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VARIATIONAL PROBLEMS WITH OBSTRUCTIONS

Roberta MUSINA CARRARO

Ph. D. Thesis 3/Sept. 88/M

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Il presente lavoro costituisce la tesi presentata dalla Dott. Roberta Musina sotto la direzione del Prof. G. Mancini, in vista di ottenere l'attestato di ricerca postuniversitaria "Doctor Philosophiae", settore di Analisi Funzionale e Applicazioni.

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I would like to dedicate this Thesis work to my husband Paolo, as a further token of love.

TESI DI "PHILOSOPHIAE DOCTOR"

VARIATIONAL PROBLEMS WITH OBSTRUCTIONS

Settore: Analisi Funzionale ed Applicazioni
Supervisore: Prof. Giovanni MANCINI
Candidato: dott. Roberta MUSINA
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Contents.

Introduction.

The purpose of this Thesis is to survey the research work I carried out under the direction of Prof. G. Mancini.

Our main object is the study of a class of problems which are usually described as "problems at critical growth". Roughly speaking, we are interested in variational problems of the form

"*find* $u \in X$ *which is stationary for* $E : X \to \mathbb{R}$" ,

when the functional E is invariant with respect to a non compact group of transformations of X into itself. In particular, our aim is to study how an obstacle condition can affect the existence - non existence phenomena which can be related to the invariance mentioned above.

Let us consider the following model problem: given a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$, find a map $u \in H^1_o(\Omega)$ which realizes the best constant S in the Sobolev imbedding $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, where $2^* = 2N/(N-2)$:

$$S := \operatorname*{Inf}_{|u|_{2^*}=1} \int_\Omega |\nabla u|^2 . \tag{0.1}$$

It is well known that this minimization problem has no solution (see for example [2]). In fact the H^1_o and the L^{2^*} norms are invariant with respect to the transformations

$$T_\varepsilon : u(x) \to \varepsilon^{-\frac{N-2}{2}} u(\frac{x}{\varepsilon}) ,$$

for any $\varepsilon > 0$. This implies that the best Sobolev constant S does not depend on the domain Ω. Therefore, it is clear that every solution u to the minimization problem (0.1) would provide an extremal on every ball B_r containing Ω, i.e. it would be a non trivial solution (up to a Lagrange multiplier) of the equation

$$- \Delta u = | u |^{2^*-2} u \quad \text{in} \quad B_r \ , \quad u \in H^1_o(B_r) , \tag{0.2}$$

contradicting the Pohozaev' identity (see [2] for details).

In many recent papers (see for example [2], [1], [5], [4], [6], [14]), it has been shown that the above non existence result for (0.2) is "highly unstable", in the sense that small perturbations in the equation or in the domain Ω may produce an existence result.

In their celebrated paper, Brezis and Nirenberg have considered a "lower order" perturbation to the equation $-\Delta u = |u|^{2^*-2} u$. More precisely, they show that the problem

$$(0.3) \qquad \begin{cases} -\Delta u = |u|^{2^*-2} u + \lambda u & \text{in } \Omega \\ u \in H^1_o(\Omega) \quad , \quad u \neq 0 \end{cases}$$

has a positive solution if $\lambda \in]0,\lambda_1[$ and $N \geq 4$, where λ_1 is the first eigenvalue of $-\Delta$ on $H^1_o(\Omega)$. In case $N = 3$, they prove that there exists a $\lambda^* \in]0,\lambda_1[$ such that, if $\lambda \in]\lambda^*,\lambda_1[$, then (0.3) has a positive solution. Following this scheme, we have perturbed the equation (0.2) by introducing a small obstacle ψ (see the paper [8] which is presented in the first part of this Thesis). The result is the following: if ψ is a given smooth obstacle, such that its maximum on Ω is positive, and $\psi < 0$ on $\partial\Omega$, then the variational inequaltity

$$(0.4) \qquad \begin{cases} u \in H^1_o(\Omega) \quad , \quad u \geq \psi \quad \text{in } \Omega \\ \displaystyle\int_\Omega \nabla u \, \nabla(v-u) \geq \int_\Omega |u|^{2^*-2} u \, (v-u) \quad \text{if } \; v \in H^1_o(\Omega), \; v \geq \psi \end{cases}$$

has two distinct solutions $\overline{u} \geq \underline{u} > 0$, provided ψ has small H^1 - norm. We point out that this result holds true in case $N = 3$ as well as in case $N \geq 4$.

More recently, Bahri and Coron [1] (see also [5]) have investigated the role of the topology of the domain Ω in the study of problem (0.2). Actually, in 1975, Kazdan and Warner [7] have observed that (0.2) has a solution whenever Ω in an annulus around the origin. In fact, in this case we can restrict our attention to radially symmetric functions, and thus we are allowed to use ODE's methods in order to have an existence result.

We can see that the argument here, as well as the non existence result given by Pohozaev' identity, depends strongly on the <u>geometry</u> of the domain. In [5], Coron was able to relate existence - non existence phenomena to the <u>topology</u> of Ω. He observed, roughly speaking, that "holes" in Ω induce a richer topology on the sublevels of the energy functional associated to (0.2), and this gives rise to the existence of non trivial solutions.

This result, which still contains a technical restriction on Ω, was generalized in [1], where Bahri and Coron perform a deep analysis on the relations between the homology groups of Ω and the topology of the energy sublevels for (0.2).

The paper [9] we present in the first part of this Thesis is somehow related to the above results. Here we are concerned with variational problems of the form:

$$
(0.5) \qquad \begin{cases} u \in H^1_o(\Omega) \ , \quad u \le \psi \ \text{ on } \ C \\ \displaystyle\int_\Omega \nabla u \, \nabla(v-u) \ge \int_\Omega |u|^{2^*-2} u\,(v-u) \quad \text{if} \ \ v \in H^1_o(\Omega) \ , \ v \le \psi \ \text{on } C. \end{cases}
$$

where C is a closed subset of Ω, and $\psi : \Omega \to \mathbb{R}$ is a given smooth, positive function. Following the methods of [5], we observe that the obstacle condition on C enriches the topology of the energy sublevels. Thus, we can prove that (0.5) has a positive solution u which can be found via a "saddle point argument", provided the set $\Omega \setminus C$ verifies a geometrical assumption (as in [5]).

This result concludes the first part of this Thesis.

Even if in the papers quoted above our main attention was referred to the limiting case, our results seem to be new as well in the sub-critical case, when the exponent 2^* is replaced by some $p \in]2,2^*[$. From this viewpoint, the results in [8], [9] could be seen as examples of existence and multiplicity results for nonlinear, anticoercive variational inequalities. Therefore, we have introduced them by a Section on variational problems on convex constraints, where we give a variational proof of the $W^{2,p}$ regularity result by Brezis-Stampacchia [3] for linear variational inequalities involving an obstacle condition of the form: $u \ge \psi$.

Variational problems at critical growth naturally arise in geometry. In the second part of this Thesis we will be concerned with two examples of problems for parametric surfaces in $\mathbb{R}^3$, which are subjected to some non-convex constraints. These problems are invariant with respect to the non compact group of dilations in $\mathbb{R}^2$.

The first problem we will describe (see [12] in the second part of this Thesis) concerns surfaces, spanned over obstacles, having prescribed boundary and prescribed mean curvature. The result achieved in this context, already announced in [11], was subsequently improved, and we present here in its final form. The variational problem consists in finding stationary points for the energy functional

$$
E(u) = \frac{1}{2} \int_D |\nabla u|^2 + \frac{2H}{3} V(u)
$$

on the constraint

$$
K = \{ u \in H^1(D, \mathbb{R}^3) \ | \ u \text{ maps } \partial D \text{ on } \Gamma \text{ monotonically}, F(x,u) \ge 0 \},
$$

where H is a positive constant which represents the curvature, V is the volume functional:

$$V(u) = \int_D \det(u, \nabla u) \qquad \text{for } u \in H^1(D) \cap L^\infty(D),$$

D is the unit disk in $\mathbb{R}^2$, Γ is a given Jordan curve in $\mathbb{R}^3$ and, finally, $F : D \times \mathbb{R}^3 \to \mathbb{R}$ represents the constraint. When F is of the form $F = u^3 - \psi(u^1, u^2)$ or concave in u, we prove that, under some additional hypothesis on Γ and F, there exists a solution $\underline{u}$ to our variational problem. This solution is a local minimum of the energy E on the set K. On the other hand, if there exists

$$u \in K \text{ such that } \quad E(u) < E(\underline{u}),$$

then we are pushed to find a second solution of mountain-pass type. However, this problem is still completely open. Some technical difficulties arise from the non-convexity of the constraint and from its non-parametric nature. In addition to that, both the energy and the constraint are invariant with respect to dilations, and this produces some lack of compactness as in the examples in the first part of the Thesis.

The last example regards a problem about minimal surfaces with obstacles; in particular, we present the paper [10] which proves a conjecture suggested in my Master Thesis ([11], Chap. I, Sect. 5).

The problem is the following: given an open connected smooth subset Ω of $\mathbb{R}^3$, and a Jordan curve Γ in $\mathbb{R}^3 \setminus \Omega$, find two disk-type minimal surfaces $\underline{u}$ and $\bar{u}$ spanned by Γ, which "enclose" Ω. Here, "minimal surface" means that $\underline{u}$ and $\bar{u}$ have to be stationary points for the area functional on the class X_Γ of maps which satisfies both the obstacle condition and the Plateau's boundary condition.

The "small" surface $\underline{u}$ can be found by minimizing the Dirichlet's integral on the class X_Γ. In 1972, Tomi proved in [15] a regularity result for such a minimizer, which in particular assures that $\underline{u}$ is conformal and $\underline{u}$ has minimal area in the class X_Γ. In order to find the second surface $\bar{u}$ one has to define in a suitable way a class X_Γ^e of admissible maps u such that u and $\underline{u}$ enclose Ω in a weak sense. Thus, the variational problem becomes:

$$\text{(0.6)} \qquad \textit{find} \quad \bar{u} \in X_\Gamma^e \,, \quad \int_D |\nabla \bar{u}|^2 = \operatorname*{Inf}_{X_\Gamma^e} \int_D |\nabla \cdot|^2 =: I_\Gamma \,.$$

Problem (0.6) has in general no solution: in the degenerate case, when Γ reduces to a point P, and consequently $\underline{u} = \text{const.} = P$, then every minimizing sequence converges weakly (up to a subsequence) to the constant P (see [11]). As in the above examples, this lack of compactness is due to the invariance of the variational problem with respect to the non compact group of dilations in $\mathbb{R}^2$. In [10] we give a necessary and sufficient condition for compactness of all minimizing sequences: namely we have to require

(0.7)
$$I_\Gamma < \int_D |\nabla \underline{u}|^2 + I_\infty ,$$

where I_Γ is the infimum in (0.6), $\underline{u}$ is the small solution, and the number I_∞ is twice the minimal area of S^2 type surfaces enclosing Ω. More precisely, I_∞ is the infimum of the Dirichlet's integral taken over a suitable class X^e of homotopically non trivial maps from S^2 into $\mathbb{R}^3 \setminus \Omega$:

(0.8)
$$I_\infty := \operatorname*{Inf}_{U \in X^e} \int_{S^2} |dU|^2 .$$

This problem was first solved in case Ω is the unit ball in $\mathbb{R}^3$, when $I_\infty = 8\pi$. In Section 2, part 2, we present this (unpublished) result, since the proof looks simpler and more direct than in the general case.

In [10] we also study the minimization problem (0.8). Because of the invariance of the energy and the constraint with respect to dilations and translations in $\mathbb{R}^2$, there exist minimizing sequences which converge almost everywhere to a constant map. On the other hand, taking advantage from this invariance, we can prove that every minimizing sequence converges to a solution of our minimization problem, up to dilations and translations in $\mathbb{R}^2$.

In a subsequent paper (see [13] in the second part of this Thesis), the case of unconnected obstacles was considered. In that case, the invariance with respect to translations in $\mathbb{R}^2$ plays a fundamental role. If X^e is the class of maps $U : S^2 \to \mathbb{R}^3 \setminus \Omega$ which enclose all the connected components of the obstacle Ω, then the infimum

$$\operatorname*{Inf}_{U \in X^e} \int_{S^2} |dU|^2$$

is achieved provided a "Douglas Criterion" is satisfied: namely, we can prove the existence of a minimizer when some strict inequalities hold true; the number of these inequalities depends on the number of connected components of the obstruction Ω.

Problems (0.6), (0.8) can be interpretated from the point of view of Differential Geometry: the first result we have stated before is in fact a multiplicity result for the Dirichlet's problem for harmonic maps from the disk into the manifold $\mathbb{R}^3 \setminus \Omega$. Similarly, a solution to problem (0.8) is a homotopically non trivial harmonic map from the unit two-sphere into $\mathbb{R}^3 \setminus \Omega$.

At present, our research is dedicated to some extensions and generalizations of the results in [10]. Looking, for example, at the problem for closed surfaces, one can first think to replace $\mathbb{R}^3 \setminus \Omega$ with a three dimensional Riemannian manifold with boundary. By replacing $\mathbb{R}^3$ by $\mathbb{R}^N$, $N \geq 4$, then it would be of interest to study the problem of existence of a non constant unstable harmonic map from S^2 into $\mathbb{R}^N \setminus \Omega$, where Ω is an open set in $\mathbb{R}^N$.

Finally, we could consider the case when Ω is a "thin" obstacle in $\mathbb{R}^3$, namely Ω is a lower dimensional subset of $\mathbb{R}^3$, and look for minimal surfaces "enclosing" Ω.

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Part 1:

VARIATIONAL INEQUALITIES ON CONVEX SETS

Abstract. We present here some existence, multiplicity and regularity results for a class of elliptic variational inequalities on convex sets.

In an Introduction, we describe the variational approach to problems of the form

$$(1)\qquad \begin{cases} u \in \mathbf{K} \quad \text{closed convex subset of } H_o^1(\Omega) \\ \int_\Omega \nabla u \, \nabla(v - u) \geq \int_\Omega g(u)\,(v - u) \qquad \forall\, v \in \mathbf{K} \,. \end{cases}$$

In case $\mathbf{K} = \{ u \in H_o^1(\Omega) \ , \ u \geq \psi \}$ we give a completely variational proof of the well known $W^{2,p}$ regularity result by Brezís - Stampacchia [4].

Part 1 will be concluded with two papers by G. Mancini and the Author of this Thesis. In those papers two examples of nonlinear, anticoervive variational inequalities of the form (1) are considered. A special attention is payed to the case in which the nonlinearity g has the limit Sobolev growth.

Introduction.

A Variational Approach and a Regularity Result.

In this Section, we describe a variational approach to problems of the form:

$$(1.1)\qquad \begin{cases} u \in \mathbf{K} \\ \displaystyle\int_\Omega \nabla u\, \nabla(v-u) \geq \int_\Omega g(u)\,(v-u) \qquad \forall\; v \in \mathbf{K}\,. \end{cases}$$

where $\mathbf{K}$ is a not empty, closed, convex subset of the Sobolev space $H^1_o(\Omega)$, Ω is a bounded, smooth region in $\mathbb{R}^N$, and $g : \mathbb{R} \to \mathbb{R}$ is a given smooth map.

For sake of simplicity, we suppose that g is a continuous map satisfying the growth condition

$$(1.2)\qquad | g(u) | \leq a\, | u |^s + b \qquad \text{for some } s \leq (N+2)/(N-2),$$

and for some constants a, b. For a more general treatment we refer to [3] and [11].

Let $G : \mathbb{R} \to \mathbb{R}$ be a primitive of g ; by virtue of (1.2), the functional

$$F(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(u)$$

is of class C^1 on $H^1_o(\Omega)$, and, by using our convexity assumption on $\mathbf{K}$, (1.1) can be rewritten as:

$$\begin{cases} u \in \mathbf{K} \\ (\nabla F(u), v-u) \geq 0 \qquad \forall\; v \in \mathbf{K}, \end{cases}$$

or

$$(1.3)\qquad \liminf_{\substack{v \in K \\ v \to u}} \frac{F(v) - F(u)}{\| v-u \|_{H^1_o(\Omega)}} \geq 0\,.$$

We take (1.3) as a definition of *stationary point* for F on $\mathbf{K}$: notice that (1.3) is equivalent to the usual definition of stationary point if $\mathbf{K} = H^1_o(\Omega)$, and that (1.3) makes sense also for less regular functionals. Actually, this is exactly the definition of stationary point given by Brezís [3], and used for example in [11] in a more general context.

In the "coercive case", one can try to solve (1.1) by studying the minimization problem

$$\operatorname*{Min}_{u \in K} F(u)\,.$$

Let us take as model problem for the coercive case the following

$$(1.1)_{C.} \qquad \begin{cases} u \in K \\ \displaystyle\int_\Omega \nabla u \,\nabla(v-u) + a_o \int_\Omega |u|^{p-2} u\,(v-u) \geq \int_\Omega f\,(v-u) \qquad \forall\ v \in K\,, \end{cases}$$

where $a_o \geq 0$ is a real number, $2 < p \leq 2N/(N-2)$ and $f \in L^q(\Omega)$, $q \geq 2N/(N+2)$. In this case the related energy functional:

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{a_o}{p}\int_\Omega |u|^p - \int_\Omega f\,u$$

is bounded from below, coercive and strictly convex. Thus, F is weakly lower semicontinuous on $H^1_o(\Omega)$, and since K is convex, the direct method of the Calculus of Variations guarantees the existence of a unique solution $\underline{u}$ to Problem $(1.1)_C$; $\underline{u}$ is characterized by:

$$F(\underline{u}) = \operatorname*{Min}_K F\,.$$

In the "anti-coercive case", as for example in

$$(1.1)_{N.C.} \qquad \begin{cases} u \in K \\ \displaystyle\int_\Omega \nabla u \,\nabla(v-u) \geq \int_\Omega |u|^{p-2} u\,(v-u) \qquad \forall\ v \in K\,, \end{cases}$$

the functional F is unbounded from below and from above. Nevertheless, there could exist solutions to $(1.1)_{N.C.}$, depending on the geometry of the convex set K. Let us consider for example the following three cases:

<u>Case 1</u> : $K = H^1_o(\Omega)$. In this case, Problem $(1.1)_{N.C.}$ is equivalent to:

$$-\Delta u = |u|^{p-2} u \quad , \quad u \in H^1_o(\Omega).$$

When $p < 2N/(N-2)$, problem $(1.1)_{N.C.}$ has infinitely many solutions. In general, in the limiting case $p = 2N/(N-2)$, no solutions exist for Problem $(1.1)_{N.C.}$ besides $u \equiv 0$. This is the case if Ω is a starshaped domain (see for example [4]).

<u>Case 2</u> : $K = \{u \in H^1_o(\Omega)\ ,\ u \geq \psi\}$, where $\psi \in H^1_o(\Omega)$. This problem was studied in [8], [9], where we prove that if $p \leq 2N/(N-2)$, and ψ is positive somewhere in Ω and "suitable small", then there exist at least two distinct solutions $\bar{u} \geq \underline{u} > 0$.

<u>Case 3</u> : $K = \{u \in H^1_o(\Omega)\ ,\ u \geq 0$ in Ω, $u \leq \psi$ on $C\}$, where $C \subset\subset \Omega$, and $\psi : \Omega \to \mathbb{R}$ is a given smooth, positive function. Using the results by Szulkin [11], it is not difficult to

prove that if $p < 2N/(N-2)$ then Problem $(1.1)_{N.C.}$ has a non trivial solution. In case $p = 2N/(N-2)$, and C satisfies a suitable geometrical assumption, then $(1.1)_{N.C.}$ has a positive solution ([10]).

In the following, we will present the existence results for Cases 2,3 mentioned above ([9], [10]). Most attention is payed here to the limit case $p = 2N/(N-2)$: when $p < 2N/(N-2)$, the compactness of the Sobolev imbedding $H^1_o \hookrightarrow L^p$ guaranteessome compactness properties of the energy functional F which allows us to apply the existence theorems in [11], while in the limit case those compactness properties fail at some energy levels.

Before ending this Introduction, we wish to present a regularity result for (1.1) in case

$$K = \{u \in H^1_o(\Omega)\ ,\ u \geq \psi \ \text{a.e. in } \Omega\}$$

where $\psi \in H^1(\Omega)$, $\psi_{|\partial\Omega} \leq 0$. The Theorem we are going to state was proved by Brezìs - Stampacchia [5] via a penalization method (see also [7], [6] - Chap. IV, [2] - 7.4.2). We follow here a completely variational approach to the problem of regularity, that is, we only use the fact that a solution $\underline{u}$ to (1.1) is the unique minimum point on K of the functional

$$\tilde{F}(v) := \frac{1}{2}\int |\nabla v|^2 - \int g(\underline{u})\, v \ .$$

The proof is based on the following elementary Lemma:

Lemma 1.1: *Let* $p \leq 2N/(N-2)$ *and let* $u \in H^1_o(\Omega)$. *If there exists a constant* L *s.t.*

$$\text{(1.4)} \qquad \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2}\int_\Omega |\nabla v|^2 \leq L\,|u-v|_p \qquad \forall\, v \in H^1_o(\Omega)$$

then $u \in W^{2,q}(\Omega)$, *where* $q = p/(p-1)$ *is the conjugate Sobolev exponent, and*

$$|-\Delta u|_q \leq L\,.$$

Proof. Testing (1.4) with $v = u - tw$, where $w \in H^1_o(\Omega)$, $t > 0$, we get

$$-\frac{t^2}{2}\int_\Omega |\nabla w|^2 + t\int_\Omega \nabla u\, \nabla w \leq L\, t\, |w|_p \ .$$

Dividing this inequality by t and letting t go to zero, we can easily obtain that the linear functional $Tv := \int \nabla u\, \nabla v$ is bounded on $H^1_o(\Omega)$ equipped with the L^p norm:

$$\| T \| = \sup_{\substack{w \in H_o^1(\Omega) \\ w \neq 0}} \frac{\int \nabla u \, \nabla w}{| w |_p} \leq L \; .$$

Thus T extends continuously on $L^p(\Omega)$, i.e.

$$\exists \, ! \;\; \alpha \in L^q(\Omega) \quad \text{s.t.} \quad T(v) = \int \alpha v \quad \forall \; v \in H_o^1(\Omega) \quad \text{and} \quad | \alpha |_q = \| T \| \leq L \, .$$

In particular, u is a weak solution of

$$\begin{cases} - \Delta u = \alpha & \text{in } \Omega \\ u \in H_o^1(\Omega) & \end{cases}$$

and hence Lemma 1.1 follows from the elliptic regularity theory. ∎

Now, let $\underline{u} \in H_o^1(\Omega)$ be a solution to Problem (1.1). If g verifies (1.2), by Sobolev imbedding theorem we get

$$g \circ \underline{u} \in L^q(\Omega) \;\; , \quad \text{where} \;\; q = \frac{1}{\alpha} \frac{2N}{N-2} \geq \frac{2N}{N+2} \; .$$

Thus, we can limit ourself to the study of the following linear variational inequality:

$$(1.1)_{L.} \qquad \begin{cases} \text{find } \underline{u} \in K = \{ \, v \in H_o^1(\Omega) \mid v \geq \psi \text{ a.e. on } \Omega \, \} \\ \int \nabla \underline{u} \, \nabla (v - \underline{u}) \;\; \geq \;\; \int g \, (v - \underline{u}) \quad \forall \; v \in K \, , \end{cases}$$

where $g \in L^q(\Omega)$. Problem $(1.1)_{L.}$ has a unique solution $\underline{u}$ which minimizes

$$F(u) = 1/2 \int |\nabla u|^2 + \int g \, u \qquad \text{on } K.$$

<u>***Theorem 1.2***</u>: *Suppose* $\psi \in W^{2,q}(\Omega)$, $g \in L^q(\Omega)$ *for some* $q \geq 2N/(N+2)$. *Then*

$$\underline{u} \in W^{2,q}(\Omega) \quad \textit{and} \qquad | - \Delta \underline{u} \, |_q \; \leq \; | - \Delta \psi \, |_q \; + \; | \, g \, |_q \, .$$

Proof. It is enough to prove that (1.4) holds for $\underline{u}$, with $L \leq | -\Delta \psi |_q + | g |_q$. Let us define the projection on the closed, convex set **K**:

$$Pw := \text{Max} \, \{w, \psi\} \qquad P : H_o^1(\Omega) \to \mathbf{K}.$$

Using standard results in Sobolev Spaces (see for example [6], Chap.1, Appendix A), we have that $Pw \in K \quad \forall\ w \in H^1_o(\Omega)$, and

$$\nabla(Pw) = \begin{cases} \nabla\psi & \text{a.e. on } \{ w < \psi \} \\ \nabla w & \text{a.e. on } \{ w > \psi \}. \end{cases}$$

In addition, we easily get

$$\int | \underline{u} - Pw |^p = \int_{\{w<\psi\}} (\underline{u} - \psi)^p + \int_{\{w>\psi\}} |\underline{u} - w|^p \leq \int_\Omega |\underline{u} - w|^p ;$$

(1.5)

$$\int | w - Pw |^p = \int_{\{w<\psi\}} (\psi - w)^p \leq \int_\Omega |\underline{u} - w|^p .$$

Since $\underline{u}$ minimizes F on K, for every $w \in H^1_o(\Omega)$ we find

$$\frac{1}{2}\int|\nabla\underline{u}|^2 - \frac{1}{2}\int|\nabla w|^2 = F(\underline{u}) - F(Pw) - \int g\,(\underline{u} - Pw) + \frac{1}{2}\Big[\int|\nabla(Pw)|^2 - \int|\nabla w|^2\Big]$$

$$\leq | g |_q\ | \underline{u} - Pw |_p + \frac{1}{2}\Big[- \int|\nabla(w - Pw)|^2 - 2\int\nabla Pw\ \nabla(w - Pw)\Big]$$

$$\leq | g |_q\ | \underline{u} - Pw |_p - \int\nabla\psi\ \nabla(w - Pw) \leq \Big(| g |_q + | -\Delta\psi |_q \Big)\ | \underline{u} - w |_p$$

by (1.5) and Hölder inequality. Thus

$$L := \operatorname*{Sup}_{w \in H^1_o(\Omega)} \frac{\frac{1}{2}\int|\nabla u|^2 - \frac{1}{2}\int|\nabla w|^2}{| u - w |_p} \leq | g |_q + | -\Delta\psi |_q$$

and the Theorem is proved. ∎

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A Free Boundary Problem Involving Limiting Sobolev Exponents

The results in this paper have been obtained in collaboration with G. Mancini.

A FREE BOUNDARY PROBLEM
INVOLVING LIMITING SOBOLEV EXPONENTS

In this paper we present existence and nonexistence results for the variational inequality:

$$0 \neq u \geq \psi^+, \ \int_\Omega \nabla u \ \nabla(v-u) \geq \int_\Omega u^{2^*-1}(v-u) \quad \forall \ v \in H_o^1(\Omega), \ v \geq \psi \ \text{ a.e. in } \Omega$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $\psi \in H^{2,s}(\Omega)$, $s>N$, $\psi<0$ on $\partial\Omega$, and $2^*=2N/(N-2)$. In particular, we show there are no solutions if ψ is too large, while there are distinct solutions $0 < \underline{u} \leq \bar{u}$ if ψ is positive somewhere and sufficiently small.

0. Introduction

Much attention has been payed in recent years to problems of the form

$$(0.1) \qquad \begin{cases} -\Delta u = u^{2^*-1} + f(x,u) & \text{in} \quad \Omega \subset \mathbb{R}^N \\ u \in H_o^1(\Omega) \ , \quad u > 0 \end{cases}$$

where Ω is a smooth bounded domain and f is a lower order perturbation (see [5], [6] and references there in). First, because (0.1) is a nice model for problems arising in geometry ([2], [3], [4], [7], [8], [9], [15], [21], [22]) and in theoretical physics ([10], [24]). Secondly, because (0.1) exibits interesting existence - non existence phenomena, related to some lack of compactness of the corrisponding energy functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2^*}\int_\Omega u^{2^*} + \int_\Omega F(x,u) \quad , \quad u \in H_o^1(\Omega), \quad F' = f .$$

Such lack of compactness explains why (0.1) does not have, in general, any solution (e.g. if f=0 and Ω is starshaped, see [12]).

In their pioneering work Brezis and Nirenberg showed that the lower order term f (e.g. $f(u)=\lambda u$ for some $\lambda>0$) allowes to regain compactness for some (P.S.) sequences and consequentely makes (0.1) solvable. Many different problems remain open in this context (see [12] and [5], [6]).

In this paper we "perturb" the equation $-\Delta u = u^{2^*-1}$, $u \in H_o^1(\Omega)$, with an "obstacle" $\psi \in H^{2,s}(\Omega)$, s>N, $\psi<0$ on $\partial\Omega$, i.e. we look for u which solves the equation just above ψ.

As a standard device, we replace this free boundary problem with the variational inequality

$$(0.2) \qquad \begin{cases} u \in H_o^1, \ u \geq \psi^+ \\ \int_\Omega \nabla u \, \nabla(v-u) \geq \int_\Omega u^{2^*-1}(v-u) \quad \forall \ v \geq \psi \end{cases}$$

Such a problem does not fit into the classical theory of variational inequalities for monotone operators, and not much is known, at our knowledge, for (0.2) even in the case when 2^* is replaced by some $p \in]2,2^*[$. Thus we will consider here inequality (0.2) for the broader class of nonlinearities u^{p-1} , $p \in]2,2^*]$, even if our main concern is the limiting case $p = 2^*$.

Solutions to (0.2) turn out to be the stationary points of an energy functional f of the form C^1 + convex proper lower semicontinuous. For such a class of functionals a critical point theory has been recentely developed by Szulkin ([23]).

In the case Ω is starshaped, $p = 2^*$, $\psi \leq 0$ in Ω, we have $f = E$ and no solutions exist for (0.2) besides $u = 0$. As soon as ψ becomes positive somewhere, but not too large, f "gaines" a critical point of saddle type: again, as in Brezis-Nirenberg [12] (or, maybe more appropiately, as in Struwe [21]), the perturbation produced by the obstacle has the effect of "lowering the Mountain Pass level" below a "magic" level.

In a forthcoming paper ([20]), following ideas from [13] and [2], [3], we will prove existence of solutions of (0.2) satisfying $u \leq \phi$ on some $A \subset \Omega$.

The paper is organized as follows.

In Section 1, besides presenting a regularity result for solutions of (0.2), we show that (0.2) is not solvable if ψ is too large.

In Section 2 we prove the existence of a "stable" solution $\underline{u}$, in case ψ is suitably small. Later, we show that "moving the obstacle" up to $\underline{u}$ might yield solutions to (0.2) which are greater than $\underline{u}$.

Finally, in Section 3 we prove the existence of a second solution $\bar{u} \geq \underline{u}$.

The results in this paper were announced in [19].

Notations. In the following, $\langle \cdot , \cdot \rangle$ and $\| \cdot \|$ will denote respectively the scalar product and the norm in $H_0^1(\Omega)$, $\rightharpoonup$ the weak convergence in $H_0^1(\Omega)$ and $| \cdot |_\infty$ will denote the L^∞ - norm.

1. Preliminaries, regularity remarks and a non existence result.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ ba a bounded domain, with smooth boundary, and $\psi \in H^{2,s}(\Omega)$, $s>N$, a given "obstacle" in Ω, satisfying $\psi<0$ on $\partial\Omega$ and positive somewhere. Set $K_\psi := \{u \in H_0^1(\Omega) \mid u \geq \psi \text{ a.e. in } \Omega\}$ and let $p \in]2,2^*]$,

$2^*=2N/(N-2)$. We are interested in the following

(1.1) <u>**Problem**</u>: <u>Given</u> Ω, ψ <u>and</u> p <u>as above</u>, <u>find</u> $u\in \mathbf{K}_\psi$, $u\geq 0$ <u>satisfying</u>

$$\int_\Omega \nabla u\, \nabla(v-u) \geq \int_\Omega u^{2^*-1}(v-u) \qquad \forall\ v\in \mathbf{K}_\psi .$$

The main purpose in this section is to show that (1.1) has in general no solution, unless ψ is assumed to be suitably small.

We first recall some well known facts on linear variational inequalities and present a regularity result for solutions of (1.1).

<u>**PROPOSITION 1.1**</u> . (see [17], Th 8.8) <u>Let</u> $\psi\in H^{2,\sigma}(\Omega)$, $\sigma\geq 2N/(N+2)$ <u>with</u> $\psi<0$ <u>on</u> $\partial\Omega$. <u>Then for every</u> $f\in L^\sigma$ <u>there is a unique</u> $h\in \mathbf{K}_\psi$ <u>satisfying</u>

(1.2) $$\int_\Omega \nabla h \nabla(v-h) \geq \int_\Omega f\,(v-h) \qquad \forall\ v\in \mathbf{K}_\psi$$

<u>Furthermore</u>, $h\in H^{2,\sigma}$.

In the sequel we will denote by k the solution of (1.2) corresponding to f = 0. Thus $k\in H^{2,s}$. Furthermore it is well known that $k>0$ in Ω and satisfies $-\Delta k\geq 0$ in Ω, $\Delta k = 0$ outside the "coincidence set" $C := \{k=\psi\}$. In view of our assumptions we have

(1.3) $$\alpha := \operatorname*{Inf}_C k = \operatorname*{Inf}_C \psi > 0 .$$

For convenience, we also recall the following

<u>**PROPOSITION 1.2**</u> (see [16], Th. 6.4, Chap. II): <u>Let</u> $u\in \mathbf{K}_\psi$ <u>satisfy</u> $-\Delta u\geq 0$ <u>in</u> H^{-1}. <u>Then</u> $u\geq k$ <u>a.e. in</u> Ω.

In the following remark we summarize a few properties of solutions of (1.1).

<u>**REMARK 1.3**</u> . Let u be a solution of Problem (1.1). Then

(i) $u\in H^{2,p'}$;

(ii) $u \geq k$ a.e. in Ω ;

(iii) $-\Delta u = u^{p-1}$ a.e. in $\Omega \setminus \{u=\psi\}$;

(iiii) $u(x) \geq \operatorname*{Inf}_{C} k \geq \alpha > 0$ for almost every $x \in \{u=\psi\}$. ∎

PROPOSITION 1.4 . Let u be a solution of Problem (1.1). Then $u \in H^{2,s}(\Omega)$.

Proof. In case $p < 2^*$, the result follows directly by Proposition 1.1. In case $p=2^*$, we write $-\Delta u = a(x)\, u$ a.e., where

$$a(x) := \begin{cases} u^{2^*-2} & \text{in } \Omega \setminus \{u=\psi\} \\ -\Delta u / u & \text{in } \{u=\psi\} \end{cases}$$

Since $\Delta u = \Delta \psi$ a.e. on $\{u=\psi\}$ (see [16], Lemma A4) one can easly see, using Remark (1.3), (iii), that $a \in L^{N/2}$. Hence $u \in L^t$ $\forall\, t \geq 1$ by a Lemma of Brezis and Kato ([12]), and the result follows again from Proposition 1.1. ∎

In order to state our non existence result, it is convenient to introduce a family of problems:

$$(1.1)_\lambda \qquad \begin{cases} \text{Given } \Omega,\ \psi,\ p \text{ as above and } \lambda > 0, \text{ find} \\ 0 \leq u \in \mathbf{K}_{\lambda\psi} := \{v \in H^1_0(\Omega) \mid v \geq \lambda\psi \text{ a.e. in } \Omega\} \text{ s.t.} \\ \int_\Omega \nabla u\, \nabla(v-u) \geq \int_\Omega u^{p-1}(v-u) \qquad \forall\, v \in \mathbf{K}_{\lambda\psi} \end{cases}$$

PROPOSITION 1.5 . Assume in addition $\Delta\psi \in L^\infty$ and suppose either $p<2^*$ or $p=2^*$ and Ω starshaped. Then, there exist λ_ψ such that if $\lambda > \lambda_\psi$ $(1.1)_\lambda$ has no solution.

Proof. The Proposition is a consequence of the following facts:

(i) if k_λ is the solution of $u \in \mathbf{K}_{\lambda\psi}$, $\int_\Omega \nabla u\, \nabla(v-u) \geq 0$ $\forall\, v \in \mathbf{K}_{\lambda\psi}$

then $k_\lambda = \lambda k$ and hence $\inf_{\{k_\lambda=\lambda\psi\}} k_\lambda = \lambda \inf_{\{k=\psi\}} k > 0$ (see (1.3)) ;

(ii) if u_λ solves $(1.1)_\lambda$, then (see Remark 1.3, (iii)),

$$-\Delta(\lambda\psi) = -\Delta u_\lambda \geq (u_\lambda)^{p-1} = (\lambda\psi)^{p-1} \qquad \text{a.e. on } \{u_\lambda = \lambda\psi\} .$$

Now, since $u_\lambda \geq \lambda k \geq \lambda\psi$, by (i) and Proposition 1.2, we see that for almost every $x \in \{u_\lambda = \lambda\psi\}$ we have $\psi(x) = k(x)$ and hence $\psi(x) \geq \alpha > 0$ for almost every $x \in \{u_\lambda = \lambda\psi\}$, in view of (1.3). Using (ii) we get

$$-\Delta\psi \geq \lambda^{p-2}\, \alpha^{p-1} \qquad \text{a.e. on } \{u_\lambda = \lambda\psi\} .$$

This implies $\{u_\lambda = \lambda\psi\}$ has zero measure if λ is large enough, i.e. $-\Delta u_\lambda = u_\lambda^{p-1}$ a.e. in Ω, by Remark 1.3, (iii). In case p=2* and Ω starshaped, this contradicts a well known non existence result related to the Pohozaev Identity (see [12]). In case p<2*, it has been proved in [14] that there exist a L^∞- a priori bound M_p for positive solutions $u \in H^1_o(\Omega)$ of the equation $-\Delta u = u^{p-1}$. Hence, from $M_p \geq |u_\lambda|_\infty \geq \lambda\, |\psi^+|_\infty$ we get the desired bound on λ. ∎

2. Existence of a stable solution and a reformulation of Problem (1.1)

We begin this Section proving the existence of a "stable" solution of (1.1), assuming ψ is suitably small. To be more precise, set

$$S_p = \inf_{\substack{u \in H^1_o(\Omega) \\ u \neq 0}} \frac{\int |\nabla u|^2}{\{\int u^p\}^{2/p}} \quad , \quad \bar{S}_p = S_p^{p/(p-2)} \quad , \quad S = S_{2^*}$$

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p} \int_\Omega |u|^p \qquad , u \in H^1_o(\Omega)$$

PROPOSITION 2.1 . <u>Assume</u> $\int |\nabla\psi^+|^2 \le \frac{p-2}{p}\, \bar{S}_p$. <u>Then (1.1) has a solution which is a local minimum for</u> E <u>on</u> K_ψ .

Proof. It is enough to prove that there exists $\underline{u} \in K_\psi$ with $\int |\nabla \underline{u}|^2 < \bar{S}_p$ such that

$$E(\underline{u}) = m := \operatorname{Inf}\ \{ E(u) \mid \int |\nabla u|^2 \le \bar{S}_p\ ,\ u \in K_\psi \}.$$

Let $E(\underline{u}) = m$. Since, by the definition of $\bar{S}_p$, we have

$$E(u) \ge {}^{(p-2)}/_{2p} \int |\nabla u|^2 \qquad \text{if} \qquad \int |\nabla u|^2 \le \bar{S}_p \tag{2.1}$$

we obtain ${}^{(p-2)}/_{2p} \int |\nabla \underline{u}|^2 \le E(\underline{u}) \le E(\psi^+) < {}^{(p-2)}/_{2p}\, \bar{S}_p$ by assumption. It remains to prove the infimum is achieved. This is trivial if $p<2^*$. Hence, let $p=2^*$. Let $u_n \rightharpoonup \underline{u}$ be a minimizing sequence, and set $v_n := u_n - \underline{u}$. Using a Lemma by Brezis and Lieb [11] we get

$$\underline{\lim}\, E(v_n) = m - E(\underline{u}) \le 0 .$$

Now, since $\underline{\lim} \int |\nabla v_n|^2 < \underline{\lim} \int |\nabla u_n|^2$, plugging v_n into (2.1) we get $(1/2 - 1/2^*)\ \underline{\lim} \int |\nabla v_n|^2 \le \underline{\lim}\ E(v_n)$ and hence $u_n \to \underline{u}$ strongly. ∎

REMARK 2.2 . Inequalities above yield

$$E(u) \ge {}^{(p-2)}/_{2p}\, \bar{S}_p \ge E(\underline{u}) \quad \text{if} \quad \| u \|^2 = \bar{S}_p .$$ ∎

In Section 3 such an inequality will be used to apply a generalized mountain pass Lemma to a functional related to E and $\underline{u}$. Actually this will lead to a solution of the following variational inequality:

$$\begin{cases} \bar{u} \in H^1_o(\Omega)\ ,\ \bar{u} \ge \underline{u} \ \text{a.e.} \\ \int_\Omega \nabla\bar{u}\, \nabla(v-\bar{u}) \ge \int_\Omega \bar{u}^{p-1}\,(v-\bar{u}) \quad \forall\, v \in H^1_o(\Omega),\ v \ge \underline{u} \end{cases} \tag{2.2}$$

Clearly enough, if $\bar{u}$ solves Problem (1.1) and furthermore $\bar{u} \ge \underline{u}$ a.e., then $\bar{u}$

solves (2.5). Conversely

LEMMA 2.3 . Assume $\bar{u}$ satisfies (2.5). Then $\bar{u}$ is also a solution of Problem (1.1).

Proof : As in Proposition 1.4, one can easly see that $\bar{u}\in H^{2,s}(\Omega)$. Thus (see Remark 1.3, (iii)) $-\Delta\bar{u} = \bar{u}^{p-1}$ on $\{\bar{u}>\underline{u}\}$. This, jointly with $-\Delta\bar{u} = -\Delta\underline{u} = \underline{u}^{p-1}$ a.e. on $\{\underline{u}=\bar{u}\}\cap\{\underline{u}>\psi\}$, implies

$$(2.3)\qquad \int_\Omega -\Delta\bar{u}\,\phi = \int_{\{\bar{u}>\underline{u}\}} \bar{u}^{p-1}\phi + \int_{\{\underline{u}=\bar{u}\}\cap\{\underline{u}>\psi\}} \underline{u}^{p-1}\phi = \int_\Omega \bar{u}^{p-1}\phi \qquad \forall\ \phi\in C_0^\infty(\{\bar{u}>\psi\})$$

Now, given $v\geq\psi$ a.e. in Ω, we can write

$$(2.4)\qquad \int_\Omega \nabla\bar{u}\,\nabla(v-\bar{u}) = \int_{\{v\geq\bar{u}\}} -\Delta\bar{u}\,(v-\bar{u}) + \int_{\{v<\bar{u}\}} -\Delta\bar{u}\,(v-\bar{u})$$

Since $-\Delta\bar{u} = \bar{u}^{p-1}$ a.e. in $\{\psi\leq v<\bar{u}\}$ by (2.3), while $-\Delta\bar{u}\geq\bar{u}^{p-1}$ a.e. (see Remark 1.3, (iii)) implies $-\Delta\bar{u}\,(v-\bar{u})\geq\bar{u}^{p-1}(v-\bar{u})$ a.e. on $\{v\geq\bar{u}\}$, we see from (2.4) that $\bar{u}$ solves (1.1). ∎

3. Existence of a second solution

In this Section we will prove the existence of a solution $\bar{u}$ of (1.1) satisfying the stronger inequality $\bar{u}\geq\underline{u}$, where $\underline{u}$ is the "small" solution given by Proposition 2.1. In view of Lemma 2.3 it amounts to solve an obstacle problem where the "old" obstacle ψ is replaced by $\underline{u}$. This new obstacle problem will be solved using a critical point theory recently developed by Szulkin for functionals of the form C^1 + convex proper lower semicontinuous. Actually, the functional we are interested in, namely

(3.1) $\qquad f(u) = E(u) + I(u) \qquad u \in H^1_o(\Omega)$

where

$$I(u) = \begin{cases} 0 & \text{if } u \in \mathbf{K}_{\underline{u}} := \{v \in H^1_o(\Omega) \mid v \geq \underline{u} \text{ a.e.}\} \\ +\infty & \text{otherwise} \end{cases}$$

has the special form: C^1 + indicatrix function of a closed, convex set and is continuous in its effective domain $\mathbf{K}_{\underline{u}}$. We will recall the definitions and the result given in [23] in this framework.

<u>**Definition 3.1**</u>. Let f be as above. Then u is stationary for f iff

$$\langle \nabla E(u), \phi - u \rangle \geq 0 \qquad \forall\ \phi \in \mathbf{K}_{\underline{u}}\ .$$

<u>**Definition 3.2**</u>. A sequence $u_n \in \mathbf{K}_{\underline{u}}$ is a (P.S.) sequence for f iff

$$\operatorname{Sup} E(u_n) < +\infty \quad \text{and}$$

$$\exists\ z_n \in H^1_o\ ,\ z_n \to 0 \text{ s.t. } \langle \nabla E(u_n), \phi - u_n \rangle \geq \langle z_n, \phi - u \rangle \quad \forall\ \phi \in \mathbf{K}_{\underline{u}}\ .$$

<u>**Definition 3.3**</u>. Let $c^* \in \mathbb{R}$. We will say that f satisfies $(P.S.)_{c^*}$ if any (P.S.) sequence satisfying $\operatorname{Sup} E(u_n) \leq c^*$ has a convergent subsequence.

The following Lemma is an extension in the above framework of the classical Mountain Pass Lemma (see [1]).

<u>**LEMMA 3.4**</u>. <u>Let</u> f <u>as above and</u> $\underline{u}$ <u>be given</u>. <u>Assume</u>

(i) $\exists\ U \ni \underline{u}$ open s.t. $f(u) \geq f(\underline{u})\ \ \forall\ u \in U$ and $\operatorname*{Inf}_{\partial U} f(u) > f(\underline{u})$

(ii) $\exists\ e \notin U$ s.t. $f(e) \leq f(\underline{u})$

(iii) $\exists\ c^* > c := \operatorname*{Inf}_{\Gamma} \operatorname*{Sup}_{\gamma} f$ s.t. f satisfies $(P.S.)_{c^*}$.

Here $\Gamma := \{\gamma \in C^o([0,1], H^1_o) \mid \gamma(0) = \underline{u} \text{ and } f(\gamma(1)) < f(\underline{u})\}$.

<u>Then</u> c <u>is a critical level for</u> f.

As a direct application of Lemma 3.4 we will get:

THEOREM 3.5. Let Ω, ψ, p be given as in Prop. 1.1. If in addition $\int|\nabla\psi^+|^2$ is suitably small, Problem (1.1) has at least two distrinct solutions $0 < \underline{u} \leq \bar{u}$.

Proof. By Proposition 2.1 we already know that (1.1) has a solution $\underline{u} > 0$ in Ω. By Lemma 2.3 we know we may look for $\bar{u}$ as a stationary point of f as given in (3.1). Clearly (see also Remark 2.2) f satisfies assumptions (i), (ii) in Lemma 3.4. Hence, to get a critical point of f via Lemma 3.4, we just need to prove a suitable $(P.S.)_{c^*}$ condition holds true for f. In case $p < 2^*$, it is easy to see, using compactness of Sobolev imbeddings, that f satisfies $(P.S.)_{c^*}$ for every c^*.

On the other hand, if $p = 2^*$, we can exibit a (P.S.) sequence $(u_n)_n$, with $f(u_n) \to E(\underline{u}) + {}^1/_N\, S^{N/2}$, which has no convergent subsequence, i.e. (P.S.) fails at level $E(\underline{u}) + {}^1/_N\, S^{N/2}$.

Nevertheless, we may apply Lemma 3.4, in view of the following Lemmata:

LEMMA 3.6. f satisfies $(P.S.)_{c^*}$ for $c^* < {}^1/_N\, S^{N/2} + E(\underline{u})$.

LEMMA 3.7. $\operatorname*{Inf}_{\Gamma} \operatorname*{Sup}_{\gamma} f < {}^1/_N\, S^{N/2} + E(\underline{u})$.

Proof of Lemma 3.6. Let $u_n \in \mathbf{K}_{\underline{u}}$ such that

(3.2) $E(u_n) = c^* + o(1) = O(1)$, $c^* < {}^1/_N\, S^{N/2} + E(\underline{u})$;

(3.3) $\exists\ z_n \to 0 : \langle z_n, v-u_n \rangle \leq \langle \nabla E(u_n), v-u_n \rangle \quad \forall\ v \in \mathbf{K}_{\underline{u}}$.

We first prove that $\| u_n \|$ is bounded. From (3.2) we get

$$\int (u_n)^{2^*} = {}^{2^*}/_2\, \| u_n \|^2 + O(1),$$

while setting $v = 2u_n$ in (3.3) we obtain $\| u_n \|^2 \geq o(1)\, \| u_n \| + \int (u_n)^{2^*}$.

From these two inequalities we get $(2^*/2 - 1) \, \| u_n \|^2 \leq o(1) \, \| u_n \| + O(1)$ and hence $\operatorname{Sup} \| u_n \| < +\infty$. Thus we can assume

$u_n \rightharpoonup u$ weakly in H_0^1 and a.e. for some $u \in K_{\underline{u}}$;

$u_n \rightharpoonup u$ weakly in L^{2^*} .

Also, setting $v_n := u_n - u$, by a Lemma in [11] we know that (eventually passing to a subsequence)

$$\lim E(u_n) = E(u) + \lim E(v_n) . \tag{3.4}$$

Furthermore, the Concentration-Compactness Lemma of P.L. Lions [18] insures the existence of numbers $a_j \geq 0$, points $x_j \in \overline{\Omega}$ and positive Radon measures ν, μ with

$$\nu = \sum_j a_j \, \delta_{x_j} \qquad\qquad \mu \geq S \sum_j (a_j)^{2/2^*} \delta_{x_j}$$

such that

$$| v_n |^{2^*} \rightharpoonup \nu \quad , \quad | \nabla v_n |^2 \rightharpoonup \mu \quad \text{weakly in the sense of measures.} \tag{3.5}$$

The proof will be now concluded in two steps:

__Step 1__ : $\lim E(u_n) \geq E(\underline{u}) + {}^1/_N \, \nu(\overline{\Omega})$;

__Step 2__ : $\nu(\overline{\Omega}) \geq S^{N/2}$ provided $a_j \neq 0$ for some j .

In fact, (3.2) and Step 1 imply $\nu(\overline{\Omega}) < S^{N/2}$ and hence $a_j = 0$ in view of Step 2, i.e. $v_n \to 0$ in L^{2^*}. Thus, since plugging $v = u$ in (3.3) we get

$$\int |\nabla v_n|^2 + \int \nabla u \, \nabla v_n + \langle z_n, u - u_n \rangle \leq \int (u_n)^{2^*-1} \, v_n$$

we conclude that $\lim \int |\nabla v_n|^2 \leq 0$, and hence Lemma 3.6 follows. ∎

__Proof of Step 1__. From (3.4), (3.5) we derive

(3.6) $\lim E(u_n) = E(u) + {}^1/_2\ \mu(\bar\Omega) - {}^1/_{2^*}\ \nu(\bar\Omega) =$

$$= E(\underline{u}) + {}^1/_2 \{ \mu(\bar\Omega) + \int|\nabla u|^2 - \int|\nabla\underline{u}|^2 \} - {}^1/_{2^*} \{ \nu(\bar\Omega) + \int u^{2^*} - \int \underline{u}^{2^*} \}.$$

Setting $v = 2u_n - \underline{u}$ in (3.3), $u = \underline{u}$ and $v = u$ in (1.1), adding and passing to the limit we get

$$\mu(\bar\Omega) + \int|\nabla u|^2 - \int|\nabla\underline{u}|^2 \geq \nu(\bar\Omega) + \int (u^{2^*-1} + \underline{u}^{2^*-1})(u-\underline{u}) \tag{3.7}$$

Inserting (3.7) in (3.6) we obtain

$$\lim E(u_n) \geq E(\underline{u}) + {}^1/_N\ \nu(\bar\Omega) + {}^1/_N \int (u^{2^*} - \underline{u}^{2^*}) - {}^1/_2 \int (u^{2^*-1}\,\underline{u} - \underline{u}^{2^*-1}\,u)$$

and Step 1 follows from the elementary inequality

$$b \geq a \geq 0 \Rightarrow {}^1/_N (b^{2^*} - a^{2^*}) \geq {}^1/_2 (b^{2^*-1}\, a - a^{2^*-1}\, b). \qquad \blacksquare$$

<u>**Proof of Step 2**</u>. Taking $v = u$ in (3.3) and passing to the limit we get

$$\mu(\bar\Omega) \leq \nu(\bar\Omega) \tag{3.8}$$

Since $\nu(\bar\Omega) = \sum_j a_j$ and $\mu(\bar\Omega) \geq S \sum_j (a_j)^{2/2^*}$ we obtain from (3.8)

$$\sum_j a_j \geq S^{N/2}\ , \text{ unless } a_j = 0\ \ \forall\, j. \qquad \blacksquare$$

<u>**Proof of Lemma 3.6**</u>. A crucial role in estimating the min-max level is played by the "non contact set" of the small solution $\underline{u}$. In fact we have

$$E(\underline{u}+v) = E(\underline{u}) + E(v) - (2^*-1)(2^*-2) \int_\Omega (\underline{u}\, v^2 \int_Q t\,(\sigma\underline{u} + tsv)^{2^*-3})\, dx \tag{3.9}$$
$$\forall\ v \in C_o^\infty(\{\underline{u}>\psi\})\ ,\ v \geq 0\ .$$

Here $Q = [0,1]^3$. Note that $2^* - 3 > -1$. To get (3.9) we recall that

$$\int_\Omega \nabla\underline{u}\, \nabla\phi = \int_\Omega \underline{u}^{2^*-1}\, \phi \qquad \forall\ \phi \in C_o^\infty(\{\underline{u}>\psi\})$$

and hence for every $v \in C_o^\infty(\{\underline{u}>\psi\})$ it results

$$E(\underline{u}+v) = E(\underline{u}) + E(v) - {}^{1}/_{2^*} \int [(\underline{u}+v)^{2^*} - \underline{u}^{2^*} - v^{2^*} - 2^*\, \underline{u}^{2^*-1}\, v].$$

Thus (3.9) follows from the elementary identity: if $a > 0$ and $h \geq 0$, then

$$(a+h)^{2^*} - a^{2^*} - 2^*\, a^{2^*-1} h = h^{2^*} + 2^*\,(2^*-1)(2^*-2)\, h^2 a \int_Q t\,(\sigma a + tsh)^{2^*-3} dt ds d\sigma.$$

From (3.9) it immediately follows:

(3.10) $\quad E(\underline{u}+v) \leq E(\underline{u}) + E(v) - \int \underline{u}\, v^{2^*-1} \qquad \forall\ v \in C_o^\infty(\{\underline{u}>\psi\})\ ,\ v \geq 0$ if $N \leq 6$.

In case $2^* < 3$ (i.e. $N > 6$) we can prove that $\exists\, c_N > 0$ s.t.

(3.11) $\quad \begin{cases} E(\underline{u}+tv) \leq E(\underline{u}) + E(tv) - c_N\, t^{2^*-1}\, |\, v\, |_\infty^{2^*-3} \int_\Omega \underline{u}\, v^2 \\ \forall\, t \geq {}^{1}/_{2} \text{ and } \forall\ v \in C_o^\infty(\{\underline{u}>\psi\})\ ,\ v \geq 0,\ |\, v\, |_\infty \geq 2|\, \underline{u}\, |_\infty \text{ if } N > 6\, . \end{cases}$

This can be seen as follows: for $\tau \geq {}^{1}/_{2}$ and $|\, v\, |_\infty \geq 2|\, \underline{u}\, |_\infty$ we have

$$\tau\ |\, v\, |_\infty \geq {}^{1}/_{2}\,(\,|\, \underline{u}\, |_\infty + \tau\,|\, v\, |_\infty) \quad \text{and hence}$$

$$(2\tau\,|\, v\, |_\infty)^{2^*-3} \leq [\sigma \underline{u}(x) + ts\tau v(x)]^{2^*-3} \quad \forall\ x \in \Omega, \quad \sigma, t, s \in [0,1].$$

From (3.9) we thus obtain (3.11) with $c_N = (2^*-1)(2^*-2)/2^{4-2^*} < 1$.

To conclude the proof of the Lemma, remark that as $t \to +\infty$,

$$f(\underline{u}+tv) \leq E(\underline{u}) + E(tv) \to -\infty \quad \text{if } v \in C_o^\infty(\{\underline{u}>\psi\})\ ,\ v \geq 0$$

and thus it is enough to prove

(3.12) $\quad \exists\ v^\circ \geq 0\ :\ \operatorname*{Max}_{t \geq 0}\ E(\underline{u}+tv^\circ) < E(\underline{u}) + {}^{1}/_{N}\, S^{N/2}\ ,\ v^\circ \in C_o^\infty(\{\underline{u}>\psi\})$.

To check (3.12) we follow closely [12].

Let $U \in H^1(\mathbb{R}^N)$ be the positive radially symmetric solution of

$$-\Delta U = U^{2^*-1}$$

Let $B_r(x^\circ) \subset\subset \{\underline{u}>\psi\}$, $\phi \in C_o^\infty(\, B_r(x^\circ))$ s.t. $\phi = 1$ in $B_{r/2}(x^\circ)$, $0 \leq \phi \leq 1$. Set

$$v_\varepsilon(x) := \varepsilon^{-(N-2)/2} \ \phi(x)\ U((x-x^\circ)/\varepsilon) = \varepsilon^{-(N-2)/2}\ z_\varepsilon(x), \qquad \varepsilon > 0\ .$$

Easy computations shows (see [12]):

$$a_\varepsilon := \int |\nabla v_\varepsilon|^2 = S^{N/2} + O(\varepsilon^{N-2})\ , \qquad b_\varepsilon := \int (v_\varepsilon)^{2^*} = S^{N/2} + O(\varepsilon^N)$$

(3.13)

$$c_\varepsilon := \int \underline{u}\ (z_\varepsilon)^{2^*-1} = O(\varepsilon^N) \qquad , \qquad d_\varepsilon := \int \underline{u}\ (z_\varepsilon)^2 = O(\varepsilon^N) \text{ if } \ N \geq 5.$$

Here $O(\varepsilon^q)$ means $| O(\varepsilon^q)/ \varepsilon^q |$ bounded and bounded away from zero. In order to handle (3.10), (3.11) altogether, we observe that

$$\exists\ \varepsilon^\circ > 0\ : \qquad \underset{0<\varepsilon\leq\varepsilon^\circ}{\mathrm{Max}}\ [\ \underset{t\leq 1/2}{\mathrm{Max}}\ E(\underline{u}+tv_\varepsilon)\ ] \ < \ E(\underline{u}) + {}^1/_N\, S^{N/2}.$$

This allowes to replace (3.12) with

(3.14) $$\exists\ \varepsilon > 0\ : \quad \underset{t\geq 1/2}{\mathrm{Max}}\ E(\underline{u}+tv_\varepsilon) < E(\underline{u}) + {}^1/_N\, S^{N/2}.$$

This in turn can be rewritten, using (3.10) or (3.11)

(3.15) $$\exists\ \varepsilon > 0\ : \quad \underset{t\geq 1/2}{\mathrm{Max}}\ E(tv_\varepsilon) - c_N\ t^{2^*-1}\ \varepsilon^{-(N+2)/2}\ h_\varepsilon \ < \ {}^1/_N\, S^{N/2}$$

where h_ε denotes either c_ε or d_ε. Here and in the sequel we denote by c_N various positive constants only dependent on N.

Now, it is easy to see that if t_ε realizes the maximum in (3.15), it has to satisfy

(3.16) $$t_\varepsilon = 1 - c_N\ h_\varepsilon\ \varepsilon^{-(N+2)/2} \ + \ O(\varepsilon^{N-2})\ .$$

Finally, using (3.13) and (3.16), we obtain

$$E(t_\varepsilon v_\varepsilon) - c_N\ t^{2^*-1}\ \varepsilon^{-(N+2)/2}\ h_\varepsilon\ =$$

$$= {}^1/_2\ (t_\varepsilon)^2\ a_\varepsilon - {}^1/_{2^*}\ (t_\varepsilon)^{2^*}\ b_\varepsilon - c_N\ t^{2^*-1}\ \varepsilon^{-(N+2)/2}\ h_\varepsilon =$$

$$= ({}^1/_2\ a_\varepsilon - {}^1/_{2^*}\ b_\varepsilon)\ (1 + O(\varepsilon^{N-2})) - c_N\ \varepsilon^{-(N+2)/2}\ h_\varepsilon + O(\varepsilon^{(N-2)/2}) =$$

$$= {}^1/_N\ S^{N/2} + O(\varepsilon^{N-2}) - c_N\ h_\varepsilon\ \varepsilon^{-(N+2)/2}\ <\ {}^1/_N\ S^{N/2}$$

if ε is small, because

$$h_\varepsilon\ \varepsilon^{-(N+2)/2}\ =\ O(\varepsilon^{(N-2)/2})\ ,$$

and $h_\varepsilon > 0$.

This completes the proof of Lemma 2.5. ∎

REMARK 3.8.. Construction above shows as well that (P.S.) does not hold at level $E(\underline{u}) + {}^1/_N\ S^{N/2}$. ∎

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Holes and Obstacles

The results in this paper have been obtained in collaboration with G. Mancini.

HOLES AND OBSTACLES

0. *Introduction.*

In a remarkable series of papers J.M. Coron and A. Bahri have been giving a complete explanation of a phenomenon previously observed by Kazdan and Warner [9], i.e. the role of the geometry of the domain with respect to existence - non existence for non linear elliptic boundary value problems of the form

$$(0.1) \qquad \begin{cases} -\Delta u = u^{2^*-1} & \text{in } \Omega \subseteq \mathbb{R}^N \text{, open and bounded} \\ u \in H_0^1(\Omega) \quad , & u \geq 0 \end{cases}$$

It is well known that (0.1) has only the trivial solution if Ω is starshaped. Conversely, A. Bahri and J.M. Coron showed, roughly speaking, that "holes" in Ω induce richer topology on the energy sublevels for (0.1). This, in turn, is responsable of the existence of non trivial critical points for the energy associated to (0.1).

In this paper we prove that a similar effect results by imposing a bilateral condition to (0.1).More precisely, we are interested to the following free boundary problem:

Given $\psi\in H^1_o(\Omega)\cap C^o(\overline{\Omega})$, $\psi\geq 0$ and a smooth closed subset $C\subset\Omega$, find $u\in H^1_o(\Omega)\cap C^o(\Omega)$ and a closed set $E\subseteq C$ such that

$$(0.2)\qquad \begin{cases} -\Delta u = |u|^{2^*-1} & \text{in } \Omega\setminus E \\ u=\psi & \text{in } E \\ u\leq\psi & \text{in } C \end{cases}$$

In case $\psi = 0$, a solution to (0.2) solves (0.1) in $\Omega\setminus C$, and hence (0.2) includes the study of (0.1) for domains with "holes".

The paper is organized as follows.

In Section 1 we discuss the behaviour of P.S. sequences for the following variational inequality:

Problem 1:
$$\begin{cases} \text{find } u\in \mathbf{K} \text{ such that} \\ \int_\Omega \nabla u\, \nabla(v-u) \geq \int_\Omega u^{2^*-1}(v-u) \quad \forall\ v\in\mathbf{K} \end{cases}$$

where $\mathbf{K}$ is the closed convex set of functions $u\in H^1_o(\Omega)$ such that $u\geq 0$ a.e. in Ω and $u\leq\psi$ on C in the sense of H^1 (see [10], Definition 5.1 pg. 35).

In Section 2 we give a variational principle for Problem 1 and prove, under additional hypothesis on C, the existence of non trivial critical points for the energy functional associated to Problem 1.

In the last section, we will prove a regularity result for Problem 1 which insures that every solution of Problem 1 solves the free boundary problem (0.2).

Notations. We denote by $\|\,.\,\|$ the norm in the Sobolev space $H^1_o(\Omega)$, and for $p\geq 1$, $|\,.\,|_p$ will denote the usual norm in $L^p = L^p(\Omega)$. If $u,w\in L^p$, we write $u\vee w = \mathrm{Max}\{u,w\}$, $u\wedge w = \mathrm{Min}\{u,w\}$.

All the inequalities between H^1 functions on the closed set C have to be regarded in the H^1 sense.

1. *The behaviour of P.S. sequences.*

DEFINITION 1.1: $u_n \in H^1_o(\Omega)$ is called a *P.S. sequence* for Problem 1 if

(i) $u_n \in \mathbf{K}$

(ii) $\sup_n \left\{ \frac{1}{2} \int_\Omega |\nabla u_n|^2 - \frac{1}{2^*} \int_\Omega |u_n|^{2^*} \right\} < +\infty$

(iii) $\exists \quad z_n \in H^1_o(\Omega),\ z_n \to 0$ in $H^1_o(\Omega)$ s.t.

$$\int_\Omega \nabla u_n \nabla(v-u_n) - \int_\Omega |u_n|^{2^*-1}(v-u_n) \geq \int_\Omega \nabla z_n \nabla(v-u_n) \qquad \forall\ v \in \mathbf{K}.$$

PROPOSITION 1.2. *Every P.S. sequence is bounded in $H^1_o(\Omega)$.*

Proof. Choosing $v_n = 2u_n - \min\{u_n, \psi\} = u_n + (u_n - \psi)^+$ in (iii), we get

$$\int |\nabla(u_n - \psi)^+|^2 + \int \nabla\psi \nabla(u_n - \psi)^+ \geq \int |u_n|^{2^*} - \int \psi^{2^*} - \int |u_n|^{2^*-1}\psi - \|(u_n - \psi)^+\|$$

for n large and hence, by Hölder inequality,

$$\| u_n \|^2 + c_1 \| u_n \| \geq | u_n |_{2^*}^{2^*} - c_2 | u_n |_{2^*}^{2^*-1} - c_3$$

Since by (ii) we have

$$| u_n |_{2^*}^{2^*} = \frac{2^*}{2} \| u_n \|^2 + O(1)$$

we readily get the boundeness of $\| u_n \|$. ∎

REMARK 1.3 : In view of Proposition 1.2, we will always assume in the sequel, that if u_n is a P.S. sequence then $u_n \rightharpoonup u$ weakly in $H^1_o(\Omega)$ for some $u \in \mathbf{K}$, and $\lim \int |\nabla u_n|^2$, $\lim \int |u_n|^{2^*}$ exist. Moreover, we can suppose that $|\nabla u_n|^2$, $|u_n|^{2^*}$ converge weakly in the sense of measures. ∎

PROPOSITION 1.4 : *Let* $u_n \rightharpoonup u$ *be a P.S. sequence. Then* u *is a solution of Problem 1.*

Proof. Choosing in (iii) $v_n = u_n + (u_n - u - \psi)^+$ we get, denoting $\vartheta_n = u_n - u$:

$$\int \nabla u_n \nabla(\vartheta_n - \psi)^+ \geq \int |u_n|^{2^*-1} (\vartheta_n - \psi)^+ + o(1), \text{ i.e.}$$

$$(1.1) \qquad \int \nabla u_n \nabla \vartheta_n - \int \nabla u_n \nabla(\vartheta_n \wedge \psi) \geq \int |u_n|^{2^*-1} (\vartheta_n - \vartheta_n \wedge \psi) + o(1).$$

We claim that

$$(1.2) \qquad \int \nabla u_n \nabla(\vartheta_n \wedge \psi) \to 0\ ;$$

$$(1.3) \qquad \int |u_n|^{2^*-1} (\vartheta_n \wedge \psi) \to 0\ .$$

From the claim it follows, using (1.1):

$$(1.4) \qquad \lim \int |\nabla u_n|^2 - \int |\nabla u|^2 \geq \lim \int |u_n|^{2^*} - \int u^{2^*}.$$

Since (iii) yields in the limit

$$\int \nabla u \nabla v - \int u^{2^*-1} v \geq \lim \left(\int |\nabla u_n|^2 - \int |u_n|^{2^*} \right)$$

we see from (1.4) that u solves Problem 1.

It remains to prove (1.2) and (1.3). Since $\vartheta_n \wedge \psi \to 0$ a.e., (1.3) follows from Lebesgue's dominated convergence Theorem. Finally, setting $v = u + (\vartheta_n - \psi)^+$ in (iii), we get

$$(1.5) \qquad \limsup \int \nabla u_n \nabla(\vartheta_n \wedge \psi) \leq \lim \int |u_n|^{2^*-1} (\vartheta_n \wedge \psi) = 0\ .$$

On the other hand, since $\vartheta_n \wedge \psi \rightharpoonup 0$ in H^1_o, we have

$$(1.6) \qquad \liminf \int \nabla u_n \nabla(\vartheta_n \wedge \psi) = \liminf \int \nabla \vartheta_n \nabla(\vartheta_n \wedge \psi).$$

But, denoted by χ_n the characteristic function of $\{ \vartheta_n \geq \psi \}$, it results

$$\lim \left| \int (\nabla \vartheta_n \nabla \psi) \chi_n \right| \leq \text{cost.} \lim \left(\int |\nabla \psi|^2 \chi_n \right)^{1/2} = 0$$

since $\chi_n \to 0$ almost for every x for which $\psi(x) > 0$. Thus (1.6) gives

(1.7) $$\liminf \int \nabla u_n \nabla (\vartheta_n \wedge \psi) \geq 0.$$

Hence, (1.7), (1.5) yield (1.2). ∎

In view of the above Lemma, we will be concerned in the following with P.S. sequences which weakly converge to zero.

REMARK 1.5 : Let $u_n \rightharpoonup 0$ be a P.S. sequence. Since (iii) implies, taking $v = 0$, $\lim \int |\nabla u_n|^2 \leq \lim \int |u_n|^{2^*}$, from (1.4) we get

$$\lim \int |\nabla u_n|^2 = \lim \int |u_n|^{2^*}$$ ∎

Let us now introduce the energy functional:

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{2^*} \int_\Omega u^{2^*} \quad \text{and} \quad S = \text{Inf} \left\{ \frac{\| u \|^2}{| u |_{2^*}^2} \mid u \in H_o^1(\Omega), u \neq 0 \right\},$$

The main result in this section is the following

THEOREM 1.6 . *Let* $u_n \rightharpoonup 0$ *be a P.S. sequence with* $E(u_n) \to c \neq 0$. *Then* $\lim E(u_n) = (k/N)\, S^{N/2}$ *for some* $k \in \mathbb{N}$.

One of the basic ingredients in the proof of Theorem 1.6 is a Lemma, essentially contained in P.L.Lions [12], concerning the local behaviour of weakly convergent sequences satisfying some kind of "reverse" inequalities.

LEMMA 1.7 : *Let* $u_n \in L^{2^*}(\mathbb{R}^N)$, $\nabla u_n \rightharpoonup 0$ *weakly in* $L^2(\mathbb{R}^N, \mathbb{R}^N)$. *Let* $U \subseteq \mathbb{R}^N$ *be a given open set, and assume*

$$(1.8) \qquad \lim \int |\nabla u_n|^2 \varphi^2 \le \lim \int | u_n |^{2^*} \varphi^2 \qquad \forall\, \varphi \in C_0^\infty(U).$$

Then there is a (possibily empty) finite set of points $x_1, \dots, x_m \in U$ *such that*

$$(1.9) \qquad \liminf \int_K |\nabla u_n|^2 = 0 \qquad \forall \text{ compact } K \subset U \setminus \{x_1, \dots, x_m\} \ ;$$

$$(1.10) \qquad \liminf \int_{B_r(x_j)} u_n^{2^*} \ge S^{N/2} \qquad \forall\, j = 1, \dots, m \text{ and } r > 0 \text{ small.}$$

Proof : We have to prove:

$$\exists\, x^\circ \in U \ : \ \liminf \int_{B_r(x^\circ)} |\nabla u_n|^2 > 0 \ \forall\, r \quad \Longrightarrow \quad \liminf \int_{B_r(x^\circ)} u_n^{2^*} \ge S^{N/2} \quad \forall\, r > 0.$$

Let $\varphi \in C_0^\infty(B_r(x^\circ))$, $\varphi = 1$ in $B_{r/2}(x^\circ)$. We have:

$$0 < \lim \int |\nabla(u_n \varphi)|^2 = \lim \int |\nabla u_n|^2 \varphi^2 \le \lim \int u_n^{2^*-2} (u_n \varphi)^2 \le$$

$$\le \liminf \Big(\int_{B_r(x^\circ)} u_n^{2^*} \Big)^{2/N} \Big(\int |u_n \varphi|^{2^*} \Big)^{2/2^*} \le \liminf \Big(\int_{B_r(x^\circ)} u_n^{2^*} \Big)^{2/N} S^{-1} \int |\nabla(u_n \varphi)|^2$$

by (1.8), Hölder and Sobolev inequalities. Thus (1.10) readily follows. ∎

REMARK 1.8 : Let $u_n \rightharpoonup 0$ be a P.S. sequence. After extending u_n to be equal to zero outside Ω, an application of Lemma 1.7, with $U = \mathbb{R}^N$, gives

there is a finite set of points, $x_1, \dots, x_m \in \overline{\Omega \setminus C}$, *such that:*

$u_n \to 0$ *in* $H^1_{loc}(\mathbb{R}^N \setminus \{x_1, \dots, x_m\})$,

$a_j := \lim \int_{B_r(x_j)} | u_n |^{2^*} \ge S^{N/2}$ $\forall\, j$ *and* $r > 0$ *small enough* (for some subsequence). In fact (1.8) is easily checked, taking $v = (1 - \varphi)\, u_n$, $\varphi \in C_0^\infty(\mathbb{R}^N)$, $0 \le \varphi \le 1$, in (iii). ∎

REMARK 1.9 : Let $u_n \rightharpoonup 0$ be a P.S. sequence. If $u_n \not\to 0$ in $H_0^1(\Omega)$, necessarily

(1.11) $$\lim E(u_n) \ge (1/N)\, S^{N/2}.$$

In fact, by the previous Remark, $\lim \int |u_n|^{2^*} \ge S^{N/2}$. On the other hand, by Remark 1.5, we have $\lim E(u_n) = (1/N) \lim \int |u_n|^{2^*}$, and (1.11) follows. ∎

Proof of Theorem 1.6 : By Remark 1.5, it amounts to prove

(1.12) $$\lim \int |u_n|^{2^*} = k\, S^{N/2} \qquad \text{for some } k \in \mathbb{N}.$$

In view of Remark 1.8, we can assume there is a finite set of "concentration points" $x_1, \ldots, x_m$ in $\overline{\Omega \setminus C}$, such that

$$\int_K |\nabla u_n|^2 \to 0 \qquad \text{if } K \subseteq \overline{\Omega} \setminus \{x_1, \ldots, x_m\},\ K \text{ compact},$$

$$\lim \int_{B_r(x^\circ)} |u_n|^{2^*} \ge S^{N/2} \qquad \forall j = 1, \ldots, m \quad \text{and } r > 0.$$

In order to prove (1.12) we will use an iteration procedure, which, at each step, reduces the energy by exactly $S^{N/2}$. This will be done "blowing" each singularity x_j. In what follows, we will use quite the same arguments as in Brezis [3] (see also [4], [13]).

To perform the "blowing up" technique, let $\delta \in]0, S^{N/2}[$ be given and let $\varepsilon_n > 0$ be such that

$$(1.13) \qquad \delta \leq \underset{x' \in \overline{B_\rho(x^\circ)}}{\mathrm{Sup}} \int_{B_{\varepsilon_n}(x')} |u_n|^{2^*} \leq S^{N/2} - \delta .$$

Here x° denotes any of the "concentration points" x_j, and ρ is chosen in order $\overline{B_{2\rho}(x^\circ)}$ contains only x° as a concentration point.

Now, let $x_n \in \overline{B_\rho(x^\circ)}$ be such that

$$(1.14) \qquad \int_{B_{\varepsilon_n}(x_n)} |u_n|^{2^*} = \underset{x' \in \overline{B_\rho(x^\circ)}}{\mathrm{Sup}} \int_{B_{\varepsilon_n}(x')} |u_n|^{2^*} .$$

Notice that $\varepsilon_n \to 0$. In fact, if (for a subsequence) $\varepsilon_n \geq \varepsilon^\circ > 0$, by (1.13) we get

$S^{N/2} - \delta \geq \int_{B_{\varepsilon_n}(x^\circ)} |u_n|^{2^*} \geq \int_{B_{\varepsilon^\circ}(x^\circ)} |u_n|^{2^*}$ while $\lim \int_{B_{\varepsilon^\circ}(x^\circ)} |u_n|^{2^*} \geq S^{N/2}$ by assumption.

Also, $x_n \to x^\circ$. In fact, $x_n \to y$ implies $\delta \leq \int_{B_{\varepsilon_n}(x_n)} |u_n|^{2^*} \leq \int_{B_r(y)} |u_n|^{2^*}$ for any given $r > 0$, provided n is sufficiently large. But, if r is small, $\lim \int_{B_r(y)} |u_n|^{2^*} = 0$ if $y \neq x^\circ$ again by assumption.

Now, define

$$\tilde{u}_n(x) = \varepsilon_n^{N/2^*} u_n(x_n + \varepsilon_n x) .$$

Remark that $\tilde{u}_n = 0$ outside

$$\Omega_n := \frac{\Omega - x_n}{\varepsilon_n} .$$

Since $\int_{\mathbb{R}^N} |\nabla \tilde{u}_n|^2 = \int_\Omega |\nabla u_n|^2$ and $\int_{\mathbb{R}^N} |\tilde{u}_n|^{2^*} = \int_\Omega |u_n|^{2^*}$ we can assume there is ω, with $\int_{\mathbb{R}^N} |\nabla \omega|^2 < +\infty$, such that

$$\nabla \tilde{u}_n \rightharpoonup \nabla \omega \qquad \text{weakly in } L^2(\mathbb{R}^N, \mathbb{R}^N), \text{ and}$$

$$u_n \rightharpoonup \omega \qquad \text{weakly in } L^{2^*}(\mathbb{R}^N).$$

Finally, let us set $U := \{ z \in \mathbb{R}^N \mid x_n + \varepsilon_n z \in \Omega \setminus C \quad \forall\, n \text{ large} \}$. Notice that

$$U = \mathbb{R}^N \quad \text{if} \qquad (1/\varepsilon_n)\ \mathrm{dist}\,(x_n, (\Omega \setminus C)^C) \to +\infty ,$$

while $\quad U = \emptyset \quad$ iff $\quad x_n \notin \Omega$ and $(1/\varepsilon_n)\ \mathrm{dist}\,(x_n, \partial\Omega) \to +\infty$ or

$\qquad x_n \in C$ and $(1/\varepsilon_n)\ \mathrm{dist}\,(x_n, \partial C) \to +\infty$.

In case $x^\circ \in \partial\Omega$ and $(1/\varepsilon_n)$ dist $(x_n, \partial\Omega) \to \ell < \infty$, or $x^\circ \in \partial C$ and $(1/\varepsilon_n)$ dist $(x_n, \partial C) \to \ell < \infty$ clearly U is an half-space.

Let us remark that $\omega=0$ a.e. in $\bar{U}^C$. In fact, if $z \notin U$, $B_r(z) \cap \bar{U} = \emptyset$ then, either $B_{r\varepsilon_n}(x_n + \varepsilon_n z) \subset \bar{\Omega}^C$ or $B_{r\varepsilon}(x_n + \varepsilon_n z) \subset \mathring{C}$. In both cases:

$$\lim_n \int_{B_r(z)} |\tilde{u}_n|^{2^*} = \lim_n \int_{B_{\varepsilon_n r}(x_n + \varepsilon_n z)} |u_n|^{2^*} = 0 \ .$$

Using Lemma 1.7 we can exclude the case $\omega \equiv 0$ in $\mathbb{R}^N$. In fact, since, as one can easily check in this case, $\tilde{u}_n$ satisfies (1.8) while (1.10) cannot be satisfied, in view of (1.13), at any point, an application of Lemma 1.7 yields $\tilde{u}_n \to 0$ in $H^1_{loc}(\mathbb{R}^N)$, contraddicting the inequality on the left in (1.13).

The first consequence is that $U \neq \emptyset$; thus, either $U = \mathbb{R}^N$ or U is an half space. Later we will rule out the second alternative.

We are now in position to prove (1.12). Il will require a few steps:

Step 1. $\quad \tilde{u}_n \to \omega \quad$ in $H^1_{loc}(U)$;

Step 2. $\quad -\Delta\omega = \omega^{2^*-1} \quad$ in U, $\omega \in H^1_0(U)$, $\omega > 0$ and hence $U = \mathbb{R}^N$;

Step 3. $\quad \lim \int |u_n - \hat{\omega}_n|^{2^*} = \lim \int ((u_n - \hat{\omega}_n)^+)^{2^*}$

$$\text{where} \quad \hat{\omega}_n(x) = \varepsilon_n^{-N/2^*}\, \omega\left(\frac{x - x_n}{\varepsilon_n}\right) ;$$

Step 4. $u_{1,n}(x) := (u_n - \hat{\omega}_n)^+$ is a P.S. sequence ;

Step 5. Proof of (1.12) concluded.

Proof of Step 1. In order to apply Lemma 1.7 to $\tilde{\eta}_n := \tilde{u}_n - \omega$, let us fix $\varphi \in C_o^\infty(U)$, $0 \le \varphi \le 1$. Notice that

$$v_n := u_n + \varphi\left(\frac{x-x_n}{\varepsilon_n}\right) \left(\varepsilon_n^{-N/2^*} \omega\left(\frac{x-x_n}{\varepsilon_n}\right) - u_n\right)$$

is admissible for (iii) in Definition 1.1, and $(v_n)_n$ is uniformly bounded in $H_o^1(\Omega)$; hence

$$\lim \int_{\mathbb{R}^N} \nabla\tilde{u}_n \nabla (\varphi\tilde{\eta}_n) \quad \le \quad \lim \int_{\mathbb{R}^N} |\tilde{u}_n|^{2^*-1}\, \tilde{\eta}_n\, \varphi \ .$$

Using a Lemma by Brezis and Lieb [5], one can verify that

$$\lim \int_{\mathbb{R}^N} \nabla\tilde{u}_n \nabla (\varphi\tilde{\eta}_n) \quad = \quad \lim \int_{\mathbb{R}^N} |\nabla\tilde{\eta}_n|^{2^*} \varphi$$

and

$$\lim \int_{\mathbb{R}^N} |\, \tilde{u}_n\, |^{2^*-1} \tilde{\eta}_n \varphi \ = \ \lim \ \int_{\mathbb{R}^N} |\tilde{\eta}_n|^{2^*} \varphi \ .$$

Thus Lemma 1.7 applies to get $\tilde{\eta}_n \to 0$ in $H^1_{loc}(U)$, since the inequality

$$\lim \int_{B_r(x)} |\tilde{u}_n - \omega|^{2^*} \ \ge \ S^{N/2}$$

cannot be satisfied for any $x \in U$, in view of (1.13) and the obvious inequality:

$$\lim \int_{B_r(x)} \tilde{u}_n^{2^*} \ \ge \ \lim \int_{B_r(x)} |\tilde{u}_n - \omega|^{2^*} \quad .$$

Proof of Step 2. Standard arguments in variational inequalities insure it is enough to prove

$$(1.15) \quad \int_{B_r(z)} \nabla\omega \, \nabla(\xi - \omega) \ \ge \int_{B_r(z)} \omega^{2^*-1} (\xi - \omega) \qquad \forall\, \xi \in H_o^1(B_r(z)) + \omega|_{B_r(z)} \ , \ \xi \ge 0$$

for every $r > 0$, $z \in U$ for which $B_{2r}(z) \subset U$.Thus, given ξ, we extend it outside $B_r(z)$, setting $\xi = \omega$. Now, given $\vartheta \in C_0^\infty(\mathbb{R}^N)$, $0 \le \vartheta \le 1$, $\vartheta = 1$ on $B_r(z)$, $\vartheta = 0$ outside $B_{2r}(z)$, we see that

$$v_n := u_n + \vartheta\left(\frac{x-x_n}{\varepsilon_n}\right)\left(\varepsilon_n^{-N/2^*}\, \xi\left(\frac{x-x_n}{\varepsilon_n}\right) - u_n\right)$$

is admissible for (iii) in Definition 1.1, and we obtain

$$\int_{\mathbb{R}^N} \nabla\tilde{u}_n \nabla(\xi - \tilde{u}_n)\,\vartheta \ \ge\ o(1) + \int_{\mathbb{R}^N} \tilde{u}_n^{2^*-1} (\xi - \tilde{u}_n)\,\vartheta$$

By $H^1_{loc}(U)$ convergence we can pass to the limit, getting

$$\int_{\mathbb{R}^N} \nabla\omega \nabla(\xi - \omega)\,\vartheta \ \ge\ \int_{\mathbb{R}^N} \omega^{2^*-1} (\xi - \omega)\,\vartheta$$

i.e. (1.15), because $\xi - \omega = 0$ outside $B_r(z)$ and $\vartheta = 1$ on $B_r(z)$.

Furthermore, since $\omega = 0$ outside U, clearly $\omega \in H^1_0(U)$. Since $\omega \not\equiv 0$, as we have noticed before, by Pohozaev identity this implies U cannot be an half space, and hence $U = \mathbb{R}^N$.

Finally, let us recall that ω is uniquely determined (up to translations and changes of scale) and satisfies

$$\int |\nabla\omega|^{2^*} = \int \omega^{2^*} = S^{N/2} .$$

Proof of Step 3. It is enough to observe that

$$\int_\Omega |u_n - \hat{\omega}_n|^{2^*} = \int_{\mathbb{R}^N} |\tilde{u}_n - \omega|^{2^*} = \int_{\mathbb{R}^N} [(\tilde{u}_n - \omega)^+]^{2^*} + \int_{\mathbb{R}^N} [(\omega - \tilde{u}_n)^+]^{2^*}$$

Since $0 \le (\omega - \tilde{u}_n)^+ \le \omega$ and $\omega - \tilde{u}_n \to 0$ a.e. in $\mathbb{R}^N$, the claim follows by Lebesgue Theorem.

Proof of Step 4. First of all, remark that $u_{1,n} \in K$. Let $\eta_n = u_n - \hat{\omega}_n$, so that $u_{1,n} = \eta_n \vee 0$. We will prove later that:

$$\lim \int_\Omega \nabla\eta_n \nabla(\varphi - \eta_n) \geq \lim \int_\Omega |\eta_n|^{2^*-2} \eta_n (\varphi - \eta_n) \tag{1.16}$$

uniformly for φ on bounded subsets of K. Choosing $\varphi = \eta_n \vee 0$ in (1.16) and setting $\tilde{\eta}_n = \tilde{u}_n - \omega$ we get

$$\lim \int_\Omega \nabla\eta_n \nabla(\eta_n \wedge 0) \leq \lim \int_{\mathbb{R}^N} |\tilde{\eta}_n|^{2^*-2} \tilde{\eta}_n (\tilde{\eta}_n \wedge 0) = 0$$

by Lebesgue Theorem, since $-\omega \leq \tilde{\eta}_n \wedge 0 \leq 0$. Thus we can replace η_n by $\eta_n \vee 0 = u_{1,n}$ in (1.16) and this completes the proof of Step 4.

Inequality (1.16) follows by Step 2, since

$$\int_\Omega \nabla\eta_n \nabla(\varphi - \eta_n) = \int_\Omega \nabla u_n \nabla(\varphi - u_n) + \int_{\mathbb{R}^N} \nabla\tilde{u}_n \nabla\omega - \int_{\mathbb{R}^N} \nabla\omega \nabla(\tilde{\varphi}_n - \tilde{\eta}_n) \geq$$

$$\geq \int_\Omega u_n^{2^*-1} (\varphi - u_n) + \int_{\mathbb{R}^N} |\nabla\omega|^2 - \int_{\mathbb{R}^N} \omega^{2^*-1} (\tilde{\varphi}_n - \tilde{\eta}_n) + o(1) =$$

$$= \int_\Omega u_n^{2^*-1} (\varphi - \eta_n) - \int_{\mathbb{R}^N} \tilde{u}_n^{2^*-1} \omega + \int_{\mathbb{R}^N} \omega^{2^*} - \int_\Omega \hat{\omega}_n^{2^*-1} (\varphi - \eta_n) + o(1) =$$

$$= \int_\Omega (u_n^{2^*-1} - \hat{\omega}_n^{2^*-1}) (\varphi - \eta_n) + o(1)$$

where the o(1) are uniform on φ and, as usual,

$$\tilde{\varphi}_n(x) = \varepsilon_n^{N/2^*} \varphi(\varepsilon_n x + x_n)$$

Since $\varphi - \eta_n$ is uniformly bounded in $L^{2^*}(\Omega)$ for φ on bounded subsets of $H_o^1(\Omega)$, it is enough to prove:

$$z_n := (\tilde{u}_n^{2^*-1} - \omega^{2^*-1}) - |\tilde{u}_n - \omega|^{2^*-2} (\tilde{u}_n - \omega) \to 0 \text{ in } L^{\frac{2N}{N+2}}(\mathbb{R}^N).$$

times, if $k\, S^{N/2} \leq \lim \int |u_n|^{2^*} < (k+1)\, S^{N/2}$, obtaining, for the k^{th} iterate $u_{k,n}$, the equality

(1.17) $$\lim \int |u_{k,n}|^{2^*} = \lim \int |u_n|^{2^*} - kS^{N/2}.$$

This implies $\lim \int |u_{k,n}|^{2^*} < S^{N/2}$. Thus $u_{k,n}$ is a P.S. sequence satisfying

$$\lim E(u_{k,n}) = (1/N) \lim \int |u_{k,n}|^{2^*} < (1/N)\, S^{N/2}.$$

An application of Remark 1.9 yields $u_{k,n} \to 0$ in $H^1_o(\Omega)$ and (1.12) follows from (1.17). ∎

REMARK 1.11: From the results in this Section it follows that if $u_n \rightharpoonup u$ is a P.S. sequence (u non necessarily zero) and $E(u_n) \to c$, then

(1.18) $$E(u_n) = E(u) + (k/N)\, S^{N/2} + o(1) \qquad \text{for some } k \in \mathbb{N} \cup \{0\}.$$

Also, $k=0$ if and only if $u_n \to u$ strongly. In order to prove (1.18) consider the sequence $\vartheta_n := u_n - u$ and use Proposition 1.1 to verify

(1.19) $$\lim \int \nabla\vartheta_n \nabla(\varphi - \vartheta_n) \geq \lim \int |\vartheta_n|^{2^*-2} \vartheta_n (\varphi - \vartheta_n)$$

uniformly with respect to φ on bounded subsets of **K**. From (1.19) follows, choosing $v = \vartheta_n \vee 0$ as test function,

(1.20) $$\lim \int |\nabla(\vartheta_n \wedge 0)|^2 \leq \lim \int |\vartheta_n \wedge 0|^{2^*} = 0$$

by Lebesgue theorem, since $-u \leq \vartheta_n \wedge 0 \leq 0$. Thus, we can replace ϑ_n by $\vartheta_n \vee 0$ in (1.19), and this proves that $(\vartheta_n \vee 0)_n$ is a P.S. sequence for Problem 1. Thus Theorem 1.6 implies that $E(\vartheta_n \vee 0) = (k/N)\, S^{N/2}$ for some $k \in \mathbb{N} \cup \{0\}$, with $k=0$ iff $\vartheta_n \vee 0 \to 0$ i.e., by (1.20), $\vartheta_n \to 0$. Now, from (1.20) and Taylor's expansion formula we easily get (1.18).

In particular, this result implies that the energy functional

$$f(u) := \begin{cases} E(u) & \text{if } u \in \mathbf{K} \\ +\infty & \text{otherwise in } H^1_0(\Omega) \end{cases}$$

verifies P.S. condition (in the sense of Szulkin [14]) at every energy level except for those of the form $E(u) + (k/N)\, S^{N/2}$, where u is a solution to Problem 1 and $k \geq 1$ an integer. ∎

2. *The existence Theorem.*

In this Section we will use a Min-Max principle in order to get the existence of a non trivial solution to Problem 1. More precisely, we prove that if $u \equiv 0$ is the only solution to Problem 1 with energy less than $(1/N)\, S^{N/2}$ and the set $\overline{\Omega \setminus C}$ verifies a geometrical assumption (as in Coron [8]), then there exists a critical point of "saddle type" for the functional f with energy in $](1/N)S^{N/2}, (2/N)S^{N/2}[$. Notice that by Remark 1.11, under this hypothesis f verifies P.S. condition in this interval .

In order to prove our existence theorem, we will construct, following Coron [8], a continuous map g° defined on an N+1 - dimensional cylinder Z with values in **K**, such that

$$c^\circ := \operatorname*{Sup}_{\partial Z} f(g^\circ) \geq \frac{1}{N}\, S^{N/2}$$

Then, we define

$$\Sigma := \{ g \in C^\circ(Z, H^1_0) \mid g_{|\partial Z} = g^\circ{}_{|\partial Z} \},$$

$$c := \operatorname*{Inf}_{\Sigma} \operatorname*{Sup}_{Z} f(g)$$

and prove that

$$c^\circ < c < (2/N)\, S^{N/2}.$$

Since f verifies P.S. condition in a neighbourhood of c, an application of the deformation Lemma by Szulkin [14] for functionals of the form C^1 + convex-proper - lower semicontinuous gives the existence of a critical point at the level c, and this will complete the proof of the following

THEOREM 2.1: *If* Ω, C *verify: there exist* $x^\circ \in \mathbb{R}^N$ *and* $R_2 > R_1 > 0$ *such that*

$$\{x \in \mathbb{R}^N \mid R_1 \le |x - x^\circ| \le R_2\} \subset \Omega \setminus C$$

$$\{x \in \mathbb{R}^N \mid |x - x^\circ| \le R_1\} \not\subset \overline{\Omega \setminus C}$$

and R_2/R_1 *is large enough, then Problem 1 has a non trivial solution.*

Proof. First of all we remark as in [8] that we can suppose

$$x^\circ = 0 \quad , \quad R_1 = \alpha^{-1} \quad , \quad R_2 = \alpha$$

for some $\alpha > 1$, so that the hypothesis " R_2/R_1 large enough" in Theorem 2.1 means " α large enough".

For the construction of the map g° we will use the functions

$$\Gamma : H^1_o(\Omega) \to \mathbb{R} \quad , \quad \Gamma(u) = \int |\nabla u|^2 - \int |u|^{2^*}$$

$$F : H^1_o(\Omega) \to \mathbb{R}^N \quad , \quad F(u) = S^{-N/2} \int x\, |\nabla u|^2\, dx$$

As an immediate consequence of the Concentration-Compactness Lemma by P.L. Lions [12], we get the following

LEMMA 2.2: *For every neighbourhood* V *of* $\overline{\Omega \setminus C}$ *there exist some* $\varepsilon > 0$ *s.t.*

$$u \ne 0 \ , \ \Gamma(u) = 0 \ , \ f(u) \le (1/N)\, S^{N/2} + 2\varepsilon \quad \Longrightarrow \quad F(u) \in V.$$

Now, fix a point $a^\circ \notin \overline{\Omega \setminus C}$, $|a^\circ| < \alpha^{-1}$ and a compact neighbourhood V of $\overline{\Omega\setminus C}$ such that $a^\circ \notin V$, and correspondingly fix $\varepsilon > 0$ as in Lemma 2.2, in such a way that

$$\sigma + \xi \neq a^\circ \ \text{ if } \ |\sigma| = 1, \ |\xi| \leq \varepsilon.$$

Let ω be the unique positive and radially symmetric (around the origin) solution of

$$(2.1) \qquad -\Delta\omega = \omega^{2^*-1} \quad \text{on } \mathbb{R}^N , \qquad \int_{\mathbb{R}^N} |\nabla\omega|^2 < +\infty$$

and let

$$\omega_t^\sigma = (1-t)^{-N/2^*}\, \omega\left(\frac{x - t\sigma}{1-t}\right)$$

for $t \in [0,1[$, $\sigma \in \partial B^N$, where $B^N = \{ \xi \in \mathbb{R}^N \mid |\xi| \leq 1 \}$. Then, ω_t^σ solves (2.1) and for every σ, t it results

$$\int_{\mathbb{R}^N} |\nabla \omega_t^\sigma|^2 = \int_{\mathbb{R}^N} (\omega_t^\sigma)^{2^*} = S^{N/2}$$

If α is large, we can find, as in [8], a cut-off function $\varphi \in C_0^\infty(\Omega)$ with support in $\Omega\setminus C$, such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on a neighbourhood of ∂B^N, and such that the functions:

$$v_t^\sigma := \frac{\|\varphi\omega_t^\sigma\|^{N/2^*}}{|\varphi\omega_t^\sigma|_{2^*}^{N/2}} (\varphi\omega_t^\sigma) , \quad v_t^\sigma \in K$$

verify:

$$(2.2) \qquad f(v_t^\sigma) \leq (2/N)\, S^{N/2} - \varepsilon \quad \forall\, \sigma \in \partial B^N , \ \forall\, t \in [0,1[,$$

$$(2.3) \qquad f(v_{t^\circ}^\sigma) \leq (1/N)\, S^{N/2} + \varepsilon \quad \forall\, \sigma \in \partial B^N ,$$

$$(2.4) \qquad |F(v_{t^\circ}^\sigma) - \sigma| < \varepsilon .$$

for t° large enough. Remark that since $\Gamma(v_t^\sigma) = 0 \quad \forall\, \sigma, \ \forall\, t$, from (2.2), (2.3) it follows

$$\frac{1}{N}\, S^{N/2} < \operatorname*{Max}_{\mu>0} f(\mu v_t^\sigma) = f(v_t^\sigma) = \frac{1}{N} \| v_t^\sigma \|^2 \le \frac{2}{N}\, S^{N/2} - \varepsilon \quad \forall\sigma \quad \forall t, \text{ and}$$

$$\frac{1}{N}\, S^{N/2} < \operatorname*{Max}_{\mu>0} f(\mu v_{t^\circ}^\sigma) = f(v_{t^\circ}^\sigma) \le \frac{1}{N}\, S^{N/2} + \varepsilon \qquad \forall\, \sigma.$$

Moreover, if $\lambda > 1$ is big enough then

$$f(\lambda v_t^\sigma) < 0 \quad \forall\, \sigma \in \partial B^N \ , \ \forall\, t \in [0,1[.$$

Now we can define our "boundary data" $g^\circ : Z := [0,1] \times B^N \to \mathbf{K}$ by setting:

$$g^\circ(s, t\sigma) := \lambda\, s\, v_{tt^\circ}^\sigma$$

for $s \in [0,1]$, $t \in [0,1]$, $\sigma \in \partial B^N$. Remark that g° is well-defined and continuous on Z

since for t = 0 v_0^σ does not depend on σ. By observations above, we have

$$c^\circ = \operatorname*{Sup}_{\partial Z} f(g^\circ) \le \frac{1}{N}\, S^{N/2} + \varepsilon \qquad \text{and} \qquad \operatorname*{Sup}_{Z} f(g^\circ) \lneqq \frac{2}{N}\, S^{N/2}$$

Thus, to conclude the proof of the Theorem is enough to verify:

$$\operatorname*{Sup}_{Z} f(g) \ge \frac{1}{N} S^{N/2} + 2\varepsilon \qquad \text{for every} \quad g \in C^\circ(Z, H_\circ^1)\ , \ g_{|\partial Z} = g^\circ{}_{|\partial Z}\, .$$

Suppose by contradiction that there exists a $g \in C^\circ(Z,\mathbf{K})$ such that $g = g^\circ$ on ∂Z,

(2.5) $$f(g(s,\xi)) \le (1/N)\, S^{N/2} + 2\varepsilon \qquad \forall\, (s,\xi) \in Z$$

and consider the map

$$\begin{cases} G : Z \to \mathbb{R}^{N+1} \ , \\ G(s,\xi) = (s, F(g(s,\xi))\, . \end{cases}$$

We claim that

$$\deg\,(G, Z, (\lambda^{-1}, a^\circ)\,) = 1$$

since the map

$$\begin{cases} H : [0,1] \times Z \to \mathbb{R}^{N+1} \\ H(\,t\,; s, \xi) := t\, G(s,\xi) + (1 - t\,)(s,\xi) = (s, \ t\, F(g(s,\xi)) + (1 - t\,)\, \xi\,) \end{cases}$$

is an admissible homotopy between G and Id_Z. In fact if $H(\,t\,; s, \xi) = (\,\lambda^{-1}, a^\circ)$

then necessarily $s = \lambda^{-1}$ and $\xi \notin \partial B^N$ since $\forall\ \sigma \in \partial B^N$

$$t\,F(g(\lambda^{-1}, \sigma)) + (1-t)\,\sigma = t\,(F(v^{\sigma}) - \sigma) + \sigma \neq a^{\circ} \qquad \text{because of (2.4).}$$

Let us define the sets:

$$Z^{+} = \{ (s, \xi) \in Z \mid \Gamma(g(s, \xi)) > 0 \} \cup \{ (0, \xi) \mid \xi \in B^N \}$$

$$Z^{-} = \{ (s, \xi) \in Z \mid \Gamma(g(s, \xi)) < 0 \}$$

$$Z^{\circ} = \{ (s, \xi) \in Z \mid \Gamma(g(s, \xi)) = 0 \, , \, s > 0 \}.$$

Notice that Z^+ is open in Z and Z° is closed in Z since $\Gamma(u) > 0$ if $u \neq 0$ and $\| u \|$ is small. Moreover

$$(2.6) \qquad \begin{cases} (s,\xi) \in Z^{+} & \text{for } (s,\xi) \in \partial Z \, , \; 0 \leq s < \lambda^{-1} \\ (\lambda^{-1},\xi) \in Z^{\circ} & \text{for } \xi \in \partial B^N \\ (s,\xi) \in Z^{-} & \text{for } (s,\xi) \in \partial Z \, , \; \lambda^{-1} < s \leq 1 \end{cases}$$

By Lemma 2.2 and (2.5) we have that $F(g(Z^{\circ})) \subset V$ and in particular

$$(2.7) \qquad F(g\,(s, \xi)) \neq a^{\circ} \qquad \forall\ (s, \xi) \in Z^{\circ}.$$

Hence, by excision property we have

$$1 = \deg\,(G, Z, (\lambda^{-1}, a^{\circ})) = \deg\,(G, Z^{+}, (\lambda^{-1}, a^{\circ})) + \deg\,(G, Z^{-}, (\lambda^{-1}, a^{\circ}))$$

while on the other hand we shall prove that

$$(2.8) \qquad \deg\,(G, Z^{+}, (\lambda^{-1}, a^{\circ})) = 0\,;$$

$$(2.9) \qquad \deg\,(G, Z^{-}, (\lambda^{-1}, a^{\circ})) = 0$$

getting in this way a contradiction which proves Theorem 2.1.

<u>Proof of (2.8)</u> : Fix $R > \lambda^{-1}$ such that $y \in \mathbb{R}^{N+1}$, $| y | \geq R \Rightarrow y \notin G(Z)$, and consider the path

$$p : [0,1] \to \mathbb{R}^{N+1} \, , \quad p(t) = (t\,R + (1-t)\,\lambda^{-1}, a^{\circ}) .$$

We claim that $p(t) \notin G(\partial Z^{+})$ for every t. Suppose this is not the case; then there exist $t \in [0,1]$ and $(s,\xi) \in \partial Z^{+}$ such that

$$(\,t\,R + (1-t)\,\lambda^{-1}, a^{\circ}) = (\,s, F(g(s,\xi))\,) .$$

We first deduce that $s \geq \lambda^{-1}$; on the other hand, from $F(g(s,\xi)) = a^\circ$ and (2.7) it follows that $(s,\xi) \notin Z^\circ$. Since $\partial Z^+ \subset \partial Z \cup Z^\circ$ we conclude that the only possibility is: $\xi \in \partial Z$ and $(s, \xi) \in Z^+$ which implies, together with (2.6), $s < \lambda^{-1}$, in contrast with $s \geq \lambda^{-1}$.

Since $p(\cdot)$ is admissible, we have that $\deg(G, Z^+, p(t))$ does not depend on t, and hence

$$\deg(G, Z^+, (\lambda^{-1}, a^\circ)) = \deg(G, Z^+, (R, a^\circ)) = 0 \quad \text{since } (R, a^\circ) \notin G(Z).$$

Formula (2.9) can be proved in the same way, observing that the path

$$q : [0,1] \to \mathbb{R}^{N+1}, \quad q(t) = (-t R + (1-t)\lambda^{-1}, a^\circ)$$

is admissible for the degree, and thus

$$\deg(G, Z^-, (\lambda^{-1}, a^\circ)) = \deg(G, Z^-, (-R, a^\circ)) = 0$$

∎

3. *A regularity remark.*

Before stating our regularity result, we point out some properties of solutions to Problem 1.

<u>**PROPOSITION 3.1**</u>. *If* u *solves Problem 1, then*

(3.1) $$\int \nabla u \, \nabla(v-u) \geq \int u^{2^*-1}(v-u) \qquad \forall\, v \in H^1_\circ(\Omega),\ v \leq \psi \text{ on } C.$$

Proof. Let us set $f := u^{2^*-1} \in L^{2N/(N+2)}(\Omega)$, and let w be the unique solution of:

(3.2) $$\begin{cases} w \in H^1_\circ(\Omega), \quad w \leq \psi \text{ on } C \\ \int \nabla w \, \nabla(v-w) \geq \int f(v-w) \qquad \forall\, v \in H^1_\circ,\ v \leq \psi \text{ on } C. \end{cases}$$

In order to prove that $w = u$, we observe first of all that $w \geq 0$ in Ω (i.e. $w \in \mathbf{K}$); in fact, choosing $v = w \vee 0$ as test function in (3.2) we get

$$\int |\nabla(w \wedge 0)|^2 = \int \nabla w \, \nabla(w \wedge 0) \leq \int f \, (w \wedge 0)$$

and hence $w \wedge 0 = 0$, since $f \geq 0$ a.e. in Ω.

Thus Proposition 3.1 follows from uniqueness for the linear variational inequality:

$$u \in \mathbf{K} \quad , \quad \int \nabla u \, \nabla(v-u) \geq \int f \, (v-u) \quad \forall \, v \in \mathbf{K} \; .$$

∎

From Proposition 3.1 it follows immediately that u is a weak solution of the equation

(3.3) $\qquad - \Delta u = u^{2^*-1} \quad$ in $\;\Omega \setminus C$.

We are now in position to state and prove our regularity result:

THEOREM 3.2: *If* $\psi \in C^0(\bar{\Omega}) \cap H^1(\Omega)$ *and* u *solves Problem 1, then u is continuous in* Ω.

Proof. We first prove that $u \in L^\infty(\Omega)$. Let $\mathcal{u}$ be the unique solution of:

$$- \Delta \mathcal{u} = 0 \;\text{ in } \Omega \setminus C, \qquad \mathcal{u} = u \;\text{ on }\; \partial(\Omega \setminus C).$$

From (3.3) it follows that the function $z := u - \mathcal{u}$ solves

$$\text{(3.4)} \qquad \begin{cases} - \Delta z = a(x) \, z + g & \text{in } \Omega \setminus C \\ u \in H^1_o(\Omega \setminus C) \end{cases}$$

where $a := u^{2^*-2} \in L^{N/2}$, $g := u^{2^*-2} \, \mathcal{u} \in L^{N/2}$, since $\mathcal{u} \in L^\infty$ by the maximum principle. The boundness of u is a consequence of the following Lemma, which is essentially contained in [6] (see also [7], Lemma 1.5):

LEMMA 3.3 : *Suppose* $a \in L^{N/2}$, $g \in L^q$ *with* $q \geq N/2$ *and* z *solves (3.4). Then* $\quad z \in L^\tau \quad \forall \, \tau < \infty$.

Applying Lemma 3.3 we easily get $u \in L^\infty(\Omega \setminus C)$ and finally, since $0 \le u \le \psi$ in C, we can conclude that $u \in L^\infty(\Omega)$.

We now set $f := u^{2^*-1} \in L^\infty(\Omega)$, $w := h - u$, where h solves

$$-\Delta h = f \quad \text{in } \Omega, \quad h = 0 \quad \text{on } \partial\Omega.$$

Using Proposition 3.1 it is easy to verify that w is the unique solution of the linear variational inequality:

$$\begin{cases} w \in H^1_\circ(\Omega) \ , \quad w \ge h - \psi \quad \text{on } C \\ \int \nabla w \, \nabla(v-w) \ \ge 0 \qquad \forall\, v \in H^1_\circ(\Omega) \ , \ v \ge h - \psi \ \text{ on } C. \end{cases}$$

Since $h - \psi$ is continuous on $\overline{\Omega}$, an application of a Theorem by Lewy-Stampacchia ([11], Part II) gives the continuity of w, and the theorem is proved.

∎

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Part 2:

VARIATIONAL PROBLEMS WITH NON CONVEX CONSTRAINTS

Abstract. In the second part of this Thesis we present two examples of Variational Problems with non convex constraints. The first example concerns the problem of the existence of a surface with prescribed boundary and prescribed mean curvature, which is spanned over an obstacle in $\mathbb{R}^3$.

Secondly, we illustrate an example of an obstacle problem for harmonic maps in $\mathbb{R}^3$, which will introduce the papers "*Surfaces of Minimal Area Enclosing a Given Body in* $\mathbb{R}^3$", "S^2 *- Type Minimal Surfaces Enclosing Many Obstacles in* $\mathbb{R}^3$".

H - SUPERFICI CON OSTACOLO

0. *Introduzione.*

Il problema dell'esistenza di superfici a curvatura media costante $H > 0$ aventi per contorno una preassegnata curva di Jordan Γ ha destato l'interesse di diversi Autori (vedi ad esempio [5], [6], [14], [9], [10], [15], [3]). Tuttavia solo agli inizi degli anni '80 Brezis e Coron in [2] e Struwe in [11] giungono a dimostrare l'esistenza di due soluzioni distinte per ogni costante H sufficientemente piccola, risolvendo così un problema proposto da Rellich e rimasto aperto per lungo tempo.

La formulazione parametrica di questo problema porta allo studio del seguente sistema di equazioni differenziali:

$$\text{Trovare } u \in C^2(U,\mathbb{R}^3) \cap C^\circ(\overline{U},\mathbb{R}^3) \quad \text{tale che}$$

$$(0.1) \qquad \begin{cases} \Delta u = 2H\, u_x \wedge u_y & \text{in } U \\ u_x \cdot u_y = 0 \quad , \quad |u_x| = |u_y| & \text{in } U \\ u(\partial U) = \Gamma \end{cases}$$

dove $U = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \}$ indica il disco unitario.

Per il corrispondente problema con ostacolo si conosce solamente un risultato di esistenza di Tomi ([12]), del 1971. Dal punto di vista geometrico si tratta di risolvere il seguente problema: trovare una superficie C^1, avente per contorno una data curva di Jordan Γ, che stia sopra un ostacolo rigido e tale che la parte di superficie che non tocca l'ostacolo abbia curvatura media H. Dovremo dunque risolvere:

Trovare $u \in C^{1,\beta}(U,\mathbb{R}^3) \cap C^0(\bar{U},\mathbb{R}^3)$ e C chiuso in $\bar{U}$ tali che

Problema 1

$$\begin{cases} \Delta u = 2H\, u_x \wedge u_y & \text{in } U \setminus C \\ u^3 = \psi(u^1,u^2) \ \text{ in } C \ , \quad u^3 \geq \psi(u^1,u^2) & \text{in } U \\ u_x \cdot u_y = 0 \ , \quad |u_x| = |u_y| & \text{in } U \\ u(\partial U) = \Gamma & \end{cases}$$

La mappa $\psi : \mathbb{R}^2 \to \mathbb{R}$ rappresenta l'ostacolo, e la curva di Jordan Γ è data in modo che

(0.2) $$z^3 \geq \psi(z^1,z^2) \qquad \forall\, z = (z^1,z^2,z^3) \in \Gamma.$$

Nella Sezione 2 presenteremo un nuovo risultato di esistenza che generalizza quello ottenuto da Wente [14] per il problema senza ostacolo (Teorema 2.4).

Definita in modo opportuno la classe $\mathcal{A}(\Gamma,\psi)$ delle funzioni ammissibili per il Problema 1 (vedi Sezione 1), dimostreremo l'esistenza di una soluzione qualora siano verificate le seguenti ipotesi:

(0.3) $$\psi \in C^3(\mathbb{R}^2,\mathbb{R}) \ , \quad |\nabla\psi| \in L^\infty(\mathbb{R}^2,\mathbb{R}) \ ;$$

(0.4) $$\operatorname*{Inf}_{u \in \mathcal{A}(\Gamma,\psi)} \int_U |\nabla u|^2 \leq \frac{2}{3} \frac{2\pi}{H^2}$$

Nel caso in cui valgano le (0.3), il problema di minimo in (0.4) ha una soluzione $h_{\Gamma,\psi} \in C^{1,\beta}$ ([12], Theorem 2) che rappresenta la superficie di area minima di bordo Γ contenuta nel sopragrafico di ψ, e che dunque risolve il Problema 1 nel caso H = 0. Inoltre, quando ψ è convessa, una applicazione del principio di massimo ([7]) mostra che $h_{\Gamma,\psi} = h_\Gamma$ è la superficie minima di bordo Γ.

Tomi in [12] dimostra l'esistenza di una soluzione del Problema 1 sotto l'ipotesi

$$H\,|\,h_\Gamma\,|_\infty \leq 1 \tag{0.5}$$

qualora valgano le (0.3) e ψ soddisfi un'ulteriore condizione di tipo geometrico. Le ipotesi (0.4), (0.5) non sono confrontabili. Tuttavia, nel caso in cui Γ rappresenti un circolo, un risultato di non esistenza di Heinz [5] mostra che la (0.5) è ottimale, mentre questo stesso esempio lascia supporre che la costante 2/3 nella (0.4) sia migliorabile.

In [13], Tomi propone una prima generalizzazione del Problema 1, provando alcuni risultati di regolarità C^1 per problemi della forma:

Trovare $u \in H^1(U,\mathbb{R}^3) \cap C^\circ(U,\mathbb{R}^3)$ e C chiuso in U tali che

Problema 1'
$$\begin{cases} \Delta u = 2H\, u_x \wedge u_y & \text{in } U \setminus C \\ F(\xi,u(\xi)) = 0 \ \text{ per } \xi \in C \quad , \quad F(\xi,u(\xi)) \geq 0 & \text{in } U \\ u_x \cdot u_y = 0 \quad , \quad |\,u_x\,| = |\,u_y\,| & \text{in } U \\ u(\partial U) = \Gamma & \end{cases}$$

dove $F : U \times \mathbb{R}^3 \to \mathbb{R}$ è una funzione assegnata.

Nell'ultima parte dell'articolo mostreremo come il metodo variazionale scelto per la dimostrazione dell'esistenza di una soluzione del Problema 1 si può estendere allo studio del Problema 1'. In questo modo, giungeremo a stabilire un risultato di esistenza per 1' nel caso in cui la funzione F sia concava nella seconda variabile.

1. *Notazioni e preliminari.*

Indichiamo con $H^1(U)$ o semplicemente con H^1 l'usuale spazio di Sobolev $H^1(U,\mathbb{R}^3)$. Nel seguito, $\|\cdot\|$ denoterà la norma in H^1 e $|\cdot|_p$ la norma in $L^p = L^p(U)$ per $p\in[1,\infty]$. Sia $\Omega \subseteq U$ un aperto; definiamo i funzionali

$$D_\Omega(u) = \int_\Omega |\nabla u|^2 \qquad \text{per } u\in H^1(\Omega)\,,$$

$$Q_\Omega(u) = \int_\Omega u\cdot u_x \wedge u_y \quad \text{per } u\in H^1 \cap L^\infty(\Omega)\,.$$

Per $\Omega = U$ scriveremo semplicemente D, Q al posto di D_U, Q_U. Per $g\in H^1$, indichiamo con $g + H^1_o(\Omega)$ lo spazio delle $u\in H^1(\Omega)$ tali che $u - g\in H^1_o(\Omega)$. Per un noto risultato di Wente [14], Q_Ω si estende con continuità ed in un unico modo su $g + H^1_o(\Omega)$, per ogni fissato dato al bordo g.

Indichiamo infine con $L(\cdot\,,\cdot)$ l'estensione ad $H^1(U)\times H^1(U)$ del funzionale

$$L(v,h) = \int_U v\cdot h_x \wedge h_y \qquad \text{per } v\in H^1\cap L^\infty\,,\ h\in H^1\,.$$

Ricordiamo brevemente alcune proprietà dei funzionali L , Q :

<u>**PROPOSIZIONE 1.1**</u> **(disuguaglianze isoperimetriche) :**

(i) $\qquad 4\sqrt{2\pi}\,|Q_\Omega(v)| \le |D_\Omega(v)|^{3/2} \qquad \forall\, v\in H^1_o(\Omega)$;

(ii) *Se* $u, h\in H^1(U)$ *hanno traccia continua su* $\partial\Omega$, *e* $u(\partial\Omega) = h(\partial\Omega)$, *allora*
$4\sqrt{2\pi}\,|Q_\Omega(u) - Q_\Omega(h)| \le |D_\Omega(u) + D_\Omega(h)|^{3/2}$;

(iii) *Se* $\overline{\Omega} \subset U$, e $u, k\in H^1(U)$, $u - k\in H^1_o(\Omega)$, *allora*
$4\sqrt{2\pi}\,|Q_\Omega(u) - Q_\Omega(k)| \le |D_\Omega(u) + D_\Omega(k)|^{3/2}$;

(iv) $\qquad 4\sqrt{2\pi}\,|L(u,v)| \le D(u)^{1/2}\, D(v)$
$\qquad 2\sqrt{2\pi}\,|L(v,u)| \le D(v)^{1/2}\, D(u) \qquad \forall\, u\in H^1,\ v\in H^1_o$.

Dimostrazione. Per la **(i)** rimandiamo a [1] (vedi anche [2], Lemma A.8).

(ii): La proposizione è dimostrata per $u, h \in H^1(\Omega) \cap C^\circ(\bar{\Omega})$ (vedi [14], Theorem 2.5). Se u,h non sono continue, fissiamo $u_n, h_n \in H^1(\Omega) \cap C^\circ(\bar{\Omega})$ tali che $u_n - u \to 0$ in $H^1_o(\Omega)$, $h_n - h \to 0$ in $H^1_o(\Omega)$. Poichè il funzionale Q_Ω è continuo su $u + H^1_o(\Omega)$, $h + H^1_o(\Omega)$, avremo:

$$Q_\Omega(u_n) = Q_\Omega(u) + o(1) \quad , \quad Q_\Omega(h_n) = Q_\Omega(h) + o(1) .$$

Possiamo pertanto passare al limite nella:

$$4\sqrt{2\pi}\, | Q_\Omega(u_n) - Q_\Omega(h_n)| \leq | D_\Omega(u_n) + D_\Omega(h_n) |^{3/2}$$

per ottenere la tesi.

(iii): Per (ii), la proposizione è dimostrata nel caso in cui $u \in C^\circ(U)$. Se u non è continua, estendiamo k con u fuori da Ω, in modo che $k \in u + H^1_o(U)$. Sia

$$u_n \in C^\circ(U) \cap H^1(U) \quad \text{t.c. } u_n - u \to 0 \text{ in } H^1_o(U) \text{ , e sia}$$

$$v_n \in C^\circ(\bar{\Omega}) \cap H^1_o(\Omega) \quad \text{t.c. } v_n \to u - k \text{ in } H^1_o(\Omega).$$

Estesa v_n con 0 fuori da Ω, si consideri la successione

$$k_n := u_n - v_n \in u + H^1_o(U).$$

Allora $k_n = u_n$ fuori da Ω, e $k_n \to k$ in $H^1(U)$. Applicando la (ii) a u_n, k_n si ha

$$4\sqrt{2\pi}\, | Q_\Omega(u_n) - Q_\Omega(k_n)| \leq | D_\Omega(u_n) + D_\Omega(k_n) |^{3/2} \tag{1.1}$$

Il funzionale Q è continuo su $u + H^1_o(U)$ e pertanto

$$| Q_\Omega(u_n) - Q_\Omega(k_n)| = | Q(u_n) - Q(k_n)| = | Q(u) - Q(k)| + o(1) =$$
$$= | Q_\Omega(u) - Q_\Omega(k)| + o(1) .$$

Passando al limite nella (1.1) si ottiene dunque la tesi.

Per la dimostrazione della **(iv)** si rimanda a [14], Theorem 3.3. ∎

Il funzionale Q non è continuo rispetto alla topologia debole di H^1. Possiamo tuttavia evidenziare alcune proprietà di compattezza che risulteranno utili in seguito.

PROPOSIZIONE 1.2: *Siano* $v^n \in H_o^1$, $h^n \in H^1 \cap L^\infty$ *due successioni date, tali che* $v^n \rightharpoonup v$ *debolmente in* H_0^1 , $h^n \to h$ *in* L^∞ *e debolmente in* H^1. *Allora:*

a) $Q(v^n) = Q(v) + Q(v^n - v) + o(1)$;

b) $L(h,v^n) = L(h,v) + o(1)$;

c) $L(v,h^n) = L(v,h) + o(1)$;

d) $L(h^n,v^n) = L(h,v) + o(1)$;

e) $L(v^n,h^n) = L(v,h) + o(1)$.

Dimostrazione. Per le **a), b)** rimandiamo a [2], Lemma A.12 e Lemma A.9.

c): Per densità (cfr. Proposizione 1.1, iv) possiamo supporre $v \in H_o^1 \cap L^\infty$. In tal caso, dal Lemma A.5 in [2] segue:

$$2\,L(v,h^n) = 2 \int v \cdot h_x^n \wedge h_y^n = \int h^n \cdot [v_x \wedge h_y^n + h_x^n \wedge v_y] .$$

Poichè $v_x \wedge h_y^n + h_x^n \wedge v_y \rightharpoonup v_x \wedge h_y + h_x \wedge v_y$ debolmente in L^1, e $h^n \to h$ in L^∞, usando nuovamente [2], Lemma A.5 otteniamo

$$2\,L(v,h^n) = \int h \cdot [v_x \wedge h_y + h_x \wedge v_y] + o(1) = 2\,L(v,h) + o(1).$$

d): Dalla proposizione b) otteniamo direttamente

$$\begin{aligned} | L(h^n,v^n) - L(h,v)| &\le | L(h,v^n) - L(h,v)| + | L(h^n - h, v^n)| \\ &\le o(1) + \text{cost. } |h^n - h|_\infty\, D(v^n) = o(1). \end{aligned}$$

e): Dalla c), e da $L(v^n,h^n) = L(v^n-v,h^n) + L(v,h^n) = L(v^n-v,h^n) + o(1)$ vediamo che non è restrittivo supporre $v^n \rightharpoonup 0$, e per densità possiamo inoltre limitarci al caso $v^n \in H_o^1 \cap L^\infty$. Facendo uso del Lemma A.5 in [2] e della disuguaglianza di Hölder otteniamo dunque

$$2\,L(v^n,h^n) = 2 \int v^n \cdot h_x^n \wedge h_y^n = \int h^n \cdot [v_x^n \wedge h_y^n + h_x^n \wedge v_y^n] =$$

$$= o(1) + \int h \cdot [v_x^n \wedge h_y^n + h_x^n \wedge v_y^n] = o(1) + \int h^n \cdot [v_x^n \wedge h_y + h_x \wedge v_y^n] .$$

La conclusione segue facilmente, poichè da $h^n \to h$ in L^∞, e $v_x \wedge h_y \rightharpoonup 0$, $h_x \wedge v_y \rightharpoonup 0$ debolmente in L^1 segue $2L(v^n,h^n) = o(1)$, i.e. la e). ∎

Il prossimo obiettivo sarà quello di definire una classe di "funzioni ammissibili" $\mathcal{A}(\Gamma,\psi)$ per il Problema 1 sulla quale il funzionale di volume risulti continuo. Su $\mathcal{A}(\Gamma,\psi)$ potremo successivamente definire il funzionale dell'energia relativo al Problema 1.

Data una curva di Jordan Γ, parametrizzata mediante una mappa iniettiva $j \in C^\circ(\partial U,\mathbb{R}^3) \cap H^{1/2}(\partial U,\mathbb{R}^3)$, fissiamo tre punti distinti P_1, P_2, P_3 su ∂U e tre punti distinti ξ_1, ξ_2, ξ_3 su Γ. Diciamo $\mathcal{A}(\Gamma)$ la classe delle mappe $h \in C^\circ(\overline{U}) \cap H^1(U)$ che verificano la "condizione dei tre punti":

$$h(P_i) = \xi_i \qquad \text{per } \ i = 1,2,3$$

e che mandano ∂U su Γ "in modo monotono":

$$h(\partial U) = \Gamma \quad , \quad j^{-1} \circ h \ : \partial U \to \partial U \ \text{ è non decrescente.}$$

Definiamo la classe delle funzioni ammissibili ponendo:

$$\mathcal{A}(\Gamma,\psi) = [\, H_0^1 + \mathcal{A}(\Gamma)\,] \cap \{\, u \in H^1 \mid u^3 \geq \psi(\hat{u}) \ \text{ q.o. in } U \},$$

dove si è indicato con $\hat{z}$ il vettore $(z^1, z^2) \in \mathbb{R}^2$, per $z = (z^1, z^2, z^3) \in \mathbb{R}^3$. Si osservi che se vale la (0.2) (i.e. se Γ è contenuta nel sopragrafico di ψ), e in più $\psi \in C^1(\mathbb{R}^2,\mathbb{R})$ allora $\mathcal{A}(\Gamma,\psi)$ non è vuota; detta infatti $h_\Gamma \in \mathcal{A}(\Gamma)$ la superficie minima di bordo Γ risulta:

$$(\, \hat{h}_\Gamma , \operatorname{Max}\{\psi(\hat{h}_\Gamma), h_\Gamma^3\}) \in \mathcal{A}(\Gamma,\psi) .$$

La Proposizione che segue assicura l'esistenza di una superficie di area minima nella classe $\mathcal{A}(\Gamma,\psi)$, e risolve il Problema 1 nel caso $H = 0$:

PROPOSIZIONE 1.3 ([12], Theorem 2): *Se* $\psi \in C^1(\mathbb{R}^2,\mathbb{R})$, *allora il problema di minimo*

(1.2) $$D(h) = \operatorname*{Min}_{\mathcal{A}(\Gamma,\psi)} D(\cdot)$$

ha almeno una soluzione $h_{\Gamma,\psi}$. Inoltre:

a) *Se* $\nabla\psi \in L^\infty$, *allora* $h_{\Gamma,\psi} \in C^0(\bar{U}) \cap C^{0,\alpha}(U)$ *per un* $\alpha = \alpha(\psi) \in]0,1[$;

b) *Se in più* $\psi \in C^3(\mathbb{R}^2,\mathbb{R})$, *allora* $h_{\Gamma,\psi} \in C^{1,\beta}(U)$ $\forall\, \beta \in [0,1[$.

Per $u \in \mathcal{A}(\Gamma,\psi)$ indichiamo con h_u l'estensione armonica di u in U. Allora $h_u \in \mathcal{A}(\Gamma)$ ed inoltre dal Lemma di Courant-Lebesgue ([4], Lemma 3.2, pg. 103) e da un teorema di Bononcini ([1]) segue :

LEMMA 1.4: *Sia* $u_n \in \mathcal{A}(\Gamma,\psi)$ *t.c.* $u_n \rightharpoonup u$ *debolmente in* H^1 . *Allora* $h_{u_n} \to h_u$ *uniformemente su* $\bar{U}$ *e debolmente in* H^1, *ed inoltre*

$$Q(h_{u_n}) = Q(h) + o(1).$$

Dal Lemma 1.4 segue in particolare che la classe $\mathcal{A}(\Gamma,\psi)$ è debolmente chiusa. Inoltre il Lemma 1.4 assicura che il funzionale di volume Q è continuo sulla classe $\mathcal{A}(\Gamma,\psi)$, in quanto risulta ([14]):

(1.3) $$Q(u) = Q(u - h_u) + Q(h_u) + 3L(h_u, u - h_u) + 3L(u - h_u, h_u).$$

Combinando il Lemma 1.4 e la Proposizione 1.2, dalla (1.3) segue facilmente:

PROPOSIZIONE 1.5: *Sia* $u_n \in \mathcal{A}(\Gamma,\psi)$ *t.c.* $u_n \rightharpoonup u$ *debolmente in* H^1 . *Siano* h_n , h *le estensioni armoniche di* u_n, u *rispettivamente. Posto* $v_n = u_n - h_n \in H^1_0$, $v = u - h \in H^1_0$, *risulta:*

$$Q(u_n) - Q(u) = Q(v_n - v) + o(1) = L(u_n, v_n - v) + o(1) .$$

Definiamo infine il funzionale

$$E(u) = \frac{1}{2} D(u) + \frac{2H}{3} Q(u) \qquad \text{per } u \in \mathcal{A}(\Gamma,\psi) .$$

La possibilità di determinare soluzioni del Problema 1 mediante lo studio dei punti critici del funzionale E è assicurata dal seguente Lemma:

LEMMA 1.6: *Siano* $u \in \mathcal{A}(\Gamma,\psi)$, $\varphi \in H^1_0 \cap L^\infty$ *tali che* $u + t\varphi \in \mathcal{A}(\Gamma,\psi)$ *per* $|t| < \varepsilon$. *Allora*

$$\textit{esiste} \quad \lim_{t \to 0} \frac{E(u + t\varphi) - E(u)}{t} = \int \nabla u \, \nabla \varphi + 2H \int \varphi \cdot u_x \wedge u_y$$

La dimostrazione del Lemma 1.6 si ottiene immediatamente dalla definizione del funzionale E.

Nella prossima Sezione dimostreremo l'esistenza, sotto opportune ipotesi su Γ e H, di un minimo locale $\underline{u}$ per E. Il Lemma 1.6, unito ad alcuni risultati di regolarità, mostreranno che $\underline{u}$ è soluzione del Problema 1.

2. *Il Teorema di Esistenza.*

D'ora in poi supporremo $\psi \in C^1(\mathbb{R}^2,\mathbb{R})$. Detta $h_{\Gamma,\psi}$ la soluzione del problema di minimo (1.2), poniamo

$$\mathcal{B} := \{ u \in \mathcal{A}(\Gamma,\psi) \mid D(u) \leq 5\, D(h_{\Gamma,\psi}) \} .$$

<u>**TEOREMA 2.1**</u>: *Se* $D(h_{\Gamma,\psi}) \leq \frac{2}{3}\frac{2\pi}{H^2}$, *allora il problema di minimo*

$$E(\underline{u}) = \operatorname*{Inf}_{\mathcal{B}} E \tag{2.1}$$

ha una soluzione $\underline{u}$ *tale che* $D(u) < 5\, D(h_{\Gamma,\psi})$. *Inoltre:*

i) *Se* $\nabla\psi \in L^\infty$ *allora* $\underline{u} \in C^0(U) \cap C^{0,\alpha}(U)$ per un $\alpha = \alpha(\psi) \in]0,1[$;

ii) *Se in più* $\psi \in C^3(\mathbb{R}^2,\mathbb{R})$, *allora* $\underline{u} \in C^{1,\beta}(U)$ $\forall\, \beta \in [0,1[$.

Dimostrazione. La dimostrazione del Teorema 2.1 si divide in tre parti:

Step 1: ogni successione minimizzante E su $\mathcal{B}$ ha una estratta convergente in norma.

Step 2: Esiste $\underline{u}$ soluzione di (2.1) tale che $D(\underline{u}) < 5\, D(h_{\Gamma,\psi})$.

Step 3: Continuità e differenziabilià di $\underline{u}$.

Step 1. Il principio di massimo e il principio di Dirichlet assicurano che $\mathcal{B}$ è limitato in H^1; per ogni $u \in \mathcal{A}(\Gamma,\psi)$ risulta infatti

$$\begin{aligned} \| u \| &\leq \| u - h_u \| + C\, | h_u |_\infty + C\, | \nabla h_u |_2 \leq C\, | \nabla(u - h_u) |_2 + C + C\, | \nabla u |_2 \\ &\leq C\, (\, | \nabla u |_2 + | \nabla h_u |_2 + 1\,) \leq C\, (\, | \nabla u |_2 + 1) \end{aligned}$$

dove le costanti C dipendono solamente da Γ. Si ha pertanto che ogni successione u_n minimizzante E su $\mathcal{B}$ è limitata in H^1 e pertanto possiamo supporre $u_n \rightharpoonup \underline{u}$ debolmente in H^1. In virtù del Lemma 1.4, $\underline{u} \in \mathcal{B}$ e quindi

$$0 \geq E(u_n) - E(\underline{u}) + o(1) = (1/2)\, \{ D(u_n) - D(\underline{u}) \} + (2H/3)\, \{ Q(u_n) - Q(\underline{u}) \} + o(1).$$

Usando la Proposizione 1.5 vediamo dunque che

$$(1/2)\,\{ D(u_n) - D(\underline{u}) \} \;\le\; (2H/3)\,|\,L(u_n, v_n - v)| + o(1),$$

e applicando a quest'ultima disuguaglianza la iv) della Proposizione 1.1, otteniamo

$$D(u_n) - D(\underline{u}) \;\le\; (H/3\sqrt{2\pi})\; D(u_n)^{1/2}\,\{ D(v_n) - D(v) \} \; + o(1).$$

Da $u_n \in \mathcal{B}$ e dall'ipotesi su $h_{\Gamma,\psi}$ segue $(H/3\sqrt{2\pi})\; D(u_n)^{1/2} \le a^\circ < 1$ per una certa costante a° non dipendente da n, da cui

$$D(u_n) - D(\underline{u}) \;\le\; a^\circ\,\{ D(v_n) - D(v) \} \; + o(1). \tag{2.2}$$

Possiamo migliorare la (2.2) ricordando che h_n, h sono armoniche in U, e quindi $D(v_n) = D(u_n) - D(h_n)$, $D(v) = D(\underline{u}) - D(h)$; la (2.2) diventa allora

$$(1 - a^\circ)\,\{ D(u_n) - D(\underline{u}) \} \;\le\; (1 + a^\circ)\,\{ D(h) - D(h_n) \} \; + o(1).$$

Infine, ricordando che $h_n \rightharpoonup h$, dalla debole semicontinuità inferiore dell'integrale di Dirichlet segue (passando eventualmente a sottosuccessioni)

$$0 \le (1 - a^\circ)\; \lim\,\{ D(u_n) - D(\underline{u}) \} \le (1 + a^\circ)\; \lim\,\{ D(h) - D(h_n) \} \le 0$$

e quindi

$$D(u_n) \to D(\underline{u}). \tag{2.3}$$

Dalla (2.3) e dalla convergenza uniforme delle h_n (Lemma 1.4) segue facilmente la convergenza in norma della successione u_n. ∎

Step 2. Osserviamo che $h_{\Gamma,\psi} \in \mathcal{B}$ e quindi $E(h_{\Gamma,\psi}) \ge E(\underline{u})$. Possiamo inoltre supporre che valga la disuguaglianza stretta (altrimenti $h_{\Gamma,\psi}$ è la soluzione cercata). Se per assurdo risulta $D(\underline{u}) = 5\, D(h_{\Gamma,\psi})$, dalla ii) della Proposizione 1.1 segue:

$$\begin{aligned} 2\,D(h_{\Gamma,\psi}) &= (1/2)\,\{ D(\underline{u}) - D(h_{\Gamma,\psi}) \} \;<\; (2H/3)\,|\,Q(\underline{u}) - Q(h_{\Gamma,\psi})\,|^{3/2} \\ &\le (H/6\sqrt{2\pi})\,|\,D(\underline{u}) + D(h_{\Gamma,\psi})|^{3/2} \;=\; \sqrt{6}\,(H/\sqrt{2\pi})\; D(h_{\Gamma,\psi})^{3/2} \end{aligned}$$

da cui $D(h_{\Gamma,\psi}) > (2/3)(2\pi/H^2)$, in contrasto con l'ipotesi fatta. ∎

Step 3. La differenziabilità di $\underline{u}$ segue dalla continuità e da un risultato di Tomi ([13], Satz 4). Dimostreremo la i) seguendo una tecnica già collaudata in questioni di regolarità per H-superfici (vedi ad esempio [14], [6], [12]). Faremo uso di un risultato di Morrey ([8], Theorems 1.10.2, 4.3.2) che qui ricordiamo per comodità. Per ogni sottodominio $\Omega \subset\subset U$ diciamo h_Ω l'unica soluzione di:

$$\begin{cases} -\Delta h_\Omega = 0 & \text{in } \Omega \\ h_\Omega = u & \text{su } \partial\Omega \end{cases}$$

<u>**LEMMA 2.2**</u> (Morrey): *Se esiste una costante* $C\ (>1/2)$ *tale che*

(2.4) $$\int_\Omega |\nabla \underline{u}|^2 \le C \int_\Omega |\nabla h_\Omega|^2 \qquad \forall\ \Omega \subset\subset U$$

allora $\underline{u} \in C^0(\bar{U}) \cap C^{0,\alpha}(U)$ per $\alpha = 1/2C$.

Per applicare il Lemma di Morrey è conveniente introdurre un nuovo problema di minimo:

(2.5) $$\inf_{\mathcal{B}_\Omega} D_\Omega(\cdot)$$

dove $\mathcal{B}_\Omega = \{ k \in H^1(\Omega) \mid k - u \in H^1_0(\Omega),\ k^3 \ge \psi(\hat{k}) \text{ q.o. in } \Omega \}$. La classe $\mathcal{B}_\Omega$ è debolmente chiusa, e quindi esiste una soluzione k di (2.5). Si osservi che:

$$h^* := (\hat{h}_\Omega, \operatorname{Max}\{\psi(\hat{h}_\Omega), h^3_\Omega\}) \in \mathcal{B}_\Omega \text{ , e quindi}$$

$$D_\Omega(k) \le D_\Omega(h^*) \le (1 + |\nabla\psi|^2_\infty)\, D_\Omega(h_\Omega)$$

Per ottenere il controllo (2.4) nel Lemma di Morrey basterà dunque stimare $D_\Omega(\underline{u})$ in termini di $D_\Omega(k)$. A tal fine osserviamo che $D_\Omega(k) \le D_\Omega(\underline{u})$ e quindi, prolungata k con $\underline{u}$ fuori da Ω, si ha $k \in \mathcal{B}$. Pertanto $E(\underline{u}) \le E(k)$, da cui

$$(1/2)\{D_\Omega(\underline{u}) - D_\Omega(k)\} \le (2H/3)\, |Q_\Omega(\underline{u}) - Q_\Omega(k)|\,.$$

Usando la (iii) della Proposizione 1.1 ne deduciamo che:

$$(1/2)\,\{\, D_\Omega(\underline{u}) - D_\Omega(k) \,\} \le (H/6\sqrt{2\pi})\ \,|\, D_\Omega(\underline{u}) + D_\Omega(k)|^{3/2}$$

$$\le (H/6\sqrt{2\pi})\,[2D(\underline{u})]^{1/2}\,[D_\Omega(\underline{u}) + D_\Omega(k)|$$

da cui

$$D_\Omega(\underline{u}) \le D_\Omega(k) + (H/3\sqrt{2\pi})\ [2D(\underline{u})]^{1/2}\,[D_\Omega(\underline{u}) + D_\Omega(k)]. \tag{2.6}$$

Poichè $\underline{u} \in \mathcal{B}$, dall'ipotesi su $h_{\Gamma,\psi}$ segue: $(H/3\sqrt{2\pi})\ [2D(\underline{u})]^{1/2} \le b° < 1$ (b° non dipendente da Ω), e quindi dalla (2.6) si ha immediatamente la tesi.

Il Teorema 2.1 è così completamente dimostrato. ∎

OSSERVAZIONE 2.3. Nel caso in cui l'ostacolo ψ sia convesso, l'ipotesi $\nabla\psi \in L^\infty$ è eliminabile: in questo caso risulta infatti $k = h_\Omega$ in quanto

$$h^3_\Omega \ge \psi(\hat{h}_\Omega) \qquad \text{in } \Omega \tag{2.7}$$

e quindi, prolungata h_Ω con $\underline{u}$ fuori da Ω, si ha $h_\Omega \in \mathcal{B}$. Pertanto $E(h_\Omega) \ge E(\underline{u})$, da cui, procedendo come nell'ultima parte della dimostrazione del Teorema 2.1, si ottiene il controllo richiesto dal Lemma di Morrey, con una costante C non dipendente da ψ. ∎

Avendo stabilito l'esistenza di una soluzione continua del problema variazionale siamo in grado di dimostrare il teorema di esistenza per il problema a frontiera libera:

<u>**TEOREMA 2.4**</u>. *Se* $\psi \in C^3(\mathbb{R}^2,\mathbb{R})$, $\nabla\psi \in L^\infty$ *ed in più i dati* Γ, H *verificano:*

$$\int_U |\nabla h_{\Gamma,\psi}|^2 \le \frac{2}{3}\,\frac{2\pi}{H^2}$$

allora esiste una soluzione $\underline{u}$, C *del Problema 1.*

Dimostrazione. Sia $\underline{u}$ la soluzione del problema variazionale del Teorema 2.1.

Poniamo $C := \{ \underline{u}^3 = \psi(\hat{\underline{u}}) \}$. Poichè $\underline{u}$ è continua, dal Lemma 1.6 segue subito che $\underline{u}$ risolve $\Delta\underline{u} = 2H\, \underline{u}_x \wedge \underline{u}_y$ in $U \setminus C$. Le condizioni di conformalità:

$$|\underline{u}_x| = |\underline{u}_y| \quad , \quad \underline{u}_x \cdot \underline{u}_y = 0$$

possono essere verificate come in [4], pp. 107-115, usando il fatto che $\underline{u}$ è "interno" a $\mathcal{B}$. ∎

Modificando opportunamente l'enunciato e la dimostrazione del Teorema 2.1 è possibile stabilire un risultato di esistenza per il seguente problema a frontiera libera:

Problema 1'

Trovare $u \in C^{1,\beta}(U,\mathbb{R}^3) \cap C^\circ(\bar{U},\mathbb{R}^3)$ e C chiuso in U tali che

$$\begin{cases} \Delta u = 2H\, u_x \wedge u_y & \text{in } U \setminus C \\ F(\xi,u(\xi)) = 0 \text{ per } \xi \in C \quad , \quad F(\xi,u(\xi)) \geq 0 & \text{in } U \\ u_x \cdot u_y = 0 \quad , \quad |u_x| = |u_y| & \text{in } U \\ u(\partial U) = \Gamma & \end{cases}$$

quando l'ostacolo $F : U \times \mathbb{R}^3 \to \mathbb{R}$ verifica

$F_0)$ $F(\xi,z) \geq 0 \quad \forall\, \xi \in \partial U \ , \ \forall\, z \in \Gamma$;

$F_1)$ $F \in C^3(U \times \mathbb{R}^3)$;

$F_2)$ $F(\xi,z) = 0 \implies \nabla_z F(\xi,z) \neq 0$;

$F_3)$ $\forall\, \xi \in U$, la funzione $F(\xi, \cdot)$ è concava.

<u>**TEOREMA 2.5**</u>. *Sia* $h_\Gamma \in \mathcal{A}(\Gamma)$ *la superficie minima di bordo* Γ. *Se* F *verifica le* $F_0), F_1), F_2), F_3)$ *ed in più*

$$\int_U |\nabla h_\Gamma|^2 \le \frac{2}{3}\frac{2\pi}{H^2}$$

allora esiste una soluzione $\underline{u}$ *del Problema 1'.*

Dimostrazione. Poniamo

$$\mathcal{A}(\Gamma,F) = [\, H_0^1 + \mathcal{A}(\Gamma)\,] \cap \{\, u \in H^1 \mid F(\xi,u(\xi)) \ge 0 \quad \text{per q.o. } \xi \in U \,\}.$$

Poichè l'immagine di ogni funzione armonica continua su U è contenuta nell'inviluppo convesso dei suoi valori al bordo ([7]), dalle $F_0)$, $F_3)$ segue

$$h_\Gamma \in \mathcal{A}(\Gamma,F),$$

e pertanto è non vuoto l'insieme

$$\mathcal{B}^* = \{\, u \in \mathcal{A}(\Gamma,F) \mid D(u) \le 5\, D(h_\Gamma) \,\}.$$

Gli stessi argomenti usati nel corso della dimostrazione del Teorema 2.1 permettono di stabilire l'esistenza di una soluzione per il problema di minimo:

$$\begin{cases} \underline{u} \in \mathcal{A}(\Gamma,F) \;,\; D(u) < 5\, D(h_\Gamma) \\ E(\underline{u}) = \inf\limits_{\mathcal{B}^*} E \;. \end{cases}$$

La continuità di $\underline{u}$ si ottiene utilizzando nuovamente il Lemma 2.2 di Morrey; è importante osservare che dalla concavità di F nella seconda variabile segue

$$F(\xi, h_\Omega(\xi)) \ge 0 \quad \text{in } \Omega$$

(vedi Osservazione 2.3). Estesa h_Ω con $\underline{u}$ fuori da Ω possiamo dunque dire che $h_\Omega \in \mathcal{B}^*$ e quindi $E(h_\Omega) \ge E(\underline{u})$. Procedendo come nella dimostrazione del Teorema 2.1 si giunge così a determinare una costante C (non dipendente da F), tale che

$$\int_\Omega |\nabla \underline{u}|^2 \le C \int_\Omega |\nabla h_\Omega|^2 \qquad \forall\, \Omega \subset\subset U,$$

da cui la continuità fino al bordo e l'holderianità di $\underline{u}$. A questo punto le $F_1)$, $F_2)$ permettono di applicare il già citato teorema di regolarità di Tomi ([13], Satz 4) per ottenere la differenziabilità di u, e la conclusione segue come nel Teorema precedente. ▮

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THE DIRICHLET PROBLEM FOR HARMONIC MAPS SUPPORTED BY A BALL IN $\mathbb{R}^3$: EXISTENCE OF A LARGE SOLUTION

0. Introduction: the general problem.

Let us denote by D the open unit disk in $\mathbb{R}^2$. Let C be a closed subset of $\mathbb{R}^3$, and $g : D \to C$ a given smooth map, say $g \in C^0 \cap H^{1/2}(\partial D, \mathbb{R}^3)$. We suppose that the set

$$X_g(C) := \{u \in H^1(D, \mathbb{R}^3) \mid u = g \text{ on } \partial D,\ u(\xi) \in C \text{ for a.e. } \xi \in D\}$$

is not empty.

The variational problem we are interested in is the following:

$$(0.1) \qquad \begin{cases} \text{find } u \in X_g(C) \\ \\ u \text{ stationary for } E(v) = \int_D |\nabla v|^2 \text{ on } X_g(C), \end{cases}$$

where, accordingly to Part 1, Introduction, $u \in X_g(C)$ is *stationary* for E on $X_g(C)$ iff

$$\liminf_{\substack{v \to u \text{ in } H^1 \\ v \in X_g(C)}} \frac{E(v) - E(u)}{\| v - u \|_{H^1}} \geq 0.$$

Notice that a solution to Problem (0.1) is in fact harmonic as a map into C.

Let us first consider a simple example: the case when C is <u>convex</u>. In this case, the set $X_g(C)$ of admissible functions turns out to be a closed, convex subset of the Sobolev Space $H^1(D, \mathbb{R}^3)$, and hence Problem (0.1) has exactly one solution h, which solves also

$$E(h) = \operatorname*{Min}_{X_g(C)} E .$$

By using the maximum principle and the convexity assumption on C, it is easy to show that h is the harmonic extension of the boundary data g, i.e. h solves:

$$-\Delta h = 0 \text{ in } D , \quad h = g \text{ on } \partial D.$$

In the general case, we can immediately prove the existence of a "small" solution $\underline{u}$, which is stable in the sense that

$$E(\underline{u}) = \operatorname*{Min}_{X_g(C)} E . \tag{0.2}$$

The existence of such a solution can be obtained via the direct method of the Calculus of Variations, since $X_g(C)$ is weakly closed in H^1 and E is weakly lower semicontinuous. Notice that the minimum problem (0.2) could have more than one solution, since in the general case $X_g(C)$ is not convex; for the same reason there might exist "unstable solutions" to (0.1).

In the following Section we consider, as an example, the case

$$C = \mathbb{R}^3 \setminus B_1(0) = \{ z \in \mathbb{R}^3 \ : \ |z| \geq 1 \},$$

and we give a sufficient condition for the existence of an unstable solution to Problem (1.1). Actually, this result was strongly generalized in [8], but since in this particular case the proof of the multiplicity result is simpler and more direct, we believe it is useful to report it here in order to clarify the main difficulties in solving the Variational Problem (0.1).

We conclude this Introduction with some remarks.

Remark 0.1. When g is a constant, then the constant map g is the unique solution to Problem (0.1): see Lemaire [7]. ∎

Remark 0.2. When $C = S^2$ is the unit sphere, and g is not a constant, then Problem (0.1) has two distinct solutions: see Brezìs-Coron [2] and Jost [6]. For the existence of a large solution of the Dirichlet Problem for harmonic maps into the n-sphere we refer to Benci-Coron [1]. ∎

Remark 0.3. When $C = \mathbb{R}^3 \setminus \Omega$, and Ω is an open, bounded, convex set, and moreover $g(\partial D) \subset \partial\Omega$, then a solution u to Problem (0.1) is a harmonic map $D \to \partial\Omega$.

In order to prove this statement we will use the projection p on the convex set Ω. If $v \in X_g(C)$, then by well known theorems about composition in Sobolev Spaces we get

$$pv \in X_g(\partial\Omega) \quad \text{and} \quad \int |\nabla(pv)|^2 \leq \int |\nabla v|^2$$

since p is not expansive. It is easy to show that

$$v_t := u + t\,(pu - u) \in X_g(C) \quad \forall\, t \in [0,1]$$

and hence, if $u \neq pu$, from the definition of stationary point we get

$$0 \le \lim_{t\to 0} \frac{E(v_t) - E(u)}{\| v_t - u \|} = \frac{\int \nabla u\, \nabla(pu - u)}{\| pu - u \|} \ .$$

This leads to:

$$\int |\nabla(pu - u)|^2 = \int |\nabla pu|^2 + \int |\nabla u|^2 - 2\int \nabla u\, \nabla pu \le 2\{ \int |\nabla u|^2 - \int \nabla u\, \nabla(pu)\} \le 0$$

that is $u = pu$, or equivalently, u maps D into $\partial\Omega$. ∎

Remark 0.4. Let $C = \mathbb{R}^3 \setminus \Omega$, where Ω is an open, bounded, smooth subset of $\mathbb{R}^3$. Using the results in [8], [11], we can give a sufficient condition (a *Douglas Criterion)* for the existence of an "unstable" solution to Problem (0.1). ∎

Remark 0.5. Regularity properties of the solutions to (0.1) were studied by several Authors: we recall here Tomi [12], Hildebrandt [5], Fuchs [4], Duzaar [3]. ∎

1. The Existence Theorem.

In This Section we study Problem (0.1) when

$$C = \{ z \in \mathbb{R}^3 \ : \ |z| \ge 1 \}.$$

Let $\underline{u}$ be a stable solution to Problem (0.1):

$$E(\underline{u}) = \operatorname*{Min}_{X_g(C)} E$$

We will look for a large solution $\bar{u}$ to Problem (0.1) such that $\underline{u}$, $\bar{u}$ "enclose" the unit ball.

If $u \in X_g(C)$ we define the map

$$u^*(\xi) := \begin{cases} u(\xi) & \text{if } |\xi| < 1 \\ \underline{u}\left(\dfrac{\xi}{|\xi|^2}\right) & \text{if } |\xi| \ge 1 \ . \end{cases}$$

Notice that $u^* \in L^2_{loc}(\mathbb{R}^2, C)$ and $\nabla u^* \in L^2(\mathbb{R}^2, \mathbb{R}^6)$. Let $p : C \to S^2$ be the radial projection on S^2:

$$pz := \frac{z}{|z|}$$

and let $\pi^{-1} : \mathbb{R}^2 \to S^2$ be the stereographic projection. For every $u \in X_g(C)$, we get $pu^*\pi \in H^1(S^2,S^2)$, and by using the Volume Functional (see [13], [2]), we can compute the degree of $pu^*\pi$:

$$\deg (pu^*\pi) = \frac{1}{4\pi} V(pu^*) = \frac{1}{4\pi} \{V_D(pu) - V_D(p\underline{u})\},$$

where

$$V(v) = \int_{\mathbb{R}^2} v \cdot v_x \wedge v_y \qquad \text{for } v \in L^\infty(\mathbb{R}^2,\mathbb{R}^3),\ \nabla v \in L^2(\mathbb{R}^2,\mathbb{R}^6) \quad \text{and}$$

$$V_D(v) = \int_D v \cdot v_x \wedge v_y \qquad \text{for } v \in H^1 \cap L^\infty(D,\mathbb{R}^3).$$

In particular, $V(pu^*)/4\pi \in \mathbb{Z}$ $\forall$ $u \in X_g(C)$, and if $u \in C^0(\bar{D},\mathbb{R}^3) \cap X_g(C)$ then $V(pu^*) \neq 0$ iff the continuous map $u^* \circ \pi$ $S^2 \to C$ is not homotopic to a constant. Thus, the class

$$X_g^e(C) := \{ u \in X_g(C) \mid V_D(pu) \neq V_D(p\underline{u}) \}$$

could be seen as the class of maps $u \in X_g(C)$ which enclose, jointly with $\underline{u}$, the unit ball.

We will look for an unstable solution to Problem (0.1) by solving

$$(1.1) \qquad \begin{cases} \textit{find} \quad u \in X_g^e(C) \quad \textit{such that} \\ E(u) = I_g := \inf\limits_{X_g^e(C)} E . \end{cases}$$

The first results we can prove are summarized in the following

Proposition 1.1: *The following estimate holds:*

$$8\pi - E(\underline{u}) \leq I_g \leq E(\underline{u}) + 8\pi.$$

If

$$I_g = E(\underline{u}) + 8\pi,$$

then there exist minimizing sequences which are not relatively compact. Moreover, if $\underline{u} = g$ *is a constant, then* $I_g = 8\pi$ *and every minimizing sequence converges weakly (up to subsequences) to* $\underline{u}$.

We don't enter into the details of the proof. The estimates on I_g can be obtained as in [8], Lemma 3.3. The non-existence result in case $g = \text{const.}$ (compare with the uniqueness result by Lemaire, see Remark 0.1) follows from an uniqueness result by Wente [14] (see also [10], Cap. I, § 5).

In this Section we will prove the following

Theorem 1.2. *I_g is achieved in $X_g^e(C)$, provided*
$$I_g < E(\underline{u}) + 8\pi.$$

Proof. Let $(u_n)_n \in X_g^e(C)$ be a minimizing sequence:

(1.2)
$$V_D(pu_n) - V_D(p\underline{u}) \in 4\pi\mathbb{Z} \setminus \{0\},$$
$$E(u_n) \to I_g \quad \text{as} \quad n \to \infty.$$

Passing eventually to a subsequence we can suppose
$$u_n \rightharpoonup u \quad \text{in} \quad H^1(D, \mathbb{R}^3) \quad \text{for some} \quad u \in X_g(C)$$
and by well-known theorems on compositions in Sobolev Spaces,
$$pu_n \rightharpoonup pu \quad \text{in } H^1(D, \mathbb{R}^3).$$
If $u \in X_g^e(C)$ then we are done, because from the weak-lower semicontinuity of the energy functional it follows
$$I_g \le E(u) \le \liminf E(u_n) = I_g.$$
Let us suppose by contradiction $u \notin X_g^e(C)$, i.e.
$$V_D(pu) = V_D(p\underline{u})$$
and hence, from (1.2),

(1.3)
$$| V_D(pu_n) - V_D(pu) | \ge 4\pi.$$

Now we use:
$$V_D(pu_n) = V_D(pu) + V_D(pu_n - pu) + o(1)$$
(see (13) in [2]), which, compared with (1.3) and the isoperimetric inequality (see [13]):
$$| V_D(v) |^{2/3} \le \frac{1}{2\,(4\pi)^{1/3}} E(v) \qquad \forall\, v \in H_o^1(D, \mathbb{R}^3)$$

gives

$$(4\pi)^{2/3} \leq \frac{E(pu_n - pu)}{2\,(4\pi)^{1/3}} + o(1) \ .$$

Since $E(pu_n - pu) = E(pu_n) - E(pu) + o(1)$, we get

$$8\pi \leq E(pu_n) - E(pu) + o(1)\,. \tag{1.4}$$

The conclusion of the proof of the existence Theorem follows from a simple Lemma which we state here in a more general framework:

Lemma 1.3. *Let* $p \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ *be a given map, and suppose* $|\nabla p|_\infty \leq 1$. *Then the functional*

$$E(v) := E(v) - E(pv)$$

is weakly lower semicontinuous on $H^1(D, \mathbb{R}^n)$.

From Lemma 1.3 it follows

$$\liminf\,[E(pu_n) - E(pu)] \leq \liminf\,[E(u_n) - E(u)\,] = I_g - E(u)$$

and (1.4) implies

$$I_g \geq E(u) + 8\pi,$$

in contrast with our hypotheses. ∎

Proof of Lemma 1.3. We have

$$E(v) = \sum_{i=1}^{n} \int f^i(v, \nabla v)\,, \quad \text{where}$$

$$f^i(v, \nabla v) = |\nabla v^i|^2 - \left[\langle \nabla p^i(v), v_x \rangle^2 + \langle \nabla p^i(v), v_y \rangle^2 \right]$$

We rewrite f^i as:

$$f^i(v, \nabla v) = g(v, v_x) + g(v, v_y),$$

where

$$g(v, \xi) = |\,\xi\,|^2 - \langle \nabla p^i(v), \xi \rangle^2 \quad \text{for } v, \xi \in \mathbb{R}^n$$

is a continuous, positive function on $\mathbb{R}^{2n}$. By computing

$$\nabla_\xi g(v, \xi) = 2\,[\, \xi - < \nabla p^i(v), \xi > \nabla p^i(v)\,],$$

from $|\nabla p|_\infty \leq 1$, it easily follows that $\nabla_\xi g(v, \cdot)$ is monotone, i.e. $f^i(v, \cdot)$ is convex, and Lemma 1.3 follows from the lower semicontinuity Theorem by Serrin (see [9], Theorem 4.1.2). ∎

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Surfaces of Minimal Area Enclosing a Given Body in $\mathbb{R}^3$

The results in this paper have been obtained in collaboration with G. Mancini.

SURFACES OF MINIMAL AREA ENCLOSING A GIVEN BODY IN $\mathbb{R}^3$

Given a body Ω in $\mathbb{R}^3$, we consider a class of surfaces parametrized by S^2 which enclose, in a weak sense, Ω. To "enclose" means, under some regularity assumption on the surface under consideration, that such a surface is not contractible in $\mathbb{R}^3 \setminus \Omega$.

The first problem we deal with, is concerned with the existence of surfaces which minimize the area integral in such a class. In case $\partial\Omega$ is of class C^2, this will lead to finding a $C^{1,\alpha}$ surface parametrized by a map $U^\infty : \mathbb{R}^2 \to \mathbb{R}^3 \setminus \Omega$ which satisfies

$$\begin{cases} -\Delta U^\infty = \chi_{(U^\infty)^{-1}(\partial\Omega)} (-b(U^\infty)(\nabla U^\infty, \nabla U^\infty))\, \nu(U^\infty) & \text{a.e. in } \mathbb{R}^2 \\ |U_x^\infty|^2 = |U_y^\infty|^2 \ , \ \ U_x^\infty \cdot U_y^\infty = 0 \quad \text{in } \mathbb{R}^2, \end{cases}$$

and which is <u>not contractible</u> in $\mathbb{R}^3 \setminus \Omega$ (i.e. "encloses" Ω in a strong sense). Here b is the second fondamental form of $\partial(\mathbb{R}^2 \setminus \Omega)$, ν is the outer normal at $\partial\Omega$, and χ_A is the characteristic function of the set $A \subseteq \mathbb{R}^2$.

This problem, which at our knowledge was not previously considered in the framework of parametric surfaces, is somehow related to the problem of minimal boundaries with obstacles (see for example [12]).

We attack our problem by means of a Dirichlet's Principle, i.e. we look for extremals of the Dirichlet integral over a suitable class of maps from S^2 into $\mathbb{R}^2 \setminus \Omega$. In a more regular setting this problem amounts to finding not homotopically trivial minimal spheres in a Riemannian manifold N with boundary. In case N has empty boundary, striking results have been obtained in a celebrated paper by Sacks and Uhlenbeck ([17], see also [10], [19]). Here we perform a blow-up technique introduced in this context by Sacks and Uhlenbeck. But, in order to avoid estimates on the solutions of Euler-Lagrange equations related to approximated problems, we follow a more direct approach based on a lemma by Brezìs, Coron and Lieb [4].

We also consider the case of disk-type minimal surfaces spanned by a given wire Γ over the obstacle Ω. The existence and regularity of an area minimizing surface $\underline{u}_\Gamma$ spanned over Ω was proved by Tomi [20] (see also [9]). We answer here the rather natural question whether it exists a second minimal surface $\bar{u}_\Gamma$ which, jointly with $\underline{u}_\Gamma$, "encloses" Ω. While this is not the case in general, we prove that this occurs provided

$$\operatorname*{Inf}_{u \in X_\Gamma^e} \int |\nabla u|^2 \lneq \int |\nabla U_\infty|^2 + \int |\nabla \underline{u}_\Gamma|^2 .$$

Here X_Γ^e is a suitable class of surfaces which, jointly with $\underline{u}_\Gamma$, "enclose" Ω.

In the first section of this paper we present preliminary remarks on the functional setting and we define precisely the class of surfaces enclosing Ω.

In Section 2 we describe a Dirichlet's principle for minimal surfaces enclosing a given body Ω and we prove the existence of a closed regular $\mathbb{R}^2$ - type minimal surface spanned over the obstacle Ω.

In Section 3 we give an existence result for pairs of minimal surfaces spanned by the same wire Γ over an obstacle Ω and enclosing it.

In an Appendix we present a result concerning continuous dependence, upon boundary datas, of minimizers for the Dirichlet integral in presence of obsacles. This result, which we didn't find in the literature, turns out to be a key tool in proving the basic inequality (see Proposition 3.4) on which our existence results rely.

Notations. $D_r(z)$ denotes the open disk of radius r and center z in $\mathbb{R}^2$, $|\cdot|$, $\cdot$ denote the norm and the scalar product in $\mathbb{R}^3$, $\rightharpoonup$ denotes weak convergence in various spaces, $|\cdot|_\infty$ and $|\cdot|_2$ denote L^∞ and L^2 norms respectively.

1. Preliminary remarks and statement of the Problem.

Let C be the closure of the unbounded connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$, where Ω is a given bounded open connected set in $\mathbb{R}^3$. We will assume throughout the paper that

(1.1) there is an open neighbourhood $\mathcal{O}$ of C and a Lipschitz retraction $\pi : \mathcal{O} \to C$.

We will denote

$$X := \{ U \in L^\infty(\mathbb{R}^2, \mathbb{R}^3) \mid \int_{\mathbb{R}^2} |\nabla U|^2 < +\infty \}$$

$$X(C) := \{ U \in X \mid U(z) \in C \quad \text{for a.e. } z \in \mathbb{R}^2 \}$$

Under assumption (1.1), and using a smoothing -by averaging- method (see [18], and [1], Appendix), one easily obtains a density result which will be useful in the sequel:

Lemma 1.1. *For every* $U \in X(C)$ *there exists a sequence* $U_n \in C^\infty \cap X(C)$ *such that* $\operatorname{Sup} |U_n|_\infty < +\infty$, $|\nabla U_n - \nabla U|_2 \to 0$ *and* $U_n \to U$ *a.e.*
Furthermore, for each n, U_n *can be taken constant far away.*

In order to give our notion of "mappings enclosing Ω" we define the Volume Functional (see for example [21]):

$$V(U) := \int_{\mathbb{R}^2} U \cdot U_x \wedge U_y \quad , \quad U \in X ,$$

which is well defined since by Hölder inequality

$$(1.2) \qquad |V(U)| \le \frac{1}{2} |U|_\infty |\nabla U|_2^2 .$$

Notice that if $U_n \in X$, $\operatorname{Sup} |U_n|_\infty < \infty$ and $\nabla U_n \to \nabla U$ in L^2 for some $U \in X$, then $V(U_n) \to V(U)$.

Now, assuming for simplicity, $0 \notin C$, we define the map

$$p\xi = \xi / |\xi| \quad \text{for } \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Since p is lipschitz continuous far away from 0, we have $pU \in X$ if $U \in X(C)$. Moreover, if a sequence $(U_n)_n \subseteq X(C)$ is bounded in L^∞ and $\nabla U_n \to \nabla U$ in L^2, $U_n \to U$ a.e., then $\nabla(pU_n) \to \nabla(pU)$ in L^2 and $V(pU_n) \to V(pU)$. We recall that if $U \in X(C) \cap C^0(\mathbb{R}^2, \mathbb{R}^3)$ and U is regular at infinity, that is

(1.3) $$\text{there exists} \quad \lim_{|z|\to\infty} U(z) = U(\infty)$$

then $$\frac{1}{4\pi} V(pU) \in \mathbb{Z}$$

and gives the degree of $pU\circ\Pi \in C^0(S^2,S^2)$, where Π denotes the stereographic projection of S^2 onto $\mathbb{R}^2$ (see [15], and [1], Lemma 1). We notice that, because of the density Lemma, and continuity properties of the volume functional we have

(1.4) $$\frac{1}{4\pi} V(pU) \in \mathbb{Z} \qquad \forall\ U\in X(C).$$

This integer still denotes the degree of pU, so that $X^e(C)$ is the set of maps which have a non zero degree with respect to the sphere $|\xi| = 1$. In particular, if $U\in X(C)\cap C^0(\mathbb{R}^2,\mathbb{R}^3)$ is regular at infinity, and $V(pU) \neq 0$, then U, as a map from S^2 into C, is not contractible.

Accordingly, we set

$$X^e(C) := \{\ U\in X(C) \mid V(pU) \neq 0\ \}.$$

<u>***Remark 1.2***</u>: If $p : C \to S^2$ induces an isomorphism between the second homotopy groups of C and S^2, then

$$\{U\in X^e(C) \mid U \text{ is continuous and regular at infinity}\},$$

can be identified with the closure of the set of smooth, non contractible maps from S^2 into C, by Hopf's Theorem. ∎

In the following section, we will study

<u>***Problem 1***</u> : *Find* $U_\infty\in X^e(C)$, *continuous and regular at infinity in the sense of (1.3), which has minimal area among all the surfaces in* $X^e(C)$.

In view of the previous remarks, U_∞ will be a closed non contractible surface in C.

Before ending this section, we wish to state a problem concerning disk-type minimal surfaces spanned by a wire Γ over the obstacle Ω.

We first recall a well known result (see [20], [9]). Let $\Gamma\subseteq C$ be a closed Jordan curve, and suppose that the class $X_\Gamma(C)$ of maps $u\in H^1(D,C)$ whose trace on ∂D is a continuous, weakly monotone parametrization of the curve Γ, is not empty. Then if ∂C is of class C^2, there is

$$\underline{u} \in H^{2,p}_{loc}(D) \cap C^{\circ}(\bar{D}) \cap X_\Gamma(C)$$

which has minimal area among all surfaces in $X_\Gamma(C)$, and which satisfies the conformality conditions

$$|\underline{u}_x|^2 - |\underline{u}_y|^2 = 0 = \underline{u}_x \cdot \underline{u}_y \quad \text{in } D,$$

i.e. its area is given exactly by $(1/2)\int|\nabla\underline{u}|^2$.

We wish to find a second surface

$$\bar{u} \in H^{2,p}_{loc}(D) \cap C^{\circ}(\bar{D}) \cap X_\Gamma(C)$$

satisfying the conformality conditions, which is harmonic where it doesn't touch $\partial\Omega$, and which "encloses jointly with $\underline{u}$", the obstacle Ω. To make more precise the last statement, let us write

$$V_D(u) := \int_D u \cdot u_x \wedge u_y \ , \quad u \in H^1 \cap L^\infty(D, \mathbb{R}^3)$$

and set

$$X^e_\Gamma(C) := \{ u \in X_\Gamma(C) \mid V_D(pu) \neq V_D(p\underline{u}) \}$$

$$\text{where} \quad p(\xi) = \frac{\xi}{|\xi|} \quad \text{if } \xi \neq 0 .$$

In order to describe the geometric property of surfaces in $X^e_\Gamma(C)$, let us first recall (see [1]) that

$$\text{(1.5)} \qquad \frac{1}{4\pi}[V_D(pu) - V_D(pv)] \in \mathbb{Z}$$

for every $u, v \in X_\Gamma(C)$ with $u - v \in H^1_o(D)$. Actually, (1.5) holds for every $u, v \in X_\Gamma(C)$ (Corollary B.4).

Furthermore, the integer in (1.5) gives the degree of $p \circ U \circ \Pi \in H^1(S^2, S^2)$, where

$$\text{(1.6)} \qquad U(z) := \begin{cases} u(z) & \text{if } |z| \leq 1 \\ v\left(\frac{z}{|z|^2}\right) & \text{if } |z| > 1 \end{cases} \qquad (\text{for } u - v \in H^1_o(D)).$$

Thus, if $u, v \in X_\Gamma(C) \cap C^o(\bar{D}, \mathbb{R}^3)$, $u = v$ on ∂D, the condition $V_D(pu) \neq V_D(pv)$ is equivalent to the non contractibility of $p \circ U \circ \Pi$, U given by (1.6).

In addition, if $\underline{u} \in C^1(\bar{D})$ (which occurs, e.g., if $\Gamma \cap \partial C = \emptyset$, see [8]), one can build, for every $u \in X_\Gamma(C)$ (see Lemma B.3) a change of variables $g_u \in C^o(\bar{D}, \bar{D})$ such that

$$\underline{u} \circ g_u = u \quad \text{on } \partial D \qquad \text{and} \qquad V_D(\, p \circ \underline{u} \circ g_u\,) = V(p\underline{u}) \ .$$

In this case, if $u \in X^e_\Gamma(C) \cap C^o(\bar{D}, \mathbb{R}^3)$ then $p \circ U \circ \Pi$ is not contractible, where U corresponds here to the pair u, $\underline{u} \circ g_u$.

After this preliminaries we are ready to state

Problem 2: *Find* $u \in X^e_\Gamma(C)$ *of class* $C^{1,\alpha}(D) \cap C^o(\bar{D})$ *which has minimal area in the class* $X^e_\Gamma(C)$.

2. *Closed minimal surfaces spanned over obstacles.*

As a standard procedure, we are going to replace the minimization Problem 1:

$$\operatorname*{Min}_{U \in X^e(C)} \int_{\mathbb{R}^2} |U_x \wedge U_y|$$

with the simpler problem

Find $U_\infty \in X^e(C)$ *such that*

$$\text{(2.1)} \qquad \int_{\mathbb{R}^2} |\nabla U_\infty|^2 = I_\infty := \operatorname*{Inf}_{U \in X^e(C)} \int_{\mathbb{R}^2} |\nabla U|^2 \ .$$

First we prove the following

Theorem 2.1. *Let* C *be as in the above Section, and assume in addition that* $\partial C \in C^2$. *Let* $U_\infty \in X^e(C)$ *be a solution of (2.1). Then*

(i) $U_\infty \in C^{1,\alpha}(\mathbb{R}^2, \mathbb{R}^3 \setminus \Omega)$, *and* $U_\infty(z)$ *has a limit as* $|z| \to +\infty$;

(ii) $|(U_\infty)_x|^2 - |(U_\infty)_y|^2 = 0 = (U_\infty)_x \cdot (U_\infty)_y$ *in* $\mathbb{R}^2$;

(iii) $\Delta U_\infty = 0$ *in* $\{\, (x,y) \mid U_\infty(x,y) \notin \Omega \,\}$;

(iv) $\int |(U_\infty)_x \wedge (U_\infty)_y| \ \le \ \int |U_x \wedge U_y| \quad \forall\ U \in X^e(C)$.

Proof. (i) In view of a result by Duzaar [7], it is enough to prove

$$(2.2)\qquad \begin{cases} \forall\ a\in\mathbb{R}^2\ \ \exists\ r>0 & \text{such that} \\ \displaystyle\int_{D_r(a)} |\nabla U_\infty|^2 \le \int_{D_r(a)} |\nabla\Phi|^2 & \forall\ \Phi\in H^1(D_r(a),C) \text{ with} \\ & \Phi - U\in H^1_o(D_r(a),\mathbb{R}^3) \end{cases}$$

According to Duzaar's result, this will imply $U_\infty\in H^{2,p}_{loc}(\mathbb{R}^2,\mathbb{R}^3)$ for every $p\in[1,\infty[$, and hence $U_\infty\in C^{1,\alpha}$ by Sobolev imbedding Theorem.

Given $a\in\mathbb{R}^2$, let $r>0$ be such that

$$(2.3)\qquad \int_{D_r(a)} |\nabla U_\infty|^2 \le I_\infty/2$$

and let $\Phi\in H^1(D_r(a),C)$, with $\Phi = U_\infty$ on $\partial D_r(a)$. Let us consider

$$\Psi(z) := \begin{cases} \Phi(z) & \text{in } D_r(a) \\ U_\infty(z) & \text{in } \mathbb{R}^2\setminus D_r(a) \end{cases} .$$

Since truncation decreases the Dirichlet integral, we can assume $|\Phi|_\infty \le |U_\infty|_\infty$, so that in particular, if Ψ is admissible (i.e. $\Psi\in X^e(C)$),

$$\int_{\mathbb{R}^2} |\nabla U_\infty|^2 \le \int_{\mathbb{R}^2} |\nabla\Psi|^2 = \int_{\{|z-a|>r\}} |\nabla U_\infty|^2 + \int_{D_r(a)} |\nabla\Phi|^2 .$$

If $\Psi\notin X^e(C)$, i.e.

$$0 = \int_{\mathbb{R}^2} \det(p\Psi, \nabla(p\Psi)) = \int_{D_r(a)} \det(p\Phi, \nabla(p\Phi)) + \int_{\mathbb{R}^2\setminus D_r(a)} \det(pU_\infty, \nabla(pU_\infty))$$

and hence

$$(2.4)\qquad \int_{D_r(a)} \det(p\Phi, \nabla(p\Phi)) = \int_{D_r(a)} \det(pU_\infty, \nabla pU_\infty) - V(pU_\infty) ,$$

let us consider

$$W(z) := \begin{cases} \Phi(z) & \text{in } D_r(a) \\ U_\infty\left(a + r^2\dfrac{z-a}{|z-a|^2}\right) & \text{for } |z-a|\ge r \end{cases} .$$

Since $\displaystyle\int_{\{|z-a|>r\}} \det(pW, \nabla(pW))\,dz = -\int_{\{|z-a|<r\}} \det(pU_\infty, \nabla(pU_\infty))\,dz$, from (2.4) we deduce

$$V(pW) = \int_{D_r(a)} \det(p\Phi, \nabla(p\Phi)) - \int_{\{|z-a|<r\}} \det(pU_\infty, \nabla(pU_\infty)) = -V(pU_\infty) \neq 0$$

and hence $W \in X^e(C)$. Thus

$$I_\infty = \int |\nabla U_\infty|^2 \leq \int |\nabla W|^2 = \int_{D_r(a)} |\nabla\Phi|^2 + \int_{D_r(a)} |\nabla U_\infty|^2 \leq \int_{D_r(a)} |\nabla\Phi|^2 + \frac{I_\infty}{2}$$

and (2.2) follows from (2.3). Finally, being

(2.5) $$U^*_\infty(z) := U_\infty\left(\frac{z}{|z|^2}\right)$$

again a minimizer, it is continuous at $z = 0$ and we find

$$\lim_{|z|\to\infty} U_\infty(z) = \lim_{z\to 0} U^*_\infty(z)$$

i.e. U_∞ is regular at infinity.

(ii) - (iii) Here we rely on the "Euler Equation" for the "energy minimizing maps" (i.e. for minima of (2.1)) established by Duzaar [7]:

$$-\Delta U_\infty = \chi_{U_\infty^{-1}(\partial\Omega)}(-b(U_\infty)(\nabla U_\infty, \nabla U_\infty))\, \nu(U_\infty) \qquad \text{a.e. in } \mathbb{R}^2 .$$

Here b is the second fondamental form of ∂C, ν is the inner normal at ∂C, and χ_A is the characteristic function of the set $A \subseteq \mathbb{R}^2$. As a consequence, U_∞ is harmonic in the open set $\{z \in \mathbb{R}^2 \mid U_\infty(z) \notin \Omega\}$. Also

$$\Delta U_\infty \cdot (U_\infty)_x = 0 = \Delta U_\infty \cdot (U_\infty)_y \quad \text{a.e. on } \mathbb{R}^2 .$$

This easily implies that, setting (in complex notation),

$$\varphi + i\psi = |(U_\infty)_x|^2 - |(U_\infty)_y|^2 - 2i\,(U_\infty)_x \cdot (U_\infty)_y$$

then $0 = \int \varphi\, \Delta\eta = \int \psi\, \Delta\eta \quad \forall\, \eta \in C^\infty_0(\mathbb{R}^2, \mathbb{R}^3)$, i.e. φ and ψ are harmonic. Since $\varphi + i\psi \in L^1(\mathbb{R}^2)$, this implies $\varphi = \psi = 0$.

(iv) From Morrey's ε - conformality result ([13], see also [16], § 226) we have that for every $U \in C^\infty \cap X^e(C)$ constant far away, and for every $\varepsilon > 0$, there exists a $V_\varepsilon \in X^e(C)$ s.t. $(1/2)\int |\nabla V_\varepsilon|^2 \leq \int |U_x \wedge U_y| + \varepsilon$. Thus, for every $U \in C^\infty \cap X^e(C)$ constant far away we have

$$(1/2)\int |\nabla U_\infty|^2 = \int |(U_\infty)_x \wedge (U_\infty)_y| \leq \int |U_x \wedge U_y| ,$$

and the conclusion follows from the density Lemma.

∎

The main result in this Section is

Theorem 2.2. $$I_\infty := \inf_{U\in X^e(C)} \int_{\mathbb{R}^2} |\nabla U|^2$$ *is achieved.*

The proof is based on a blow up technique, introduced in this class of problems by Sacks and Uhlenbeck [17]. A crucial step in the proof of Theorem 2.2 is the description of the behaviour of sequences $(U_n)_n \subseteq X^e(C)$, which, in the limit, jump out of the class. To this extent, we first recall a result in [4] (see also [23]).

Let $(U_n)_n \subseteq X(C)$ satisfy

(2.6) $$\nabla U_n \rightharpoonup \nabla U \text{ in } L^2 \text{ and } \operatorname{Sup} |U_n|_\infty < \infty.$$

Then, eventually passing to a subsequence,

(2.7) $$\det(pU_n, \nabla(pU_n)) \rightharpoonup \det(pU, \nabla(pU)) + 4\pi \sum_{i=1}^{n} d_i\, \delta_{a_i}$$

weakly in the sense of measures, for some $d_i \in \mathbb{Z}$, $a_i \in \mathbb{R}^2$. Here δ_{a_i} denotes the Dirac measure concentrated at a_i.

We first show that if $\det(pU_n, \nabla(pU_n))$ "concentrates" at some a_i, then U_n loses, in the limit, at least as much energy as I_∞.

Proposition 2.3. *Let* $(U_n)_n \subseteq X^e(C)$ *satisfy (2.6). Assume* $U \notin X^e(C)$ *and* $d_i \neq 0$ *for some index* i *in (2.7). Then*

$$\forall \rho > 0 \textit{ small enough,} \qquad \liminf \int_{B_\rho(a_i)} |\nabla U_n|^2 \geq \int_{B_\rho(a_i)} |\nabla U|^2 + I_\infty .$$

Proof. Fix $\rho>0$ such that $a_j \notin D_{2\rho}(a_i)$ if $j \neq i$. We can assume, eventually passing to a subsequence, that there exists

$$\lim_n \int_{D_\rho(a_i)} |\nabla U_n|^2 .$$

For almost every $r < \rho$, there exists a subsequence U_n (depending on r) such that

$$\operatorname{Sup}_k \int_{\partial D_r(a_i)} [\, |\nabla U_{n_k}|^2 + |U_{n_k}|^2 \,] < +\infty$$

and hence $U_n \rightharpoonup U$ weakly in $H^1(\partial D_r(a_i),\mathbb{R}^3)$. Now, we denote by h_k a solution of

$$\operatorname{Inf}\ \{ \int_{D_r(a_i)} |\nabla v|^2 \mid v \in H^1(D_r(a_i),C) \ ,\ v - U_{n_k}\big|_{D_r(a_i)} \in H^1_o(D_r(a_i)) \} \ .$$

Because of the good behaviour of h_k on $\partial D_r(a_i)$, one can prove (see Proposition A.1) that up to subsequences

$$h_k \to h \quad \text{in} \quad H^1(D_r(a_i)) \tag{2.8}$$

where h minimizes the Dirichlet integral with constraint C and boundary data U. In particular

$$\int_{D_r(a_i)} |\nabla h|^2 \le \int_{D_r(a_i)} |\nabla U|^2 \ . \tag{2.9}$$

Now, let us define

$$\tilde{U}_k(z) := \begin{cases} U_{n_k}(z) & \text{if } |z-a| \le r \\ h_k\left(a_i + \dfrac{z-a_i}{|z-a_i|^2} r^2 \right) & \text{if } |z-a| > r \ . \end{cases}$$

We claim that if r is chosen small enough, then $\tilde{U}_k \in X^e(C)$ for $k \ge k(r)$ big enough. In fact, using (1.2), (2.8), (2.9) we find

$$|V(p\tilde{U}_k)| = \left| \int_{D_r(a_i)} \det(pU_{n_k}, \nabla(pU_{n_k})) - \int_{D_r(a_i)} \det(ph_k, \nabla(ph_k)) \right| \ge$$

$$\ge o(1) + 4\pi |d_i| - \left| \int_{D_r(a_i)} \det(pU, \nabla(pU)) \right| - \frac{1}{2} \int_{D_r(a_i)} |\nabla h|^2 > 0$$

for $k \ge k(r)$. Hence, by using again (2.8), (2.9) we get

$$I_\infty \le \liminf \int_{\mathbb{R}^2} |\nabla \tilde{U}_k|^2 = \liminf \left[\int_{D_r(a_i)} |\nabla U_{n_k}|^2 + \int_{D_r(a_i)} |\nabla h_k|^2 \right] \le$$

$$\le \liminf \int_{D_r(a_i)} |\nabla U_{n_k}|^2 + \int_{D_r(a_i)} |\nabla U|^2 \ .$$

Thus, for such good r's , we have

$$\lim_n \int_{D_\rho(a_i)} |\nabla U_n|^2 = \lim_k \left[\int_{D_r} |\nabla U_{n_k}|^2 + \int_{D_\rho \setminus D_r} |\nabla U_{n_k}|^2 \right] \ge$$

$$\ge I_\infty - \int_{D_r} |\nabla U|^2 + \int_{D_\rho \setminus D_r} |\nabla U|^2 \ge I_\infty + \int_{D_\rho} |\nabla U|^2 - 2\int_{D_r} |\nabla U|^2$$

and letting r go to zero we conclude the proof of Proposition 2.3. ∎

In particular, we get

Proposition 2.4. *Let* $(U_n)_n \subseteq X^e(C)$ *satisfy (2.6). Assume* $U \notin X^e(C)$. *Then*

$$\liminf \int_{\mathbb{R}^2} |\nabla U_n|^2 \geq I_\infty + \int_{\mathbb{R}^2} |\nabla U|^2 .$$

Proof. Let us first remark that, in case $d_i = 0$ for every i, then $U_n^* \rightharpoonup U^*$ (U_n^*, U^* defined as in (2.5)) satisfy the same assumptions as U_n, U and (compare with (2.7)):

(2.10) $$\det(pU_n^*, \nabla(pU_n^*)) \rightharpoonup \det(pU^*, \nabla(pU^*)) + 4\pi d\, \delta_0$$

where $d = -V(pU_n)/4\pi \in \mathbb{Z} \setminus \{0\}$ (since $U \notin X^e(C)$ by assumption) does not depend on n for n large. Since $\int |\nabla U_n^*|^2 = \int |\nabla U_n|^2$, eventually replacing U_n by U_n^*, we can assume U_n satisfies the assumptions in Proposition 2.3 and hence, for some a_i:

$$\liminf \int_{\mathbb{R}^2} |\nabla U_n|^2 \geq \liminf \int_{D_r(a_i)} |\nabla U_n|^2 + \liminf \int_{\{|z-a_i|>r\}} |\nabla U_n|^2 \geq$$

$$\geq I_\infty + \int_{D_r(a_i)} |\nabla U|^2 + \int_{\{|z-a_i|>r\}} |\nabla U|^2 . \qquad \blacksquare$$

In case U_n is, in addition, minimizing, i.e. $\int |\nabla U_n|^2 \to I_\infty$, we can say more:

Proposition 2.5. *Let* $(U_n)_n \subseteq X^e(C)$ *satisfy (2.6) and, in addition,* $\int |\nabla U_n|^2 \to I_\infty$.

If $U \notin X^e(C)$, *then* U *is a constant, and either*

(i) *There is (exactly one)* $a \in \mathbb{R}^2$ *s.t.*

$$\int_{D_r(a)} |\nabla U|^2 \to I_\infty \quad \forall r>0 \quad \text{and} \quad \nabla U_n \to 0 \quad \text{in} \quad L^2_{loc}(\mathbb{R}^2 \setminus \{a\}, \mathbb{R}^6) \text{ , or}$$

(ii) $\nabla U_n \to 0$ *in* $L^2_{loc}(\mathbb{R}^2, \mathbb{R}^6)$

Proof. First, $U = \text{const.}$ by Proposition 2.4. Furthermore, if $d_i \neq 0$ for some i in (2.7), there is just one $d_i \neq 0$, by Proposition 2.3, which, at the same time, implies

$$I_\infty \geq \lim \int_{D_r(a_i)} |\nabla U_n|^2 \geq I_\infty \qquad \text{and hence}$$

$$\lim \int_{\{|z-a_i|>r\}} |\nabla U_n|^2 = \lim \int_{\mathbb{R}^2} |\nabla U_n|^2 - \lim \int_{D_r(a_i)} |\nabla U_n|^2 = 0\ .$$

In case $\det(pU_n, \nabla(pU_n)) \rightharpoonup 0$ we have (see (2.10)):

$$\det(pU_n^*, \nabla(pU_n^*)) \rightharpoonup 4\pi d\, \delta_0 \qquad \text{with } d \in \mathbb{Z} \setminus \{0\} \text{ for n large.}$$

Again by Proposition 2.3 we have, as in the previous case,

$$\int_{\{|z|<R\}} |\nabla U_n|^2 = \int_{\{|z|>1/R\}} |\nabla U_n^*|^2 \to 0 \quad \forall R>0\ . \qquad \blacksquare$$

Proof of Theorem 2.2. Let $(U_n)_n \subseteq X^e(C)$ be minimizing : $\int |\nabla U_n|^2 \to I_\infty$. Since truncation does not increase the Dirichlet integral and Ω is bounded, we can assume the U_n have a common L^∞ bound and, eventually passing to a subsequence, $\nabla U_n \rightharpoonup \nabla U$ in L^2 , $U_n \to U$ a.e. for some $U \in X(C)$. If $U \in X^e(C)$, U is a minimizer by the lower semicontinuity of the Dirichlet integral. If $U \notin X^e(C)$, we want to show that, after rescaling and translating U_n, we can construct a new minimizing sequence weakly converging in $X^e(C)$. Let us introduce the concentration function (see for example [11], [3]):

$$Q_n(t) = \operatorname{Sup}_{z \in \mathbb{R}^2} \int_{D_t(z)} |\nabla U_n|^2\ .$$

This is a continuous, non decreasing function, with $Q_n(0) = 0$ and $\lim \operatorname{Sup}\{Q_n(t) : t>0\} = I_\infty$. Thus, given $\delta \in]0, I_\infty[$, there are, for n large, $t_n > 0$, $z_n \in \mathbb{R}^2$ s.t.

$$\delta = \int_{D_{t_n}(z_n)} |\nabla U_n|^2 = Q_n(t_n)\ .$$

Set $\tilde{U}_n(z) := U_n(t_n z + z_n)$. Notice that $\tilde{U}_n \in X^e(C)$, $\int |\nabla \tilde{U}_n|^2 = \int |\nabla U_n|^2 \to I_\infty$ and $\operatorname{Sup} |\tilde{U}_n|_\infty = \operatorname{Sup} |U_n|_\infty < +\infty$. Again we can find a subsequence $(\tilde{U}_n)_n$ and $U_\infty \in X(C)$ s.t.

$$\nabla \tilde{U}_n \rightharpoonup \nabla U_\infty \quad \text{in } L^2\ , \quad \tilde{U}_n \to U_\infty \text{ a.e.}$$

We claim that $U_\infty \in X^e(C)$. Otherwise, by Proposition 2.5, either

$$\int_{D_r(a)} |\nabla \tilde{U}_n|^2 \to I_\infty \quad \text{for some } a \in \mathbb{R}^2 \text{ and } \forall r>0, \text{ or } \quad \int_{D_R(0)} |\nabla \tilde{U}_n|^2 \to 0 \quad \forall R>0\ .$$

But the first alternative cannot occour, since

$$\int_{D_r(a)} |\nabla\tilde{U}_n|^2 = \int_{D_{t_n r}(t_n a + z_n)} |\nabla U_n|^2 \leq Q_n(t_n) = \delta < I_\infty$$

for $r \leq 1$. Finally, the second alternative cannot occour either, because

$$\int_{D_1(0)} |\nabla\tilde{U}_n|^2 = \int_{D_{t_n}(z_n)} |\nabla U_n|^2 = \delta > 0 \ . \qquad \blacksquare$$

Remark 2.6. It may happen that the "coincidence set" $\{U_\infty \in \partial\Omega\}$ is the all plane $\mathbb{R}^2$. Since projections on convex sets reduce the Dirichlet integral, this is for example the case when Ω is a convex set. Moreover in this case it results that the image through the map U_∞ is exactly $\partial\Omega$ (otherwise the map U_∞ would be contractible in C) and, identifying U_∞ with its composition with the stereographic projection, our solution U_∞ is in fact a non constant harmonic map from the sphere onto $\partial\Omega$.

In order to avoid this phenomena, we could use an observation by Duzaar [7]: since the minimizer U_∞ satisfies

$$- b(U_\infty(z))(\nabla U_\infty(z), \nabla U_\infty(z)) \geq 0$$

for almost every $z \in U_\infty(\partial\Omega)$, the obstacle $\partial\Omega$ has to satisfy a "concavity condition" (when viewed from C) in order to be "essentially touched" by the enveloping surface U_∞. In other words, if b is positive defined somewhere on $\partial\Omega$, U_∞ cannot lie enterely on $\partial\Omega$, and as we have previously noticed, it is harmonic outside the coincidence set. $\blacksquare$

3. *Pairs of solutions of the Plateau Problem for disk-type minimal surfaces with obstructions.*

Given the obstacle Ω, we assume as in the previous Sections that C, the unbounded connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$, is of class C^2 and satisfies (1.1).

Let $\Gamma \subseteq C$ be a given Jordan curve, parametrizable with a diffeomorphism $\gamma^\circ: \partial D \to \mathbb{R}^3$. Let us denote by $\mathcal{A}_\Gamma$ the class of $H^{1/2} \cap C^0(\partial D, \mathbb{R}^3)$ - weakly monotone parametrizations of Γ which are normalized by a three-point condition. We suppose that the class of "admissible functions":

$$X_\Gamma(C) := \{ u \in H^1(D,\mathbb{R}^3) \mid u_{|\partial D} \in \mathcal{A}_\Gamma ,\ u(z) \in C \ \text{ for a.e. } z \in D\},$$

is not empty.

The "small solution" $\underline{u}$ obtained by Tomi [20] is just a solution of the minimum problem:

$$d_\Gamma := \underset{u \in X_\Gamma(C)}{\mathrm{Min}} \int_D |\nabla u|^2$$

and its existence is easily proved using weakly lower semicontinuity of the Dirichlet integral and the Courant-Lebesgue Lemma [5]. In order to find a second solution as an extremal for the Dirichlet integral, we will first prove that the set

$$X^e_\Gamma(C) := \{ u \in X_\Gamma(C) \mid V_D(pu) \neq V_D(p\underline{u}) \}$$

(see Section 1), is not empty whenever $X_\Gamma(C) \neq \emptyset$. Then we will consider the following minimization problem:

Find $\bar{u} \in X^e_\Gamma(C)$ *such that*

$$\int_D |\nabla \bar{u}|^2 = I_\Gamma := \underset{u \in X^e_\Gamma(C)}{\mathrm{Inf}} \int_D |\nabla u|^2 . \tag{3.1}$$

As in Section 2 one can prove the following Dirichlet's Principle:

Theorem 3.1. *Let* C *be as above, and* $\partial C \in C^2$. *Let* $u \in X^e_\Gamma(C)$ *be a solution of (3.1). Then*

(i) $\bar{u} \in C^0(\bar{D},\mathbb{R}^3) \cap C^{1,\alpha}(D,\mathbb{R}^3)$ *for every* $\alpha \in]0,1[$;

(ii) $\bar{u}$ *is conformal:* $\bar{u}_x \cdot \bar{u}_y = 0 = |\bar{u}_x| - |\bar{u}_y|$;

(iii) $\Delta \bar{u} = 0$ *in* $\{ (x,y) \in D \mid \bar{u}(x,y) \notin \partial\Omega\}$;

(iv) $\int |\bar{u}_x \wedge \bar{u}_y| \leq \int |u_x \wedge u_y|$ *for every* $u \in X^e_\Gamma(C)$.

The regularity result in (i) follows from a Theorem by Hildebrandt ([9], see also Tomi [20], Satz 6) and Duzaar [7], via arguments similar to those used in the proof of Theorem 2.1. Propositions (ii), (iii), (iv) can be obtained in a standard way (see [5], pg. 105 for (ii) and the Morrey's ε-conformality result - [13], Theorem 1.2 - for (iv)), using the invariance of the volume functional under reparametrizations of the domain.

Remark 3.2. We notice that in case $\Gamma \subseteq \mathring{C}$, the conformal map u is harmonic in a neighbourhood of ∂D, and thus $u \in C^1(\bar{D},\mathbb{R}^3)$ (see [8]). At our knowledge, it is not known a C^1-regularity result up to the boundary in the general case. ∎

In order to solve (3.1) we first prove

<u>*Lemma 3.3*</u>. *The set* $X_\Gamma^{e}(C)$ *is not empty and* $\quad I_\Gamma \le \int |\nabla \underline{u}|^2 + I_\infty$.

Actually, we want to prove a more general result:

$$(3.2)\qquad \begin{cases} \forall\ u \in H^1(D,C) \cap C^o(D,C) \quad \text{and} \quad \forall\ \varepsilon>0\,, \\ \exists\ v_\varepsilon \in H^1(D,C) \quad \text{s.t.} \quad v_\varepsilon = u \quad \text{on } \partial D, \quad V_D(pv_\varepsilon) \neq V_D(pu) \quad \text{and} \\ \int |\nabla v_\varepsilon|^2 \le \int |\nabla u|^2 + I_\infty + o(1) \qquad \text{as} \quad \varepsilon \to 0\,. \end{cases}$$

Proof of (3.2). Denoted by U a solution of Problem 1, notice that under our regularity assumptions on ∂C, U is continuous and regular at infinity, that is

$$\exists \lim_{|z| \to \infty} U(z) =: U(\infty)\,.$$

We set

$$U^\varepsilon(z) := U\left(\frac{z}{\varepsilon^5}\right) \qquad \text{for} \quad \varepsilon > 0\,.$$

Let $\lambda : [0,1] \to C$ be a Lipschitz map with $\lambda(0) = u(0)$, $\lambda(1) = U(\infty)$. Our map v^ε is given by

$$v^\varepsilon(z) := \begin{cases} u(z) & \text{if } \varepsilon \le |z| \le 1 \\ \pi\big(u(z) - u(0) + \lambda(\Phi_\varepsilon(|z|))\big) & \text{if } \varepsilon^2 \le |z| \le \varepsilon \\ \pi\big(\Phi_{\varepsilon^2}(|z|)\,[U^\varepsilon(z) - u(z) + u(0) - U(\infty)] + u(z) - u(0) + U(\infty)\big) & \text{if } \varepsilon^4 \le |z| \le \varepsilon^2 \\ U^\varepsilon(z) & \text{if } |z| \le \varepsilon^4. \end{cases}$$

where π is the Lipschitz retraction in (1.1) and $\Phi_\varepsilon = (\lg r - \lg\varepsilon)/\lg\varepsilon$ if $\varepsilon < 1$, $\varepsilon^2 \le r \le \varepsilon$. For ε small, $u(z) - u(0)$ is small if $|z| \le \varepsilon$ since u is continuous, and hence $u(z) - u(0) + \lambda(\Phi_\varepsilon(|z|)$ belongs to a small neighbourhood of C. Similarly, if $\varepsilon^4 \le |z| \le \varepsilon^2$, $U^\varepsilon(z) = U(z/\varepsilon^5)$ is close to $U(\infty)$ and hence $\Phi_{\varepsilon^2}(|z|)\,[U^\varepsilon(z) - U(\infty) + u(0) - u(z)] + u(z) - u(0) + U(\infty)$ belongs to a neighbourhood of C as well. Thus v^ε is well defined and $v^\varepsilon \in H^1(D,C)$, $v^\varepsilon = u$ on ∂D. A direct computation shows that

$$(3.3)\qquad \int_{\{\varepsilon^4 < |z| < \varepsilon\}} |\nabla v^\varepsilon|^2 \to 0 \qquad \text{as} \quad \varepsilon \to 0$$

while

$$\int_{\{\varepsilon < |z| < 1\}} |\nabla v^\varepsilon|^2 \to \int_D |\nabla u|^2\,,$$

$$\int_{\{|z|<\varepsilon^4\}} |\nabla v^\varepsilon|^2 = \int_{\{|z|<1/\varepsilon\}} |\nabla U|^2 \to I_\infty$$

and thus
$$\int_D |\nabla v^\varepsilon|^2 = I_\infty + \int_D |\nabla u|^2 + o(1) .$$

To end the proof, it is enough to observe that

$$V_D(pv^\varepsilon) - V_D(pu) = -\int_{D_\varepsilon} \det(pu, \nabla(pu)) + \int_{\{\varepsilon^4<|z|<\varepsilon\}} \det(pv^\varepsilon, \nabla(pv^\varepsilon)) + \int_{\{|z|<1/\varepsilon\}} \det(pU, \nabla(pU)) =$$

$$= V(pU) + o(1)$$

by (3.3), and hence, if ε is small enough, $V_D(pv^\varepsilon) - V_D(pu) \in 4\pi\mathbb{Z} \setminus \{0\}$, since $V(pU) \neq 0$. ∎

Remark 3.4. Equality in Lemma 3.3 cannot be excluded, in general. Moreover, the sequence $(v^\varepsilon)_\varepsilon$ in the proof of (3.2) shows that, whenever equality occurs, there exsist minimizing sequences for Problem (3.1) which weakly converge to the small solution $\underline{u}$ and hence do not have strongly convergent subsequences.

Equality occours, for example, in the "degenerate case", i.e. when Γ reduces to a point z°. In this case the set of "admissible functions" is $X_\Gamma^e(C) = \{u \in H^1(D,C) \mid u = z^\circ \text{ on } \partial D, V_D(pu) \neq 0\}$ and $I_\Gamma \leq I_\infty$. Actually, $I_\Gamma = I_\infty$, since in this case $X_\Gamma^e(C)$ is embedded in a natural way in $X^e(C)$.

It is quite likely that I_Γ is not achieved whenever equality holds. This is the case if Ω is the unit ball and Γ reduces to a point, e.g. in $\partial\Omega = S^2$. In fact a minimizer for the Dirichlet integral would be a non constant harmonic map from the disk into S^2 with constant boundary data; but this cannot occour in view of a uniquess result due to Lemaire [10].

Notice also that such a minimizer would also be an extremal for the Bononcini-Wente isoperimetric inequality:

$$|V_D(v)|^{2/3} \leq \frac{1}{(32\pi)^{1/3}} \int |\nabla v|^2 \qquad \text{for every } v \in H^1_\cdot(D,\mathbb{R}^3) \quad \text{constant on } \partial D ,$$

which is known not to exists ([22]). ∎

The main result in this Section is

Theorem 3.5. *Let* C, Γ *be as above. Then* I_Γ *is achieved, provided*

(3.4) $$I_\Gamma < \int |\nabla \underline{u}|^2 + I_\infty .$$

Proof. We split the proof into two steps:

Step 1: There is $v^\circ \in X_\Gamma(C)$ such that

$$I_\Gamma = \inf_{\substack{u \in X^e_\Gamma(C) \\ u - v^\circ \in H^1_o}} \int |\nabla u|^2 \quad .$$

Step 2: $I_{v^\circ} := \operatorname{Inf} \{ \int |\nabla u|^2 \mid u \in X^e_\Gamma(C) \ , \ u - v^\circ \in H^1_o \}$ is achieved in $X^e_\Gamma(C)$.

Proof of Step 1. Here we don't make use of assumption (3.4). Let $(u_n)_n \subseteq X^e_\Gamma(C)$ be such that $\int |\nabla u_n|^2 \to I_\Gamma$. We can assume $\operatorname{Sup} | u_n |_\infty < \infty$, and since $(u_{n|\partial D})_n$ is equicontinuous on ∂D by Courant-Lebesgue Lemma, we can also assume $u_n \rightharpoonup v^\circ$ weakly in H^1 and $u_n \to v^\circ$ uniformly on ∂D for some $v^\circ \in X_\Gamma(C)$. Thus, if $\Delta h_n = 0$, $h_n = u_n - v^\circ$ on ∂D, $h_n \to 0$ uniformly and weakly in H^1. As a consequence $w_n := \pi(u_n - h_n)$ is well defined for large n (here π is the retraction of some neighbourhood of C onto C) and, with easy computations

$$\liminf \int |\nabla w_n|^2 \le \liminf \int |\nabla u_n|^2 = I_\Gamma . \tag{3.5}$$

Since $w_n - v^\circ \in H^1_o$, it is enough to prove, in view of (3.5), that $V_D(pw_n) \neq V_D(p\underline{u})$. But this readily follows, because $| pw_n - pu_n |_\infty \le \text{const.} | h_n |_\infty \to 0$ and $pw_{n\,|\partial D} = v^\circ$, so that, by Lemma B.1, $V_D(pu_n) = V_D(pu_n - pw_n + pw_n) = V_D(pw_n) + o(1)$. Since

$$V_D(pu_n) - V_D(p\underline{u}) \in 4\pi\mathbb{Z} \setminus \{0\}$$

(see Corollary B.4) the proof is complete. ∎

Proof of Step 2. The argument we present here applies to the solvability of Dirichlet problems, and hence we give it in this more general form. Let $v \in X_\Gamma(C)$ and let

$$w_n - v \in H^1_o \ , \ w_n \in X^e_\Gamma(C) \ , \ w_n \rightharpoonup w.$$

If $w \notin X^e_\Gamma(C)$ then, applying Proposition 2.4 to the sequence

$$U_n(z) := \begin{cases} w_n(z) & \text{if } |z| \le 1 \\ w\left(\dfrac{z}{|z|^2}\right) & \text{if } |z| > 1 \end{cases}$$

we immediatly get

$$\liminf \int_D |\nabla w_n|^2 \ge I_\infty + \int_D |\nabla w|^2 . \tag{3.6}$$

From (3.6) it follows that the infimum

$I_v := \operatorname{Inf} \{ \int |\nabla w|^2 \mid w - v \in H^1_o \, , \ w \in X^e_\Gamma(C) \}$ is achieved provided (compare with (3.2)):

$$I_v < I_\infty + \operatorname*{Inf}_{\substack{w \in X_\Gamma(C) \\ w - v \in H_0^1}} \int |\nabla w|^2 . \tag{3.7}$$

This ends the proof of Step 2 since (3.7) holds, with $v = v°$ given by Step 1, in view of assumption (3.4). ∎

Remark 3.6. It is interesting to reformulate Theorem 3.5 from the point of view of relaxation. If we define the energy associated to the minimum problem

$$\operatorname*{Min}_{u \in X_\Gamma^e(C)} \int |\nabla u|^2$$

as

$$E(u) := \begin{cases} \int |\nabla u|^2 & \text{if } u \in X_\Gamma^e(C) \\ +\infty & \text{otherwise in } X_\Gamma(C) \end{cases}$$

then the relaxed functional in the weak H^1 topology is defined by

$$(sc^- E)(u) := \operatorname{Inf} \{ \liminf \int |\nabla u_n|^2 \mid u_n \in X_\Gamma(C) \,,\ u_n \rightharpoonup u \text{ weakly in } H^1 \}.$$

Slight modifications in our arguments show that

$$(sc^- E)(u) := \begin{cases} \int |\nabla u|^2 & \text{if } u \in X_\Gamma^e(C) \\ \int |\nabla u|^2 + I_\infty & \text{otherwise in } X_\Gamma(C) . \end{cases}$$

Now, let u be a minimum point for the functional $sc^- E$, that is

$$(sc^- E)(u) = \operatorname*{Inf}_{X_\Gamma(C)} E = \operatorname*{Inf}_{X_\Gamma^e(C)} \int |\nabla u|^2 .$$

If (3.4) holds, then necessarily $u \in X_\Gamma^e(C)$, and hence u is also a solution of our minimization Problem 2. ∎

Remark 3.7. A simple variant of Problem 1 arises if we drop the connectivity assumption on the obstacle Ω; related results are presented in [14], where are also considered extensions to higher dimensions.

It would be of interest to describe the limit problem as the connected components of Ω become infinite while their size go to zero. This could also be a way to deal with a much deeper variant of Problem 1, namely the case of thin obstacles. Problems of this kind have been considered in the framework of minimal boundaries (see [6]). ∎

Appendix A.

We present here a result concerning continuous dependence of minimizers for the Dirichlet integral, subjected to obstacle conditions, with respect to H^1 weak convergence of boundary values.

Let $C \subseteq \mathbb{R}^3$ be a closed set satisfying

(A.1) There is $\delta > 0$ and a Lipschitz retraction $\pi : \{\xi \in \mathbb{R}^3 \mid d(\xi, C) < \delta\} \to C$

Let us denote

$$H^1(D,C) = \{\, u \in H^1(D,\mathbb{R}^3) \mid u(z) \in C \text{ for a.e. } z \in D \,\}.$$

Let $h_n \in H^1(D,C)$ satisfy

$$\text{(A.2)} \qquad \int |\nabla h_n|^2 = \operatorname*{Min}_{\substack{v \in H^1(D,C) \\ v - h_n \in H^1_0}} \int |\nabla v|^2 \quad \text{and} \quad \operatorname*{Sup}_n \| h_n \|_{H^1(D)} < +\infty$$

$$\text{(A.3)} \qquad h_n|_{\partial D} \in H^1(\partial D, \mathbb{R}^3) \quad \text{and} \quad \operatorname*{Sup}_n \| h_n|_{\partial D} \|_{H^1(\partial D)} < +\infty .$$

Proposition A.1. *Let h_n satisfy (A.2), (A.3). Then if $h_n \rightharpoonup h$, we have*

$$\text{(i)} \qquad \int |\nabla h|^2 = \operatorname*{Min}_{\substack{v \in H^1(D,C) \\ v - h \in H^1_0}} \int |\nabla v|^2 ;$$

$$\text{(ii)} \qquad \int |\nabla h_n|^2 \to \int |\nabla h|^2 .$$

Proof. Since $\int |\nabla h|^2 \leq \liminf \int |\nabla h_n|^2$ it is enough to prove

$$\text{(A.4)} \qquad \limsup \int |\nabla h_n|^2 \leq \operatorname*{Inf}_{\substack{v \in H^1(D,C) \\ v - h \in H^1_0}} \int |\nabla v|^2 .$$

To prove (A.4), let us consider, for $r \in]0,1[$:

$$v_n^r(s,\vartheta) := \frac{\log r/s}{\log r} \left[h_n(1,\vartheta) - h(1,\vartheta) \right] + h(1,\vartheta)$$

(in polar coordinates). Since

$$\sup_{\substack{0\le\vartheta<2\pi\\ r\le s\le 1}} |v_n^r(s,\vartheta) - h(1,\vartheta)| \le |h_n - h|_{L^\infty(\partial D)} \to 0 \quad \text{as} \quad n\to\infty ,$$

πv_n is well defined on $D\setminus D_r$, where π is the retraction given by (A.1). Moreover

$$\text{(A.5)} \qquad \int_{D\setminus D_r} |\nabla(\pi v_n^r)|^2 \le L^2 \int_{D\setminus D_r} |\nabla v_n^r|^2$$

if L denotes the Lipschitz constant for π. Now, let $\hat{h}\in H^1(D,C)$ be such that

$$\hat{h} - h \in H_o^1 \quad \text{and} \quad \int |\nabla\hat{h}|^2 = \min_{\substack{v\in H^1(D,C)\\ v-h\in H_o^1}} \int |\nabla v|^2$$

and we define

$$\omega_n(z) := \begin{cases} \pi v_n^r(z) & \text{if } z\in D\setminus D_r \\ \hat{h}\left(\frac{z}{r}\right) & \text{if } z\in D_r \end{cases} .$$

Since $\omega_n\in H^1(D,C)$ and $\omega_n = h_n$ on ∂D, we have

$$\int_D |\nabla h_n|^2 \le \int_D |\nabla \omega_n|^2 \qquad \text{while} \qquad \int_D |\nabla\omega_n|^2 \le L^2 \int_{D\setminus D_r} |\nabla v_n^r|^2 + \int_D |\nabla\hat{h}|^2 .$$

An easy computation gives, using (A.3)

$$\int_{D\setminus D_r} |\nabla v_n^r|^2 = O(|\log r|)$$

and hence $\limsup \int |\nabla h_n|^2 \le O(|\log r|) + \int |\nabla\hat{h}|^2$ for every $r\in]0,1[$, i.e. (A.4). ∎

<u>***Corollary A.2.***</u> *Let* h_n *satisfy (A.2). If* $h_n \rightharpoonup h$, *then*

$$\text{(i)} \qquad \int |\nabla h|^2 = \min_{\substack{v\in H^1(D,C)\\ v-h\in H_o^1}} \int |\nabla v|^2 ;$$

$$\text{(ii)} \qquad h_n \to h \qquad \text{in } H^1_{loc}(D,\mathbb{R}^3) .$$

Proof. For a.e. $r<1$ we have $\sup \| h_{n\,|\partial D} \|_{H^1(\partial D_r)} < +\infty$. Since clearly

$$\int_{D_r} |\nabla h_n|^2 = \min_{\substack{v\in H^1(D_r,C)\\ v-h_n\in H_o^1(D_r)}} \int |\nabla v|^2 ,$$

Proposition A.1 applies to obtain $h_n \to h$ in $H^1(D_r)$ and $\int_D |\nabla h|^2 \le \int_D |\nabla v|^2$ for every $v \in H^1(D_r, C)$ with $v - h \in H^1_o(D_r)$. Thus, if $\omega \in H^1(D,C)$, $\omega - h \in H^1_o(D)$, setting

$$\omega_r(z) := \begin{cases} h\left(\frac{z}{|z|^2} r^2\right) & \text{if } r^2 \le |z| \le r \\ \omega(z/r^2) & \text{if } |z| \le r^2 \end{cases}$$

we see that, for a.e. $r < 1$:

$$\int_{D_r} |\nabla h|^2 \le \int_{D_r} |\nabla \omega_r|^2 \tag{A.6}$$

because $\omega_r - h \in H^1_o(D_r)$. But

$$\int_{D_r} |\nabla \omega_r|^2 = \int_{D_r} |\nabla \omega|^2 + \int_{\{r<|z|<1\}} |\nabla h|^2$$

and hence, sending r to 1 in (A.6) we get (i). ∎

Remark A.3. To complete these continuous dependence results, it would be of interest to prove that, if in addition to the assumptions in Corollary A.2, one also assumes $h_n \to h$ uniformly on ∂D, then $h_n \to h$ uniformly on D. Since we don't need this result, we don't go into details. ∎

Appendix B.

For convenience of the reader we list here a few simple properties of the volume functional (see [21] and [2], Appendix).

Lemma B.1. *Let* $k^n, \psi^n, \psi \in H^1 \cap L^\infty(D, \mathbb{R}^3)$. *Assume that* $k^n \to 0$ *in* L^∞ *and weakly in* H^1, *and* $\psi^n - \psi \rightharpoonup 0$ *in* $H^1_o(D,\mathbb{R}^3)$. *Then*

$$\int_D (\psi^n + k^n) \cdot (\psi^n + k^n)_x \wedge (\psi^n + k^n)_y = \int_D \psi^n \cdot (\psi^n)_x \wedge (\psi^n)_y + o(1).$$

Proof. From (1.2) one sees that

$$\int_D (\psi^n + k^n) \cdot (\psi^n + k^n)_x \wedge (\psi^n + k^n)_y = \int_D \psi^n \cdot (\psi^n)_x \wedge (\psi^n)_y +$$
$$+ \int_D \psi^n \cdot [(\psi^n)_x \wedge (k^n)_y + (k^n)_x \wedge (\psi^n)_y] + \int_D (\psi^n - \psi) \cdot (k^n)_x \wedge (k^n)_y + o(1)$$

because $k^n \to 0$ in L^∞ and weakly in H^1. Now, the second integral in the right hand side goes to zero by Lemma A.7 in [2], while the third one goes to zero by Lemma A.6 in [2]. ∎

__*Lemma B.2*__. *Let* $g \in C^0(\bar{D},\bar{D})$ *be an orientation preserving bilipschitz homeomorphism. Then*

$$V_D(u) = V_D(u\circ g) \qquad \textit{for every} \quad u \in H^1 \cap L^\infty(D,\mathbb{R}^3).$$

This follows from the chain rule:

$$\int \det(u\circ g, \nabla(u\circ g)) = \int \det(u(g(z)),(\nabla u)(g(z))) \det J_g . \qquad ∎$$

__*Lemma B.3*__. *Let* $\alpha : [0,2\pi] \to [0,2\pi]$ *be a nondecreasing function, with* $\alpha(0) = 0$, $\alpha(2\pi) = 2\pi$. *Let* $g(r,\vartheta) := r\, e^{i\,\alpha(\vartheta)}$. *Then*

$$V_D(u\circ g) = V_D(u) \qquad \textit{for every} \quad u \in C^1(\bar{D}).$$

Proof. Since $J_g = r\,\alpha'$, $\det(u\circ g, \nabla(u\circ g)) \in L^1$. After properly extending α, we can regularize it to get $\alpha_n \in C^\infty$, $\alpha_n \to \alpha$ uniformly, $\alpha'_n \to \alpha'$ in L^1 , $\alpha_n(2\pi) = 2\pi + \alpha_n(0)$ and $\alpha_n \geq 0$ in $[0,2\pi]$. Then we set

$$\tilde{\alpha}_n(\vartheta) := \frac{n}{n+1}\,\alpha_n(\vartheta) + \frac{1}{n+1}\,\vartheta ,$$

$$\tilde{g}_n(r,\vartheta) = r\, e^{i\,\tilde{\alpha}_n(\vartheta)} .$$

By the previous Lemma we get $V_D(u\circ\tilde{g}_n) = V_D(u)$. But

$$V_D(u\circ\tilde{g}_n) = \int_D \det(u\circ\tilde{g}_n, \nabla(u\circ\tilde{g}_n)) = \int_D \det(u(\tilde{g}_n(z)), (\nabla u)(\tilde{g}_n(z)))\, J_{\tilde{g}_n}\, dz =$$

$$= \int_D \det(u(g(z)),(\nabla u)(g(z)))\det J_g + o(1) = \int_D \det(u\circ g, \nabla(u\circ g)) + o(1)$$

because $\tilde{\alpha}_n \to \alpha$ uniformly and $u \in C^1(\bar{D})$ imply

$$\det(u(\tilde{g}_n(z)),(\nabla u)(\tilde{g}_n(z))) \to \det(u(g(z)),(\nabla u)(g(z)))$$

uniformly, while

$$\det J_{\tilde{g}_n} \to \det J_g \quad \text{in } L^1 \qquad ∎$$

<u>***Corollary B.4.***</u> *Assume Γ is parametrizable with a diffeomorphism $\gamma^\circ : \partial D \to \mathbb{R}^3$. Then*

$$\frac{1}{4\pi}\left\{V_D(pu) - V_D(pv)\right\} \in \mathbb{Z} \qquad \forall\ u, v \in X_\Gamma(C)\ .$$

Proof. Given $\delta > 0$, let $h \in C^1(\bar{D}, \mathbb{R}^3 \setminus B_\delta)$ with $h_{|\partial D} = \gamma^\circ$ (assuming for simplicity $0 \notin C$). It is enough to prove

$$\frac{1}{4\pi}\left\{V_D(pu) - V_D(ph)\right\} \in \mathbb{Z} \qquad \forall\ u \in X_\Gamma(C)\ .$$

Since, for a given $u \in X_\Gamma(C)$, $u_{|\partial D}$ is a weakly monotone reparametrization of Γ, there is a map $\alpha_u : [0,2\pi] \to [0,2\pi]$, continuous and nondecreasing, with $\alpha_u(0) = 0$, $\alpha_u(2\pi) = 2\pi$, such that

$$u\left(e^{i\vartheta}\right) = \gamma^\circ\left(e^{i\,\alpha_u(\vartheta)}\right)\ .$$

By Lemma B.3, setting $g_u(r,\vartheta) = r\, e^{i\,\alpha_u(\vartheta)}$, we have $V_D(p \circ h) = V_D(p \circ h \circ g_u)$ and hence

$$\frac{1}{4\pi}\left\{V_D(pu) - V_D(ph)\right\} = \frac{1}{4\pi}\left\{V_D(pu) - V_D(ph \circ g_u)\right\} \in \mathbb{Z}\ ,$$

because $u(z) = h(g_u(z))$ for every $z \in \partial D$, so that Lemma 1 in [1] applies. ∎

<u>***Remark B.5.***</u> Actually, the above result holds for any Γ such that $X_\Gamma(C) \neq \emptyset$ ∎

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S^2 - TYPE MINIMAL SURFACES ENCLOSING MANY OBSTACLES IN $\mathbb{R}^3$

In a joint paper with G. Mancini [8] we have discussed the problem of existence of a closed surface having minimal area among all surfaces which are parametrized by S^2 and which enclose a given connected body Ω in $\mathbb{R}^3$. Taking advantage of the fact that the second homotopy group of $\mathbb{R}^3 \setminus \Omega$ is not trivial, we can prove our existence result by showing that there exists a minimum for the energy functional on the class of maps from S^2 into $\mathbb{R}^3 \setminus \Omega$ which are not homotopic to a constant.

The argument above suggests the possibility of finding more solutions if, for example, we enrich the topology of the target space by letting the obstacle to be more complex.

In this paper we drop out the connectivity assumption on Ω, and we study the behaviour of minimizing sequences for the energy functional on the class of closed surfaces which "envelope" every connected component of Ω. The goal is to find a harmonic map from S^2 into $\mathbb{R}^3 \setminus \Omega$ which is not homotopic to a constant in the complement of each connected component of Ω.

We first go back to a minimization problem in $\mathbb{R}^2$ via composition with the stereograhpic projection. Using the same tools as in [8] we are able to define a suitable class X^e of maps $\mathbb{R}^2 \to \mathbb{R}^3 \setminus \Omega$ which "enclose", in a weak sense, every connected component of the obstruction Ω, and we study the minimization problem

$$I := \operatorname*{Min}_{U \in X^e} \int_{\mathbb{R}^2} |\nabla U|^2 . \tag{0.1}$$

In the case of a connected obstacle, the infimum in (0.1) is always achieved, while in the general case the invariance of the Dirichlet integral and the constraint X^e with respect to translations and dilations in $\mathbb{R}^2$ may produce the phenomena of dichotomy for minimizing sequences (see Proposition 2.2). Consequently, there might exist minimizing sequences which are not compact up to translations and changes of scale.

In order to guarantee compactness of all minimizing sequences we need a "Douglas criterion", i.e. we require a family of strict inequalities hold true, the number of these inequalities depending on the number of connected component of Ω (see Theorem 2.3).

Similar hypothesis appear often in problems where the invariance with respect to the non compact group of dilations in $\mathbb{R}^N$ might produce lack of compactness at some energy levels: see for example the collection of problems in [6], [7], [5].

<u>Notations</u>. $D_r(z)$ will denote the open disk of radius r and center z in $\mathbb{R}^2$, and we will often write $D_r = D_r(0)$. With $|\_|$, $\_\cdot\_$, $\_\wedge\_$, we denote respectively the norm, the scalar and the vector product in $\mathbb{R}^3$, while $|\_|_2$, $|\_|_\infty$ are the L^2 , L^∞ norms respectively.

If $(U_n)_n \subseteq L^\infty(\mathbb{R}^2,\mathbb{R}^3)$, $\nabla U_n \in L^2(\mathbb{R}^2,\mathbb{R}^6)$, we will say that U_n converges (weakly) to U, if

$U_n \to U$ a.e. and $\nabla U_n \to \nabla U$ in L^2 (respectively: weakly in L^2),

and we shall write $U_n \to U$ (resp. $U_n \rightharpoonup U$).

1. The Variational Problem.

Let $(\Omega_i)_{i\in F}$ be a finite collection of disjoint subsets of $\mathbb{R}^3$. We suppose that for each index $i\in F$, Ω_i is an open, bounded, connected set, and we define C as the closure of the unbounded connected component of $\mathbb{R}^3\setminus\overline{\cup\Omega_i}$.

Throughout the paper we assume that

(1.1) $\qquad$ C *is a lipschitz neighbourhood retract,*

that is there exists a lipschitz retraction of an open neighbourhood of C into C. We set

$$X(C) := \{\ U\in L^\infty(\mathbb{R}^2,\mathbb{R}^3)\ \mid\ U(z)\in C \text{ for a.e. } z\in\mathbb{R}^2,\ \nabla U\in L^2(\mathbb{R}^2,\mathbb{R}^6)\ \}.$$

Notice that $X(C)$ is closed with respect to weak convergence in Notations.

Assumption (1.1) allows us to prove, as in [11], the following density result:

LEMMA 1.1. *For every* $U\in X(C)$, *there exists a sequence* $U_n\in X(C)\cap C^0(\mathbb{R}^2,\mathbb{R}^3)$ *s.t.*

(i) *for each* n, U_n *is constant far out;*

(ii) $U_n\to U$ *a.e.*, $\nabla U_n\to\nabla U$ *in* $L^2(\mathbb{R}^2,\mathbb{R}^6)$.

In order to give our definition of maps $U\in X(C)$ which enclose each of the Ω_i' s, we will use, as in [8], the Volume Functional

$$V(w)=\int w\cdot w_x\wedge w_y \qquad \text{for } w\in L^\infty(\mathbb{R}^2,\mathbb{R}^3),\ \nabla w\in L^2(\mathbb{R}^2,\mathbb{R}^6)$$

(see for example [12], [1], [2]). For every $i\in F$, we fix a point $\xi_i\in\Omega_i$, and we define the map

$$p_i : C\to S^2 \quad , \quad p_i u = \frac{u-\xi_i}{|u-\xi_i|} .$$

Since p_i is lipschitz continuous on C, it turns out that for every $U\in X(C)$, $V(p_iU)$ is well defined by Hölder inequality.

Moreover, if $U\in X(C)$ is continuous and regular at infinity, that is

$$\text{there exists} \qquad \lim_{|z|\to\infty} U(z) =: U(\infty) ,$$

then $V(p_iU)/4\pi$ is an integer, and gives the degree of $p_iU\circ\Pi : S^2\to S^2$, where Π is the stereographic projection (see [9], [1]). In particular, since Ω_i is connected, $V(p_iU)$ does not depend on the choiche of the point ξ_i in Ω_i. In addition, if $V(p_iU)\neq 0$ then $U\circ\Pi$ is not homotopic to a constant as a map $S^2\to\mathbb{R}^3\setminus\Omega_i$.

Thanks to the density Lemma, the set

$$X^e(c) := \{ U \in X(C) \mid V(p_i U) \neq 0 \quad \forall\, i \in F \}$$

can be regarded as the class of admissible functions which "enclose" each obstacle Ω_i. In fact, if $(U_n)_n \subseteq X(C)$, $U_n \to U$ (see Notations in the Introduction), then

$$V(p_i U_n) \to V(p_i U) \qquad \text{for every } i \in F.$$

Remark 1.2. Since $V(p_i U)/4\pi \in \mathbb{Z}$ $\forall\, U \in X(C)$ and $\forall\, i \in F$, Hölder inequality implies:

$$4\pi \leq |V(p_i U)| \leq \frac{1}{2d(\xi_i, \partial\Omega_i)^2} |\nabla U|_2^2 \qquad \text{if } V(p_i U) \neq 0 .$$

Thus, there exists $\delta° > 0$ such that

$$|\nabla U|_2^2 \geq \delta° \qquad \text{if } U \in X(C) ,\quad V(p_i U) \neq 0 \quad \text{for some } i \in F . \qquad ∎$$

The aim of this paper is to study the minimization Problem:

$$(1) \qquad \begin{cases} \textit{find} \quad U_\infty \in X^e(C) \quad \textit{such that} \\ \displaystyle\int_{\mathbb{R}^2} |\nabla U_\infty|^2 = I^F(C) := \operatorname*{Inf}_{U \in X^e(C)} \int_{\mathbb{R}^2} |\nabla U|^2 . \end{cases}$$

Before ending this Section we notice that a solution to Problem 1 corresponds to a harmonic map $S^2 \to C$. In fact, it turns out that the map $U_\infty \circ \Pi \in H^1(S^2, C)$ minimizes the energy functional on its homotopy class.

In addition we can prove as in [8], Theorem 2.1, that in case ∂C is regular enough, then U_∞ is a closed, regural surface and it has minimal area in the class $X^e(C)$.

We summarize these results in the following

THEOREM 1.3. *Suppose* $U_\infty \in X^e(C)$ *solves Problem 1. Then*

(i) $U_\infty \circ \Pi$ *is harmonic as a map* $S^2 \to C$.

(ii) *If* ∂C *is of class* C^2, *then* $U_\infty \in C^{1,\alpha}(\mathbb{R}^2, \mathbb{R}^3)$, U_∞ *is regular at infinity and conformal on* $\mathbb{R}^2$:

$$|(U_\infty)_x|^2 - |(U_\infty)_y|^2 = 0 = (U_\infty)_x \cdot (U_\infty)_y.$$

Moreover, U_∞ *minimizes the area functional on* $X^e(C)$:

$$\int |(U_\infty)_x \wedge (U_\infty)_y| \leq \int |U_x \wedge U_y| \qquad \forall\, U \in X^e(C).$$

2. *The Existence Result.*

This Section is divided into two parts: in the first one we describe our "Douglas Criterion", and in the second one we state and prove the existence result.

The Douglas Criterion.

In This Section we only require the constraint C satisfies (1.1). If $A \subseteq F$ is any subset of F, we set

$$I^A(C) := \begin{cases} \mathrm{Inf}\left\{ \int |\nabla U|^2 \mid U \in X(C)\,,\ V(p_i U) \neq 0 \quad \forall\, i \in A \right\} & \text{if } A \neq \varnothing\,, \\ 0 & \text{if } A = \varnothing\,. \end{cases}$$

Clearly, $I^F(C) \geq I^A(C)$ and $I^A(C) > 0$ if $A \neq \varnothing$ (see Remark 1.2). We shall prove our existence result whenever the following "Douglas Criterion" holds true:

$$\text{(D)} \qquad I^F(C) \lneqq \sum_{i=1}^{m} I^{A_i}(C) \qquad \text{for any family of } m \geq 2 \text{ not empty subsets of } F, \text{ s.t. } \bigcup_i A_i = F\,.$$

Notice that in case of one connected obstacle no restrictions are required, i.e. (D) is automatically verified.

Condition (D) has a simpler form when $|F| \leq 3$, by means of the following

PROPOSITION 2.1. *Suppose* $|F| \leq 3$. *Then*

$$\text{(2.1)} \qquad I^{A \cup B} \leq I^A + I^B \qquad \forall\, A, B \subseteq F\,.$$

From (2.1) follows that $I^F \leq I^{F\setminus A} + I^A \quad \forall\, A \subset F$ and thus condition (D) is equivalent to

$$\text{(D')} \qquad I^F < I^{F\setminus A} + I^A \qquad \forall\, A \subset F\,,\ A \neq \varnothing\,.$$

It is quite likely that 2.1 holds for any number of obstacles; we limit ourself to the case $|F| \leq 3$ since the proof in the general case seems to be more complicated.

The construction we will use in the proof of Proposition 2.1 shows that the Douglas criterion (D) is a necessary condition for compactness, up to translations and changes of scale in $\mathbb{R}^2$, of all minimizing sequences:

PROPOSITION 2.2. *Suppose* $|F| \leq 3$, *and suppose there exists* $A \subset F$ *not empty, s.t.*

$$I^F = I^{F \setminus A} + I^A.$$

Then there exists a minimizing sequence for Problem 1 such that none of those obtained from it by translations and dilations in $\mathbb{R}^2$ *is compact.*

Proof of Proposition 2.1. We can suppose A, B to be not empty, and

$$I^A(C) < I^{A \cup B}(C) \quad , \quad I^B(C) < I^{A \cup B}(C). \tag{2.2}$$

Let $(U_n)_n$, $(W_n)_n \subset X(C)$ be minimizing sequences for $I^A(C)$, $I^B(C)$ respectively, that is

$$\begin{cases} V(p_i U_n) \neq 0 \quad \forall\, i \in A \quad , & \int |\nabla U_n|^2 = I^A + o(1)\,; \\ V(p_i W_n) \neq 0 \quad \forall\, i \in B \quad , & \int |\nabla W_n|^2 = I^B + o(1)\,. \end{cases} \tag{2.3}$$

Because of the density Lemma, and the invariance of the Dirichlet integral and the Volume functional with respect to dilations in $\mathbb{R}^2$, we can choose U_n, W_n to be continuous and constant outside the unit disk. Using the sequences $(U_n)_n$, $(W_n)_n$, we will construct a sequence $(\Phi_n)_n \subset X(C)$ such that

$$\int |\nabla \Phi_n|^2 = I^A(C) + I^B(C) + o(1)\,; \tag{2.4}$$

$$V(p_i \Phi_n) \neq 0 \qquad \forall\; i \in A \cup B \tag{2.5}$$

and this will conclude the proof of Proposition 2.1.

Hypotheses (2.2) immediately implies:

$$\exists\; a \in A\,,\; \exists\; b \in B \;\text{ s.t. }\; a \neq b \;\text{ and (for a subsequence)}$$

$$V(p_a W_n) = 0 \quad , \quad V(p_b U_n) = 0,$$

otherwise $I^B(C) = I^{A \cup B}(C)$ or $I^A(C) = I^{A \cup B}(C)$. Let us define the sets

$$N := \{\, i \in A \cup B \mid V(p_i U_n) = V(p_i W_n) \,\}$$

$$\tilde{N} := \{\, i \in A \cup B \mid V(p_i U_n) = -V(p_i W_n) \,\}\,.$$

Since $|V(p_i U_n)| + |V(p_i W_n)| \neq 0 \;\; \forall\; i \in A \cup B$, then $N, \tilde{N}$ are disjoint (eventually empty), and $a, b \notin N \cup \tilde{N}$. Since $\{a,b\} \cup N \cup \tilde{N} \subset A \cup B$ and $A \cup B$ contains at most three indexes, then either $N = \emptyset$ or $\tilde{N} \neq \emptyset$, and in this case $N = \emptyset$.

Now, suppose $N = \emptyset$, and define

$$\Phi_n(z) := \begin{cases} U_n(n^2 z) & \text{if } |z| \le 1/n \\ \gamma\left(\frac{\lg n\,|z|}{2\lg n}\right) & \text{if } 1/n < |z| < n \\ W_n\left(\frac{n^2 z}{|z|^2}\right) & \text{if } |z| \ge n \end{cases}$$

where $\gamma : [0,1] \to C$ is a smooth (lipschitz) path joining $U_n(\infty)$ with $W_n(\infty)$. Notice that Φ_n is well defined and continuous on $\mathbb{R}^2$. Easy computations show that $\Phi_n \in X(C)$ and

$$\int_{\{|z|<1/n\}} |\nabla\Phi_n|^2 = \int_{\{|z|<n\}} |\nabla U_n|^2 = I^A(C) + o(1)\,;$$

$$\int_{\{|z|>n\}} |\nabla\Phi_n|^2 = \int_{\{|z|<n\}} |\nabla W_n|^2 = I^B(C) + o(1)\,;$$

because U_n, W_n are constant outside the unit disk. Moreover,

$$\int_{\{1/n<|z|<n\}} |\nabla\Phi_n|^2 = o(1) \quad \text{as } n \to \infty, \tag{2.6}$$

and (2.4) follows immediately. Using again (2.6), we can prove that (for a subsequence)

$$V(p_i;\Phi_n) = V(p_i;U_n) - V(p_i;W_n) \qquad \forall\, i \in F$$

and thus $(\Phi_n)_n$ verifies (2.5), since by hypotheses $N = \emptyset$.

If $N \ne \emptyset$, and hence $\tilde{N} = \emptyset$, we replace W_n by

$$\tilde{W}_n(x,y) := W_n(y,x)$$

in the definition of $(\Phi_n)_n$. Since

$$\int |\nabla\tilde{W}_n|^2 = \int |\nabla W_n|^2 \quad , \quad V(p_i;\tilde{W}_n) = -V(p_i;W_n) \quad \forall\, i \in F,$$

and $\tilde{N} = \emptyset$, we can prove as before that the new sequence $(\Phi_n)_n$ satisfies (2.4), (2.5), and the proof is complete. ∎

Proof of Proposition 2.2. Let us go back to the proof of Proposition 2.1; if equality holds in (2.1) for some not empty subset $A \subset F$, then the sequence $(\Phi_n)_n$ is minimizing for the minimum problem $I^F(C)$. Suppose by contradiction that there exist sequences $(z_n)_n \subset \mathbb{R}^2$, $(t_n)_n \subset]0,+\infty[$ s.t.

$$\Phi_n \to \Phi\ , \quad \nabla\Phi_n \to \nabla\Phi \ \text{ in } \ L^2(\mathbb{R}^2,\mathbb{R}^6) \tag{2.7}$$

for some $\Phi \in X(C)$, where

$$\Phi_n(z) = \Phi_n\left(\frac{z - z_n}{t_n}\right).$$

From (2.7), and from the continuity properties of the Volume Functional it follows in particular that $\Phi \in X^e(C)$, and hence

$$\int_{\mathbb{R}^2} |\nabla\Phi|^2 = I^A(C) + I^{F\setminus A}(C) = I^F(C).$$

Taking into account the definition of the sequence $(\Phi_n)_n$ we easly find

$$\int_{\{|z-z_n|<t_n/n\}} |\nabla\Phi_n|^2 = I^A + o(1)\ , \qquad \int_{\{|z-z_n|>t_n n\}} |\nabla\Phi_n|^2 = I^B + o(1).$$

We can immediately exclude the case $(z_n)_n$ bounded, since in this case (2.7) would imply $t_n/n \geq \varepsilon > 0$, $t_n\, n \leq \text{const.} < \infty$, which is impossible. On the other hand, by simple computations we see that in case $|z_n| \to \infty$, then both the sequences $|z_n| - t_n/n$, $|z_n| - t_n n$ must be bounded, and this is again an absurd. ∎

The Existence Theorem.

Proposition 2.2 shows that the Douglas criterion (D) is a necessary condition for compacness of all minimizing sequences. Actually, (D) it is also sufficient to prevent the phenomena of dichotomy:

THEOREM 2.3. $I^F(C)$ *is achieved in* $X^e(C)$, *provided* (D) *holds.*

Remark 2.4. In case of one connected obstacle: $|F| = 1$, no conditions are required. Thus, Theorem 2.3 includes the existence Theorem in [8]. ∎

The proof of Theorem 2.3 is based on the analysis, via a blow-up technique, of sequences $(U_n)_n \subset X^e(C)$ whose weak limit (see Notations) does not belong to $X^e(C)$. This technique was introduced in this framework by Sacks and Uhlenbeck [10].

To perform the blow-up technique, we will use a Lemma in [4] (see also [13]), which applies to sequences $(U_n)_n \subset X(C)$, s.t. $(U_n)_n$ is bounded in L^∞ and $(|\nabla U_n|)_n$ is bounded in L^2. For such a sequence, we can find a subsequence $(U_n)_n$, a map $U \in X(C)$, and for any index j we can find a finite set of distinct points $a^j_1, ..., a^j_{k_j}$ in $\mathbb{R}^2$, and a finite set of integers $d^j_1, ..., d^j_{k_j}$ such that

(2.8) $$U_n \to U \text{ a.e.}, \quad \nabla U_n \rightharpoonup \nabla U \quad \text{weakly in } L^2,$$

(2.9) $$\det(p_jU_n, \nabla(p_jU_n)) \rightharpoonup \det(p_jU, \nabla(p_jU)) + 4\pi \sum_{h=1}^{k_j} d^j_h \, \delta_{a^j_h} \qquad \forall\, j \in F,$$

weakly in the sense of measures.

If a is any point in $\mathbb{R}^2$, we can consider the set of indexes j such that p_jU_n "concentrates" at a:

$$A_a = \{\, j \in F \mid a^j_{h_j} = a \;,\; d^j_{h_j} \neq 0 \text{ for some } h_j \,\}.$$

Notice that if $A_a \neq \emptyset$, then $\forall\, j \in A_a$ there exist a unique h_j s.t. $a^j_{h_j} = a$, $d^j_{h_j} \neq 0$.

Similarly, $A_\infty \subset F$ will be the set of indexes j such that p_jU_n "concentrates" at infinity:

$$A_\infty = \{\, j \in F \mid c_j \neq 0\}, \quad \text{where}$$

$$c_j := \frac{1}{4\pi} \lim \left[V(p_j U_n) - V(p_j U) \right] - \sum_{h=1}^{k_j} d_h^j \in \mathbb{Z} .$$

The sets A_a, A_∞ could possibly be empty.

The following Proposition is a crucial step in the proof of Theorem 2.3:

PROPOSITION 2.5. *Let* $(U_n)_n \subset X(C)$ *be as above. Then*

(i) $$\liminf_n \int_{D_\rho(a)} |\nabla U_n|^2 \geq \int_{D_\rho(a)} |\nabla U|^2 + I^{A_a}(C) \quad \forall\, a \in \mathbb{R}^2,\ \forall\, \rho \text{ small};$$

(ii) $$\liminf_n \int_{\mathbb{R}^2 \setminus D_R} |\nabla U_n|^2 \geq \int_{\mathbb{R}^2 \setminus D_R} |\nabla U|^2 + I^{A_\infty}(C) \qquad \forall\, R \text{ large.}$$

Proof. We follow here the outline of the proofs of Propositions 2.3, 2.4 in [8]. Remark that in case $A_a = \emptyset$, or $A_\infty = \emptyset$ the estimates in Proposition follow directly from weak lower semicontinuity of the Dirichlet integral.

(i). Suppose $A_a \neq \emptyset$ for some $a \in \mathbb{R}^2$. Fix $\rho > 0$ s.t. $D_{2\rho}(a)$ does not contain any concentration point different from a. Passing eventually to a subsequence, we can suppose $\int_{D_\rho(a)} |\nabla U_n|^2$ has a limit as $n \to \infty$. For almost every $r < \rho$, we can find a subsequence $(U_{n_m})_m$ for which the sequence of traces on $\partial D_r(a)$ is bounded in $H^1(\partial D_r(a), \mathbb{R}^3)$. Thus, by using Proposition A.1 in [8] we can prove that

$\exists\ h_m \in H^1(D_r(a), C)$, $h_m = U_{n_m}$ on $\partial D_r(a)$, such that (for some subsequence)

$$\lim_m \int_{D_r(a)} |\nabla h_m|^2 \leq \int_{D_r(a)} |\nabla U|^2 . \tag{2.10}$$

Next, we define the map

$$\tilde{U}_m(z) := \begin{cases} U_{n_m}(z) & \text{if } |z-a| < r \\ h_m\left(\dfrac{z-a}{|z-a|^2}\, r^2 + a \right) & \text{if } |z-a| \geq r . \end{cases}$$

By a simple computation, it turns out that the map $\widetilde{U}_m$ belongs to X(C), and using (2.10) we get that for every $j \in A_a$ and k large enough

$$|V(p_j\widetilde{U}_m)| \geq \Big| \int_{D_r(a)} \det(p_jU_{n_m}, \nabla(p_jU_{n_m})) \Big| - \Big| \int_{D_r(a)} \det(p_jh_m, \nabla(p_jh_m)) \Big| \geq$$

$$\geq \Big| 4\pi d^j_{h_j} \Big| - \text{const.} \int_{D_r(a)} |\nabla U|^2 + o(1) \geq 4\pi - \varepsilon \gneq 0$$

where h_j is the unique index s.t. $d^j_{h_j} \neq 0$, $a^j_{h_j} = a$ (see the definition of the set A_a). Again from (2.10), and from the definition of $I^{A_a}(C)$, we infer that

$$I^{A_a}(C) \leq \int_{\mathbb{R}^2} |\nabla\widetilde{U}_m|^2 = \int_{D_r(a)} |\nabla U_{n_m}|^2 + \int_{D_r(a)} |\nabla h_m|^2 .$$

Thus, we have proved that for a.e. $r < \rho$

$$\lim \int_{D_\rho(a)} |\nabla U_n|^2 = \lim \Big[\int_{D_\rho \setminus D_r} |\nabla U_{n_m}|^2 + \int_{D_r} |\nabla U_{n_m}|^2 \Big] \geq \int_{D_\rho(a)} |\nabla U|^2 + I^{A_a}(C) - 2\int_{D_r(a)} |\nabla U|^2$$

and (i) follows by letting r go to zero.

(ii) We can apply the same argument to the sequence $U^*_n(z) := U_n(z/|z|^2)$. Notice that $U^*_n \rightharpoonup U^*$, where $U^*(z) = U(z/|z|^2)$. Using again the Lemma in [4] we get:

$$\text{(2.11)} \qquad \det(p_jU^*_n, \nabla(p_jU^*_n)) \rightharpoonup \det(p_jU^*, \nabla(p_jU^*)) - 4\pi \sum_{\substack{h=1 \\ a^j_h \neq 0}}^{k_j} d^j_h \, \delta_{b^j_h} - 4\pi c_j \delta_0$$

where $b^j_h = a^j_h / |a^j_h|^2$ if $a^j_h \neq 0$, and c_j are the integers defined above. In particular $c_j \neq 0$ iff $j \in A_\infty$, that is, using notations above, $A_\infty = A^*_0$. Thus, from the first part of the Proposition we infer

$$\liminf_n \int_{D_\rho} |\nabla U^*_n|^2 \geq \int_{D_\rho} |\nabla U^*|^2 + I^{A_\infty}(C) \qquad \forall\, \rho \text{ small},$$

which immediately implies (ii), with $R = 1/\rho$. ∎

Applying Proposition 2.5 to a minimizing sequence $(U_n)_n$ we get the following

PROPOSITION 2.6. *Suppose that the Douglas criterion* (D) *is satisfied. Let* $(U_n)_n$ *be a minimizing sequence for Problem 1, and suppose* $U_n \rightharpoonup U$ *for some* $U \in X(C)$. *If* $U \notin X^e(C)$ *then* U *is* <u>*constant*</u> *and*

either $\quad \exists\,!\ a \in \mathbb{R}^2 \quad$ *s.t.* $\quad \liminf \int_{D_\rho(a)} |\nabla U_n|^2 = I^F(C) \quad \forall\, \rho > 0,$

or $\quad \nabla U_n \to 0 \quad$ *in* $\quad L^2_{loc}(\mathbb{R}^2, \mathbb{R}^6).$

Proof. By hypothesis, the set

$$B := \{j \in F \mid V(p_j U) = 0\}$$

is not empty. Comparing (2.9), (2.11), we get the existence of a finite (maybe empty) set of points $a_1, \ldots, a_m$ in $\mathbb{R}^2$ such that

$$A_\infty \cup \bigcup_{i=1}^{m} A_{a_i} = B\,.$$

If r is small enough, from Proposition 2.5 we infer

$$I^F(C) = \lim \Big[\int_{D_{1/r} \setminus \bigcup_i D_r(a_i)} |\nabla U_n|^2 + \int_{\mathbb{R}^2 \setminus D_{1/r}} |\nabla U_n|^2 + \sum_{i=1}^{m} \int_{D_r(a_i)} |\nabla U_n|^2 \Big] \geq$$

$$\geq \int_{\mathbb{R}^2} |\nabla U|^2 + I^{A_\infty}(C) + \sum_{i=1}^{m} I^{A_{a_i}}(C)$$

that is, using the definition of $I^{F \setminus B}(C)$,

$$I^F(C) \geq \int_{\mathbb{R}^2} |\nabla U|^2 + I^{A_\infty}(C) + \sum_{i=1}^{m} I^{A_{a_i}}(C) \geq I^{F\setminus B}(C) + I^{A_\infty}(C) + \sum_{i=1}^{m} I^{A_{a_i}}(C)\,. \tag{2.12}$$

From (2.12) and hypothesis (D) it follows that at most one of the sets $F \setminus B$, A_∞, A_a is not empty. Since $B \neq \emptyset$ we first get

$$F = B \quad \text{and } U = \text{constant}.$$

Moreover, either there exists a unique $\underline{a} \in \mathbb{R}^2$ s.t. $A_{\underline{a}} \neq \emptyset$, and in this case $A_{\underline{a}} = F$, or $A_\infty = F$. In view of Proposition 2.5 these two cases correspond to those in the statement of Proposition 2.6, and the proof is complete. ∎

Proof of Theorem 2.3. Let (U_n) be a minimizing sequence for Problem 1. Since projections on convex sets reduce the Dirichlet integral and since ∂C is bounded, we can suppose $(U_n)_n$ bounded in L^∞. Hence, passing eventually to a subsequence, we may assume

$(U_n)_n$ verifies (2.8), (2.9). (2.11) for some $U \in X(C)$. If $U \in X^e(C)$ then

$$I^F(C) \leq \int |\nabla U|^2 \leq \liminf \int |\nabla U_n|^2 = I^F(C)$$

and we are done.

In case $U \notin X^e(C)$, we can use the "concentration function" (see for example [6], [7], [3]):

$$Q_n(t) := \sup_{z \in \mathbb{R}^2} \int_{D_t(z)} |\nabla U_n|^2$$

in order to find a sequence $(t_n)_n$ of positive real numbers, and a sequence $(z_n)_n$ of points in $\mathbb{R}^2$, such that

$$\delta \leq \int_{D_{t_n}(z_n)} |\nabla U_n|^2 \leq Q_n(t_n) \leq I^F(C) - \delta$$

where $\delta \in]0, I^F/2[$ is a fixed number. We claim that the minimizing sequence

$$\tilde{U}_n(z) := U_n(t_n z + z_n)$$

converges (up to subsequences) to a solution of Problem 1. Notice that $\tilde{U}_n \in X^e(C)$, $\int |\nabla \tilde{U}_n|^2 = \int |\nabla U_n|^2 = I^F(C) + o(1)$, and moreover $(\tilde{U}_n)_n$ is bounded in L^∞. Thus, we can assume

$$\tilde{U}_n \to U_\infty \text{ a.e.}, \quad \nabla \tilde{U}_n \rightharpoonup \nabla U_\infty \quad \text{weakly in } L^2$$

for some $U_\infty \in X(C)$. If $U_\infty \notin X^e(C)$, arguments above show that U_∞ is constant, and $(\tilde{U}_n)_n$ satisfies one of the possibilities in Proposition 2.6. But since for every R large, for every r small, and for every $a \in \mathbb{R}^2$,

$$\int_{D_R} |\nabla \tilde{U}_n|^2 \geq \int_{D_{t_n R}(z_n)} |\nabla U_n|^2 \geq \delta > 0, \text{ and}$$

$$\int_{D_r(a)} |\nabla \tilde{U}_n|^2 \leq \int_{D_{t_n r}(t_n a + z_n)} |\nabla U_n|^2 \leq Q_n(t_n) \leq I^F(C) - \delta,$$

we see that those two possibilities cannot occur. Thus, we have proved that $U_\infty \in X^e(C)$, and the conclusion follows from weak semicontinuity of the Dirichlet integral. ∎

__Remark 2.7__. The method above, and in particular Proposition 2.5, allows us to prove a multiplicity result, similar to Theorem 3.5 in [8], for the Plateau problem for disk-type minimal surfaces with many obstacles. ∎

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