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NON-CONVEX PROBLEMS FOR MULTIFUNCTIONS

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INDEX

PART I : PRELIMINARY CONSIDERATIONS AND SELECTION THEOREMS

PART II : EXISTENCE RESULTS FOR DIFFERENTIAL INCLUSIONS

PART I

PRELIMINARY CONSIDERATIONS AND SELECTION THEOREMS

1. INTRODUCTION

In the theory of multivalued maps, the assumption that the values of a multifunction F are convex subsets of a normed space is often crucial: many results which can be proved under the convexity hypothesis are indeed false when F has not convex values. In some sense, the convexity discriminates those multifunctions which behave like single-valued functions. For instance, a lower semicontinuous multifunction with closed and convex values can be represented by continuous (single valued) selections (Michael's Theorem: see [3,Theorem 1.11.1]), while if the convexity fails continuous selections may not exist (see [3, Examples 1.6.1 and 2]); an analogue of Peano's theorem for ordinary differential equations holds for a differential inclusion with upper semicontinuous and closed convex valued right-hand side (see [3,Theorem 2.1.3]), while examples (see Example **4**.1.1 in Part II) show that the convexity hypothesis cannot be eliminated.

Multivalued maps with non-convex values have however been studied by many authors, and many developments have been obtained by using new techniques. The present thesis is devoted to discuss some recent results concerning the existence of continuous selections and of solutions to differential inclusions without convexity.

2. NOTATIONS AND BASIC DEFINITIONS

Let X be any set: by 2^X we denote the family of all the nonempty subsets of X. A multifunction F, having X as domain and taking values in a set Y, is a map $F:X \to 2^Y$. In the present thesis, X and Y are always supposed to be at least metric (or Banach) spaces, with distances d_X and d_Y, respectively; in the product space $X \times Y$, the distance is $d_X + d_Y$.

If $y \in Y$ and $A, B \subseteq Y$, we define $d_Y(y,A) = \inf\{d_Y(y,z) : z \in A\}$, the open ε-neighbourhood of A as $B(A,\varepsilon) = \{y \in Y : d_Y(y,A) < \varepsilon\}$ and the separation between A and B as $h^*(A,B) = \sup\{d_Y(y,B) : y \in A\}$. The Hausdorff distance between A and B is $h(A,B) = \max\{h^*(A,B), h^*(B,A)\}$; h is actually a metric on the space of all closed and bounded subsets of Y. The separation between A and B can be equivalently defined as $h^*(A,B) = \inf\{\varepsilon : A \subseteq B(B,\varepsilon)\}$. By using these definitions, some continuity concepts can be stated.

DEFINITION **2**.1. *The multivalued map* $F:X \to 2^Y$ *is*

a) *lower semicontinuous (l.s.c.) in* X *iff* $\forall x_o \in X$, $\forall y_o \in F(x_o)$, $\forall \varepsilon > 0$, $\exists \delta > 0$ *such that*

$$d_X(x,x_o) < \delta \quad \textit{implies} \quad F(x) \cap B(y_o,\varepsilon) \neq \emptyset \ ;$$

b) *upper semicontinuous (u.s.c.) in* X *iff* $\forall x_o \in X$ *and for every open set* $U \subseteq Y$ *containing* $F(x_o)$, $\exists \delta > 0$ *such that*

$$F(B(x_o,\delta)) \subseteq U \ ;$$

c) *continuous in* X *iff it is both l.s.c. and u.s.c..*

The following equivalent formulations of a) and b) will be also used:

a) $\Leftrightarrow$ the set $F^+(C) = \{x \in X : F(x) \subseteq C\}$ is closed in X for every closed set $C \subseteq Y$;

b) $\Leftrightarrow$ the set $F^-(C) = \{x \in X : F(x) \cap C \neq \emptyset\}$ is closed in X for every closed set $C \subseteq Y$.

DEFINITION **2**.2. *A multivalued map* $F:X \to 2^Y$ *is*

a') *Hausdorff-lower semicontinuous in* X *(h-l.s.c.) iff* $\forall x_o \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$ *such that*

$$x \in B(x_o, \delta) \quad \textit{implies} \quad h^*(F(x_o), F(x)) < \varepsilon \quad ;$$

b') *Hausdorff-upper semicontinuous in* X *(h-u.s.c.) iff* $\forall x_o \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$ *such that*

$$x \in B(x_o, \delta) \quad \textit{implies} \quad h^*(F(x), F(x_o)) < \varepsilon \quad ;$$

c') *Hausdorff-continuous iff it is both h-l.s.c. and h-u.s.c., i.e. iff* $\forall x_o$, $\forall \varepsilon > 0$, $\exists \delta > 0$ *such that*

$$x \in B(x_o, \delta) \quad \textit{implies} \quad h(F(x_o), F(x)) < \varepsilon \quad .$$

The graph of the multivalued map F, graph{F}, is the set $\{(x,y) \in X \times Y : y \in F(x)\}$. We recall the following implications among the above concepts:

PROPOSITION **2**.3. *Let* $F:X \to 2^Y$ *be a multifunction. Then:*

1) a') $\Rightarrow$ a) *and* b) $\Rightarrow$ b');

2) *if* F *has compact values, then* a) $\Leftrightarrow$ a') *and* b) $\Leftrightarrow$ b'), *and therefore* c) $\Leftrightarrow$ c');

3) *if* F *is u.s.c. with closed values, then it has closed graph; conversely, if* Y *is compact a map with closed graph is u.s.c..*

Proof. See [3,§1.1].

We remark also that if f is a continuous function from Y into a space Z and $F:X\to 2^Y$ is u.s.c., then the multifunction $f\circ F:X\to 2^Z$, $(f\circ F)(x) = \{f(y):y\in F(x)\}$, is u.s.c..

A selection from the multivalued map $F:X\to 2^Y$, is a single-valued function $f:X\to Y$ such that $f(x)\in F(x)$ for every $x\in X$.

Let now E be a Banach space, Ω an open subset of $\mathbb{R}\times E$ and $(t_o,x_o)\in\Omega$. By a solution of the Cauchy problem for differential inclusions

$$x'(t) \in F(t,x(t)) \qquad (2.1)$$

$$x(t_o) = x_o \qquad (2.2)$$

we mean a function $x:[t_o,t_o+T]\to E$, $T>0$, which is absolutely continuous and satisfies (2.1) for almost every $t\in I=[t_o,t_o+T]$, and (2.2).

In what follows, we consider also a measure space $(T,\mathcal{F},\mu)$, where $\mathcal{F}$ is a σ-algebra of subsets of T and μ is a non-atomic probability measure on $\mathcal{F}$. We recall that an atom of the measure μ is a set $A\in\mathcal{F}$ such that $\mu(A)>0$ and, for every measurable $B\subseteq A$, either $\mu(B)=0$ or $\mu(B)=\mu(A)$; μ is non-atomic if it has no atoms. Given a μ-integrable function $f:T\to\mathbb{R}$, we write $f\cdot\mu$ for the measure having density f with respect to μ. We denote by $\sigma\{A_\lambda:\lambda\in\Lambda\}$ the σ-algebra generated by a family of measurable sets $A_\lambda\in\mathcal{F}$; if $(S,\mathcal{G})$ is another measure space, on the product $S\times T$ the σ-algebra $\mathcal{G}\otimes\mathcal{F} = \sigma\{A\times B:A\in\mathcal{G},\ B\in\mathcal{F}\}$ will be considered. The characteristic function of a set $A \in \mathcal{F}$ is written χ_A. If E is a Banach space with norm $\|\cdot\|_E$ and unit ball B, $L^1(T,E)$ denotes the Banach space of Bochner μ-integrable functions $u:T \to E$ [49,p.132], with norm $\|u\|_1 = \int_T \|u\|_E d\mu$. The Lebesgue measure in $\mathbb{R}^d$ is indicated by meas(·); the σ-algebras of Lebesgue measurable and Borel sets in $\mathbb{R}^d$ are written, respectively, $\mathcal{L}$ and $\mathcal{B}$.

Following Hiai and Umegaki [30], we now introduce an important concept which will be discussed later.

DEFINITION 2.4. *A set* $K \subseteq L^1(T,E)$ *is decomposable iff*

$$u \cdot \chi_A + v \cdot \chi_{T \setminus A} \in K \quad \textit{whenever } u,v \in K,\ A \in \mathcal{F}.$$

The collection of all nonempty decomposable subsets of $L^1(T,E)$ *is denoted by* $\mathrm{Dec}(L^1(T,E))$.

A multivalued map $F:T \to 2^E$ is said measurable iff the set $F^-(C) \in \mathcal{F}$ for every $C \subseteq E$ closed, and integrably bounded iff there exists a positive L^1 function $p:T \to \mathbb{R}$ such that $\|F(t)\| := \sup\{\|x\|_E : x \in F(t)\} \le p(t)$ a.e. in T. We recall that the set $S_F = \{u \in L^1(T,E) : u(t) \in F(t)$ a.e. in $T\}$ of all the L^1 selections from a measurable integrably bounded multifunction $F:T \to 2^E$ is decomposable. When F is integrably bounded, the set S_F can be used to define a concept of integral of F, the Aumann integral: $\int_T F d\mu = \{\int_T u d\mu : u \in S_F\}$.

Finally, given a set $A \subseteq X$, its diameter is indicated by diam $(A) = \sup\{d_X(x,x') : x,x' \in A\}$, and if A is a subset of the Banach space E, the closed convex hull of A is written $\overline{\mathrm{co}}A$, while, for $\lambda \in \mathbb{R}$ and $B \subseteq E$, $\lambda A = \{\lambda x : x \in A\}$, $A+B = \{x+y : x \in A, y \in B\}$. If A is convex, define extA as the set of all the extreme points of A, i.e. the set of points $x \in A$ such that no nondegenerate segment in A exists which contains x in its relative interior; its closure is indicated by $\overline{\text{ext}}A$. We write also $\overline{A}$ and A^c for the closure and the complement of A, respectively, while $A \setminus B$ denotes the set-theoretic difference between A and B; their symmetric difference is $A \Delta B = (A \setminus B) \cup (B \setminus A)$ and $\#A$ stands for the cardinality of A. If $f : X \to \mathbb{R}$ is a single-valued function, by supp(f) we mean the closure in X of the set $\{x \in X : f(x) \neq 0\}$. If $a, b \in \mathbb{R}$, we write $a \wedge b$ for $\min\{a,b\}$ and $a \vee b$ for $\max\{a,b\}$.

3. SELECTIONS THEOREMS FOR MULTIFUNCTIONS WITH DECOMPOSABLE VALUES

3.1 INTRODUCTION

In [1], Antosiewicz and Cellina presented an analogue of Michael's selection theorem for a special non-convex valued multifunction related to differential inclusions: the image of a point was the set of all measurable selections from a given multivalued map. The main newness of their proof was the idea of continuously interpolating among measurable functions by using a method of "cutting and piecing", which substituted convex combinations.

More recently [28], Fryszkowski introduced explicitely the decomposability in the multivalued maps framework and stated an abstract version of the Antosiewicz-Cellina selection theorem: the decomposable sets are precisely those which are closed with respect to the operation of cutting and piecing devised in [1]. Fryszkowski's result lies on a corollary of Liapunov's Convexity Theorem on the range of vector measures [28, Proposition 1.1], which allows to substitute convexity by decomposability in a fairly general setting. An application of Fryszkowski's theorem to differential inclusions will be shown in Chapter 2 of Part II. In [10], the authors remove the compactness assumption on the domain of F in [28], by improving Proposition 1.1 and 1.2 of [28] (see Lemmas 3.3.5 and 3.3.6 below).

The present chapter gives the proof of one of the selections theorems contained in [10], together with some similar statements. For a general discussion of the analogies between decomposable and convex sets we refer to [41], while for the measure theory details one can see [10] or [19].

3.2 SELECTION THEOREMS

We consider first two classical theorems concerning multifunctions with convex values.

THEOREM 3.2.1 (Michael's Selection Theorem [3,Theorem 1.11.1]). *Let* X *be a paracompact topological space and* Y *a Banach space, and let* F *from* X *into the closed convex subsets of* Y *be a l.s.c. multifunction. Then there exists* $f:X\to Y$, *a continuous selection from* F.

THEOREM 3.2.2 (Cellina's Approximate Selection Theorem [3, Theorem 1.12.1]). *Let* X *be a metric space,* Y *a Banach space and* F *a h-u.s.c. map from* X *into the convex subsets of* Y. *Then, for every* $\varepsilon>0$ *there exists a locally Lipschitzean map* $f_\varepsilon:X\to Y$ *such that*

$$\text{graph}\{f_\varepsilon\} \subseteq B(\text{graph}\{F\},\varepsilon)$$

(i.e. f_ε *is an* ε*-approximate selection of* F*), and* $f_\varepsilon(X)\subseteq \text{co}F(X)$.

In this paragraph the analogues of the above results are stated, in the case where Y is the Banach space $L^1(T,E)$ and, in the assumptions, convexity is replaced by decomposability.

THEOREM 3.2.3 ([10, Theorem 3]). *Let* X *be a separable metric space and let* $F:X\to \text{Dec}(L^1(T,E))$ *be a l.s.c. multifunction with closed, decomposable values. Then* F *has a continuous selection.*

THEOREM 3.2.4 ([10,Theorem 2]). *Let* X *be a metric space and let* $F:X\to \text{Dec}(L^1(T,E))$ *be a h-u.s.c. multifunction with decomposable values. If either* X *or* $L^1(T,E)$ *is separable, then for every* $\varepsilon>0$ *there exists a continuous map* $f_\varepsilon:X\to L^1(T,E)$ *such that*

$$\text{graph}\{f_\varepsilon\} \subseteq B(\text{graph}\{F\},\varepsilon)\ .$$

Moreover, $f_\varepsilon(X)$ *is contained in the smallest decomposable subset of* $L^1(T,E)$ *containing* $F(X)$.

We report only the proof of Theorem 3.2.3, since for Theorem 3.2.4 the basic ideas are the same.

3.3. PROOF OF THEOREM 3.2.3.

We state first three technical lemmas of measure theory.

LEMMA 3.3.1 ([10, Lemma 4.1]). *Let* $(T,\mathcal{F},\mu)$ *be a measure space with a* σ*-algebra* $\mathcal{F}$ *of subsets of* T *and a non-atomic probability measure* μ *on* $\mathcal{F}$. *Let* $(g_n)_{n\geq 0}$ *be a sequence of non-negative functions in* $L^1(T,\mathbb{R})$ *with* $g_0\equiv 1$. *Then there exists a map* Φ: $\mathbb{R}^+\times[0,1]\to\mathcal{F}$ *with the following properties:*

a) $\Phi(\tau,\lambda_1) \subseteq \Phi(\tau,\lambda_2)$ *if* $\lambda_1 \leq \lambda_2$,

b) $\mu(\Phi(\tau_1,\lambda_1)\Delta\Phi(\tau_2,\lambda_2)) \leq |\lambda_1 - \lambda_2| + 2|\tau_1 - \tau_2|$,

c) $\displaystyle\int_{\Phi(\tau,\lambda)} g_n \, d\mu = \lambda\int_T g_n \, d\mu \qquad \forall n \leq \tau$,

for all λ, λ_1, $\lambda_2 \in [0,1]$, τ, τ_1, $\tau_2 \geq 0$.

LEMMA 3.3.2 ([10,LEMMA 4.2]). *Let* X *be a separable metric space, and let* $\phi_n:X\to L^1(T,\mathbb{R})$, $h_n:X\to[0,1]$ $(n\geq 1)$ *be two sequences of continuous functions, with* $\phi_n(x)(t) \geq 0$ $\forall x\in X$, $\forall t\in T$, *and such that* $\{\mathrm{supp}(h_n);n\geq 1\}$ *is a locally finite (closed) covering of* X. *Then, for every* $\varepsilon>0$ *and every* *continuous, strictly positive function* $\ell: X \to \mathbb{R}^+$*, there exist a continuous function* $\tau: X \to \mathbb{R}^+$ *and a map* $\Phi: \mathbb{R}^+ \times [0,1] \to \mathcal{F}$ *which satisfy conditions* a), b) *in* Lemma 3.3.1 *together with*

c') *for all* $x \in X$, $\lambda \in [0,1]$ *and* $n \geq 1$, *if* $h_n(x)=1$ *then*

$$\left| \int_{\Phi(\tau(x),\lambda)} \phi_n(x)\ d\mu - \lambda \int_T \phi_n(x)\ d\mu \right| < \varepsilon/4\ell(x) \quad .$$

LEMMA 3.3.3 ([39,p.121]). *For every family* $\mathcal{K}$ *of non-negative measurable functions* $u: T \to \mathbb{R}^+$*, there exists a measurable function* $v: T \to \mathbb{R}^+$ *such that*

i) $v \leq u$ μ*-a.e. for all* $u \in \mathcal{K}$,

ii) *if* w *is a measurable function such that* $w \leq u$ μ*-a.e. for all* $u \in K$*, then* $w \leq u$ μ*-a.e..*

Furthermore, there exists a sequence (u_n) *in* $\mathcal{K}$ *such that*

$$v(t) = \inf\{u_n(t) : n \geq 1\} \quad \text{for a.e. } t \text{ in } T.$$

If the family $\mathcal{K}$ *is directed downwards, (i.e., if for every* $u, u' \in \mathcal{K}$ *there exists* $w \in \mathcal{K}$ *such that* $w \leq u$ *and* $w \leq u'$ μ*-a.e.), then the sequence* (u_n) *can be chosen to be decreasing.*

By ii), the function v is unique up to μ-equivalence. It represents the greatest lower bound of $\mathcal{K}$ in the sense of μ-a.e. inequality, and is denoted by ess $\inf\{u : u \in \mathcal{K}\}$.

The actual proof of Theorem 3.2.3 begins with some Propositions concerning decomposability.

PROPOSITION 3.3.4. *Let* K *be a nonempty, closed decomposable subset of* $L^1(T,E)$ *and let* $\psi(t)$ = ess $\inf\{\|u(t)\|_E : u \in K\}$. *Then,* *for every* $v_o \in L^1(T,\mathbb{R})$ *such that* $v_o(t) > \psi(t)$ *a.e., there exists an element* $u_o \in K$ *such that*

$$\|u_o(t)\|_E < v_o(t) \quad \mu\text{-}a.e.. \qquad (3.3.1)$$

Proof. Notice that the set $\mathcal{K}=\{\|u(\cdot)\|_E : u\in K\}$ is a decomposable subset of $L^1(T,\mathbb{R})$. Therefore it is directed downwards. Using Lemma 3.3.3, take a sequence $(u_n)_{n\geq 1}$ in K such that

$$\|u_m(t)\|_E \geq \|u_n(t)\|_E \quad \forall\ m < n\ ,\ t \in T\ ,$$

$$\psi(t) = \lim_{n\to\infty} \|u_n(t)\|_E \qquad \mu\text{-a.e.}\ .$$

Let now v_o be given, with $v_o(t) > \psi(t)$ a.e., and define the increasing sequence of sets : $T_o = \emptyset$, $T_n = \{t\in T : \|u_n(t)\|_E < v_o(t)\}$, $n\geq 1$. Observe that $\mu(T \setminus \bigcup_o^\infty T_n) = 0$. Define the sequence (w_n) by setting

$$w_n(t) = \begin{cases} u_k(t) & \text{if } t\in T_k\setminus T_{k-1},\ k=1,\ldots,n-1\ , \\ u_n(t) & \text{if } t\in T\setminus \bigcup_{k<n} T_k\ . \end{cases}$$

Since K is decomposable, each w_n belongs to K. Moreover, the sequence $w_n(t)$ is eventually constant for a.e. $t\in T$, and $\|w_n(t)\|_E \leq \|u_1(t)\|_E$ μ-a.e.; hence, by the Dominated Convergence Theorem, w_n converges in $L^1(T,E)$ to some function u_o. Clearly, $u_o\in K$ because K is closed. Finally, if $t\in T_n\setminus T_{n-1}$ for some n, then $\|u_o(t)\|_E = \|u_n(t)\|_E < v_o(t)$. Therefore, u_o satisfies (3.3.1). ∎

PROPOSITION 3.3.5. *Let* X *ba a metric space and let* $F:X\to Dec(L^1(T,E))$ *be a l.s.c. map with decomposable values.*

For all $x \in X$, *set* $\psi_x(t) = \text{ess inf}\{\|u(t)\|_E ; u \in F(x)\}$. *Then the multivalued map* $P : X \to L^1(T, \mathbb{R})$ *defined as*

$$P(x) = \{ v \in L^1(T, \mathbb{R}) : v(t) > \psi_x(t) \ \mu\text{-a.e.}\} \qquad (3.3.2)$$

is lower semicontinuous.

Proof. Let C be an arbitrary closed subset of $L^1(T, \mathbb{R})$. It suffices to show that, if $P(x_n) \subseteq C$ for some sequence $(x_n)_{n \geq 1}$ converging to x_o, then also $P(x_o) \subseteq C$. To this purpose, fix any $v_o \in P(x_o)$ and take, by Proposition **3**.3.4, a function $u_o \in F(x_o)$ such that $\|u_o(t)\|_E < v_o(t)$ μ-a.e.. Because of the lower semicontinuity of F, there exists a sequence $u_n \in F(x_n)$ such that $u_n \to u_o$ in $L^1(T, E)$. Then, for every $n \geq 1$, the function $v_n = \|u_n\|_E + v_o - \|u_o\|_E$ belongs to $P(x_n)$, which is contained in C. Since the sequence (v_n) converges to v_o in the norm of $L^1(T, \mathbb{R})$ and C is closed, this implies $v_o \in C$. ∎

PROPOSITION **3**.3.6. *Let* X *be a metric space and let* $G : X \to \text{Dec}(L^1(T, E))$ *be a l.s.c. map with closed decomposable values. Assume that* $g : X \to L^1(T, E)$ *and* $\phi : X \to L^1(T, \mathbb{R})$ *are continuous functions such that, for every* $x \in X$, *the set*

$$H(x) = \{u \in G(x) : \|u(t) - g(x)(t)\|_E < \phi(x)(t) \ \mu\text{-a.e.}\}$$

is nonempty. Then the map $H : X \to \text{Dec}(L^1(T, E))$ *is l.s.c. with decomposable values.*

Proof. For every $x \in X$, $H(x)$ is the intersection of two decomposable sets, hence it is decomposable. To check the lower semicontinuity of H, let C be any closed subset of $L^1(T, E)$. It suffices to show that, for any sequence (x_n) in

X converging to a point x_o, if $H(x_n)\subseteq C$ for all $n\geq 1$, then $H(x_o)\subseteq C$. To this purpose, fix any $u_o\in H(x_o)$. Because of the lower semicontinuity of G, there exists a sequence $u_n\in G(x_n)$ such that $u_n\to u_o$ in $L^1(T,E)$. By possibly taking a subsequence, we can assume that $u_n(t)$, $g(x_n)(t)$, $\phi(x_n)(t)$ converge to $u_o(t)$, $g(x_o)(t)$, $\phi(x_o)(t)$ respectively, μ-a.e. in T. Applying Egorov's theorem to these sequences with respect to the measure $\phi(x_o)\cdot\mu$, for each $i\geq 1$ we obtain a measurable set $T_i\subseteq T$ such that $u_n, g(x_n)$, and $\phi(x_n)$ converge uniformly on T_i and $\int_{T\setminus T_i} \phi(x_o)\, d\mu < 1/i$. For each $k\geq 1$, consider the sets

$$T_i^k = \{t\in T_i : \|u_o(t)-g(x_o)(t)\|_E < \phi(x_o)(t) - 1/k\}.$$

Notice that $\bigcup_{k\geq 1} T_i^k = T_i$ and $T_i^k \subseteq T_i^{k+1}$. Hence, for every $i\geq 1$, there exists a k(i) such that

$$\int_{T_i\setminus T_i^{k(i)}} \phi(x_o)\, d\mu < 1/i.$$

Define $T_i^{'} = T_i^{k(i)}$. The sets $T_i^{'}$ have the following properties:

$$\int_{T\setminus T_i^{'}} \phi(x_o)\, d\mu < 2/i, \tag{3.3.3}$$

$$\|u_o(t) - g(x_o)(t)\|_E < \phi(x_o) - 1/k(i) , \quad \forall t\in T_i^{'} . \tag{3.3.4}$$

By (3.3.4) and by the uniform convergence on $T_i^{'}$, for all $i\geq 1$ there exists some n_i such that

$$\|u_n(t) - g(x_n)(t)\|_E < \phi(x_n)(t) \quad \forall\ t\in T_i^{'} ,\ n\geq n_i . \tag{3.3.5}$$

We can also assume that the sequence $(n_i)_{i\geq 1}$ is strictly increasing. For each n, choose an arbitrary $w_n \in H(x_n)$ and set, for $n_i \leq n < n_{i+1}$, $v_n = u_n \cdot \chi_{T_i'} + w_n \cdot \chi_{T\setminus T_i'}$. Since $H(x_n)$ is decomposable, $v_n \in H(x_n)$. We claim that $v_n \to u_o$ in $L^1(T,E)$, which implies $u_o \in C$. Indeed, for $n_i \leq n < n_{i+1}$, (3.3.3) and (3.3.5) yield

$$\|v_n - u_o\|_1 \leq \int_{T\setminus T_i'} \|w_n - g(x_n)\|_E \, d\mu + \int_{T\setminus T_i'} \|g(x_n) - g(x_o)\|_E \, d\mu +$$

$$+ \int_{T\setminus T_i'} \|g(x_o) - u_o\|_E \, d\mu + \int_{T_i'} \|u_n - u_o\| \, d\mu$$

$$\leq \int_{T\setminus T_i'} \phi(x_n) \, d\mu + \|g(x_n) - g(x_o)\|_1 + \int_{T\setminus T_i'} \phi(x_o) \, d\mu + \|u_n + u_o\|_1$$

$$\leq \left[2/i + \|\phi(x_n) - \phi(x_o)\|_1\right] + \|g(x_n) - g(x_o)\|_1 + 2/i + \|u_n - u_o\|_1 \, .$$

As $n \to +\infty$, we also have $i \to +\infty$, hence our claim is proved.∎

The next result, concerning the existence of approximate selections, is the core of the whole proof of Theorem 3.2.3.

PROPOSITION 3.3.7. *Let* X *be a separable metric space and let* $G: X \to Dec(L^1(T,E))$ *be a l.s.c. map with closed decomposable values. Then, for every* $\varepsilon > 0$, *there exist continuous maps* $f_\varepsilon : X \to L^1(T,E)$ *and* $\phi_\varepsilon : X \to L^1(T,\mathbb{R})$ *such that* f_ε *is an* ε*-approximate selection of* G, *in the sense that, for each* $x \in X$, *the set*

$$G_\varepsilon(x)=\{u\in G(x):\|u(t) - f_\varepsilon(x)(t)\|_E < \phi_\varepsilon(x)(t) \quad \mu\text{-a.e.}\} \qquad (3.3.6)$$

is non-empty, and $\|\phi_\varepsilon(x)\|_1 < \varepsilon$. *Moreover, the map* $x\to G_\varepsilon(x)$ *is l.s.c. with decomposable values.*

Proof. Fix $\varepsilon>0$. For every $\bar{x}\in X$ and $\bar{u}\in G(\bar{x})$, the multivalued map Q defined as

$$Q(x) = \{v\in L^1(T,\mathbb{R}):v(t)\geq \text{ess inf}\{\|u(t)-\bar{u}(t)\|_E:u\in G(x)\} \text{ for a.e. } t\in T\} \qquad (3.3.7)$$

is l.s.c. with closed convex values. To see this, define $F(x)=\{u-\bar{u}:u\in G(x)\}$. Then the map F is also l.s.c. with closed decomposable values. By Proposition 3.3.5, the multivalued map P defined in (3.3.2) is l.s.c.. Hence Q is also l.s.c., because Q is the closure of P(x), for all $x\in X$. It is therefore possible to apply Michael's theorem to Q and obtain a continuous selection $\phi_{\bar{x},\bar{u}}$ such that $\phi_{\bar{x},\bar{u}}(x)\in Q(x)$ for all $x\in X$ and $\phi_{\bar{x},\bar{u}}(\bar{x})\equiv 0$. The family of sets

$$\{\{x\in X:\|\phi_{\bar{x},\bar{u}}(x)\|_1 < \varepsilon/4\}:\bar{x}\in X,\bar{u}\in G(\bar{x})\}$$

is an open covering of the separeble metric space X, therefore it has a countable nbd-finite open refinement $\{V_n:n\geq 1\}$. Let $\{p_n(\cdot)\}$ be a continuous partition of unity subordinate to the covering $\{V_n\}$ and let $\{h_n(\cdot)\}$ be a family of continuous functions from X into [0,1] such that $h_n\equiv 1$ on $\text{supp}(p_n)$ and $\text{supp}(h_n)\subseteq V_n$. For every $n\geq 1$, choose x_n,u_n such that $V_n \subseteq \{x:\|\phi_{x_n,u_n}(x)\|_1 < \varepsilon/4\}$ and set $\phi_n = \phi_{x_n,u_n}$. The functions ϕ_n have the following properties:

$$\phi_n(x)(t) \geq \text{ess inf}\{\|u(t) - u_n(t)\|_E : u\in G(x)\} , \qquad (3.3.8)$$

$$p_n(x)\cdot\|\phi_n(x)\|_1 \le p_n(x)\cdot\varepsilon/4 \qquad (x\in X, n\ge 1). \qquad (3.3.9)$$

Lemma 3.3.2, applied to the sequences $\{\phi_n\}$ and $\{h_n\}$, and to the function ℓ : $\ell(x) = \sum_{n=1}^{\infty} h_n(x)$, yields a continuous function $\tau:X\to\mathbb{R}^+$ and a family $\{\Phi(\tau,\lambda)\}$ of measurable subsets of T satisfying a),b) and c'). It is now possible to construct the functions f_ε and ϕ_ε. Set $\lambda_0\equiv 0$, $\lambda_n(x) = \sum_{m=1}^{n} p_m(x)$, and define

$$f_\varepsilon(x) = \sum_{n\ge 1} u_n\cdot\chi_{\Phi(\tau(x),\lambda_n(x))\setminus\Phi(\tau(x),\lambda_{n-1}(x))},$$

$$\phi_\varepsilon(x) = \varepsilon/4 + \sum_{n\ge 1} \phi_n(x)\cdot\chi_{\Phi(\tau(x),\lambda_n(x))\setminus\Phi(\tau(x),\lambda_{n-1}(x))}.$$

Clearly, f_ε and ϕ_ε are continuous, because the above summations are locally finite. Let G_ε be defined by (3.3.6). To check that the values of G_ε are non-empty, fix any $x\in X$. For every $n\ge 1$ use Proposition 3.3.4 and select $u_x^n\in G(x)$ such that

$$\|u_x^n(t) - u_n(t)\|_E < \varepsilon/4 + \text{ess inf}\{\|u(t) - u_n(t)\|_E : u\in G(x)\}$$

μ-a.e. in T. (3.3.10)

Then

$$u_x = \sum_{n\ge 1} u_x^n\cdot\chi_{\Phi(\tau(x),\lambda_n(x))\setminus\Phi(\tau(x),\lambda_{n-1}(x))}$$

lies in G(x), because G(x) is decomposable. We claim that $u_x\in G_\varepsilon(x)$. Indeed, (3.3.8) and (3.3.10) yield

$$\|u_x(t) - f_\varepsilon(x)(t)\|_E \le \sum_{n\ge 1} \|u_x^n(t)-u_n(t)\|_E\cdot\chi_{\Phi(\tau(x),\lambda_n(x))\setminus\Phi(\tau(x),\lambda_{n-1}(x))}(t)$$

$$< \phi_\varepsilon(t) \quad \mu\text{-a.e. in } T.$$

Hence $G_\varepsilon(x) \neq \emptyset$. Being the intersection of two decomposable sets, $G_\varepsilon(x)$ is also decomposable. The lower semicontinuity of G_ε follows from Proposition **3**.3.6. To conclude the proof of Proposition **3**.3.7, it now suffices to show that $\|\phi_\varepsilon(x)\|_1 < \varepsilon$ for every x. Set $I(x) = \{n\geq 1 : p_n(x) > 0\}$ and notice that $1 \leq \#I(x) \leq \ell(x)$. From c') in Lemma **3**.3.2 and (**3**.3.9) we deduce

$$\|\phi_\varepsilon(x)\|_1 = \varepsilon/4 + \sum_{n\geq 1} \int_T \phi_n(x) \cdot \chi_{\Phi(\tau(x),\lambda_n(x))\setminus\Phi(\tau(x),\lambda_{n-1}(x))} \, d\mu$$

$$< \varepsilon/4 + \sum_{n\in I(x)} \Big[p_n(x)\cdot\|\phi_n(x)\|_1 + \frac{\varepsilon}{2\cdot\ell(x)} \Big] \leq \varepsilon/4 + \Big[\varepsilon/4 + \frac{\#I(x)\cdot\varepsilon}{2\cdot\ell(x)}\Big] \leq \varepsilon. \blacksquare$$

At this stage, everything is ready for the completion of the proof of Theorem **3**.2.3.

Let the function F be given. Construct two sequences of continuous maps $f_n : X \to L^1(T,E)$ and $\phi_n : X \to L^1(T,\mathbb{R})$, and a sequence of l.s.c. multifunctions G_n with decomposable values, such that, for all $x\in X$ and $n\geq 1$,

i) $G_n(x) = \{u\in F(x) : \|u(t) - f_n(x)(t)\|_E < \phi_n(x)(t) \ \mu\text{-a.e.}\} \neq \emptyset$,

ii) $\|f_n(x)(t) - f_{n-1}(x)(t)\|_E \leq \phi_n(x)(t) + \phi_{n-1}(x)(t)$

μ-a.e. in T $\quad (n\geq 2)$,

iii) $\|f_n(x)\|_1 < 2^{-n}$.

To do this, define f_1 and ϕ_1 by applying Proposition **3**.3.7 with $G = F$, $\varepsilon = 1/2$. Let now f_m, ϕ_m and G_m be defined so that i)÷iii) hold for all $m = 1,\ldots,n-1$. To construct f_n and ϕ_n, apply again Proposition **3**.3.7 with $\varepsilon = 2^{-n}$, defining $G(x)$ to be the closure of $G_{n-1}(x)$, for all x. By induction, the maps f_n, ϕ_n and G_n can be defined for all $n\geq 1$. By ii), the sequence $(f_n)_{n\geq 1}$ is Cauchy in the L^1-norm, hence it converges uniformly to some continuous function $f:X\rightarrow L^1(T,E)$. By i) and iii), $d_{L^1}(f_n(x),F(x)) < 2^{-n}$. Since $F(x)$ is closed, this implies that $f(x)\in F(x)$ for all $x\in X$, hence f is a selection of F.

PART II

EXISTENCE RESULTS FOR DIFFERENTIAL INCLUSIONS

1. INTRODUCTION

Consider the differential inclusion

$$x'(t) \in F(t,x(t)) \qquad (1.1)$$

with initial conditions

$$x(0) = x_o \in \mathbb{R}^n \, . \qquad (1.2)$$

Under the assumption of lower or upper semicontinuity, this problem is rather easily solvable when the values of F are convex and compact. In fact, in the l.s.c. case, by Michael's selection theorem the differential inclusion (1.1) contains a family of ordinary differential equations, which have solutions by Peano's theorem, while in the u.s.c. case problem (1.1) can be approximated, via Cellina's approximate selection theorem, by ordinary differential equations with continuous right-hand side (see [3,§2.1]). In the first case, any solution of the equations contained in the problem is automatically a solution of (1.1). In the second case, one more step is needed in order to establish the existence of solutions to (1.1),(1.2). In fact, the uniform limit of approximate solutions might not be a solution of a differential inclusion, contrary to the ordinary differential equations, where the continuity of the right-hand side insures this property. The reason of this behaviour lies on the fact that the uniform convergence of approximate solutions of (1.1) to a function $x(\cdot)$ does not imply the pointwise convergence of their derivatives to $x'(\cdot)$, and, hence, there is no way to establish that $x'(t) \in F(t,x(t))$. However, when the values of F are convex, the limit is indeed a solution, because it is always possible to obtain the convergence of the derivatives. To prove this claim, consider a sequence $(f_n)_{n\geq 1}$ of 1/n-approximate selections of F, i.e. a sequence of continuous maps $f_n:[0,T]\times\mathbb{R}^n\to\mathbb{R}^n$ such that

$$\text{graph}\{f_n\} \subseteq B(\text{graph}\{F\},1/n) \quad ,$$

and a sequence $(x_n)_{n\geq 1}$ of solutions of the Cauchy problems

$$x_n' = f_n(t,x_n)$$
$$x_n(0) = x_o \ ,$$

defined on a same interval I. Since F has compact values, it is not restrictive to assume the sequence (x_n) to be uniformly convergent to a function x, and the sequence (x_n') to be weakly-$L^2(I)$ convergent to x'. By Mazur's theorem, a sequence $(y_n)_{n\geq 1}$ of convex combinations of the x_n' converges strongly in $L^2(I)$, and hence pointwise a.e., to x'. When the values of F are convex, it can be proved [3, Theorem 1.4.1] that the convex combinations

$$z_n=x_o+\int_o^t y_n(s)\,ds$$

of the x_n are still approximate solutions of (**1**.1), in the sense that

$$d((t,z_n(t),z_n'(t)),\text{graph}\{F\}) \to 0 \quad \text{when } n\to\infty;$$

moreover, $z_n\to x$ uniformly.

Since the graph of F is closed, the convergence of $z_n(t),z_n'(t)$ implies that $(t,x(t),x'(t))\in\text{graph}\{F\}$, that is $x'(t)\in F(t,x(t))$.

When F has compact convex values, the existence of solutions to (**1**.1),(**1**.2) is thus essentially a corollary of Peano's existence theorem for ordinary differential equations. In the non-convex case, the two preceding arguments clearly fail, and in fact the study of differential inclusions without convexity has given impulse to several new techniques.

The first result of this type was Filippov's theorem (see [3,Theorem 2.3.1]), in which F is supposed to be continuous with compact values. His proof is an astute refinement of the polygonal method for ordinary differential equations: he constructs piecewise linear functions such that not only are approximate solutions, but also have derivatives remaining in a compact subset of $L^{\infty}(I)$. Then, by the convergence of the polygonals together with their derivatives, the existence of solutions is easily deduced. The method of polygonal approximations was later used by other authors (among the others [40,33,32]); however, in general weaker assumptions of continuity on F make it of difficult use. Here, an argument of this type is used in the proof of Theorem **4**.3.1.

Another argument which has been used for the existence problem is the "fixed point approach", which is due to Antosiewicz and Cellina, and appears in [1,4,18]. It turns (**1**.1),(**1**.2) into the research of a fixed point of a kind of integral operator, analogously to the Volterra method for differential equations. It is mainly suitable when F is lower semicontinuous and it has reached a remarkable degree of simplicity, thanks to the theory of multifunctions with decomposable values. Here we use this approach for Theorem **2**.2.1.

A third method we want to mention here is of a different nature, because it does not come directly from ordinary differential equations. It consists in studying first the differential inclusion

$$x' \in \overline{co}F(x), \ x(0)=x_0 \qquad (\mathbf{1}.3)$$

and then in showing that the set of the solutions of (**1**.3) which are not solutions of (**1**.1), (**1**.2) is meager in the sense of Baire category. As a corollary one then obtains that the set of solutions of (**1**.1),(**1**.2) is nonempty. This argument had its origin in a paper by A.Cellina [14] and was used first by De Blasi and Pianigiani [22,23,24]. It holds when F is Hausdorff-continuous, and it is of interest for differential inclusions in an infinite dimensional space. In

Chapter **3**, we present a further development of this method.

The case in which F is l.s.c. and has compact values, where there is always existence of solutions, has been widely explored (there is actually another approach, due to A.Bressan [7]). Here the lack of convexity does not affect the existence, but the structure of the set of solutions is more complicate than in the convex case (see [44,8]), and still not entirely clear. On the contrary, if F is u.s.c. and has compact values, solutions to (**1**.1),(**1**.2) may fail to exist (see Examples **4**.1.1, **4**.2.2 and **4**.2.3 below and [9]) and positive results are, up to now very rare. Indeed, upper semicontinuity allows a behaviour of the function F which is more irregular - from the viewpoint of differential inclusions (and also of continuous selections) - than that one permitted by the lower semicontinuity; actually, even in the convex case sketched at the beginning of this paragraph, the proof is much more delicate for the upper semicontinuity. Here, Theorem **4**.3.1 in Chapter **4** is an existence result for a particular kind of differential inclusion with u.s.c. non-convex right-hand side.

Chapter **2** contains a theorem for the problem $x' \in -Ax + F(t,x)$, $x(0)=x_o$, where A is a maximal monotone operator and F is l.s.c., which appears in [20]. Chapter **3** comes from [11], where the problem $x' \in F(x)$, $x(0)=x_o$ is solved, for x in a Banach space and F Hausdorff-continuous, under assumptions which exclude the compactness of the values of F. The last chapter contains a discussion of problems with u.s.c. right-hand side, with an existence result [15] for $F(x)=f(\pi(x))$, where $\pi(x)$ is the projection of x on a closed set and x belongs to $\mathbb{R}^2$, and a generalization of a theorem of Wazewski which compares the inclusions $x' \in F(x)$ and $x' \in \overline{co}F(x)$.

2. LOWER SEMICONTINUOUS PERTURBATIONS OF MAXIMAL MONOTONE DIFFERENTIAL INCLUSIONS

2.1. INTRODUCTION

In [18], Cellina and Marchi proved an existence result for differential inclusions of the form

$$x' \in -Ax + F(t,x), \tag{2.1.1}$$

where A is a maximal monotone operator and F is a continuous map with compact (not necessarily convex) values which verifies a sublinear growth condition. The main tool used in their proof is a continuous selection theorem for the map

$$x \to \{u \in L^1(I) : u(t) \in F(t,x(t)) \text{ a.e.}\} , \tag{2.1.2}$$

defined on a compact subset of $L^1(I,\mathbb{R}^n)$. This approach goes back to a paper of Antosiewicz and Cellina [1], where the special case A=0 is considered.

It is well known that the existence of solutions for the Cauchy problem

$$x'=f(t,x), \quad x(0)=x_0$$

is equivalent to the existence of fixed points for the integral operator

$$x \to x_0 + \int_0^t f(s,x(s))ds ,$$

in a suitable set of functions $x(\cdot)$. For the initial value problem (1.1), (1.2) one can analogously consider the multivalued operator

$$T : x \to x_0 + \int_0^t F(s,x(s))ds .$$

Provided the definition of T makes sense, it is clear that a fixed point x* of T, that is a function x* such that

$x^*(t) \in T(x^*)$, is a solution to (**1**.1),(**1**.2) (see [3,§§ 2.5,2.6]). To establish the existence of fixed points, Antosiewicz and Cellina proved a continuous selection theorem for the multivalued map $x \to Tx$ and then used Schauder's theorem on a suitable space of functions $x(\cdot)$. The main features of their selection procedure have been discussed in Chapter **3** of Part I. For technical reasons, the map (**2**.1.2) is preferred to the integral operator T in [18] and also here; notice that it has decomposable values.

The results in [1] were generalized by Bressan [4] and Łojasiewicz [36], by assuming the map F to be:

i) jointly measurable in (t,x),

ii) lower semicontinuous in x.

We show here that (**2**.1.1) still has solutions if A is a maximal monotone operator and F satisfies only (a) and (b) above and the same sublinear growth condition. Our proof follows the same fixed point argument of [18] and it is based on a selection theorem of Fryszkowski (see Theorem **3**.2.3 in Part I), which contains the selection theorems used in [1], [4] and [18]. Comparing these three papers with the present proof, one can see that the abstract setting of Fryszkowski's theorem has led to a considerable simplification.

2.2 ASSUMPTIONS AND STATEMENT OF THE MAIN RESULT

In what follows, A is a maximal monotone operator in $\mathbb{R}^n$, i.e. a set-valued map from a subset D(A) of $\mathbb{R}^n$ into the subsets of $\mathbb{R}^n$, with the following two properties:

(A1) $\forall\, x_1, x_2 \in D(A),\ \forall\, v_i \in Ax_i,\ i=1,2,$

$$< x_1 - x_2\ ,\ v_1 - v_2 > \ \geq\ 0\ ;$$

(A2) the range of I + A is all of $\mathbb{R}^n$.

It is known that $\overline{D(A)}$ is convex, and that Ax is convex closed for any $x \in D(A)$ (see [2]).

We will consider a map F from $[a,+\infty) \times \overline{D(A)}$ into the compact subsets of $\mathbb{R}^n$ with the following properties:

(F1) $F(\cdot,\cdot)$ is $\mathcal{L}\otimes\mathcal{B}$-measurable;

(F2) for each $t \geq a$, $F(t,\cdot)$ is lower semicontinuous;

(F3) there exist two non-negative locally integrable functions $\alpha, \beta: [a,+\infty) \to \mathbb{R}$ such that, for every $(t,x) \in [a,+\infty) \times \overline{D(A)}$, $\|F(t,x)\| \leq \alpha(t)|x| + \beta(t)$.

We want to study the existence of solutions to the initial value problem

$$x' \in -Ax + F(t,x) \ , \ x(a) = x_o \in \overline{D(A)} \ . \tag{P}$$

By a solution of (P) we mean a function $x \in C([a,+\infty),\mathbb{R}^n)$ which is absolutely continuous on every compact subset of $(a,+\infty)$, is such that $x(a) = x_o$ and $x(t) \in D(A)$ for a.e. $t>a$, and, for some measurable selection $f(\cdot)$ from $F(\cdot,x(\cdot))$,

$$x' \in -Ax + f(t) \quad \text{for a.e. } t \geq a \tag{Pf}$$

(see [2] and [18]).

The main result is the following:

THEOREM 2.2.1. *If* A *is a maximal monotone operator and* (F1)÷(F3) *hold, then problem* (P) *has a solution for any* $x_o \in \overline{D(A)}$.

2.3. PROOF OF THEOREM 2.2.1

We state first some known facts about maximal momotone differential inclusions which will be used in the following. For any compact interval I in $[a,+\infty)$, we denote by $|\cdot|_{i,I}$ the usual norm in $L^i(I):=L^i(I,\mathbb{R}^n)$, and set $L^i_{loc}([a,+\infty)):=$

$L^i_{loc}([a,+\infty),\mathbb{R}^n)$ ($i=1$ or $i=\infty$).

LEMMA **2**.3.1 ([2,Theorem 2.2]). *For any* $f\in L^1_{loc}([a,+\infty))$ *and any initial value* $x_0\in\overline{D(A)}$ *there exists a unique solution* u_f *to* (P_f). *For every* $t\geq a$,

$$|u_f(t) - u_g(t)| \leq \int_a^t |f(s) - g(s)|\, ds$$

and, given any interval $I:=[\tau,\tau+T]$, *there exists a constant* C *depending only on* A *such that*

$$|u_f'|_{i,I} \leq C\ [(1 + T + |f|_{i,I})\cdot(1 + |u_f|_{\infty,I}) + |u_f(t)|^2]\ .$$

As a straightforward consequence of Lemma **2**.3.1 we have that the map $i:L^1_{loc}([a,+\infty)) \to L^1_{loc}([a,+\infty))$, $f \to u_f$, is well defined and continuous. The next lemma gives a kind of *a priori* estimate on the solutions of (P_f). We denote by $u_0(\cdot)$ the solution of (P_f) with $f=0$.

LEMMA **2**.3.2 ([18, Lemma 2.1]). *Set*

$$\psi(t) = \int_a^t \left(\alpha(s)|u_0(s)| + \beta(s)\right)\cdot \exp\left(\int_s^t \alpha(l)\,dl\right) ds\ .$$

Fix a function $w:[a,+\infty)\to\overline{D(A)}$ *and let* $f(\cdot)$ *be a measurable selection from* $F(\cdot,w(\cdot))$. *The following holds:*

if $|w(t) - u_0(t)| \leq \psi(t)$, *then also* $|u_f(t) - u_0(t)| \leq \psi(t)$.

The following result makes Fryszkowski's selection theorem applicable to our problem.

PROPOSITION 2.3.3. *Let* F *be as in Section* 2.2, I *be a compact interval in* $[a,+\infty)$ *and let* K *be a compact subset of* $L^1(I)$*, bounded in* $L^\infty(I)$*. Then the operator*

$$G:K \to Dec(L^1(I))$$

$$x \to \{u\in L^1(I): u(t)\in F(t,x(t)) \text{ for a.e. } t\in I\}$$

is well defined and lower semicontinuous.

Proof. It is easily seen that $G(x_1)=G(x_2)$ whenever $x_1(\cdot)=x_2(\cdot)$ a.e.. Moreover $G(x)$ is clearly decomposable, for any $x\in K$. In order to prove the lower semicontinuity, let C be a closed subset of $L^1(I)$ and let $(x_n)_n$ be a sequence in K converging in $L^1(I)$ to some x_o in K and such that $G(x_n)\subseteq C$. We just need to prove that $G(x_o)\subseteq C$ or, since C is closed, that

$$h^*(G(x_o),G(x_n)) \to 0 \quad \text{as } n \to +\infty \quad . \tag{2.3.1}$$

Let $\hat{x}_n$ (resp. $\hat{x}_o$) be Borel functions such that $\hat{x}_n=x_n$ a.e. (resp. $\hat{x}_o=x_o$ a.e.). We begin by proving the following

CLAIM:

$$\int_I h^*(F(t,\hat{x}_o(t)),F(t,\hat{x}_n(t)))\, dt \quad \to \quad 0$$

as $n \to \infty$.

Proof of the Claim. Set

$$h_n(t) = h^*(F(t,\hat{x}_o(t)),F(t,\hat{x}_n(t))) \quad , \quad \eta_n = \int_I h_n(t)dt \quad .$$

First of all we remark that the maps

$$t \to F(t,\hat{x}_o(t)) \quad , \quad t \to F(t,\hat{x}_n(t))$$

are measurable. Next we show that $h_n(\cdot)$ is measurable. By Theorem 3.5 e) in [31], the map

$$(t,z) \to d(z,F(t,\hat{x}_n(t))$$

is Carathéodory, and by Theorem 6.5 in the same paper the multivalued map given by

$$\Phi(t) = \left\{ d(z,F(t,\hat{x}_n(t))) \; : \; z \in F(t,\hat{x}_o(t)) \right\}$$

is measurable. Hence Theorem 6.6 again in [31] gives the measurability of $h_n(\cdot)$.

Now we will prove that every subsequence (η_{n_k}) of (η_n) has a subsequence converging to 0. In fact, $(x_{n_k}(\cdot))$ contains a subsequence (still denoted by $(x_{n_k}(\cdot))$) converging to $x_o(\cdot)$ a.e.. Then the lower semicontinuity of $F(t,\cdot)$ together with the fact that the values of F are compact imply that $h_{n_k}(t) \to 0$ for a.e. $t \in I$. Moreover, by (F3),

$$\begin{aligned}
h_{n_k}(t) &= h^*\left(F(t,\hat{x}_o(t)),F(t,\hat{x}_{n_k}(t))\right) \\
&\leq h^*\left(F(t,\hat{x}_o(t)),\{0\}\right) + h^*\left(\{0\},F(t,\hat{x}_{n_k}(t))\right) \\
&\leq \|F(t,\hat{x}_o(t))\| + \|F(t,\hat{x}_{n_k}(t))\| \\
&\leq \alpha(t)\cdot\{|\hat{x}_o(t)| + |\hat{x}_{n_k}(t)|\} + 2\beta(t) \\
&\leq 2\cdot\{M\alpha(t) + \beta(t)\}
\end{aligned}$$

for a suitable constant M. The Lebesgue Dominated Convergence Theorem gives

$$\eta_{n_k} = \int_I h_{n_k}(t)\, dt \to 0 \qquad \text{as} \quad k\to\infty$$

and this proves the claim.

Finally, in order to prove (2.3.1), fix $u_o \in G(x_o)$ and consider the multivalued function

$$\Gamma_n \; : \; t \to \overline{B}(u_o(t),h_n(t) + 1/n) \cap F(t,\hat{x}_n(t)) \;,$$

which clearly has closed nonempty values. To show that Γ_n is measurable, it is enough to prove the measurability of the map

$$\psi : t \to \overline{B}(u_o(t),h_n(t) + 1/n)$$

(see [31, Theorem 4.1]). But ψ is the composition of the measurable map $t \to (u_0(t), h_n(t)+1/n)$ with the continuous map $(x,r) \to B(x,r)$, hence it is measurable. Therefore we can choose a L^1 selection un from Γ_n. Clearly $u_n \in G(x_n)$ and we have

$$|u_n - u_o|_{1,I} \leq \int_I (h_n(t) + 1/n)\, dt .$$

By the above claim, the right-hand side of this inequality converges to 0 as $n \to \infty$, uniformly in u_o. Hence (2.3.1) is proved. ∎

Completion of the proof of Theorem 2.2.1.

We will follow essentially the proof of Theorem 2.2 in [18]. Define K as the closure in $L^1_{loc}([a,+\infty))$ of the set of those absolutely continuous functions v having the following properties:

i) $v(a) = x_o$ and $v(t) \in \overline{D(A)}$ $(t \geq a)$;

ii) $|v(t) - u_o(t)| \leq \psi(t)$ $(t \geq a)$;

iii) for every interval $I = [\tau, \tau+T]$ $(\tau \geq a)$,

$$|v'|_{1,I} \leq C[(1+T+N(I))\cdot(1+M(I)) + r^2(t)],$$

where

$$M(I) = \exp\left(\int_I \alpha(t)\,dt\right)\cdot\int_I (\alpha(t)\,|u_o(t)| + \beta(t))\,dt + |u_o|_{\infty,I} ,$$

$$N(I) = M(I)\cdot\int_I \alpha(t)\,dt + \int_I \beta(t)\,dt$$

$$r(\tau) = |u_o(t)| + 2\int_a^\tau (\alpha(t)\,|u_o(t)| + \beta(t))\cdot\exp\left(2\int_t^\tau \alpha(s)\,ds\right) dt .$$

It is easily seen (as in [18]) that K is nonempty, convex, compact in $L^1_{loc}([a,+\infty))$ and bounded in $L^\infty_{loc}([a,+\infty))$.

Set, for $n = 1,2,\ldots$, $I_n:=[a, a+n]$, $K_n:=\{v_{|I_n}:v\in K\}\subseteq L^1(I_n)$. We will construct recursively a sequence of continuous maps $g_n:K_n\to L^1(I_n)$ verifying, for each $x\in K_n$,

$$g_n(x)(t) \in F(t,x(t)) \quad \text{for a.e. } t\in I_n \quad , \tag{2.3.2}$$

$$\text{if } n>1,\ g_n(x)(t) = g_{n-1}(x)(t) \quad \text{for a.e. } t\in I_{n-1} . \tag{2.3.3}$$

Define the operator G_1 in the same way as G with K_1 in place of K, and, for n>1, assuming that g_{n-1} has already been defined, define the operator

$$G_n : K_n \to \mathrm{Dec}(L^1(I^n))$$

$$x \to \{u\in L^1(I_n):u(t)\in F(t,x(t)) \text{ for a.e. } t\in I_n$$

$$\text{and } u(t)=g_{n-1}(x)(t) \text{ for a.e. } t\in I_{n-1}\}.$$

By Proposition **2**.3.3 and Theorem **3**.2.3 in Part I, the operator G_1 has a continuous selection g_1. Therefore we can consider the operator G_2, and it is not difficult to see, in view of Proposition **2**.3.3, that it is lower semicontinuous. Applying again Theorem **3**.2.3 of Part I we see that G_2 admits a continuous selection g_2 which by construction satisfies (**2**.3.2) and (**2**.3.3). Similarly, for any n>2, we obtain g_n from g_{n-1} satisfying (**2**.3.2) and (**2**.3.3). Now we define

$$g : K \to L^1_{loc}([a,+\infty))$$

by setting $g(x)_{|I_n} = g_n(x)$, $n=1,2,\ldots$. Using (**2**.3.2) and (**2**.3.3), it easy to see that g is well defined and continuous and satisfies

$$g(x)(t) \in F(t,x(t)) \quad \text{for a.e. } t\geq a .$$

To conclude the proof we define, as in [18],

$$s : K \to L^1_{loc}([a,+\infty))$$

$$x \rightarrow i(g(x)) \quad .$$

The map s is continuous and, by Lemma 2.3.2, $s(K) \subseteq K$. Since K is compact and convex, the theorem of Schauder-Tichonov yields a fixed point of s, which is a solution to (P). ∎

3. GENERALIZED BAIRE CATEGORY AND DIFFERENTIAL INCLUSIONS IN BANACH SPACES

3.1. INTRODUCTION

Let E be a separable reflexive Banach space and consider the Cauchy problem

$$\dot{x}(t) \in F(x(t)), \quad x(0) = x_0 \in E, \tag{3.1.1}$$

where F is a Hausdorff continuous multifunction with closed, bounded values. In this chapter we prove the local existence of a solution of (3.1.1), assuming that the convex closure of $F(x_o)$ has finite codimension. More precisely, we assume the existence of a closed affine subspace $E_o \subseteq E$ with finite codimension, such that the interior of $E_o \cap \overline{co}F(x_o)$ relative to E_o is nonempty. Two special cases deserve mention. If the interior of $\overline{co}F(x_o)$ is nonempty, the above condition holds with $E_o = E$. On the other hand, if E is finite-dimensional, every continuous multifunction F satisfies our condition. Indeed, one can always select an element $y_o \in F(x_o)$ and set $E_o = \{y_o\}$. The present result therefore contains the theorems of De Blasi and Pianigiani [24] and of Filippov [27], both as special cases. Remark that the autonomous form of the differential inclusion in (3.1.1) is not restrictive here. In fact, if the set $\overline{co}F(0,x_o)$ has finite codimension in E, then so does in $\mathbb{R}\times E$ and the Cauchy problem $x'(t) \in F(t,x(t))$, $x(0)=x_o$, by setting $y=(t,x)\in\mathbb{R}\times E$ and $G(y)=(1,F(y))\in\mathbb{R}\times E$, is equivalent to $y'(t)\in G(y(t))$, $y(0)=y_o=(0,x_o)$.

For a map F whose values are convex sets with finite codimension, the Cauchy problem (3.1.1) was recently studied by A. Cortesi [21]. To remove the convexity assumption, we rely on a generalized version of Baire's category theorem, which will also be proved in this chapter. Together with (3.1.1), we consider the problem

$$\dot{x}(t) \in \overline{co}\, F(x(t)), \quad x(0) = x_0 \,. \tag{3.1.2}$$

The family S of all solutions of (3.1.2), on a suitable interval [0,T], is nonempty and closed in the metric of uniform convergence. We will show that the set S_F of all $x \in S$ which are solutions of (3.1.1) is a subset of second category in S (in a generalized sense). This will imply that S_F is nonempty.

The possibility of using a category argument, in order to prove the existence of solutions of a Cauchy problem, was first suggested by a paper of Cellina [14], in which it is shown that the set of solutions of the differential inclusion $x' \in \{-1,+1\}$ is of second Baire category in the set of solutions of $x' \in [-1,+1]$, with respect to the uniform convergence topology. This research program was pursued in a series of articles by De Blasi and Pianigiani [22,23,24]. A remarkable feature of their results is that the compactness assumptions on F, which are present in most of the previous papers [6,37,45,46], can be here entirely avoided. Notice also that, when E is infinite dimensional, no ordinary differential equation can satisfy our assumptions.

In the current literature, solutions of a Cauchy problem are often obtained by means of the Contraction Mapping Principle or by an application of Schauder's fixed point theorem. Compared with these other techniques for proving existence, Baire's theorem reaches a much stronger conclusion: the solution set is not only nonempty, but everywhere dense. In practice, this additional feature is often an inconvenience, because it severely restricts the range of applicability of the method. For instance, it is known [44] that the set S_F of solutions of (3.1.1) may not be dense on the set S of solutions of (3.1.2). Therefore, S_F is not of second category in S, in general, and a straightforward application of Baire's theorem is doomed to fail. To avoid this difficulty, in [24] the authors choose a closed subset $S^* \subseteq S$ (by an educated guess) and prove that S_F contains a subset of second category in S^*. In the present chapter, this same technical problem is solved by using a more flexible version of Baire's theorem. Under suitable assumptions, we prove that a sequence of open sets has a nonempty intersection, even in cases where the intersection is not everywhere dense. Moreover, no preliminary guessing of a set $S^* \subseteq S$ contained in the closure of S_F will be needed.

The chapter consists of eleven sections. Notations and basic definitions are contained in §2. Our main theorem, together with an equivalent result, is stated in §3. A "multivalued" version of Baire's theorem, with compact sets playing the role of points, is proved in §4. §5 contains a review of techniques and results from [24], for later use. An outline of the basic ideas involved in the proof of the main theorem can be found at the beginning of §6. The actual proof is given in §11, while §§7÷10 contain a number of preliminary technical results.

3.2. NOTATIONS AND BASIC DEFINITIONS

Throughout this chapter, E is a separable and reflexive Banach space, with norm $\|\cdot\|$. If X is a closed and convex subset of E, a point $x_o \in X$ is strongly exposed if there exists a continuous linear functional
$\psi: E \to \mathbb{R}$ such that:
(i) $\psi(x) < \psi(x_o)$ for every $x \in X$ with $x \neq x_o$; (ii) if $(x_n)_{n\geq 1}$ is a

sequence in X and $\psi(x_n)\to\psi(x_o)$, then $x_n\to x_o$. If X is also bounded, the set expX of its strongly exposed points is nonempty and X is the closed convex hull of expX (see Theorem 4 in [35]). Notice that every strongly exposed point is also an extremal point of X.

Let E_o be a closed affine subspace of E and let X be a subset of E_o. By $\mathrm{int}_{E_o}X$ we denote the set $\{x\in X:\exists\,\varepsilon>o$ such that $B(x,\varepsilon)\cap E_o\subseteq X\}$. If E_X is the intersection of all closed affine subspaces of E which contain X, we define the relative interior of X as rel int X = $\mathrm{int}_{E_X}X$. When $X\subseteq E$ is convex, we say that X has finite codimension if there exists an affine subspace E_o with finite codimension such that $\mathrm{int}_{E_o}(X\cap E_o)\neq\emptyset$.

3.3. STATEMENT OF THE MAIN RESULT

Theorem 3.3.1. *Let E be a separable reflexive Banach space. Let* $F:E\to E$ *be a Hausdorff continuous multifunction with nonempty, closed, bounded values. If* $\overline{co}F(x_o)$ *has finite codimension, then the Cauchy problem (3.1.1) has a Carathéodory solution on some positive interval* [O,T].

A more precise version of this result will be actually proved. By the assumption on $\overline{co}F(x_o)$, there exists a closed affine subspace $E_o\subseteq E$ with finite codimension and a point $y_o\in\mathrm{int}_{E_o}(E_o\cap\overline{co}F(x_o))$. Define the vector space $E'=E_o-y_o\subseteq E$. Let E" be an algebraic supplement of E' in E, and denote by π',π'' the canonical projections of E onto E', E" respectively. Since E' is closed and E" is finite dimensional, the projections π',π'' are continuous, hence $E=E'\oplus E''$ is actually a topological sum.

Clearly, $\pi'(y_o) \in \text{int}_{E'}(\pi' \circ \overline{co}F(x_o))$. We claim that $\pi'(y_o) \in \text{int}_{E'}(\pi' \circ \overline{co}F(x))$ for every x in a neighborhood of x_o.

LEMMA 3.3.1. *Let* Y_o *be a closed convex subset of a Banach space* E'. *If* $B(y_o,\rho) \subseteq Y_o$, *then* $B(y_o,\rho/3) \subseteq Y$ *for every closed convex set* $Y \subseteq E'$ *with* $h(Y,Y_o)<\rho/3$. *In particular,* Y *has nonempty interior.*

Proof. Assume, on the contrary, that $h(Y,Y_o)<\rho/3$, but $\omega \notin Y$ for some $\omega \in B(y_o,\rho/3)$. Let ϕ be a continuous linear functional on E', with unit norm, which separates ω from Y:

$$\phi(y_o) + \rho/3 > \phi(\omega) > \phi(y) \qquad \forall y \in Y.$$

Choose $y_1 \in B(y_o,\rho)$ such that $\phi(y_1) > \phi(y_o)+2\rho/3$. This implies

$$\|y_1-y\| \geq \phi(y_1)-\phi(y) > (\phi(y_o)-2\rho/3)-\phi(\omega) > \rho/3,$$

for every $y \in Y$, hence $h(Y,Y_o) \geq \rho/3$, a contradiction.

Since the map F is Hausdorff continuous, the same holds for the maps $x \to \overline{co}F(x)$ and $x \to \pi' \circ \overline{co}F(x)$. The previous lemma thus implies that the set of points $\{x \in E : \pi'(y_o) \in \text{int}_{E'}(\pi' \circ \overline{co}F(x))\}$ is open. Consider the following set of assumptions:

A1) $x_o = 0 \in E$,

A2) $E = E' \oplus E''$, with continuous projections $\pi' : E \to E'$, $\pi'' : E \to E''$, A3) $||x|| = \max\{||\pi'(x)||_{E'}, ||\pi''(x)||_{E''}\}$, E" being a finite-dimensional euclidean space.

A4) $F(x) \subset B(0,M-1) \qquad \forall x \in B(0,2\tilde{\rho})$,

A5) $\exists\ \omega' \in E' : \omega' \in \mathrm{int}_{E'}(\pi' \circ \overline{co}F(x)) \quad \forall x \in B(0,2\tilde{\rho})$.

A6) $0 < T \leq \tilde{\rho}/M$.

In the setting of Theorem 3.3.1, by possibly translating the origin and using an equivalent norm, the previous remarks indicate that it is always possible to choose suitable $E', E'', \omega', \tilde{\rho}, M$ and T such that A1) ÷ A6) hold. Therefore, Theorem 3.3.1 is an immediate consequence of the following more precise result.

THEOREM 3.3.3. *Let* $F : E \to E$ *be a Hausdorff continuous multifunction with closed bounded values, on the separable reflexive Banach space* E. *If the assumptions* A1) ÷ A6) *hold, then* (3.1.1) *has a Carathéodory solution defined on* [0,T].

3.4. A GENERALIZED CATEGORY THEOREM

Let S be a complete metric space, $\mathcal{F} = \{K_i : i \in I\}$ ($I \neq \emptyset$) be a family of nonempty compact subsets of S.

DEFINITION 3.4.1. *A set* $R \subseteq S$ *is* $\mathcal{F}$-rare *iff for every* $K \in \mathcal{F}$, $\varepsilon > 0$ *there exists* $K' \in \mathcal{F}$ *such that* $K' \subseteq B(K,\varepsilon) \setminus \overline{R}$.

DEFINITION 3.4.2. *A set* $M \subseteq S$ is $\mathcal{F}$-meager *iff it is the union of countably many* $\mathcal{F}$-rare *sets.*

In the special case where $\mathcal{F} = \{\{x\} : x \in S\}$ is the family of all singletons, a subset $V \subseteq S$ is $\mathcal{F}$-rare ($\mathcal{F}$-meager) iff V is rare (meager) in the usual sense. The following is thus an extension of Baire's Category Theorem.

THEOREM 3.4.1. *The complement of an $\mathcal{F}$-meager set* M *in* S *is nonempty. More precisely,*

$$\overline{S\setminus M} \cap K \neq \emptyset, \ \forall K \in \mathcal{F} . \tag{3.4.1}$$

Proof. It suffices to show that for every $K\in\mathcal{F}$, $\varepsilon > o$,

$$\overline{S\setminus M} \cap B(K,\varepsilon) \neq \emptyset. \tag{3.4.2}$$

Let $M = \cup R_n$, each R_n being $\mathcal{F}$-rare. Choose $K_1 \in \mathcal{F}$ such that $K_1 \subseteq B(K,\varepsilon)\setminus\overline{R}_1$. Since K_1 is compact and the complement of $B(K,\varepsilon)$ is closed, the distance

$$\delta = \inf \{d(x,y): x\in K_1, y\in B^c(K,\varepsilon) \cup \overline{R}_1\}$$

is strictly positive. Therefore, setting $\delta_1=\min\{\delta/3,1\}$ we have

$$B(K_1,2\delta_1) \subseteq B(K,\varepsilon)\setminus\overline{R}_1 . \tag{3.4.3}$$

By induction, construct a sequence of compact sets $K_n\in\mathcal{F}$ and radii δ_n such that $0 < \delta_n \leq 1/n$ and

$$B(K_n,2\delta_n) \subseteq B(K_{n-1},\delta_{n-1})\setminus\overline{R}_n . \tag{3.4.4}$$

Of course (3.4.4) implies

$$\overline{B}(K_n,\delta_n) \subseteq B(K_{n-1},\delta_{n-1})\setminus\overline{R}_n \subseteq B(K_{n-1},\delta_{n-1}) . \tag{3.4.5}$$

Denoting by $\alpha(V)$ the Kuratowski measure of non-compactness of a set V, we have

$$\alpha(\overline{B}(K_n,\delta_n)) \leq 2\delta_n \leq 2/n.$$

By Kuratowski's theorem [34, p. 412], the intersection D = $\cap \overline{B}(K_n,\delta_n)$ is a compact nonempty set. From (3.4.3) and (3.4.4) it now follows

$$D \subseteq B(K,\varepsilon) \setminus (\bigcup_{n=1}^{\infty} R_n),$$

proving the theorem.

EXAMPLE. Let $S=\mathbb{R}$ and let $\mathcal{F}$ be the family of compact sets consisting of the interval [0,1] together with all singletons $\{x\}$ with $x\geq 1$. Then the set $R=(-\infty,1]$ is $\mathcal{F}$-rare, according to Definition 3.4.1. Its complement $R^c=(1,+\infty)$ is not dense on $\mathbb{R}$, but the intersection $\overline{R^c}\cap K$ is nonempty, for all $K\in\mathcal{F}$.

3.5. A SET OF UPPER SEMICONTINUOUS FUNCTIONALS

We now introduce a family of upper semicontinuous functionals $\phi:E\times E\to[-\infty,1]$ which, loosely speaking, measure the distance of a vector $u\in E$ from the set of extreme points of the convex set $\overline{co}F(x)$. Both the definition and the main properties of these functionals are taken from [24].

By $\mathcal{K}(E)$ we denote the family of all nonempty closed, bounded subsets of E. Let C be any closed ball in E. If $X\in\mathcal{K}(E)$ and $u\in\overline{co}X$, define

$$\alpha(C,X) = \inf\{\|y-x\| : y\in C, x\in X\} \wedge 1, \tag{3.5.1}$$

$$\gamma_C(u,X) = \begin{cases} 0 \quad , & \text{if } C \cap \mathrm{co}X = \emptyset, \\ \sup\{\lambda\in[0,1] : u\in\lambda(C\cap\overline{\mathrm{co}}X)+(1-\lambda)\overline{\mathrm{co}}X\}, & \text{if } C\cap\overline{\mathrm{co}}X\neq\emptyset, \end{cases} \tag{3.5.2}$$

$$d_C(u,X) = \alpha(C,X).\gamma_C(u,X). \tag{3.5.3}$$

LEMMA 3.5.1. *If* $X\in\mathcal{K}(E)$ *and* $u\in\exp\overline{\mathrm{co}}X$, *then* $d_C(u,X)=0$. *Moreover,* $\exp\overline{\mathrm{co}}X\neq\emptyset$ *and* $\overline{\mathrm{co}}\exp\overline{\mathrm{co}}X=\overline{\mathrm{co}}X$.

For the proof, see Proposition 4.7 in [24].

LEMMA 3.5.2. *Let* $X_o\in\mathcal{K}(E)$, $\varepsilon>0$ *be given. Then there exists* $\delta>0$ *such that*

$$d_C(u,X_o)\geq d_C(u,X)-\varepsilon$$

for every $X\in\mathcal{K}(E)$ *satisfying* $\overline{\mathrm{co}}X \subseteq \overline{\mathrm{co}}X_o$, $h(X,X_o)<\delta$ *and every* $u\in\overline{\mathrm{co}}X$.

Indeed, the assumption $\overline{\mathrm{co}}X \subseteq \overline{\mathrm{co}}X_o$ implies $\gamma_C(u,X) \leq \gamma_C(u,X_o)$. This, together with the continuity of the map $X\to\alpha(C,X)$ w.r.t. the Hausdorff metric, yields the result. ■

Let F be the multifunction considered at (3.1.1). For any closed ball $C \subseteq E$, define

$$\phi_C(u,X) = \begin{cases} d_C(u,F(x)) & \text{if } u\in\overline{\mathrm{co}}F(x), \\ -\infty & \text{if } u\notin\overline{\mathrm{co}}F(x). \end{cases} \tag{3.5.4}$$

LEMMA 3.5.3. *The map* $\phi_C: E \times E \to [-\infty, 1]$ *is upper semicontinuous. For any fixed* x, *the map* $u \to \phi_C(u,x)$ *is concave.*

Indeed, the upper semicontinuity of γ_C and the concavity of the map $u \to \gamma_C(u,X)$ (for $u \in \overline{co}X$) were proved in Propositions 4.3 and 4.2 in [24]. Since the map F is Hausdorff-continuous and $\alpha(C,X)$ depends continuously on X, the above statements are clear. ∎

Let $(u_i)_{i \geq 1}$ be a sequence of points dense on E. The family of all closed balls centered at some u_i with positive rational radius is countable, hence it can be arranged into a sequence, say $(C_j)_{j \geq 1}$. Define the functional

$$\phi(u,x) = \sum_{j=1}^{\infty} 2^{-j} \phi_{C_j}(u,x) .$$

LEMMA 3.5.4. *The map* $\phi: E \times E \to [-\infty, 1]$ *is upper semicontinuous. Moreover*

$$\phi(u,x) > 0 \quad \forall u \in \overline{co}F(x) \setminus F(x) . \tag{3.5.6}$$

Proof. The first statement follows from Lemma 3.5.3. To prove (3.5.6), let $u \in \overline{co}F(x) \setminus F(x)$. Since F(x) is closed, there exists a rational $\rho > 0$ such that $B(u,4\rho) \cap F(x) = \emptyset$. The density of the sequence $(u_i)_{i \geq 1}$ implies that $\|u_k - u\| < \rho$ for some integer k. By construction, $\overline{B}(u_k, 2\rho) = C_j$ for some j. Observe that $\overline{B}(u_k, 2\rho) \subseteq B(u, 3\rho)$, hence $\alpha(C_j, F(x)) \geq \min\{\rho, 1\} > 0$. Moreover, $u \in C_j \cap \overline{co}F(x)$ hence $\gamma_{C_j}(u, F(x)) = 1$. This implies $\phi_{C_j}(u,x) > 0$, hence $\phi(u,x) > 0$.

3.6. A FAMILY OF DIFFERENTIAL INCLUSIONS WITH MEMORY

To motivate the definition of "n-level feedback" given below, a few words of introduction are in order. In a naive attempt to prove the existence of solutions of (3.1.1), one may consider the closed, nonempty set S of all solutions of (3.1.2), and try to show that the subset S_F of all $x \in S$ which are solutions of (3.1.1) is of second Baire category in S. This will be the case if, for every integers $j,n \geq 1$, the set of solutions

$$S_{jn} = \left\{ x \in S : \int_0^T \phi_{C_j} (\dot{x}(t), x(t))\, dt < 1/n \right\} \qquad (3.6.1)$$

is an open dense subset of S. Indeed, (3.5.6) implies $S_F \supseteq (\bigcap_{j,n} S_{jn})$.

Unfortunately, this naive approach fails when F is not Lipschitz continuous : a well known counterexample by Plis (see [44]) shows that S_F need not be dense on S. In [24], this difficulty is overcome by considering not the whole set of solutions S of (3.1.2), but a suitable closed subset $S^* \subseteq S$ with the property that $S_{jn} \cap S^*$ is dense in S^* for all $j,n \geq 1$. S^* is defined as the closure of the set of all polygonal solutions of (3.1.2) whose derivative lies a.e. in the interior of $\overline{co}F(x)$.

In the present case, however, the interior of $\overline{co}F(x)$ may be empty, and a substantially different line of proof is needed. We shall define a family $\{ \mathcal{E}_i : i \in I\}$ of differential inclusions with memory, with the following properties. Each $\mathcal{E}_i$ has a nonempty compact set K_i of solutions, with $K_i \subseteq S$. If $\mathcal{F} = \{K_i : i \in I\}$ is the collection of all such sets of solutions, for every

$j,n \geq 1$ the set $S \setminus S_{jn}$ is $\mathcal{F}$-rare. Our "multivalued" category theorem will therefore imply

$$\hat{S} = \bigcap_{j,n} S_{jn} = \left\{ x \in S : \int_0^T \phi(\dot{x}(t), x(t))\, dt = 0 \right\} \neq \emptyset . \qquad (3.6.2)$$

By (**3**.5.6), every $x \in \hat{S}$ is a solution of (**3**.1.1). This will establish Theorem **3**.3.3.

The next definition was inspired by Filippov's construction of piecewise linear approximate solutions [3,p.112]. Here $\tilde{B}$ denotes the closed ball $\overline{B}(0,\tilde{\rho}) \subseteq E$.

DEFINITION **3**.6.1. *A n-level feedback* $\mathcal{E}$ *on* $[0,T] \times \tilde{B}$ *is a triple* $(\mathcal{P}, \mathcal{G}, f)$ *consisting of*

1) $n+1$ *finite partitions of* $[0,T]$

$$\mathcal{P}_k = \left\{ t_0^k, t_1^k, \ldots, t_{p_k}^k \right\}, \quad k=1,\ldots,n+1, \text{ with } t_i^k = iT/p_k ,$$

p_{k+1} *being an integer multiple of* p_k , *for all* $k \leq n$.

2) n *finite open coverings of* $\tilde{B}$

$$\mathcal{G}_k = \left\{ V_1^k, \ldots, V_{q_k}^k \right\} \qquad k=1,\ldots n .$$

3) *A finite set of continuous vector fields on* E

$$x \to f[i_1, \ldots, i_n, j](x)$$

with $i_k \in \{1,\ldots,q_k\}$, $1 \leq k \leq n$, $j \in \{0,\ldots,p_{n+1}-1\}$, *each vector field* $f[\ldots]$ *being defined on some open subset of* E.

For $t \in [0,T]$, $1 \leq k \leq n+1$, define

$$\tau^k(t) = \max\left\{ t_i^k : 0 \le i \le p_k \,,\, t_i^k \le t \right\} \,, \qquad (3.6.3)$$

$$\sigma_{n+1}(t) = \max\left\{ \ell : 0 \le \ell \le p_{n+1} \,,\, t_\ell^{n+1} \le t \right\} . \qquad (3.6.4)$$

DEFINITION **3**.6.2. *An absolutely continuous map* $y : [0,T] \to E$ *is a solution of* $\mathcal{E}$ *if there exist* n *functions*

$\sigma_k : \left\{ t_0^k, \dots, t_{p_k-1}^k \right\} \to \{1, \dots, q_k\}$, $1 \le k \le n$, *such that*

i) $y(t_i^k) \in \overline{V}^k_{\sigma_k(t_i^k)}$ $\qquad \forall\ i,k$

ii) $\dot{y}(t) = f[\sigma_1(\tau^1(t)), \sigma_2(\tau_2(t)), \dots, \sigma_n(\tau^n(t)), \sigma_{n+1}(t)](y(t))$

a.e. *on* $[0,T]$.

Intuitively, the velocity y(t) of a solution of the feedback $\mathcal{E}$ is determined by two things : 1) the time t, measured by the value $j = \sigma_{n+1}(t)$ of a digital clock, 2) the position of y at the nodes $\tau^1(t), \dots, \tau^n(t)$ of the partitions $\mathcal{P}_1, \dots, \mathcal{P}_n$ immediately preceding t. More precisely, on the interval $[t_i^k, t_{i+1}^k)$ of the k-th partition $\mathcal{P}_k$, y(t) depends on the set V_ℓ^k of the covering $\mathcal{G}_k$ whose closure contains $y(t_i^k)$. If $y(t_i^k)$ is contained in more than one set, for example $y(t_i^k) \in \overline{V_2^k} \cap \overline{V_5^k}$, then different choices are possible (in the example, $\sigma_k(t_i^k) = 2$ or $\sigma_k(t_i^k) = 5$). However, one has to stick with the same choice throughout the interval $[t_i^k, t_{i+1}^k)$. In the following, the short notation $[\underline{i}, j] =$

$[i_1, \ldots, i_n, j]$ will be used.

DEFINITION 3.6.3. *Under the assumptions* A1) ÷ A6) *in* §3, *the* n-level feedback $\mathcal{E} = (\mathcal{P}, \mathcal{G}, f)$, *on* $[0,T]\times\tilde{B}$, *is admissible for* (3.1.1) *if, for all* $\underline{i}, j$,

i) $f[\underline{i},j](x) \in \text{rel int } \overline{\text{co}}F(x) \quad \forall x \in \text{Dom } f[\underline{i},j]$;

ii) *the domain of* $f[\underline{i},j](\cdot)$ *is an open neighborhood of*

$$\bar{B}(V^n_{i_n}, MT/p_n) \cap \tilde{B} ;$$

iii) *the projection of* $f[\underline{i},j](x)$ *on* E' *is constant and lies in the interior of* $\pi' \circ \overline{\text{co}}F(x)$ *, i.e.*

$$f[\underline{i},j](x) = \pi' \circ f[\underline{i},j](x) + \pi'' \circ f[\underline{i},j](x)$$

$$= f'[\underline{i},j] + f''[\underline{i},j](x) \quad , \quad \textit{with}$$

$$f'[\underline{i},j] \in \text{int}_{E'} (\pi' \circ \overline{\text{co}}F(x)) \qquad \forall x \in \text{Dom } f[\underline{i},j] .$$

3.7. SOLUTIONS OF ADMISSIBLE FEEDBACKS

DEFINITION 3.7.1. *Let* $\mathcal{E} = (\mathcal{P}, \mathcal{G}, f)$ *be a* n-level feedback. *We call* $S_{\mathcal{E}}$ *the set of all solutions* $y(\cdot)$ *of* $\mathcal{E}$ *such that* $y(0) = 0$. *For* $\delta \in [0,1]$, $S^{\delta}_{\mathcal{E}}$ *denotes the set of all absolutely continuous* $z : [0,T] \to E$ *such that* $z(0) = 0$ *and there exist* n *functions* $\beta_k : \{t^k_0, \ldots, t^k_{p_k-1}\} \to \{1, \ldots q_k\}$, $1 \le k \le n$, *such that*

$$z(t_i^k) \in \overline{V}^k_{\beta_k(t_i^k)} \qquad \forall\ i,k \tag{3.7.1}$$

$$\dot{z}(t) \in f[\beta_1(\tau^1(t)),\dots,\beta_n(\tau^n(t)),\beta_{n+1}(t)](z(t)) + (\overline{B}(0,\delta) \cap E'') \quad a.e.\ on\ [0,T]\ . \tag{3.7.2}$$

Here, as in (3.6.4), $\beta_{n+1}(t) = \max\{i : t_i^{n+1} \le t\}$. Intuitively, $S^\delta_{\mathcal{E}}$ represents a set of δ-approximate solutions of the feedback $\mathcal{E}$. Notice that the "error" $\dot{z}(t) - f[\dots](z(t))$ must be inside the finite dimensional space E". Of course, $S_{\mathcal{E}} = S^0_{\mathcal{E}}$.

PROPOSITION 3.7.1. *Let $\mathcal{E} = (\mathcal{P},\mathcal{G},f)$ be an admissible* n-level *feedback for* (3.1.1), *according to Definition* 3.6.3. *Then the set $S_{\mathcal{E}}$ of its solutions is nonempty and compact. Moreover, the multivalued map $\delta \to S^\delta_{\mathcal{E}}$ from* [0,1] *into* $C^0([0,T];E)$ *has closed graph and compact values (hence it is Hausdorff-upper semicontinuous).*

Proof. To show that $S_{\mathcal{E}} \ne \emptyset$, we shall construct a solution y of E piecewise on the intervals $[t_j^{n+1}, t_{j+1}^{n+1}] = I_j$ of the partition $\mathcal{P}_{n+1}$, by induction on j. Since every $\mathcal{G}_k$ is a covering, for each $k \in \{1,\dots,n\}$ we can choose an index $\sigma_k(0) \in \{1,\dots,q_k\}$ such that $0 \in V^k_{\sigma_k(0)}$. By conditions ii) and iii) in Definition 3.6.3 together with A4) in §3, the vector field $f[\sigma_1(0),\dots,\sigma_n(0),0](\cdot)$ is defined on an open neighborhood of the set $\overline{B}(V^n_{\sigma_n(0)}, MT/p_n) \cap \tilde{B}$ and takes values in a finite-dimensional subset of the ball $B(0,M-1)$. Therefore, the Cauchy problem

$$\dot{u}(t) = f[\sigma_1(0),\dots,\sigma_n(0),0](u(t)) \quad , u(0) = 0 \in E$$

has a solution y(t), defined on the interval $[0,t_1^{n+1}]$. Now assume, by induction, that the function y has been defined up to the nodal point $t_j^{n+1} \in \mathcal{P}_{n+1}$, $0 < t_j^{n+1} < T$. In addition, assume that the values $\sigma_k(t_i^k)$ have already been chosen for all nodal points $t_i^k < t_j^{n+1}$. Define $\ell(j) = \min\{k : t_j^{n+1} \in \mathcal{P}_k\}$. If $k < \ell(j)$, then $\tau_k(t_j^{n+1}) < t_j^{n+1}$ and $\sigma_k(\tau_k(t_j^{n+1}))$ has already been defined. If $\ell(j) \le k \le n$, choose $\sigma_k(t_j^{n+1})$ such that $y(t_j^{n+1}) \in V^k_{\sigma_k(t_j^{n+1})}$. This is possible because $\mathcal{G}_k$ is a covering. The admissibility of the feedback $\mathcal{E}$ implies that the Cauchy problem

$$\begin{cases} \dot{u}(t) = f[\sigma_1(\tau_1(t)),\ldots,\sigma_n(\tau_n(t)),\sigma_{n+1}(t)](u(t)) \\ u(t_j^{n+1}) = y(t_j^{n+1}) \end{cases}$$

has a solution $u(\cdot)$ on the interval $I_j = [t_j^{n+1}, t_{j+1}^{n+1}]$. Extend the function y by setting $y(t) = u(t)$ on I_j. By A4) and A6) in §3, y cannot escape from the ball $\tilde{B}$ within time T. By induction, the function y can be defined on the whole interval [0,T]. Our construction implies that i) and ii) in Definition 3.6.2 are both satisfied, hence y is a solution of $\mathcal{E}$.

To prove the second statement, assume that $\delta_m \to \delta$ and that $z_m \to z$ in $C^0([0,T];E)$ with $z_m \in S_{\mathcal{E}}^{\delta_m}$ for all $m \ge 1$. By possibly taking a subsequence we can assume that the discrete maps $\beta_1,\ldots,\beta_n$ which appear in Definition 3.7.1 are the same for all m. In particular, since $z_m(t_i^k) \in \overline{V^k_{\beta_k(t_i^k)}}$, the uniform convergence of the sequence z_m to z implies

$$z(t_i^k) \in \overline{V}^k_{\beta_k(t_i^k)} \tag{3.7.3}$$

at all nodal points $t_i^k \in \mathcal{P}_k$, $k = 1,\ldots,n$. Moreover, for $t \in [t_j^{n+1}, t_{j+1}^{n+1}]$,

$$z_m(t) = \int_0^{t_1^{n+1}} f[\beta_1(0),\ldots,\beta_n(0),0](z_m(s))\,ds + \ldots +$$

$$\int_{t_j^{n+1}}^{t} f[\beta_1(\tau_1(t)),\ldots,\beta_n(\tau_n(t)),j](z_m(s))\,ds + \phi_m(t) \quad ,$$

where ϕ_m is a Lipschitz continuous function from [0,T] into the finite-dimensional space E", with $\phi_m(0) = 0$ and with Lipschitz constant δ_m. Therefore, Ascoli's theorem provides a subsequence, say $\phi_{m'}$, converging to some function ϕ, with Lipschitz constant δ. Using the continuity of the vector fields $f[\underline{i},j]$ and recalling (3.7.3), we conclude that $z \in S^\delta_{\mathcal{E}}$. This implies that the map $\delta \to S^\delta_{\mathcal{E}}$ has closed graph. In particular, each set $S^\delta_{\mathcal{E}}$ is closed, and such is also $S_{\mathcal{E}} = S^\circ_{\mathcal{E}}$. Since all functions $z \in S^\delta_{\mathcal{E}}$ are uniformly Lipschitz continuous and take values inside a finite-dimensional subspace of E, the compactness of $S^\delta_{\mathcal{E}}$ follows again from Ascoli's theorem. ∎

3.8. A CLOSURE THEOREM

Let S be the family of all solutions of (3.1.2). We will show that the sets S_{jn} defined at (3.6.1) are relatively open in S.

THEOREM 3.8.1. *For any closed ball* $C \subseteq E$ *and any* $\delta>0$, *the set*

$$R_{C,\delta} = \left\{ x \in S : \int_0^T \phi_C(\dot{x}(t), x(t))\,dt \geq \delta \right\}$$

is closed in $C^0([0,T];E)$.

Proof. Let $(x_n)_{n\geq 1}$ be a sequence in $R_{C,\delta}$ such that $x_n \to x$ uniformly on $[0,T]$. The corresponding sequence of derivatives $\dot{x}_n$ converges weakly to $\dot{x}$ in $L^2([0,T];E)$. By Mazur's theorem, there exists a sequence of convex combinations

$$v_n = \sum_{k=1}^{r_n} \lambda_{n,k}\, \dot{x}_{n+k}$$

of the $\dot{x}_n$ which converges to $\dot{x}$ strongly in $L^2([O,T];E)$ and pointwise a.e. on $[O,T]$. Fix $\varepsilon>0$. Using the theorems of Lusin and Egorov we obtain a compact subset $J\subseteq[O,T]$ such that meas(J) $> T-\varepsilon$, the restriction of $\dot{x}$ to J is continuous and v_n converges to $\dot{x}$ uniformly on J. By the upper semicontinuity of d_C and the compactness of the set $\{(\dot{x}(t),x(t)):t\in J\}$, there exists $\eta>0$ such that

$$d_C(\dot{x}(t),F(x(t))) \geq d_C(v,\overline{B}(F(x(t)),\eta))-\varepsilon \tag{3.8.1}$$

whenever $t\in J$, $\|v-\dot{x}(t)\| < \eta$. Using Lemma 3.5.2 and the continuity of F, we can find $\rho>o$ such that $t\in J, \|y-x(t)\| \leq \rho$ imply $F(y) \subseteq \overline{B}(F(x(t)),\eta)$ and

$$d_C(u,\overline{B}(F(x(t)),\eta)) \geq d_C(u,F(y)) - \varepsilon \tag{3.8.2}$$

for all $u \in \overline{co}F(y)$. Assume that $\|v_n(t)-\dot{x}(t)\| < \eta$ and $\|x_n(t)-x(t)\| < \rho$ for all $n \geq N$. The concavity of the map $u \rightarrow d_C(u,\overline{B}(F(x(t)),\eta))$ together with (3.8.1),(3.8.2) yields

$$\begin{aligned} d_C(\dot{x}(t),F(x(t))) &\geq d_C(v_n(t),\overline{B}(F(x(t)),\eta)) - \varepsilon \\ &\geq \sum_{k=1}^{r_n} \lambda_{n,k} d_C(\dot{x}_{n+k}(t),\overline{B}(F(x(t)),\eta)) - \varepsilon \\ &\geq \sum_{k=1}^{r_n} \lambda_{n.k} d_C(\dot{x}_{n+k}(t),F(x_{n+k}(t))) - 2\varepsilon \end{aligned} \tag{3.8.3}$$

whenever $n \geq N$, $t \in J$. Since $\phi_C(\dot{x}_{n+k}(t),x_{n+k}(t)) = d_C(x_{n+k}(t),F(x_{n+k}(t))) \in [0,1]$ for a.e. $t \in [0,T]$, from (3.8.3) we obtain

$$\int_0^T \phi_C(\dot{x}(t),x(t))dt \geq \int_J d_C(\dot{x}(t),F(x(t)))dt$$

$$\geq \int_J \sum_{k=1}^{r_n} \lambda_{n,k}\Big[d_C(\dot{x}_{n+k}(t),F(x_{n+k}(t))) - 2\varepsilon\Big]dt$$

$$\geq \sum_{k=1}^{r_n} \lambda_{n.k}\left[\int_0^T \phi_C(\dot{x}_{n+k}(t),x_{n+k}(t))dt - \varepsilon\right] - 2\varepsilon T$$

$$\geq \delta - \varepsilon - 2\varepsilon T .$$

Since ε was arbitrary, the theorem is proved. ∎

3.9. SOME GEOMETRIC LEMMAS.

Let $Q \subseteq B(O,M)$ be a closed, convex subset of the Banach space E with finite codimension. Observe that the relative interior of Q is nonempty and dense on Q. Moreover, if V is a closed convex neighborhood of a point $\omega \in Q$, then $\omega \in$ rel int Q if and only if $\omega \in$ rel int$(V \cap Q)$. Throughout this section, we refer to the decomposition $E = E' \oplus E''$, whith E' closed and E'' finite-dimensional, assuming that A2), A3) in §3 hold. We remark that, if $A \subseteq \mathbb{R}^d$ is compact and convex, then rel int A is the smallest convex subset of A whose closure is A.

LEMMA 3.9.1. *If* $y_0 \in$ rel int$(Q \cap (y_0 + E''))$ *and* $\pi'(y_0) \in$ $\text{int}_{E'}(\pi'(Q))$, *then* $y_0 \in$ rel int$Q$. *Conversely, if* $y_0 \in$ rel int$Q$ *and* $\text{int}_{E'}(\pi'(Q)) \neq \emptyset$, *then* $\pi'(y_0) \in \text{int}_{E'}(\pi'(Q))$.

Proof. The set $\Lambda = (y_0 + E'') \cap$ rel int$Q$ is clearly a convex subset of $(y_0 + E'') \cap Q$. By our previous remark, to prove the first statement it suffices to show that Λ is dense on $(y_0 + E'') \cap Q$. Let $\omega_0 \in (y_0 + E'') \cap Q$, $\varepsilon > 0$. By the second assumption, there exists $\delta > 0$ such that $B(\pi'(y_0),\delta) \cap E' \subseteq \pi'(Q)$. Choose $\omega_1 \in$ rel int Q such that $\|\omega_1 - \omega_0\| < \varepsilon$. Since the projections π', π'' have norm 1, this implies $\pi'(\omega_0 + (\delta/\varepsilon)(\omega_0 - \omega_1)) \in B(\pi'(\omega_0),\delta) \cap E' \subseteq \pi'(Q)$. Hence $\pi'(\omega_0 + (\delta/\varepsilon)(\omega_0 - \omega_1)) = \pi'(\omega_2)$ for some $\omega_2 \in Q$. Consider the convex combination $\omega = (\delta\omega_1 + \varepsilon\omega_2)/(\varepsilon + \delta)$. An easy computation shows that $\omega \in \Lambda$ and $\|\omega - \omega_0\| \leq (\delta\|\omega_1 - \omega_0\| + \varepsilon\|\omega_2 - \omega_0\|)/(\varepsilon + \delta) \leq \varepsilon + 2M\varepsilon/(\varepsilon + \delta)$. Since δ and M are fixed while $\varepsilon > 0$ is arbitrary, the density of Λ on $(y_0 + E'')\cap Q$ is proved.

Conversely, assume $y_0 \in$ rel int Q and $B(\pi'(y_1),\rho) \cap E' \subseteq \pi'(Q)$ for some $y_1 \in Q$, $\rho > 0$. Choose $y_2 \in Q$ and $\lambda > 0$ such that $y_0 = \lambda y_1 + (1 - \lambda)y_2$. The convexity of $\pi'(Q)$ now implies $\pi'(Q) \supseteq (\lambda B(\pi'(y_1),\rho)) \cap E') + (1 - \lambda)\pi'(y_2) = B(\pi'(y_0),\lambda\rho) \cap E'$. ∎

In the following, if Ω is a bounded subset of $\mathbb{R}^d$ with positive Lebesgue measure, we consider the barycenter of Ω:

$$\mathrm{bar}(\Omega) = \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega x \, dx \quad .$$

If $A \subseteq \mathbb{R}^d$ is compact and convex, $\mathrm{bar}(B(A,1)) \in A$. Moreover, the map $A \to \mathrm{bar}(B(A,1))$ is Lipschitz continuous w.r.t. the Hausdorff metric [3,p.77].

LEMMA **3**.9.2. *Let* $A \subseteq \mathbb{R}^d$ *be compact and convex. Then* $b = \mathrm{bar}(B(A,1)) \in \mathrm{rel\,int} A$.

Proof. It suffices to prove that, if ψ is a linear functional on $\mathbb{R}^d$ and

$$\psi(b) = \max\{\psi(x) : x \in A\} , \qquad (\mathbf{3}.9.1)$$

then ψ is constant on A. Call Γ the hyperplane $\{x \in \mathbb{R}^d : \psi(x) = \psi(b)\}$ and let $S(x)$ be the point symmetric to x w.r.t. Γ, i.e. $S(x) = x + 2(\pi(x) - x)$, $\pi(x)$ being the perpendicular projection of x on Γ. Define

$$B^+ = \{x \in B(A,1) : \psi(x) \geq \psi(b)\},$$

$$B_1 = B^+ \cup S(B^+) \quad , \quad B_2 = B(A,1) \setminus B_1.$$

Notice that, if (**3**.9.1) holds, then $B_1 \subseteq B(A,1)$.

If meas $(B_2) > 0$, using well-known properties of the barycenter and the linearity of ψ , we obtain

$$\psi(b) = \frac{\mathrm{meas}(B_1)}{\mathrm{meas}(B(A,1))} \cdot \psi(b_1) + \frac{\mathrm{meas}(B_2)}{\mathrm{meas}(B(A,1))} \cdot \psi(b_2) \quad , \qquad (\mathbf{3}.9.2)$$

where b_1 , b_2 are the barycenters of B_1 , B_2 respectively. By symmetry, $\psi(b_1) = \psi(b)$. Since $\psi(x) < \psi(b)$ for every $x \in B_2$, we have $\psi(b_2) < \psi(b)$ and (**3**.9.2) yields a contradiction. Therefore $\mathrm{meas}(B_2) = 0$, from which one easily deduces $B_2 = \emptyset$ and $A \subseteq \Gamma$, completing the proof. ∎

LEMMA **3**.9.3. *Let* $x \to Q(x)$ *be a Hausdorff continuous multifunction defined on an open set* $U \subseteq E$ *, with closed convex values contained in the ball* $B(0,M) \subseteq E = E' \oplus E''$. *Assume* $y_o \in Q(x_o)$ *and* $\pi'(y_o) \in \mathrm{int}_{E'} \pi'(Q(x_o))$. *Then, for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\pi'(y_o) \in \mathrm{int}_{E'} \pi'(Q(x) \cap \bar{B}(y_o,\varepsilon)) \qquad \forall x \in B(x_o,\delta). \qquad (\mathbf{3}.9.3)$$

Proof. By Lemma **3**.3.2 and the Hausdorff continuity of Q, there exist ρ', $\delta' > 0$ such that

$$E' \cap \bar{B}(\pi'(y_o), 2\rho') \subseteq \pi'(Q(x)), \qquad \forall x \in B(x_o,\delta').$$

Choose $\rho \in (0,\rho')$ such that $\rho + 2M\rho/\rho' < \varepsilon$, and $\delta \in (0,\delta')$ such that

$$Q(x) \cap B(y_o,\rho) \neq \emptyset, \qquad \forall x \in B(x_o,\delta).$$

Let $z \in E'$, $\|z-\pi'(y_o)\| \leq \rho$. The Lemma will be proved by showing that, if $\|x-x_o\| < \delta$, then $z = \pi'(y)$ for some $y \in Q(x) \cap \bar{B}(y_o,\varepsilon)$. Choose any $y_1 \in Q(x) \cap \bar{B}(y_o,\rho)$. If $\pi'(y_1) = z$ we are done. Otherwise, define

$$z' = z + \frac{z-\pi'(y_1)}{\|z-\pi'(y_1)\|}\rho' . \qquad (\mathbf{3}.9.4)$$

Since $z' \in \bar{B}(\pi'(y_1),\rho'+\rho)$, $z' = \pi'(y_2)$ for some $y_2 \in Q(x)$. From (**3**.9.4) we deduce $z = \pi'(y)$ with

$$y = (\|z-\pi'(y_1)\| y_2 + \rho' y_1)/(\rho' + \|z-\pi'(y_1)\|) \quad .$$

The choice of ρ implies

$$\|y-y_o\| \le \frac{\|z-\pi'(y_1)\|}{\rho'} \|y_2-y_o\| + \|y_1-y_o\| \le \frac{2M\rho}{\rho'} + \rho < \varepsilon,$$

proving (3.9.3). ∎

LEMMA 3.9.4. *With the same assumptions of Lemma 3.9.3, for every* $\varepsilon>0$ *there exists a neighborhood* V *of* x_o *such that the map*

$$x \to A(x) = Q(x) \cap \bar{B}(y_o,\varepsilon) \cap (y_o + E'')$$

is Hausdorff continuous on V.

Proof. The map A has closed graph and its values are compact subsets of the finite-dimensional affine space $y_o + E''$. Hence A is Hausdorff upper semicontinuous on its domain. To show that A is also lower semicontinuous, we use Lemma 3.9.3 and choose an open neighborhood V of x_o such that

$$\pi'(y_o) \in \text{int}_{E'}\pi'(Q(x) \cap \bar{B}(y_o,\varepsilon/2)), \qquad \forall x \in V. \tag{3.9.5}$$

Fix any $\tilde{x} \in V$ and any $\tilde{y} \in A(\tilde{x})$. Since $A(\tilde{x})$ is compact, it is enough to prove that, for every $\eta>0$, there exists $\delta>0$ such that

$$d(\tilde{y},A(x)) \le \eta \qquad \forall x \in B(\tilde{x},\delta). \tag{3.9.6}$$

By (3.9.5), there exists $y \in Q(\tilde{x}) \cap \bar{B}(y_o,\varepsilon/2)$ such that $\pi'(y) = \pi'(\tilde{y}) = \pi'(y_o)$. Choose $\lambda > 0$ sufficiently small so that, setting $y_\lambda = \lambda y + (1-\lambda)\tilde{y} \in B(y_o,\varepsilon)$, one has $\|y_\lambda-\tilde{y}\| < \eta/2$. Choose $\eta' \in (0,\eta/2)$ so small that $B(y_\lambda,\eta') \subset B(y_o,\varepsilon)$. A new application of Lemma 3.9.3 provides the existence of $\delta > 0$

such that $\|x-\tilde{x}\| < \delta$ implies

$$\pi'(y_o) \in \pi'(Q(x) \cap \bar{B}(y_\lambda,\eta')) . \tag{3.9.7}$$

From (3.9.7) we deduce that, whenever $\|x-\tilde{x}\| < \delta$, there exists y such that

$$y \in Q(x) \cap \bar{B}(y_\lambda,\eta') \cap (y_o+E'') \subseteq A(x) .$$

Therefore $d(\tilde{y},A(x)) \leq \|\tilde{y}-y_\lambda\| + \|y_\lambda-y\| < \eta/2 + \eta' < \eta$, proving (3.9.6). ∎

COROLLARY 3.9.5. *If the assumptions* A1)÷A6) *in* §3 *hold, then there exists a continuous vector field* $\tilde{g}:B(0,2\tilde{\rho})\to E$ *such that, for all* x,

$$\tilde{g}(x) \in \text{rel int}\,\overline{\text{co}}F(x) , \tag{3.9.8}$$

$$\pi'(\tilde{g}(x)) = \omega', \tag{3.9.9}$$

where ω' *is the point considered at* A5).

Proof. For $x\in B(0,2\tilde{\rho})$, set $A(x) = \overline{\text{co}}F(x) \cap (\omega'+E'')$, and define $\tilde{g}(x) = \text{bar}(B(A(x),1) \cap (\omega'+E''))$, the barycenter being taken with respect to the Lebesgue measure on the affine, finite-dimensional space $\omega'+E''$. The Lemmas 3.9.1 and 3.9.2 now imply (3.9.8), while the continuity of $\tilde{g}$ follows from Lemma 3.9.4.

LEMMA 3.9.6. *Let all of the assumptions in Theorem* 3.3.3 *hold. Let* $\varepsilon > 0$, $\xi \in B(0,2\tilde{\rho})$, $v \in \text{rel int}\,\overline{\text{co}}F(\xi)$ *be given. For a fixed closed ball* $C \subset E$, *consider the function* d_C *defined at* (3.5.3). *Then there exist a neighborhood* V *of* ξ, *finitely many* *continuous functions* $g_1,\dots,g_N : V \to E$ *and rational coefficients* $\lambda_1,\dots,\lambda_N > 0$ *with the following properties:*

i) $\sum_{k=1}^{N} \lambda_k = 1$,

ii) $y_k(x) \in \text{rel int } \overline{co}F(x)$,

iii) $g'_k = \pi'(g_k(x)) \in \text{int}_{E'}(\pi' \circ \overline{co}F(x))$ *is independent of* x ,

iv) $\sum_{k=1}^{N} \lambda_k g'_k = \pi'(v)$,

v) $\|v - \sum_{k=1}^{N} \lambda_k g_k(x_k)\| < \varepsilon$,

vi) $d_C\,(g_k(x)\ ,\ F(x)) < \varepsilon$,

for every $x, x_1, \dots, x_N \in V$, $k = 1,\dots,N$.

Proof. Call E_ξ the closed affine subspace generated by $\overline{co}F(\xi)$. Let z^* be a strongly exposed point of $\overline{co}F(\xi)$. By Lemma 3.5.1, $d_C(z^*, F(\xi)) = 0$. Using the upper semicontinuity of d_C , we can find a point $z \in \text{rel int}\overline{co}F(\xi)$, close to z^* , and $\delta > 0$ such that

$$B(z,\delta) \cap E_\xi \subset \overline{co}\,F(\xi) \ , \tag{3.9.10}$$

$$d_C(y, F(\xi)) < \varepsilon \ , \quad \forall y \in B(z,\delta) \cap E_\xi \ . \tag{3.9.11}$$

Since $v \in \text{rel int } \overline{co}F(\xi)$, $\pi'(v) \in \text{int}_{E'}\pi'(\overline{co}F(x))$. Moreover, there exist $y' \in \overline{co}F(\xi)$ and a rational number $\eta \in (0,1]$ such that

$$v = \eta z + (1-\eta)y' \ . \tag{3.9.12}$$

By Lemma 3.5.1, there exist finitely many elements $y'_2,\dots,y'_N \in$

exp $\overline{co}F(\xi)$ and rational coefficients $\eta_2,\dots,\eta_N > 0$ such that

$$\sum_{k=2}^{N} \eta_k = 1 \quad , \quad \|y' - \sum_{k=2}^{N} \eta_k y_k'\| < \delta\eta \quad .$$

Since $d_C(y_k') = 0$ and d_C is upper semicontinuous, by choosing $y_k \in \text{rel int}\ \overline{co}F(\xi)$ sufficiently close to y_k', we still have

$$\|y - \sum_{k=2}^{N} \eta_k y_k\| < \delta\eta \quad , \tag{3.9.13}$$

$$d_C(y_k \ , \ F(\xi)) \ < \varepsilon \quad , \qquad k = 2,\dots,N \ . \tag{3.9.14}$$

Define

$$y_1 = z + \frac{1}{\eta}\,(v - \eta z - (1-\eta)\sum_{k=2}^{N} \eta_i y_i) \in B(z,\delta) \cap \overline{co}\,F(\xi) \ ,$$

$$\lambda_1 = \eta \ , \ \lambda_k = (1 - \eta)\eta_k \ , \qquad k = 2,\dots,N \ .$$

By (3.9.13),(3.9.14) and the upper semicontinuity of d_C, there exist $\rho > 0$ and a neighborhood V' of ξ such that

$$\|v - \sum_{k=1}^{N} \lambda_k v_k\| < \varepsilon, \tag{3.9.15}$$

$$d_C(v_k, F(x)) < \varepsilon \tag{3.9.16}$$

for every $x \in V'$, $k \in \{1,\dots,N\}$, $v_k \in \overline{B}(y_k,\rho)$. Using Lemma 3.9.4, we can find a neighborhood $V \subseteq V'$ of ξ such that the multifunctions

$$x \to A_k(x) = \overline{co}F(x) \cap \overline{B}(y_k,\rho) \cap (y_k + E'')$$

are defined and continuous on V . For $x \in V$, $k = 1,\dots,N$, define

$$g_k(x) = \text{bar}\,(B(A_k(x),1) \cap (y_k + E'')) \quad ,$$

the barycenter being taken with respect to the Lebesgue measure on the finite dimensional, affine space $y_k + E''$.

The continuity of g_k now follows from the Hausdorff continuity of A_k, condition ii) is a consequence of Lemmas 3.9.1 and 3.9.2, while v) and vi) follow from (3.9.15), (3.9.16) respectively. By construction, the remaining conditions i), iii) and iv) clearly hold. This completes the proof. ∎

3.10. AN APPROXIMATION THEOREM

Theorem 3.10.1. *Let* $\mathcal{E} = (\mathcal{P}, \mathcal{G}, f)$ *be an admissible* n-level feedback *for* (3.1.1). *Let* C *be a closed ball in* E, $\delta > 0$. *Then there exists an admissible* (n+1)-level feedback $\hat{\mathcal{E}} = (\hat{\mathcal{P}}, \hat{\mathcal{G}}, \hat{f})$ *such that*

$$S_{\hat{\mathcal{E}}} \subseteq B(S^{\delta}_{\mathcal{E}}, \delta) \qquad \text{and}$$

$$\int_0^T \phi_C(\dot{y}(t), y(t))\,dt \quad \leq \quad \delta T \qquad \forall\, y \in S_{\hat{\mathcal{E}}} \; .$$

Proof. Let E^* be the finite dimensional space spanned by E'' and by the vector fields $f[\underline{i},j]$ of $\mathcal{E}$. Set $\Omega = E^* \cap \tilde{B}$, where $\tilde{B} = \bar{B}(o,\tilde{\rho}) \subseteq E$, as usual. For every fixed $\xi \in \Omega$, denote by $\Gamma(\xi)$ the set of multiindexes $\underline{i} = (i_1, \ldots, i_n)$ such that

$$\xi \in \overline{B}(V^{n}_{i_n},\ MT/p_n)\ . \tag{3.10.1}$$

By the definition of admissible feedback, $\underline{i} \in \Gamma(\xi)$ implies that, for all j, the domain of $f[\underline{i},j]$ is an open neighborhood of ξ. Applying Lemma (3.9.6) to all vector fields $f[i,j]$ with $i \in \Gamma(\xi)$, we deduce the existence of a radius $\rho_\xi > 0$, a finite set of vector fields $g_1,\ldots,g_N$ defined on $B(\xi,\rho_\xi)$ and rational coefficients $\lambda_1[\underline{i},j],\ldots,\ \lambda_N[\underline{i},j] \geq 0$ satisfying the following conditions:

C1) $B(\xi,\rho_\xi) \subseteq \text{Dom } f[\underline{i},j]$;

C2) $\|f[\underline{i},j](x) - f[\underline{i},j](\xi)\| < \delta/2$;

C3) $g_k(x) \in \text{rel int } \overline{co}F(x)$;

C4) $g'_k = \pi' \circ g_k(x) \in \text{int}_{E'}(\pi' \circ \overline{co}F(x))$ is independent of x ;

C5) $d_C(g_k(x), F(x)) < \delta$;

C6) $\sum_{k=1}^{N} \lambda_k[\underline{i},j] = 1$;

C7) $\sum_{k=1}^{N} \lambda_k[\underline{i},j]g'_k = \pi \circ f[\underline{i},j]x = f'[\underline{i},j]$;

C8) $\|f[\underline{i},j](\xi) - \sum_{k=1}^{N} \lambda_k[\underline{i},j]g_k(x_k)\| < \delta/2$

whenever $x, x_1,\ldots,x_N \in B(\xi,\rho_\xi)$, $\underline{i} \in \Gamma(\xi)$, $k = 1,\ldots,N$, together with

C9) $\forall i \in \{1,\ldots,q_n\}$, if $\xi \notin \overline{B}(V^n_i, MT/p_n)$, then

$$B(\xi,\rho_\xi) \cap \overline{B}(V^n_i, MT/p_n) = \emptyset\ .$$

Repeat the above construction for every $\xi \in \Omega$. The family $\{B(\xi,\rho_\xi/2) : \xi \in \Omega\}$ is an open covering of the compact set Ω. Let $\{B(\xi_\ell,\rho_\ell/2) : \ell = 2,\dots,\hat{q}_{n+1}\}$ be a finite subcovering. Choose an integer multiple $\hat{p}_{n+1}$ of p_{n+1} such that

$$T/\hat{p}_{n+1} < \min\{\rho_\ell/2M \ ; \ \ell = 2,\dots,\hat{q}_{n+1}\} \wedge \delta/2M . \tag{3.10.2}$$

Call g_k^ℓ, $\lambda_k^\ell[\underline{i},j]$ respectively the vector fields and the rational coefficients constructed in connection with the point ξ_ℓ, $\ell = 2,\dots,\hat{q}_{n+1}$, $k = 1,\dots,N_\ell$.

We are now ready to define the feedback $\hat{\mathcal{E}}$.

Partitions : Set $\hat{\mathcal{P}}_k = \mathcal{P}_k$ for $1 \le k \le n$ and define

$$\hat{\mathcal{P}}_{n+1} = \{t_i^{n+1} : i=0,\dots,p_{n+1}\}$$

with $\hat{t}_i^{n+1} = iT/\hat{p}_{n+1}$. Recalling that the coefficients λ_k^ℓ are all rational, there exist integers p and $m_k^\ell[\underline{i},j]$ such that

$$\lambda_k^\ell[\underline{i},j] = m_k^\ell[\underline{i},j]/p \tag{3.10.3}$$

for all $k,\ell,\underline{i},j$. Set $\hat{p}_{n+2} = p \cdot \hat{p}_{n+1}$ and define

$$\hat{\mathcal{P}}_{n+2} = \{\hat{t}_i^{n+2} : i=0,\dots,\hat{p}_{n+2}\} , \text{ with } \hat{t}_i^{n+2} = iT/\hat{p}_{n+2} .$$

Coverings : Set $\hat{\mathcal{G}}_k = \mathcal{G}_k$ if $k=1,\dots,n$. To construct the last covering $\hat{\mathcal{G}}_{n+1}$, for $\ell = 2,\dots,\hat{q}_{n+1}$ define $\hat{V}_\ell^{n+1} = B(\xi_\ell,\rho_\ell/2)$. The above sets $\hat{V}_\ell^{n+1}$ now cover Ω. To obtain a covering of $\tilde{B}$, one more set must be added. The compactness of Ω implies that the distance

$$\rho_1 = \inf\{\|x-y\| \ : \ x \in \Omega, \ \ y \notin \bigcup_\ell B(\xi_\ell, \rho_\ell/2)\}$$

is strictly positive. Define

$$\hat{V}_1^{n+1} = \{x \in E : \|x\| < 2\tilde{\rho} \ , \ d(x,\Omega) > \rho_1/2\} \qquad (3.10.4)$$

Then $\hat{\mathcal{G}}_{n+1} = \{\hat{V}_\ell^{n+1} \ ; \ \ell = 1,\ldots,\hat{q}_{n+1}\}$ is an open covering of $\tilde{B}$.

Functions : Set $\lambda_0^\ell[\underline{i},j] = 0$ for every $\underline{i},j,\ell$, and define $\hat{\tau}^k(t) = \max\{\hat{t}_i^k : 0 \le i \le p_k, \ \hat{t}_i^k \le t\}$ as in (3.6.3). Call $\omega = T/\hat{p}_{n+2}$ the length of the intervals of the partition $\hat{\mathcal{P}}_{n+2}$. If $\underline{i} = (i_1,\ldots,i_n)$ is a multiindex, $\ell \in \{1,\ldots,\hat{q}_{n+1}\}$ and $r \in \{0,\ldots,\hat{p}_{n+2}-1\}$ to define $\hat{f}[\underline{i},\ell,r]$ we consider two cases.

Case 1: $\ell \ge 2$ and $\underline{i} \in \Gamma(\xi_\ell)$, i.e. $\xi_\ell \in \overline{B}(V_{i_n}^n, MT/p_n)$. Since the new partition $\hat{\mathcal{P}}_{n+1}$ is finer than the old partition $\mathcal{P}_{n+1}$, there exists a unique $j = j(r) \in \{0,\ldots,p_{n+1}-1\}$ such that $I = [\hat{\tau}^{n+1}(r\omega), \hat{\tau}^{n+1}(r\omega) + T/\hat{p}_{n+1}) \subseteq [t_j^{n+1}, t_{j+1}^{n+1})$ (i.e.: $t_j^{n+1} = \tau^{n+1}(r\omega)$). By conditions C1) and C8), on the ball $B(\xi_\ell, \rho_\ell)$ the vector field $f[\underline{i},j]$ can be approximated by a convex combination of the vector fields g_k^ℓ with coefficients $\lambda_k^\ell[\underline{i},j]$, $k=1,\ldots,N_\ell$. To "track" a solution of the differential equation $\dot{x}(t)=f[\underline{i},j](x(t))$ on the interval I, a solution y(.) of the new feedback $\hat{\mathcal{E}}$ should satisfy the equations $\dot{y}(t) = g_k^\ell(y(t))$ on subintervals $I_k \subseteq I$ whose lengths are proportional to $\lambda_k^\ell[i,j]$. With this in mind, we define

$$\gamma[\underline{i},\ell,r] = \max \{ k\geq 1 : \hat{t}^{n+1}(r\omega) + \frac{T}{\hat{p}_{n+1}}\sum_{m=0}^{k-1} \lambda_m^\ell[\underline{i},j] \leq r\omega \}, \quad (3.10.5)$$

$$\hat{f}[\underline{i},\ell,r](x) = g^\ell_{\gamma[\underline{i},\ell,r]}(x) \quad , \quad \forall x \in B(\xi_\ell,\rho_\ell) \quad . \quad (3.10.6)$$

Otherwise stated, $\hat{f}[\underline{i},\ell,r] = g_k^\ell$ whenever $[r\omega,(r+1)\omega] \subseteq I_k$, with

$$I_k = \left[\hat{t}^{n+1}(r\omega) + \frac{T}{\hat{p}_{n+1}}\sum_{m=0}^{k-1} \lambda_m^\ell[\underline{i},j(r)], \hat{t}^{n+1}(r\omega) + \frac{T}{\hat{p}_{n+1}}\sum_{m=0}^{k} \lambda_m^\ell[\underline{i},j(r)] \right] \quad (3.10.7)$$

Case 2: $\ell = 1$ or $\underline{i} \notin \Gamma(\xi_\ell)$. In this case we set

$$\hat{f}[\underline{i},\ell,r](x) = \tilde{g}(x) \qquad \forall x \in B(0,2\tilde{\rho}) \quad ,$$

where $\tilde{g}$ is the vector field constructed in Corollary 3.9.5.

Let us show that the (n+1)-level feedback $\hat{\mathcal{E}}$ is admissible for (3.1.1). If $\hat{f}[\underline{i},\ell,j] = g_k^\ell$, as in Case 1, then the properties i) and iii) in Def.3.6.3 follow from C3) and C4) respectively. Moreover, (3.10.2) implies

$$\mathrm{Dom}(g_k^\ell) = B(\xi_\ell,\rho_\ell) \supseteq \overline{B}(B(\xi_\ell,\rho_\ell/2),MT/\hat{p}_{n+1}) \quad ,$$

which proves ii). In Case 2, $\hat{f}[\underline{i},\ell,r] = \tilde{g}$ and all conditions i) ÷ iii) are clearly satisfied. Hence $\hat{\mathcal{E}}$ is admissible. By Proposition 3.7.1, the set of solutions $S_{\hat{\mathcal{E}}}$ is therefore nonempty and compact.

Now consider any $y \in S_{\hat{\mathcal{E}}}$. Define $\sigma_{n+2}(t) = \max\{r : t_r^{n+2} \leq t\}$

and let $\sigma_k : \{\hat{t}^k_0, \ldots, \hat{t}^k_{p_k-1}\} \to \{1, \ldots, \hat{q}_k\}$, $k=1,\ldots,n+1$ be such that

$$y(\hat{t}^k_i) \in \overline{\hat{V}^k_{\sigma_k(\hat{t}^k_i)}} \qquad 1 \le k \le n+1 \ , \ 0 \le i \le \hat{p}_k \ , \tag{3.10.8}$$

$$\dot{y}(t) = \hat{f}[\sigma_1(\hat{\tau}^1(t)), \ldots, \sigma_{n+1}(\hat{\tau}^{n+1}(t)), \sigma_{n+2}(t)](y(t)) \tag{3.10.9}$$

a.e. on $[0,T]$. We claim that the vector fields $\hat{f}[.,.]$ which actually occur in (3.10.9) are all defined according to Case 1. This will be a consequence of the next lemma. To shorten the notation, we write $\sigma_k(t) = \sigma_k(\hat{\tau}^k(t))$.

LEMMA 3.10.2. *In the above setting, one has*

$$y(\hat{t}^{n+1}_i) \in \Omega \ , \qquad \forall\ i \in \{0,\ldots,\hat{p}_{n+1}\} \tag{3.10.10}$$

$$\xi_{\sigma_{n+1}(t)} \in \bar{B}(V^n_{\sigma_n(t)}, MT/\hat{p}_n) \ , \quad \forall\ t \in [0,T] \ . \tag{3.10.11}$$

The statements (3.10.10), (3.10.11) will be proved together, by induction on $i \in \{0,\ldots,\hat{p}_{n+1}\}$. Assume that (3.10.10) holds for all $i \le \iota$ and that (3.10.11) holds whenever $t \in [0, \hat{t}^{n+1}_\iota)$. In particular, for $t \in I_\iota = [\hat{t}^{n+1}_\iota, \hat{t}^{n+1}_{\iota+1})$, (3.10.8) and (3.10.10) imply $\sigma_{n+1}(t) = \sigma_{n+1}(\hat{t}^{n+1}_\iota) \ge 2$, because the closure of V^{n+1}_1 does not intersect Ω. Assume that (3.10.11) fails for some $t \in I_\iota$. Then the condition C9) implies

$$\bar{B}(\xi_{\sigma_{n+1}(t)},\rho_{\sigma_{n+1}(t)}) \quad \cap \quad \bar{B}(V^n_{\sigma_n(t)},MT/\hat{p}_n) = \emptyset \quad .$$

We now have

$$y(\hat{\tau}^n(t)) \in \overline{V^n_{\sigma_n(t)}} \quad ,$$

$$y(\hat{\tau}^{n+1}(t)) \in \bar{B}(\xi_{\sigma_{n+1}(t)},\rho_{\sigma_{n+1}(t)}/2) \subseteq B(\xi_{\sigma_{n+1}(t)},\rho_{\sigma_{n+1}(t)}) \quad ,$$

together with the Lipschitz condition

$$\|y(\hat{\tau}^{n+1}(t)) - y(\hat{\tau}^n(t))\| \leq M\cdot(\hat{\tau}^{n+1}(t) - \hat{\tau}^n(t)) \leq MT/\hat{p}_n \quad ,$$

reaching a contradiction. This establishes (3.10.11) for all $t \in [0,\hat{t}^{n+1}_{\iota+1})$. In particular, we now know that, for $t \in [\hat{t}^{n+1}_{\iota},\hat{t}^{n+1}_{\iota+1})$, the vector fields $\hat{f}[.,.]$ occurring in (3.10.9) are defined according to Case 1. To prove (3.10.10), observe that the conditions C4) and C7) together with the definitions (3.10.5), (3.10.6) imply

$$\pi'(y(\hat{t}^{n+1}_{\iota+1}) - y(\hat{t}^{n+1}_{\iota})) = \frac{T}{\hat{p}_{n+1}} \sum_{k=1}^{N_\ell} \lambda^\ell_k[\underline{i},j]g'_k = \frac{T}{\hat{p}_{n+1}} f'[\underline{i},j] \in E, \quad (3.10.12)$$

where $[\underline{i},j] = [\sigma_1(\hat{t}^{n+1}_{\iota}),\dots,\sigma_n(\hat{t}^{n+1}_{\iota}),\tau^{n+1}(\hat{t}^{n+1}_{\iota})]$, $\ell = \sigma_{n+1}(\hat{t}^{n+1}_{\iota})$.

Because of the inductive hypothesis, (3.10.12) implies $y(\hat{t}^{n+1}_{\iota+1}) \in \Omega$ because no solution $y \in S_{\hat{\mathcal{E}}}$ can escape from $\tilde{B} = \bar{B}(0,\tilde{\rho})$ within time T. By induction, the Lemma is proved. ∎

We now return to the proof of Theorem 3.10.1. By Lemma 3.10.2, Case 2 never applies, hence for a.e. $t \in [0,T]$, $\dot{y}(t) =$

$g_k^\ell(y(t))$ for some k, ℓ, depending on t. Condition C5) now yields

$$\int_0^T \phi_C(\dot{y}(t), y(t))\,dt \;=\; \int_0^T d_C(g_{k(t)}^{\ell(t)}(y(t)), F(y(t)))\,dt \;\le\; \delta T \quad .$$

It remains to prove that $y \in B(S_{\mathcal{E}}^{\delta}, \delta)$. Define $z(.)$ as the polygonal function which coincides with y at each node $\hat{t}_i^{n+1}$ of the partition $\hat{\mathcal{P}}_{n+1}$ and is linear on each interval $[\hat{t}_{i+1}^{n+1}, \hat{t}_i^{n+1}]$; $i \in \{1, \dots, \hat{p}_{n+1}\}$. Since y has Lipschitz constant M, (**3**.10.2) implies

$$\|y(t) - z(t)\| \le \delta \qquad \forall\, t \in [0,T] \quad .$$

We claim that $z \in S_{\mathcal{E}}^{\delta}$. For $1 \le k \le n$, set $\beta_k = \sigma_k$. By construction, for $k \le n$ the coverings $\mathcal{G}_k$ in the feedbacks $\mathcal{E}$ and $\hat{\mathcal{E}}$ coincide. Hence (**3**.7.1) is an immediate consequence of (**3**.6.5). To prove (**3**.7.2), fix an arbitrary interval $I_\iota = [\hat{t}_\iota^{k+1}, t_{\iota+1}^{k+1})$ of the partition $\hat{\mathcal{P}}_{n+1}$. For $t \in I_\iota$, call

$$[\underline{i}, j] = [i_1, \dots, i_n, j] = [\beta_1(\tau^1(t)), \dots, \beta_n(\tau^n(t)), \beta_{n+1}(t)],$$

$$[\underline{i}, \ell, \sigma_{n+2}(t)] = [\sigma_1(\hat{\tau}^1(t)), \dots, \sigma_n(\hat{\tau}^n(t)), \sigma_{n+1}(\hat{\tau}^{n+1}(t)), \sigma_{n+2}(t)].$$

For $t \in I_\iota$, both $y(t)$ and $z(t)$ remain inside $B(\xi_\ell, \rho_\ell)$. By the definitions, we have

$$\dot{y}(t) = f[\underline{i},\ell,\sigma_{n+2}(t)](y(t)), \tag{3.10.13}$$

$$\dot{z}(t) = \frac{\hat{p}_{n+1}}{T}\int_{I_\iota} f[\underline{i},\ell,\sigma_{n+2}(\sigma)](y(s))ds = \sum_{k=1}^{N_\ell}\lambda_k^\ell[\underline{i},j]\left(\frac{1}{\mathrm{meas}(I_k)}\int_{I_k} g_k^\ell(y(s))ds\right) \tag{3.10.14}$$

with I_k defined at (3.10.7).

Using C7) together with (3.10.5) and (3.10.6), from (3.10.14) we deduce

$$\pi'(\dot{z}(t)) = \sum_{k=1}^{N_\ell}\lambda_k^\ell[\underline{i},j]\cdot\pi'(g_k^\ell(y(t))) = f'[\underline{i},j] \quad . \tag{10.15}$$

Moreover C2) and C8) imply

$$\|f[\underline{i},j](z(t)) - \dot{z}(t)\| \leq \|f[\underline{i},j](z(t)) - f[\underline{i},j](\xi_\ell)\|$$
$$+\|f[\underline{i},j](\xi_\ell) - \sum_{k=1}^{N_\ell}\lambda_k^\ell[\underline{i},j]\left(\frac{1}{\mathrm{meas}(I_k)}\int_{I_k} g_k^\ell(y(s))ds\right)\|$$
$$\leq \delta/2 + \delta/2. \tag{3.10.16}$$

Together, (3.10.15) and (3.10.16) yield (3.7.2). This completes the proof of Theorem 3.10.1.

3.11. COMPLETION OF THE PROOF

Let $\{\mathcal{E}_i : i \in I\}$ be the family of all admissible n-level feedbacks ($n\geq 1$) for the Cauchy problem (3.1.1), according to Definition 3.6.3. Notice that this family is nonempty. Indeed, using the assumptions A1) ÷ A6) in §3, an admissible 1-level feedback $\mathcal{E} = (\mathcal{P},\mathcal{G},f)$ on $[0,T]\times\tilde{B}$ can be easily constructed by setting $\mathcal{P}_1 = \mathcal{P}_2 = \{0,T\}$, $\mathcal{G}_1 = \{V_1\} = \{B(0,2\tilde{\rho})\}$ and letting f consist of the single vector field $f[1,0] = \tilde{g}$ defined in Corollary 3.9.5. For every $i \in I$, define the set of solutions $K_i = S_{\mathcal{E}_i}$, which is nonempty and compact by Proposition 3.7.1.

Set $\mathcal{F} = \{K_i : i \in I\}$. As in §6, let S be the family of all solutions of (3.1.2) and consider the sets S_{jn} defined at (3.6.1). By Theorem 3.8.1, the sets $R_{jn} = S \backslash S_{jn}$ are all closed. To prove that they are $\mathcal{F}$-rare, let $\mathcal{E}$ be any admissible n-level feedback and let $\varepsilon > 0$, $j,n \geq 1$ be given. By proposition 3.7.1 there exists $\delta > 0$ such that $S^{\delta}_{\mathcal{E}} \subseteq B(S_{\mathcal{E}}, \varepsilon/2)$. Moreover, Theorem 3.10.1 provides an admissible (n+1)-level feedback $\hat{\mathcal{E}}$ such that $S_{\hat{\mathcal{E}}} \subseteq B(S^{\delta}_{\mathcal{E}}, \varepsilon/2) \subseteq B(S_{\mathcal{E}}, \varepsilon)$ and

$$\int_0^T \phi_{C_j}(\dot{y}(t), y(t))\,dt < 1/n \qquad \forall y \in S_{\hat{\mathcal{E}}} .$$

This shows that the sets R_{jn} are $\mathcal{F}$-rare. Therefore, Theorem 3.4.1 implies that

$$\hat{S} = \bigcap_{j,n} S_{jn} \neq \emptyset \quad .$$

Using Lemma 3.5.4, we conclude that every $y \in \hat{S}$ is actually a solution of (3.1.1). This completes the proof of Theorem 3.3.3.

4. SOME REMARKS FOR THE UPPER SEMICONTINUOUS CASE

4.1. INTRODUCTION

It has already been remarked that the Cauchy problem (**1**.1),(**1**.2) may not have solutions if the right-hand side is u.s.c. with non-convex values. A simple counterexample is, in fact, the following

EXAMPLE **4**.1.1. Define $F:\mathbb{R}\to 2^{\mathbb{R}}$ by setting $F(x) = -1$ if $x>0$, $F(x) = +1$ if $x<0$, $F(0) = \{-1,+1\}$ and consider the initial value problem

$$x' \in F(x),\ x(0) = 0 . \qquad (4.1.1)$$

Notice first that $x\equiv 0$ is not a solution. Then suppose that a solution $x(\cdot)$ exists and is such that $x(t_1)>0$ for some $t_1>0$. Define $t_o=\sup\{0\le t<t_1 : x(t)\le 0\}$: clearly $t_o<t_1$, $x(t_o)=0$ and $x(t)>0$ if $t_o<t\le t_1$. By the absolute continuity of x,

$$x(t_1) - x(t_o) = \int_{t_o}^{t_1} x'(s)ds = -1\cdot(t_1 - t_o) < 0 ,$$

a contradiction. A similar argument shows that no $t_1>0$ can exist such that $x(t_1)<0$, and therefore (**4**.1.1) has no solutions.∎

The preceding situation is rather typical for differential inclusions with u.s.c. right-hand side. A general feature of these equations is that, contrary to the l.s.c. case, polygonal approximate solutions do not exist, at least in the sense that

$$s_\varepsilon'(t) \in B(F(t,s_\varepsilon(t)),\varepsilon) : \qquad (4.1.2)$$

this behaviour can be understood by looking at the definitions of continuity in Chapter **2** of Part I. However, approximate solutions in a weaker sense can be always constructed, provided

F has compact values. In fact, let $r, M \in \mathbb{R}^+$ be such that for $(t,x) \in K=[0,r]\times B(x_0,r)$ we have $\|F(t,x)\| \le M$ (see [3, Proposition 1.1.3]), and let $k \in \mathbb{N}$. Write $h_k=r/k$ and $t_k^i=i\cdot h_k$ $(i=0,\dots,k)$. Define $s_k:[0,r]\to\mathbb{R}^n$ by setting $s_k(0)=x_0$, and, for $t\in[t_k^i, t_k^{i+1}]$,

$s_k(t)=s_k(t_k^i) + (t-t_k^i)v_k^i$, where $v_k^i \in F(t_k^i, s_k(t_k^i))$. Then we have that $|s_k(t)-s_k(t_k^i)| \le Mh_k$ $\forall t\in[0,r]$ and that $|s_k'(t)|\le M$ a.e. in $[0,r]$. The polygonal $s_k(\cdot)$ is an approximate solution, in the sense that

$$d((t,s_k(t),s_k'(t)),\text{graph}\{F\}) \le h_k(1+M).$$

(approximate solutions in the sense of graph).

Also in the non-convex case, therefore, the only problem for the existence of solutions is the pointwise convergence of the derivatives $s_k'(\cdot)$. In Example **4**.1.1, any sequence $(x_k'(\cdot))_k$ constructed using the preceding rule cannot have this property, because it is oscillating together with all its subsequences. By reasoning as in Chapter **1**, we see that $(s_n(\cdot))_n$ converges uniformly to a solution of the "relaxed" problem

$$x'(t) \in \overline{\text{co}}\,F(t,x(t)), \quad x(0)=x_0. \qquad (\mathbf{4}.1.3)$$

We prove here (Theorem **4**.4.2) that the converse is also true: every solution of the problem (**4**.1.3) is uniform limit of approximate solutions in the sense of graph of (**4**.1.1), (**4**.1.2).

In the existence result we are going to present (Theorem **4**.3.1), the convergence problems are solved, for a particular F, first by imposing a kind of geometrical compatibility condition between x_0 and $F(x_0)$, and then by using a monotonicity property of F. We observe that a multifunction which is both u.s.c. and monotone has better continuity properties than a simply u.s.c. map: in fact we construct polygonal lines s_k which are approximate solutions in the stronger sense of (**4**.1.2). Finally, the monotonicity of F allows the derivatives $s_k'(\cdot)$ to remain in a compact set of L^1 and hence to converge pointwise a.e..

4.2. STATEMENT OF AN EXISTENCE PROBLEM AND EXAMPLES

We consider a closed subset K of $\mathbb{R}^2$ and a continuous map $f:K\to\mathbb{R}^2$, satisfying the tangential condition

$$\liminf_{h\to 0^+} \frac{d(x+hf(x),K)}{h} = 0 \tag{T}$$

for every $x\in K$. Let π be the upper semicontinuous, compact valued map of nearest projection on K : $\pi(x) = \{y\in K : d(x,y) = d(x,K)\}$. The Cauchy Problem

$$\begin{cases} x' \in f(\pi(x)) \\ x(0) = x_o \end{cases} \tag{CP}$$

has a solution for $x_0\in K$. In fact, for a known result of Nagumo [38], the problem

$$\begin{cases} x' = f(x) \\ x(0) = x_o \end{cases}$$

has a solution taking values in K, and this is a solution to (CP). In general, given f and K, (CP) has or has not solutions, depending on x_0. Remark that the set-valued map at the right-hand side of (CP) is upper semicontinuous but not convex valued.

In this section we impose a condition on x_0, Condition (C), that is sufficient to insure the existence of local solutions (in the future). To discuss the sharpness of this condition to discriminate the initial points giving existence of solutions, we will present two examples.

The following is the condition on x_0.

CONDITION (C). *The point* x_0 *satisfies Condition (C) if there exists* $y^+ \in \pi(x_0)$ *such that :*

$$\text{either } f(y^+) = 0 \text{ or, for every } z \in \pi(x_0), \qquad (4.2.1)$$

$$z \neq y^+ \text{ implies } <y^+ - z, f(y^+)> \; > 0.$$

This condition is automatically verified whenever $\pi(x_0)=\{y\}$. We have also

PROPOSITION **4**.2.1. *Assume that* $y, z \in \pi(x_0)$, $y \neq z$ *imply*

$$< y - x_0, f(y) - f(z) > \; > \; 0 \; . \qquad (4.2.2)$$

Then Condition (C) holds.

Proof. Assume that $f(y) \neq 0$ for every $y \in \pi(x_0)$, and let y^+ be such that the continuous function of y, $<y - x_0, f(y)>$, attains a maximum on $\pi(x_0)$ at y^+. We claim this y^+ satisfies (C). In fact, assume this is not true : there exists $y_1 \in \pi(x_0)$, $y_1 \neq y^+$, such that

$$< y_1 - x_0, f(y^+) > \; \geq \; < y^+ - x_0, f(y^+) > .$$

From (**4**.2.2) we have

$$<y_1 - x_0, f(y_1)> \; > \; <y_1 - x_0, f(y^+)> \; \geq \; <y^+ - x_0, f(y^+)> ,$$

contradicting the choice of y^+. ∎

In what follows we set $B[a,\varepsilon] = \{ b \in \mathbb{R}^2 : \|a-b\| \le \varepsilon\}$ and $S[a,\varepsilon] = \{ b \in \mathbb{R}^2 : \|a-b\| = \varepsilon \}$. Moreover, let $(\rho(a), \theta(a))$ be the polar coordinates of a point a : then $P[a,\varepsilon] := \{ b \in \mathbb{R}^2 : |\rho(b) - \rho(a)| \le \varepsilon$ and $|\theta(b) - \theta(a)| \le \varepsilon \}$.

EXAMPLE **4**.2.2. Set $O^+ = (1,0)$, $O^- = (-1,0)$, $S^+ = S[O^+,1]$, $S^- = S[O^-,1]$ and $K = S^+ \cup S^-$. Define, for $(x,y) \in K$,

$$f(x,y) = \begin{cases} (y, 1-x) & \text{for } x \ge 0 \\ (-y, 1+x) & \text{for } x \le 0 \end{cases}.$$

On each point of the two half lines $Y^+ = \{(0,y): y>0\}$ and $Y^- = \{(0,y): y<0\}$ the projection on K consists of the two intersections with K of the segments joining $(0,y)$ with O^- and O^+. For $y > 0$, Condition (C) is satisfied by each of the two points in $p(0,y)$, and (PC) with $(0,y)$ as initial point has solutions , while for $y < 0$ Condition (C) is not satisfied, and indeed (CP) has no solutions. For $y = 0$, (CP) has solutions, by Nagumo's theorem. For any other x_0, $x_0 \ne O^+, O^-$, there is exactly one solution to (CP) and our condition is verified, since the map $x \to \pi(x)$ is single-valued. Our condition is not verified at O^+ and O^-, since for every y there are z in $\pi(O^+)$ or $\pi(O^-)$ with $< y - z, f(y) > < 0$ and there are no solutions to (CP) with $x_0 = O^+$ or $x_0 = O^-$. ∎

Purpose of the next example is to show that the strict inequality of Condition (C) cannot be replaced by a non-strict inequality : there exist K, f and x_0 such that the scalar product in (C) is zero and there are no solutions to (CP) .

EXAMPLE **4**.2.3. Set $\mathbb{R}^2_+ = \{(x,y) : y \geq 0\}$, $P_1 = (-1,1)$, $P_2 = (1,1)$ and define $C = \{(x,y) : y \geq |x|\}$, $K_0 = C \cap B[O,1] \setminus B[O,\sqrt{2} - 4/3]$, $K_1 = (\mathbb{R}^2_+ \setminus C) \cap B[P_1,4/3] \cap B[O,1]$, $K_2 = (\mathbb{R}^2_+ \setminus C) \cap B[P_2,4/3] \cap B[O,1]$, and finally $K = K_0 \cup K_1 \cup K_2$.

Let (x,y) be in K, described in polar coordinates by ρ and θ. Let us define f at (x,y) . For $x \neq 0$, let $T(x,y)$ be the unit tangent vector to the circle centered in O and radius ρ, having positive scalar product with the y axis. Set $f(x,y)$ to be $|\pi/2 - \theta| \cdot T(x,y)$ whenever $x \neq 0$ and zero when $x = 0$.

At the point $x_0 = (0, 1 - 3/\sqrt{7})$, $\pi(x_0)$ consists of the two points $Q_1 = (1 - \sqrt{7}/3, 0)$ and $Q_2 = (\sqrt{7}/3 - 1, 0)$; $f(Q_1) = f(Q_2) = (0, \pi/2)$, hence $< Q_1 - Q_2, f(Q_1) > = 0$ and so Condition (C) is not verified. Our Cauchy Problem has no solutions from x_0.

However, note that all the points P, of coordinates $(0,y)$ with $y < 1 - 3/\sqrt{7}$, violate Condition (C), but still solutions exist. These points are characterized by the property that the set K is not tangent to the ball $B[P,R]$, with $R = d(P,K)$. In this case the map $t \to \pi(s(t))$ is (locally) constant along a solution $s(t)$. ∎

We conclude this paragraph with a geometrical proposition and a compactness criterion which will be used in the following.

PROPOSITION **4**.2.4. a) *Let* S' *and* S'' *be two circles centered at* O' *and* O'', $O' \neq O''$. *Then* $S' \cap S''$ *consists at most of two points, symmetric with respect to the line* $(O'' - O')\cdot \mathbb{R}$.

b) *Let* y' *and* y^* *be such that* $d(y^*,O') = d(y',O')$ *and* $d(y^*,O'') \leq d(y',O'')$. *Then*

$$< O'' - O', y^* > \;\geq\; < O'' - O', y'>. \qquad (\mathbf{4}.2.3)$$

c) *Assume moreover that* $y' \neq y^*$ *and that the absolute value of the angle between* $y' - O'$ *and* $O'' - O'$, β, *is positive. Whenever the angle between* $y^* - O'$ *and* $y' - O'$ *is in absolute value not larger than* β , *we have*

$$< (y^* - y')/\| y^* - y'\| , (O'' - O')/\| O'' - O'\| > \;\geq \sin(\beta/2) . \qquad (\mathbf{4}.2.4)$$

HELLY'S COMPACTNESS THEOREM [2,Thm. 16.3, p.29]. *Let* g_n : $[a,b] \to \mathbb{R}$, $n = 1,2, \ldots$, *be a sequence of functions, uniformly bounded and of uniformly bounded variation. Then there exists a function* g_* on $[a,b]$, *of bounded variation, and a subsequence* $\{g_{n_k}\}$ *such that*

$$\lim_{k\to\infty} g_{n_k}(t) = g_* (t)$$

for every $t \in [a,b]$.

2. AN EXISTENCE THEOREM

Our purpose is to prove the following

THEOREM 4.3.1. *Let* $K \subseteq \mathbb{R}^2$ *be closed,* $f : K \to \mathbb{R}^2$ *be continuous and satisfy the tangential condition (T). Then the Cauchy Problem (CP) has at least one solution on some interval* $[0,T]$, *provided that the initial point* x_o *satisfies Condition (C).*

Proof. We can assume that $0 \notin f(\pi(x_o))$; that $R = d(x_o,K) > 0$, otherwise the theorem is proved by Nagumo's result, and that $x_o = 0$.

(A) Let y^+ be as given by Condition (C). In case $y^+ = \lambda f(y^+)$, $\lambda > 0$, there exists the solution $tf(y^+)$ on some interval $[0,T]$. Hence, in the sequel we assume that the absolute value of the angle between y^+ and $v = f(y^+)/\|f(y^+)\|$ equals $2|\alpha|$, $|\alpha| > 0$. The tangentiality condition (T) on f implies that $2|\alpha|$ is not larger than $\pi/2$. There exist positive ε' and ε'' such that : y' in $B[y^+,\varepsilon']$ and v' in $B[v,\varepsilon'']$, $\|v'\| = 1$, imply that the absolute value of the angle between y' and v' is at least $|\alpha|$.

Let us consider polar coordinates with center O and angles counterclockwise from the half line $v \cdot \mathbb{R}^+$. Call y^- the point described by $\rho(y^-) = R$ and $\theta(y^+) = -2\alpha$. To simplify our inequalities we assume from now on that $\alpha > 0$. Remark that no point y with $\rho(y) = R$, $-2\alpha \le \theta(y) < 2\alpha$ belongs to K, otherwise it would violate Condition (C). Since K is closed, for every such y we can associate a neighbourhood $P[y,\varepsilon(y)]$ having empty intersection with K.

We will consider the cone

$$C = \{ \lambda v' : \lambda \geq 0 \ , \|v'\| = 1 \text{ and } |\theta(v')| \leq \eta \} , \tag{4.3.1}$$

where $\eta = \min\{ \varepsilon'', \varepsilon(y^-)/2, \ \alpha/3, \ \sin(\alpha/2)/2 \}$.

By the continuity of f at y^+, there exists a neighborhood $P[y^+,\varepsilon(y^+)]$ such that y in it implies that $f(y) \in \mathcal{C}$. We can assume that $P[y^+,\varepsilon(y^+)] \subseteq B[y^+,\varepsilon']$ and that $\varepsilon(y^+) < \alpha/3$. Consider the compact set $S = \{y : \rho(y) = R$ and $\theta(y^-) \leq \theta(y) \leq \theta(y^+) - \varepsilon(y^+)\}$: the collection $\{P[y,\varepsilon(y)] : y \in S\}$ covers S . Let $\{P[y_i,\varepsilon_i]\}$ be a finite subcover and set $2\cdot\varepsilon = \inf\{\varepsilon_i\}$ (for future use we also assume $\varepsilon \leq R/2$, $\varepsilon \leq \alpha/4$, $\varepsilon \leq R\alpha/6$ and $\varepsilon/(R^2-\varepsilon^2)^{1/2} \leq 1$).

The above considerations imply that there are no points of K satisfying

$$-2\alpha - 2\varepsilon(y^-) \leq \theta \leq 2\alpha - 2\varepsilon(y^+)$$
$$\rho \leq R + 2\varepsilon \; . \qquad (4.3.2)$$

Remark that for any z, $d(z,\pi(z)) = d(z,K) \leq d(z,y^+)$ and $d(O,\pi(z)) \leq d(O,z) + d(z,\pi(z)) \leq d(O,z) + d(z,y^+) \leq d(O,y^+) + 2d(O,z)$. Hence

$$z \in B[O,\varepsilon] \text{ and } y \in \pi(z) \text{ imply } \rho(y) \leq R + 2\varepsilon . \qquad (4.3.3)$$

The set $\pi(z)$ is contained in $B[z,d(z,y^+)]$ but not in the interior of $B[O,R]$ (that cannot contain points of K) : $\pi(z) \subseteq B[z,d(z,y^+)] \cap (\mathbb{R}^2 \setminus \mathrm{int}B[O,R])$. The two circles $S[O,R]$ and $S[z,d(z,y^+)]$ intersect at y^+, hence they intersect at a point $\bar{y}$, symmetric to y^+ with respect to the direction of z. In particular, consider z in the cone $\mathcal{C}$: any point y of $\pi(z)$ satisfies $|2\theta(z)| - \theta(y^+) \leq \theta(y) \leq \theta(y^+)$, i.e.

$$-2\alpha - 2\eta \leq \theta(y) \leq 2\alpha . \qquad (4.3.4)$$

For $z \in \mathcal{C} \cap B[O,\varepsilon]$, (**4**.3.3) and (**4**.3.4) compared with (**4**.3.2) yield, by (**4**.3.1):

$$\pi(z) \subseteq P[y^+,\varepsilon(y^+)] . \qquad (4.3.5)$$

(B) Let $M = \sup\{\|f(y)\| : y \in P[y^+, \varepsilon(y^+)] \cap K\}$ and $T = \varepsilon/M$. We wish to define a solution s on $I = [0,T]$ as a limit of polygonal lines s_n. Given a sequence σ_n decreasing to zero, we claim that: i) for each n we can define a finite sequence of net points $0 = \tau_0 < \tau_1 < \ldots < \tau_i < \ldots \tau_M = T$ and a continuous function s_n, linear on each $]\tau_i, \tau_{i+1}[$, such that, except at the net points,

$$s_n'(t) \in B[f(\pi(s_n(t))), \sigma_n] . \qquad (4.3.6)$$

We also claim : ii) the limit s of an uniformly converging subsequence of $\{s_n\}$ is a solution to (CP).

Both claims are a consequence of the following remark :

Remark : let $s : I \to \mathcal{C} \cap B[0,\varepsilon]$ be such that for $t', t'' \in I$, $t'' > t'$, we have $s(t'') - s(t') \in \mathcal{C} \setminus \{0\}$. Then the set-valued maps

$$t \to P(t) = \{ \rho(y) : y \in \pi(s(t)) \}$$

$$t \to \Theta(t) = \{ \theta(y) : y \in \pi(s(t)) \}$$

are monotonic (P is increasing and Θ decreasing) .

Proof of the remark.

a) Monotonicity of P.

Let y' be in $\pi(s(t'))$ and $y'' \neq y'$ be in $\pi(s(t'))$. By construction, $\theta(y') \geq 2\alpha - \varepsilon(y^+)$; since $s(t') \in B[0,\varepsilon]$, the absolute value of the angle between $s(t') - y'$ and $0 - y'$ is at most $2\varepsilon/R$; hence the angle between $y' - s(t')$ and v is at least $2\alpha - \varepsilon(y^+) - 2\varepsilon/R$. From $s(t'') - s(t') \in \mathcal{C}$, $\theta(s(t'') - s(t')) \leq \eta$ and the angle between $y' - s(t')$ and $s(t'') - s(t')$

is at least $2\alpha - \varepsilon(y^+) - 2\varepsilon/R - \eta \geq \alpha$.

Since $d(y'',s(t')) \geq d(y',s(t'))$, the segment from y'' to O joins a point in the complement of the convex set $\text{int}B[s(t'),d(y',s(t'))]$ with a point in it, hence it meets the boundary at a point y^* .

We wish to apply part c) of Proposition **4**.2.4, setting O' to be $s(t')$, O" to be $s(t'')$. We have first to check that $d(y^*,s(t'')) \leq d(y',s(t''))$. By definition of nearest point, $d(y'',s(t'')) \leq d(y',s(t''))$, i.e. y'' belongs to $B[s(t'),d(y',s(t''))]$. By our choice of ε, so does O; since this set is convex, y^*, a convex combination, also belong to it. In order to estimate the angle between $y^* - s(t')$ and $y' - s(t')$, consider the diagonals of the quadrilateral of vertices y^*, y', $s(t')$ and O, and call z their intersection. Of the four triangles having z as a vertex, consider the one having a vertex in O and as third vertex either y^* or y'. The angle in O is at most $2\varepsilon(y^+)$; the angle in the third vertex at most $2\varepsilon/R$: hence the angle in z is at least $\pi - (2\varepsilon(y^+) + 2\varepsilon/R)$. In the triangle opposed in vertex at z, the sum of the two angles in $s(t')$ and in either y^* or y' is at most $\pi - (\pi - (2\varepsilon(y^+) + 2\varepsilon/R)) = 2\varepsilon(y^+) + 2\varepsilon/R \leq \alpha$. In particular the angle between $y^* - s(t')$ and $y' - s(t')$ is at most α. We have

$$<(y^* - y')/\|y^* - y'\|, (s(t') - O)/\|s(t') - O\|> =$$
$$<(y^* - y')/\|y^* - y'\|, (s(t'') - s(t'))/\|s(t'') - s(t')\|>$$
$$+ <(y^* - y')/\|y^* - y'\|, (s(t') - O)/\|s(t') - O\| -$$
$$(s(t'') - s(t'))/\|s(t'') - s(t')\|> .$$

Since $s(t') - O$ and $s(t'') - s(t')$ belong to $\mathcal{C}$, and by inequality (**4**.2.4),

$$<(y^* - y')/\|y^* - y'\|, (s(t') - O)/\|s(t') - O\|>$$

$$\geq \sin(\alpha/2) - 2\eta > 0,$$

i.e.

$$< y^* - y' , s(t') > \ > 0 . \qquad (4.3.7)$$

Consider the isosceles triangle $y^*, y', s(t')$. We have $<y^* - y', s(t')> = <y^* - y', (y^* + y')/2>$. The line orthogonal to $y^* - y'$ and passing through $(y^* + y')/2$ divides the plane into two half spaces. By (4.3.7), O belongs to the half space containing y', hence

$$d(O,y') \leq d(O,y^*) \leq d(O,y'') , \qquad (4.3.8)$$

proving part a) of our remark .

b) The monotonicity of Θ.

Since, by a) , $d(y'',O) \geq d(y',O)$, the line from y'' to O meets the boundary of $B[O,d(y',O)]$ at a point $\tilde{y}$. The same reasoning as used for y^* shows that $\tilde{y} \in B[s(t''), d(y',s(t''))]$. By part b) of Proposition 4.2.4,

$$< s(t'') , \tilde{y} > \ \geq \ < s(t'') , y' > ,$$

i.e. the angle $\theta(y'') = \theta(\tilde{y}) \leq \theta(y')$. This concludes the proof of the Remark.

Ad i) . Since f is locally uniformly continuous, there exists ε_n such that $f(P[y,\varepsilon_n]) \subseteq B[f(y),\sigma_n/2]$ for every y in $P[y^+,\varepsilon(y^+)] \cap K$. We can assume that for some N , $\varepsilon_n = \varepsilon(y^+)/N$. Consider $t\cdot f(y^+)$ and let τ_1 be

$$\tau_1 = \sup\{\tau \in [0,T] : \pi(t\cdot f(y^+)) \subseteq \operatorname{int}P[y^+,\varepsilon_n], \text{ for } t \in [0,\tau]\}.$$

We observe that τ_1 is positive. In fact, by the upper semicontinuity of the map π, for t sufficiently small and y in $\pi(tf(y^+))$ we must have $\rho(y) < R + \varepsilon_n$, $\theta(y) > \theta(y^+) - \varepsilon_n$, while by the monotonicity of $t \to P(t) = \{\rho(y) : y \in \pi(t\cdot f(y^+))\}$ and $t \to \Theta(t) = \{\theta(y) : y \in \pi(t\cdot f(y^+))\}$ we must have $\rho(y) \geq R$, $\theta(y) \leq \theta(y^+)$, i.e. $y \in \operatorname{int}P[y^+,\varepsilon_n]$. The same considerations imply that we cannot have $\pi(\tau_1 f(y^+)) \subseteq \mathrm{int}P[y^+,\varepsilon_n]$: hence there is y_1 in $\pi(\tau_1 f(y^+))$ such that either

$$\theta(y_1) \le \theta(y^+) - \varepsilon_n$$

or

$$\rho(y_1) \ge \rho(y^+) + \varepsilon_n .$$

On $[0,\tau_1[$ set $s_n(t)$ to be $t\cdot f(y^+)$ and $y_n(t)$ to be y^+: we have $s_n'(t) = f(y_n(t))$ and

$$d(s_n'(t)\ ,\ f(\pi(s_n(t)))) \le \sigma_n . \qquad (4.3.9)$$

Assume that we have defined the polygonal line s_n and the piecewise constant map y_n up to τ_k , where τ_k is defined as

$$\tau_k = \sup\{\tau\in[\tau_{k-1},T] : \pi(s_n(t)) \subseteq \mathrm{int}P[y_n(\tau_{k-1}),\varepsilon_n],$$
$$\text{for } t\in[\tau_{k-1},\tau]\} \qquad (4.3.10)$$

If $\tau_k = T$, s_n is defined on $[0,T[$; otherwise set $s_n(\tau_k) = \lim_{t\to\tau_k^-} s_n(t)$ and consider $\pi(s_n(\tau_k))$. As a consequence of (4.3.10), there are points y in $\pi(s_n(\tau_k))$ such that either

$$\theta(y) \le \theta(y_n(\tau_{k-1})) - \varepsilon_n \qquad (4.3.11)$$

or

$$\rho(y) \ge \rho(y_n(\tau_{k-1})) + \varepsilon_n . \qquad (4.3.12)$$

Choose $y_n(\tau_k)$ in $\pi(s_n(\tau_k))$ such that $\theta(y_n(\tau_k)) = \min\{\theta(y) : y \in \pi(s_n(\tau_k))\}$: we claim that $y_n(\tau_k))$ is one such point. It is clear from the definition that if (4.3.11) is satisfied by some y, (4.3.11) is satisfied by $y_n(\tau_k)$. We wish to show that the same statement holds for (4.3.12). To see this we prove that $\rho(y_n(\tau_k)) \geq \rho(y)$. Notice first that, since $\theta(y_n(\tau_k)) \leq \theta(y)$, the angle $\beta_1 = s_n(\tau_k)Oy$ is, in absolute value, not smaller than the angle $\beta_2 = s_n(\tau_k)Oy_n(\tau_k)$. Moreover, the generalized Pythagorean theorem yields

$$\rho(y_n(\tau_k)) = \|y_n(\tau_k)\| =$$

$$= \|s_n(\tau_k)\|\cos\beta_2 + [d(s_n(\tau_k))^2 - \|s_n(\tau_k)\|^2\sin^2\beta_2]^{1/2}$$

and

$$\rho(y) = \|y\| = \|s_n(\tau_k)\|\cos\beta_1 + [d(s_n(\tau_k))^2 - \|s_n(\tau_k)\|^2\sin^2\beta_1]^{1/2}.$$

By our choice of ε in part (A) of the proof, the function

$$\beta \to \|s_n(\tau_k)\|\cos\beta + [d(s_n(\tau_k))^2 - \|s_n(\tau_k)\|^2\sin^2\beta]^{1/2}$$

is decreasing for $\beta \in [0,\pi]$, and therefore $\rho(y_n(\tau_k)) \geq \rho(y)$, proving our claim.

By the above remarks, we have that the sum of $\theta(y_n(\tau_{k-1}))-\theta(y_n(\tau_k))$ and of $\rho(y_n(\tau_k))-\rho(y_n(\tau_{k-1}))$ is at least ε_n, and, computing from τ_o,

$$\sum_{i=1}^{k}[\theta(y_n(\tau_{i-1}))-\theta(y_n(\tau_i))] + \sum_{i=1}^{k}[\rho(y_n(\tau_i))-\rho(y_n(\tau_{i-1}))] \geq k\varepsilon_n. \qquad (4.3.13)$$

Since $\theta(y_n(\tau_k)) = \min\{\theta(y) : y \in \pi(s_n(\tau_k))\}$, we see, exactly as for τ_1, that $\tau_{k+1} = \sup\{\tau \in [\tau_k,T] : \pi(s_n(t)) \subseteq \text{int } P[y_n(\tau_k),\varepsilon_n]$, for $t \in [\tau_k,\tau]\}$ is larger than τ_k. On $[\tau_k, \tau_{k+1}[$ set $s_n(t)$ to be $s_n(\tau_k) + (t-\tau_k)\, f(y_n(\tau_k))$. Inequality

(4.3.13) holds. Since $N\varepsilon_n = \varepsilon(y^+)$, after at most 2N steps one of the two sums in (17) is at least $\varepsilon(y^+)$. By our choice of T this implies that for some $k \le 2N$, $\tau_k = T$.

Ad ii) . For every n , each of the maps $t \to \theta(y_n(t))$ and $t \to \rho(y_n(t))$ is monotonic. By the Ascoli-Arzelà theorem applied to $\{s_n\}$ and Helly's theorem applied to $\{\rho(y_n)\}$ and $\{\theta(y_n)\}$, we can assume that (along a subsequence) $s_n \to s_*$ uniformly and $y_n \to y_*$ pointwise in [0,T[. Since $s_n'(t) = f(y_n(t))$, t not a nodal point, and f is continuous, s_n' converges pointwise a.e. to a function that is on one hand a.e. s_*' and on the other $f(y_*)$. We wish to show that $s_*'(t) \in f(\pi(s_*(t)))$ or , equivalently, that

$$y_*(t) \in \pi(s_*(t))$$

for almost every t .

Fix t outside the exceptional set. Fix $\eta > 0$. For some $\delta>0$,

$\pi(B[s_*(t),\delta]) \subseteq B[\pi(s_*(t)), \eta/3]$. For n large we have at once $d(s_n(t),s_*(t)) < \delta$, $d(y_n(t),y_*(t)) < \eta/3$ and $d(y_n(t), \pi(s_n(t))) < \eta/3$, so that from

$$\pi(s_n(t)) \subseteq B[\pi(s_*(t)), \eta/3] ,$$

we infer

$$y_*(t) \in B[\pi(s_*(t)), \eta] .$$

Since η is arbitrary, this completes the proof .∎

REMARK. The tangential condition (T) can be replaced by the following assumption on x_o :

$$x_o \notin K \text{ and, if } y^+ \text{ is as given by Condition (C) ,}$$

$$f(y^+) \neq \lambda(y^+ - x_o) \quad \text{for every } \lambda < 0 \; .$$

In fact every argument used in the proof holds also in this case, i.e. when $2|\alpha| \neq \pi$.

4.4. APPROXIMATION OF SOLUTIONS TO THE CONVEXIFIED PROBLEM

A comparison between the problems

$$x' \in F(t,x) \qquad (4.4.1)$$

and

$$x' \in \overline{co}\, F(t,x) \qquad (4.4.2)$$

has been carried out in many papers. In particular, Ważewski [47] proved that, for a continuous F, every solution of (4.4.2) is uniform limit of functions $y_k(\cdot)$ such that

$$d(y_k'(t), \overline{ext}\,\overline{co}\, F(t, y_k(t))) \to 0 \qquad \text{for a.e. } t.$$

(according to Ważewski's definitions: every trajectory of $\overline{co}F$ is a quasitrajectory of $\overline{ext}$ $\overline{co}$ F). A result in a similar direction was proved later by Filippov [26]: he showed that, if $F:\Omega \to 2^{\mathbb{R}^n}$ is Lipschitzean (with respect to the Hausdorff distance) and compact valued, the set S_F of solutions to (4.4.1) is dense in the set S of solutions to (4.4.2), for the uniform convergence topology. Filippov's theorem was generalized by Pianigiani [43] and Ornelas [42], and, from other viewpoints, by Bressan [5] and Cellina [14], always under strong assumptions of continuity (or l.s.c.) for F.

In this paragraph we essentially prove the same statement as Ważewski, for F only u.s.c.; of course the notion of quasitrajectory must be modified. Notice also that the map $(t,x)\to \mathrm{ext}\,\overline{\mathrm{co}}F(t,x)$ may not be u.s.c..

DEFINITION **4**.4.1. *Let* $G:\Omega\to 2^{\mathbb{R}^n}$ *be a multifunction and* I *be an interval. A solution of* $x'\in G(t,x)$ *is said a trajectory of* G*; a map* $x:I\to\mathbb{R}^n$*, uniform limit of a sequence of absolutely continuous functions* $(y_k)_k$ *such that*

$$\lim_{k\to\infty} d\Big((t,y_k'(t)),\mathrm{graph}\{G(\cdot,y_k(\cdot))\}\Big) = 0 \quad \textit{for a.e. } t\in I$$

is called a quasitrajectory of G*. We say also that an absolutely continuous function* y *on* I *is a quasipolygonal iff the derivative* y' *is a simple function with respect to the* σ*-algebra* $\mathcal{L}$ *of Lebesgue measurable subsets of* I.

THEOREM **4**.4.2. *Let* $F:\Omega\to 2^{\mathbb{R}^n}$ *be an u.s.c. multifunction with compact values and let* $(t_o,x_o)\in\Omega$*. Then every trajectory of* $\overline{\mathrm{co}}F(t,x)$ *defined on a compact interval* I *is a quasitrajectory of* ext $\overline{\mathrm{co}}$ F(t,x) *(hence of* F(t,x)*) on* I*. More precisely, for every solution* $x(\cdot)$ *of* (**4**.4.2) *such that* $x(t_o)=x_o$ *and for every* $\varepsilon>0,\eta>0$ *there exists a quasipolygonal function* $y:I\to\mathbb{R}^n$ *such that* $y(t_o)=x_o$, $|x(t)-y(t)|<\varepsilon$ *for every* $t\in I$ *and*

$$d\Big((t,y'(t)),\mathrm{graph}\{\mathrm{ext}\,\overline{\mathrm{co}}F(\cdot,x(\cdot))\}\Big) \le \eta \quad \textit{for a.e. } t\in I.$$

The proof of Theorem **4**.4.2 is an adaptation of that one presented in [3,Theorem 2.4.2] of Filippov's theorem. We begin by stating a lemma contained in [13,Theorem 1], which is itself of interest, because it provides a kind of uniform upper semicontinuity for a map defined on a compact space.

PROPOSITION **4**.4.3 (Cellina). *Let* $(X,d_X),(Y,d_Y)$ *be two metric spaces, with* X *compact, and* $G:X\to 2^Y$ *be a h-u.s.c. multivalued map. Then, for every* $\varepsilon>0$ *there exists* $\delta>0$ *such that*

$$\forall x\in X \quad \exists x'\in B(x,\delta) \ : \ G(B(x,\delta)) \subseteq B(G(x'),\varepsilon).$$

Proof. Fix $\varepsilon>0$ and, for each $x\in X$, define the function $\rho(\cdot):X\to\mathbb{R}$ as

$$\rho(x) = \sup\{\delta>0: \exists x'\in B(x,\delta):G(B(x,\delta))\subseteq B(G(x'),\varepsilon)\}. \qquad (\mathbf{4}.4.3)$$

To prove our thesis we are going to show that $\rho(x)$ is positive and bounded away from zero on X.

By the h-u.s.c. of G, for every $x\in X$ there exists $\eta(x)>0$ such that $G(B(x,\eta(x)))\subseteq B(G(x),\varepsilon)$. Therefore, setting $x'=x$ in (**4**.4.3), we see that $0<\eta(x)\leq\rho(x)$. Assume now, by contradiction, that there exist two sequences $(\zeta_n)_n$ and $(x_n)_n$ such that $\zeta_n\in\mathbb{R}$, $\zeta_n\downarrow 0$, $x_n\in X$ and $\rho(x_n)<\zeta_n$. By the compactness of X, we can suppose that $x_n\to x_o\in X$. Consider the number $\eta_o=\eta(x_o)$: when $d_X(x_n,x_o)<\eta_o/2$, we have

$$G(B(x_n,\eta_o/2)) \subseteq G(B(x_o,\eta_o)) \subseteq B(G(x_o),\varepsilon),$$

and therefore $\rho(x_n)\geq\eta_o/2$, a contradiction. ∎

Proof of Theorem **4**.4.2.

Let x be a solution of $x'\in\overline{co}F(t,x)$, $x(t_o)=x_o$, on the interval $I=[t_o,t_o+T]$ and fix $\varepsilon>0$. Then the function $G:I\to 2^{\mathbb{R}^n}$, $G(t)=\overline{co}F(t,x(t))$ is u.s.c. and we can suppose $\|F(\cdot,x(\cdot)\|\leq M$. By Proposition **4**.4.3 there exists a δ such that

$\forall t\in I\ \exists t'\in B(t,\delta)$:

$$|t-s|<\delta \Rightarrow \overline{co}F(s,x(s)) \subseteq B(\overline{co}F(t',x(t')),\varepsilon/6T). \tag{4.4.4}$$

Partition I into N intervals $I_i=[t_i,t_{i+1}]$ of lenghth $T/N<\delta\wedge\eta$. Further requirements on N will be made later.Choose, for $i=0,\dots,N-1$, a point $t_i'\in B(t_i,\delta)$ such that (**4**.4.4) holds for $t=t_i$, $t'=t_i'$ and define $\Phi_i = \overline{co}F(t_i',x(t_i'))$.

Fix now $i\in\{0,\dots,N-1\}$ and select a partition of the set

$$S^i = \bigcup_{t\in I_i} \overline{co}\,F(t,x(t))$$

made of a finite number of Borel subsets S^i_j having diameter not larger than $\varepsilon/6T$; choose moreover a subset $J_i\subseteq I_i$ such that $meas(J_i)=meas(I_i)$ and $x'(t)$ exists for each $t\in J_i$. Set $H^i_j = \{t\in J_i : x'(t)\in S^i_j\}$ and $\chi_j(\cdot)=\chi_{H^i_j}(\cdot)$, and let ξ_j be some point in S^i_j. Since $\xi_j\in S^i$, by (**4**.4.4) and by our choice of the interval I_i,

$$d(\xi_j,\Phi_i) < \varepsilon/6T\ . \tag{4.4.5}$$

Define the map $\xi:J_i\to\mathbb{R}^n$ as $\xi(t) = \sum_j \xi_j\chi_j(t)$; then ξ is a simple function such that $|\xi(t)-x'(t)|\le\varepsilon/6T$ for every $t\in J_i$. The derivative of the quasitrajectory we are looking for will be obtained from this first approximation of x'.

By Carathéodory's theorem and by (**4**.4.5), for each j there exist n+1 points $y_{jk}\in ext\Phi_i$ and positive coefficients α_{jk} such that $\sum_k \alpha_{jk} = 1$ and

$$|\xi_j - \sum_{k=1}^{n+1} \alpha_{jk} y_{jk}| < \frac{\varepsilon}{6T} \ . \qquad (4.4.6)$$

The function $y_{|I_i}$ will be constructed by assigning the vectors y_{jk} as derivatives on suitable subsets of I_i. To this purpose, select for each j, by Liapunov's Convexity Theorem [28,Proposition 1.1], a family $(A_j(\alpha))_{\alpha\in[0,1]}$ of Lebesgue measurable subsets of H_j^i such that

i) $A_j(\alpha) \subseteq A_j(\beta)$ if $\alpha \le \beta$,

ii) $\text{meas}\,(A_j(\alpha)) = \alpha\cdot\text{meas}(H_j^i)$ $(\alpha\in[0,1])$,

and set

$$p_o=0,\ p_k=\sum_{\ell=1}^{k}\alpha_{j\ell} \text{ and } \chi_{jk}=\chi_{A_j(p_k)\setminus A_j(p_{k-1})} \qquad (k = 1,\dots,n+1).$$

Define the simple function $\rho: I\to\mathbb{R}^n$ as

$$\rho(t) = \begin{cases} \sum_{k=1}^{n+1} y_{jk}\chi_{jk}(t) & \text{for } t\in H_j^i \\ 0 & \text{for } t\in I\setminus\bigcup_{ij} H_j^i \end{cases}$$

and set

$$y(t) = x_o + \int_{t_o}^{t} \rho(s)\,ds \ .$$

We claim that, for a suitable choice of N, y is the desired approximation. Notice first that $y'(t) \in \bigcup_1^N S^i$ a.e., and therefore y is Lipschitzean with the same constant M as x. Then fix $t\in I$. For some i, $t\in I_i$ and we have

$$|y(t)-x(t)| \le |y(t)-y(t_i)| + |y(t_i)-x(t_i)| + |x(t_i)-x(t)|.$$

The first and the last term of the right-hand side are smaller than $\varepsilon/3$ if $MT/N < \varepsilon/3$. To estimate the second term, remark that, on each I_i,

$$\int_{I_i} \xi(s)\,ds = \sum_j \text{meas}(H_j^i)\,\xi_j$$

and

$$\int_{I_i} \rho(s)\,ds = \int_{I_i} \sum_{j,k} y_{jk}\chi_{jk}(s)\,ds = \sum_{j,k} \alpha_{jk}\,\text{meas}(H_j^i)\,y_{jk}\,.$$

Thus, by the preceding remark and (**4**.4.6),

$$|\int_{I_i} \rho(s)ds - \int_{I_i} \xi(s)ds| = |\sum_j \text{meas}(H_j^i)\cdot(\xi_j - \sum_{k=1}^{n+1} \alpha_{jk}y_{jk})|$$

$$\leq \sum_j \text{meas}(H_j^i)\,|\xi_j - \sum_{k=1}^{n+1} \alpha_{jk}y_{jk}| \leq \text{meas}(I_i)\cdot\frac{\varepsilon}{6T}\,. \tag{4.4.7}$$

Then, for each end point of the intervals t_h we have, in view of our choice of ξ and of (**4**.4.7),

$$|y(t_h) - x(t_h)| \leq |x(t_h) - x_o - \int_{t_o}^{t_h} \xi(s)\,ds| + |\int_{t_o}^{t_h} (\xi(s) - \rho(s))\,ds|$$

$$\leq \frac{\varepsilon}{6} + \sum_{i<h} \text{meas}(I_i)\cdot\frac{\varepsilon}{6T} \leq \frac{\varepsilon}{3}\,,$$

and by consequence, if N satisfies the preceding conditions,

$$\sup_{t\in I} |y(t) - x(t)| \leq \varepsilon\,.$$

Finally, by construction

$$y'(t) \in \text{ext}\,\overline{\text{co}}\,F(t_i', x(t_i')) \qquad \text{a.e. in } I_i$$

and thus

$$d\big((t, y'(t)), \text{graph}\{\text{ext}\,\overline{\text{co}}\,F(\cdot, x(\cdot))\}\big) \leq |t - t_i'| \leq \eta.$$

The proof of Theorem **4**.4.2 is concluded. ∎

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