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A THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR PHILOSOPHIAE

EXISTENCE AND MULTIPLICITY RESULTS FOR NONLINEAR DIRICHLET PROBLEMS

CANDIDATE

Dott. D. LUPO

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INTRODUCTION

Considering the problem

$$
(P) \qquad \begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}
$$

where Ω is a bounded open regular domain of $\mathbb{R}^n$, several questions can be posed: the existence of solutions of (P), the problem of possibly multiple solutions, the regularity of such solutions, the qualitative properties of the same, ect.

In this thesis we report results obtained in [A-Lu], [Lu-So-S] (chapter 1), [Lu-So], [C-Lu-So] (chapter 2), [Lu-M] (chapter 3), [Lu] (chapter 4) which all concern the question of existence and multiplicity of solutions of (P) under various assumptions of particular interest on the nonlinearity f (in the result [Lu-So-S] only the case n=1 is considered); to get these results one has to make use of several different methods of nonlinear analysis.

More precisely, in the first chapter we are concerned with two different problems of multiplicity of solutions, both produced by the interaction of the nonlinearity f with the spectrum of the linear part $-\Delta$, it is easy in fact to show that if such an interaction does not exists we have the unicity of the solution; to obtain the first result, that is relative to a sublinear problem with a suitable growth of the nonlinearity at zero, we make use of the Morse theory, after having reduced, via a Liapounov-Schmidt reduction, our problem to a finite dimensional one.
In the second problem considered in this chapter we treat the case of an even nonlinearity with superlinear growth, the proof strongly depends on the fact that this time we are considering an ordinary differential equation, and it is given by means of a combination of the shooting method and of a topological method (more exactly of a suitable version of the Miranda's theorem): in fact we construct, via the shooting method, functions which have as many positive parts as we want and that are piecewise solutions of our problem; then,

by a suitable version of the Miranda's theorem, we prove that the initial value can be arranged in such a way to make us get a solution in the whole domain.

In the second chapter we consider the problem (P) in a resonance case, namely when the nonlinearity f grows asymptotically as a linear function whose coefficient is given by one of the eigenvalues of the linear part. The problem of existence of solution for any suitable forcing term and that of multiplicity of solutions are treated via min-max tecniques, but we shall be concerned with some cases known in the litterature as periodic and strong resonance in which the resonance produces a "lack of compactness" which makes the [P-S] condition not true. This difficulty (that one has to overcome also in different kind of problems, for instance when the nonlinearity has a critical growth, or if one works in an unbounded domain), can not be overcome avoiding the bad critical level (as it was for instance done in the critical case by [B-N2], where one can prove that some solutions appear at a level low enough to make the [P-S] condition hold), because it is possible to show by the simple analysis of the linear case that all the solutions can exactly lie at that bad level. Therefore we have to work with suitable classes of min-max sets introduced by Lazer and Solimini in [L-S], in such a way that we can construct a Palais-Smale sequence such that its terms have a bounded component in the kernel of the linear part and this is enough to say that the sequence has a limit point that of course is a solution of (P). Then we show how, by using different classes of min-max sets, also introduced in [L-S] and which require a more technical definition, one can get a nontrivial solution result from an estimate of the Morse index.

In the third chapter we consider the problem (P) in the case in which the nonlinearity f is such that there exist $\limsup$ and $\liminf$ of $\frac{f(t)}{t}$ at plus and minus infinity laying between two consecutive eigenvalues. We study conditions for which some of them can exactly be equal to the eigenvalue and we still have existence of solution for every forcing term. This problem has previously been studied by de Figueiredo and Gossez by means of topological degree, on the contrary we treat the problem by variational methods via the Rabinowitz's saddle point theorem. Utilizing then the same classes of min-max considered in chapter two we also get a multiplicity result for the same problem.

Finally we treat a case in which the nonlinearity f is assumed to be not continuous. In this case the variational methods can not be straightforwardly used because the functional associated to the problem does not have a good regularity; the problem has been treated in [A-B], [A-T] utilizing the Clarke's dual action principle and existence and multiplicity results have been obteined. We adopt a different point of view: we imbed this problem in a multivalued one, defining a suitable multivalued nonlinearity, then we prove, utilizing as bifurcation parameter the point in which the discontinuity happen, that a global branch of positive solutions, bifurcating from the trivial one, exists for the multivalued problem assocated to (P). The proof follows the lines of the Rabinowitz's global bifurcation theorem, where one has to utilize the degree for compact multivalued maps introduced by Cellina and Lasota in [C-L]instead of the usual Leray-Schauder degree.

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CONTENTS

CHAPTER 1.

Multiplicity results

In this chapter we will consider problems of the type

$$\text{(DP)} \qquad \begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded open regular domain of $\mathbb{R}^n$ and we will give two examples of how the interaction of the derivative of the nonlinearity f with the spectrum σ of the Laplace operator $-\Delta$ (with Dirichlet boundary conditions) produce phenomena of multiplicity of solutions. It is in fact easy to show, by the Banach-Caccioppoli contraction Theorem that, on the contrary, if f' does not interact with $\sigma(-\Delta)$ then one has unicity of the solution. Suppose, for example (indeed weaker conditions are needed), that:

1) $f \in C^1(\mathbb{R})$,
2) there exist $j \geq 0$ and $a,b \in \mathbb{R}$ such that $[a,b] \subset]\lambda_j,\lambda_{j+1}[$, (where $\lambda_j,\lambda_{j+1} \in \sigma(-\Delta)$) such that $f'(s) \in [a,b]$ for each $s \in \mathbb{R}$,

then (DP) admits exactly one solution.

It is hence natural to see if, imposing such an interaction, one gets multiplicity of solutions. In this field many authors have worked in the past years (see for instance for sublinear problems [A-M1] and furthermore [B] and [Be] for exact multiplicity results); in this chapter we present two examples in this direction.
The first example we treat (see [A-Lu]) is a sublinear problem where we suppose that $\lim\limits_{|s| \to +\infty} \dfrac{f(s)}{s} < \lambda_1$. The multiplicity is produced in this case by imposing a suitable slope to the nonlinearity at zero and the result is obtained utilizing the Morse theory.
The second paragraph is a result ([C-Lu-So]) of arbitrarily many solutions for an O.D.E. (n=1) with even nonlinearity (i.e. the nonlinearity cross all the spectrum); the result is obtained combining the shooting method with a suitable variant of Miranda's theorem.
Here and in the following we work with the Laplace operator but obviously most of the results are true for more general uniformely elliptic operators and if the nonlinearity depend also from the space's variable $x \in \Omega$. Moreover we suppose known all the classical results on elliptic operators and on their eigenvalues (see [Co-H]) we will need in the following.

§ 1. Nonlinearity with sublinear growth and prescribed behaviour at zero.

We consider the nonlinear Dirichlet boundary value problem:

$$(1.1)\quad \begin{cases} -\Delta u = \lambda u - g(u) & x\in\Omega \\ u = 0 & x\in\partial\Omega \end{cases}$$

where λ is a real parameter and $g:\mathbb{R}\to\mathbb{R}$ satisfies the following assumptions:

(I) $\qquad g\in C^1, \quad g(0)=g'(0)=0$

(II) $\qquad \lim\limits_{t\to\mp\infty} \dfrac{g(s)}{s} = +\infty.$

Since $g(0)=0$, then $u\equiv 0$ is a solution of (1.1) for all λ and we are looking for nontrivial (namely $u\not\equiv 0$) solutions of (1.1).

Let us denote by λ_j, $0<\lambda_1<\lambda_2\leq\lambda_3\leq\ldots$, the eigenvalues of

$$(1.2)\quad \begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with corresponding eingenfunctions ϕ_j, normalized by $\int_\Omega \phi_j^2 = 1.$

Remark explicitly that we do not assume λ_2 is simple.
Our goal will be to prove the following theorem of [A-Lu]

Theorem 1.1. Suppose (I) and (II) hold. Then:

(i) for $\lambda>\lambda_1$, (1.1) has at least two nontrivial solutions u_1, u_2, with $u_1>0$ and $u_2<0$ in Ω;

(ii) for $\lambda>\lambda_2$, (1.1) has at least a third nontrivial solution u_3, different from u_1 and u_2.

Remarks 1.3
(a) If $\lambda\leq\lambda_1$, (1.1) could have only the trivial solution $u=0$. This happens, for example, if $sg''(s) > 0$ for every $s\neq 0$ (cf. [A1]);
(b) the existence of a positive and a negative solution, for all $\lambda>\lambda_1$ has been obtained for example in [R2];

(c) if g is odd then a much stronger result is true: for all $\lambda>\lambda_k$, (1.1) has at least k pairs of nontrivial solutions (cf. for example [A1]);

(d) assuming further that $\frac{g(s)}{s}$ is increasing Struwe [Sw] proved (ii); our result is hence an improvement of the Struwe's one. Moreover, essentially under the same assumptions, namely if we suppose:

$$(1.4) \qquad \frac{g(s)}{s} < g'(s) \qquad \text{for each } s\in \mathbb{R},\ s\neq 0$$

the result can be sharpened, by showing that for $\lambda_1<\lambda\leq\lambda_2$ (1.1) has precisely two nontrivial solutions [A-M2];

(e) it is trivial to see that g could depend on x, with obvious modifications in (I-II).

The proof will be carried out by looking for the solutions of (1.1) as critical points of a suitable functional f on $H_0^1(\Omega)$, which will be studied by means of the Morse theory. By using the Morse inequalities the proof of the Struwe's result (assuming (1.4)) reduces just to a few lines: (1.4) implies, by a simple comparison argument, that u_1 and u_2 are nondegenerate local minima of f on $H_0^1(\Omega)$. To handle the general case, the proof requires a Liapunov-Schmidt reduction and a slight modification of the classical Morse inequalities (rather obvionus, in fact), but is quite straight forward as well.

We state two result of the Morse theory which will use later. For more details we refer, for example, to [P] or to Section 4 of [S], and more specifically, for lemma 1.8 to [M-P], Section2.

Let E be a Hilbert space and $f\in C^2(E,\mathbb{R})$. Let us suppose that f satisfies the [P-S] condition i.e.

[P-S]
every sequence $(u_n)_n\in E^{\mathbb{N}}$ such that
(a) $|f(u_n)|\leq c$
(b) $\nabla f(u_n)\to 0$ in E'
has a converging subsequence.

A critical point u of f, namely an $u\in E$ such that $\nabla f(u)=0$, is nondegenerate with Morse index q if $\nabla^2 f(u)$ is invertible and q is the dimension of the linear manifold where $\nabla^2 f(u)$ is negative-defined. Let C_0 be the number of the isolated, local minima of f and, for q>0 let C_q be the number of nondegenerate critical points of f with Morse index q. The relationships between the C_q and the rank of the Homology groups of E are given by the Morse inequalities: suppose $f\in C^2(E,\mathbb{R})$ is bounded from below on E and satisfies [P-S]. Moreover, suppose f has only isolated local minima and nondegenerate critical points of finite Morse index. Then

$$(1.5)\qquad \begin{array}{l} 1\le C_0 \\ -1\le C_1-C_0 \\ 1\le C_2-C_1+C_0 \\ \cdots\cdots \\ (-1)^k\le C_k-C_{k-1}+\ldots+(-1)^k C_0. \end{array}$$

Of course, in (1.5) we have used the fact that rank $H_q(E)=1$ for $q=0$ and zero otherwise.

Remark 1.6. The fact that it is possible to include in C_0 the possibly degenerate, isolated, local minima was first observed in [A2]. Roughly, the justification relises on the following: let u_0 be a local, isolated minimum of f; put $U^-:=\{\ u\in E : \|u-u_0\| < \varepsilon,\ f(u)\le f(u_0)\}$; to evaluate the Morse groups $H_q(U^-,U^-\backslash\{u_0\})$ u_0 need not be nondegenerate of finite Morse index (see, for example [M-P], theorem 1.2), but it sufficies to take ε in such a way that $U^-=\{u_0\}$. It follows that the ranks r_q of the above Morse groups are simply: $r_0=1$, $r_q=0$ for each $q>1$. See [A2] for some more details.
We remark also that this fact can be used to show the Leray-Schauder index of such a point is 1 (see [A]).

From the Morse inequalities (1.5) we deduce:

Lemma 1.7. Suppose $f\in C^2(E,\mathbb{R})$ is bounded from below and satisfies [P-S]. Assume further that: (i) $u=0$ is a nondegenerate critical point of Morse index $q_0\ge 2$; (ii) f has two local minima. Then f has at least an other critical point.

Proof. Otherwise the local minima are isolated and we could apply (1.5) with $C_0=2$ and $C_q=0$ for every $q\ge 2$, $q\ne q_0$. Since $q_0\ge 2$, we have in particular $C_1=0$, in contradiction with the second inequality of (1.5). ■

In some cases it is possible to deal with functionals having degenerate critical points, by using a perturbation argument due to Marino and Prodi jointly with the Sard lemma [M-P], [Mo]. The lemma below is what we need later and is almost the same as [M-P], lemma 1.2.

Lemma 1.8 Suppose $f\in C^2(E,\mathbb{R})$ satisfies [P-S], has $u=0$ as an isolated, possibly degenerate, critical point and $\nabla^2(0)$ is Fredholm of index 0. Then there exist $\varepsilon>0$ and $\tilde{f}\in C^2(E,\mathbb{R})$ such that: $\tilde{f}$ satisfies [P-S], $\tilde{f}(u)=f(u)$ for $\|u\|\geq\varepsilon$, and has a finite number of nondegenerate critical points in $\|u\|<\varepsilon$. Moreover:

$$(1.9) \qquad \|\nabla^2\tilde{f}(u) - f(u)\| < \varepsilon \qquad \text{for every } u\in E.$$

Proof of part (i) of the theorem 1.1. We will shortly sketch, for completeness, the proof of part (i) of the theorem. Let $E:=H_0^1(\Omega)$ with norm $\|u\|:=\int_\Omega|\nabla u|^2$. Denote by $(\cdot,\cdot)$ and $|\cdot|$ the scalar product (the norm, respectively) in $L^2(\Omega)$. Since we have not assumed any growth condition on g, we first truncate the right-hand side in (1.1). Let $s^-< 0 < s^+$ be such that $\lambda s^+-g(s^+)\leq 0\leq\lambda s^--g(s^-)$: such $s^\pm$ exist by (II). Let $p(s)$ be C^1 and such that: $p(s)=\lambda(s)-g(s)$ for $s^-\leq s\leq s^+$, $sp(s)\leq 0$ for every $s\notin[s^-,s^+]$, $|p'(s)|\leq c$ for every $s\in\mathbb{R}$, c=const. By the maximum principle, every solution $u(x)$ of

$$(1.10) \qquad \begin{cases} -\Delta u = p(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies $s^-\leq u(x)\leq s^+$ for every $x\in\Omega$ and hence solves (1.1). Therefore we will look for the critical points of

$$f(u)=\frac{1}{2}\|u\|^2-\int_\Omega P(u), \qquad P(s)=\int_0^s p(t)dt$$

on E. It is easy to verify that the boundeness of p implies f is bounded from below and satisfies [P-S].

To find $u_1>0$ and $u_2<0$ solutions of (1.1) we need an other truncation. Denote by p^+ the positive part of p, $p^-:=p-p^+$ and $P^+:=\int_0^s p^+(t)dt$. Of course $f^+(u)=\frac{1}{2}\|u\|^2-\int P^+(u)$ are bounded from below and satisfy the [P-S] condition. The minima, say u_1 and u_2, of f^+ and f^- respectively solve (1.10) with p replaced by p^+ and p^- respectively. By the maximum principle $u_1>0$ and $u_2<0$: thus u_1 and u_2 are solutions of (1.10) and hence of (1.1).

Remark 1.11. In other words we have that $f(u_1)<f(u)$ for all $u>0$ in Ω, $0<\|u-u_1\|$ small enough. Remark also that if u_1 is not an isolated minimum, we have infinitely many solutions of (1.1), and we have done. Similar remark holds for u_2.

We can deduce the Struwe result from lemma 1.7. In fact define,

$$h(s)=\begin{cases} \dfrac{g(s)}{s} & \text{if } s\neq 0 \\ h(0)=0 \end{cases}$$

and denote by $\mu_j(q)$, for $q\in L^\infty(\Omega)$, the j-th eigenvalue of

$$\begin{cases} -\Delta u = \mu q(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Lemma 1.12. Suppose (I-II) and (1.4) hold. Then u_1, u_2 are nondegenerate critical points of f of Morse index 0.

Proof. Since u_1 solves (1.1) we get

$$\begin{cases} -\Delta u_1 = (\lambda-h(u_1))u_1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

This implies $\mu_j(\overline{q})=1$ for $\overline{q}=\lambda-h(u_1)$, with associate eingenfunction u_1. Since $u_1>0$ in Ω then $\mu_1(\overline{q})=1$. By (1.4), $\overline{q}>\lambda-g'(u_1)$. The comparison property of the eigenvalues yelds $\mu_1(\lambda-g'(u_1))>1$. Hence u_1 is nondegenerate of Morse index 0. The argument for u_2 is the same.■

The following is quite direct:

Lemma 1.13. If $\lambda>\lambda_2$ $\lambda\neq\lambda_k$ then $u=0$ is nondegenerate of Morse index $q_0\geq 2$.

As this point, a straight application of lemma 1.7, shows the existence of a third nontrivial solution for $\lambda>\lambda_2$, $\lambda\neq\lambda_k$. To eliminate the restriction $\lambda\neq\lambda_k$, the use of lemma 1.8 is needed. We will see this in more detail later.

Proof of part (ii) of the theorem 1.1. The proof will be carried out in several steps.

Step 1. Liapunov-Schmidt reduction

Since $p'(s)$ is bounded, it is possible to take a positive integer N such that

(1.14) $\qquad p'(s)<\lambda_N \qquad$ for every $s\in\mathbb{R}$.

Set $V:=\mathrm{span}\{\phi_1, \ldots, \phi_N\}$ and W its L^2-orthogonal complement and denote by Q and $I-Q$ the projections onto W and V respectively. Every $u\in E$ can be put in the form $u=v+w$, $v\in V$, $w\in W$; we will use the notation $v=\alpha\phi$, $\alpha=(\alpha_1, \ldots, \alpha_N)\in\mathbb{R}^N$, to indicate $\sum_1^N \alpha_i\phi_i$.

Problem (1.1) is equivalent to:

(1.15)' $\qquad -\Delta w=Qp(v+w)$

(1.15)" $\qquad -\Delta v=(I-Q)p(v+w)$.

From (1.14) it follows that for every $v=\alpha\phi$, (1.15)' has a unique solution $w=w(\alpha)$, which is C^1 with respect to α. Moreover, taking into account that $p\in C^1$ and bounded and L has smooth coefficients, usual bootstrap arguments imply $w(\alpha)$ is continuous with respect to the C^1 topology, too.

Substituting $w=w(\alpha)$ into (1.15)" we obtain the following system in V

(1.16) $\qquad \lambda_j\alpha_j=\int_\Omega p(\alpha\phi+w(\alpha))\phi_j \qquad (j=1, \ldots, N).$

Let $\psi:V\to\mathbb{R}$ be defined as:

$$\psi(\alpha)=\frac{1}{2}\|w(\alpha)\|^2+\frac{1}{2}\sum_1^N\lambda_j\alpha_j^2-\int_\Omega P(\alpha\phi+w(\alpha)).$$

A direct calculation, taking into account that $w(\alpha)$ solves (1.15)' for $v=\alpha\phi$, yelds:

(1.17) $\qquad \dfrac{\partial\psi}{\partial\alpha_j}=\lambda_j\alpha_j-\int_\Omega p(\alpha\phi+w(\alpha))\phi_j.$

Hence $\psi'(\alpha)=0$ if and only if (1.16) are satisfied.

Remark that $\psi \in C^2$ and

(1.18) $\psi(\alpha)=f(\alpha\phi+w(\alpha))$.

Among other things, ψ satisfies [P-S] and is bounded from below on V.

Step 2. Study of ψ

Let $\bar{\alpha}\in \mathbb{R}^N$ be such that $(I-Q)u_1=\bar{\alpha}$ and $\hat{\alpha}\in \mathbb{R}^N$ such that $(I-Q)u_2=\hat{\alpha}$.

L*emma 1.19.* $\bar{\alpha}$ and $\hat{\alpha}$ are local minima for ψ on V.

Proof. As we have seen in step 1, $\|w(\alpha)-w(\bar{\alpha})\|_{C^1}\to 0$ provided $|\alpha-\bar{\alpha}|^2\equiv\sum_{i=1}^{N}(\alpha_j-\bar{\alpha}_j)^2\to 0$. Recall that $u_1>0$ in Ω; then, when $|\alpha-\bar{\alpha}|<\varepsilon$, ε small enough, $\alpha\phi+w(\alpha)>0$ in Ω as well. Using remark 1.11 we get

$$f(u_1)<f(\alpha\phi+w(\alpha)) \text{ for every } 0<|\alpha-\bar{\alpha}|<\varepsilon.$$

By (1.18), we have $\psi(\bar{\alpha})=f(u_1)$ and $f(\alpha\phi+w(\alpha))$ and the lemma is proved for $\bar{\alpha}$. Same argument for $\hat{\alpha}$. ∎

Step 3. Proof of the theorem completed for $\lambda\neq\lambda_j$

Let $\lambda>\lambda_2$ and $\lambda\neq\lambda_j$. With a view to applying lemma 1.7 we shall show $\alpha=0$ is a nondegenerate critical point of ψ on V of Morse index $q_0\geq 2$. In fact, differentiating in (1.17) and setting $\alpha=0$ we have

$$\frac{\partial^2\psi(0)}{\partial\alpha_i\partial\alpha_j}=\lambda_j\delta_{ij}-\int_\Omega p'(w(0))\phi_j\left(\phi_i+\frac{\partial w(0)}{\partial\alpha_i}\right)$$

where $\delta_{ij}=1$ if $i=j$ and 0 otherwise.
Since $w(0)=0$ and $p'(0)=\lambda$, we deduce:

$$\frac{\partial^2\psi(0)}{\partial\alpha_i\partial\alpha_j}=(\lambda_j-\lambda)\delta_{ij} \qquad i,j=1,2,\ldots,N. \tag{1.20}$$

Since $\lambda>\lambda_2$, $\lambda\neq\lambda_j$ the claim follows.
Now, using lemma 1.19 and applying lemma 1.7 to ψ the result follows.

Step 4. The case $\lambda=\lambda_j$

We complete the proof by considering the case $\lambda=\lambda_j$ for some $j>2$. Suppose again that ψ has only $\bar{\alpha}$, $\hat{\alpha}$ and 0 as critical points. Remark $\psi'(0)$ is trivially Fredholm of index 0. Applying lemma 1.8 to ψ, we find $\tilde{\psi}\in C^2$ and $\varepsilon>0$ such that $\tilde{\psi}$ has in $|\alpha|\geq\varepsilon$ the critical points $\bar{\alpha}$ and $\hat{\alpha}$ only. In $|\alpha|<\varepsilon$ $\tilde{\psi}$ has only a finite number of nondegenerate critical points, say $\beta_1, \ldots, \beta_k$. By (1.9) and (1.20) it follows that, if ε is small enough, then for the Morse index q_j of β_j one has: $q_j>2$ $(j=1, \ldots, k)$. Applying lemma 1.7 to $\tilde{\psi}$ we get again a contradiction, because $C_1=0$. ∎

Remark 1.21. Applying lemma 1.8 to f (remark that $\nabla f(0)$ is of the type identity-compact), the same arguments permit us to complete the proof of Struwe's result as well.

Remark 1.22. In a later paper (see [Ho]) Hofer has shown, using a sharp Mountain-Pass argument in the positive and negative cone combined with a degree argument in [A], that in fact if $\lambda_i<\lambda<\lambda_{i+1}$, $i\geq 2$, then problem (1.1) has at least four nontrivial solutions. For this reason one could think that for $\lambda>\lambda_k$ (1.1) admits 2k nontrivial solutions. This has been shown by Dancer [D2] to be false in general, i.e. there exist examples of nonlinearities for which $\lambda>\lambda_k$ and with the prescribed behaviour at infinity, such that one has exactly four nontrivial solutions.

§ 2. The case of even nonlinearity.

As second case consider the problem

$$(2.1)\qquad \begin{cases} -\Delta u = g(u)+h & x\in\Omega \\ u = 0 & x\in\partial\Omega \end{cases}$$

where as before $\Omega\subseteq\mathbb{R}^N$ is a bounded open set with smooth boundary $\partial\Omega$. Here h is a given function and $g\in C(\mathbb{R},\mathbb{R})$ is such that

$$g_\pm := \lim_{s\to\pm\infty} \frac{g(s)}{s}$$

exist and are finite. In such a case, see [So3], many existence and multiplicity results have been obtained for (2.1), depending on the number of eigenvalues present in the interval (g_-, g_+).
In particular if $h=-te_1$, where t is a positive parameter and e_1 is the positive eigenfunction corresponding to the first eigenvalue then in [L-M1] it was conjectured by Lazer and McKenna that if

$$g_-<\lambda_1<\lambda_2\leq\lambda_3\leq\ldots\leq\lambda_k<g_+$$

then (2.1) admits at least 2k solutions for large values of the parameter t. This conjecture has been shown to be true in the case when N=1, i.e. for the O.D.E. in [L-M2].
Keeping in mind the above stated facts it seems natural to conjecture that if the nonlinearity g is such that $g_-=-\infty$ and $g_+=+\infty$, then the problem (2.1) with $h=-te_1$ admits arbitrarly many solutions depending on the largeness of t. Problems of this type where first considered by Kazdan and Warner [K-W] and by Dancer in [D1]. Solimini in [So4] showed the existence of at least two solutions for large values of the parameter t.

The results of the kind we are expecting were obtained for the O.D.E. in [C-S], [Sc], [Ru-So], but in all these cases the problem considered is of the type

$$(2.2)\qquad \begin{cases} -u''=u^2-t & \text{in}(0,\pi) \\ u(0)=u(\pi)=0 \end{cases}$$

that is in the autonomous case. Moreover, the proofs depend very strongly on the fact that the equation under consideration is autonomous. Hence, it seems to be interesting to study the nonautonomous case. Actually more general nonlinearities than u^2 are considered in [C-S], [Ru-So]. We consider the problem

$$(2.3) \qquad \begin{cases} -u''=u^2-t\cdot\sin x & \text{in } (0,\pi) \\ u(0)=u(\pi)=0 \end{cases}$$

and prove the following theorem

Theorem 2.1. Given any $N_0 \in \mathbb{N}$ there there exists $t_{N_0}>0$ such that for all $t \geq t_{N_0}$, (2.3) has at least N_0 solutions.

The method we employ are a combination of shooting and topological arguments. We prove that we can obtain solutions with as many positive parts as we want depending on the largeness of t.

First of all we prove various estimates we will require in the following for both negative and positive solutions of the equation

$$(2.4) \qquad \begin{cases} -u''=u^2-t\phi & \text{in } (a,b) \\ u(a)=u(b)=0 \end{cases}$$

where $[a,b] \subseteq [0,+\infty)$ and from now on we assume that $\phi:[0,+\infty] \to \mathbb{R}$ is a nonzero positive decresing Lipschitz function.

Lemma 2.5. Let $[a,b] \subseteq [0,+\infty)$. Suppose u satisfies (2.4). Then $u \geq 0$ ($u \leq 0$) implies $|u'(b)| \geq |u'(a)|$ ($|u'(b)| \leq |u'(a)|$) respectively.

Proof. Suppose $u \geq 0$. Multiply (2.4) by u' and integrate, we have

$$(2.6) \qquad \frac{1}{2}\left(u'(b)^2-u'(a)^2\right)=-t\int_a^b \phi u'.$$

Hence, a simple integration by parts on the right-hand side of (2.6) shows that $u'(b)^2 \geq u'(a)^2$. A similar computation in the case $u \leq 0$ shows $u'(b)^2 \leq u'(a)^2$. Hence the lemma. ∎

Now we digress to discuss qualitatively the solutions of the Cauchy problem:

$$(2.7) \qquad \begin{cases} -u''=u^2-t\phi \\ u(a_0)=0 \\ u'(a_0) \geq 0 & \text{where } a_0 \in [0,+\infty). \end{cases}$$

Near a_0 it is clear that u is convex and that u starts increasing (except in the case when $\phi(a_0)=0$ and $u'(a_0)=0$, in which case however $u\equiv 0$). As u increases, since ϕ is decreasing, u" decreases and must become zero in a finite time, since as far as it is bigger or equal to zero, u increases with a derivate bounded from below by a positive number, so $u^2 \geq \max\phi$ in a finite time. At this time u has positive derivative so u still increases for a while and u" becomes strictly negative. As far as u increases u" becomes more and more negative so u' becomes zero in a finite time. So there exists x_0 such that $u'(x_0)=0$ and since $u''(x_0)<0$ we see that x_0 is a local maximum. Moreover, we claim that u will meet again the $(0,+\infty)$ axis at some point b.

Proof of the claim. Let y_1 and y_2 be two points to the left and the right of the point x_0 such that $u(y_1)=u(y_2)$, and that $u'\leq 0$ in $[x_0,y_2]$. Then we prove $|u'(y_2)|\geq|u'(y_1)|$. To this end consider the equation u satisfies on $[y_1,y_2]$ i.e.

$$\text{(2.8)}\qquad \begin{cases} -u''=u^2-t\phi & \text{in } (y_1,y_2) \\ u(y_1)=u(y_2). \end{cases}$$

Multiply (2.8) by u' and integrate to obtain

$$\text{(2.9)}\qquad \frac{(u'(y_1))^2}{2}-\frac{(u'(y_2))^2}{2}=-t\int_{y_1}^{y_2}\phi u'.$$

Now consider

$$\int_{y_1}^{x_0}\phi u' \geq \phi(x_0)\int_{y_1}^{x_0}u'=\phi(x_0)(u(x_0)-u(y_1))$$

$$=\phi(x_0)(u(x_0)-u(y_2))=\phi(x_0)\int_{x_0}^{y_2}(-u')$$

$$\geq -\int_{x_0}^{y_2}\phi(s)u'(s)ds.$$

Hence

$$(2.10)\qquad \int_{y_1}^{x_0}\phi u' + \int_{x_0}^{y_2}\phi u' \geq 0 \text{ i.e. } \int_{y_1}^{y_2}\phi u' \geq 0.$$

Combining (2.9) and (2.10) we have that

$$(2.11)\qquad |u'(y_2)|\geq|u'(y_1)|.$$

From the above estimate it is clear that the fall on the right of x_0 is faster then the rate of the increase on the left. Hence it is obvious that u hits the 0-axis again at some point b. ■

Let a, x_0 and b be the points as in the above discussion. Then it is clear that the solution of the Cauchy problem (2.7) satisfies

$$(2.12)\qquad \begin{cases} -u''=u^2-t\phi \\ u(a)=u(b)=0. \end{cases}$$

Lemma 2.13. The solution u obtained for (2.12) with the previous arguments, satisfies

$$\int_a^b u dx \leq 2\int_a^{x_0} u.$$

Proof. This is clear from the estimate (2.11). ■

We now proceed to prove some estimates on maximum of positive solutions which we have been discussing above. Also from now on whenever we refer to a positive solution it is to be assumed that it is obtained by considering a Cauchy problems like (2.7). Also in most cases we assume that all the numerical constants we get are absorbed in the constants like c_1, c_2 which we have used.

Lemma 2.14. Suppose $0\leq u'(a)\leq c_1\cdot t^{3/4}$ in (2.7) then

$$(2.15)\qquad \max_{x\in[a,b]} u(x) \leq c_2\sqrt{t}$$

where $c_2:=\max\left(\sqrt[3]{3c_1^2}, \sqrt{6\phi(a)}\right)$.

Proof. Since u satisfies (2.12), multiplying by u' and integrating from a to x_0, x_0 being the point where the maximum is achieved, we have

$$\frac{(u'(a))^2}{2}=\frac{u^3(x_0)}{3}-t\int_a^{x_0}u'\phi$$

i.e.

$$u'(a)\geq\frac{2}{3}u^3(x_0)-2t\phi(a)\int_a^{x_0}u'\geq\frac{2}{3}u^3(x_0)-2t\phi(a)u(x_0)$$

i.e.

$$u^3(x_0)\leq\frac{3}{2}u'(a)^2+3t\phi(a)u(x_0).$$

Then either

$$u^3(x_0)\leq 3u'(a)^2$$

or

$$u^3(x_0)\leq 6t\phi(a)u(x_0).$$

In the former case it follows from our hypothesis that

$$(2.16)\qquad u(x_0)\leq 3^{1/3}c_1^{2/3}t^{1/2}.$$

In the latter case it follows that

$$(2.17)\qquad u(x_0)\leq\sqrt{6t\phi(a)}.$$

Now the lemma follows from (2.16) and (2.17). ■

Lemma 2.18. Under the same hypotheses as in lemma 2.13, we have

$$(2.19)\qquad (x_0-a)\leq k_0\left[\sqrt{\left(\frac{c_2}{\phi(x_0)}\right)}+\left(\frac{c_2}{\phi(x_0)^{3/4}}\right)\right]t^{-1/4}$$

where k_0 is some fixed positive constant and c_2 is given by lemma 2.14.

Proof. We estimate separately in $[a,x_0]$ the length s_1 of the interval in which $u^2\leq(1/2)t\phi$, the length s_2 of the interval in which $u^2/t\phi\in(1/2,2)$ and finally that of the interval in which $u^2\geq 2t\phi$, length denoted by s_3.

Estimate for s_1: that is the case in which $u^2\leq(1/2)t\phi$. Since u satisfies (2.12), we have

$$(2.19)\qquad u''\geq\frac{1}{2}t\phi.$$

By the Taylor formula we have that if x is a point where $u^2/t\phi=1/2$, then

$$u(x)=u(a+s_1)=u(a)+u'(a)s_1+\frac{u''(\xi)}{2}s_1^2 \tag{2.20}$$

where $\xi\in(a,a+s_1)$. Combining (2.15), (2.19) and (2.20) we have

$$\frac{1}{2}t\phi s_1^2\le c_2\sqrt{t}$$

or

$$s_1\le\sqrt{\frac{2c_2}{(x_0)}}\cdot t^{-1/4}.$$

<u>Estimate for s_2</u>: that is in the case $u^2/t\phi\in(1/2,2)$. From

$$\frac{u'(a)^2}{2}-\frac{u'(y)^2}{2}=\frac{u^3(y)}{3}-t\int_a^y\phi u'$$

where y is such that

$$\frac{u^2(y)}{t\phi(y)}\in(\frac{1}{2},\,2),$$

we have

$$u'(a)^2-u'(y)^2\le-\frac{2}{3}t\phi(y)u(y)$$

hence

$$u'(y)^2\ge\frac{2}{3}t\phi(y)u(y)$$

$$\ge\frac{2}{3}t\phi(y)\frac{\sqrt{t\phi(y)}}{\sqrt{2}}$$

since $u^2(y)\ge(t\phi(y))/2$. Therefore, we have,

$$u'(y)\ge\frac{2^{1/4}}{\sqrt{3}}\phi(y)^{3/4}t^{3/4}.$$

By the mean value theorem we have

$$u(a+s_1+s_2)-u(a+s_1)=u'(y)s_2 \quad \text{where } y\in(a+s_1,a+s_1+s_2).$$

Hence, using (2.14) and (2.15) we have:

$$c_2\sqrt{t} \geq s_2 \frac{2^{1/4}}{\sqrt{3}} \phi(y)^{3/4}\, t^{3/4}$$

i.e.

$$s_2 \leq \frac{\sqrt{3}}{2^{1/4}} \cdot \frac{c_2}{\phi(x_0)^{3/4}} \cdot t^{-1/4}.$$

Estimate for s_3: in this case $u^2 \geq 2t\phi$. We have from (2.12) that $-u'' \geq t\phi$. A simple application of Taylor with x_0 as base point gives

$$u(x_0)-u(x_0-s_3) \leq \frac{t\phi(x_0)s_3^2}{2}.$$

Using (2.15) in the above leads to

$$c_2\sqrt{t} \geq \frac{t\phi(x_0)}{2}\, s_3^2$$

or

$$s_3 \leq \sqrt{\frac{2c_2}{\phi(x_0)}} \cdot t^{-1/4}.$$

Hence $(x_0-a)=s_1+s_2+s_3 \leq c_3 t^{-1/4}$, where

$$c_3 := k_0\left[\sqrt{\left(\frac{c_2}{\phi(x_0)}\right)} + \frac{c_2}{\phi(x_0)^{3/4}}\right].$$

Here k_0 is some fixed number. ■

Lemma 2.21. Under the same hypotheses of lemma 2.14 one has

$$\|u\|_{L^1} \leq 2c_2c_3t^{1/4}.$$

Proof. This is a direct consequence of lemmas 2.13-2.18. ■

We now proceed to estimate the difference in the slopes at a and b. Here we make distinction between the case when ϕ remains bounded away from zero in [a,b] and the case when ϕ becomes zero somewhere in this interval.

Lemma 2.22. Suppose [a,b] is such that ϕ remains bounded away from zero, then under the same hypotheses as in lemma 2.14, it holds:

(2.23) $\quad (|u'(b)|^2-|u'(a)|^2) \leq 2\left(\sup_{[a,b]}\phi'\right)c_2c_3t^{5/4}.$

Proof. This is a direct consequence of earlier proved lemmas. ■

Lemma 2.24. Let $u \geq 0$ satisfy (2.12). In addition suppose $u'(a)=0$. Then we have the following estimate for $u'(b)$:

(2.25) $\quad |u'(b)| \leq \sqrt{2}\, 6^{1/4}(\phi(a))^{3/4}t^{3/4}.$

Proof. We known from the above argument that:

$$|u'(b)|^2 = -2t\int_a^b \phi' u \leq 2t \max_{[a,b]} u \int_a^b -\phi' \leq 2\phi(a)t \max_{[a,b]} u.$$

This estimate and lemma 2.14 immediately give the proof of the statment. ■

We now turn to some estimates of the negative solution of

(2.26) $$\begin{cases} -u''=u^2-t\phi & \text{in } (c,d) \\ u(c)=u(d)=0. \end{cases}$$

where $(c,d) \subseteq \mathbb{R}^+ \cup \{0\}$. We will not prove the existence and uniqueness for the negative solution of (2.26) under the given hypotheses on ϕ, since it is easy prove this via a sub-supersolution argument (see [K-W]).

Lemma 2.27. Suppose u is the negative solution of (2.26) then

$$|u'(c)|+|u'(d)| \leq c_4 t(d-c)$$

where c_4 is a numerical constant, which depends only on the maximum of ϕ.

Proof. This is trivial and we omit the details. ■

Before we prove further estimates for negative solutions we consider the following autonomous O.D.E.

(2.28) $$\begin{cases} -u''=u^2-t\alpha & \text{in } (c,d) \\ u(c)=u(d)=0. \end{cases}$$

with $\alpha>0$ and we will prove some estimates for its negative solutions.

Lemma 2.29. Suppose $u \le 0$ satisfies (2.28), then

$$u'(d) \ge \min\left(\frac{t\alpha(d-c)}{4},\ \left(\frac{2\sqrt{2}}{3}\right)^{1/2}\alpha^{3/4}t^{3/4}\right)$$

Proof. We distinguish two cases, namely

(i) $$\int_c^d u^2 \le \frac{t}{2}\alpha(d-c)$$

and

(ii) $$\int_c^d u^2 > \frac{t}{2}\alpha(d-c).$$

Supposing we are in the case (i), then we have

$$2u'(d)=u'(d)-u'(c)=\int_c^d u^2 \ge \frac{t}{2}\alpha(d-c)$$

and we are done.

In the case (ii), since $\int_c^d u^2 > \frac{t}{2}\alpha(d-c)$, there exists

(2.30) $$x_0 \in (c,d) \text{ such that } u^2(x_0) > \frac{t\alpha}{2}\ .$$

Multiply (2.28) by u' and integrate between x_0 and d to obtain

(2.31) $$u'(d) \ge \frac{2}{3}u^3(x_0)-2t\alpha u(x_0) \ge 2u(x_0)\left[\frac{u^2(x_0)}{3} - t\alpha\right]$$

$$\ge 2|u(x_0)|\cdot\left[t\alpha - \frac{u^2(x_0)}{3}\right].$$

By the maximum principle, $u^2(x_0) \le t\alpha$, hence we have from (2.30) and (2.31) that $u^2(d) \ge \frac{2\sqrt{2}}{3}\alpha^{3/2}t^{3/2}$. ∎

Now let $u \le 0$ be the solution [c,d] of (2.4) and ϕ satisfy the same hypothesis as before.

Lemma 2.32. The following estimates hold

(2.33) $$u'(d) \ge \min\left(\frac{t\phi(d)(d-c)}{4},\ \left(\frac{2\sqrt{2}}{3}\right)^{1/2}\phi(d)^{3/4}t^{3/4}\right)$$

Moreover, if η is such that $0<\eta<d-c$

$$(2.34)\qquad -u'(c) \geq \min\left(\frac{t\phi(c+\eta)(d-c)}{4},\ \left(\frac{2\sqrt{2}}{3}\right)^{1/2}\phi(c+\eta)^{3/4}t^{3/4}\right).$$

Proof. We will first prove (2.34). Let η be given, since ϕ is decreasing $\phi(c+\eta)$ is the minimum of ϕ on $[c,c+\eta]$. Let now v be the function defined by the negative solution of (2.28), when $\alpha=\phi(c+\eta)$, in $[c,c+\eta]$ extended by zero in $[c+\eta,d]$. In this way v is a supersolution for our problem and so by the arguments of [K-W] since u is the only negative solution of our problem, then: $u\leq v$. Since $v(c)=u(c)=0$, we will get $|u'(c)|>|v'(c)|$. For the restriction of v to $[c, c+\eta]$ the estimates of lemma 2.29 will hold. This proves (2.34); (2.33) is analogously proved by sostituting ϕ by the constant function $\phi(d)$ in all $[c,d]$ and by estimating the supersolution obtained by solving that autonomous equation. ∎

Corollary 2.35. Under the hypotheses of lemma 2.32, $u'(d)\neq 0$ as long as $\phi\not\equiv 0$.

Proof. Suppose $\phi(d)\neq 0$, then the corollary follows from lemma 2.32. Suppose $\phi(d)=0$, then since $\phi\not\equiv 0$, we can choose $c_0\in(c,d)$ such that $\phi\geq\phi(c_0)=\delta>0$ on $[c, c_0]$. Define

$$\Psi:=\begin{cases}\delta \text{ on } [c,c_0]\\ 0 \text{ on } (c_0,d]\end{cases}$$

and consider the problem

$$(2.36)\qquad \begin{cases} -v''=v^2-t\Psi & \text{in } (c,d)\\ v(c)=v(d)=0.\end{cases}$$

It is clear from a sub-supersolution argument that problem (2.36) admits a negative solution. Also it is easy to see that such a solution is unique. Let v_0 be such a solution. It is clear that $v_0\in C^1([c,d])$ and that v_0 is C^2 except at c_0. We claim $v_0'(d)\neq 0$. If $v_0'(d)=0$, then we are lead to a contradiction to the fact that v_0 is a solution of (2.36) by considering the initial value problem

$$\begin{cases} -v''=v^2-t\Psi & \text{in } (c,d)\\ v(d)=v'(d)=0\end{cases}$$

and following the flow backwards. Hence $v_0'(d)\neq 0$. clearly a solution of (2.36) is a supersolution for

$$\begin{cases} -u''=u^2-t\phi \\ u(c)=u(d)=0. \end{cases}$$

It then follows from [K-W] that the unique negative solution of (2.4) is smoller than v_0. Hence the result. ∎

We prove now our main theorem for the problem

$$\text{(2.37)} \quad \begin{cases} -u''=u^2-t\phi \\ u(0)=u(\pi)=0. \end{cases}$$

where ϕ is as before.

Theorem 2.2. Given any $N_0\in \mathbb{N}$ there there exists $t_{N_0}>0$ such that for all $t\geq t_{N_0}$, (2.37) has at least N_0 solutions.

Before to go to the proof of theorem 2.2 we explain how we plan to procede and we introduce a number of functions which we shall imploy in the course of the proof.
We denote by $(x_i)_{i=1}^k$ a partition of $(0,\pi)$. Let $(a_i)_{i=1}^k$ denote k numbers which we use to shoot, i.e. at each x_i we consider the Cauchy problem,

$$\text{(2.38)}_i \quad \begin{cases} -u'' = u-t\phi \\ u(x_i) = 0 \\ u'(x_i)= a_i \end{cases}$$

with $t>0$ fixed, $i=1, \ldots, k$.
From the previous discussion, the solution u_i of $(2.38)_i$ meets the axis $(x_i,+\infty)$ at some point v_i. We call our slope $(a_i)_{i=1}^k$ good if for each of the i one has that the points $v_i\in(x_i,x_{i+1})$ $(i=1,\ldots,k-1)$ and $v_k\in(x_k,\pi)$, i.e. we are in a situation like fig.1.

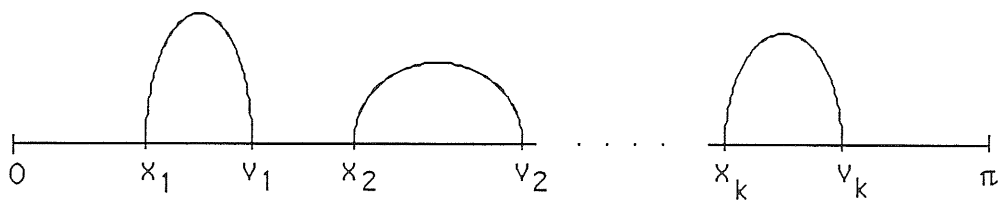

Fig. 1.

From the previonus section we also know that given any interval $(c,d)\subseteq(0,\pi)$ we can find an unique negative solution of:

$$\begin{cases} -u''=u^2-t\phi \\ u(c)=u(d)=0. \end{cases}$$

Use $(0,x_1)$, (v_i,x_{i+1}) and (v_k,π) for (c,d) and join the respective intervals by the unique negative solution, to obtain a configuration as in fig. 2

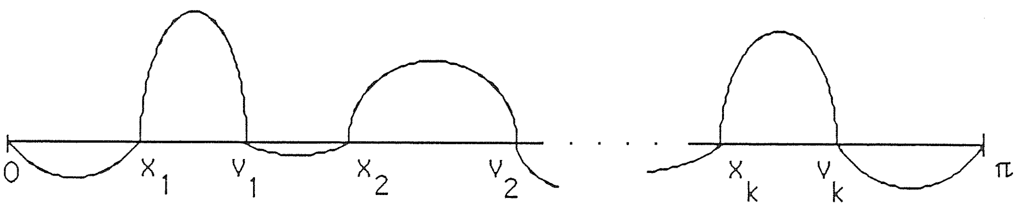

Fig. 2.

We now define functions $(\delta_i)_{i=1}^{k}$ in the good cases as follows:

δ_{2i-1}:= $a_i -$ |Slope of negative solution terminating at x_i| for i=1,...,k;
δ_{2i} := |Slope of positive solution terminating at v_i| –
– |Slope of negative solution originating at v_i| for i=1,...,k.

In the above definitions and in what follows the symbol $|\cdot|$ indicates that the slopes are taken in their absolute values.

Suppose the slopes $(a_i)_{i=1}^{k}$ are not good in the sense of the definition before, that is, for some i, v_i falls after x_{i+1} or π, then we define for these indices i

δ_i:= | Slope of the positive solution terminating at v_i |

and

$\delta_{2i+1}:=a_{i+1}$ if i<k.

Notice that the functions $(\delta_i)_{i=1}^{2k}$ are continuous with regard to the variables x_i and a_i. This is because when $v_i \to x_{i+1}$, the corrisponding negative solution gets flat (lemma 2.27).

Note that if we can find x_i's and a_i's such that $(\delta_i)_{i=1}^{2k}$ are all zero the this combination of positive and negative solutions give a solution of problem (2.37). Moreover, this solution will have k positive parts for large values of the parameter t.

Lemma 2.39. A solution u obtained by the above discussed arguments has k parts of positivity.

Proof. Let us distinguish two cases.
If $x_1>0$, since $\phi(0)\neq 0$ the left derivate in x_1 is nonzero (lemma 2.32 and corollary 2.35). Therefore, as one easily verifies by using lemma 2.5, all the derivatives in all points x_i's and v_i's are not zero. By our definition of the δ_i's in the bad cases this shows that none of the possible collapsing between v_i and x_{i+1} if $i<k$ or between v_k and π can occur. Assume now $x_1=0$. If $a_1>0$ proceed like in the previous case, if $a_1=0$ then the derivative of u in the x_i's is zero as far as ϕ is constant. In this case $v_i=x_{i+1}$ but one still counts a positive part for any i. As soon as ϕ decreses from $\phi(0)$ the left and the right derivatives of u becomes nonzero in the next v_i point. Then one repeats the arguments in the previous case. ■

Remark 2.40. In all what follows we work to prove theorem 2.2. However, note than an easy modification of the ideas involved lead to the proof of theorem 2.1 (in fact for any positive symmetric function decreasing in $(\pi/2,\pi)$). In fact the modification to be carried out is the following: work only in the interval $(\pi/2,\pi)$ and use symmetry to complete the argument in the sense that if x_i's are a partition of $(\pi/2,\pi)$, then consider in $(0,\pi/2)$ the partition given by the points $(\pi-x_i)$. Join $\pi-x_1$ and x_1 by the negative solution. The only modification in the δ_i's occurs when i=1. In this case we take it as the slope by which we shoot from x_1 minus the absolute value of the slope at x_1 of the negative solution in $(\pi-x_1,x_1)$.

We now state and prove an abstract theorem which we shall employ to prove the existence of a common zero for the δ_i's. This theorem, which is a variant of the theorem due to Miranda [Mi], is taken from [So3].

Theorem 2.3. Let $f_i:[0,1]^n \to \mathbb{R}$ $i=1, \ldots, n$ be continuous functions satisfying the following

(1) on the face $z_i=0$, $f_j(z)=0$, for every $j<i \Rightarrow f_i(z)\leq 0$

(2) on the face $z_i=1$, $f_j(z)=0$, for every $j<i \Rightarrow f_i(z)\geq 0$

then there exists $z \in [0,1]^n$ such that $f_i(z)=0$ for all $i=1, \ldots, n$.

Proof. Let Z_i be $\{z\in [0,1]^n \mid \text{for every } j<i\text{: } f_j(z)=0\}$. By Tietze's theorem one can extend the restriction of f_i to $Z_i\cap\{z_i=0\}$ on $\{z_i=0\}$ taking values in $(-\infty,0]$. Then one extends the restriction of f_i to $Z_i\cap\{z_i=1\}$ on $\{z_i=1\}$ taking values in $[0,+\infty)$. In this way one gets a continuous function on $\{z_i=0\}\cup Z_i\cup\{z_i=1\}$ and extends it to a continuous function g_i defined on $[0,1]^n$, by using again Tietze's theorem. It is immediate to recognize that $g_1, \ldots, g_n$ verify the assumption of Miranda's theorem [Mi] and, therefore, there exists $z\in[0,1]^n$ such that: for every $i=1, \ldots, n$: $g_i(z)=0$. It is also immediate to recognize by induction on i that: $f_i(z)=0$. ■

We now introduce new variables and functions from the already introduced x_i's, a_i's and δ_i's. Notice we have in fact 2k variables in $(x_i)_{i=1}^k$ and $(a_i)_{i=1}^k$. Though we have used δ_i with a slightly confusing index, our arguments so far clearly point to the case that the index i for δ_i varies from 1 to 2k.

We introduce a variable $\underline{a}$ using the already defined a_i's.

We define

$$\underline{a}:=\sum_{i=1}^{k} a_i$$

with this definition of $\underline{a}$, given $\underline{a}$ we can think of the variables $(a_i)_{i=1}^k$ as defining a (k-1)-simplex given by $\sum_{i=1}^{k} a_i=\underline{a}$.

Also x_i's $i=1, \ldots, k$ being a partition of an interval can be thought of as defining a k-simplex.

Since the theorem 2.3 which we have stated and proved requires that in order to apply it one needs to work on cubes and since we

have observed that the x_i's and $\sum_{i=1}^{k} a_i=\underline{a}$ define simplexes, we make the following observation.
Let $[0,1]^n$ be the n-cube with its coordinates system being the usual space coordinates. Let Σ_n denote an n-simplex and use barycentric coordinates to represent points in Σ_n. Then there exists a continuous map $p_n:[0,1]^n \rightarrow \Sigma_n$ such that it is onto and satisfies

$$\{z_i=0\} \xrightarrow{p_n} \{y_i=0\} \qquad i=1, \ldots, n$$

and

$$\{z_i=1\} \rightarrow \left\{\sum_{j=1}^{i} y_j=1\right\}$$

where any $z \in [0,1]^n$ is written $z=(z_1, \ldots, z_n)$ and $y \in \Sigma_n$ is written $y=(y_1, \ldots, y_{n+1})$.
Keeping in mind the above discussion we now introduce a change of variables to have our δ_i's defined on the 2k-dimensional cube $[0,1]^{2k}$.
Let $z=(z_1, z_2, \ldots, z_k, \ldots, z_{2k}) \in [0,1]^{2k}$ and set $\underline{a}=M_0.z_1$ where M_0 is large enough. We shall specify the exact M_0 we require later on.
We set

$$x_i=\pi \sum_{j=1}^{i} y_j \qquad i=1, \ldots, k.$$

with $y=p_k(z_2, \ldots, z_{k+1})$.
Finally we define $a_1=\underline{a}y'_k$ and

$$a_i=\underline{a}y'_{k-1} \qquad \text{for } i=2, \ldots, k$$

where

$$y'=(y'_1, \ldots, y'_k)=p_{k-1}(z_{k+2}, \ldots, z_{2k}).$$

Through these changes of variables we understand that δ_i's are defined on the cube $[0,1]^{2k}$ with z being the variable. We construct from δ_i's new functions $(f_i)_{i=1}^{2k}$ which play the role of the functions of theorem 2.3.

We define

$$
(2.41)\quad \begin{cases} f_1=\sum_{i=2}^{2k}\delta_i+c.\delta_1, & c>0 \text{ is a suitable constant} \\ f_2=-\delta_1 \\ f_i=-\delta_{2(i-2)}-\delta_{2i-3} & i=3,\ldots,k+1 \\ \text{and finally} \\ f=\delta_{2(i-k)-1} & i=k+2,\ldots,2k. \end{cases}
$$

It is clear that finding a common zero of these f_i's is equivalent to finding a common zero for δ_i's.

We now proceed to prove some more lemmas which we shall need later. We use all the earlier introduced notation and also lemmas proved in the last section. We use these now with (x_i,v_i) standing for (a,b) of the preecending estimates and $(0,x_1)$, (v_i,x_{i+1}) or (v_k,π) for (c,d) of the same previous results. We shall use these without standing this explicitly in what follows.

Lemma 2.42. Let $k\in\mathbb{N}$ be given and $c_1>0$. There exist constants $t_0>0$, $\eta>0$ such that for any $(x_i)_{i=1}^k$, $(a_i)_{i=1}^k$ with $a_i\le c_1t^{3/4}$ $i=1,\ldots,k$, for $t>t_0$ one of the negative intervals intersected with the support of ϕ has length bigger than 2η.

("Negative interval" in the lemma means an interval like $(0,x_1)$, (v_i, x_{i+1}) or (v_k,π) in which we consider the negative solution).

Proof. Let $[0,\xi]$ be the support of ϕ intersected with $[0, \pi]$. Divide the interval $[0,\xi]$ into two equal parts, than in the half of the second part truncate ϕ by ψ,

$$
(2.43)\quad \psi:=\begin{cases} \phi & \text{in } [0, \frac{3}{4}\xi) \\ \phi(\frac{3}{4}\xi) & \text{in } [\frac{3}{4}\xi,\infty) \end{cases}
$$

and substitute for a moment ϕ with ψ in our problem. Observe that $\inf\psi>0$. Assume there exist some x_i's of our partition in $(0,\xi/2)$, with slopes $a_i\le c_1t^{3/4}$ (where c_1 is as in lemma 2.14). Then we know from lemma 2.18 and lemma 2.21 that the distances $|x_i-v_i|\to 0$ uniformly with respect to x_i as $t\to\infty$. For t large enough we will have all the v_i's falling before $\frac{3}{4}\xi$ and, therefore, $|x_i-v_i|\to 0$ uniformly for $x_i\le\xi/2$ also with the original ϕ. At this point we do not need any more the truncation on ϕ. Take t large enough such that if l of the k x_i's are in $(0, \xi/2)$ then

$$(2.44) \qquad \sum_{i=1}^{l} |x_i - v_i| < \frac{\xi}{4}.$$

Take $0<\eta< \dfrac{\xi}{8(k+1)}$. Hence the lemma. ■

<u>*Lemma 2.45.*</u> Assume k fixed. There exist constants t_0 and c such that for any partition $(x_i)_{i=1}^{k}$ if the a_i's are zero, then if $t>t_0$

$$c\delta_1 + \sum_{i=2}^{2k} \delta_i < 0.$$

<u>*Proof.*</u> We distinguish two cases.
<u>Case 1</u>: suppose that the negative interval given by lemma 2.42, is not the first interval. In this case we consider the slope of the negative solution starting at v_j and terminating at x_{j+1} (π if $j=k$), $|v_j - x_{j+1}|>2\eta$. We know from lemma 2.32, (2.34) that the slope with which the negative solution starts from v_j, which we write $u_r'(v_j)$, is such that

$$(2.46) \qquad |u_r'(v_j)| \geq c_0 t^{3/4}$$

where c_0 is independent of the partition as also t for $t\geq t_0$ of lemma 2.42.
In the case in which $\inf\limits_{x\in(0,\pi)} \phi = 0$, it is clear we can take $p_0 \in [0,\pi]$ such that

$$(2.47) \qquad 3^{1/4}\phi(p_0) \leq \frac{c_0}{2k}$$

where c_0 is the same as in (2.46).
Distinguish positive parts with x_i's$\geq p_0$. Then by (2.25), we have the slopes of the positive part terminating at v_i, which we shall denote $u_l'(v_i)$, are such that

$$(2.48) \qquad \sum |u_l'(v_i)| \leq \frac{c_0}{2} t^{3/4}.$$

For the other x_i's we use a truncation argument as before to obtain using (2.23)

(2.49) $|u'_l(v_i)| \leq c'_0 t^{5/8}$.

From (2.46), (2.47), (2.48) and (2.49) the conclusion follows whatever is $c \geq 0$.

Case 2: if the interval given as in lemma 2.42, happens to be the first one, that is the one starting from 0 and terminating at x_1, then, from lemma 2.32 the negative solution terminating at x_1 as slope

$$|u'_l(x_1)| \geq \left(\frac{2\sqrt{2}}{3}\right)^{1/2} \phi(x_1)^{3/4} t^{3/4}.$$

However, the positive solutions terminating at v_i are such that their slopes are as below, (by lemma 2.24),

$$|u'_l(v_i)| \leq 2^{3/4} 3^{1/4} (\phi(x_i))^{3/4} t^{3/4},$$

hence, since $\phi(x_i) \leq \phi(x_1)$ by putting a large constant c in front of δ_1 we have the required result. Hence the lemma. ∎

Lemma 2.50. Assume $a_1=0$ and $\delta_1=0$. Suppose all the even δ_i's are positive and that all the odd δ_i's are negative. Then t is bounded from above. (i.e. this can't happen if t is large enough).

Proof. Since $a_1=0$ and $\delta_1=0$, it is clear that the positive solution is originating from zero at the first instance, namely $x_1=0$. Using the hypothesis of the lemma and also lemma 2.5, all the slopes we get for x_i's and v_i's are bounded by a constant times $t^{3/4}$. Now using lemma 2.42, we have an interval $[v_j, x_{j+1}]$ in which the slope of the negative solution originating at v_j and terminating at x_{j+1} has

(2.51) $|u'_r(v_j)| \geq c_0 t^{3/4}$

indipendent of t and the partition. However, we will show that under the hypothesis of our lemma there exists a constant c'_0 such that

(2.52) $|u'_r(v_i)| \leq c'_0 t^{5/8}$

for all i such that $i \leq j$ which contradicts (2.51) unless t is bouded. Notice that by lemma 2.18 and lemma 2.22, we can choose t large enough such that the positive solution terminating at v_1 has its slope bounded as given below.

$$|u_l'(v_1)| \leq c_0'' t^{5/8}.$$

Now by using the hypothesis of the lemma, lemma 2.5 and lemma 2.22, it is trivial to verifies that (2.52) holds for all $i \leq j$, by repeating j times the same kind of argument. ∎

Now we proceede to prove the main theorem. In the following we shall assume that the f_i's are defined as in (2.41) with the constant c appearing in the definition of f_1 has been now fixed with its value given by lemma 2.45. We also assume that we let a vary from $[0, M_0]$, where M_0 is large such that whatever the a_i's such that

$$\sum_{i=1}^{k} a_i = M_0,$$

then $f_1 > 0$. Note that such a choice of M_0 (once we have fixed up all the previously mentioned constants) is possible because of lemma 2.27. Note that during the course of the proof one always works with a fixed t but we shall ask that this t is large enough, the largeness being determined by the lemmas.

Proof of theorem 2.2.

We verify the hypotheses of theorem 2.3, with n=2k and f_i's being given by (2.41). First we start with the case $z_1=0$, i.e. all the a_i's are zero. In this case, by lemma 2.45 we have $f_1<0$. Also by our choice of M_0, if $z_1=1$, then $f_1>0$. Hence we have verified the hypothesis of theorem 2.3 for f_1. Let us now consider $i \in \{2, \ldots, k+1\}$. Suppose $z_i=0$ then by definition we have $y_{i-1}=0$ where $y=p_k(z_2, \ldots, z_{k+1})$. By the way we have defined our change of variables, this means $x_1=0$ if i=2 or $x_{i-1}=x_{i-2}$ if i>2. For i=2 this means $f_2=-\delta_1 \leq 0$. If i>2, then by our definition of δ_i's it is clear that δ_{2i-2} and δ_{2i-3} are positive and hence $f_i \leq 0$. Let us now consider $z_i=1$, $i \in \{2, \ldots, k+1\}$. This time we use the hypothesis of theorem 2.3 i.e. we verify what happens to $f_i(z)$ with $z_i=1$, $i \in \{2, \ldots, k+1\}$ when $f_j(z)=0$ for j<i.
We know that if $z_i=1$, then $p_k(z_2, \ldots, z_{k+1})=y$ satisfies

$$\sum_{\lambda=1}^{i-1} y_\lambda = 1.$$

So we have $y_\mu=0$ for $\mu=i, \ldots, k+1$. By the previous analysis this implies that $f_{\mu+1}(z)\le 0$ ($\mu=i, \ldots, k$). Moreover $\mu=k+1$ gives $\delta_{2k}\ge 0$. This follows directly from the definition of the δ_i's. In fact $y_{k+1}=0$ by the change of variables introduced earlier, implies that

$$x_k=\pi\sum_{j=1}^{k} y_j=\pi\sum_{j=1}^{k+1} y_j=\pi.$$

Hence it follows from

$$f_1(z) = c\delta_1(z) - \sum_{m=3}^{i} f_m(z) - \sum_{m=i+1}^{k+1} f_m(z) + \delta_{2k}(z) = 0$$

that

$$(2.53) \qquad f_i(z) = c\delta_1(z) - \sum_{m=3}^{i-1} f_m(z) - \sum_{m=i+1}^{k+1} f_m(z) + \delta_{2k}(z).$$

However, under our hypothesis, the first two terms in the right side of (2.53) are zero. On the other hand we have shown $f_m(z)$, $i+1\le m\le k+1$ and $-\delta_{2k}(z)$ are negative. Hence $f_i(z)\ge 0$. Thus, we have verified the hypothesis of the theorem for f_i, $i=1, \ldots, k+1$.
We now procede to verify the hypothesis for $i\in\{k+2, \ldots, 2k\}$. Let $z_i=0$, which means $y'_{i-k-1}=0$ where $y'=p_{k-1}(z_{k+2}, \ldots, z_{2k})$.
This means $a_{i-k}=0$ by definition. Hence it follows directly from our definiton that $\delta_{2(i-k)-1}=f_i\le 0$.
We now consider the case $z_i=1$, $i\in\{k+2, \ldots, 2k\}$. This means

$$\sum_{s=1}^{i-k-1} y'_s = 1.$$

This implies $y'_s=0$ for $s>i-k-1$. By the previous analysis this implies $f_\mu\le 0$ for $\mu>i$, which means that $\delta_{2(\mu-k)-1}\le 0$, $\mu>i$. Suppose now $f_j(z)=0$ for each $j<i$. In particular $f_2(z)=\delta_1(z)=0$. Also $\delta_{2(j-k)-1}(z)=f_j(z)=0$ for every j such that $k+2\le j<i$. Suppose by contraddiction that $f_i=\delta_{2(i-k)-1}(z)<0$. Then all the odd δ_i's are negative in z. From $f_j(z)=0$ for $j=3, \ldots, k+1$, we have that all the even δ_i's are positive for $i<2k$. Moreover, $\delta_{2k}+c\delta_1=0$ by the assumption $f_1(z)=0$ and $f_j(z)=0$ for $j\in\{2,\ldots,k+1\}$. But $\delta_1=0$ also implies $\delta_{2k}=0$.
Thus, we are in the situation of lemma 2.50, for since $y'_k=0$, $a_1=0$. Thus we never meet this situation when t is large enough. Hence we

have verified all the requirements of theorem 2.3, from which theorem 2.2 follows. ∎

Remark 2.61. Using the remark 2.40 and discussions as above it is clear that theorem 2.1 follows.

CHAPTER 2.

Resonance problems

In this chapter we are concerned with problems "at resonance" i.e. problems of the type

$$(2.1)\quad \begin{cases} -\Delta u = \lambda_k u + g(u) + h(x) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where Ω is an open, bounded, regular domain in $\mathbb{R}^n$, g is a continuous sublinear function and λ_k is, as usual, an eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

It is well-known (Fredholm alternative) that for $g \equiv 0$ problem (2.1) has solution if and only if h is orthogonal to the eigenspace V_k corresponding to λ_k.
The nonlinear case has been studied by several authors, first of all by Landesmann and Lazer in [L-L]. They assumed that:

$$(2.2)\quad \begin{cases} \text{there exists } \lim\limits_{t\to\pm\infty} \dfrac{g(t)}{t} = g(\pm\infty) \\ \text{and} \\ \text{for each } t\in\mathbb{R} \text{ it holds: } g(-\infty)<g(t)<g(+\infty) \end{cases}$$

Under these conditions they proved that (2.1) has a solution if and only if the condition:

$$\text{(LL)}\quad \int_\Omega [g(-\infty)\phi^+ - g(+\infty)\phi^-] < \int_\Omega h\phi < \int_\Omega [g(+\infty)\phi^+ - g(-\infty)\phi^-]$$

is satisfied for each ϕ belonging to V_k. Actually, only the case λ_k simple was treated in [L-L] (for the proof see also [H1]), while the general situation has been treated, between others, in [A-L-P], [B-N1] and [R3].
It is clear that if the above defined limits do not exists, for instance if the nonlinearity g is a periodic function with mean value zero (the fact that g has mean value zero is not a restriction since one can add a constant to h) or if they are equal, e.g. when $g(+\infty)=g(-\infty)=0$, the (LL) condition has no meaning.
These cases have been studied by several authors (see for instance [M-W], [W], [C] for the first one and [B-B-F], [A-M], [W], [H2] for the other one, that is also known as "strong resonance" if $\int_{+\infty}^{-\infty} g(t)dt=0$).

In the following we report an existence theorem (see [So2] for the case λ_k simple and [Lu-So] for the general case) and a multiplicity result (see [C-Lu-So]) for a class of problems which include the previous two ones as particular cases.
All our results are obtained by means of variational methods, taking into account the fact that the resonance produce a "lack of compactness", therefore the usual min-max tecniques are not sufficient to give the existence of solutions. The multiplicicy results (Theorems 2.2, 2.3) are obteined combining the informations on Morse-index and on the min-max characterization of the critical level given by the approach developped in [L-S]. The result of theorem 2.2 is of Amann-Zehnder type (see [A-Z]) the extra difficulty of the lack of compactness, while the result in Theorem 2.3 is produced by resonance like in [A-M1] and [H2], in which the multiplicity comes out by the choice of suitable forcing terms (while we don't need it), and in [B-B-F], whose theorem 2.2 can be seen as a particular case of theorem 2.3 below. The fact that [A-Z] should be extended to the resonance case when the (LL) conditions holds is remarked in [A-Z]; the extension to the strong resonance case require a method which can work without a good compactness assumption.

From now on let g be a continuous real function and let G denote a primitive of it. Then it is well-known that g defines a Nemitskji operator $g_{\#}$ from the set of real measurable functions on Ω (with the metric of the convergence in measure) which we denote by $M(\Omega)$, into $L^{\infty}(\Omega)$. Moreover $g_{\#}$ is uniformely continuous for every L^p topology, $p<+\infty$, in the target space $L^{\infty}(\Omega)$. We denote by $\mathcal{G}$ the continuous functional on $M(\Omega)$ which sends u into $\int_{\Omega} G(u)$.

We say that g satisfies the condition (g_0) if and only if:

If $\psi, \psi_n \in C^1(\Omega)$, $\psi = \lim_{n\to+\infty} \psi_n$ in $C^1(\bar{\Omega})$ and $\nabla\psi = 0$ almost everywhere, if U is a precompact subset of $H_0^1(\Omega)$, then:

(g_0) a) $\lim_{n\to+\infty} g_{\#}(u+n\psi_n) = 0$ weakly in $H^{-1}(\Omega)$

b) $\lim_{n\to+\infty} \mathcal{G}(u+n\psi_n) = 0$,

both a) and b) holding uniformely for $u \in U$.

Remark 2.3. The hypothesis (g_0) is a quite indirect condition on g. Consider then, the following explicit assumptions :

(g_1) g is a continuous periodic real function with mean value zero (i.e. G is periodic too).

(g_2) g is a bounded continuous real function such that: $\lim_{|s|\to+\infty} g(s) = \lim_{|s|\to+\infty} G(s) = 0.$

It is possible to show that if g satisfies (g_i) for i=1 or 2, then g satisfies (g_0). For the proof see [So2] or [Lu-So]. Assumptions (g_1)-(g_2) are two explicit examples of the hypothesis under which we are working, however these do not cover all the cases in which (g_0) holds; for instance (g_1) could be weakened asking simply that G is sublinear and has a sublinear primitive. It is not our interest to characterize completely the explicit form of (g_0). The fact that the periodicity assumption on g leads to an abstract condition of the type of (g_0) was previously observed by Ward [W] who first used it in order to apply the Rabinowitz saddle point theorem.

Suppose, by simplicity of notation, that the sequence of eigenvalues of $-\Delta$ is numbered in such a way that $0<\lambda_0<\lambda_1\leq\ldots\leq\lambda_2\leq\ldots$. Moreover suppose that in (2.1) λ_k is an eigenvalue of multiplicity n-k and that $\lambda_{k-1}<\lambda_k$.

Our results are

Theorem 2.1. Let k be given and let $h\in H^{-1}(\Omega)$ be orthogonal, in the duality between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, to the eigenspace corresponding to λ_k. Let g satisfies (g_0). Then problem (2.1) has at least one (weak) solution.

Theorem 2.2. Let $h\equiv 0$ and suppose that g satisfies (g_0), g(0)=0 and

$$(g_3)\qquad \begin{cases} g'(0)>\lambda_n-\lambda_k \\ \text{or} \\ g'(0)<\lambda_{k-1}-\lambda_k \text{ if } k>0 \end{cases}.$$

Then (2.1) has a nontrivial solution.

Theorem 2.3. Let $h \equiv 0$ and suppose that g satisfies (g_0), $g(0)=0$ and

(g_4) $g'(0) \cdot G(0) \geq 0$, $g'(0) \neq 0$.

Then (2.1) has a nontrivial solution.

Remark 2.4. Theorems 2.2 and 2.3 are natural results under hypothesis (g_2), while they are not what can be reasonably expected under the hypothesis (g_1); in fact in this case one can conjecture stronger multiplicity results without any extra assumptions of the type (g_3)-(g_4).

Remark 2.5. Conditions (g_3) are in some sense natural conditions for the existence of nontrivial solutions of (2.1). In fact easy estimates show that if $h \equiv 0$, $g(0)=0$ and:

$$0 \leq \frac{g(s)}{s} \leq \lambda_n - \lambda_k \quad \text{for any } s \in \mathbb{R} \setminus \{0\},$$

where the strict inequality holds near zero, then (2.1) has only the trivial solution $u \equiv 0$. One also easily recognize that a weak violation of the previous condition is not sufficient to produce the existence of a nontrivial solution.

Let us give some notation and abstract results that will be useful in the following. Let E be a Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\| \cdot \|$. Let B^i be the ball centred at zero with fixed radius r (or any isomorphic set) of an i-dimensional subspace E_i of E. We denote by P_i the orthogonal projection on E_i and by Σ^{i-1} the relative boundary of B^i in E_i. Let U^i be a given set containing Σ^{i-1}, we define for every $A \subset E$

$$S_i(A) := \{\sigma \in C(A, E_i) \mid \sigma = P_i \text{ on } A \cap U^i\}$$

and

$$\Gamma_i^* := \{A \subset E \mid A \text{ is compact and for every } \sigma \in S_i(A)\colon 0 \in \sigma(A)\}$$

We shall say that $A \subset E$ satisfies condition (γ_i) if and only if

there exists $c>0$ and $(c_h)_h \in \mathbb{N}^{\mathbb{N}}$ such that $\lim\limits_{n \to +\infty} \frac{c_h}{h^{i+1}} = 0$ and,

(γ_i) for infinitely many $h \in \mathbb{N}$, A can be covered by the union of c_h

sets, of diameter less than $\frac{c}{h}$.

We define $\bar{\Gamma}_i := \{A \in \Gamma_i^* \mid A \text{ satisfies } (\gamma_i)\}$.

Proposition 2.6. The following result holds:

a) B^i satisfies (γ_i)

b) If A satisfies (γ_i) and $\eta: A \to E$ is a Lipscitz map then $\eta(A)$ satisfies (γ_i)

c) If $A \subset \mathbb{R}^j$, $j>i$ and A satisfies (γ_i) then $\mathring{A} = \varnothing$.

Proof. a) and b) easily follow by the definition of (γ_i). c) follows from the fact that one can get, as an upper bound for the Lebesgue exterior measure of A, a constant times $\frac{c_h}{h^j}$ for infinitely many h. ■

Remark 2.7. The classes Γ_i^* and $\bar{\Gamma}_i$ have essentially been introduced in [L-S] to the aim of giving a min-max characterization of critical levels with critical points of prescribed Morse index.

Fixed $k \in \mathbb{N}$, $n \in \mathbb{N}$, $k<n$ and $\alpha \in \mathbb{R}$, $\alpha>0$; let W be a (n-k)-dimensional subspace of E, which we shall assume to be orthogonal to E_k. We denote by P the ortogonal projector on W and by $\tilde{B}$ the unit ball in W centred at zero. Let $C_\alpha := B^k + \alpha\tilde{B}$; for any α C_α is isomorphic to B^n so we can use it in the definition of the classes Γ_n^* and $\bar{\Gamma}_n$ which in this case will be denoted by $\Gamma_{n,\alpha}^*$ and $\bar{\Gamma}_{n,\alpha}$ respectively, where the set U^n is chosen as $U_\alpha^n :=$ the boundary of C_α in $E_k + W$. Let U^k be a neighborhood of Σ^{k-1}. This means that a real positive number ε exists such that

(2.8) for any $\bar{x} \in \varepsilon B$, for any $\alpha>0$ $U_\alpha^n \cap (\bar{x} + W^\perp) \subset U^k$.

In (2.8) and in the sequel, we assume that B is the unit ball in E, then in this situation, with the notation fixed above and with $\varepsilon>0$ fixed as in (2.8), $\varepsilon \leq 1$, the following result holds:

Proposition 2.9. For any $A \in \Gamma_{n,\alpha}^*$, for any $x \in \varepsilon B$ one has:

(2.10) $A \cap (x + W^\perp) \in \Gamma_k^*$

Moreover for any $A \in \bar{\Gamma}_{n,\alpha}$ a dense subset $D \subset W$ exists such that for any $x \in \varepsilon B \cap D$ one has

$$(2.11) \qquad A\cap(x+W^{\perp})\in\bar{\Gamma}_k.$$

Proof. See Appendix 1. ∎

We recall the result of [L-S], about the existence of critical points with a prescribed Morse-index, which will be used to prove Theorems 2.2 and 2.3; first of all we recall that a functional $I\in C^1(E)$ is said to satisfy the $[P\text{-}S]_c$ condition at the level $c\in(-\infty,+\infty)$ if and only if:

$$[P\text{-}S]_c \qquad \begin{array}{l} \text{every sequence } (u_n)_n\in E^{\mathbb{N}} \text{ such that} \\ \text{(a) } I(u_n)\to c \\ \text{(b) } \nabla I(u_n)\to 0 \text{ in } E' \\ \text{has a converging subsequence.} \end{array}$$

Moreover let $I\in C^2(E)$ and x be a critical point of I. Suppose that the Hessian matrix $\nabla^2 I(x)$ has only a finite number of negative eigenvalues, then let denote by $m_-(x)$ the number of strictly negative eigenvalues of $\nabla^2 I(x)$ and by $m_0(x)$ the dimension of the kernel of $\nabla^2 I(x)$. Define

$$(2.12) \qquad c_i^* := \inf_{\Gamma_i^*} \sup_A I$$

and

$$(2.13) \qquad \bar{c}_i := \inf_{\bar{\Gamma}_i} \sup_A I,$$

then the following result essentially comes from [L-S]:

Proposition 2.14.

A. Assume

$$(2.15) \qquad c_i^* > \sup_{U^i} I$$

then for every sequence $(A_j)_j\in \Gamma_i^{*\,\mathbb{N}}$ such that $\lim_{j\to\infty} \sup_{A_j} I= c_i^*$, there exists a sequence $(x_j)_j\in E^{\mathbb{N}}$ such that

$$(2.16) \qquad \begin{array}{l} d(x_j,A_j)\to 0 \\ I(x_j)\to c_i^* \\ \nabla I(x_j)\to 0 \end{array}$$

Moreover if $[P\text{-}S]_c$ (with $c=c_i^*$) and (2.15) hold and if $\nabla^2 I(x)$ is a Fredholm operator for every $x\in K_{c_i^*}$, then a critical point $\bar{x}\in K_{c_i^*}$ exists such that

(2.17) $\quad m_-(\bar{x}) + m_0(\bar{x}) \geq i.$

B. Assume:

(2.18) $\quad \bar{c}_i > \sup_{U^i} I$

then for every sequence $(A_j)_j \in \bar{\Gamma}_i^{\mathbb{N}}$ such that $\lim_{j\to\infty} \sup_{A_j} I = \bar{c}_i$, there exists a sequence $(x_j)_j \in E^{\mathbb{N}}$ such that

(2.19) $$\begin{aligned} &d(x_j,A_j)\to 0\\ &I(x_j)\to \bar{c}_i\\ &\nabla I(x_j)\to 0 \end{aligned}$$

Moreover if $[P\text{-}S]_c$ (with $c=\bar{c}_i$) and (2.18) hold and if $\nabla^2 I(x)$ is a Fredholm operator for every $x\in K_{\bar{c}_i}$, then a critical point $\bar{x}\in K_{\bar{c}_i}$ exists such that

(2.20) $\quad m_-(\bar{x}) \leq i \leq m_-(\bar{x}) + m_0(\bar{x}).$

C. The condition (2.15) and (2.18) are satisfied provided

(2.21) $\quad \inf_{E_i^\perp} I > \sup_{U^i} I.$

Proof.

C. (2.21) implies (2.15) and (2.18). In fact for any $A\in\Gamma_i^*$, $A\in\bar{\Gamma}_i$: $P_i\in S_i(A)$ therefore $0\in P_i(A)$. This means that $A\cap E_i^\perp\neq\varnothing$ that implies $\inf_{E_i^\perp} I \leq \sup_A I$ and therefore

$$c_i^* \geq \inf_{E_i^\perp} I > \sup_{U^i} I \quad \text{and} \quad \bar{c}_i \geq \inf_{E_i^\perp} I > \sup_{U^i} I.$$

The first part of *A.* and *B.* follows by a standard deformation argument (see for instance [So2]). The second part of *A.* is contained in [L-S], Theorem 2.5 . The proof of the second part of *B.* is contained in [L-S], Theorem 2.6, with the differences we are going to point out. First of all in [L-S] is essentially asked (2.21), however that is used only in order to prove (2.18) that we assume from the beginning. Moreover the condition (γ_i) is assumed in [L-S] in a slightly simpler version (there denoted by $(\bar{c})$). However all what was used of condition $(\bar{c})$ was the fact that $(\bar{c})$ satisfies properties of proposition 2.6 (with of course (γ_i) replaced by $(\bar{c})$). Since Proposition 2.6 establish the same properties for (γ_i) the proof of Theorem 2.6 of the [L-S] directly applies in our case. ∎

Remark 2.22. We have modified condition $(\bar{c})$ in the (γ_i) to the aim of having a more flexibile version in order to prove (2.11).

Remark 2.23. If one consider C_α defined as above and the corresponding classes $\Gamma^*_{i,\alpha}$ and $\bar{\Gamma}_{i,\alpha}$ one can define

(2.24) $$c_i^*(\alpha) := \inf_{\Gamma^*_{i,\alpha}} \sup_A I$$

(2.25) $$\bar{c}_i(\alpha) := \inf_{\bar{\Gamma}_{i,\alpha}} \sup_A I,$$

and the analogous results of Proposition 2.14 still hold.

Let us apply these abstract results to the resonance problem. Let $E = H_0^1(\Omega)$. It is well-known that solutions of (2.1) are critical points of the functional $I(u)$ defined as:

(2.26) $$I(u) := J(u) + \mathcal{G}(u)$$

where

(2.27) $$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda_k}{2}\int_\Omega |u|^2 - \int_\Omega hu\ .$$

Let us consider E_k to be the subspace spanned by the eigenvectors associated to the eigenvalues $\lambda_0, \lambda_1, \dots, \lambda_{k-1}$, W the eigenspace associated to λ_k, $E_n = E_k + W$ and by P the orthogonal projection on W. In the following we will denote by u^* a solution in E of

$$-\Delta u - \lambda_k u = h(x)$$

which exists because $h \in W^\perp$; moreover we can assume $u^* \in W^\perp$.

Proposition 2.28. Let g satisfy (g_0). Then the functional I satisfies the $[P\text{-}S]_c$ for every $c \in (-\infty, +\infty) \setminus \{J(u^*)\}$. Moreover it verifies the condition [P-S]':

[P-S]' every sequence $(u_n)_n \in E^{\mathbb{N}}$ such that
(a) $P(u_n)$ is bounded
(b) $\nabla I(u_n) \to 0$ in E'
has a converging subsequence.

Proof. We sketch the main steps. Suppose $(u_n)_n \in E^{\mathbb{N}}$ be such that $\nabla I(u_n) \to 0$, write $u_n = v_n + w_n$ with $v_n = (I-P)(u_n) \in W^{\perp}$, $w_n = P(u_n) \in W$. By standard arguments one gets $v_n \to v$ in E (see [So2] for further details), therefore if w_n are bounded we have finished, and this proves in particular [P-S]'. We can hence suppose $\|w_n\| \to +\infty$ and we want to show that then $I(u_n) \to J(u^*)$. For infinitely many $n \in \mathbb{N}$ there exists k_n such that $\lim\limits_{n \to +\infty} \frac{1}{n} \|w_{k_n}\| = 1$, for these n we define the functions $\psi_n := \frac{1}{n} w_{k_n}$; obviously, passing to a subsequence, $\psi_n \to \psi$ by compacteness and, since ψ is a normalized eigenfunction, $\nabla \psi \neq 0$ almost everywhere. This implies, by $(g_0\text{-a})$, that $g_{\neq}(v_n + n\psi) \to 0$ weakly (this is immediately seen taking $\psi_n = \psi$ for those n for which ψ_n has not yet been defined). Projecting by I-P and passing to the weak limit one gets $-\Delta v - \lambda_k v = h(x)$ i.e. $v = u^*$. If we pass to the limit in (2.26) we get by $(g_0\text{-b})$

$$I(u_n) = J(v_n) + \mathcal{G}((u_n) \to J(v) = J(u^*). \qquad \blacksquare$$

Remark 2.29. The functional I does not satisfy [P-S] at $c = J(u^*)$. In fact, let $w \in W \setminus \{0\}$ and consider the sequence $u_n = u^* + nw$. It is clear that, by condition (g_0), u_n is such that $I(u_n) \to J(u^*)$ and $\nabla I(u_n) \to 0$, but, obviously, u_n does not contain a converging subsequence.

It is easily seen that the following estimate on J holds:

$$(2.30) \qquad J(u^*) = \max_{u^* + E_k} J$$

and that we can fix $\bar{r} > 0$ such that

$$(2.31) \qquad \sup_{\partial B_k(u^*, \bar{r})} I < \inf_{u^* + E_k^{\perp}} I$$

In fact (2.30) follows as J is concave in $u^* + E_k$ and $\nabla J(u^*) = 0$, while we can fix $\bar{r} > 0$ such that (2.31) holds because I is bounded from below on $u^*+E_k^\perp$ and goes quadratically to $-\infty$ on E_k. Indeed we can strengthen (2.31) as

(2.32) $$\sup_{\partial B_k(u^*,\bar{r})+W} I < \inf_{u^*+E_k^\perp} I$$

because the component belonging to W gives only a bounded contribution in order to determine the value of the functional I. We can also choose a neighbourhood U^k of $\Sigma^{k-1} := \partial B_k(u^*,\bar{r})$ such that, by continuity, the following estimates still holds:

(2.33) $$\sup_{U^k} I < \inf_{u^*+E_k^\perp} I$$

We consider C_α as defined before and recalling that U_α^n denotes the relative boundary of C_α in E_n we define

(2.34) $$b(\alpha) := \sup_{U_\alpha^n} I$$

It is possible to show that

(2.35) $$\lim_{\alpha\to+\infty} b(\alpha) \le J(u^*).$$

In fact by (2.32) $b(\alpha) = \sup_{B_k+\partial(\alpha B)\cap W} I$; hence to prove (2.35) it is enough to show that for every $v \in B_k$, $w \in \partial(\alpha B)\cap W$ $I(v+w)$ converges uniformely to $J(v)$ and then by (2.30), (2.35) will follow. To get the uniform convergence it is sufficient to argue by contradiction and use (g_0).

Proof of Theorem 2.1.

By 2.33 we can utilize Proposition 2.14 *A*. From that result and Proposition 2.28 it is clear that if $c_k^* \neq J(u^*)$ then c_k^* is a critical level; therefore from now on suppose $c_k^* = J(u^*)$.

Consider then C_α defined as before and the corresponding $c_n^*(\alpha)$; consider now $\alpha\to+\infty$. Two cases can occur:

1) $$\limsup_{\alpha\to+\infty} c_n^*(\alpha) > J(u^*),$$

2) $$\limsup_{\alpha\to+\infty} c_n^*(\alpha) \le J(u^*).$$

If 1) holds then take a sequence $\alpha_i \to +\infty$ and choose i such that $c_n^*(\alpha_i) > J(u^*)$ and $c_n^*(\alpha_i) > b(\alpha)$ (this is clearly possible because

$\lim_{\alpha \to +\infty} b(\alpha) = J(u^*)$).

Hence we are in a position to apply proposition 2.14 (more precisely the variation suggested by remark 2.23) and find a critical point.
If 2) holds take a sequence $\alpha_i \to +\infty$. Then for each j there exists $A_j \in \Gamma^*_{n,\alpha_j}$ such that:

$$\sup_{A_j} I \le c_n^*(\alpha_j) + \frac{1}{j},$$

therefore, for each j, for each $x \in \varepsilon\bar{B}$

$$\sup_{A_j \cap (x+W^\perp)} I \le c_n^*(\alpha_j) + \frac{1}{j},$$

hence one has

$$\sup_{A_j \cap (x+W^\perp)} I \le c_n^*(\alpha_j) + \frac{1}{j} \le b(\alpha) + \frac{1}{j} \to J(u^*) = c_k^*,$$

therefore the sequence $A_j \cap (x+W^\perp)$ is such that one can apply Proposition 2.14 <u>A</u> to find the sequence $(x_j)_j \in E^{\mathbb{N}}$ who satisfies [P-S]', thus we get the critical point. ∎

Before proving our multiplicity results let us note that if $h \equiv 0$ then one can take $u^*=0$ and $J(0)=0$.

<u>*Proof of Theorem 2.2.*</u>

Fix a neighbourhood U^k of Σ^{k-1} in such a way that (2.33) holds. Define $\bar{\Gamma}_k$ and $\bar{c}_k$ as before. If $\bar{c}_k \neq 0$ by proposition 2.28 the $[P\text{-}S]_{\bar{c}_k}$ holds, hence we can apply proposition 2.14 to find a critical point $x_0 \in K_{\bar{c}_k}$ such that (2.20) holds. By (g_3) one can easily show that $x_0 \not\equiv 0$. In fact if $g'(0) > \lambda_n - \lambda_k$ then $m_-(0) > n > k$ or if $g'(0) < \lambda_{k-1} - \lambda_k$, $m_-(0) + m_0(0) < k$ and therefore zero can not be the critical point found with Proposition 2.14.

If $\bar{c}_k=0$ consider $\bar{c}_n(\alpha)$ defined as in Remark 2.23. As in Theorem 2.1 distinguish two different cases:

1) $\displaystyle \limsup_{\alpha\to+\infty} \bar{c}_n(\alpha) > 0,$

or

2) $\displaystyle \limsup_{\alpha\to+\infty} \bar{c}_n(\alpha) \leq 0,$

If 1) holds one can choose an α such that $\bar{c}_n(\alpha)>0$ and $\bar{c}_n(\alpha)>b(\alpha)$, by 2.35. In this case we are again in a position to apply Proposition 2.14 in order to find a critical point $x_0\in K_{\bar{c}_n(\alpha)}$, with Morse-index such that (2.20) holds. Again such a critical point cannot be zero because if $g'(0)>\lambda_n-\lambda_k$ then $m_-(0)>n$, while if $g'(0)<\lambda_{k-1}-\lambda_k$, $m_-(0)+m_0(0)<k<n$.

Suppose now that 2) holds. Fix a sequence $(\alpha_i)_i$ such that $\alpha_i\to+\infty$, then for every i there exists $A\in\bar{\Gamma}_{n,\alpha_i}$ such that

(2.36) $\displaystyle \sup_{A_i} I \leq \bar{c}_n(\alpha_i) + \frac{1}{i}.$

By Proposition 2.9 for every i one can find a dense subset $D_i\subset W$ such that for any $x\in\varepsilon B\cap D_i$ one has $A_i\cap(x+W^{\perp})\in\bar{\Gamma}_k$. Let $\bar{x}\in\varepsilon\bar{B}$ be arbitrarily choosen. One can find a sequence $(x_i)_i$ such that $x_i\in\varepsilon B\cap D_i$ and $x_i\to\bar{x}$. Then for such a sequence one has, in view of 2):

(2.37) $\displaystyle \sup_{A_i\cap(x_i+W^{\perp})} I \leq \sup_{A_i} I \leq \bar{c}_n(\alpha_i) + \frac{1}{i} \to \leq 0=\bar{c}_k.$

We are now in a position to apply again Proposition 2.14 *B* in order to find a sequence $(y_i)_i\in E^{\mathbb{N}}$ such that $d(y_i,A_i\cap(x_i+W^{\perp}))\to 0$ (hence $P(y_i)\to\bar{x}$) and $\nabla I(y_i)\to 0$. Therefore by Proposition 2.28 $(y_i)_i$ has a converging subsequence which will converge to a critical point of I. Such a critical point will have a component in W equal to $\bar{x}$ which has been arbitrarily choosen in $\varepsilon\bar{B}$, therefore one has, in this case, infinitely many solution of (2.1). ■

In the previous proof the characterization of the Morse index was enough in order to conclude, while in the following proof it will be combined with estimates on the level.

Proof of Theorem 2.3.

Fix a neighbourhood U^k of Σ^{k-1} in such a way that (2.33) holds. Define $\bar{\Gamma}_k$ and $\bar{c}_k$ as before. If $\bar{c}_k \neq 0$ then by Proposition 2.28 the [P-S] holds at level $\bar{c}_k$, therefore recalling (2.33) and applying Proposition 2.14 B one sees that $K_{\bar{c}_k} \neq \varnothing$. We can now disinguish the two cases which produces (g_4) namely

(2.38) $\quad g'(0)>0$ and $G(0) \geq 0$,

(2.39) $\quad g'(0)<0$ and $G(0) \leq 0$.

If (2.38) holds then the fact that the critical point $\bar{x}$ obtained by applying Proposition 2.14 *B* cannot be zero follows by a Morse-index argument since $m_-(0)>k$ while $m_-(\bar{x}) \leq k$.
If (2.39) holds the Morse-index argument cannot directly apply, therefore let us suppose that zero is the only critical point of I. By (2.39) and by the definition of I one has

(2.40) $\quad I(0)=-|\Omega|\cdot G(0) \geq 0$,

(2.41) $\quad \bar{c}_k>0$ (since $\bar{c}_k \neq 0$).

where in (2.40) $|\cdot|$ denotes the Lebesgue measure of Ω.

By Proposition 2.9 and by (2.41) for every α one has $\bar{c}_n(\alpha) \geq \bar{c}_k>0$, therefore by (2.35) it is possible to choose α in such a way that we also have $\bar{c}_n(\alpha)>b(\alpha)$. Since $[P\text{-}S]_c$ holds at level $\bar{c}_n(\alpha)>0$ we can apply Proposition 2.14 in order to find a critical point $x_0 \in K_{\bar{c}_n(\alpha)}$ such that $m_-(x_0)+m_0(x_0) \geq n$ and this implies that $x \not\equiv 0$ because one has that $m_-(0)+m_0(0)<n$.

Suppose now $\bar{c}_k=0$. As in the proof of the previous Theorem one has to consider two different cases. If 2) holds, then exactly as in the former proof we can find infinitely many solutions of (2.1), (each one with a fixed component $\bar{x} \in \varepsilon B \cap W$).

If 1) holds then $\alpha>0$ exists such that $\bar{c}_n(\alpha)>0$ and $\bar{c}_n(\alpha)>b(\alpha)$. We have already shown that this produces a nontrivial solution if (2.39). If (2.38) holds we get a nontrivial solution because $\bar{c}_n(\alpha)>0$ while $I(0) \leq 0$. ∎

Remark 2.42. The problem of the research of T-periodic solutions of the equation

(2.43) $\quad -x''-\lambda_k x = \nabla V(x,t)$

where V(x,t) is T-periodic in t and satisfies the following assumptions

$$(2.44) \qquad \left\{ \begin{array}{l} V(x,t) \to 0 \\ \qquad\qquad \text{uniformely with respect to t as } |x| \to +\infty \\ \nabla V(x,t) \to 0 \end{array} \right.$$

can be handled using the same arguments emploied in the proves of the previous theorems. We obtain the following result:

Theorem 2.4. Suppose that V satisfies the following assumptions

(V_0) $\quad V \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$

(V_1) $\quad V_{xx}(0,t) \geq (\lambda_n - \lambda_k)I$ for any $t \in \mathbb{R}$, with the strict inequality for some t.

(V_2) $\quad V(0,t)=0$ for any $t \in \mathbb{R}$ and (2.44) holds

where λ_n is the first eigenvalue strictly greater than λ_k, then (2.43) has at least a nontrivial solution. ■

APPENDIX 1.

In this appendix we are concerned with the proof of Proposition 2.9 of chapter 2. We therefore will adopt the same notations and definitions in there.

We begin with a remark which makes the definition of the class Γ_i^* more flexible.

Lemma A.1. If $U^i=\Sigma^{i-1}$, $A\in\Gamma_i^*$, $\sigma\in S_i(A)$ then $B^i\subset\sigma(A)$.

Proof. Let $\bar{x}\in B^i$ be given and fix $\varepsilon>0$. We consider the function $h_\varepsilon:\mathbb{R}^+\to[0,1]$:

$$h_\varepsilon(t):=\begin{cases} 0 & \text{if } \ 0\le t\le\varepsilon \\ \dfrac{t}{\varepsilon} & \text{if } \ \varepsilon\le t\le 2\varepsilon \\ 1 & \text{if } \ \varepsilon\le t \end{cases}$$

and we set

$$(A.2) \qquad \sigma_\varepsilon(x):=\sigma(x)-h_\varepsilon(d(x,\Sigma^{i-1}))\bar{x}$$

where $d(x,\Sigma^{i-1})$ denotes the distance of x from Σ^{i-1}.

Obviously $\sigma_\varepsilon=\mathrm{id}$ on Σ^{i-1}. Then $\sigma_\varepsilon\in S_i(A)$ and therefore a $x_\varepsilon\in A$ exists such that $\sigma_\varepsilon(x_\varepsilon)=0$,which implies $\sigma(x_\varepsilon)=h_\varepsilon(d(x,\Sigma^{i-1}))\bar{x}$.

By the definition of h_ε one has

$$(A.3) \qquad \|\sigma(x_\varepsilon)\|\le\|\bar{x}\|<1 \quad \text{for any } \varepsilon>0$$

If $d(x_\varepsilon,\Sigma^{i-1})\to 0$ as $\varepsilon\to 0$ then a suitable sequence $(\varepsilon_j)_j$ and $x_0\in A\cap\Sigma^{i-1}$ exist such that

$$(A.4) \qquad x_{\varepsilon_j}\to x_0 \ \text{ as } j\to+\infty\ .$$

By (A.4) we get $\|\sigma(x_{\varepsilon_j})\|\to\|\sigma(x_0)\|=\|x_0\|=1$ which contradicts (A.3). Thus we deduce that ε exists such that $2\varepsilon\le d(x_\varepsilon,\Sigma^{i-1})$ which implies that $h_\varepsilon(d(x_\varepsilon,\Sigma^{i-1}))=1$ and finnally $\sigma(x_\varepsilon)=\bar{x}$. ∎

Lemma A.1 allows us to prove the first part of Proposition 2.9. We have

Lemma A.5. If $A\in\Gamma^*_{n,\alpha}$, $\varepsilon\le 1$ is such that (R.8) of Chapter 2 holds and if $\bar{x}\in\varepsilon B$ then one has

(A.6) $\quad A\cap(x+W^{\perp})\in\Gamma^*_k$

Proof. Let $\sigma\in S_k(A\cap(x+W^{\perp}))$, we have that $0\in\sigma(A\cap(\bar{x}+W^{\perp}))$. By definition $\sigma:A\cap(\bar{x}+W^{\perp})\to E_k$ and we extend σ from $A\cap(\bar{x}+W^{\perp})$ on $A\cap(\bar{x}+W^{\perp})\cup(A\cap U^n_\alpha)$ to be equal to P_k on $(A\cap U^n_\alpha)\setminus(A\cap(\bar{x}+W^{\perp}))$. This

extention is continuous since $A\cap U^n_\alpha$ and $A\cap(\bar{x}+W^{\perp})$ are closed. Moreover by (R.8) of Chapter 2:

$$(A\cap U^n_\alpha)\cap(A\cap(\bar{x}+W^{\perp}))=(A\cap(\bar{x}+W^{\perp}))\cap(U^n_\alpha\cap(\bar{x}+W^{\perp}))\subset A\cap(\bar{x}+W^{\perp})\cap U^k$$

and on U^k $\sigma=P_k$ by definition. By Dugundji theorem we can consider an extension σ_1 of σ, $\sigma_1:A\to E_k$. Let us define

$$\sigma':A\to E_n \qquad \text{by } \sigma':=\sigma_1+P$$

It is easy to verify that $\sigma'\in S_n(A)$. In fact $\sigma'=P_k+P=P_n$ on $A\cap U^n_\alpha$. Since $A\in\Gamma^*_{n,\alpha}$ from Lemma A.1 we get (for our choice of U^n_α) that:

there exists $x_0\in A$ such that $\bar{x}=\sigma'(x_0)$.

This implies that

(A.7) $\quad \sigma_1(x_0)=0$ and $P(x_0)=\bar{x}$.

The second equality of (A.7) implies that $x_0\in A\cap(\bar{x}+W^{\perp})$ and therefore $\sigma(x_0)=\sigma_1(x_0)=0$. ■

Lemma A.8. If A satisfies (γ_n) and $k<n$ is fixed, then a dense subset $D\subset W$ exists such that

(A.9) $\quad$ for any $x\in D$: $A\cap(\bar{x}+W^{\perp})$ satisfes (γ_k).

Proof. Let $H=\{h\in\mathbb{N} \mid A$ is contained in the union of c_h sets of diameter less or equal to $\frac{c}{h}\}$. Let $Q_0\subset W$ be a cube with side as small as we want that we can assume to be equal to $\frac{c}{h_0}$. We can choose $h_1\in H$ in such a way that the following inequalities hold:

$$\left[\frac{h_1}{2h_0}\right]\geq\frac{h_1}{3h_0}$$

(where $[\cdot]$ denotes the integer part of a real number);

$$(3h_0)^{n-k}\frac{c_{h_1}}{h_1{}^{n+1}}<1.$$

Q_0 contains a cube Q_0' with side $2\left[\frac{h_1}{2h_0}\right]\frac{c}{h_1}<\frac{c}{h_0}$.

Divide now each side of Q_0' in $2\left[\frac{h_1}{2h_0}\right]$ parts with length $\frac{c}{h_1}$. Therefore Q_0' will be divided in $2^{n-k}\left[\frac{h_1}{2h_0}\right]^{n-k}$ small cubes. We choose $\left[\frac{h_1}{2h_0}\right]^{n-k}$ of them, which have distance greater or equal than $\frac{c}{h_1}$ from each other. By hypothesis $A\cap(Q_0'+W)\subset A$ is covered by c_{h_1} sets of diameter less than $\frac{c}{h_1}$, then no one of them can touch more than one of the small cubes which we have choosen. For this reason at least one small cube can be covered by at most

$$c_{h_1}':=\frac{c_{h_1}}{\left[\frac{h_1}{2h_0}\right]^{n-k}}\text{ sets of diameter less than }\frac{c}{h_1}.$$

By repeating the same argument we can assume that we have fixed a cube Q_n with side $\frac{c}{h_\mu}$ and choose $h_{\mu+1}\in H$ in such a way that

$$\text{(A.10)}\quad\left[\frac{h_{\mu+1}}{2h_\mu}\right]\geq\frac{h_{\mu+1}}{3h_\mu}$$

$$(A.11) \qquad (3h_\mu)^{n-k}\frac{c_{h\mu+1}}{{h_{\mu+1}}^{n+1}} < \frac{1}{\mu+1}.$$

We can now divide Q_μ in $2^{n-k}\left[\frac{h_{\mu+1}}{2h_\mu}\right]^{n-k}$ small cubes with side $\frac{c}{h_{\mu+1}}$.

Among them we can fix $\left[\frac{h_{\mu+1}}{2h_\mu}\right]^{n-k}$ with reciprocal distance greater or equal than $\frac{c}{h_{\mu+1}}$. By the hypothesis $A\cap(Q_\mu+W)$ is covered by $c_{h_{\mu+1}}$ sets of diameter less than $\frac{c}{h_{\mu+1}}$. Therefore at least one of these small cubes is covered at most by $c'_{h\mu+1} := \dfrac{c_{h\mu+1}}{\left[\frac{h_{\mu+1}}{2h_\mu}\right]^{n-k}}$ of such sets.

One has by (A.10) and (A.11)

$$\frac{c'_{h\mu+1}}{h^{k+1}_{\mu+1}} = \frac{c_{h\mu+1}}{\left[\frac{h_{\mu+1}}{2h_\mu}\right]^{n-k} h^{k+1}_{\mu+1}} \leq \frac{c_{h\mu+1}}{\left[\frac{h_{\mu+1}}{3h_\mu}\right]^{n-k} h^{k+1}_{\mu+1}} < \frac{1}{\mu+1}\ .$$

Therefore we define

$$c'_h := \begin{cases} c'_{h\mu} & \text{if there exists } \mu \text{ such that } h=h_\mu \\ 0 & \text{otherwise} \end{cases}$$

For every h of the kind h_μ for some μ $(Q_\mu+W^\perp)\cap A$ is covered by c'_h sets of diameter less than $\frac{c}{h}$. Since the Q_μ's are a sequence of closed bounded sets decreasing by inclusion a $\bar{x}\in\bigcup_\mu Q_\mu\subset W$ will exists, clearly $A\cap(\bar{x}+W^\perp)$ satisfies (γ_k). Therefore if we call D the set of the points of W which satisfy (A.9) the above argument shows that $D\cap Q_0 \neq \varnothing$ and therefore, for the arbitrarity of Q_0, D is dense in W. ■

The proof of proposition 2.9 follows from lemmas A.5 and A.8 . ■

CHAPTER 3.

Nonresonance conditions

In this chapter we are concerned with the problem of finding non-resonance conditions on the nonlinearity f in order to have solutions for problem of the type

$$(3.1)\qquad \begin{cases} \Delta u + f(x,u) = h & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

for any h. In a classical result Hammerstein [H] proved that if f satisfies a linear growth and if

$$(3.2)\qquad \limsup_{t\to+\infty} \frac{2F(x,t)}{t^2} < \lambda_1 \text{ uniformely in } x$$

(where F denotes a primitive of f) then (3.1) admits a solution. Susequently Mawhin-Ward-Willem in [M-W-W] proved that one still get an existence result if one weakens (3.2) in the following way:

there exists $\alpha \in L^{\infty}(\Omega)$ such that

$$(3.3)\qquad \limsup_{t\to+\infty} \frac{2F(x,t)}{t^2} \leq \alpha(x) \leq \lambda_1 \text{ uniformely in } x \text{ and } \alpha(x) < \lambda_1 \text{ on a}$$

set of positive measure.

In [D-G2] this condition has been furthermore weakened; in fact De Figueiredo and Gossez allow:

$$\limsup_{t\to+\infty} \frac{2F(x,t)}{t^2} = \lambda_1$$

provided that the nonlinearity f does not touch $\lambda_1 t$ "too much" (for a precise sense see later).
The same kind of problem (let say in a rough way how f'(t) can "touch" the eigenvalues) has been also studied for problems of the type "jumping nonlinearity" i.e. problems in which the nonlinearity admits different limits at plus and minus infinity (or $\limsup_{t\to-\infty} \frac{f(x,t)}{t}$ is different from $\liminf_{t\to+\infty} \frac{f(x,t)}{t}$). Problems with jumping nonlinearity have been studied by several authors, see for instance [Fu1], [So6],[A-P],(see [So5] and [Fu2] for more references), moreover the same kind of nonlinearity has been also considered for the telegraph equation (see [F-M] and a sligth improvement in [D-Lu]).

The nonresonance problem in the jumping contest has been considered in [M-W] (also for the Neumann problem). Later, analogously to the previous situation, De Figueiredo and Gossez in

[D-G1] gave a weaker condition of nonresonance. In this chapter we will see a different approach (see [Lu-M]) to the problem of nonresonance for jumping nonlinearities which leads to a distinct condition to that of [D-G1] or [D-G2]; this approach obviously, works also in the previous situation as it will subsequently be remarked.

Let us therefore consider the problem

$$(3.4)\qquad \begin{cases} \Delta u + g(u) = h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is an open, bounded, regular domain in $\mathbb{R}^n$ and h belongs to $L^p(\Omega)$. From now on we suppose that g satisfies:

(g_1) If $\alpha' := \liminf_{t\to+\infty} \frac{g(t)}{t}$, $\alpha'' := \limsup_{t\to+\infty} \frac{g(t)}{t}$

$\beta' := \liminf_{t\to-\infty} \frac{g(t)}{t}$, $\beta'' := \limsup_{t\to-\infty} \frac{g(t)}{t}$

and λ_i, λ_{i+1}, $i\geq 1$ are two consecutive distinct eigenvalues of $-\Delta$ (with zero boundary conditions) then:

$$\lambda_i \leq \alpha' \leq \alpha'' \leq \lambda_{i+1}, \qquad \lambda_i \leq \beta' \leq \beta'' \leq \lambda_{i+1}.$$

De Figueiredo and Gossez (see [D-G1]) proved that (1) has at least one solution for each $h \in L^p(\Omega)$ provided that there exists an $\eta>0$ such that

(H_1) $\liminf_{n\to+\infty} \frac{\mu(F_n)}{n} >0$ where $F_n := \left\{ t\in\,]-n,n[\mid t\neq 0,\ \frac{g(t)}{t} > \lambda_i + \eta \right\}$

and

(H_2) $\liminf_{n\to+\infty} \frac{\mu(G_n)}{n} >0$ where $G_n := \left\{ t\in\,]-n,n[\mid t\neq 0,\ \frac{g(t)}{t} < \lambda_{i+1} - \eta \right\}$

(μ denotes the Lebesgue measure).

They get the result by a degree argument and (H_1)-(H_2) are employed in proving an a priori bound for solutions of a suitable problem, in order to have the homotopy invariance of the topological degree.

In this chapter we study the existence of at least one solution for the problem (1) by means of variational methods, more exactly we look for critical points of the functional $f_h : H^1_0(\Omega) \to \mathbb{R}$ defined by

$$f_h(u):=\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega (G(u) - hu)$$

where $G(u):= \int_0^u g(t)dt$, and we apply the Rabinowitz saddle-point theorem [R4]. In order to prove the Palais-Smale condition (see chapter 1) and to study the behaviour of the functional on the subspaces of the saddle point theorem we replace conditions (H_1)-(H_2) by the following hypothesis:

(g_2) g is a Lipschitzian function with Lipschitz constant M_1

(g_3) when i=1 : if $\alpha'=\lambda_1$ then $\lambda_1<\alpha''$; if $\beta'= \lambda_1$ then $\lambda_1<\beta''$.

when i>1 : if $\alpha'=\beta'=\lambda_i$ then or $\lambda_i<\alpha''$ or $\lambda_i<\beta''$.

if $\alpha''=\beta''=\lambda_{i+1}$ then or $\alpha'<\lambda_{i+1}$ or $\beta'< \lambda_{i+1}$

From (g_1)-(g_2) and (g_3) in the case i>1, $\alpha'=\beta'=\lambda_i$, $\lambda_i<\alpha''$ it is clear that for each $\eta>0$ sufficiently small there exist two sequences $(a_j(\eta))_j$, $(b_j(\eta))_j$ in $\mathbb{R}$ such that

a). $a_j \to +\infty$, $b_j \to +\infty$

b). $0 < a_1(\eta) < b_1(\eta) < a_2(\eta) < \ldots < a_n(\eta) < b_n(\eta) < \ldots$

c). $\frac{g(t)}{t} \leq \lambda_i + \eta$ if and only if t belongs to $\bigcup_j [a_j(\eta), b_j(\eta)]$

Remark 3.5. Obviously in the other cases considered in (g_3) one has analogous sequences. For instance in the case i>1, $\alpha''=\beta''=\lambda_{i+1}$, $\beta'<\lambda_{i+1}$ for each $\eta>0$ sufficiently small there exist two sequences $(\alpha_j(\eta))_j$, $(\beta_j(\eta))_j$ such that

a). $\alpha_j \to -\infty$, $\beta_j \to -\infty$

b). $0 > \beta_1(\eta) > \alpha_1(\eta) > \ldots > \alpha_{n-1}(\eta) > \beta_n(\eta) > \ldots$

c'). $\frac{g(t)}{t} \geq \lambda_{i+1} - \eta$ if and only if t belongs to $\bigcup_j [\alpha_j(\eta), \beta_j(\eta)]$

In the case i>1, $\alpha'=\beta'=\lambda_i$, $\lambda_i<\alpha''$ we ask the further condition :

(g_4) There exists an $\eta>0$ such that $\frac{a_j(\eta)}{b_j(\eta)} \to c \neq 0$ when $j \to +\infty$.

For the other sequences constructed by the other cases considered in (g_3) we have to ask the correspondig conditions (g_4).

Remark 3.6. We explicetely remark that condition (g_4) is weaker than conditions (H_1)-(H_2) of De Figueiredo-Gossez. In fact it is possible to exhibit a function which satisfies (g_4) and does not satisfies, for example, (H_1). Define for instance :

$$h(t):=\begin{cases} -1 & \text{if } t\in I_1(k) \\ \sin\log(t+e_1e^{2\pi}-(k^2+1)) & \text{if } t\in I_2(k) \end{cases}$$

where $I_1(k):=[k^2,(k^2+1)-e_1(e^{2\pi}-1)]$, $I_2(k):=[(k+1)^2-e_1(e^{2\pi}-1),(k+1)^2)]$, $k\in\mathbb{N}$, $e_1:=e^{3\pi/2}$ and consider $g(t):=\lambda_i th(t)$. It is easy to show that $g(t)$ satisfies (g_4) but not (H_1). On the other hand it is possible to prove that conditions (g_1)-(g_4) imply that the sets F_n and G_n have positive density at infinity (i.e. they satisfy (H_1)-(H_2)), therefore our result provides a different proof of [D-G1] and [D-G2]) for nonlinearities g which have a more regular behaviour.

We obtain the following result

Theorem 3.1. Let us assume (g_1)-(g_2)-(g_3) and (g_4). Then for each $h\in L^p(\Omega)$ problem (1) admits at least one solution.

Remark 3.7. We explicitly note that all our considerations in the following are independent on the particular choice of h, and therefore from the existence of critical points of f_h, for a given h, we get the existence of solutions of (1) for any h.

We denote by $H:=H_0^1(\Omega)$ and by $\|\cdot\|$ its norm. Let H^- be the subspace of H spanned by all the eigenvectors corresponding to the eigenvalues $\lambda_1,\dots,\lambda_i$ and let H^+ be the one associated to the eigenvalues greater or equal to λ_{i+1}. It is obvious that $H=H^-\oplus H^+$ and that $\dim(H^-)<+\infty$, while $\dim(H^+)=+\infty$. All what we have to prove, in order to apply Rabinowitz theorem, is that

a. $\inf_{\|u\|\to+\infty} f_h(u) = -\infty$ for $u\in H^-$

b. $\inf_{u\in H^+} f_h(u) > -\infty$

c. f_h satisfies [P-S],

and this will be done by successive steps.

Step 1: proof of a.

Let $(u_k)_k$ be a sequence in H^- such that $\|u_k\| \to +\infty$, write $\tau_k:=\|u_k\|$ and $v_k:=\dfrac{u_k}{\tau_k}$. Then there exists an $\bar{u}\in H^-$ such that, for a subsequence, $v_k\to \bar{u}$ strongly in H, strongly in $L^2(\Omega)$ and a.e. in Ω, but $\dim(H^-) < +\infty$ implies $v_k\to \bar{u}$ in $C^2(\Omega)$, $\bar{u}\not\equiv 0$ and $\|\bar{u}\| =1$.

Let $\Omega^-:=\{ x\in\Omega \mid \bar{u}(x)<0\}$, $\Omega^0:= \{ x\in\Omega \mid \bar{u}(x)=0\}$, $\Omega^+ :=\{ x\in\Omega \mid \bar{u}(x)>0\}$.

Lemma 3.8. It is possible to define a function m such that: $\dfrac{g(u_k)}{\|u_k\|} \to m\bar{u}$ weakly in $L^2(\Omega)$, moreover if g is a Lipschitzian function then $\dfrac{g(u_k)}{\|u_k\|} \to m\bar{u}$ strongly in $L^2(\Omega)$ and a.e. in Ω.

Furthermore the function m can be chosen in such a way that:

$$(3.9)\qquad \begin{array}{l} \alpha' \le m(x) \le \alpha'', \text{ for all } x\in\Omega^+ \\ \beta' \le m(x) \le \beta'', \text{ for all } x\in\Omega^- \\ m(x)=0, \text{ for all } x\in\Omega^0 \end{array}$$

■

To prove a. it is sufficent to show that f is bounded on bounded sets and that

$$(3.10)\qquad \frac{f'_h(u)[u]}{\|u\|^2} \to -\infty \text{ when } u\in H^- \text{ and } \|u\| \to +\infty$$

It will be therefore enough to prove that $\limsup_{\|u\|\to+\infty} \dfrac{f'_h(u)[u]}{\|u\|^2} < 0$ for $u\in H^-$. Let $(u_k)_k$ be a sequence such that $\|u_k\| \to +\infty$, by lemma 3.7

$$\frac{f'_h(u_k)[u_k]}{\|u_k\|^2} \to 1 - \int_\Omega m\,\bar{u}^2 \text{ when } k \to +\infty;$$

by the Poincaré inequality one has: $1 - \int_\Omega m\bar{u}^2 \le \int_\Omega (\lambda_i - m)\bar{u}^2 \le 0;$ hence the only thing to be proved is that the strict inequality holds i.e. $m(x) \not\equiv \lambda_i$. Let us assume by contradiction that $m(x) \equiv \lambda_i$, then it follows that $\bar{u}$ belongs to the eigenspace spanned by the eigenvectors associated to λ_i: hence if $i>1$, $\bar{u}$ changes sign, then by (3.9) we have $\alpha'=\beta'=\lambda_i$, thus by (g_3) or $\alpha''>\lambda_i$ or $\beta''>\lambda_i$; on the other hand if $i=1$ by (3.9) if $\bar{u}>0$, $\alpha'=\lambda_1$ thus by (g_3) $\alpha''>\lambda_1$ while $\bar{u}<0$ implies $\beta'=\lambda_1$ thus by (g_3) $\beta''>\lambda_1$. We consider, for the sake of semplicity, only the case $i>1$, $\alpha'=\beta'=\lambda_i$ and $\alpha''>\lambda_i$ and we will utilize the sequences $(a_j(\eta))_j$, $(b_j(\eta))_j$ already defined.

Let $\varepsilon>0$ be fixed; define $\Omega^+_\varepsilon := \{ x \in \Omega \mid \bar{u}(x) > \varepsilon \}$.

<u>*Lemma 3.11.*</u> If (g_3) holds and $m(x) \equiv \lambda_i$ then:

$$(3.12) \qquad \frac{g(\tau_k \bar{u}(x))}{\tau_k \bar{u}} \to \lambda_i \text{ a.e. } x \in \Omega^+_\varepsilon$$

<u>Proof</u>. Since $m(x) \equiv \lambda_i$, by lemma 3.9 $\dfrac{g(u_k)}{u_k} \to \lambda_i$ a.e. $x \in \Omega^+_\varepsilon$, hence:

$$\frac{g(\tau_k v_k)}{\tau_k \bar{u}} = \frac{g(\tau_k v_k)}{\tau_k v_k}\,\frac{v_k}{\bar{u}} = \frac{g(u_k)}{u_k}\,\frac{v_k}{\bar{u}} \to \lambda_i \text{ a.e. } x \in \Omega^+_\varepsilon;$$ on the other hand,

on Ω^+_ε, $\dfrac{g(\tau_k \bar{u}(x))}{\tau_k \bar{u}}$ behaves as $\dfrac{g(\tau_k v_k)}{\tau_k \bar{u}}$ when $k \to +\infty$ because, by (g_2):

$$\left| \frac{g(\tau_k \bar{u}) - g(\tau_k v_k)}{\tau_k} \right| \le M_1 \,|\bar{u} - v_k|$$

and when $k \to +\infty$, $|\bar{u} - v_k| \to 0$ in $C^0(\Omega)$. ∎

Remark 3.13. The lemma holds if $|\bar{u} - v_k| \to 0$ in $C^0(\Omega)$ and this is immediate if one works in a finite dimensional space.

Remark 3.14. Let $A^+ := \{ x \in \Omega_\varepsilon^+ \mid \frac{g(\tau_k \bar{u}(x))}{\tau_k \bar{u}} \not\to \lambda_i \}$ then $\mu(A)=0$.

Define $\Pi := \{ p \in [\varepsilon, M] \mid \frac{g(\tau_k p)}{\tau_k p} \not\to \lambda_i \}$, where $M := \max_\Omega \bar{u}$.

Lemma 3.15. $\Pi = \varnothing$.

Proof. Claim 1. The interior of Π is empty. Infact take any open interval $I \subset \Pi$, let $p \in I$. Then for each $x \in \Omega_\varepsilon^+$ such that $\bar{u}(x)=p$ one has that $\frac{g(\tau_k \bar{u})}{\tau_k \bar{u}} \not\to \lambda_i$; therefore $\bar{u}^{-1}(I) \subset A$. By remark 3.14 this implies $\mu(\bar{u}^{-1}(I))=0$, but $\bar{u}^{-1}(I)$ is an open set, therefore $\bar{u}^{-1}(I)=\varnothing$, that is $I=\varnothing$.

Claim 2. Π is an open set. Infact by (g_1) there exist constants $a>0$ and $b>0$ such that $|g(t)| \le a+b|t|$ for all $t \in \mathbb{R}$, therefore for any sequence $\tau_k \to +\infty$ and for any $p \in \mathbb{R}$, $\frac{g(\tau_k p)}{\tau_k p}$ is a bounded set. Hence for any element $\bar{p} \in \Pi$, there will exists a $\bar{\lambda} \neq \lambda_i$ such that, for a subsequence, $\frac{g(\tau_k \bar{p})}{\tau_k p}$ converges to $\bar{\lambda}$. Let $\delta = |\bar{\lambda} - \lambda_i|$. As before there exists a $\bar{\lambda}_p$ such that, for a subsequence $\frac{g(\tau_k p)}{\tau_k p} \to \bar{\lambda}_p$ if p is in a suitable neighboroud of $\bar{p}$. Following the ideas of lemma 3.11 it is easy to show $\bar{\lambda}_p \neq \lambda_i$, therefore $p \in \Pi$, and this proves claim 2. From claim 1 and claim 2 the lemma follows. ∎

Corollary 3.16. For any $p \in [\varepsilon, M]$ $\frac{g(\tau_k p)}{\tau_k p} \to \lambda_i$.

By means of corollary 3.15 we see that for each $p \in [\varepsilon, M]$ there exists a $k(p)$ such that for each $k > k(p)$: $\frac{g(\tau_k p)}{\tau_k p} < \lambda_i + \eta$ i.e. $\tau_k p \in \bigcup_j [a_j, b_j]$, in particular we have that there exists $(j_k)_k$ such that $\tau_k M \in [a_{j_k}, b_{j_k}]$ for $k > k(M)$.

Lemma 3.17. $k_0(\varepsilon):=\sup_{[\varepsilon,M]} k(p)<+\infty$.

Proof. Suppose by contradiction that for each $N>0$ there exists a $p_N\in[\varepsilon,M]$ such that $k(p_N)>N$, therefore $\tau_N p_N\in \mathbb{R}\setminus\bigcup_j [a_j,b_j]$. On the other hand, for a subsequence, $p_N\to p_0\in[\varepsilon,M]$. Following the ideas of lemma 3.12 it is easy to show that

$$\frac{g(\tau_N p_0)}{\tau_N p_0}\geq\lambda_i+\eta \text{ for each } N\in\mathbb{R} \text{ suitable great}$$

this implies $p_0\in\Pi$ but this is impossible by lemma 3.15. ∎

By corollary 3.16 and lemma 3.17 for each $k>k_0$ $\tau_k\cdot[\varepsilon,M]$ must be contained in the interval $[a_{j_k},b_{j_k}]$, in fact one has $\tau_k\cdot[\varepsilon,M]\subset\bigcup_j[a_j,b_j]$, $\tau_k M\in[a_{j_k},b_{j_k}]$ and the intervals $[a_j,b_j]$ are disjoint. This means that $\frac{a_{j_k}}{\tau_k}\leq\varepsilon<M\leq\frac{b_{j_k}}{\tau_k}$ which is impossible by (g_4): a contradiction. By its definition f_h is bounded on bounded sets. The other cases in (g_3) are treated in the same way. ∎

Step 2: proof of b.

As before it is sufficient to prove that $\liminf_{\|u\|\to+\infty}\frac{f'_h(u)[u]}{\|u\|^2}>0$ for $u\in H^+$.

Let $(u_k)_k$ be a sequence in H^+ such that $\|u_k\|\to+\infty$. By lemma 3.8

$$\frac{f'_h(u_k)[u_k]}{\|u_k\|^2}\to 1-\int_\Omega m\,\bar{u}^2 \text{ when } k\to+\infty;$$

Let us show that $1-\int_\Omega m\,\bar{u}^2>0$. This is obvious if $\bar{u}\equiv 0$. Suppose $\bar{u}\not\equiv 0$.

Since $\bar{u}\in H^+$ we have:

$$\lambda_{i+1}\int_\Omega \bar{u}^2\leq\|\bar{u}\|\leq 1. \tag{3.18}$$

If one of the inequalities in (3.17) is strict, we have done. Therefore it remains to consider the case in which

$$\lambda_{i+1}\int_\Omega \bar{u}^2=\|\bar{u}\|=1, \tag{3.19}$$

that is when $\bar{u}=v+z$ where v corresponds to the eigenspace corresponding to λ_{i+1} and z belongs the one corresponding to eigenvalues greater of λ_{i+1}. From their definitions v and z are orthogonal in $L^2(\Omega)$ and in $H_0^1(\Omega)$ therefore from (3.19) it is immediate that $z\equiv 0$. In such a way we reduce to the finite dimensional space corresponding to λ_{i+1} and, as in step 1, we can get a contradiction if $m(x)\equiv\lambda_{i+1}$. ■

Remark 3.20. The same ideas of the proof of step b provide a different proof of [D-G2]

Step 3: proof of [P-S].

Let $(u_n)_n$ be a Palais-Smale sequence (i.e. such that $|f_h(u_n)|\leq c$ and $\nabla f_h(u_n)\to 0$). By classical results it is sufficient to show that $(u_n)_n$ is bounded in H. Let us suppose by contradiction that $\|u_n\|\to+\infty$, let $v_n:=\frac{u_n}{\|u_n\|}$. Therefore there exists a $v\in H$ such that, for a subsequence, $v_n\to v$ weakly in H, strongly in $L^2(\Omega)$ and a. e. in Ω. By lemma 3.8 and by standard arguments on elliptic equations from the fact that $\nabla f_h(u_n)\to 0$, we get that $v_n\to v$ strongly in H and that v satisfies

$$\Delta v + m(x)v=0. \tag{3.21}$$

Since $v\not\equiv 0$ by a result equivalent to the unique continuation property we have that $m(x)\equiv\lambda_i$ or $m(x)\equiv\lambda_{i+1}$. From now on, denote by E the space spanned by eigenvectors corresponding to $\lambda_1,..,\lambda_{i-1}$ and by F the one corresponding to λ_i. Let P,Q and P^+ be the orthogonal projections onto E, F and H^+ respectively. Let us suppose $m(x)\equiv\lambda_i$; this implies that v belongs to the space F. Since $v_n\to v\in F$ we have that :

$$Pv_n\to 0 \quad \text{and} \quad P^+v_n\to 0 \tag{3.22}$$

therefore, obviously:

$$\frac{\|Q(u_n)\|^2}{\|P(u_n)\|^2+\|Q(u_n)\|^2+\|P^+(u_n)\|^2}\to 1 \tag{3.23}$$

By (3.22) one has $\frac{Q(u_n)}{\|u_n\|} \to v$ and this implies $w_n := \frac{Q(u_n)}{\tau_n} \to v$ where $\tau_n := \|Q(u_n)\|$; since g is Lipschitz and by (3.23) one has $\frac{g(\tau_n w_n)}{\tau_n} \to \lambda_i v$ weakly in H, strongly in $L^2(\Omega)$ and a.e. in Ω. We are now in a finite dimensional space, therefore like in step 1 we get the contradiction. The case $m(x) \equiv \lambda_{i+1}$ can be treated as in step 2. ■

Remark 3.24. Conditions a. and b. are in fact sufficient to show that it is possible to substitute to the usual min-max class of Rabinowitz, in the saddle point theorem, the class Γ_n^* of min-max sets defined by Lazer-Solimini in [L-S], for the definitions and the abstract results on Γ_n^* see chapter 2. Utilizing their characterization of Morse index of critical point obteined with Γ_n^* it is possible to get a multiplicity result for our problem when $h \equiv 0$.

Therefore we get the following multiplicity result:

Theorem 3.2. Let $h \equiv 0$, $g \in C^1(\Omega)$, $g(0)=0$. Suppose that g satisfies $(g_1),(g_2),(g_3),(g_4)$ and $g'(0)<\lambda_i$, $g'(0) \notin \sigma(-\Delta)$. Then there exists a nontrivial solution of (3.4).

Proof. It follows immediately by Proposition 2.14 A of chapter 2, taking into account the fact that the augmented Morse index of zero is strictly less of n by the hypotheses $g'(0)<\lambda_i$ and $g'(0) \notin \sigma(-\Delta)$. ■

Remark 3.25. The previous result is of the Amann-Zehnder type [A-Z] with the difference that we are not asking that there exist $\varepsilon>0$ and $\rho>0$ such that

$$\lambda_i + \varepsilon \leq \frac{f(\xi)}{\xi} \leq \lambda_{i+1} - \varepsilon \text{ for each } |\xi|>\rho$$

Remark 3.26. It is clear that $(-\Delta)$ can be substituted in problem (3.4) by any uniformely elliptic, simmetric operator $L := \sum_{|\alpha|,|\beta| \leq m} (-1)^{\beta} D^{\beta}(a_{\alpha\beta}(x) D^{\alpha} u)$ whose coefficent are in $C^{|\alpha|+|\beta|}(\Omega)$. No hypothesis are done on the sign of the eigenfunctions associated to λ_i, λ_{i+1}.

CHAPTER 4.

A bifurcation result for a problem with a discontinuous nonlinearity

Problems with discontinuous nonlinearities have been studied by several authors both in a general setting (e.g. [A-B], [A-T], [Ce], [Ch], [Ku], [St]) and related to applications (e.g. [A-T], [Ci], [F-B]). In particular in this chapter (see [Lu]) we focus on the following problem, discussed in [A-T] and motivated by a problem arising in plasma physics (cfr. §4 [A-T], [Ci]) :

$$(1) \quad \begin{cases} -\Delta u = h(u-a)\cdot p(u) & \text{in} \quad \Omega \\ \quad u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where Ω is an open bounded regular domain in $\mathbb{R}^n$; h is the Heaviside's function (i.e. $h(s-a)=0$ if $s\leq a$, $h(s-a)=1$ if $s>a$), $a\in\mathbb{R}$ and p(s) satisfies:

(p_0) $p\in C^0(\mathbb{R})$, p is a nondecreasing positive function,

(p_1) $p(s) \leq \alpha s + c$ with $\alpha < \lambda_1$ and c given constants,

where $\lambda_1 = \lambda_1(\Omega)$ is the lowest eigenvalue of $-\Delta$ in $H_0^1(\Omega)\cap H^2(\Omega)$.

Let us denote, by simplicity, $f_a(s):=h(s-a)p(s)$. In [A-T] it has been shown, beside other results, that there exists an a^* such that problem (1) admits at least two nontrivial solutions for any $0<a<a^*$. The proof was carried out by means of variational methods, utilizing an idea due to [A-B] for which the problems tied to the discontinuity of the function f_a (i.e. the nonregularity of the functional associated to problem (1)) can be overcome by using the Clarke's dual action principle [Cl].

In this chapter we study problem (1) from a different point of view, considering it as a bifurcation problem for "multivalued mappings". More precisely we introduce the multivalued function $F_a(s):\mathbb{R} \to 2^{\mathbb{R}}$ defined as

$$F_a(s) := \begin{cases} 0 & \text{if} \quad s<a \\ T_a & \text{if} \quad s=a \\ p(s) & \text{if} \quad s>a \end{cases}$$

where T_a denotes the closed interval $[0,b(a)]$, with $b(a):=p(a)$ denoting the "jump" of the function f_a at the point s=a.

Denote by $\mathcal{F}_a$ the Nemitskji operator associated to F_a, i.e. $\mathcal{F}_a: L^2(\Omega) \to 2^{L^2(\Omega)}$ is defined as follows:

$$\text{for each } u \in L^2(\Omega) \ \mathcal{F}_a(u) := \{ v \in L^2(\Omega) \mid v(x) \in F_a(u(x)) \text{ a. e. } x \in \Omega \};$$

furthermore denote by $K := (-\Delta)^{-1}$, $K: L^2(\Omega) \to H_0^1(\Omega) \cap H^2(\Omega)$ and recall that K is compact from $L^2(\Omega)$ into $L^2(\Omega)$.

We consider the problem

(2) $\qquad 0 \in u - K\mathcal{F}_a(u).$

We say that $u \in H_0^1(\Omega)$ is a solution of (2) if and only if there exists a $v \in L^2(\Omega)$ such that $-\Delta u = v$ in Ω and for a.e. $x \in \Omega$ one has $v(x) \in F_a(u(x))$. In this sense we say that a solution of (2) is a solution of the problem

$$(3) \qquad \begin{cases} -\Delta u \in \mathcal{F}_a(u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} .$$

In the following we consider (2) as a bifurcation problem with bifucation parameter a.

Remark 1.1. Note that if $u \geq 0$ and $a < 0$ then $\mathcal{F}_a(u) := \{p(u(x))\}$, therefore in this case if u is a solution of (2) with $a<0$, one has that u satisfies, actually, $-\Delta u = p(u)$, i.e. it is a solution of (1).

It is obvious that for any $a \geq 0$, $u \equiv 0$ is a solution of our problem. We show that from $(0,0) \in \mathbb{R} \times L^2(\Omega)$ a global branch of nontrivial solutions of (2) bifurcates. To this aim we apply a modification of the Rabinowitz's global bifurcation theorem [R], that has already been used in [A-H]. We recall that in [R], [A-H] the principal tool in the proof of the global bifurcation theorem is the Leray-Schauder topological degree; in this case it will be substituted by the multivalued degree defined in [C-L].

Let $\mathcal{S} := \{(a,u) \in \mathbb{R} \times L^2(\Omega) \mid a \geq 0, u \text{ is a solution of (2)}\}$. Our main result is:

Theorem A. Let $\alpha < \lambda_1(\Omega)$. $\mathcal{S}$ contains at least a connected component $\mathcal{C}$ such that:

1. $(0,0) \in \mathcal{C}$
2. There exist two constants $\beta>0$ and $R>0$ such that $\mathcal{C} \subset [0,\beta] \times B_R$, where $B_R := \{ u \in L^2(\Omega) \mid \| u \| < R \}$
3. There exists $\tilde{a}<\beta$ such that, for each $0<a<\tilde{a}$, $\mathcal{C}_a := \{u \in L^2(\Omega) \mid (a,u) \in \mathcal{C}\}$ contains at least two points.

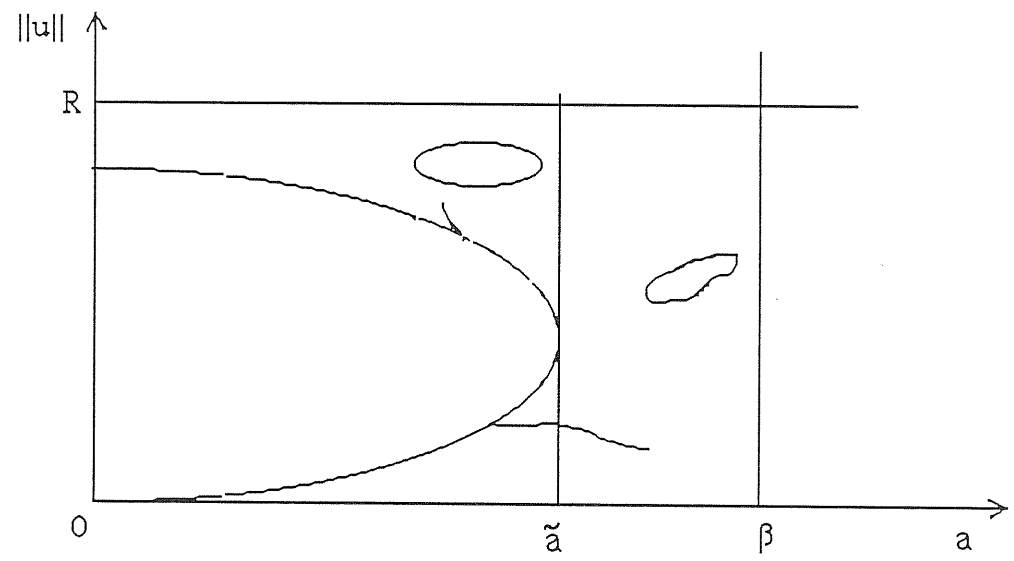

Fig. 1

§ 1. Notation and basic definitions on multivalued maps.

Suppose that X is a Banach space, let $\| \cdot \|$ denotes its norm, 2^X the class of all the subsets of X; a multivalued function T from X into X can be thought as a map $T: X \to 2^X$. Define the graph of T as:

$$grT := \{(x,y) \mid x \in X,\ y \in T(x)\}$$

We have the following:

Definition 1.1. Let grT_1, grT_2 be the graphs of two multivalued functions T_1, T_2. We call separation between T_1, T_2 the number:

$$d^*(T_1, T_2) := \sup_{z \in grT_1} d(z, grT_2)$$

where d is the metric defined as $d((x,y),(u,v)) := \max(\|x-u\|, \|y-v\|)$.

Moreover we recall:

Definition 1.2. $T : X \rightarrow 2^X$ is upper semi continuous (u.s.c.) at $x \in X$ if $T(x) \not\equiv \varnothing$ and if, given $\varepsilon>0$, there exists a $\delta>0$ such that $T(B(x,\delta)) \subset N_\varepsilon(T(x))$, where $B(x,\delta)$ is the open ball about x of radius δ and N_ε is an ε–neighbourhood of $T(x)$. T is u.s.c. in X if it is u.s.c. at each point $x \in X$.

It is clear from the definition that $F_a : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is u.s.c. . For known results (see [A-C]), for any $u \in L^2(\Omega)$ there exists a measurable selection $\xi : x \rightarrow F_a(u(x))$. Furthermore by the definition of $F_a(s)$ and because $\mu(\Omega) < +\infty$, it is easy to show that $\xi \in L^2(\Omega)$. Therefore $\mathcal{F}_a(u) \neq \varnothing$ for any $u \in L^2(\Omega)$.

Definition 1.3. $T: X \rightarrow 2^X$ is said to be compact if it maps bounded sets into relative compact sets.

Let CK(X) be the set of all closed convex subsets of X; let A be a subset of X and denote by co(A) its convex hull. Given a sequence of multivalued (singlevalued) maps T_n , we say that T_n converges to T, $T_n \rightarrow\rightarrow T$, if $d^*(T_n,T) \rightarrow 0$. It is known ([C1], [L] §7 theorem 3.3) that given any multivalued function $T: X \rightarrow CK(X)$ there exists a sequence of continuous singlevalued functions T_n (which can be also taken to be compact if T is compact) such that $T_n \rightarrow\rightarrow T$, i. e. T_n approximate T; moreover the range of T_n, $R(T_n)$, is such that $R(T_n) \subset co(R(T))$. Using this approximation result, in [C-L] a topological degree for a multivalued function of the type T := id-G, with G compact, has been defined. More precisely let D be an open bounded subset of X, and $T: \bar{D} \rightarrow CK(X)$ be a multivalued function such that T = id-G, G compact; let G_n be a sequence of compact singlevalued functions defined in $\bar{D}$, whose range is contained in $co(G(\bar{D}))$, such that $G_n \rightarrow\rightarrow G$. Let $T_n :=$ id - G_n.

Definition 1.4. Let $p \in D$ such that $p \notin T(\partial D)$. Then define the degree:

$$d(T,D,p) := \lim_{n \to +\infty} \deg(T_n,D,p)$$

where $\deg(T_n,D,p)$ denotes the Leray-Schauder degree.

Remark 1.5. This definition is meaningful because: i) for n large $p \notin T(\partial D)$; ii) the definition is independent on the particular choice of the approximating functions G_n; iii) for n large $\deg(T_n,D,p)$ is constant.

It is possible to show that all the usual properties of a degree hold, i.e. invariance by homotopy, additivity, excision and the solution property.

2. Preliminary results and proof of main theorem.

It will be convenient to introduce some notation; from now on let $X:=L^2(\Omega)$ and for any $a \in \mathbb{R}$, for any $u \in X$ define the sets:

$\Omega_a^-(u):=\{x \in \Omega \mid u(x)<a\}$, $\Omega_a(u):=\{x \in \Omega \mid u(x)=a\}$ and $\Omega_a^+(u):=\{x \in \Omega \mid u(x)>a\}$.

From the definitions it is clear that for any $a \in \mathbb{R}$ for any $u \in X$, for any $v \in \mathcal{F}_a(u)$ $v \equiv 0$ on $\Omega_a^-(u)$, $v(x) \in T_a$ for $x \in \Omega_a(u)$ and $v(x)=p(u(x))$ for any $x \in \Omega_a^+(u)$.

Lemma 2.1. For any $a \in \mathbb{R}$, $\mathcal{F}_a$ and $K\mathcal{F}_a$ are u.s.c. multivalued functions.

Proof. The u.s.c. of $\mathcal{F}_a$ follows from the upper semicontinuity result ([C2]) for the Nemitskji operator associated to any u.s.c. $F:\mathbb{R}^m \rightarrow 2^{\mathbb{R}^m}$ such that there exist a>0, b>0 such that $|F(x)| \leq a+b|x|$. For the reader convenience we give the proof in appendix 2. The second part of the assertion follows from the continuity of K: $X \rightarrow X$. ∎

Lemma 2.2. For any $a \in \mathbb{R}$, $K\mathcal{F}_a$ is compact.

Proof. $\mathcal{F}_a$ sends bounded sets of X into bounded sets: in fact let B a bounded set of X and let $M_1>0$ be a constant such that for each $u \in B$

$\|u\|\leq M_1$. Consider $\mathcal{B} := \bigcup_{u\in B} \mathcal{F}_a(u)$ and let $v\in \mathcal{B}$. Then, clearly, there exists an $u\in B$ such that $v\in \mathcal{F}_a(u)$. It is immediate to estimate that $\|v\|\leq M_2$, where M_2 is a constant depending only on M_1, $|\Omega|$, T_a, c and α. Finally K is a compact operator from X into X, therefore it sends bounded sets into precompact sets. ■

Lemma 2.3. For any $a\in \mathbb{R}$, for each $u\in X$, $K\mathcal{F}_a(u)$ is closed and convex.

Proof. First of all for each $u\in X$ $\mathcal{F}_a(u)$ is closed and convex. In fact let $v_n\in \mathcal{F}_a(u)$ such that $v_n\to v^*$ in X, therefore, for a subsequence, $v_n\to v^*$ a.e. $x\in \Omega$. By definition F_a has closed values, hence $v^*(x)\in F_a(u(x))$ for a.e. $x\in \Omega$, i.e. $v^*\in \mathcal{F}_a(u)$. Let $v_1,v_2\in \mathcal{F}_a(u)$ and consider $tv_1+(1-t)v_2$; for a.e. $x\in\Omega$ $tv_1(x)+(1-t)v_2(x) \in F_a(u(x))$ since for any $x\in\Omega$ $F_a(u(x))$ is convex; this implies that $\mathcal{F}_a(u)$ is convex. The convexity of $K\mathcal{F}_a(u)$ is immediate. Let now consider $v_n\in K\mathcal{F}_a(u)$ such that $v_n\to v^*$ in X. There exists $w_n\in \mathcal{F}_a(u)$ such that $v_n=Kw_n$ for any n, moreover by lemma 2.2 $\mathcal{F}_a(u)$ is a bounded set, therefore $w_n\to w^*$ weakly in X . On the other hand, by the first part of this proof, $\mathcal{F}_a(u)$ is weakly closed, hence $w^*\in \mathcal{F}_a(u)$. Finally $Kw_n\to Kw^*$ strongly in X because K is compact, hence, by the uniqueness of the limit, $v^*=Kw^*$. ■

Consider the multivalued function map $\Phi:\mathbb{R}\times X\to 2^X$ defined by $\Phi(a,u) := u-K\mathcal{F}_a(u)$. By the previous results we can define for any open set $D\subset X$ the degree of Φ with respect to any point $p\notin(\partial D)$. Before proving our result we need some preliminary remarks and lemmas.

Remark 2.4. By definition $F_a(s)\geq 0$ for each $s\in \mathbb{R}$ therefore the maximum principle implies that any solution of (2) (or (3)) is such that $u\geq 0$ in Ω (see [G-T]).

Remark 2.5 If $a<0$, then for each $u\geq 0$, $u\in X$, $K\mathcal{F}_a(u)$ is a singlevalued operator (i.e. solutions of (2) are indeed solutions of (1)); moreover if $a<0$ $u\equiv 0$ is not a solution of problem (1) because $f_a(0)>0$.

Lemma 2.6. For each $0<A<B$ there exists $\delta>0$ such that for each $a\in[-B,-A]$ one gets: $d(\Phi(a,\cdot),B_\delta,0) = 0$.

Proof. It sufficies to prove that $0\notin \Phi(a,u)$ for each $\|u\|<\delta$. By contradiction let us suppose that there exist $0<A<B$ such that for each $n\in \mathbb{N}$ there exist an $a_n\in[-B,-A]$ and an $u_n\in X$, $u_n\geq 0$ such that $\|u_n\|\leq 1/n$ and $u_n-K\mathcal{F}_{a_n}(u_n)=0$ (recalling remarks 1, 2.4 and 2.5). Hence $u_n\to 0$ weakly in X, but K is a compact operator, hence $u\to 0$ strongly in X, moreover $a_n\to b$, $b\in[-B,-A]$ therefore one has $0= K\mathcal{F}_b(0)$ which is impossible by remark 2.5. ■

Remark 2.7. As in [A-T] it is possible to show that there exists a $\beta>0$ such that for any $a>\beta$ problem (2) has no solution. In fact for $\alpha<\lambda_1$ take $\beta := \dfrac{\alpha}{\lambda_1-\alpha}$. It is clear that for $a>\beta$ $f_a(s)<\lambda_1 s$, therefore taking the inner product of each side of (3) with the eigenfunction associated to λ_1 one gets a contradiction.

Lemma 2.8.. Let $\alpha<\lambda_1$. Then there exists an $R>0$ such that for each $a\in[0,\beta]$, for each solution u of (3) one has $\|u\| < R$.

Proof. First of all, remark that for each $a\in \mathbb{R}$, for each $u\in X$, for each $v\in \mathcal{F}_a(u)$ one has $v(x) \leq \alpha u(x)+c$ for a.e. $x\in \Omega$, from (3) one has $-\Delta u \leq \alpha u(x) + c$. Therefore taking the inner product with u one gets

$$\int_\Omega |\nabla u|^2 \leq \int_\Omega u^2 + c \int_\Omega u$$

and from the Poincare' inequality one has $(\lambda_1-\alpha)\int_\Omega u^2 \leq c\int_\Omega u$ and this gives the thesis. ■

Lemma 2.9. Let $\Lambda:=[\alpha_1,\beta]$ with $\alpha_1>0$. Then there exists $\delta > 0$ such that for each $u\in X$ $0<\|u\|\leq\delta$ and for each $a\in\Lambda$: $0 \notin u - K\mathcal{F}_a(u)$.

Proof. Let us suppose by contradiction that there exist $u_n\in X$ and $a_n\in\Lambda$ such that $0<\|u_n\|\leq 1/n$ and $0 \in u_n - K\mathcal{F}_{a_n}(u_n)$, i.e.:

(5) $\quad -\Delta u_n \in \mathcal{F}_{a_n}(u_n)$.

It is clear that for $n \to +\infty$, $u_n \to 0$ in X and $a_n \to \bar{a}$. Let us consider any $w_n \in \mathcal{F}_a(u_n)$. We now show that:

(6) $\quad w_n \to 0$ in X.

In fact for a.e. $x \in \Omega'$, (where $\Omega' := \Omega^-_{a_n}(u_n) \cup \Omega^+_{a_n}(u_n)$ by (p_1) $0 \le w_n(x) \le \alpha u_n(x) + c$, and from this it is easy to show that $\int_{\Omega'} |w_n|^2 \to 0$; on the other hand for a.e. $x \in \Omega \setminus \Omega'$ $w_n(x) \le b(a_n) < +\infty$, hence $\int_{\Omega \setminus \Omega'} |w_n|^2 \le b(a_n)^2 |\Omega_{a_n}|$. Since $a_n \to \bar{a}$ $b(a_n)$ is bounded uniformely with respect to n and moreover $|\Omega_{a_n}| \to 0$ (otherwise it is immediate to get a contradiction with $u_n \to 0$ in X). By (5) and (6) one has $-\Delta u_n \to 0$ in X hence $u_n \to 0$ in $H^2(\Omega)$; by the Sobolev's immersion theorem we know that $H^{k,p}(\Omega)$ is embedded in $L^q(\Omega)$ with $q \le \frac{np}{n-kp}$ furthermore by the same ideas of lemma 2.2 one proves that $\mathcal{F}_a$ sends bounded sets of $L^q(\Omega)$ into bounded sets of $L^q(\Omega)$ and, as before, one can improve the convergence of u_n; by this kind of bootstrap argument one sees that $u_n \to 0$ uniformely in Ω. As a consequence, for n large $u_n(x) < \bar{a}$ a.e. $x \in \Omega$ and hence $F_{a_n}(u_n(x)) \equiv 0$. Let us consider $v_n := \frac{u_n}{\|u_n\|}$, for n large the preceding remark implies

$$-\Delta v_n \in \frac{1}{\|u_n\|} \mathcal{F}_{a_n}(u_n) \equiv 0$$

which is impossible because $\|v_n\| = 1$. ∎

Corollary 2.10. There exists $\delta > 0$ such that for each $a \in \Lambda$:

$$d(\Phi(a,\cdot), B_\delta, 0) = 1.$$

Proof. It suffices to show that there exists a $\delta > 0$ such that the homotopy $H(t,u) := u - tK\mathcal{F}_a(u)$ is admissible then, by the homotopy invariance of the multivalued degree, the result follows. The admissibility of H can be proved following exactly the ideas of lemma 2.9. ∎

Proof of theorem A.

Taking into account that the multivalued degree satisfies the homotopy and the excision properties as the Leray-Schauder degree, part one of the proof follows, as in [A-H], by a simple modification of the Rabinowtz's global bifurcation theorem. In fact it is possible to apply the Rabinowitz's argument since by the lemma 2.6 it is possible to choose a δ such that for each $a<0$ $d(\Phi(a,\cdot),B_\delta,0) =0$ while corollary 2.10 implies that $d(\Phi(a,\cdot),B_\delta,0) =1$ for each $a>0$.
Since $\alpha< \lambda_1(\Omega)$ by remark 2.7 and lemma 2.8 it is clear that part 2 follows.
The last assertion follows because the continuoum $\mathcal{C}$, by the properties of the continuoum of the Rabinowtz's theorem and by part 2 of this theorem, must touch again the $(0,u)$-axis. ∎

We explicetly remark that we are not able to prove that the solutions on $\mathcal{C}$ satisfy (1) a.e. . In fact we do not know whether $|\Omega_a(u)|=0$ for any solution of (2) as it was shown, on the contrary, for the solutions of minimum in [A-B] and [A-T].

APPENDIX 2

Let us denote by B the ball in $\mathbb{R}^m$ centered at zero with radius one, by B_2 the unitary ball in L^2 and by $d\!I$ the Hausdorff distance.

Lemma . Let Ω in $\mathbb{R}^n$, $\mu(\Omega)<+\infty$; let F from $\mathbb{R}^m$ into the non-empty closed subsets of $\mathbb{R}^m$ be upper semicontinuous and such that, for some $a,b\geq 0$:

$$|F(x)| \leq a+b|x|.$$

Then the map $\mathcal{F}$ from $L^2(\Omega)$ into the non-empty closed subsets of $L^2(\Omega)$ such that for each $u\in L^2(\Omega)$

$$\mathcal{F}(u):= \{\ v\in L^2(\Omega) \mid v(x)\in F(u(x)) \text{ a. e. } x\in\Omega\ \}$$

is upper semicontinuous.

Proof:

a) Let u and v be in $L^2(\Omega)$ and let ξ be in $\mathcal{F}(u)$, ζ in $\mathcal{F}(v)$. Then:

$$|\xi(x)-\zeta(x)| \leq 2a + b(|u(x)| + |v(x)|) \leq 2a + b(|v(x)-u(x)| + 2|u(x)|)=$$
$$=2a + 2b|u(x)| + b|v(x)-u(x)|$$

and

(1) $\quad |\xi-\zeta|^2 \leq 4a^2 + 4b^2|u|^2 + 8ab|u| + b^2|v-u|^2 + 4ab|v-u| + 4b^2|u|\cdot|v-u|$

b) Fix u and $\varepsilon>0$. Let λ be such that $E\subset\Omega$, $\mu(E)\leq 3\lambda$ implies

$$\int_E\{4a^2 + 4b^2|u(x)|^2 + 8ab|u(x)|\}dx < \frac{\varepsilon^2}{3}.$$

Since F is u.s.c., for every x in Ω there exists $\eta(x)>0$: $|v-u(x)|\leq\eta(x)$ implies: $F(v) \subset F(u(x)) + \dfrac{\varepsilon}{2\sqrt{\mu(\Omega)}}B$ (where B denotes here the unitary ball). Let $A\subset\Omega$,$\mu(\Omega/A)<$be such that the restriction to A of the maps $x\to u(x)$ and $x\to F(u(x))$ is continuous. Denote by $C[x,\delta]$ a cube centered in x with sides of length δ. Then for every x in A there exists $\delta>0$: x' in $A\cap C[x,\delta(x)]$ implies

$$d(F(u(x),F(u(x'))< \frac{\varepsilon}{4\sqrt{\mu(\Omega)}} \quad\text{and}\quad |u(x')-u(x)|< \frac{\eta(x)}{2}.$$

The family $(C[x,\delta])_{x\in A,\delta\leq\delta(x)}$ is a Vitali covering of A. Let $C_n=C[x_n,\delta_n]$ be a countable family of disjoint sets such that $A\subset N^\circ \bigcup_n (C_n)$, where N° is a null set, and such that the series $\sum_n \mu(C_n)$ is convergent. Let v be such that

$$\sum_{n=v+1}^{\infty}\mu(C_n)\leq\lambda$$

and set $\eta>0$ to be $\eta:=\min\{\eta(x_n): n\leq v\}$.

c) Let $\sigma>0$ be such that $\sigma\leq\sqrt{\lambda}\,\frac{\eta}{2}$ and

(2) $4b^2\sigma^2+8ab\sigma\mu(\Omega)+4b^2\|u\|\sigma\leq\frac{\varepsilon^2}{3}$.

We wish to show that $\|v-u\|\leq\sigma$ implies

$\mathcal{F}(v)\subset\mathcal{F}(u(x))+\varepsilon B_2$ (here B_2 denotes the ball in $L^2(\Omega)$)

Fix any v in $u+\sigma B_2$ and let ζ be a measurable selection from the map $x\rightarrow F(v(x))$. Choose a measurable selection from the map $x\rightarrow F(u(x))$ such that $|\xi(x)-\zeta(x)|=d(\xi(x), F(u(x))$ a.e. in Ω. We have to estimate $\int_\Omega|\xi(x)-\zeta(x)|^2dx$.

Set $E=E(u,v):=\{x\in\Omega|\ |v(x)-u(x)|\geq\frac{\eta}{2}\}$. Since

(3) $$\left(\frac{\eta}{2}\right)^2\mu(E)\leq\int_\Omega|v-u|^2dx\leq\sigma^2\leq\lambda\left(\frac{\eta}{2}\right)^2,$$

we have that $\mu(E)\leq\lambda$.

d)Set:

$$\Omega=E\cup[(\Omega\backslash E)\backslash A]\cup\{[(\Omega\backslash E)\cap A](\bigcup_{n\leq v}C\}\cup\{[(\Omega\backslash E)\cap A](N^\circ\cup(\bigcup_{n\geq v+1}B)\}=$$
$$=E\cup E_1\cup E_2\cup E_3,$$

and

$$\int_\Omega|\xi(x)-\zeta(x)|^2dx=\int_{E\cup E_1\cup E_3}|\xi(x)-\zeta(x)|^2dx+\int_{E_2}|\xi(x)-\zeta(x)|^2dx$$

From (1),

$$(4) \qquad \int_{E\cup E_1\cup E_3}|\xi-\zeta|^2dx \le \int_{E\cup E_1\cup E_3} 4a^2+4b^2|u(x)|^2+8ab|u(x)|dx \ +$$

$$+ \int_\Omega (b^2|v(x)-u(x)|^2+8a|v(x)-u(x)|+4b^2|u(x)|\cdot|v(x)-u(x)|)dx \ .$$

We have: $\mu(E)\le\lambda$; since $E_1 \subset \Omega\backslash A$, also $\mu(E_1)\le\lambda$; from our choice of ν, $\mu(E_3)\le\lambda$. From the choice of in b), the first integral at the right hand side of (4) is bounded by $\frac{\varepsilon^2}{3}$. Moreover, since

$$\int_\Omega |u-v| \ \le \left(\int_\Omega |u-v|^2 \right)^{1\backslash 2} (\mu(\Omega))^{1\backslash 2} \le \sigma\cdot(\mu(\Omega))^{1\backslash 2},$$

the second integral to the right of (4) is bounded by:

$$b^2\sigma^2+8a\sigma(\mu(\Omega))^{1\backslash 2}+4b^2\|u\|\,(\mu(\Omega))^{1\backslash 2}\sigma$$

and by the choice of σ in (c) this therm is bounded by $\frac{\varepsilon^2}{3}$. Hence the integral to the left of (4)is not larger than $2\,\frac{\varepsilon^2}{3}$.

e) We are left with the estimate on E. Whenever $x\in A\cap C_n$, $n\le\nu$, then $|u(x)-u(x_n)|\le\frac{1}{2}\eta(x_n)$ and

$$(5) \qquad d(F(u(x)),F(u(x_n))) \le \frac{\varepsilon}{4\sqrt{\mu(\Omega)}}.$$

Whenever $x\in \Omega\backslash E$, $|v(x)-u(x)|\le\frac{\eta}{2}$ hence when x is in $(\Omega\backslash E)\cap(A\cap C_n)$,

$$|v(x)-u(x)|\le \frac{\eta(x)}{2}+\frac{\eta}{2} \le \eta(x_n)$$

and by the definition η of in b),

$$F(v(x)) \subset F(u(x_n)) + \frac{\varepsilon}{4\sqrt{\mu(\Omega)}}B$$

From (5) then,

$$F(v(x)) \subset F(u(x)) + \frac{\varepsilon}{2\sqrt{\mu(\Omega)}}B$$

The above estimate is independent of η, $n \leq \nu$, hence it holds on E_2.

We have then $\int_{E_2} |\xi-\zeta|^2 dx = \int_{E_2} d(\zeta(x),F(u(x)))^2 dx \leq \frac{\varepsilon}{4\mu(\Omega)}\mu(\Omega) = \frac{\varepsilon^2}{4}$

f) By adding the estimates in d)and e),

$$\int_{\Omega} |\xi(x)-\zeta(x)|^2 dx \leq \frac{2}{3} + \frac{\varepsilon^2}{4} < \varepsilon^2.$$

■

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