2d N=2 Landau–Ginzburg Models

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To my parents

CONTENTS

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This Ph. D. thesis is based on work done in collaboration with Sergio Cecotti and Luciano Girardello which began in early 1989 and which continued in 1990 also with Paolo Soriani. Part of the results I am going to illustrate have already been published in [1] *and* [2].

I apologize for the uncompleteness of the references, but I have quoted only those publications to which I am most indebted.

Finally, some remarks on the general structure of this work. I have tried to keep as readable as possible this thesis separating from the main corpus the purely technical parts, like long computations or extremely subtle problems, by confining them to the numerous appendices or in paragraphs which can be skipped at a first reading.

INTRODUCTION

Recently there has been much interest in the 2d N=2 Wess–Zumino supersymmetric models with polynomial superpotential mainly in connection with Superstring Theories.

One of the open problem in the construction of a four dimensional Superstring theory with N=1 spacetime supersymmetry is that of the classical vacuum. A possibility is to start from the ten dimensional heterotic string. In ten dimensions the structure of the vacuum is well understood: there are only two models, one with gauge group $E_8 \times E_8$ and the other with $SO(32)$. To get a four dimensional model one can choose a different vacuum, for example of the form $M \times K$ where M is the usual four dimensional Minkowski space and K is an internal manifold. K turns out to be a complex manifold of vanishing first Chern class. These manifolds are called "Calabi–Yau manifolds" and the four dimensional superstring models are said to be "compactifications on Calabi–Yau manifolds" of the heterotic 10d superstring.

It is also possible to construct superstring models directly in four dimensions adding to the string sector an internal sector made by 2d N=2 Superconformal Theories. Indeed, two dimensional conformal theories turn out to be the building blocks of the classical solutions in (super–) string theory. Moreover, conformal theories with N=2 supersymmetry are the most interesting since they lead to solutions which are spacetime supersymmetric.

Gepner [3] conjectured the existence of a relation between Calabi–Yau compactifications and N=2 superconformal classical vacua. He claimed that all the N=2 Superconformal theories with $c = 3n$ are equivalent to some σ–model on a (possibly singular) Calabi–Yau n–fold. Indeed, a σ–model on a Calabi–Yau 3–manifold can be seen as the effective theory of a four dimensional superstring model compactified on this Calabi–Yau manifold.

Martinec, Vafa and collaborators [4,5,6,7] proposed a way of connecting Calabi–Yau spaces and 2d N=2 Superconformal theories. An abstract conformal field theory should always correspond to a field theory at a critical point. Zamolodchikov [8] noticed that the 2d non–supersymmetric field theory models with potential $V = \phi^n$ at the infrared critical point correspond to the minimal models of the conformal field theory. In other words, these models are the Landau–Ginzburg effective lagrangians for the conformal minimal models. The same connection has been established in the supersymmetric N=1 and N=2 cases [9] by comparing the algebra of the primary fields. The N=2 supersymmetric field theory with superpotential $W = \frac{1}{\mu+1} X^{\mu+1}$ (X is a N=2 chiral superfield) at the infrared critical point has the same algebra of the primary chiral fields as that of the N=2 superconformal minimal model with $c = 3 - \frac{6}{\mu+1}$ [7,9]. On the other hand, the superpotentials, seen as analytic functions, are exactly the functions appearing in the construction of some Calabi–Yau manifolds [5,6]. Indeed, the crucial observation of these authors is that the critical behaviour of these models is encoded in the behaviour of the holomorphic function $W(X)$ around its critical points.

Thus, it seems that the 2d N=2 supersymmetric field theories with polynomial superpotential (also called 2d N=2 Landau–Ginzburg models) can play an interesting role in connection with 2d N=2 Superconformal theories, Calabi–Yau manifolds and, eventually, String Theories.

But the study of these field theories has a relevance "per se". In fact 2d N=2 Landau–Ginzburg theories are very interesting from a pure field theoretical point of view since they are one of the few cases where it is possible to get exact, non perturbative results. Much is known and can be done at the non perturbative level, for example in the study of the critical properties of these models. Moreover, as the name Landau–Ginzburg suggests, these models can be relevant also in statistical mechanics since they can be considered as Landau–Ginzburg effective theories for 2d critical phenomena.

In this thesis the possible connections of the 2d N=2 Landau–Ginzburg models with critical phenomena or Calabi–Yau manifolds will not be considered. Indeed, the subject of this thesis is the study from a field theoretical point of view of two dimensional N=2 supersymmetric models constructed with (anti–) chiral superfields and with a polynomial superpotential. The main aim is to find the general properties of these models and to study their behaviour in the infrared and/or in the strong coupling limit.

In Constructive Quantum Field Theory [10] it has been shown that the 2d N=2 Landau–Ginzburg models *exist* on a two dimensional Euclidean cylinder. Thus, I will choose as spacetime a 2d Euclidean cylinder.

Among all the Green's functions of a 2d N=2 Landau–Ginzburg model, two classes are of major interest: the chiral and antichiral Green's functions. They are the correlation functions of only chiral or only antichiral fields, respectively. The interest in these classes of Green's functions is given not only for their properties but also because they give very interesting informations on the critical behaviour of the model.

In this thesis only the (anti–) chiral Green's functions are considered. The fact that it will be possible to exactly compute the chiral Green's functions does not mean that the theories are necessary solvable but only that a part of them is computable.

The starting point of this thesis is the observation that the (anti–) chiral Green's functions of a 2d N=2 Landau–Ginzburg model on an Euclidean cylinder do not depend on the coordinates on the cylinder and on the kinetic term in the Lagrangian (this last result is true only if the change of the kinetic term does not bring outside the given "topological class"). To explore the consequences of these two results the simplest examples, the "A_μ models", are first considered. The A_μ models are the 2d N=2 Landau–Ginzburg theories with superpotential $V = \frac{g}{\mu+1}X^{\mu+1}$, $\mu \geq 2$, which, at the infrared critical point, should correspond to the A_μ minimal models of the 2d N=2 Superconformal Field Theory. With simple arguments it is possible to get the exact dependence of the chiral Green's functions on the coupling constant g and to prove that the chiral Green's functions are equal to the Supersymmetric Quantum Mechanical amplitudes in a Super Quantum Mechanical theory with superpotential $V = \frac{1}{\mu+1}\phi^{\mu+1}$. In other words, the chiral Green's functions do not change under a dimensional reduction, i.e. from two dimensional field theory to quantum mechanics. Moreover, the chiral correlation functions in the N=2 Super Quantum Mechanical theory with superpotential $V = \frac{1}{\mu+1}\phi^{\mu+1}$ have been exactly computed [11] so that it is possible to get exactly and non–perturbatively the Chiral Green's functions of the A_μ models. For these models, with a (para–) stochastic reconstruction, it is possible to show the existence of the infrared critical point and to check that the chiral Green's functions at the critical point coincide with the correlation functions of the corresponding 2d N=2 Superconformal model.

From the example of the A_μ models the general strategy for studying a 2d N=2 Landau–Ginzburg model can be outlined. First one has to look for the dependence of the chiral Green's functions on the overall (dimensional) coupling constant g. Then, it should be possible to reformulate the problem of the explicit computation of the chiral Green's functions as a "simpler problem", something similar to the reduction from a field theoretical to a quantum mechanical problem of the A_μ case. Obviously, for a generic model all that is more difficult. The simple physical arguments which give the dependence of the chiral Green's functions on the overall dimensional coupling constant in the A_μ case work also for all the quasihomogeneous superpotentials but they are not a general procedure. Moreover, the Schroedinger equation of the associated Quantum Mechanical problem is solvable only for the A_μ models. Thus, it is necessary to give a different formulation to these problems.

The first remark is that the Chiral Green's functions, being independent from the space-time coordinates, can be seen as $\Delta \times \Delta$ matrices (Δ is the Witten index = number of vacua). More exactly, the Chiral Green's functions form a *Ring* of matrices. This ring, called "Chiral ring", is computable, it gives the OPE (or "fusion rules") of the chiral fields and is the more important characteristic of a 2d N=2 Landau–Ginzburg model.

To proceed, it is necessary to give a mathematical interpretation of the chiral Green's functions. Following an idea of Witten [12], it is possible to prove that the chiral Green's functions are equal to the intersection numbers in a certain cohomological problem. They are the intersection numbers in the cohomology group $H^n_{\overline{\partial}_V}$ of the $\overline{\partial}_V$–differential complex, where the differential operator $\overline{\partial}_V$ is defined by $\overline{\partial}_V = \overline{\partial} + \partial V \wedge$ and V is the superpotential. It is possible to show that the cohomology group $H^n_{\overline{\partial}_V}$ is isomorphic to the chiral ring and to explicitly compute the intersection numbers. The intersection numbers turn out to be proportional to the normalization constants of the forms which represent the cohomology classes and which correspond to the field theory vacua.

Following the example of the A_μ models, I will first get the dependence of the chiral Green's function on the overall coupling constant g. To do that, one starts from the observation of Martinec, Vafa and collaborators [4,5] on the connection between N=2 supersymmetry and "Singularity Theory" where the "singular" function is the superpotential. Indeed, having interpreted the chiral Green's functions as intersection numbers in a cohomological problem, it is possible to make a rigorous connection between the chiral Green's functions of a 2d N=2 Landau–Ginzburg model and Singularity Theory. To get this, it is necessary to introduce some standard material in Algebraic Geometry. With the vocabulary of translation between N=2 supersymmetry and Singularity Theory it is possible to face the open problems.

It turns out that the dependence on the overall coupling constant g is related to the "monodromy" of the Singularity. The main problem which arises is a problem of basis. The physical basis is in general different from the mathematical one. This has the consequence that the dependence on the overall coupling constant g (and its complex conjugate $\overline{g}$) is described by a complicated linear differential equation. In the simplest case (the "quasihomogeneous" superpotentials) this equation is solvable and dictates the dependence of the chiral Green's functions on the overall dimensional coupling constant. Since the dimensional parameters in the chiral Green's functions are only g and the size of the cylinder L, this result also implies the knowledge of the scaling dimension of the chiral fields in the infrared limit. Assuming the existence of a critical point in the infrared limit, this gives the conformal dimension of the fields.

All what is left to do to get in an exact and non perturbative way the chiral Green's functions is to compute the absolute normalization of the forms representing the cohomology classes. The normalization condition comes from the identification of the "real structure" on the cohomology. In physical terms this corresponds to the action of the "Spectral Flow". The condition is explicitly formulated for the quasihomogeneous models. Then it is possible (in principle) to compute the chiral Green's functions of all quasihomogeneous models. Notice that to obtain this result it is not necessary to solve any Schroedinger equation.

The thesis ends with two examples, the computation of the chiral Green's functions for the A_μ and D_μ models. Obviously the results agree with the previous computations and with the known correlation functions in 2d N=2 Superconformal Field Theory.

CHAPTER 1

General properties of two–dimensional N=2 Landau–Ginzburg field theories

The subject of this thesis is the study of a class of field theories in two dimensions with N=2 supersymmetry (i.e. two supersymmetry charges).

In the first part of the thesis this will be done by means of (almost standard) non–perturbative field theoretical techniques. I will study some general and peculiar properties of these field theories and their behaviour in the infrared (IR) (i.e. in the large distance) or in the strong coupling limit.

The 2d N=2 field theories I am interested in are scalar field theories with polynomial interactions (polynomials superpotentials), these models are also called "2d N=2 Landau–Ginzburg models".

1.1 Definition of the models

The 2d N=2 field theories I consider are constructed with scalar supermultiplets and described by a lagrangian with a polynomial superpotential. Let $Q_\pm$ be the supercharges [1], they satisfy $\{Q^*_\pm, Q_\pm\} = 2P_\pm$. The smallest representation of supersymmetry is given by chiral and/or antichiral superfields. A chiral superfield satisfies:

$$[Q^*_\pm, \Phi] = 0 \tag{1.1}$$

and an antichiral one satisfies:

$$[Q_\pm, \Phi] = 0 \quad . \tag{1.2}$$

The most general lagrangian which can be written using these fields is (in superfield notations):

$$\mathcal{L} = \int d^4\theta\, K(\Phi_i, \overline{\Phi}_i) + \left(\int d^2\theta\, W(\Phi_i) + h.c.\right) \quad . \tag{1.3}$$

[1] See Appendix A for notations, conventions and a brief introduction to N=2 supersymmetry in two dimensions.

Supersymmetry requires $K(\Phi,\overline{\Phi})$ to be a Kähler potential. The free field theory is described by the flat metric without interaction, that is $K(\Phi,\overline{\Phi}) = \overline{\Phi}\Phi$, $W = 0$.

To define the theory one should also specify the space ("spacetime") where the coordinates $\vec{x} = (x,t)$ take value. Usually one would take $\mathbb{R}^2$ or the sphere S^2 (the plane compactified). $\mathbb{R}^2$ is excluded a priori because of IR divergencies. Instead my choice is to define the theory on an *Euclidean cylinder*, the space is a circle of length L. There are three principal reasons for this choice. Consider the models with minimal Kähler potential $K(\Phi,\overline{\Phi}) = \overline{\Phi}\Phi$ and superpotential $W(\Phi) = \Phi^n$, it has been shown that:

1. these theories are UV finite [13,14] (instead in perturbation theory there are bad IR singularities);
2. these theories were shown to exist by A. Jaffe and collaborators in the sense of constructive QFT [10] [2]. In particular the models are finite even *non–perturbatively*;
3. with this choice it is possible to apply supersymmetric non–perturbative techniques, like Witten index [15], stochastic interpretation [13,16], Nicolai map [17].

I will assume from now on that the models with polynomial superpotential which will be studied in this thesis are UV finite even non–perturbatively.

Since I will be mainly interested in the critical properties of these models, it is natural to make a comparison with the properties of the 2d N=2 Superconformal field theories. For Conformal theories, different spacetime geometries are related by conformal maps, thus at criticality every geometry should be good as well, but off criticality they are quite different.

1.2 The Witten Index

One of the major tools in the study of supersymmetric field theories (in a box) or supersymmetric quantum mechanics is the concept of *Witten Index* [15].
In a supersymmetric theory the lowest energy states (or vacua) are the zero energy states. The Witten Index is a topological number which gives some information on the structure of the vacua of the theory. I will briefly review the definition and some properties of the Witten Index Δ, referring to the literature [15] for more information.

Witten defined a topological index in supersymmetric quantum mechanics as the difference between the number of bosonic and fermionic states:

$$\Delta := n_b - n_f \quad . \tag{1.4}$$

Since in supersymmetric theories there is an equal number of non–zero energy bosonic and fermionic states, it follows that the Witten Index is equal to the difference between the number of bosonic and fermionic vacua:

$$\Delta = n_b^0 - n_f^0 \quad . \tag{1.5}$$

Formally

$$\Delta = \mathrm{tr}(-1)^F e^{-\beta H} =: \mathrm{Str}\, e^{-\beta H} \quad . \tag{1.6}$$

[2] Notice that the $L \to \infty$ limit may have problems due to IR divergencies.

Under general conditions (for example the presence of a mass gap) it can be proven that Δ is independent of β and of the dimensions of the box.

Thus, turning to functional integral formalism, one can define the Witten index as

$$\Delta = \int_{\mathrm{PBC}} [d\Phi] e^{-S(\Phi)} \tag{1.7}$$

where the integral is evaluated in a box with periodic boundary conditions (PBC) on all fields (bosons and fermions).

Known results for the Witten Index Δ in supersymmetric field theories are the following.

1. If $\Delta \neq 0$ then supersymmetry cannot be broken.
2. Formally Δ is written as an integer, but generically it can be any number. If it is not an integer, its interpretation as the difference between the number of bosonic and fermionic vacua is lost, but its value as topological index still remains.
3. $\Delta \geq 0$.
4. In a massive free theory $\Delta = 1$.
5. If the theory has not mass gap then the Witten Index can be ill defined or infinite.
6. When Δ does not depend on β and the size of the box, it can be computed in the limit of vanishing β or vanishing box ("*ultralocal limit*"). Only the constant configurations survive this limit. For Wess–Zumino models (N=2 in d=2 or N=1 in d=4) with minimal kinetics term the ultralocal prescription gives:

$$\Delta = \int \frac{d\phi d\bar{\phi}}{2\pi s^2} s^4\, W''(\phi)\overline{W''(\phi)} \exp\left[-s^2 W'(\phi)\overline{W'(\phi))}\right] \tag{1.8}$$

   where s is the size of the box.
   If the map $\eta = W'(\phi)$ is locally invertible (always if $W'(\phi)$ is a polynomial) [3], then

$$\Delta = \int \frac{d\phi d\bar{\phi}}{2\pi s^2} \exp\left[-s^2 \eta\bar{\eta}\right] = \quad \text{degree of the polynomial } W'(\phi)\ . \tag{1.9}$$

   This is the same as stating that Δ is the winding number of the Nicolai mapping [13,17], i.e. the number of distinct configurations of the original bosonic fields which are mapped in a given configuration of the gaussian field η.
7. For 2d N=2 Wess–Zumino models with polynomial superpotential, Δ (if well defined) is an integer.
8. The condition

$$\mathrm{Det}\left[\frac{\partial^2 W}{\partial\phi_a \partial\phi_b}\right] \neq 0 \quad \text{almost everywhere} \tag{1.10}$$

   guarantees the (almost–) surjectivity of the Nicolai mapping $\eta_a = \frac{\partial W}{\partial \phi_a}$ and then the fact that the Witten Index is well defined.

Thus, in the following I will limit myself to the study of 2d N=2 Wess–Zumino (or Landau–Ginzburg) theories with polynomial superpotential such that the Witten index Δ is well defined, finite and different from zero.

[3] In the case of more then one superfield, $W_i'(\phi)$ can have non–isolated zeros. In this situation Δ can be not well defined.

In the simplest situation the Witten index Δ can be computed noticing that it is just the number of zeros of the equation $(\partial W/\partial \phi_a) = 0$, i.e. of classical zero–action vacua (each counted with its algebraic multiplicity).

(1.11) Example : Consider the case of one field Φ and superpotential

$$W = \frac{1}{3} g_1 \Phi^3 + \frac{1}{5} g_2 \Phi^5 \quad . \tag{1.12}$$

In this case $\Delta = 4$ $((\partial W/\partial \Phi) = g_1\Phi^2 + g_2\Phi^4 = 0$ has four roots) for $g_2 \neq 0$ and every value of g_1. For $g_2 = 0$ and $g_1 \neq 0$, $\Delta = 2$. Thus a deformation of the theory which leads to $g_2 = 0$ changes the "topological structure". From this example it is immediately obvious which is the physical meaning and importance of the concept of "topological structure". ∎

Notice that the free massless theory has Witten index $\Delta = 0$. This has the strong consequence that the free massless theory does not belong to the class of models which I consider. This is the first signal of a very important fact (as we will see in the following): the (weak–coupling) perturbation theory cannot be used to compute any quantity for the models I am considering. In fact all quantities I will compute are just identically zero — as they should — order by order in perturbation theory. Thus the weak coupling $(g \to 0)$ limit is, at least, a not well defined limit and in the following I will not consider anymore this problem.

Strictly connected to the Witten Index another non–perturbative tool is rather well known, it is the **Nicolai mapping**. Nicolai [17,13] noticed that for a large class of supersymmetric theories in two and four dimensions, 2d N=2 Landau–Ginzburg models included, there exists a non–linear and (generally speaking) non–local transformation of the bosonic fields $\phi_i(x) \to \xi_i(x)$ such that the bosonic part of the Euclidean Lagrangian (after the elimination of the auxiliary fields) is given simply by

$$\mathcal{L}_B = \frac{1}{2} \sum_i \xi_i^2 + \text{ total divergence} \tag{1.13}$$

and the Jacobian of the transformation, $\det\left(\frac{\delta\xi}{\delta\phi}\right)$, is the Matthews–Salam determinant for fermions. This means that

$$[d\phi] \int [d\psi d\overline{\psi}] e^{-\int (\mathcal{L}-\mathcal{L}_B) d^\mu x} = [d\phi] \det\left[\frac{\delta\xi}{\delta\phi}\right] = [d\xi] \ . \tag{1.14}$$

Since $\mathcal{L}_B = \frac{1}{2}\sum_i \xi_i^2$, this implies that the bosonic measure, after the integration over the fermionic fields, becomes, in terms of the fields ξ_i, Gaussian with zero mean and covariance one. Indeed, the winding number of the map $\phi \to \xi$ is the Witten index. All the dynamical informations are encoded in the Nicolai mapping since, after the transformation, the theory becomes trivial.

The Nicolai mapping for the 2d N=2 LG models is explicitly constructed in ref. [13]. As we will see in the following, it will be sufficient to consider the N=2 SQM system with superpotential $W(\phi)$. The Nicolai map in the N=2 SQM case is

$$\xi_i(z,\overline{z}) = \frac{\partial}{\partial z}\phi_i(z,\overline{z}) + \left[\frac{\partial W^*}{\partial \phi_i}\right](z,\overline{z}) \quad . \tag{1.15}$$

Since ξ_i is a Gaussian field, an useful remark is that the topological properties of the Nicolai map do not change under a linear perturbation of the superpotential, $W(\phi_i) \to W_\lambda(\phi_i) = W(\phi_i) + \sum_i \lambda_i \phi_i$. This remark will be very useful in the future developments of the theory.

1.3 Spacetime independence of the (anti–) Chiral Green functions

The simplest Green's functions of the models defined by (1.3) are the following three classes:

i. "chiral" Green's functions (CGFs), constructed only with chiral fields

$$\begin{aligned} \mathcal{G}^{m,n}_{chiral}(z_1,\ldots,z_k,\bar{z}_1,\ldots,\bar{z}_k) :=& \\ & T\langle m|\phi_{i_1}(z_1,\bar{z}_1)\ldots\phi_{i_k}(z_k,\bar{z}_k)|n\rangle \end{aligned} \tag{1.16}$$

where $|m\rangle$ and $|n\rangle$ are supersymmetric vacua, ϕ_i is the first (bosonic) component of the chiral supermultiplet Φ_i and $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$.

ii. "antichiral" Green's functions (ACGFs), constructed only with antichiral fields:

$$\begin{aligned} \mathcal{G}^{m,n}_{antichiral}(z_1,\ldots,z_k,\bar{z}_1,\ldots,\bar{z}_k) :=& \\ & T\langle m|\bar{\phi}_{i_1}(z_1,\bar{z}_1)\ldots\bar{\phi}_{i_k}(z_k,\bar{z}_k)|n\rangle \end{aligned} \tag{1.17}$$

iii. "mixed" Green's functions, constructed with both chiral and antichiral fields:

$$\begin{aligned} &\mathcal{G}^{m,n}_{mixed}(z_1,\ldots,z_k,\bar{z}_1,\ldots,\bar{z}_k;z_{k+1},\ldots,z_{k+l},\bar{z}_{k+1},\ldots,\bar{z}_{k+l}) := \\ &\quad T\langle m|\phi_{i_1}(z_1,\bar{z}_1)\ldots\phi_{i_k}(z_k,\bar{z}_k)\bar{\phi}_{i_{k+1}}(z_{k+1},\bar{z}_{k+1})\ldots\bar{\phi}_{i_{k+l}}(z_{k+l},\bar{z}_{k+l})|n\rangle\ . \end{aligned} \tag{1.18}$$

Notice that the "propagator" $T\langle m|\phi_i(z,\bar{z})\bar{\phi}_j(w,\bar{w})|n\rangle$ belongs to this class.

The antichiral Green's functions have the same properties of the chiral ones (up to conjugation), thus one needs only to study the chiral and mixed ones.

Purpose of this thesis is the study of the properties of the chiral (and then also of the antichiral) Green's functions.

Following the techniques already introduced by Novikov et al. and Amati et al. [18] in the study of supersymmetric Yang–Mills theories in four dimensions, it is easy to prove the first statement on the CGFs:

the (anti–) chiral Green's functions do not depend on the coordinates $z_i, \bar{z}_i$.

Let me briefly show the proof of this statement. The supersymmetric vacua on the cylinder $|m\rangle$ satisfy the conditions

$$Q_\pm|m\rangle = Q^*_\pm|m\rangle = H|m\rangle = 0 \qquad m = 1,\ldots,\Delta \tag{1.19}$$

and the chiral fields satisfy $[Q^*_\pm, \phi_i] = 0$. The N=2 supersymmetric algebra gives $\{Q^*_\pm, Q_\pm\} = 2P_\pm$ where $P_+ = -i\frac{\partial}{\partial z}$, $P_- = -i\frac{\partial}{\partial \bar{z}}$. Then

$$\begin{aligned} -2i\frac{\partial}{\partial z}\phi_i &= \left[\{Q_+,Q^*_+\},\phi_i\right] = \left\{[Q^*_+,\phi_i],Q_+\right\} - \left\{[\phi_i,Q_+],Q^*_+\right\} = \\ &= \left\{Q^*_+,[Q_+,\phi_i]\right\} \end{aligned} \tag{1.20}$$

and analogously

$$-2i\frac{\partial}{\partial\bar z}\phi_i = \{Q^*_-, [Q_-, \phi_i]\} \quad . \tag{1.21}$$

Now:

$$\begin{aligned}
-2i\frac{\partial}{\partial z_k} T\langle m|\phi_1(z_1,\bar z_1),\ldots,\phi_k(z_k,\bar z_k),\ldots,\phi_l(z_l,\bar z_l)|n\rangle &= \\
= -2iT\langle m|\phi_1(z_1,\bar z_1),\ldots,\frac{\partial\phi_k(z_k,\bar z_k)}{\partial z_k},\ldots,\phi_l(z_l,\bar z_l)|n\rangle &= \\
= T\langle m|\phi_1(z_1,\bar z_1),\ldots,\{Q^*_+,[Q_+,\phi_k]\},\ldots,\phi_l(z_l,\bar z_l)|n\rangle &= \\
= T\langle m|\{Q^*_+,\phi_1(z_1,\bar z_1),\ldots,[Q_+,\phi_k],\ldots,\phi_l(z_l,\bar z_l)\}|n\rangle &= 0
\end{aligned} \tag{1.22}$$

where the first equality follows from the fact that there is no T–order contribution since the theory is UV finite.

In the same way one shows that the CGFs do not depend on $\bar z_i$. ∎

This in some sense trivial fact is the milestone of all the future construction. It allows to translate the problem in an algebraic language and gives the possibility of interpreting the CGFs as "topological correlation functions".

Notice that the independence of these Green's functions from the coordinates implies that short–distance singularities do not exist in the product of two chiral fields, or at least that any such singular term would cancel in (anti–) chiral Green's functions.

1.4 Independence of the chiral Green functions from the Kähler potential

The chiral Green's functions are also independent from the choice of the Kähler potential $K(\Phi_i,\overline{\Phi}_i)$ in (1.3) . The statement which I want to prove is:

the chiral Green's functions are invariant under "small" deformations of K, i.e. deformations which do not change the topology of the Kähler space.

A "small" deformation is intended to be a perturbation of the Kähler potential $K'_\epsilon(\Phi_i,\overline{\Phi}_i) = K(\Phi_i,\overline{\Phi}_i) + \epsilon G(\Phi_i,\overline{\Phi}_i)$ which does not change the Witten index Δ, nor the number of Witten vacua neither their quantum numbers.

(1.23) Example : consider the case of one field Φ (n=1) and minimal Kähler potential $K(\Phi,\overline{\Phi}) = \Phi\overline{\Phi}$. Then by usual Witten index arguments, K'_ϵ is R–invariant (see the end of this chapter for the notion of *R*–invariance), i.e. a function of $\Phi\overline{\Phi}$ alone and the corresponding Kähler metric is a singularity–free *complete* metric for the complex plane $\mathbb{C}$. ∎

Generally it is expected that the explicit vacuum wave functions do depend on the particular choice of the Kähler potential. Indeed only the complete chiral Green's function is independent under "small" deformations of the Kähler potential, not its components.

I will now give a *formal* proof of this statement, a more rigorous proof will be an almost trivial statement using the mathematical formalism which will be introduced in the second part of this thesis. Indeed in the second part of this thesis much stronger results will be needed, results which allow for deformations of the metric which make the Witten index to jump discontinuosly.

Consider the CGFs $\langle m|\phi_i(z,\bar{z})|n\rangle$. From what follows it will be obvious that the independence from the Kähler potential of this CGF is a sufficient condition for the independence of any CGF. Anyway, it is a trivial matter to generalize to any CGF the following formal proof. From the action principle in the path integral notation, one gets

$$\begin{aligned}\delta_\epsilon\langle m|\phi_i(z,\bar{z})|n\rangle &= -\int d^2z d^4\theta T\langle m|\phi_i\delta_\epsilon K(\phi_j,\bar{\phi}_k)(z,\bar{z})|n\rangle = \\ &= -\epsilon\int d^2zT\langle m|\phi_i\left[G(\phi_j,\bar{\phi}_k)\right]_D(z,\bar{z})|n\rangle\end{aligned} \tag{1.24}$$

where $[...]_D$ means taking the D–component. In this equation it was used the fact that supersymmetry implies $\delta_\epsilon Z = 0$. The assumptions on the perturbation $G(\phi_j,\bar{\phi}_k)$ guarantee that the asymptotical boundary conditions on the path integral are obeyed. Now

$$[G]_D = \{Q^*_\alpha, [G]^\alpha_\lambda\} \tag{1.25}$$

where $[G]^\alpha_\lambda$ is the λ–component of the composite superfield G. Then

$$\delta_\epsilon\langle m|\phi_i(z,\bar{z})|n\rangle = -\epsilon\int d^2zT\langle m|\phi_i\{Q^*_\alpha,[G]^\alpha_\lambda\}(z,\bar{z})|n\rangle = 0 \tag{1.26}$$

commuting again Q^*_α across ϕ_i. ∎

In the following, unless explicitly stated, thanks to this independence, I will consider always models with minimal kinetics term, $K=\sum_i\overline{\Phi}_i\Phi_i$.

From the independence of the Kähler potential it follows that the CGFs are also independent from the UV cut–off. Indeed, if the theory has a kinetic term which is a "small" deformation of the minimal one, this follows from the fact that the minimal theory is UV finite [10,14]. More generally, the independence from the Kähler potential suffice to show directly that these Green's functions are independent of the masses of the Pauli–Villars regulators. The non–renormalization theorem [19] implies that the only renormalization of the N=2 model is a finite wave–function renormalization, which is essentially conventional. Wave–function renormalization is unmaterial since it can be undone by a (continuous) deformation of the kinetic term which cannot change the CGFs. In fact, all the equation I will write down are valid in any renormalization scheme consistent with supersymmetry, since different schemes can be related by adding local counterterms in the Kähler potential. For semplicity I shall work with bare quantities (which are finite) and UV (over–) subtraction, but the results take the same form in terms of renormalized quantities, as it is easy to check.

Up to now I have assumed that the lagrangian is of the standard form. However it is possible to relax this condition and add some non–standard couplings, as for example higher derivative couplings. This is equivalent to let the function $K(\Phi,\overline{\Phi})$ to depend also on the spinorial covariant derivatives of the chiral fields. It is evident that the CGFs do not depend on this exotic couplings too, since they have the general form $\int d^4\theta(...)$.

Thus it is sufficient to consider 2d N=2Wess–Zumino models with (almost) arbitrary polynomial superpotential in a finite number of superfields and with minimal kinetics terms.

For these models the CGFs do not depend on the spacetime coordinates but only on the length of the circle (space manifold), or IR cut–off, L [4], on the coupling constants g_i, on the various parameters which appear in the superpotential and on the vacua which are chosen to evaluate the Green's functions.

The following convention will be adopted: let $X_i(z,\theta)$ be a chiral superfield ($i = 1,\ldots,n$), then $X^{\underline{m}} := X_1^{m_1}(z)X_2^{m_2}(z)\cdots X_n^{m_n}(z)$ where m_i are integers and $m_i \geq 0$. Thus, it is possible to write in the general case the superpotential as

$$W(X_i) = \sum_{i=1}^{k} g_i X^{\underline{m}_i} \quad . \tag{1.27}$$

It is useful to introduce an overall dimensionful complex coupling constant g by defining:

$$W(X_i) = g\sum_{i=1}^{k} c_i X^{\underline{m}_i} \tag{1.28}$$

where c_i are adimensional coupling constants. With these notations only $g,\bar{g}$ and L are dimensional parameters. In two dimensions the bosonic fields (and the superfields) have mass zero dimension and then $[g] = +1$, $[L] = -1$. This follows from the bosonic potential $W_{bos} = g\bar{g}\frac{\partial F}{\partial X_i}\frac{\partial \bar{F}}{\partial \bar{X}_{\bar{i}}}g^{i\bar{i}}$ or from the action written in superfield notations remembering that $[d^2\theta] = -1$, $S = \int d^2z d^2\theta\, gF(X)$, where F is defined by $W =: gF$. This implies that (gL) is adimensional. Thus the CGFs have the following dependence on the parameters of the theory [5]

$$\langle p|X^{\underline{\alpha}}|q\rangle =: F_{p,q}(g,\bar{g},L;\Delta;c_1,\ldots,c_k;\bar{c}_1,\ldots,\bar{c}_k;\underline{m}_1,\ldots,\underline{m}_k;\underline{\alpha}) \tag{1.29}$$

where $0 \leq p,q \leq \Delta - 1$ and $\Delta = \Delta(c_1,\ldots,c_k;\bar{c}_1,\ldots,\bar{c}_k;\underline{m}_1,\ldots,\underline{m}_k)$.

The main purpose of this thesis is the computation of the dependence on $g,\bar{g}$ (and then on L) of the CGFs, i.e. of $F_{p,q}(g,\bar{g},L)$. This goal will be reached for all the class of theories I have defined thanks to the use of the mathematical theory of Singularities which will be the subject of the second part of this thesis. In the first part of the thesis the study of this problem will be done with standard field theory methods which give the physical insight of the problem and are so powerful to completely solve the problem at least for the simplest models.

1.5 The simplest models: the A_μ series

The name to these models come from the fact that in the IR (strong coupling) limit they are described by the A_μ N=2 Superconformal minimal models, as I will explicitly show in the next chapter.

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[4] There is no dependence on the UV cut–off since the theory is UV finite.

[5] I implicitly assume that all the chiral superfields can be expressed as integral powers of the fields. This will be shown in the following chapters.

These models are defined by the following lagrangian in one (anti–) chiral superfield ($n = 1$):

$$\mathcal{L} = \int d^4\theta K(\overline{X}, X) + \left(\frac{g}{\mu+1}\int d^2\theta X^{\mu+1} + \quad \text{h.c.}\right) \tag{1.30}$$

where K is in the class of the minimal kinetic term $K(\overline{X}, X) = \overline{X}X$, g is the (complex) coupling constant, μ an integer bigger than 1. (In the notation of the preceding section $k = 1$, $n = 1$, $m_1 = \mu + 1$ and $c_1 = 1/m_1 = 1/(\mu + 1)$).

It is easy to compute the value of the Witten index:

$$\Delta = \mu \quad . \tag{1.31}$$

These models possess a very important global U(1) symmetry called R–symmetry. This symmetry does not commute with supersymmetry or, in other words, acts also on the fermionic supercoordinates θ. Under an R–symmetry transformation $\theta \to \theta e^{-i\alpha/2}$, thus for the A_μ series:

$$\begin{aligned}
X(z) &\longrightarrow e^{i\frac{\alpha}{\mu+1}} X(z) \\
\overline{X}(z) &\longrightarrow e^{-i\frac{\alpha}{\mu+1}} \overline{X}(z) \\
\psi(z) &\longrightarrow e^{-i\frac{\mu-1}{2(\mu+1)}\alpha} \psi(z) \\
\tilde{\psi}(z) &\longrightarrow e^{i\frac{\mu-1}{2(\mu+1)}\alpha} \tilde{\psi}(z) \\
\theta &\longrightarrow e^{-i\frac{\alpha}{2}} \theta \\
\bar{\theta} &\longrightarrow e^{i\frac{\alpha}{2}} \bar{\theta}
\end{aligned} \tag{1.32}$$

where $X(z)$ and $\psi(z)$ are respectively the first and the second component of the $X(z, \theta)$ chiral superfield. It follows that $d^4\theta$ is invariant but $d^2\theta \to e^{-i\alpha} d^2\theta$ which is compensated by $X^{\mu+1} \to e^{i\alpha} X^{\mu+1}$.

This symmetry is very important because at criticality (as we will see) will be identified with the U(1) belonging to the N=2 superconformal group.

A constraint which follows from the existence of this symmetry is that the CGFs must be neutral. I will not directly impose this constraint in the following computation but just verify at the end that it is satisfied. Nevertheless, in future developments it will be very useful.

1.5.1 *Dependence on the coupling constant g of the CGFs*

Let's start by computing the dependence on g of the CGFs. I will use here a very intuitive argument but in the following I will prove this result at least two other times with rigorous arguments. Let me euristically prove the following statement:

the CGFs of the A_μ models have the following dependence on the (complex) coupling constants g, $\overline{g}$ and on the IR cut–off L:

$$\langle h|X^p|k\rangle =: F_{h,k}(g, \overline{g}, L; \mu; p) = \frac{1}{(gL)^{\frac{p}{\mu+1}}} f_{h,k}(\mu; p) \tag{1.33}$$

I will now prove this statement in the case $p = 1$, the generalization to the case $p \geq 2$ is trivial.

Consider the lagrangian

$$\mathcal{L} = |\rho|^2 \int d^4\theta \overline{X} X + \left(\frac{g}{\mu+1} \int d^2\theta X^{\mu+1} + \text{ h.c.} \right) \quad . \tag{1.34}$$

As already stated $\langle h|X|k\rangle =: F_{h,k}(g, \overline{g}, L; \mu; 1)$ does not depend on ρ. Make the field redefinition $Y = \rho X$, then $\mathcal{L}$ becomes:

$$\begin{aligned} \mathcal{L} &= \int d^4\theta \overline{Y} Y + \left(\frac{g}{\mu+1} \left(\frac{1}{\rho}\right)^{\mu+1} \int d^2\theta Y^{\mu+1} + \text{ h.c.} \right) \\ &= \int d^4\theta \overline{Y} Y + \left(\frac{g'}{\mu+1} \int d^2\theta Y^{\mu+1} + \text{ h.c.} \right) \end{aligned} \tag{1.35}$$

where $g' := g/(\rho)^{\mu+1}$. It is convenient to introduce the notation $\langle h|X|k\rangle_{\rho,g}$ which means that the $\langle h|X|k\rangle$ Green's function is computed with the $|\rho|^2$ term in front of the kinetic term and with g as coupling constant. Now

$$\rho F_{h,k}(g, \overline{g}, L; \mu; 1) = \rho \langle h|X|k\rangle_{\rho,g} = \langle h|Y|k\rangle_{1,g'} = F_{h,k}(g', \overline{g'}, L; \mu; 1) \tag{1.36}$$

where the change of variables $X \to Y$ in the lagrangian $\mathcal{L}$ has been used. On the other hand, the CGF does not depend on the kinetic term (i.e. on ρ), then

$$\rho \langle h|X|k\rangle_{\rho,g} = \rho \langle h|X|k\rangle_{1,g} \tag{1.37}$$

Now, choosing $\rho = g^{1/(\mu+1)}$ and equating the last two formulæ one gets:

$$\langle h|X|k\rangle_{1,1} = g^{\frac{1}{\mu+1}} \langle h|X|k\rangle_{1,g} \quad . \tag{1.38}$$

The same argument shows that the CGFs do not depend on $\overline{g}$, then:

$$F_{h,k}(g, \overline{g}, L; \mu; 1) = \frac{1}{(gL)^{\frac{1}{\mu+1}}} f_{h,k}(\mu; 1) \tag{1.39}$$

where the dependence on L is obtained by dimensional considerations and f is up to now an unknown function. ∎

1.5.2 *Reduction to a Quantum Mechanical problem*

One of the most (or the most) important tool in the study of the properties of the 2d N=2 LG models is the possibility of reducing to the study of an associated (super–) quantum mechanical problem. In this section I will give the physical motivation and technique to do that. In the following section I will use this reduction to explicitly and exactly compute the CGFs for the A_μ models.

Consider the function

$$(gL)^{\frac{p}{\mu+1}}\langle h|X^p(z)|k\rangle = f_{h,k}(\mu;p) \quad . \tag{1.40}$$

Since the left hand side (as it is obvious) is independent of g and L, it can be computed in the most convenient limit. For instance consider:

$$L \longrightarrow 0 \,, \qquad |z| \gg L \,,\ \frac{1}{g} \quad . \tag{1.41}$$

In this limit, the problem of computing (1.40) reduces to that of the computation of the *quantum mechanical* amplitude

$$\langle h|X^p(z)|k\rangle_{SQM} \tag{1.42}$$

computed in the N=2 supersymmetrical quantum mechanical (SQM) system [20,11] [6] with superpotential

$$W_{SQM} = \frac{1}{\mu+1}X^{\mu+1} \tag{1.43}$$

that is the WZ models reduced to one dimension and with unit coupling constant. In eq. (1.42) $|h\rangle$ are the SQM Witten vacua.

To substantiate this statement let consider the $L \to 0$ limit. For $|z|$ large, only vacuum ($\equiv$ supersymmetric) states contribute to the amplitude (1.40) . Decompose $X(x,t)$ into Fourier components

$$X(x,t) = \frac{1}{\sqrt{L}}\sum_n e^{2\pi i n \frac{x}{L}} X_n(t) \quad . \tag{1.44}$$

As $L \longrightarrow 0$ only the constant mode $X_0(t)$ has bounded energy. Therefore all the other modes should decouple from the given amplitude and hence can be neglected. The computation reduces to the one–dimensional one for $X_0(t)$. Notice that the same argument [7] which guarantees (in non–gauge theories) that the Witten index and the quantum numbers of the supersymmetric vacua are invariant under dimensional reduction, and so can be computed in SQM by solving the relevant Schroedinger equation. Moreover all critical quantities (c, h, C_{ijk}, ...) which are related by the superconformal algebra to the CGFs and/or to the

[6] See also Appendix B.

[7] This argument can be made mathematically more rigorous using the uniform continuity proven in ref. [10].

quantum numbers of the supersymmetric vacua (see next chapter) can be computed exactly from the Schröedinger equation.

Although the fact that the relevant informations are encoded in the reduced dimensional (SQM) problem at a first time can leave the reader sceptical, in the second part of this thesis there will be shown a completely different approach which leads to a rigorous mathematical interpretation of the meaning and the extent of validity of this procedure.

1.5.3 *Explicit and exact solution of the associated SQM problem*

To compute explicitly the quantum mechanical amplitude (1.42) it is sufficient to know the expressions of the vacuum wave functions. The SQM vacuum wave functions for these models were computed some years ago by Elitzur and Schwimmer [11] (see Appendix B). The normalized wave functions for their Witten vacua are [8]:

$$\begin{aligned}
|m\rangle =& \frac{1}{\pi}\sqrt{\frac{2\sin\left(\frac{\pi m}{\mu+1}\right)}{\mu+1}}\left\{\partial\overline{W}_{SQM}\left(\frac{W_{SQM}}{\overline{W}_{SQM}}\right)^{\frac{m}{2(\mu+1)}} K_{\frac{m}{\mu+1}}\left(2|W_{SQM}|\right)\psi^\dagger|0\rangle + \right.\\
&\left. -\partial W_{SQM}\left(\frac{\overline{W}_{SQM}}{W_{SQM}}\right)^{\frac{\mu+1-m}{2(\mu+1)}} K_{1-\frac{m}{\mu+1}}\left(2|W_{SQM}|\right)\tilde{\psi}^\dagger|0\rangle\right\} = \\
=& \frac{1}{\pi}\sqrt{\frac{2\sin\left(\frac{\pi m}{\mu+1}\right)}{\mu+1}}\left\{e^{i(m-\mu)\theta}\rho^\mu K_{\frac{m}{\mu+1}}\left(\frac{2\rho^{\mu+1}}{\mu+1}\right)\psi^\dagger|0\rangle + \right.\\
&\left. -e^{i(m-1)\theta}\rho^\mu K_{1-\frac{m}{\mu+1}}\left(\frac{2\rho^{\mu+1}}{\mu+1}\right)\tilde{\psi}^\dagger|0\rangle\right\}\\
& 1 \le m \le \Delta = \mu
\end{aligned} \tag{1.45}$$

where $X = \rho e^{i\theta}$, $|0\rangle$ is the tensor product of the Schrödinger standard ket with the Clifford vacuum for fermions and $K_\nu(x)$ are the Bessel functions.

From this explicit expression it is easy to obtain the R–charge of a vacuum wave function. Indeed, under an R–transformation $\theta \to \theta + \alpha/(\mu+1)$ and then

$$|m\rangle \longrightarrow e^{i\left[\frac{m}{\mu+1}-\frac{1}{2}\right]}|m\rangle \tag{1.46}$$

or

$$R|m\rangle = \left[\frac{m}{\mu+1} - \frac{1}{2}\right]|m\rangle \quad . \tag{1.47}$$

From Appendix B one gets:

$$\begin{aligned}
\langle m_2|X^p|m_1\rangle_{SQM} =& \frac{1}{\pi}\delta_{m_2,p+m_1}\sqrt{\sin\left(\frac{\pi m_2}{\mu+1}\right)\sin\left(\frac{\pi m_1}{\mu+1}\right)}\\
&(\mu+1)^{\frac{p}{\mu+1}}\,\Gamma\left(\frac{m_2}{\mu+1}\right)\Gamma\left(1-\frac{m_1}{\mu+1}\right) \quad .
\end{aligned} \tag{1.48}$$

[8] In Appendix B $\psi^\dagger$ is called $\psi^{\bar{1}}$ and $\tilde{\psi}^\dagger$ is called ψ^1.

Thus

$$f_{h,k}(\mu,p) = \delta_{h,p+k}\frac{1}{\pi}\sqrt{\sin\left(\frac{\pi(k+p)}{\mu+1}\right)\sin\left(\frac{\pi k}{\mu+1}\right)} \\ (\mu+1)^{\frac{p}{\mu+1}}\Gamma\left(\frac{k+p}{\mu+1}\right)\Gamma\left(1-\frac{k}{\mu+1}\right) \quad . \tag{1.49}$$

Notice that $\delta_{h,p+k}$ ensures the R–charge conservation.

Moreover, using the identity $\sin(\pi x)\Gamma(x)\Gamma(1-x) = \pi$, it is easy to show that

$$\sum_{m=1}^{\Delta}\langle h|X^{p_1}|m\rangle\langle m|x^{p_2}|k\rangle = \langle h|X^{p_1+p_2}|k\rangle \tag{1.50}$$

This fact is a general property of the CGFs and will be deeply studied in the next chapters.

Thus, it has been possible to exactly compute for the A_μ models the CGFs whose final expression is:

$$\langle h|X^p|k\rangle = \frac{1}{(gL)^{\frac{p}{\mu+1}}}\delta_{h,p+k}\frac{1}{\pi}\sqrt{\sin\left(\frac{\pi(k+p)}{\mu+1}\right)\sin\left(\frac{\pi k}{\mu+1}\right)} \\ (\mu+1)^{\frac{p}{\mu+1}}\Gamma\left(\frac{k+p}{\mu+1}\right)\Gamma\left(1-\frac{k}{\mu+1}\right) \quad . \tag{1.51}$$

It is worth to stress that these Green's functions have been computed without imposing any condition on the coupling constants. They always hold, at criticality and outside criticality, in any ("regular") point of the space of coupling constants and in any phase of the theory [9].

1.6 The quasihomogeneous models

A quasihomogeneous model is a LG model with a superpotential which is an holomorphic quasihomogeneous function in the superfields. A quasihomogeneous function of degree d and indices (weights) $(\alpha_1,\ldots,\alpha_n)$ is an holomorphic function f in n–variables X_i which satisfies the property [21]

$$f(\lambda^{\alpha_i}X_i) = \lambda^d f(X_i) \quad . \tag{1.52}$$

In what follows we shall usually consider quasihomogeneous superpotentials of degree 1.

The condition of quasihomogeneity for a superpotential is equivalent to that of having an unbroken R–symmetry [10]. Thus, all the reasoning done for the A_μ models based on the R–symmetry can be repeated for all quasihomogeneous models. For example, it is possible using the same techniques as in the previous section (or that used in the following chapter based on the supersymmetric Ward identities) to determine the exact dependence of the CGFs on

[9] Indeed, in the technical computation the opposite limit ($L \to 0$) to that I will be interested in for the critical behaviour ($L \to \infty$) has been made.

[10] These are two names for the same thing !

(almost all) the coupling constants of the theory. Let me first consider the following two examples.

(1.53) Example : consider the D_μ models $W = g_1 X^2 Y + g_2 Y^{\mu-1}$. Applying the argument of the previous section one gets

$$\langle h|X^\alpha Y^\beta|k\rangle_{g_1,g_2} = \frac{\langle h|X^\alpha Y^\beta|k\rangle_{1,1}}{(Lg_1)^{\frac{\alpha}{2}}(Lg_2)^{\frac{2\beta-\alpha}{2(\mu-1)}}} \quad . \tag{1.54}$$

Following my general conventions, let $g_1 = gc_1$ and $g_2 = gc_2$, where g is an overall dimensionful coupling constant and c_1, c_2 two adimensional coupling constants. Obviously one of c_1, c_2 is not an independent variable since it is fixed by $g_1/g_2 = c_1/c_2$; for example we can always choose $c_1 = 1$. Anyway, one gets

$$\begin{aligned}\langle h|X^\alpha Y^\beta|k\rangle_{g_1,g_2} &= \frac{\langle h|X^\alpha Y^\beta|k\rangle_{1,1}}{(Lg)^{\alpha\frac{(\mu-2)}{2(\mu-1)}+\beta\frac{1}{\mu-1}}(c_1)^{\frac{\alpha}{2}}(c_2)^{\frac{2\beta-\alpha}{2(\mu-1)}}} \\ &= \frac{f_{h,k}(\alpha,\beta;\mu)}{(Lg)^{\alpha\frac{(\mu-2)}{2(\mu-1)}+\beta\frac{1}{\mu-1}}(c_1)^{\frac{\alpha}{2}}(c_2)^{\frac{2\beta-\alpha}{2(\mu-1)}}} \qquad \blacksquare\end{aligned} \tag{1.55}$$

(1.56) Example : consider the modality 1 [21] superpotential $W = g_1X^3 + g_2Y^3 + g_3Z^3 + \hat{a}XYZ$. Reasoning as before one gets

$$\langle h|X^\alpha Y^\beta Z^\gamma|k\rangle_{g_1,g_2,g_3,\hat{a}} = \frac{\langle h|X^\alpha Y^\beta Z^\gamma|k\rangle_{1,1,1,a}}{(Lg_1)^{\frac{\alpha}{3}}(Lg_2)^{\frac{\beta}{3}}(Lg_3)^{\frac{\gamma}{3}}} \tag{1.57}$$

where $a = \hat{a}/(g_1g_2g_3)^{(1/3)}$. Introducing $g_1 = gc_1$, $g_2 = gc_2$ and $g_3 = gc_3$ (of c_1, c_2 and c_3 only two are independents) one finally obtains

$$\begin{aligned}\langle h|X^\alpha Y^\beta Z^\gamma|k\rangle_{g_1,g_2,g_3,\hat{a}} &= \frac{\langle h|X^\alpha Y^\beta Z^\gamma|k\rangle_{1,1,1,a}}{(Lg)^{\frac{\alpha+\beta+\gamma}{3}}(c_1)^{\frac{\alpha}{3}}(c_2)^{\frac{\beta}{3}}(c_3)^{\frac{\gamma}{3}}} \\ &= \frac{f_{h,k}(\alpha,\beta,\gamma;\mu=8;a)}{(Lg)^{\frac{\alpha+\beta+\gamma}{3}}(c_1)^{\frac{\alpha}{3}}(c_2)^{\frac{\beta}{3}}(c_3)^{\frac{\gamma}{3}}} \qquad \blacksquare\end{aligned} \tag{1.58}$$

The generalization of these two examples is the following. Let W be a quasihomogeneous superpotentials in n fields and with $k \geq n$ monomials, i.e. with k coupling constants g_i: $W = W(X_1,\ldots,X_n;g_1,\ldots,g_k)$. By means of the scaling argument of the previous section one can determine the exact dependence of the CGFs on n coupling constants g_i, that is

$$\begin{aligned}\langle h|X^{\underline{a}}|l\rangle_{g_1,\ldots,g_k} &= \frac{\langle h|X^{\underline{a}}|l\rangle_{1,\ldots,1,g_{n+1},\ldots,g_k}}{\phi(\underline{a};g_1,\ldots,g_n)} = \\ &= \frac{f_{h,l}(\underline{a};\mu;c_{n+1},\ldots,c_k;\underline{\alpha})}{\phi(\underline{a};Lg;c_1,\ldots,c_n;\underline{\alpha})}\end{aligned} \tag{1.59}$$

where ϕ is a *known* function of $(\underline{a};Lg;c_1,\ldots,c_n;\underline{\alpha})$ whereas $f_{h,k}$ is an unknown function.

Again using the argument of the previous section and the definition of quasihomogeneity one can easily compute the dependence of the CGFs on the overall coupling constant g:

$$\phi(\underline{a};Lg;c_1,\ldots,c_n;\underline{\alpha}) = (Lg)^{\sum_{i=1}^n a_i\alpha_i}\varphi(\underline{a};c_1,\ldots,c_n;\underline{\alpha}) \quad . \tag{1.60}$$

One can summarize all that in the following statement:

let W be a quasihomogeneous superpotential of degree 1 and weights $\underline{\alpha}$, $W = W(X_1, \ldots, X_n; g_1, \ldots, g_k)$, $k \geq n$. Let $g_i = gc_i$ where g is an overall dimensionful coupling constant and c_i are adimensional coupling constants (only $k-1$ of them are linearly independent), $W = gF(X_1, \ldots, X_n; c_1, \ldots, c_k)$. Then

$$\langle h|X^{\underline{a}}|k\rangle = \frac{f_{h,k}(\underline{a}; \mu; c_{n+1}, \ldots, c_k; \underline{\alpha})}{(gL)^{\sum_{i=1}^{n} a_i\alpha_i}\varphi(\underline{a}; c_1, \ldots, c_n; \underline{\alpha})} \tag{1.61}$$

where $\varphi(\underline{a}; c_1, \ldots, c_n; \underline{\alpha})$ is a known function.

In principle one could use the arguments of the previous section also to compute $f_{h,k}$. Again the reduction to a SQM problem is formally possible, but now the explicit solution of the associated Schroedinger equation is not known. The computation of $f_{h,k}$ can be explicitly done at least for the D_μ series (see the first example in this section) using the approach developed in the second part of this thesis.

CHAPTER 2

THE A_μ MODELS IN THE IR LIMIT

As an introduction to the general discussion of the IR limit which will be done in the following, in this chapter I will discuss the case of the A_μ models. This case is very instructive also because is exactly solvable, from it one can have some insight in the general situation.

2.1 Preliminary considerations

The Zamolodchikov conjecture [8] has been extended by some authors [9,4,5,6,7,22] to the case of 2d N=2 supersymmetric field theories. It states that the 2d N=2 Wess Zumino (WZ) model [23] with superpotential

$$W = \frac{g}{\mu+1} X^{\mu+1} \qquad \mu \geq 2 \tag{2.1}$$

(which I call the A_μ models) is the Landau–Ginsburg (LG) effective theory [8] for the μ–th model in the N=2 superconformal minimal series [22,24]. If this is indeed the case, these LG lagrangians allow for the exact computation of many properties of the critical theory, thanks to the algebraic properties of the chiral primary fields (for the notion of *Chiral Ring* see [7], next chapter and Appendix C). The aim of this chapter is to prove rigorously this conjecture.

The following discussion has an heuristic nature and should be not seen as an adequate description of the dynamical mechanism, but just an apetizer for the more accurate analysis of the following sections.

Consider the simplest case, the model with cubic superpotential

$$W = \frac{g}{3} X^3 \quad . \tag{2.2}$$

According to refs. [4,5,6,7] this model should reduce — in the critical regime — to the N=2 supersymmetric Gaussian model [22]. In particular, in this limit the light sector of the WZ theory should consist of just one free scalar.

At first, such a claim may sound odd. We know that the model has unbroken supersymmetry since its Witten index Δ [15] is 2. The classical vacuum is $X = 0$. The West theorem [25] guarantees that this vacuum will persist to all orders in perturbation theory. Non–perturbative violation of this theorem are known to be possible [26], but here the situation is more subtle since the persistence of the $X = 0$ vacuum is independently guaranteed by th R–symmetry together with the Coleman theorem [27]. This persistence of the vacuum, (the non–renormalization of the superpotential W [19]), seems to imply that the full superfield X should be massless.

This naive analysis would be correct in four dimensions. In 4d it follows from a symmetry argument. The theory cannot have a mass–gap since it has a chiral invariance: one needs a massless state to saturate the two–current function ('t Hooft criterion [28]). Then, by the irreducibility of the chiral supermultiplet, the whole superfield should be massless.

In two dimensions this irreducibility does not hold. In fact it is known [22] that a free massless scalar alone can lead to an N=2 supersymmetric theory. Since the scalar field X itself has not definite supersymmetric partner, this possibility is connected to the IR divergencies which cause X not to be a well defined operator in the quantum theory. The well–defined operators — those creating the quantum states — are the exponentials $\exp(i\alpha X)$which do have supersymmetric partners [22]. The non–standard dynamical realizations of supersymmetry are thus related to the very IR singularities which make perturbation theory not to be reliable and spoil the naive version of the West's theorem.

To understand euristically the physical origin of these IR subtleties, let consider the following aspect of the problem. If the WZ model should become (in the IR region) the Gaussian model, some component of the "original" fermion should acquire a dynamical mass. This may seem paradoxical, since the model has an *unbroken* chiral symmetry. This "paradox" was solved long ago by Witten [29]. In 2d one can have massive fermions *together* with unbroken chiral symmetry. This peculiar possibility is related to the same IR divergencies needed to *restaurate* supersymmetry after a mass-splitting inside the WZ superfield. In the fermionic sector of the 2d WZ model Witten's arguments apply (at least qualitatively). Write the complex scalar of the WZ multiplet as $X = \rho \exp[i\theta]$ and use the scalar σ to bosonize its superpartner ψ. If in the relevant regime the theory behaves as if ρ had a non–trivial v.e.v., then the linear combination $\theta + \sigma$ would correspond to a massive fermion inert under chirality [29]. The other linear combination φ will look as a free field (in the IR) carrying by itself a representation of supersymmetry in the sense explained before.

Witten remarks [29] on the reliability of perturbation theory apply to the present models as well. Since perturbation theory easily leads to fallacious conclusions, it is important to have some non–perturbative tool to study the dynamics of the theory. With such a non–perturbative control at our disposal, it is possible to easily solve the apparent inconsistencies one finds in perturbation theory. Such a tool is the (local) Nicolai map [17,13,16] that will be discussed in detail in the following sections.

2.2 Computation of the relevant critical indices

The Zamolodchikov conjecture states that at the IR critical point the A_μ models are described by the N=2 A_μ superconformal minimal models. More precisely, at criticality the correlation functions for the chiral fields which have been computed in the previous Chapter should correspond to that of the chiral primary fields of the corresponding N=2 SCFT. Indeed I will identify (up to normalization) the (anti–) chiral fields in field theory with the primary (anti–) chiral NS SCFT fields:

$$X^k(z,\bar{z}) \sim \frac{1}{\alpha_\mu(k)} N_k(z,\bar{z}) \qquad k = 0,\ldots,\mu-1 \tag{2.3}$$

where $\alpha_\mu(k)$ are suitable normalization coefficients. (For a detailed discussion of the choice of normalization in field theory and superconformal theories see Appendix A and Appendix C.) Obviously $\alpha_\mu(0) = 1$. Moreover the condition $k \leq \mu - 1$ is imposed in field theory by the conservation of the R–charge in the Green's functions (see (1.51)), in fact $\langle m_1|X^{\mu+k}|m_2\rangle = 0$ for $k \geq 0$.

I will now check that the identification between chiral fields in field theory and superconformal theory is indeed correct. First, the chiral fields are in the same number as the primary chiral SCFT fields. It is easy to check that they have also the same dimensions (see (1.51)) [11]

$$h(X^k) = \frac{k}{2(\mu+1)} \qquad 0 \leq k \leq \mu - 1 \tag{2.4}$$

and identifying the R–charges with the U(1) charges, they have also the same charges

$$\chi(X^k) = \frac{k}{\mu+1} \qquad 0 \leq k \leq \mu - 1 \quad . \tag{2.5}$$

From Appendix C it follows that

$$\frac{c}{6} = h_{max} = \frac{1}{2}\left(\frac{\mu-1}{\mu+1}\right) \tag{2.6}$$

or

$$c = 3 - \frac{6}{\mu+1} \tag{2.7}$$

as it should be.

The central charge can be computed also from the Witten vacua. Notice that the Witten vacua (eq. (1.45)) correspond in the critical limit to the Ramond operators R_m with $h = \bar{h} = c/24$, creating the supersymmetric Ramond states of the critical theory

$$|m\rangle \simeq R_m(0,0)|0\rangle_{SL(2,C)} \quad . \tag{2.8}$$

In particular their chiral charge is given by eq (1.47) . The states $|\mu\rangle$ ($|1\rangle$) with the maximal (minimal) R–charge are created out of the $SL(2,C)$ vacuum by the action of primary fields which are pure exponentials of the free field bosonizing the superconformal U(1) current. This statement can be justified from the WZ point of view by an analysis of the WZ models on the lattice or by comparing their dimensions with their U(1) charge. This also follows

[11] By definition the anomalous dimension at the fixed point $h(X)$ satisfies $L\frac{\partial}{\partial L}\langle p|X^k|q\rangle = -2h\langle p|X^k|q\rangle$, see for example Chap. 3 [30].

from the spectral flow argument [7]. Writing the superconformal U(1) current as $J(z) = i\partial\tilde{\phi}$ with the following normalization $\langle\tilde{\phi}(z)\tilde{\phi}(0)\rangle = (c/3)\log(z)$, and using

$$\langle J(z)J(0)\rangle = \frac{c}{3}\frac{1}{z^2} \tag{2.9}$$

one can see that the dimension of $\exp(3i\frac{\chi}{c}\tilde{\phi})$ (U(1) charge χ) is

$$h_\chi = \frac{3}{2c}(\chi)^2 \quad . \tag{2.10}$$

Using this formula for the operator R_μ with $\chi_{max} = \frac{\mu-1}{2(\mu+1)}$ (eq. (1.47)) one gets

$$\frac{c}{24} = \frac{1}{4}\chi_{max} = \frac{1}{8}\left(\frac{\mu-1}{\mu+1}\right) \tag{2.11}$$

which is the correct result.

But it is possible to do much more. In the previous chapter I have exactly computed the correlation functions of the chiral fields in field theory and they do not depend on the regime of the theory. Thus, these correlation functions are correct also at criticality and up to the normalization (the $\alpha_\mu(k)$) must coincide with those of primary chiral SCFT fields. Thus

$$\langle m_2|X^k|m_1\rangle \tag{2.12}$$

must be compared with

$$\langle R^{\mu-m_2}_{\mu-m_2,1}(\infty)N_k(1)R^{\mu-m_1}_{\mu-m_1,1}(0)\rangle =: C_\mu(k, m_1-1) \tag{2.13}$$

which is nothing else that the coefficient of the operator product expansion defined by

$$N_k(z,\bar{z})N_h(z,\bar{z}) \sim C_\mu(k,h)N_{k+h}(z,\bar{z}) \quad . \tag{2.14}$$

The expression of $C_\mu(k,h)$ has been computed in literature and is given in Appendix C.

Thus, all we need to do is to compute the $\alpha_\mu(k)$. Indeed I want to check if the following formula is satisfied

$$C_\mu(k,h) = \langle h+k|N_k|h\rangle \stackrel{?}{=} \alpha_\mu(k)\langle h+k|X^k|h\rangle \quad . \tag{2.15}$$

From (1.50) it follows

$$X^k \cdot X^h \sim 1 \cdot X^{k+h} \quad . \tag{2.16}$$

Thus, if $X^k \sim \frac{1}{\alpha_\mu(k)}N_k$ then

$$C_\mu(k,h) \stackrel{?}{=} \frac{\alpha_\mu(k)\alpha_\mu(h)}{\alpha_\mu(k+h)} \quad . \tag{2.17}$$

By consistency of (2.15) and (2.17) one gets

$$\frac{\alpha_\mu(h-1)}{\alpha_\mu(k+h-1)} = \langle h+k|X^k|h\rangle \quad . \tag{2.18}$$

This equation is consistent because of the property

$$F_\mu(m, k+h) = F_\mu(k+m, h)F_\mu(m, k) \tag{2.19}$$

where $F_\mu(m,k) := \langle m+k|X^k|m\rangle$.
Its solution is

$$\alpha_\mu(k) = \frac{\alpha_\mu(0)}{F_\mu(1,k)} = \frac{1}{F_\mu(1,k)} \tag{2.20}$$

and hence

$$C_\mu(k,h) \stackrel{?}{=} \frac{F_\mu(1,k+h)}{F_\mu(1,k)F_\mu(1,h)} \tag{2.21}$$

$0 \leq k, h, k+h \leq \mu - 1$. This relation holds true since

$$F_\mu(1,k)\Sigma(\mu,k) = (\mu+1)^{\frac{k}{\mu+1}} \tag{2.22}$$

(see Appendix C).

Thus it has been checked that also the OPE coefficients and the chiral correlation functions of the superconformal field theory and of the LG theory agree in the case of the A_μ models.

2.3 Infrared critical point and Stochastic reconstruction of the WZ models

In the previous sections it has been shown that some correlation functions computed exactly in field theory (also at the critical point) are equal (up to the normalization) to those of the corresponding N=2 SCFT as indicated by the Zamolodchikov conjecture. This is not a sufficient argument to prove that the field theory has a critical point. In this section I want to show that the A_μ models have a critical fixed point in the IR where the Zamolodchikov conjecture holds true.

2d WZ models have an interpretation as stochastic systems [13,16]. Indeed, all the bosonic correlation functions can be computed from the Langevin equation

$$\partial\phi + \left[\frac{\partial W}{\partial \phi}\right]^* = h(z,\bar{z}) \tag{2.23}$$

where h is a "*white noise* " (i.e. a Gaussian variable with zero expectation value and covariance 1). The stochastic identities (the statement that h is Gaussian) can be shown to be equivalent to the supersymmetric Ward identities.

The stochastic interpretation is correct only on the *torus* and only if the fermionic fields are periodic on both cycles [13]. This fact will be important below.

The stochastic process can be seen as an infinite set of sum rules for the Green's functions of some composite operators

$$\begin{aligned}
&\langle(\partial\phi + F)\rangle = \langle(\bar\partial\bar\phi + \bar F)\rangle = 0 \\
&\langle(\partial\phi + F)(z)(\partial\phi + F)(w)\rangle = 0 \\
&\langle(\bar\partial\bar\phi + \bar F)(z)(\bar\partial\bar\phi + \bar F)(w)\rangle = 0 \\
&\langle(\partial\phi + F)(z)(\bar\partial\bar\phi + \bar F)(w)\rangle = \delta^2(z-w) \\
&\cdots \quad \cdots \\
&\cdots \quad \cdots
\end{aligned} \tag{2.24}$$

where $F = [\partial W/\partial\phi]^*$. These sum rules give some non–perturbative control over the theory. This would be enough for our present purposes. But it is convenient to go one step further, i.e. use the stochastic process (2.23) to solve the WZ model.

Instead of looking for the general solution of the stochastic problem, it is convenient to try with a physical ansatz for the solution and to check if this ansatz is a solution (obviously, this is a simpler problem). The ansatz will be constructed starting from the Zamolodchikov conjecture that the WZ theory becomes at the critical point a known N=2 SCFT.

For simplicity I consider the simplest model with cubic superpotential ($\mu = 2$)

$$W = \frac{g}{3}\phi^3 \quad . \tag{2.25}$$

which at the critical IR point should become the Gaussian model.

A solution to its Langevin equation consists of a Hilbert space $\mathbf{H}$ and two special states $|0,\pm\rangle \in \mathbf{H}$ (the two vacua on the cylinder predicted by $\Delta = \mu = 2$) together with a scalar 2d field operator $\Phi(z)$ ($\Phi^\dagger \neq \Phi$) acting on $\mathbf{H}$, such that the composite fields

$$\begin{aligned}
\eta[\Phi](z,\bar z) &= \partial\Phi(z,\bar z) + g_0 : [\Phi^\dagger(z,\bar z)]^2 : \\
\eta^\dagger[\Phi](z,\bar z) &= \bar\partial\Phi^\dagger(z,\bar z) + g_0 : [\Phi(z,\bar z)]^2 :
\end{aligned} \tag{2.26}$$

(: : denotes some "natural" normal product) satisfy the Gaussian identities

$$\begin{aligned}
&\langle\eta(z)\rangle_\pm = \langle\eta^\dagger(z)\rangle_\pm = 0 \\
&\langle\eta(z)\eta(w)\rangle_\pm = 0 \\
&\langle\eta^\dagger(z)\eta^\dagger(w)\rangle_\pm = 0 \\
&\langle\eta(z)\eta^\dagger(w)\rangle_\pm = \delta^2(z-w) \\
&\langle\eta(z_1)\ldots\eta(z_k)\eta^\dagger(w_1)\ldots\eta^\dagger(w_l)\rangle_\pm^{conn} = 0 \qquad \text{for } n+m>2
\end{aligned} \tag{2.27}$$

($\langle\cdots\rangle_\pm$ is the Euclidean continuation of $T\langle 0,\pm|\cdots|0,\pm\rangle$).

The expectation value $\langle\cdots\rangle$ can be expressed as functional integrals on the torus [13] and with this formulation one can explicitly show that the knowledge of such a Φ would give a complete solution to the 2d WZ model [12].

[12] I refer to [13] for every detail on the stochastic reconstruction of 2d N=2 supersymmetric WZ models.

I will now construct the operator Φ. If the Zamolodchikov conjecture that the WZ theory becomes at the critical point the Gaussian model is true, in this regime one should be able to construct the field Φ out of the Gaussian model. Let start with a (real) free massless scalar [22] φ

$$\begin{aligned} S[\varphi] &= \frac{6}{\pi}\int d^2z\partial\varphi\bar\partial\varphi \\ \varphi(z,\bar z) &= \varphi_l(z)+\varphi_r(\bar z) \\ \varphi_a &= 6[\varphi_l(z)-\varphi_r(\bar z)] \end{aligned} \tag{2.28}$$

where $\varphi \sim \varphi + 2\pi$.

According to ref. [22] the exponentials of φ give rise to a chiral superfield [13]

$$e^{2i\varphi}+\frac{\theta_+}{\sqrt3}e^{-4i\varphi_l(z)+2i\varphi_r(\bar z)}+\frac{\bar\theta_+}{\sqrt3}e^{2i\varphi_l(z)-4i\varphi_r(\bar z)}+\frac{\theta_+\bar\theta_+}{3}e^{-4i\varphi}\ . \tag{2.29}$$

With the identifications

$$\begin{aligned} &\Phi = e^{2i\varphi} \\ &\Phi^\dagger = e^{-2i\varphi} \\ &:\Phi^2:\equiv:(e^{2i\varphi})^2:=\lim_{w\to z}\left(\frac{a}{|z-w|}\right)^{\frac13}e^{2i\varphi(w)}e^{2i\varphi(z)} \end{aligned} \tag{2.30}$$

I am led to the ansatz

$$\begin{aligned} \eta &= \partial(e^{2i\varphi})+ge^{-4i\varphi} \\ \eta^\dagger &= \bar\partial(e^{-2i\varphi})+ge^{4i\varphi} \end{aligned} \tag{2.31}$$

(here $g = 1/(3a)$ where a is the scale used to define the free propagator $\langle\varphi(z)\varphi(0)\rangle = -(1/12)\ln(|z|^2/a^2)$).

All what I have to check is that on the torus (periodic sector) eqs. (2.27) hold for the operators η and $\eta^\dagger$ of the N=2 Gaussian model.

First of all, I will prove the following result:

$$\langle\eta(z_1)\ldots\eta(z_n)\eta^\dagger(w_1)\ldots\eta^\dagger(w_m)\rangle_{torus,\ periodic}=0 \tag{2.32}$$

if all the points $z_1,\ldots,z_n,w_1,\ldots,w_m$ are distinct.

The proof of this equation goes as follows. Consider a parallel section of the bundle of $(-1/2)$ – differentials ε (Killing spinors). They exist only on the torus with the odd spin structure. In a suitable parametrization ε is just the number 1. Then consider the *closed* 1–form

$$\omega(z,\bar z)=i\left[\varepsilon T_F^+(z)+\bar\varepsilon\widetilde{T}_F^-(\bar z)\right] \tag{2.33}$$

[13] The notation of ref. [22] for the Gaussian model are adopted.

where

$$T_F^{\pm}(z) = \frac{1}{\sqrt{3}} e^{\pm i(3\varphi + \frac{1}{2}\varphi_a)} \tag{2.34}$$
$$\widetilde{T}_F^{\pm}(\bar{z}) = \frac{1}{\sqrt{3}} e^{\pm i(3\varphi - \frac{1}{2}\varphi_a)}$$

One has

$$\frac{1}{\sqrt{3}} \oint_C \omega(w,\bar{w}) e^{-i(\varphi+\frac{1}{2}\varphi_a)}(z,\bar{z}) = \eta(z,\bar{z}) \tag{2.35}$$
$$\frac{1}{\sqrt{3}} \oint_C \omega(w,\bar{w}) e^{+i(\varphi-\frac{1}{2}\varphi_a)}(z,\bar{z}) = \eta^{\dagger}(z,\bar{z})$$

where the contour C encloses the point z.

From the OPE

$$\omega(z,\bar{z})\omega(w,\bar{w}) = \text{ finite as } z \longrightarrow w \tag{2.36}$$

and the anticommuting nature of the supercurrents $T_F^{\pm}$ one gets

$$\oint_C \omega(z,\bar{z})\eta(w,\bar{w}) = \oint_C \omega(z,\bar{z})\eta^{\dagger}(w,\bar{w}) = 0 \tag{2.37}$$

which correspond in the operator formalism to the identities

$$\begin{aligned}
&-[(\overline{Q}_+ + Q_-), \{(\overline{Q}_+ + Q_-), \psi_+\}] = \\
&= [(\overline{Q}_+ + Q_-), \{\psi_+, (\overline{Q}_+ + Q_-)\}] + [\psi_+, \{(\overline{Q}_+ + Q_-), (\overline{Q}_+ + Q_-)\}] = \\
&= [(\overline{Q}_+ + Q_-), \{(\overline{Q}_+ + Q_-), \psi_+\}] \Rightarrow \\
&\Rightarrow [(\overline{Q}_+ + Q_-), \{(\overline{Q}_+ + Q_-), \psi_+\}] = 0 \\
&\;[(\overline{Q}_+ + Q_-), \{(\overline{Q}_+ + Q_-), \overline{\psi}_-\}] = 0.
\end{aligned} \tag{2.38}$$

Suppose that in equation (2.32) $z_1 \neq z_i$ ($i \neq 1$) and $z_1 \neq w_i$. Use for $\eta(z_1)$ the representation given in equation (2.35) with C a small circle around z_1. Deforming C one can reduce the given Green's function to a sum over contours around the points z_i ($i \neq 1$) and w_j. Then it vanishes by eq. (2.37) . Thus, the Green's functions in eq. (2.32) can be non–vanishing only if *all* the points z_i are equal to some w_j. ∎

Now it remains to check that the behaviour at coinciding points is the one required by eq. (2.27) . The δ–functions come from the T–product. If the derivatives acts inside the T–product, the above Green's functions vanish identically. The δ contribution to the n–point function comes from the derivation of the T–product prescription. For instance, for the two–point function the δ–term is

$$-\partial\bar{\partial}\langle e^{2i\varphi(z)} e^{-2i\varphi(w)}\rangle \simeq \delta(t-t')\langle [e^{2i\varphi}, e^{-2i\varphi}\partial\varphi]_{ET}\rangle \tag{2.39}$$

which *naively* would be the correct result, but it is not since the product $e^{2i\varphi} \cdot e^{-2i\varphi}$ is singular and gives an additional divergent factor. In fact, the major problem is that the product $e^{2i\varphi} \cdot e^{-2i\varphi}$ does not corresponds to the one introduced by the path integral with which (2.27) are defined.

For these reasons, the above ansatz is not the exact non–perturbative solution of the 2d WZ model. This is expected on physical grounds. The WZ model becomes the Gaussian model only at criticality and hence the Gaussian ansatz could be a solution of the WZ Langevin equation only in a specific limit, corresponding to the critical case. Since our ansatz give a fairly good approximate solution to the stochastic process (up to short–distance singularities), let me try to correct their problems to get the true solution.

Physically it is rather clear what is going on. The original model has two (physical) scalars and a Dirac fermion. There are good reasons to believe that in the IR regime the theory looks like a single massless scalar. This implies that some of the original degrees of freedom get a dynamical mass. In the UV region the massive degrees of freedom are also relevant. Of course, it is the UV behaviour which should be used to compute the coincidence limit for the operators. This behaviour is just the free one, and, at a sufficiently short scale, one gets canonical scaling with no singularity in the product of any two chiral superfields at coinciding points [14]

$$(-\partial\bar{\partial}\langle\varphi(z)\varphi^{\dagger}(w)\rangle)_{\delta-singularity} = \delta^{(2)}(z-w) \quad . \tag{2.40}$$

One can use these considerations to construct an *approximate* solution to the Langevin equation. The limit in which it becomes exact is the critical limit for the WZ model. Consider, just as an example, the Lagrangian

$$\beta\partial_\mu\phi\partial^\mu\phi^* + \lambda^2[|\phi|^2 - 1]^2 \tag{2.41}$$

interpolating between a free complex scalar at $\lambda = 0$ and the Gaussian model (for the field φ, $\phi = \exp[i\varphi]$) at $\lambda = \infty$. The field $\rho = |\phi|$ gets a mass $O(\lambda)$ and it decouples in the strong coupling regime. For a large value of the coupling λ and a suitable β the field ϕ would be a fairly good approximation for our stochastic process. It gives the correct value for the η, $\eta^\dagger$ correlation function both for separations $\gg \lambda^{-1}$ where it reduces to the Gaussian model, as well as in the coincidence limit, where it reduces to the free field ϕ. At the same time, it solves the problem with the definition of the product in the ansatz. Of course, with this approximation, there are violations of the stochastic identities at length scales $O(\lambda^{-1})$. However, at least pointwise, the difference with the correct stochastic Green's functions vanishes in the limit $\lambda \to \infty$, that is for infinite mass. So, for very large (but finite) λ this seems to be a good approximation.

Since the coupling constant g is the only scale in the WZ theory, the limit in which everything (but 't Hooft's scalars) gets an infinite mass corresponds to $g \to \infty$, i.e. the strong coupling. This is also clear from the Gaussian approximation to the stochastic process (i.e. the ansatz). The divergent factor $\varepsilon^{-1/3}$ one needs in the point splitting definition of the drift term $\Phi(z+\varepsilon)\cdot\Phi(z-\varepsilon)$ can be seen physically as a divergent coupling $g = g_0\varepsilon^{-1/3}$ needed to make the Gaussian ansatz for η not trivial. However the physical, measurable, coupling is finite in the limit.

[14] Because of the non–renormalization theorem.

The evidence from the Langevin process is fully consistent with the idea that the spectrum of the theory consists of a massless scalar, the rest of the chiral multiplet getting a mass of order g. The Gaussian model is an approximate solution to the WZ model which becomes exact in the strong coupling limit. This implies that the critical behaviour of the WZ model is that of the Gaussian model.

The case of a general model with a superpotential of the form

$$V = \frac{g}{(\mu+1)}\phi^{\mu+1} \tag{2.42}$$

is similar. For each model of the N=2 discrete series, one can consider the would be Wess–Zumino field [15]

$$\Phi \simeq \phi_1^{(1)} \exp\left\{ i\frac{1}{\sqrt{2(\mu^2-1)}}\varphi(z)\right\} \tag{2.43}$$

and the corresponding "white noise"

$$\eta_{(n)}[\Phi] = \partial\Phi + g' : [\Phi]^n : \quad . \tag{2.44}$$

Using the same arguments as in eq. (2.32) one verifies that the (Ramond sector) correlation functions for the fields $\eta_{(n)}[\Phi]$ and $\eta^\dagger_{(n)}[\Phi]$ satisfy the stochastic identities (2.27) as a consequence of the operator algebra of the N=2 minimal models [16]. Again, the δ singularity for coinciding points is not the correct one. At length scales $\ll g^{-1}$ the WZ theory is essentially free, since the couplings are super–renormalizable and the theory is free of UV divergencies. Nevertheless, the N=2 superconformal minimal model is a solution of the Langevin equation in the infrared limit, i.e. at criticality. And it becomes exact for strong coupling.

Some remarks are in order. The correctness of the above picture for the IR regime follows from the uniqueness of the solution to the stochastic equation. This can also be seen as follows. The Green's functions for ϕ and ϕ^n cannot be trivial in the IR since the model has a continuous chiral symmetry acting non–trivially on the fermions. Then the functions

$$\langle \phi(z_1)\ldots\phi(z_n)\phi^\dagger(w_1)\ldots\phi^\dagger(w_m)\phi^n(y_1)\ldots\phi^n(y_p)\phi^{\dagger n}(u_1)\ldots\phi^{\dagger n}(u_q)\rangle \tag{2.45}$$

should scale for large distances like a power (up to logarithms). The short distance behaviour

$$\langle\phi(z)\phi^\dagger(0)\rangle = -2\ln|z|^2 + O(|z|^2\ln^2|z|^2) \tag{2.46}$$

is not a solution for large distances. Thus, there must be a scale at which the theory changes from the short–scale canonical behaviour, to the long–range one. This scale must be of order g. The stochastic identities give an infinite set of equations for the IR scaling dimensions and the operator algebra coefficients. Since this algebra should be supersymmetric, after a little thought one realizes that the above solution is the only possible one.

These results should be compared with the analysis of refs. [4,5,6]. There one (essentially) neglects the kinetic terms on the ground that they are irrelevant in the critical regime. Here we find that the critical theory is approached in the limit $g \to \infty$, which is the limit in which the superpotential dominates the kinetic term.

[15] $\phi_1^{(1)}$ is a parafermionic field, see Appendix C for details.

[16] Compare with the argument in ref. [8].

CHAPTER 3

ALGEBRAIC PROPERTIES OF 2D N=2 LANDAU–GINZBURG MODELS

In this chapter, always with standard QFT methods, I will study some general properties of LG models. The conclusions drawn in this chapter, often obtained through formal arguments, will be rigorously proven and developed in the second part of this thesis.

3.1 The matrix isomorphism

Let me first introduce a definition

(3.1) Definition : *A chiral composite operator* $\Phi\,(X_i(z))$ *is called* **primary** *if there exist two Witten vacua* $|m\rangle$ *and* $|n\rangle$ *such that*

$$\langle m|\Phi(X_i)|n\rangle \neq 0 \quad . \tag{3.2}$$

In the first chapter we have seen that the independence of CGFs from the coordinates implies that short distance singularity does not exist in the product of two chiral fields. The absence of singular contributions in the OPE of two chiral fields can be easily proven in the critical case, see [7]. From the definition of a chiral field we see that all the operators in the OPE of two chiral fields are also chiral. Taking advantage of the absence of singularities, one can define a natural product in the space of all chiral fields

$$X_iX_j(z) := \lim_{z_1,z_2\to z} X_i(z_1)X_j(z_2), \tag{3.3}$$

the point-wise product.

Consider

$$\langle k|X_i(z)X_j(w)|h\rangle, \tag{3.4}$$

since this function does not depend on z, w, its values in the coincidence limit $z\to w$ and in the limit $|z-w|\to\infty$ should agree. In the first limit one gets just the point-wise product of the two fields

$$\langle k|X_i(z)X_j(w)|h\rangle \underset{w\to z}{\longrightarrow} \langle k|X_iX_j(z)|h\rangle \quad . \tag{3.5}$$

In the infinite distance limit only the zero-energy intermediate states give a non-vanishing contribution (on the cylinder and under our assumptions on W there is an energy-gap):

$$\langle k|X_i(z)X_j(w)|h\rangle \underset{|w-z|\to\infty}{\longrightarrow} \sum_l \langle k|X_i|l\rangle\langle l|X_j|h\rangle \quad . \tag{3.6}$$

Thus,

$$\langle k|X_iX_j|h\rangle = \sum_l \langle k|X_i|l\rangle\langle l|X_j|h\rangle \quad . \tag{3.7}$$

This equation has an obvious meaning. Consider the algebra $\mathcal{A}$ of the chiral fields with the point-wise product. It is a subalgebra of the operator algebra of the theory. Let

$$\varpi\colon \mathcal{A} \to M \qquad (M = \{\Delta \times \Delta \text{ complex matrices}\}) \tag{3.8}$$

be the map

$$\varpi(\Phi) = \langle k|\Phi|h\rangle, \tag{3.9}$$

then eq. (3.7) states that ϖ is a $\mathbb{C}$–Algebra homomorphism. In other words, ϖ maps the point-wise product of field operators into the usual matrix product.

The kernel of ϖ (i.e. the Φ such that $\varpi(\Phi) = \langle k|\Phi|h\rangle = 0$) is an ideal $\mathcal{S}$ of $\mathcal{A}$. I shall call the elements of $\mathcal{S}$ the **secondary** chiral fields. Then the algebra $\mathcal{R} = \mathcal{A}/\mathcal{S}$ will be called the **primary chiral algebra**, and its elements **primary** chiral fields. $\mathcal{R}$ is a close relative, defined even outside criticality, of the chiral ring introduced in ref. [7] for the critical theory. Notice that here a "primary operator" is an *equivalence class*, not just a *single* operator as it is in the conformal theory. This is the price to pay to extend the chiral ring to the general situation.

As a motivation for the definition of primary and secondary chiral fields, notice that if a chiral field can be written as $X' = \{\overline{Q}, \Psi\}$, for some operator Ψ, then it is necessarily secondary since by sandwiching it between any two susy vacua one gets zero. So the primary fields are those which cannot be written as supersymmetric transformations of some other field. Obviously, $\varpi : \mathcal{R} = \mathcal{A}/\mathcal{S} \longrightarrow \mathcal{M} := \varpi(\mathcal{A}) \subset M$ *is an algebra isomorphism.*

Since the point-wise product is commutative, then $\mathcal{M}$ *is a commuting algebra of* $\Delta \times \Delta$ *matrices.*

This isomorphism gives an explicit matrix realization of the chiral algebra which is very convenient for practical computations. $\mathcal{R}$ is an algebra with unity, since 1 is a chiral field and does not belong to $\mathcal{S}$.

A commuting algebra of $\Delta \times \Delta$ matrices ($\Delta < \infty$) has always a finite dimensional basis (see Appendix F), thus $\mathcal{M}$ has a finite basis. Thank to the isomorphism ϖ, this implies that *the chiral ring* $\mathcal{R}$ *is finite dimensional, i.e. there exists a finite number of chiral primary fields which constitute a basis of the primary chiral algebra. Moreover, it can be shown that there exists a* **monomial** *basis of* $\mathcal{R}$.

(For other details on commutative algebras see Appendix F.)

Many properties of the chiral ring found in ref. [7] can be easily recovered from the isomorphism ϖ. In particular, (under mild assumptions) we can recover the spectral flow [31]. The remarkable fact is that one gets these results also outside criticality.

At this algebraic level, a "spectral–flow" is just the following. Let $\mathcal{W}$ be the vector space spanned by the supersymmetric vacua $|k\rangle$. It is an $\mathcal{R}$–module [17]. Obviously, $\mathcal{R}$ is also

[17] For the definition of $\mathcal{R}$–module see Appendix F.

an $\mathcal{R}$–module. The spectral flow states that $\mathcal{W}$ and $\mathcal{R}$ are isomorphic as $\mathcal{R}$–modules. Let $\varrho : \mathcal{R} \to W$ be this isomorphism. Any isomorphism between $\mathcal{R}$–modules has the property $\varrho(a) = a\varrho(1)$, $\forall a \in \mathcal{R}$. Then, if $|0\rangle \equiv \varrho(1)$, the isomorphism should read as $\varrho(a) = a|0\rangle$. So the spectral flow requires that $\mathcal{W}$ is a cyclic module (i.e. that there exist a cyclic vector $|0\rangle$ such that $\mathcal{R}|0\rangle$ spans $\mathcal{W}$) and that $\dim_{\mathbb{C}}\mathcal{R} = \dim_{\mathbb{C}}\mathcal{W}$.

3.1.1 *The $n = 1$ case*

To illustrate the above ideas and to motivate further developments, I give a detailed discussion of the simplest case, the LG models with just one chiral field. In this case, the algebra $\mathcal{A}$ is generated by a single element X, so it is isomorphic to $\mathbb{C}[X]$ and then it is an Euclidean ring. If $\Delta < \infty$, $\mathcal{W}$ is a finitely generated module over an Euclidean ring, and therefore, in the most general situation, it is a direct sum of cyclic modules $\mathcal{W} = \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_N$ ($\mathcal{W}_i$ cyclic). There are cyclic vectors

$$|0, i\rangle \in \mathcal{W}_i \qquad \text{with} \quad \mathcal{W}_i \subset \mathcal{A}|0, i\rangle \quad . \tag{3.10}$$

Identifying $\mathcal{R}$ with its isomorphic image $\varpi(\mathcal{R})$, one sees that $\mathcal{R} = \bigoplus_i \mathcal{R}_i$ where $\mathcal{R}_i\colon \mathcal{W}_i \to \mathcal{W}_i$ and $\mathcal{R}_i \xrightarrow{\sim} \mathcal{W}_i$ as $\mathcal{R}_i$–modules. Each couple $(\mathcal{R}_i, \mathcal{W}_i)$ has the structure of a chiral ring for a critical theory (possibly trivial) and in fact, it corresponds to some critical theory related to the given LG model. In the rest of this section I want to substantiate the following statement

the critical behaviour of the LG models is characterized by the Jordan normal form of the matrices $\langle k|X|h\rangle$.

To see this, let me give a closer look to the matrix algebra. Consider $\langle k|X|h\rangle$ as a $\Delta \times \Delta$ matrix with complex entries. The Caley–Hamilton theorem [18] states that there exists an unique monic (i.e. with "director" coefficient 1) polynomial of minimum degree $\leq \Delta$, $p(x)$, such that $\langle k|X|h\rangle$ is a root of $p(x) = 0$. This polynomial is called the **minimal polynomial** of the matrix $\langle k|X|h\rangle$, this means that $p(\langle X\rangle) = 0$ in $\mathcal{R}$ or that $p(X) = \{\overline{Q}, \psi\}$ for some ψ.

On the other hand, let me consider the following equation which is a natural consequence of the supersymmetric Ward identities

$$0 = \langle m|\{\overline{Q}_+, [\overline{Q}_-, \overline{X}]\}|n\rangle = \langle m|\frac{\partial W}{\partial X}|n\rangle \tag{3.11}$$

which also follows from the equation of motion

$$\overline{D}^2\overline{X} + \frac{\partial W}{\partial X} = 0 \quad . \tag{3.12}$$

Thus the polynomial $\frac{\partial W}{\partial X}(x) =: q(x)$ admits $\langle X\rangle$ as a root. Moreover $q(x)$ has degree Δ by definition of the Witten index and, by definition of $\mathcal{W}$, $\dim\mathcal{W} = \Delta$. Thus, by purely algebraic reasoning, $q(x) = r(x)p(x)$ since $p(x)$ is the minimal polynomial. But $q(\langle X\rangle) = 0$ are the

[18] For some general reference on the theory of matrices and λ–matrices see ref. [32] and Appendix F.

equations of motion and if the degree of $p(x)$ would be less than Δ it would imply that there is an equation $\overline{D}^2($ something $) = p(X)$ more basic than the equation of motion itself, since this implies the equation of motion but it is not implied.

Thus $q(x) = \alpha\, p(x)$ and the minimal polynomial has degree Δ. From the theory of λ–matrices (see Appendix F) it is known that

$$p(x) = \pm \frac{\det[\langle X \rangle - x\mathbf{1}]}{gcd\{M_{ij}[\langle X \rangle - x\mathbf{1}]\}} \tag{3.13}$$

where $M_{ij}[\langle X \rangle - x\mathbf{1}]$ are the minors of degree $\Delta - 1$ of the matrix $(\langle X \rangle - x\mathbf{1})$ and gcd is the greatest common divisor. $\det[\langle X \rangle - x\mathbf{1}]$ is the **characteristic polynomial** associated to the matrix $\langle X \rangle$ (usually denoted by $\det(A - \lambda\mathbf{1})$) and has degree Δ for a $\Delta \times \Delta$ matrix. Since also $p(x)$ has degree Δ then $gcd\{M_{ij}\}$ has degree 0, i.e. is a constant. Thus, the minimal polynomial coincides with the characteristic polynomial (up to a constant)

$$p(x) = \alpha' \det\left[\langle X \rangle - x\mathbf{1}\right] \quad . \tag{3.14}$$

Another proof of the fact that $\frac{\partial W}{\partial X}(x) = \alpha' \det[\langle X \rangle - x\mathbf{1}]$ is the following. Recall from §1.2 the definition and properties of the Nicolai map. If the map is surjective, and this always applies in the cases I consider by the restrictions on W, then the dynamics is continuous under a "*linear*" perturbation of the superpotential, $W \to W + \lambda X$. In fact, this amounts to a constant linear shift in the Gaussian field $h(z) \to h(z) + \lambda^*$. Now, the characteristic polynomial of $W + \lambda X$ for almost all λ has no multiple roots, thus there are no minors and from (3.13) it follows that the characteristic polynomial coincides with the minimal polynomial of $W + \lambda X$. By the continuity of the Nicolai map under linear perturbations, this holds also in the limit $\lambda \to 0$ and then the minimal polynomial $p(x)$ is proportional to the characteristic polynomial also for W, $p(x) = \alpha' \det[\langle X \rangle - x\mathbf{1}]$. Since the characteristic polynomial has degree Δ, also $p(x)$ has degree Δ. But $q(x) = \frac{\partial W}{\partial X}(x)$ has degree Δ too, thus $q(x) = \beta\, p(x) = \alpha \det[\langle X \rangle - x\mathbf{1}]$.

From these considerations and the formulae which are recalled in Appendix F one immediately gets the explicit form of the matrix $\langle X \rangle$ up to conjugacy.

$$X = S^{-1} \begin{pmatrix} J(n_1, x_1) & 0 & \cdots & \cdots & 0 \\ 0 & J(n_2, x_2) & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & J(n_k, x_k) \end{pmatrix} S \tag{3.15}$$

where x_i ($x_i \neq x_j$ for $i \neq j$) are the roots of $p(x)$, n_j their multiplicities, and $J(n_j, x_j)$ is a $(n_j \times n_j)$ Jordan block corresponding to the eigenvalue x_j. The statement that the characteristic polynomial is minimal just means that the Jordan blocks have their maximal dimension, equal to the multiplicity of the corresponding root.

The following remark will be very useful for the future developments:

each Jordan block corresponds to a "critical point" *of the superpotential considered as a holomorphic function.*

Indeed, a "critical point" of a holomorphic function is by definition a common zero of its first derivatives and the roots of $p(x)$, x_j are exactly these points.

I will now give a physical proof of the following two statements, these results will be rediscussed at length within the mathematical contest which will be introduced in the following chapters.

Each Jordan block of X corresponds to a cyclic submodule $\mathcal{R}_i$ of $\mathcal{R}$, and in turn each such submodule correspond to an allowed critical limit for the theory. The Witten index of the corresponding critical theory is equal to the dimension of the submodule, $\Delta_j = n_j$.

Up to now I have formulated the theory on a cylinder, i.e. the space is a circle of length L. On the cylinder the theory is known to exist as a quantum theory [10]. Then, taking the thermodynamic limit $L \to \infty$, one constructs an (Euclidean) QFT [19]. However, not all the Osterwalder–Schrader axioms [33] need to be fulfilled in the resulting theory: there is no reason why one should get a unique vacuum, given that one has Δ of them for finite volume. However, if only the uniqueness of the vacuum axiom fails, the functional measure is a convex combination of measures satisfying all the Euclidean axioms (see, e.g., Theorem 19.7.7 in ref. [34]). The ergodic components of the measure are characterized by different values of the vacuum condensates. In particular, at infinite volume, states with different expectation values of the field X should belong to distinct ergodic components of the measure. Consider a vacuum state in the j–th block $|k,j\rangle$. Obviously, one has $\langle k,j|X|k,j\rangle = x_j$. Since this is true for all L, it is true also in the thermodynamical limit. Hence, different blocks lead to different theories in the IR limit $L \to \infty$. Generally, they have very different physical properties. The possibility of having different large volume limits is typical of supersymmetric theories. In a non-supersymmetric theory it is rather unlikely that different would-be vacuum states are degenerate in energy as $L \to \infty$, but, in a supersymmetric theory this is quite natural, provided all vacua do not break supersymmetry.

Instead, there is no contradiction with the idea that the various vacua in the a Jordan block $\mathcal{W}_i$ go over to one and the same ergodic component of the functional measure. This requires that the expectation value of the operator $\tilde{X} = X - x_j$ (restricted to $\mathcal{W}_j$) goes to zero as $L \to \infty$. This implies that the dimension of $\tilde{X}$ should be positive. This is quite possible, since the dependence on L is hidden in the conjugacy matrix S. Indeed, the computation of the first chapter shows that S is

$$S|_{\mathcal{W}_j} = \mathrm{diag}[1, L^{-1/(n_j+1)}, L^{-2/(n_j+1)}, \dots, L^{-n_j/(n_j+1)}] \tag{3.16}$$

and then $\langle k,j|\tilde{X}|h,j\rangle = O(L^{-1/(n_j+1)})$. In the sector corresponding to $\mathcal{W}_j$, the chiral algebra is the polynomial algebra generated by $\tilde{X}$ with the relation $\tilde{X}^{n_j} = 0$ i.e. it is the chiral ring corresponding to the n_j–th N=2 minimal superconformal model [24]. This algebra has the same dimension as the corresponding irreducible cyclic submodule, showing the "spectral–flow" isomorphism (outside criticality).

There are some general lessons to be drawn from this simple example. In the above example the chiral algebra is $\mathcal{R} = \bigoplus_j \mathcal{R}_j$, before taking any kind of critical limit. In particular, the theory with superpotential

$$V = X^n + a_1 X^{n+1} + \cdots + a_m X^{n+m} \tag{3.17}$$

[19] For the models of the present section the existence of the limit is checked *a posteriori*.

has a chiral algebra which is the direct sum of that for the simple monomial $V = X^n$ with "something" (generically, "something" is trivial). In this sense the equivalence of the two superpotentials holds even before taking the critical limit. There is a broader notion of universality than in the non–supersymmetric case. This fact allows for the computation of the critical quantities in more treatable regimes. In the following chapters we shall see that the equivalence is even stronger.

The Witten index of the critical theory is generally different from the one of the LG model. The Witten index of each critical limit is given by the dimension of the corresponding irreducible submodule $\mathcal{W}_j$, $\Delta_j = n_j \leq \Delta$. The fact that it is always less than the UV Witten index has important implications, as mentioned in the introduction. Once one has the explicit form of the matrix $\langle X \rangle$, to compute Δ_j is trivial.

The most important lesson is that all the dependence of the matrix $\langle X \rangle$ on the various parameters of the theory (except for those appearing in the superpotential), and in particular on the UV cut-offs and the IR scale L, are hidden in the conjugacy matrix S. Moreover, if $V = gF(X_i)$, all the dependence on the overall coupling g is hidden in S, since $p(x)$ is independent of g. In particular, a scaling transformation should act on the matrix $\langle X \rangle$ by conjugation, or, in other words, the renormalization group acts on $\langle X \rangle$ by similarity transformations. This is a non-perturbative version of the non–renormalization theorem [19]. Therefore, we must have

$$L\frac{\partial}{\partial L}\langle X \rangle = [M(L), \langle X \rangle] \tag{3.18}$$

for a $\Delta \times \Delta$ matrix $M(L)$.

(3.19) Example : Let $V = \frac{g_1}{\mu+1}X^{\mu+1} + \frac{g_2}{m+1}X^{m+1}$ with $1 < \mu < m$; the UV Witten index is $\Delta_{UV} = m$. Then $p(x) = x^\mu[(g_1/g_2) + x^{m-\mu}]$ and the Jordan matrix is

$$J = \begin{array}{c@{}c} & \mu \\ \mu & \left[\begin{array}{cccccc|cccc}
0 & 1 & 0 & \cdots & & & & & & \\
 & 0 & 1 & & & 0 & & & & \\
 & & 0 & \ddots & & & & & 0 & \\
 & & & \ddots & 1 & 0 & & & & \\
 & 0 & & & 0 & 1 & & & & \\
 & & & \cdots & 0 & 0 & & & & \\ \hline
 & & & & & & \lambda_1 & & & \\
 & & & & & & & \lambda_2 & & 0 \\
 & & 0 & & & & & & \ddots & \\
 & & & & & & & 0 & & \\
 & & & & & & & & & \lambda_{m-\mu}
\end{array}\right]_{m\times m}\end{array} \tag{3.20}$$

since the characteristic roots are: $\lambda = 0$ with molteplicitly μ, $\lambda_k = (-g_1/g_2)^{1/(m-\mu)} \exp[2\pi i \frac{k}{m-\mu}]$ for $0 \leq k \leq m-\mu$ with molteplicity 1. Thus there are $m - \mu + 1$ Jordan blocks, the first of dimension μ (W_1 with

$\Delta_1 = \mu$), the other of dimension 1 (W_k with $\Delta_k = 1$ $k > 1$). Since the W_k blocks are trivial at criticality ($\Delta_k = 1$ means free massive theory) at criticality only W_1 with $\Delta_1 = \mu$ survives and the effective superpotential is $V = \frac{g_1}{\mu+1} X^{\mu+1}$ as it should be. ∎

3.1.2 *The $n > 1$ case*

The general case of n chiral fields is similar to the $n = 1$ case. The only new complication is that $\mathbb{C}[X_1, \ldots, X_n]$ is not longer an Euclidean ring. This leads to new phenomena for $n > 1$. However from the *matricial* equations

$$\langle \frac{\partial V}{\partial X_i} \rangle = 0 \tag{3.21}$$

one can still eliminate algebraically the variables X_i, $i \geq 2$, ending with an algebraic equation for the matrix $\langle X_1 \rangle$ of the form $p(\langle X_1 \rangle) = 0$ of degree $\leq \Delta$. As before, submodules corresponding to different roots of this polynomial leads to different theories. Candidate conformal limits are related factors of $p(x)$ of the form $(x - \lambda)^n$ in this equation. The critical index satisfies $\Delta_{cr} \leq n \leq \Delta$. Fixing our attention to such a submodule one can assume $\lambda = 0$. However, now $\langle X_1 \rangle$ is not in general a single Jordan block of dimension n, since now the Jordan structure refers to the whole algebra generated by the operators X_i. As before, the criterion to find the $\langle X_i \rangle$ is that they should be solutions to the given equations of motion, and that they should be not solution to simpler (smaller degree) equations. In this way one can easily construct the solution. However in practice it is often more convenient to use the fact that the solution to eq. (3.21) statisfying the above maximality condition is given by the matrices of the regular representation of the algebra $\mathcal{R}$. These, as we will see below, can be written down (up to conjugacy) immediately. Then it is easy to find the irreducible submodules corresponding to the given large volume behaviour. Restricting the matrices to these submodules one finds the candidate critical algebra which, in general, is different from the "naive" one given by the relations $\partial V = 0$. The dimensions of these submodules give the critical Witten index.

To conclude this section I will sketch a practical method to compute the structure of the matrices, i.e. their Jordan form. First one should compute the general (UV–) Witten index of the model and a basis for the algebra $\mathcal{R}$. To find the Jordan decomposition of a matrix given by the expectation value of an operator in $\mathcal{R}$ one can proceed for example following Baumgärtel in ref. [32] pagg. 93–97. The starting observation is that a matrix A and its commutant in $\mathcal{R}$ have the same Jordan decomposition. With the help of a computer it is not too difficult to compute the common commutant of all the elements of the basis of $\mathcal{R}$ and then of $\mathcal{R}$. (Notice that one can choose the commutant matrices to be triangular.) Having the general structure of matrices in the commutant, by a similarity transformation it is possible to obtain the Jordan form of a matrix in the commutant and then that of a matrix in $\mathcal{R}$.

3.2 General considerations on the IR limit

In this paragraph I will do some general and naive considerations (see ref. [4]) on the IR (scaling) limit, which summarize what we have seen and are the starting point for the future developments. The following reasoning is not to be considered rigorous, indeed the aim of the rest of this thesis will be to prove these statements or to show to what extent they are true.

We have seen that the (anti–) chiral Green's functions do not depend on the explicit form of the kinetic term and that the Renormalization Group (RG) acts on the corresponding matrices by similarity transformations. This means that the structure of the CGFs, i.e. their Jordan form, does not change and it is fixed by the superpotential $W(X_i)$. In this sense the superpotential is an invariant of the RG flow, since it dictates the Jordan structure of the matrices along all the RG flow.

In the IR limit, it is reasonable that the models I consider have a fixed point. This for the moment is only an assumption taken just as a starting point for the following reasoning, which indeed holds independently of it. (Really, for a large class of models the existence of a critical point in the IR or strong coupling limit is at least to be doubt and it is one of the most interesting open problems.) At a critical point dilatational invariance of the theory is regained [35], thus the theory must be invariant under a rescaling of the two dimensional metric g: $g \to \lambda^2 g$.

Indeed, start with a general superpotential [20] and rescale the metric $g \to \lambda^2 g$, then the F term scales as λ (because $\int d^2 z d\theta \to \lambda \int d^2 z d\theta$). One can re–define the superfields by scaling them in such a way as to compensate for this. If one takes the limit $\lambda \to \infty$, only the combinations of fields with the lowest scaling dimension will survive. At the fixed point the theory must be scale invariant, and so one ends up with a superpotentials which has the property that if one scales the superfields according to ($\rho = (1/\lambda)$):

$$X_i \longrightarrow \rho^{\omega_i} X_i$$

for some (rational) number ω_i, then the superpotential scales by ρ:

$$W(\rho^{\omega_i} X_i) = \rho W(X_i) \quad . \tag{3.22}$$

Functions W which have this property are called quasihomogeneous with weights ω_i.

It is a trivial fact to show that a model with quasihomogeneous superpotential has an R–symmetry (see chapter §1.5), indeed quasihomogeneity and R–symmetry coincide. Thus at criticality the theory gains the expected U(1) symmetry needed to reconstruct the N=2 superconformal algebra. From this it is obvious that the left–right U(1) charge in the conformal theory of the field X_i is $(\omega_i, -\omega_i)$ and the scaling dimension is $(\omega_i/2, \omega_i/2)$.

(3.23) Example : consider $W = X^m + X^{m+2}$. After rescaling $X \to \lambda^{-1/m} X$ and taking $\lambda \to \infty$, one ends up with $W = X^m$ which is quasihomogeneous:

$$\begin{aligned} \int d^2x d^2\theta \, W &= \int d^2x d^2\theta \left(X^m + X^{m+2}\right) = \text{rescaling by } \lambda = \\ &= \int \left(d^2x d^2\theta \cdot \lambda\right) \cdot \frac{1}{\lambda} \left(X^m + \frac{1}{\lambda^{\frac{2}{m}}} X^{m+2}\right) \overset{\lambda\to\infty}{\longrightarrow} \\ &= \int d^2x d^2\theta \, (X^m) \qquad \blacksquare \end{aligned} \tag{3.24}$$

[20] This argument is certainly true for semi–quasihomogeneous superpotentials but not in the very general case.

Without loss of generality, one can assume that $W(X_i)|_{X_i=0} = 0$ and $\partial_j W(X_i)|_{X_i=0} = 0$. If $\partial_i\partial_j W(X_k)|_{X_k=0} \neq 0$, it implies that some of the fields have masses. But if there is a mass gap in the theory, the propagators for massive fields will fall off exponentially with distance and in the IR limit one is left with a trivial theory in which the massive fields are frozen out. Therefore it must be $\partial_i\partial_j W(X_k)|_{X_k=0} = 0$ for a non–trivial theory, i.e. W has a completely degenerated critical point at $X_i = 0$, or it has a "singularity" at the origin. Similarly, if one adds to the theory new fields X_j that have a quadratic mass term in the superpotential, this does not affect the RG flow and one still arrives at the same conformal theory.

By general principles of QFT, two theories can be considered the same if and only if they are related one another by field redefinitions. Thus the (quasihomogeneous) superpotentials are holomorphic functions which should be considered equivalent if under a holomorphic change of variables they become identical. In addition, two superpotentials in m and n variables should be considered the equivalent in the scaling limit if, after adding a quadratic form in $k-m$ variables to the first and a quadratic form in $k-n$ new variables to the second, they become equivalent (i.e. identical after a change of variables). This notion of equivalence is known in the mathematical literature as "*stable equivalence*".

In the study and classification of the singularities of an holomorphic map usually one demands the singularities to be isolated. If they were not, this would mean that the superpotential has exact flat directions, which would mean that those direction lead to strings propagation on an infinite radius flat space. This is what one could reasonably expect from the behaviour in the IR critical limit.

But, as we have already seen, one can be more ambitious and study the CGFs even outside criticality. Thus in general, one can go beyond the study of the quasihomogeneous superpotentials and consider more general classes of superpotentials. One interesting (and up to now not completely answered) question is to check if from the rigorous treatment of the following chapters one is able to prove in complete generality the previous statement on the behaviour in the IR limit.

(3.25) Example : the superpotential

$$W = X^4 + Y^4 + Z^4 + XYZ \tag{3.26}$$

is the simplest example of superpotential of which it is not known the behaviour in the scaling limit. Indeed its critical Witten index is $\Delta = 11$ and in the scaling limit, following the previous argument, it should become $W = XYZ$. This superpotential does not belong to the class of superpotentials I consider and it seems that it does not describe a critical theory. Moreover, from Witten index considerations and the arguments on the matricial structure of the CGFs, it seems reasonable that in the scaling limit W would not go into XYZ (otherwise the Witten index and the matrix properties of the CGFs will be violated). The unique possibilities are that there does not exist a critical limit of W (the CGFs in that limit do not converge to something but "oscillate") or that the limit exists but is much more complicated. ■

3.3 Holomorphic dependence on the chiral couplings: subtleties

The only parameters, on which the chiral matrix elements may depend, are the couplings in the superpotential g_a and the length L (which can be seen as the IR cut–off). There is a standard (albeit quite formal) argument showing that the dependence on the couplings g_a should be holomorphic. Indeed,

$$\begin{aligned} \frac{\partial}{\partial \bar{g}_a}\langle k|X_i|h\rangle &= -\int d^2z d^2\bar{\theta} T\langle k|X_i \frac{\partial \overline{V}(z,\theta)}{\partial \bar{g}_a}|h\rangle = \\ &= \int d^2z T\langle k|X_i\{\overline{Q}_+, \cdots\}|h\rangle = 0 \quad . \end{aligned} \tag{3.27}$$

However, in general, this result is just *not true*. To understand why this may happen, I have to discuss what this equation is supposed to mean. First of all, recall that I am interested in the off–diagonal elements of the matrix $\langle k|X_i|h\rangle$. In fact, in a suitable basis, the diagonal elements are just the eigenvalues $x_{i,j}$ of the algebraic equations of the previous sections, and they depend holomorphically on g_a. For the off–diagonal elements, the validity of eq. (3.27) is, at best, a question of conventions. In fact, multiplying each vacuum state $|k\rangle$ for a different g_a–dependent phase, one can destroy any holomorphic dependence. So, at best, eq. (3.27) may be a (heuristic) existence argument for a "*special basis*" in which the holomorphic dependence does hold.

(In the case of the A_μ models considered in the previous chapters I exploited the fact that (for those models) different vacua are completely characterized by their R–charge, and used the fact that one can insert in the path integral g–independent projectors to the various R–eigenspaces to give an unambigous path integral representation of each matrix elements. Then the holomorphic basis is automatically selected. This argument is, however, not valid for the general case in which the different vacua can mix.)

Since the ambiguity is due to the possibility of making a g_a–dependent linear transformation of the basis of $\mathcal{W}$, I am forced to substitute the ordinary derivatives $\partial/\partial g_a$ and $\partial/\partial \bar{g}_a$ with *covariant* derivatives D_{g_a} and $D_{\bar{g}_a}$. These derivatives transform under changes of basis as $GL(\Delta, \mathbf{C})$ connections. Eq. (3.27) , correctly interpreted, defines a natural connection in coupling–constant space. This connection is, in general, non–trivial. Write the superpotential in the standard form $V = gF(X_i, c_k)$, and consider the connection D_g with respect to the overall coupling constant g. A direct computation (see Chapter 6) will show that the connection is flat for quasihomogeneous superpotentials, but not in general.

In the quasi–homogeneous case, the connection is flat, but it is still non–trivial. One cannot extend the Δ supersymmetric vacua $|k\rangle$ to all the complex g–plane, because for $g = 0$ one gets the free theory, and the Witten index collapses to zero. So one can define the connection only on the space $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. This space is not simply connected, and going round the origin one can pick up a non–trivial monodromy ($\equiv$ Wilson loop).

This monodromy is trivial (i.e. $= \pm 1$) only if the corresponding critical theory is trivial. In fact, it turns out that this operator is equal to the monodromy h_* of the singularity class of V. A theorem by A' Campo [36], states that [21]

$$\operatorname{tr} h_* = (-1)^{n-1}. \tag{3.28}$$

[21] n is the number of chiral fields.

Then, if the monodromy is ± 1, $\Delta = 1$, and there is only one primary chiral operator, the identity. Since its $U(1)$ charge is zero, at criticality one gets $c = 0$ (see [7]), i.e. the trivial theory. The non–triviality of the monodromy for general models can be seen as an obstruction to the uni–valuedness of the chiral matrix elements as (holomorphic) functions of g.

In the standard parametrization only the overall coupling g is dimensional. Then, the matrix elements $\langle k|X_i|h\rangle$ are (in a suitable basis) holomorphic functions of the adimensional quantities gL and c_k. In particular, to get the behaviour as $L \to \infty$ one needs to know only the dependence on g.

3.4 Ward identities and critical behaviour

In many interesting cases, the dependence of the chiral matrix elements on the *phase* of the coupling constants can be computed using Ward identities. If, in addition, one knows that the matrix elements are holomorphic, one can reconstruct the Green functions up to some numerical coefficients (see Chapter 1). I discuss this point just to show how many of the results of the general analysis can be already obtained by simple "physical" arguments. For this reason I shall not over–emphasize questions of rigour although, under appropriate circumstances, the arguments can be made rigorous.

For definiteness, I shall consider the simple superpotential

$$V = \frac{1}{n}\lambda_1 X^n + \frac{1}{m}\lambda_2 X^m \qquad m > n \quad . \tag{3.29}$$

This models has $\Delta = m - 1$, but at criticality the index jumps to $\Delta' = n - 1$.

This model has a discrete R–symmetry given by

$$\begin{aligned} X &\longrightarrow e^{\frac{2\pi i}{m-n}k} X \\ V &\longrightarrow e^{\frac{2\pi i}{m-n}nk} V \\ k &= 0, 1, \dots, m-n-1 \end{aligned} \tag{3.30}$$

(i.e. $\alpha = \frac{2\pi}{m-n}nk$) and $\dim(R) = m - n$. Notice the following property of the Witten index deficency: $\Delta - \Delta' = m - n = \dim(R)$.

The idea of the Ward–identities approach is that all the couplings in the superpotential have dimension less than 2, and hence the symmetries which are (explicitly) broken only by the superpotential are just softly broken and we can use standard PCAC methods. Then, the method is similar to the one for the mass–dependence in 4D super–QCD [18].

Consider the $U(1)$ symmetry $X(z,\theta) \to e^{i\alpha} X(z,\theta)$, and let J_μ the corresponding partially conserved current. One has

$$\begin{aligned} &\left(n\lambda_1 \frac{\partial}{\partial \lambda_1} - n\bar\lambda_1 \frac{\partial}{\partial \bar\lambda_1} + m\lambda_2 \frac{\partial}{\partial \lambda_2} - m\bar\lambda_2 \frac{\partial}{\partial \bar\lambda_2} \right) \langle h|X|k\rangle = \\ &= \int d^2z \langle h|\delta_{U(1)}\mathcal{L}(z) X|k\rangle = \int d^2z \langle h|\partial^\mu J_\mu X|k\rangle = \\ &= -\langle h|[Q, X]|k\rangle + \int d^2z\, \partial^\mu \langle h|J_\mu X|k\rangle = \\ &= -\langle h|X|k\rangle + \int d^2z\, \partial^\mu \langle h|J_\mu X|k\rangle . \end{aligned} \tag{3.31}$$

The first equality is subjected to the same basis specifications as the holomorphic dependence in §3.3 . So, before using the results of the present section, one has to check that, in the basis at hand, this equality holds.

On the cylinder, one has

$$\int d^2 z \partial^\mu \langle h|J_\mu X|k\rangle = \\ = \lim_{\tau\to\infty}\Big[\int d\sigma \langle h|J_0(\sigma,\tau)X|k\rangle - \int d\sigma\langle h|X J_0(\sigma,\tau)|k\rangle\Big] \tag{3.32}$$

and

$$\lim_{\tau\to\infty}\int d\sigma\langle h|J_0(\sigma,\tau)X|k\rangle = L\sum_j \langle h|J_0|j\rangle\langle j|X|k\rangle \quad . \tag{3.33}$$

Now, $\langle j|J_0|k\rangle = 0$. This follows from a lemma in Kählerian geometry: $U(1)$ acts by holomorphic isometries on the Kähler manifold. Since a Kähler space is symplectic, each holomorphic isometry is generated by a Hamiltonian field, i.e. it is the canonical transformation generated by some Hamilton function H. In our case, $H = i(\overline{X}K_{\overline{X}} - XK_X)$. The N=2 Noether theorem states that the $U(1)$ conserved current is the vector component of the superfield H,

$$J_\pm = \{\overline{Q}_\pm,[Q_\pm,H]\} - (\overline{Q}\leftrightarrow Q) + \text{derivatives}, \tag{3.34}$$

and then $\langle h|J_\pm|k\rangle = 0$ by supersymmetry. Finally one gets,

$$\left(n\lambda_1\frac{\partial}{\partial\lambda_1} - n\bar{\lambda}_1\frac{\partial}{\partial\bar{\lambda}_1} + m\lambda_2\frac{\partial}{\partial\lambda_2} - m\bar{\lambda}_2\frac{\partial}{\partial\bar{\lambda}_2}\right)\langle h|X|k\rangle = -\langle h|X|k\rangle \quad . \tag{3.35}$$

If the conditions for holomorphic dependence are fulfilled, this equation implies [22]

$$\langle h|X|k\rangle = \frac{1}{(\lambda_1 L)^{1/n}} f_{hk}\left(\frac{\lambda_2 L^{1-m/n}}{\lambda_1^{m/n}}\right) \tag{3.36}$$

with $f(\cdot)$ holomorphic. Let me show that this equation holds true independently of this assumption, except that $f_{hk}(\cdot)$ needs not to be holomorphic [23]. Consider the Lagrangian

$$|\mu|^2\int d^4\theta\overline{X}X + \left(\int d^2\theta\Big[\frac{1}{n}\lambda_1 X^n + \frac{1}{m}\lambda_2 X^m\Big] + \text{h.c.}\right) \quad . \tag{3.37}$$

We know that $\langle h|X|k\rangle \equiv F_{hk}(\lambda_1,\lambda_2,L)$ is independent of μ. Make the field redefinition $Y = \mu X$. In terms of Y eq. (3.37) becomes the Lagrangian with the canonical kinetic term and a superpotential of the form in eq. (3.29) with $\lambda_1' = \lambda_1\mu^{-n}$ and $\lambda_2' = \lambda_2\mu^{-m}$. Then it must be

$$\mu F_{hk}(\lambda_1,\lambda_2,L) = \mu\langle h|X|k\rangle = \langle h|Y|k\rangle = F_{hk}(\lambda_1\mu^{-n},\lambda_2\mu^{-m},L) \tag{3.38}$$

[22] Indeed, another solution of this equation is admitted: $\langle h|X|k\rangle = (1/(\lambda_2 L)^{1/m})\hat{f}_{hk}\left((\lambda_1 L^{1-(n/m)})/\lambda_2^{n/m}\right)$. In the sense that will be soon explained this is the UV solution.

[23] This is the same argument already used in §1.5 .

Taking $\mu = \lambda_1^{1/n}$ one gets eq. (3.36) with $f_{hk}(\cdot) = F_{hk}(1, \cdot)$.

Let me show the possible consequences of this equation. Suppose that the (matrix valued) function f of eq. (3.36) as $L \to \infty$ goes to zero, that is, denoting $z := (\lambda_2 L^{1-(m/n)})/\lambda_1 = (\lambda_2/\lambda_2)L^{-(m-n)/n}$, then for $L \to \infty$, $z \to 0$. Suppose that

$$f(z) \stackrel{z\to 0}{\longrightarrow} f^0 + zf^1 + \cdots \quad , \tag{3.39}$$

then from (3.36)

$$\langle h|X|k\rangle \stackrel{L\to\infty}{\sim} \frac{1}{(\lambda_1 L)^{\frac{1}{n}}} f^0_{hk} \quad . \tag{3.40}$$

This means that the IR limit is effectively as if $\lambda_2 \equiv 0$. The same reasoning can be made for the "UV limit" $L \to 0$ interchanging the role of m and n and that of f and $\hat{f}$. Thus the "UV limit" is effectively as if $\lambda_1 \equiv 0$ and

$$\langle h|X|k\rangle \stackrel{L\to 0}{\sim} \frac{1}{(\lambda_2 L)^{\frac{1}{m}}} \hat{f}^0_{hk} \quad . \tag{3.41}$$

To get more information on the function $f_{hk}(\cdot)$, I consider the other *softly* broken symmetry, the R–symmetry $X(z,\theta) \to e^{i\alpha/m}X(z, e^{-i\alpha/2}\theta)$. By the same argument used before one gets

$$\begin{aligned}\left(\frac{n}{m} - 1\right)\left(\lambda_1 \frac{\partial}{\partial\lambda_1} - \bar{\lambda}_1 \frac{\partial}{\partial\bar{\lambda}_1}\right)\langle h|X|k\rangle &= \\ = -\frac{1}{m}\langle h|X|k\rangle + \sum_j \Big[\langle h|R|j\rangle\langle j|X|k\rangle - \langle h|X|j\rangle\langle j|R|k\rangle\Big]&,\end{aligned} \tag{3.42}$$

where R is the R–charge. This equation is again subjected to the basis conditions. Writing $\langle X\rangle$ for the matrix $\varpi(X)$ and putting $\mathbf{R} = \varpi(R)$, this equation becomes

$$\left(\frac{n}{m} - 1\right)\left(\lambda_1 \frac{\partial}{\partial\lambda_1} - \bar{\lambda}_1 \frac{\partial}{\partial\bar{\lambda}_1}\right)\langle X\rangle = -\frac{1}{m}\langle X\rangle + [\mathbf{R}, \langle X\rangle] \quad . \tag{3.43}$$

From the above results one can get many informations on the critical theory. First of all, if one can find a basis in which the holomorphicity and the above Ward identities hold true, one has a complete set of equations giving the chiral matrix elements in a closed form. These conditions can be fulfilled at least in an approximate sense. In fact, one can easily check that, near a fixed point, such a basis exists. However, one cannot follow the renormalization flow between the two fixed points by this simple–minded method, since away from the critical points one must pay attention to the above subtleties.

A more general argument goes as follows. A necessary condition for having a fixed point is that the matrix $\langle X\rangle$ scales (after restriction to the relevant Jordan block) as a power of L. Considering eq. (3.36) , one sees that $\langle X\rangle \sim L^{-k}$ whenever $[\mathbf{R}, X] = kX$. I want to argue that this happens if the current

$$J^k_\mu = R_\mu + \frac{1}{m}(k-1)J_\mu \tag{3.44}$$

becomes effectively conserved at the fixed point. (Here R_μ is the partially conserved current of eq. (3.42)). That is, a necessary condition for a fixed point is that some *continuous* R–symmetry gets restored. Indeed, the PCAC Ward identity for the current J^k_μ is

$$-k\langle h|X|k\rangle + m\Big([\mathbf{J}^k, X]\Big)_{h\,k} = m\int d^2z\langle h|\partial^\mu J^k_\mu(z)X|k\rangle \tag{3.45}$$

where $\mathbf{J}^k$ is the matrix $\varpi(J^k)$. Now, $\mathbf{J}^k = \mathbf{R}$ since the $U(1)$ current J_μ has vanishing matrix elements between supersymmetric invariant states. So, if $[\mathbf{R}, X] = kX$, one gets

$$\int d^2z\langle h|\partial^\mu J^k_\mu(z)X|k\rangle = 0 \tag{3.46}$$

which shows that the current J^k_μ is effectively conserved in the chiral sector.

In all the above equations, the only thing that changes when one passes from the UV fixed point to the IR one is the (asymptotically) conserved R–charge, i.e. the linear combination in eq. (3.44) . Then, the difference of scaling dimensions for the two cases is equal to the difference of their conserved charges q,

$$2h - 2h' = q - q', \tag{3.47}$$

from which it is easy to get the formula (valid for any chiral primary operator) $2h = q$.

(3.48) Example : consider again eq. (3.29) and suppose that the dependence on the coupling constants λ_1 and λ_2 is holomorphic. For physical reasons, one expects that at the IR critical point $[\mathbf{R}, X] = \frac{1}{n}X$ and at the UV–point $[\mathbf{R}, X] = \frac{1}{m}X$. Thus in the IR limit from eq. (3.43) one gets

$$\lambda_1\frac{\partial}{\partial\lambda_1}\langle X\rangle = -\frac{1}{n}\langle X\rangle \Longrightarrow \langle X\rangle = \frac{1}{\lambda_1^{\frac{1}{n}}}F^0(\lambda_2) \quad . \tag{3.49}$$

Inserting this equation in (3.35) , it follows that

$$\lambda_2\frac{\partial}{\partial\lambda_2}\langle X\rangle = 0 \Longrightarrow \langle X\rangle = \frac{1}{\lambda_1^{\frac{1}{n}}}f^0 \tag{3.50}$$

i.e. $\langle X\rangle$ is independent from λ_2 (effectively as if $\lambda_2 \equiv 0$).

Analogously, in the "UV limit" from (3.43) one gets

$$\lambda_1\frac{\partial}{\partial\lambda_1}\langle X\rangle = 0 \Longrightarrow \langle X\rangle = \hat{F}^0(\lambda_2) \tag{3.51}$$

i.e. $\langle X\rangle$ is independent from λ_1 (effectively as if $\lambda_1 \equiv 0$). Substituting this in (3.35) , it follows that

$$\lambda_2\frac{\partial}{\partial\lambda_2}\langle X\rangle = -\frac{1}{m}\langle X\rangle \Longrightarrow \langle X\rangle = \frac{1}{\lambda_2^{\frac{1}{m}}}\hat{f}^0 \quad . \quad \blacksquare \tag{3.52}$$

CHAPTER 4

CHIRAL GREEN'S FUNCTIONS AS A COHOMOLOGICAL PROBLEM

In this chapter I introduce a different approach to the study of the CGFs of the class of 2d N=2 LG theories here considered. The main result is that the CGFs can be seen as intersection numbers in a cohomological problem. From this, all the results obtained in the previous chapters can be re–obtained in a more elegant and easy way. Moreover old and new results, included the exact and explicit computation of the CGFs, can be formulated in a more rigorous and firm basis.

I will start by reviewing the idea of this approach in SQM which goes back to Witten [12]. Then I will apply this idea to our case and state the "*Main Lemma*" of translation of the field theory problem in a cohomological problem. From the properties of the cohomological groups, I will prove the independence of the CGFs from the Kähler potential and I will explicitly compute (up to the normalization constants) the CGFs as intersection numbers in cohomology.

4.1 N=1 supersymmetry and Morse theory 'd'après' Witten

In this section I review some ideas and results mainly obtained by Witten in ref. [12].

Witten considered N=1 SQM problems. He noticed some basic and fundamental facts of supersymmetry. The Hilbert space H can be decomposed as $H = H^+ \oplus H^-$ where H^+ and H^- are the bosonic and fermionic subspaces respectively, Q_i, $i = 1, \ldots, N$ are the (Hermitian) supersymmetry operators which anticommute with $(-)^F$, the operator which distinguishes between H^+ and H^- and counts the number of fermions modulo two.

The supersymmetry operators have the following properties

$$\{Q_i, Q_j\} = \delta_{i,j} H \quad . \tag{4.1}$$

This algebra must be generalized when one comes to relativistic quantum field theory because the Lorentz transformations relate the Hamiltonian H to the momentum operators P [24]. In

[24] I restrict myself to a world with one time and one space direction.

the simplest situation with 2 supersymmetry operators the algebra is

$$(Q_1)^2 = H + P \qquad (Q_2)^2 = H - P \qquad \{Q_1, Q_2\} = 0 \tag{4.2}$$

and

$$H = \frac{1}{2}\left((Q_1)^2 + (Q_2)^2\right) \quad . \tag{4.3}$$

Thus $H \geq 0$ and the vacua are the lowest energy states (if they exist) with $E = 0$. The vacuum states $|\omega\rangle$ are obviously characterized by the equations

$$Q_i|\omega\rangle = 0 \quad . \tag{4.4}$$

The existence of such a state can be seen as a well known mathematical problem, the computation of the index of an operator, that is the number of independent solution of this equation. Indeed, considering the relativistic case, if the equation $Q_i|\omega\rangle = 0$ has a solution then $P|\omega\rangle = 0$ by the supersymmetry algebra. Then, in looking for states $|\omega\rangle$ which satisfy $Q_i|\omega\rangle = 0$, one can restrict himself to the subspace H_0 of H with momentum eigenvalue zero. Within H_0 the SQM algebra $(Q_i)^2 = H$, $\forall i$ is satisfied. Thus, if it exists an $|\omega\rangle$ such that $Q_i|\omega\rangle = 0$ for a fixed i, then it holds for every Q_i. The problem of the existence of $|\omega\rangle$ can be formulated as an index problem as follows. Let Q one of the Q_i and $Q_+ + Q_-$ be the restriction of Q to H_0 where Q_+ maps H_0^+ to H_0^- and Q_- its adjoint. A non zero index for Q_+ would ensure that Q does have a zero eigenvalue in H_0.

This problem can be easily formulated as a problem in differential geometry. Let M be a Riemannian manifold of dimension n (it is the space where the fields take value, for N=1 supersymmetry is a real space). Let V_p, $p = 0, 1, \ldots, n$ be the space of p–forms. Let d and $\bar{d}$ be the usual exterior derivative and its adjoint. Define

$$Q_1 := d + \bar{d} \qquad Q_2 := i(d - \bar{d}) \qquad H := d\bar{d} + \bar{d}d \tag{4.5}$$

where H is the usual Laplacian acting on forms. It is easy to see that these operators satisfy the SQM algebra where one must interpret the p–forms as being bosonic or fermionic depending on whether p is even or odd so that the Q_i map bosonic states into fermionic states and vice–versa.

The Betti numbers B_p of d is by definition the number of linearly independent p–forms which obey $d\psi = 0$ but cannot be written as $\psi = d\chi$ for any χ (i.e. $B_p = \dim(\mathrm{H}_d^p(M))$). The Betti number B_p is, in physical language, the number of vacua of the model.

Witten introduced in ref. [12] some generalization of this model which lead to the introduction of the interaction between the fields and to the case of quantum field theory.

The interaction between physical fields (which are the coordinates of M) can be seen as a modification by conjugation of the operator d in this mathematical approach. Let h be a smooth (real–valued) function on M and t a real number. h will play the role of the (super–) potential and t that of the (overall) coupling constant. Define

$$d_t := e^{-ht}\, d\, e^{ht} \qquad \bar{d}_t := e^{ht}\, \bar{d}\, e^{-ht} \quad . \tag{4.6}$$

Evidently d_t and $\bar{d}_t$ satisfy the same algebra as d and $\bar{d}$ for any t. h plays the role of a Morse function and Witten was able to deduce from the supersymmetry algebra the Morse inequalities. His main observation is that, since d and d_t differ only by conjugation by the invertible operator e^{th} so that the mapping $\psi \to e^{th}\psi$ is an invertible mapping from p–forms which are closed but not exact for d to p–forms which are closed but not exact in the sense of d_t, then the Betti numbers $B_p(t)$ are equal to B_p for any t.

In the large t limit, the spectrum of H_t simplifies dramatically and $B_p(t)$ becomes computable. Indeed what happens is that the "potential energy" $V(\phi) = t^2(dh)^2$ becomes very large except in the vicinity of the critical points of h where $dh = 0$. Therefore the eigenfunction of H_t are, for large t, concentrated near the critical points of h. This mechanism is exactly the same mechanism which works in the LG models and give, as we will see, the connection between the N=2 LG models and "Singularity Theory".

A second generalization of (4.5) introduced by Witten has the purpose of obtaining differential operators which satisfy the relativistic algebra (4.2) instead of the quantum mechanical one (4.1) . The resulting structure has been called "*Witten Complex*".

Let M be a compact Riemannian manifold of dimension n which admits the action of a continuous group of isometries. Let K be a Killing vector field — the infinitesimal generator of an isometry of M. Let N be the space of zeros of K. K can be seen as an operator i_K on differential forms acting by interior multiplication. Witten now modified the usual exterior derivative d defining

$$d_s := d + s\, i_K \tag{4.7}$$

where s is an arbitrary real number which physically can be interpreted as the inverse of the radius of the cylinder. One can easily show that

$$(d_s)^2 = -(\bar{d}_s)^2 = s\; \mathcal{L}_K \tag{4.8}$$

where $\bar{d}_s$ is the adjoint of d_s and $\mathcal{L}_K$ is the Lie derivative along K. The Hamiltonian is as usual $H_s = d_s\, \bar{d}_s + \bar{d}_s\, d_s$.

Witten showed that, as long as $s \neq 0$, the problem of finding the "vacua" of H_s can be reformulated as in the SQM case. First, one can restrict himself to the subspace of the De Rham complex consisting of states which are annihilated by $\mathcal{L}_K$. Moreover, again by conjugation arguments, the dimension of (Ker(d_s) / Im(d_s)) does not depend on s as long as $s \neq 0$. Finally, the numbers of zero eigenvalues of H_s equals the sum of the Betti numbers of N.

To get the supersymmetry algebra (4.2) Witten defines

$$\begin{aligned} Q_{1,s} &:= \sqrt{i}\, d_s + \frac{1}{\sqrt{i}}\, \bar{d}_s \qquad & H_s &:= d_s\, \bar{d}_s + \bar{d}_s\, d_s \\ Q_{2,s} &:= \frac{1}{\sqrt{i}}\, d_s + \sqrt{i}\, \bar{d}_s \qquad & P &:= 2i\, s\, \mathcal{L}_K \quad . \end{aligned} \tag{4.9}$$

The final generalization which leads to the structure of the Witten Complex of a N=1 LG theory is obtained putting together the two structures given by (4.6) and (4.7) . Let h be any function invariant under the action of K, that is $i_K dh = 0$. Let $d_{s,t} := e^{-ht}\, d_t\, e^{ht}$, then define

$$\begin{aligned} Q_{1,s,t} &:= \sqrt{i}\, d_{s,t} + \frac{1}{\sqrt{i}} \bar{d}_{s,t} \qquad & H_{s,t} &:= d_{s,t}\, \bar{d}_{s,t} + \bar{d}_{s,t}\, d_{s,t} \\ Q_{2,s,t} &:= \frac{1}{\sqrt{i}} d_{s,t} + \sqrt{i}\, \bar{d}_{s,t} \qquad & P &:= 2i\, s\, \mathcal{L}_K \quad . \end{aligned} \tag{4.10}$$

This is the final formulation of the Witten Complex which applies to 2d N=1 supersymmetric field theory.

(In the case of Field Theory some difficulties arise in the correct mathematical formulation of this construction. In fact, in Quantum Field Theory M is infinite dimensional and N is finite dimensional. M is the Loop Space of the fields, that is, let $\phi_i : S \to B$, $\phi_i \in \Omega(B, S) = M$ where S is the circle of length L which is taken as space. The group of U(1) rotations on the circle S can be considered to act on Ω (the action being simply $\sigma(x) \to \sigma(x+a)$ for any loop σ in Ω), and K is taken as the corresponding Killing vector field.)

In the case of N=1 LG with one field $\Phi : S \to \mathbb{R}$, one defines $h(\Phi) := \int_S dx\, W(\Phi(x))$ where W is the superpotential. The study of the zero eigenvalue problem of the associated Hamiltonian $H_{s,t}$ can be now simplified in a relevant way. In fact Witten's arguments guarantee that the structure of the vacua are independent of s and t. By consideration of the large s behaviour of the spectrum, one may reduce the index problem on Ω to an immensely simpler index problem on the space of the zeros of the Killing vector field N which in this case is $\mathbb{R}$ [25]. By restriction of $d_{s,t}$ to $\mathbb{R}$, one reduces to consider the SQM problem described by (4.6) and d_t is equivalent to the ordinary differential operator

$$D := \frac{d}{d\phi} + t\, \frac{dW}{d\phi} \quad . \tag{4.11}$$

Before constructing the generalization of the Witten Complex to N=2 LG Theories, I will make the summary of the main ideas now exposed. Given a 2d supersymmetric field theory one construct the associated Witten Complex and consider those objects in field theory which are correctly described in the Witten Complex, like for example the supersymmetric vacua. Using the independence on the parameter s one can reduce to a simpler complex associated to a quantum mechanical problem [26], and thus restricting to the case of finite–dimensional spaces. This strategy will be adopted and carried out in the next section.

4.2 The Main Lemma

As we have seen in §3.1 , there exists a basis of the chiral primary fields made up of monomials in the fields X_i and to study the CGFs one can limit himself to considering $\langle \underline{k} | X^{\underline{q}} | \underline{h} \rangle$ where $X^{\underline{q}}$ is a monomial basis in $\mathcal{R}$. In this section I will prove the following statement

[25] Since $\mathbb{R}$ is not compact, many of Witten's results become less mathematically rigorous.
[26] This is the correct generalization of the reduction to a SQM problem seen in §1.5 .

(4.12) The Main Lemma : *the CGFs are equal to the intersection numbers in the following (double) complex*

$$\partial_V := \partial + \overline{\partial V} \qquad \overline{\partial}_V := \overline{\partial} + \partial V \qquad \delta_V := - * \overline{\partial}_V * \qquad \overline{\delta}_V := - * \partial_V * \tag{4.13}$$

that is

$$\langle \underline{k} | X^{\underline{q}} | \underline{h} \rangle = \int_{\mathbb{C}} \overline{*\omega_{\underline{k}}} \wedge X^{\underline{q}} \omega_{\underline{h}} \tag{4.14}$$

where $\omega_{\underline{h}}$ is a representative of a cohomology class in this (double) complex.

The explicit construction of the double complex ∂_V and the description of its principal properties are given in Appendix B and Appendix D.

In the remainder of this section I will prove the previous statement. I will give two proofs of the equivalence with the SQM problem. The first, although mathematically rigorous, has a more transparent physical meaning. The second is rather abstract but useful for practical computations.

4.2.1 *First proof of the Main Lemma*

The main idea is that the 2 dimensional supercharges can be seen as operators in loop–space cohomology, so many computations can be done using algebraic–topology techniques. This point of view has been useful in other contexts, see, e.g. refs. [12,37]. The crucial point can be easily stated in terms of the Wilson renormalization group [38]. In this approach, the renormalization transformation $R[\Lambda, \Lambda']$, mapping the theory defined at the cut–off Λ into one defined at Λ', is obtained by integrating away the degrees of freedom with frequencies in the range (Λ', Λ). The explicit form of $R[\Lambda, \Lambda']$ is quite involved. However, at the cohomological level, it is just a *homotopy.* Then all homotopic–invariant quantities are also RG invariant — this is the origin of non–renormalization theorems in supersymmetry. This has already been shown in refs. [10] for the Witten index. Here I need an extension of this result to more general invariants.

I work on a cylinder (of unit radius) and, for definiteness, I assume minimal kinetic terms. Let me expand the chiral fields as [27]

$$X_i(t, x) = \sum_k e^{ikx} X_{i,k}(t) \quad . \tag{4.15}$$

As a (rather crude) cut–off, I truncate the RHS to a finite sum $\sum_{|k|<\Lambda}$. The cut–off theory is just a SQM model with a number of degrees of freedom of order Λ. In fact, it can be seen [13] as the N=1 model with superpotential

$$V(X_{i,k}) = \sum_{|k|<\Lambda} k \overline{X}_{i,-k} X_{i,k} + \Re e\, \mathcal{V}(X_{i,k}) \tag{4.16}$$

[27] For notational simplicitly, in this section I will indicate the chiral fields as X_i instead of $X^{\underline{q}}$.

where

$$\mathcal{V}(X_{i,k}) = \int_0^{2\pi} ds\, V\left(\sum_{|k|<\Lambda} e^{iks} X_{i,k}\right) \quad . \tag{4.17}$$

The supercharges of this SQM model can be seen as the operators of a differential complex (known as the Witten Complex [12], see also §4.1) strongly related to Weil's de Rham model for equivariant cohomology [39].

Write the supercharges as

$$\begin{aligned} \bar\partial_W &= \bar\partial_{\mathcal{V}} + i_Y \\ \partial_W &= \partial_{\mathcal{V}} + i_{\overline{Y}} \end{aligned} \tag{4.18}$$

where $\bar\partial_{\mathcal{V}}$ is the supercharge for the N=2 model defined by the holomorphic superpotential $\mathcal{V}(X_{i,k})$. It is as a modified version of the Dolbeault operator (see §4.1 and Appendix D for details on the formalism). $\bar\partial_W$ commutes with the holomorphic functions of $X_{i,k}$. In eq. (4.18) , i_Y is the inner product with the holomorphic Killing field

$$Y^i = i\partial^i\left[\sum_{|k|<\Lambda} k\overline{X}_{i,-k} X_{i,k}\right] \quad . \tag{4.19}$$

Note that $i_Y\, d\mathcal{V} = 0$. The other two supercharges, δ_W and $\bar\delta_W$, are the Hermitean conjugates to $\bar\partial_W$ and ∂_W, respectively. The algebra is

$$\begin{aligned} &\bar\partial_W^2 = \partial_W^2 = 0 \\ &\bar\partial_W \partial_W + \partial_W \bar\partial_W = \mathcal{L}_Y + \mathcal{L}_{\overline{Y}} \\ &\delta_W \bar\partial_W + \bar\partial_W \delta_W = \bar\delta_W \partial_W + \partial_W \bar\delta_W = H_W \end{aligned} \tag{4.20}$$

where H_W is the "Laplacian" (the Hamiltonian) and $\mathcal{L}_Y$ is the Lie derivative. In eq. (4.20) I used the fact that $\mathcal{L}_Y\mathcal{V} = \mathcal{L}_{\overline{Y}}\mathcal{V} = 0$.

Following Witten [12], I put a free parameter s in the definition of the supercharges

$$\bar\partial_s = \bar\partial_{\mathcal{V}} + s i_Y \tag{4.21}$$

(For a discussion from the viewpoint of equivariant cohomology, see Atiyah and Bott [39]; for the infinite–dimensional case see ref. [40]). For $s = 1$ it is the theory coming from two dimensions, but it is a meaningful SQM system for all real values of s.

(4.22) Lemma : *the matrix elements*

$$\left(\mathcal{M}_s^i\right)_{h\,k} \stackrel{\text{def}}{=} \langle h|X_{i,0}|k\rangle_s \quad , \tag{4.23}$$

where $|k\rangle_s$ *are the vacuum states of the SQM model so constructed, are independent of* s, *as long as* $s \neq 0$.

The proof is deferred to Appendix E. ∎

The same Lemma can be shown to hold also for the $X_{i,k}$ matrix elements. Since we know that the CGFs do not depend on the ultraviolet cut–off Λ, one can always choose $\Lambda < k$ showing that $\langle k|X_i|h\rangle = \langle k|X_{i,k}|h\rangle$. This is indeed a consequence of a much stronger result.

Let $s \to \infty$. In this limit the wave–functions get concentrated on the zero set of the Killing vector Y^i, i.e. on the submanifold $X_{i,k\neq 0} = 0$ of the fixed–points for the Y–flow. The cohomology of $\bar{\partial}_\infty$ reduces to the cohomology of $\bar{\partial}_V$ on this submanifold [12], as it is known from equivariant cohomology. In more physical terms, as $s \to \infty$, the non–zero modes get decoupled by the Appelquist–Carazzone mechanism [41], i.e. the wave–function goes to the product of the wave–function for the zero mode times harmonic oscillator ground–states (with frequencies $O(ks)$) for the other modes. Then the chiral matrix elements are equal to those computed in the dimensional reduced model, where in eq. (4.16) one keeps only the zero mode. These matrix elements are sufficient to make contact with singularity theory as well as to compute critical quantities in the 2 dimensional theory (see §1.5).

Thus I have shown the following statement

the CGFs $\langle k|X_i|h\rangle$ are equal to the same correlation functions but computed in the dimensional reduced model. i.e. keeping only the zero oscillatory modes of the fields.

We know from §4.1 that to this SQM problem is associated a cohomological problem which is described by the complex defined in (4.13) .

A fundamental result in differential geometry is that the harmonic forms, i.e. the linearly independent solutions of the equation $H\psi = 0$ which physically we call "vacua", are in one to one correspondence with the cohomology classes of the differential complex (4.13) . In mathematical terms this is the "*Hodge Theorem*" which states that the harmonic space $\mathcal{H}^{p,q}$ is isomorphic to the Dolbeault cohomology group $H^{p,q}_{\bar{\partial}_V}$ [42,43] [28]. It is easy to show that if $|h\rangle$ is a representative of a cohomology class and $f(X_i)$ is a chiral operator, then $f(X_i)|h\rangle$ is again a representative of a cohomology class. Thus the CGFs can be seen as intersection numbers in the $\bar{\partial}_V$ cohomology. This completes the first proof of the "Main Lemma". ∎

4.2.2 *Second proof of the Main Lemma*

A direct computation of the QFT cohomology is possible and also very useful for practical purposes. I use the following notation: M_Λ is the Kähler manifold parametrized by the modes $X_{i,k}$ with $|k| \leq \Lambda$, and M is the Kähler manifold parametrized by the zero modes $X_{i,0}$. There is a natural structure of holomorphic vector bundle

$$\pi: M_\Lambda \to M \quad . \tag{4.24}$$

Notice that M is nothing else than the fixed submanifold for the Y–flow. Let $s: M \to M_\Lambda$ be the zero section. π and s are inverses in homotopy, and hence in de Rham cohomology [44]. I want to find the anologous statement for the supersymmetric cohomologies.

<u>(4.25) Lemma</u> : *Let $\beta \in \Lambda^*(M_\Lambda)$. Then*

[28] For the fundamental material in differential and algebraic geometry I refer to [43].

$$\bar{\partial}_V s^* \beta = s^* \bar{\partial}_W \beta \tag{4.26}$$

where $\bar{\partial}_W$ is the cut–off two–dimensional supercharge and $\bar{\partial}_V$ is the corresponding SQM supercharge (see Appendix D*).*

Proof : the proof of this lemma follows from the fact that the zero section is also the fixed–point set for the Y–flow. However,

$$\pi^* \bar{\partial}_V \alpha \neq \bar{\partial}_W \pi^* \alpha \qquad \alpha \in \Lambda^*(M) \quad . \tag{4.27}$$

Then, one has to replace π^* by another map $\Pi^* : \Lambda^*(M) \to \Lambda^*(M_\Lambda)$ which is a chain map between the SQM and QFT differential complesses, i.e.

$$\Pi^* \bar{\partial}_V \alpha = \bar{\partial}_W \Pi^* \alpha \qquad \alpha \in \Lambda^*(M) \quad . \qquad \blacksquare \tag{4.28}$$

(4.29) Lemma : *There exists $\sigma \in \Lambda^*(M_\Lambda)$ such that*

$$\begin{aligned} & s^* \sigma = 1 \\ & \bar{\partial}_W \sigma = (\pi^* dV) \wedge \sigma. \end{aligned} \tag{4.30}$$

σ can be chosen holomorphic. Alternatively, it can be chosen to have compact support in the vertical direction.

The proof is deferred to Appendix E. ∎

Now, define

$$\Pi^* \alpha := \pi^* \alpha \wedge \sigma \quad . \tag{4.31}$$

A simple computation gives

(4.32) Lemma : *it holds*

$$\begin{aligned} & s^* \Pi^* = 1 \\ & \Pi^* \bar{\partial}_V \alpha = \bar{\partial}_W \Pi^* \alpha \quad \forall \alpha \in \Lambda^*(M) \quad . \end{aligned} \tag{4.33}$$

The basic result is

(4.34) Theorem : *Let $\psi \in \Lambda^*(M_\Lambda)$ be a $\bar{\partial}_W$–closed form. Then there exists $\eta \in \Lambda^*(M_\Lambda)$ such that*

$$(1 - \Pi^* s^*) \psi = \bar{\partial}_W \eta \quad . \tag{4.35}$$

See Appendix E for the proof of this statement. ∎

(4.36) Corollary : *The $\bar{\partial}_W$–cohomology in $\Lambda^*(M_\Lambda)$ (i.e. in the 2 dim. QFT) and the $\bar{\partial}_V$–cohomology in $\Lambda^*(M)$ (i.e. in SQM)* are *isomorphic.*

Proof : by the lemmas (4.25) and (4.32) , Π^* and s^* descend to maps between the relevant cohomologies, and they are each other inverses (in cohomology) by eqs. (4.33) , (4.35) . ■

Remark : a priori we are interested not in the generic $\bar{\partial}_W$–cohomology, but in the cohomology taking value in the Hilbert space $\mathcal{H}$. However, one can show that all the classes in $\Lambda^*(M_\Lambda)$ have representatives in $\mathcal{H}$. Conversely, if a class is trivial in $\mathcal{H}$, it is also trivial in $\Lambda^*(M_\Lambda)$. The argument is an extension of that which will be presented for the SQM case. ■

Given that the chiral matrix elements are the product coefficients in the cohomology ring $\overline{\partial}_W$, the "Main Lemma" follows from the isomorphism Π^*, which is a *ring* isomorphism. In particular, this isomorphism proves the N=2 non–renormalization theorem. This completes the second proof of (4.12) . ■

The above equivalence with the one–dimensional theory should not be supposed to have a deeper dynamical content than it actually has. Morally speaking, the theories in 1D and 2D are only "homotopically equivalent" much in the same way as $\mathbb{R}^n$ is homotopically equivalent to a point. For a standard theory, there are no physically relevant quantities which are "homotopic invariants". Instead, in N=2 Landau–Ginsburg models such quantities exist.

4.3 The structure of the $\bar{\partial}_V$–cohomology groups

The supersymmetric vacua are the V–harmonic representatives of the $\bar{\partial}_V$–cohomology classes in the Hilbert space $\mathcal{H}$ since they satisfy $\bar{\partial}_V|k\rangle = 0 = \delta_V|k\rangle$. The harmonic forms have not definite (p,q) type since the differential operators themselves have not definite type.

To study the properties of the vacua and then of the CGFs which are the intersection numbers in the $\bar{\partial}_V$–cohomology, it is necessary to begin with a detailed study of the $H^*_{\bar{\partial}_V}$ cohomology group.

I start with the following lemma

(4.37) $\overline{\partial}_V$–Poincaré Lemma : *let ω be a $\overline{\partial}_V$–closed k–form in $\mathbb{C}^n$ (or in a n–dimensional Stein space). Then there exist a $(k-1)$–form η and a* holomorphic *k–form α such that*

$$\omega = \overline{\partial}_V \eta + \alpha \tag{4.38}$$

Conversely, any such ω is $\overline{\partial}_V$–closed.

The proof is given in Appendix D. ■

An immediate corollary of this lemma is the following [29]

(4.39) Corollary : *let $H^k_{\bar{\partial}_V}$ be the k–th $\bar{\partial}_V$–cohomology group and $H^k(K_{\partial V})$ the k–th cohomology group of the Koszul Complex $K_{\partial V}$ defined by ∂V (see* Appendix D*); then*

$$H^k_{\bar{\partial}_V} \overset{\text{iso}}{=} \begin{cases} 0 & \text{if } k > n \\ H^k(K_{\partial V}) & \text{if } k \le n \end{cases} . \tag{4.40}$$

[29] See also ref. [45].

From this corollary the next important proposition follows.
Let Ω^k be the space of holomorphic k–forms, let $p_i \in M$ be the critical points (they are isolated by assumption) of V (i.e. the points such that $\partial V|_{p_i} = 0$) and let $\widetilde{\mathcal{R}}_i := \Omega^n_i/[\partial V \wedge \Omega^{n-1}_i]$ where Ω^k_i is the space of the germs of holomorphic k–forms at p_i. Let $\widetilde{\mathcal{R}} := \Omega^n/[\partial V \wedge \Omega^{n-1}]$. It is well known that, as vector spaces, $\widetilde{\mathcal{R}} \overset{\text{iso}}{=} \oplus_i \widetilde{\mathcal{R}}_i$ [46].
Let $\mathcal{R}_i := \mathcal{O}_i/I_V$ where $\mathcal{O}_i$ is the space of germs of holomorphic functions at p_i and I_V is the *local* Jacobian ideal generated by the partial derivatives $\partial_j V \in \mathcal{O}_i$. $\mathcal{R} := \oplus_i \mathcal{R}_i$ is the *Chiral Ring* of section §3.1 . It is well known that $\widetilde{\mathcal{R}}_i \overset{\text{iso}}{=} \mathcal{R}_i$ [47] and the isomorphism $\phi \,:\, \mathcal{R}_i \rightarrow \widetilde{\mathcal{R}}_i$ is explicitly given by $\phi[h(X)] = h(X)dX_1 \wedge \ldots \wedge dX_n$.

(4.41) Proposition :

$$H^k_{\bar{\partial}_V} \overset{\text{iso}}{=} \begin{cases} 0 & \text{if} \quad k < n \\ \widetilde{\mathcal{R}} \overset{\text{iso}}{=} \mathcal{R} & \text{if} \quad k = n \end{cases} . \tag{4.42}$$

The proof is given in Appendix D. ∎

This proposition states that all the cohomology is concentrated in the middle dimension. Thus, the vacua are closed n–forms in $H^*_{\bar{\partial}_V}$. The fact that $H^n_{\bar{\partial}_V} \overset{\text{iso}}{=} \widetilde{\mathcal{R}}$ is obvious from the $\bar{\partial}_V$–Poincaré Lemma. Indeed ω is $\bar{\partial}_V$–exact if and only if there exists an η such that $\alpha = 0$. Moreover ω is invariant under

$$\eta \longrightarrow \eta - \tau \qquad\qquad \alpha \longrightarrow \alpha + \partial V \wedge \tau \tag{4.43}$$

for all holomorphic $(n-1,0)$ forms τ. Then the $\bar{\partial}_V$–cohomology in $\bigwedge^*(C^n)$ is isomorphic to the space of all holomorphic n–forms α modulo this equivalence, that is to $\widetilde{\mathcal{R}}$.

The chain of isomorphisms $\mathcal{R} \rightarrow \widetilde{\mathcal{R}} \rightarrow H^n_{\bar{\partial}_V}$ is of fundamental importance since give the possibility of an explicit construction of the cohomology groups. Moreover it embeds the *ring* structure (which has been studied in §3.1 and Appendix F) directly in the cohomology groups. Indeed, given a superpotential V one can associate to it its ring $\mathcal{R}$, compute its dimension and a basis of monomials. Then, using the isomorphism ϕ, one can explicit construct a holomorphic representative for each cohomological class in $H^n_{\bar{\partial}_V}$.

Another consequence of proposition (4.41) is that the cohomology is localized at the critical points p_i. This has various consequences. For V a polynomial, there is a compact $U \subset \mathbb{C}^n$ such that there is no critical points outside U. Since the cohomology is localized inside U, for each cohomology class in $\mathbb{C}^n$, one can find a representative with compact support in U. In other words, all the $\bar{\partial}_V$–classes *in* $\Lambda^*(\mathbb{C}^n)$ have a representative in the Hilbert space $\mathcal{H}$. Conversely, all cohomology in $\mathcal{H}$, represents cohomology classes in $\Lambda^*(\mathbb{C}^n)$. This fact, that the cohomology in $\Lambda^*(\mathbb{C}^n)$ and in $\mathcal{H}$ are equivalent, is important for practical computations: sometimes it is convenient to see the cohomology in $\mathcal{H}$, in order to apply Hodge–theoretical arguments, sometimes in $\Lambda^*(\mathbb{C}^n)$ to work without bothering if the actual representative is in $\mathcal{H}$ or not. The explicit construction of the representative with compact support will be given below.

As a corollary, the Witten index, which is the dimension of the cohomology space, is equal to the sum of the Milnor numbers $\mu_i = \dim\mathcal{R}_i$ of each critical point p_i . Since the cohomology is "localized" at p_i, we shall call the Milnor number μ_i the *local* Witten index at p_i. In view of §3.2 , it is the critical index.

The correspondence between supersymmetric vacua and the class of α in $\widetilde{\mathcal{R}}$ is an isomorphism. I denote this isomorphism by

$$G|k\rangle = [\alpha_k] \in \Omega^n/[dV \wedge \Omega^{n-1}] \quad . \tag{4.44}$$

This isomorphism is very useful for practical computations. The "local" version of the map G will be denoted by G_{p_i}. It maps $|k\rangle$ into the class $[\alpha_k] \in \widetilde{\mathcal{R}}_i$.

The physical relevance of the holomorphic germ–forms α_k stems from the fact that we can compute the chiral matrix elements $\langle h|X_i|k\rangle$ out of them. Indeed, these matrix elements are intersection forms in cohomology. One has

$$\alpha_h = \langle h|X_i|k\rangle \ \alpha_k \quad (\text{mod. } dV \wedge \beta) \quad . \tag{4.45}$$

4.4 Independence of the chiral Green's functions from the Kähler potential

The interpretation of the supersymmetric vacua as $\bar{\partial}_V$–classes has an important consequence. $\bar{\partial}_V$ does not depend on the Kähler metric. So, everything that can be written just in term of these classes, does not depend on the details of the Kähler metric (as long as the change of the metric does not imply a basic change in the geometry of $\mathcal{H}$). This is, in particular, true for

$$\int_M *\overline{\omega}_h \wedge f(X)\omega_k \tag{4.46}$$

where ω_k are cohomology classes representing supersymmetric vacua and $f(X)$ is any chiral operator. Indeed, this equation can be seen as an intersection form between the $\bar{\partial}_V$–class $[f(X)\omega_k]$ and the class $[*\overline{\omega}_h]$ belonging to the "time–reversed" cohomology $\bar{\partial}_{-V}$ [13] (which is the natural dual space to $H^n_{\bar{\partial}_V}$). Note that the map $\omega_h \to *\overline{\omega}_h$ is independent of the Kähler metric since, for primitive n–forms, the Hodge dual is simply [42,43]

$$* = (-1)^{n(n-1)/2} C \quad , \tag{4.47}$$

where $C = i^{\tilde{F}}$ is the Weil operator and $\tilde{F}$ is the operator which acting on a (p,q) form gives $(p-q)$.

In fact, one can give even a stronger notion of independence from the Kähler potential. Consider first the $n = 1$ case. In this case, the vacuum wave–forms are independent of the Kähler potential as forms, not just as cohomology classes.

Indeed, a vacuum 1–form ω satisfies the equations

$$\bar{\partial}_V\omega = \delta_V\omega = 0 \quad . \tag{4.48}$$

Using the Kähler–Hodge relation $[\partial_V, \Lambda] = -\delta_V$, the second equation can be rewritten as

$$\Lambda\partial_V\omega = 0 \quad . \tag{4.49}$$

For $n = 1$, Λ is an isomorphism of the 2–forms into the 0–forms, by Lefschetz SU(2) (see Appendix B). Then one can rewrite (4.48) as

$$\bar{\partial}_V \omega = 0 \qquad \partial_V \omega = 0 \tag{4.50}$$

where the Kähler metric no longer appears. Notice that this is just the conformal–invariance of the zero–energy Schrödinger equation. (For $n = 1$ all Kähler metrics are conformally equivalent).

Of course, this "universality" property does not hold for $n > 1$. However, there is a weaker version of it which holds for all n, and which is exactly what I need below.

(4.51) Proposition : *the $(n,0)$ (resp. $(0,n)$) parts of the vacuum wave–forms are independent of the Kähler metric.*

Proof : fix a reference Kähler metric. Consider an arbitrary n–form ω. It can be written as [42,43]

$$\omega = \omega_P + L\beta \tag{4.52}$$

where ω_P is *primitive*, that is $\Lambda\omega_P = 0$. If ω satisfies the vacuum equations, then so do ω_P and $L\beta$. By Kähler–Hodge, on a primitive n–form,

$$\delta_V \omega_P = 0 \Leftrightarrow \Lambda \partial_V \omega_P = 0 \Rightarrow \partial_V \omega_P = 0 \quad . \tag{4.53}$$

Take the space $\mathcal{S}$ of all the n–forms Ξ such that $\bar{\partial}_V \Xi = \partial_V \Xi = 0$. The subspace $\mathcal{P} \in \mathcal{S}$ of the primitive n–forms is then the space of the vacuum wave–forms [30]. A generic element $\omega \in \mathcal{S}$ can be written as $\omega_P + L\beta$ with $\omega_P \in \mathcal{P}$. The space $\mathcal{S}$ does not depend on the metric, but — of course — $\mathcal{P}$ does. Consider another Kähler form $\tilde{L}$. The new vacuum wave–forms span $\tilde{\mathcal{P}} \subset \mathcal{S}$. But any element $\tilde{\omega}_P \in \tilde{\mathcal{P}}$, being in $\mathcal{S}$, can be written as

$$\tilde{\omega}_P = \omega_P + L\beta \quad . \tag{4.54}$$

Since the $(n,0)$ (resp. $(0,n)$ forms are not contained in the image of L one gets

$$\tilde{\omega}_P|_{(n,0)} = \omega_P|_{(n,0)} \qquad \tilde{\omega}_P|_{(0,n)} = \omega_P|_{(0,n)} \tag{4.55}$$

which is the universality property which was to be proved. ■

This is the V–analogue of the fact that a holomorphic (respectively anti–holomorphic) n–form is harmonic for all the Kähler metrics. The relevance of this universality property stems from the fact that all the $\bar{\partial}_V$–cohomology is concentrated in the $(n,0)$ component, in the sense that any two $\bar{\partial}_V$–closed forms, whose $(n,0)$ projections agree, are in fact cohomologous (see §4.3). So the universal part of the wave–function is enough to completely characterize the vacuum state. Thus, for the dependence on the parameters in the Kähler potential, there is not the ambiguity one has for the dependence on the couplings in the superpotential. Alternatively, one can see the independence of the chiral matrix elements from the parameters

[30] For a non–degenerate potential the vacua have zero Lefschetz angular momentum, that is, are primitive n–forms.

in K just as an angular momentum selection rule of the kind common in atomic physics. Here the rôle of angular momentum is played by the Lefschetz $SU(2)$.

This leads to the following result

the CGFs are invariant under deformations of the Kähler space which do not change its topology.

This result was already obtained in §1.4 by physical arguments.

4.5 The computation of CGFs as intersection numbers in cohomology

In this section I will show two major results: the existence of a representative in each cohomology class with compact support and the computation up to the normalization constants of the CGFs. To get these results I will need some fairly standard material in differential and algebraic geometry. I will not recall here in all details these results. For definiteness, I will refer to [43] for a complete introduction to the subject and the conventions here adopted.

4.5.1 *Compact support representative of a cohomology class*

As I have already noticed, a vacuum wave form ω to be physical must be in the Hilbert space $L^2[\bigwedge^n(\mathbb{C}^n)]$. I will now show that in any cohomology class in $H^n_{\overline{\partial}_V}$ there is a representative with compact support and, a fortiori, in the Hilbert space.

I begin with the following fundamental lemma.

Let

$$\begin{aligned}
\lambda_i &= \frac{\overline{\partial_i V}}{|\partial_1 V|^2 + \ldots + |\partial_n V|^2} \\
\beta_0 &= \sum_i (-1)^{i-1} \lambda_i dX_i \wedge \ldots \wedge \widehat{dX_i} \wedge \ldots \wedge dX_n \\
\beta_k &= \sum_i (-1)^{i-1} \sum_{j_1 \neq i} \sum_{j_2 \neq i} \cdots \sum_{j_k \neq i} \Lambda_i dX_1 \wedge \ldots \wedge \widehat{dX_i} \wedge \ldots \\
&\quad \ldots \wedge \bar{\partial}\lambda_{j_1} \wedge \ldots \wedge \bar{\partial}\lambda_{j_k} \wedge \ldots \wedge dX_n
\end{aligned} \tag{4.56}$$

where $\widehat{dX_i}$ means omitted and each $\bar{\partial}\lambda_{j_s}$ is in the j_s–th position counting the omitted $\widehat{dX_i}$ as present.

<u>(4.57) Lemma</u> : *let* $\mu := \sum_{k=0}^{n-1} (-)^k \beta_k$ *then* $\overline{\partial}_V \mu = dX_1 \wedge \ldots dX_n$.

The proof is given in Appendix D. ∎

Notice that μ is an $(n-1)$–form of not definite (p,q) type.

Let $K \subset K'$ be compacts in $\mathbb{C}^n$ such that all the critical points of V belong to the interior of K. Let $\rho(X)$ be a smooth function which vanishes in K and equals 1 outside K'. Let $\omega = \overline{\partial}_V \eta_\alpha + \alpha$ a vacuum germ form where α is a $(n,0)$–holomorphic form $\alpha = \alpha(X) dX_1 \wedge \ldots \wedge dX_n$ and η_α any $(n-1)$–form. Choose η_α as follows

$$\eta_\alpha := -\rho(X)\alpha(X) \cdot \mu \tag{4.58}$$

then the vacuum wave form $\omega = \overline{\partial}_V \eta_\alpha + \alpha$ *has compact support in* K'.

Thus, given a cohomology class specified by an holomorphic $(n,0)$ representative α which is explicitly known through the isomorphism between $\mathcal{R}$ and $\widetilde{\mathcal{R}}$, one has also the explicit expression of the compact supported vacuum wave form $\omega = \overline{\partial}_V \eta_\alpha + \alpha$. This expression gives the possibility of explicitly compute the intersection numbers in cohomology.

4.5.2 *The computation of the intersection numbers*

The CGFs are given by

$$\langle \underline{p} | X^{\underline{k}} | \underline{q} \rangle = \int_{\mathbb{C}^n} * \overline{\omega_{2,\underline{p}}} \wedge X^{\underline{k}} \omega_{1,\underline{q}} \quad . \tag{4.59}$$

I begin to compute the following intersection number in cohomology:

$$\int * \overline{\omega_{2,\underline{p}}} \wedge \omega_{1,\underline{q}} \quad . \tag{4.60}$$

The two forms ω_1 and ω_2 belong to different although anti–isomorphic cohomologies. Indeed ω_1 is a representative of a closed class of the $\bar{\partial}_V$ cohomology and ω_2 of the ∂_V cohomology. Write

$$\omega_{1,\underline{q}} = \alpha_{\underline{q}} + \bar{\partial}_V \eta_{\underline{q}} \tag{4.61}$$

$$\omega_{2,\underline{p}} = \widetilde{\beta}^*_{\underline{p}} + \partial_V \widetilde{\eta}^*_{\underline{p}} \tag{4.62}$$

where α is a $(n,0)$ form and β^* is a $(0,n)$ form. It follows that:

$$* \overline{\omega_{2,\underline{p}}} = (-1)^{n(n-1)/2} i^n \left(\widetilde{\beta}_{\underline{p}} + \bar{\partial}_{-V} \widetilde{\eta}_{\underline{p}} \right) \tag{4.63}$$

(n in the number of fields).

I adopt the following convention for the holomorphic forms on $\widetilde{\mathcal{R}}$

$$\alpha_{\underline{i}} = \lambda_{\underline{i}} X^{\underline{i}-\underline{1}} dX_1 \wedge \ldots \wedge dX_n =: \alpha_{\underline{i}}(X) dX_1 \wedge \ldots \wedge dX_n \tag{4.64}$$

where $\underline{1} = (1, \ldots, 1)$, $(i)_j \geq 1$ and $\alpha_{\underline{i}}(X)$ belongs to $\mathcal{R}$. Moreover, $\widetilde{\beta}_{\underline{p}}(X) = \widetilde{\lambda}_{\underline{p}} X^{\underline{p}-\underline{1}}$.

The first result is the following

<u>(4.65) Lemma :</u>

$$\begin{aligned} \langle \underline{p} | \underline{q} \rangle &= \int_{\mathbb{C}^n} * \overline{\omega_{2,\underline{p}}} \wedge \omega_{1,\underline{q}} \\ &= (-2\pi)^n \int_{S_{2n-1}} F^* K \cdot \widetilde{\beta}_{\underline{p}}(X) \alpha_{\underline{q}}(X) \end{aligned} \tag{4.66}$$

where K *is the Bochner–Martinelli kernel and* $F(z_i) = (z_i + \partial_i V, z_i)$.

The definition of the Bochner–Martinelli kernel and the details of the proof are given in Appendix D, see also refs. [42,43].

The Bochner–Martinelli kernel gives the extension of the Cauchy integral formula of residue to the case of many complex variables. Indeed, there is a generalization of the "Residue Theorem" as follows (see for example [43] pag. 649)

(4.67) Residue Theorem : *let*

$$\omega = \frac{g(z)dz_1 \wedge \cdots \wedge dz_n}{\partial_1 V(z) \cdots \partial_n V(z)} \tag{4.68}$$

and Γ *a real n–cycle defined by* $\Gamma = \{ z : \ |\partial_i V(z)| = R_i \}$ *and oriented by* $d(\arg \partial_1 V) \wedge \cdots \wedge d(\arg \partial_n V) \geq 0$*, then*

$$Res_{\{0\}}(\omega) := \left(\frac{1}{2\pi i}\right)^n \int_\Gamma \omega = \frac{g(0)}{\det(\partial_i \partial_j V)(0)} \quad . \tag{4.69}$$

Notice, moreover, that if $g \in$ Ideal$(\partial_i V)$ then $Res_{\{0\}}\omega = 0$.
The next result I need is the

(4.70) Holomorphic Lefschetz fixed point formula :

$$\int_{S_{n-1}} F^* K \cdot g(z) = \frac{g(0)}{\det\left(\partial_i \partial_j V\right)(0)} \quad . \tag{4.71}$$

See for example [43] pag. 426.

From this it is an immediate consequence the following

(4.72) Corollary :

$$\begin{aligned} \langle \underline{p} | \underline{q} \rangle &= \int_{\mathbb{C}^n} * \overline{\omega_{2,\underline{p}}} \wedge \omega_{1,\underline{q}} \\ &= (-2\pi)^n Res_{\{0\}} \left(\frac{\widetilde{\beta}_{\underline{p}}(X) \alpha_{\underline{q}}(X) dX_1 \wedge \cdots \wedge dX_n}{\partial_1 V(X) \cdots \partial_n V(X)} \right) \quad . \end{aligned} \tag{4.73}$$

This surely holds for all $\alpha, \widetilde{\beta} \in \widetilde{\mathcal{R}}$ and $\langle \underline{p} | \underline{q} \rangle = 0$ if $\widetilde{\beta}$ or $\alpha \in$ Ideal. One can now define a symmetric pairing as follows

(4.74) Definition : *let* res_V *be the pairing*

$$\begin{aligned} res_V \ &: \mathcal{R} \otimes \mathcal{R} \longrightarrow \mathbb{C} \\ res_V(\widetilde{\beta}_{\underline{p}}, \alpha_{\underline{q}}) &:= Res_{\{0\}} \left(\frac{\widetilde{\beta}_{\underline{p}}(X) \alpha_{\underline{q}}(X) dX_1 \wedge \cdots \wedge dX_n}{\partial_1 V(X) \cdots \partial_n V(X)} \right) \quad . \end{aligned} \tag{4.75}$$

A fundamental result is the following

(4.76) Grothendieck Local Duality Theorem : res_V *is a symmetric, non–degenerate pairing.*

See for example [43] pag. 659. This theorem means that if $res_v(\beta, \alpha) = 0$ for all $\beta \in \mathcal{O}$ where $\mathcal{R} = \mathcal{O}/I$, then $\alpha \in I$.

From the "Lefschetz Fixed Point Theorem" (see [43] pag. 419) and theorem 3.7 of ref. [48] (also lemma 4.5.3 of ref. [48] and theorem 14.5 pag. 412 of ref. [49]) it follows that the intersection number $\langle \underline{p}|\underline{q}\rangle$ is a scalar product in the Hilbert space. Moreover,

the intersection number $\langle \underline{p}|\underline{q}\rangle = \int_{\mathbb{C}^n} *\overline{\omega_{2,\underline{p}}} \wedge \omega_{1,\underline{q}}$ *is different from zero* **only if** $\widetilde{\beta}_{\underline{p}}(X) \cdot \alpha_{\underline{q}}(X)$ *is proportional to the Hessian of* V*, that is*

$$X^{\underline{p}-\underline{1}} \cdot X^{\underline{q}-\underline{1}} = C \det \left[\frac{\partial^2 V}{\partial X_i \partial X_j} \right] =: C \cdot J(X) \tag{4.77}$$

(this equation should be understood in the ring $\mathcal{R}$*) where* C *is a constant and* $\widetilde{\beta}_{\underline{p}}(X) = \widetilde{\lambda}_{\underline{p}} X^{\underline{p}-\underline{1}}$*,* $\alpha_{\underline{q}}(X) = \lambda_{\underline{q}} X^{\underline{q}-\underline{1}}$*.*

This result gives the possibility of explicitly compute the intersection numbers.

Let $\alpha_{\underline{q}}$ and $\widetilde{\beta}_{\underline{p}}$ be such that $\langle \underline{p}|\underline{q}\rangle \neq 0$. Then

$$\widetilde{\beta}_{\underline{p}}(X) \cdot \alpha_{\underline{q}}(X) = \widetilde{\lambda}_{\underline{p}} \lambda_{\underline{q}} C \cdot J(X) \quad . \tag{4.78}$$

By substituting this in (4.73) one gets

$$\langle \underline{p}|\underline{q}\rangle = (-2\pi)^n Res_{\{0\}} \left(\frac{d(\partial_1 V) \wedge \ldots \wedge d(\partial_n V)}{\partial_1 V \ldots \partial_n V} \right) \widetilde{\lambda}_{\underline{p}} \lambda_{\underline{q}} C \quad . \tag{4.79}$$

The following standard results are known (see for example [43] pag. 663)

<u>(4.80) Lemma :</u>

$$Res_{\{0\}} \left(\frac{d(\partial_1 V) \wedge \ldots \wedge d(\partial_n V)}{\partial_1 V \ldots \partial_n V} \right)$$

is by definition the "Local Intersection Number"*, it is equal to an integer that depends only on the ideal and not on the choice of the generators* $\partial_i V$*. In particular, it depends only on the divisor* D_i *and not on their defining functions. Its numerical value is equal to the topological degree of* $\partial_i V$*, that is*

$$Res_{\{0\}} \left(\frac{d(\partial_1 V) \wedge \ldots \wedge d(\partial_n V)}{\partial_1 V \ldots \partial_n V} \right) = \text{Milnor number } \mu = \text{Witten index } \Delta \quad .$$

This Lemma can be taken as another definition of the Witten Index (see §1.2). From this lemma one gets the following result

<u>(4.81) Proposition :</u>

$$\langle \underline{p}|\underline{q}\rangle = \begin{cases} (-2\pi)^n C \Delta \, \widetilde{\lambda}_{\underline{p}} \lambda_{\underline{q}} & \text{if } \widetilde{\beta}_{\underline{p}}(X) \cdot \alpha_{\underline{q}}(X) = \widetilde{\lambda}_{\underline{p}} \lambda_{\underline{q}} C \cdot J(X) \\ 0 & \text{otherwise} \end{cases} \quad . \tag{4.82}$$

Since, as already said, $\langle \underline{p}|\underline{q}\rangle$ is a scalar product in the Hilbert space one can choose an orthonormal basis in $H^n_{\bar{\partial}_V}$. Indeed, physically one normalizes the vacua imposing that

$$\langle \underline{p}|\underline{q}\rangle = \delta_{\underline{p},\underline{q}} \tag{4.83}$$

One can proceed as follows. Choose a basis in $\mathcal{R}$ (and then also in $\widetilde{\mathcal{R}}$ and in $H^n_{\bar{\partial}_V}$) fixing a set of monomials $\alpha_{\underline{q}}(X)$. From formula (4.77) one can construct an orthogonal basis $\widetilde{\beta}_{\underline{p}(\underline{q})}(X) = \widetilde{\lambda}_{\underline{p}(\underline{q})} X^{\underline{p}(\underline{q})-\underline{1}}$, which will be called the *dual* basis such that (4.78) holds. Next one can normalize the vacua using proposition (4.81) and the condition (4.83) . Thus, having chosen a basis $\alpha_{\underline{q}}(X)$ in $\mathcal{R}$, the dual basis $\widetilde{\beta}_{\underline{p}(\underline{q})}(X)$ is completely characterized by the conditions

$$\begin{aligned} &\widetilde{\beta}_{\underline{p}(\underline{q})}(X) \cdot \alpha_{\underline{q}}(X) = \widetilde{\lambda}_{\underline{p}(\underline{q})} \lambda_{\underline{q}} C \cdot J(X) \\ &\widetilde{\lambda}_{\underline{p}(\underline{q})} \cdot \lambda_{\underline{q}} = \frac{1}{(-2\pi)^n C \Delta} \quad . \end{aligned} \tag{4.84}$$

For notational convenience I will always denote the dual basis by the index $\underline{p}(\underline{q})$, i.e. $\langle \underline{p}(\underline{q})|\underline{p}\rangle = 1$ or $\langle \underline{p}|\underline{q}\rangle = \delta_{\underline{p},\underline{p}(\underline{q})}$.

From (4.84) one immediately obtains the following observation. If one should know the ratio $\lambda_{\underline{q}}/\widetilde{\lambda}_{\underline{p}(\underline{q})}$, then from (4.84) one would be able to compute the *"absolute normalization"* of the vacuum germ form $\alpha_{\underline{q}}$. The computation of the absolute normalization of the vacuum germ forms is the technical problem which will be addressed and solved in the rest of this thesis. Its relevance stems from the following result:

(4.85) *the knowledge of the absolute normalization $\lambda_{\underline{q}}$ of the vacuum germ form $\alpha_{\underline{q}}$ is sufficient to exactly compute the CGFs since*

$$\langle \underline{p}|X^{\underline{k}}|\underline{q}\rangle = \int_{\mathbb{C}^n} *\overline{\omega_{2,\underline{p}}} \wedge X^{\underline{k}} \omega_{1,\underline{q}} = \delta_{\underline{p},\underline{p}(\underline{k}+\underline{q})} \frac{\lambda_{\underline{q}}}{\lambda_{\underline{k}+\underline{q}}} \quad . \tag{4.86}$$

<u>Proof :</u> notice first that

$$X^{\underline{k}} \alpha_{\underline{q}} = \frac{\lambda_{\underline{q}}}{\lambda_{\underline{k}+\underline{q}}} \cdot \alpha_{\underline{k}+\underline{q}} \tag{4.87}$$

(all these equations should be understood in $\mathcal{R}$). From formula (4.58) it follows that $\omega_{1,\underline{q}} = \bar{\partial}_V \eta_{\alpha,\underline{q}} + \alpha_{\underline{q}}$ where $\eta_{\alpha,\underline{q}} = -\rho(X)\alpha_{\underline{q}}(X) \cdot \mu$. Then $X^{\underline{k}} \omega_{1,\underline{q}} = (\lambda_{\underline{q}}/\lambda_{\underline{k}+\underline{q}}) \cdot \omega_{1,\underline{k}+\underline{q}}$ or $X^{\underline{k}}|\underline{q}\rangle = (\lambda_{\underline{q}}/\lambda_{\underline{k}+\underline{q}})|\underline{k}+\underline{q}\rangle$. Recalling (4.83) the statement is immediately proven. ∎

CHAPTER 5

N=2 Supersymmetry and Singularity Theory

In this chapter I will make contact with "*Singularity Theory*". The aim is twofold: I want to establish the connection between the N=2 LG models and Singularity Theory and, at the same time, I will introduce the mathematical structure which will be needed for the future developments. Thus, I will alternate physical reasoning to mathematical definitions. For what concerns mathematics, the main reference is the book of Arnold, Gusein–Zade and Varchenko, "*Singularities of differentiable maps*", specially volume II [21,49].

In the previous chapter I have noticed many times how the properties of the "*singular*" points of the superpotential (i.e. the points p_i where $\partial V|_{p_i} = 0$) characterize the physics and the mathematics of the N=2 LG models. From the Witten Index Δ to the cohomology groups $H^n_{\bar{\partial}_V}$, everything depends on the singularities of the superpotential. Since in mathematics this is a well known and deeply studied subject, it is worth to make an explicit connection between physics and mathematics. Indeed, this purpose has been first undertaken by Martinec, Vafa and collaborators [4,5,7] who noticed that the results of Singularity Theory (see ref. [21]) can be applied to physics with success. Their physical arguments allowed them to use the simplest results of Singularity Theory. But since my aim is more ambitious, it is necessary to find the deep connection between physics and mathematics. For this, all the machinery of Singularity Theory must be introduced.

5.1 Critical points of smooth functions

Before entering in the details of Singularity Theory, it is better to recall some definitions and results mainly of algebraic nature (see also Appendix F and [21] for a general review).

<u>(5.1) Definition</u> : *let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a holomorphic map germ in zero, then the quotient algebra of the functions by the ideal generated by the components of the map*

$$Q_f := \frac{\mathcal{E}_X}{I_f} \tag{5.2}$$

where $\mathcal{E}_f$:= { the algebra of germs of infinitely differentiable functions at 0} and I_f := { ideal in $\mathcal{E}_f$ generated by $(f_1, \ldots, f_n)$} is called the **Local Algebra of the map f at zero.**

Let μ be the dimension of this algebra, it is called the **algebraic local multiplicity of** f **at zero.**

(5.3) Theorem : *the number of preimages near zero of a generic point near zero for the map f is equal to the dimension of the local algebra $\mu = \dim_{\mathbb{C}} Q_f$.*

(For this theorem see §4.3 [21].) This is indeed another way of defining the Witten Index (see §1.2).

(5.4) Example : the example, which is indeed the case of interest here, is when the map f is the gradient of another holomorphic map V:
$V\ :\ (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $f = (f_1, \ldots, f_n)$, $f_i\ :\ \mathbb{C}^n \to \mathbb{C}$, $f_i := \partial_i V$. It is obvious that in this case $Q_f = Q_{\partial V} = \mathcal{R}$ defined in §3.1 . From now on I will consider only this case. ■

(5.5) Definition : *let $V :\ \mathbb{C}^n \to \mathbb{C}$ be a smooth map, a point x is called* **a critical point** *for V if at that point the derivatives of V are zero.*

One can always assume that V has a critical point in zero with value zero. Singularity Theory studies the properties of the holomorphic map germ V defined in a neighborhood of the critical point (chosen conventionally as zero), $V :\ (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$.

In Singularity Theory a very important problem is the classification of the singularities, in this ambitus an important place belongs to the quasihomogeneous and semi–quasihomogeneous singularities. As we will immediately see, these correspond also to the superpotentials which are more interesting from a physical point of view.

(5.6) Definition : *a holomorphic function $V :\ (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is said to be* **quasihomogeneous** *of degree d and indices $\alpha_1, \ldots, \alpha_n$ if for any $\lambda > 0$ it holds*

$$V(\lambda^{\alpha_1} X_1, \ldots, \lambda^{\alpha_n} X_n) = \lambda^d V(X_1, \ldots, X_n) \quad . \tag{5.7}$$

The indices α_s are also called weights of the variable X_s.

(5.8) Definition : *a quasihomogeneous function V is said to be* **non–degenerate** *if zero is an* **isolated** *critical point (that is if the multiplicity of zero is finite).*

(5.9) Definition : *a monomial $X^{\underline{k}} = X_1^{k_1} \cdots X_n^{k_n}$ has (generalized)* **degree** *(or weight) d if $\sum_{i=1}^n \alpha_i k_i = d$.*

(5.10) Definition : *a polynomial (power series, germ, function) has* **order** *d if all of its monomials have degree d or higher.*

(5.11) Definition : *a polynomial (power series, germ) is said to be* **semi–quasihomogeneous** *of degree d with weights $\alpha_1, \ldots, \alpha_n$ if it is of the form $V = V_0 + V'$ where V_0 is a non–degenerate quasihomogeneous polynomial of degree d and weights $\underline{\alpha}$ and V' a polynomial (power series, germ) of order strictly greater than d.*

(5.12) Definition : *a monomial is said to be* **upper** *or* **lying above the diagonal** *(respectively* **lower** *or* **diagonal***) if it has degree greater than d (respectively less than d or equal to d) for the given indices of quasihomogeneity.*

Let $(e_1, \ldots, e_s)$ be the system of all **upper** basis monomials of a fixed basis for the local algebra of the functions V_0.

(5.13) Theorem : *every semi–quasihomogeneous function with quasihomogeneous part V_0 is equivalent to a function of the form $V_0 + \sum_k c_k e_k$ where c_k are constants.*

For the proof see §12.6 [21].

(5.14) Theorem : *the multiplicity of the point zero of a semi–quasihomogeneous function V is equal to the multiplicity of the critical point zero of it quasihomogeneous part V_0: $\mu(V) = \mu(V_0)$.*

(5.15) Corollary : *suppose that the system of monomials $(e_1, \ldots, e_\mu)$ is a basis for the local algebra of the quasihomogeneous part V_0 of the semi–quasihomogeneous function V. Then the same system of monomials also gives a basis for the local algebra of the function V.*

For the proofs see §12.2 [21]. These results are the mathematical analogue of the physical statement that in the IR regime the physics is described by the monomials in the potentials which have the lowest power, or, in other words, that the terms in the superpotential with higher degree are irrelevant in the IR regime.

(5.16) Remark : if V is a semi–quasihomogeneous function of degree d and type $\underline{\alpha}$ then the map $X \to \text{grad } V(X)$ is semi–quasihomogeneous of degree $d_s := d - \alpha_s$.

(5.17) Definition : *the* **Poincaré Polynomial** *of a semi–quasihomogeneous map V (where $\alpha_s = A_s/N$, A_s and N being integral) is the polynomial $p_V(t) := \sum_i \mu_i t^i$, where $t \in \mathbb{R}$, μ_i is the number of basis monomials of the local algebra of V having quasidegree i/N.*

(5.18) Remark : the dimension of the local algebra of V is given by $\mu = p_V(1)$.

(5.19) Theorem : *the Poincaré polynomial of a semi–quasihomogeneous map V of degree d and type $\underline{\alpha}$ for which $\alpha_s = A_s/N$, $d_s = D_s/N$ (A_s, D_s and N integral) is given by*

$$p_V(t) = \prod_{s=1}^{n} \frac{t^{D_s} - 1}{t^{A_s} - 1} \quad . \tag{5.20}$$

(5.21) Corollary :

$$\mu = \prod_{s=1}^{n} \left(\frac{d}{\alpha_s} - 1 \right) \quad . \tag{5.22}$$

(5.23) Corollary : *a monomial basis for the local algebra of a semi–quasihomogeneous function V of type $\underline{\alpha}$ of degree d has exactly one generator of degree*

$$d_{max} := \sum_{s=1}^{n} (d - 2\alpha_s) \tag{5.24}$$

and all the monomials of higher degree belong to the ideal $(\partial_1 V, \ldots, \partial_n V)$.

(5.25) Corollary : *the Poincaré polynomial of a semi–quasihomogeneous polynomial map V is always recurrent* $\mu_i = \mu_{k-i}$ *where* $k = \sum_{i=1}^{n} D_s - \sum_{i=1}^{n} A_s$.

(5.26) Corollary : *the degree of the penultimate monomial in a monomial basis for the local algebra of a quasihomogeneous function of type* $\underline{\alpha}$ *of degree 1 is equal to* $d_{max} - \alpha_{min}$ *where* $\alpha_{min} = \min(\alpha_1, \ldots, \alpha_n)$.

For the proofs of these results see §12.3 [21].

These results give the possibility of easily compute all the main properties of the ring $\mathcal{R}$ of the semi–quasihomogeneous superpotentials. Indeed, it is obvious that $\Delta = \mu$ and that with these results it is very easy to explicitly construct a basis in $\mathcal{R}$. In the future I will show how other physical informations can be obtained from these results.

Another important important concept has an immediate physical analogue, it is the concept of "*modality*". For semplicitly I consider the modality only in the case of semi–quasihomogeneous superpotentials.

(5.27) Definition : *the* **modality** *of a semi–quasihomogeneous function is the total number of diagonal and superdiagonal basis monomials of a monomial basis for the local algebra.*

Thus, the "*modality*" in physical language is the total number of marginal and irrelevant chiral primary operators for the given 2d N=2 LG model.

In mathematics, modality is very important, for example in the classification of singularities. In ref. [21] §15 all singularities with modality $m \leq 2$ and for which $\mu \leq 16$ are completely classified.

As an example of physical consequences of these results, remember that the quasihomogeneity property coincides with what in physics is called R–symmetry. The weights α_s of the variable X_s in physics are the R–charges of the chiral primary fields X_s, $q_s = \alpha_s$. From the corollary (5.23) one obtains the maximum value between the R–charges of the chiral primary fields. Since the central charge of the N=2 superconformal theory which is obtained at the IR critical limit is $c/3 = q_{max}$, one has [7]

$$c = 3 \sum_{s=1}^{n} (d - 2q_s) \quad . \tag{5.28}$$

This first result is a clear indication of how many informations on the physics one can obtain once the connection between N=2 LG models and Singularity Theory has been correctly established.

5.2 N=2 Supersymmetry and Picard–Lefschetz theory

As we havw seen, Witten pointed out that N=1 supersymmetry is deeply related to Morse theory [12]. It is quite natural that, when one specializes to the particular case of N=2 supersymmetry, Morse theory gets replaced by its holomorphic counterpart, the Picard–Lefschetz (PL) theory. In this section I sketch the connection between N=2 supersymmetry and PL theory.

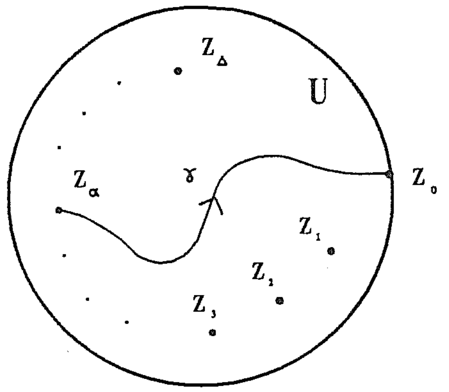

Figure 1
The path in the V plane defining the homology cycle vanishing at the critical point X_α. Here $Z_\alpha = V(X_\alpha)$ is the corresponding critical value

The Picard–Lefschetz theory deals with functions V which are both holomorphic and Morse (i.e. all the critical points of V are non–degenerated [31] and the values of V at the critical points are all distinct, $V_\alpha \neq V_\beta$ if $\alpha \neq \beta$). In the case of interest here, the superpotential V is typically not a Morse function. It has a degenerate singularity. However, for the class of N=2 models I am interested in, the dynamics is smooth under a *linear* perturbation of the superpotential $V(X) \to V_\lambda(X) = V + \sum_i \lambda_i X_i$ (see §1.2) For almost all λ_i, V_λ is a Morse function. Let me start by recalling some standard results in PL theory and then applying them to the physical situation.

Let $f : M \to \mathbb{C}$ be a holomorphic function on a n–dimensional complex manifold M with a smooth boundary ∂M (in the real sense). Let U be a contractible compact region in the complex plane, $U \subset \mathbb{C}$, with smooth boundary ∂U. Let f have a finite number of critical points p_i, $i = 1, \ldots, \mu$, with critical values $z_i = f(p_i)$ lying inside the region U. Denote by F_z the **Level Set** of the function f, that is $F_z := f^{-1}(z)$. If $z \in U$ is a non-critical value of the function f, then the corresponding level set F_z is a compact $(n-1)$–dimensional complex manifold with smooth boundary $\partial F_z = F_z \cap \partial M$.

Let f be a Morse function (that is $\frac{\partial^2 F}{\partial z_i \partial z_j}|_{z=0} \neq 0$) and fix a non–critical value z_0 at the boundary of a (compact) region $U \subset \mathbb{C}$ containing all the critical values of f (see Figure 1). Take a path $\gamma(t) : [0,1] \to U$ joining the critical value z_α to the non–critical one z_0 ($\gamma(0) = z_\alpha, \gamma(1) = z_0$) and not passing through critical values of f for $t \neq 0$. By lifting the homotopy, the map $\gamma(t)$ defines a continuous family of mappings $\Gamma_t : F_{z_0} \to \mathbb{C}^n$. Γ_t maps the level manifold F_{z_0} into the level manifold $F_{\gamma(t)}$. For t small, one can construct a sphere in the level manifold $F_{\gamma(t)}$ using the Morse coordinates. Indeed, by the Morse lemma [50], there exist coordinates centered at the critical point p_α so that $f = p_\alpha + \sum_i z_i^2$. Lifting the homotopy, one gets a family of $(n-1)$–spheres $S(\gamma(t)) \subset F_{\gamma(t)}$ generating a homology class in the level manifolds $F_{\gamma(t)}, \forall t \in (0,1]$. As $t \to 0$ the sphere $S(\gamma(t))$ reduces to the critical point p_α. For this reason, the homology class $\delta \in H_{n-1}(F_{z_0})$ corresponding to the above

[31] I.e. at the zeros X_α of the gradient ∂V one has $\det[\partial^2 V/\partial X_i \partial X_j](X_\alpha) \neq 0$.

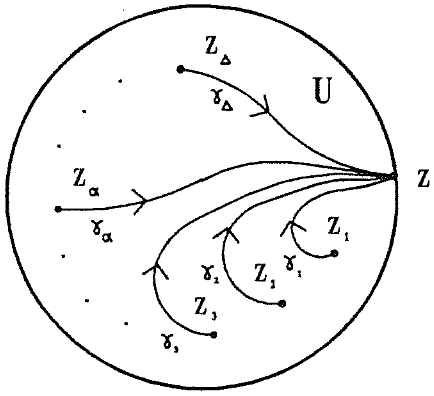

Figure 2
The set of paths defining the Δ vanishing cycles for the Morsified function

sphere is called the "*Picard–Lefschetz cycle vanishing along the path* γ".

There are as many vanishing Picard–Lefschetz cycles as there are critical points of the Morse function f. They are associated to the paths of Figure 2. For a Morse superpotential the number of critical points ($\equiv$ classical vacua) is equal to the Witten index Δ, so on F_{z_0} one has Δ vanishing cycles, $\delta_1, \ldots, \delta_\Delta$.

To each path γ_α connecting a critical value z_α to the regular value z_0 one can associate a *simple loop* τ_α. It is a loop going from z_0 to the point z_α along the path γ_α, then going around the critical value z_α in the anticlockwise direction and returning to z_0 along the path γ_α (see Figure 3). The fundamental group of the complement of the set of critical values, $\pi_1(U \setminus \{z_\alpha\}, z_0)$, is freely generated by the Δ simple loops τ_α corresponding to the paths γ_α. As above, by lifting the homotopy along the closed paths τ_α one gets maps $m_\alpha : F_{z_0} \to F_{z_0}$ from the non–singular level manifold F_{z_0} into itself. The **Monodromy Operator** along the simple loop τ_α

$$h_\alpha := m_{\alpha *} : H_{n-1}(F_{z_0}) \to H_{n-1}(F_{z_0}) \tag{5.29}$$

is called the **Picard–Lefschetz operator corresponding to the path** γ_α (or to the vanishing cycle δ_α). Thus, to each supersymmetric vacuum there is associated a Picard–Lefschetz (PL) operator h_α. The Δ PL operators generate the **Monodromy Group** of the Morse function f.

The action of the monodromy group on the homology of the reference level manifold is given by the *Picard–Lefschetz formula*

$$h_\alpha(a) = a + (-1)^{n(n+1)/2}(a \circ \delta_\alpha)\delta_\alpha \qquad \forall a \in H_{n-1}(F_{z_0}) \tag{5.30}$$

where $(a \circ b)$ is the intersection form in homology.

Take now a closed ball, $\overline{B}(z_\alpha, \varrho) := \{(z_1, \ldots, z_n) : |z - z_\alpha| \leq \varrho\}$, centered at the critical point z_α and so small that it does not contain any other critical point. A simple calculation shows that the space $\widetilde{F}_z := F_z \cap \overline{B}(z_\alpha, \varrho)$ is diffeomorphic to the total space of the disk

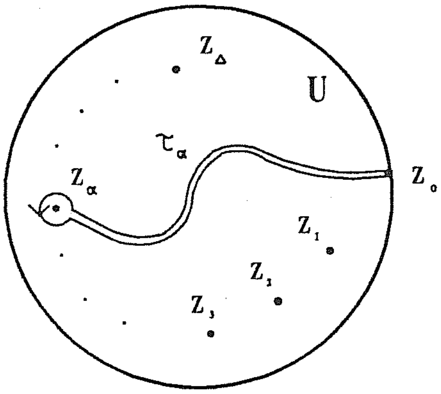

Figure 3
The loop τ_α associated to the PL vanishing cycle δ_α

subbundle of the tangent bundle of the $(n-1)$–sphere S^{n-1} [49]. From this it follows that the cohomology groups of the level manifold are

$$H^k(\widetilde{F}_z, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{for } k = n-1 \\ 0 & \text{otherwise} \end{cases} . \tag{5.31}$$

Indeed, by the Morse lemma, there exist coordinates centered at the critical point z_α so that $f = z_\alpha + \sum_i X_i^2$ in $\overline{B}(z_\alpha, \varrho)$. In these coordinates, the non–trivial homology in $\widetilde{F}_z$ is generated by the sphere $S(z) = \sqrt{z - p_\alpha} \cdot S^{n-1}$, where $S^{n-1} = \{(z_1, \ldots, z_n) : \sum_j z_j^2 = 1, \Im m\, z_j = 0\}$ is the standard unit sphere [49].

Consider now the case of a function f with degenerate isolated critical points z_α.

(5.32) Definition : *let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a* **singularity**, *that is the germ of a holomorphic function with an isolated singularity at the origin.*

(5.33) Definition : *the* **non–singular level set** *of the singularity f near the critical point 0 is the set*

$$V_\epsilon := f^{-1}(\epsilon) \cap \overline{B}_\rho \tag{5.34}$$

where $\overline{B}_\rho := \{x \in \mathbb{C}^n \mid f(X) = \epsilon, ||X|| < \epsilon\}$ for $0 < |\epsilon| \leq \epsilon_0$.

Milnor [51] has shown that the manifold V_ϵ has the homotopy type of a bouquet of spheres of dimension $(n-1)$. The number $\mu = \mu(f)$ of these spheres is called the **multiplicity** or the **Milnor number** of the singularity f. This implies the following

(5.35) Corollary : *the homology group $H_k(V_\epsilon)$ of the non–singular level manifold is zero for $k \neq (n-1)$, $H_{(n-1)}(V_\epsilon) \overset{\text{iso}}{=} \mathbb{Z}^\mu$ is a free abelian group with μ generators.*

In the following I will need to explicit construct a basis of $H_{n-1}(V_\epsilon)$. This is possible with the following construction (see §2.1 [49]).

Let $\widetilde{f} := f_\lambda := f + g_\lambda$ a linear perturbation of the function f, i.e. g is a linear function $: \mathbb{C}^n \to \mathbb{C}$, $g = \sum_i \lambda_i z_i$. For almost all λ_i, $\widetilde{f}$ is a Morse function. Now, $\widetilde{f}^{-1}(\epsilon) \cap \overline{B}_\rho$ is diffeomorphic to V_ϵ for $|\epsilon| \leq \epsilon_0$.

Let $F_z := \tilde{f}^{-1}(\epsilon) \cap \overline{B}_\rho$ for $|z| \leq \epsilon_0$.

(5.36) Theorem : $H_{n-1}(F_{z_0}) \stackrel{\text{iso}}{=} H_{n-1}(V_\epsilon)$ *where* $|z_0| = \epsilon_0$. *In particular, the number of non–degenerate critical points of* $\tilde{f}$ *(into which the critical point of* f *decomposes) is equal to the multiplicity* $\mu(f)$ *of the singularity* f [32].

(For this theorem see §2.1 [49].) Since $\tilde{f}$ is a Morse function, the results of PL theory can be directly applied to it.

(5.37) Definition : *a set of cycles* $\Gamma_1, \ldots, \Gamma_\mu$ *from the* $(n-1)$*st homology group* $H_{n-1}(F_{z_0})$ *of the non–singular level set* F_{z_0} *is called* **distinguished** *if*

i *the cycle* Γ_i $(i = 1, \ldots, \mu)$ *are vanishing along non–self–intersecting paths* u_i*, joining the critical value* z_i *with the non–critical value* z_0*;*

ii *the paths* u_i *and* u_j *have, for* $i \neq j$*, a unique common point* $u_i(1) = u_j(1) = z_0$*;*

iii *the paths* $u_1, \ldots, u_\mu$ *are numbered in the same order in which they enter the point* z_0*, counting clockwise, beginning at the boundary* ∂U *of the region* U *(this implies that* $z_0 \in \partial U$*).*

(5.38) Theorem : *a distinguished set of vanishing cycles* $\{\Gamma_i\}$ *forms a basis of the (free abelian) homology group* $H_{n-1}(F_{z_0}) \stackrel{\text{iso}}{=} H_{n-1}(V_\epsilon)$.

In 2d N=2 LG theories, the superpotential W is (under my assumptions, see §1.1 , §1.2) a holomorphic function with degenerate isolated critical points. Thus, it gives rise to a singularity $f = W$.

For my purpose, the relevant element of the monodromy group is the **Classical Monodromy**, i.e. the operator $h : H_{n-1}(F_{z_0}) \rightarrow H_{n-1}(F_{z_0})$ corresponding to the path $\gamma_0 := z_0 \exp[2\pi i t]$, $t \in [0, 1]$. h is equal to the product (in a certain natural order) of all the Picard–Lefschetz operators h_α.

5.3 Gelfand–Leray wave–forms

In what follows I will be mainly interested in the cohomological properties of Singularity Theory. A direct connection between the cohomology of Singularity Theory and the (physical) $\overline{\partial}_V$–cohomology is through the Gelfand–Leray (GL) forms.

The vacuum wave–forms are representatives of the $\overline{\partial}_V$–cohomology in Hilbert space. However, if one relaxes the condition that the wave–forms should be everywhere non–singular, the cohomology becomes trivial.

Let $\omega_k = \overline{\partial}_V \eta_k + \alpha_k$ be a vacuum wave form and β_k a Gelfand–Leray $(n-1, 0)$–form for α_k (see §10.2.2 ref. [49]),

$$\alpha_k = dV \wedge \beta_k \tag{5.39}$$

and let $\chi_k := \eta_k + \beta_k$. Then from the $\overline{\partial}_V$–Poincaré Lemma (4.37) one has

$$\omega_k = \bar{\partial}_V \chi_k \quad . \tag{5.40}$$

[32] This result is the mathematical formulation of what stated in physical terms at the beginning of this section, $\tilde{f} = V_\lambda$.

I call χ_k the **Gelfand–Leray (GL) wave–form** of the k–th vacuum. The motivation for the name is the following. As one approaches a critical point X_i of V, the GL wave–forms approach the GL forms in the standard sense. In fact, in $\omega_k = \overline{\partial}\eta_k + \alpha_k$, η_k is regular at the critical point, whereas β_k is singular. Eq. (5.39) does not fix uniquely β_k, since one can add to it any expression of the form $\partial V \wedge \tau$. However, if one restricts this form to a *non–critical* level hypersurface

$$F_{V_0} = \{X \in \mathbb{C}^n \mid V(X) = V_0\} \tag{5.41}$$

(i.e. the set of points in $\mathbb{C}^n$ in which the superpotential takes the value V_0) β_k, and hence χ_k, is uniquely fixed. The Gelfand–Leray forms represent (in a sense to be specified) the cohomology classes of the $(n-1)$–dimensional manifold F_{V_0}. Our "Gelfand–Leray" wave–form coincides with the mathematical GL form only asymptotically near a classical–vacuum. However, they have the same topological content and, for many computations, are more convenient. This topological interpretation of the wave functions is crucial in making contact between the quantum theory and the classical one (Singularity Theory).

Of course, the form χ_k is not unique. As a "gauge condition" I shall require

$$\delta_V \chi = 0 \quad . \tag{5.42}$$

In this gauge, the second property of a vacuum wave–form $\delta_V \omega = 0$ implies

$$H\chi = 0 \tag{5.43}$$

that is χ is a zero–energy "state". A state in the physical Hilbert space cannot have energy–zero and not be supersymmetric invariant. Since $\bar{\partial}_V \chi = \omega \not\equiv 0$, χ should not belong to the Hilbert space. χ is single–valued in $\mathbb{C}^n$, and at infinity approaches zero exponentially (because so does ω in a non–degenerate theory). The only reason for χ not to be in the Hilbert space is the presence of a singularity at some point in $\mathbb{C}^n$. Since ω should be regular everywhere, such singularities should be located at the zeros of ∂V that is in the *classical* vacua of the model, as well as along some Dirac–Hartogs "string" emanating from these points.

To a vacuum wave–form, one can assign *two* "Gelfand–Leray" $(n-1)$–forms by considering the $\overline{\partial}_V$ or ∂_V cohomology. They are defined by the formulae

$$\omega = \bar{\partial}_V \chi = \partial_V \tilde{\chi} \quad . \tag{5.44}$$

Since the GL forms will be very important in the future developments, it is convenient to give a better description of the relation between the (physical) GL wave–forms χ_k and the (mathematical) GL forms $\beta_k = \alpha_k / \partial V$.

A first argument is the following. Remember the isomorphism G between supersymmetric vacua and classes α in $\mathcal{R}$. It is not difficult to show that the $(n,0)$–forms $\alpha_k = G_0(|k\rangle)$ are a "*basis trivialization*" for the Singularity V at zero (see §12.1 [49]). It is well known that the GL forms $\beta_k = \alpha_k/\partial V$, restricted to the level manifold $f_t := \{V(X) = t\} \cap V^{-1}(S)$, S a suitably small punctured neighborhood of a critical value of V conventionally assumed to be zero, give a basis of $H^{n-1}(f_t)$ (see §12.1 [49]). Thus there is an isomorphism between the space of GL wave forms χ_k and GL forms β_k.

Another argument that substantiate this result is the following. Consider the Morsified superpotential V_λ (the non–singular level sets of V and V_λ are diffeomorphic). We know already that the homology of F_{z_0} is generated by the cycles vanishing along the paths of Figure 2. What remains to check is that the GL wave–form χ_α associated to the quantum vacuum wave–function which is picked on a given critical point X_α of V_λ, asymptotically represents (up to normalization) the cohomology class dual to the PL cycle δ_α, vanishing as one approaches X_α. It is enough to show that on the corresponding vanishing sphere S^{n-1} the integral of the GL wave–form χ_α is not zero. Consider the wave–function near the critical point. In a sufficiently small neighborhood of the given non–degenerated critical point, one can use the Morse Lemma to put V_λ in the form $V_\alpha + \sum_i X_i^2$. Then the leading term of the GL wave–form is just that of a free theory. The free vacuum wave–form is $\omega = \text{const}\; dX_1 \wedge \ldots \wedge dX_n[1 + O(|X|^2)]$, where const $\neq 0$. Then, one can choose the GL form as

$$\chi_\alpha = \text{const}\frac{1}{\sum_j X_j^2}\sum_i(-1)^{i-1}X_i dX_1 \wedge \ldots \wedge \widehat{dX_i} \wedge \ldots \wedge dX_n \left(1 + O(|X|^2)\right) \quad . \tag{5.45}$$

Writing $X_i = u_i + iv_i$, the Picard–Lefschetz vanishing cycle is represented by the standard $(n-1)$–dimensional sphere

$$S_\alpha^{n-1}(\varepsilon) = \{(X_1, \ldots, X_n) : \sum u_i^2 = \varepsilon, v_i = 0\} \tag{5.46}$$

(see §1.3 ref. [49]). Now the result follows from the fact that χ_α reduces on $S_\alpha^{n-1}(\varepsilon)$ to a multiple of the standard volume form. ∎

(5.47) Example : consider the case of one field, i.e. $n = 1$. Let $F_t = V^{-1}(t)$ be the level set of the holomorphic function $V(X) = \sum_{i=1}^{D+1}$ and let $\chi(X)$ be the GL wave form associated to a supersymmetric state. Let $\gamma = \sum_{i=1}^{D+1} a_i X_i$ (where $\{X_i\} = V^{-1}(t)$) be a chain in $H_0(F_t)$, and define

$$\chi[\gamma] := \sum_{i=1}^{D+1} a_i \chi(X_i) \quad . \tag{5.48}$$

Then $\chi[\ ]$ is a representative of a cohomology class in the reduced cohomology group $\tilde{H}^0(F_t)$. The proof of this statement goes as follows. By construction $\chi[\ ]$ is a linear form on $H_0(F_t)$. For belonging to the reduced cohomology group $\tilde{H}^0(F_t)$ it must hold $\chi[\sum_{i=1}^{D+1} X_i] = 0$. Indeed, consider the function

$$\Xi(V) := \sum_{X \in V^{-1}(V)} \chi(X) \quad . \tag{5.49}$$

On the complement of the critical values, $\Xi(V)$ is univalued and satisfies the Helmoltz equation. Moreover, the form $\Xi dV + \bar{\Xi} d\bar{V}$ is regular everywhere and square sommable. Therefore, $\Xi(V) = 0$. This property is easily checked in the explicit wave–functions (see Appendix D and §1.5). ∎

However, for a complex singularity the identification between supersymmetric vacua and the elements of $H^{n-1}(f_t)$ is more involved than this. These two vector spaces are isomorphic, since both have dimension Δ, but, in general, there is no canonical isomorphism between them. Anyway, for most "*physical*" considerations an informal correspondence suffices.

The most important conclusion of this section is the following statement

the cohomology described by a vacuum wave form ω is equivalently described by a $(n-1)$–form, the (physical) Gelfand–Leray wave–form, which instead of being defined on $\mathbb{C}^n$ or on a subspace $U \subset \mathbb{C}^n$ is defined over $\mathbb{C}^n - \{0\}$ or $U' := U - \{0\}$, i.e. the space without the critical point.

This fact is very important since it is one of the milestone of Singularity Theory, as we will see soon.

5.4 Milnor (co–) homology of a singularity

This section will be fully devoted to the introduction of some mathematical techniques (see refs. [49,48,51,52,53]).

5.4.1 *Milnor (co–) homological fibration*

(5.50) Definition : *let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with an isolated critical point at zero. Fix ϵ and δ such that $0 < \delta \ll \epsilon \ll 1$. Set $B_\epsilon := \{x \in \mathbb{C}^n \mid \sum_j |x_j|^2 < \epsilon^2\}$, $T := \{t \in \mathbb{C} \mid |t| < \delta\}$, $T' := T - \{0\}$. Let $X := B_\epsilon \cap f^{-1}(T)$, $X' := X - f^{-1}(0)$, $X_t := X \cap f^{-1}(t)$, $X_- := X \cap f^{-1}(T_-)$ and $T_- := \{t \in \mathbb{C} \mid \Re e\, t < 0\} \cap T$.*
The map $f : X' \to T'$ is the projection map of a smooth, locally trivial fibration called the **Milnor fibration**.

The fibre of the Milnor fibration is an $(n-1)$–dimensional complex analytic manifold with boundary. According to Milnor's theorem [51] the fibre X_t (called f_t or F_{z_0} in the previous sections) is homotopy equivalent to a bouquet of spheres of the middle dimension, the number μ of these sphere is equal to the multiplicity of the critical point of f.

In Singularity Theory one is interested in constructing over the Milnor fibration a (co–) homological fibration constructed with the $(n-1)$st homology and cohomology groups of the fibre of the Milnor fibration. These groups are the same as for a bouquet of spheres, their dimension is equal to the multiplicity of the original critical point if $n-1 > 0$ and to 1 more than the multiplicity in the case $n-1 = 0$ (a bouquet of μ zero–dimensional spheres is $\mu + 1$ points).

(5.51) Definition : *associated to the Milnor fibration there are two vector bundles:*

— *the* **Milnor cohomology bundle** *$f^* : H^{n-1} \to T'$, whose fibers are the reduced cohomology groups $H^{n-1}(X_t)$;*

— *the* **Milnor homology bundle** *$f_* : H_{n-1} \to T'$, whose fibers are the reduced homology groups $H_{n-1}(X_t)$.*

(These bundles are also called the "*(co–) homological Milnor fibration*".) All these definitions can be generalized to the case of a "*versal deformation* " of a singularity (see §10.3

[49]). Since I am not interested in proving all the mathematical results which I will need, I will not enter so much in the technical details of Singularity Theory.

It is indeed worth to do some remarks on the Milnor (co–) homology bundles. $H^{n-1}(X_t)$ is generated by the μ classes Poincaré dual (see Appendix D) to the PL vanishing cycles (see §5.2). The cohomological Milnor fibration is **locally trivial** by construction, but it is **not globally trivial.** Indeed, going around the origin in T lifting the homotopy, one does not return to the original class σ, but instead to $h^*\sigma$, where h^* is the classical monodromy, as follows from the Picard Lefschetz theory.

The following remark will be very useful. From the long exact sequence in (co–) homology it is possible to show (§11.1.1 [49]) that $H_{n-1}(X_t) \overset{\text{iso}}{=} H_n(X, X_-)$ where $t \in T_-$ (see (5.50) for the definitions of X_t, X_- and T_-).

Before studying the properties of the monodromy in the Milnor (co–) homology bundles, I want to obtain some more general informations on these bundles. To do that it is necessary to introduce the "*Poincaré Residue Map*".

5.4.2 *Poincaré Residue Map*

Let $P(X_1, \ldots, X_n)$ and $Q(X_1, \ldots, X_n)$ be polynomials in $\mathbb{C}^n$ and $\omega := (P(X)/Q(X))dX_1 \wedge \ldots \wedge dX_n$ a rational n–form in $\mathbb{P}_n :=$ { the complex projective space of dimension n }.

Let V be the **polar locus** of ω, that is the algebraic variety in $\mathbb{P}_n$ defined by $Q(X) = 0$. Let Γ be a cycle in $H_n(\mathbb{P}_n - V)$.

(5.52) Lemma (of the tube mapping): (see ref. [54]) *every n–cycle $\Gamma \in H_n(\mathbb{P}_n - V)$ is homologous to a tube over an $(n-1)$–cycle γ on V, $\gamma \in H_{n-1}(V)$; i.e. there exists a map $\tau : H_{n-1}(V) \to H_n(\mathbb{P}_n - V)$ such that $\Gamma = \tau(\gamma)$ which satisfies $\partial\tau(\gamma) = \tau(\partial\gamma)$ and which is given by taking tubes over circles.*

(5.53) Definition : *consider the following integral*

$$\int_\Gamma \omega \quad . \tag{5.54}$$

Since $\Gamma = \tau(\gamma)$ one can define the **Poincaré Residue Form** *$R(\omega)$ and the* **Poincaré Residue Map** *R as follows* [55,54,56] :
Let $R(\omega)$ be a cohomology class in $H^{n-1}(V)$ defined by

$$\langle R(\omega), \gamma \rangle := \int_\gamma R(\omega) := \frac{1}{2\pi i} \int_{\Gamma = \tau(\gamma)} \omega \tag{5.55}$$

and R the map $R : H^n(\mathbb{P}_n - V) \to H^{n-1}(V)$ such that $\omega \to R(\omega)$, or defining $\mathcal{H}^n(V)$ as the space of n–forms on $\mathbb{P}_n$ with poles of arbitrary order on V, $R : \mathcal{H}^n(V) \to H^{n-1}(V)$.

(5.56) Example : let $n = 2$ and $\omega = (P(X,Y)/Q(X,Y))dX \wedge dY$. Then $R(\omega) = (P(X,Y)/\frac{\partial Q}{\partial Y}(X,Y))dX$ (obviously $R(\omega)$ is defined on V, which means that it is restricted on $Q(X,Y) = 0$ from which follows $Y = Y(X)$.). ∎

The Poincaré Residue map will be a very useful tool in the following sections. As a very instructive example (in some sense the proof is more important than the statement itself which is almost obvious) I will now prove the following

(5.57) Proposition : *let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a singularity, $\mathcal{R}$ be the ring $\mathcal{R} = \mathcal{O}/I$ where $\mathcal{O}$ is the sheaf of germs of holomorphic functions in zero and $I \in \mathcal{O}$ the Ideal generated by $\{\partial_i f\}$. Let $H^{n-1}(X_t)$ be the reduced cohomology group of f, then*

$$H^{n-1}(X_t) \overset{\text{iso}}{=} \mathcal{R} \tag{5.58}$$

and the isomorphism is explicitly given as follows:
let $X^{\underline{p}}$ be a basis monomial in $\mathcal{R}$, then

$$\sum_j (-)^{j-1} X^{\underline{p}} \left. \frac{dX_1 \wedge \ldots \wedge \widehat{dX_j} \wedge \ldots \wedge dX_n}{\frac{\partial f}{\partial X_j}} \right|_{f(X_1 \ldots X_n) = t} \tag{5.59}$$

(where $\widehat{dX_j}$ means omitted) is a representative of a cohomology class in $H^{n-1}(X_t)$ which gives rise to a basis of $H^{n-1}(X_t)$.

Proof : it is very well known that each de Rham class in $H^{n-1}(X_t)$ can be represented by a holomorphic form (see for example ref. [43] pagg. 445 — 449). Now, consider the exact sequence of the "*Logarithm Complex*" [42,43,57]

$$0 \longrightarrow \mathcal{O}^k_{\mathbb{C}^n} \longrightarrow \mathcal{O}^k_{\mathbb{C}^n}[X_t] \xrightarrow{R} \mathcal{O}^{k-1}_{X_t} \longrightarrow 0 \tag{5.60}$$

where $\mathcal{O}^k_M$ is the sheaf of germs of holomorphic k–forms on the complex manifold M and $\mathcal{O}^k_M[N]$ is the sheaf of germs of holomorphic k–forms α on M where α and $d\alpha$ have at most single poles along the hypersurface $N \subset M$ and are holomorphic elsewhere. A generic element in $\mathcal{O}^k_{\mathbb{C}^n}[X_t]$ can be written as $(g(X) dX_1 \wedge \ldots \wedge dX_n)/(f(X) - t)$. R is the Poincaré Residue map. Since the cohomology of $\mathbb{C}^n$ is trivial, this means that $H^n_{\mathbb{C}^n}[X_t] \overset{\text{iso}}{=} H^{n-1}(X_t)$. The isomorphism is explicitly given by

$$\begin{aligned} R \left[\frac{g(X) dX_1 \wedge \ldots \wedge dX_n}{f(X) - t} \right] &= \\ &\sum_j (-)^{j-1} g(X) \left. \frac{dX_1 \wedge \ldots \wedge \widehat{dX_j} \wedge \ldots \wedge dX_n}{\frac{\partial f}{\partial X_j}} \right|_{f(X_1 \ldots X_n) = t} \end{aligned} \tag{5.61}$$

(where, in the definition of the Poincaré Residue, $Q(X)$ has been taken equal to $f(X) - t$ and the polar locus V is given by $f(X) - t = 0$).

Let $A^k_l(X_t)$ denote the meromorphic k–forms on $\mathbb{C}^n$ with at most a pole of order l along X_t and holomorphic elsewhere. By definition

$$H^n_{\mathbb{C}^n}[X_t] = \frac{A^n_1}{dA^{n-1}_1 \cap A^n_1} \quad . \tag{5.62}$$

Now, it is not difficult to show that the map $\lambda : \mathcal{R} \to H^n_{\mathbb{C}^n}[X_t]$, $\lambda(X^{\underline{p}} = (X^{\underline{p}} dX_1 \wedge \ldots \wedge dX_n)/(f(X) - t)$, is an isomorphism. Indeed, let

$$(-)^{j-1} g_j(X) \frac{dX_1 \wedge \ldots \wedge \widehat{dX_j} \wedge \ldots \wedge dX_n}{f(X) - t} =: a_j \in A_1^{n-1} \quad , \tag{5.63}$$

then

$$da_j = \frac{(f(X) - t) \cdot \frac{\partial g_j}{\partial X_j} - g_j \frac{\partial f}{\partial X_j}}{(f(X) - t)^2} dX_1 \wedge \ldots \wedge dX_n \quad . \tag{5.64}$$

$da_j \in A_1^n$ only if $(f(X) - t) \cdot \frac{\partial g_j}{\partial X_j} - g_j \frac{\partial f}{\partial X_j} = h_j(X)(f(X) - t)$. The unique solution to this equation is $g_j = (f(X) - t) \cdot \log(f(X) - t)$ and $h_j(X)$ turns out to be $h_j(X) = \frac{\partial f}{\partial X_j}$. Thus, $da_j = \left(\frac{\partial f}{\partial X_j}\right) / (f(X) - t) \cdot dX_1 \wedge \ldots \wedge dX_n$ and from this it is obvious that λ is an isomorphism. Thus, composing λ and the Poincaré Residue Map R one gets the proposition. ∎

Thus $\mathcal{R} \overset{\text{iso}}{=} H^{n-1}(X_t)$. However this identification is physically unsatisfactory since, although it is an isomorphism of vector spaces, it is by no means a **canonical** isomorphism. Moreover, the above identification does not induce a structure of $\mathcal{R}$–module on $H^{n-1}(X_t)$ (in fact $f(X) - t$ is a multiple of the identity in $\mathcal{R}$ but vanishes after restriction to X_t). Since the physics is encoded in the ring structure, I have to make a better correspondence between the (physical) cohomological problem and Singularity Theory. This will be done in the following sections. For the moment I will go on in the study of the properties of $H^{n-1}(X_t)$.

5.4.3 *Gauss–Manin connection*

I have already observed that the (co–) homological Milnor bundle is **locally** trivial, that is it is a locally trivial complex vector bundle. Speaking generically, let $\Pi_* : H_k \to B$ a locally trivial complex vector bundle and $U \subset B$ a contractible open subset of B. Then $\Pi_*^{-1}(U)$ is homeomorphic to the direct product of the fiber and the set U. Thus, on every subset U_i the fibers of the bundle are constant which implies that the *transition functions are locally constant.* In this way, it is defined **naturally** a operation of parallel transport of fibres over curves in the basis.

<u>(5.65) Definition</u> : *the operation of translation of (co–) homologies so defined is called the* **Gauss–Manin connection** *in the (co–) homological fibration.*

(See for example §12.3 [49], [56,58].) The Gauss–Manin connection possesses the following property: *the map does not depend on the choice of the curve in the homotopy class of curves with fixed ends.*

<u>(5.66) Definition</u> : *a section of the (co–) homological fibration over an open subset of the base is said to be* **covariantly constant** *if its values are invariant relative to parallel translation along any curves lying in this open set.*

Notice that the operation of translation by means of the Gauss–Manin connection on the fibres of the cohomology groups of the Milnor fibration is the operation of translation which defines the Monodromy Operator in cohomology. This monodromy operator is obviously the adjoint of that defined in homology in §5.2 (see also below).

5.4.4 *Geometric Sections*

(5.67) Definition : *let α be a holomorphic $(n,0)$–form on X, for any $t \in T'$ the restriction of the GL–form of α to the fibre X_t of the Milnor fibration defines the cohomology class $\left[\frac{\alpha}{df}|_{X_t}\right] \in H^{n-1}(X_t)$. the section*

$$t \longrightarrow \left[\frac{\alpha}{df}|_{X_t}\right] \tag{5.68}$$

of the cohomological Milnor fibration is called the **Geometric Section** *of the form α and is denoted by*

$$s[\alpha] \quad . \tag{5.69}$$

The dependence on the value of $t \in T'$ of $s[\alpha]$ will be denoted by $s[\alpha](t)$. Notice that $\alpha|_{X_t} = dV \wedge s[\alpha](t)$.

In Singularity Theory the following integrals are of fundamental importance.
Let $\delta(t)$ be a continuous family of integral PL vanishing cycles in X_t. Let

$$\langle s[\alpha], \delta\rangle := \int_\delta \frac{\alpha}{df} \quad . \tag{5.70}$$

This integral is holomorphic and it is a many–valued function in $t \in T'$.

The following theorem is one of the milestone of Singularity Theory (see refs. [52], §10.3.4 [49]).

(5.71) Theorem : *in each sector $a \le t \le b$, $\langle s[\alpha], \delta\rangle$ can be expanded in series*

$$\langle s[\alpha], \delta\rangle = \int_{\delta(t)} \frac{\alpha}{df}(t) = \sum_{\hat\alpha, k} a_{\hat\alpha,k} t^{\hat\alpha} (\ln t)^k \frac{1}{k!} \tag{5.72}$$

This series converges if $|t|$ is sufficiently small (i.e. for $|t| \to 0$). All the numbers $\hat\alpha$ are rational. Each number $\hat\alpha$ is greater than -1. Each number $\hat\alpha$ possesses the property: $\exp(2\pi i\hat\alpha)$ is an eigenvalue of the classical monodromy operator in the homology. The coefficients $a_{\hat\alpha,k}$ are equal to zero at any time that the classical monodromy operator does not have Jordan blocks of dimensions $k+1$ or greater associated with the eigenvalue $\exp(2\pi i\hat\alpha)$.

(5.73) Definition : *the (covariantly constant many–valued) section $A^\alpha_{\hat\alpha,k}$ (for fixed k and $\hat\alpha$) of the cohomological Milnor fibration are defined by the formula*

$$\langle A^\alpha_{\hat\alpha,k}(t), \delta(t)\rangle := a_{\hat\alpha,k}(\alpha, \delta) \quad . \tag{5.74}$$

By definition

$$s[\alpha](t) = \sum_{\hat\alpha,k} t^{\hat\alpha} (\ln t)^k A^\alpha_{\hat\alpha,k}(t) \frac{1}{k!} \quad . \tag{5.75}$$

(5.76) Proposition : *let* $M : H^{n-1}(X_t) \to H^{n-1}(X_t)$ *be the monodromy operator of the Gauss–Manin connection generated by a "counterclockwise" circuit around* $0 \in \mathbb{C}$.

1. *for any* $\hat{\alpha}, k, \alpha$, *the values of the sections* $A^{\alpha}_{\hat{\alpha},k}$ *belong to the root subspace of* M *corresponding to the eigenvalue* $\exp(-2\pi i\hat{\alpha})$;
2. *let* $M =: M_s M_u$ *be the decomposition of* M *into the semisimple and unipotent parts; set* $N := \log M_u$. *Then, for any* $\hat{\alpha}, k, \alpha$,

$$\begin{aligned} A^{\alpha}_{\hat{\alpha},k} &= \left(-\frac{N}{2\pi i}\right)^{k} A^{\alpha}_{\hat{\alpha},0} \\ M_s A^{\alpha}_{\hat{\alpha},k} &= \exp(-2\pi i\hat{\alpha}) A^{\alpha}_{\hat{\alpha},k} \end{aligned} \tag{5.77}$$

Notice that if M_{hom} is the monodromy operator defined in an analogous fashion in the homologies $H_{n-1}(X_t)$, then $M^* = M^{-1}_{hom}$, where * denotes the dual operator.

(5.78) Corollary : *it is obvious from the previous proposition that*

1. *the geometric section* $s[\alpha]$ *is determined by the sections* $A^{\alpha}_{\hat{\alpha},k}$ *with* $k = 0$;
2. *for any* $\hat{\alpha}$ *(fixed), consider*

$$\mathcal{A}^{\alpha}_{\hat{\alpha}}(t) := t^{\hat{\alpha}} \left(A^{\alpha}_{\hat{\alpha},0}(t) + \ldots + (\log t)^{n-1} A^{\alpha}_{\hat{\alpha},n-1}(t) \frac{1}{(n-1)!} \right) \tag{5.79}$$

(notice that the dimension of the Jordan blocks of the monodromy operator are not greater than n, *therefore the coefficients* $A^{\alpha}_{\hat{\alpha}}$ *with* $k \geq n$ *are zero). Then*

i.
$$\mathcal{A}^{\alpha}_{\hat{\alpha}}(t) = \exp\left(\log t \left[\hat{\alpha}\,\mathrm{Id} - \frac{N}{2\pi i}\right]\right) \cdot A^{\alpha}_{\hat{\alpha},0}(t) \quad ; \tag{5.80}$$

ii. *the section* $\mathcal{A}^{\alpha}_{\hat{\alpha}}(t)$ *is a* **holomorphic single–valued** *section of the cohomological Milnor fibration*

(5.81) Definition : *let* $\hat{\alpha}$ *be a rational number with the property that* $\lambda = \exp(-2\pi i\hat{\alpha})$ *is an eigenvalue of the monodromy operator in the cohomology and* A *be a section of the Milnor cohomology bundle. The section*

$$s[A, \hat{\alpha}] := \exp\left[\log t \left(\hat{\alpha}\,\mathrm{Id} - \frac{N}{2\pi i}\right)\right] \cdot A \tag{5.82}$$

is called an **Elementary Section** *of order* $\hat{\alpha}$ *generated by the section* A.

Obviously, $s[\alpha] = \sum_{\hat{\alpha}} s\left[A^{\alpha}_{\hat{\alpha},0}, \hat{\alpha}\right]$ and $s[A, \hat{\alpha}]$ is a single–valued holomorphic section of the cohomological fibration.

Let denote ∇_t the differentiation with respect to the Gauss–Manin connection along the vector field $\frac{\partial}{\partial t}$ on X_t (that is the differentiation of the coordinates of a section with respect to a covariantly constant frame).

(5.83) Proposition : *the action of the Gauss–Manin connection on the geometric section* $s[\alpha]$ *are characterized by the following formulas* (see §13.1.2 [49], [59,47])

1.
$$t\nabla_t \cdot s[A,\hat{\alpha}](t) = \hat{\alpha}\, s[A,\hat{\alpha}](t) + s[-\frac{N\cdot A}{2\pi i},\hat{\alpha}](t) \tag{5.84}$$

this means that $t\nabla_t$ acts on the explicit expression (5.82) as $t\frac{\partial}{\partial t}$;

2. *from 1. and the previous observation on the linearity of the relation between elementary and geometric sections, it follows that $t\nabla_t$ acts on $s[\alpha]$ as $t\frac{\partial}{\partial t}$ on its explicit expression $s[\alpha] = \sum_{\hat{\alpha},k} t^{\hat{\alpha}}(\log t)^k A^{\alpha}_{\hat{\alpha},k}\frac{1}{k!}$;*

3.
$$\begin{aligned} s[d\sigma](t) &= \nabla_t s[df\wedge\sigma](t) \\ s[f\cdot\alpha](t) &= t\ s[\alpha](t) \end{aligned} \tag{5.85}$$

(5.86) Definition : *the* **Order of the form** *α or of the geometric section $s[\alpha]$ is the* **smallest** *number $\hat{\alpha}$ for which the coefficient $A^{\alpha}_{\hat{\alpha},0}$ is different from zero. The order will be denoted by $\hat{\alpha}(\alpha)$.*

This index will have much to do with the physical critical indices in the strong coupling regime. Indeed, the leading behaviour of $s[\alpha](t)$ for $t\to 0$ is given by $t^{\hat{\alpha}(\alpha)}$. Consider the zero class in $H^{n-1}(X_t)$, $[0]\in H^{n-1}(X_t)$. Obviously, $\langle s[0],\delta\rangle = \int_\delta \frac{[0]}{df} = 0$. The order of any representative of $[0]$ is by definition $\hat{\alpha}(0) = +\infty$, since zero can be considered as the most rapidly power convergent to zero as $t\to 0$.

(5.87) Definition : *the* **Principal Part** *of the form α or of the geometric section $s[\alpha]$ is the single–valued holomorphic section of the cohomological Milnor fibration*

$$\begin{aligned} s_{max}[\alpha] :=& s\left[A^{\alpha}_{\hat{\alpha},0},\hat{\alpha}(\alpha)\right] \\ =& t^{\hat{\alpha}(\alpha)}\left(A^{\alpha}_{\hat{\alpha}(\alpha),0} + \ldots + (\log t)^{n-1}A^{\alpha}_{\hat{\alpha}(\alpha),n-1}\frac{1}{(n-1)!}\right) \quad . \end{aligned} \tag{5.88}$$

This corresponds to extract the leading behaviour of $s[\alpha](t)$ for $t\to 0$.

(5.89) Example : let f be a quasihomogeneous polynomial of weights q_i and degree 1, i.e. $f(\lambda^{q_i}X_i) = \lambda f(X)$. Let $l(\underline{m}) := [\sum_{i=1}^n (m_i+1)q_i] - 1$, and $X^{\underline{m}}$ be a basis monomial of $\mathcal{R}$. Let $\alpha_{\underline{m}} := X^{\underline{m}}dX_1\wedge\ldots\wedge dX_n$, then

$$s[\alpha_{\underline{m}}](t) = s_{max}[\alpha_{\underline{m}}](t) = t^{\hat{\alpha}(\alpha_{\underline{m}})}A^{\alpha_{\underline{m}}}_{\hat{\alpha}(\alpha_{\underline{m}}),0} \tag{5.90}$$

where $\hat{\alpha}(\alpha_{\underline{m}}) = l(\underline{m})$. Notice that $\hat{\alpha}(\alpha_{\underline{m}}) = l(\underline{m})$ is the degree of quasihomogeneity of the GL form $(\alpha_{\underline{m}}/df)$. ∎

(5.91) Definition : *let denote α_{min} the smallest number among all the orders of holomorphic n–forms on X. The number*

$$-(1+\alpha_{min}) \tag{5.92}$$

is called the **Complex Oscillation Index** *of the singularity f. The number*

$$\frac{n}{2} - (1+\alpha_{min}) \tag{5.93}$$

is called the **Complex Singular Index** *of the singularity f.*

5.4.5 *Oscillatory integrals*

I now will introduce some other results in Singularity Theory which concern integrals of holomorphic $(n,0)$–forms over n–cycles. The asymptotic behaviour of these integrals was the problem which led Arnold to develop the Singularity Theory [60]. Here I will just summarize some results which I will need in the following chapters; for a complete introduction to the subject see [21,49].

(5.94) Definition : *let X, X', X_t and X_- as in definition* (5.50) . *The following integral*

$$\int_\Gamma e^{tf(X)}\alpha \tag{5.95}$$

where Γ is a chain in $H^n(X, X_-)$ and α a holomorphic $(n,0)$–form, is called a **(complex) Oscillatory Integral** *or an Integral of the saddle–point method.*

In Singularity Theory one is interested in the asymptotic behaviour of the oscillatory integrals as the (real) parameter t goes to infinity (see §11 [49], [52]).

Notice the strong analogies between these integrals and the (physical) "path integrals". Indeed consider the integral

$$\int_\Gamma e^{\frac{i}{\hbar}f(X)}\alpha \tag{5.96}$$

where $t \to (i/\hbar)$ (called "(real) oscillatory integral"); thus, studying the behaviour as $|t| \to \infty$ corresponds to consider the limit $\hbar \to 0$. Morally speaking, the oscillatory integrals are the (informal) semi–classical version of the Schroedinger picture integrals giving the chiral matrix elements.

Anyway, this is *only an analogy* since these integrals do not correspond to path integrals. As we will see, one is lead to interpret t as the coupling constant so that $tf(X) = gF(X)$, thus one can think that these integrals give the correct dependence on the (overall) coupling constant of the CGFs. Again, this is not true for the presence, in the non–quasihomogeneous case, of what can be called a "holomorphic anomaly". This subject will be discussed at length in the following chapter.

Let me go back to the oscillatory integrals. It is not difficult to show that if Γ and Γ' are two representative of the same class in $H_n(X, X_-)$, then for all α the difference of the integrals along Γ and Γ' is exponentially small as $|t| \to +\infty$ [59]. (Complex oscillatory integrals which differs by an exponentially small quantity as $|t| \to +\infty$ are considered to be equal in Singularity Theory.)

The following theorem is the counterpart of theorem (5.71) (see theorem 11.1 §11.1 [49]).

(5.97) Theorem : *as $|t| \to +\infty$ the integral*

$$\int_\Gamma e^{tf}\alpha \tag{5.98}$$

can be expanded in an asymptotic series

$$\sum_{\gamma,k} a_{\gamma,k} t^\gamma (\log t)^k \tag{5.99}$$

where the parameter γ runs through a finite set of arithmetic progressions, depending only on the phase and consisting of negative *rational numbers. Moreover, each number γ possesses the property:* $\exp(-2\pi i\gamma)$ *is an eigenvalue of the classical monodromy operator of the critical point of the phase. The coefficient $a_{\gamma,k}$ is equal to zero whenever the classical monodromy does not have Jordan blocks of dimension $k+1$ or more associated with the eigenvalue* $\lambda = \exp(-2\pi i\gamma)$.

As we have already seen, $H_n(X,X_-)$ is isomorphic to $H_{n-1}(X_t)$. Thus there should be a direct connection between the oscillatory integral $\int_\Gamma e^{tf}\alpha$ and the integral $\int_\gamma \frac{\alpha}{df}$ where γ is a cycle in $H_{n-1}(X_t)$. This is given by the following (see §11.1.2 [49], [52])

(5.100) Lemma : *Let α be a holomorphic $(n,0)$–form on X and Γ a cycle in $H_n(X,X_-)$. Then*

$$\int_\Gamma e^{tf}\alpha \cong \int_o^\infty e^{-t\tau}\left(\int_{\partial_{-\tau}\Gamma}\frac{\alpha}{df}\right)d\tau \tag{5.101}$$

where $\partial_{-\tau}\Gamma$ is a cycle in $H_{n-1}(X_\tau)$ constructed taking the boundary of the chain Γ and contracting it in the set X_- to the fibre X_τ (∂_τ is the isomorphism between $H_n(X,X_-)$ and $H_{n-1}(X_\tau)$), the symbol $\cong$ means that the right hand side differs from the left hand side by terms going to zero exponentially as $|t| \to +\infty$.

Thus, the oscillatory integrals are related to the integrals over the vanishing cycles of the GL forms by a Laplace transform. The following formula for the Laplace transform will be very useful

$$\int_o^\infty e^{-t\tau}\tau^{\hat\alpha}(\log\tau)^k d\tau = \left(\frac{d}{d\hat\alpha}\right)^k\left(\frac{\Gamma(\hat\alpha+1)}{t^{\hat\alpha+1}}\right) \quad . \tag{5.102}$$

Notice for example that in the case $k=0$, γ of the previous theorem is equal to $\gamma = -(\hat\alpha+1)$.

In practice, one is usually able to compute explicitly integrals over vanishing cycles of GL forms and then, using formula (5.102) , one gets the value for the corresponding oscillator integral.

(5.103) Definition : *consider a singularity f and a holomorphic $(n,0)$ form α. let*

$$\int_\Gamma e^{tf}\alpha = \sum_{\gamma,k} a_{\gamma,k}t^\gamma(\log t)^k \tag{5.104}$$

as $t\to\infty$ where $\Gamma\in H_n(X,X_-)$. Denote by $\beta(\alpha,\Gamma)$ the largest *γ occurring in this series, i.e. $\beta(\alpha,\Gamma)$ is the* **index of the principal term of the asymptotic series**.

(5.105) Lemma : *for any chain $\Gamma\in H_n(X,X_-)$ the following inequality holds*

$$\beta(\alpha,\Gamma) \le -(\hat\alpha(\alpha)+1) \tag{5.106}$$

where $\hat\alpha(\alpha)$ is the order of α. Furthermore, there exists a $\Gamma\in H_n(X,X_-)$ for which

$$\beta(\alpha,\Gamma) = -(\hat\alpha(\alpha)+1) \quad . \tag{5.107}$$

The following result is the last one I will need on the oscillatory integrals (see [59], §11.3.1 [49]).

(5.108) Theorem : *let $f(x) : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and $g(y) : (\mathbb{C}^l, 0) \to (\mathbb{C}, 0)$ two singularities in independent variables; the map $f + g : (\mathbb{C}^{n+l}, 0) \to (\mathbb{C}, 0)$ is again a singularity. Let α be a holomorphic $(n, 0)$ form in the x coordinates and η a holomorphic $(l, 0)$ form in the y coordinates. Let $\Gamma_1 \in H_n(X, X_-)$ be a cycle for f and $\Gamma_2 \in H_l(Y, Y_-)$ be a cycle for g, $\Gamma_1 \otimes \Gamma_2$ is a chain for the function $f + g$. Then*

$$\int_{\Gamma_1 \otimes \Gamma_2} e^{t(f+g)} \alpha \wedge \eta = \left(\int_{\Gamma_1} e^{tf} \alpha \right) \cdot \left(\int_{\Gamma_2} e^{tg} \eta \right) \quad . \tag{5.109}$$

This "direct product" property of the oscillatory integral is obviously a physical property, in the sense that every Green's function satisfies an analogous property. Thus it will be useful in the following chapters to make contact with the cohomological formulation of the CGFs.

CHAPTER 6

G–DEPENDENCE OF THE CHIRAL GREEN'S FUNCTIONS

In this chapter I reconsider the physical problem of the computation of the CGFs using the techniques of Singularity Theory. In this chapter I will obtain one of the main results, a differential first order equation which gives the dependence on the overall coupling constant (and its complex conjugate) of the chiral Green's functions. Before getting to this problem, another general subject must be considered which can go under the name of "*Physical Localization*".

6.1 Physical localization

Singularity Theory applies to functions defined in a (sufficiently small) neighborhood of the critical point and only one critical point at the time is considered. In physics, at a first glance, the situation is totally different. Indeed, a priori, the wave functions depend on the detailed form of the superpotential everywhere. Then, one has to show that it is possible to "*localize*" the problem around the critical points. In section §3.1 it has already been shown with physical arguments that in the strong coupling regime the theory behaves according to the decomposition in Jordan blocks of the CGFs seen as $\Delta \times \Delta$ matrices. Moreover, each Jordan block corresponds to a critical point of the superpotential. Thus, what follows can be considered as the continuation of the arguments in §3.1 .

6.1.1 *The $n = 1$ case*

I start by showing the above claims in the simpler case of one field. Let $V = \sum_{k=1}^{D+1} g_k X^k$ be a polynomial of degree $D+1$. As usual, it is convenient to factor out the overall coupling constant and write the superpotential as $V = gF(X, c_k)$ where $g \equiv g_1$, $c_k = g_k/g$.

From §4.2 we know that one can effectively reduce the computation to supersymmetric quantum mechanics. The effective (adimensional) SQM coupling g is related to the overall coupling in 2d by $g = g_{2d}L$. So, finding the dependence on g, one gets the scaling law too.

The (off–criticality) Witten index is D. Consider the GL wave–forms $\chi_a(X)$ ($a = 1, \ldots, D$) of the supersymmetric vacua $|a\rangle$. They are univalued on the (non–critical) level set F_t. It is easy to see that they are linearly independent over the field of the univalued functions of t. Then, they generate the *reduced* cohomology group of the level manifold F_t. This interpretation of the GL wave–forms gives the dictionary with Singularity Theory. One needs just one more step before getting complete agreement with mathematics. χ_a give the cohomology of the *full* level manifold. As I have already said, the problem is now to *localize* the computation around a particular critical point.

This can be done exactly as in Singularity Theory. Let X_0 be an isolated critical point of V. Without loss of generality, one can take $X_0 = 0$ and $V(0) = 0$. Take a sufficiently small disk D_ϱ in the complex V–plane centered at the critical value 0, and a sufficiently small disk D_r in the X–plane centered at the critical point 0. Let $Y = D_r \cap V^{-1}(D_\varrho)$ and $f_t = F_t \cap Y$. The reduced cohomology group generated by the Picard–Lefschetz cycles vanishing at 0, $H^{\rm red}(f_t)$, is defined by the exact sequence

$$0 \longrightarrow H^0(Y) \xrightarrow{i^*} H^0(f_t) \longrightarrow H^{\rm red}(f_t) \longrightarrow 0 \tag{6.1}$$

where i is the inclusion $f_t \to Y$. The inclusion $j : f_t \to F_t$ induces a map $j^* : H^0(F_t) \to H^0(f_t)$. The maps give the "local" cohomology group, $H^{\rm red}(f_t)$, out of the "global" one, $H^{\rm red}(F_t)$. In practice, one takes a basis for the $\chi_a(X)$ such that the holomorphic germs at 0 of first Δ wave–forms span $\tilde{\mathcal{R}}_0$, and those of the last $(D - \Delta)$ belong to the local Jacobian ideal $dV \wedge \Omega_0^{n-1}$. ($\Delta \equiv \dim \mathcal{R}_0$ is the critical Witten index at 0). Such a basis exists by the Poincaré $\bar{\partial}_V$–lemma (4.37) .

The above construction is rather abstract and hides some important aspect of the theory. So it is better to look for a physical and explicit "localization" of the quantum theory. At each critical point of V, I want to construct a *bona fide* quantum theory, having the same superpotential and chiral Green functions as the original model, but such that the vacuum wave–forms have support in a given neighborhood of the relevant critical point. The only free choice I have is the Kähler potential. Let me choose the disk $\overline{D}_\varrho$ so small that the connected component of $V^{-1}(\overline{D}_\varrho)$ containing the origin, $\overline{\Omega}$, satisfies $\overline{\Omega} \subset D_r$. As "localized" theory I take the N=2 σ–model on Ω with the Bergman Kähler metric and with the original superpotential V. In the path integral I impose vanishing boundary conditions on $\partial\overline{\Omega}$. In other words, I just restrict the domain in which the chiral field X takes value from $\mathbb{C}$ to Ω, changing the Kähler metric in such a way that Ω is now a *complete* manifold. Completeness is necessary if the SQM wave–functions should have support on Ω, without spoiling unitarity. This is a trick that allows to make the "localization" of the cohomology classes to correspond to an actual localization of the support of the wave–forms to a small neighborhood Ω of the origin.

The crucial properties of the localized model are:

a. *the physical localization corresponds to the map in cohomology*

$$j^* : H^0(F_t) \longrightarrow H^0(f_t) \quad ; \tag{6.2}$$

b. *the relevant chiral Green functions for the localized model differ from those of the original model only for terms which are exponentially small as $g \to \infty$. (The limit I am interested in);*

c. *the localized theory is R–invariant.*

Proof : I start by showing c. By the Riemann theorem, one can map $\overline{\Omega}$ conformally into the unit disk $\overline{D}_1$. One can choose the conformal map so that the origin is mapped into the origin. Let Z be the transformed chiral field, taking value in D_1. In this coordinate the Bergman metric is simply the Poincaré metric. The superpotential $V(Z)$ has a zero of order $(\Delta+1)$ at the origin and no other zeros in D_1. Then, it can be written has $V(Z) = Z^{\Delta+1}\psi(Z)$ with $\psi(Z)$ never vanishing in D_1. By definition of Ω, one has $|V(e^{it})| = \varrho$, which implies $|\psi(Z)| = \varrho$ for $Z \in \partial\overline{D}_1$. Then $\psi(Z)$ is a constant. Rescaling the field Z, one writes the superpotential as $V = gZ^{\Delta+1}$ and Ω becomes a disk of radius R, D_R. The domain D_R, the Kähler metric, and the superpotential are all invariant under the R–transformation $Z \to e^{i\alpha}Z$.

Let now turn to points a. and b. The Schrödinger equation for the localized model on the disk is explicitly solved in Appendix G. There it is shown that the Witten index is ∞, but only Δ supersymmetric vacua have a real cohomological meaning [33]. The wave–forms of these special vacua, $\omega_k(D_R)$, are given — as forms not just as classes — by the pull back of the corresponding vacuum wave–forms in the original model, $\omega_k(\mathbb{C})$,

$$\omega_k(D_R) = \iota^*\omega_k(\mathbb{C}) \tag{6.3}$$

where $\iota\colon D_R \to \mathbb{C}$ is the inclusion. That the forms $\iota^*\omega_k(\mathbb{C})$ are solutions of the Schrödinger equation follows from the universality of the vacuum wave–forms, proven in §4.4 . The only tricky point is that they, indeed, satisfy the boundary condition on $\partial\overline{D}_R$. This follows from the completeness of the Bergman metric. This shows point a.

Point b. is eq. (G.16) of Appendix G. The exponentially small deviations can be understood topologically as follows. The germ argument $\alpha_h = \langle h|X_i|k\rangle\ \alpha_k \quad (\text{mod. } dV\wedge\beta)$ shows that only local properties at the singularity are needed to compute the matrix elements. However, in the disk case, the relevant matrix elements are the product coefficients of a different cohomology ring: two closed forms are equivalent if their difference is equal to $\bar{\partial}_V\eta$ with η *vanishing* on the boundary. Since on the boundary the wave–forms are exponentially small, one has only small deviations. ∎

The localization procedure gives a firm basis to the idea that the critical quantities are independent of the details of the D–terms, and do not change under a deformation of them, even if this deformation corresponds to a *topological change* of the Kähler manifold and the Witten index jumps discontinuously.

The fact that one can restrict the domain in which the quantum fields take value as much as one wants, without loosing any information about the critical theory, shows that, even at the quantum level, all the information is contained in the germ of the superpotential at its critical points. In this sense, as $L \to \infty$, N=2 QFT reduces to Singularity Theory.

In the localized theory the chiral matrix elements are

$$\begin{aligned}\langle h_2|Z^k|h_1\rangle &= \frac{1}{g^{k/m}}\delta_{k,(h_2-h_1)}\,T[k;h_1,h_2] + O(e^{-2|g_{2d}|LR^m})\\ T[k;h_1,h_2] &= \frac{1}{\pi}\sqrt{\sin\left(\frac{\pi\,h_2}{m}\right)\sin\left(\frac{\pi\,h_1}{m}\right)}\ \Gamma\left(\frac{h_2}{m}\right)\ \Gamma\left(1-\frac{h_1}{m}\right)\quad .\end{aligned} \tag{6.4}$$

[33] Compare with the $\bar{\partial}_V$–Poincaré lemma (4.37) .

On D_R, the field Z and the original one X are related by a biholomorphic map

$$X = Z + a_2(c_i)Z^2 + \ldots + a_\Delta(c_i)Z^\Delta + \ldots \tag{6.5}$$

whose coefficients do not depend on g. Then,

$$\langle h_2|X|h_1\rangle = \frac{a_{(h_2-h_1)}(c_i)}{g^{(h_2-h_1)/m}} T[(h_2-h_1);h_1,h_2] + O(e^{-2|g_2|LR^m}) \tag{6.6}$$

from which one sees that, off–criticality, in general there are power-law deviations from scaling.

6.1.2 *The general $n \geq 1$ case*

The general case is rather similar to the $n = 1$ one. We know that the GL forms of the holomorphic germs α_k, when restricted to a non–singular level manifold $F_t = \{X \in \mathbb{C}^n : V(X) = t\}$, are well–defined. We also know that they span the Picard–Lefschetz vanishing cohomology $H^{n-1}(F_t)$. One can localize the theory by considering a small neighborhood Ω of a critical point X_0. For technical reasons, Ω is chosen to be Stein. Again, one chooses (say) the Bergman metric on Ω. This guarantees completeness and the maximal symmetry compatible with the holomorphic geometry of Ω and the given superpotential V. From the universality of the $(n,0)$–part of the wave–forms and the arguments in Appendix G, one gets

$$\chi_k^{(n-1,0)}(X_j;\mathbb{C}^n) = \iota^*\chi_k^{(n-1,0)}(X_j;\Omega) \tag{6.7}$$

where again $\iota : \Omega \to \mathbb{C}^n$ is the inclusion. Taking Ω sufficiently small, one can transform V to the normal form of the singularity at X_0 by a holomorphic transformation. If the corresponding normal form is quasihomogeneous, one can further restrict Ω to a polydisk $\tilde{\Omega}$, invariant under $Z_i \to \lambda^{\alpha_i} Z_i$ for $\lambda \in \mathbb{C}^*$. In this way, one gets a localized model which is exactly R–invariant. Obviously, for the non–quasihomogeneous case, there exists no such choice of Ω.

Again, the chiral matrix elements of the localized theory are equal to those of the original theory, up to terms which are exponentially small for $g \to \infty$. This, in particular, means that they can be computed using any superpotential which, in the vicinity of X_0, has the given behaviour, *e.g.* the *normal* form of the singularity at X_0. This shows the *universality* of the IR behaviour. As $L \to \infty$ the matrix elements become equal for all the theories differing only by *irrelevant* couplings (in the RG sense).

6.2 Monodromy and g–dependence

Now that the correspondence between N=2 LG theory and Singularity Theory has been fully exploited, I consider the problem of the dependence of the CGFs on the overall coupling constant.

The idea is to use the topological interpretation of the GL wave–forms to compute the dependence of the Green's functions on the coupling constants. In doing this, one is inspired by the asymptotic theory of oscillatory integrals (see [49]). Let first consider the case of one field, $n = 1$. Assume that $X = 0$ is a critical point of multiplicity Δ. Then

$$V(X) = gX^{\Delta+1}\Big[1 + c_{\Delta+2}X^1 + \ldots + c_{D+1}X^{D-\Delta}\Big] \quad . \tag{6.8}$$

I am interested in the dependence on the overall coupling g at *fixed* c_i. We already know that the dependence on g becomes holomorphic as $g \to \infty$ (see also below). Varying g the wave–forms change continuously. Let consider the following path in the complex g–plane

$$g(t) = e^{-it}g \quad . \tag{6.9}$$

As $t \to 2\pi$ one gets back to the original theory. However, the wave–forms do not return, in general, to their original values. Indeed, this path is not contractible, since for $g = 0$ we cannot define the supersymmetric vacua.

However, $|a, e^{-2\pi i}g\rangle$ is still a solution to the zero–energy Schröedinger equation. Then one must have

$$|a, e^{-2\pi i}g\rangle = \sum_b M_a{}^b|b, g\rangle \tag{6.10}$$

with M some numerical matrix. The matrix M depends on the particular way I have defined the vectors $|a, g\rangle$. If one takes different linear combinations (with g–dependent coefficients), one gets different matrices. This is the "*basis problem*" again (see §3.3).

For $n = 1$ there is a (unique) natural choice such that the dependence on g is asymptotically holomorphic. This choice can be seen as arising from geometry.

What is to be shown is that, in this natural basis, the matrix M is just the matrix H of the classical monodromy of the singularity $(V, 0)$.

Consider the GL functions $\chi_a(X)$ (see §6.1). They can be seen as *multi–valued* functions of V. One can construct a basis for the GL wave–functions such that $G_0(\bar{\partial}_V\chi_k)$ $(k = 1, \ldots, \Delta)$ span the local ring $\tilde{\mathcal{R}}_0$ whereas the last $(D - \Delta)$ GL–forms have the property $G_0(\bar{\partial}_V\chi_\alpha) = 0$ $(\alpha = \Delta + 1, \ldots, D)$. Obviously, the χ_k are well-defined only modulo the χ_α's. Up to normalization, the first Δ GL–functions are asymptotic to "universal" functions of V

$$\chi_k = \text{const.}\, V^{-k/(\Delta+1)}\big[1 + O([V\overline{V}]^{k/(\Delta+1)})\big] \qquad (\text{as } X \to 0) \quad , \tag{6.11}$$

$(k = 1, \ldots, \Delta)$. Asymptotically, (i.e. at the germ level) these functions depend on g only implicitly, through the dependence on the superpotential. This means that, for small X, the wave–functions are invariant under the transformation

$$g \to e^{-it}g \qquad\qquad F(X, c_k) \to e^{it}F(X, c_k) \tag{6.12}$$

since it leaves V fixed. Requiring no *explicit* dependence on g of the germ α_k, one fixes the ambiguity in M. One has still to show that this is the natural basis. Using the results of

§4.5 and Appendix D, it is easy to see that for a vacuum wave–form $\omega = [\chi dV + \tilde{\chi} d\overline{V}]$, one has

$$\bar{g}\frac{\partial}{\partial \bar{g}}\omega = \bar{\partial}_V \left[\overline{V}\tilde{\chi}\right] + \text{“explicit dependence”} \tag{6.13}$$

where “explicit dependence” means any dependence on $\bar{g}$ not already implicit in the superpotential. Given that $\overline{V}\tilde{\chi}$ is regular as $X \to 0$, from the last equation it follows that, if there is no explicit dependence, varying $\bar{g}$ does not change the $\bar{\partial}_V$–cohomology class of the vacuum ω. The chiral matrix elements are intersection forms in this cohomology. Therefore, they are independent of $\bar{g}$, i.e. holomorphic [34]. Alternatively, the germ, $G_0(\omega_k) = \text{const}[V^{(1-k)/(\Delta+1)}dV]$, depends holomorphically on g and hence the holomorphicity of the chiral matrix elements follows from $\alpha_h = \langle h|X_i|k\rangle\ \alpha_k$ (mod. $dV \wedge \beta$). Then eq. (6.11) gives the natural basis.

In the previous equations the loop $F \to e^{it}F$ is just the one which defines the classical monodromy of the Milnor fibration for the singularity F at 0. Since the GL functions χ_k span the (reduced) cohomology of the fiber, and since following continuously the change in the superpotential amounts to homotopy lifting, one gets

$$\chi_k(g)|_{e^{2\pi i}F} = \sum_h H_k{}^h \chi_h(g)|_F \qquad \text{mod. } \chi_\alpha|_F \quad (\alpha = \Delta+1,\ldots,D) \tag{6.14}$$

where H is the matrix of the monodromy of F acting on the cohomology of the fiber. Since

$$|k, e^{-2\pi i}g\rangle|_F = |k,g\rangle|_{e^{2\pi i}F} \quad , \tag{6.15}$$

one sees that the matrix M in eq. (6.10) (in the natural basis) is equal to the monodromy matrix of the singularity F at $X = 0$. The wave functions change under $g \to e^{-it}g$ as

$$\begin{aligned} &|k, e^{-2\pi i}g\rangle = \sum_h H_k{}^h|h,g\rangle \qquad \text{mod. } |\alpha, g\rangle \\ &|\alpha, e^{-2\pi i}g\rangle = |\alpha, g\rangle \qquad k,h = 1,\ldots,\Delta;\ \alpha = \Delta+1,\ldots,D \quad . \end{aligned} \tag{6.16}$$

In fact, this is just the way the monodromy of the critical point $X = 0$ acts on the cohomology of the *full* level manifold F_t, according to the Picard–Lefschetz formula.

The matrix elements of the chiral operators are

$$(X^k)_g := \frac{\langle h_2, g|X^k|h_1, g\rangle}{\sqrt{\langle h_2,g|h_2,g\rangle\ \langle h_1,g|h_1,g\rangle}} \quad . \tag{6.17}$$

Then, acting with the monodromy operator one gets

$$(X^k)_{e^{2\pi i}g} = (H^T)^\dagger (X^k)_g H^T \quad . \tag{6.18}$$

In writing this, I used the fact that, for $n = 1$, the matrix H is unitary and so the denominator in $(X^k)_g$ does not change.

[34] This is true if the Kähler space is the full complex plane. On a bounded domain the story is subtler. This will be discussed in detail below. Anyhow, the subtleties do not spoil the result.

Since the dependence on g is holomorphic (asymptotically), one gets

$$(X^k)_g = \sum_{\kappa} \frac{1}{g^{\kappa}} A_{\kappa}^{(k)} \qquad \text{as } g \to \infty \tag{6.19}$$

where $A_{\kappa}^{(k)}$ do not depend on g, $\bar{g}$, and $\exp[2\pi i\kappa]$ is an eigenvalue of the adjoint action of the monodromy H. Using the explicit form of H, one gets back the results of §1.5 .

Let consider now the general case, $n \geq 1$.

After the localization, the GL–forms $\chi_k^{(n-1,0)}(X_j;\Omega)$ generate $H^{n-1}(f_t)$, where $f_t = \Omega \cap F_t$. In view of the results for the $n = 1$ case, it is natural that these forms transform, as $g \to e^{2\pi i} g$, according the classical monodromy of the given singularity. Let me show this. The previous argument cannot be directly extended to $n > 1$. However, one can reduce the general case to the previous one by a two step argument typical of Singularity Theory. First of all, consider superpotentials of the form

$$V = g\sum_{i=1}^{n} X_i^m \quad . \tag{6.20}$$

In this case, there is no coupling between the chiral fields X_i, and the wave–forms are simply the exterior products of the wave–forms for each field X_i separately. Then one can apply the previous result. Along the closed path $g \to e^{2\pi i} g$, the wave–forms transform according to the tensor product of the monodromy for each one–dimensional theory. But this tensor product is just the monodromy of the singularity $V = g\sum_{i=1}^{n} X_i^m$, as it follows from a theorem of Sebastiani and Thom [61]. So the claim is true in the direct product case.

The general case is reconducted to this one by using the following principle (§12.1.4 [49]): any singularity in n complex variables with Milnor number (local Witten index) Δ, belongs to the versal family of the superpotential $V = g\sum_{i=1}^{n} X_i^m$ for all $m \geq \Delta + 2$. In physical terms, this means the following. Consider a family of superpotentials

$$V = gF(X_i; c_s) \quad , \tag{6.21}$$

depending on the coupling constants gc_s, and such that $F(X_i;0) = \sum_{i=1}^{n} X_i^m$. I require that, for c_s small enough, $F(X_i;c_s)$ has a non–degenerate critical point at $X_0 \equiv 0$, with vanishing critical value. Such a family defines a deformation of the singularity $F(\cdot,0)$ or, in other words, a perturbation of the superpotential V. Then, the theorem states that, restricting to a sufficiently small domain Ω, any non–degenerate superpotential can be obtained by an *arbitrarily small* perturbation of a decoupled superpotential of the form $V = g\sum_{i=1}^{n} X_i^m$ with a large enough Witten index.

At first sight, the assertion that the perturbation can be arbitrarily small sounds non–sense. However, it is not so. Indeed, by rescaling the fields one can make the coupling in the superpotential as small as one wishes. Usually, doing this will accomplish nothing, since the big couplings will appear in the kinetic terms, and nothing changes. However, here I am interested only in quantities which are invariant under deformations of the kinetic term, and then one can freely rescale the kinetic term back to its original value. Thus one can make the perturbation as small as one wishes without changing its physical implications. Of course,

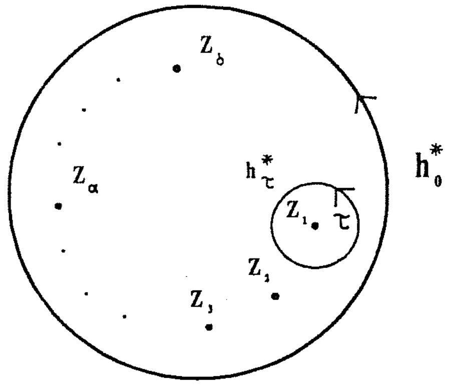

Figure 4
The paths defining the "unperturbed" monodromy h_0^*
and the monodromy of the deformed singularity h_τ^*

this argument is quite crude. Fixing up its details gives conditions on the "unperturbed" theory, which are encoded in the bound for its Witten index. I shall not discuss the details of the argument, since it is a word–for–word repetition of the proof of Lemma 12.4 in ref [49].

Under a perturbation, the classical monodromy does not change. This is just the fact that it is a topological quantity and hence invariant under smooth deformation. However, this is true at the level of $H^{n-1}(F_t)$. The critical point of the unperturbed theory, with index $(m-1)^n$, splits under perturbation into many critical points for $F(X_i; c_k)$ (Figure 4). Physically, as well as mathematically, I am not interested in the monodromy h_0^* (equal to the "unperturbed" one), but in the monodromy of the new critical point at the origin, that is to the monodromy h_τ^* associated to the path τ in Figure. The Picard–Lefshetz formula (see §5.2) reads in cohomology

$$h_a^*(\omega_b) = \omega_b - (-1)^{n(n-1)/2} \delta_{ab} S_{ac} \omega_c \tag{6.22}$$

where ω_c is a basis of $H^{n-1}(F_t)$ dual to the vanishing PL cycles δ_a, and $S_{ab} = (\delta_a \circ \delta_b)$ is the intersection matrix. Let choose the above basis such that ω_i $(i = 1, \ldots, \Delta)$ is a basis for $H^{n-1}(f_t)$ and ω_α $(\alpha = \Delta + 1, \ldots, (m-1)^n)$ span the space $\ker \iota^* \subset H^{n-1}(F_t)$. Since $h_\tau^* = \prod_i^\Delta h_i^*$, and $h_0^* = \prod_a h_a^*$, one gets

$$\begin{aligned} h_\tau^* \omega_i &= h_0^* \omega_i \quad \text{mod. } \omega_\alpha \\ h_\tau^* \omega_\alpha &= \omega_\alpha . \end{aligned} \tag{6.23}$$

This result holds in supersymmetry since the $(n, 0)$ part of the GL wave–forms span $H^{n-1}(F_t)$, in the sense of the previous sections. Let me show it explicitly. First of all, by the continuity of the perturbation, the monodromy does not change. One way to see this is the following. As the fields X_i go to ∞ the unperturbed superpotential dominates the perturbation (because of the bounds on the Witten index). So, for large $|X|$ the wave–forms are asymptotic to the unperturbed ones. Consider the change as $g \to e^{2\pi i} g$ of the GL wave–forms, restricted to the level manifold F_V for $V \to \infty$. One gets the same result as in the unperturbed theory, just because the wave–functions of these two situations are asymptotically equal. Then

$$\langle X \sim \infty | a, e^{2\pi i} g \rangle = \langle X \sim \infty | \sum_b H_a{}^b | b, g \rangle \tag{6.24}$$

where H is the unperturbed Milnor monodromy. This equation (up to the "basis problem") implies

$$| a, e^{2\pi i} g \rangle = \sum_b H_a{}^b | b, g \rangle \tag{6.25}$$

which shows the invariance under perturbations of the "physical" monodromy *defined* as the transformation of the vacuum states as $g \to e^{2\pi i} g$.

To show that eq. (6.23) holds in the supersymmetric case, one has to restrict the theory to a Stein (bounded) domain Ω which contains only the critical point at the origin. The localization theory implies that the *relevant* vacuum wave–forms in Ω are just the pull–backs of the ones in $\mathbb{C}^n$. Hence, they transform under $g \to \infty$ as the original ones, *modulo* vacua whose GL forms are regular in Ω (i.e. which under G_0 project to the Jacobian ideal). Then, the action of the transformation $g \to e^{2\pi i} g$ on the localized vacua is given by

$$| i, e^{2\pi i} g \rangle = \sum_j H_i{}^j | j, g \rangle \qquad \mathrm{mod.}\, | \alpha, g \rangle \quad , \tag{6.26}$$

where each localized state is denoted by the same symbol as the corresponding vacuum in $\mathbb{C}^n$. Comparing eqs. (6.23) and (6.26) , one sees that the relevant localized vacua transform under $g \to e^{2\pi i} g$ by the monodromy of the singularity at the origin h_T^*, modulo terms which are irrelevant (in the sense that give only exponentially small corrections as $g_{2d} L \to \infty$). This completes the argument. ∎

It is generally believed that, adding some relevant operator to a non–degenerate conformal theory, one gets a convergent perturbation theory. The above argument can be seen as a proof of this conjecture for a particular case.

The situation looks quite similar to the one for the $n = 1$ case. However, before getting any conclusion about the chiral matrix elements, one should solve the basis problem, i.e. determine under which circumstances there exists a basis in which: *i.*) the above arguments applies, and *ii.*) the dependence on g is holomorphic. In fact, we shall see that a basis having **simultaneously** these two properties does not exist if V is not quasi–homogeneous. In this last case new phenomena appear which deserve a careful discussion.

To proceed I have to consider in full detail the "*basis problem*" for the vacuum states. There are three distinguished set of bases:

i. The "holomorphic" bases (*if* they exist). In such a basis the chiral matrix elements are holomorphic functions of g.
ii. The "constant" bases. The chiral matrix elements are independent of the *phase* of the coupling g.
iii. The "topological" bases. The states transform under $g \to e^{2\pi i} g$ according to the monodromy of the singularity (this is the basis used in mathematics).

The holomorphic basis is uniquely defined, up to a linear transformation independent of g, $\bar{g}$. The constant one always exists, and it is defined up to a linear transformation depending only on $|g|$. The "topological" bases has been already shown to exist and are defined only up to a linear transformation which is an uni–valued function of g.

Giving the transformation intertwining between the holomorphic and the constant bases is enough to compute the dependence on g (and hence on L), and is exactly what I did in §3.4 using the Ward Identities.

Instead, constructing an intertwining with the topological basis, one can make contact with Singularity Theory, as before. The usual statements, about the connection between Singularity Theory and the N=2 Landau–Ginsburg models, can be restated as the fact that, for *semi–quasihomogeneous* superpotentials, the holomorphic basis is also a topological basis. Such a statement would be self–contradictory in the non–quasihomogeneous case, since the monodromy acts in a non–unitary way on the states.

6.3 Master equations for the g–dependence

After having considered the general features of the connection between the Monodromy and the g dependence of the CGFs, I will now look for an equation explicitly describing the dependence of the CGFs on the overall coupling constant.

The dependence of the intersection numbers in Singularity Theory on the parameter t (see §5.4) is holomorphic and described by the Gauss–Manin connection. In physics, as we have just seen, things are not so due to the "*basis problem*".

I will now begin to study the dependence of the vacuum wave forms ω_k on the overall coupling constant g. The vacuum wave forms satisfy the equations (see Appendix B)

$$\bar{\partial}_V \omega_k = 0, \quad \partial_V \omega_k = 0, \quad \Lambda \omega_k = 0 \quad . \tag{6.27}$$

Taking the derivatives of these equations with respect to the overall coupling constant g, one gets the following

(6.28) Lemma : *let V be a superpotential with a singular point in zero and (local) Witten index Δ. Let A_k^h and $\widetilde{A}_k^h$ be two $\Delta \times \Delta$ arbitrary matrices which, under a g–dependent change of basis*

$$\omega_k \longrightarrow S_k^h(g,\bar{g})\omega_h \tag{6.29}$$

transform as $GL(\Delta, \mathbb{C})$ connections. Define the **Natural covariant derivatives**

$$\begin{aligned} D_g &:= g\frac{\partial}{\partial g} - A \\ D_{\bar{g}} &:= \bar{g}\frac{\partial}{\partial \bar{g}} - \widetilde{A} \quad . \end{aligned} \tag{6.30}$$

Then

$$D_g \omega_k = \eta_k \qquad D_{\bar{g}} \omega_k = \tilde{\eta}_k \tag{6.31}$$

where η_k and $\tilde{\eta}_k$ are orthogonal to the vacuum wave forms ω_k and are given by

$$\eta_k = \partial_V \lambda_k \qquad \tilde{\eta}_k = \overline{\partial}_V \tilde{\lambda}_k \tag{6.32}$$

where λ_k is the unique solution to

$$\begin{aligned} &\overline{\partial}_V \lambda_k = V\,\omega_k - B_k^h \omega_h \qquad \partial_V \lambda_k = 0 \\ &B_k^h := C_{hl} \int \overline{*\omega_l} \wedge V\,\omega_k \\ &\left(C^{-1}\right)_{kj} := \int \overline{*\omega_k} \wedge \omega_j \end{aligned} \tag{6.33}$$

and $\tilde{\lambda}_k$ is the unique solution to

$$\begin{aligned} &\partial_V \tilde{\lambda}_k = \overline{V}\,\omega_k - \widetilde{B}_k^h \omega_h \qquad \overline{\partial}_V \tilde{\lambda}_k = 0 \\ &\widetilde{B}_k^h := C_{hl} \int \overline{*\omega_l} \wedge \overline{V}\,\omega_k \quad . \end{aligned} \tag{6.34}$$

The proof of this lemma and more details on the covariant derivative D_g are given in Appendix H. ∎

The covariant derivatives D_g and $D_{\bar{g}}$ are specified by their curvature. It is not difficult to show the following

(6.35) Corollary : *the curvature of the connection D_g is*

$$\begin{aligned} &[D_{\bar{g}}, D_g]\omega_k = \mathcal{F}_k^h \omega_h \\ &\mathcal{F} = [B, \widetilde{B}] \quad . \end{aligned} \tag{6.36}$$

The proof is given in Appendix H.

Up to now the covariant derivative D_g is a formal object. The following result gives the first explicit informations on it.

(6.37) Proposition : *the curvature $\mathcal{F}$ of the D_g covariant derivative is zero (possibly up to non–universal exponentially small corrections)* **if and only if** *the (isolated) singularity of V at the origin is holomorphic equivalent to a quasihomogeneous one.*

Proof : in the quasi–homogeneous case $V \in I_V$ (the local Jacobian ring of the singularity) by Euler's theorem. Hence it belongs to the kernel of the map ϖ of §3.1 . Since $B^T = \varpi(V)$, one gets $B = \tilde{B} = 0$. On the contrary, if the singularity is non–quasihomogeneous, the matrix B is *nihilpotent* [62], but non–vanishing. Since B is nilpotent and $\widetilde{B}$ is its Hermitean conjugate, $[B, \widetilde{B}] = 0 \Leftrightarrow B = 0$. Then in the non–quasihomogeneous case $\mathcal{F} \neq 0$. ∎

This result has an interesting corollary. Notice that a basis in which $A = \widetilde{A} = 0$ exists if and only if the natural connection is flat, i.e. if V is quasihomogeneous. In general, this is a multi–valued basis since its monodromy is not trivial. Thus, one gets the following

(6.38) Corollary : *in the quasihomogeneous case there exists a* **Special Basis** *in which* D_g *and* $D_{\bar{g}}$ *are replaced by the ordinary derivatives* $g\frac{\partial}{\partial g}$ *and* $\bar{g}\frac{\partial}{\partial \bar{g}}$. *In the general case, there exists a* **Unitary Basis** *in which* $C_{hk} = \delta_{hk}$ *and* A *is a* $U(\Delta)$ *connection.*

Since

$$D_{\bar{g}} B = D_g \tilde{B} = 0 \tag{6.39}$$

(where the covariant derivatives are those of the adjoint representation), one has an **improved connection** which is actually flat

$$\begin{aligned} \nabla_g &= D_g + B \\ \nabla_{\bar{g}} &= D_{\bar{g}} + \tilde{B} \\ \left[\nabla_{\bar{g}}, \nabla_g\right] &= 0 \quad . \end{aligned} \tag{6.40}$$

The natural and the improved connections agree in the quasihomogeneous case.

I will now reconsider the three basis defined in the previous section, §6.2 .

6.3.1 *The holomorphic basis*

The problem is to find under what circumstances, in the given basis, the dependence on g of the chiral matrix elements is holomorphic. A sufficient (but not necessary) condition is that, in the given basis, $A = \tilde{A} = 0$. This is just the condition under which the variation of the matrix elements is equal to the insertion of the $\overline{Q}$ supersymmetry transformation of "something". In the general case, the variation is still $\overline{Q}$ closed, but not $\overline{Q}$ exact since it represents the non–trivial cohomology class corresponding to the V–harmonic part of $A^h_k \omega_k + \eta_k$.

In more technical terms, such a "holomorphic basis" is the one in which the $\bar{\partial}_V$–cohomology classes, to which the forms ω_k and $\overline{\omega}_h$ belong, are independent of $\bar{g}$. Then the holomorphicity of the chiral matrix elements is obvious, in view of the fact that they are defined as intersection forms in cohomology.

The existence of a curvature in the non–quasihomogeneous case may seem at first to be an obstruction to the holomorphic dependence on g of the chiral matrix elements. However, it is not necessarily so. Denote by $\langle Y \rangle$ the matrix $\varpi(Y)$ for all chiral operator $Y \in \mathcal{R}$. From Lemma (6.28) one has

$$D_{\bar{g}} \langle Y \rangle = 0 \tag{6.41}$$

or, explicitly,

$$\bar{g}\frac{\partial}{\partial \bar{g}} \langle Y \rangle = \langle Y \rangle \tilde{A}^T + A^* \langle Y \rangle \quad . \tag{6.42}$$

If we are in an orthonormal basis, $A^\dagger = -\tilde{A}$ and the RHS of this last equation becomes $[\langle Y \rangle, \tilde{A}^T]$. Then, in order the chiral matrix elements to be holomorphic, it suffices that $\tilde{A}^T$ commutes with all the matrices in $\varpi(\mathcal{R})$. Since the commutant of this matrix algebra is quite big, this condition may have solutions. It is very difficult to prove (or disprove) the existence of an *unitary* holomorphic basis in the non–quasihomogeneous case. (A *non–unitary* holomorphic basis is easily constructed).

6.3.2 *The constant basis*

The constant basis is constructed the following way. Let return to the usual notations $dX_i \to \psi_i$, $d\overline{X}_i \to \bar{\psi}_i$. In these notations the Schrödinger equation is invariant under the transformation

$$g \to e^{i\alpha} g, \quad \psi_i \to e^{-i\alpha/2}\psi_i, \quad \bar{\psi}_i \to e^{i\alpha/2}\bar{\psi}_i \quad . \tag{6.43}$$

Returning to the form notation, this means that one can choose the basis for the vacuum wave–forms in such a way that

$$\omega_k(g,\bar{g})|_{const} = \left(\frac{\bar{g}}{g}\right)^{\frac{1}{4}\tilde{F}} \omega_k(|g|) \tag{6.44}$$

where $\tilde{F}$ is the operator which acting on a (p,q) form gives $(p-q)$. In this basis one has

$$\langle k|X_i|h\rangle\Big|_{const.} = \int *\overline{\omega}_k(|g|) \wedge X_i\, \omega_h(|g|) \tag{6.45}$$

from which it is manifest that they do not depend on the phase of g. It is also obvious that the constant basis can be chosen orthonormal.

In the quasihomogeneous case, there is the special holomorphic basis with $A = \tilde{A} = 0$. Let find the intertwining between this basis and the constant one. In this last basis one has

$$g\frac{\partial \omega_k|_{const}}{\partial g} - \bar{g}\frac{\partial \omega_k|_{const}}{\partial \bar{g}} = -\frac{1}{2}\tilde{F}\omega_k = \frac{1}{2}\sum_h \mathbf{R}_k{}^h \omega_h|_{const} + \partial_V \lambda_k - \bar{\partial}_V \tilde{\lambda}_k \tag{6.46}$$

where

$$\mathbf{R}_k{}^h(|g|) = -\int *\overline{\omega}_h|_{const} \wedge \tilde{F}\omega_k|_{const} \quad . \tag{6.47}$$

$\mathbf{R}_k{}^h$ is the intertwining matrix: if $\omega'_k(g)$ is the special holomorphic basis, then

$$\omega_k|_{const} = \left(\exp\left\{\frac{1}{4}\mathbf{R}(|g|)[\ln(g) - \ln(\bar{g})]\right\}\right)_k^{\ h} \omega'_h(g) \tag{6.48}$$

is an (orthonormal) constant basis.

From this one gets the chiral matrix element. If the matrix $\mathbf{R}$ is constant, then

$$\langle k|X_i|h\rangle = \left(e^{-\ln(g)\mathbf{R}/2}\right)_{kj} \langle j|X_i|l\rangle\Big|_{g=1} \left(e^{\ln(g)\mathbf{R}/2}\right)_{lh} \quad . \tag{6.49}$$

This dependence of the chiral matrix elements on g, as well as the matrix $\mathbf{R}$, is just what was obtained in §3.4 from physical arguments. Of course, one can write similar intertwinners for the non–quasihomogeneous case, but they are more involved.

6.3.3 *Comparison with the topological basis*

To complete the circle of ideas, one has to find the relation between the "topological" basis (i.e. the one used in Singularity Theory) and the "holomorphic" one. The procedure is to study the relationship between the connection D_g and the Gauss–Manin connection of the singularity. The idea is to obtain the action of the covariant derivative D_g on the holomorphic $(n,0)$ vacuum germ forms $\alpha_k \in \widetilde{\mathcal{R}}$ and on their geometric sections $s[\alpha_k]$ expressed in terms of the Gauss–Manin connection.

To obtain this, one starts applying D_g on ω_k written as $\omega_k = \overline{\partial}\eta_k + \alpha_k$. At the end, one gets the following

(6.50) Lemma : *let T_k^h be the* **torsion** *defined by the equations*

$$(V\delta_k^h - B_k^h)\partial_V\eta_h = \overline{\partial}_V\xi_k^l + T_k^h\alpha_h + \overline{V}dV \wedge \sigma \quad . \tag{6.51}$$

Then $D_g\alpha_k = d\sigma_k + T_k^h\alpha_h$ and the **Master Equations** *are*

$$\begin{aligned} &D_g s[\alpha_k](t) = \left[(t\delta_k^h - B_k^h)\nabla_t + T_k^h\right] s[\alpha_h](t) \\ &D_{\bar{g}} s[\alpha_k](t) = 0 \end{aligned} \tag{6.52}$$

where ∇_t is the Gauss–Manin connection of the singularity.

The proof and other details are given in Appendix H. ∎

The equations (6.52) can be solved (at least in principle) to give the explicit dependence on g and $\bar{g}$ of the germ forms α_k and hence (via the G isomorphism) of the states and of the CGFs.

As in the case of the covariant derivative D_g where the matrices A and B are very difficult to compute in a general case, also the "torsion" T is very difficult to compute in a general case. But in the case of a quasihomogeneous superpotential the result is again simple.

(6.53) Proposition : *let V be a quasihomogeneous superpotential, then*

$$T_h^k = 0 \quad . \tag{6.54}$$

The proof is given in Appendix H. ∎

(6.55) Corollary : *in the quasi–homogeneous case, one has*

$$\begin{aligned} &g\frac{\partial}{\partial g} s[\alpha_k](t) = t\nabla_t\, s[\alpha_k](t) \\ &\bar{g}\frac{\partial}{\partial \bar{g}} s[\alpha_k](t) = 0 \quad . \end{aligned} \tag{6.56}$$

Note that this equation implies that the monodromy of the D_g connection is equal to the Gauss-Manin one. This shows that — in the quasihomogenous case — the "holomorphic" basis is also "topological".

6.4 The quasihomogeneous case

Let $V(X) = gF(X)$ be a quasihomogeneous superpotential of degree 1 and weights $(q_1, \ldots, q_n)$. Let $\{X^{\underline{m}}\}$ be a set of monomials giving a basis of $\mathcal{R}$. Let $\hat{\alpha}(\alpha_{\underline{m}}) = l(\underline{m}) = [\sum_{i=1}^{n}(m_i+1)q_i] - 1$ be the order of the form $\alpha_{\underline{m}} := \lambda_{\underline{m}} X^{\underline{m}} dX_1 \wedge \ldots \wedge dX_n$ (see §5.4). From §5.4 we know that

$$s[\alpha_{\underline{m}}](t) = t^{l(\underline{m})} A^{\alpha_{\underline{m}}}_{l(\underline{m}),0} \tag{6.57}$$

and hence

$$t\nabla_t\, s[\alpha_{\underline{m}}](t) = l(\underline{m}) \cdot s[\alpha_{\underline{m}}](t) \quad . \tag{6.58}$$

The equations (6.56) now become

$$\begin{aligned} g\frac{\partial}{\partial g} s[\alpha_{\underline{m}}](t) &= l(\underline{m}) \cdot s[\alpha_{\underline{m}}](t) \\ \bar{g}\frac{\partial}{\partial \bar{g}} s[\alpha_k](t) &= 0 \quad . \end{aligned} \tag{6.59}$$

Since $\alpha_{\underline{m}}|_t = \partial V \wedge s[\alpha_{\underline{m}}]$ and $V(X) = gF(X)$ one gets

$$g\frac{\partial}{\partial g}\alpha_{\underline{m}} = [l(\underline{m}) + 1]\,\alpha_{\underline{m}} \tag{6.60}$$

which implies

$$\alpha_{\underline{m}}(g) = g^{l(\underline{m})+1}\alpha_{\underline{m}}(g=1) \quad . \tag{6.61}$$

Let me go back to §4.5 and proposition (4.85) . Let define a new normalization constant $\nu_{\underline{m}}$ by

$$\lambda_{\underline{m}} =: g^{l(\underline{m})+1}\nu_{\underline{m}} \quad . \tag{6.62}$$

From (6.61) $\nu_{\underline{m}}$ does not depend on g. Repeating the proof of proposition (4.85) one easily gets

$$\langle \underline{p}|X^{\underline{k}}|\underline{r}\rangle = \delta_{\underline{p},\underline{p}(\underline{k}+\underline{r})} \cdot \frac{\nu_{\underline{r}}}{\nu_{\underline{k}+\underline{r}}} \cdot \frac{1}{(gL)^{\sum_i k_i q_i}} \tag{6.63}$$

where I used the fact that the CGFs depend on L only through the adimensional combination (gL).

As $L \to +\infty$, i.e. in the IR limit, from this result one can directly read the scaling dimension of the chiral primary fields $X^{\underline{k}}$. One has

$$h(X^{\underline{k}}) = \frac{1}{2}\sum_i q_i k_i \tag{6.64}$$

in agreement with refs. [6,5].

The explicit knowledge of the scaling dimensions of the chiral primary fields in the IR scaling regime for the quasihomogeneous models has very interesting consequences. Indeed, it is not hard to believe that the (semi–) quasihomogeneous 2d N=2 LG models have an IR fixed point (a rigorous proof, as that in §2.3 for the A_μ models, as far as I know, is still missing, but the less rigorous arguments of §3.2 are enough to be quite sure of the existence of the IR fixed point).

At the IR fixed point, a (semi–) quasihomogeneous model becomes a N=2 SCFT theory (supersymmetry and U(1) — R symmetry are not broken). The R symmetry of the quasihomogeneous model (or its "quasihomogeneity") become the U(1) symmetry of the N=2 SCFT algebra (see [24,31] and Appendix C). Thus, from (6.64) one can directly read the "*conformal dimension*" of the chiral primary field $X^{\underline{k}}$ and its U(1)–charge:

$$h_{scft}(X^{\underline{k}}) = \frac{1}{2}Q(X^{\underline{k}}) = \frac{1}{2}\sum_i q_i k_i \quad . \tag{6.65}$$

Moreover, it is well known that the "*central charge*" c of the N=2 SCFT theory is related to the maximal U(1) charge by $c/3 = Q_{max}$ [7] and that $Q_{max} = \sum_{i=1}^{n}(1-2q_i)$ (see §5.1), thus

$$c = 3\sum_{i=1}^{n}(1-2q_i) \quad . \tag{6.66}$$

A useful remark is the following one. The dimension of the primary chiral field $X^{\underline{k}}$ can be written as

$$2h(X^{\underline{k}}) = \left[(l(\underline{k})+1) + \left(\frac{c}{6}-\frac{n}{2}\right)\right] \tag{6.67}$$

where c is the central charge given by the previous formula and n is the number of fields. Since $l(\underline{k})$ is equal to the order (or spectral number) of the form $\alpha_{\underline{k}}$, $\hat{\alpha}(\alpha_{\underline{k}})$, this formula can be rewritten as:

$$2h(X^{\underline{k}}) = \left[(\hat{\alpha}(\alpha_{\underline{k}})+1) + \left(\frac{c}{6}-\frac{n}{2}\right)\right]$$

or (6.68)

$$Q(X^{\underline{k}}) = \left[(\hat{\alpha}(\alpha_{\underline{k}})+1) + \left(\frac{c}{6}-\frac{n}{2}\right)\right]$$

where $Q(X^{\underline{k}})$ is the U(1) R–charge of the chiral primary field $X^{\underline{k}}$ (I am still considering only the quasihomogeneous case). (6.68) can be interpreted as follows: the U(1) R–charge of the chiral primary field $X^{\underline{k}}$ is given by its order (or spectral number) up to a "*shift*".

It is very interesting to understand from where this shift comes from. The "+1" comes only for technical reasons from the fact that the order is defined for the geometric section $s[\alpha]$ and $\alpha = \partial V \wedge s[\alpha]$. The other terms are more interesting and arise from a "*background*" R–charge.

Since the spectral numbers are associated to n–forms, which corresponds to the supersymmetric vacua, it is more appropriate to compare the $\hat{\alpha}_{\underline{k}}$'s with the $U(1)$ charges of the Ramond operators R_m creating the supersymmetric vacua out of the $SL(2,\mathbb{C})$–invariant vacuum. The $U(1)$ charges of these operators, $Q_{\underline{k},ram}$, are equal to the R–charges of the corresponding supersymmetric vacua. The spectral flow [7,31] relates these $U(1)$ charges with those of the chiral primary operators $Q_{\underline{k}}$,

$$Q_{\underline{k},ram} = Q_{\underline{k}} - c/6 \quad . \tag{6.69}$$

Thus the shift by $c/6$ is explained, it remains to explain the shift by $-n/2$ between $Q_{\underline{k},ram}$ and $(\hat{\alpha}_{\underline{k}}+1)$. This arises as follows. Consider the basic identification in §4.1 and Appendix B,

$$\psi_1^\dagger \cdots \psi_n^\dagger|0\rangle \to dX_1 \wedge \ldots \wedge dX_n \quad , \tag{6.70}$$

which maps the physical vacua into differential forms. Under the R–transformation

$$X_r(z,\theta) \to e^{iq_r\alpha} X_r(z, e^{-i\alpha/2}\theta) \quad , \tag{6.71}$$

$\psi_1^\dagger \cdots \psi_n^\dagger$ picks up the phase

$$\exp[+i\alpha \sum_{r=1}^{n} (q_r - 1/2)] \quad . \tag{6.72}$$

To agree with eq. (6.69) , the R–charge of the operator $\psi_1^\dagger \cdots \psi_n^\dagger$ should be $-c/6$ (the R–charge of the identity operator is zero, $Q_{\underline{0}} = 0$). Indeed, the supersymmetric vacuum obtained from the monomial $X^{\underline{k}}$ by spectral flow has a wave–function

$$X^{\underline{k}}\psi_1^\dagger \cdots \psi_n^\dagger|0\rangle + \ldots \qquad (\text{as } X \to 0) \tag{6.73}$$

(this is $\alpha_{\underline{k}}$ in physical notation) and it has a $U(1)$ charge $Q(X^{\underline{k}}) - c/6$.

On the other hand, $dX_1 \wedge \ldots \wedge dX_n$ under the R–transformation picks up a phase

$$\exp[+i\alpha \sum_{r=1}^{n} q_r] \quad . \tag{6.74}$$

if the R–charge of $\psi_1^\dagger \cdots \psi_n^\dagger|0\rangle$ and $dX_1 \wedge \ldots \wedge dX_n$ should agree, one has to shift the "*background*" R–charge in one of the two formulations by $(\sum_{i=1}^{n} 1/2) = n/2$. If one is interested in the mathematical functoriality of the formulation, one should shift the Clifford vacuum $|0\rangle$ by $+n/2$ units of R–charge. In this case one gets $Q_{\underline{k},ram} = (\hat{\alpha}(\alpha_{\underline{k}}) + 1)$, the mathematical properties are evident but the usual physical interpretation of $\psi_1^\dagger \cdots \psi_n^\dagger|0\rangle$ is lost. However such a shift does not change the R–charge of the physical quantities, like Green's functions.

On the other hand, I am most interested in the physical interpretation of the objects and thus I will shift the background R–charge of $dX_1 \wedge \ldots \wedge dX_n$ by $-n/2$ to get the R–charge of $\psi_1^\dagger \cdots \psi_n^\dagger|0\rangle$. In this case $Q_{\underline{k},ram} = (\hat{\alpha}(\alpha_{\underline{k}}) + 1) - n/2$ and it is a true R–charge of a Ramond supersymmetry vacuum associated to the form $X^{\underline{k}}dX_1 \wedge \ldots \wedge dX_n$. Thus, from $Q_{\underline{k},ns} = Q_{\underline{k},ram} + c/6$ one gets (6.68) . Obviously this formulation spoils the "functoriality" of the wave–forms under R–transformations ($\equiv$ monodromy) which is built in the form language, but the physical properties are evident: for example the charge conjugation acts as $Q_{\underline{k},ram} \leftrightarrow -Q_{\underline{k},ram}$.

In other words, the fact is that $\psi_1^\dagger \cdots \psi_n^\dagger|0\rangle$ and $dX_1 \wedge \ldots \wedge dX_n$ have different R–charges (this is obvious, see §1.5), thus to get the correct physical R–charges in the mathematical formulation (using $dX_1 \wedge \ldots \wedge dX_n$ as "vacuum") one is obliged to introduce a background R–charge.

These considerations are quite general, they depend only on the facts that the model under consideration has a fixed point in the IR limit and that its superpotential is quasihomogeneous.

CHAPTER 7

EXACT COMPUTATION OF THE CHIRAL GREEN'S FUNCTIONS

In this chapter I will compute the CGFs in an exact way without explicitly solving the Schrödinger equation. To get this it is sufficient to compute the "*absolute normalization of the vacuum germ forms*". The absolute normalization of the vacuum germ forms is given by the "*real structure*" in the cohomology. This has a clear physical interpretation since the action of the real structure corresponds to that of the "*Spectral Flow*".

Finally I will explicitly do two examples computing the CGFs for the models of the A_μ and D_μ series.

7.1 Spectral Flow and Real Structure

I start by making a brief summary of the results obtained in the previous chapters. The CGFs have been interpreted as intersection numbers in cohomology and to each vacuum it has been associated a cohomology class $[\omega]$ in $H^n_{\bar{\partial}_V}$. To this class it is naturally associated a class $[\alpha]$ in $\widetilde{\mathcal{R}}$. The value of the CGFs seen as intersection numbers in cohomology is deeply related to the absolute normalization of the vacuum germ forms $\alpha \in \widetilde{\mathcal{R}}$ representing the vacua. Indeed ((4.85))

$$\langle \underline{p}|X^{\underline{k}}|\underline{q}\rangle = \delta_{\underline{p},\underline{p}(\underline{k}+\underline{q})} \cdot \frac{\lambda_{\underline{q}}}{\lambda_{\underline{k}+\underline{q}}} \tag{7.1}$$

where the coefficients $\lambda_{\underline{q}}$ satisfy the following relation (see (4.84))

$$\widetilde{\lambda}_{\underline{p}(\underline{q})} \cdot \lambda_{\underline{q}} = \frac{1}{(-2\pi)^n C\Delta} \quad . \tag{7.2}$$

With the results of Singularity Theory, it has been possible to compute the dependence on the overall coupling constant g of the CGFs. The computation has been done explicitly for a quasihomogeneous superpotential of degree 1 and weights q_i. One has (see §6.4)

$$\langle \underline{p}|X^{\underline{k}}|\underline{q}\rangle = \delta_{\underline{p},\underline{p}(\underline{k}+\underline{q})} \cdot \frac{\nu_{\underline{q}}}{\nu_{\underline{k}+\underline{q}}} \cdot \frac{1}{(gL)^{\sum_{i=1}^n k_i q_i}} \quad . \tag{7.3}$$

From now on I will consider *only quasihomogeneous superpotentials.* What is left to do is to compute $\lambda_{\underline{q}}/\widetilde{\lambda}_{\underline{p}(\underline{q})}$. The computation of the absolute normalization of the vacuum wave forms reduces to the identification of the "real structure" in the cohomology. Indeed, the vacuum wave forms can be seen as representatives of $\bar{\partial}_V$ and ∂_V classes, $\omega_k = \alpha_k + \bar{\partial}_v \eta_k = (\widetilde{\beta}_k)^* + \partial_V \widetilde{\eta}_k^*$. The correspondence $\alpha_k \to \widetilde{\beta}_k$ gives an antilinear involution (anti–isomorphism) $\widetilde{\ }$. This map corresponds to the way complex conjugation acts on $\mathcal{R}$ and then it is the real structure on the cohomology. Physically, this corresponds to the action of the Spectral Flow.

The formulation of the reality condition requires the introduction of a filtration on the cohomology. This because otherwise the integrals which will be introduced would be ill–defined since they would depend on the particular representative, α_k, chosen in a class of $\mathcal{R}$.

Generally speaking, the filtrations in the Milnor cohomology bundle arise naturally in mathematics. These filtrations gives rise to a "*Mixed Hodge structure*". I will not introduce this mathematical tool since I will not need it in a relevant way. I refer to the literature the interested reader [49,59,63,64,65,66,2].

Anyway, this structure has a clear physical meaning. One needs a natural identification between the space of zero–energy states with the PL cohomology. The first space is canonically isomorphic to the chiral ring $\mathcal{R}$, so the problem is to give a concrete correspondence between elements of the chiral ring and the vanishing cohomology. This requires selecting out a "natural" representative of each class in $\mathcal{R}$. In the quasihomogeneous case, this is quite easy, since there is a quantum number — the R–charge — and one can select the natural representative as a monomial in $\mathcal{R}$ having the correct charge. In the general case, there is not such a simple prescription. Physically, the problem is that different operators get mixed under a scaling transformation, since there is no symmetry to protect them. It is well known from renormalization theory how one can define a basis in this situation. One takes the leading infrared behaviour of each operator and then define each operator *modulo* less relevant ones. One takes as representative of each class in $\mathcal{R}$, the *less* relevant operator in it, always defined up to operators which are even less relevant. The space of all the such defined equivalence classes of operators is naturally graded according to relevance (in the IR). This is, informally, the physical definition of the filtrations in $\mathcal{R}$.

I will now formulate the reality condition.
Let [2]

$$\psi[\alpha] := \left(\int_{\Gamma_1} e^{tF}\alpha, \cdots, \int_{\Gamma_\mu} e^{tF}\alpha \right) \tag{7.4}$$

where $\{\Gamma_i\}$ is a basis of cycles in $H_n(X, X_-)$ and ψ is defined modulo exponentially small terms. Notice that these integrals have the "*direct product*" property which physically is required (see §5.4). Thus, they are the most natural objects to consider.

However, it is very easy to see that these integrals depend on the particular representative chosen in a class of $\widetilde{\mathcal{R}}$. To avoid this, following the mathematicians, one can introduce a filtration by the charge. Denote

$$\begin{aligned} \mathcal{K}[r] &:= \{ \text{ span of } \psi[\alpha] \text{ with } l(\alpha) \leq r\} \\ \Psi[r] &:= \frac{\mathcal{K}[r]}{\mathcal{K}[r-1]} \\ w[\alpha] &:= [\psi[\alpha]] \in \Psi[l(\alpha)] \quad . \end{aligned} \tag{7.5}$$

It is not difficult to see the $w[\alpha]$ is well defined in the sense that it does not depend on the particular representative of a class in $\widetilde{\mathcal{R}}$. This procedure is very similar to the one used to introduce a mixed structure in $\widetilde{\mathcal{R}}$, see for example [49,47,59,48]. In technical terms one is associating to each class in $\widetilde{\mathcal{R}}$ the Laplace transform of its "*original coefficient.*

In ref. [2] it is proven that the reality condition is

$$w[\alpha] = (w[\widetilde{\alpha}^*])^* \quad . \tag{7.6}$$

To prove this equation one uses the real structure given by the Spectral Flow or, equivalently, by the 1d Schroedinger equation [2]. Indeed, since the Schroedinger equation is real, one has $\omega_k^* = M_k{}^h \omega_h$ for some matrix $M_k{}^h$ ($MM^* = 1$). M is the matrix describing the real structure. This implies that $\tilde{\alpha}_k = M_k{}^h \alpha_h$. I will not repeat here the proof of this equation but I will now give two examples of its application.

7.2 Two explicit examples

In this section I will give almost all details of the computation of the CGFs in the case of the A_μ and D_μ series.

For convenience I change notation for the vacuum germ forms $\alpha_{\underline{q}}$ setting

$$\alpha_{\underline{q}} := \lambda_{\underline{q}} X^{\underline{q}-\underline{1}} dX_1 \wedge \ldots \wedge dX_n \tag{7.7}$$

where $\underline{1} = (1, \ldots, 1)$ and $q_i \geq 1$.

7.2.1 *The A_μ series*

The A_μ series (see §1.5) are the models described by the superpotential

$$V = \frac{g}{\mu+1} X^{\mu+1} \tag{7.8}$$

(obviously $n = 1$ and $\mu \geq 2$). The Witten Index is $\Delta = \mu$. A basis of $\widetilde{\mathcal{R}}$ is given by $\alpha_q = \lambda_q X^{q-1} dX$, $1 \leq q \leq \mu$. Moreover (see (4.77))

$$\begin{aligned} \det\left(\frac{\partial^2 V}{\partial X_i \partial X_j}\right) &= \mu g X^{\mu-1} \\ C &= \frac{1}{\mu g} \quad . \end{aligned} \tag{7.9}$$

Since the A_μ models are the simplest quasihomogeneous models with weightd of quasihomogeneity $q = \frac{1}{\mu+1}$, one has

$$\begin{aligned} l(q) &= \frac{q}{\mu+1} - 1 \\ \lambda_q &= g^{\frac{q}{\mu+1}} \nu_q \quad . \end{aligned} \tag{7.10}$$

The dual of a vacuum germ form α_q is given by

$$\begin{aligned}\widetilde{\alpha}_{p(q)} &= \widetilde{\lambda}_{p(q)} X^{p(q)-1} dX \\ p(q) &= \mu + 1 - q \quad .\end{aligned} \tag{7.11}$$

In the case of one field ($n = 1$) the reduced homology group $H_0(F_t)$ has some peculiarities (see §5.2). It is composed by $\mu + 1$ points which are given by the $\mu + 1$ roots of the equation $F(X) = t$ (where $F(X) = \frac{1}{\mu+1} X^{\mu+1}$ and t is less than zero, $t < 0$). Thus, $H_0(F_t) = \{X_j\}$, where $X_j(t) = |(\mu+1)t|^{1/(\mu+1)} \exp(i\pi \frac{(2j-1)}{\mu+1})$.

The computation of a basis of vanishing cycles is given in §2.9 [49]. There it is shown that a basis of vanishing cycles is given by the 0–cycles $\epsilon_{j+1} := X_j(t) - X_{j+1}(t)$, $j = 1, \ldots, \mu$.

The GL wave 0–forms α_q/dF are

$$\frac{\alpha_q}{dF} = \lambda_q X^{q-1-\mu} \quad . \tag{7.12}$$

The computation of the integrals over the vanishing cycles of the GL forms is given in §11.1.3 [49]. One has

$$\int_{\epsilon_{j+1}} \frac{\alpha_q}{dF} = \lambda_q \left[(X_j(t))^{q-(1+\mu)} - (X_{j+1}(t))^{q-(1+\mu)} \right] \tag{7.13}$$

which gives

$$\int_{\epsilon_{j+1}} \frac{\alpha_q}{dF} = \lambda_q |(\mu+1)t|^{(\frac{q}{\mu+1}-1)} \cdot 2i \sin\left(\frac{\pi q}{\mu+1}\right) e^{2\pi i \frac{jq}{\mu+1}} \quad . \tag{7.14}$$

Using formula (5.102) with $t = -1$ one gets

$$\int_{\Gamma_{j+1}} e^{-F} \alpha_q = \Gamma\left(\frac{q}{\mu+1}\right) \lambda_q (\mu+1)^{(\frac{q}{\mu+1}-1)} \cdot 2i \sin\left(\frac{\pi q}{\mu+1}\right) e^{2\pi i \frac{jq}{\mu+1}} \quad . \tag{7.15}$$

Analogously one can compute the integral of the dual form:

$$\begin{aligned}\int_{\epsilon_{j+1}} \frac{\widetilde{\alpha}^*_{p(q)}}{d\overline{F}} &= \widetilde{\lambda}^*_{\mu+1-q} \left[\left(\overline{X}_{\mu+1-j}(t)\right)^{-q} - \left(\overline{X}_{\mu-j}(t)\right)^{-q} \right] \\ &= \widetilde{\lambda}^*_{\mu+1-q} |(\mu+1)t|^{-q} \cdot 2i \sin\left(\frac{\pi q}{\mu+1}\right) e^{-2\pi i \frac{jq}{\mu+1}} \quad .\end{aligned} \tag{7.16}$$

Then

$$\int_{\Gamma_{j+1}} e^{-\overline{F}} \widetilde{\alpha}^*_{p(q)} = \Gamma\left(1 - \frac{q}{\mu+1}\right) \widetilde{\lambda}^*_{\mu+1-q} (\mu+1)^{-\frac{q}{\mu+1}} \cdot 2i \sin\left(\frac{\pi q}{\mu+1}\right) e^{-2\pi i \frac{jq}{\mu+1}} . \tag{7.17}$$

The reality condition simply states that

$$\int_{\Gamma_{j+1}} e^{-F} \alpha_q = \left(\int_{\Gamma_{j+1}} e^{-\overline{F}} \widetilde{\alpha}^*_{p(q)} \right)^* \quad . \tag{7.18}$$

Then, one has

$$\frac{\lambda_q}{\widetilde{\lambda}_{\mu+1-q}} = \frac{-\pi(\mu+1)^{1-\frac{2q}{\mu+1}}}{\sin(\pi\frac{q}{\mu+1})\left(\Gamma(\frac{q}{\mu+1})\right)^2} \tag{7.19}$$

where the relation $\sin(\pi x)\Gamma(x)\Gamma(1-x) = \pi$ has been used.

Now,

$$\widetilde{\lambda}_{\mu+1-q}\cdot\lambda_q = -\frac{1}{2\pi g} \tag{7.20}$$

and using $(\lambda_q)^2 = (\widetilde{\lambda}_{\mu+1-q}\cdot\lambda_q)\cdot\left(\frac{\lambda_q}{\widetilde{\lambda}_{\mu+1-q}}\right)$ one gets

$$\lambda_q = \frac{g^{\frac{q}{\mu+1}}(\mu+1)^{\frac{1}{2}-\frac{q}{\mu+1}}}{\sqrt{2\sin(\pi\frac{q}{\mu+1})}\left(\Gamma(\frac{q}{\mu+1})\right)} \quad . \tag{7.21}$$

Finally, one has

$$\begin{aligned}\langle q|X^k|q\rangle =& \frac{1}{(gL)^{\frac{k}{\mu+1}}}\delta_{p,p(k+q)}\frac{1}{\pi}\sqrt{\sin(\pi\frac{q}{\mu+1})\sin(\pi\frac{q+k}{\mu+1})}\cdot \\ &\cdot(\mu+1)^{\frac{k}{\mu+1}}\Gamma\left(\frac{q+k}{\mu+1}\right)\Gamma\left(1-\frac{q}{\mu+1}\right) \quad .\end{aligned} \tag{7.22}$$

This is exactly the same result obtained in §1.5 . Finally, in §2.2 it is shown that, with a change of the normalization of the primary chiral fields, these CGFs become those of the A_μ N=2 minimal SCFT models.

7.2.2 *The D_μ series*

I will now come to the case of the D_μ models [4,5,21] (see also §1.6). The D_μ models are models constructed with two fields ($n = 2$) and with superpotential

$$V = g\left(c_1X^2Y + c_2Y^{\mu-1}\right) \quad . \tag{7.23}$$

The Witten Index is $\Delta = \mu$ and the spectrum of the primary chiral fields and their algebra correspond to those of the D_μ minimal superconformal models with central charge $c = 3\sum_{i=1}^{2}(1-2q_i) = 3 - \frac{3}{\mu-1} = 3 - \frac{6}{p+2}$ where $\mu = \frac{1}{2}p + 2$ and $p \in 2\mathbb{Z}$. The weights (R–charges) of the fields X and Y are $q_x = \frac{\mu-2}{2(\mu-1)}$, $q_y = \frac{1}{\mu-1}$.

A basis of $\widetilde{\mathcal{R}}$ is given by

$$\alpha_{k_x,k_y} = \lambda_{k_x,k_y}X^{k_x-1}Y^{k_y-1}dX\wedge dY \tag{7.24}$$

where $k_x = 1,2$, $k_y = 1,\ldots,\mu-1$ and $k_x = 2$ only if $k_y = 1$. The equation defining the ideal $\mathcal{I}$ are

$$\begin{aligned}&\frac{\partial}{\partial X}V = 0 = 2gc_1XY \\ &\frac{\partial}{\partial Y}V = 0 = gc_1X^2 + (\mu-1)gc_2Y^{\mu-2} \quad .\end{aligned} \tag{7.25}$$

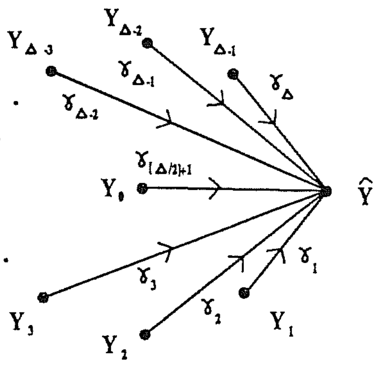

Figure 5
Vanishing cycles for μ odd

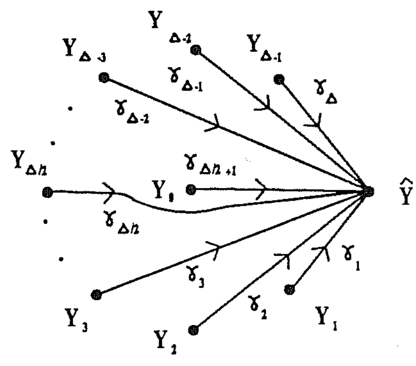

Figure 6
Vanishing cycles for μ even

Thus, $gc_1X^2 \cong -(\mu-1)gc_2Y^{\mu-2}$ where $\cong$ means equal in $\mathcal{R}$. Moreover

$$\det\left(\frac{\partial^2 V}{\partial X_i \partial X_j}\right) \cong 2g^2\mu(\mu-1)c_1c_2Y^{\mu-2} \cong -2g^2\mu(c_1)^2X^2 \quad . \tag{7.26}$$

It follows that the constant C is $C = (2g^2\mu(\mu-1)c_1c_2)^{-1}$ or $C = (-2g^2\mu(c_1)^2)^{-1}$.

It holds

$$\begin{aligned} l(k_x, k_y) &= \frac{(\mu-2)}{2(\mu-1)}k_x + \frac{k_y}{\mu-1} - 1 \\ \lambda_{k_x,k_y} &= g^{\frac{(\mu-2)}{2(\mu-1)}k_x + \frac{k_y}{\mu-1}}\nu_{k_x,k_y} \quad . \end{aligned} \tag{7.27}$$

The dual of the vacuum germ form α_{k_x,k_y} is given by

$$\widetilde{\alpha}_{p_x(k),p_y(k)} = \widetilde{\lambda}_{p_x(k),p_y(k)}X^{p_x(k)-1}Y^{p_y(k)-1}dX \wedge dY \tag{7.28}$$

where $p_y(k) = \{1 \text{ if } k_x > 1,\ \mu - k_y \text{ if } k_x = 1\}$ and $p_x(k) = \{1 \text{ if } k_x = 1,\ 2 \text{ if } k_x = 2,\ 1 \text{ if } k_x = 3\}$.

The GL wave forms $\alpha_{k_x,k_y}/dF$ (where $V(X,Y) = gF(X,Y)$) are defined on the non-critical level hypersurface F_t ($F_t := \{X, Y \mid F(X,Y) = t,\ t < 0\}$). A useful parametrization of the GL wave form is the following one

$$\frac{\alpha_{k_x,k_y}}{dF} = \frac{1}{2gc_1}\frac{\alpha_{k_x,k_y}(X,Y)}{XY}dY \quad . \tag{7.29}$$

It is very convenient to consider the level hypersurface F_t as described by the (complex) coordinate Y, while X on F_t is given by

$$X = \pm\sqrt{\frac{t - c_2Y^{\mu-1}}{c_1Y}} \quad . \tag{7.30}$$

The manifold F_t can also be seen as a Riemann surface whose branching points are the image of the critical points of the function F.

As already stated t is less than zero, $t < 0$. On F_t the critical points are given by the equations

$$F(X,Y) = t \qquad \frac{\partial}{\partial X}F(X,Y) = 0 \tag{7.31}$$

which have the solutions

$$Y_0 = 0 \qquad Y_j = e^{-i\pi\frac{(2j-1)}{(\mu-1)}}\left|\frac{t}{c_2}\right|^{\frac{1}{\mu-1}} \tag{7.32}$$

where $j = 1, \ldots, \mu - 1$.

I choose the μ vanishing cycle in the following way. Let $\widehat{Y}$ be the non critical value (see §5.2), I choose it to be real and positive and such that $(\widehat{Y} - \left|\frac{t}{c_2}\right|^{\frac{1}{\mu-1}})$ is larger than zero and sufficiently small so that the μ vanishing cycles can be taken as in figures 5 and 6 (it always exists a finite ρ such that for $0 < (\widehat{Y} - \left|\frac{t}{c_2}\right|^{\frac{1}{\mu-1}}) < \rho$ one can construct these vanishing cycles since μ is, by hypothesis, finite).

That these are the correct vanishing cycles can be explicitly checked by taking a linear perturbation of F, F_λ (for example the perturbation can be $\lambda_x X + \lambda_y Y$) and constructing the Picard–Lefschetz vanishing cycles of the Morse function F_λ (see §5.2).

Let denote by ϵ_j, $j = 0, \ldots, \mu - 1$, the vanishing cycles instead that with γ_i, where $\epsilon_j = Y_j - \widehat{Y}$, that is: $\epsilon_j = \gamma_j$ for $1 \le j \le [\mu/2]$, $\epsilon_j = \gamma_{j+1}$ for $[\mu/2] + 1 \le j \le \mu - 1$ and $\epsilon_0 = \gamma_{[\mu/2]+1}$.

For simplicity, it is convenient to take a linear combination of the vanishing cycles ϵ_j. Let η_j be the cycle $\eta_j := \epsilon_j - \epsilon_0$ if $j > 0$ and $\eta_0 := \epsilon_0$. Notice that for $0 < j < \mu - 1$, $\eta_j = Y_j - Y_0$ (see Figure 7).

I can now compute the integrals of the GL wave forms over the vanishing cycles. Consider the case of α_{k_x,k_y} with $k_x = 1$, i.e. Y^{k_y-1}, integrated over the vanishing cycles η_j with $j > 0$, that is

$$\int_{\eta_j}\frac{\alpha_{1,k_y}}{dF} = \frac{\lambda_{1,k_y}}{2gc_1}\int_{\eta_j}\frac{Y^{k_y-1}}{XY}dY = \frac{\lambda_{1,k_y}}{2g\sqrt{c_1}}\int_{\eta_j}\frac{Y^{k_y-\frac{3}{2}}}{\sqrt{t-c_2Y^{\mu-1}}}dY \quad . \tag{7.33}$$

This is an integral in the complex plane from the origin to Y_j. Thus one has

$$\int_{\eta_j}\frac{\alpha_{1,k_y}}{dF} = -\frac{\lambda_{1,k_y}}{2g\sqrt{c_1}}\int_0^{Y_j}\frac{Y^{k_y-\frac{3}{2}}}{\sqrt{t-c_2Y^{\mu-1}}}dY \quad . \tag{7.34}$$

Make the change of variable

$$Y = \left(\frac{Z}{c_2}\right)e^{-i\pi\frac{(2j-1)}{(\mu-1)}} \quad , \tag{7.35}$$

then

$$\int_{\eta_j}\frac{\alpha_{1,k_y}}{dF} = -\frac{\lambda_{1,k_y}e^{i\pi/2}}{2g(\mu-1)\sqrt{c_1}}\left[\frac{e^{-i\pi\frac{(2j-1)}{(\mu-1)}}}{c_2^{\frac{1}{\mu-1}}}\right]^{k_y-\frac{1}{2}}\int_0^{|t|}Z^{\frac{k_y-1/2}{\mu-1}-1}(|t|-Z)^{\frac{1}{2}-1}dZ. \tag{7.36}$$

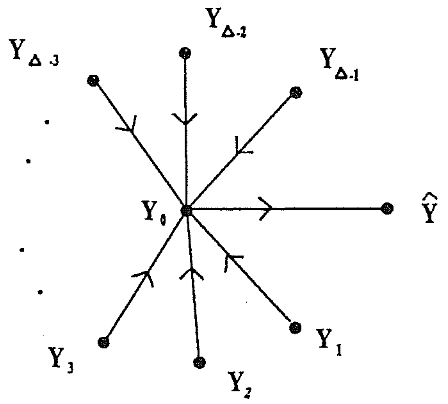

Figure 7
The cycles η_j

Using formula §3.191.1 [67] one gets

$$\int_{\eta_j} \frac{\alpha_{1,k_y}}{dF} = -\frac{\lambda_{1,k_y} e^{i\pi/2} e^{-i\pi\frac{(2j-1)}{(\mu-1)}(k_y-\frac{1}{2})} |t|^{\frac{k_y-1/2}{\mu-1}-\frac{1}{2}}}{2g(\mu-1)\sqrt{c_1}(c_2)^{\frac{k_y-1/2}{\mu-1}}} B\left(\frac{1}{2}, \frac{k_y-1/2}{\mu-1}\right) \tag{7.37}$$

where $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$.

Using formula (5.102) with $t=-1$ one gets

$$\int_{\Gamma_j} e^{-F}\alpha_{1,k_y} = -\frac{\lambda_{1,k_y} e^{i\pi/2} e^{-i\pi\frac{(2j-1)}{(\mu-1)}(k_y-\frac{1}{2})}}{2g(\mu-1)\sqrt{c_1}(c_2)^{\frac{k_y-1/2}{\mu-1}}} B\left(\frac{1}{2}, \frac{k_y-1/2}{\mu-1}\right)\cdot\Gamma\left(\frac{k_y-1/2}{\mu-1}+\frac{1}{2}\right) \tag{7.38}$$

and for the conjugate form

$$\left(\int_{\Gamma_j} e^{-\overline{F}}\tilde{\alpha}^*_{p(1,k_y)}\right)^* = -\frac{\tilde{\lambda}_{p(1,k_y)} e^{i\pi/2} e^{-i\pi\frac{(2j-1)}{(\mu-1)}(k_y-\frac{1}{2})}}{2g(\mu-1)\sqrt{c_1}(c)_2^{1-\frac{k_y-1/2}{\mu-1}}} B\left(\frac{1}{2}, 1-\frac{k_y-\frac{1}{2}}{\mu-1}\right)\cdot \tag{7.39}$$

$$\cdot\,\Gamma\left(\frac{3}{2}-\frac{k_y-1/2}{\mu-1}\right) \quad .$$

As in the A_μ case the "reality" equation is now very easy to interpret. Equating the two integrals one gets

$$\frac{\lambda_{1,k_y}}{\tilde{\lambda}_{1,\mu-k_y}} = \frac{\pi\,(c_2)^{\frac{2k_y-1}{\mu-1}-1}}{\sin\left(\pi\frac{k_y-1/2}{\mu-1}\right)\cdot\left(\Gamma\left(\frac{k_y-1/2}{\mu-1}\right)\right)^2} \tag{7.40}$$

and finally

$$(\lambda_{1,k_y})^2 = \frac{2g^{\frac{(\mu-2)}{(\mu-1)}k_x+\frac{2k_y}{\mu-1}}(\mu-1)\pi c_1(c_2)^{\frac{2k_y-1}{\mu-1}}}{(2\pi)^2\sin\left(\pi\frac{k_y-1/2}{\mu-1}\right)\cdot\left(\Gamma\left(\frac{k_y-1/2}{\mu-1}\right)\right)^2} \quad . \tag{7.41}$$

With these normalizations I can compute all the CGFs involving only Y fields:

$$\langle p(1,k_y+l)|Y^l|1,k_y\rangle = \frac{1}{(gL)^{\frac{l}{\mu-1}}}\cdot\frac{1}{\pi(gLc_2)^{\frac{l}{\mu-1}}}\cdot \cdot\sqrt{\sin\left(\pi\frac{k_y-1/2}{\mu-1}\right)\sin\left(\pi\frac{k_y+l-1/2}{\mu-1}\right)}\cdot \cdot\Gamma\left(1-\frac{k_y-1/2}{\mu-1}\right)\cdot\Gamma\left(\frac{k_y+l-1/2}{\mu-1}\right) \quad . \tag{7.42}$$

In the same way one can compute the absolute normalization of the vacuum germ form $\alpha_{2,1}$ which is associated to the monomial X of the basis of $\mathcal{R}$.

Following §2.2 and Appendix C, it is possible to check that these CGFs coincide (up to the normalization) with the correlation functions of the corresponding model in the D_μ series of the N=2 SCFT minimal models. From the argument of §2.2 , one has that the N=2 SCFT correlation functions should be given by

$$\frac{F_\mu(\underline{1};\underline{k}+\underline{h})}{F_\mu(\underline{1};\underline{k})\cdot F_\mu(\underline{1};\underline{h})} \tag{7.43}$$

where $F_\mu(\underline{m};\underline{k}) := \langle \underline{p}(\underline{m}+\underline{k})|X^{\underline{k}}|\underline{m}\rangle$. Let consider the Y^h case. One has

$$F^2(\underline{1};(0,h)) = \left[\frac{1}{(gLc_2)^{\frac{h}{\mu-1}}}\right]^2 \frac{\Gamma\left(1-\frac{1}{2(\mu-1)}\right)\cdot\Gamma\left(\frac{h+1/2}{\mu-1}\right)}{\Gamma\left(\frac{1}{2(\mu-1)}\right)\cdot\Gamma\left(1-\frac{h+1/2}{\mu-1}\right)} \quad . \tag{7.44}$$

Since

$$F(\underline{1};(0,h))\cdot\Sigma_D(\mu;(0,h)) = \frac{1}{(gLc_2)^{\frac{h}{\mu-1}}} \tag{7.45}$$

(see Appendix C), one has that

$$\frac{F_\mu(\underline{1};\underline{k}+\underline{h})}{F_\mu(\underline{1};\underline{k})\cdot F_\mu(\underline{1};\underline{h})} \tag{7.46}$$

is indeed equal to the N=2 SCFT correlation function $C_\mu(\underline{k},\underline{h})$.

In the same way one can check that also the correlation functions involving the X field do agree.

In this section I have explicitly computed all the CGFs of the A_μ and D_μ models and I have checked that they coincide (after a change of normalizations) with those of the corresponding N=2 SCFT models.

APPENDIX A

2D N=2 SUPERSYMMETRIC QUANTUM FIELD THEORY

In this appendix I will make a very brief review of 2d N=2 supersymmetric field theory just to state the conventions adopted in this thesis. For some general references on supersymmetry (mainly in four dimensions) see [68,69]. I have quoted many references on four dimensional N=1 supersymmetry since, formally, N=2 supersymmetry in two dimensions can be seen as the dimensional reduction of 4d N=1 supersymmetry. Many formulas are identical in 4d N=1 and 2d N=2 supersymmetry.

The 2d N=2 supersymmetric theories I consider are defined on an Euclidean space. To pass from a Minkowskian to an Euclidean space one can do the usual analytic continuation in time. For example, if the Minkowskian metric is $\eta_{\mu\nu} = (-1,1)$, one should make the following substitutions to get the Euclidean theory: $\eta_{\mu\nu} \to \delta_{\mu\nu}$, $t \to -i\,t$, $\gamma^0 \to i\gamma^0$.

Let $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$, $\mu,\nu = 1,2$ be the Dirac algebra. Let α,β be the spinor indices ($\alpha,\beta = 1,2$). I choose the gamma matrices as follows: $\gamma_\mu = (\sigma^1,\sigma^2)$ where σ^i are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{A.1}$$

It turns out that

$$\gamma_3 = -i\gamma_1\gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^3 \quad . \tag{A.2}$$

A 2d Dirac spinor is a two component spinor with complex entries. The Dirac conjugate of a spinor χ is given by $\overline{\chi} := \chi^\dagger\sigma^1 = (\chi_2^*, \chi_1^*)$. The conjugate spinor is defined by $\chi^c := C\overline{\chi}$ and $\overline{\chi}^c := C^{-1}\chi$ where

$$C := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^2 \quad . \tag{A.3}$$

A 2d Majorana spinor satisfies $\chi^c = \chi$ which implies $\chi_1 = \chi_1^*$ and $\chi_2 = -\chi_2^*$. A 2d Weyl spinor is defined by $\chi = \gamma_3\chi$ and a 2d anti–Weyl spinor by $\chi = -\gamma_3\chi$. From a 2d Dirac spinor one can get two 2d Weyl spinors using the chiral projectors $\frac{1}{2}(1 \pm \gamma_3)$. Thus,

$$(\chi)_+ = \frac{1+\gamma_3}{2}\chi = \begin{pmatrix} \chi_1 \\ 0 \end{pmatrix} \quad \text{and} \quad (\chi)_- = \frac{1-\gamma_3}{2}\chi = \begin{pmatrix} 0 \\ \chi_2 \end{pmatrix} \quad . \tag{A.4}$$

Notice, moreover, that $(\overline{\chi})_+ = (\chi_2^*, 0)$ and $(\overline{\chi})_- = (0, \chi_1^*)$.

The minimal supersymmetric extension of the Poincarè algebra can be constructed using only one supersymmetry charge (N=1 supersymmetry) which is a 2d Weyl or 2d Majorana spinor.

In the 2d N=2 supersymmetry algebra the supercharge is a 2d Dirac spinor, Q_α. The algebra is given by

$$\begin{aligned} \{Q_\alpha, \overline{Q}_\beta\} &= -2i(\gamma^\mu)_{\alpha\beta}P_\mu \qquad & [Q_\alpha, R] &= \gamma_3 Q_\alpha \\ \{Q_\alpha, Q_\beta\} &= 0 = \{\overline{Q}_\alpha, \overline{Q}_\beta\} \qquad & [Q_\alpha, P_\mu] &= 0 \end{aligned} \tag{A.5}$$

plus the usual commutators of the Poincaré group. Here R is the chiral rotation, or R–charge, and the momentum operator is $P_\mu := \frac{\partial}{\partial x_\mu}$, $\mu = 1,2$. Notice that in the field theory conventions the supercharges Q_α have R–charge 1. This must be kept in mind when one has to make the connection with the 2d N=2 Superconformal models, see Appendix C.

It is convenient to express the N=2 supersymmetry algebra in a more convenient way in which, moreover, it is immediately clear that it is N=2. This convention has been more often adopted in the thesis. Let χ_+ be the first component of a 2d Dirac spinor and χ_- the second one, that is

$$\chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} \quad . \tag{A.6}$$

In other words, $\chi_+ = \chi_1$ and $\chi_- = \chi_2$. χ_+ and χ_- are the non zero components of the 2d (anti–) Weyl spinors in which a 2d Dirac spinor can be decomposed. Thus, the N=2 supersymmetry algebra can be rewritten as

$$\{Q_\pm, Q_\pm^*\} = 2P_\pm \qquad [Q_\pm, R] = \pm Q_\pm \tag{A.7}$$

where

$$\begin{aligned} P_+ &:= -i\frac{\partial}{\partial z} = -i\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right) \\ P_- &:= -i\frac{\partial}{\partial \overline{z}} = -i\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) \quad . \end{aligned} \tag{A.8}$$

The most general scalar multiplet which is a representation of the N=2 supersymmetry algebra is

$$(\phi(x), \chi(x), F(x)) \tag{A.9}$$

where ϕ and F are two complex bosonic fields and χ is a 2d Dirac fermion. In superfield notation it can be written as

$$\Phi(x, \theta_+, \theta_-) = \phi(x) + \theta_-^*\psi(x) + \theta_+^*\widetilde{\psi}(x) + \theta_-^*\theta_+ G(x) + \theta_+^*\theta_- iH(x) \tag{A.10}$$

where the last conventions have been used, ψ, $\widetilde{\psi}$ are the components of the 2d Dirac fermion χ and $F(x) = G(x) + iH(x)$.

As in 4d N=1 supersymmetry, one can impose a constraint on the general superfield Φ and obtain a minimal representation of the 2d N=2 supersymmetry algebra. The **chiral** multiplet, or superfield, satisfies

$$[Q_\pm^*, \Phi_c] = 0 \tag{A.11}$$

and the **antichiral** one $[Q_\pm, \Phi_{ac}] = 0$. Obviously, a chiral superfield has half the degrees of freedom of the general (unconstrained) superfield Φ. Φ_c is composed by a complex scalar ϕ and a 2d Weyl fermion ψ.

APPENDIX B

N=2 SUPERSYMMETRIC QUANTUM MECHANICS

In this appendix I will give a brief review of N=2 supersymmetric quantum mechanics and explicitly do some computation needed in §1.5 . Then I will explicitly give the translation between the formalism used in quantum mechanics and that used in differential geometry.

B.1 N=2 SQM, definition and conventions

The N=2 supersymmetric quantum mechanics is a quantum mechanical system which is the realization of the following operator algebra [20,11]

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= \{\overline{Q}_\alpha, \overline{Q}_\beta\} = 0 \\ \{Q_\alpha, \overline{Q}_\beta\} &= 2\delta_{\alpha,\beta} H \end{aligned} \tag{B.1}$$

where $\{Q_\alpha\}$, $\alpha = 1, 2$ are two *complex* [35] supersymmetry charges. These models are conveniently written in term of a complex bosonic coordinate z and two–component complex fermion $(\psi)_\alpha$ (α is the spinor index). The lagrangian is

$$\begin{aligned} \mathcal{L} = &\dot{\bar{z}}\dot{z} - \overline{W}' W' + i\bar{\psi}_\alpha \dot{\psi}_\alpha + \\ &+ \tfrac{1}{2}\epsilon_{\alpha\beta}\left[W'' \psi_\alpha \psi_\beta - \overline{W}'' \bar{\psi}_\alpha \bar{\psi}_\beta\right] \end{aligned} \tag{B.2}$$

where the superpotential $W(z)$ is an (a priori) arbitrary holomorphic function of only z, $W' = \frac{\partial W}{\partial z}$, $W'' = \frac{\partial^2 W}{\partial z^2}$ and $\epsilon_{12} = -\epsilon_{21} = 1$.

The supersymmetric charges are given by

$$\begin{aligned} Q_\alpha &= \sqrt{2}\left(\psi_\alpha p + i\epsilon_{\alpha\beta}\bar{\psi}_\beta \overline{W}'\right) \\ \overline{Q}_\alpha &= \sqrt{2}\left(\bar{\psi}_\alpha \bar{p} - i\epsilon_{\alpha\beta}\psi_\beta W'\right) \end{aligned} \tag{B.3}$$

[35] In some literature this is called N=4 supersymmetric quantum mechanics since the number of real supersymmetry charges is counted.

in which $p = \dot{\bar{z}}$ and $\bar{p} = \dot{z}$ are the momenta conjugate to z and $\bar{z}$, respectively. Using the canonical algebra

$$[z,p] = [\bar{z},\bar{p}] = i \qquad \{\psi_\alpha, \bar{\psi}_\beta\} = \delta_{\alpha,\beta} \tag{B.4}$$

it is easy to verify that the supersymmetry algebra (B.1) is satisfied with the Hamiltonian

$$H = \bar{p}p + \overline{W}'W' - \frac{1}{2}\epsilon_{\alpha\beta}\left[W''\psi_\alpha\psi_\beta - \overline{W}''\bar{\psi}_\alpha\bar{\psi}_\beta\right] \quad . \tag{B.5}$$

Since $H\delta_{\alpha,\beta} = \frac{1}{2}\{Q_\alpha, \overline{Q}_\beta\}$, the energy spectrum is semi–positive definite and the zero energy states are those that satisfy $H|\omega\rangle = 0$ or $Q_\alpha|\omega\rangle = 0 = \overline{Q}_\alpha|\omega\rangle$ [20, Witten in ref. 15].

Let $|0\rangle$ be the product of the Schroedinger standard ket with the Clifford vacuum for fermions such that ψ_α are annihilation operators ($\psi_\alpha|0\rangle = 0$) and $\bar{\psi}_\alpha$ are creation operators. A general wave function is

$$|\omega\rangle = f(z,\bar{z})_{\alpha\beta}\bar{\psi}^\alpha\bar{\psi}^\beta|0\rangle + f_\alpha(z,\bar{z})\bar{\psi}^\alpha|0\rangle + f(z,\bar{z})|0\rangle \quad . \tag{B.6}$$

Following Claudson and Halpern [11] it is easy to show that the vacua (or ground states or zero energy states) are bosonic. All explicit computations can be easily done using these conventions, but to be more close to the conventions used in field theory and to easily translate supersymmetry in the *cohomological* language, I will use a totally different convention.

Indeed, the naming of fermionic and bosonic states as used in this contest is at least merely a convention. If one relabels the fermions by $\psi_2 \leftrightarrow \bar{\psi}_2$ throughout the model, the names "bosonic" and "fermionic" would be interchanged. In particular the Hamiltonian becomes

$$H = \bar{p}p + \overline{W}'W' + W''\bar{\psi}_2\psi_1 + \overline{W}''\bar{\psi}_1\psi_2 \tag{B.7}$$

and the supersymmetric vacua are now fermionic, i.e. they take the form $f_\alpha(z,\bar{z})\bar{\psi}^\alpha|0\rangle$.

Let me introduce the conventions I use throughout this thesis. Consider a model with n complex coordinates $(z_i, \bar{z}_i)$, $i = 1,\ldots,n$ and canonical kinetic terms (for the moment I restrict myself to this case following the considerations done in §1.4). Each supermultiplet has a Dirac fermion, its components will be denoted in the following way: the two creator fermionic operators are ψ^i and $\psi^{\bar{i}}$ (upper index = creator) and the two annihilation fermionic operators are ψ_i and $\psi_{\bar{i}}$ (lower index = annihilator). This convention is chosen since the supersymmetry transformations suggest to see the creation operators as forms on M and the annihilation operators as vectors. The basic commutation relation reads as the natural pairing of vectors and forms, $\{\psi^i, \psi_j\} = \delta^i_j$, and similarly for barred indices. The general wave function reads

$$|\omega\rangle = \sum_{(p,q)} f(z_l,\bar{z}_l)_{i_1,\ldots,i_p,\bar{i}_1,\ldots,\bar{i}_q}\psi^{i_1}\ldots\psi^{i_p}\psi^{\bar{i}_1}\ldots\psi^{\bar{i}_q}|0\rangle \quad . \tag{B.8}$$

In these conventions the supersymmetry charges (B.3) read

$$\begin{aligned}
\overline{Q}_1 &= \sqrt{2}\left[\psi^{\bar{i}}p_{\bar{i}} - i\psi^i\partial_i V\right]\\
\overline{Q}_2 &= \sqrt{2}\left[\psi^i p_i - i\psi^{\bar{i}}\partial_{\bar{i}}\overline{V}\right]\\
Q_1 &= \sqrt{2}\left[\psi_{\bar{i}}g^{\bar{i}i}p_i + i\psi_i g^{i\bar{i}}\partial_{\bar{i}}\overline{V}\right]\\
Q_2 &= \sqrt{2}\left[\psi_i g^{i\bar{i}}p_{\bar{i}} + i\psi_{\bar{i}}g^{\bar{i}i}\partial_i V\right]
\end{aligned} \tag{B.9}$$

where $p_i = -i\frac{\partial}{\partial z_i}$, $p_{\bar{i}} = -i\frac{\partial}{\partial \bar{z}_{\bar{i}}}$, $V(z)$ is the superpotential and $g^{i\bar{i}}$ is (for the moment) the diagonal metric.

The supersymmetric algebra (B.1) reads

$$\begin{aligned} \{Q_a, Q_b\} &= 0 = \{\overline{Q}_a, \overline{Q}_b\} \\ \{\overline{Q}_a, Q_b\} &= 2\delta_{a,b} H \end{aligned} \tag{B.10}$$

where $a, b = 1, 2$ and the Hamiltonian (B.7) reads

$$H = g^{\bar{i}i} p_{\bar{i}} p_i + g^{\bar{i}i}\partial_{\bar{i}}\overline{V}\partial_i V + g^{i\bar{j}}\partial_{\bar{i}}\partial_{\bar{j}}\overline{V}\psi^{\bar{i}}\psi_i + g^{\bar{i}j}\partial_i\partial_j V\psi^i\psi_{\bar{j}} \quad . \tag{B.11}$$

Let me introduce some other useful operators. Let F be the *naive* fermion number operator, i.e.

$$F := \psi^{\bar{i}}\psi_{\bar{i}} + \psi^i\psi_i \tag{B.12}$$

and N a *new* fermion number operator

$$N := \psi^{\bar{i}}\psi_{\bar{i}} + \psi_i\psi^i = F - n \tag{B.13}$$

where n is the number of coordinates (of superfields in Field Theory). In the usual quantum mechanical conventions a state is called fermionic if $(-)^F|\omega\rangle = -|\omega\rangle$ and bosonic if $(-)^F|\omega\rangle = +|\omega\rangle$. In my conventions, this definition of "bosonic" and "fermionic" states still holds but using the "new" fermion number operator, that is a state is "fermionic" if $(-)^N|\omega\rangle = -|\omega\rangle$ and "bosonic" if $(-)^N|\omega\rangle = +|\omega\rangle$. Notice that in the case of one coordinate $n = 1$, $|\omega\rangle := f_a(z,\bar{z})\psi^a|0\rangle$ $(a = (1,\bar{1}))$ is a "bosonic" state and $|\eta\rangle := f(z,\bar{z})|0\rangle + f_{ab}(z,\bar{z})\psi^a\psi^b|0\rangle$ is a "fermionic" state. Notice moreover that N (like F) commutes with H and

$$[N, Q_a] = -Q_a \qquad [N, \overline{Q}_a] = +\overline{Q}_a \quad . \tag{B.14}$$

Let me introduce now other two operators, called the Lefschetz–SU(2) operators

$$\begin{aligned} \Lambda &:= g^{\bar{i}i}\psi_{\bar{i}}\psi_i \\ L &:= g_{i\bar{i}}\psi^i\psi^{\bar{i}} \quad . \end{aligned} \tag{B.15}$$

They satisfy

$$\begin{array}{lll} [L,\Lambda] = N & [N,L] = 2L & [N,\Lambda] = -2\Lambda \\ [\overline{Q}_1,\Lambda] = Q_2 & [\overline{Q}_1, L] = 0 & [\overline{Q}_2,\Lambda] = -Q_1 \\ [\overline{Q}_2, L] = 0 & [Q_1, L] = -\overline{Q}_2 & [Q_1,\Lambda] = 0 \\ [Q_2, L] = \overline{Q}_1 & [Q_2,\Lambda] = 0 & \\ [H, L] = 0 & [H,\Lambda] = 0 & \end{array} \tag{B.16}$$

B.2 Explicit computation of the wave functions

I turn now to the explicit computation of the wave functions in the case of models with one coordinate ($n = 1$). I want first to obtain the explicit expression of the massless and massive states.

The massless states $|\omega\rangle$ satisfy the equations

$$\overline{Q}_1|\omega\rangle = 0 = Q_1|\omega\rangle \qquad \overline{Q}_2|\omega\rangle = 0 = Q_2|\omega\rangle \quad . \tag{B.17}$$

Let a general state $|k\rangle$ be of the form

$$|k\rangle = f(z,\bar{z})|0\rangle + f_1(z,\bar{z})\psi^1|0\rangle + \overline{f}_{\bar{1}}(z,\bar{z})\psi^{\bar{1}}|0\rangle + f_{1\bar{1}}(z,\bar{z})\psi^1\psi^{\bar{1}}|0\rangle \quad . \tag{B.18}$$

Then, applying the four equations (B.17) to the general state $|k\rangle$ one gets the following equations for the coefficients

$$\begin{aligned} &f = 0 \\ &\partial\overline{f}_{\bar{1}} - f_1\overline{\partial V} = 0 \\ &\overline{\partial} f_1 - \overline{f}_{\bar{1}}\partial V = 0 \\ &f_{1\bar{1}} = 0 \end{aligned} \tag{B.19}$$

(where the indices have been suppressed where this does not make confusion). Thus a massless state $|\omega\rangle$ has the form

$$|\omega\rangle = f_1(z,\bar{z})\psi^1|0\rangle + \overline{f}_{\bar{1}}(z,\bar{z})\psi^{\bar{1}}|0\rangle \quad . \tag{B.20}$$

As it was expected it is a "bosonic" state.

Non–zero energy states appear in quartets. Indeed a general non zero energy state has the following expression [36] ($E > 0$)

$$|E\rangle = f(z,\bar{z})|0\rangle \quad . \tag{B.21}$$

The quartet is given by the following states: $|E\rangle$, $\overline{Q}_1|E\rangle$, $\overline{Q}_2|E\rangle$ and $\overline{Q}_1\overline{Q}_2|E\rangle$, explicitly

$$\begin{aligned} |E\rangle &= f(z,\bar{z})|0\rangle \\ \overline{Q}_1|E\rangle &= \left(-i\overline{\partial} f\psi^{\bar{1}} - i\partial V\cdot f\psi^1\right)|0\rangle \\ \overline{Q}_2|E\rangle &= \left(-i\partial f\psi^1 - i\overline{\partial V}\cdot f\psi^{\bar{1}}\right)|0\rangle \\ \overline{Q}_1\overline{Q}_2|E\rangle &= \left(\partial\overline{\partial} f - \partial V\overline{\partial V}\right)\psi^1\psi^{\bar{1}}|0\rangle \quad . \end{aligned} \tag{B.22}$$

The Schroedinger equation $H|E\rangle = E|E\rangle$ on the wave function f becomes

$$\left(-\partial\overline{\partial} + \partial V\overline{\partial V}\right) f = Ef \quad . \tag{B.23}$$

[36] There is another equivalent possibility: $|E\rangle = f_{1\bar{1}}(z,\bar{z})\psi^1\psi^{\bar{1}}|0\rangle$. With this choice the role of $\overline{Q}_a$ and Q_a and that of $f_{1\bar{1}}$ and f are interchanged.

Let me go back to study more deeply the equations (B.19) for the wave functions f_1 and $\overline{f}_{\overline{1}}$ in the massless case. Make the following change of variables

$$f_1 = \beta \partial V \qquad \overline{f}_{\overline{1}} = \alpha \overline{\partial V} \quad . \tag{B.24}$$

The equations (B.19) become

$$\begin{aligned} &\overline{\partial V}\left[\partial\alpha - \beta\partial V\right] = 0 \\ &\partial V\left[\overline{\partial\beta} - \alpha\overline{\partial V}\right] = 0 \quad . \end{aligned} \tag{B.25}$$

Acting formally, one can reduce this system to the following one

$$\begin{aligned} &\partial\alpha - \beta\partial V = 0 \\ &\left(\partial\overline{\partial} - \partial V\overline{\partial V}\right)\alpha = 0 \end{aligned} \tag{B.26}$$

If a "good" solution for α is known, one can recover f_1 and $\overline{f}_{\overline{1}}$ by

$$f_1 = \partial\alpha \qquad \overline{f}_{\overline{1}} = \alpha\overline{\partial V} \quad . \tag{B.27}$$

I will now study the equation for α

$$\left(\partial\overline{\partial} - \partial V\overline{\partial V}\right)\alpha = 0 \quad . \tag{B.28}$$

Solutions of this equation are zero energy solutions to an ordinary bosonic hamiltonian with positive potential, and in general one does not expect the solution to such an equation to be well–behaved. On the other hand, I am not directly concerned with singularities in α since in the wave function f_1 and $\overline{f}_{\overline{1}}$ appear. Thus, I will require that f_1 and $\overline{f}_{\overline{1}}$ are analytic in z and $\bar{z}$, i.e. I demand that f_1 and $\overline{f}_{\overline{1}}$ have not branch cuts nor singularities and that they are single valued for $|\arg z| \leq \pi$. Moreover, I will formally (as I have done until now) act as if $\partial V \neq 0$. Thus, I can make the change of variable $z \to V(z)$ in eq. (B.28) , obtaining

$$\left(-\frac{\partial^2}{\partial V\overline{\partial V}} + 1\right)\alpha = 0 \tag{B.29}$$

so that the problem is mapped onto that of a free boson with energy $E = -1$. This problem has a trivial "plane–wave" solution $\alpha \sim e^V$. However, these solutions are ill–behaved for large $|V|$. The correct procedure is to go into polar coordinates $V = Re^{i\Phi}$, the equation becomes

$$\left[R^2\frac{\partial^2}{\partial R^2} + R\frac{\partial}{\partial R} + \frac{\partial^2}{\partial\Phi^2} - 4R^2\right]\alpha = 0 \quad . \tag{B.30}$$

Letting $\alpha(R,\Phi) = e^{im\Phi}\alpha_m(R)$, it becomes

$$\left[R^2\frac{\partial^2}{\partial R^2} + R\frac{\partial}{\partial R} - (m^2 + 4R^2)\right]\alpha_m(R) = 0 \quad . \tag{B.31}$$

From formula 8.491.2 and 8.494.1, page 971 – 972 of [67] one gets

$$\alpha_m(R,\Phi) = e^{im\Phi} K_m(2R) = \left(\frac{V}{\overline{V}}\right)^{\frac{m}{2}} K_m\left(2\sqrt{V\overline{V}}\right) \tag{B.32}$$

where m is arbitrary and where I have chosen the Macdonald function solution $K_\nu(x)$ since it is well–behaved for large $|V|$. The allowed values of m are selected from the requirement that the wave functions

$$\begin{aligned} f_1 &= -\partial V \left(\frac{\overline{V}}{V}\right)^{\frac{1-m}{2}} K_{|1-m|}\left(2\sqrt{V\overline{V}}\right) \\ \overline{f}_{\overline{1}} &= \overline{\partial V} \left(\frac{V}{\overline{V}}\right)^{\frac{m}{2}} K_m\left(2\sqrt{V\overline{V}}\right) \end{aligned} \tag{B.33}$$

are free of singularities. For a polynomial superpotential the only possible singularities occur at the zeros z_i of V. Near one of the zeros, the leading behaviour of f_1 and $\overline{f}_{\overline{1}}$ is

$$\begin{aligned} f_1 &\sim \begin{cases} -\partial V \cdot V^{-(1-m)} & m<1 \\ -\partial V \cdot \overline{V}^{-|1-m|} & m>1 \end{cases} \\ \overline{f}_{\overline{1}} &\sim \begin{cases} \overline{\partial V} \cdot \overline{V}^{-m} & m>0 \\ \overline{\partial V} \cdot V^{-|m|} & m<0 \end{cases} \end{aligned} \tag{B.34}$$

and to avoid singularities it is necessary to choose m to lie in the range $0 < m < 1$.

Let consider an explicit example, that is the simplest superpotential $V = \frac{1}{\mu+1} z^{\mu+1}$ (A_μ models). Let $z = \rho e^{i\theta}$, then $V = \frac{1}{\mu+1}\rho^{\mu+1} e^{i(\mu+1)\theta}$, which means

$$R = \frac{\rho^{\mu+1}}{\mu+1} = \sqrt{V\overline{V}} \qquad e^{i\Phi} = e^{i(\mu+1)\theta} = \left(\frac{V}{\overline{V}}\right)^{\frac{1}{2}} \quad . \tag{B.35}$$

It follows that

$$\begin{aligned} f_1 &= -\rho^\mu e^{i(m\mu+m-1)\theta} K_{|1-m|}\left(2\frac{\rho^{\mu+1}}{\mu+1}\right) \\ \overline{f}_{\overline{1}} &= \rho^\mu e^{i(m\mu+m-\mu)\theta} K_m\left(2\frac{\rho^{\mu+1}}{\mu+1}\right) \quad . \end{aligned} \tag{B.36}$$

Now, under $\theta \to \theta + 2\pi$ the wave functions must go into themselves since they should be univalued in z as to have a chance to be regular and normalizable. This is possible if and only if $m = k/(\mu+1)$ with k integer.

Let consider now the regularity at the origin $z = 0$ which is the only singular point. Since

$$K_\nu\left(\rho^{\mu+1}\right) \underset{\rho\to 0}{\sim} \left(\rho^{\mu+1}\right)^{-\nu} \tag{B.37}$$

then

$$\begin{aligned} f_1 &\sim \rho^\mu \rho^{-|1-m|(\mu+1)} \\ \overline{f}_{\overline{1}} &\sim \rho^\mu \rho^{-m(\mu+1)} \end{aligned} \tag{B.38}$$

whose solution is $1 \le k \le \mu$. This implies that there are exactly μ regular solutions of the equations. These solutions give the $\mu = \Delta$ vacuum wave functions that I was looking for. The non–normalized vacuum wave functions are

$$\begin{aligned}|k\rangle = A_k \Bigg(& e^{i(k-\mu)\theta}\rho^\mu K_{\frac{k}{\mu+1}}\left(2\frac{\rho^{\mu+1}}{\mu+1}\right)\psi^{\bar{1}}|0\rangle + \\ & - e^{i(k-1)\theta}\rho^\mu K_{1-\frac{k}{\mu+1}}\left(2\frac{\rho^{\mu+1}}{\mu+1}\right)\psi^1|0\rangle\Bigg) \quad .\end{aligned} \tag{B.39}$$

The next thing to do is to normalize these vacuum wave functions. This is possible with the help of the following integral (from [67] 6.576.4)

$$\begin{aligned}&\int_0^\infty \rho d\rho \cdot \rho^{2\mu}\rho^k K_{\frac{p}{\mu+1}}\left(\frac{2\rho^{\mu+1}}{\mu+1}\right) K_{\frac{q}{\mu+1}}\left(\frac{2\rho^{\mu+1}}{\mu+1}\right) = \\ &= \frac{[(\mu+1)]^{1+\frac{k}{\mu+1}}}{8\Gamma(2+\frac{k}{\mu+1})}\Gamma\left(1+\frac{k+p+q}{2(\mu+1)}\right) \cdot \\ &\cdot \Gamma\left(1+\frac{k-p+q}{2(\mu+1)}\right)\Gamma\left(1+\frac{k+p-q}{2(\mu+1)}\right)\Gamma\left(1+\frac{k-p-q}{2(\mu+1)}\right) \quad .\end{aligned} \tag{B.40}$$

The normalization condition is

$$\langle p|q\rangle = \delta_{p,q} \quad . \tag{B.41}$$

Since

$$\begin{aligned}\langle p| = A_p \Bigg(& \langle 0|\psi_{\bar{1}} e^{-i(p-\mu)\theta}\rho^\mu K_{\frac{p}{\mu+1}}\left(2\frac{\rho^{\mu+1}}{\mu+1}\right) + \\ & - \langle 0|\psi_1 e^{-i(p-1)\theta}\rho^\mu K_{1-\frac{p}{\mu+1}}\left(2\frac{\rho^{\mu+1}}{\mu+1}\right)\Bigg)\end{aligned} \tag{B.42}$$

one gets

$$\begin{aligned}\langle p|q\rangle = \int dz d\bar{z} \Bigg(& \rho^{2\mu} e^{i(q-p)\theta} K_{\frac{p}{\mu+1}}\left(2\frac{\rho^{\mu+1}}{\mu+1}\right) K_{\frac{q}{\mu+1}}\left(2\frac{\rho^{\mu+1}}{\mu+1}\right) + \\ & + \rho^{2\mu} e^{i(q-p)\theta} K_{1-\frac{p}{\mu+1}}\left(2\frac{\rho^{\mu+1}}{\mu+1}\right) K_{1-\frac{q}{\mu+1}}\left(2\frac{\rho^{\mu+1}}{\mu+1}\right)\Bigg) \\ = & A_q^2 \delta_{p,q} \frac{\pi^2(\mu+1)}{2\sin\left(\frac{\pi q}{\mu+1}\right)}\end{aligned} \tag{B.43}$$

which means

$$A_q = \frac{1}{\pi}\sqrt{\frac{2\sin\left(\frac{\pi q}{\mu+1}\right)}{\mu+1}} \quad . \tag{B.44}$$

In the same way one can compute che CGFs which turn out to be

$$\begin{aligned}\langle m_2|z^k|m_1\rangle = \frac{1}{\pi}\delta_{k,(m_2-m_1)} & \sqrt{\sin\left(\frac{\pi m_2}{\mu+1}\right)\sin\left(\frac{\pi m_1}{\mu+1}\right)} \cdot \\ & \cdot (\mu+1)^{\frac{k}{\mu+1}}\Gamma\left(\frac{m_2}{\mu+1}\right)\Gamma\left(1-\frac{m_1}{\mu+1}\right) \quad .\end{aligned} \tag{B.45}$$

The same exercise can be carried out for the superpotential $V = z^{\mu+1} + z^{m+1}$, $m > \mu$. One can easily compute μ vacuum wave functions which correspond to the wave functions of the critical theory $V = z^{\mu+1}$ obtained in the IR limit. However, there are other $m-\mu$ critical points different from zero which should give rise in the UV critical limit to the missing $m-\mu$ vacuum wave functions. In general, it turns out that only the wave functions dominant in the IR limit are computable with these techniques. Indeed, only in the case just discussed one is able to explicitly solve the Schroedinger equations for the wave functions. In other cases, it is possible to compute only some of the Δ vacuum wave functions and just *at criticality*.

B.3 Some considerations on the vacua of N=2 SQM

After having considered the $n = 1$ case, I go back to (B.16) and to the Lefschetz SU(2) operators. The operators N, L, Λ act as the generators of a representation of a conserved SU(2) since they all commute with H. From the commutation relations and its definition one can see Λ as an annihilator operator and L as a creation operator, and one can organize the eigenstate of H in representations of this SU(2) starting from the states $|\gamma\rangle$ which satisfy

$$\Lambda|\gamma\rangle = 0 \tag{B.46}$$

Let start considering the first states, that is those satisfying also to

$$L|\gamma\rangle = 0 \tag{B.47}$$

From (B.16) it follows that also $N|\gamma\rangle = 0 = (F-n)|\gamma\rangle$.

Consider the case of one coordinate z, then the conditions $\Lambda|\gamma\rangle = 0 = L|\gamma\rangle$ when applied on the general wave function (B.18) imply $f = 0 = f_{1\bar{1}}$. This is the same result coming from imposing the condition $N|\gamma\rangle = 0$. Indeed, if $|\eta\rangle$ is such that $\Lambda|\eta\rangle = 0$ and $L|\eta\rangle \neq 0$, since $N|\eta\rangle \neq 0$ it follows that $|\eta\rangle$ cannot be massless.

A similar result can be obtained for an arbitrary number n of coordinates z_i. Let me first recall the following result which has been proven in §4.3 :

(B.48) : *the vacua $|\omega\rangle$, i.e. the states which satisfy $H|\omega\rangle = 0$ or $\overline{Q}_a|\omega\rangle = 0 = Q_a|\omega\rangle$, have eigenvalue zero of the "new" Fermi number operator, that is*

$$N|\omega\rangle = 0 \quad . \tag{B.49}$$

The previous remark on the role of Lefschetz SU(2) in the $n = 1$ case leads to consider more deeply the connection between supersymmetry charges and the Lefschetz SU(2) operators Λ, L. Since the operator $\overline{Q}_a, Q_a$ and Λ, L do not commute (see (B.16)) it is not possible to organize the tower of states in common eigenstates.

Anyway, the following result will be very useful:

a vacuum state is completely determined by the following equations $N|\omega\rangle = 0$, $\Lambda|\omega\rangle = 0$, $Q_a^\dagger|\omega\rangle = 0$, or

$$|\omega\rangle \text{ vacuum state} \iff N|\omega\rangle = 0,\ \Lambda|\omega\rangle = 0,\ \overline{Q}_a|\omega\rangle = 0 \quad . \tag{B.50}$$

Proof : ($\Rightarrow$) I will first prove that if $|\omega\rangle$ is a vacuum state then $N|\omega\rangle = \Lambda|\omega\rangle = \overline{Q}_a|\omega\rangle = 0$. Statement (B.48) implies that $N|\omega\rangle = 0$. From (B.16) it follows that $\overline{Q}_a\Lambda|\omega\rangle = 0$ and $Q_a\Lambda|\omega\rangle = 0$, thus $\Lambda|\omega\rangle$ should be a vacuum state, but $N\Lambda|\omega\rangle \neq 0$ implying that $\Lambda|\omega\rangle = 0$. ($\Leftarrow$) I will prove now that if $N|\omega\rangle = \Lambda|\omega\rangle = \overline{Q}_a|\omega\rangle = 0$ then $|\omega\rangle$ is a vacuum state. From (B.16) we know that $Q_a = \epsilon_{ab}[\overline{Q}_b, \Lambda]$ where $\epsilon_{21} = -\epsilon_{12} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$. Thus $Q_a|\omega\rangle = 0$ and then $H|\omega\rangle = 0$. ∎

This way of expressing the equations defining a vacuum state is very convenient for the translation of the SQM problem into the language of differential geometry.

B.4 N=2 SQM and differential geometry

To conclude this appendix I shall discuss the connection between the formalism used until now and that used in differential geometry. The first observation is that the supersymmetry algebra

$$\begin{aligned} &\{Q_a, Q_b\} = 0 = \{\overline{Q}_a, \overline{Q}_b\} \\ &\{\overline{Q}_a, Q_b\} = 2\delta_{a,b}H \end{aligned} \tag{B.51}$$

is the same algebra as that satisfied by the exterior derivatives ∂, $\bar{\partial}$ and their adjoint operators δ and $\bar{\delta}$ in complex differential geometry:

$$\Box = \{\bar{\partial}, \delta\} = \{\partial, \bar{\delta}\} \tag{B.52}$$

with the obvious correspondence $\Box \leftrightarrow 2H$, $\bar{\partial} \leftrightarrow \overline{Q}_1$, $\delta \leftrightarrow Q_1$, $\partial \leftrightarrow \overline{Q}_2$, $\bar{\delta} \leftrightarrow Q_2$. Notice moreover that in a flat space (following the conventions of [70])

$$\Box = -2\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} \tag{B.53}$$

and that the first term of the Hamiltonian is

$$H = -\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} + \dots \quad . \tag{B.54}$$

To make a more precise correspondence one should look more closely to these operators.

Let $|\alpha\rangle_{pq}$ be a generic state with p ψ^i operators and q $\psi^{\bar{i}}$ operators acting on the Clifford vacuum [37]

$$|\alpha\rangle_{pq} := \sum_{\substack{0\leq i_1 \leq \dots \leq i_p \leq n \\ 0 \leq \bar{i}_1 \leq \dots \leq \bar{i}_q \leq n}} f_{i_1 \dots i_p \bar{i}_1 \dots \bar{i}_q} \psi^{i_1} \dots \psi^{i_p} \psi^{\bar{i}_1} \dots \psi^{\bar{i}_q} |0\rangle \quad . \tag{B.55}$$

[37] The choice of convention on the sum is the simplest one for what follows.

Now

$$Q_1^\dagger|\alpha\rangle_{pq} = -i\sqrt{2} \sum_{\substack{0\le i_1\le\dots\le i_p\le n \\ 0\le \bar{i}_1\le\dots\le \bar{i}_q\le n}} \Big[(-)^p \partial_{\bar{i}} f_{i_1\dots i_p \bar{i}_1\dots\bar{i}_q} \psi^{i_1}\dots\psi^{i_p}\psi^{\bar{i}}\psi^{\bar{i}_1}\dots\psi^{\bar{i}_q}|0\rangle + \\ +\partial_i V f_{i_1\dots i_p \bar{i}_1\dots\bar{i}_q}\psi^i\psi^{i_1}\dots\psi^{i_p}\psi^{\bar{i}_1}\dots\psi^{\bar{i}_q}|0\rangle\Big] \quad . \tag{B.56}$$

Introduce (following again the conventions of [70]) the operators

$$\begin{array}{ll} \bar{\partial}_V := \bar{\partial} + \partial V\wedge & \partial_V := \partial + \overline{\partial V}\wedge \\ \delta_V := -*\bar{\partial}_V * & \bar{\delta}_V := -*\partial_V * \end{array} \tag{B.57}$$

and the Lefschetz SU(2) operators defined by

$$L := \omega_K\wedge \qquad \Lambda := (L)^* \tag{B.58}$$

where $\omega_K = g_{i\bar{i}}dz^i\wedge d\bar{z}^{\bar{i}}$ is the Kähler form ($d\omega_K = 0$) corresponding to the Kähler metric $g_{i\bar{i}} = \partial_i\partial_{\bar{i}}K$ on the complex manifold M described by the coordinates $(z^i, \bar{z}^{\bar{i}})$. The explicit expression of Λ is $\Lambda = g^{i\bar{i}}\psi_i\psi_{\bar{i}}$. It is easy to verify that the usual relations hold

$$\begin{array}{ll} \bar{\partial}_V^2 = \partial_V^2 = 0 & \bar{\partial}_V\partial_V + \partial_V\bar{\partial}_V = 0 \\ \delta_V^2 = \bar{\delta}_V^2 = 0 & \bar{\delta}_V\delta_V + \delta_V\bar{\delta}_V = 0 \quad . \end{array} \tag{B.59}$$

and

$$[N, L] = 2L \qquad [N,\Lambda] = -2\Lambda \qquad [L,\Lambda] = N \tag{B.60}$$

(Lefschetz SU(2)). The Kähler–Hodge relations are

$$\begin{array}{ll} [\bar{\partial}_V,\Lambda] = \bar{\delta}_V & [\partial_V,\Lambda] = -\delta_V \\ [\delta_V, L] = -\partial_V & [\bar{\delta}_V, L] = \bar{\partial}_V \quad , \end{array} \tag{B.61}$$

from which one easily shows the remaining relations of Kähler geometry (Kähler identities)

$$\begin{array}{l} \partial_V\delta_V + \delta_V\partial_V = 0 = \bar{\partial}_V\bar{\delta}_V + \bar{\delta}_V\bar{\partial}_V \\ \partial_V\bar{\delta}_V + \bar{\delta}_V\partial_V =: \Box_V := \bar{\partial}_V\delta_V + \delta_V\bar{\partial}_V \quad , \end{array} \tag{B.62}$$

where the complex Laplacian $\Box_V$ corresponds in this identification to twice the Hamiltonian, $\Box_V \leftrightarrow 2H$.

Consider a general (p,q)–form which I write as

$$|\beta\rangle_{pq} := \sum_{\substack{0\le i_1\le\dots\le i_p\le n \\ 0\le \bar{i}_1\le\dots\le \bar{i}_q\le n}} f_{i_1\dots i_p \bar{i}_1\dots\bar{i}_q} dz^{i_1}\wedge\dots\wedge dz^{i_p}\wedge d\bar{z}^{\bar{i}_1}\wedge\dots\wedge d\bar{z}^{\bar{i}_q} \quad . \tag{B.63}$$

Then

$$
\begin{aligned}
\overline{\partial}_V|\beta\rangle_{pq} = \sum_{\substack{0\le i_1\le\ldots\le i_p\le n\\ 0\le\bar{i}_1\le\ldots\le\bar{i}_q\le n}} \Big[&(-)^p\partial_{\bar{i}}f_{i_1\ldots i_p\bar{i}_1\ldots\bar{i}_q}\,dz^{i_1}\wedge\ldots\wedge dz^{i_p}\wedge\\
&\wedge d\bar{z}^{\bar{i}}\wedge d\bar{z}^{\bar{i}_1}\wedge\ldots\wedge d\bar{z}^{\bar{i}_q}+\\
&+\partial_i V f_{i_1\ldots i_p\bar{i}_1\ldots\bar{i}_q}\,dz^{i}\wedge dz^{i_1}\wedge\ldots\wedge dz^{i_p}\wedge\\
&\wedge d\bar{z}^{\bar{i}}d\bar{z}^{\bar{i}_1}\wedge\ldots\wedge d\bar{z}^{\bar{i}_q}\Big] \quad .
\end{aligned}
\tag{B.64}
$$

Thus, formally one can write

$$\overline{Q}_1|\alpha\rangle_{pq}\cong(-i\sqrt{2})\;\overline{\partial}_V|\beta\rangle_{pq} \quad . \tag{B.65}$$

Analogously, one can show that

$$
\begin{aligned}
\overline{Q}_2|\alpha\rangle_{pq}&\cong(-i\sqrt{2})\;\partial_V|\beta\rangle_{pq}\\
Q_1|\alpha\rangle_{pq}&\cong(i\sqrt{2})\;\delta_V|\beta\rangle_{pq}\\
Q_2|\alpha\rangle_{pq}&\cong(i\sqrt{2})\;\overline{\delta}_V|\beta\rangle_{pq}
\end{aligned}
\tag{B.66}
$$

so that

$$2H=\{\overline{Q}_1,Q_1\}=\{\overline{\partial}_V,\delta_V\}=\Box_V \tag{B.67}$$

(a similar formula holds for Q_2 and ∂_V).

Thus, as already observed by Witten [12], the fermions can be considered as differentials in the tangent space to M, $\psi^i\leftrightarrow dz^i$. This identification must be handled with care because there are indeed (little) differencies between them. One has just to verify that every property of fermions (or differentials) goes without modifications to the differentials (fermions) and take care of the possible differencies [38].

Thus one can reformulate the N=2 SQM in a differential geometry language. This has the well known consequence that the problem of computing the vacuum states (or vacuum wave functions) is mapped into a cohomological problem, that associated to the $\bar{\partial}_V$–complex, since

$$\overline{Q}_1|\omega\rangle=0=Q_1|\omega\rangle \qquad \sim \qquad \overline{\partial}_V|\omega\rangle=0=\delta_V|\omega\rangle \tag{B.68}$$

(a similar formula holds for Q_2 and ∂_V).

A concluding remark. I have considered until now the case of models with canonical kinetic term (or flat Kähler metric). It is easy now to generalize everything to the case of non–canonical kinetic terms, which means $g_{i\bar{i}}=\partial_i\partial_{\bar{i}}K(z,\bar{z})\neq\delta_{i,\bar{i}}$. For example, letting $V=0$ for simplicity, one gets

$$
\begin{aligned}
\overline{Q}_2&=\sqrt{2}\left[-i\psi^i\partial_i\right]\\
\overline{Q}_1&=\sqrt{2}\left[-i\psi^{\bar{i}}\partial_{\bar{i}}\right]\\
Q_2&=\sqrt{2}\left[-i\psi_i D^i\right]\\
Q_1&=\sqrt{2}\left[-i\psi_{\bar{i}} D^{\bar{i}}\right]
\end{aligned}
\tag{B.69}
$$

where $D^i=g^{i\bar{i}}\left(\partial_{\bar{i}}-\Gamma_{\bar{i}}\right)$ is the Riemann covariant derivative with metric $g_{i\bar{i}}$. The interaction can be easily introduced again by conjugation.

[38] One example of that is the R–charge, as explained in §6.4

APPENDIX C

2D N=2 SUPERCONFORMAL FIELD THEORY

In this Appendix I will recall some known results in 2d N=2 Superconformal Field Theory (SCFT) and fix the notations which are used in this thesis. I will not enter in any way in the very interesting world of the 2d (super–) conformal theories, for this I just refer to the very large literature. The references from which a big part of the following material is taken are [7,22,24,35,71].

The Virasoro N=2 extended superconformal algebra is:

$$\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\
&[L_n, J_m] = -mJ_{n+m} \\
&[L_n, G^{\pm}_{\alpha}] = (\frac{n}{2}-\alpha)G^{\pm}_{n+\alpha} \\
&[J_n, J_m] = \frac{c}{12}n\delta_{n+m,0} \\
&[J_n, G^{\pm}_{\alpha}] = \pm\frac{1}{2}G^{\pm}_{n+\alpha} \\
&\{G^{+}_{\alpha}, G^{-}_{\beta}\} = 2L_{\alpha+\beta} + 2(\alpha-\beta)J_{\alpha+\beta} + \frac{c}{3}(\alpha^2-\frac{1}{4})\delta_{\alpha+\beta,0} \\
&\{G^{+}_{\alpha}, G^{+}_{\beta}\} = 0 \qquad \{G^{-}_{\alpha}, G^{-}_{\beta}\} = 0
\end{aligned} \tag{C.1}$$

(conventions of Mussardo et al. [24]) where the L_n are the Virasoro generators, i.e. the Laurent expansion modes of the stress energy tensor $T(z)$, the G^{+}_{α} and G^{-}_{β} are the Laurent expansion modes of the two supercurrents, $G^{\pm}(z) = \frac{1}{\sqrt{2}}(G^1 \pm iG^2)(z)$, the J_m are the Laurent expansion modes of the $U(1)$–current $J(z)$ and c in the central charge. One can choose periodic or antiperiodic boundary conditions for the supercurrents, these correspond to the Neveu–Schwartz (NS) and Ramond (R) sectors. In the NS sector $\alpha, \beta \in \mathbb{Z} + \frac{1}{2}$, whereas in the R sector $\alpha, \beta \in \mathbb{Z}$. Since $[J_n, G^{\pm}_{\alpha}] = \pm\frac{1}{2}G^{\pm}_{n+\alpha}$, this conventions are those in which the supercurrents have $U(1)$–charge $\frac{1}{2}$. There exists another widely used convention (see for example [7]) in which the supercurrents have $U(1)$–charge 1:

$$\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\
&[L_n, J_m] = -mJ_{n+m} \\
&[L_n, G^{\pm}_{\alpha}] = (\frac{n}{2}-\alpha)G^{\pm}_{n+\alpha}
\end{aligned}$$

$$\begin{aligned}
&[J_n, J_m] = \frac{c}{3} n \delta_{n+m,0} \\
&[J_n, G_\alpha^\pm] = \pm G_{n+\alpha}^\pm \\
&\{G_\alpha^+, G_\beta^-\} = 2L_{\alpha+\beta} + (\alpha-\beta) J_{\alpha+\beta} + \frac{c}{3}(\alpha^2 - \frac{1}{4})\delta_{\alpha+\beta,0} \\
&\{G_\alpha^+, G_\beta^+\} = 0 \qquad \{G_\alpha^-, G_\beta^-\} = 0
\end{aligned} \tag{C.2}$$

In this section I will use the conventions (C.1) , but the conventions adopted in Field Theory are that of (C.2) . It is very easy to pass from one to the other.

The representations of the (super–) Virasoro algebra are organized in "*conformal families*". Each family is generated from its "*primary superfield*" acting with the creation operators L_{-n}, J_{-p}, $G^\pm_{-\alpha}$, $n, p, \alpha \geq 0$. The lowest weight representation, and then the primary superfields, for each fixed value of c are labelled by two parameters: the conformal dimension d and $U(1)$ charge q.

(C.3) Definition : *the* **primary states** $|h,q\rangle$ *satisfy the conditions*

$$\begin{aligned}
&L_0|h,q\rangle = h|h,q\rangle \qquad J_0|h,q\rangle = q|h,q\rangle \\
&L_n|h,q\rangle = G_\alpha^\pm|h,q\rangle = J_m|h,q\rangle = 0\,, \quad \alpha, n, m > 0\,.
\end{aligned} \tag{C.4}$$

The $OSp(2|2)$ invariant vacuum $|0\rangle$ of the theory belongs to the NS sector and satisfies

$$L_n|0\rangle = J_M|0\rangle = G_\alpha^\pm|0\rangle = 0 \tag{C.5}$$

for $n \geq -1$, $m \geq 0$, $\alpha \geq -\frac{1}{2}$. A NS primary state $|h,q\rangle$ can be realized in terms of a **NS primary superfield** defined by

$$N_h^q(z,\theta^+,\theta^-) := \phi_h^q(z) + \theta^+ \psi_{h+1/2}^{q-1/2}(z) + \theta^- \overline{\psi}_{h+1/2}^{q+1/2}(z) + \theta^-\theta^+\overline{\phi}_{h+1}^q(z)\,. \tag{C.6}$$

It holds

$$\begin{aligned}
&|h,q\rangle = \phi_h^q(0)|0\rangle & &G_{-1/2}^+|h,q\rangle = \overline{\psi}_{h+1/2}^{q+1/2}(0)|0\rangle \\
&G_{-1/2}^-|h,q\rangle = \psi_{h+1/2}^{q-1/2}(0)|0\rangle & &G_{-1/2}^+G_{-1/2}^-|h,q\rangle = \overline{\phi}_{h+1}^q(0)|0\rangle\,.
\end{aligned} \tag{C.7}$$

Analogously one can introduce the **Ramond primary states** and the associated Ramond operators, creating Ramond states from the vacuum.

The Kac (determinant) formula gives some relations for the "degenerate unitary representations" of the superconformal algebra between the central charge c, the dimension d and the $U(1)$–charge q in the case of the "minimal models". For the NS sector one has

$$\begin{aligned}
&c = 3 - \frac{6}{p+2} \qquad p = 1,2\ldots \\
&q_s = \frac{s}{2(p+2)} \\
&h_{n,1}^s = \frac{(p+2-n)^2 - s^2 - 1}{4(p+2)} \qquad n = 0,1,\ldots,p+1; \quad |s| \leq p-n+1; \\
&h_{n,0}^s = \frac{(n+|s|)^2 - s^2 - 1}{4(p+2)} \qquad n = 1,3,5,\ldots \quad |s| + n \leq p+1\,.
\end{aligned} \tag{C.8}$$

In the R sector one has

$$
\begin{aligned}
&c = 3 - \frac{6}{p+2} \qquad p = 1, 2 \ldots \\
&q_s = \frac{s-r}{2(p+2)} + \frac{r}{4} \qquad r = \pm 1 \\
&h^{s,r}_{n,1} = \frac{(p+2-n)^2 - (s-r)^2 - 1}{4(p+2)} + \frac{1}{8} \quad n = 0, 1, \ldots, p+1; \quad |s| \le p-n+1; \\
&h^{s,r}_{n,0} = \frac{(n+|s-r|)^2 - (s-r)^2 - 1}{4(p+2)} + \frac{1}{8} \qquad n = 0, 2, 4, \ldots \quad |s-r| + n \le p\,.
\end{aligned} \tag{C.9}
$$

A convenient explicit representation of the primary operators $N^l_m(z)$ and $R^l_{m,\alpha}(z)$ is given by the parafermionic construction:

$$
\begin{aligned}
&N^l_m(z) = \phi^l_m(z) : \exp\left\{ i \frac{m}{\sqrt{2p(p+2)}}\varphi(z) \right\} : \\
&l = 0, 1, \ldots, p \qquad m = -l, m-l+2, \ldots, l \\
&q^l_m = \frac{m}{2(p+2)} \\
&h^l_m = \frac{l(l+2) - m^2}{4(p+2)}
\end{aligned} \tag{C.10}
$$

where $\phi^l_m(z)$ are the parafermions and $\varphi(z)$ is a free scalar field with propagator $\langle\varphi(z_1)\varphi(z_2)\rangle = -2\ln(z_{12})$, and

$$
\begin{aligned}
&R^l_{m,\alpha}(z) = \phi^l_m(z) : \exp\left\{ i \frac{m - \alpha p/2}{\sqrt{2p(p+2)}}\varphi(z) \right\} : \\
&l = 0, 1, \ldots, p \qquad m = -l, m-l+2, \ldots, l \qquad \alpha = \pm 1 \\
&q^l_m = \frac{2m - \alpha p}{4(p+2)} \\
&h^l_m = \frac{l(l+2) - (m+\alpha)^2}{4(p+2)} + \frac{1}{8}
\end{aligned} \tag{C.11}
$$

(the change of notations with respect to the previous formulas for the parameters in the dimensions and $U(1)$–charges is the following

$$
\begin{aligned}
&m = s, \quad r = -\alpha, \quad n = p - l + 1 \qquad \text{for } d^{r,s}_{n,1} \\
&m = s, \quad r = -\alpha, \quad n = l - |m| + 1 \qquad \text{for } d^{r,s}_{n,0} \quad .)
\end{aligned} \tag{C.12}
$$

In the NS sector it is possible to define a subclass of primary superfields (or states) which are simpler and have peculiar properties.

(C.13) Definition : *the* **primary Chiral states** *are those primary states satisfying also the condition*

$$
G^+_{1/2}|h, q\rangle = 0 \tag{C.14}
$$

whereas the **primary antichiral states** *satisfy* $G^-_{1/2}|h, q\rangle = 0$.

The corresponding primary (anti–) chiral operators are

$$\begin{aligned} N_h^{q+}(z) &= \phi_h^q(z) + \theta^+ \psi_{h+1/2}^{q-1/2}(z) + \theta^-\theta^+\partial\phi_h^q(z) \qquad D^+N^+ = 0 \\ N_h^{q-}(z) &= \phi_h^q(z) + \theta^- \overline{\psi}_{h+1/2}^{q+1/2}(z) - \theta^-\theta^+\partial\phi_h^q(z) \qquad D^-N^- = 0 \end{aligned} \tag{C.15}$$

where $D^\pm := \frac{\partial}{\partial\theta^\mp} + \theta^\pm\partial$. Using the N=2 Virasoro algebra it is easy to show that a primary state is chiral *if and only if*

$$h = q \tag{C.16}$$

and antichiral *if and only if* $h = -q$ (in the conventions (C.2) $h = \pm q/2$). Moreover, the dimension of a primary chiral field satisfies

$$h \leq \frac{c}{6} \tag{C.17}$$

and there always exists a unique primary chiral state which saturates this bound. Notice that, if a theory is non–degenerate, this implies that there is only a finite number of primary chiral operators.

Since the product of two primary chiral fields is again chiral and non singular, it follows that the primary chiral fields form a ring $\mathcal{R}$. This is the same as the usual operator algebra of chiral fields, modulo setting to zero the descendant chiral fields [39].

Notice that the knowledge of the primary chiral fields and of their ring $\mathcal{R}$, although it gives a proper characterization of a N=2 Superconformal model, it is not sufficient to completely specify it.

It is well known [31] that one can continuously connect the NS sector to the R sector. One considers a collection of Hilbert spaces $\mathcal{H}_\theta$ parametrized by θ which differ by the original one, $\mathcal{H}_0$, only by the fact that their $U(1)$ charges are shifted by $-\frac{c}{6}\theta$ (in the conventions (C.2) $-\frac{c}{3}\theta$). Thus, there is a map

$$\mathcal{U}_\theta : \ \mathcal{H}_0 \longrightarrow \mathcal{H}_\theta \tag{C.18}$$

which acts on every operator $\mathcal{O}$ defined on $\mathcal{H}_0$ as $\mathcal{O}_\theta = \mathcal{U}_\theta \mathcal{O} \mathcal{U}_\theta^{-1}$. This mapping is called **Spectral Flow**. The spectral flow acting on the N=2 Virasoro algebra gives a isomorphic algebra:

$$\begin{aligned} \mathcal{U}_\theta L_n \mathcal{U}_\theta^{-1} &= L_n + 2\theta J_n + \frac{c}{6}\theta^2\delta_{n,0} \\ \mathcal{U}_\theta J_m \mathcal{U}_\theta^{-1} &= J_m + \frac{c}{6}\theta\delta_{m,0} \\ \mathcal{U}_\theta G_\alpha^+ \mathcal{U}_\theta^{-1} &= G_{\alpha+\theta}^+ \\ \mathcal{U}_\theta G_\beta^- \mathcal{U}_\theta^{-1} &= G_{\beta-\theta}^- \quad . \end{aligned} \tag{C.19}$$

A flow with $\theta = \frac{1}{2}$ interpolates between the NS and the R sectors and one with $\theta = 1$ takes NS in NS and R in R.

[39] Indeed, there are four distinct rings depending on whether one considers left and right movers and chiral or antichiral primary fields.

In the Ramond sector from the Virasoro algebra it follows that the conformal dimension of the primary states must satisfy $h \geq \frac{c}{24}$. It is possible to show that under a flow with $\theta = \frac{1}{2}$ all chiral primary states go into the ground states of the Ramond sector, i.e. the states with $c = \frac{h}{24}$, and viceversa. Moreover, $-\frac{c}{12} \leq q_{Ramond} \leq \frac{c}{12}$ ($-\frac{c}{6} \leq q_{Ramond} \leq \frac{c}{6}$ in (C.2) conventions).

Thus, for chiral primary fields the parameters l and m in N^l_m and l, m, α in $R^l_{m,\alpha}$ cannot assume all possible values. Indeed, for the NS fields N^l_m the relation $h = q$ implies that there are only $p+1$ fields; they are conveniently labelled by the parameter m, $m = 0, 1, 2 \ldots, p$ and $m = l$. Thus, the primary chiral fields can be unambigously written as $N_m(z)$. Analogously, in the case of the primary chiral Ramond fields the relation $h = \frac{c}{24}$ implies that there are only $p+1$ fields conveniently labelled again by m. In this case their parafermionic expression is

$$R^{p+1-m}_{p+1-m,1}(z) \tag{C.20}$$

with $m = 0, 1, 2, \ldots, p$. It is easy to see that the parameter p, which labels the N=2 SCFT minimal models, corresponds to the Witten Index of the LG theory minus one, i.e. $p+1 = \Delta = \mu$.

All what I have said up to now works having chosen $\mathbb{C}$ (or S^2) as spacetime with a "*radial quantization*" procedure. What one usually does in CFT is to define the conformal theory on an *Euclidean Cylinder* and then to conformally map it on the complex plane. In this way the usual quantization on the cylinder becomes the radial quantization on the complex plane.

But I am interested in making the comparison between the N=2 SCFT and LG models at the critical point defined on an Euclidean cylinder. Thus I will map back the N=2 SCFT to the Euclidean cylinder. Let $w = t + i\sigma$ and $\overline{w} = t - i\sigma$ be the complex coordinates on the Euclidean cylinder. The conformal map between the complex plane and the cylinder is $z = e^w$. Passing from the plane to the cylinder the fields change boundary conditions, the periodic fields ($\phi(e^{2\pi i}z) = \phi(z)$), also called Ramond fields, become antiperiodic ($\phi(w+2\pi i) = -\phi(w)$, and viceversa. I will continue to call Ramond fields also on the cylinder those fields which were called Ramond on the plane. Thus, in my conventions, the Ramond fields on the cylinder are antiperiodic and the NS fields are periodic.

Let now consider the LG field theory. As we have seen in this thesis (see for example Appendix B), the CGFs are vacuum expectation functions of chiral fields on fermionic vacua. From the properties of the chiral fields and of the vacua of the LG theories, it follows immediately that, at the critical point, one should compare the CGFs with the following correlation functions in the N=2 SCFT on the cylinder

$$\langle 0 | R^{p+1-m'}_{p+1-m',1}(\infty) N_k(w) R^{p+1-m}_{p+1-m,1}(0) | 0 \rangle \quad . \tag{C.21}$$

Thus N_k should correspond to the field X^k and $R^{p+1-m}_{p+1-m,1}(0)|0\rangle$ to the vacuum $|m\rangle$ in Field Theory.

In the SCFT on the cylinder the momentum operator is L_0, $L_0 = \frac{\partial}{\partial w}$. But, from the Virasoro algebra it follows

$$L_0 = \{G^+_{1/2}, G^-_{-1/2}\} = \{G^+_{-1/2}, G^-_{1/2}\} \quad . \tag{C.22}$$

Applying L_0 to the correlation functions (C.21) and using this relation one gets zero; this means that the correlation function (C.21) (on the cylinder) does not depend on the (space-time) coordinates. This is exactly the same result (and the same proof) already seen in §1.3 for the CGFs.

Since the connection between a SCFT theory defined on the plane and that defined on the cylinder is a conformal mapping, the general properties of the states and of the fields do not change. What surely does change, but in a known way, is their dependence on the spacetime coordinates. Indeed, generically

$$\phi_{cyl}(w) = \left(\frac{dz}{dw}\right)^h \phi_{plane}(z) = z^h \phi_{plane}(z) \quad . \tag{C.23}$$

Let me consider more closely the OPE and the correlation functions of the SCFT fields. It is well known [35] that the conformal fields (let denote them generically by A_I) make up an associative algebra through the OPE. Indeed one can formally write the OPEs as

$$A_I\, A_J = \sum_K C_{IJ}^K\, A_K \tag{C.24}$$

(the index I combines space coordinates and indices labelling the fields). Moreover, it is defined a symmetric bilinear form

$$D_{IJ} := \langle A_I\, A_J \rangle \quad . \tag{C.25}$$

One can define the coefficients

$$C_{IJK} := \sum_{K'} D_{K\,K'} C_{IJ}^{K'} \quad . \tag{C.26}$$

Evidently, they coincide with the three–point correlation functions

$$C_{IJK} = \langle A_I\, A_J\, A_K \rangle \quad . \tag{C.27}$$

Since I am interested in the correlation functions of primary fields on the cylinder, these are numerical coefficients independent from the spacetime coordinates.

In the parafermionic conventions (see Mussardo et al. in ref. [24]), if N_h denotes the (first) bosonic component of a primary chiral field, one has

$$\langle N_h\, N_{h'} \rangle = \delta_{h,h'} \quad . \tag{C.28}$$

Thus, $D_{k,k'} = \delta_{k,k'}$ and the three–point functions coincide with the OPE coefficients. Moreover, since the two–point function of the parafermionic fields is normalized in such a way that the coefficient in front is exactly one, one gets the same result for NS and R fields. This implies that the three–point functions are exactly equal to the structure constants of the associated $SU(2)$–Wess–Zumino–Witten theory.

Notice that I have to compare the three-point functions (C.21) where $R^n_{m,\alpha}$ is a primary Ramond field with $h = \frac{c}{24}$ and N^l_m is a primary chiral field, with the CGF $\langle m+k|X^k|m\rangle$ at the critical point in field theory. Thus, what I have finally to compare are the CGFs at the critical point with the OPE coefficients of the primary chiral fields in the corresponding N=2 SCFT (or the three–point functions of the associated $SU(2)$–Wess–Zumino–Witten model).

Between the LG field theory and the N=2 SCFT there is anyway a change of notations. Indeed, in LG field theory

$$X^k \cdot X^h = 1 \cdot X^{h+k} \quad , \tag{C.29}$$

instead in N=2 SCFT

$$N_m \cdot N_{m'} \sim C^{m+m'}_{m,m'}\, N_{m+m'} \quad . \tag{C.30}$$

Thus the correct identification between LG fields and N=2 SCFT fields is

$$X^k \sim \frac{1}{\alpha_\mu(k)} N_k \tag{C.31}$$

where μ is the Witten Index. The normalization constants $\alpha_\mu(k)$ can be computed by comparing the formulas:

$$\begin{aligned} &N_m \cdot N_{m'} \sim C^{m+m'}_{m,m'}\, N_{m+m'} \\ &X^k \cdot X^h \sim 1 \cdot X^{h+k} \\ &C^{m+m'}_{m,m'} = \langle 0|R^{p+1-k}_{p+1-k,1}(\infty) N_m R^{p+1-m'}_{p+1-m',1}(0)|0\rangle = \\ &\qquad\quad = \alpha_\mu(m)\, \langle m+m'|X^m|m'\rangle \; . \end{aligned} \tag{C.32}$$

The C_{IJK} have been computed for all the minimal models through the $SU(2)$–Wess–Zumino–Witten model. I am interested in the results for the OPE coefficients for the primary chiral fields for the A_μ and D_μ series. They can be summarized as follows (see Mussardo et al. [24] and ref. [72]). Let define

$$\Sigma^2(k_f; i) := (i+1) \frac{\Gamma\left(1+\frac{1}{k_f+2}\right) \cdot \Gamma\left(1-\frac{i+1}{k_f+2}\right)}{\Gamma\left(1-\frac{1}{k_f+2}\right) \cdot \Gamma\left(1+\frac{i+1}{k_f+2}\right)} \tag{C.33}$$

and denote the OPE coefficients by $C_{k_f}(i,j)$. Then

$$C_{k_f}(i,j) = \frac{\Sigma(k_f;i) \cdot \Sigma(k_f;j)}{\Sigma(k_f;i+j)} \quad . \tag{C.34}$$

For the A_μ models $k_f = \mu_A - 1$ where μ_A is the Witten index of the A_μ series ($\mu_A = p_A + 1$) and the identification with the parameters labelling the field theory states is $i = h$ and $j = k$. For the D_μ models $k_f + 2 = 2(\mu_D - 1)$ where μ_D is the Witten index of the D_μ series, i and j assume only even values [72] and the identification with the parameters labelling the field theory states is $i = 2h$ and $j = 2k$.

APPENDIX D

SOME NOTIONS IN ALGEBRAIC GEOMETRY

In this appendix I will recall some notions in Algebraic Geometry extensively used throughout this thesis and give the proofs of some lemmas which, since they are purely technical and fairly standard, I have chosen not to put in the main text.

D.1 Some definitions and results in Algebraic Geometry

The Algebraic Geometrical material which I am going to recall is almost standard [42,43]. For notations and conventions I follow mainly ref. [43].

(D.1) Definition : *a* **Differential Complex** *is a direct sum of vector spaces* $C = \bigoplus_{q \in \mathbb{Z}} C^q$ *indexed by the integer if there are homomorphisms*

$$\cdots \longrightarrow C^{q-1} \xrightarrow{d} C^q \xrightarrow{d} C^{q+1} \longrightarrow \cdots \tag{D.2}$$

such that $d^2 = 0$. *d is the* **differential operator** *of the complex C. The* **Cohomology** *of C is the direct sum of vector spaces* $H(C) := \bigoplus_{q \in \mathbb{Z}} H^q(C)$ *where* $H^q(C) := (\mathrm{Ker}\ d \cap C^q)/(\mathrm{Im}\ d \cap C^q)$.

The simplest differential complex is the De Rham complex. The one in which I am more interested is the Dolbeault complex.

(D.3) Definition : *let* $\bigwedge^{p,q}(X)$ *be the space of smooth (p,q)–forms on X and X be a (Stein [40]) space of complex dimension n. Let $\bar{\partial}$ be the usual anti–holomorphic exterior derivative or Dolbeault operator. Let* $\bigwedge^k := \bigoplus_{p+q=k} \bigwedge^{p,q}$, *then the following complex*

$$\overset{0}{\bigwedge} \xrightarrow{\bar{\partial}} \overset{1}{\bigwedge} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \overset{2n}{\bigwedge} \tag{D.4}$$

is called the **Dolbeault Complex**.

In this thesis the space X is always a compact subspace of $\mathbb{C}^n$ and thus a Stein space. For notational convenience and following a physical tradition more than the more rigorous

[40] For the definition of a "Stein space" see for example [73].

mathematical notation, I have usually considered the cohomological problem as defined over all $\mathbb{C}^n$. However, has I have shown in §4.5 , in every cohomological class there is at least one representative with compact support and I can use this representative in the computations.

In the following I will denote by $\Omega^p(X)$ the space of holomorphic p–forms on X.

Notice that an equivalence class in $H^n_{\bar\partial}$ is represented by the solutions of the two first order equations

$$\bar\partial\psi = 0 \qquad \delta\psi = 0 \quad . \tag{D.5}$$

Another interesting differential complex is the Koszul complex which I define in a different (although isomorphic) way with respect to ref. [43].

(D.6) Definition : *let V be an holomorphic function on X, i.e. $V \in \Omega^0(X)$, let $\partial V \in \Omega^1(X)$ act on $\bigoplus_{k=0}^{2n} \bigwedge^k(X)$ by exterior multiplication. Since $(\partial V)^2 = 0$, one can define the following complex*

$$\Omega^0 \xrightarrow{\partial V} \Omega^1 \xrightarrow{\partial V} \cdots \xrightarrow{\partial V} \Omega^n \tag{D.7}$$

which is called the **Koszul Complex** *and denoted by $K_{\partial V}$.*

Using the Dolbeault and Koszul complexes, one can define a *double complex* whose vertical coboundary operator is $\bar\partial$ and whose orizontal coboundary operator is $\partial V \wedge$.

(D.8) Definition : *the $\bar\partial_V$–* **Complex** *is the differential complex defined by the operator $\bar\partial_V := \bar\partial + \partial V\wedge$ acting on $\bigwedge^k$, that is*

$$\overset{0}{\bigwedge} \xrightarrow{\bar\partial_V} \overset{1}{\bigwedge} \xrightarrow{\bar\partial_V} \cdots \xrightarrow{\bar\partial_V} \overset{2n}{\bigwedge} \quad . \tag{D.9}$$

Notice that the Koszul complex appears as the bottom row of the double $\bar\partial_V$–complex. I refer to Appendix D for all the properties of the operators $\bar\partial_V$, ∂_V, δ_V and $\bar\delta_V$.

D.2 Deferred proofs of Algebraic–Geometrical Lemmas from Chapter 4

I will now prove the lemmas stated in §4.5 .

(4.37) $\bar\partial_V$–Poincaré Lemma : *let ω be a $\bar\partial_V$–closed k–form in $\mathbb{C}^n$ (or in a n–dimensional Stein space). Then there exist a $(k-1)$–form η and a* holomorphic *k–form α such that*

$$\omega = \bar\partial_V \eta + \alpha \tag{D.10}$$

Conversely, any such ω is $\bar\partial_V$–closed.

Proof : let decompose ω according to type

$$\omega = \sum_p \omega_{(p,k-p)} \quad . \tag{D.11}$$

Then $\bar{\partial}_V \omega = 0$ decompose into

$$\bar{\partial}\omega_{(p,k-p)} + \partial V \wedge \omega_{(p-1,k-p+1)} = 0 \qquad 0 \le p \le k \quad . \tag{D.12}$$

By the Dolbeault lemma $H^{p,q}_{\bar{\partial}} = \{0 \ \text{ if } q \geq 1, \ \Omega^p \ \text{ if } q = 0\}$, there exists a $(0, k-1)$ form $\eta_{(0,k-1)}$ such that $\omega_{(0,k)} = \bar{\partial}\eta_{(0,k-1)}$. Then eq. (D.12) yields

$$\begin{aligned} &\bar{\partial}\omega_{(1,k-1)} + \partial V \wedge \bar{\partial}\eta_{(0,k-1)} = 0 \\ &\Rightarrow \bar{\partial}\left[\omega_{(1,k-1)} - \partial V \wedge \eta_{(0,k-1)}\right] = 0 \\ &\Rightarrow \omega_{(1,k-1)} - \partial V \wedge \eta_{(0,k-1)} = \bar{\partial}\eta_{(1,k-2)} \\ &\Rightarrow \omega_{(1,k-1)} = \bar{\partial}\eta_{(1,k-2)} + \partial V \wedge \eta_{(0,k-1)}, \end{aligned} \tag{D.13}$$

where again I used the Dolbeault lemma. Then the argument goes on by induction: once proven that

$$\omega_{(p,k-p)} = \bar{\partial}\eta_{(p,k-p-1)} + \partial V \wedge \eta_{(p-1,n-p)} \quad , \tag{D.14}$$

one obtains the same result with $p \to p+1$ by the same manipulations. Eventually, one arrives at the equation

$$\bar{\partial}\left[\omega_{(k,0)} - dV \wedge \eta_{(k-1,0)}\right] = 0, \tag{D.15}$$

showing that the k–form $\alpha \equiv \omega_{(k,0)} - dV \wedge \eta_{(k-1,0)}$ is holomorphic. Writing $\eta = \sum_p \eta_{(p,k-1-p)}$, one gets the lemma. ∎

(4.41) Proposition :

$$H^k_{\bar{\partial}_V} \overset{\text{iso}}{=} \begin{cases} 0 & \text{if} \quad k < n \\ \tilde{\mathcal{R}} \overset{\text{iso}}{=} \mathcal{R} & \text{if} \quad k = n \end{cases} \quad . \tag{D.16}$$

Proof : From corollary (4.39) we know that $H^k_{\bar{\partial}_V} \overset{\text{iso}}{=} H^k(K_{\partial V})$ for $k \leq n$. The Koszul complex as previously defined differs slightly from that defined for example in ref. [43] pag. 688. Nevertheless it is immediate to check that the two Koszul complexes $K_{\partial V}$ and $E(\partial V)$ are indeed isomorphic. The isomorphism on the cohomology groups is $H^k(K_{\partial V}) \overset{\text{iso}}{=} H_{n-k}(E(\partial V))$. By the lemma on the Koszul complex, pag. 688 [43], we know that

$$H_{n-k}(E(\partial V)) = \{0 \ \text{ for } \ k < n, \ \ \mathcal{R} \ \text{ for } \ k = n\} \quad . \tag{D.17}$$

From this the proposition follows. ∎

Let

$$\begin{aligned} \lambda_i &= \frac{\overline{\partial_i V}}{|\partial_1 V|^2 + \ldots + |\partial_n V|^2} \\ \beta_0 &= \sum_i (-1)^{i-1} \lambda_i dX_i \wedge \ldots \wedge \widehat{dX_i} \wedge \ldots \wedge dX_n \\ \beta_k &= \sum_i (-1)^{i-1} \sum_{j_1 \neq i} \sum_{j_2 \neq i} \cdots \sum_{j_k \neq i} \lambda_i dX_1 \wedge \ldots \wedge \widehat{dX_i} \wedge \ldots \\ &\quad \ldots \wedge \bar{\partial}\lambda_{j_1} \wedge \ldots \wedge \bar{\partial}\lambda_{j_k} \wedge \ldots \wedge dX_n \end{aligned} \tag{D.18}$$

where $\widehat{dX_i}$ means omitted and each $\bar{\partial}\lambda_{j_s}$ is in the j_s–th position counting the omitted $\widehat{dX_i}$ as present. Notice that β_k is an $(n-1-k, k)$ form.

(4.57) Lemma : *let* $\mu := \sum_{k=0}^{n-1}(-)^k\beta_k$ *then* $\overline{\partial}_V\mu = dX_1 \wedge \dots dX_n$.

Proof : the proofs of this and the following lemmas are inspired by [43] pagg. 653–655. First, let me recall the definition of **Divisor** ([43] pag. 130). Let M be a complex manifold of dimension n and V an analytic subvariety $V \supset M$ of dimension $n-1$. V is an **analytic hypersurface** if for any point $p \in V \supset M$, V can be given in a neighborhood of p as the zeros of a single holomorphic function f. f is called a **local defining function** for V near p and is unique up to multiplication by function non–zero at p.

(D.19) Definition : *a* **Divisor** *D on M is a locally finite formal linear combination*

$$D = \sum_i a_i V_i \tag{D.20}$$

of irreducible analytic hypersurfaces V_i of M.

("Locally finite" means that for any $p \in M$ there exists a neighborhood of p meeting only a finite number of V_i's appearing in D; of course if M is compact this means that the sum is finite.)

Now let

$$\begin{aligned} dV :=& \sum_i \partial_i V dX_i \\ D_i :=& (\partial_i V) = \text{ divisor of } \partial_i V \\ U :=& \text{ sufficiently large ball in } \mathbb{C}^n \text{ of radius } R \\ U_i :=& U - D_i \\ U^* :=& \bigcup_i U_i \end{aligned} \tag{D.21}$$

Moreover let

$$\rho_i = \frac{|\partial_i V|^2}{|\partial_1 V|^2 + \dots + |\partial_n V|^2} \tag{D.22}$$

so that $\rho_i = \partial_i V \lambda_i$. Obviously:

$$\begin{aligned} &\rho_i \quad \text{is } \mathbb{C}^\infty \text{ in } U^* \\ &\sum_i \rho_i = 1 \quad \text{(it is a partition of unity)} \\ &\sum_i \partial_i V \cdot \bar{\partial}\lambda_i = \bar{\partial}(\sum_i \partial_i V \cdot \lambda_i) = 0 \end{aligned} \tag{D.23}$$

Now in U^*

$$dV \wedge \beta_0 = \left(\sum_i \partial_i V \cdot \lambda_i\right) dX_1 \wedge \dots \wedge dX_n = dX_1 \wedge \dots \wedge dX_n \tag{D.24}$$

and

$$\alpha := \alpha(X)dX_1 \wedge \ldots \wedge dX_n = \alpha(X) \cdot \partial V \wedge \beta_0 = \partial V \wedge (\alpha(X)\beta_0) \quad . \tag{D.25}$$

It is a standard although lengthy matter to verify that

$$dV \wedge \beta_k = \bar{\partial}\beta_{k-1} \qquad \text{for} \quad k > 0 \tag{D.26}$$

Now, from the definition $\mu = \sum_{k=0}^{n-1}(-1)^k\beta_k$ it follows that

$$\bar{\partial}_V\mu = \bar{\partial}\mu + dV \wedge \mu = dX_1 \wedge \cdots \wedge dX_n \quad . \quad \blacksquare \tag{D.27}$$

Before proving the last lemma I give the definition of the Bochner–Martinelli kernel.

(D.28) Definition : *let z_i and ζ_i, $i = 1, dots, n$, be complex coordinates on a n-dimensional manifold M. The* **Bochner–Martinelli kernel** *is*

$$K(z,\zeta) = C_n \frac{\sum_i(-1)^{i-1}\overline{(z_i - \zeta_i)} \bigwedge_{j\neq i}(d\bar{z}_j - d\bar{\zeta}_j) \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n}{[|z_1 - \zeta_1|^2 + \cdots + |z_n - \zeta_n|^2]^n} \tag{D.29}$$

where $C_N = (-1)^{n(n-1)/2}(n-1)!/(2\pi i)^n$.

The Bochner–Martinelli kernel gives the extension of the Cauchy integral formula of residues to the case of many complex variables. In fact, the following formula holds:

$$\phi(z) = \int_{B[R]} \bar{\partial}\phi(\zeta) \wedge K(z,\zeta) + \int_{\partial B[R]} \phi(\zeta)K(z,\zeta) \tag{D.30}$$

where $B[R]$ is a ball of radius R in $\mathbb{C}^n$. In the case in which ϕ is holomorphic, then

$$\phi(z) = \int_{S_{2n-1}(R)} \phi(\zeta)K(z,\zeta) \tag{D.31}$$

where $S_{2n-1}(R)$ is a circle of radius R in $\mathbb{C}^n$. Obviously this integral in independent of R.

(4.65) Lemma :

$$\begin{aligned}\langle \underline{p}|\underline{q}\rangle &= \int_{\mathbb{C}^n} *\overline{\omega_{2,\underline{p}}} \wedge \omega_{1,\underline{q}} \\ &= (-2\pi)^n \int_{S_{2n-1}} F^*K \cdot \beta_{\underline{p}}(X)\alpha_{\underline{q}}(X)\end{aligned} \tag{D.32}$$

where K is the Bochner–Martinelli kernel and $F(z_i) = (z_i + \partial_i V, z_i)$.

Proof : let

$$\omega_{1,q} = \alpha_q + \bar{\partial}_V\eta_q \tag{D.33}$$

$$\omega_{2,p} = \beta_p^* + \partial_V\eta_{2,p}^* \tag{D.34}$$

$$*\overline{\omega_{2,p}} = (-1)^{n(n-1)/2} i^n \left(\beta_p + \bar{\partial}_{-V}\eta_{2,p}\right) \tag{D.35}$$

where $\alpha = \alpha(X)dX_1 \wedge \ldots \wedge dX_n$ is a (n,0) form, β^* is a (0,n) form, $\beta = \beta(X)dX_1 \wedge \ldots \wedge dX_n$ and $* = (-1)^{n(n-1)/2} i^{\hat{F}}$ (see §4.4). Now

$$\begin{aligned} \langle \underline{p}|\underline{q}\rangle &= \int_{\mathbb{C}^n} *\overline{\omega_{2,\underline{p}}} \wedge \omega_{1,\underline{q}} = \\ &= (-1)^{n(n-1)/2} i^n \int_{\mathbb{C}^n} \beta_{\underline{p}} \wedge \left[\alpha_{\underline{q}} + \bar{\partial}_V \eta_{\underline{q}}\right] = \\ &= (-1)^{n(n-1)/2} (-i)^n \int_{\mathbb{C}^n} \bar{\partial} \left(\beta_{\underline{p}} \wedge \eta_{\underline{q}}|\right) = \\ &= (-1)^{n(n-1)/2} (-i)^n \int_{\mathbb{C}^n} d \left(\beta_{\underline{p}} \wedge \eta_{\underline{q}}|\right) \end{aligned} \tag{D.36}$$

where $|$ means to take the $(0, n-1)$ part of the form.

Now from (4.58) we know that $\eta_{\underline{q}} = -\rho(X)\alpha(X)_{\underline{q}} \cdot \mu$. It is easy to compute $\mu| = \mu|_{(0,n-1)}$:

$$\begin{aligned} \mu| &= (-)^{n-1}(n-1)! \sum_i (-1)^{i-1} \lambda_i \bar{\partial}\lambda_1 \wedge \cdots \wedge \widehat{\bar{\partial}\lambda_i} \wedge \cdots \wedge \bar{\partial}\lambda_n = \\ &= (-)^{n-1}(n-1)! \frac{\sum_i (-1)^{i-1} \rho_i \bar{\partial}\rho_1 \wedge \cdots \wedge \widehat{\bar{\partial}\rho_i} \wedge \cdots \wedge \bar{\partial}\rho_n}{\prod_k \partial_k V} = \\ &= (-)^{n-1}(n-1)! \frac{\sum_i (-1)^{i-1} \overline{\partial_i V}\ \overline{d\partial_1 V} \wedge \cdots \wedge \widehat{\overline{d\partial_i V}} \wedge \cdots \wedge \overline{d\partial_n V}}{[|\partial_1 V|^2 + \cdots + |\partial_n V|^2]^n} \end{aligned} \tag{D.37}$$

Define $F : U \to \mathbb{C}^n \otimes \mathbb{C}^n$, $F(z_i) = (z_i + \partial_i V, z_i)$. It is immediate to check that

$$\begin{aligned} \mu| \wedge dX_1 \wedge \cdots \wedge dX_n &= -(-1)^{n(n-1)/2}(-2\pi i)^n K(\{z_i + \partial_i V\}, \{z_i\}) = \\ &= -(-1)^{n(n-1)/2}(-2\pi i)^n F^* K \quad . \end{aligned} \tag{D.38}$$

Thus

$$\begin{aligned} \langle \underline{p}|\underline{q}\rangle &= (-1)^{n(n-1)/2}(-i)^n \int_{\mathbb{C}^n} d\left(\beta_{\underline{p}} \wedge \eta_{\underline{q}}|\right) = \\ &= (-1)^{n(n-1)/2}(-i)^n \lim_{R\to\infty} \int_{S_{2n-1}(R)} \beta_{\underline{p}} \wedge \eta_{\underline{q}}| = \\ &= -(-1)^{n(n-1)/2}(-i)^n \int_{S_{2n-1}} \beta_{\underline{p}} \wedge \alpha_{\underline{q}}(X)\mu| = \\ &= -(-1)^{n(n-1)/2}(-i)^n \int_{S_{2n-1}} \beta_{\underline{p}}(X)\alpha_{\underline{q}}(X) \cdot dX_1 \wedge \ldots \wedge dX_n \wedge \mu| = \\ &= (-2\pi)^n \int_{S_{2n-1}} F^* K \cdot \beta_{\underline{p}}(X)\alpha_{\underline{q}}(X) \end{aligned} \tag{D.39}$$

where the following relation has been used $dX_1 \wedge \ldots \wedge dX_n \wedge \mu| = \mu| \wedge dX_1 \wedge \ldots \wedge dX_n$. ∎

APPENDIX E

DEFERRED PROOFS FROM CHAPTER 4

In this appendix I will prove some lemmas and the theorem stated in §4.2 . I start to prove the following

(4.22) Lemma : *the matrix elements*

$$\left(\mathcal{M}^i_s\right)_{h\,k} \stackrel{\text{def}}{=} \langle h|X_{i,0}|k\rangle_s \quad , \tag{E.1}$$

where $|k\rangle_s$ are the vacuum states of the SQM model so constructed, are independent of s, as long as $s \neq 0$.

Proof : by standard Hodge theory, the vacuum states $|k\rangle_s$ are identified with $\bar{\partial}_s$–cohomology classes in the Hilbert space [15,12]. They are invariant under the Y–flow. Let $\Psi_{k,\,s}$ be the corresponding (orthonormal) wave functions, seen as elements of the differential complex (i.e. as forms). Since $X_{i,0}\Psi_{k,\,s}$ is $\bar{\partial}_s$–closed, it also represents some cohomology class. From [41]

$$\langle h|X_{i,0}|k\rangle_s = \int *\bar{\Psi}_{h,\,s} \wedge X_{i,0}\Psi_{k,\,s} \tag{E.2}$$

it is manifest that

$$X_{i,0}|k\rangle_s = \sum_h |h\rangle_s \langle h|X_{i,0}|k\rangle_s + \bar{\partial}_s|\text{some state}\rangle_s \tag{E.3}$$

i.e. the relevant matrix elements give the multiplication table in the $\bar{\partial}_s$–cohomology ring. From this it follows already that the matrix $\mathcal{M}^i_s$ is independent of s, *up to conjugacy* [42]. But this is too weak a result for my purposes, and one has to do some more work.

Consider $S_\lambda = \exp[\ln(\lambda)F]$, where F is the operator which counts the degree of a form. S_λ is bounded in the Hilbert space. One has

$$\bar{\partial}_s S_\lambda = \lambda^{-1} S_\lambda \bar{\partial}_{\lambda^2 s} \quad , \tag{E.4}$$

[41] In this section for notational simplicitly I denote the chiral fields by X_i instead of $X^{\underline{q}}$.

[42] It follows also from the matrix formulation, since the minimal polynomial $p(x)$ is independent of s.

so S_λ maps $\bar\partial_{\lambda^2 s}$ classes into $\bar\partial_s$ classes. Moreover $S_\lambda X_{i,0} = X_{i,0} S_\lambda$. Then one must have

$$S_{\lambda^{-1}}|k\rangle_s = \sum_h \left(A^T\right)_{k\,h}|h\rangle_{\lambda^2 s} + \bar\partial_{\lambda^2 s}|\text{some state}\rangle \tag{E.5}$$

where A_{kh} is some non–singular matrix.

A useful remark is the following. Eq. (E.4) holds also for ∂_s. Then

$$S_{\lambda^{-1}}|k\rangle_s = \sum_h \left(A^T\right)_{k\,h}|h\rangle_{\lambda^2 s} + \partial_{\lambda^2 s}|\text{some state}\rangle \quad , \tag{E.6}$$

where I used the fact that the harmonic projection of $S_{\lambda^{-1}}|k\rangle_s$ is equal in the two cases since the $\bar\partial_s$– and ∂_s–Laplacians are equal.

On the other hand by the same token, $X_{i,0}|k\rangle_s$ can be seen as a representative of a $\bar\delta_s$–class. But now,

$$\bar\delta_s S_{\lambda^{-1}} = \lambda^{-1} S_{\lambda^{-1}} \bar\delta_{\lambda^2 s} \tag{E.7}$$

and so

$$S_\lambda|k\rangle_s = \sum_h (B^T)_{k\,h}|h\rangle_{\lambda^2 s} + \bar\delta_{\lambda^2 s}|\text{some state}\rangle \quad . \tag{E.8}$$

Taking the inner product of eqs. (E.6) , (E.8) , one gets

$$\begin{aligned}\delta_{h\,k} &= \langle h|S_\lambda S_{\lambda^{-1}}|k\rangle_s = \\ &= \Big(\sum_j \left(B^\dagger\right)_{h\,j}\langle j| + \langle \dots|\partial_{\lambda^2 s}\Big)\Big(\sum_l \left(A^T\right)_{k\,l}|l\rangle_{\lambda^2 s} + \partial_{\lambda^2 s}|\dots\rangle\Big) \\ &= \sum_j \left(B^\dagger\right)_{h\,j}\left(A^T\right)_{k\,j} = \left(B^\dagger A\right)_{h\,k}\end{aligned} \tag{E.9}$$

i.e. $B = (A^\dagger)^{-1}$.

Multiplying both sides of eq. (E.5) by $X_{i,0}$ and using eq. (E.3) , one gets (modulo $\bar\partial_{\lambda^2 s}$–exact forms)

$$\begin{aligned}&\sum_{h,j} \left(A^T\right)_{h\,j}|j\rangle_{\lambda^2 s}\langle h|X_{i,0}|k\rangle_s = \sum_h S_{\lambda^{-1}}|h\rangle_s\langle h|X_{i,0}|k\rangle_s = \\ &= S_{\lambda^{-1}}X_{i,0}|k\rangle_s = \sum_h \left(A^T\right)_{k\,h}X_{i,0}|h\rangle_{\lambda^2 s} = \sum_{h,j}\left(A^T\right)_{k\,h}|j\rangle_{\lambda^2 s}\langle j|X_{i,0}|h\rangle_{\lambda^2 s}\end{aligned} \tag{E.10}$$

$$\Rightarrow \qquad A\mathcal{M}^i_s = \mathcal{M}^i_{\lambda^2 s}A.$$

By the same manipulations, but viewing $X_{i,0}|k\rangle_s$ as a $\bar\delta_s$–class and using eq. (E.8) instead of (E.5) , one gets

$$\left(A^\dagger\right)^{-1}\mathcal{M}^i_s = \mathcal{M}^i_{\lambda^2 s}\left(A^\dagger\right)^{-1} \quad . \tag{E.11}$$

Comparing eqs. (E.10) , (E.11) one gets

$$\begin{aligned}&A\mathcal{M}^i_s A^{-1} = \left(A^\dagger\right)^{-1}\mathcal{M}^i_s A^\dagger \\ &\Rightarrow \Big[(A^\dagger A), \mathcal{M}^i_s\Big] = 0, \qquad A\mathcal{M}^i_s\left(A^\dagger A\right)^{-1} = \left(A^\dagger\right)^{-1}\mathcal{M}^i_s \quad .\end{aligned} \tag{E.12}$$

Consider the *unitary* matrix $U = A(A^\dagger A)^{-1/2}$. Then

$$\begin{aligned} U\mathcal{M}^i_s U^\dagger &= A(A^\dagger A)^{-1/2}\mathcal{M}^i_s(A^\dagger A)^{-1/2}A^\dagger = \\ &= A\mathcal{M}^i_s(A^\dagger A)^{-1}A^\dagger = (A^\dagger)^{-1}\mathcal{M}^i_s A^\dagger = \mathcal{M}^i_{\lambda^2 s} \end{aligned} \tag{E.13}$$

which is the lemma since one can get rid of U by a change of basis. Replacing in the argument F with the operator $\tilde{F}$, which acting on a (p,q) form gives $(p-q)$, one gets an operator T_λ which relates SQM models with different values of the overall coupling g in the superpotential. In many cases this is enough to compute the chiral Green functions. ∎

To prove the other lemmas of §4.2 I start with the following

(E.14) Lemma : *if $\mu \in \Lambda^*(M_\Lambda)$ satisfies*

$$s^*\mu = i_Y\mu = 0 \quad , \tag{E.15}$$

then there exists $\xi \in \Lambda^*(M_\Lambda)$ such that

$$\mu = i_Y\xi \qquad \text{and} \qquad s^*\xi = 0 \quad . \tag{E.16}$$

Moreover, if μ is holomorphic, ξ can be chosen to be holomorphic.

To prove this, I use a little dirty trick. I define a "new" external derivative $\hat{d}$ by the rules:

$$\begin{aligned} &\hat{d}X_{i,k} = \frac{1}{k}dX_{i,k} \qquad &&\text{if } k \neq 0 \\ &\hat{d}X_{i,0} = 0 &&\hat{d}(dX_{i,k}) = 0. \end{aligned} \tag{E.17}$$

Clearly, $\hat{d}^2 = 0$ and $s^*\hat{d} = \hat{d}s^* = 0$. Consider the "new" Lie derivative

$$\hat{\mathcal{L}}_Y \stackrel{\text{def}}{=} \hat{d}i_Y + i_Y\hat{d} \quad . \tag{E.18}$$

Using the explicit form of Y, eq. (4.19) , I get

$$\begin{aligned} \hat{\mathcal{L}}_Y X_{i,k} &= (1-\delta_{k,0})X_{i,k} \\ \hat{\mathcal{L}}_Y dX_{i,k} &= (1-\delta_{k,0})dX_{i,k} \quad . \end{aligned} \tag{E.19}$$

$\hat{\mathcal{L}}_Y$ commutes with i_Y. Then one can find a basis for $\ker[i_Y]$ in which $\hat{\mathcal{L}}_Y$ is diagonal. Let ϕ be a form such that

$$i_Y\phi = 0 \quad \text{and} \quad \hat{\mathcal{L}}_Y\phi = \lambda\phi \quad . \tag{E.20}$$

If $\lambda \neq 0$, from eq. (E.18) one gets

$$\phi = i_Y[\lambda^{-1}\hat{d}\phi] \qquad \text{and} \qquad s^*[\lambda^{-1}\hat{d}\phi] = 0 \quad . \tag{E.21}$$

So the only obstruction to find a form ξ as in eq. (E.16) is the component of μ belonging to the kernel of $\hat{\mathcal{L}}_Y$. Let me show that this kernel consists of the forms which can be written as $\pi^*\tau$ for some $\tau \in \Lambda^*(M)$. Write μ as a sum of monomials of the form (recall that μ is holomorphic)

$$\prod_{(i,k)\in S} X_{i,k} \bigwedge_{(j,h)\in T} dX_{i,h} \quad . \tag{E.22}$$

From eq. (E.19) one sees that, acting on this product, $\hat{\mathcal{L}}_Y$ just counts how many factors $X_{i,k}$ and $dX_{j,k}$ with $k \neq 0$ are present. Then, $\ker \hat{\mathcal{L}}_Y$ is just the space of the forms which contain only $X_{i,0}$ and $dX_{i,0}$, i.e. $\pi^*\Lambda^*(M)$. Since $s^*\pi^* = 1$, $s^*\mu = 0$ is precisely the condition for μ not to have a component in $\ker \hat{\mathcal{L}}_Y$. Now the existence of ξ follows from eq. (E.21) . ■

I will now prove Lemma (4.29) .

(4.29) Lemma : *There exists $\sigma \in \Lambda^*(M_\Lambda)$ such that*

$$\begin{aligned} &s^*\sigma = 1 \\ &\bar{\partial}_W \sigma = (\pi^* dV) \wedge \sigma. \end{aligned} \tag{E.23}$$

and σ can be chosen holomorphic. Alternatively, it can be chosen to have compact support in the vertical direction.

Proof : for simplicity I look for an *holomorphic* σ. Decomposing σ according to the degree, $\sigma = \sum_k \sigma_k$, the equations one has to solve read

$$\begin{aligned} &s^*\sigma_k = \delta_{k,0} \\ &i_Y \sigma_k = [(\pi^* dV) - d\mathcal{V}] \wedge \sigma_{k-2} \end{aligned} \tag{E.24}$$

with $k = 0, \ldots, \dim M_\Lambda$. I shall proceed by induction on k. Assuming to have solved this equation for a given k I shall find the solution for $k+2$. Notice that it is consistent to put $\sigma_{2p+1} = 0$, so only even k matters. I choose $\sigma_0 = 1$. Consider the form

$$\mu_{k+1} \equiv [(\pi^* dV) - d\mathcal{V}] \wedge \sigma_k \quad . \tag{E.25}$$

Using the induction hypothesis, eq. (E.24) , one has

$$i_Y \mu_{k+1} = -[(\pi^* dV) - d\mathcal{V}] \wedge [(\pi^* dV) - d\mathcal{V}] \wedge \sigma_{k-2} = 0 \quad . \tag{E.26}$$

(recall that $d\mathcal{V}\, i_Y + i_Y d\mathcal{V} = 0$). Moreover, $s^*\mu_{k+1} = 0$, since $s^* d\mathcal{V} = dV$. Then, by the lemma (E.14) , there exists a holomorphic form ξ_{k+2} such that $\mu_{k+1} = i_Y \xi_{k+2}$ and $s^*\xi_{k+2} = 0$. Then take $\sigma_{k+2} = \xi_{k+2}$. ■

To prove theorem (4.34) , I need two lemmas.

(E.27) Lemma : *let $\psi \in \Lambda^*(M_\Lambda)$ and $\bar{\partial}_W \psi = 0$. Then there exists a form η and a holomorphic form α with*

$$d\mathcal{V} \wedge \alpha + i_Y \alpha = 0 \tag{E.28}$$

such that

$$\psi = \bar{\partial}_W \eta + \alpha \quad . \tag{E.29}$$

The proof is a word–for–word repetition of the $\bar{\partial}_V$–Poincaré lemma (4.37) . ∎

(E.30) Lemma : *let $\psi \in \Lambda^*(M_\Lambda)$ and*

$$\bar{\partial}_W \psi = s^* \psi = 0 \quad . \tag{E.31}$$

Then there exists $\eta \in \Lambda^*(M_\Lambda)$ such that

$$\psi = \bar{\partial}_W \eta \quad . \tag{E.32}$$

Proof : by lemma (E.27) , one can assume ψ to be holomorphic and

$$d\mathcal{V} \wedge \psi + i_Y \psi = 0 \quad . \tag{E.33}$$

It is enough to show that there exists a holomorphic form η such that

$$\psi = d\mathcal{V} \wedge \eta + i_Y \eta \tag{E.34}$$

or, decomposing according to the degree,

$$\psi_k = i_Y \eta_{k+1} + d\mathcal{V} \wedge \eta_{k-1} \quad . \tag{E.35}$$

I use this equation as the induction hypothesis. Let me solve it for $k = 0, 1$. I choose $\eta_0 = 0$. From eq. (E.33) one gets $i_Y \psi_0 = i_Y \psi_1 = 0$. Since, by hypothesis, $s^*\psi_0 = s^*\psi_1 = 0$, one can apply lemma (E.14) to write $\psi_0 = i_Y \eta_1$ and $\psi_1 = i_Y \eta_2$, for certain holomorphic η_1, η_2 such that $s^*\eta_1 = s^*\eta_2 = 0$. Thus, eq. (E.35) is solved for $k = 0, 1$. Let us now suppose that eq. (E.35) is solved up to k and that $s^*\eta_r = 0$ for $r \le (k+1)$. Let me prove the induction hypothesis for $k+2$. From eq. (E.33) one has

$$i_Y \psi_{k+2} + d\mathcal{V} \wedge \psi_k = 0 \quad . \tag{E.36}$$

Substituting the induction hypothesis (E.35) one gets

$$\begin{aligned} i_Y[\psi_{k+2} - d\mathcal{V} \wedge \eta_{k+1}] &= 0 \\ s^*[\psi_{k+2} - d\mathcal{V} \wedge \eta_{k+1}] &= 0 \end{aligned} \tag{E.37}$$

where I used the fact that $s^*\psi_k = 0 \ \forall k$. Now, in eq. (E.37) one can use lemma (E.14) to find a holomorphic form η_{k+3} such that

$$\begin{aligned} s^*\eta_{k+3} &= 0 \\ \psi_{k+2} - d\mathcal{V} \wedge \eta_{k+1} &= i_Y \eta_{k+3} \qquad ∎ \end{aligned} \tag{E.38}$$

(4.34) Theorem : *Let $\psi \in \Lambda^*(M_\Lambda)$ be a $\bar{\partial}_W$–closed form. Then there exists $\eta \in \Lambda^*(M_\Lambda)$ such that*

$$(1 - \Pi^* s^*)\psi = \bar{\partial}_W \eta \quad . \tag{E.39}$$

Proof : if $\bar{\partial}_W \psi = 0$, then one has also $\bar{\partial}_W (1 - \Pi^* s^*)\psi = 0$. On the other hand, $s^*(1 - \Pi^* s^*)\psi = 0$. Apply the last lemma to the form $(1 - \Pi^* s^*)\psi$, then the theorem is proved. ∎

APPENDIX F

Some notions in Commutative Algebra

In this section I will recall some very standard material in (commutative) algebra which is often used throughout the thesis. For an introduction to the subject and some general references see [32].

(F.1) Definition : *a* **Ring** *A is a set with two binary operations (addition and multiplication) such that*

1) *A is an abelian group with respect to the addition (so that A has a zero element, denoted by 0, and every $x \in A$ has an (additive) inverse, $-x$.*
2) *Multiplication is associative ($(xy)z = x(yz)$) and distributive over the addition ($x(y+z) = xy + xz$, $(y+z)x = yx + zx$).*

(F.2) Definition : *a Ring is* **commutative** *if*

3) *$xy = yx$ for all $x, y \in A$.*

(F.3) Definition : *a ring has an* **identity element** *(denoted by 1) if:*

4) *$\exists\, 1 \in A$ such that $x1 = 1x = x$ for all $x \in A$.*

(The identity element is unique.) I will always consider commutative rings with an identity element.

(F.4) Definition : *an* **Ideal** *$\mathcal{I}$ of a ring A is a subset of A which is an additive subgroup and is such that $A\mathcal{I} \subseteq \mathcal{I}$.*

The quotient group $A/\mathcal{I}$ inherits a uniquely defined multiplication from A which makes it into a ring called the **quotient ring**.

(F.5) Definition : *an ideal $\mathcal{P}$ in A is* **prime** *if $\mathcal{P} \neq (1)$ and if $xy \in \mathcal{P} \Rightarrow x \in \mathcal{P}$ or $y \in \mathcal{P}$.*

(F.6) Definition : *an ideal m in A is* **maximal** *if $m \neq (1)$ and if there is no ideal $\mathcal{I}$ such that $m \subset \mathcal{I} \subset (1)$ (strict inclusions).*

(F.7) Definition : *a ring with exactly one maximal ideal is called a* **local ring**.

(F.8) Definition : *the multiples ax of an element $x \in A$ form an ideal called the* **principal ideal** *and denoted by (x) or Ax.*

(F.9) Definition : *let A be a ring, $A[x_1, \ldots, x_n]$ is the ring of polynomials in n indeterminates $x_1, \ldots, x_n$ with coefficients in A.*

(F.10) Definition : *let A be a ring, $A[[x_1, \ldots, x_n]]$ is the ring of formal power series $f = \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n} x_1^{m_1} \ldots x_n^{m_n}$ with coefficients in A.*

In this thesis I always consider A as the field of complex numbers $\mathbb{C}$.

Let $V(X_1, \ldots, X_n)$ be a polynomial in n–indeterminates with coefficients in $\mathbb{C}$. Let $\mathbb{C}[\partial_1 V, \ldots, \partial_n V]$ be the ring generated by its first derivatives, obviously it is an ideal of $\mathbb{C}[X_1, \ldots, X_n]$. Let $\mathcal{R}$ be the quotient ring $\mathbb{C}[\partial_1 V, \ldots, \partial_n V]/\mathbb{C}[X_1, \ldots, X_n]$. This is one of the main object introduced in this thesis and it is also one of the simplest examples of ring.

(F.11) Definition : *let A be a ring. An A–***module** *is an abelian group M (written additively) on which A acts linearly: more precisely, it is a pair (M, μ) where M is an abelian group and μ is a mapping of $A \times M$ into M such that, if one writes ax for $\mu(a, x)$ ($a \in A$, $x \in M$), the following axioms are satisfied:*

$$\begin{aligned} a(x+y) &= ax + ay \qquad & (a+b)x &= ax + bx \\ (ab)x &= a(bx) \qquad & 1x &= x \quad . \end{aligned} \tag{F.12}$$

Equivalently, M is an abelian group together with a ring homomorphism $A \to E(M)$ where $E(M)$ is the ring of homomorphisms of the abelian group M.

The notion of module is a common generalization of several familiar concepts like: an ideal, a K–vector space where K is a field, an abelian group ($\equiv$ a $\mathbb{Z}$ module), a ring, ...

(F.13) Definition : *let A be a ring and S a multiplicative subset of A (i.e. a subset containing 1 and such that if $x, y \in S$ then $xy \in S$). The* **ring of fractions** *of A by S is the ring of pairs (a, s) with $a \in A$ and $s \in S$ denoted by A/S such that two pairs are equivalent $(a, s) \sim (a', s')$ if there exists an element $s_1 \in S$ such that*

$$s_1(s'a - sa') = 0 \quad ; \tag{F.14}$$

an equivalence class is denoted by a/s; the multiplication in A/S is defined by $(a/s)(a'/s') := aa'/ss'$ and the addition by $\frac{a}{s} + \frac{a'}{s'} := \frac{s'a+sa'}{ss'}$.

(F.15) Definition : *let A be a ring and M an A–module. M is* **Noetherian** *if it satisfies one of the following three conditions:*

1) *every submodule of M is finitely generated;*
2) *every ascending sequence of submodules of M, $M_1 \subset M_2 \subset \ldots \subset M_n \subset \ldots$ such that $M_i \neq M_{i+1}$ is finite;*
3) *every non empty set S of submodules of M has a maximal element (i.e. a submodule M_0 such that for any element N of S which contains M_0 one has $M_0 = N$).*

(F.16) Definition : *let A be a ring and M an A–module. M is* **Artinian** *if it satisfies one of the two following conditions:*

1) *every descending sequence of submodules of M, $M_1 \supset M_2 \supset M_3 \supset \ldots$ such that $M_{i+1} \neq M_i$ is finite;*

2) *every non empty set S of submodules of M has a minimal element.*

(F.17) Proposition : *an Artinian ring is also Noetherian.*

(F.18) Remark : the opposite is not always true. This proposition is not true in general for modules.

(F.19) Proposition : *let A be a Noetherian (resp. Artinian) ring, M a finitely generated A–module. Then M is Noetherian (resp. Artinian).*

From this it follows

(F.20) Proposition : *let A be a Noetherian (resp. Artinian) ring, $\mathcal{I}$ and ideal of A. Then $A/\mathcal{I}$ is a Noetherian (resp. Artinian) ring.*

(F.21) Hilbert's basis Theorem : *if A is Noetherian then the polynomial ring $A[x]$ is Noetherian.*

The same result holds for $A[x_1, \ldots, x_n]$ and $A[[x_1, \ldots, x_n]]$.

The main results for Noetherian and Artinian rings are on the primary decomposition. This is the generalization of the concept of factorization in irreducible components of polynomials. I will not review this standard material which can be found for example in ref. [32].

In the study of the cohomology groups $H^n_{\bar{\partial}_V}$ a ring arises naturally. It is the ring of holomorphic functions, or better the ring of germs of analytic functions defined in some neighborhood U of the origin in $\mathbb{C}^n$. Let call it $\mathcal{O}$. This ring is isomorphic to the ring of convergent powers series. The ring $\mathcal{O}$ is a Local ring, its unique maximal ideal is given by the functions $f \in \mathcal{O}$ such that $f(0) = 0$. Moreover $\mathcal{O}$ is Noetherian.

The central object in my discussion is the quotient ring $\mathcal{O}/\mathcal{I}$ where $\mathcal{I} := \{f_1, \ldots, f_n\}$ is an ideal in $\mathcal{O}$ and $f_k := \partial_k V$. $\mathcal{O}/\mathcal{I}$ is an Artinian ring.

It is clear now how all the structures and results of commutative algebra can be applied to the study of the cohomological groups $H^n_{\bar{\partial}_V}$.

In §3.1 I have shown that the CGFs form a ring in $\mathbb{C}$ which is isomorphic to the quotient ring of formal power series $\mathcal{R} := \mathbb{C}[[x_1, \ldots x_n]]/\mathcal{I}$. Remember that the CGFs can be thought as $\Delta \times \Delta$ matrices with complex coefficients. In general, a square $\Delta \times \Delta$ matrix in a ring $\mathcal{R}$ (i.e. whose elements belong to $\mathcal{R}$) is an $\mathcal{R}$–module and also a ring. A square matrix can also always be seen as a linear map applied to a basis of a space E over $\mathcal{R}$.

Thus, one can make the following construction. From §3.1 we know the ring of matrices of the CGFs. On the other side, one can consider $X^{\underline{k}}$ as an operator acting on a space M whose basis is given by the $|\underline{q}\rangle$. $X^{\underline{k}}$ in general belongs to the ring of formal power series $\mathbb{C}[[X_1, \ldots, X_n]]$. In physics it is defined an action of the $X^{\underline{k}}$ into M. This action gives to M the structure of an $\mathcal{R}$–module where $\mathcal{R}$ is the minimal ring generated by the $X^{\underline{k}}$. Since from the physical point of view I am interested in the CGFs, $\mathcal{R}$ is by definition the ring of all the formal power series elements which have a non zero matrix (i.e. a non zero CGF). Thus $\mathcal{R} := \mathbb{C}[[X_1, \ldots, X_n]]/\mathcal{I}$ where $\mathcal{I}$ is the ideal generated by $\{\partial_1 V, \ldots, \partial_n V\}$ in $\mathbb{C}[[X_1, \ldots, X_n]]$.

By definition, all the properties of the ring $\mathcal{R}$, for example its primary decomposition, can be studied directly on the matrices. This, obviously, is much more easy to do. Thus, I will now recall some standard definitions and results in the theory of matrices and polynomials.

From the Galois theory, one can introduce in general a "*determinant*" mapping which can be seen as a map det : $\mathrm{Mat}_n(R) \to R$ where R is a ring.

(F.22) Definition : *the theory of λ–***matrices** *is the theory of matrices in R where R is the ring of polynomials in one indeterminate λ, $A = \mathbb{C}[\lambda]$.*

(F.23) Definition : *let A be a $\Delta \times \Delta$ λ–matrix or in general a $\Delta \times \Delta$ matrix in R. The* **characteristic polynomial** *is the determinant* $\det(\lambda 1_\Delta - A)$ *where 1_Δ is the unit $\Delta \times \Delta$ matrix.*

(F.24) Definition : *let A be a $\Delta \times \Delta$ matrix in R. The* **minimal polynomial** *is the monic (i.e. with director coefficient 1) polynomial in $R[\lambda]$ of minimal degree which admits A as a root.*

I now recall some results of the theory of λ–matrices. Similar matrices have the same minimal polynomial and the same characteristic polynomial. Similar matrices have the same eigenvalues with the same degeneracies. The characteristic matrices of two similar matrices are λ–equivalent. If K is a complete field and all the roots of the characteristic matrix of a matrix A belong to K, then the matrix A is similar to a Jordan matrix.

(F.25) Definition : **a Jordan matrix** *is a block diagonal matrix where each block is* **a Jordan block***:*

$$\begin{bmatrix} J(n_1,\lambda_1) & 0 & \cdots & 0 \\ 0 & J(n_2,\lambda_2) & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & J(n_k,\lambda_k) \end{bmatrix} \qquad (F.26)$$

where $J(n,\lambda)$ is a Jordan block:

$$J(n,\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & & & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix} \qquad (F.27)$$

where λ_i are the characteristic root of A.

The canonical λ–form of the characteristic matrix of a Jordan block is

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & (\lambda - \lambda_i)^n \end{bmatrix}_{n \times n} \qquad (F.28)$$

Let me now reconsider these results from the point of view of Singularity Theory (for a general review see ref. [21]).

Let f be a holomorphic map–germ in zero, $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$.

(F.29) Definition : *the* **local ring of the map** f **at zero** *is the Artinian ring* $\mathcal{R} := \mathcal{O}/\mathcal{I}$ *where* $\mathcal{I} := (\partial_1 f, \ldots, \partial_n f)$.

(F.30) Theorem : *the number of preimages near to zero of a generic point near to zero for the map* f *is equal to the dimension of the local algebra:* $\mu = \dim_{\mathbb{C}}\mathcal{R}$. μ *is called the* **algebraic local multiplicity of** f **at zero**.

(F.31) Corollary : *the local ring* $\mathcal{R}$ *of a smooth map* f *of finite multiplicity always has a basis consisting of monomials.*

(F.32) Weierstrass preparation theorem : *let* $y = f(x)$ *be a map of finite multiplicity and let* $(e_1, \ldots, e_\mu)$ *be a system of generators of its local algebra* $\mathcal{R}$. *Then for every function* α *there exists a decomposition*

$$\alpha = c_1(f(x))e_1(x) + \cdots c_\mu(f(x))e_\mu(x) \quad , \tag{F.33}$$

where the c_k *are functions of* y.

The consequences of this theorem are of fundamental importance in the study of the properties of $\mathcal{R}$.

(F.34) Theorem : a *holomorphic map germ fails to be of finite multiplicity at a point* x, *if and only if* x *is a non–isolated inverse image of zero of the germ.*

(F.35) Definition : *two germs* f *and* g *at a point* x *are said to be* **algebraically equivalent** *or, briefly,* **A–equivalent** *if there is a germ of a holomorphic family of linear non degenerate maps* $A(z) \in GL(n, \mathbb{C})$ *such that* $f(z) = A(z)g(z)$.

(F.36) Proposition : *the multiplicities of A–equivalent germs are equal.*

The known classification of holomorphic map–germs of finite multiplicity can be found in ref. [21] (see also §5.1).

APPENDIX G

N=2 Supersymmetric Quantum Mechanics on a disk

In this appendix I discuss the quantum mechanics of the model with a superpotential $V = gX^m$ defined on the open unit disk $D_R = \{|X| < R\}$ and having the $SU(1,1)$–invariant Kähler potential

$$-\ln[1 - |X|^2/R^2] \quad . \tag{G.1}$$

The quantum mechanics of such models has some peculiar property. I start with the simpler case $V = 0$ [74]. The Witten index of this model is ∞. Indeed, in the case of no superpotential the theory is $SU(1,1)$ invariant. Since the vacuum is in a non–trivial unitary representation of this non–compact group, there are an infinite number of vacua, all with the same statistics. In my notation they read [74]

$$\begin{array}{ll} Z^{m-1}dZ & m > 0 \\ \overline{Z}^{n-1}d\overline{Z} & n > 0 \quad . \end{array} \tag{G.2}$$

These functions belong to the Hilbert space $\mathcal{H}(D_R)$, in fact they vanish on the boundary $\partial\overline{D}_R$. To see this, I revert to the standard notation

$$dZ \to (1 - |Z|^2/R^2)\psi^\dagger|0\rangle \tag{G.3}$$

(the new factor comes from the zweibein of the Poincaré metric). For $V = 0$, $\bar{\partial}_V = \bar{\partial}$ and it is just the Dolbeault complex. From eq. (G.1) one sees that *all* the vacua correspond to the *trivial* de Rham class. Thus, one has an infinite number of normalizable vacua, but they do not represent any real topology of the disk. They are the harmonic representatives of the $\bar{\partial}_V$ "cohomology classes"

$$\alpha \sim \alpha + \bar{\partial}_V\beta \qquad \text{with} \quad \beta|_{\partial D_R} = 0 \quad . \tag{G.4}$$

That the Witten index is infinite can also be understood from the formula [75]

$$\text{Witten Index } = \frac{1}{2\pi}\left|\int_{D_R}\sqrt{g}R\,d^2x\right| = \infty \tag{G.5}$$

which is different from the *topological* index because of the surface term in the Gauss–Bonnet theorem.

Now add a superpotential $V = gX^m$ or any other which has a *unique* critical point in D_R of multiplicity $(m-1)$. The vacuum wave–forms can be written in terms of Bessel functions as ($k \in \mathbf{Z}$)

$$\begin{aligned} &\left(\frac{V}{\overline{V}}\right)^{k/2m} I_{k/m}(|V|)dV + \left(\frac{V}{\overline{V}}\right)^{(k+m)/2m} I_{(k+m)/m}(|V|)d\overline{V} \qquad k \geq -(m-1) \\ &\left(\frac{V}{\overline{V}}\right)^{k/2m} I_{-k/m}(|V|)dV + \left(\frac{V}{\overline{V}}\right)^{(k+m)/2m} I_{-(k+m)/m}(|V|)d\overline{V} \qquad k \leq -1 \end{aligned} \tag{G.6}$$

which vanish at the boundary in the sense of eq. (G.3) . The Witten index is again ∞. This could also be inferred from the fact that the superpotential is a *bounded* perturbation in the disk, and that the $V = 0$ theory has a finite energy–gap.

However, not all the vacuum wave–forms are $\bar{\partial}_V$–cohomologous to zero [43]. There are $\Delta = (m-1)$ of them which cannot be written as $\bar{\partial}_V \alpha$ for any regular $\alpha \in \Lambda^*(D_R)$. They read

$$\left(\frac{V}{\overline{V}}\right)^{(k-m)/2m} K_{(m-k)/m}(|V|)dV - \left(\frac{V}{\overline{V}}\right)^{k/2m} K_{k/m}(|V|)d\overline{V} \qquad k = 1, \ldots, m-1. \tag{G.7}$$

Consider the natural inclusion $i \colon D_R \to \mathbf{C}$. Denote by ω_k the vacuum wave–forms of the model with the same superpotential defined on the full complex plane $\mathbf{C}$ (see 1.5 and Appendix B). The wave–forms in eq. (G.7) are simply $i^*\omega_k$.

This is the "localization" theorem I need. For $n = 1$ it is valid at the form level, not just at the cohomological level as it is natural in view of the independence from the kinetic term proven in §4.4 .

Let me note that on the disk the formula $\langle k|\partial V/\partial X|h\rangle = 0$ does not hold. Indeed, although $\partial V/\partial X$ can be written as the supersymmetry transformation of some operator Ψ, Ψ has a singular behaviour as $|X| \to R$ and hence it is not a legitimate quantum operator since it maps functions vanishing on the boundary into functions which do not vanish. Then the arguments of the §3.1 do not apply on the disk. From the cohomological point of view, this can be understood from the formula

$$\bar{\partial}(\alpha \wedge *\bar{\beta}) = \bar{\partial}_V \alpha \wedge *\bar{\beta} - \alpha \wedge *\overline{\delta_V \beta} \tag{G.8}$$

integrating this equation on the disk, from the LHS one gets a surface term which vanishes for the good operators but not for the "bad" ones.

G.1 Matrix elements and the "holomorphic anomaly"

The next problem is to compute the matrix elements

$$\langle h+k|X^k|h\rangle_{(gL,R)} \tag{G.9}$$

[43] Of course, as in eq. (G.4) , all vacua represent non–trivial cohomology classes.

as functions of the radius R and the 1d effective coupling gL. Here $|h\rangle$ are the states in eq. (G.7) . First of all, by the rescaling $X \to \lambda X$ one sees that

$$\langle h+k|X^k|h\rangle_{(gL,R)} = \lambda^k \langle h+k|X^k|h\rangle_{(gL\lambda^m,R/\lambda)} \quad . \tag{G.10}$$

The theory is R–invariant. If the Ward–identity arguments of §3.4 would apply, $\langle m + k|X^k|m\rangle_{(gL,R)} = f(R)g^{-k/m}$ which together with eq. (G.10) would imply that the matrix elements in eq. (G.9) are independent of R. However, the situation is more subtle. What is not longer true is the holomorphic dependence on g.

For the wave–functions, one has

$$\frac{\partial}{\partial g}\omega_k = \partial_V \left[\frac{\partial V}{\partial g}\chi_k\right] \qquad \frac{\partial}{\partial \bar{g}}\omega_k = \bar{\partial}_V \left[\frac{\partial \overline{V}}{\partial \bar{g}}\tilde{\chi}_k\right] \quad . \tag{G.11}$$

Then, one has

$$\begin{aligned}\frac{\partial}{\partial \bar{g}}\int_{D_R} *\overline{\omega_{h+k}} \wedge X^k \omega_h &= -\int_{D_R} \bar{\partial} * \left\{X^k \frac{\partial \overline{V}}{\partial \bar{g}}[\overline{\chi_{h+k}}\omega_h + \tilde{\chi}_h \overline{\omega_{h+k}}]\right\} = \\ &= i\oint_{\partial D_R}\left[X^k \frac{\partial \overline{V}}{\partial \bar{g}}\right][\overline{\chi_{h+k}}\,\chi_h + \tilde{\chi}_h\,\tilde{\chi}^*_{h+k}]dV \neq 0 \quad .\end{aligned} \tag{G.12}$$

The formal argument is not valid because of the "surface term" in the integration by parts. Of course, this term is absent in the plane. Such "spurious" terms in formal identities are common in theories in which the fields take value in "bounded domains". In the path–integral approach they may be seen as due to a failure of the functional integration by parts because of *surface contribution* in field configuration space. They are typical of the finite–volume situation, for not too patological quantities they go to zero as the volume of space go to infinity.

What is wrong with the argument based on the equation

$$\frac{\partial}{\partial \bar{g}}\langle k+h|X^k|h\rangle = \int_{-\infty}^{+\infty} dt d^2\theta T\,\langle k+h|X^k \frac{\partial \overline{V}}{\partial \bar{g}}(t,\theta)|h\rangle \quad ? \tag{G.13}$$

The integrand should be zero by supersymmetry, since V is a good operator. In fact, formally the problem comes from the integration in dt. As it is well known from old–fashioned perturbation theory, this integration is equivalent to the insertion of the resolvent of the Hamiltonian $(H - E)^{-1}$. For a supersymmetric vacuum $E = 0$. In fact, I already noticed that one can rewrite eq. (G.11) as

$$\frac{\partial}{\partial g}\omega_k = \frac{1}{H}\partial_V \delta_V \left[\frac{\partial V}{\partial g}\omega_k\right] \tag{G.14}$$

now, H^{-1} is not a good operator on the Hilbert–space, even projecting out the subspace generated by the wave–forms ω_k. H^{-1} diverges like $(1 - |X|^2/R^2)^{-2}$ as $|X| \to R$. Since this "anomaly" is associated with the vanishing of the inverse propagator one can say that the usual argument is spoiled because of an IR divergence (or, better, an IR ambiguity). There is a third interpretation which shows that there is nothing anomalous about this phenomenon. In fact, it is just the basis problem again. Indeed, since H is strongly elliptic, the fact that H^{-1} is not well–defined on the subspace of the Hilbert space $\mathcal{H}(D_R)$ orthogonal to the ω_k is equivalent to say that there are additional supersymmetry vacua. Moreover, only vacua with the same R–charge as ω_k may contribute in the above formulae. From eq. (G.6) it follows that for each ω_k there is a second supersymmetric vacuum with the same quantum numbers. Varying g, in general, these vacua mix together. Then, what it happens is just that there is a basis in which the naive argument is valid, but it is not the one useful for my purposes.

However, this is just a minor nuisance. In computing the matrix elements for the states one has the very same integrals as in the case of the plane, (see eq. (B.40)) except that now one integrates in ϱ up to R. Taking the derivative of this integrals with respect R one gets

$$\frac{\partial(\text{Integral})}{\partial R} = O(R^{a}e^{-2|g|LR^{m}}) \quad ; \tag{G.15}$$

The same result follows from the general argument in eq. (G.12) . In the limit $R \to \infty$ one gets

$$\langle h+k|X^{k}|h\rangle_{(gL,R)} = \langle h+k|X^{k}|h\rangle_{(gL,\infty)} + O(e^{-2|g|LR^{m}}) \quad . \tag{G.16}$$

Therefore in the limit $L \to \infty$ one obtains the same results as in the theory defined on the plane (§1.5) up to terms which are exponentially small. This proves the universality in the critical limit of the numbers computed in §1.5 with a special choice of the kinetic term.

G.2 Generalization to bounded domains

Most of the above discussion applies to any bounded domain $\Omega \in \mathbb{C}^{n}$ equipped with the Bergmann metric (or any other complete Kähler metric). In particular, the Witten index is ∞ by the same token.

Moreover, by the "universality" proven in §4.4 the vacuum wave–forms in Ω which represent *true* cohomology, are essentially the pull-back of the corresponding wave–forms in $\mathbb{C}^{n}$. More precisely, $\omega_{\Omega}|_{(n,0)} = i^{*}\omega_{\mathbb{C}^{n}}|_{(n,0)}$ and an analogous equation for the $(0,n)$ component holds true. This "functorial" property of the wave–form under (holomorphic) inclusions is exactly what is needed in §6.1 to make contact with Singularity Theory. Again, these are "good" states on Ω because they vanishes, in the physical sense, on the boundary by the usual asymptotics of the Bergman kernel.

APPENDIX H

DEFERRED PROOFS FROM CHAPTER 6

In this appendix I will give some proof deferred from Chapter 6.

(6.28) Lemma : *let V be a superpotential with a singular point in zero and (local) Witten index Δ. Let A_k^h and $\widetilde{A}_k^h$ be two $\Delta \times \Delta$ arbitrary matrices which, under a g–dependent change of basis*

$$\omega_k \longrightarrow S_k^h(g,\bar{g})\omega_h \tag{H.1}$$

transform as $GL(\Delta,\mathbb{C})$ connections. Define the **Natural covariant derivatives**

$$\begin{aligned} D_g &:= g\frac{\partial}{\partial g} - A \\ D_{\bar{g}} &:= \bar{g}\frac{\partial}{\partial \bar{g}} - \widetilde{A} \quad . \end{aligned} \tag{H.2}$$

Then

$$D_g\omega_k = \eta_k \qquad D_{\bar{g}}\omega_k = \tilde{\eta}_k \tag{H.3}$$

where η_k and $\tilde{\eta}_k$ are orthogonal to the vacuum wave forms ω_k and are given by

$$\eta_k = \partial_V\lambda_k \qquad \tilde{\eta}_k = \overline{\partial}_V\tilde{\lambda}_k \tag{H.4}$$

where λ_k is the unique solution to

$$\begin{aligned} &\overline{\partial}_V\lambda_k = V\,\omega_k - B_k^h\omega_h \qquad \partial_V\lambda_k = 0 \\ &B_k^h := C_{hl}\int \overline{*\omega_l} \wedge V\,\omega_k \\ &\left(C^{-1}\right)_{kj} := \int \overline{*\omega_k} \wedge \omega_j \end{aligned} \tag{H.5}$$

and $\tilde{\lambda}_k$ is the unique solution to

$$\begin{aligned} &\partial_V\tilde{\lambda}_k = \overline{V}\,\omega_k - \widetilde{B}_k^h\omega_h \qquad \overline{\partial}_V\tilde{\lambda}_k = 0 \\ &\widetilde{B}_k^h := C_{hl}\int \overline{*\omega_l} \wedge \overline{V}\,\omega_k \quad . \end{aligned} \tag{H.6}$$

Proof : the wave–forms ω_k satisfy the equations (see Appendix B)

$$\bar{\partial}_V \omega_k = \partial_V \omega_k = \Lambda \omega_k = 0 \quad . \tag{H.7}$$

Taking the derivative of these equations with respect the overall coupling constant g, one gets (notice that $\partial_V[V\omega_k] = \partial V \wedge \omega_k$)

$$\begin{aligned} &\bar{\partial}_V g \frac{\partial}{\partial g}\omega_k + \partial_V \left[V \omega_k\right] = 0 \\ &\partial_V g \frac{\partial}{\partial g}\omega_k = 0 \\ &\bar{\partial}_V \bar{g} \frac{\partial}{\partial \bar{g}}\omega_k = 0 \\ &\partial_V \bar{g} \frac{\partial}{\partial \bar{g}}\omega_k + \bar{\partial}_V \left[V \omega_k\right] = 0 \\ &\Lambda \bar{g} \frac{\partial}{\partial \bar{g}}\omega_k = \Lambda g \frac{\partial}{\partial g}\omega_k = 0 \quad . \end{aligned} \tag{H.8}$$

Using the definition $H := \delta_V \bar{\partial}_V + \bar{\partial}_V \delta_V$, it is easy to rewrite these formulae in the standard form of (old fashioned) perturbation theory, e.g.

$$\begin{aligned} H\left(g \frac{\partial}{\partial g}\omega_k\right) &= -\delta_V \partial_V \left[V \omega_k\right] \\ H\left(\bar{g} \frac{\partial}{\partial \bar{g}}\omega_k\right) &= -\bar{\delta}_V \bar{\partial}_V \left[\overline{V} \omega_k\right] \quad . \end{aligned} \tag{H.9}$$

The general solution to the above equations is

$$\begin{aligned} g \frac{\partial \omega_k}{\partial g} &= A_k^{\ h} \omega_h + \eta_k \\ \bar{g} \frac{\partial \omega_k}{\partial \bar{g}} &= \tilde{A}_k^{\ h} \omega_h + \tilde{\eta}_k \end{aligned} \tag{H.10}$$

where A, $\tilde{A}$ are arbitrary matrices [44], and η_k, $\tilde{\eta}_k$ are particular solutions, orthogonal to the vacuum forms ω_k. The basis problem stems from the arbitrariness of the V–harmonic terms in the RHS of this equation. Under a g–dependent change of basis,

$$\omega_k \to S_k^{\ h}(g, \bar{g})\, \omega_h \quad , \tag{H.11}$$

one has

$$\begin{aligned} &A \longrightarrow A' = SAS^{-1} + g \frac{\partial S}{\partial g} S^{-1} \\ &\eta \to \eta' = S\eta, \end{aligned} \tag{H.12}$$

that is, A transforms as a $GL(\Delta, \mathbb{C})$ connection. For this reason, I define the natural covariant derivatives

$$\begin{aligned} D_g &= g \frac{\partial}{\partial g} - A \\ D_{\bar{g}} &= \bar{g} \frac{\partial}{\partial \bar{g}} - \tilde{A}, \end{aligned} \tag{H.13}$$

[44] In a real basis, A and $\tilde{A}$ are complex conjugate.

so that

$$D_g\omega_k = \eta_k \qquad D_{\bar{g}}\omega_k = \tilde{\eta}_k \quad . \tag{H.14}$$

This connection is uniquely defined (up to gauge transformations) by the condition that the covariant derivatives are orthogonal to all the V–harmonic forms ω_k.

Let P be the projector onto the space orthogonal to the V–harmonic forms,

$$\begin{aligned} P(V\omega_k) =& V\omega_k - \sum_{h,j} \omega_h C_{hj} \int *\overline{\omega_j} \wedge V\omega_k \\ (C^{-1})_{kj} =& \int *\overline{\omega_k} \wedge \omega_j \quad . \end{aligned} \tag{H.15}$$

By the Hodge theorem, there exists a *unique* (regular) λ_k in $\mathcal{H}$ such that

$$P(V\omega_k) = \bar{\partial}_V\lambda_k \qquad \delta_V\lambda_k = 0 \quad . \tag{H.16}$$

The first of eqs. (H.8) implies,

$$\bar{\partial}_V\left[Pg\frac{\partial\omega_k}{\partial g}\right] = -\partial_V\left[P(V\omega_k)\right] \tag{H.17}$$

from which

$$\bar{\partial}_V\left[Pg\frac{\partial\omega_k}{\partial g} - \partial_V\lambda_k\right] = 0 \quad , \tag{H.18}$$

again by Hodge-Kähler, there is a *unique* ϱ_k in $\mathcal{H}$ such that

$$P\left(g\frac{\partial\omega_k}{\partial g}\right) - \partial_V\lambda_k = \bar{\partial}_V\varrho_k \quad \text{and} \quad \delta_V\varrho_k = 0 \quad . \tag{H.19}$$

On the other hand, eqs. (H.8) imply

$$\delta_V\left[P\left(g\frac{\partial\omega_k}{\partial g}\right)\right] = 0 \quad . \tag{H.20}$$

Then, $\delta_V\bar{\partial}_V\varrho_k = 0 \;\Rightarrow\; H\varrho_k = 0 \;\Rightarrow\; \varrho_k = 0$. $P(g\partial\omega_k/\partial g) = \partial_V\lambda_k$ is the particular solution I am looking for. So,

$$D_g\omega_k = \partial_V\lambda_k \quad , \tag{H.21}$$

where λ_k is the unique solution to

$$\bar{\partial}_V\lambda_k = V\omega_k - \sum_{hj}\omega_h C_{hj}\int *\overline{\omega_j}\wedge V\omega_k, \qquad \delta_V\lambda_k = 0 \quad . \tag{H.22}$$

Analogously,

$$D_{\bar{g}}\omega_k = \bar{\partial}_V\tilde{\lambda}_k \tag{H.23}$$

where $\tilde{\lambda}_k$ is the unique solution to

$$\partial_V \tilde{\lambda}_k = \bar{V}\omega_k - \sum_{hj} \omega_h C_{hj} \int *\overline{\omega_j} \wedge \bar{V}\omega_k, \qquad \bar{\delta}_V \tilde{\lambda}_k = 0 \quad . \quad \blacksquare \tag{H.24}$$

It is worth to notice that the covariant derivative D_g satisfies the following relations:

$$\begin{aligned}
&D_g V = 0 \qquad\qquad D_{\bar{g}}\overline{V} = \overline{V} \\
&D_g \overline{V} = 0 \qquad\qquad D_{\bar{g}} V = 0 \\
&\left[D_g, \overline{\partial}_V\right] \omega_k = \partial V \wedge \omega_k \\
&\left[D_{\bar{g}}, \partial_V\right] \omega_k = \overline{\partial V} \wedge \omega_k \\
&\left[D_{\bar{g}}, \overline{\partial}_V\right] \omega_k = 0 \\
&\left[D_g, \partial_V\right] \omega_k = 0
\end{aligned} \tag{H.25}$$

where ω_k is a physical vacuum wave form, i.e. satisfies $\bar{\partial}_V \omega_k = \partial_V \omega_k = \Lambda \omega_k = 0$.

(6.35) Corollary : *the curvature of the connection D_g is*

$$\begin{aligned}
&[D_{\bar{g}}, D_g]\omega_k = \mathcal{F}_k^h \omega_h \\
&\mathcal{F} = [B, \widetilde{B}] \quad .
\end{aligned} \tag{H.26}$$

Proof : it is easy to show that

$$\Big[D_{\bar{g}},\, D_g\Big]\omega_k = \mathcal{F}_k{}^h \omega_h \tag{H.27}$$

for some matrix $\mathcal{F}_k{}^h$, i.e. that the LHS of this equation is V–harmonic. A simple computation shows that

$$\begin{aligned}
\Big[D_{\bar{g}},\, D_g\Big]\omega_k = &\ \partial_V D_{\bar{g}} \lambda_k - \bar{\partial}_V D_g \tilde{\lambda}_k + \bar{\partial}_V(\bar{V}\lambda_k) - \\
&- \partial_V(V\tilde{\lambda}_k) + [V\bar{B}_k{}^h \omega_h - \bar{V} B_k{}^h \omega_h] \quad .
\end{aligned} \tag{H.28}$$

The RHS is easily computed using standard Hodge–theoretical arguments giving the result:

$$\mathcal{F} = \left[B, \widetilde{B}\right] \quad . \quad \blacksquare \tag{H.29}$$

(6.50) Lemma : *let T_k^h be the* **torsion** *defined by the equations*

$$(V\delta_k^h - B_k^h)\partial_V \eta_h = \overline{\partial}_V \xi'_k + T_k^h \alpha_h + \overline{V} dV \wedge \sigma \quad . \tag{H.30}$$

Then $D_g \alpha_k = d\sigma_k + T_k^h \alpha_h$ and the **Master Equations** *are*

$$\begin{aligned}
&D_g s[\alpha_k](t) = \left[(t\delta_k^h - B_k^h)\nabla_t + T_k^h\right] s[\alpha_h](t) \\
&D_{\bar{g}} s[\alpha_k](t) = 0
\end{aligned} \tag{H.31}$$

where ∇_t is the Gauss–Manin connection of the singularity.

Proof : to extract the action of D_g on $G_0(\omega_k)$ (or $s[\alpha_k](t)$) is a bit delicate since one should pay attention to the representatives one uses. Of course, it is not important what they actually are, but that one uses them consistently. One check of the consistency of the procedure is that the resulting curvature is equal to the one computed in the previous corollary. Let

$$\omega_k = \bar{\partial}_V \eta_k + \alpha_k \tag{H.32}$$

with α_k holomorphic. Here $\eta_k \in \Lambda^*(\mathbb{C}^n)$ and α_k are defined only up to the equivalence $\alpha \to \alpha + \partial V \wedge \tau$ (see §4.3). From Lemma (6.28) one has

$$D_{\bar{g}}\omega_k = \bar{\partial}_V D_{\bar{g}}\eta_k + D_{\bar{g}}\alpha_k = \bar{\partial}_V \tilde{\lambda}_k \quad . \tag{H.33}$$

It is consistent to choose the representatives so that $D_{\bar{g}}\eta_k = \tilde{\lambda}_k$, and then

$$D_{\bar{g}}\alpha_k = 0 \quad . \tag{H.34}$$

On the other hand

$$D_g\alpha_k = \partial_V \lambda_k - dV \wedge \eta_k - \bar{\partial}_V D_g \eta_k \quad . \tag{H.35}$$

From the definition of the matrix B one sees that

$$\begin{aligned} &G_0\big[V\omega_k - \sum_h B_k{}^h\omega_h\big] = 0 \\ \Rightarrow \qquad &V\alpha_k - \sum_h B_k{}^h\alpha_h = dV \wedge \sigma_k \quad , \end{aligned} \tag{H.36}$$

for some *holomorphic* $(n-1,0)$ form σ_k. I adopt the following short–hand notation. If μ is a k–form, I shall use $\mu|$ to denote its $(k,0)$ projection. Projecting on the $(n,0)$ component the defining equation of λ_k (see Lemma (6.28)) one gets

$$\begin{aligned} &dV \wedge \lambda_k\big| = dV \wedge \Big(V\eta_k\big| - B_k{}^h\eta_h\big| + \sigma_k\Big) \\ \Rightarrow \quad &\lambda_k\big| = \sigma_k + (V\eta_k - B_k{}^h\eta_h)\big| + dV \wedge \varrho_k \quad , \end{aligned} \tag{H.37}$$

for some $(n-2,0)$ form ϱ_k. Then, one has

$$D_g\alpha_k = d\sigma_k + (V\partial\eta_k - B_k{}^h\partial\eta_h)\big| - dV \wedge \Big(\partial\varrho_k + D_g\eta_k\big|\Big) \quad . \tag{H.38}$$

Consider the expression

$$(V\delta_k{}^h - B_k{}^h)\partial_V\eta_h - \overline{V}dV \wedge \sigma_k \quad . \tag{H.39}$$

It is obviously $\bar{\partial}_V$–closed. Then one can write

$$\begin{aligned} (V\delta_k{}^h - B_k{}^h)\partial_V\eta_h &= \bar{\partial}_V\xi_k + T_k{}^h\omega_h + \overline{V}dV \wedge \sigma_k = \\ &= \bar{\partial}_V\xi_k' + T_k{}^h\alpha_h + \overline{V}dV \wedge \sigma_k \end{aligned} \tag{H.40}$$

for some coefficients ${T_k}^h$. Then

$$D_g\alpha_k = d\sigma_k + {T_k}^h\alpha_h + dV \wedge \left(\xi'_k\big| - \partial\varrho_k - D_g\eta_k\big| + \overline{V}\sigma_k\right) \quad . \tag{H.41}$$

Again, it is consistent to choose the representatives so that the term multiplying dV in the RHS cancels. Finally one has

$$D_g\alpha_k = d\sigma_k + {T_k}^h\alpha_h \tag{H.42}$$

where now only *holomorphic* forms appear. Let take the geometric section of both sides of this equation (see §5.4). Since by definition $\alpha_k = dV \wedge s[\alpha_k]$, one has ($V = gF$)

$$s[D_g\alpha_k](t) = \big(D_g + 1\big)s[\alpha_k](t) \quad , \tag{H.43}$$

and then

$$\big(D_g + 1\big)s[\alpha_k](t) = s[d\sigma_k](t) + {T_k}^h s[\alpha_h](t) \quad . \tag{H.44}$$

This expression can be simplified using standard properties of the geometrical sections (see (5.83))

$$\begin{aligned} s[d\sigma_k](t) &= \frac{\partial}{\partial t}s[dV \wedge \sigma_k](t) = \\ &= \frac{\partial}{\partial t}s[V\alpha_k - {B_k}^h\alpha_h](t) = \\ &= \frac{\partial}{\partial t}\left\{\left[t{\delta_k}^h - {B_k}^h\right]s[\alpha_h](t)\right\} \quad . \end{aligned} \tag{H.45}$$

So the final formulae (Master equations) read

$$\begin{aligned} &D_{\bar g}s[\alpha_k](t) = 0 \\ &D_g s[\alpha_k](t) = \left[\left(t{\delta_k}^h - {B_k}^h\right)\nabla_t + {T_k}^h\right]s[\alpha_h](t) \quad . \end{aligned} \tag{H.46}$$

These equations (at least in principle) can be solved to give the explicit dependence on g, $\bar g$ of the germ–forms α_k and hence, *via* the G–isomorphism of the states and then of the chiral matrix elements $\langle h|X_i|k\rangle$.

As a further check of the consistency of the "η–α splitting" procedure, let compute the curvature of the D_g connection through the Master equations. First operate $D_{\bar g}$ on both sides of eq. (H.40) . Recalling that $D_{\bar g}B = D_{\bar g}\sigma_k = 0$, one gets

$$\begin{aligned} &(V{\delta_k}^h - {B_k}^h)D_{\bar g}\partial_V\eta_h - \overline{V}dV \wedge \sigma_k = \\ &\qquad = \bar\partial_V D_{\bar g}\xi_k + \left(D_{\bar g}{T_k}^h\right)\omega_k + {T_k}^h\bar\partial_V\tilde\lambda_h \quad , \end{aligned} \tag{H.47}$$

expanding the first term and equating the two sides in cohomology, one gets

$$D_{\bar g}T = \mathcal{F} = [B, \tilde B] \quad . \tag{H.48}$$

Now,

$$[D_{\bar g}, D_g]s[\alpha_k] = D_{\bar g}D_g s[\alpha_k] = (D_{\bar g}{T_k}^h)s[\alpha_h] = {\mathcal{F}_k}^h s[\alpha_h] \tag{H.49}$$

as it should. ∎

<u>(6.53) Proposition :</u> *let V be a quasihomogeneous superpotential, then*

$$T_h^k = 0 \quad . \tag{H.50}$$

<u>Proof :</u> consider the expression $(\partial_V \eta_k - \overline{V}\alpha_k)$ where $\omega_k = \bar{\partial}_V \eta_k + \alpha_k$. It is $\bar{\partial}_V$–closed. So there exists a matrix ${Z_k}^h$ such that

$$(\partial_V \eta_k - \overline{V}\alpha_k) = {Z_k}^h \omega_h + \bar{\partial}_V(\cdots) \quad . \tag{H.51}$$

From the definitions one sees that

$$T = [Z,\, B] \qquad D_{\bar{g}} Z = -\widetilde{B} \quad . \tag{H.52}$$

Hence $B = 0 \Rightarrow T = 0$. ∎

Notice that the matrix Z is not a cohomological object since it depends on the representative of the vacuum germ form chosen. Indeed, under $\eta_k \to \eta_k + \tau$, $\alpha_k \to \alpha_k - \partial V \wedge \tau$, ω_k does not change but Z does.

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SUBJECT INDEX