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A mio padre e mia madre

CONTENTS

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1. INTRODUCTION

In this thesis we present an algebraic approach to gauge theories where individual events of space-time, which are usually idealized as points of a differentiable manifold, play no role. The framework for our algebraic theory is an extension of (possibly infinite dimensional) Lie algebras $0 \to A \to E \to B \to 0$. Any such extension is endowed with the analogue of a gauge potential (connection) and all usual machineries of a gauge theory can be developed. Although we don't use any manifold structure at all, by realizing a sequence in terms of vector fields associated with suitable fibre bundles, we can recover a usual gauge configuration. Our approach seems to extract the essential algebraic axpects of gauge theories. This allows for generalizations since the sequences are not necessarily associated with any fibre bundle. A first, straightforward generalization is obtained by simply starting with an extension of Lie superalgebras (or $\mathbb{Z}_2$-graded Lie algebras) and by replacing all quantities with graded analogues. The result is an algebraic graded gauge theory.

As a preliminar step, in sections 2.1. we shall associate an 'exterior calculus' with any Lie algebra E, namely we shall introduce objects like an exterior derivative, a Lie derivative and so on. The algebra E will be a Lie algebra over a field of numbers $\mathfrak{R}$ and a left module over a commutative $\mathfrak{R}$-algebra $\mathfrak{F}$ with unit (this means that E has coefficients in $\mathfrak{F}$). In addition, E will act on $\mathfrak{F}$ as an algebra of derivations. Following Nelson [Ne], such a Lie algebra E will be called a Lie module over $\mathfrak{F}$. The algebraic setting we shall get is quite general. If $\mathfrak{F}$ is the algebra $C^\infty(M)$ of smooth functions on a smooth manifold M and E is the module $\aleph(M)$ of vector fields on M, then one gets an algebraic description of differential geometry and recovers the usual calculus on M. However, the setting goes beyond this case and includes situations where there is no corresponding manifold M. For instance, if E is a finite dimensional Lie algebra we get the cohomology of E with coefficients in the representation space $\mathfrak{F}$. Also, if E is the infinite dimensional Lie algebra $\aleph(M)$ of vector fields on a manifold M and $\mathfrak{F}$ is the field of real numbers, on which E acts trivially, we get the Gel'fanf-Fuks cohomology of $\aleph(M)$.

The possibility of algebrizing standard differential geometry stems from the observation that it is possible to algebrize the notion of space by replacing the description of a C^∞ differentiable manifold M in terms of charts and points, with the algebra $\mathfrak{F} = C^\infty(M)$ of real valued C^∞ functions on M [DeR]. The main point in reconstructing M from $C^\infty(M)$ is the fact that the evaluation map $ev : M \to Hom(C^\infty(M), \mathbb{R})$, $ev(p)(f) = f(p)$, is bijective, so that one has the identification $Hom(C^\infty(M), \mathbb{R}) \equiv M$ (see for instance [KM]). However, although one knows that the differentiable structure of M is uniquely determined by giving the algebra $C^\infty(M)$ [Hel], the problem of a complete algebrization of the theory of differentiable manifolds, and in particular the problem of reconstructing the differentiable structure starting from abstract properties of $C^\infty(M)$ is, to our knowledge, still unsolved. The best one can do is to reconstruct M as a topological space, starting from the collection $\mathcal{M}$ of all maximal ideals of $C^\infty(M)$ endowed with the topology of Stone-Jacobson-Zariski [GJ],[TeS1] (the resulting topological space is known as the maximal spectrum of $C^\infty(M)$). Indeed, it is possible to show that all maximal ideals of the algebra $C^\infty(M)$ are of the form $M_p = \{ f \in C^\infty(M) : f(p) = 0 \}$ [TeS1], [Yo]. Therefore, the map $\tau : M \to \mathcal{M}$, $p \to M_p$, is surjective. It is also injective because smooth functions, being in particular continuous, separate points, namely $p \neq q$ implies $M_p \neq M_q$. A basis for the closed sets in $\mathcal{M}$ is provided by the family of sets $\mathcal{M}(f) = \{ I \in \mathcal{M} : f \in I \}$ with f any element in $C^\infty(M)$. We see that the ideal M_p belongs to $\mathcal{M}(f)$ if and only if $f(p) = 0$. Consider also the family of subsets of M of the form $M(f) = \{ p \in M : f(p) = 0 \}$ with f any element in $C^\infty(M)$. The subsets M(f) are closed (one easily shows that their complements are open) and form a basis for the closed sets in M. The map τ carries the family of M(f)' s to the family of $\mathcal{M}(f)$' s and is therefore a homeomorphism.

As regards the relation between $\mathrm{Hom}(C^\infty(M), \mathbb{R})$ and $\mathcal{M}$, one has that for each point p, the quotient $C^\infty(M) / M_p$ is isomorphic with the field of real numbers $\mathbb{R}$. The natural homomorphism from $C^\infty(M)$ to $C^\infty(M) / M_p$ defines a unique homomorphism from $C^\infty(M)$ to $\mathbb{R}$ namely an element of $\mathrm{Hom}(C^\infty(M), \mathbb{R})$. Viceversa, any element of $\mathrm{Hom}(C^\infty(M), \mathbb{R})$ can be got from a maximal ideal M_p in the way described above.

Given the space $\mathrm{Hom}(C^\infty(M), \mathbb{R})$, a tangent vector at the 'point' $\alpha \in \mathrm{Hom}(C^\infty(M),\mathbb{R})$ is a derivation of $C^\infty(M)$ relative to α, namely a linear map β_α from $C^\infty(M)$ to $\mathbb{R}$ such that $\beta_\alpha(fg) = \beta_\alpha(f)\alpha(g) + \alpha(f)\beta_\alpha(g)$ for any f, g in $C^\infty(M)$. A vector field on M can be given as a derivation of $C^\infty(M)$ relative to the identity, namely as a linear map X from $C^\infty(M)$ to $C^\infty(M)$ such that $X(fg) = X(f)\, g + f\, X(g)$ for any f, g in $C^\infty(M)$. The collection $\mathrm{Der}C^\infty(M)$ of all derivations of $C^\infty(M)$ is a Lie algebra over $\mathbb{R}$ with Lie brackets defined by $[X, Y] =: X \circ Y - Y \circ X$. It is well known that the Lie algebra $\mathrm{Der}C^\infty(M)$ is isomorphic with the Lie algebra of all vector fields on M (see for instance [AMR]). The Lie algebra $\mathrm{Der}C^\infty(M)$ is also a left module over $C^\infty(M)$. All tensor calculus on M as well as the de Rham theory for M [TeS2] can then be reconstructed from the module $\mathrm{Der}C^\infty(M)$ together with the dual module. Finally, we mention that the group DiffM of diffeomorphisms of M is isomorphic with the group $\mathrm{Aut}C^\infty(M)$ of automorphisms of $C^\infty(M)$ (when M is paracompact) [AMR]. Here $\mathrm{Aut}C^\infty(M)$ is the collection of all invertible linear mappings ϕ from $C^\infty(M)$ to itself such that $\phi(fg) = \phi(f)\phi(g)$ for any f, g in $C^\infty(M)$; $\mathrm{Aut}C^\infty(M)$ is a group under composition.

It is worth mentioning that given the algebra $C^\infty(M)$ it is possible to construct the algebras of bundles associated with M. Indeed, if $\mathcal{A}$ is the algebra of dual or Study numbers, namely the algebra over $\mathbb{R}$ generated by two elements 1 and ε with $\varepsilon^2 = 0$, then one has that $\mathrm{Hom}(C^\infty(M), \mathcal{A}) \equiv TM$; also $\mathrm{Hom}(C^\infty(M), \mathcal{A} \otimes \mathcal{A}) \equiv TTM$ [KM], [Ok]. Moreover, if n is the dimension of M and $A = \mathbb{R}[T_1, ..., T_n]/(T_1, ..., T_n)^{k+1}$, namely the algebra of polynomials over $\mathbb{R}$ in n variables and truncated at the order $k + 1$, then the open of $\mathrm{Hom}(C^\infty(M), \mathcal{A})$ of maximal rank is the space of k-frames over M [Ok].

In a true non commutative generalization (see later for a $\mathbb{Z}_2$-graded commutative extension) one could start with a non commutative algebra $\mathcal{F}$ with unit [Co]. Although there is no reconstruction theorem in this case (to this respect see [KW]), one could think of the set of all maximal ideal of $\mathcal{F}$ or of the space $\mathrm{Hom}(\mathcal{F}, \mathbb{R})$ as a 'noncommutative space' (noncommutative spaces would be objects of the category dual to the category of non commutative algebras). Another difference which is at the origin of possible different generalizations of the exterior calculus, is the fact that the Lie algebra $\mathrm{Der}\mathcal{F}$ of all derivations of $\mathcal{F}$ is not an $\mathcal{F}$-module in the noncommutative case (see for instance [Du]). For many related ideas see [Ma].

A 'geometry without points' had already been proposed by von Neumann to deal with Quantum Mechanics [vNe]. Recently there has been an attempt to use a 'non commutative geometry' in connection with string theories [Wi]. Moreover, objects like Hopf algebras or quantum groups [Dr] and pseudogroups [Wor], which are based on a non commutative geometry seem to play a preminent role in the study of quantum integrable systems

Given a Lie module E over $\mathcal{F}$, we can introduce the notion of metric. By this we mean an isomorphism from E into the $\mathcal{F}$-dual module E^* of E. Given a metric, it is possible to associate with it a connection (Levi-Civita connection) and construct in a purely algebraic manner the (analogue of the) usual Riemannian calculus. This is done in section 2.2.. Similar ideas have led Geroch [Ge] to introduce the notion of Einstein algebras which he uses for an algebraic description of General Relativity. In section 2.3. the existence of a metric is used to introduce a codifferential. It is interesting to notice that for this one does not need an orientation nor a volume form but only a metric and the associated Levi-Civita connection.

In section 2.5. the algebraic calculus developed in previous sections is generalized to situations where the Lie algebra E is an extension of Lie algebras. The reason for considering such algebras is that, as will be clear later, they allow to develop an algebraic description of gauge theories. A very simple example to illustrate this fact is the following. Let us consider the following exact sequence of vector spaces

$$0 \to \mathbb{R}^k \to \mathbb{R}^{k+4} \to \mathbb{R}^4 \to 0 . \tag{1}$$

We use coordinates $\{ z^a , a = 1,..., k \}$, $\{ z^a, y^\mu , \mu = 1,..., 4 \}$ and $\{ x^\mu , \mu = 1,..., 4 \}$ on $\mathbb{R}^k$, $\mathbb{R}^{k+4}$ and $\mathbb{R}^4$ respectively. Next, let us take the three infinite dimensional Lie algebras of first order differential operators (vector fields) given by : $\aleph^v = \{ h^a(y,z)\, \partial / \partial z^a \}$, $\aleph^\pi = \{ f^\mu(y)\, \partial / \partial y^\mu + k^a(y,z)\, \partial / \partial z^a \}$ and $\aleph^\pi(\mathbb{R}^4) = \{ g^\mu(x)\, \partial / \partial x^\mu \}$ (here and after we sum over repeated indeces). We have then an extension of infinite dimensional Lie algebras

$$0 \rightarrow \aleph^v \rightarrow \aleph^\pi \rightarrow \aleph(\mathbb{R}^4) \rightarrow 0 \ . \tag{2}$$

The Lie algebras in the sequence are left modules over the algebra of functions on $\mathbb{R}^4$. On the sequence (2) we define a map ρ wich is linear over functions on $\mathbb{R}^4$ in the following way

$$\rho : \aleph(\mathbb{R}^4) \rightarrow \aleph^\pi ,$$

$$\rho(\partial / \partial x^\mu) =: \partial / \partial y^\mu - A_\mu(x)_a{}^b\, z^a\, \partial / \partial z^b , \quad \mu = 1,..., 4 . \tag{3}$$

One finds that

$$\rho([\partial / \partial x^\mu , \partial / \partial x^\mu]) - [\rho(\partial / \partial x^\mu) , \rho(\partial / \partial x^\mu)] =$$

$$= \{ \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\nu , A_\mu] \}_a{}^b\, z^a\, \partial / \partial z^b \quad . \tag{4}$$

We see from (3) and (4) that ρ can be interpreted as a gauge potential, while its deviation from being a Lie algebra homomorphism is the field strength. If in (1) we take $k = 1$ and in (3) we take $\rho(\partial / \partial x^\mu) = \partial / \partial y^\mu - A_\mu(x) \partial / \partial z^b$, $\mu = 1,..., 4$, then we get an abelian theory, namely, the commutator at the right hand side of (4) vanishes.

Given a fibre bundle $\pi : P \rightarrow M$ (without any additional structure) we can naturally associate with it an exact sequence of infinite dimensional Lie algebras of vector fields like (2). To this aim let us consider the algebra of functions $\mathcal{F}(P)$ and the subalgebra $\pi^*\mathcal{F}(M)$. By identifying vector fields with derivations on these algebras we can introduce few infinite dimensional Lie algebras. Firstly, we have the subalgebra $\aleph^\pi(P)$ of derivations of $\mathcal{F}(P)$ which are also derivations for $\pi^*\mathcal{F}(M)$, i.e. which take $\pi^*\mathcal{F}(M)$ into itself; $\aleph^\pi(P)$ is nothing but the algebra of projectable vector fields on P. Secondly, the subalgebra of $\aleph^\pi(P)$ of derivations which are zero when applied to $\pi^*\mathcal{F}(M)$; this is the invariant subalgebra in $\aleph^\pi(P)$, call it $\aleph^v(P)$, made of vertical vector fields on P. The quotient algebra $\aleph^\pi(P)/\aleph^v(P)$ is the algebra $\aleph(M)$ of vector fields on M (when M is paracompact) and we get an exact sequence of Lie algebras $0 \rightarrow \aleph^v(P) \rightarrow \aleph^\pi(P) \rightarrow \aleph(M) \rightarrow 0$ associated with the fibre bundle $\pi : P \rightarrow M$. This sequence is similar to what Nelson calls a Lie bundle [Ne] and

Atiyah-Bott associate with a principal bundle in their treatment of Yang-Mills fields [AB]. One can also construct the sequence starting from $\aleph^v(P)$ and $\mathcal{F}(M)$ and defining $\aleph^\pi(P)$ as the normalizer of $\aleph^v(P)$ in the algebra $\aleph(P)$. It is worth mentioning that there exist sequences of Lie algebras of vector fields which are not associated with any fibre bundle. In section 2.4. we shall give an example of such a situation.

In section 2.5. the basic object is a sequence of Lie algebras $0 \to A \to E \to B \to 0$ abstractly given, i.e. not necessarily associated with a fibre bundle. The extension will be a Lie bundle over the couple $(\mathfrak{F}, \mathfrak{R})$ where $\mathfrak{F}$ is a commutative $\mathfrak{R}$-algebra with unit. This means that A , E and B are Lie algebras over $\mathfrak{R}$ and left module over $\mathfrak{F}$ and act on $\mathfrak{F}$ as Lie algebras of derivations. On such an extension we introduce the analogue of a connection and define all ingredients which constitute the kinematic of a gauge theory like curvature, covariant differential and codifferential and so on. In section 2.6. we construct a Chern-Weil homomorphism and Chern-Simons secondary characteristic classes for any extension of Lie algebras endowed with a connection.

The algebraic framework developed in section 2.5. is used in chapter 4. to develop the dynamical part of our algebraic theory. In particular, in section 4.1., given a metric on the module B , we introduce field equations and obtain them from a lagrangian. The main point here is that everything is algebraic and there is no reference to any manifold structure at all. In addition, in sections 4.3., 4.4. and 4.5. we introduce gauge transformations, conserved quantities and BRST transformations. In section 4.2. we give few examples by constructing sequences of Lie algebras which provide algebraic analogues of known solutions like monopoles and instantons. Our sequences extract the basic algebraic properties of these solutions. By realizing them in terms of vector fields associated with suitable fibre bundles we get back the usual solutions of gauge theory.

It is also possible to include Geroch's notion of Einstein algebra in our setting and construct what may be called an algebraic Kaluza-Klein theory. In particular in section 4.7. we shall introduce an extension of Lie algebras which, by adding the structure of Einstein algebra, gives an algebraic description of the Kaluza-Klein monopole [So], [GP].

A $\mathbb{Z}_2$-graded commutative generalization is given in chapter 3. In section 3.1.–3.3. we shall generalize the results of sections 2.1.– 2.3. by starting with a Lie superalgebra (or $\mathbb{Z}_2$-graded Lie algebra) E over a $\mathbb{Z}_2$-graded commutative algebra of 'constants' $\mathfrak{R}$. The Lie superalgebra E will be a left graded module over a $\mathbb{Z}_2$-graded commutative $\mathfrak{R}$-algebra $\mathfrak{F}$ with unit and will act on $\mathfrak{F}$ as a Lie superalgebra of graded derivations. The result is a 'graded exterior calculus'. As for the possibility of recovering, in specific cases, all information and notably the cohomology theory for a supermanifold, the situation is now more complicate. If $\mathfrak{F}$ is the graded algebra of 'supersmooth' functions on a graded manifold [BeLe], [Kon] or a De Witt supermanifold [DeW] M , and E is the Lie superalgebra of all derivations of $\mathfrak{F}$, one recovers all possible information about M . On the other hand, this is not the case if M is a more general supermanifold admitting nontrivial topology in the odd directions [Rog], [Rot]. Indeed, if this is the case, the structure sheaf of M has a nontrivial Cech cohomology in degree higher than zero [BB]. As a consequence, and in contrast to what happens for ordinary manifolds or for graded manifolds [HM] and De Witt supermanifolds [Br], the structure sheaf of M cannot be recovered from the graded algebra of global sections, the latter being the algebra of supersmooth functions on M . However, so far, the only supermanifolds which seem to be relevant in physics are the ones with trivial topology in the odd directions. Finally, we mention that in the case of a graded manifold it is possible to recover, as for an ordinary manifold, the topological structure of the underlying space in terms of the maximal spectrum of the graded algebra of supersmooth functions on M [HM].

The kinematic for a graded algebraic gauge theory is developed in section 3.5. where the framework is an extension of Lie superalgebras endowed with a connection. The dymamics for the theory is sketched in section 4.8. where as examples we construct the sequences for a graded electromagnetism and a graded abelian monopole. In order to realize the latter in terms of a principal superbundle, we also construct a Grassmann extension of the Hopf fibration of the Dirac monopole. The base space of the superfibration is a (2, 2) dimensional (two even dimensions and two odd ones) sphere in a Grassmann algebra, while the structure group is the Grassmann version of U(1). The canonical connection on the resulting bundle turns out to be a Grassmann extension of the Dirac monopole.

Finally, in chapter 5. we give the ingredients for an algebraic description of Lagrangian dynamical systems. If we are on a tangent bundle, then our formalism gives the usual intrinsic (i.e. coordinate free) formulation of dynamical systems. In section 5.2. we exhibit a theorem which gives sufficient conditions in order that a manifold carries a tangent bundle structure. We hope to use this algebraic setting to construct graded generalizations of Lagrangian dynamics.

2. ALGEBRAIC DIFFERENTIAL CALCULUS

In this chapter we shall introduce the basic tools which will be used for an algebraic description of gauge theories. Following [Kos2], [Ne] we will first construct an exterior calculus for any Lie algebra acting as an algebra of derivations on a commutative algebra with unit. As we shall see the resulting framework is very general and includes, as particular cases, the exterior calculus over a manifold as well as the cohomology theory of finite and infinite dimensional Lie algebras. By introducing a metric we shall develop the (algebraic counterpart of the) usual Riemannian calculus. Similar ideas have been used by Geroch [Ge] for an algebraic description of General Relativity. The presence of a metric allow also to introduce a codifferential. All operations will be specialized (and generalized) by taking the Lie algebra to be an extension of Lie algebras. This will allow to introduce all machineries, like connection and curvature, for an algebraic description of gauge theories. In particular the generalization of the codifferential will be used in section 4.1. to introduce field equations. It would be interesting to introduce different kind of structures on the Lie algebra and analyse the corresponding outcomes. However we shall not do it in order not to go astray from the main path of the thesis. In section 2.6. we shall construct a set of characteristic classes and a Chern-Weil homomorphism for an extension of Lie algebras endowed with a connection.

2.1. EXTERIOR CALCULUS

In this section the starting object is a commutative ring $\mathfrak{F}$ with unit 1 which contains a field $\mathfrak{R}$ of characteristic zero (isomorphic to the field of real, complex, p-adic,... numbers); then $\mathfrak{F}$ can be made a commutative $\mathfrak{R}$-algebra. A *derivation* of $\mathfrak{F}$ is any $\mathfrak{R}$-linear map $X : \mathfrak{F} \to \mathfrak{F}$ with the property that $X(fg) = X(f)g + fX(g)$, for any $f, g \in \mathfrak{F}$. From $X(1) = X(1) + X(1)$ one has that $X(1) = 0$; by $\mathfrak{R}$-linearity, $X(a) = 0$ for any $a \in \mathfrak{R}$. The collection $\mathrm{Der}\mathfrak{F}$ of all derivations of $\mathfrak{F}$ is a Lie algebra over $\mathfrak{R}$: the Lie bracket $\mathcal{L}_X Y \equiv [X, Y]$ of any two derivations X and Y is defined by $\mathcal{L}_X Y =: X \circ Y - Y \circ X$, and is easily seen to be a derivation. Furthermore, $\mathrm{Der}\mathfrak{F}$ is a left module over $\mathfrak{F}$ by the obvious definition $(gX)f =: g(Xf)$ and with property $\mathcal{L}_X(f\,Y) = f\,\mathcal{L}_X Y + (Xf)\,Y$, for any X, Y in $\mathrm{Der}\mathfrak{F}$, f in $\mathfrak{F}$.

Let E be any Lie algebra over $\mathfrak{R}$ which is also a left $\mathfrak{F}$-module. E will be called a *Lie* $\mathfrak{F}$*-module* if there is a representation of E into $\mathrm{Der}\mathfrak{F}$, $X \to X\cdot$. In this manner $\mathfrak{F}$ becomes a left E module. Obviously $\mathrm{Der}\mathfrak{F}$ itself is a Lie module in a natural way.

Given any $\mathcal{F}$-module E, the dual module E* is the collection of all $\mathcal{F}$-linear mappings $\varphi : E \to \mathcal{F}$, $X \to \varphi(X)$. At this point one defines *'tensors'* of any rank (m,n). In particular, a tensor m times contravariant and n times covariant is any $\mathcal{F}$-multilinear map $u : E^* \times \cdots \times E^* \times E \times \cdots \times E \to \mathcal{F}$ (m E* factors and n E factors). The collection of all rank (m, n) tensors is naturally a vector space over $\mathbb{R}$ and a left $\mathcal{F}$-module. One can also defines the tensor product $u \otimes v$ of any two tensors u and v in an obvious way. We shall consider only situations in which the algebra $\mathcal{F}$ and the Lie algebra E are such that E is totally reflexive namely the dual of E* is isomorphic with E and tensors can be identified with tensor products [Ne].

Let us denote by $\Lambda^p(E, \mathcal{F})$ the collection of all skew-symmetric covariant tensors of rank p . Elements in $\Lambda^p(E, \mathcal{F})$ will be also called *p-forms* or p-$\mathcal{F}$-*cochains* . In particular $\Lambda^1(E, \mathcal{F}) = E^*$, $\Lambda^0(E, \mathcal{F}) = \mathcal{F}$. If E is a Lie module, one defines on the direct sum $\Lambda^*(E,\mathcal{F}) = \oplus_p \Lambda^p(E, \mathcal{F})$ an *exterior derivative* d, by

$$d: \Lambda^p(E, \mathcal{F}) \to \Lambda^{p+1}(E, \mathcal{F}) ,$$

$$(d\varphi)(X_1,\ldots,X_{p+1}) =: \Sigma_i (-1)^{i+1} X_i \cdot \varphi(X_1,\ldots,\hat{\imath},\ldots,X_{p+1}) +$$

$$+ \Sigma_{i<j} (-1)^{i+j} \varphi([X_i , X_j], X_1,\ldots,\hat{\imath},\ldots,\hat{j},\ldots,X_{p+1}) , \quad \forall X_i \in E ; \qquad (1)$$

here a hat ^ means omission. Since the action of E is a representation it follows, by standard methods, that d is a coboundary operator, namely $d^2 = 0$. An element φ in $\Lambda^p(E, \mathcal{F})$ will be called a *cocycle* if $d\varphi = 0$, a *coboundary* if $\varphi = d\psi$ for some ψ in $\Lambda^p(E, \mathcal{F})$. The coboundaries clearly form a subspace of the cocycles. It is then possible to introduce the *cohomology* of the Lie algebra E with coefficients in the representation space $\mathcal{F}$. The p-th cohomology group is defined by

$$H^p(E, \mathcal{F}) = (\ker d: \Lambda^p(E, \mathcal{F}) \to \Lambda^{p+1}(E, \mathcal{F})) / (\mathrm{im}\, d : \Lambda^{p-1}(E, \mathcal{F}) \to \Lambda^p(E, \mathcal{F})) . \qquad (2)$$

We see that $H^0(E, \mathcal{F}) = \ker d: \Lambda^0(E, \mathcal{F}) \to \Lambda^1(E, \mathcal{F})$, namely, $H^0(E, \mathcal{F})$ is the subset of $\mathcal{F}$ made of elements which are invariant under the action of E .

The next ingredient we introduce is an *exterior product*

$$\wedge: \Lambda^p(E, \mathcal{F}) \times \Lambda^q(E, \mathcal{F}) \to \Lambda^{p+q}(E, \mathcal{F}) ,$$

$$\varphi \wedge \psi(X_1,..,X_{p+q}) =: 1/p!q! \Sigma_\sigma \chi(\sigma)\varphi(X_{\sigma(1)},\ldots,X_{\sigma(p)})\, \psi(X_{\sigma(p+1)},..,X_{\sigma(p+q)}), \ \forall X_i \in E . \ (3)$$

This product makes $\Lambda^*(E, \mathcal{F})$ a commutative graded algebra over $\mathcal{R}$. The derivative d is of degree one with respect to it, namely, $d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^{\deg\varphi}\, \varphi \wedge d\psi$. It follows that if φ and ψ are two cocycles, then $\varphi \wedge \psi$ is a cocycle as well; furthermore, if either φ or ψ is a coboundary, then $\varphi \wedge \psi$ is a coboundary. Therefore, the product (3) defines a product from $H^p(E,\mathcal{F}) \times H^q(E,\mathcal{F})$ to $H^{p+q}(E,\mathcal{F})$ and the direct sum $H^*(E, \mathcal{F}) = \oplus_p H^p(E, \mathcal{F})$ is made into a ring (in fact an $\mathcal{R}$-algebra) called the *cohomology ring* of the Lie algebra E with coefficients in $\mathcal{F}$.

We continue by introducing two more derivations of the algebra $\Lambda^*(E, \mathcal{F})$. The first one is the *Lie derivative* $\mathcal{L}_{(\cdot)}$ and is of degree zero; it is defined as

$$\mathcal{L}_{(\cdot)}: \quad E \times \Lambda^p(E, \mathcal{F}) \rightarrow \Lambda^p(E, \mathcal{F}),$$

$$(\mathcal{L}_X\varphi)(X_1,\dots,X_p) =: X\cdot\varphi(X_1,\dots,X_p) - \Sigma_i\varphi([X, X_i],X_1,\dots,\hat{\imath},\dots,X_p)\,,\ \forall\, X, X_i \in E\,, \qquad (4)$$

and one has that $\mathcal{L}_{(\cdot)}(\varphi \wedge \psi) = \mathcal{L}_{(\cdot)}\varphi \wedge \psi + \varphi \wedge \mathcal{L}_{(\cdot)}\psi$. The second derivation is the *inner product* $i_{(\cdot)}$, and is of degree minus one; it is given by

$$i_{(\cdot)}: \quad E \times \Lambda^p(E, \mathcal{F}) \rightarrow \Lambda^{p-1}(E, \mathcal{F}),$$

$$(i_X\varphi)(X_1,\dots,X_{p-1}) =: \varphi(X,X_1,\dots,X_{p-1})\,, \qquad \forall\, X, X_i \in E\,; \qquad (5)$$

in addition, $i_{(\cdot)}f = 0 \ \ \forall f \in \mathcal{F}$. One has that $i_{(\cdot)}(\varphi \wedge \psi) = i_{(\cdot)}\varphi \wedge \psi + (-1)^{\deg\varphi}\, \varphi \wedge i_{(\cdot)}\psi$. From definitions one can prove the Cartan identity $\mathcal{L}_{(\cdot)} = d \circ i_{(\cdot)} + i_{(\cdot)} \circ d$, so that $\mathcal{L}_{(\cdot)}$ and d commute. Moreover, all usual algebraic relations among the quantities defined in (1), (3), (4) and (5) hold true.

If $\mathcal{F}$ is the ring of smooth functions on a manifold M, and E is the Lie module of all derivations of $\mathcal{F}$, namely the module of smooth vector fields on M, then $\Lambda^p(E, \mathcal{F})$ is nothing but the space $\Omega^p(M)$ of smooth differential p-forms on M and the definitions (1), (3)-(5) give the usual exterior calculus on M. In addition, $H^*(E, \mathcal{F})$ is the de Rham cohomology ring of M. In this case $\Lambda^p(E, \mathcal{F})$ and $H^p(E, \mathcal{F})$ are both zero if $p > \dim M$.

Let us consider the particular case in which $\mathcal{F} = \mathcal{R}$ (on which E acts trivially). In this case, in the definition (1) of the coboundary operator and in the definition (4) of the Lie derivative, the first term of the right-hand side vanishes. Also $H^0(E, \mathcal{R}) = \mathcal{R}$. Moreover, there are no nontrivial 1-coboundary; a 1-cochain which is closed (namely, a cocycle) cannot be exact (namely, a coboundary), and the space $H^1(E, \mathcal{R})$ is the space of all

1-cocycles. If $\varphi \in \Lambda^1(E, \mathcal{R})$, then $(d\varphi)(X_1, X_2) = -\varphi([X_1, X_2]$, and φ is a cocycles if and only if φ vanishes on the subalgebra $[E, E]$ of E spanned by commutators of any two elements of E. As a consequence, $H^1(E, \mathcal{R}) = \mathrm{Hom}(E / [E, E], \mathcal{R})$ (see for instance [HeRu]) and $H^1(E, \mathcal{R}) = 0$ if $E = [E, E]$ (for instance, this is the case if E is semisimple [CE]).

If E is a finite dimensional Lie algebra, then the groups $H^p(E, \mathcal{R})$ are nothing but the usual cohomology spaces of the Lie algebra E [Ce], [Kos1]. In this case $\Lambda^p(E, \mathcal{R})$ and $H^p(E, \mathcal{R})$ are both zero if $p > \dim E$ since $\Lambda^*(E, \mathcal{R})$ is the Grassmann algebra over E and $\Lambda^p(E, \mathcal{R})$ is the space of elements of order p in $\Lambda^*(E, \mathcal{R})$. Among other things one knows [CE] that if E is the (real) algebra of a compact connected Lie group G (E can be identified with the algebra of left invariant vector fields on G) then the algebraic cohomology of E as defined before is isomorphic with the cohomology of the left invariant forms on G wich in turn coincides with the de Rham cohomology of G, namely, the cohomology groups $H^p(E, \mathbb{R})$ are isomorphic with the real cohomology groups $H_{dR}{}^p(G)$. For many results on the cohomology of Lie algebras see [CE], [Kos].

The Gel'fand-Fuks cohomology of the Lie algebra $\aleph(M)$ of smooth vector fields on a smooth manifold M provides examples of cohomologies of infinite dimensional Lie algebras [GF1], [Gel]. The cohomology groups $H^p(\aleph(M), \mathbb{R})$ are finite dimensional, although very complicate to compute in general. For instance, if M is the unit sphere S^1, the cohomology ring $H^p(\aleph(S^1), \mathbb{R})$ is generated by a generator of degree two and a generator of degree three which have a very simple expression [Gel].

We say that the cochain $\varphi \in \Lambda^p(E, \mathcal{R})$ is invariant if for any $X \in E$ it happens that $\mathcal{L}_X\varphi = 0$; from (3) this is equivalent to $\Sigma_i\varphi([X, X_i], X_1, ..., \hat{\imath}, ..., X_p) = 0$, $\forall X, X_i \in E$. By explicit calculations one can show that any invariant cochain is a cocycle, namely $d\varphi = 0$. In [CE] it has been proved that if E is a semisimple (finite dimensional) Lie algebra, then every cohomology class of $H^p(E, \mathcal{R})$ contains exactly one invariant cocycle. The invariant cocycles form a ring which is isomorphic with the cohomology ring $H^*(E, \mathcal{R})$. Notice that it make sense to consider an invariant cohomology because the Lie derivative commutes with the exterior differential. If E is the algebra of a compact connected Lie group G, then invariant cochains on E correspond to left and right invariant forms on G. In this case it is known [CE] that every cohomology class of differential forms on G contains precisely one invariant form and that invariant forms span a ring which is isomorphic with the de Rham cohomology ring of the manifold G.

As for the more general case in which $\mathcal{F}$ does not coincide with $\mathcal{R}$, examples of cohomology of infinite dimensional Lie algebras are given by the Gel'fand-Fuks cohomology of the algebra $\aleph(M)$ of smooth vector fields on a manifold M, with coefficients in the smooth differential forms $\Omega^*(M)$; $\aleph(M)$ acting on them with the Lie derivative [Lo], [GF2], [FL], [DL].

Very interesting examples for physics are provided by the cohomology of the infinite dimensional Lie algebra and group of gauge transformations and diffeomorphisms; such cohomology has been intensively used in the problem of anomalies in quantum field theories [Zu], [Sto], [BC], [FS]. Other examples are the semi-infinite cohomology of Kac-Moody and Virasoro algebras recently introduced by Feigin [Fe]. They have been used in the BRST cohomological approach to string theories [FGZ], [KoSt]. This algebraic cohomology has an equivalent geometric counterpart in the $\mathrm{Diff}S^1$-invariant cohomology of semi-infinite differential forms on $\mathrm{Diff}S^1/S^1$, the latter being the space of complex structures used by Bowick and Rajeev in their geometric quantization approach to string theories [BR].

We continue with the general framework by defining the Lie derivative of any tensor. This is done simply by requiring Leibnitz rule. If u is any (m, n) tensor, namely any $\mathcal{F}$-multilinear map $u : E^*\times\cdots\times E^*\times E\times\cdots\times E \to \mathcal{F}$ (m E^* factors and n E factors), then, for any $X \in E$, the Lie derivative $\mathcal{L}_X u$ is the tensor of the same type given by

$$
\begin{aligned}
(\mathcal{L}_X u)(\omega^1,\dots,\omega^m,X_1,\dots,X_n) &=: X\cdot(u(\omega^1,\dots,\omega^m,X_1,\dots,X_n)) \\
&- \Sigma_i u(\omega^1,\dots,\mathcal{L}_X\omega^i,\dots,\omega^m,X_1,\dots,X_n) \\
&- \Sigma_i u(\omega^1,\dots,\omega^m,X_1,\dots,[X,X_i],\dots,X_n), \quad \forall\, X, X_i \in E,\ \omega^i \in E^* . \qquad (6)
\end{aligned}
$$

An algebraic approach to the classical operators of differential geometry based on a couple $(\mathcal{F}, E)$, ($\mathcal{F}$ a commutative algebra with unit and E a Lie module over $\mathcal{F}$) has been presented also in [KaSt1] where the couple $(\mathcal{F}, E)$ is called a Lie-Cartan Pair.

2.2. RIEMANNIAN CALCULUS AND EINSTEIN ALGEBRAS

In [Ge] Geroch has showed how one can algebraize Einstein theory of General Relativity. To this aim we now introduce additional structures to the framework developed in the previous section. Following [Kos2], we first introduce the notion of affine connection on a Lie module. Then, by giving a metric on the module, we shall construct in a canonical way the Levi-Civita connection associated with it.

The framework is a Lie module E over the algebra $\mathcal{F}$. An *affine connection* on E is a map $\nabla : E \to Hom_{\mathbb{R}}(E, E)$, $X \to \nabla_X$ with properties

$$\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z \ ,$$

$$\nabla_{fX+Y} Z = f \nabla_X Z + \nabla_Y Z \ ,$$

$$\nabla_X (f Y) = f \nabla_X Y + (X{\cdot}f) Y \ , \qquad \forall \ X, Y, Z \in E \, , \ f \in \mathcal{F} \ . \tag{7}$$

The element of E given by $\nabla_X Y$ is called the *covariant derivative* of Y in the direction of X . As for elements in E* , their covariant derivative is defined by requiring Leibnitz rule. If $X \in E$ and $\varphi \in E^*$, the covariant derivative $\nabla_X \varphi$ is the element in E* given by

$$(\nabla_X \varphi)(Y) =: X{\cdot}(\varphi(Y)) - \varphi(\nabla_X Y) \ , \qquad \forall \ Y \in E \ . \tag{8}$$

In general, if u is a tensor of type (m, n), the covariant derivative $\nabla_X u$ is the tensor of the same type defined by

$$(\nabla_X u)(\omega^1,..., \omega^m, X_1,...,X_n) =: X{\cdot}(u\ (\omega^1,..., \omega^m, X_1,...,X_n))$$

$$- \Sigma_i u(\omega^1,...,\nabla_X \omega^i,...,\omega^m, X_1,...,X_n)$$

$$- \ \Sigma_i u(\omega^1,..., \omega^m, X_1,..., \nabla_X X_i,...,X_n) \, , \qquad \forall \ X, \ X_i \in E \, , \ \omega^i \in E^* \ . \tag{9}$$

From this definition it follows that $\nabla_X (u \otimes v) = (\nabla_X u) \otimes v \ + \ u \otimes \nabla_X v$.

Given an affine connection ∇ on the Lie module E , the *torsion* of ∇ is the map T from $E \times E$ into E defined by

$$T(X,Y) =: \nabla_X Y - \nabla_Y X - [X, Y] \ , \qquad \forall \ X, \ Y \in E \, . \tag{10}$$

It is easy to verify that T is $\mathcal{F}$-bilinear. Therefore it determines a skew-symmetric *torsion tensor* of type (1, 2) by the rule $T(\omega,X,Y) =: \omega(T(X,Y))$.

The *curvature* of the affine connection ∇ on the Lie module E is the map R from $E \times E$ into $Hom_{\mathbb{R}}(E, E)$ defined by

$$R(X, Y) =: \nabla_X \nabla_Y \ - \nabla_Y \nabla_X \ - \nabla_{[X,Y]} \ , \qquad \forall \ X, Y \in E \, . \tag{11}$$

The *curvature tensor* R of the affine connection ∇ is the tensor of type (1,3) given by

$$R(\omega, Z, X, Y) =: \omega(R(X, Y)(Z))$$

$$= \omega([\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}](Z)) \quad , \quad \forall\ X, Y, Z \in E\ , \quad \omega \in E^* \ . \qquad (12)$$

One verifies by explicit computation that R is a tensor, namely that it is $\mathfrak{F}$-linear. Moreover, all usual (anti)symmetry properties of R as well as the usual algebraic identities (Bianchi identities) relating R and T are true [Ne].

On a finite dimensional Lie algebra E over the field $\mathbb{R}$, there are interesting examples of affine connections [Hel]. In this cases an affine connection is the same as an $\mathbb{R}$-linear map B from $E \times E$ into E . Affine connections such that $B(X, Y) = - B(Y, X)$ give $B(X, X) = 0$. If E is the real Lie algebra of a Lie group G , such connections imply that any left invariant vector field on G is autoparallel and the geodesics starting from the identity of G are just the 1-parameter subgroups (one can prove that any bilinear map from $E \times E$ into E determines a left invariant affine connection on the manifold of G and viceversa [Hel]). Well known examples of this situation are: $B(X, Y) = 0$ (the (−)-connection), $B(X, Y) = 1/2[X, Y]$ (the (0)-connection and $B(X, Y) = [X, Y]$ (the (+)-connection). As for the torsion and the curvature of these connections, they are given by $T(X, Y) = - [X, Y]$, $R = 0$ for (−) , $T(X, Y) = 0$, $R(X, Y)Z = -1/4\ [[X, Y], Z]$ for (0) , $T(X, Y) = - [X, Y]$, $R = 0$ for (+) .
The previous affine connections can be defined for any Lie algebra E over $\mathfrak{F}$.

We now introduce the notion of a (Riemannian) metric on a Lie module E and specialize accordingly the notion of affine connection. A *Riemannian metric* on the Lie module E is an isomorphism g from the module E to the module E* which is symmetric, namely which satisfies $g(X, Y) = g(Y, X)$, for any $X, Y \in E$, where $g(X,Y) =: [g(X)](Y)$ .With any metric there is naturally associated a covariant tensor of rank 2.

Given a metric we may define its Levi-Civita connection. If $X \in E$ and $\varphi \in E^*$, the covariant derivative $\nabla_X \varphi$ associated with the metric g is defined by

$$\nabla_X \varphi\ (Y) =: 1/2\ \{\ X \cdot \varphi(Y) - Y \cdot \varphi(X) - \varphi([X, Y])$$

$$+ [g^{-1}(\varphi)] \cdot (g(X, Y)) - g(\mathcal{L}_{g^{-1}(\varphi)} X, Y) - g(X, \mathcal{L}_{g^{-1}(\varphi)} Y)\ \}$$

$$= 1/2\ \{\ (d\varphi)(X, Y) + (\mathcal{L}_{g^{-1}(\varphi)}\, g)(X, Y)\ \} \quad , \quad \forall\ Y \in E\ . \qquad (13)$$

From last equality it follows that $\nabla_X\varphi(Y)$ is $\mathcal{F}$-linear in both X and Y. Moreover, one easily checks that all other defining properties (7) are satisfied. The connection (13) is the Levi-Civita connection of the metric g in the sense that one has : i) $\nabla_X g = 0$; ii) the torsion vanishes. One can prove that, given a Riemannian metric on a module E there is a unique Levi-Civita connection associated with it [Ne].

If R is the curvature as defined in (10), then the *Riemann tensor* of the Levi-Civita connection ∇ is the covariant rank four tensor Riem defined by

$$\mathrm{Riem}(X_1, X_2, X_3, X_4) =: g(X_1, R(X_3, X_4)X_2) , \quad \forall\ X_i \in E . \tag{14}$$

From (14) one gets the usual properties :

i) $\mathrm{Riem}(X, Y, Z, W) = -\mathrm{Riem}(Y, X, Z, W) = -\mathrm{Riem}(X, Y, W, Z) = \mathrm{Riem}(Z, W, X, Y)$;

ii) $\mathrm{Riem}(X, Y, Z, W) + \mathrm{Riem}(Y, Z, X, W) + \mathrm{Riem}(Z, X, Y, W) = 0$. (15)

In order to define the Ricci tensor we need to introduce a *contraction operation* C on tensors. By the assumption of total reflexivity it suffices to define C on rank two tensors. If α is a rank two tensor then $C(\alpha) \in \mathcal{F}$, and C is linear and satisfies $C(\varphi \otimes \psi) = \varphi(g^{-1}\psi)$, $\forall\ \varphi, \psi \in E^*$. We assume the ring $\mathcal{F}$, the module E, and the metric g to be such that there exists a unique operation C which fulfils previous properties (contraction properties).

With Y, W fixed, $S(X, Z) =: \mathrm{Riem}(X, Y, Z, W)$ is bilinear in X, Z. Then C(S) is bilinear in W, Y and defines the covariant rank two *Ricci tensor* Ric(W, Y). Ric can be defined equivalently as

Ric(Y, W) = trace of the map $V \to R(V, Y)W$ of the Lie module E,

where the trace is with respect to the contraction operator [KN1]. In the particular case in which the Lie module E is of finite type as an $\mathcal{F}$-module, namely it is generated by a finite number of elements $\{e_i, i = 1,...,n\}$, we can write

$$\mathrm{Ric}(Y, W) = \Sigma_i\ \mathrm{Riem}(e_i, W, e_i, Y) .$$

If E is a semisimple Lie algebra (over the real), so admitting a non degenerate

(Killing-Cartan) metric, then one can show that its (0)-connections, which is torsionless, it is also metric and is therefore the Levi-Civita connection. Moreover, the Ricci tensor turns out to be proportional (the proportional constant being positive) to the metric itself. Therefore, if E is the Lie algebra of a compact semisimple Lie group G , the latter is an Einstein space.

With all previous ingredients we can slightly modify Geroch's definition of *Einstein algebra* [Ge]. An *Einstein algebra* consists of : i) a commutative ring $\mathcal{F}$ with unity which contains a subring $\mathcal{R}$ isomorphic to the real numbers, ii) a Lie $\mathcal{F}$-module E with a metric g which is such that the contraction properties are satisfied and the Ricci tensor vanishes (in the original definition the module E was just the module Der$\mathcal{F}$ of all derivations of $\mathcal{F}$). An Einstein algebra is then a generalized theory of vacuum Einstein's equations. The main advantage is that there is no reference to a manifold structure, though any space-time which is a solution of Einstein's equations does define an Einstein algebra. In section 4.7. we shall give an example of Einstein algebra.

2.3. THE CODIFFERENTIAL

Given a riemannian metric on a Lie module, one can introduce the codifferential. As we shall see, to this aim one does not need an orientation nor a volume form but only a metric and the associated Levi-Civita connection. The general construction is a little bit involved although it can be carried over by using, for instance, an abstract index notation [Ne]. For the sake of simplicity we shall assume that the Lie module E is of finite type as an $\mathcal{F}$-module, namely that it is generated by a finite number of elements $\{e_i, i = 1,...,n\}$ (by the way, by Whitney embedding theorem this is always the case for the algebra of vector fields on paracompact manifolds). The dual basis will be indicated with $\{\varepsilon^i, i = 1,...,n\}$, $\varepsilon^i(e_j) = \delta^i{}_j$. Then the *codifferential* is defined in the following way

$$\delta : \Lambda^p(E, \mathcal{F}) \to \Lambda^{p-1}(E, \mathcal{F}) ,$$

$$(\delta\varphi)(X_1,...,X_{p-1}) =: - \Sigma_i (\nabla_{e_i}\varphi)(g^{-1}(\varepsilon^i), X_1,..., X_{p-1}) \ , \quad \forall\, X_i \in E \ . \qquad (16)$$

In particular, if $\varphi \in E^*$, its *divergence* is the element of $\mathcal{F}$ given by

$$\mathrm{div}\varphi =: \delta\varphi \ = \ - \Sigma_i (\nabla_{e_i}\varphi)(g^{-1}(\varepsilon^i)). \qquad (17)$$

By explicit calculation one can prove that $\delta^2 = 0$.

Given a metric on the Lie module E, there is also an induced metric on the vector spaces $\Lambda^p(E, \mathcal{F})$. Indeed one can define

$$g^{-1} : \Lambda^p(E, \mathcal{F}) \times \Lambda^p(E, \mathcal{F}) \rightarrow \mathcal{F} ,$$

$$g^{-1}(\varphi, \psi) =: 1/p!\, \Sigma\, \varphi(e_{i_1},\ldots, e_{i_p})\, \psi(g^{-1}(\varepsilon^{i_1}),\ldots, g^{-1}(\varepsilon^{i_p})) \ . \tag{18}$$

Here the summation is over repeated indeces. Again, by using the abstract index notation the previous definition can be extended to modules which are not of finite type.

Proposition 2.1. The codifferential δ is the adjoint operator of d in the sense that, for any $\varphi \in \Lambda^{p-1}(E, \mathcal{F})$ and any $\psi \in \Lambda^p(E, \mathcal{F})$ one has

$$g^{-1}(d\varphi , \psi) - g^{-1}(\varphi, \delta\, \psi) = \mathrm{div}(\mathrm{something}) \ . \tag{19}$$

A preliminar result that we need for the proof is the following proposition which can be proved by explicit calculations (see also [Kos2])

Proposition 2.2. The connection ∇ being torsionless it follows that for any $\varphi \in \Lambda^p(E, \mathcal{F})$

$$(d\varphi)(X_1,\ldots,X_{p+1}) = \Sigma_i\, (-1)^{i+1}(\nabla_{X_i}\varphi)(X_1,\ldots,\hat{\imath},\ldots, X_{p+1}) \ . \tag{20}$$

We can now prove proposition 2.1. For the sake of simplicity we give the proof with the assumption that E is of finite type. From definition (18) and (20) one has

$$g^{-1}(d\varphi, \psi) = 1/p!\, \Sigma\, (d\varphi)(e_{i_1},\ldots, e_{i_p})\, \psi\circ g^{-1}(\varepsilon^{i_1},\ldots, \varepsilon^{i_p})$$

$$= 1/p!\, \Sigma\, \Sigma_k\, (-1)^{k+1}(\nabla_{e_k}\varphi)\, (e_{i_1},\ldots,\hat{\imath}_k,\ldots, e_{i_p})\, \psi\circ g^{-1}(\varepsilon^{i_1},\ldots, \varepsilon^{i_p})$$

$$= 1/p!\, \Sigma\, \Sigma_k\, (\nabla_{e_{ik}}\varphi)\, (e_{i_1},\ldots,\hat{\imath}_k,\ldots, e_{i_p})\, \psi\circ g^{-1}(\varepsilon^{i_k}, \varepsilon^{i_1},\ldots,\hat{\imath}_k,\ldots, \varepsilon^{i_p})$$

$$= 1/(p-1)!\, \Sigma\, \Sigma_j\, (\nabla_{e_j}\varphi)\, (e_{i_1},\ldots,e_{i_{p-1}})\, \psi\circ g^{-1}(\varepsilon^j, \varepsilon^{i_1},\ldots,\varepsilon^{i_{p-1}})$$

$$= 1/(p-1)! \, \Sigma \, \Sigma_j \{ - \varphi(e_{i_1},...,e_{i_{p-1}}) \, (\nabla_{e_j}\psi \circ g^{-1})(\varepsilon^j, \varepsilon^{i_1},...,\varepsilon^{i_{p-1}}) +$$

$$+ \, [\nabla_{e_j} (\varphi(e_{i_1},...,e_{i_{p-1}}) \, \psi \circ g^{-1} (\cdot , \varepsilon^{i_1},...,\varepsilon^{i_{p-1}}))] \, (\varepsilon^j) \}$$

$$= - g^{-1}(\varphi, \delta \psi) + \mathrm{div} \, \{ 1/(p-1)! \, \Sigma \, \varphi(e_{i_1},...,e_{i_{p-1}}) \, \psi \circ g^{-1} (\cdot , \varepsilon^{i_1},...,\varepsilon^{i_{p-1}}) \} .$$

This ends the proof of (19).

Having the codifferential (together with the exterior differential), one could also introduce the Laplace operator on the algebra $\Lambda^*(E, \mathfrak{F})$ and construct an algebraic theory of harmonic forms [Ne]. However we shall not need such machineries in the following.

In section 2.5. the codifferential will be generalized to a covariant codifferential. The latter will be used in section 4.1. to introduce field equations for our algebraic theory.

2.4. EXTENSIONS OF LIE ALGEBRAS AND FIBRED STRUCTURES

In the next section we shall show how to generalize the algebraic calculus developed in sections 2.1.–2.2. to cases in which the Lie algebra E is an extension of Lie algebras. These will turn out to be the appropriate framework for gauge theories. In this section we recall the essential facts of the theory of extensions of Lie algebras. There are many mathematical papers on the subject [CE], [Cal], [Jaco], [MacL].

The Lie algebra E is called an extension of the Lie algebra B by the Lie algebra A if there exists a short exact sequence of Lie algebras

$$0 \rightarrow A \overset{i}{\rightarrow} E \overset{\pi}{\rightarrow} B \rightarrow 0 \tag{21}$$

Maps i and π are respectively the injection of A in E and the projection of E onto B with $\mathrm{im}\, i = \ker \pi$. For the time being we take the Lie algebras A , E and B to be Lie algebras over a field of numbers $\mathfrak{R}$ (to be definite one could think of $\mathfrak{R}$ as the field of real numbers). The extension (21) will be denoted with E(B, A).

Given any extension E(B, A) there is canonically associated an exact sequence of algebras of derivations as in the following diagrams

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & C(A) & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & A & \xrightarrow{i} & E & \xrightarrow{\pi} & B & \longrightarrow & 0 \\
 & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\
0 & \longrightarrow & \mathrm{In\,D}(A) & \longrightarrow & D(A) & \longrightarrow & \mathrm{Out\,D}(A) & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array} \qquad (22)$$

Here $C(A)$ is the centre of A. Moreover, $D(A)$ is the Lie algebra of all derivations of A. As a vector space, $D(A)$ is the collection of all maps φ from A into itself such that $\varphi([X,Y]) = [\varphi(X), Y] + [X, \varphi(Y)]$, for any $X, Y \in A$; the Lie bracket of any two derivations φ and ψ is defined as usual by $[\varphi, \psi] = \varphi \circ \psi - \psi \circ \varphi$. $\mathrm{InD}(A)$ is the Lie algebra of inner derivations of A, i.e. the adjoint action of A on itself $D_a(a') =: [a, a']$ for some $a \in A$; this also defines the homomorphism α. Finally, $\mathrm{OutD}(A)$ is the Lie algebra of outer derivations of A, that is of equivalence classes of derivations defined modulo inner ones. If we consider the cohomology of A with coefficient into itself and with respect to the adjoint action, we see from their definitions that $D(A)$, $\mathrm{InD}(A)$ and $\mathrm{OutD}(A)$ are the space of 1-cocycles, the space of 1-coboundaries and the first cohomology group respectively. Identifying A with its image in E, β is defined as

$$\beta : E \to D(A), \quad e \to D_e, \quad D_e(a) = [e, a] \in A. \qquad (23)$$

Finally, γ is defined to make the last diagram in (22) commute. The map γ is called the *character* of the extension. Two extensions $A \xrightarrow{i} E \xrightarrow{\pi} B$ and $A \xrightarrow{i'} E' \xrightarrow{\pi'} B$ with the same character are said to be equivalent if there exists an isomorphism $f : E \to E'$ such that $i' = f \circ i$ and $\pi = f \circ \pi'$.

The extension $E(B, A)$ is called: *inessential*, if E is a semidirect sum of A with a supplementary Lie algebra L of A in E (L is then isomorphic with B); *trivial*, if it is inessential and L is an ideal, i.e. E is a direct sum $E = L \oplus A$; *central*, if the character $\gamma = 0$ or equivalently if the image of β is in $\mathrm{InD}(A)$. For A abelian, A is in the centre of E if and only if the extension is central. For A not abelian, if A is in the center of E then $E(B, A)$ is central but the converse is not true any longer.

Central extensions of Lie algebras (and Lie groups) are familiar in physics since Bargmann's proof that a central extension of the Galilei group is a symmetry group of the non relativistic Schroedinger equation while the Galilei group itself is not [Ba]. Similar situations are also present in anomalous quantum systems where it may happen that only an extension of a classical algebra of symmetry acts as a symmetry at quantum level. An example is the Faddeev's formulation of anomalous theories where the commutator of Gauss law constraints exhibits a central term (Schwinger term) [Fa]. Central extensions are also used in the geometric quantization program [Si], [Wood] and are present in the problem of quasi-invariance of lagrangian dynamical systems [MMSS]. For relations between central extensions of the Lie algebra of a Lie group and central extensions of this Lie group, together with many examples of their use in physics see [TW]. Virasoro algebra, which is a central extension of the Lie algebra of vector fields on the unit sphere S^1, Kac-Moody algebras, which are central extensions of loop algebras, and central extension of current algebras (where the central term is known as Schwinger term) have been recently studied expecially in connection with string theory and two-dimensional conformal quantum theories; for a rewiev see [GO]; for many mathematical results see [Se]. The theory of (not necessarily central) extensions of Lie algebras ang groups has also been used in connection with the problem of mixing internal and space time symmetry, mainly by studying extensions of the Poincaré group and of its algebra [Mi], [Ga], [Cat].

Before we proceed with our scheme, we give few elements of the theory of classification of Lie algebra extensions. What follows is based mainly on [Cal]. Let then $E(B, A)$ be an extension like (21). A section σ of $E(B, A)$ is any $\mathcal{R}$-linear map

$$\sigma : B \to E, \quad \text{such that} \quad \pi \circ \rho = id_B . \tag{24}$$

Given any section σ, we have a decomposition of vector space $E = A \oplus \sigma(B)$, namely any element $e \in E$ can be uniquely written as $e = i(a)+\sigma(b)$ with $a \in A$ and $b \in B$. In general $\sigma(B)$ is not a subalgebra of E. The factor set f_2 of σ measures how far σ is from be a Lie algebra homomorphism. f_2 is the A-valued $\mathcal{R}$-linear skew map on $B \times B$ defined by

$$f_2(b, b') =: \sigma([b, b']) - [\sigma(b), \sigma(b')], \quad \forall \ b, b' \in B . \tag{25}$$

Next one defines a prerepresentation of B into $D(A)$ lifted over γ and associated with f_2, namely the mapping

$$\phi = B \to D(A), \quad \phi =: \beta \circ \sigma, \quad \phi(b) \cdot a =: [\sigma(b), a], \quad \forall \ b \in B, a \in A, \tag{26}$$

which has the property that

$$\phi(b)\,\phi(b') - \phi(b')\,\phi(b) = \phi([b, b']) + \mathrm{ad}_{f_2(b,\, b')} \quad \forall\ b, b' \in B . \tag{27}$$

If g_2 is any A-valued $\mathbb{R}$-linear skew mapping on $B\times B$ (not necessarily associated with a section like in (25)) and $\phi = B \to D(A)$ any prerepresentation of B into $D(A)$, lifted over $\gamma : B \to \mathrm{OutD}(A)$, and associated with g_2 like in (27), the couple (ϕ, g_2) is called a 2-pseudocochain of (B, A, γ). Given the couple (ϕ, g_2) one defines a 3-pseudocochain by

$$\delta_\phi g_2(b_1, b_2, b_3) =: \phi(b_1)\cdot g_2(b_2, b_3) - g_2([b_1, b_2], b_3) + \text{c. p.} \ , \ \forall\ b_1, b_2, b_3 \in B . \tag{28}$$

A couple (ϕ, g_2) such that $\delta_\phi g_2 = 0$ is called a 2-pseudocycle of (B, A, γ). As for the pseudocochain (ϕ, f_2) constructed in (25) and (26) one can show that Jacoby identity implies $\delta_\phi f_2 = 0$. In addition, if σ' is another section for the extension $E(B, A)$, then $\sigma' - \sigma = f_1$ is an $\mathbb{R}$-linear map from B to A; the corresponding pseudococycles (ϕ, f_2) and (ϕ', f_2') are related by

$$f_2'(b, b') = f_2(b, b') + (\delta_\phi f_1)(b, b') + [f_1(b), f_1(b')] \ ,$$

$$\phi'(b) = \phi(b) + \mathrm{ad}_{f_1(b)} . \tag{29}$$

Any two pseudococycles (not necessarily associated with a section) which are related as in (29) are said to be equivalent (one can easily show that (29) determines an equivalence relation). The vector space of equivalence classes of 2-pseudococycles is denoted by $\mathcal{H}^2{}_\gamma(B, A)$ and is called the Calabi pseudocohomology space of order 2, of the Lie algebra B with values in A, and associated with the character γ. This pseudocohomology can be defined only at order 2 and is deeply related with extensions of Lie algebras.
From (29) we see that two pseudococycles (ϕ, f_2) and (ϕ', f_2') associated with two sections of the extension (21) belong to the same equivalence class in $\mathcal{H}^2{}_\gamma(B, A)$. More generally, one can show that pseudococycles associated with equivalent extensions determine the same equivalence class in $\mathcal{H}^2{}_\gamma(B, A)$. Conversely, given a pseudococycle (ϕ, f_2), it is possible to construct an extension $E(B, A)$ of B by A with character γ. As a vector space $E = B \times A$ and π and σ are defined by $\pi(b, a) = b$ and $\sigma(b) = (b, 0)$. The Lie bracket in E is defined by

$$[(b_1, a_1), (b_2, a_2)] =: ([b_1, b_2],\ [a_1, a_2] + \phi(b_1)a_2 - \phi(b_2)a_1 + f_2(b_1, b_2)) \ ,$$

$$\forall\ b_1, b_2 \in B, a_1, a_2 \in A . \tag{30}$$

The Jacoby identity follows from the cocycle condition $\delta_\phi f_2 = 0$. Moreover one finds that $\phi(b)\cdot a = [\,\sigma(b), a\,]$. It turns out that cohomologous pseudocycles give equivalent extensions. As a consequence the set of equivalence classes of extensions of B by A with character γ is in a bijective correspondence with $\mathcal{H}^2{}_\gamma(B, A)$.
If A is abelian, one has few simplifications. Firstly $\mathrm{InD}(A) \equiv 0$ makes $D(A) \equiv \mathrm{OutD}(A)$ so that γ defines an action of B on A (in fact a representation). In addition, $\mathcal{H}^2{}_\gamma(B, A)$ is a true cohomology space and is nothing but the second cohomology group $H^2(B, A, \gamma)$ associated with the representation γ. In this case any extension is completely characterized by its factor set as defined in (25) [CE] . If A is not abelian, for some γ the space $\mathcal{H}^2{}_\gamma(B,A)$ can also be empty. In general, there is an obstruction to the possibility of constructing extensions. Using the theory of Lie algebra kernels [Mo], [Ho], one can show that the obstruction to constructing an extension of the Lie algebra B by the Lie algebra A, once given the character $\gamma : B \to \mathrm{OutDA}$, is an element in $H^3(B, C(A), \gamma)$ the third group of the cohomology of B with values in the centre of A. Moreover, if $\mathcal{H}^2{}_\gamma(B, A)$ is not empty, the set of equivalence classes of extensions of B by A is isomorphic with the space $H^3(B, C(A), \gamma)$ (not in a canonical way, namely one has to fix an extension). If A is abelian the obstruction vanishes automatically.

Cocycles of groups and algebras are also familiar to physicists nowadays; for a rewiew see [Jack2]. Of course 2-cocycles are responsible for central extensions and give rise to projective or ray representations. More recently, in [Car] it has been shown that a non trivial 3-cocycles of the algebra of infinitesimal gauge transformations is an obstruction to the existence of an extension of this algebra acting as a symmetry algebra at quantum level. Finally, in [LP] for any manifold M, it has been constructed an extension of the Lie algebra of vector field $\aleph(M)$ by the abelian Lie algebra of 2-forms $\Omega^2(M)$. This is possible because there is a privileged element in $H^2(\aleph(M), \Omega^2(M), \mathcal{L})$, the second group of the cohomology of the Lie algebra $\aleph(M)$ with values in the abelian algebra $\Omega^2(M)$.

We hope to use the theory of classification of Lie algebra extensions in the study of the classification of the gauge fields which will be introduced in the next section.

As we want to recover results on fibre bundles when our algebras are identified with algebras of vector fields, we endow the extension E(B, A) in (21) with additional structures. We take E(B, A) to be a *Lie bundle* over the couple $(\mathcal{F}, \mathcal{R})$ where $\mathcal{F}$ is a commutative $\mathcal{R}$-algebra with unit. This means that A, E and B are Lie algebras over $\mathcal{R}$ and left Lie modules over $\mathcal{F}$. Therefore there is a representation of A, E and B into the Lie algebra $\mathrm{Der}\mathcal{F}$ of all derivations of $\mathcal{F}$. Since A is an ideal in E it is forced to act trivially on $\mathcal{F}$ and A is a Lie algebra also over $\mathcal{F}$.

Given any fibre bundle $\pi : P \to M$ we can naturally associate with it a sequence of infinite dimensional Lie algebras. Indeed, let us consider the algebra $\mathcal{F}(P)$ of smooth real valued functions on P and the subalgebra $\pi^*\mathcal{F}(M)$. Let $\aleph^{\pi}(P)$ be the subalgebra of derivations of $\mathcal{F}(P)$ which are also derivations for $\pi^*\mathcal{F}(M)$, i.e. which take $\pi^*\mathcal{F}(M)$ into itself. The subalgebra of $\aleph^{\pi}(P)$ of derivations which are zero when applied to $\pi^*\mathcal{F}(M)$ is an invariant subalgebra in E, which we call $\aleph^{v}(P)$. The algebra $\aleph^{\pi}(P)$ is nothing but the algebra of projectable vector fields on P whereas $\aleph^{v}(P)$ is the algebra of vertical vector fields on P. Then the quotient algebra $\aleph^{\pi}(P) / \aleph^{\pi}(P) = \aleph(M)$ is the algebra of smooth vector fields on M (when M is paracompact) and we get an extension of algebras $0 \to \aleph^{v}(P) \to \aleph^{\pi}(P) \to \aleph(M) \to 0$ which is a Lie bundle over the algebra of functions $\mathcal{F}(M)$.

It is worth mentioning that there are extensions of Lie algebras of vector fields which are not associated with any fibre bundle. We give now a simple example of such a situation. Let us consider the two dimentional torus T^2 parametrized by two angles θ_1 and θ_2. A sequence like (21) is constructed as follows. With a and b real numbers, E is generated by the vector fields $X_1 = a\,\partial/\partial\theta_1 + b\,\partial/\partial\theta_2$ and $X_2 = a\,\partial/\partial\theta_1 - b\,\partial/\partial\theta_2$ and A is generated by X_1 alone. As a consequence B is generated by X_2. If a/b is an irrational number the only functions which are left invariant by A are the constant ones so that the resulting sequence is a Lie bundle only over $\mathbb{R}$. Moreover, there is no manifold which X_2 is tangent to, so that the sequence is not associated with any fibre bundle.

If $\pi : P \to M$ is a vector bundle, one can construct another natural sequence associated with it which is a subsequence of the one considered above, in a sense which will be clear in few lines [Lec1]. Let Hom(P, P) be the bundle of endomorphisms of P; it is a bundle over M. The space gl(P) of smooth sections of Hom(P,P) is made a Lie algebra by a pointwise bracketing. The *vertical lift* X^v of any $X \in gl(P)$ is the vector field on P whose flow is $(t, u) \in R \times P \to (\exp tX)(u) \in P$. The space $gl(P)^v$ of all vertical lifts is clearly a Lie subalgebra of the algebra of vertical fields $\aleph^{v}(P)$. The vertical lift associated with the identity $\mathbb{I} \in gl(P)$ is denoted by Δ and is called the Liouville's field of P; Δ is the generator of dilatations along the fibres. One can show that any vertical vector field on P is a vertical lift iff it commutes with Δ. The Lie algebra of *infinitesimal automorphisms* $\mathcal{U}_P$ of P is the commutant of Δ in $\aleph(P)$ that is $\mathcal{U}_P =: \{ X \in \aleph(P) \text{ such that } [\Delta, X] = 0 \}$. One can show that $\mathcal{U}_P$ is a subalgebra of $\aleph^{\pi}(P)$, that $gl(P)^v$ is an ideal in $\mathcal{U}_P$ and that $\mathcal{U}_P/gl(P)^v = \aleph(M)$. As a consequence we get an extension of infinite dimensional Lie algebras $0 \to gl(P)^v \to \mathcal{U}_P \to \aleph(M) \to 0$ which is again a Lie bundle over the functions $\mathcal{F}(M)$. In [Lec1] it has been shown that if the first Cech cohomology space $H^1(M, \mathbb{Z}_2)$ of M vanishes, then two vector bundles P and P' over M are isomorphic if and only if the Lie algebras $\mathcal{U}_P$ and $\mathcal{U}_{P'}$ are isomorphic.

2.5. EXTERIOR CALCULUS OVER LIE ALGEBRA EXTENSIONS

As we have pointed out in the introduction, given an extension of Lie algebras of vector fields, one can introduce on it the analogue of a gauge potential. The latter is just a way to lift vector fields from a Lie algebra to another in the sequence. In general this lifting does not preserves the Lie algebra structure. The deviation from a Lie algebra homomorphism is to be interpreted as a field strength. In this section we shall generalize this fact to any extension of Lie algebras, not necessarily associated with a fibre bundle. The outcome will be the 'kinematic' for an algebraic gauge theory. In chapter 4. we shall give the 'dynamics' for such a theory, together with few examples.

The basic object in this section is an extension of Lie algebras $0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$ like (21) which is a Lie bundle over the commutative $\mathfrak{R}$-algebra with unity $\mathfrak{F}$. We remind that this means that A, B and E are Lie algebras over $\mathfrak{R}$; in addition they are left Lie module over $\mathfrak{F}$ and there is a representation of A, E and B into the Lie algebra Der$\mathfrak{F}$ of derivations of $\mathfrak{F}$.

We call *connection* on the Lie bundle $0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$, any $\mathfrak{F}$-module homomorphism or, equivalently, any splitting of E as an $\mathfrak{F}$-module

$$\rho : B \rightarrow E \qquad \text{such that} \qquad \pi \circ \rho = id_B \ . \tag{31}$$

Notice that in contrast to what happens for a section like (24), here we are requiring $\mathfrak{F}$-linearity and not merely $\mathfrak{R}$-linearity. If ρ and ρ' are any two connections, their difference $\rho - \rho'$ is an $\mathfrak{F}$-linear map from B to A. The set of all connections is then an affine space modelled on $Lin_{\mathfrak{F}}(B, A)$.

Equivalently we could give an A-valued $\mathfrak{F}$-linear *connection 1-form* ω on E

$$\omega : E \rightarrow A \qquad \text{such that} \qquad \omega \circ i = id_A \ , \tag{32}$$

the relation between the two being

$$\omega = id_E - \rho \circ \pi \ . \tag{33}$$

Any connection allows to write E as the module direct sum $E = A \oplus HorE$, with the horizontal $\mathcal{F}$-module in E defined as HorE =: ker $\omega = \rho(B)$. In general HorE is not a subalgebra of E or equivalently ρ is not a Lie algebra homomorphism. The *curvature* F measures the extent to which ρ fails to be a Lie algebra homomorphism

$$F(X_1, X_2) =: \rho([X_1, X_2]) - [\rho(X_1), \rho(X_2)] \ , \quad \forall \ X_1, X_2 \in B \ . \tag{34}$$

Since π is a homomorphism and $\pi \circ \rho = 1_B$, then $\pi \circ F = 0$ and $F \in \Lambda^2(B, A)$, i.e F is an A-valued $\mathcal{F}$-linear skew map from B×B to A. The horizontal module HorE is a subalgebra of E if and only if the connection is flat, i.e. if and only if $F \equiv 0$; then E is the semidirect sum of A and B once the latter is identified with its image in E through ρ.

In the introduction we have given a simple example of an extension of Lie algebras of vector fields endowed with a connection. Other, physically relevant examples will be given in section 4.2..

Given a connection, the *curvature 2-form* Ω is the A-valued $\mathcal{F}$-linear skew map on E×E defined by

$$\Omega(Y_1, Y_2) =: F(\pi Y_1, \pi Y_2) \ , \quad \forall \ Y_1, Y_2 \in E \ , \tag{35}$$

and in terms of ω

$$\Omega(Y_1, Y_2) = [Y_1, \omega(Y_2)] - [Y_2, \omega(Y_1)] - \omega([Y_1,Y_2]) - [\omega(Y_1), \omega(Y_2)] \ . \tag{36}$$

By its definition Ω is horizontal, i.e. $\Omega(Y_1,Y_2) = 0$ whenever one of the Y_i's is in A.

We continue by introducing the *covariant differential* D_ρ. Let $\Lambda^p(B, A)$ be the vector space of $\mathcal{F}$-linear, p-linear skew maps from $B\times\cdots\times B$ (p times) to A. The covariant differential is defined by

$$D_\rho : \Lambda^p(B, A) \to \Lambda^{p+1}(B, A) \ ,$$

$$(D_\rho\varphi)(X_1,\ldots,X_{p+1}) =: \Sigma_i (-1)^{i+1}[\rho(X_i), \varphi(X_1,\ldots,\hat{\imath},\ldots,X_{p+1})] +$$

$$+ \Sigma_{i<j} (-1)^{i+j}\varphi([X_i, X_j],X_1,\ldots,\hat{\imath},\ldots,\hat{j},\ldots,X_{p+1}) \ , \quad \forall \ X_i \in B \ . \tag{37}$$

In particular it is easy to prove Bianchi identity

$$D_\rho F = 0 . \tag{38}$$

In order to introduce a *covariant codifferential* we need a metric on B, namely a symmetric non degenerate map $g : B\times B \to \mathcal{F}$. We shall indicate with ∇ the corresponding Levi-Civita connection as constructed in section 2.2. We first define a generalization of the covariant derivative (13). Given any $X \in B$, we define an operator

$$\nabla^\rho{}_X : \Lambda^p(B, A) \to \Lambda^p(B, A) ,$$

$$(\nabla^\rho{}_X \varphi)(X_1,..,X_p) =: [\rho(X), \varphi(X_1,..,X_p)] - \Sigma_i \varphi(X_1,..,\nabla_X X_i,..,X_p), \quad \forall X_i \in B . \tag{39}$$

Then, by direct calculations one can prove an analogue of proposition 2.2.

Proposition 2.3. The connection ∇ being torsionless it follows that

$$(D_\rho\varphi)(X_1,..,X_{p+1}) = \Sigma_i (-1)^{i+1}(\nabla^\rho{}_{X_i} \varphi)(X_1,..,\hat{\imath},..X_{p+1}) , \quad \forall \varphi \in \Lambda^p(B, A) . \tag{40}$$

We are now ready to define the covariant codifferential $\mathcal{D}_\rho$. As it happens for the codifferential, the general construction is involved although it may be managed, again by means of abstract index notations [Ne], [Pe], [AHM]. For the sake of simplicity we take B to be a module of finite type with basis $\{b_i, i = 1,..,n\}$ and dual basis $\{\beta^i, i = 1,..,n\}$, $\beta^i(b_j) = \delta^i{}_j$. Then we define the codifferential to be

$$\mathcal{D}_\rho : \Lambda^p(B, A) \to \Lambda^{p-1}(B, A) ,$$

$$(\mathcal{D}_\rho\varphi)(X_1,...,X_{p-1}) =: - \Sigma_i (\nabla^\rho{}_{b_i}\varphi)(g^{-1}(\beta^i), X_1,..., X_{p-1}) , \quad \forall X_i \in B . \tag{41}$$

Let us now assume there is a metric also on the Lie algebra A, $h : A\times A \to \mathcal{F}$ which is symmetric non degenerate. For later convenience we take it to be adjoint invariant, namely

$$h([Z , Z_1] , Z_2) + h(Z_1 , [Z , Z_2]) = 0 , \quad \forall Z, Z_1, Z_2 \in A . \tag{42}$$

In addition h is 'invariant' under the action of B ; by this we mean that

$$X\cdot h(Z_1, Z_2) - h([\rho(X), Z_1], Z_2) - h(Z_1, [\rho(X), Z_2]) , \quad \forall\, X \in B, Z_1, Z_2 \in A . \quad (43)$$

Given the metrics g and h, we shall indicate with (hg^{-1}) the natural metric induced on $\Lambda^p(B, A)$. If B is a module of finite type, (hg^{-1}) is given by

$$(hg^{-1}) : \Lambda^p(B, A) \times \Lambda^p(B, A) \rightarrow \mathcal{F} ,$$

$$(hg^{-1}) (\varphi, \psi) =: 1/p!\ \Sigma\ h \left(\varphi(b_{i_1},..., b_{i_p}) , \psi(g^{-1}(\beta^{i_1}),...,g^{-1}(\beta^{i_p})) \right) . \quad (44)$$

This definition can be extended to arbitrary module B by using the abstract index notation. By using (40) one can prove that the covariant codifferential $\mathcal{D}_\rho$ is the dual of D_ρ with respect to the metric (hg^{-1}). One has indeed the following proposition.

Proposition 2.4. The covariant codifferential $\mathcal{D}_\rho$ is the dual of D_ρ with respect to the metric (hg^{-1}) in the sense that for any $\varphi \in \Lambda^p(B, A)$ and any $\psi \in \Lambda^{p+1}(B, A)$ one has

$$(hg^{-1})(D_\rho\varphi, \psi) - (hg^{-1}) (\varphi, \mathcal{D}_\rho\varphi) = \mathrm{div}(\text{something}) . \quad (45)$$

The proof of proposition 2.4. is similar to the proof of proposition 2.1. and we shall omit it. In the simpler case in which B is of finite type (45) reduces to

$$(hg^{-1})(D_\rho\varphi, \psi) - (hg^{-1}) (\varphi, \mathcal{D}_\rho\varphi) =$$

$$= \mathrm{div} \left\{1/(p-1)!\ \Sigma\ h (\varphi(b_{i_1},...,b_{i_{p-1}}) , \psi\circ g^{-1} (\cdot , \beta^{i_1},...,\beta^{i_{p-1}})) \right\} . \quad (46)$$

In chapter 4. we shall use the operations defined in this section to develop a full gauge theory. In particular the codifferential $\mathcal{D}_\rho$ will be used to introduce field equations, while relation (45) will allow us to derive them from a lagrangian.

2.6. CHARACTERISTIC CLASSES AND CHERN-WEIL HOMOMORPHISM

In this section we shall construct a Chern-Weil homomorphism and Chern-Simons secondary characteristic classes for an extension of Lie algebras endowed with a connection. As an example we shall recover the usual construction for a principal fibre bundle. This will also allow to construct characteristic classes for the Gel'fand-Fuks cohomology of the Lie algebra of vector fields over the base manifold. Similar constructions have been also presented in [TeN], [Lec2]. In section 4.6 we shall apply the techniques developed in this section to the problem of anomalies in Yang-Mills gauge theories. For the use of Chern-Simons forms in physics, notably in quantum field theory see [Jack1].

The basic object is an extension of Lie algebras $0 \to A \to E \to B \to 0$ like (21), which is a Lie bundle over an $\mathcal{R}$-algebra with unit $\mathcal{F}$. Moreover there is a connection ρ as defined in (31) with connection form ω and curvature form Ω given respectively by (32) and (35).

Let us start by taking the tensor algebra over A

$$A^* = \oplus_k A^{\otimes k} , \quad A^{\otimes k} = A \otimes \cdots \otimes A \quad (k \text{ times}), \quad A^0 = \mathcal{F} \ . \tag{47}$$

Since A is an ideal in E it is possible to give a representation r of E in the Lie algebra of derivations of $A^{\otimes k}$. Take

$$r : \ E \times A^{\otimes k} \to A^{\otimes k} ,$$

$$r(Y)\cdot(Z_1 \otimes \cdots \otimes Z_k) =: \Sigma_i \ Z_1 \otimes \cdots \otimes [Y , Z_i] \otimes \cdots \otimes Z_k \tag{48}$$

and extend r as a derivation to all of A^*. One can prove that r is a representation, i.e.,

$$r([X, Y]) = r(X) \circ r(Y) - r(Y) \circ r(X) \ , \quad \forall \ X ,Y \in E \ . \tag{49}$$

Furthermore, since A acts trivially on $\mathcal{F}$, r is $\mathcal{F}$-linear.

Next we take $\Lambda^{p,k} =: \Lambda^p(E, A^{\otimes k})$ the space of $\mathcal{F}$-linear, p-linear skew maps from $E \times \cdots \times E$ (p times) to $A^{\otimes k}$ (for short $A^{\otimes k}$-valued p-forms on E). By using representation (48) we define the operator

$$d_r : \Lambda^{p,k} \to \Lambda^{p+1,k} ,$$

$$d_r \varphi(Y_1,...,Y_{p+1}) =: \Sigma_i (-1)^{i+1} r(Y_i)\cdot\varphi(Y_1,...,\hat{\imath},...,Y_{p+1}) +$$

$$+ \Sigma_{i<j} (-1)^{i+j} \varphi([Y_i, Y_j],Y_1,...,\hat{\imath},...,\hat{j},...,Y_{p+1}) , \quad \forall\ Y_i \in E . \qquad (50)$$

Since r is a representation it follows by standard techniques that d_r is a coboundary operator, that is $(d_r)^2 = 0$.

We can define an exterior product by

$$\wedge : \Lambda^{p,k} \times \Lambda^{q,h} \to \Lambda^{p+q,k+h}$$

$$\varphi \wedge \psi (Y_1,...,Y_{p+q}) =: 1/p!q!\ \Sigma_\sigma \chi(\sigma)\ \varphi(Y_{\sigma(1)},...,Y_{\sigma(p)}) \otimes \psi(Y_{\sigma(p+1)},...,Y_{\sigma(p+q)}) ,$$

$$\forall\ Y_i \in E ; \qquad (51)$$

here $\chi(\sigma)$ is the sign of the permutation σ . It is also possible to define a bracket operation

$$[\ ,\ ] : \Lambda^{p,1} \times \Lambda^{q,1} \to \Lambda^{p+q,1} ,$$

$$[\varphi , \psi](Y_1,...,Y_{p+q}) =: 1/p!q!\ \Sigma_\sigma \chi(\sigma)\ [\varphi(Y_{\sigma(1)},...,Y_{\sigma(p)}) , \psi(Y_{\sigma(p+1)},...,Y_{\sigma(p+q)})] ,$$

$$\forall\ Y_i \in E . \qquad (52)$$

Using definition (50), (51) and (52) one can prove the following

Proposition 2.5. If $\varphi \in \Lambda^{p,k}$ and $\psi \in \Lambda^{q,h}$ then

$$d_r(\varphi \wedge \psi) = (d_r \varphi) \wedge \psi + (-1)^p \varphi \wedge d_r\psi , \qquad (53a)$$

moreover, if $\varphi \in \Lambda^{p,1}$, $\psi \in \Lambda^{q,1}$ and $\lambda \in \Lambda^{r,1}$ then

$$d_r([\varphi , \psi]) = [d_r \varphi , \psi] + (-1)^p [\varphi , d_r \psi] , \qquad (53b)$$

$$[\varphi , \psi] = - [\psi , \varphi] , \qquad (53c)$$

$$(-1)^{pr} [\varphi , [\psi , \lambda]] + (-1)^{qp} [\psi , [\lambda , \varphi]] + (-1)^{rq} [\lambda , [\varphi , \psi]] = 0 . \qquad (53d)$$

By using (50) and (52) the curvature Ω in (35) may be written in the form

$$\Omega = d_r \omega - 1/2 [\omega, \omega], \tag{54}$$

and by using (53b-d), the Bianchi identity (38) is equivalent to

$$d_r\Omega - [\omega, \Omega] = 0 . \tag{55}$$

Definition. A p-form $\varphi \in \Lambda^p(E, \mathcal{F})$ is projectable onto a p-form $\underline{\varphi} \in \Lambda^p(B, \mathcal{F})$ if

$$\varphi(Y_1,\ldots,Y_p) = \underline{\varphi}(X_1,\ldots,X_p), \quad \forall\ Y_i \in E,\ X_i \in B \text{ such that } X_i = \pi Y_i . \tag{56}$$

The following lemma is immediate

Lemma 2.1. A p-form φ on E is projectable onto a unique p-form $\underline{\varphi}$ on B if φ is horizontal, that is if $\varphi(Y_1,\ldots,Y_p) = 0$ whenever one of the Y_i's belong to A.

Let now $I(A^{\otimes k}, \mathcal{F})$ be the space of symmetric $\mathcal{F}$-multilinear maps $W : A^{\otimes k} \to \mathcal{F}$ which are such that

$$r(Y)\, W = 0 , \quad \forall\ Y \in E . \tag{57}$$

Here $r(Y)W$ is defined by

$$r(Y)W\ (Z_1 \otimes\cdots\otimes Z_k) =: Y\cdot W(Z_1 \otimes\cdots\otimes Z_k) - W(r(Y)\cdot(Z_1 \otimes\cdots\otimes Z_k)), \quad \forall\ Z_i \in A . \tag{58}$$

Condition (57), together with the trivial action of A on $\mathcal{F}$ implies 'adjoint invariance'

$$\Sigma_i W(Z_1 \otimes\cdots\otimes [Z, Z_i] \otimes\cdots\otimes Z_k) = 0 , \quad \forall\ Z,\ Z_i \in A . \tag{59}$$

We take $I(A, \mathcal{F}) =: \oplus_k I(A^{\otimes k},\mathcal{F})$ and make it into an algebra by defining, for $W \in I(A^{\otimes k},\mathcal{F})$ and $V \in I(A^{\otimes h}, \mathcal{F})$ the product $WV \in I(A^{\otimes(k+h)}, \mathcal{F})$ as follows

$$WV(Z_1 \otimes\cdots\otimes Z_{k+h}) = 1/k!h!\ \Sigma_\sigma W(Z_{\sigma(1)} \otimes\cdots\otimes Z_{\sigma(k)}) \times V(Z_{\sigma(k+1)} \otimes\cdots\otimes Z_{\sigma(k+h)}), \quad \forall\ Z_i \in A . \tag{60}$$

If $\varphi \in \Lambda^{p,k}$ and $W \in I(A^{\otimes k}, \mathcal{F})$ then $W(\varphi) =: W \circ \varphi \in \Lambda^p(E, \mathcal{F})$. Property (57) implies

$$d_r W(\varphi) = W(d_r \varphi), \tag{61}$$

while condition (59) gives

$$\Sigma_k (-1)^{p_1+\cdots+p_i} W(\psi_1 \wedge \cdots \wedge [\psi_i, \varphi] \wedge \cdots \wedge \psi_k) = 0 \ , \ \forall \ \psi_i \in \Lambda^{p_i, 1}, \varphi \in \Lambda^{1,1} . \tag{62}$$

We are now ready to decribe the Chern-Weyl homomorphism. We shall closely parallel the steps used in the standard situation [CS], [KN2]. We have the following

Proposition 2.6. Given an extension of Lie algebras (21) with a connection, let Ω be the corresponding curvature form and let $W \in I(A^{\otimes k}, \mathcal{F})$. Then

I. the 2k-form $W(\Omega^k) \in \Lambda^{2k}(E, \mathcal{F})$ projects onto a closed 2k-form $\underline{W(\Omega^k)} \in \Lambda^{2k}(B, \mathcal{F})$;

II. if $w(W)$ is the element of the cohomology group $H^{2k}(B, \mathcal{F})$ defined by $\underline{W(\Omega^k)}$, then $w(W)$ does not depend on the connection and $w : I(A, \mathcal{F}) \to H^*(B, \mathcal{F})$ is an algebra homomorphism (Weil homomorphism).

Proof of part I. Since Ω is horizontal so it is $W(\Omega^k)$. Lemma 2.1. assures the existence of a form $\underline{W(\Omega^k)} \in \Lambda^{2k}(B, \mathcal{F})$ to which $W(\Omega^k)$ projects. To prove that $d\,\underline{W(\Omega^k)} = 0$ it suffices to show that $d_r W(\Omega^k) = 0$. By using (61), (53a) and (55) respectively we have $d_r W(\Omega^k) = W(d_r \Omega^k) = k\, W(d_r \Omega \wedge \Omega^{k-1}) = k\, W([\omega, \Omega] \wedge \Omega^{k-1})$ and this gives $d_r W(\Omega^k) = 0$ by condition (62).

In order to prove part II. we first state the following lemma, whose proof is very simple.

Lemma 2.2. Let ω_0 and ω_1 be two connection forms on the sequence (2.13) and define $\omega_t = \omega_0 + t\,\alpha$, $\alpha = \omega_1 - \omega_0$, $t \in [0,1]$; then

a) $\alpha \in \Lambda^1(B; A)$ and α is horizontal,

b) ω_t is a 1-parameter family of connection forms,

c) $(d/dt)\, \Omega_t = d_r \alpha - [\omega_t, \alpha]$.

We shall also need another result.

Proposition 2.7.

$$W(\Omega_1^k) - W(\Omega_0^k) = k\, d_r \int_0^1 W(\alpha \wedge \Omega_t^{k-1})\, dt \ . \qquad (63)$$

Proof: $(d/dt)W(\Omega_t^k) = kW((d/dt\, \Omega_t) \wedge \Omega_t^{k-1}) = kW(d_r\, \alpha \wedge \Omega_t^{k-1}) - kW([\omega_t, \alpha] \wedge \Omega_t^{k-1})$ by c) of lemma 2.2..

On the other hand,

$$k\, d_r W(\alpha \wedge \Omega_t^{k-1}) = k\, W(d_r(\alpha \wedge \Omega_t^{k-1}))$$

$$= k\, W(d_r\alpha \wedge \Omega_t^{k-1}) - k(k-1)\, W(\alpha \wedge d_r\Omega_t \wedge \Omega_t^{k-2})$$

$$= k\, W(d_r\alpha \wedge \Omega_t^{k-1}) - k(k-1)\, W(\alpha \wedge [\omega_t, d_r\Omega_t] \wedge \Omega_t^{k-2}) \quad \text{by (55)}$$

$$= k\, W(d_r\alpha \wedge \Omega_t^{k-1}) - k\, W([\omega_t, \alpha] \wedge \Omega_t^{k-1}) \quad \text{by (62)}$$

and by integrating with respect to the parameter t one gets (63).

Now, the (2k−1) form $\Phi = k\int_0^1 W(\alpha \wedge \Omega_t^{k-1})\, dt$, being horizontal, projects onto a form $\underline{\Phi} \in \Lambda^{2k-1}(B, \mathcal{F})$ and equation (63) in turn projects to

$$\underline{W(\Omega^k)} - \underline{W(\Omega^k)} = d\, \underline{\Phi} \ . \qquad (64)$$

This completes the proof of part II. of proposition 2.6.

The Chern-Simons (secondary characteristic) classes are provided by the following

Proposition 2.8. The 2k-form $W(\Omega^k)$ is exact on E and one has

$$W(\Omega^k) = k\, d \int_0^1 W(\omega \wedge \Psi_t^{k-1})\, dt \ , \qquad (65)$$

where

$$\Psi_t = t\, d\omega - 1/2\, t^2 [\omega, \omega] \ . \qquad (66)$$

The proof is similar to the one of proposition 2.7. and we shall omit it.

Proposition 2.8. will be used in section 2.8. in relation with gauge anomalies.

As an example we shall apply the previous analysis to the extension of infinite dimensional Lie algebras canonically associated with a principal fibre bundle and we shall describe how to recover the usual Chern-Weil construction [CS], [KN]. Let then $\pi : P \to M$ be a principal fibre bundle with structure group G ; $\mathfrak{g}$ will denote the Lie algebra of G. Let AutP be the group of automorphisms of P (diffeomorphisms of P which commute with the action of G) and $\mathcal{G} \equiv \mathrm{Aut}_V P$ its normal subgroup of automorphisms which map any fibre into itself. One has a short exact sequence of groups [Tr2]

$$\mathbb{I} \to \mathcal{G} \to \mathrm{Aut}\,P \to \mathrm{Diff}\,M \to \mathbb{I} \quad . \tag{67}$$

The group $\mathcal{G}$ is the group of gauge transformations of a pure gauge theory. The Lie algebra of AutP can be identified with the algebra $\aleph_G$ of G-invariant vector fields on P while the Lie algebra of $\mathcal{G}$ is the algebra $\mathcal{L}$ of G-invariant vertical vector fields on P. If $\aleph(M)$ is the algebra of vector fields on M we have an extension of Lie algebras [AB]

$$0 \to \mathcal{L} \to \aleph_G \to \aleph(M) \to 0 . \tag{68}$$

The extension (68) is a Lie bundle over the algebra $C^\infty(M)$ of smooth function on M.
It is well known that a connection on P can be given by means of a ($C^\infty(M)$-module) splitting of the sequence (68) $\rho : \aleph(M) \to \aleph_G$ or equivalently by means of a connection 1-form $\omega \in \Lambda^1(\aleph_G, \mathcal{L})$, $\omega(Y) = Y - \rho(\pi_*(Y))$ [AB]. Proposition 2.6. applied to sequence (68) is the statement that given any $W \in I(\mathcal{L}^{\otimes k}, C^\infty(M))$, the 2k-form $W(\Omega^k)$ on P projects onto a closed 2k-form $\underline{W(\Omega^k)}$ on M and the element of the De Rham cohomology group $H^{2k}(\aleph(M), C^\infty(M))$ defined by $\underline{W(\Omega^k)}$ does not depend on the connection. All this sounds very much like the usual Chern-Weil construction for a principal bundle. The possibility of recovering the usual Chern-Weil construction by means of our methods follows from the following

Proposition 2.9. The space $I(\mathfrak{g}^{\otimes k}, \mathbb{R})$ of symmetric, adjoint invariant mappings from $\mathfrak{g}^{\otimes k}$ to $\mathbb{R}$ is isomorphic with the space of local mappings $I_{loc}(\mathcal{L}^{\otimes k}, C^\infty(M)) \subset I(\mathcal{L}^{\otimes k}, C^\infty(M))$. Here an element $W \in I(\mathcal{L}^{\otimes k}, C^\infty(M))$ is local if the support of $W(V_1 \otimes \cdots \otimes V_k)$ is contained in the intersection of the supports of the V_j's for any $V_1, \ldots, V_k \in \mathcal{L}$.

Remark. A relation like $\mathrm{Supp}W(V_1 \otimes \cdots \otimes V_k) \subset \bigcap_j V_j$ makes sense because any invariant and vertical vector field on P can be identified with a section of the bundle over M associated with P by means of the adjoint action of the group G on its Lie algebra $\mathfrak{g}$ (see later).

In order to prove proposition 2.9. we need some additional machineries. First of all, it is easy to show that the adjoint action of AutP on $\mathcal{L}$ is given by

$$\mathrm{Ad}_\varphi V = \varphi_* V \in \mathcal{L} \ , \ \forall \varphi \in \mathrm{AutP} , \ V \in \mathcal{L} \ . \tag{69}$$

while the adjoint action of $\aleph_G$ on $\mathcal{L}$ is given by

$$[\, Y, V \,]_{gr} \equiv \mathrm{ad}_Y V = -\mathcal{L}_Y V \in \mathcal{L} \ , \ \forall Y \in \aleph_G , \ V \in \mathcal{L} \ . \tag{70}$$

Let us consider the space of equivariant mappings

$$C(P, G) = \{ \tau : P \to G \ : \ \tau(pg) = \mathrm{Ad}_{g^{-1}} \circ \tau(p) \} , \tag{71}$$

which is a group under pointwise multiplication $(\tau \cdot \tau')(p) =: \tau(p)\, \tau'(p)$; and the space

$$C(P, \mathfrak{g}) = \{ H : P \to \mathfrak{g} \ : \ H(pg) = \mathrm{Ad}_{g^{-1}} \circ H(p) \} , \tag{72}$$

which is a Lie algebra under pointwise bracketing $[\, H , H' \,] =: [\, H(p), H'(p) \,]$. It is possible to construct an exponential map which obeys the usual properties [Bl]

$$\exp_C : \ C(P, \mathfrak{g}) \to C(P, G) \ , \quad H \to \exp_C(H)(p) =: \exp H(p) \ , \tag{73}$$

where the last exp is in the group G . One can prove that $\mathcal{G}$ and C(P, M) are canonically isomorphic via the following mapping [Bl]

$$\tau : \mathcal{G} \to C(P, G) , \quad \varphi \to \tau_\varphi \quad \text{such that} \quad \varphi(p) = p\, \tau_\varphi(p) \ ; \tag{74}$$

and that $\mathcal{L}$ and $C(P, \mathfrak{g})$ are isomorphic (antiisomorphic with the bracket $[\, Y, V \,]_{gr}$) via

$$H : \mathcal{L} \to C(P, \mathfrak{g}) , \quad H \to H_V \quad \text{such that} \quad \tau_{\exp V} = \exp_C H_V \tag{75}$$

(here expV indicates the flow of the vector field V) . By using isomophism (75) and the well known identification of equivariant mappings with section of associated bundles (see for instance [Tr5]) one has also the identification of the Lie algebra $\mathcal{L}$ with the space of sections of the bundle associated with P by means of the adjoin action og G on $\mathfrak{g}$; this space of sections is made a Lie algebra by pointwise bracketing.

The isomorphism (75) is also used to prove the

Lemma 2.3.

$$H_{Ad\varphi(V)} = \varphi_* H \equiv H \circ \varphi^{-1} \quad , \ \forall \varphi \in \mathrm{AutP} \, , \ V \in \mathcal{L} \ ; \tag{76}$$

$$H_{\mathcal{L}_Y (V)} = \mathcal{L}_Y H_V \quad , \ \forall Y \in \aleph_G , \ V \in \mathcal{L} \ . \tag{77}$$

We can now prove proposition 2.9. We first show that $I(\mathfrak{g}^{\otimes k}, \mathbb{R}) \subset I_{loc}(\mathcal{L}^{\otimes k}, C^\infty(M))$. For any element $\mathcal{W}$ in $I(\mathfrak{g}^{\otimes k}, \mathbb{R})$ we define an element W in $I_{loc}(\mathcal{L}^{\otimes k}, C^\infty(M))$ by

$$W(V_1, ..., V_k)(p) =: \mathcal{W} (H_{V_1}(p), ..., H_{V_k}(p)) , \tag{78}$$

where, for any $V_j \in \mathcal{L}$, $H_{V_j} \in C(P, \mathfrak{g})$ as given by (75) . That $W(V_1,..., V_k)$ is in $C^\infty(M)$ follows from the defining property of the H_{V_j} 's and by Ad-invariance of $\mathcal{W}$. Moreover, by using lemma 3.2. one can prove that W in (78) obeys (57), that is, for any $Y \in \aleph_G$ one has $\mathcal{L}_Y W(V_1,..., V_k) = \Sigma_j W(V_1,...,[Y, V_k],..., V_k)$. This shows the inclusion $I(\mathfrak{g}^{\otimes k}, \mathbb{R}) \subset I_{loc}(\mathcal{L}^{\otimes k}, C^\infty(M))$. To prove the converse let $\mathcal{W}$ be an element of $I_{loc}(\mathcal{L}^{\otimes k}, C^\infty(M))$. By Peetre's theorem [CDS], [Lec2], $\mathcal{W}$ is a multidifferential operator. It can be proved that $\mathcal{W}$ is of zero order and locally with constant cofficients. By using identification (75), property (59) amounts to the invariance of $\mathcal{W}$ under the adjoint action of $\mathfrak{g}$ on $C(P, \mathfrak{g})$. Then, it turns out that $\mathcal{W}$ determines an element in $I(\mathfrak{g}^{\otimes k}, \mathbb{R})$ [Lec2].

By applying proposition 2.6. to sequence (68) one can also construct characteristic classes for the Gel'fand-Fuks cohomology of the Lie algebra $\aleph(M)$ of vector fields on M [TeN]. Indeed, the sequence (68) being a Lie bundle over $C^\infty(M)$ it will also be a Lie bundle over $\mathbb{R}$. Moreover, any splitting of (68) as a Lie $C^\infty(M)$-bundle will also provide a splitting of (68) as an $\mathbb{R}$ bundle. Then, given any $W \in I(\mathcal{L}^{\otimes k}, \mathbb{R})$, from proposition 2.6. it follows that the 2k-form $W(\Omega^k) \in \Lambda^{2k}(\aleph_G, \mathbb{R})$ projects onto a closed 2k-form $\underline{W(\Omega^k)}$ $\in \Lambda^{2k}(\aleph(M), \mathbb{R})$ and the element of the Gel'fand-Fuks [GF] cohomology group $H^{2k}(\aleph_G,\mathbb{R})$ defined by $\underline{W(\Omega^k)}$ does not depend on the connection.

3. GRADED DIFFERENTIAL CALCULUS

In this chapter we shall give a $\mathbb{Z}_2$-graded generalization of the algebraic calculus developed in chapter 2.. Firstly, we will construct an exterior calculus for any Lie superalgebra or $\mathbb{Z}_2$-graded Lie algebra (for Lie superalgebras see [CNS], [Ka]) which acts as a superalgebra of derivations on a $\mathbb{Z}_2$-graded commutative algebra with unit. By giving a metric we shall develop a graded Riemannian calculus and introduce the notion of graded Einstein algebra. All operations will be generalized by taking the Lie superalgebra to be an extension of Lie superalgebras. In analogy with the non graded situation we shall use extensions of Lie superalgebras as the framework for algebraic graded gauge theories. Finally, in section 3.6. we shall construct a graded Chern-Weil homomorphism for an extension of Lie superalgebras endowed with a connection and shall give a sistematic algebraic derivation of graded Chern-Simons terms.

Throughout all the chapter by graded we shall mean $\mathbb{Z}_2$-graded. In addition, $p(\cdot)$ will denote the $\mathbb{Z}_2$-parity of any graded object, while $V^\bullet$ will be the set of all homogeneous elements of any $\mathbb{Z}_2$-graded vector space $V = V^{(0)} \oplus V^{(1)}$. As a rule, all operations concerning V will be given on $V^\bullet$ and then extended to V by linearity.

3.1. GRADED EXTERIOR CALCULUS

The starting object is a $\mathbb{Z}_2$-graded commutative algebra $\mathfrak{F} = \mathfrak{F}^{(0)} \oplus \mathfrak{F}^{(1)}$, over a $\mathbb{Z}_2$-graded commutative algebra of 'constants' $\mathfrak{R}$ which is assumed to contain a field isomorphic with the field of real or complex numbers (more precisely $\mathfrak{F}$ is a graded $\mathfrak{R}$-module). $\mathfrak{F}$ will be assumed to containe a unit. A *graded derivation* of $\mathfrak{F}$ of parity $p(X)$ is any $\mathfrak{R}$-linear map $X : \mathfrak{F} \to \mathfrak{F}$ such that $X(fg) = (X(f))g + (-1)^{p(f)p(X)} f\, X(g)$, for any $f, g \in \mathfrak{F}^\bullet$. The parity of X is defined by $p(X) =: p(X(f)) - p(f) \mod \mathbb{Z}_2$, for any f in $\mathfrak{F}^\bullet$. Since $X(1) = X(1) + X(1)$ one has that $X(1) = 0$; by $\mathfrak{R}$-linearity it follows that $X(a) = 0$ for any a in $\mathfrak{R}$. The collection $\mathrm{GDer}\mathfrak{F}$ of all derivations of $\mathfrak{F}$ is a Lie superalgebra over $\mathfrak{R}$. The graded Lie bracket $\mathfrak{L}_X Y \equiv [X, Y]_G$ of any two homogeneous derivations X and Y is defined by $\mathfrak{L}_X Y =: X \circ Y - (-1)^{p(X)p(Y)} Y \circ X$; $\mathfrak{L}_X Y$ is a graded derivation of parity $p(\mathfrak{L}_X Y) = p(X) + p(Y)$. The algebra $\mathrm{GDer}\mathfrak{F}$ is made a graded left module over $\mathfrak{F}$ by defining $(gX)f =: g(Xf)$. Furthermore, one has the property $\mathfrak{L}_X(fY) = (-1)^{p(f)p(X)} f\, \mathfrak{L}_X Y + (Xf)\, Y$, for any $X, Y \in \mathrm{GDer}\mathfrak{F}^\bullet$, $f \in \mathfrak{F}^\bullet$.

Let E be any Lie superalgebra over $\mathfrak{R}$ which is also a graded left module over $\mathfrak{F}$. E will be called a *Lie supermodule* if there is an even representation of E into the Lie superalgebra GDer$\mathfrak{F}$, $X \to X\cdot$. By this, $\mathfrak{F}$ will also be a left E-module. Obviously, GDer$\mathfrak{F}$ itself is a Lie supermodule.

Given any $\mathfrak{F}$-module E the dual module E* is the collection of all graded $\mathfrak{F}$-linear mappings $\varphi : E \to \mathfrak{F}$, $X \to \varphi(X)$, $\varphi(fX) = (-1)^{p(f)p(\varphi)}\varphi(X)$, with the parity $p(\varphi)$ defined by the usual rule $p(\varphi) = p(\varphi(X)) - p(X)$, for any $X \in E^{\bullet}$. In general, an (m, n) graded tensor is any graded $\mathfrak{F}$-multilinear map $\alpha : E^{*}\times\cdots\times E^{*}\times E\times\cdots\times E \to \mathfrak{F}$ (m E* factors and n E factors). The collection of all rank (m, n) tensors is a graded left $\mathfrak{F}$-module. One also defines the graded tensor product $\alpha\otimes\beta$ of any two tensors α and β. As in the nongraded situation, we shall be interested only in cases in which the algebra $\mathfrak{F}$ and the module E are such that E is totally reflexive namely, the dual of E* is isomorphic with E and tensors can be identified with tensor products.

We denote by $\Lambda^p(E, \mathfrak{F})$ the collection of all skew-symmetric covariant graded tensors of rank p. If $\varphi \in \Lambda^p(E, \mathfrak{F})$, then $\varphi(\ldots,Y_{i+1},Y_i,\ldots) = (-1)^{1+p(Y_i)p(Y_{i+1})}\varphi(\ldots,Y_i,Y_{i+1},\ldots)$, for any $Y_i \in E^{\bullet}$. In particular $\Lambda^1(E, \mathfrak{F}) \equiv E^*$, $\Lambda^0(E, \mathfrak{F}) \equiv \mathfrak{F}$. Elements in $\Lambda^p(E, \mathfrak{F})$ will be also called graded *p-forms* or graded *p-$\mathfrak{F}$-cochains*.

If E is a Lie supermodule, on the direct sum $\Lambda^*(E, \mathfrak{F}) = \oplus_p \Lambda^p(E, \mathfrak{F})$ we define an *exterior derivative* d, by

$$d : \Lambda^p(E, \mathfrak{F}) \to \Lambda^{p+1}(E, \mathfrak{F}) ,$$

$$d\varphi(X_1,\ldots,X_{p+1}) =: \Sigma_i (-1)^{i+1+a(\varphi,i)} X_i\cdot\varphi(X_1,\ldots,\hat{\imath},\ldots,X_{p+1}) +$$

$$+ \Sigma_{i<j}(-1)^{i+j+b(i,j)}\varphi([X_i, X_j]_G, X_1,\ldots,\hat{\imath},\ldots,\hat{j},\ldots,X_{p+1}) , \quad \forall\ X_i \in E^{\bullet}, \qquad (1)$$

where

$$a(\varphi,i) = p(X_i)\,[\, p(\varphi) + \Sigma_{k<i}\, p(X_k)\,] ,$$

$$b(i,j) = [\, p(X_i) + p(X_j)\,]\Sigma_{k<i}\, p(Y_k) + p(Y_j)\, \Sigma_{i<k<j}\, p(Y_k) . \qquad (2)$$

From definition one has that $p(d) = 0$. Since the action of E is a representation, one can prove that d is a coboundary operator, namely $d^2 = 0$. An element φ in $\Lambda^p(E,\mathfrak{F})$ will be called a graded *cocycle* if $d\varphi = 0$, a graded *coboundary* if $\varphi = d\psi$ for some ψ in $\Lambda^p(E,\mathfrak{F})$. Then one introduces the graded cohomology of the Lie superalgebra E with coefficients in the representation space $\mathfrak{F}$. The p-th graded cohomology group is defined by

$$H_G{}^p(E, \mathcal{F}) = (\ker d : \Lambda^p(E, \mathcal{F}) \to \Lambda^{p+1}(E, \mathcal{F})) \,/\, (\mathrm{im}\, d : \Lambda^{p-1}(E, \mathcal{F}) \to \Lambda^p(E, \mathcal{F})) \,. \qquad (3)$$

As for the cohomology at order zero, one has $H^0(E, \mathcal{F}) = \ker d : \Lambda^0(E, \mathcal{F}) \to \Lambda^1(E, \mathcal{F})$, namely, $H^0(E, \mathcal{F})$ is made of elements of $\mathcal{F}$ which are invariant under the action of E.
The cohomology of Lie superalgebras was introduced in [Lei]. It has been applied mainly to study deformations of Lie algebras [Lec3] and superalgebras [Bi].

Next, we define an exterior product

$$\wedge : \ \Lambda^p(E, \mathcal{F}) \times \Lambda^q(E, \mathcal{F}) \to \Lambda^{p+q}(E, \mathcal{F}) \,,$$

$$\varphi \wedge \psi(X_1,..,X_{p+q}) =: 1/p!q! \,\Sigma_\sigma\, \chi(\sigma)(-1)^{c(\psi,\sigma,p)+d(\sigma)}\, \varphi(X_{\sigma(1)},...,X_{\sigma(p)}) \times$$

$$\times\, \psi(X_{\sigma(p+1)},...,X_{\sigma(p+q)}) \,, \quad \forall \ Y_i \in E^\bullet \,; \qquad (4)$$

here, $c(\psi,\sigma,p)$ and $d(\sigma)$ are given by

$$c(\psi,\sigma,p) = p(\psi)\Sigma_{i\leq p}\, p(X_{\sigma(i)}) \,,$$

$d(\sigma)$ = numbers of minus signs that occur by going from the ordered sequence $X_1, ...,X_{p+q}$ to the ordered sequence $X_{\sigma(1)}, ...,X_{\sigma(p+q)}$. (5)

The derivative d is of degree one with respect to the exterior product, namely, one has $d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^{\deg\varphi}\, \varphi \wedge d\psi$.

We introduce two more operators on $\Lambda^*(E, \mathcal{F})$. Firstly, the *Lie derivative* $\mathcal{L}_{(\cdot)}$ given by

$$\mathcal{L}_{(\cdot)} : \ E \times \Lambda^p(E, \mathcal{F}) \to \Lambda^p(E, \mathcal{F}) \,,$$

$$(\mathcal{L}_X\varphi)(X_1,...,X_p) =: X\cdot\varphi(X_1,...,X_p) \,+$$

$$-\, \Sigma_i\, (-1)^{p(\varphi)p(X)+p(X_i)\Sigma_{k<i}\, p(X_k)}\, \varphi([X, X_i], X_1,...,\hat{\imath},...,X_p) \,, \quad \forall\, X, X_i \in E^\bullet \,. \qquad (6)$$

We have also the *inner product* $\iota_{(\cdot)}$, defined by

$$\iota_{(\cdot)} : \; E \times \Lambda^p(E,\mathcal{F}) \to \Lambda^{p-1}(E,\mathcal{F}),$$

$$(\iota_X\varphi)(X_1,\dots,X_{p-1}) =: (-1)^{p(\varphi)p(X)}\, \varphi(X,X_1,\dots,X_{p-1}), \qquad \forall\, X, X_i \in E^\bullet ; \tag{7}$$

and $\iota_{(\cdot)}f = 0 \;\; \forall\, f \in \mathcal{F}$. From they definitions one has that $p(\mathcal{L}_X) = p(X)$, $p(\iota_X) = p(X)$. Moreover, $\mathcal{L}_X(\varphi \wedge \psi) = \mathcal{L}_X\varphi \wedge \psi + (-1)^{p(\varphi)p(X)}\, \varphi \wedge \mathcal{L}_X$ and $\iota_X(\varphi \wedge \psi) = \iota_X\varphi \wedge \psi + (-1)^{\deg\varphi + p(\varphi)p(X)}\, \varphi \wedge \iota_X\psi$. One has also the identity $\mathcal{L}_{(\cdot)} = d \circ \iota_{(\cdot)} + \iota_{(\cdot)} \circ d$.

Finally, we define the Lie derivative of any tensor. If u is any (m, n) tensor of definite parity $p(u)$, then, given any $X \in E^\bullet$, the Lie derivative $\mathcal{L}_X u$ is the tensor of the same type and of parity $p(u) + p(X)$, given by

$$(\mathcal{L}_X u)(\omega^1,\dots,\omega^m,X_1,\dots,X_n) =: X\cdot(u\,(\omega^1,\dots,\omega^m,X_1,\dots,X_n))$$

$$- \Sigma_i\, (-1)^{p(X)[p(u)+\Sigma_{k<i}\, p(\omega^k)}\, u(\omega^1,\dots, \mathcal{L}_X\omega^i,\dots,\omega^m, X_1,\dots,X_n)$$

$$- \;\Sigma_i\, (-1)^{p(X)[p(u)+\Sigma p(\omega^k)+\Sigma_{k<i}\, p(X_k)}\, u(\omega^1,\dots,\omega^m,X_1,\dots,[X, X_i],\dots,X_n),$$

$$\forall\, X,\; X_i \in E^\bullet,\; \omega^i \in E^{*\bullet}. \tag{8}$$

A 'fermionic' differential calculus based on a couple $(\mathcal{F}, E)$, ($\mathcal{F}$ a graded commutative algebra with unit and E a Lie supermodule over $\mathcal{F}$) has been presented also in [JK] where the couple $(\mathcal{F}, E)$ is called a graded Lie-Cartan Pair.

3.2. GRADED RIEMANNIAN CALCULUS AND GRADED EINSTEIN ALGEBRAS

In analogy with section 2.1., we introduce the notion of affine connection on a Lie supermodule. Then by giving a metric on the module, we shall construct the Levi-Civita connection associated with it.

The framework is a Lie supermodule E over the superalgebra $\mathcal{F}$ with unit. A *graded affine connection* on E is a degree zero map $\nabla : E \rightarrow \mathrm{Hom}_{\mathbb{R}}(E, E)$, $X \rightarrow \nabla_X$ such that

$$\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z \ ,$$

$$\nabla_{fX+Y} Z = f \nabla_X Z + \nabla_Y Z \ ,$$

$$\nabla_X (f Y) = (X \cdot f) Y + (-1)^{p(X)p(f)} f \nabla_X Y \ , \qquad \forall \ X, Y, Z \in E^{\bullet} , \ f \in \mathcal{F}^{\bullet} \ . \qquad (9)$$

We call $\nabla_X Y$ the *covariant derivative* of Y in the direction of X . The covariant derivative of elements in E^* is defined by requiring Leibnitz rule. If $X \in E$ and $\varphi \in E^*$, the covariant derivative $\nabla_X \varphi$ is the element in E^* given by

$$(\nabla_X \varphi)(Y) =: X \cdot (\varphi(Y)) - (-1)^{p(X)p(\varphi)} \varphi(\nabla_X Y) \ , \qquad \forall \ Y \in E \ . \qquad (10)$$

In general, if u is a tensor of type (m, n), the covariant derivative $\nabla_X u$ is the tensor of the same type and parity the sum $p(u) + p(X)$, defined by

$$(\nabla_X u)(\omega^1, \ldots, \omega^m, X_1, \ldots, X_n) =: X \cdot (u \, (\omega^1, \ldots, \omega^m, X_1, \ldots, X_n))$$

$$- \Sigma_i (-1)^{p(X)[p(u) + \Sigma_{k<i} p(\omega^k)} u(\omega^1, \ldots, \nabla_X \omega^i, \ldots, \omega^m, X_1, \ldots, X_n)$$

$$- \Sigma_i (-1)^{p(X)[p(u) + \Sigma p(\omega^k) + \Sigma_{k<i} p(X_k)} u(\omega^1, \ldots, \omega^m, X_1, \ldots, \nabla_X X_i, \ldots, X_n) \ ,$$

$$\forall \ X, \ X_i \in E^{\bullet} , \ \omega^i \in E^{*\bullet} \ . \qquad (11)$$

Given an affine connection ∇ on the Lie supermodule E , the *torsion* of ∇ is the grade zero map T from $E \times E$ into E defined by

$$T(X,Y) =: \nabla_X Y - (-1)^{p(X)p(Y)} \nabla_Y X - [X, Y] \ , \qquad \forall \ X, \ Y \in E^{\bullet} . \qquad (12)$$

One verifies that T is $\mathcal{F}$-bilinear. Therefore, it determines a graded skew-symmetric *torsion tensor* of type $(1, 2)$ by the rule $T(\omega, X, Y) =: \omega(T(X,Y))$.

The *curvature* of the affine connection ∇ on the Lie supermodule E is the map R from $E \times E$ into $\mathrm{Hom}_{\mathbb{R}}(E, E)$ given by

$$R(X, Y) =: \nabla_X \nabla_Y - (-1)^{p(X)p(Y)} \nabla_Y \nabla_X - \nabla_{[X,Y]} \,, \qquad \forall\ X, Y \in E^{\bullet} . \tag{13}$$

The *graded curvature tensor* R of the affine connection ∇ is the tensor of type $(1, 3)$ given by

$$R(\omega, Z, X, Y) =: (-1)^{p(Z)[p(X)+p(Y)]}\, \omega(R(X, Y)(Z)) \,, \quad \forall\ X, Y, Z \in E^{\bullet}, \quad \omega \in E^{*} . \tag{14}$$

One verifies by explicit computation that R is a tensor, namely that it is $\mathcal{F}$-linear. Moreover, graded versions of all usual properties of the Riemann and torsion tensor are true, provided one pays attention to the 'minus signs'.

A metric on the Lie supermodule E is an isomorphism g from the module E to the module E^*, which is graded symmetric, namely $g(X,Y) = (-1)^{p(X)p(Y)} g(Y,X)$, for any $X, Y \in E^{\bullet}$, where $g(X,Y) =: [g(X)](Y)$ (g is even in $\mathbb{Z}_2$-grading). To any metric there is associated a covariant tensor of rank 2.

Given a metric we may construct its Levi-Civita connection. If $X \in E^{\bullet}$ and $\varphi \in E^{*\bullet}$, the covariant derivative $\nabla_X \varphi$ associated with the metric g is defined by

$$\nabla_X \varphi\,(Y) =: 1/2\,\{X \cdot \varphi(Y) - (-1)^{p(\varphi)p(X)+p(Y)[p(\varphi)+p(X)]}\, Y \cdot \varphi(X) - (-1)^{p(\varphi)p(X)} \varphi([X,Y]_G)$$

$$+ (-1)^{p(\varphi)p(X)} [g^{-1}(\varphi)] \cdot (g(X,Y)) - (-1)^{p(X)p(\varphi)}\, g(\mathcal{L}_{g^{-1}(\varphi)} X, Y) - g(X, \mathcal{L}_{g^{-1}(\varphi)} Y)\,\}$$

$$= 1/2\, (-1)^{p(\varphi)p(X)} \{\, (d\varphi)(X, Y) + (\mathcal{L}_{g^{-1}(\varphi)}\, g)(X, Y)\,\} \,, \quad \forall\ Y \in E . \tag{15}$$

From last equality it follows that $\nabla_X \varphi(Y)$ is graded $\mathcal{F}$-linear in both X and Y. Moreover, all other defining properties (9) are satisfied. The connection (15) is the Levi-Civita connection of the metric g in the sense that one has : i) $\nabla_X g = 0$; ii) the torsion vanishes.

If R is the curvature as defined in (13), then the *graded Riemann tensor* of the Levi-Civita connection ∇ is the even covariant rank four tensor Riem defined by

$$\mathrm{Riem}(X, Y, Z, W) =: g(X, R(Z,W)Y\,) \,, \quad \forall\ X, Y, Z, W \in E^{\bullet} \ ; \tag{16}$$

The graded Riemann tensor obeys properties analogous to that of the Riemann tensor (2.16) provided one pays attention to the proper 'minus signs'.

The last ingredient we need to define the graded Ricci tensor is a *contraction operation* C. By the assumption of total reflexivity it suffices to define C on rank two tensor. If α is a rank two tensor then $C(\alpha) \in \mathcal{F}$, and C is linear and satisfies $C(\varphi \otimes \psi) = \varphi(g^{-1}\psi)$, $\forall\, \varphi$, $\psi \in E^*$. We assume the algebra $\mathcal{F}$, the supermodule E and the metric g to be such that there exists a unique operation C which fulfils the previous contraction properties.

With Y, W fixed, $S(X, Z) =: \mathrm{Riem}(X, Y, Z, W)$ is bilinear in X, Z. Then C(S) is bilinear in W, Y and defines the covariant rank two *Ricci tensor* Ric(W, Y). Ric can be defined equivalently as

Ric(Y, W) = trace of the map $V \to R(V, Y)W$ of the Lie supermodule E.

We have all ingedients to generalize the notion of Einstein algebra introduced in section 2.2. We define a *Graded Einstein algebra* to consist of : i) a $\mathbb{Z}_2$-graded commutative algebra $\mathcal{F}$ over a $\mathbb{Z}_2$-graded commutative algebra of 'constants' $\mathcal{R}$, the latter containing a field isomorphic to the real numbers, ii) a graded Lie $\mathcal{F}$-module E with a metric g which is such that the contraction properties are satisfied and the graded Ricci tensor vanishes.

3.3. THE GRADED CODIFFERENTIAL

Given a metric on a Lie supermodule E, we can introduce a graded codifferential. For this we only need a metric and the associated Levi-Civita connection as constructed in (15). For the sake of simplicity we take the Lie supermodule E to be of finite type as an $\mathcal{F}$-module with basis given by $\{e_i, i = 1,\dots,n\}$ and dual basis $\{\varepsilon^i, i = 1,\dots,n\}$, $\varepsilon^i(e_j) = \delta^i_j$ (we take corresponding elements in the two basis to have the same parity). The codifferential is defined as

$$\delta : \Lambda^p(E, \mathcal{F}) \to \Lambda^{p-1}(E, \mathcal{F}),$$

$$(\delta\varphi)(X_1,\dots,X_{p-1}) =: -\Sigma_i (\nabla_{e_i}\varphi)(g^{-1}(\varepsilon^i), X_1,\dots, X_{p-1}) \;, \quad \forall\, X_i \in E \;. \qquad (17)$$

In particular if $\varphi \in E^*$, its *divergence* is the element of $\mathcal{F}$ given by

$$\mathrm{div}\varphi =: \delta\varphi = -\Sigma_i (\nabla_{e_i}\varphi)(g^{-1}(\varepsilon^i)). \tag{18}$$

As in the non graded case one shows that $\delta^2 = 0$. Moreover, one has an analogue of proposition 2.1., namely, the codiferential δ is the adjoint operator of the exterior derivative d in the sense that , given any $\varphi \in \Lambda^{p-1}(E, \mathcal{F})$ and any $\psi \in \Lambda^p(E, \mathcal{F})$ one has

$$g^{-1}(d\varphi , \psi) - g^{-1}(\varphi, \delta \psi) = \mathrm{div(something)} . \tag{19}$$

Here g^{-1} is the natural extension of the metric to the space of graded forms, and is defined in a way similar to (2.18). The proof of (19) is analogous to that of proposition 2.1..

3.4. EXTENSIONS OF LIE SUPERALGEBRAS

Extensions of Lie superalgebras will be used in the next section in order to generalize the results of section 2.5. and construct a framework for a graded gauge theory. Here we give few basic notions. Lie superalgebra extensions have been considered in [Ti] too.

An extension of the Lie superalgebra B by the Lie superalgebra A is a Lie superalgebra E togetherr with a short exact sequence of Lie superalgebras

$$0 \rightarrow A \xrightarrow{i} E \xrightarrow{\pi} B \rightarrow 0 \tag{20}$$

For the time being, A , B and E are Lie superalgebras over a $\mathbb{Z}_2$-graded commutative algebra of 'constants' $\mathcal{R}$. The maps i and π are respectively an even injective and an even surjective Lie superalgebras homomorphisms with the condition that $\mathrm{im}\, i = \ker \pi$.

With any extension like (20) there is associated an exact sequence of Lie superalgebras of derivations by means of the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & E & \xrightarrow{\pi} & B & \longrightarrow & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\ 0 & \longrightarrow & InG(A) & \longrightarrow & GD(A) & \longrightarrow & OutGD(A) & \longrightarrow & 0 \end{array} \tag{21}$$

Here GD(A) is the Lie superalgebra of all derivations of A . As a vector space, D(A) is the collection of all graded maps φ from A into itself such that $\varphi([X,Y]) = [\varphi(X), Y] + (-1)^{p(\varphi)p(X)}[X, \varphi(Y)]$, for any $X, Y \in A$. The Lie bracket of any two homogeneous derivations φ and ψ is defined as usual by $[\varphi, \psi] = \varphi \circ \psi - (-1)^{p(\varphi)p(\psi)}\psi \circ \varphi$.
InGD(A) is the Lie superalgebra of inner derivations of A , which also gives the adjoint action of A on itself $D_a(a') = [a, a']_G$ for some $a \in A$. This gives the homomorphism α as $a \to \alpha(a) =: D_a$. Finally, OutGD(A) is the Lie superalgebra of equivalence classes of derivations of A modulo inner ones. If we consider the cohomology of A with coefficient into itself and with respect to the adjoint action, we see from their definitions that GD(A) , InG(A) and OutGD(A) are the space of graded 1-cocycles, the space of graded 1-coboundaries and the first cohomology group respectively .
The map β is defined as $e \to \beta(e), \beta(e)a = [e, a]_G$ which is an element of A because the latter is an invariant subalgebra . Finally, the map γ , called the *character* of the extension (20), is defined to make the last diagram in (21) commute.
Two extensions $A \xrightarrow{i} E \xrightarrow{\pi} B$ and $A \xrightarrow{i'} E' \xrightarrow{\pi'} B$ with the same character are said to be equivalent if there is an isomorphism $f : E \to E'$ such that $i' = f \circ i$ and $\pi = f \circ \pi'$.
If the superalgebra A is abelian, then γ defines a representation of B in A . In analogy with what happens for extensions of ordinary Lie algebras, one can prove that the set of equivalence classes of extensions of B by A is in a bijective correspondence with the space $H^2(B, A, \gamma)$ of the cohomology of B with coefficients in A .

3.5. CALCULUS OVER LIE SUPERALGEBRA EXTENSIONS

In this section we shall introduce the basic ingredients we need for the graded gauge theory which will be developed in chaptev 4.. The framework is an extension of Lie superalgebras $0 \to A \to E \to B \to 0$ like (20). In addition, we take the extension to be a *Lie superbundle* over the couple $(\mathcal{F}, \mathcal{R})$; here $\mathcal{F}$ is a graded commutative algebra with unit, over a graded commutative algebra of 'constants' $\mathcal{R}$. By this we means that A , B and E are Lie algebras over $\mathcal{R}$ and left Lie supermodules over $\mathcal{F}$. Therefore there is a representation of A , E and B into the Lie superalgebra GDer$\mathcal{F}$ of all graded derivations of $\mathcal{F}$. Since A is an ideal in E it is forced to act trivially on $\mathcal{F}$ and A is a Lie superalgebra also over $\mathcal{F}$.

We call *connection* on the sequence (20) any even $\mathcal{F}$-module homomorphism

$$\rho : B \to E \qquad \text{such that} \qquad \pi \circ \rho = id_B . \tag{22}$$

The map ρ is required to be $\mathfrak{F}$-linear. If ρ and ρ' are any two connections, their difference $\rho - \rho'$ is an $\mathfrak{F}$-linear map from B to A. The set of all connections is then an affine space modelled on $\mathrm{Lin}_{\mathfrak{F}}(B, A)$.
Equivalently, we could give a connection by means of an even A-valued $\mathfrak{F}$-linear *connection 1-form* ω on E

$$\omega : E \to A \qquad \text{such that} \qquad \omega \circ i = id_A . \tag{23}$$

The relation between ω and ρ is

$$\omega = id_E - \rho \circ \pi . \tag{24}$$

In general ρ is not a homomorphism of Lie superalgebas. The extent to which ρ fails to be so is measured by the *curvature* F. The latter is the A-valued $\mathfrak{F}$-linear skew map of zero parity on $B \times B$ defined by

$$F(X_1, X_2) =: \rho([X_1, X_2]_G) - [\rho(X_1), \rho(X_2)]_G , \quad \forall \; X_1, X_2 \in B . \tag{25}$$

In section 4.8. we shall give few examples of extensions of Lie superalgebras with connection.

The *curvature 2-form* Ω of the connection is the A-valued skew map on $E \times E$ defined by

$$\Omega(Y_1, Y_2) =: F(\pi Y_1, \pi Y_2) , \quad \forall \; Y_1, Y_2 \in E , \tag{26}$$

and in terms of ω

$$\Omega(Y_1, Y_2) = [Y_1, \omega(Y_2)]_G - (-1)^{p(Y_1)p(Y_2)} [Y_2, \omega(Y_1)]_G + \\ - \omega([Y_1, Y_2]_G) - [\omega(Y_1), \omega(Y_2)]_G . \tag{27}$$

The form Ω is horizontal, i.e. $\Omega(Y_1, Y_2) = 0$ whenever one of the Y_i's belongs to A .

Let $\Lambda^p(B, A)$ be the graded vector space of graded $\mathfrak{F}$-linear, p-linear skew maps from $B \times \cdots \times B$ (p times) to A. We define the graded *covariant differential* D_ρ by the rule

$$D_\rho : \Lambda^p(B, A) \to \Lambda^{p+1}(B, A) ,$$

$$(D_\rho\varphi)(X_1,\ldots,X_{p+1}) =: \Sigma_i (-1)^{i+1+a(\varphi,i)} [\rho(X_i), \varphi(X_1,\ldots,\hat{\imath},\ldots,X_{p+1}]_G +$$

$$+ \Sigma_{i<j} (-1)^{i+j+b(i,j)}\varphi([X_i, X_j]_G, X_1,\ldots,\hat{\imath},\ldots,j^\wedge,\ldots, X_{p+1}) \ , \forall \ X_i \in B^\bullet, \qquad (28)$$

where $a(\varphi,i)$ and $b(i,j)$ are given in (2).

If F is the curvature as defined in (25), one can prove the Bianchi identity

$$D_\rho F = 0 . \qquad (29)$$

In order to introduce a graded *covariant codifferential* we need a metric on B, namely a graded symmetric non degenerate map $g : B\times B \to \mathcal{F}$. We shall indicate with ∇ the corresponding Levi-Civita connection as constructed in (15). Given any $X \in B$, we first define an operator

$$\nabla^\rho{}_X : \Lambda^p(B, A) \to \Lambda^p(B, A) \quad ,$$

$$(\nabla^\rho{}_X \, \varphi)(X_1,..,X_p) =: [\rho(X), \varphi(X_1,..,X_p)] +$$

$$- (-1)^{p(\varphi)p(X)}\Sigma_i(-1)^{p(X)\Sigma_{k<i}\, p(X_k)}\, \varphi(X_1,..,\nabla_X X_i,..,X_p), \quad \forall \ X_i \in B^\bullet . \qquad (30)$$

To define the covariant codifferential $\mathcal{D}_\rho$, for the sake of simplicity, we take B to be of finite type with basis $\{b_i, i = 1,..,n\}$ and dual basis $\{\beta^i, i = 1,..,n\}$, $\beta^i(b_j) = \delta^i{}_j$. Then the covariant codifferential is given by

$$\mathcal{D}_\rho : \Lambda^p(B, A) \to \Lambda^{p-1}(B, A) ,$$

$$(\mathcal{D}_\rho\varphi)(X_1,\ldots,X_{p-1}) =: - \Sigma_i (\nabla^\rho{}_{b_i}\varphi)(g^{-1}(\beta^i), X_1,\ldots, X_{p-1}) \ , \quad \forall \ X_i \in B \ . \qquad (31)$$

Finally, we assume there is a metric also on the Lie superalgebra A , $h : A\times A \to \mathcal{F}$ which is graded symmetric and non degenerate, and of degree zero. In addition, we take it to be adjoint invariant, namely

$$h([\,Y, Z_1\,], Z_2) + (-1)^{p(Y)p(Z_1)}\, h(Z_1, \;[Y, Z_2]) = 0 \;, \quad \forall\, Y \in E^{\bullet} \,,\; Z_1, Z_2 \in A^{\bullet} , \qquad (32)$$

and invariant under the action of B ,

$$X{\cdot}h(\,Z_1, Z_2) - h([\rho(X), Z_1], Z_2) - (-1)^{p(X)p(Z_1)}\, h(Z_1, [\rho(X), Z_2]) = 0 \,,$$

$$\forall\, X \in B^{\bullet} \,,\; Z_1, Z_2 \in A^{\bullet} . \qquad (33)$$

Let us indicate with (hg^{-1}) the natural metric induced on $\Lambda^p(B, A)$ by g and h , and constructed as in (2.44). One can prove as for the nongraded case that $\mathcal{D}_\rho$ is the dual of D_ρ with respect to the metric (hg^{-1}) , namely, relations analogous to (2.45) and (2.46) are true.

3.6. GRADED CHARACTERISTIC CLASSES AND GRADED CHERN-WEIL HOMOMORRRPHISM

In this section we shall generalize the construction of section 2.6. by constructing a Chern-Weil homomorphism for extensions of Lie superalgebras. We shall also give a sistematic algebraic derivation of graded Chern-Simons secondary characteristic classes.
A construction similar to the one presented here has been used in [BBL1] to construct a Weil homomorphism for a principal superbundle which, in turn, has been used to construct in terms of curvature forms some invariants associated to the superbundle and, in particular, to define Chern-Simons forms. Graded Chern-Simons terms are presented in several supergravity models (see for instance [vNe]) . Moreover they are used in the analysis of anomalies in supersymmetric theories [BPT], [BBL2], [BrLa].

The basic object is an extension of Lie superalgebras $0 \to A \to E \to B \to 0$ which is a Lie superbundle over a graded commutative algebra $\mathcal{F}$ with unit. Moreover, on the extension there is a connection ρ as defined in (22) with connection form ω and curvature form Ω given respectively by (23) and (27).

Let us consider the tensor algebra over A

$$A^* = \oplus_k A^{\otimes k} \;, \quad A^{\otimes k} = A \otimes \cdots \otimes A \quad (k \text{ times}) \,, \quad A^0 = \mathcal{F} \;. \qquad (34)$$

We make A^* a $\mathbb{Z}_2$-graded algebra with the tensor product $\otimes$ as multiplication and grading defined by $p(Z_1 \otimes \cdots \otimes Z_k) = \Sigma p(Z_i)$, mod $\mathbb{Z}_2$, whenever the $Z_i \in A^\bullet$.

We can define a representation of E in the graded Lie algebra of derivations of $A^{\otimes k}$. Take

$$r: \ E \times A^{\otimes k} \rightarrow A^{\otimes k},$$

$$r(Y)\cdot(Z_1 \otimes \cdots \otimes Z_k) =: \Sigma_i (-1)^{p(Y)\Sigma_{k<i} p(Z_k)} Z_1 \otimes \cdots \otimes [Y , Z_i]_G \otimes \cdots \otimes Z_k , \quad (35)$$

and extend r as a graded derivation to the whole of A^*. One can show that r is a representation, namely

$$r([X, Y]_G) = r(X) \circ r(Y) - (-1)^{p(X)p(Y)} r(Y) \circ r(X) , \quad \forall \ X, Y \in E^\bullet . \quad (36)$$

Since A acts trivially on $\mathcal{F}$, r is $\mathcal{F}$-linear.

Let $\Lambda^{p,k} =: \Lambda^p(E, A^{\otimes k})$ be the space of $\mathcal{F}$-linear, p-linear graded skew maps from $E\times\cdots\times E$ (p times) to $A^{\otimes k}$ ($A^{\otimes k}$-valued graded p-forms on E). If $\varphi \in \Lambda^{p,k}$ then

$$\varphi(\ldots , Y_{i+1}, Y_i, \ldots) = (-1)^{1+p(Y_i)p(Y_{i+1})} \varphi(\ldots , Y_i, Y_{i+1}, \ldots) , \quad \forall \ Y_i \in E^\bullet . \quad (37)$$

By using representation (35) we define an operator

$$d_r: \ \Lambda^{p,k} \rightarrow \Lambda^{p+1,k},$$

$$d_r \varphi(Y_1,\ldots,Y_{p+1}) =: \Sigma_i (-1)^{i+1+a(\varphi,i)} r(Y_i)\cdot\varphi(Y_1,\ldots,\hat{\imath},\ldots,Y_{p+1}) +$$

$$+ \Sigma_{i<j} (-1)^{i+j+b(i,j)}\varphi([Y_i, Y_j]_G, Y_1,\ldots,\hat{\imath},\ldots,\hat{\jmath},\ldots,Y_{p+1}) , \quad \forall \ Y_i \in E^\bullet . \quad (38)$$

with $a(\varphi,i)$ and $b(i,j)$ as in (2). As it happens for the operator in (1) one can show that d_r is a coboundary operator, namely $(d_r)^2 = 0$.

Next, we define an exterior product by

$$\wedge \ : \Lambda^{p,k} \times \Lambda^{q,h} \rightarrow \Lambda^{p+q,k+h} ,$$

$$\varphi \wedge \psi (Y_1,\ldots,Y_{p+q}) =: 1/p!q! \ \Sigma_\sigma \chi(\sigma)(-1)^{c(\psi,\sigma,p)+d(\sigma)} \varphi(Y_{\sigma(1)},\ldots,Y_{\sigma(p)}) \otimes$$

$$\otimes \psi(Y_{\sigma(p+1)},\ldots,Y_{\sigma(p+q)}) , \quad \forall \ Y_i \in E^\bullet ; \quad (39)$$

here $c(\psi,\sigma,p)$ and $d(\sigma)$ are given by (5).

One also defines a bracket operation

$$[\ ,\ ]_G : \Lambda^{p,1} \times \Lambda^{q,1} \rightarrow \Lambda^{p+q,1},$$

$$[\varphi , \psi](Y_1,...,Y_{p+q}) =: 1/p!q! \ \Sigma_\sigma \chi(\sigma)(-1)^{c(\psi,\sigma,p)+d(\sigma)} [\varphi(Y_{\sigma(1)},...,Y_{\sigma(p)}) ,$$

$$, \psi(Y_{\sigma(p+1)},...,Y_{\sigma(p+q)})]_G , \quad \forall \ Y_i \in E^\bullet . \qquad (40)$$

with $c(\psi,\sigma,p)$ and $d(\sigma)$ again as in (5).

From definitions (38), (39) and (40) we have a proposition analogous to proposition 2.5.

Proposition 3.1. If $\varphi \in \Lambda^{p,k}$ and $\psi \in \Lambda^{q,h}$ then

$$d_T (\varphi \wedge \psi) = (d_T \varphi) \wedge \psi + (-1)^p \varphi \wedge d_T \psi \ , \qquad (41a)$$

moreover, if $\varphi \in \Lambda^{p,1}$, $\psi \in \Lambda^{q,1}$ and $\lambda \in \Lambda^{r,1}$, then

$$d_T ([\varphi , \psi]_G) = [d_T \varphi , \psi]_G + (-1)^p [\varphi , d_T \psi]_G \ , \qquad (41b)$$

$$[\varphi , \psi]_G = (-1)^{1+ pq + p(\varphi)p(\psi)} [\psi , \varphi]_G \ , \qquad (41c)$$

$$(-1)^{PR}[\varphi, [\psi, \lambda]_G]_G + (-1)^{QP}[\psi, [\lambda, \varphi]_G]_G + (-1)^{RQ}[\lambda, [\varphi, \psi]_G]_G = 0 \ ; \qquad (41d)$$

here P, R, Q are the total degrees, namely $P = r + p(\varphi)$, $Q = q + p(\psi)$, $R = r + p(\lambda)$.

By using (38) and (40) the curvature Ω defined in (2.7) may be equivalently written as

$$\Omega = d\omega - 1/2 \, [\omega, \omega]_G , \qquad (42)$$

and, by (41b-d), the Bianchi identity (29) may be written as

$$d\Omega - [\omega, \Omega]_G = 0 \ . \qquad (43)$$

Let now $I_G(A^{\otimes k}, \mathcal{F})$ be the space of symmetric $\mathcal{F}$-multilinear maps $W : A^{\otimes k} \to \mathcal{F}$, graded symmetric

$$\varphi(\cdots \otimes Z_{i+1} \otimes Z_i \otimes \cdots) = (-1)^{p(Z_i)p(Z_{i+1})} \varphi(\cdots \otimes Z_i \otimes Z_{i+1} \otimes \cdots), \quad Z_i \in A^\bullet, \tag{44}$$

and such that

$$r(Y)\, W = 0 , \quad \forall\ Y \in \mathcal{E}^\bullet . \tag{45}$$

Here $r(Y)W$ is defined by

$$r(Y)W(Z_1 \otimes\cdots\otimes Z_k) =: \ Y\cdot W(Z_1 \otimes\cdots\otimes Z_k) - W(\ r(Y)\cdot(Z_1 \otimes\cdots\otimes Z_k)) ,$$

$$\forall\ Z_i \in A . \tag{46}$$

Condition (45), together with the trivial action of A on $\mathcal{F}$ implies 'adjoint invariance'

$$\Sigma_i(-1)^{p(Z)\Sigma_{j<i}\, p(Z_j)}\ W(Z_1 \otimes\cdots\otimes [\,Z, Z_i\,]_G \otimes\cdots\otimes Z_k) = 0 , \quad \forall\ Z,\ Z_i \in A^\bullet . \tag{47}$$

The space $I_G(A^{\otimes k}, \mathcal{F}) =: \oplus_k I_G(A^{\otimes k}, \mathcal{F})$ is made into a graded algebra by defining, for $W \in I_G(A^{\otimes k}, \mathcal{F})$ and $V \in I_G(A^{\otimes h}, \mathcal{F})$ the product $WV \in I_G(A^{\otimes(k+h)}, \mathcal{F})$ as follows

$$WV(Z_1 \otimes\cdots\otimes Z_{k+h}) = 1/k!h!\ \Sigma_\sigma(-1)^{c(W,\sigma,p)+d(\sigma)}W(Z_{\sigma(1)} \otimes\cdots\otimes Z_{\sigma(k)}) \times$$

$$\times\, V(\,Z_{\sigma(k+1)} \otimes\cdots\otimes Z_{\sigma(k+h)}) , \quad \forall\ Z_i \in A^\bullet, \tag{48}$$

with $c(W,\sigma,p)$ and $d(\sigma)$ given as in (5).

If φ is any element in $\Lambda^{p,k}$ and W any element in $I_G\,(A^{\otimes k},\mathcal{F})$ we may define a form in $\Lambda_G{}^p(E; \mathcal{F})$ by $W(\varphi) =: W\circ\varphi$. Property (45) implies

$$d_r\, W(\varphi) = W\,(d_r\varphi) , \tag{49}$$

while condition (47) gives

$$\Sigma_k(-1)^{p_1+..+p_i\,+p(\varphi)\Sigma_{j<i}\, p(\psi_j)}W(\psi_1 \wedge\cdots\wedge[\psi_i\,,\,\varphi]_G \wedge\cdots\wedge \psi_k) = 0 ,$$

$$\forall\ \psi_i \in \Lambda^{p_i,1} , \varphi \in \Lambda^{1,1} . \tag{50}$$

We are ready to construct the graded Chern-Weil homomorphism.

Proposition 3.2. Given an extension of Lie superalgebras $0 \to A \to E \to B \to 0$, with a connection, let Ω be the corresponding curvature form and let $W \in I_G(A^{\otimes k}, \mathcal{F})$. Then

I. the 2k–form $W(\Omega^k) \in \Lambda^{2k}(E, \mathcal{F})$ projects onto a closed 2k–form $\underline{W(\Omega^k)} \in \Lambda^{2k}(B, \mathcal{F})$;

II. if $w(W)$ is the element of the graded cohomology group $H^{2k}(B,\mathcal{F})$ defined by $\underline{W(\Omega^k)}$, then $w(W)$ does not depend on the connection and $w : I_G(A, \mathcal{F}) \to H^*(B, \mathcal{F})$ is a graded algebra homomorphism (graded Weil homomorphism).

The proof goes on as the one of proposition 2.6. and we shall omit it; it can be found in [LaMa3]. The graded Chern-Simons forms are provided by the following

Proposition 3.3. The 2k–form $W(\Omega^k)$ is exact on E and one has

$$W(\Omega^k) = k\, d\int_0^1 W(\omega \wedge \Psi_t^{k-1})\, dt \quad , \tag{51}$$

with

$$\Psi_t = t\, d\omega - 1/2\, t^2 [\omega, \omega]_G \quad . \tag{52}$$

The graded Chern-Simons forms $\int_0^1 W(\omega \wedge \Psi_t^{k-1})$ are used in the study of anomalies of supersymmetric gauge theories [BPT].

4. ALGEBRAIC GAUGE THEORY

In section 2.4. we have shown how an extension of Lie algebras can be endowed with concepts which constitute the kinematic of a gauge theory like connection, curvature, covariant derivative and codifferential and so on. What we would like to do in this chapter is to introduce the dynamical counterparts. Our framework will be an extension of Lie algebras $0 \to A \to E \to B \to 0$ as in (2.21) which in addition is a Lie bundle over a commutative algebra $\mathcal{F}$ with unit (for details see section 2.5.) and carrying a connection ρ as defined in (2.31). Firstly, by using the covariant codifferential defined in (2.41), we shall introduce fields equations. These equations will be derived from a lagrangian by means of a suitable variational principle. Then, we shall give few examples. We shall construct sequences of Lie algebras which provide algebraic abstract analogues of electromagnetism, monopoles and instantons. The reason for the names is that by realizing the sequences in terms of vector fields associated with proper fibre bundles, we get back the usual solutions. However, our sequences go beyond these realizations and could be applied to situations where the Lie algebras are not Lie algebras of vector fields. We stress that in our construction we don't need any manifold stucture. In our formalism there is also room for gauge transformations and symmetries and conserved quantities; they are described in sections 4.3. and 4.4. respectively. In section 4.5. we shall describe BRST transformations for our sequences. Finally, in section 4.7. we shall construct a sequence which, when endowed with the structure of Einstein algebra, gives an algebraic Kaluza-Klein monopole; this will also provide an example of Einstein algebra.

In section 4.8. we shall sketch the algebraic graded gauge theory. This will be a straightforwad generalization of the nongraded theory. As examples we shall construct a graded algebraic electromagnetism and a graded algebraic abelian monopole.

4.1. FIELD EQUATIONS AND LAGRANGIAN

In section 2.4. a connection, if it exists, is completely arbitrary. We shall now restrict it by requiring that a set of algebraic equations be satisfied. Suppose we are given a connection ρ with curvature F_ρ as in (2.34), on the sequence $0 \to A \to E \to B \to 0$.
The *'Yang-Mills equations'* for ρ are the following algebraic equations

$$\mathcal{D}_\rho F_\rho = 0, \tag{1}$$

where $\mathcal{D}_\rho$ is the covariant codifferential associated with ρ and defined in (2.41).

We call *'Yang-Mills lagrangian'* for the connection ρ the element of $\mathcal{F}$ defined by

$$L_\rho = (hg^{-1})(F_\rho, F_\rho) . \tag{2}$$

Here the metric (hg^{-1}) is the metric induced on the space $\Lambda^p(B, A)$ of A-valued forms on B by a metric g on B and a metric h on A as shown in section 2.4.

We can define a variational problem so that equations (1) are an extremum of the lagrangian (2). Since the space of all connections is an affine space modelled on the vector space $\Lambda^1(B,A)$, it suffices to vary the connection ρ along lines of connections $\rho_t = \rho + t\eta$, $\eta \in \Lambda^1(B,A)$, $t \in \mathbb{R}$. As a preliminar result we have

Proposition 4.1. Given the line of connections $\rho_t = \rho + t\,\rho$, $\eta \in \Lambda^1(B, A)$, the corresponding curvature is $F_{\rho_t} = F_\rho - t\,D_\rho\eta - 1/2\, t^2\, [\eta, \eta]$.

By using this proposition one has the expansion

$$L_{\rho_t} = (hg^{-1})(F_{\rho_t}, F_{\rho_t}) = L_\rho - 2t\,(hg^{-1})(D_\rho\eta, F_\rho) + O(t^2) .$$

Then we require that at an extremum

$$(hg^{-1})(D_\rho\eta, F_\rho) = 0 , \tag{3}$$

which in turn, by using proposition 2.4., gives

$$(hg^{-1})(\eta, \mathcal{D}_\rho F_\rho) + \text{div(something)} = 0 , \quad \forall\, \eta \in \Lambda^1(B; A) . \tag{4}$$

If we identify elements of $\mathcal{F}$ which differ by a divergence we see that up to divergences, equation (4) says that the quantity $(hg^{-1})(\eta, \mathcal{D}_\rho F_\rho)$ can be put equal to zero for any η in $\Lambda^1(B, A)$. Since the metric (hg^{-1}) is nondegenerate we may infer that $\mathcal{D}_\rho F_\rho = 0$.
If the algebra B is of finite type we can write explicitly the divergence in (4). Indeed, with $\{b_i, i = 1,..., n\}$ a basis of B and $\{\beta^i, i = 1,..., n\}$ the dual basis, from (2.46) we get

$$(hg^{-1})(D_\rho\eta, F_\rho) - (hg^{-1})(\eta, \mathcal{D}_\rho F_\rho) = \text{div}\,(\Sigma\, h\,(\eta \circ g^{-1}(\beta^k), F(b_k, \cdot))) . \tag{5}$$

The previous relation will be used in section 4.4. to construct the conserved currents associated with symmetries of the lagrangian (2).

4.2. EXAMPLES

In this section we shall construct the sequences of Lie algebras for the algebraic electromagnetism, monopoles, instantonic solutions, and so forth. For any example we shall show how to recover the usual solutions by properly realizing the sequences in terms of vector fields.

The basic ingredient is an extension $0 \to A \to E \to B \to 0$ with the structure of Lie bundle over a commutative algebra $\mathcal{F}$. Furthermore, we shall assume that B generates over $\mathcal{F}$ the all of the module Der$\mathcal{F}$ (B and Der$\mathcal{F}$ are the same as $\mathcal{F}$-modules). In any example we specialize in a suitable manner the sequence and give a particular connection on it.

4.2.1. Algebraic electromagnetism [LaMa1]

Let us take an extension of Lie algebras $0 \to A \to E \to B \to 0$ with the structure of Lie bundle over a commutative algebra $\mathcal{F}$. We make the following assumptions on it:

1. $\mathcal{F}$ is taken to be 'generated' by four elements f_1, f_2, f_3, h ; generated here really means that the $\mathcal{F}$-module (Der$\mathcal{F}$)* is freely generated as an $\mathcal{F}$-module by the elements df_1, df_2, df_3, dh . The operator d is the algebraic operator defined by (2.1) ;

2. B is an abelian algebra generated as an $\mathcal{F}$-module by four elements $X_1, \ldots, X_4$ whose action on $\mathcal{F}$ is given by $X_\mu \cdot f_\nu = \delta_{\mu\nu}$; $\mu, \nu = 1, \ldots, 4$.

3. E is an abelian algebra generated by five elements $Y_1, \ldots, Y_4, Z$ such that $\pi(Y_i) = X_i$, $\pi(Z) = 0$. Then the ideal A is generated by Z .

We take a connection ρ so that the corresponding connection form ω gives

$$\omega(Z) = Z , \quad \omega(Y_i) = 0 . \tag{6}$$

For a given choice of Z, any A-valued form φ can be considered as an $\mathcal{F}$-valued form $\underline{\varphi}$ via $\varphi = Z \otimes \underline{\varphi}$. In particular $\omega = Z \otimes \underline{\omega}$ with $\underline{\omega}(Z) = 1$.

As for the curvature F defined by (2.34), it may be written equivalently as

$$F(X, X') = \omega([\rho(X), \rho(X')]) , \quad \forall \ X, X' \in B , \tag{7}$$

which in turn gives

$$\underline{F}(X, X') = \underline{\omega}([\rho(X), \rho(X')]) = -d\underline{\omega}(\rho(X), \rho(X')) , \tag{8}$$

since $\rho(X)\cdot\underline{\omega}(\rho(X')) = \rho(X')\cdot\underline{\omega}(\rho(X)) = 0$. From (8) we have

$$d\underline{\omega} = -\pi^* \underline{F} \tag{9}$$

and in turn the homogeneous Maxwell equations follow

$$d\underline{F} = 0 . \tag{10}$$

If we define $\underline{F}_{\mu\nu} =: \underline{F}(X_\mu, X_\nu)$, then previous equations read

$$X_\lambda\cdot\underline{F}_{\mu\nu} + X_\mu\cdot\underline{F}_{\nu\lambda} + X_\nu\cdot\underline{F}_{\lambda\mu} = 0 , \quad \lambda, \mu, \nu = 1,\dots, 4. \tag{11}$$

In order to obtain the 'inhomogeneous equations', we assume there is a metric η on B, such that $\eta(X_\mu, X_\nu) = \eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. Then the codifferential of $\underline{F}$ as defined in (2.16) (in this example the covariant codifferential of F can be identified with the codifferential of $\underline{F}$) is given by

$$\delta\underline{F}(X) = -\Sigma\, \eta^{\mu\nu} X_\mu\cdot\underline{F}(X_\nu, X) \in \mathcal{F} , \quad \forall \ X \in B . \tag{12}$$

with $\eta^{\mu\nu}$ the inverse matrix of $\eta_{\mu\nu}$. The inhomogeneous vacuum Maxwell equations are

$$\delta\underline{F} = 0 \tag{13}$$

and in components

$$\Sigma\, \eta^{\mu\nu} X_\mu\cdot\underline{F}_{\nu\lambda} = 0 , \quad \lambda = 1, \dots, 4 . \tag{14}$$

We can obtain a realization of the previous construction in terms of vector fields associated with a principal fibration. Let us consider the trivial fibre bundle $U(1) \rightarrow M^4 \times U(1) \rightarrow M^4$ where M^4 is the Minkowski space. Then we take for $\mathcal{F}$ the algebra of smooth $\mathbb{R}$-valued functions on M^4 with (f_1, f_2, f_3, h) the coordinates $(x^\mu) = (x^1, x^2, x^3, t)$. We take for B the Lie algebra of smooth vector fields on M^4 with $X_\mu = \partial/\partial x^\mu$. Finally, E is the Lie algebra of smooth vector fields on $M^4 \times U(1)$ with $Y_\mu = \partial/\partial x^\mu$ and Z the generator of the right action of U(1) on itself. Equations (9) through (14) are nothing but the usual electromagnetism.

4.2.2. Algebraic abelian monopole [LaMa2]

Let us take an extension of Lie algebras $0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$ with the structure of Lie bundle over a commutative algebra $\mathcal{F}$. We make the following assumptions on it:

1. $\mathcal{F}$ is taken to be generated by three elements f_1, f_2, f_3.

2. The Lie algebra E is a free $\mathcal{F}$-module of rank three generated, as an $\mathcal{F}$-module, by three elements Y_1, Y_2, Y_3 which close the Lie algebra of the rotation group

$$[Y_i, Y_j] = \Sigma \varepsilon_{ijk} Y_k , \quad i, j = 1, 2, 3. \tag{15}$$

and action on $\mathcal{F}$ given by

$$Y_j \cdot f_k - \Sigma \varepsilon_{jkh} f_h = 0 \quad , \quad j, k = 1, 2, 3. \tag{16}$$

This action leaves invariant the quantity $f^2 = (f_1)^2 + (f_2)^2 + (f_3)^2$. In the sequel we shall assume that f^2 is a non vanishing 'constant', that is a non vanishing element of $\mathbb{R}$.

3. The Lie algebra A is generated over $\mathcal{F}$ by the element

$$Z = 1/f\,(f_1 Y_1 + f_2 Y_2 + f_3 Y_3). \tag{17}$$

The particular action (16) implies that Z commute with the Y_j's

$$[Y_j, Z] = 0. \tag{18}$$

As a consequence of assumptions 2. and 3. we have that

4. the Lie algebra B is an $\mathcal{F}$-module generated by three elements $X_j = \pi(Y_j)$, $j = 1, 2, 3$, with the constraint $\Sigma f_j X_j = 0$ (so B is not free).

Let now E* be the $\mathcal{F}$-dual module of E . E* is automatically a free module of rank three. We take its basis to consist of elements θ^i, $i = 1, 2, 3$, such that $\theta^i (Y_j) = \delta^i_j$.

Consider the combination

$$\theta = 1/f \; \Sigma f_j \theta^j \;, \quad f \neq 0 \;, \tag{19}$$

then $\theta(Z) = 1$, and the A -valued 1-form on E

$$\omega = Z \otimes \theta \tag{20}$$

is a connection form for the sequence we are considering. The corresponding connection is given by

$$\rho : B \to E \;, \quad \rho(X_j) = Y_j - (1/f)\, f_j Z \;, \; j = 1, 2, 3 \;. \tag{21}$$

Remark : For a given choice of Z, any A-valued form $\underline{\varphi}$ can be considered as an $\mathcal{F}$ valued form $\underline{\varphi}$ via $\varphi = Z \otimes \underline{\varphi}$.

As A is an abelian ideal in E, the curvature Ω of the connection (20) becomes

$$\Omega(X, Y) = \{ X{\cdot}\theta(Y) - Y{\cdot}\theta(X) - \theta([X, Y]) \}\, Z$$

$$= \{ d\theta\,(X, Y) \}\, Z \;, \quad \forall \; X, Y \in E \;. \tag{22}$$

Now $\underline{\Omega} = d\theta$, being horizontal, projects onto a 2-form $\underline{F} \in \Lambda^2(B, \mathcal{F})$ which defines a non trivial cohomology class, i.e. it is closed but not exact. For, suppose $\underline{F} = d\alpha$ for some $\alpha \in \Lambda^1(B,\mathcal{F})$, then its pullback $\pi^*\alpha$ is a horizontal form on E such that $d(\theta - \pi^*\alpha) = 0$. Since $(\theta - \pi^*\alpha)(Z) = 1$, the 1-form $\theta - \pi^*\alpha$ would give a connection whose curvature vanishes, i.e. we could find a rotationally invariant 2 + 1 dimensional splitting of the three dimensional vector space generated by Y'_1, Y'_2, Y'_3. These vector fields close on the rotation Lie algebra, are annihilated by $\theta - \pi^*\alpha$ and have the form $Y'_i = Y_i - (f_i - g_i)\, Z$

where the g_i's are the coefficients of $\pi^*\alpha = \Sigma\, g_i\, \theta^i$ (we have taken $f^2 = 1$ which is possible because we assume that f^2 is a non vanishing number).

By explicit calculations one can show that the connection (21) solves the Yang-Mills equations (1) when the metric g on B is just taken to be the 'natural' metric

$$g = \Sigma\, (df_j \otimes df_j) \ , \text{restricted by } \ \Sigma\, f_j\, f_j = 1 \text{ and } \Sigma\, f_j\, df_j = 0 \ . \tag{23}$$

We shall give now a realization of our Lie algebra extension and show that all previous defining assumptions are met; in doing this we also recover the usual abelian monopole [LaMa1]. Let us consider the U(1) Hopf fibration over the two dimensional unit sphere S^2

$$S^1 \to S^3 \to S^2 \ . \tag{24}$$

Then we take:

1. $\mathfrak{F}$ to be the algebra of smooth $\mathbb{R}$-valued functions on S^2 with f_1, f_2 and f_3 the cartesian coordinate in $\mathbb{R}^3$ and with the condition $\Sigma\, f_j\, f_j = 1$.
2. E the Lie algebra of right invariant vector fields on $SU(2) \equiv S^3$ with basis vectors Y_j just the three canonical right invariant vector fields .
3. Z the left invariant vector fields on SU(2) which generate the right action of U(1) on S^3 . Then B is the Lie algebra of smooth vector fields on S^2, with a basis given by the three generators of the action of SU(2) on S^2, i.e. $X_j = \Sigma\, \varepsilon_{jkh}\, f_k\, \partial / \partial f_h$, $j = 1, 2, 3$.

One checks that all conditions from (15) through (18) are satisfied. Then, the connection ω given in (20) describes the Dirac monopole of lower strength [Di].

4.2.3. Algebraic Instantonic Solutions [LaMa6]

Let us start by considering a commutative $\mathfrak{R}$-algebra with unit $\mathfrak{G}$ 'generated' by five elements $f_1,\ldots, f_5$. Let us also take a Lie algebra L generated over $\mathfrak{G}$ by five elements $X_1,\ldots, X_5$, and assume there is a representation of L in the Lie algebra Der $\mathfrak{G}$. The action of the X_j's on the f_k's is taken to be

$$X_j \cdot f_k = \delta_{jk} - f_j\, f_k \quad , \qquad j, k = 1,\ldots, 5 \ . \tag{25}$$

If we consider the ten elements in L given by

$$X_{jk} = [X_j, X_k], \qquad j,k = 1,\dots,5 \ , \tag{26}$$

from (25) if follows that

$$[X_{jk}, X_{hm}] = \delta_{kh} X_{jm} - \delta_{jh} X_{km} - \delta_{km} X_{jh} + \delta_{jm} X_{kh} \ , \tag{27}$$

so that the X_{jk}'s provide a representation of the Lie algebra of SO(5) in Der$\mathfrak{G}$. From (26) and (25) one has also

$$X_{jk} \cdot f_n = f_j \delta_{kn} - f_k \delta_{jn} = \{f_j X_k - f_k X_j\} \cdot f_n \ , \tag{28}$$

so that we may take

$$X_{jk} = f_j X_k - f_k X_j \ , \qquad j,k = 1,\dots,5 \ , \tag{29}$$

Let us now take $f^2 = \Sigma_j f_j f_j$. From (25) it follows that f^2 is constant under the action of the X_j's, namely, $X_j \cdot f^2 = 0$. We assume that $f^2 \neq 0$. We now construct an extension of Lie algebras $0 \to A \to E \to B \to 0$ with the structure of Lie bundle over a commutative algebra $\mathfrak{F}$ in the following manner. We take the algebra $\mathfrak{F}$ to be the subalgebra of $\mathfrak{G}$ generated by the elements $\mathfrak{f}_j = f_j / f^2$, $j = 1,\dots,5$; notice that $\mathfrak{F}$ contains also the unit element since $\Sigma_j \mathfrak{f}_j \mathfrak{f}_j = 1$. The X_j 's and the X_{jk} 's will act on $\mathfrak{F}$ but now the corresponding representation on $\mathfrak{F}$ (thought of as the restriction of the representation of L on $\mathfrak{G}$) is no more a free one since, from $\Sigma \mathfrak{f}_j \mathfrak{f}_j = 1$, one has

$$(\Sigma \mathfrak{f}_j X_j) \cdot \mathfrak{f}_k = \mathfrak{f}_{jk}(1 - \Sigma \mathfrak{f}_j \mathfrak{f}_j) = 0 \tag{30}$$

which, in turn, gives the condition

$$(\Sigma \mathfrak{f}_j X_j) \cdot \mathfrak{f}_k = 0 \ , \quad k = 1,\dots,5 \ . \tag{31}$$

We take the Lie algebra B to be the subalgebra of L which provides the restricted representation in Der$\mathfrak{F}$. Condition (31) then amounts to say that B is not free as an $\mathfrak{F}$-module. Equations (27) and (29) are true even when restricted to $\mathfrak{F}$. By using the condition $\Sigma \mathfrak{f}_j \mathfrak{f}_j = 1$ we have in addition that

$$\{\Sigma \mathfrak{f}_j X_{jk}\} \cdot \mathfrak{f}_n = X_k \cdot \mathfrak{f}_j \ , \quad j,k = 1,\dots,5 \ . \tag{32}$$

Finally, we give conditions on the algebra E. The latter is taken to be a ten dimensional free $\mathcal{F}$-module generated, as an $\mathcal{F}$-module, by ten elements $Y_{jk} = -Y_{kj}$, $j, k = 1,..., 5$, which close the Lie algebra of $Sp(2) = Spin(5)$

$$[Y_{jk}, Y_{hm}] = \delta_{kh} Y_{jm} - \delta_{jh} Y_{km} - \delta_{km} Y_{jh} + \delta_{jm} Y_{kh} , \tag{33}$$

and project onto the elements X_{jk}, namely

$$\pi(Y_{jk}) = X_{jk} , \quad j, k = 1,..., 5 . \tag{34}$$

The action of the Y_{jk}'s on the f_j's is just taken to be

$$Y_{jk} \cdot f_n = \pi(Y_{jk}) \cdot f_n = f_j \delta_{kn} - f_k \delta_{jn} , \quad j, k, n = 1,..., 5 \tag{35}$$

That much for the assumptions on the sequence. As a consequence the Lie algebra A is six dimensional as an $\mathcal{F}$-module. A possible redundant basis is given by the ten elements

$$Z_{jk} = Y_{jk} - f_j (\Sigma f_h Y_{hk}) + f_k (\Sigma f_h Y_{hj}) , \quad j, k = 1,..., 5 , \tag{36}$$

with the four conditions

$$\Sigma f_j Z_{jk} = 0 , \quad k = 1,..., 5 , \tag{37}$$

Notice that (37) gives only four independent conditions due to condition $\Sigma f_j f_j = 1$. From (35) it follows that $Z_{jk} \cdot f_n = 0$ so that $\pi(Z_{jk}) = 0$ as it should be. Moreover, from (33)

$$[Y_{jk}, Z_{hm}] = \delta_{kh} Z_{jm} - \delta_{jh} Z_{km} - \delta_{km} Z_{jh} + \delta_{jm} Z_{kh} , \tag{38}$$

which explicitly shows that A is an ideal in E.

We are now ready to give a connection which will be the algebraic counterpart of the instanton cum antiinstanton configuration in usual Yang-Mills theories. We take

$$\rho : B \to E , \quad \rho(X_j) =: \Sigma f_k Y_{kj} , \quad j = 1,..., 5 , \tag{39}$$

where the Y_{kj}'s are just the basis element of E. One can show that $\pi \circ \rho(X_j) = X_j$. From definition (39), and by using (33) and (35) it easily follows that

$$[\,\rho(X_j)\,,\,\rho(X_k)\,] = Y_{jk}\ , \quad j,k = 1,\dots,5\ . \tag{40}$$

As for the curvature of ρ defined in (2.34), by using (40) and (29) and (32) one finds that

$$F(X_j\,,X_k) = Y_{jk} - f_j\,\rho(X_k) + f_k\,\rho(X_j) = Y_{jk} - f_j\,(\Sigma f_h Y_{hk}) + f_k\,(\Sigma f_h Y_{hj}) = Z_{kj}\ . \tag{41}$$

By explicit calculations one can check that the connection (39) solves the Yang-Mills equations (1) when the metric g on B is just taken to be the 'natural' metric

$$g = \Sigma\,(df_j \otimes df_j)\,, \quad \text{restricted by} \quad \Sigma f_j f_j = 1 \text{ and } \Sigma f_j\ df_j = 0\ . \tag{42}$$

We shall give now a realization of our Lie algebra extension and show that all previous defining assumptions are met. This will also account for the name of the connection (39).

Consider a Spin(4)-principal bundle over the four dimensional sphere S^4

$$\mathrm{Spin}(4) \rightarrow \mathrm{Spin}(5) \rightarrow S^4\ . \tag{43}$$

Notice that $\mathrm{Spin}(5) \equiv \mathrm{Sp}(2)$ and $\mathrm{Spin}(4) \equiv \mathrm{Sp}(1) \times \mathrm{Sp}(1) \equiv \mathrm{SU}(2) \times \mathrm{SU}(2)$. Then we take:
1. $\mathcal{F}$ to be the algebra of smooth $\mathbb{R}$-valued functions on S^4 with $f_1,\dots, f_5$ the cartesian coordinate in $\mathbb{R}^5$ and with the condition $\Sigma f_j f_j = 1$.
2. B the Lie algebra of smooth vector fields on $\mathbb{R}^5$ which are tangent to the unit sphere S^4; a basis for them is given by the vector fields $X_j = \partial/\partial f_j - f_j \Sigma_k f_k \partial/\partial f_k$, $j = 1,\dots, 5$.
3. E the Lie algebra of Spin(4)-invariant vector fields on Spin(5) with the basis vectors Y_{jk} just the ten generators of the left action of Spin(5) onto itself.
The Lie algebra A is the algebra of vertical and Spin(4)-invariant vector fields on Spin(5). One easily verifies that all assumptions from (25) through (38) are satisfied. The resulting extension of Lie algebras coincides with the extension considered in [AB] for a general gauge theory. As for the connection defined in (39) one can verify that it gives the instanton along with the antiinstanton solution of Yang-Mills equations on S^4 [JR], [AHS].

Next thing we would like to consider is the split of the connection (39) into 'self-dual' and 'antiself-dual' part. To this aim we need few more machineries. From the assumption that B^* is generated as an $\mathcal{F}$-module by the elements $df_1, \ldots, df_5$ (remember we have the conditions $\Sigma f_j f_j = 1$ and $\Sigma f_j df_j = 0$), it follows that the $\mathcal{F}$-module $\Lambda^2(B, \mathcal{F})$ is generated by the elements $df_j \wedge df_k$, $j<k = 1, \ldots, 5$. We define an operator

$$* : \Lambda^2(B, \mathcal{F}) \rightarrow \Lambda^2(B, \mathcal{F}) ,$$

$$*(df_{i_1} \wedge df_{i_2}) =: 1/2\, \delta_{i_1 j_1} \delta_{i_2 j_2} \varepsilon_{j_1 \ldots j_5} f_{j_5} df_{j_3} \wedge df_{j_4} . \tag{44}$$

From now on repeated indeces are meant to be summed over. One can show that

$$* \circ * = \mathrm{id} . \tag{45}$$

If we write the curvature in (41) as

$$F = 1/2\, F_{jk}\, df_j \wedge df_k , \tag{46}$$

it turns out that $F_{jk} = Z_{jk}$. Moreover, condition (45) allows to write F as

$$F = F^+ + F^- , \quad *(F^\pm) = \pm F^\pm . \tag{47}$$

Condition $*F = F$ turns out to be equivalent to the following conditions

$$Y_{jk} - f_j (\Sigma f_h Y_{hk}) + f_k (\Sigma f_h Y_{hj}) = 1/2\, \varepsilon_{jkhmn} f_h Y_{mn} , \quad j, k = 1, \ldots, 5 . \tag{48}$$

One can check that they constitute only three independent conditions.

Let us now take a Lie bundle $0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$ over the algebra $\mathcal{F}$, with the same properties as before from (25) through (38) plus additional conditions (48). Now the algebra E is seven dimensional as an $\mathcal{F}$-module and is generated in a redundant manner by the ten vector fields Y_{jk} while the algebra A is three dimensional as an $\mathcal{F}$-module. The connection defined in (39) will be 'self-dual' on this sequence, i.e. its curvature will satisfy the condition $*F = F$. We shall call it the *algebraic instantonic connection*.

As for an explicit realization in terms of vector fields, consider an SU(2)-principal bundle over S^4

$$SU(2) \rightarrow S^7 \equiv Spin(5) / Sp(1) \rightarrow S^4 . \tag{49}$$

Now the Lie algebra E is the Lie algebra of SU(2)-invariant vector fields on S^7. A basis for them consists of the ten generators Y_{jk} of the left action of Spin(5) on S^7. One verifies that these vector fields Y_{jk} satisfy condition (48). Then, the connection (39) is nothing but the instanton connection on S^4.

Finally we give a straightforward generalization of the previous construction (in the following $n > 1$). We specialize an $\mathfrak{F}$ Lie bundle $0 \to A \to E \to B \to 0$ as follows:

a. take $\mathfrak{F}$ to be generated by n+1 elements $f_1,..., f_{n+1}$ with the condition $\Sigma f_j f_j = 1$;

b. take B to be generated as an $\mathfrak{F}$-module by n+1 elements $X_1,..., X_{n+1}$, whose action on the f_i's is given by

$$X_j \cdot f_k = \delta_{jk} - f_j f_k \quad , \quad j, k = 1,...,n+1 . \tag{50}$$

The condition $\Sigma f_j f_j = 1$ gives $\Sigma f_j X_j = 0$ so that B is only n-dimensional as $\mathfrak{F}$-module. From (50) the 1/2 (n+1)n commutators $X_{jk} = [X_j, X_k]$, j, k = 1,..., n+1, give a representation of the Lie algebra of SO(n+1) in Der$\mathfrak{F}$.

c. take E a 1/2 (n+1)n dimensional free $\mathfrak{F}$-module generated by elements $Y_{jk} = -Y_{kj}$, j, k = 1,...,n+1, which close the Lie algebra of Spin(n+1) and are such that $\pi(Y_{jk}) = X_{jk}$.

As a consequence of assumptions a. b. c. the Lie algebra A is 1/2 (n−1)n dimensional as $\mathfrak{F}$-module. A possible redundant basis is given by the 1/2 (n+1)n elements

$$Z_{jk} = Y_{jk} - f_j (\Sigma f_h Y_{hk}) + f_k (\Sigma f_h Y_{hj}) \ , \quad j, k = 1,...,n+1, \tag{51}$$

with the n conditions $\Sigma f_j Z_{jk} = 0$, k = 1,..., n+1. We take the following connection

$$\rho : B \to E \ , \quad \rho(X_j) =: \Sigma f_k Y_{kj} \ , \quad j = 1,...,n+1, \tag{52}$$

whose corresponding curvature turns out to be

$$F(X_j, X_k) = Y_{jk} - f_j (\Sigma f_h Y_{hk}) + f_k (\Sigma f_h Y_{hj}) \ , \quad j, k = 1,...,n+1. \tag{53}$$

A realization of the previous construction in terms of vector fields is given by means of the Spin(n)-principal bundle over the n-dimensional sphere S^n, Spin(n) $\to$ Spin(n+1) $\to S^n$.

The algebra $\mathcal{F}$ is the algebra of smooth $\mathbb{R}$-valued functions on S^n with $f_1,\ldots, f_{n+1}$ as cartesian coordinates in $\mathbb{R}^{n+1}$ and condition $\Sigma f_j f_j = 1$. B is the Lie algebra of smooth vector fields on $\mathbb{R}^{n+1}$ which are tangent to S^n and with basis given by the vector fields $X_j = \partial/\partial f_j - f_j \Sigma_k f_k \partial/\partial f_k$, $j = 1,\ldots, n+1$. E is the Lie algebra of Spin(n)-invariant vector fields on Spin(n+1) with the basis vectors Y_{jk} the 1/2 (n+1)n generators of the left action of Spin(n+1) onto itself. Then the connection in (52) is just the natural spinor connection on S^n [Tr4], [TD] . This connection is a solution of Yang-Mills equation on S^n for any n>1 [La2]. For the lower values of n it gives well known solutions [Tr4], [TD] : for n = 2 is the Dirac monopole [Di] ; for n = 3 the meron [DFF] ; for n = 4 the instanton cum antiinstanton [JR], [AHS] ; for n = 8 [LA1] the GKS solution [GKS].
Another possible realization is by means of the principal bundle $SO(n) \to SO(n+1) \to S^n$. In this case the connection (52) is a multiple of the Levi-Civita connection with the latter thought of as an SO(n) gauge connection.

4.3. GAUGE TRANSFORMATIONS

In this section we will show how to introduce '*gauge transformations*' in our algebraic setting. In particular it will turn out that the Lie algebra A in the extension (2.21) plays the role of the Lie algebra of infinitesimal gauge transformations. The basic object is an extension of Lie algebras $0 \to A \to E \to B \to 0$ which is a Lie bundle over the $\mathcal{R}$-algebra $\mathcal{F}$ with unit. Moreover there is a connection ρ as defined in (2.31) with connection form ω and curvature form Ω given by (2.32) and (2.35) respectively .

Let $\Lambda^p(E, A)$ be the space of A-valued p-forms on E , $\Lambda^0(E, A) \equiv A$. We recall that in section 2.6. (where the space $\Lambda^p(E, A)$ has been denoted by $\Lambda^{p,1}$) we have introduced an operator $d : \Lambda^p(E, A) \to \Lambda^{p+1}(E, A)$ (here and after we shall drop the suffix r from d_r) given by (2.50) and an operation $[\ ,\ ] : \Lambda^p(E, A) \times \Lambda^q(E, A) \to \Lambda^{p+q}(E, A)$ in (2.52).

We now define a 'Lie derivative' by

$$\mathcal{L}_{(\cdot)} : E \times \Lambda^q(E, A) \to \Lambda^p(E, A),$$

$$(\mathcal{L}_Y\varphi)(Y_1,..,Y_p) =: [Y , \varphi(Y_1,.., Y_{p+1})] - \Sigma_j \varphi(Y_1,..,[Y, Y_j],..,Y_{p+1}), \forall\ Y, Y_i \in E ; \quad (54)$$

and an 'inner product' by

$$\iota_{(\cdot)} \; : \; E \times \Lambda^{q}(E, A) \rightarrow \Lambda^{p-1}(E, A),$$

$$(\iota_Y \varphi)(Y_1,..,Y_{p-1}) =: \varphi(Y, Y_1,..., Y_{p+1}) \quad , \forall \; Y, Y_i \in E \; . \qquad (55)$$

The next object we introduce is a covariant derivative D_ω on $\Lambda^p(E, A)$ which is just the analogue on $\Lambda^p(E, A)$ of the operator D_ρ defined in (2.37) :

$$D_\omega \; : \; \Lambda^p(E, A) \rightarrow \Lambda^{p+1}(E, A) \; ,$$

$$D_\omega \varphi \; =: \; (d_r \varphi) \circ H \; , \qquad (56)$$

where H is the horizontal projector defined by

$$H \; : \; E \rightarrow Hor_\rho E \; , \quad H \; =: \; id_E \; - \; \omega \; = \; \rho \circ \pi \; . \qquad (57)$$

In particular, one can prove that the curvature form Ω in (2.35) can be written as

$$\Omega \; = D_\omega \, \omega \; , \qquad (58)$$

while the Bianchi identity (2.38) is

$$D_\omega \, \Omega \; = \; 0 \; . \qquad (59)$$

By explicit calculation one shows that the 'infinitesimal action' of the Lie algebra E on the connection form ω is given by

$$\mathcal{L}_Y \omega \; = \; \iota_Y \Omega \; + D(\iota_Y \omega) \; , \; \forall \; Y \in E \; . \qquad (60)$$

If $V = Y \in A$, then $\iota_Y \Omega = 0$ because Ω is horizontal and we have the 'infinitesimal gauge transformation'

$$\mathcal{L}_V \omega \; = \; D(\iota_V \omega) \; = \; DV \quad , \; \forall \; V \in A \; . \qquad (61)$$

The infinitesimal gauge transformation of the curvature Ω turns out to be

$$\mathcal{L}_V \Omega \; = \; [\, \iota_V \omega \, , \Omega \,] \; = \; [\, V \, , \Omega \,] \; , \; \forall \; V \in A \; . \qquad (62)$$

From (61) and (62) we see that the Lie algebra A can be interpreted as the *Lie algebra of infinitesimal gauge transformations* . As for the connection ρ and its curvature F , their changes under an infinitesimal gauge transformation parametrized by $V \in A$ turn out to be

$$(\delta_V \rho)(X) = [V , \rho(X)] , \quad \forall\ X \in B , \tag{63}$$

$$(\delta_V F)(X,Y) = [V , F(X, Y)] , \quad \forall\ X , Y \in B . \tag{64}$$

Proposition 4.2. The lagrangian $L_\rho = (hg^{-1})(F_\rho , F_\rho)$ is invariant under infinitesimal gauge transformations.

Proof. $\delta_V L_\rho = (hg^{-1})(\delta_V F_\rho , F_\rho) + (hg^{-1})(F_\rho , \delta_V F_\rho) = (hg^{-1})([V, F_\rho] , F_\rho) + (hg^{-1})(F_\rho , [V , F_\rho]) = 0$ by adjoint invariance (2.42) of the metric h .

4.5. SYMMETRIES AND CONSERVED CURRENTS

In this section we briefly describe the role of simmetries in our algebraic formalism.

Let us assume that on the Lie bundle $0 \to A \to E \to B \to 0$ over $\mathcal{F}$, there is a connection ρ . Then, let us take the one parameter family of connections

$$\rho_t = \rho + t\eta , \quad \eta \in \mathrm{Lin}(B, A) , \ t \in \mathbb{R} . \tag{65}$$

We say that the family of transformations (65) is a *symmetry transformation* for the lagrangian $L_\rho = (hg^{-1})(F_\rho , F_\rho)$ if there exist a 1-form $\varphi \in B^*$ such that

$$(d/dt)\, L_{\rho_t} \big|_{t=0} = \mathrm{div}\ \varphi . \tag{66}$$

Here the divergence of φ is defined as in (2.17).

We can construct a *Noether-like theorem* , namely to any symmetry of the lagrangian L_ρ we can associate a *conserved current*. The construction is very explicit if the Lie algebra B in the sequence $0 \to A \to E \to B \to 0$ is of finite type. Let us then assume that this is the case and let $\{b_i , i = 1,...,n \}$ be a basis for B with dual basis $\{ \beta^i , i = 1,...,n \}$.
We have the following

Proposition 4.3. Let the transformation (65) be of symmetry for the lagrangian L_ρ as in (66) and assume the connection ρ solves the Yang Mills equations (1). Then the 1-form

$$C(\eta) =: \Sigma\, h(\eta \circ g^{-1}(\beta^j), F_\rho(b_j, \cdot)) - \varphi \tag{67}$$

is a conserved current in the sense that

$$\operatorname{div} C(\eta) = 0 \ . \tag{68}$$

Proof. From relation (2.46), if quations (1) are satisfied, one has that $\varphi = (d/dt) L_{\rho_t}|_{t=0}$ $= 2(hg^{-1})(D_\rho\eta, F_\rho) = 2(hg^{-1})(\eta, \mathcal{D}_\rho F_\rho) + 2 \operatorname{div}\{\Sigma\, h(\eta \circ g^{-1}(\beta^j), F_\rho(b_j, \cdot))\} =$ $2\operatorname{div}\{\Sigma\, h(\eta \circ g^{-1}(\beta^j), F_\rho(b_j, \cdot))\}$ and (68) follows.

Since the codifferential δ has vanishing square (see section 2.3), any current which is the codifferential of a 2-form will be automatically conserved. Such currents are not interesting in physics and are called trivial currents. Two conserved currents are equivalent if their difference is a trivial current and one is really interested in equivalence classes of conserved currents modulo trivial ones.

Proposition 4.4. The currents associated to infinitesimal gauge transformations are trivial.

Proof. Since the lagrangian L_ρ is invariant, the φ in (66) is zero in this case. If $V \in A$ parametrize the gauge transformation and $X \in B$, from (63) we have

$$C(V)(X) = \Sigma\, h([V, \rho \circ g^{-1}(\beta^j)], F(b_j, X))$$

$$= -\Sigma\, (g^{-1}(\beta^j)) \cdot (h(V, F(b_j, X))) + \Sigma\, h(V, [\rho \circ g^{-1}(\beta^j), F(b_j, X)]) \quad \text{by (2.43)}$$

$$= -\Sigma\, (g^{-1}(\beta^j)) \cdot (h(V, F(b_j, X))) + \Sigma\, h(V, F(\nabla_{g^{-1}(\beta^j)} b_j, X)) +$$

$$+ \Sigma\, h(V, F(b_j, \nabla_{g^{-1}(\beta^j)} X)) \quad \text{by (1)}$$

$$= (\delta\, h(V, F(\cdot,\cdot)))(X) \ .$$

and this ends the proof.

A very general algebraic approach to symmetries and conservation laws, based on the theory of $\mathcal{C}$-spectral sequences has been developed in [Vi].

4.5. BRST TRANSFORMATIONS

In this section we shall show how to construct a BRST-like operator and BRST transformations for the $\mathcal{F}$ Lie bundle $0 \to A \to E \to B \to 0$ endowed with a connection. As an example we shall study the anomaly problem in Yang-Mills gauge theories.

Let $\Lambda^\alpha(A, \mathcal{F})$, $\Lambda^0(A, \mathcal{F}) \equiv \mathcal{F}$, be the space of α-linear, $\mathcal{F}$-linear skew maps from A to $\mathcal{F}$. Let us also consider the space of α-linear, $\mathcal{F}$-linear skew maps from A to $\Lambda^p(E, \mathcal{F})$

$$V_\alpha{}^p =: \Lambda^\alpha(A\,;\Lambda^p(E\,,\mathcal{F})) \equiv \Lambda^p(E,\mathcal{F}) \otimes \Lambda^\alpha(A,\mathcal{F})\,,$$

$$U \equiv \mu \otimes \varphi\ ,\quad U \in V_\alpha{}^p\,,\ \mu \in \Lambda^p(E,\mathcal{F})\,,\ \varphi \in \Lambda^\alpha(A,\mathcal{F}) \quad \text{iff}$$

$$U(Z_1,\ldots,Z_\alpha) = \varphi(Z_1,\ldots,Z_\alpha)\mu\,, \quad \forall\ Z_i \in A\ . \tag{69}$$

We call p the order of form and α the ghost number. We give $V_\alpha{}^p$ the structure of a bicomplex by introducing two cohomology operators. The first one is defined by means of the operator d given by (2.1) and increases by one the order of form

$$d:\ V_\alpha{}^p \to V_\alpha{}^{p+1}\ ,\qquad dU\,(Z_1,\ldots,Z_\alpha) =:\ d\,[U(Z_1,\ldots,Z_\alpha)]\ ,\ \ \forall\ Z_i \in A\,. \tag{70}$$

The second is a Chevalley cohomology operator which increases by one the ghost number

$$s:\ V_\alpha{}^p \to V_{\alpha+1}{}^p\ ,$$

$$(sU)(Z_1,\ldots,Z_{\alpha+1}) =:\ (-1)^{p+1}\{\Sigma_i(-1)^{i+1}\,\mathcal{L}_{Z_i}\,U(\,Z_1,\ldots,\hat{\imath},\ldots,Z_{\alpha+1}) +$$

$$+\,\Sigma_{i<j}\,(-1)^{i+j}\,U([\,Z_i,\,Z_j\,],\,Z_1,\ldots,\hat{\imath},\ldots,\hat{j},\ldots,\,Z_{\alpha+1})\}\ ,\ \forall\ Z_i \in A\,. \tag{71}$$

Here $\mathcal{L}$ is defined as in (2.4). We call s the BRST operator.
One verifies that

$$d^2 = s^2 = sd + ds = 0\,. \tag{72}$$

By using again the identification in (69) it is possible to extend the bracket (2.52) to an operation

$$[\ ,\ ] : V_\alpha^p \times V_\beta^q \to V_{\alpha+\beta}^{p+q} ,$$

$$[U, V](Z_1,\ldots,Z_{\alpha+\beta}) =: (-1)^{\alpha q} 1/\alpha!\beta!\ \Sigma_\sigma\, \chi(\sigma)[\, U(Z_{\sigma(1)},\ldots,Z_{\sigma(\alpha)}) , V(_{\sigma(\alpha+1)},\ldots,Z_{\sigma(\alpha+\beta)})\,]$$

$$\forall\ Z_i \in A . \qquad (73)$$

Then, by using definitions (70), (71) and (73) and Proposition 2.5. one proves

Proposition 4.5. If $U \in V_\alpha^p$ and $V \in V_\beta^q$, then

$$[V, U] = (-1)^{1+(\alpha+p)(\beta+q)}\, [U, V] ,$$

$$d([V, U]) = [dV, U] + (-1)^{\alpha+p} [V, dU] ,$$

$$s([V, U]) = [sV, U] + (-1)^{\alpha+p} [V, sU] . \qquad (74)$$

Any connection 1-form as defined in (2.32) can be thought of as an element $\omega \in V_0^1$. On the other hand, the Maurer-Cartan form ν is the identity map on A , $\nu(Z) = Z$ for any $Z \in A$. This implies that $\nu \in V_1^0 \simeq A \otimes \Lambda^\alpha(A, \mathcal{F})$. Given such maps, one has the following BRST relations

$$s\omega = d\nu - [\omega, \nu] \equiv D_\omega \nu , \qquad (75a)$$

$$s\nu = -1/2\, [\nu, \nu] . \qquad (75b)$$

Here the covariant derivative of any elements U in V_α^p is defined by using identification (69) : $D_\omega U\, (Z_1,\ldots,Z_\alpha) =: D_\omega\, [U(Z_1,\ldots,Z_\alpha)]$, $\forall\ Z_i \in A$.

Proof of (75a) : from (71) and (70) and (73), with Z in A and Y in E , it follows that $(s\omega(Z))(Y) = (L_Z\omega)(Y) = [Z, \omega(Y)] - \omega([Z, Y]) = [Z, \omega(Y)] - [Z, Y] = -[\omega, Z](Y) + (dZ)(Y)$, from which one has $s\omega(Z) = dZ - [\omega, Z]$. On the other hand, (70) and (73) give $(d\nu)(Z) = d(\nu(Z)) = dZ$; $[\omega, \nu](Z) = [\omega, \nu(Z)] = [\omega, Z]$.
By comparison, (75a) follows.

Proof of (75b) : from (71), with Z_1, Z_2 in A , one has $(s\,\nu)(Z_1, Z_2) = -[Z_1, \nu(Z_2)] + [Z_2, \nu(Z_1)] + \nu([Z_1, Z_2]) = -[Z_1, Z_2]$. On the other hand, from (73), one has that $[\nu, \nu](Z_1, Z_2) = 2[\nu(Z_1), \nu(Z_2)] = 2[Z_1, Z_2]$, and (75b) follows.

As an example we shall apply the previous machineries to the problem of anomalies of Yang-Mills gauge theories[LaMa]. For similar constructions see [KaSt2]. Let $\pi : P \to M$ be a principal fibre bundle with stucture group G . We know from section 2.6. that there is a canonical sequence associated with the bundle, namely $0 \to \mathcal{L} \to \aleph_G \to \aleph(M) \to 0$, where $\mathcal{L}$ is the Lie algebra of G-invariant vertical vector fields on P ($\mathcal{L}$ can be identified with the Lie algebra of infinitesimal gauge transformations) and $\aleph_G$ the Lie algebra of G-invariant vector fields on P. The sequence is a Lie bundle over the functions $C^\infty(M)$. It is well known [AB] that a principal connection can be given as a linear map $\rho : \aleph(M) \to \aleph_G$.

In order to get the homotopy formula for the anomalies we need a further generalization of the technique previously developed in this section. Let us take the set of $\Lambda^p(\aleph_G, \mathcal{L}^{\otimes k})$-valued α-linear, $C^\infty(M)$-linear skew maps on $\mathcal{L}$

$$V_\alpha^{p,k} = \Lambda^\alpha(\mathcal{L} ; \Lambda^{p,k}(\aleph_G , \mathcal{L})) \equiv \Lambda^p(\aleph_G , \mathcal{L}^{\otimes k}) \otimes \Lambda^\alpha(\mathcal{L} , C^\infty(M)) \ , \tag{76}$$

where the last identification is made as in (69). It is straightforword to generalize the coboundary operators (70) and (71) to the complex of $V_\alpha^{p,k}$. Notice that V_α^p in (69) is just $V_\alpha^{p,1}$. It is also possible to extend the exterior product (2.51) to a product

$$\wedge : V_\alpha^{p,k} \times V_\beta^{q,h} \to V_{\alpha+\beta}^{p+q,k+h} \quad ,$$

$$U \wedge V\,(Z_1,\dots,Z_{\alpha+\beta}) =: 1/\alpha!\beta!\ \Sigma_\sigma\, \chi(\sigma) U(Z_{\sigma(1)},\dots,Z_{\sigma(\alpha)}) \wedge V(_{\sigma(\alpha+1)},\dots,Z_{\sigma(\alpha+\beta)}) \ ,$$

$$\forall\ Z_i \in \mathcal{L}, \tag{77}$$

and the operators s and d are graded derivations for this product.

Let us now take the total complex ($^*V$, Δ) where $^*V =: \oplus_n {}^nV$, ${}^nV =: \oplus_{p+\alpha=n} V_\alpha^p$, $\Delta = d + s$, with connection form $\mathcal{A} =: \omega - \nu \in {}^1V$, $\mathcal{A} : \aleph_G \oplus \mathcal{L} \to \mathcal{L}$. Its curvature form $\mathcal{F} = \Delta(\omega - \nu) - 1/2[\,\omega - \nu\ ,\ \omega - \nu\,]$ reduces, as a conseguence of relations (75), to $\Omega = d\omega - 1/2\,[\omega, \omega]$. If W is a symmetric $C^\infty(M)$-multilinear map from $\mathcal{L}^{\otimes k}$ to $C^\infty(M)$ obeying property (2.57), (mutatis mutandis) we may apply proposition 2.8. and get

$$W(\Omega^k) = k\,\Delta\int_0^1 W(\mathcal{A} \wedge \mathcal{F}_t^{k-1})\,dt =: \Delta Q^{2k-1} \;, \quad \mathcal{F}_t =: t\,D\mathcal{A} - 1/2\, t^2[\,\mathcal{A}, \mathcal{A}\,] \;. \tag{78}$$

By writing Q^{2k-1} as a sum of homogeneous elements in ghost number and degree of form $Q^{2k-1} = Q_0{}^{2k-1} + Q_1{}^{2k-2} + \cdots + Q_{2k-1}{}^{0}$, equation (78) gives a chain of equations

$$dQ_0{}^{2k-1} = W(\Omega^k) \;,$$

$$sQ_{2k-p-1}{}^{p} + dQ_{2k-p}{}^{p-1} = 0 \;, \quad p = 1, \ldots, 2k-1 \;,$$

$$sQ_{2k-1}{}^{0} = 0 \;, \tag{80}$$

and $Q_1{}^{2k-2}$ provides a solution of the Wess-Zumino consistency condition in $2k-2$ dimensions [Zu], [Sto], [DTV].

4.6. REDUCTION OF SEQUENCES. THE HOLONOMY SEQUENCE

In the framework of fibre bundle description of gauge fields, the reduction of the structure group of a principal fibre bundle is a very useful concept [KN2]. The reduction is used, for instance, to provide a geometrical description of gauge fields [GM], [Tr1,3].

In this section we shall define the notion of reduction of an extension of Lie algebras. Then, given a connection for the extension, we define the holonomy algebra and the holonomy sequence of the connection and prove that it is always possible to reduce the sequence we start with to the holonomy sequence of the connection. As an example we shall describe a reduction of a 't Hooft-Polyakov monopole-like sequence to the monopole sequence constructed in 4.2.2. [LaMa4].

Definition: A *reduction* of the extension $A \to E \to B$ is an extension $A' \xrightarrow{i'} E' \xrightarrow{\pi'} B$ together with an injective Lie algebra homomorphism $I : E' \to E$ so that the following diagram commutes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\;i\;} & E & \xrightarrow{\;\pi\;} & B & \longrightarrow & 0 \\
 & & I\big\uparrow & & I\big\uparrow & & \big\| & & \\
0 & \longrightarrow & A' & \xrightarrow{\;i'\;} & E' & \xrightarrow{\;\pi'\;} & B & \longrightarrow & 0
\end{array}$$

Suppose we are given the extension $A \to E \to B$ with a connection ρ. Let L be the smallest Lie subalgebra of E which contains all elements of the form $\rho(X)$, $X \in B$ and all possible commutators $[\rho(X_1), \rho(X_2)]$, $X_1, X_2 \in B$. We define the *holonomy module* $Hol(\rho)$ of the connection ρ to be the intersection of L with A (see also [Ne])

$$Hol(\rho) = L \cap A\ . \tag{81}$$

One has then, the following

Proposition 4.6. Let ρ be a connection fot the extension $A \to E \to B$ with holonomy module $Hol(\rho)$. Then,

1. $Hol(\rho)$ is a Lie algebra generated by all possible elements of the form $F(X_1, X_2)$, $X_1, X_2 \in B$,

2. the following is a reduction

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle inc} & & \uparrow{\scriptstyle inc} & & \| & & \\ 0 & \longrightarrow & Hol(\rho) & \longrightarrow & L & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

3. ρ is a connection on $Hol(\rho) \to L \to B$ with holonomy algebra just $Hol(\rho)$.

We call the sequence $Hol(\rho) \to L \to B$ the *holonomy sequence* of the connection ρ. If L is a proper subalgebra of E we say that the connection ρ reduces the extension we start with to its holonomy sequence. The connection ρ is called irreducible if $Hol(\rho) = A$. From 1. it also follows the ρ is flat if and only if $Hol(\rho) \equiv 0$.

The next object we would like to describe is the *'t Hooft-Polyakov sequence '*. We start with an extension $A \to E \to B$ with a Lie bundle structure over a commutative algebra $\mathfrak{F}$ with unit. In addition, B is assumed to coincide with the Lie algebra $Der\mathfrak{F}$ of all derivations of $\mathfrak{F}$.

We take $\mathfrak{F}$ to be generated by three elements f_1, f_2, f_3. The Lie algebra A is taken to be a free $\mathfrak{F}$-module of finite type generated, by three elements V_1, V_2, V_3, whose commutation relations are those of the algebra of the rotation group $[V_i, V_j] = \Sigma\, \varepsilon_{ijk} V_k$.

The Lie algebra E is taken to be generated over $\mathcal{F}$ by six elements V_1, V_2, V_3, Y_1, Y_2, Y_3, such that also the Y_i's close the algebra of the rotation group $[Y_i, Y_j] = \Sigma\, \varepsilon_{ijk} Y_k$ and in addition $[Y_i, V_j] = 0$. Furthermore, we assume the combination $Z = \Sigma\, f_i\, Y_i$, to be such that Z commutes with all Y_i's

$$[Y_j, Z] = 0 \quad , \quad j = 1, 2, 3. \tag{82}$$

Previous conditions are equivalent to requiring that the f_i's transform, under the action of the Y_i's, in the following manner

$$Y_j \cdot f_k - \Sigma\, \varepsilon_{jkh}\, f_h = 0 \quad , \quad j,\ k = 1, 2, 3\,. \tag{83}$$

From (83) it follows that the Y_i's and the V_i's are not independent (i.e. E is not free as an $\mathcal{F}$-module). Indeed, from (83) one has that $Z \cdot f_k = 0$, $k = 1, 2, 3$, and this means that $Z \in \mathrm{span}(V_1, V_2, V_3) = A$ (remember that $\mathcal{F}$ is generated by f_1, f_2, f_3). As a consequence of previous assumptions, the algebra $B = E/A$ is generated by three elements $X_i = \pi(Y_i)$, $i = 1, 2, 3$, with the condition $\Sigma\, f_i\, X_i = 0$ (B is not free as well).

Remark. If $f^2 =: (f_1)^2 + (f_2)^2 + (f_3)^2$, from (83) it follows that $Y_j \cdot f^2 = 0$ so that f^2 is a 'constant' which we assume to be non vanishing. In the sequel we shall rescale the f_i's so as to have $f^2 = 1$.

Remark. The name 't Hooft-Polyakov for this sequence come from the fact that the sequence of Lie algebras associated to the SU(2)-principal bundle over S^2 used for the 't Hooft-Polyakov monopole ['tHo], [Po] is algebraically similar to the one under discussion and stems from the fact that S^2 is an orbit of SU(2) over $\mathbb{R}^3$, so that the total space of the SU(2) bundle is given by $(SU(2) \times SU(2)) / U(1)$ which in turn is diffeomorphic with the cartesian product $S^2 \times SU(2)$. So much to justify the name given to the sequence. Later on we shall give an explicit realization of the SU(2) fibration in terms of the momentum map associated with the action of SU(2) on T*SU(2) and we will show that the associated sequence of Lie algebras fulfils all assumptions we have made.

We continue by giving a connection whose holonomy sequence is the monopole sequence which we have used in section 4.2.2. to describe the Dirac monopole. Take

$$\rho : B \rightarrow E \quad , \quad \rho(X_j) = Y_j - f_j\, Z \quad , \quad j = 1, 2, 3\,. \tag{84}$$

It is easy to see that

$$[\,\rho(X_i), \rho(X_j)\,] = \Sigma\, \varepsilon_{ijk}\, \{\, \rho(X_k) - f_k\, Z\, \}\,, \tag{85}$$

$$F(X_i, X_j) = \Sigma\, \varepsilon_{ijk}\, f_k\, Z \quad . \tag{86}$$

As a consequence, in the holonomy sequence

$$0 \to \mathrm{Hol}(\rho) \to L \to B \to 0 \tag{87}$$

the Lie algebra L is generated by the three elements Y_i together with Z and in fact, since $Z = \Sigma\, f_i Y_i$, only by the Y_i's which of course close the algebra of SU(2); moreover the algebra of holonomy Hol(ρ) is generated by the element Z which from (82) commutes with the all of Y_i's . We see that the holonomy sequence (87) is just the monopole sequence constructed in section 4.2.2.. As for the connection ρ defined in (84), which according to proposition 4.6. is a connection on the sequence (87), it is easy to see that it coincides with the Dirac monopole connection given in section 4.2.2..

The mechanism we have described so far could be called algebraic reduction from the 't Hooft-Polyakov monopole to the Dirac Monopole. In the usual approach to the problem [GM], [Tr1,3] the reduction of the structure group is accomplished by a Higgs field. The role of Higgs fields in our formalism deserves further investigations.

As we mentioned before we now give an explicit realization of all previous assumptions. Consider the cotangent bundle T*SU(2) of the group SU(2). On T*SU(2) there is a left and a right symplectic action of SU(2) (these are nothing but the lifts to T*SU(2) of the left and right action of SU(2) on itself). Then we have an action of $SU(2)^L \otimes SU(2)^R$ on T*SU(2). Since the actions are lifts, equivariant momentum maps J^L and J^R can be constructed [AM], [MSSV]. To be definite let us consider the momentum map J^L associated with the left action and which is therefore invariant under the right action. We recall that

$$J^L \;:\; T^*SU(2) \;\to\; su(2)^* \approx\; su(2) \approx \mathbb{R}^3\;,\;\; J^L = \Sigma\, J_k\, \sigma_k \;, \tag{88}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices which generate the Lie algebra su(2) and su(2)* is the dual. The inverse image through J^L of the sphere $S^2 = \{\, (J^L)^2 = \mathrm{const}\, \}$ is five dimensional since in T*SU(2) both $SU(2)^L$ and $SU(2)^R$ preserve the square of the angular momentum $(J^L)^2 = (J_1)^2 + (J_2)^2 + (J_3)^2$. A basis of vector fields tangent to

$(J^L)^{-1}(S^2)$ are the generators of $SU(2)^R$, say V_1, V_2, V_3, and of $SU(2)^L$, say Y_1, Y_2, Y_3. The group $SU(2)^R$ defines a true action and its generators project through the tangent map (Jacobian) of J^L onto the zero vector field. The group $SU(2)^L$ projects onto the sphere S^2. In the kernel of $T(J^L)$ there is also the vector field $Z = J_1Y_1 + J_2Y_2 + J_3Y_3$. Summing up, $(J^L)^{-1}(S^2)$ provides the total space of an $SU(2)$ principal bundle over S^2,

$$SU(2) \to (J^L)^{-1}(S^2) \to S^2 .$$

The extension of Lie algebra associated with this fibration as in section 4.2.2. or 4.2.3. is an example of what we have called t'Hooft-Polyakov extension once we take for $\mathcal{F}$ the algebra generated by the momentum map and in particular we identify the f_i's with the J_i's.

Remark. Since all $SU(2)$ principal fibrations over S^2 are trivial (they are classified by the first homotopy group of $SU(2)$ [Ste] which is trivial in this case), the total space $(J^L)^{-1}(S^2)$ is diffeomorphic to $S^2 \times SU(2)$ which is just the total space of the t'Hooft-Polyakov monopole.

4.7. ALGEBRAIC KALUZA-KLEIN MONOPOLE [LaMa5]

From examples of previous sections it should be clear that Lie algebra extensions can be used to construct an algebraic scheme for gauge theories. On the other hand the Einstein algebras of Geroch [Ge] can be used for an algebraic formulation of General Relativity in which space-time events play no role. It seems natural to investigate possible relations between our scheme and Einstein algebras. It turns out that it is possible to include the notion of Einstein algebra in our setting and that it is possible to construct what we may call algebraic Kaluza-Klein theory. In particular in this section we shall construct an extension of Lie algebras which, by adding the structure of Einstein algebra, provides an algebraic description of the Kaluza-Klein monopole found by Sorkin [So] and Gross and Perry [GP].

As in all the examples discussed above, our framework is an extension of Lie algebras $A \to E \to B$ which is also a Lie bundle over a commutative algebra with unit $\mathcal{F}$. The algebra B is assumed to coincide with the Lie algebra $\mathrm{Der}\mathcal{F}$ of all derivations of $\mathcal{F}$. As usual we specialize the sequence in a proper manner and give a particular connection on it. The latter will be the algebraic counterpart of the Kaluza-Klein monopole [So], [GP].

We make the following requirements :

1. $\mathcal{F}$ is taken to be generated by four elements f_1, f_2, f_3, h (remember that by this we mean that the dual module Der$\mathcal{F}^*$ is a free $\mathcal{F}$-module generated by the elements df_1, df_2, df_3, dh).

2. The Lie algebra E is a free $\mathcal{F}$-module of rank five. We take its basis to consist of five elements $Y_1, Y_2, Y_3, \Gamma, \Delta$ with Lie brackets

$$[Y_j, Y_k] = \Sigma \varepsilon_{jkh} Y_h , \tag{89}$$

$$[Y_j, \Delta] = [Y_j, \Gamma] = [\Gamma, \Delta] = 0 , \tag{90}$$

and action on $\mathcal{F}$ given by

$$Y_j \cdot f_k = \Sigma \varepsilon_{jkh} f_h , \quad \text{this implies that} \quad r^2 =: \Sigma f_j f_j \quad \text{is 'invariant',} \quad Y_j \cdot r^2 = 0 , \tag{91}$$

$$Y_j \cdot h = 0 , \tag{92}$$

$$\Delta \cdot f_j = f_j , \quad \text{this implies that} \quad \Delta \cdot r^2 = 2r^2 \quad \text{and} \quad \Delta \cdot (f_j / r) = 0 , \quad r \neq 0 , \tag{93}$$

$$\Delta \cdot h = 0 , \tag{94}$$

$$\Gamma \cdot f_j = 0 , \quad \text{this implies that} \quad \Gamma \cdot r^2 = 0 , \tag{95}$$

$$\Gamma \cdot h = 1 . \tag{96}$$

3. The Lie algebra A is generated over $\mathcal{F}$ by the element

$$Z = \Sigma (f_j / r) Y_j , \tag{97}$$

then (89-93) imply that Z commutes with everything else

$$[Z, Y_i] = [Z, \Gamma] = [Z, \Delta] = 0 . \tag{98}$$

As a consequence of 2. and 3. we have that

4. the Lie algebra B is an $\mathcal{F}$-module generated over $\mathcal{F}$ by five elements $\bar{\Gamma} = \pi(\Gamma)$,

$\overline{\Delta} = \pi(\Delta)$, $X_j = \pi(Y_j)$, $j = 1, 2, 3$, with the relation $\Sigma f_j X_j = 0$ (namely B is not free).

Let E* be the $\mathcal{F}$-dual of E . Then E* is automatically a free $\mathcal{F}$-module of rank five . We take its basis to consist of five elements θ^j , $j = 1, 2, 3$, $\theta^\Gamma, \theta^\Delta$ such that

$$\theta^i (Y_j) = \delta^i_j \quad , \quad \theta^\Gamma(\Gamma) = 1 \ , \ \theta^\Delta(\Delta) = 1 \ , \qquad \text{any other pairing} = 0 \ . \tag{99}$$

By using again 2. and 3. one shows that the 'action' of $Y = (Y_1, Y_2, Y_3, \Gamma, \Delta)$ on θ^j , $j = 1, 2, 3$, θ^Γ , θ^Δ , is given by

$$\mathcal{L}_{Y_j} \theta^k = \Sigma \varepsilon_{jkh} \theta^h \quad , \quad \mathcal{L}_\Delta \theta^k = \mathcal{L}_\Gamma \theta^k = 0 \ , \quad \mathcal{L}_Y \theta^\Delta = \mathcal{L}_Y \theta^\Gamma = 0 \ . \tag{100}$$

As a preliminary step to define a connection on the sequence $A \to E \to B$ let us consider for a while the 'subsequence' over $\mathcal{F}$ '

$$0 \to A \to E' \to B' \to 0 \ , \tag{101}$$

in which $\mathcal{F}$ ' is generated by f_1 , f_2 , f_3 , with the constraint $\Sigma f_j f_j = \text{'const'}$, and E' is generated by Y_1, Y_2, Y_3 . We know from section 4.2.2. that

$$\omega = Z \otimes \theta \ , \quad \theta = \Sigma (f_j / r)\, \theta^j \quad , \qquad r \neq 0 \ , \tag{102}$$

is a connection for the extension (101) and describes the algebraic abelian monopole. The ω in (102) can be extended to a connection on the sequence $A \to E \to B$ by simply defining $\omega(\Delta) = \omega(\Gamma) = 0$. We will show that this extended connection is the algebraic counterpart of the Kaluza-Klein monopole [So], [GP] . To this aim we will make the couple (E , $\mathcal{F}$) an Einstein algebra by giving a metric on E for which the contraction properties are satisfied and the Ricci tensor vanishes (see section 2.2).

Let us restrict again for a while to the sector Y_j , $j = 1, 2, 3$. The most general symmetric tensor of rank two made out of the θ^j 's and with coefficients in $\mathcal{F}$, is of the form

$$g' = \Sigma g_{jk}\, \theta^j \otimes \theta^k \quad , \qquad g_{jk} = g_{kj} \in \mathcal{F} \quad , \tag{103}$$

and is automatically invariant under the action of Z , $\mathcal{L}_Z g' = 0$. Invariance under the action of the Y_j's requires

$$Y_h \cdot g_{jk} + \Sigma\, \varepsilon_{hmj}\, g_{mk} + \Sigma\, \varepsilon_{hmk}\, g_{jm} = 0 \,, \qquad h, j, k = 1, 2, 3 \,, \tag{104}$$

whose general solution is

$$g_{jk} = \beta(r)\, \delta_{jk} + \gamma(r)\, (f_j / r)(f_k / r) \quad . \tag{105}$$

Here $\beta(r)$ means that $d\beta(r) = \sigma(r)\, dr$ for some $\sigma(r) \in \mathcal{F}$ and the same for $\gamma(r)$.

Finally we require a 'Kaluza-Klein splitting' namely we require that g' is the sum of a term of the form $\theta \otimes \theta$ (remember that θ is the connection), and a term which is horizontal. One shows that any g' as in (105) is horizontal if $\beta(r) = -\gamma(r)$. Therefore, the most general form of a 'Kaluza-Klein splitted' g' which is also invariant under the action of Y_1, Y_2, Y_3 is

$$\begin{aligned} g' &= \beta(r)\ \Sigma\, [\delta_{jk} - (f_j / r)(f_k / r)]\, \theta^j \otimes \theta^k + \Sigma\, \gamma(r)\, (f_j / r)(f_k / r)\, \theta^j \otimes \theta^k \\ &= \beta(r)\ \Sigma\, d(f_j / r) \otimes d(f_j / r) + \gamma(r)\ \theta \otimes \theta \quad . \end{aligned} \tag{106}$$

As for a metric g on E we take it to be invariant under the action of Y_1, Y_2, Y_3, Γ, and fix $g(\Gamma, \Gamma)$ to be a negative real number (remember that the algebra $\mathcal{F}$ is assumed to contain the ring of real numbers). A metric fulfilling all these requirements is the following

$$\begin{aligned} g = -\theta^{\Gamma} \otimes \theta^{\Gamma} + \alpha(r)^2\, \theta^{\Delta} \otimes \theta^{\Delta} + \Sigma\{\, \beta(r)^2\, [\delta_{jk} - (f_j / r)(f_k / r)] \,+ \\ + \gamma(r)^2\, (f_j / r)(f_k / r)\}\theta^j \otimes \theta^k \,, \end{aligned} \tag{107}$$

with $\alpha(r)$, $\beta(r)$ and $\gamma(r)$ elements in $\mathcal{F}$.
In order to have an Einstein algebra we must impose the 'field equations'

$$\mathrm{Ricci}(g) = 0 \tag{108}$$

and this gives the following equations for $\alpha(r)$, $\beta(r)$ and $\gamma(r)$

$$\frac{1}{\beta^2}\left[\, 1 - \left(\frac{\beta'}{\alpha}\right)^2 - \frac{1}{2}\,\frac{\gamma^2}{\beta^2}\, \right] - \frac{1}{\alpha^2}\left(\frac{\beta''}{\beta} - \frac{\alpha'}{\alpha}\,\frac{\beta'}{\beta} + \frac{\beta'}{\beta}\,\frac{\gamma'}{\gamma} \right) = 0 \;,$$

$$\frac{2}{\beta}\left(\beta'' - \frac{\alpha'}{\alpha}\beta'\right) + \frac{1}{\gamma}\left(\gamma'' - \frac{\alpha'}{\alpha}\gamma'\right) = 0 \ ,$$

$$2\left(\frac{1}{\alpha^2}\frac{\beta'}{\beta}\frac{\gamma'}{\gamma} - \frac{1}{4}\frac{\gamma'^2}{\beta^4}\right) + \frac{1}{\alpha^2\gamma}\left(\gamma'' - \frac{\alpha'}{\alpha}\gamma'\right) = 0 \ , \qquad (109)$$

where $\alpha' = \Delta\cdot\alpha$, $\alpha'' = \Delta\cdot(\Delta\cdot\alpha)$ and the same for β and γ.

Finally, we give an example which verifies all assumptions we have made. Let us consider the U(1)-fibration $\mathbb{R}\times\mathbb{R}\times S^3 \to \mathbb{R}\times\mathbb{R}\times S^2$ with radius(S^3) = radius(S^2) = r. We realize the isomorphism $SU(2) \equiv S^3$ in terms of 2 x 2 complex matrices and use the Pauli matrices σ_1, σ_2, σ_3 as a basis for the Lie algebra. The projection $SU(2) \to S^2$ is

$$s \ \to\ s\,\sigma_3\, s^{-1} =: \Sigma\,(x_k / r)\,\sigma_k\,, \quad (x_1)^2 + (x_2)^2 + (x_3)^2 = r\,. \qquad (110)$$

We take the generators (f_1, f_2, f_3, h) of $\mathcal{F}$ to be just (x_1, x_2, x_3, t) with the x_j's as defined in (110) and t a time coordinate. As for the generators $(Y_1, Y_2, Y_3, \Delta, \Gamma)$ of the Lie algebra E we take (Y_1, Y_2, Y_3) to be the three canonical right invariant vector fields on SU(2), $\Delta = r\,(\partial/\partial r)$ the generator of dilatations and Γ the generator of time translations. With this assumption one finds that $Z = \Sigma\,(x_j/r)Y_j$ is nothing but the left invariant vector field on SU(2) corresponding to the infinitesimal generator σ_3 (and generating the right action of U(1) on SU(2)). The previous hypothesis are enough to completely determine the extension we have constructed and indeed one easily verifies that all the conditions (89-97) are satisfied. As for equation (109), an explicit solution is given by

$$\alpha(r) = r\,(1+ m/\rho)\,,\ \beta(r) = r\,,\ \gamma(r) = 2\,m\,r/(\rho + m)$$

$$\text{with } \rho^2 = r^2 + m^2\,, \quad m = \text{const}\,. \qquad (111)$$

The corresponding metric can be written as

$$g = -\,dt\otimes dt + (1+ m/\rho)^2\, dr\otimes dr + r^2\,\Sigma\, d(x_j/r)\otimes d(x_k/r) + [\,2mr/(\rho+m)]^2\ \theta\otimes\theta$$

$$= -\,(dt)^2 + (1+ m/\rho)^2\,(dr)^2 + r^2/2\ \mathrm{Tr}\,[\,d(s\,\sigma_3\,s^{-1})]^2 - 4[\,2mr/(\rho+m)]^2\,(\mathrm{Tr}\,\sigma_3\,s^{-1}ds)^2$$

and is the five dimensional metric which gives the Kaluza-Klein monopole [So],[GP].

4.8. A GRADED GAUGE THEORY

It is straightforward to construct a graded generalization of the gauge theory constructed in section 4.1. All machineries of the calculus over an extension of Lie superalgebras endowed with a connection ρ which has been developed in section 3.5., will constitute the kinematic of the graded gauge theory. The graded Yang-Mills equations will have the form (1) provided one takes now for the graded curvature F_ρ and the graded covariant codifferential $\mathcal{D}_\rho$ the ones defined by equations (3.25) and (3.31) respectively. It is also possible to define a lagrangian like in (2) and derive field equations from it exactly like in the non graded situation. One could also easily give a graded generalization of all topics introduced in sections 4.3.–4.6.

What we will rather do in this section is to give few examples of the graded gauge theory. The framework will be an extension of Lie superalgebras $0 \to A \to E \to B \to 0$ as in (3.20) which is a Lie superbundle over a $\mathbb{Z}_2$-graded algebra with unit $\mathcal{F}$ (see section 3.5) and carrying a connection ρ as defined in (3.22). Furthermore we shall assume that B generates over $\mathcal{F}$ the all of the graded module GDer$\mathcal{F}$ (B and GDer$\mathcal{F}$ are the same as graded $\mathcal{F}$-modules). In all examples we shall specialize the extension in a proper manner.

4.8.1. Graded Algebraic Electromagnetism [LaMa2]

For the sake of simplicity we shall construct directly an extension of Lie superalgebras $0 \to A \to E \to B \to 0$ of vector fields. More precisely we shall construct the extension of Lie superalgebras which is associated to a GU(1)-bundle (GU(1) is a graded version of U(1)) over 'superspace' $M^{4,4}$ [SS].

The graded algebra $\mathcal{F}$ on which the extension is a Lie superbundle, is taken to be the algebra of 'supersmooth' functions on $M^{4,4}$ with coordinates given by four bosonic variables x^μ, $\mu = 1, ..., 4$, and four fermionic ones θ^α, $\alpha = 1, ..., 4$ (the θ^α's are the components of an anticommuting Majorana spinor).

The Lie algebra B is the Lie superalgebra generated by the infinitesimal generators of motion on $M^{4,4}$ [SS]

$$X_\mu = \partial / \partial x^\mu \quad , \quad \mu = 1, ..., 4 ,$$

$$X_\alpha = \partial / \partial \bar{\theta}^\alpha + i/2 \, \partial / \partial x^\mu (\gamma^\mu \theta)_\alpha \quad , \quad \alpha = 1, ..., 4 , \qquad (112)$$

with γ^μ, $\mu = 1, ..., 4$, Dirac matrices, $\bar{\theta} = \theta^T C$, and C the charge conjugation matrix.

The vector fields (112) obey the following (anti)commutation relations

$$[X_\mu, X_\nu]_- = [X_\mu, X_\alpha]_- = 0 \ ,$$

$$[X_\alpha, X_\beta]_+ = 2i\, \partial/\partial x^\mu \, (\gamma^\mu C)_{\alpha\beta} \ . \qquad (113)$$

Finally we take A to be generated by an even element $\partial/\partial z$ and an odd one $\partial/\partial\eta$ which (anti)commute among themselves and with everything else (one may think of $\partial/\partial z$ and of $\partial/\partial\eta$ as the infinitesimal generators of translations along the fibres).

We take the following connection $\rho : B \to E$,

$$\rho(X_\mu) = X_\mu - A_\mu(x,\theta)\, \partial/\partial z \ , \quad \mu = 1, ..., 4 \ ,$$

$$\rho(X_\alpha) = X_\alpha - A_\alpha(x,\theta)\, \partial/\partial\eta \ , \quad \alpha = 1, ..., 4 \ , \qquad (114)$$

where $A_\mu(x,\theta)$ and $A_\alpha(x,\theta)$ are functions (the components of the connection). As for the curvature F defined in (3.25) one has

$$F_{\mu\nu} =: F(X_\mu, X_\nu) = (X_\mu \cdot A_\nu - X_\nu \cdot A_\mu)\, \partial/\partial z \ ,$$

$$F_{\mu\alpha} =: F(X_\mu, X_\alpha) = (X_\mu \cdot A_\alpha)\, \partial/\partial\eta - (X_\alpha \cdot A_\mu)\, \partial/\partial z \ ,$$

$$F_{\alpha\beta} =: F(X_\alpha, X_\beta) = -2i\, A_\mu (\gamma^\mu C)_{\alpha\beta}\, \partial/\partial z - (X_\alpha \cdot A_\beta + X_\beta \cdot A_\alpha)\, \partial/\partial\eta \ . \qquad (115)$$

It is easy to verify that, since A is abelian and acts trivially on $\mathcal{F}$, the Bianchi identity (3.29) can be written as

$$(-)^{p(X_a)p(X_b)} \{[X_a, F(X_b,X_c)]_G - F([X_a, X_b]_G, X_c)\} + \text{g.c.p.} = 0 \ , \qquad (116)$$

with $X_a = (X_\mu, X_\alpha)$. Equations (116) have a homogeneous Maxwell like form

$$dF = 0 \qquad (117)$$

where d is the coboundary operator of the cohomology of the superalgebra B with coefficient in A.

In order to obtain the inohomogeneous equations, we need a metric g on B. A possible choice for g is the Nath-Arnowitt metric [NA]

$$g = (dx^{\mu} - i/2 \ \bar{\theta} \gamma^{\mu} d\theta)^2 + d\bar{\theta} \ d\theta \ . \tag{118}$$

This metric is flat in the bosonic sector but is not in the fermionic one and it gives rise to non vanishing Christoffel simbols [NA].

The covariant codifferential defined in (3.31) is now

$$(\mathcal{D}_{\rho}F)(X) = - \Sigma \, g^{ab} \, (\nabla^{\rho}{}_{X_a} F)(X_b, X) \in A \ , \quad \forall \ X \in B \ . \tag{119}$$

Here g^{ab} are the components of g^{-1}, and ∇^{ρ}, given in general by (3.30), reduces (again because A is abelian and acts trivially on $\mathcal{F}$) to

$$(\nabla^{\rho}{}_{X_a} F)(X_b, X_c) = [X_a , F (X_b, X_c)]_G - F(\nabla_{X_a} X_b , X_c) +$$

$$- (-1)^{p(X_a)p(X_b)} F(X_b , \nabla_{X_a} X_c) \ . \tag{120}$$

∇ is the Levi-Civita connection for the metric g. This connection has been explicitly constructed in [Woo] for the metric (118).

Then, the inohomogeneous equations for the curvature F are just the graded version of (1), namely the algebraic equations

$$\mathcal{D}_{\rho} F_{\rho} = 0 \, . \tag{121}$$

One could easily give an abstract version of the whole construction of this section.

4.8.2. A Graded Algebraic Monopole [LaMa2]

In this section, we construct a monopole-like sequence of superalgebras similar to the one constructed in section 4.2.2. for ordinary algebras. As an example, the sequence is realized by means of vector fields on a principal superbundle which is a Grassmann extension of the Hopf fibration of the Dirac monopole.

We start with an extension of Lie superalgebras $0 \to A \to E \to B \to 0$ which is a Lie superbundle over a graded algebra with unit $\mathcal{F}$. We make the following assumptions :

1. The algebra $\mathcal{F}$ is taken to be generated by three even elements f_1, f_2, f_3, $p(f_j) = 0$, and by two odd ones g_1, g_2, $p(g_\alpha) = 1$.

2. We take E to be a free $\mathcal{F}$-module of finite type generated as an $\mathcal{F}$-module by three even elements Y_1, Y_2, Y_3, and by two odd ones V_1, V_2, whose graded commmutation relations are those of the graded extension uosp(1,2) of the rotation algebra [PR], [BT]

$$[\, Y_j \,, Y_k \,]_- \;=\; \Sigma\, \varepsilon_{jkm} Y_m \;,$$

$$[\, Y_j \,, V_\alpha \,]_- \;=\; i/2\, \Sigma\, (\sigma_j)_{\beta\alpha} V_\beta \;,$$

$$[\, V_\alpha \,, V_\beta \,]_+ \;=\; i/2\, \Sigma\, (C\, \sigma_j)_{\alpha\beta}\, Y_j \;, \qquad j, k, m = 1, 2, 3 \,, \quad \alpha,\beta = 1, 2 \;. \tag{122}$$

Here σ_1, σ_2, σ_3 are Pauli matrices and C is the charge conjugation matrix $C = -i\sigma_2$.

3. We take A to be an even algebra generated over $\mathcal{F}$ by an element Z of the form

$$Z \;=\; f_1\, Y_1 \;+\; f_2\, Y_2 \;+\; f_3\, Y_3 \;+\; g_1\, V_1 \;+\; g_2 V_2 \;\;. \tag{123}$$

Moreover, we force Z to (anti)commute with all elements $Y_1, \dots , V_2$. One finds that conditions $[\, Z \,, Y_j] = [\, Z \,, V_\alpha \,] = 0$ are equivalent to the following action of E on $\mathcal{F}$

$$Y_k \cdot f_m \;-\; \Sigma\, \varepsilon_{kmn}\, f_n \;=\; 0 \;, \tag{124a}$$

$$Y_k \cdot g_\alpha \;+\; i/2\, \Sigma\, (\sigma_k)_{\alpha\beta}\, g_\beta \;=\; 0 \;, \tag{124b}$$

$$Z_\alpha \cdot f_m \;-\; i/2\, \Sigma\, (C\sigma_m)_{\alpha\beta}\, g_\beta \;=\; 0 \;, \tag{124c}$$

$$Z_\alpha \cdot g_\beta \;-\; i/2\, \Sigma\, (\sigma_m)_{\beta\alpha}\, f_m \;=\; 0 \;, \qquad k, m \;=\; 1, 2, 3 \,, \;\; \alpha,\, \beta \;=\; 1, 2 \;. \tag{124d}$$

We assume that the action of E on $\mathcal{F}$ is such that a non trivial solution to the previous equations does exist. Equations (124) also imply that Z acts trivially on the all of $\mathcal{F}$. Moreover, it is easy to check that the action (124) leaves invariant the quantity

$$r^2 =: \Sigma \delta_{jk} f_j f_k + \Sigma g_\alpha C_{\alpha\beta} g_\beta = \Sigma f_j f_j + 2 g_1 g_2 . \tag{125}$$

In the following we shall assume that r^2 is a non vanishing real number (remember from assumptions of section 3.1. that $\mathcal{F}$ contains a copy of the real numbers) and rescale the elements $f_1, \dots, g_2$ so as to have $r^2 = 1$.

As a consequence of assumptions 1. 2. and 3. the Lie superalgebra B is not free as an $\mathcal{F}$-module being generated by five elements $X_j = \pi(Y_j)$, $j = 1, 2, 3$, $T_\alpha = \pi(V_\alpha)$, $\alpha = 1, 2$, with the constraint $\Sigma f_j X_j + \Sigma g_\alpha T_\alpha = 0$ (coming from the fact that $\pi(Z) = 0$).

Let E be the $\mathcal{F}$-dual of E. E* is automatically a free $\mathcal{F}$-module of rank (3, 2). We take its basis to consist of three even 1-forms θ^j, j =1,2,3, such that $\theta^j(Y_k) = \delta^j{}_l$ and $\theta^j(V_\alpha) = 0$ and two more odd 1-forms $\tilde{\theta}^\alpha$, $\alpha = 1,2$, such that $\tilde{\theta}^\alpha(V_\beta) = \delta^\alpha{}_\beta$ and $\tilde{\theta}^\alpha(Y_j) = 0$.
If we take the combination

$$\theta = f_1\theta^1 + f_2\theta^2 + f_3\theta^3 + g_2\,\tilde{\theta}^1 - g_1\,\tilde{\theta}^2 , \tag{126}$$

then it follows that $\theta(Z) = 1$ and the A-valued 1-form on E

$$\omega = Z \otimes \theta \tag{127}$$

is a connection form for the sequence we are considering. The corresponding connection is given by

$$\rho : B \rightarrow E ,$$

$$\rho(X_j) = Y_j - f_j Z , \quad j = 1, 2, 3, \quad \rho(T_1) = V_1 - g_2 Z , \quad \rho(T_2) = V_2 + g_2 Z . \tag{128}$$

Since A acts trivially on $\mathcal{F}$ and is generated by an element which commutes with all elements in E, any A-valued form φ can be identified with an $\mathcal{F}$-valued form $\underline{\varphi}$ via the equality $\varphi = Z \otimes \underline{\varphi}$.

For the curvature 2-form Ω defined in (3.26) we have

$$\Omega(X, Y) = \{ X\cdot\theta(Y) - (-1)^{p(X)p(Y)}\, Y\cdot\theta(X) - \theta([X, Y]_G) \} Z \cdot$$

$$= \{ d\theta(X, Y) \} Z \quad , \quad \forall\ X, Y \in E . \tag{129}$$

and the Bianchi identity (2.29) has now the form

$$d\,\underline{\Omega} = 0 . \tag{130}$$

On the algebra B there is a natural metric given by

$$g = \Sigma\, \delta_{jk}\, df_j \otimes df_k + \Sigma\, C_{\beta\alpha} dg_\alpha \otimes dg_\beta$$

$$= df_1 \otimes df_1 + df_2 \otimes df_2 + df_3 \otimes df_3 - dg_1 \otimes dg_2 + dg_2 \otimes dg_1 , \tag{131}$$

with relations

$$\Sigma f_j f_j + 2\, g_1 g_2 = 1 , \quad \Sigma f_j\, df_j + 2\, dg_1\, g_2 + 2\, g_1\, dg_2 = 0 . \tag{132}$$

By explicit calculation one can show that connection (128) solves the (graded version of) fields equation (1) with the metric (131-2).

We shall now give a specific realization of the previous sequence of superalgebras in terms of vector fields associated with a principal superbundle which is a graded version of the Hopf fibration of the two dimensional sphere. All previous defining assumptions are met in this example [LaMa2].

The total space of the fibration is the Lie supergroup UOSP(1,2) [BT] which can be realized as follows. Let osp(1,2) be the real Lie superalgebra of dimension (3, 2) with even generators A_j, $j = 1,2,3$, and odd generators R_α, $\alpha = 1,2$, given in the matrix representation by

$$A_1 = i/2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} , \quad A_2 = i/2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix} , \quad A_3 = i/2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$R_1 = 1/2 \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad R_2 = 1/2 \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} . \tag{133}$$

Moreover, with L an even integer, let C_L be the exterior algebra over $\mathbb{C}^L$ ($\mathbb{C}$ is the field of complex numbers). The algebra C_L has naturally a $\mathbb{Z}_2$-graduation $C_L = (C_L)_0 \oplus (C_L)_1$. Consider an even graded involution $^{\times} : C_L \to C_L$, verifying

$$(xy)^{\times} = x^{\times} y^{\times} , \quad (x^{\times})^{\times} = (-1)^{p(x)} x^{\times}, \quad (\alpha x)^{\times} = \bar{\alpha} x^{\times} , \tag{134}$$

with x and y homogeneous elements in C_L and α a complex number. The existence of such a map is assured by the fact that L is even [RS].

The Lie superalgebra uosp(1,2) is the subalgebra of $C_L \otimes_{\mathbb{R}} osp(1,2)$ made of elements of the form

$$X = \Sigma_j a_j A_j + \eta R_1 + \eta^{\times} R_2 , \quad a_j \in (C_L)_0 , \; a_j^{\times} = a_j , \; \eta \in (C_L)_1 . \tag{135}$$

Notice that elements in uosp(1,2) are even and such that $X^{\times} = -X$. The Lie supergroup UOSP(1,2) is the exponential map of uosp(1,2) . An element $s \in$ UOSP(1,2) can be parametrized as follows

$$s = \begin{bmatrix} (1 + 1/4\,\eta^{\times}\eta) & -1/2\,\eta^{\times} & 1/2\,\eta \\ -1/2(z_0^{\times}\eta + z_1\eta^{\times}) & z_0^{\times}(1 - 1/8\,\eta^{\times}\eta) & z_1(1 - 1/8\,\eta^{\times}\eta) \\ 1/2(z_1^{\times}\eta - z_0\eta^{\times}) & -z_1^{\times}(1 - 1/8\,\eta^{\times}\eta) & z_0(1 - 1/8\,\eta^{\times}\eta) \end{bmatrix} , \tag{136}$$

where z_0, z_1 are in $(C_L)_0$ with $z_0 z_1^{\times} + z_1 z_1^{\times} = 1$, and η is in $(C_L)_1$.

The structure supergroup of the fibration is $\mathcal{U}(1)$, realized as

$$\mathcal{U}(1) = \{ w \in (C_L)_0 \text{ such that } ww^{\times} = 1 \} . \tag{137}$$

The supergroup $\mathcal{U}(1)$ can be embedded into UOSP(1,2) by

$$w \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w^{\times} \end{bmatrix} ,$$

so that we may think of A_3 as the generator of $\mathcal{U}(1)$ i. e.

$$\mathcal{U}(1) \simeq \{ \exp(\lambda A_3) , \; \lambda \in (C_L)_0 \text{ s. t. } \lambda^{\times} = -\lambda \} . \tag{138}$$

By taking the right action of $\mathcal{U}(1)$ on UOSP(1,2) one gets a principal super bundle

$$\mathcal{U}(1) \rightarrow \text{UOSP}(1,2) \rightarrow S^2{}_* , \tag{139}$$

where $S^2{}_* = \text{UOSP}(1,2) / \mathcal{U}(1)$. The projection $\pi : \text{UOSP}(1,2) \rightarrow S^2{}_*$ can be given as

$$\pi(s) =: s(2/i\, A_k)s^\dagger =: \Sigma\, x_k (2/i\, A_k) + \Sigma\, \xi_\alpha (2\, R_\alpha) , \tag{140}$$

with $s^\dagger$ the adjoint of s (the transpose conjugated, with the conjugation given by the map $^\times$ [BT]). One finds explicitly

$$x_1 = -(1 - 1/4\ \eta^\times \eta)(z_0 z_1^\times + z_1 z_0^\times) ,$$

$$x_2 = -i(1 - 1/4\ \eta^\times \eta)(z_0 z_1^\times - z_1 z_0^\times) ,$$

$$x_3 = -(1 - 1/4\ \eta^\times \eta)(z_1 z_1^\times - z_0 z_0^\times) ,$$

$$\xi_1 = 1/2 (z_1 \eta^\times - z_0^\times \eta) ,$$

$$\xi_2 = 1/2 (z_0 \eta^\times + z_1^\times \eta) . \tag{141}$$

One sees from (141) that the x_k's are real ($x_k = x_k^\times$) even elements in C_L while the ξ_α's are odd ones with $\xi_2 = -(\xi_1)^\times$. Furthermore one finds the relation

$$(x_1)^2 + (x_1)^2 + (x_1)^2 + 2\, \xi_1 \xi_2 = 1 , \tag{142}$$

so that $S^2{}_*$ may be thought of as a (2, 2)-dimensional 'sphere' in C_L.

As for the realization of the sequence of Lie superalgras $0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$ as specified by 1. 2. and 3. above, we take $\mathcal{F}$ to be the graded algebra of supersmooth functions on $S^2{}_*$ with the f_k's and the g_α's in condition 1. just the (pullbacks of the) coordinate functions on $S^2{}_*$ as given by (141). E is the Lie superalgebra of right invariant vector fields on UOSP(1,2) with basis elements Y_i's and V_α's the three canonical right invariant vector fields on UOSP(1,2) corresponding to the basis (133) for its Lie superalgebra. Finally we take for Z the generator of the right action of $\mathcal{U}(1)$ on UOSP(1,2). One checks that all assumptions from (122) through (124) are satisfied. In particular the functions defined in (141) give a solution of (124). The connection given in (126) is a connection on the bundle (139). It has been studied in more details in [BBL1].

We close this section with few more remarks on the principal superbundle (139). In [BBL1] it has been shown that the base space $S^2{}_*$ is a DeWitt supermanifold [DeW] with body the ordinary two dimensional sphere S^2. This means that $S^2{}_*$ is itself a fibre bundle with fibre the space of nilpotent elements $(N_L)^{3,2}$ in $(R_L)^{3,2} = \mathbb{R} \oplus (N_L)^{3,2}$ and R_L the exterior algebra on $\mathbb{R}^L$. Furthermore it has been shown that the bundle (139) is nontrivial by explicitly checking that its first Chern class does not vanish.

We would like to mention that it is possibile to construt a graded version of the reduction mechanism described in section 4.6.. As usual, one only needs to replace ordinary algebras with Lie superalgebras. In particular, one can construct a graded 't Hooft-Polyakov sequence and reduce it to the sequence of the graded monopole of section 4.8.2. An explicit realization of this reduction can be given again in terms of a momentum map as for the nongraded case. One takes the cotangent bundle T*UOSP(1,2) of the supergroup UOSP(1,2) and the left and right action of UOSP(1,2) on T*UOSP(1,2) . Then, one constructs the associated momentum maps and repeats all steps done in section 4.6. for T*SU(2) .

5. AN ALGEBRAIC LAGRANGIAN SETTING

In this chapter we shall sketch the basic ingredients for an algebraic description of Lagrangian dynamical systems. Our formalism does not need a manifold structure. However, a possible realization is in terms of proper tensor fields (see later) on a tangent bundle. In this case we get an intrinsic formulation of Lagrangian systems. We shall also present a theorem which gives sufficient conditions in order that a manifold admits a tangent bundle structure. This could be useful in situations where one would like to know if a given dynamical system is a Lagrangian one. Of course, if this is the case, the carrier space must be a tangent space. We also hope to use our algebraic formalism to construct graded generalizations of Lagrangian dynamics.

5.1. AN ALGEBRAIC FORMALISM FOR DYNAMICAL SYSTEMS

The basic tool we need in this section, is the algebraic calculus developed in section 2.1. where the initial datum was a Lie module E over a commutative $\mathfrak{R}$-algebra with unit $\mathfrak{F}$. We remind that this means that there is a representation of E in the Lie algebra $\mathrm{Der}\mathfrak{F}$ of all derivations of $\mathfrak{F}$.

In general, given any (1,1) tensor T as defined there, $T : E \times E^* \to \mathfrak{F}$, we can define two endomorphisms, that we shall denote with the same letter,

$$T : E \to E, \quad T : E^* \to E^* ,$$

$$T(X, \alpha) =: \alpha(T(X)) =: T(\alpha)(X), \quad \forall X \in E, \ \alpha \in E^* . \tag{1}$$

The Nijenhuis tensor of T is the (1, 2) tensor defined by

$$N_T(X,Y) =: [TX, TY] + T^2[X, Y] - T[TX, Y] - T[X, TY]$$

$$= \{ \mathfrak{L}_{TX}(T) - T \circ \mathfrak{L}_X(T) \}(Y), \quad \forall X, Y \in E. \tag{2}$$

Furthermore, we can associate associate with T an operator

$$d_T : \Lambda^p(E, \mathcal{F}) \rightarrow \Lambda^{p+1}(E, \mathcal{F}) ,$$

$$(d_T\varphi)(X_1,\dots,X_{p+1}) =: \Sigma_i (-1)^{i+1} L_{T(X_i)} (\varphi(X_1,\dots,\hat{\imath},\dots,X_{p+1}))+$$

$$+\Sigma_{i<j}(-1)^{i+j}\varphi([TX_i, X_j] + [X_i, TX_j] - T[X_i, X_j],X_1,\dots,\hat{\imath},\dots,\hat{j},\dots,X_{p+1}), \quad \forall\ X_i \in E . \qquad (3)$$

For instance, if $f \in \mathcal{F}$ and $\alpha \in E^*$ and d is the exterior derivative defined in section 2.1., we have

$$d_T f = df \circ T , \qquad (4)$$

$$d_T\alpha (X, Y) = (L_{TX}\, \alpha)(Y) - (L_{TY}\, \alpha)(X) + \alpha(T[X , Y]) . \qquad (5)$$

One can prove that d_T is a coboundary operator, namely $(d_T)^2 = 0$, iff $N_T = 0$ [MFLMR].

In order to define a lagrangian carrier space we need additional structures. To start, we assume the Lie module E to be such that there exists a (1, 1) tensor S with the properties:

$$\ker S = \mathrm{Im}\, S , \qquad (6a)$$

$$N_S = 0 . \qquad (6b)$$

Condition (6a) implies the vanishing of the square of the endomorphism $S : E \rightarrow E$. This fact and (6b) imply that ImS and then KerS are closed under Lie brackets.

Now we have $\mathcal{F}$, E , S . We can select a natural subalgebra $\mathcal{F}_S$ of $\mathcal{F}$ and construct a natural extension of Lie algebras of derivations which is a Lie bundle over $\mathcal{F}_S$. We define

$$\mathcal{F}_S =: \{ f \in \mathcal{F} \text{ such that } d_S f = 0 \}. \qquad (7)$$

Let now $E^\pi \subset E$ be the subalgebra of E which take $\mathcal{F}_S$ into itself and E^v the ideal of E^π consisting of vector acting trivially on $\mathcal{F}_S$. Then we have the Lie bundle over $\mathcal{F}_S$

$$0 \rightarrow E^v \rightarrow E^\pi \rightarrow E^\pi / E^v \rightarrow 0 . \qquad (8)$$

The algebra E^{π} / E^{v} is clearly a subalgebra of the algebra $\mathrm{Der}\mathcal{F}_S$, while KerS is a subalgebra of E^{v}.
Another natural subalgebra of E we shall deal with is the algebra E^S of vectors which preserve S, i.e. $E^S =: \{ X \in E$ such that $\mathcal{L}_X(S)= 0 \}$. The algebra E^S is also a subalgebra of E^{π}.

The last assumption we make is the existence of an element $\Delta \in E$ with properties

$$\Delta \in \ker S \quad , \tag{9a}$$

$$L_{\Delta} S = - S \quad . \tag{9b}$$

Remark. Given a vector Δ which fulfils (9), it is easy to see that for any $V \in E^S \cap \mathrm{KerS}$, the element of E given by $\Delta + V$ will also fulfil them. Moreover, if Δ_1 and Δ_2 are as in (9), their difference $\Delta_1 - \Delta_2$ will belong to $E^S \cap \mathrm{KerS}$. We conclude that if in E there is at least an element Δ obeying (9), then the set of all such elements is an affine space modelled on the vector space $E^S \cap \mathrm{KerS}$.

For a given choice of S and Δ, we call *second order vector* any element $\Gamma \in E$ which is such that

$$S(\Gamma) = \Delta . \tag{10}$$

Consider now an element $L \in \mathcal{F}$; the element $\theta_L \in E^*$ defined by

$$\theta_L =: d_S L \tag{11}$$

is the *Cartan 1-form* of the *Lagrangian* L. The *Lagrangian 2-form* of L is the element $\Omega_L \in \Lambda^2(\mathcal{D}, \mathcal{F})$ defined by

$$\Omega_L =: - d\theta_L . \tag{12}$$

L will be called a *regular Lagrangian* if Ω_L is nondegenerate, namely if $i_X \Omega_L = 0$ implies $X = 0$. If Ω_L is nondegenerate it gives rise to an isomorphism from E to E*.

The *energy* E_L of L is defined as

$$E_L =: \mathcal{L}_\Delta L - L \,. \tag{13}$$

We can now introduce *field equations* . Given a lagrangian L , the *Euler-Lagrange equations* for L are the following algebraic equations

$$\mathcal{L}_\Gamma \theta_L - dL = 0 \,, \tag{14}$$

where the unknown quantity is the second order vector Γ . By using (10) and the identity $\mathcal{L}_{(\cdot)} = d \circ i_{(\cdot)} + i_{(\cdot)} \circ d$, one writes equations (14) in an equivalent hamiltonian form

$$i_\Gamma \Omega_L = dE_L \,. \tag{15}$$

In our formalism there is also room for a *Noether-like theorem* . A *Noether symmetry* for the couple (L, Γ) which solves Euler-Lagrange equations (14), is a vector X^N such that

$$X^N \in E^S \,,$$

$$[\, X^N , \Delta \,] = 0 \,,$$

$$\mathcal{L}_{X^N} L = \mathcal{L}_\Gamma f \,, \quad \text{for some } f \in \mathcal{F}_S \,. \tag{16}$$

Proposition 1. If the vector X^N is a Noether symmetry for the couple (L, Γ) , then

$$\mathcal{L}_\Gamma (\, i_X \theta_L - f \,) = 0. \tag{17}$$

and the quantity $i_X \theta_L - f$ is a conserved quantity for Γ , namely it is constant along the action of Γ .

Proof. By using the first two of (16) and definition (10) one finds that $0 = \mathcal{L}_{X^N} (S)(\Gamma) = [\, X^N , S(\Gamma)] - S([X^N, \Gamma]) = [\, X^N , \Delta] - S([X^N, \Gamma]) = - S([X^N, \Gamma])$ from which it follows that $S([X^N, \Gamma]) = 0$. By using this, the identity $\mathcal{L}_\Gamma i_X = i_X \mathcal{L}_\Gamma + i_{[\Gamma, X]}$, and field equations (14) one has $\mathcal{L}_\Gamma i_{X^N} \theta_L = i_{X^N} \mathcal{L}_\Gamma \theta_L + i_{[\Gamma, X^N]} dL \circ S = i_{X^N} \mathcal{L}_\Gamma \theta_l = i_{X^N} dL = \mathcal{L}_{X^N} L = \mathcal{L}_\Gamma f$, and (17) follows.

Remark. It may appear that the f as in (16) is defined up to quantities that vanish under the action of Γ, namely up to conserved quantities. One may however show that there are no such objects in $\mathcal{F}_S$. For this point see the discussion in [MSSV].

5.2. A THEOREM DEFINING A TANGENT BUNDLE STRUCTURE

If the algebra $\mathcal{F}$ we start with in section 5.1., is the algebra of smooth functions on a manifold M and E is just the module $\mathrm{Der}\mathcal{F}$ of derivations of $\mathcal{F}$, then in [DLMV] it has been shown that conditions (6) and (9) essentially imply that M has a tangent bundle structure. If M is a tangent bundles, then the algebra $\mathcal{F}_S$ defined in (7) is the algebra of smooth functions on the base space of the bundle and the sequence (8) is the sequence associated with the bundle as in section 2.4.; in particular, the algebra E^{π}/E^{v} in (8) is the Lie algebra of smooth vector fields defined over the base space. What has been proved is the following proposition.

Proposition 2. Let M be a smooth manifold (in fact it is enough to take M to be C^2), S and Δ respectively a (1,1) tensor field and a complete vector field on M fulfilling the hypotheses

1. $\ker S = \mathrm{Im}\, S$,

2. $N_S = 0$,

3. $\Delta \in \ker S$,

4. $L_{\Delta} S = - S$,

5. $\lim_{t \to -\infty} \exp(t\Delta)(p)$ exists for any point p in M ,

where $\exp(\cdot\Delta)$ denotes the flow of Δ. Then M has a unique tangent bundle structure whose dilatation operator is Δ and whose vertical endomorphism is S [YI] [MSSV]. Notice that condition 1. implies that M is even dimensional. If $\{ q^i , u^i , i = 1,\ldots, n \}$ is an adapted coordinate system for M, then the local expressions for S and Δ are

$$S = \Sigma\, dq^i \otimes \partial /\partial u^i \quad , \quad \Delta = \Sigma\, u^i \partial /\partial u^i \quad . \tag{18}$$

In [DLMV], proposition 2. is proved in two steps. Firstly, hypotheses 1. through 4. are used to show that the manifold has locally a tangent bundle structure: then one extends charts in the neighbourhood of singular points of Δ (the set of which forms the base manifold of the tangent bundle) all along integral curves of Δ itself so as to generate the whole corresponding fibres. One can give counterexamples if any one of hypotheses, and in particular hypothesis 5. is released. However, the proposition does not allow to exclude that a manifold carries a tangent bundle structure: it may indeed happen that one has just missed the two correct tensors.

On a tangent bundle manifold all constructions from (10) through (17) are the natural structures associated with a usual Lagrangian dynamics. In particular, Euler-Lagrange equations (14) are, in coordinates, the usual equations

$$(d/dt)(\partial L/\partial u^i) - (\partial L/\partial q^i) = 0 \ , \quad (d/dt)\, q^i = u^i \ , \ i = 1,\dots, n \, . \tag{18}$$

A possible physical application of proposition 2. could be in the context of constrained dynamical systems. Indeed, if one insists that the reduced dynamics be a Lagrangian one, one should first be sure that the reduced space carries a tangent bundle structure.

CONCLUSIONS

We have presented an algebraic framework for gauge theories which avoids the use of manifolds. The scheme is very general and allows to extract the basic algebraic structures beneath a gauge configuration. In our approach a gauge configuration is a short exact sequence $0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$ of Lie algebras endowed with a connection (a way to lift vectors in B to vectors in E). The Lie algebras are left modules over a commutative algebra with unit $\mathfrak{F}$ and act on $\mathfrak{F}$ as algebras of derivations. Any such an extension could be realized in many different ways. By realizing it in terms of vector fields on a principal fibre bundle (in this case $\mathfrak{F}$ is an algebra of functions) one recovers an usual gauge configuration. As examples we have constructed the sequences for abelian and nonabelian monopoles and instantonic solutions.

The algebraic setting allows to introduce (algebraic) field equations and derive them from a Lagrangian by means of a proper variational problem. We have also introduced gauge transformations as well as symmetries and conserved quantities.

An immediate consequence of our approach is a $\mathbb{Z}_2$-graded generalization of gauge theories. One simply takes sequences of Lie superalgebras and substitutes all objects with graded ones. A graded gauge configuration can also be realized in terms of vector fields associated with a principal superbundle. In particular we have constructed a monopole-like sequence of Lie superalgebras. In order to realize such a sequence in terms of vector fields, we have constructed a Grassmann extension of the monopole fibration of the two dimensional unit sphere.

We have also presented an algebraic setting for lagrangian dynamical systems which we hope to use to construct graded generalization of lagrangian systems.

From the 'computational' point of view, it seems that our scheme is manageable only for Lie algebras which are finitely generated (freely or not) as $\mathfrak{F}$-module. We have been able to carry over computations only for those Lie algebra sequences that in terms of fibre bundles would correspond to gauge theories over homogeneous spaces.

Another open problem is that of boundary conditions for gauge fields. It is not clear how (if possible) to implement them in our scheme. In all examples we have given this problem was absent because the boundary conditions were, in a sense, hidden in the sequences. This fact is more explicit in the realization in terms of fibre bundles since all our examples correspond to gauge configurations for which the relevant physical content is on spaces which are compact without boundary.

Finally, we mention the possibility of using the theory of Lie algebra extensions to get a classification of gauge field. However, the theory of classification of Lie algebras which we have sketched in section 2.4. is too coarse for this. Indeed, it appears that inequivalent gauge configurations may correspond to the same equivalence class of extensions of Lie algebras. It may be interesting to use ideas from physics to get a finer classification of extensions.

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