in $\mathcal{S}$ definitively. Moreover, let us give the following definitions.

Definition 5.5. *Let $\mathcal{M}$ be a* D*-dimensional Riemannian manifold and let $\mathcal{U}$ be an open subset of $\mathcal{M}$. A foliation of $\mathcal{U}$ defined by a (involutive) distribution Δ of codimension* M *, is said to be strongly $\mathcal{U}$-*M*-fit if there exists a system of coordinates on $\mathcal{U}$ which is Δ-adapted and strongly* M*-fit.*

Definition 5.6. *Let $\mathcal{M}$ and Δ be as in Definition 5.5. We say that the foliation of $\mathcal{M}$ defined by Δ is strongly* M*-fit if it is strongly $\mathcal{U}$-*M*-fit for every $\mathcal{U}\subseteq\mathcal{M}$ on which a system of Δ-adapted coordinates can be defined.*

Theorem 5.3. *Let Σ, $\mathcal{M}$, A, Δ, and $\mathcal{F}$ be as in theorem 5.2, and let us assume hypothesis* H.2 *(see sect. 3) on the forces applied to Σ.*
Then, for each $m\in\mathcal{M}$, the following conditions are equivalent:

i) *there exists a neighbourhood $\mathcal{U}$ of* m *such that $\mathcal{F}$ is strongly $\mathcal{U}$-*M*-fit*;

ii) *there exists a neighbourhood* $\mathcal{W}$ *of* m *, such that the restriction of the metric* **A** *to* $\mathcal{W}$ *is bundle-like with respect to* $\mathcal{F}$*, and the orthogonal distribution* $\Delta^{\perp}$ *is integrable on* $\mathcal{W}$;

iii) *there exists a neighbourhood* $\mathcal{Z}$ *of* m *, such that* $\Delta^{\perp}$ *is integrable on* $\mathcal{Z}$ *and each integral manifold of* $\Delta^{\perp}$ *is totally geodesic.*

6. Proofs of Theorems 5.2 and 5.3

In this section, theorems 5.2 and 5.3 will be proved. Two different proofs of Theorem 5.2 will be given.The first one has an algebraic character, whereas the second one is based on the dynamical characterization of bundle-like metrics provided by theorem 5.1.

First proof of Theorem 5.2. Let (q^i,γ^α) be a Δ-adapted system of coordinates on $\mathcal{U}$. As in [34], let us choose N convector fields $\omega^1,\dots,\omega^N$, and M vector fields $v_1, \dots, v_M$ such that, $\forall m \in \mathcal{U}$:

i) the $\omega^1(m), \dots, \omega^N(m)$ are zero on the orthogonal space $\Delta^\perp_m$;

ii) $\left(\omega^1(m), \dots, \omega^N(m), d\gamma^1(m), \dots, d\gamma^M(m)\right)$ is a basis for the cotangent space $T^*_m\mathcal{M}$;

iii) $\left(\frac{\partial}{\partial q^1}(m), \dots, \frac{\partial}{\partial q^N}(m), v_1(m), \dots, v_M(m)\right)$ is the dual basis for the tangent space $T_m\mathcal{M}$.

It follows that

$$\omega^i = dq^i + \sum_{\beta=1}^{M} b^i_\beta \, d\gamma^\beta , \qquad v_\alpha = \frac{\partial}{\partial\gamma^\alpha} + \sum_{j=1}^{N} a^j_\alpha \frac{\partial}{\partial q^j} ,$$

$\forall i = 1, \dots, N$, and $\forall \alpha = 1, \dots, M$.

Moreover, the metrics A and A^{-1} turn out to have the following local expressions:

$$A(m(q,\gamma)) = \sum_{i,j=1}^{N} \hat{A}_{ij}(q,\gamma)\, \omega^i \otimes \omega^j + \sum_{\alpha,\beta=1}^{M} \hat{A}_{N+\alpha,N+\beta}(q,\gamma) d\gamma^\alpha \otimes d\gamma^\beta$$

(6.1)

$$A^{-1}(m(q,\gamma)) = \sum_{i,j=1}^{N} \hat{A}^{ij}(q,\gamma) \frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial q^j} + \sum_{\alpha,\beta=1}^{M} \hat{A}^{N+\alpha,N+\beta}(q,\gamma) v_\alpha \otimes v_\beta ,$$

where $\left(\hat{A}^{\mathcal{RS}}\right)_{\mathcal{R},\mathcal{S}=1,\dots,N+M\equiv D}$ is the inverse of the matrix $\left(\hat{A}_{\mathcal{RS}}\right)_{\mathcal{R},\mathcal{S}=1,\dots,N+M\equiv D}$. In particular $\left(\hat{A}^{N+\alpha,N+\beta}\right)_{\alpha,\beta=1,\dots,M}$ is the inverse of $\left(\hat{A}_{N+\alpha,N+\beta}\right)_{\alpha,\beta=1,\dots,M}$.

Referred to the basis $\left(\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^N}, \frac{\partial}{\partial \gamma^1}, \dots, \frac{\partial}{\partial \gamma^M}\right)$, A^{-1} has an expression of the form

$$(6.2) \qquad A^{-1}(m(q,\gamma)) = \sum_{i,j=1}^{N} A^{ij}(q,\gamma)\frac{\partial}{\partial q^i}\otimes\frac{\partial}{\partial q^j} + 2\sum_{i=1}^{N}\sum_{\alpha=1}^{M} A^{i,N+\alpha}(q,\gamma)\frac{\partial}{\partial q^i}\otimes\frac{\partial}{\partial \gamma^\alpha} +$$

$$+ \sum_{\alpha,\beta=1}^{M} A^{N+\alpha,N+\beta}(q,\gamma)\frac{\partial}{\partial \gamma^\alpha}\otimes\frac{\partial}{\partial \gamma^\beta} .$$

Replacing $v^1, ..., v^M$ in $(6.1)_2$ with their expressions in terms of the basis $\left(\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^N}, \frac{\partial}{\partial \gamma^1}, \dots, \frac{\partial}{\partial \gamma^M}\right)$, and comparing the resulting expression with (6.2), one trivially obtains

$$A^{N+\alpha,N+\beta}(q,\gamma) = \hat{A}^{N+\alpha,N+\beta}(q,\gamma) , \qquad \alpha,\beta = 1, ..., M .$$

This implies that the matrix $\left(A^{N+\alpha,N+\beta}\right)_{\alpha,\beta=1,\dots,M}$ is the inverse of the matrix $\left(\hat{A}_{N+\alpha,N+\beta}\right)_{\alpha,\beta=1,\dots,M}$. Hence, the elements of the former are independent of the q^i, $i=1,\dots,N$, if and only if the same holds for the elements of the latter. This proves the theorem, since the conditions $\frac{\partial A^{N+\alpha,N+\beta}}{\partial q^i} = 0$, $\frac{\partial \hat{A}_{N+\beta,N+\beta}}{\partial q^i} = 0$, $\alpha,\beta = 1,\dots, M$, $i = 1,\dots, N$, characterize M-fit coordinates and bundle-like metrics, respectively.

Q.E.D.

Second proof of Theorem 5.2. Let $(q^1,\dots,q^N,\gamma^1,\dots,\gamma^M)$ be a Δ-adapted system of coordinates on $\mathcal{U}$. For each $m\in\mathcal{U}$, let us consider the isometry ι_m of $T_m\mathcal{M}$ onto $T^*_m\mathcal{M}$, defined in Section 2. Given a vector $X\in T_m\mathcal{M}$, let $\iota_m(X)_{\mathcal{R}}$ denote the $\mathcal{R}$-component of the 1-form $\iota_m(X)$ with repect to the basis $\left(dq^1(m), ..., dq^N(m), d\gamma^1(m), ..., d\gamma^M(m)\right)$. Let us note that the identities

$$\iota_m(X)_i = \left(\frac{\partial}{\partial q^i}(m), \iota_m(X)\right)_m = A(m)\left(\frac{\partial}{\partial q^i}(m), X\right)$$

imply that

(S) X *is orthogonal to* $\Delta_m \equiv \text{span}\left\{\frac{\partial}{\partial q^1}(m), \ldots, \frac{\partial}{\partial q^N}(m)\right\}$ *if and only if* $\iota_m(X)_i = 0$, *for* $i = 1, \ldots, N$.

As it is well-known, a geodesic of the Riemannian manifold $(\mathcal{U}, A)$ is a solution of the Hamiltonian system obtained from (2.12) by setting $Q_{\mathcal{R}} \equiv 0 \ \forall \mathcal{R} = 1, \ldots, D$. By (2.9), this Hamiltonian system has the following explicit form:

$$(6.3) \quad \begin{aligned}
\dot{q}^i &= \sum_{r=1}^{N} A^{ir} p_r + \sum_{\beta=1}^{M} A^{i,N+\beta} p_\beta \\
\dot{\gamma}^\alpha &= \sum_{r=1}^{N} A^{N+\alpha,r} p_r + \sum_{\beta=1}^{M} A^{N+\alpha,N+\beta} p_\beta \\
\dot{p}_i &= -\frac{1}{2}\sum_{r,s=1}^{N} \frac{\partial A^{rs}}{\partial q^i} p_r p_s - \sum_{r=1}^{N}\sum_{\beta=1}^{N} \frac{\partial A^{r,N+\beta}}{\partial q^i} p_r p_\beta - \frac{1}{2}\sum_{\beta,\delta=1}^{M} \frac{\partial A^{N+\beta,N+\delta}}{\partial q^i} p_\beta p_\delta \\
\dot{p}_\alpha &= -\frac{1}{2}\sum_{r,s=1}^{N} \frac{\partial A^{rs}}{\partial \gamma^\alpha} p_r p_s - \sum_{r=1}^{N}\sum_{\beta=1}^{M} \frac{\partial A^{r,N+\beta}}{\partial \gamma^\alpha} p_r p_\beta - \frac{1}{2}\sum_{\beta,\delta=1}^{M} \frac{\partial A^{N+\beta,N+\delta}}{\partial \gamma^\alpha} p_\beta p_\delta ,
\end{aligned}$$

where $i=1, \ldots, N$ and $\alpha=1, \ldots, M$.

The theorem will be proved by using theorem 5.1, where bundle-like metrics are characterized in terms of geodesics. Let us observe that, by (S), condition ii) in theorem 5.1 reads:

a) *for each initial condition*

$$(6.4) \qquad (q,\gamma,p,\mathfrak{p})(0) = (\bar{q},\bar{\gamma},0,\bar{\mathfrak{p}})$$

in the range of the coordinates of $T^*\mathcal{U}$, *the solution* $(q,\gamma,p,\mathfrak{p})(\cdot)$ *of the Cauchy Problem (6.3)-(6.4) satisfies* $p(t) \equiv 0$, *for each* t *belonging to its interval of definition.*

On the other hand, by theorem 3.1, M-fitness is characterized by

b) $\frac{\partial A^{N+\alpha,N+\beta}}{\partial q^i} \equiv 0$, for $i = 1, \ldots, N$, and $\alpha,\beta = 1, \ldots, M$.

Then, one has to prove that condition **a)** is equivalent to condition **b)**.
Let us assume **a)**. In particular, by imposing $p(t) \equiv 0$ in (6.3), one obtains

$$0 = \frac{1}{2}\frac{\partial A^{N+\alpha,N+\beta}}{\partial q^i}\bar{p}_\alpha \bar{p}_\beta \ .$$

which implies **b)**, since $(\bar{q},\bar{\gamma})$ is arbitrary in the range of the (q,γ), and $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_M)$ is arbitrary in $\mathbb{R}^M$.

Conversely, let us assume **b)**, and let $(\gamma^*(\cdot),p^*(\cdot))$ be the solution of the Cauchy problem

$$\begin{cases} \dot{\gamma}^\alpha = \sum_{\beta=1}^{M} A^{N+\alpha,N+\beta}(\gamma)\, p_\beta \\ \dot{p}_\alpha = -\frac{1}{2}\sum_{\beta,\delta=1}^{M} \frac{\partial A^{N+\beta,N+\delta}(\gamma)}{\partial \gamma^\alpha} p_\beta p_\delta \qquad \alpha = 1, \ldots, M \\ (\gamma(0),p(0)) = (\bar{\gamma},\bar{p}) \in \gamma(\mathcal{U}) \times \mathbb{R}^M . \end{cases}$$

Furthermore, let $q^*(\cdot)$ be the solution of the Cauchy problem

$$\begin{cases} \dot{q}^i(t) = \sum_{\beta=1}^{M} A^{i,N+\alpha}(q,\gamma^*(t))\, p_\alpha{}^*(t) \qquad i = 1, \ldots, N \\ q(0) = \bar{q} \in q(\mathcal{U}) \ . \end{cases}$$

Then $(q^*(\cdot),\gamma^*(\cdot),0,p^*(\cdot))$ is trivially a solution of (6.3) with initial condition $(q^*,\gamma^*,p^*,p^*)(0) = (\bar{q},\bar{\gamma},0,\bar{p})$. By the (assumed) uniqueness of the solution of a Cauchy problem for (6.3), and by the arbitrariness of $(\bar{q},\bar{\gamma},\bar{p})$, this yields **a)**.

Q.E.D.

Proof of Theorem 5.3. Let $m \in \mathcal{M}$, and let (q,γ) be a Δ-adapted, strongly M-fit system of coordinates defined on a neighbourhood $\mathcal{U}$ of m. Theorem 3.2 implies that A may be locally expressed by

$$(6.5)\qquad A(n(q,\gamma)) = \sum_{i,j=1}^{N} A_{ij}(q,\gamma)\, dq^i \otimes dq^j + \sum_{\alpha,\beta=1}^{M} A_{N+\alpha,N+\beta}(\gamma)\, d\gamma^\alpha \otimes d\gamma^\beta ,$$

where $n(q,\gamma)$ denotes the point of $\mathcal{U}$ having (q,γ) as coordinates.

It follows that $\Delta_n^\perp = \text{span}\left\{\frac{\partial}{\partial\gamma^1}(n), \ldots, \frac{\partial}{\partial\gamma^M}(n)\right\}$, for every $n \in \mathcal{U}$, i.e. $\Delta^\perp$ is integrable on $\mathcal{U}$. Hence i) implies ii), with $\mathcal{W} = \mathcal{U}$, because, by theorem 5.2, A is bundle-like with respect to $\mathcal{F}$.

Conversely, let us assume ii). Since Δ and $\Delta^\perp$ are integrable and complementary, there exist local coordinates (q,γ) on a neighbourhood $\mathcal{V}$ of m, $\mathcal{V} \subseteq \mathcal{W}$, such that $\Delta_n = \text{span}\left\{\frac{\partial}{\partial q^1}(n), \ldots, \frac{\partial}{\partial q^N}(n)\right\}$, $\Delta_n^\perp = \text{span}\left\{\frac{\partial}{\partial\gamma^1}(n), \ldots, \frac{\partial}{\partial\gamma^M}(n)\right\}$ for each $n \in \mathcal{V}$ (see e.g. [22] p. 182). Since, $\forall n \in \mathcal{V}$, the 1-forms $dq^1(n), \ldots, dq^N(n)$ are zero on $\Delta_n^\perp$, the bundle-like metric A may be locally expressed as in (6.5). Hence, by theorem 3.2, one concludes that ii) implies i), with $\mathcal{U} = \mathcal{V}$.

Let us show that ii) is equivalent to iii), with $\mathcal{U} = \mathcal{Z}$. Actually, if ii) holds, a straightforward application of theorem 5.1 implies that the leaves of $\Delta^\perp$ are totally geodesic. Conversely, if iii) holds, condition ii) in theorem 5.1 is trivially satisfied. Hence, A is bundle-like.

Q.E.D.

7. Some examples

This section is mainly devoted to provide examples of M-fit coordinate systems and M-fit foliations.The cases of spherical-polar coordinates and cylindrical coordinates for a material point, which have been already treated in [11], are here tackled from the point of view of foliations. Afterwards, longitude-latitude coordinates (for a material point constrained to move on a sphere) and the corresponding foliations are considered. Lastly, some facts concerning the constraint manifold of a rigid body are illustrated.

Before introducing these examples, let us make some simple remarks about the order and the subsets of an M-fit system of coordinates. It is trivial to observe that a system of coordinates $(q^1,...,q^N, \gamma^1,...,\gamma^M)$ is M-fit [resp. strongly M-fit] if and only if $(q^{\sigma_1},...,q^{\sigma_N},\gamma^{\nu_1},...,\gamma^{\nu_M})$ is M-fit [resp. strongly M-fit], where $(\sigma_1, ..., \sigma_N)$ and $(\nu_1,..., \nu_N)$ are arbitrary permutations of $(1, ..., N)$ and $(1, ..., M)$, respectively. Hence, given a set $\mathcal{S} = \{\chi^1, ..., \chi^D\}$ of coordinates and a subset $\mathcal{A} = \{\gamma^1, ..., \gamma^M\} \subseteq \mathcal{S}$, it does not generate any confusion to speak of the fitness [resp. strong fitness] of $\mathcal{A}$ to mean the fitness [resp. strong fitness] of $(q^1,...,q^N, \gamma^1,...,\gamma^M)$, where $\{q^1, ..., q^N\} = \mathcal{S} \setminus \mathcal{A}$. In particular, in [11] the following problem is investigated: given two subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{S} = \{\chi^1, ..., \chi^D\}$, is there any correspondence between the separate fitnesses of $\mathcal{A}$ and $\mathcal{B}$ and the fitness of $\mathcal{A} \cup \mathcal{B}$? In [11] it is proved that, in the case of positional forces, the separate strong fitnesses of $\mathcal{A}$ and $\mathcal{B}$ imply the strong fitness of $\mathcal{A} \cup \mathcal{B}$. Moreover, by means of an example, it is shown that the converse is false. More generally, on the bases of theorems 3.1, 3.2 one can state that

the separate fitnesses of $\mathcal{A}$ and $\mathcal{B}$ do not imply the fitness of $\mathcal{A} \cup \mathcal{B}$. Conversely, from the fitness of $\mathcal{A} \cup \mathcal{B}$, it does not follow either the fitness of $\mathcal{A}$ or the fitness of $\mathcal{B}$.

As in the previous sections, henceforth one of the following hypotheses on the forces will be considered:

H.1. *The forces applied to Σ are affine functions in the velocities.*

H.2 *The forces applied to Σ are positional, i.e. they depend only on time and the configuration of Σ.*

In [11], the cases of cylindrical and polar-spherical coordinates for a material point are studied. Here, thanks to the results obtained in Section 5, those examples will be re-examined from a global point of view.

EXAMPLE 1.

Let Σ be merely formed by a material point P of unit mass, and refer it to Cartesian coordinates (x,y,z). In this case $\mathcal{M} = \mathbb{R}^3$ and **A** is the Euclidean metric, i.e.

$$A(x,y,z) = dx\otimes dx + dy\otimes dy + dz\otimes dz\ .$$

In order to avoid degeneracy of the foliations which are being considered, we shall reduce $\mathcal{M}$ either to $\mathcal{M}' = \mathbb{R}^3 \setminus \{(0,0,0)\}$ or to $\mathcal{M}'' = \mathbb{R}^3 \setminus \{(0,0,z)\ ,\ z\in\mathbb{R}\}$. Then the following holds:

Proposition 7.1. *The (1-dimensional) distribution*

$$\Delta : (x,y,z) \to \text{span}\ \{-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\}$$

defines a foliation $\mathcal{F}$ *of* $\mathcal{M}''$ *which is formed by the leaves* $L_{c,k} = \{(x,y,z)\in\mathcal{M}'',\ z = c,\ x^2 + y^2 = k\}$, $c,k\in\mathbb{R}$, $k > 0$. *If H.1 is assumed, then* $\mathcal{F}$ *is* 2-fit *(see def. 5.2) . Moreover, under hypothesis H.2 on the applied forces, the foliation* $\mathcal{F}$ *is strongly 2-fit (see def. 5.6).*

Indeed, since **A** is (the restriction to $\mathcal{M}''$ of) the Euclidean metric, geodesics (i.e. spontaneous motions) are nothing but straight lines. Then it is trivial to check that Corollary 5.1 applies to the foliation $\mathcal{F}$. Moreover, the orthogonal distribution $\Delta^\perp$ has the maximal integrable manifolds property, and its leaves are the intersections of $\mathcal{M}''$ with the planes $ax + by = 0$. Hence, Theorem 5.3 yields the existence of strongly M-fit coordinates. More precisely, each coordinate system (q,γ^1,γ^2) such that $\Delta = \text{span}\left\{\frac{\partial}{\partial q}\right\}$ and $\Delta^\perp = \text{span}\ \left\{\frac{\partial}{\gamma^1},\frac{\partial}{\gamma^2}\right\}$ is strongly 2-fit.

Proposition 7.2. *Let* Γ *be the distribution defined at each point* $(x,y,z) \in \mathcal{M}'$ *by* $\Gamma(x,y,z) = \text{span}\,\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\}$. *The 2-dimensional, orthogonal distribution* $\Delta := \Gamma^{\perp}$ *has the maximal integral manifolds property.* Δ *defines a stongly 1-fit foliation formed by the spheres with center in the origin* $(0,0,0)$.

Also in this case it is straightforward to verify that $\mathcal{F}$ satisfies the assumption in Corollary 5.1. Furthermore there is no problem with the existence of strongly 1-fit coordinates, since it is guaranteed by Theorem 4.1.

Analogously, one can show that

Proposition 7.3. *The (nonsingular) distribution*

$$\Delta : (x,y,z) \to \text{span}\,\{-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$$

on $\mathcal{M}''$ *has the maximal integral manifolds property.* Δ *defines a strongly 1-fit foliation formed by the cylinders having equations* $x^2 + y^2 = c,\ c > 0$.

Remark : The above examples have global characters. On the other hand, they lead to some local consequences which have been directly proved in [11]. In particular, if (r,φ,θ) denotes a system of polar-spherical coordinates, proposition 7.2 implies that the subset $\{r,\theta\}$ is strongly 2-fit, and proposition 7.3 implies that $\{r\}$ is strongly 1-fit. Moreover, if (ρ,φ,z) denote a system of cylindrical coordinates, proposition 7.2 implies that the subset $\{\rho,z\}$ is strongly 2-fit, and proposition 7.4 implies that $\{\rho\}$ is strongly 1-fit. Similarly, by using corollary 5.1 and theorem 5.3, one can replace the following proposition (which is in [11]) with statements concerning foliations of $\mathcal{M}'$ and $\mathcal{M}''$

Proposition 7.5(see [11]). *Both* $\{\rho,\varphi\}$ *and* $\{z\}$ *are strongly fit in* (ρ,φ,z), *whereas* $\{\varphi\}$ *and* $\{\varphi,z\}$ *are not fit in* (ρ,φ,z). *Furthermore* ,$\{\varphi\}$, $\{\theta\}$, $\{r,\varphi\}$, *and* $\{\varphi,\theta\}$ *are not fit in* (r,φ,θ).

For instance, the global situation implying the lack of fitness of $\{\theta\}$ is illustrated by fig.1: it is evident that the pictured foliation (of $\mathcal{M}''$) does not satisfy the hypothesis in Corollary 5.1.

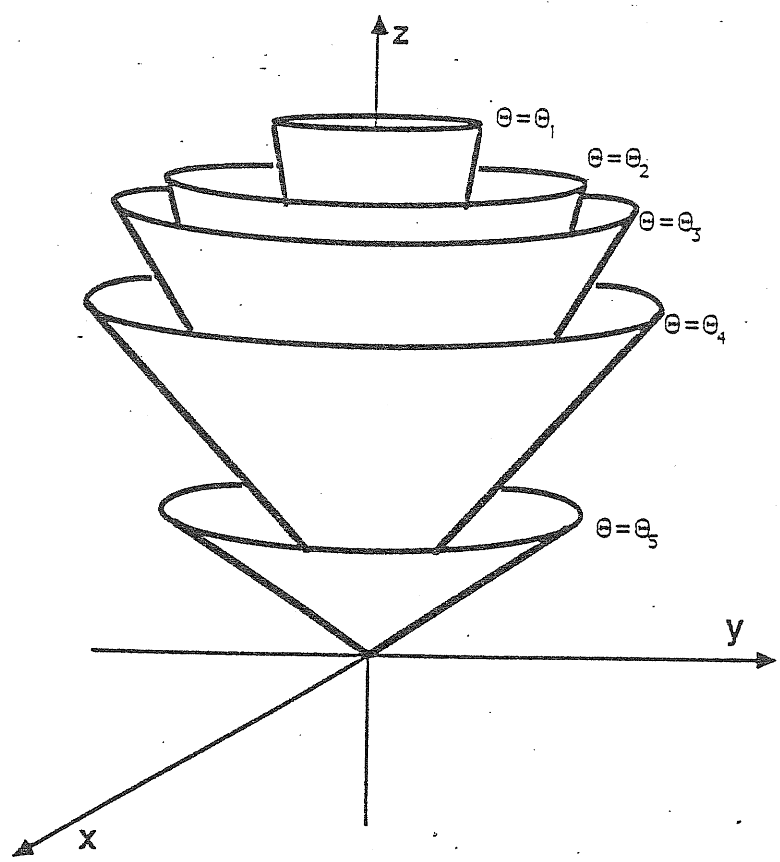

fig.1

EXAMPLE 2.

Let Σ consist of a material point P of mass m, which is constrained without friction to keep a fixed distance 1 from the origin of a Cartesian system of coordinates (x,y,z) for $\mathbb{R}^3$. In this case the constraint manifold $\mathcal{M}$ is merely the sphere $\mathcal{S}^2_r$ with center in 0, and radius r.

A pair (φ,θ), $0 < \varphi < 2\pi$, $0 < \theta < \pi$, of spherical coordinates can be defined on $\mathcal{U} = \mathcal{M} \setminus C$ where C is a closed arc of minimal lenght -i.e. an arc of *meridian*- joining (0, 0, r) with (0,0,−r). Without loss of generality, one can suppose that m and r coincide with the unit of mass and the unit of lenght, respectively. Then, in the coordinates (φ,θ), the kinetic energy $\mathcal{T}$ of Σ is expressed by

$$\mathcal{T} = \frac{1}{2}(\dot{\varphi}^2 \sin\theta + \dot{\theta}^2),$$

so that the kinetic matrix and its inverse are expressed by

$$\left(A_{\mathcal{R}\mathcal{S}}(\varphi,\theta)\right)_{\mathcal{R},\mathcal{S}=1,2} = \begin{pmatrix} \sin\theta & 0 \\ 0 & 1 \end{pmatrix}, \qquad \left(A^{\mathcal{R}\mathcal{S}}(\varphi,\theta)\right)_{\mathcal{R},\mathcal{S}=1,2} = \begin{pmatrix} \frac{1}{\sin\theta} & 0 \\ 0 & 1 \end{pmatrix},$$

respectively.
By theorem 3.1 it follows that,

under hypothesis H.1, the latitude $\{\theta\}$ *is* 1*-fit ; furthermore, if hypothesis H.2 is assumed,* $\{\theta\}$ *is strongly* 1*-fit.*

On the contrary, theorem 3.1 yields the *lack of fitness of the longitude* $\{\varphi\}$.

Moreover, theorems 3.1-3.2 imply that,

under hypothesis H.1, each coordinate system of the form $(q,\gamma) = (q(\varphi,\theta), \gamma(\theta))$ *is 1-fit . Moreover, if hypothesis H.2 is assumed, each coordinate system of the form* $(q,\gamma) = (q(\varphi,), \gamma(\theta))$ *is strongly 1-fit.*

In order to avoid degeneracy of the foliation we are going to consider, let us reduce $\mathcal{M}$ to the submanifold $\mathcal{M}' = \mathcal{M} \setminus \{(0,0,1) , (0, 0,-1)\}$. Let $\mathcal{F}$ be the 1-dimensional foliation of $\mathcal{M}'$ formed by the intersections of $\mathcal{M}'$ with the planes $z = c$, $|c| < 1$. Since a geodesic of $\mathcal{M}'$ is nothing but the intersection of $\mathcal{M}'$ with a plane passing through the origin , the foliation $\mathcal{F}$ trivially satisfies the hypotheses in Corollary 5.1. This implies that

Proposition 7.5. *$\mathcal{F}$ is a (strongly) 1-fit foliation of* $\mathcal{M}'$.

It is easy to verify that this proposition is the geometrical counterpart of the above statements on the coordinate systems of $\mathcal{M}'$.

EXAMPLE 3.A.

Let us consider a rigid body Σ . Let $I = (0, (\mathbf{i},\mathbf{j},\mathbf{k}))$ be an inertial frame and let $R = (P, (\mathbf{c}_1,\mathbf{c}_2,\mathbf{c}_3))$ be a frame in which Σ is at rest. Let us assume that i) P is fixed in I , and ii) the axes of R are aligned with the principal axes of Σ at P. Let A, B, C be the principal moments of inertia at P.

The three-dimensional constraint manifold of Σ is SO(3), which, as it is known (see e.g.[1]), may be naturally identified with the product $\mathcal{S}^2 \times \mathcal{S}^1$, $\mathcal{S}^i$, $i \in \mathbb{N}$, being the i-dimensional sphere. Eulerian angles (θ,φ,ψ), $0 \le \theta \le \pi$, $0 < \varphi \le 2\pi$, $0 < \psi \le 2\pi$, are often used as parameters: the *nutation -angle* θ is the angle between **k** and $\mathbf{c}_3$, the *precession -angle* φ is the angle between the **i**-axis and the node-line **n**, and the *rotation-angle* ψ is the angle between **n** and the $\mathbf{c}_1$-axis. Nevertheless (θ,φ,ψ) are not global coordinates, since they degenerate at $\theta = 0$ and $\theta = \pi$. In fact, the manifold $\mathcal{M} = \mathcal{S}^2 \times \mathcal{S}^1$ cannot be parametrized with only one chart. Still, Eulerian angles are (local) coordinates as soon as one restricts their domain to a suitable open subset $\mathcal{U}$ of $\mathcal{M}$. For instance, one can consider the subset $\mathcal{U} = (\mathcal{S}^2 \setminus \mathcal{C}) \times (\mathcal{S}^1 \setminus \{k\})$ where $\mathcal{C}$ is a closed semimeridian as in Example 2, and k is an arbitrary point of $\mathcal{S}^1$. Then the Eulerian angles θ,φ,ψ take values in $]0,\pi[$, $]0,2\pi[$, $]0,2\pi[$, respectively. Moreover, the kinetic matrix has the following expression:

$$(A_{\mathcal{R}\mathcal{S}})_{\mathcal{R},\mathcal{S}=1,2,3} = \begin{pmatrix} A\sin^2\psi + B\cos^2\psi & (B-A)\sin\psi\cos\psi\sin\theta & 0 \\ (B-A)\sin\psi\cos\psi\sin\theta & \begin{array}{c} A\sin^2\theta\cos^2\psi + \\ B\sin^2\theta\sin^2\psi + \\ C\cos^2\theta \end{array} & C\cos\theta \\ 0 & C\cos\theta & C \end{pmatrix}$$

Since the matrix $(A_{\mathcal{R}\mathcal{S}})$ is independent of φ, the same holds for its inverse $(A_{\mathcal{R}\mathcal{S}})^{-1} = (A^{\mathcal{R}\mathcal{S}})$. Hence, Theorem 3.1 yields the following

Proposition 7.6. *Under hypothesis H_1 on the forces, the coordinates* $(q,\gamma^1,\gamma^2) = (\varphi,\theta,\psi)$ *are 2-fit.*

This result has a geometrical counterpart. Indeed proposition 7.7 is still true if in place of $\mathcal{U}$ one considers an open subset $\mathcal{U}' = (\mathcal{S}^2 \setminus \mathcal{C}') \times (\mathcal{S}^1 \setminus \{k\})$ where $\mathcal{C}'$ is a closed semimeridian different from $\mathcal{C}$ but having the same end points, and $k' \in \mathcal{S}^1 \setminus \{k\}$. The union of $\mathcal{U}$ and $\mathcal{U}'$ is the manifold $\mathcal{M}' = (\mathcal{S}^2 \setminus \{N,S\}) \times \mathcal{S}^1$ where N and S are the end points of both $\mathcal{C}$ and $\mathcal{C}'$. Furthermore, $\mathcal{M}'$ can naturally be identified with $\mathcal{J} \times \mathcal{P} \times \mathcal{R}$, where $\mathcal{J}$ is an open interval of $\mathbb{R}$ whose elements represent the nutation-angles of Σ, while $\mathcal{P}$ and $\mathcal{R}$ are copies of $\mathcal{S}^1$ whose elements represent the precession- and rotation-angles, respectively. Then, proposition 7.7 implies

Proposition 7.7. *The foliation of $\mathcal{M}$ formed by the leaves $\mathcal{J}\times\{p\}\times\mathcal{R}$, $p\in\mathcal{P}$, is 2-fit.*

EXAMPLE 3.B.

Let Σ be as in the previous example, and consider the mechanical system Σ' obtained from Σ by constraining the axis $\mathbf{c}_3$ to rotate on a palne π fixed in the frame I and passing through **k**.

It is easy to treat this problem directly from a global point of view. The constraint manifold is the two-torus $\mathcal{M}' = \mathcal{N}\times\mathcal{R}$, where $\mathcal{N}$ and $\mathcal{R}$ are copies of $\mathcal{S}^1$ whose elements represent the *nutation-angles* –i.e. the angles formed by $\mathbf{c}_3$ and a straight line fixed in π and passing through P– and the *rotation-angles* of Σ about $\mathbf{c}_3$, respectively. Moreover, the nutation and rotation movements are mutually orthogonal, i.e., for each $m = (n,r)\in \mathcal{N}\times\mathcal{R}$ and for each couple $(v,w)\in (T_n\mathcal{N}\times\{0\})\times(\{0\}\times T_r\mathcal{R})\subseteq T_m\mathcal{M}'\times T_m\mathcal{M}'$, the vector **v** turns out to be orthogonal to **w** . Indeed, this follows from the fact that when Σ' does not rotate about $\mathbf{c}_3$ the points of Σ' move on planes which are parallel to π. Hence such planes are orthogonal (in the Euclidean frame I) to the planes on which the points of Σ' move when Σ' merely rotates about $\mathbf{c}_3$.
Then a curve on $\mathcal{M}'$ is orthogonal to the foliation $\mathcal{F} = \{\mathcal{N}\times\{r\}\ ,\ r\in\mathcal{R}\}$ iff it represents a purely rotational motion of Σ about the axis $\mathbf{c}_3$. Since uniform rotations about $\mathbf{c}_3$ are spontaneous motions of Σ' , by Corollary 5.1 one may conclude that

Proposition 7.8. *$\mathcal{F}$ is a (strongly) 1-fit foliation of $\mathcal{M}'$.*

In particular, if q and γ are coordinates on $\mathcal{N}$ and $\mathcal{R}$ respectively, then by Theorem 5.3, the system of coordinates (q,γ) is strongly 1-fit, since $\{n\}\times\mathcal{R}$ is totally geodesic for each $n\in\mathcal{N}$.

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