AN EUCLIDEAN APPROACH TO THE CONSTRUCTION AND THE ANALYSIS OF THE SOLITON SECTORS

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1. INTRODUCTION

The soliton in Classical Field Theory is a topologically stable finite energy solution of the Lagrange equation of motion.
Its counterpart in a Relativistic Quantum Field Theory is the existence of a sector in the Hilbert space of states, orthogonal to the vacuum sector [1].
The quantum soliton appears as an operator in this "large" Hilbert space, but it maps out of the vacuum sector, and it corresponds to a particle in the theory.
Let us better define and analyze this situation.
In the algebraic approach to Quantum Field Theory a state ρ is called "of interest in particle physics" [2,3] if the corresponding Hilbert space $\mathcal{H}_\rho$, obtained via the Gel'fand Naimark Segal construction, carries a unitary representation of the group of translations, whose generators P^μ satisfy the relativistic spectrum conditions

$$\text{spec. } P^0 \geq 0 \qquad \text{spec } P^\mu P_\mu \leq 0$$

(our metric signature is $-+++$).
A physical Hilbert space can then be defined as a direct sum of Hilbert spaces $\mathcal{H}_\rho$ (superselection sectors) corresponding to (inequivalent) states "of interest in particle physics":

$$\mathcal{H}_{phys} = \bigoplus_\rho \mathcal{H}_\rho$$

If $\mathcal{H}_\rho$ carries a unitary representation of the full Poincarè group and there exists a Poincarè-invariant vector $\Omega_\rho \in \mathcal{H}_\rho$ (vacuum), then $\mathcal{H}_\rho$ is called a vacuum sector, otherwise it is called charged.
One can divide the charged sectors into three classes:

1) Sectors which can be obtained by applying (charged) local field to the vacuum sectors [4] (e.g. charged sectors in the Yukawa theory);
2) Sectors which are labelled by unconfined charges obeying a Gauss law [5] (e.g. charged sectors in Q.E.D.)

3) Sectors labelled by topological charges.

Whereas in case 1) it is known that these sectors carry a unitary representation of $P_+^\uparrow$, in cases 2) and 3) this cannot hold in general. For example, this is impossible for the charged sectors of QED [5,6].
General results of the algebraic approach [3], however, suggest that this is always true if there is a mass gap in the theory.
Sometimes the sectors in class 3) are called soliton sectors, and we will follow this convention.

The euclidean approach to the construction of the vacuum sector is now by far the most powerful [7,8]. From an axiomatic point of view its basis lies in the Osterwalder-Schrader (O.S.) axioms and their reconstruction theorem [9], which can be generalized also to gauge theories [10, 11].
In contrast, to construct charged (and in particular soliton) sectors, one has followed an algebraic line [1,4,5,6,12] so far. For (local) soliton sectors a purely euclidean approach has been sketched only in a few papers (e.g. [13,14]; see also [15] for more general ideas).
The purpose of this thesis is to outline how to construct, classify and analyze the soliton sectors (in boson theories) within the euclidean description of Quantum Field Theory, making use of the so called disorder fields [16].
Since the introduction of the disorder fields involves, in the continuum, strong singularities and technical problems, we confine our attention here to the lattice approximation. One believes, however, that the essential features are not lost using this ultraviolet cutoff, since the construction of the soliton sectors is really an infrared problem.
Our approach is based on a reconstruction theorem applied to joint correlation functions of ordinary fields (with support on a lattice), and disorder fields (with support in the dual lattice) carrying a charge with value in a discrete abelian group $(\mathcal{Z})$.
There is a dimension (depending on the explicit model) in which the correlation functions of disorder fields depend on a set of points in the dual lattice and on charges with values in $\mathcal{Z}$.

(This dimension (d) is e.g. $d=2$ for ϕ^4, $d=3$ for (U(1) Higgs), $d=4$ for U(1) gauge theory).
It turns out that in this situation

1) if the correlation functions of non vanishing total charge vanish, then the Hilbert space obtained from the reconstruction theorem factorizes into orthogonal subspaces labelled by the elements of $\mathcal{Z}$;
2) if the cluster property holds the vacuum is unique and belongs to the sector labelled by the trivial element of $\mathcal{Z}$, and the other sectors are (lattice) soliton sectors.

In particular, this construction will provide us, naturally, with soliton operators (non-local, in general) which, when applied to the vacuum sector, produce the soliton sectors.

The disorder field can, in general, be localized on a string like region in the dual lattice (eventually coupled to a Coulomb-like field near its boundary) and the dependence of the correlation functions on this string allows us to divide the (massive bosonic) soliton sectors into three classes:

A) Local solitons
Mixed correlation functions of disorder field and ordinary fields invariant under $\mathcal{Z}$ gauge transformations are independent of the string (e.g. kink sector in ϕ^4_2, vortex sector in $(\mathrm{YM+Higgs})_3$).
B) Stringlike solitons
Mixed correlation functions will depend explicitly on the string (e.g. $(\mathbb{Z}_N\ \mathrm{gauge+Higgs})_3$ models).
C) Solitons with Coulomb field
In this case a Coulomb-like field is attached to the boundary of the string, and the mixed correlation functions (with $\mathcal{Z}$ gauge invariant ordinary fields) do not depend on the string but do depend on the Coulomb field (e.g. monopoles in $\mathrm{U(1)}_4$ gauge theory and in the Georgi Glashow model).

The particle structure of sectors in class A) and B) can be analyzed when suitable convergent cluster expansions exist (in the region of the parameters of the model which allows the soliton sector to exist) by means of the excitation expansion of Bricmont-Fröhlich [17]. (See sect. 3). In these cases one can show that the two point function of the soliton operator has Ornstein-Zernike decay [17,18], proving that the soliton is a massive particle in those theories, and one can also estimate mass gap and upper gap.
From heuristic considerations and results of the algebraic approach [3] one expects that if the continuum limit of the mixed lattice correlation functions exists it is euclidean invariant for (massive) solitons in class A) and B) so that by "analytic continuation" [11] one obtains a unitary representation of the full Poincarè group on the soliton sectors .
This should not be the case for the monopole sectors, where an argument analogous to that used for charged "Gauss" sectors in [5,6] applies, based essentially on the long range of the Coulomb tail. For these sectors one expects that the Lorentz group (in particular the boosts) is not unitarily implementable.

Let us end this section with a description of the plan of the thesis.
In sections 2. and 3., we discuss the general techniques of the reconstruction theorem and excitation expansion.
In sect. 4., we present a complete analysis of the soliton sector of $(\mathrm{Ising})_2$, the easiest example which shares many of the essential features of the general case.
In sect. 5. and 7., we analyze the sectors corresponding to local solitons (class A).
The intermediate section 6. will provide the main tool for the particle analysis: a suitable high-low temperature cluster expansion.
String-like solitons are analyzed in sect. 8 and monopole solitons in sect. 9.
We end by mentioning some open problems.

2. LATTICE QUANTUM MECHANICS

In this section, we discuss the reconstruction theorem for lattice field theories.
This theorem will be the main tool, when suitably adapted, in the construction of the soliton sectors.
We consider, in particular, the case of the Wilson loops of a lattice gauge theory; more general situations will require just straightforward modifications.
(The reader familiar with the reconstruction techniques may skip this section).
Let $\mathbb{L}$ denote the lattice whose sites are given by

$$\{ x : x^i + \tfrac{1}{2} \in \mathbb{Z} \ , \quad i = 0, \dots, d-1 \} \equiv \mathbb{Z}^d_{\frac{1}{2}}$$

where d is the dimension of the lattice.
The 0-th coordinate will be our time coordinate. By abuse of language, we simply write $\mathbb{L} \equiv \mathbb{Z}^d_{\frac{1}{2}}$ and in the same way for the dual $\mathbb{L}^* \equiv \mathbb{Z}^d$.+)
The starting point in the reconstruction theorem for the Wilson loops is a sequence of complex valued functions (correlation functions):

$$\{ S_n (C_1, \dots C_n) \}_{n=0}^{\infty}$$

where C_i are finite loops in $\mathbb{L}$ not intersecting each other.

+) Here the symbol $*$ denotes the dual, both on cells, (e.g. if C_k denotes a k dimensional cell in $\mathbb{L}$, $(C_k)^*$ is a $d-k$ dimensional cell of $\mathbb{L}^*$) and on fields (e.g. if ϕ is defined on k -cells $C_k \in \mathbb{L}$, ϕ^* is defined on $d-k$ cells $(C_k)^* \in \mathbb{L}^*$ by $\phi^*((C_k)^*) = \phi(C_k)$).

Given any loop C in $\mathbb{L}$ we associate to it, in a unique way, lattice coordinates $X(C) \in \mathbb{Z}^d_{\frac{1}{2}}$ and a loop C° in the dual lattice, touching the point $O \equiv (0,\ldots,0)$ in such a way that the support of C° is contained in the positive time lattice and the loop C is given by the " $X(C)$-translate" of C°, formally denoted by $C^\circ + X(C)$. (See Fig. 2.1)
To construct $X(C) = X^\circ, \ldots, X^{d-1}$ consider the set of sites contained in C with the lowest time coordinate and take as X° this value. Between the above selected sites consider the set with the lowest first space coordinate and take this value as X^1 and so on.

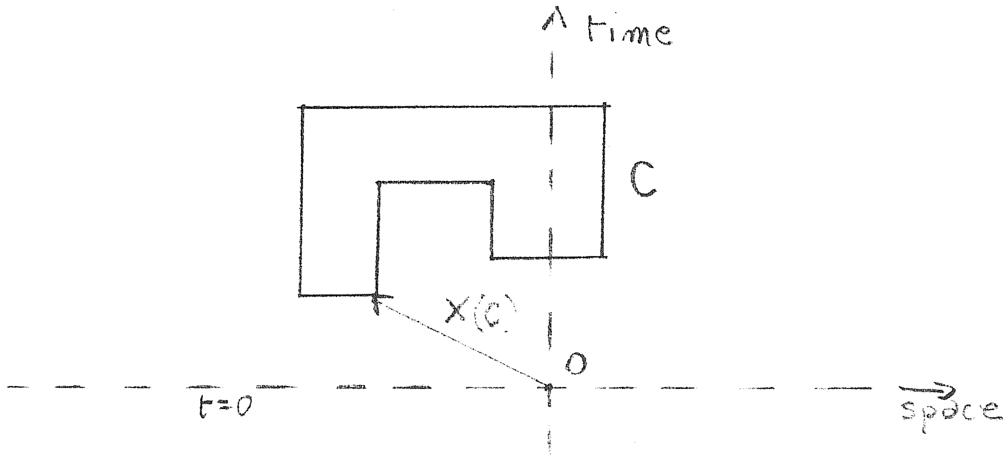


fig. 2.1

The loop C° is now defined by $C = C^\circ + X(C)$. Now let E denote the set of all loops C° constructed as above from the set of all finite loops C and define the functions $S_n(\cdot)$ on $(\mathbb{Z}^d_{\frac{1}{2}} \times E)^{\times n}$ by

$$S_n(X_1 C_1^\circ \ldots X_n C_n^\circ) \equiv S_n(C_1 \ldots C_n) \tag{2.1}$$

with $X_i \equiv X(C_i)$.
This redefinition extracts the translational degrees of freedom of the loops and although this is not strictly necessary on the lattice it is needed in the continuum contruction [10].

Moreover, it allows a unified discussion of different theories, simply by suitably changing the definition of E . For example to the correlation functions of a spin theory with N-components ϕ^α $\alpha = 1, \ldots N$ applies the same procedure, as above, with $E = \{1 \ldots N\}$.
We now consider the space $\underline{S}_E$ of all finite sequences $\underline{f} = \{f_n\}$ of complex valued functions f_n on $(\mathbb{Z}^d_1 \times E)^{\times n}$ vanishing except at finitely many points.
We turn $\underline{S}_E$ into an algebra introducing the sum, the multiplication by complex numbers, in a natural way, and a (non-commutative) product by

$$\underline{f} \times \underline{g} = \{ f_0 g_0 \,,\, f_0 g_1 + f_1 g_0 , \ldots , \sum_{k=0}^{n} f_k \, g_{n-k} , \ldots \}$$

On $\underline{S}_E$ a representation R of the lattice translation group

$$R(a)\, \underline{f} = \{ f_{n,a} \}$$

$$f_{n,a} (x_1 C_1^0 , \ldots x_n C_n^0) = f_n (x_1 - a \, C_1^0 , \ldots , x_n - a \, C_n^0)$$

and of the reflection r with respect to the time zero plane

$$R(r)\, \underline{f} = \{ f_{n,r} \}$$

$$f_{n,r} (x_1 C_1^0 \ldots x_n C_n^0) =$$

$$= f_n (x(rC_1) \, rC_1^0 , \ldots x(rC_n) \, rC_n^0)$$

are defined in a natural way.
As usual we define

$$\Theta \underline{f} \equiv \overline{R(r)\, \underline{f}}$$

where the upper bar denotes complex conjugation.
Using the sequences $\{ S_n (x_1 C_1^\circ \ldots x_n C_n^\circ) \}$ we can define the functional

$$S \; : \; \underline{f} \longmapsto S(\underline{f}) = \sum_{n=0}^{\infty} S_n \, (f_n)$$

$$S_n \, (f_n) = \prod_{i=1}^{n} \sum_{x_i \in \mathbb{Z}^d_{\frac{1}{2}}} \; \sum_{C_i^\circ \in E} S_n \, (x_1 C_1^\circ \ldots x_n C_n^\circ) \, f_n (x_1 C_1^\circ \ldots x_n C_n^\circ) \tag{2.2}$$

Now denote by $\underline{S}_E^+$ the subspace of $\underline{S}_E$ given by all the sequences of functions having support in the positive time lattice. Then we can state

<u>Theorem</u> 2.1 (Reconstruction Theorem)

If the functional S satisfies

1) Lattice translation invariance

$$S(R(a)\underline{f} \times R(a)\underline{g}) = S(\underline{f} \times \underline{g}) \qquad \forall \, \underline{f}, \underline{g} \in \underline{S}_E \; , \quad a \in \mathbb{Z}^d \; ; \tag{2.3}$$

2) Osterwalder-Schrader positivity

$$S(\Theta \underline{f} \times \underline{f}) \geqslant 0, \qquad \forall \, \underline{f} \in \underline{S}_E^+ \; ; \tag{2.4}$$

3) $S(\Theta \underline{f} \times R(t)\underline{g})$ $\forall \, \underline{f}, \underline{g} \in \underline{S}_E^+$ is bounded uniformly in $t \in \mathbb{Z}_+$.
Then S determines :

i) a separable Hilbert space $\mathcal{H}_E$
ii) a distinguished vector $\Omega_E \in \mathcal{H}_E$ of unit norm, called the vacuum;
iii) a contractive, selfadjoint representation T_E of the positive time lattice translations and a unitary representation U_E of the spatial lattice translations, which satisfy

$$T_E(t)\,\Omega_E = \Omega_E, \quad \forall t \in \mathbb{Z}_+ ,$$
$$U_E(\vec{a})\,\Omega_E = \Omega_E, \quad \forall \vec{a} \in \mathbb{Z}^{d-1} . \tag{2.5}$$

If, moreover, S satisfies
4) Cluster property

$$\lim_{a \to \infty} S(\underline{f} \times R(a)\,\underline{g}) = S(\underline{f})\,S(\underline{g}) \tag{2.6}$$

then

iv) Ω_E is the unique vector in $\mathcal{H}_E$ satisfying (2.5).
The structure $(\mathcal{H}_E, \Omega_E, T_E, U_E)$ satisfying i-iii), is called a Lattice Quantum Mechanics, by analogy with the relativistic QFT in the continuum.

Proof. The proof is fairly standard (see e.g. [11] for its continuum analogue), so we only give an outline of the explicit construction of $(\mathcal{H}_E, \Omega_E, T_E, U_E)$. Let $\underline{N}_E = \{ \underline{f} \in \underline{S}_E^+ \; ; \; S(\Theta \underline{f} \times \underline{f}) = 0 \}$ and take the quotient of $\underline{S}_E^+$ by $\underline{N}_E$ w.r.t. the addition.
Denote the associated projection by

$$\wedge : \underline{S}_E^+ \longrightarrow [\underline{S}_E^+]^\wedge \equiv \underline{S}_E^+ / \underline{N}_E$$

and define an inner product < . , . > on $[\underline{S}_E^+]^\wedge$ by

$$\langle \underline{f}^\wedge, \underline{g}^\wedge \rangle \equiv S(\Theta \underline{f} \times \underline{g})$$

From 2) it follows that < . , .> is indeed a scalar product.

Hence one can define a Hilbert space through

$$\mathcal{H}_E \equiv \overline{\underline{S}_E^+ / \underline{N}_E} \quad ,$$

where the bar denotes the closure in the norm induced by <.,..>.
The distinguished vector is given by

$$\Omega_E = \underline{1\!\!1}^\wedge$$

where $\underline{1\!\!1} = \{1, 0, \ldots, 0, \ldots\}$.
On $[\underline{S}_E^+]^\wedge$, dense in $\mathcal{H}_E$ by construction, we define

$$T_E(t)\, \underline{f}^\wedge = [R(t)\, \underline{f}\,]^\wedge \qquad t \in \mathbb{Z}_+$$

$$U_E(\vec{a})\, \underline{f}^\wedge = [R(\vec{a})\, \underline{f}\,]^\wedge \qquad \vec{a} \in \mathbb{Z}^{d-1}$$

By 1) and 3) these operators can be extended to the whole $\mathcal{H}_E$ and satisfy iii).

Remark 2.2 If Osterwalder Schrader positivity holds even for reflections in the $x^0 = \frac{1}{2}$ plane, then T_E is positive.

Remark 2.3 On the dense set $[R(t)\, \underline{S}_E^+]^\wedge$ the operators $\phi(\underline{f})$ are well defined (with

$$\operatorname{supp} f_n \subset \left([0,t] \times \mathbb{Z}^{d-1} \times E \right)^{\times n} \Big) \tag{2.7}$$

through the equation

$$\phi(\underline{f})\,[R(t)\,\underline{g}]^{\wedge} = [\,\underline{f} \times R(t)\,\underline{g}\,]^{\wedge} \qquad (2.8)$$

Clearly one has

$$S(\theta \underline{g} \times R(a=(t,\vec{a}))\underline{f}) = \langle \phi(\underline{g})\Omega,\, T(t)\,U(\vec{a})\,\phi(\underline{f})\Omega \rangle$$

(We will omit the symbol E if this does not cause confusion).
In particular the operator corresponding to the sequence $\underline{f}_{\mathcal{L}}$ (where $\mathcal{L}$ is a loop) given by

$$f_n = 0 \qquad n \neq 1$$
$$f_1\,(x\,C^0) = \delta(x\,, x(\mathcal{L}))\;\delta(C^0, \mathcal{L}^0)$$

is the Wilson loop operator usually denoted $W(\mathcal{L})$, i. e.

$$\phi\,(\underline{f}_{\mathcal{L}}) \equiv W\,(\mathcal{L})$$

The state

$$\phi\,(\,\underline{f}_{\mathcal{L}_1 \ldots \mathcal{L}_m})\,\Omega$$

with $\underline{f}_{\mathcal{L}_1 \ldots \mathcal{L}_m}$ defined by

$$f_n = 0 \qquad n \neq m$$
$$f_m\,(x_1\,C_1^0 \ldots x_m\,C_m^0) = \prod_{i=1}^{m} \delta(x_i\,, x(\mathcal{L}_i))\;\delta(C_i^0, \mathcal{L}_i^0)$$

will be denoted, for short, by $|\,\mathcal{L}_1 \ldots \mathcal{L}_m \rangle$.
Finally, let us remark that the set of operators

$$\{\phi(\underline{f})\,T(t)\;;\; t \in \mathbb{Z}_+\,,\; \underline{f} \text{ satisfying (2.7)}\} \qquad (2.9)$$

forms an algebra, as linear combinations and products are well defined.

<u>Definition 2.4</u> The operator algebra $\mathfrak{A}$ generated by the fields (2.9) is called the field algebra associated to the Lattice Quantum Mechanics $(\mathcal{H}, \Omega, T, U)$.
Notice that, since $\{\phi(\underline{f})\Omega, \underline{f} \in \underline{S}^+\}$ is dense in $\mathcal{H}$, Ω is cyclic for the algebra $\mathfrak{A}$.

<u>Remark 2.5</u> in concrete models the sequences:

$$\{ S_n (C_1 \ldots C_n) \}_{n=0}^{\infty}$$

are usually given as thermodynamic limits of the expectation values of product of fields in a finite volume Gibbs state.
For example, consider the case of a lattice gauge theory described in term of a link variable $g_{\langle xy\rangle}$ taking value in a compact group G. Then the correlation functions of the Wilson loop are defined as follows. Let $\Lambda(L)$ denote the finite lattice

$$\Lambda(L) = \{ x \in \mathbb{L} \; : \; |x^i| \leq L \quad i = 0, \ldots d-1 \}$$

χ denotes a character of G, $d g_{\langle xy\rangle}$ the Haar measure on G and $\prod_{\langle xy\rangle \in C} g_{\langle xy\rangle} \equiv g_C$ the oriented product of the link variables $g_{\langle xy\rangle}$ belonging to the loop C.
Then define the finite volume Gibbs state through

$$\langle (\cdot) \rangle = \int d\mu_\Lambda(g) \; (\cdot) =$$

$$= \frac{1}{Z_\Lambda} \int \prod_{\langle xy\rangle \in \Lambda} d g_{\langle xy\rangle} \, (\cdot) \prod_{p \in \Lambda} \exp\left[\mathrm{Re}\, \chi(g_{\partial p}) - 1 \right]$$

where Z_Λ is the normalization factor.
The functions $S_n(C_1 \ldots C_n)$ are then defined as

$$\lim_{\Lambda \uparrow \mathbb{L}} \int d\mu_\Lambda(g) \prod_{i=1}^{m} \chi(g_{C_i}) \equiv$$
$$\equiv \int d\mu(g) \prod_{i=1}^{m} \chi(g_{C_i})$$

One can substitute, in the Reconstruction Theorem, properties 1), 2) and 4) by

1') The measure $d\mu(g)$ is lattice translation invariant;
2') The measure $d\mu(g)$ is Osterwalder-Schrader positive.
i.e. $\forall$ polynomial functions of the Wilson loops $\Psi(\{C_i\})$ with C_i localized in the positive time lattice

$$\int d\mu(g)\, \Psi(C_1 \dots C_m)\, \Theta\, \Psi(C_1 \dots C_m) \geq 0$$

where

$$\Theta\, \Psi(C_1 \dots C_m) = \overline{\Psi(rC_1 \dots rC_m)}$$

and the upper bar denotes complex conjugation.
4') Cluster property

$$\lim_{a \to \infty} \int d\mu(g)\, \Psi(C_1 \dots C_m)\, \Psi'(C_1' + a \dots C_n' + a) =$$
$$= \int d\mu(g)\, \Psi(C_1 \dots C_m) \int d\mu(g)\, \Psi'(C_1' \dots C_n')$$

(For all models discussed here 2') and 4') hold for every function belonging to $L_2(d\mu(g))$, provided it is gauge invariant).

<u>Remark 2.6</u> Since we have discussed lattice field theories, no analytic continuation in time is possible, however, if the correlation functions S_m

admit the time continuum limit and in this limit property 3) of Theorem 11 is still satisfied, then

$$T(t) = e^{-tH} \qquad ,$$

with H selfadjoint and positive, and we can define a unitary representation of time translations by

$$T^*(t) = e^{-itH}$$

If, moreover, the correlation functions admit the (full) continuum limit and in this limit are euclidean invariant (i.e. property 2) of theorem 1.1 holds, replacing the lattice translation group with the euclidean group) then a unitary representation of the full proper Poincaré group can be reconstructed, using the techniques of [11].

We close this section by defining what we mean by a charged sector in Lattice Quantum Mechanics by analogy with the "continuum relativistic case".
Let $\mathcal{A}$ be the operator field algebra of a Lattice Quantum Mechanics ($\mathcal{H}$, Ω , T,V) as in def. 2.4.

<u>Definition 2.7</u> Let the Hilbert space $\mathcal{H}$ of a Lattice Quantum Mechanics decompose into orthogonal sectors $\mathcal{H}_\alpha$ invariant under lattice translations. Let $\mathcal{H}_0$ be the sector containing the vacuum Ω and $\mathcal{A}_0 \subset \mathcal{A}$ a field algebra which applied to Ω generates $\mathcal{H}_0$.
Then a sector $\mathcal{H}_\alpha$ is called a charged sector if it is invariant under $\mathcal{A}_0$ (i.e. $\mathcal{A}_0 \mathcal{H}_\alpha \subseteq \mathcal{H}_\alpha$) and if there exists no lattice translation invariant vector $\psi \in \mathcal{H}_\alpha$.
Finally, a charged sector $\mathcal{H}_\alpha$ is called a (lattice) soliton sector if it is indexed by a charge which is carried by a field (in $\mathcal{A}$) with support in the dual lattice.

In classes A) and C), defined in the introduction, the charge is of topological nature, as explained respectively in remark 5.9 and 9.4.

3. PARTICLE STRUCTURE ON THE LATTICE

In this section we will show how, using the Reconstruction Theorem sketched in Section 2, one can analyze the particle structure of lattice field theories. In the relativistic quantum field theories an isolated eigenvalue m in the spectrum of the mass operator

$$M = \sqrt{H^2 - \vec{P}^{\,2}} \tag{3.1}$$

with $H, \vec{P}$ the generators of the translations, corresponds to a one-particle-state of mass m.
Unfortunately, on the lattice we have no Lorentz covariance and therefore we cannot use definition (3.1) of the mass operator. However, there is a slightly different definition of M which works also for theories which are not Lorentz covariant, provided there exists a contractive (strictly) positive representation T of time translations and a unitary representation of space translations.
In fact, from (3.1), see fig. 3.1,

$$\text{Spec } M = \text{Spec } H \restriction \mathcal{H}^0$$

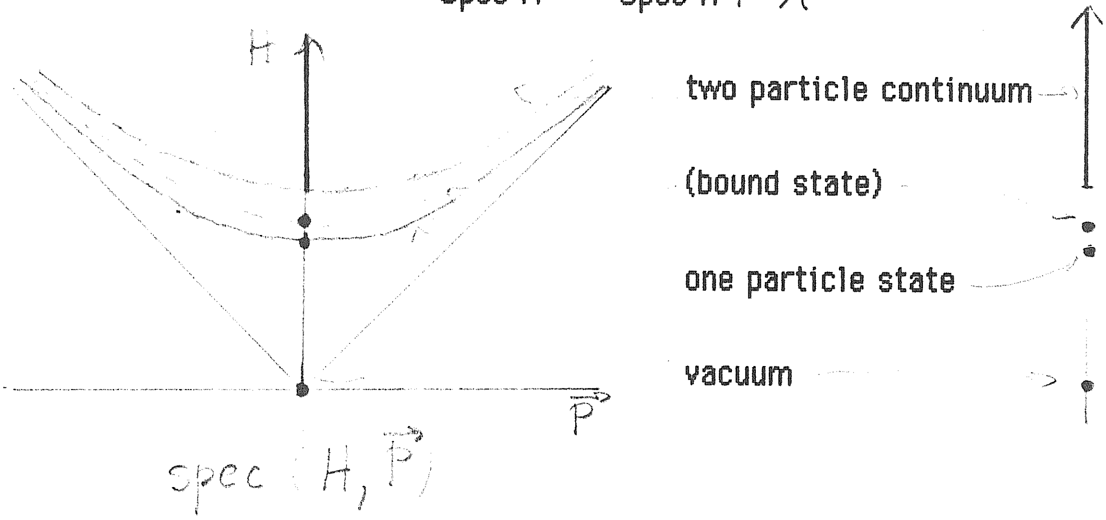


fig. 3.1

where $\mathcal{H}^{\circ}$ is the generalized subspace of the (rigged) Hilbert space [41] $([\underline{S}^{+}]^{\wedge}, \mathcal{H})$ given by the vectors of vanishing total momentum.(+)

Since in the continuum $T(t) = e^{-tH}$, with $T = T(1)$, it is natural to define

$$H = - \ln T$$

and

$$M = - \ln (T \restriction \mathcal{H}^{\circ}) \tag{3.2}$$

We already know that M has an eigenvalue 0, and if the cluster property holds its multiplicity is one. So we are interested in analyzing spec $M \setminus \{0\}$. This spectrum can be related easily to the behaviour in the time coordinate of the truncated correlation functions of fields:

$$\begin{aligned} &< \phi(\underline{f}) ; T(t) U(\vec{a}) \phi(\underline{f}) > \equiv \\ &\equiv < \phi(\underline{f}) \Omega , T(t) U(\vec{a}) \phi(\underline{f}) \Omega > - \\ &- < \phi(\underline{f}) \Omega , \Omega > < \Omega , \phi(\underline{f}) \Omega > \end{aligned}$$

as $t \nearrow \infty$.

In fact, using the spectral theorem we can write

(+) [i.e. if we denote $([\underline{S}^{+}]^{\wedge})^{*}$ the space of linear functionals F on $[\underline{S}^{+}]^{\wedge}$ continuous in the topology of $[\underline{S}^{+}]^{\wedge}$

$$\mathcal{H}^{\circ} = \{ F \in ([\underline{S}^{+}]^{\wedge})^{*} : \mathcal{P} F = 0 \}$$

$$\langle \phi(\underline{f}) ; T(t)\, U(\vec{a})\, \phi(\underline{f}) \rangle = \int d\rho_{\underline{f}}(\lambda, \vec{k})\, \lambda^t e^{i\vec{k}\cdot\vec{a}} \tag{3.3}$$

with $\lambda \in [0,1]$, $\vec{k} \in [-\pi,\pi]^{d-1}$,
and $d\rho_{\underline{f}}(\lambda,\vec{k})$ a finite positive measure.
By comparing with eq. (3.2) we immediately obtain

$$\operatorname{supp} \, d\rho_{\underline{f}}(\lambda, \vec{0}) \subseteq \operatorname{spec} e^{-M} . \tag{3.4}$$

Moreover, using the fact that the set of vectors

$$\{ \phi(\underline{f})\Omega , \underline{f} \in \underline{S}^+ \}$$

is dense in $\mathcal{H}$, we have

$$\overline{\operatorname{supp} \{ d\rho_{\underline{f}}(\lambda, \vec{0}) \}_{\underline{f}\in \underline{S}^+} \cup \{1\}} = \operatorname{spec} e^{-M} \tag{3.5}$$

where the bar denotes the closure in $\mathbb{R}$.
It is immediate from (3.3) that

$$\sum_{\vec{a} \in \mathbb{Z}^{d-1}} \langle \phi(\underline{f}) ; T(t)\, U(\vec{a})\, \phi(\underline{f}) \rangle = \int d\rho_{\underline{f}}(\lambda, \vec{0})\, \lambda^t \tag{3.6}$$

Hence, using (3.4)-(3.5) one realizes that a behaviour of the truncated correlation function (3.6) like

$$e^{-\bar{m}(\underline{f})t} \left(1 + \text{const}\, e^{-\bar{\mu}(\underline{f})t} \right) \tag{3.7}$$

as $t \nearrow \infty$,

with $\overline{m}(\underline{f})$ and $\overline{\mu}(\underline{f})$ strictly positive constants, means that $\phi(\underline{f})$ couples the vacuum to a one particle state of mass $\overline{m}(\underline{f})$.
Let m denote the mass gap, i.e. the distance between 0 and the first eigenvalue of M, and let μ denote the upper gap, i.e. the distance between m and the next eigenvalue of M.
From (3.5) we see that

$$m = \{ \inf \overline{m}(\underline{f}), \underline{f} \in \underline{S}^{+} \}$$

and, supposing $m > 0$,

$$\mu = \{ \inf \overline{\mu}, \underline{f} \in \underline{S}^{+} : \overline{m}(\underline{f}) = m \}.$$

Following the Bricmont-Fröhlich papers [17], we now show how to analyze the behaviour (3.7) by means of statistical mechanical methods.
For simplicity, let us now discuss the case of a two-point function of a spin system which admits a representation in terms of random walks ω whose boundary is $\{x, y\}$:

$$\langle \phi_x \phi_y \rangle = \sum_{\omega : x \to y} z(\omega)$$

where $z(\cdot)$ is a complex function on random walks.
The method discussed here will later be adapted to the analysis of solitons.
We decompose every path ω into two parts

1) the regular part whose maximally connected components are called straight lines and have a point as projection in the hyperplane orthogonal to the time axis; (the bonds which belong to the regular part of ω are called regular bonds).

2) the complement, a set of lines whose projection on the time axis are maximally connected and hence mutually disjoint.

The equivalence class w.r.t. lattice translations perpendicular to the time axis of such lines are called excitations and denoted by $\mathcal{E}$ (see fig. 3.2).

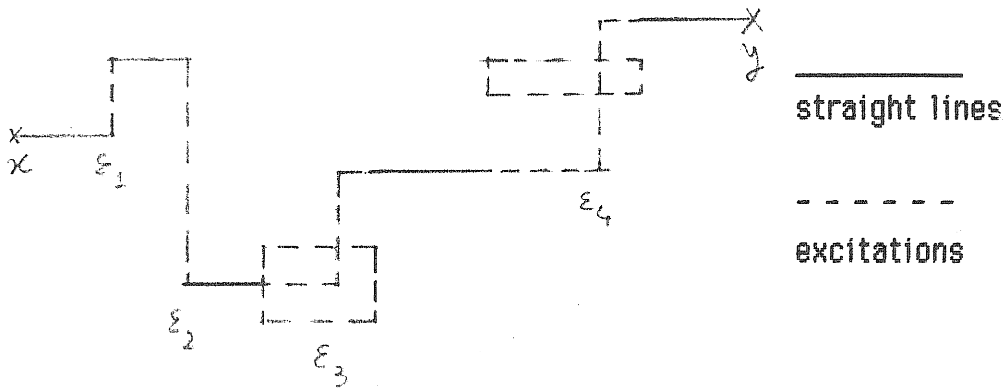


fig. 3.2

Since an excitation is, by definition, adjacent to a straight line on the left and a straight line on the right, we define its height $\vec{h}(\varepsilon)$ to be the vector difference between the projection of the two lines in hyperplane perpendicular to the time axis. We also define $\pi(\varepsilon)$ the projection of ε on the time axis,

$|\pi(\varepsilon)|$ = length of $\pi(\varepsilon)$

$l(\varepsilon)$ = length of ε

$$|\varepsilon| = l(\varepsilon) - |\pi(\varepsilon)|$$

If ε is an excitation of a path ω joining a point with a time coordinate a to a point with time coordinate b, then one says that ε is "allowed" in $[a,b]$.
Finally two excitations $\varepsilon_1, \varepsilon_2$ are called compatible if

$$\pi(\varepsilon_1) \cap \pi(\varepsilon_2) = \phi$$

It is now clear that there is a one to one correspondence between sets of compatible excitations $\{\varepsilon_1 \ldots \varepsilon_n\}$, allowed in $[0,t]$ with $\sum_{i=1}^{n} \vec{h}(\varepsilon_i) = \vec{y}$ and paths ω joining $(0,\vec{0})$ to (t, y)
We denote the correspondence by

$$\omega \sim \{\varepsilon_1 \ldots \varepsilon_n\}$$

Let now $\{\varepsilon_i\}$ denote any set of compatible excitations allowed in $[0,t]$ and denote by ω_0 the straight path $(0,\vec{0}) \longrightarrow (t,\vec{0})$.
Then

$$\langle \phi(0,\vec{0})\, \phi(t,\vec{y}) \rangle = Z(\omega_0) \cdot$$
$$\cdot \left[\sum_{\{\varepsilon_i\}} \exp\left[-U(\{\varepsilon_i\})\right] \delta\left(\sum_i \vec{h}(\varepsilon_i) - \vec{y}\right) \right] \tag{3.8}$$

with

$$U(\{\varepsilon_i\}) = \ln \frac{Z(\omega)}{Z(\omega_0)}$$

$$\omega \sim \{\varepsilon_i\}$$

In all the cases we deal with here, there is an important factorization property: $\exists$ a function $\zeta(\varepsilon)$ s.t.

$$e^{-U(\{\varepsilon_i\})} = \prod_i \zeta(\varepsilon_i) \tag{3.9}$$

so that U is a hard core exclusion potential and ζ 's are the activities of the excitations.
More generally, we require weak and short-range interactions between excitations.
Formula (3.8) expresses the two-point functions of a lattice spin theory as a partition function of a gas of extended particles, the excitations, moving in the one-dimensional interval $[0,t]$ and weakly interacting.
If this gas is dilute, its pressure can be expanded in a convergent Mayer series and the two point function takes the form

$$< \phi(0,\vec{0})\ \phi(t,\vec{y}) > = Z(\omega_0)\,\frac{1}{(2\pi)^{d-1}}\int d\vec{k}^{d-1}\, e^{i\vec{k}\cdot\vec{y}}\, e^{t\,P_{[0,t]}(\vec{k})} \tag{3.10}$$

Since the gas is confined to a finite interval, $[0,t]$,we can write

$$t\ P_{[0,t]}(\vec{k}) = t\ P(\vec{k}) + P_{bd}(\vec{k}) + P_{int}(\vec{k}) \tag{3.11}$$

where

$P(\vec{k})$ is the thermodynamic pressure,

$P_{bd}(\vec{k})$ is the correction due to the interaction of the particles with the boundary of $[0,t]$,

$P_{int}(\vec{k})$ describes the effective interaction between the boundaries of $[0,t]$ mediated by the particles in the gas.

With the above assumptions

$$Z(\omega_0) \underset{t\to\infty}{\sim} e^{-ct}$$

$$P(\vec{k}) \underset{t\to\infty}{\sim} const \qquad\qquad P_{int}(\vec{k}) \underset{t\to\infty}{\sim} e^{-\mu(\vec{k})t} \tag{3.12}$$

$$P_{bd}(\vec{k}) \underset{t\to\infty}{\sim} const$$

($f(x) \underset{x\to\infty}{\sim} g(x) \equiv \exists\, c_1>0, c_2<\infty$: as $x\to\infty$, $c_1 f(x) \le g(x) \le c_2 f(x)$)

Such behaviour is most easily seen if (3.9) holds. In fact, the Mayer expansion now converges if

$$|\zeta(\varepsilon)| \leq e^{-c|\varepsilon|} \tag{3.13}$$

with C large enough.

Let $\underset{\sim}{\mathcal{E}}$ denote a collection of excitations allowed in $[0,t]$ in which a single excitation can occur an arbitrary number of times. Then the two point function takes the more explicit form:

$$< \phi(0,\vec{0})\, \phi(t,\vec{y}) > = Z(\omega_0) \cdot$$

$$\cdot \frac{1}{(2\pi)^{d-1}} \int d\vec{k}^{d-1}\, e^{i\vec{k}\cdot\vec{y}} \exp\Big[\sum_{\underset{\sim}{\mathcal{E}}} a_T(\underset{\sim}{\mathcal{E}}) \prod_{\varepsilon\in\underset{\sim}{\mathcal{E}}} \zeta(\varepsilon)\, e^{i\vec{k}\cdot\vec{h}(\varepsilon)}\Big] \tag{3.14}$$

where $a_T(\underset{\sim}{\mathcal{E}})$ is a suitable combinatorial factor.

Then it is clear that the terms in the exponential which, for instance, contribute to $P_{int}(\vec{k})$ are families of excitations $\underset{\sim}{\mathcal{E}}$ such that $\bigcup_{\varepsilon\in\underset{\sim}{\mathcal{E}}} \pi(\varepsilon) \supset [0,t]$, so that

$$\prod_{\varepsilon\in\underset{\sim}{\mathcal{E}}} \zeta(\varepsilon)\; a_T(\underset{\sim}{\mathcal{E}}) \underset{t\to\infty}{\sim} e^{-\mu t}$$

Now we insert (3.11), (3.12) in (3.9) and we obtain

$$< \phi(0,\vec{0})\, \phi(t,\vec{y}) > = \int d\rho(\lambda,\vec{k})\, \lambda^t e^{i\vec{k}\cdot\vec{y}} \sim$$

$$\underset{t\to\infty}{\sim} \frac{1}{(2\pi)^{d-1}} \int d\vec{k}^{d-1}\, e^{i\vec{k}\cdot\vec{y}}\, e^{-ct}\, e^{tP(\vec{k})}\, e^{-\mu(\vec{k})t} \sim$$

$$\underset{t\to\infty}{\sim} \frac{1}{(2\pi)^{d-1}} \int d\vec{k}^{d-1}\, e^{i\vec{k}\cdot\vec{y}}\, e^{-m(\vec{k})t}\big(1+\mathrm{const}\, e^{-\mu(\vec{k})t}\big)$$

with $m(\vec{k}) = c - P(\vec{k})$.

By comparison with (3.7)

$$m(\vec{0}) = \overline{m}$$
$$\mu(\vec{0}) = \overline{\mu}$$

Hence the decay rate of the straight line ω_0 corrected by the thermodynamic pressure of the excitation gas for $\vec{k} = \vec{0}$ gives the mass of the one particle state $\phi\Omega$, whereas the decay rate of the pressure due to the interaction between the boundaries of $[0,t]$ mediated by the excitations, gives an upper gap.

<u>Remark 3.1</u> We have a decay condition weaker than (3.7) which ensures that $\phi(\underline{f})$ couples to a one particle state of mass $\overline{m}$, i.e. the Ornstein-Zernike decay

$$< \phi(\underline{f}) ; T(t)\, \phi(\underline{f}) > \underset{t\to\infty}{\sim} \frac{e^{-\overline{m}t}}{t^{\frac{d-1}{2}}}$$

Also this condition can be analyzed with the above method using the $\vec{k}$ dependence of $P_{[0,t]}(\vec{k})$. For details see [17,18].

4. THE ISING MODEL

In this section we discuss the soliton sector in the low temperature phase of the $(\text{Ising})_2$ model with $+bd$ conditions. It falls in class A of the introduction, i.e. that of local solitons.
Although almost everything about this soliton is known, we utilize the full procedure of analysis as a guide for the more complicated models.
In the algebraic approach both vacuum and soliton sectors (for local solitons) can be obtained by applying ordinary field operators and (the so called) soliton operators polynomials (which are almost local [1,19]) to the vacuum.
In the Euclidean strategy, however, one obtains the field operators from a sequence of correlation functions of euclidean fields via the reconstruction theorem.
Therefore, it is natural to expect that the euclidean analog of the soliton operator will arise from considering correlation functions involving some special kinds of euclidean fields somehow related to the soliton.
Which fields are the desired ones is suggested by both, a common (mainly heuristic) argument and by topological (better cohomological, see section 5) considerations dictated by the analogy with the classical soliton.
Without going into details (see e.g. [14]) we just say here that those fields are the disorder fields [16].
Historically the first example of such a field was constructed in 1971 by Kadanoff and Ceva [20] precisely in the $(\text{Ising})_2$ model. Let us briefly review their construction.

The state with + boundary conditions $\langle(\cdot)\rangle^+$ of the $(\text{Ising})_d$ model is constructed as the thermodynamic limit of the finite volume Gibbs state

$$\langle(\cdot)\rangle_\Lambda = \int d\mu_\Lambda(\sigma)\,(\cdot) = \Big\{ \sum_{\sigma_x=\pm 1,\, x\in\Lambda} \prod_{\langle xy\rangle\in\Lambda} \exp\Big[-\frac{\beta}{2}(\sigma_x-\sigma_y)^2\Big] \cdot \prod_{x\in\partial\Lambda}\delta(\sigma_x-1)\,(\cdot)\Big\}\, Z_\Lambda^{-1} \tag{4.1}$$

where σ is the spin field, $Z_\Lambda = \Big\{ \sum_{\sigma_x=\pm 1,\, x\in\Lambda} \prod_{\langle xy\rangle\in\Lambda} e^{-\frac{\beta}{2}(\sigma_x-\sigma_y)^2} \prod_{x\in\partial\Lambda}\delta(\sigma_x-1)\Big\}$.

To define the disorder fields we introduce a $\mathbb{Z}_2$ "gauge" field ω which has support on links of $\Lambda \setminus \partial\Lambda$.
Let $C = C(y_1 \dots y_{2n})$ be a curve in the dual lattice whose boundary is given by $\{y_1 \dots y_{2n}, y_i \in \Lambda^* \subset \mathbb{Z}^d\}$.
Then we define the disorder field through

$$D(\omega) \equiv D_C(y_1 \dots y_{2n}) = \prod_{\langle xy \rangle \in \Lambda} \exp\left\{ \frac{\beta}{2} \left[(\sigma_x - \omega_{\langle xy \rangle}\sigma_y)^2 - (\sigma_x - \sigma_y)^2 \right] \right\} \tag{4.2}$$

with

$$\begin{aligned} \omega_{\langle xy \rangle} &= 1 \qquad \text{if} \quad \langle xy \rangle \notin C^* \\ &= -1 \qquad \text{if} \quad \langle xy \rangle \in C^* \end{aligned}$$

i.e.

$$D_C(y_1 \dots y_{2n}) = \prod_{\langle xy \rangle \in C^*} e^{-2\beta \sigma_x \sigma_y}$$

The expectation value $\langle D_C(y_1 \dots y_{2n}) \rangle$ depends only on $\{y_1 \dots y_{2n}\}$ and not on C.
The easiest way to see this is to realize that

$$\langle D(\omega) \rangle_\Lambda = \langle D_C(y_1 \dots y_{2n}) \rangle = \frac{Z_\Lambda(\omega)}{Z_\Lambda} \tag{4.3}$$

where $Z_\Lambda(\omega)$ can be obtained from Z_Λ by substituting, in the action, for the derivative

$$(d\sigma)_{\langle xy \rangle} = \sigma_y - \sigma_x$$

the $\mathbb{Z}_2$ - covariant derivative

$$(\nabla_\omega \sigma)_{\langle xy \rangle} = \sigma_y - \omega_{\langle xy \rangle} \sigma_x ,$$

so that the modified action is invariant under the (local) gauge transformation

$$\sigma_x \longrightarrow \xi_x \sigma_x \qquad \xi_x = \pm 1$$

$$\omega_{\langle xy\rangle} \longrightarrow \xi_y \, \omega_{\langle xy\rangle} \, \xi_x \qquad \xi_x = 1 \text{ for } x \in \partial\Lambda \tag{4.4}$$

In fact, this gauge transformation corresponds just to change C in $D_c(y_1 \ldots y_{2n})$ leaving fixed its boundary.
By abuse of language we then call $\{y_1 \ldots y_{2n}\}$ the support of the disorder field and often drop the symbol C.

Even correlation functions of the disorder fields are defined by

$$S_{2n}(y_1 \ldots y_{2n}) \equiv \langle D(y_1 \ldots y_{2n}) \rangle^{+} \equiv$$

$$\equiv \lim_{\substack{\Lambda \nearrow \mathbb{L} \\ (\Lambda \nearrow \mathbb{Z}^2)}} \langle D(y_1 \ldots y_{2n}) \rangle_{\Lambda} \tag{4.5}$$

Odd correlation functions are defined as limits of even correlation functions when a point is removed to infinity (if it exists of course), i.e.

$$S_{2n+1}(y_1 \ldots y_{2n+1}) = \lim_{y_{2n+2} \to \infty} S_{2n+2}(y_1 \ldots y_{2n+1}\, y_{2n+2}) \tag{4.6}$$

As we will see non-trivial soliton sectors exist only if the limit (4.6) is zero.
Some care is needed if we now want to define joint correlation functions of spins and disorder fields. In fact one immediately sees that

e.g. $\langle D_C(y_1 \cdots y_{2n})\, \sigma_{x_1} \cdots \sigma_{x_m} \rangle^+ , \quad x_i \in \Lambda$
depends explicitly on C .
This is obvious if we remark that this expectation value is not invariant under the gauge transformation (4.4).
To have dependence only on the support of the disorder field and not on the full curve C , we thus need to use $\mathbb{Z}_2$-gauge invariant fields, like

$$\{ v_{\langle xy \rangle} = \sigma_y \, \omega_{\langle xy \rangle} \, \sigma_x \quad | \quad \langle xy \rangle \in \mathbb{Z}^d_{\frac{1}{2}} \} \tag{4.7}$$

Mixed correlation functions for $\{y_j\} \in \Lambda^*$ and $\{\langle xz \rangle_i\} \in \Lambda$ are then defined by

$$S_{2n,m}(y_1 \ldots y_{2n} \;\langle xz \rangle_1 \ldots \langle xz \rangle_m) \equiv$$

$$\equiv \langle D_C(y_1 \cdots y_{2n}) \prod_{i=1}^{m} v_{\langle xz \rangle_i} \rangle^+ \equiv$$

$$\equiv \lim_{\Lambda \uparrow \mathbb{Z}^2} \frac{1}{Z_\Lambda} \Big[\sum_{\sigma_x = \pm 1} \prod_{x \in \Lambda} \prod_{\langle xy \rangle \in \Lambda} e^{-\frac{\beta}{2}(\sigma_x - \omega_{\langle xy \rangle}\sigma_y)^2} \tag{4.8}$$

$$\cdot \prod_i \sigma_{x_i} \, \omega_{\langle xz \rangle_i} \, \sigma_{z_i} \prod_{x \in \partial\Lambda} \delta(\sigma_x - 1) \Big]$$

with

$$\omega_{\langle xy \rangle} = -1 \quad \text{on } C^*.$$

<u>Remark 4.1</u> Since the algebra generated by the fields $v_{\langle xy \rangle}$ is even (w.r.t. $\sigma \to -\sigma$) it is not able to distinguish between + and - bd conditions.
If we really want to distinguish between + and - b.c. we can introduce a field depending on a string of the external ω field which reaches, for

every fixed lattice Λ , its boundary. For instance, we can define the field

$$\psi_\Lambda(x) = \sigma_x \prod_{\langle yz\rangle \in \Gamma_x(\Lambda)} \omega_{\langle yz\rangle} \quad , \; x \in \Lambda$$

where $\Gamma_x(\Lambda)$ is the straight line in the negative space direction joining x to the boundary of Λ .
Such fields are invariant under (4.6) since $\xi = 1$ on the boundary.
By standard methods, using cluster expansions, one easily sees that

$$\lim_{\Lambda \uparrow \mathbb{Z}^2} \left\langle D_c(y_1 \dots y_{2n}) \prod_{i=1}^{m} \psi_{x_i}(\Lambda) \right\rangle_\Lambda \equiv$$

$$\equiv S_{2n,m}(y_1 \dots y_{2n}, x_1 \dots x_m)$$

exists and since the ψ are odd in σ they are able to distinguish between + and - bd conditions.
In particular

$$\langle \psi_x \rangle^+ \equiv \lim_{\Lambda \uparrow \mathbb{Z}^2} \langle \psi_\Lambda(x) \rangle_\Lambda = \langle \sigma_x \rangle^+$$

▨

We have now a sequence $\{ S_{n,m} \}_{n,m=0}^{\infty}$ of correlation functions, defined by (4.5), (4.6), (4.7), to which the reconstruction theorem can be applied .
To do this we really have to check that this sequence is lattice translation invariant and O-S positive.
This follows easily from the following two facts (but see also next section):

1) The state $\langle \; \rangle^+$ is lattice translation invariant and O-S positive;

2) a disorder field $D(\{y_i\}\{z_j\})$ having support on a set of points y, living in the positive time lattice and a set of points z, living in the negative time lattice can always be rewritten using the θ operation as

$$D_+(\{y_i\})\ \theta\ D_+(\{z_j\})$$

where D_+ is a suitable field with support on the positive time lattice. Then a slight modification of the Reconstruction Theorem 2.1 gives us a Lattice Quantum Mechanics ($\mathcal{H}$, Ω, T , U).
In particular, following the notation introduced at the end of Remark 2.3, a state in $\mathcal{H}$ corresponding to m ordinary fields $\mathcal{V}_{\langle xy\rangle}$ and the n-point field $D_+(z_1 \dots z_n)$, will be written

$$| z_1 \dots z_n ; \langle xy\rangle_1 \dots \langle xy\rangle_m \rangle$$

The Hilbert space $\mathcal{H}$ has quite a different structure in the high (small β) and low (large β) temperature region.
For large β , in fact, using the cluster expansion in Peierls contours one immediately shows that

$$\left| S_{n+1,m}(z_1 \dots z_n, z ; \langle xy\rangle_1 \dots \langle xy\rangle_m)\right| \underset{z\to\infty}{\leq} e^{-2\beta|z|} \tag{4.9}$$

Hence, the cluster property holds also w.r.t. the support of the disorder fields and in particular all odd correlation functions are zero from the definition (4.6). (Here odd means in the number of disorder fields).
This second feature implies that all the states with an even number of disorder fields are orthogonal to all the states with an odd number of disorder fields:

$$\langle z_1 \dots z_{2m+1} ; \dots | z'_1 \dots z'_{2n} ; \dots \rangle = 0$$

Hence $\mathcal{H}$ splits into two orthogonal sectors $\mathcal{H}_{even}$ and $\mathcal{H}_{odd}$ both lattice translation invariant.

Since the cluster property holds and the vacuum Ω belongs to $\mathcal{H}_{even}$, $\mathcal{H}_{odd}$ does not contain any lattice translation (T,U) invariant vector.

Moreover, both sectors are invariant w.r.t. the operator field algebra $\mathcal{A}_0$ generated by the ordinary field operators (or more generally by the field operator which generates $\mathcal{H}_{even}$).

Therefore, $\mathcal{H}_{odd}$ is a charged state.

The charge labelling the sectors carried by a field with support in the dual lattice and can be shown to be of topological nature (see remark 5.9), therefore $\mathcal{H}_{odd}$ is a soliton sector.

For low β , it is well known, using the high temperature expansion that e.g.

$$\lim_{z \to \infty} S_{z,0}(x,z) = \text{const} > 0$$

Therefore we see that there is no orthogonal splitting of $\mathcal{H}$, and no charged soliton sectors exist.

Having constructed, for large β , the soliton sector we wish now to analyze its particle content.

Let $S(x^1)$ denote the soliton operator defined by

$$S(x^1) \mid X_1 \ldots X_n ; \quad \rangle =$$

$$= \mid X = (0, x^1) , X_1 \ldots X_n ; . \ \rangle \qquad (4.10)$$

Clearly

$$S(x^1)\ \mathcal{H}_{even} \subseteq \mathcal{H}_{odd}$$

$$S(x^1)\ \mathcal{H}_{odd} \subseteq \mathcal{H}_{even} \tag{4.11}$$

Moreover in $\mathcal{H}$

$$S(x^1) = S(x^1)^+ \quad \text{and}$$

$$S(x^1)^+ S(x^1) = 1 \tag{4.12}$$

Equations (4.11), (4.12) show that $S(x^1)$, $x^1 \in \mathbb{Z}$ is an intertwiner operator.

Let us now show that it couples the vacuum to a one particle state of mass $2\beta + O(e^{-\beta})$ for large β.

Consider the two point function

$$S_{2,0}(x,y) = \langle S(x^1)\Omega,\ T(t)\, U(y^1 - x^1)\, S(y^1)\Omega \rangle$$

where $x = (0, x^1)$, $y = (t, y^1)$

Using (4.1), (4.7), (4.8) and defining $(dv)_p = \prod_{\langle xy\rangle \in \partial p} v_{\langle xy\rangle}$ we have

$$S_{2,0}(x,y) = \lim_{\Lambda \uparrow \mathbb{Z}^2} \frac{\sum_{\substack{v \subset \Lambda\setminus\partial\Lambda \\ dv = dw}} \prod_{\langle xy\rangle' \in \Lambda} e^{\beta(v_{\langle xy\rangle'} - 1)}}{\sum_{\substack{v \subset \Lambda\setminus\partial\Lambda \\ dv = 1}} \prod_{\langle xy\rangle' \in \Lambda} e^{\beta(v_{\langle xy\rangle'} - 1)}} \tag{4.13}$$

with supp.$(d\omega)^* = \{x, y\}$.
We use the standard techniques of the low temperature expansion to express the quotient in (4.13) as an expansion, convergent for large β, in curves γ, which are the support of the field v^*.
By denoting $\underset{\sim}{\gamma}$ a collection of (even repeated) γ one obtains:

$$S_{2,0}(x,y) = \sum_{\underset{\sim}{\gamma},\, \gamma_{xy}} a_T(\underset{\sim}{\gamma}, \gamma_{xy}) \prod_{\gamma \in \underset{\sim}{\gamma}} e^{-2\beta|\gamma|} e^{-2\beta|\gamma_{xy}|} \tag{4.14}$$

where: 1) γ are maximally connected closed curves in the dual lattice, and γ_{xy} is a curve with $\{x, y\}$ boundary.

2) $a_T(\underset{\sim}{\gamma}, \gamma_{xy})$ is a combinatorial factor which vanishes unless all the curves in its argument touch each other.
Due to these properties, a $\underset{\sim}{\gamma}, \gamma_{xy}$ configuration giving a non vanishing contribution appears, as in fig. 4.1.

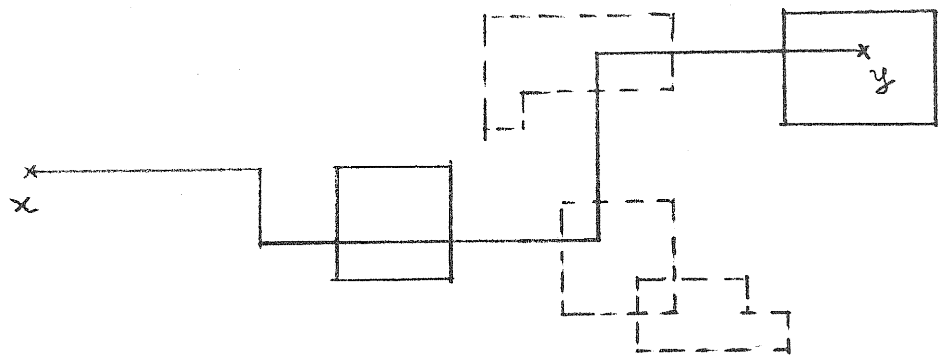


fig. 4.1

Therefore, (4.14) can be seen as an expansion in random walks joining x to y, similar to the one analyzed by means of the excitation gas in sect. 3 .

So we now identify as regular bonds in the curve $\underset{\sim}{\gamma} \cup \gamma_{xy}$ the ones which are parallel to the time axis, occur with multiplicity one and there is no other bond in $\underset{\sim}{\gamma} \cup \gamma_{xy}$ having the same projection on the time axis.
The definition of straight lines and excitations goes as in sect. 3.
We now note an important factorization property of the combinatorial factor $a_{\pi}(\underset{\sim}{\gamma}, \gamma_{xy})$ which can be seen by inspection of its explicit form (see also sect. 6).
Let ε be an excitation and define

$$\underset{\sim}{\gamma}(\varepsilon) = \underset{\sim}{\gamma} \cap \varepsilon$$

$$\gamma_{xy}(\varepsilon) = \gamma_{xy} \cap \varepsilon$$

then, if the path $\underset{\sim}{\gamma} \cup \gamma_{xy}$ is in one to one correspondence with the set of excitations $\{\varepsilon_1 \dots \varepsilon_n\}$

$$a_{\pi}(\underset{\sim}{\gamma}, \gamma_{xy}) = \prod_{i=1}^{n} a_{\pi}(\underset{\sim}{\gamma}(\varepsilon_i), \gamma_{xy}(\varepsilon_i))$$

Therefore, if we define the activity of the excitation ε through

$$\zeta(\varepsilon) = e^{2\beta|\pi(\varepsilon)|} a_{\pi}(\underset{\sim}{\gamma}(\varepsilon), \gamma_{xy}(\varepsilon)) \prod_{\gamma \in \underset{\sim}{\gamma}} e^{-2\beta|\gamma|} e^{-2\beta|\gamma_{xy}(\varepsilon)|}$$

we have

$$\sum_{y^1 \in \mathbb{Z}} S_{2,0}(x,y) = e^{-2\beta t} \sum_{\{\varepsilon_1 \dots \varepsilon_n\}} \prod_{i=1}^{n} \zeta(\varepsilon_i)$$

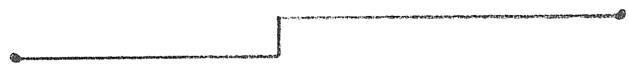
where the excitations in each term of the sum are allowed in $[0,t]$ and are compatible.
From the explicit form of a_{π} one derives the bound

$$|\zeta(\varepsilon)| \leq e^{-|\varepsilon|\, O(2\beta)}$$

for large β . Therefore, we are now in a situation similar to (3.12), (3.14) and we easily get the estimate

$$\overline{m}(\beta) = 2\beta + O(e^{-2\beta})$$

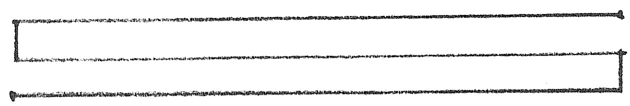
where the first term corresponds to the straight line term $Z(\omega_0)$ in (3.14)and the correction is due to the smallest excitation: the jump.



Similarly,

$$\pi(\beta) \simeq 3\,\overline{m}(\beta) > 0$$

corresponding to the leading excitation with $\pi(\varepsilon) \supset [0,t]$, i.e.



So the soliton really appears as a massive particle in $\mathcal{H}$ and it is easy to show that it is the lowest one particle state in $\mathcal{H}_{odd}$.

<u>Remark 4.2</u> It is interesting to notice that the fact that the soliton is really a (lattice) particle implies that to have asymptotic completeness we need to consider also $\mathcal{H}_{odd}$ in the physical Hilbert space i.e. $\mathcal{H}_{phys} = \mathcal{H}_{even} \oplus \mathcal{H}_{odd}$.
Moreover, for large β, the spin-spin correlation functions are known [21] not to possess an Ornstein Zernike behaviour; i.e. the spin state is not a one particle state, for large β, rather it corresponds to a two-soliton state, and the soliton is really the physical particle of the theory.

5. LOCAL SOLITON: construction of sectors

In the previous section we have seen that the expectation values of disorder fields and the $\mathbb{Z}_2$-gauge invariant ordinary fields $\nu_{\langle xy\rangle} = \sigma_x \omega_{\langle xy\rangle} \sigma_y$, do not depend on the whole support of the external gauge field ω , but just on $d\omega$.
This independence property of mixed correlation functions of disorder and suitably chosen order fields, is the characteristic feature of the lattice field theories which possess local soliton sectors (class A).
The dependence on $d\omega$ only in the Ising model was ensured by the invariance under the gauge transformation (4.4), which allows to absorb every gauge transformation of the ω field in a redefinition of the spin field σ .
This suggests a class of gauge theories to which the same mechanism should apply and eventually produce local soliton sectors.
For the sake of compactness of notation, let us first introduce some definitions.
A field with support on k dimensional cells, will be said of rank k. Given a rank-k-field ϕ taking values in a group G, one naturally defines a rank k + 1 field $d\phi$ through

$$d\phi(c_{k+1}) = \begin{cases} \sum\limits_{c_k \in \partial c_{k+1}} \phi(c_k) & \text{if G is additive} \\ \\ \#\limits_{c_k \in \partial c_{k+1}} \phi(c_k) & \text{if G is multiplicative} \end{cases}$$

(if G is abelian we just use $\prod$).
Moreover, suppose there exists a group of gauge transformations taking values in a group G' and acting on ϕ by means of a representation R .
Let ω denote a rank-k+1, G'-valued gauge field. Then one defines the rank-k+1 field $\nabla_\omega \phi$, the lattice covariant derivative of ϕ relative to ω , by

$$(\nabla_\omega \phi)(c_{k+1}) = d\phi(c_{k+1}) + \omega(c_{k+1})$$

if G and G' are additive,

$$(\nabla_\omega \phi)(c_{k+1}) = R(\omega(c_{k+1}))(d\phi(c_{k+1}))$$

if G and G' are multiplicative.
Finally, for k=0 and G additive, we have the usual definition

$$(\nabla_\omega \phi)_{\langle xy \rangle} = \phi_y - R(\omega_{\langle xy \rangle})\, \phi_x \quad .$$

The desired class of lattice field theories is then described by a state, given as the thermodynamic limit of the finite volume Gibbs states

$$\begin{aligned} \langle (\cdot) \rangle_\Lambda &= \int d\mu_\Lambda(\phi)\, (\cdot) = \\ &= \frac{1}{Z_\Lambda} \int d\nu_\Lambda(\phi)\, e^{-S_\Lambda(d\phi)}\, (\cdot) \qquad (5.1) \\ Z_\Lambda &= \int d\nu_\Lambda(\phi)\, e^{-S_\Lambda(d\phi)} \end{aligned}$$

where

1) the measure $d\nu_\Lambda(\phi)$ is invariant, except (at most) for a bd term, under a group for local gauge transformation (of rank k) with values in a discrete abelian group $\mathcal{Z}$.(+) Let ω denote the associated gauge field.

(+) If ϕ is a G'-valued gauge field, then also a non $\mathcal{Z}$ invariant G' gauge fixing term is allowed if the state is defined on G' invariant fields. (See e.g. later on the Landau gauge fixing in the U(1) Higgs model with gaussian action.)

2) The function S_Λ is local and $S_\Lambda(\nabla_\omega \phi)$ is $\mathcal{Z}$ gauge invariant. I.e. if $R(\xi)$ denotes the action of the element ξ of the gauge group on ϕ

$$S_\Lambda (\nabla_{\omega + d\xi} R(\xi) \phi) = S_\Lambda (\nabla_\omega \phi) \tag{5.2}$$

Then in general a disorder field can be defined as

$$D(\omega) = e^{-[S_\Lambda (\nabla_\omega \phi) - S_\Lambda (d\phi)]} \tag{5.3}$$

Using eq. (5.2) one sees immediately that

$$\langle D(\omega) \rangle_\Lambda$$

depends only on ω modulo $d\xi$. If Λ is convex we know that there is a bijective correspondence

$$\omega \ \mathrm{mod}\ d\xi \longleftrightarrow d\omega$$

so $\langle D(\omega) \rangle_\Lambda$ depends on $d\omega$.

Since the field $d\omega$ is discrete and closed $d(d\omega)=0$ by general arguments it has support in the dual of a set of closed $d-k-2$ dimensional surfaces.

To pursue the analogy with the $(\mathrm{Ising})_2$ model we need $d=k+2$ so that these surfaces are pointlike.

The case $d>k+2$ will be briefly discussed later.

In $d=k+2$ the field $d\omega$, since it is $\mathcal{Z}$ valued, can be completely defined in terms of its support, a set of points $\{y_i\}$ in the dual lattice, and its value at these points, a set of charges $\{q_i\}$, $q_i \in \mathcal{Z} \setminus \{0\}$ (here 0 means the identity of $\mathcal{Z}$).

So in this context we set

$$D(\omega) \equiv D(y_1 q_1 \ldots y_n q_n) \tag{5.4}$$

$$\text{for} \quad d\omega\,(c_{k+1}) = q_i \qquad c_{k+1}^{*} = y_i$$
$$= 0 \qquad \text{otherwise}$$

By explicit construction, since ω is of compact support, necessarily $\sum_i q_i = 0$.

Remark 5.1 In the $\mathcal{Z} = \mathbb{Z}_2$ case we can omit the q's and identify ω^* with its support, call it C , since $\mathbb{Z}_2$ has only one non trivial element. Then we are left with the definition used in the Ising model.
Notice that $\sum_i q_i = 0$ coincides with the requirement of an even number of points for the disorder field.

It is obvious that if we take expectation value of a disorder field and $\mathcal{Z}$ gauge-invariant ordinary fields (eventually ω -dependent), this expectation will depend on $d\omega$ and not on ω .
Denote by $\mathcal{J}$ a translation and reflection invariant set of connected sets of cells (e.g. loops, paths, etc.). Let $\Gamma \in \mathcal{J}$ and $\psi(\Gamma)$ denote a $\mathcal{Z}$ gauge invariant field with support on Γ .
Then we can define the correlation functions

$$S_{n,m}(x_1 q_1 \ldots x_n q_n ; \Gamma_1 \ldots \Gamma_m) \equiv \tag{5.5}$$

$$\equiv \lim_{\Lambda \nearrow \mathbb{Z}^d} \langle D(x_1 q_1 \ldots x_n q_n)\, \psi(\Gamma_1) \ldots \psi(\Gamma_m) \rangle$$

where $x_i \in \Lambda^*$, $q_i \in \mathcal{Z} \setminus \{0\}$, the Γ 's are non-intersecting.
Correlation functions with non vanishing total charge are defined by a limiting procedure which removes a total compensating charge to infinity.

More precisely for $\sum_{i=1}^{n} q_i = Q$

$$S_{n,m}(x_1 q_1 \dots x_n q_n ; \Gamma_1 \dots \Gamma_m) \equiv$$

$$\equiv \lim_{z\to\infty} S_{n+1,m}(x_1 q_1 \dots x_n q_n, z-Q ; \Gamma_1 \dots \Gamma_m) \tag{5.6}$$

if the limit exists.
(For convenience we set

$$S_{n+1,m}(x_1 \dots q_n ; z\ 0, \Gamma_1 \dots \Gamma_m) \equiv S_{n,m}(x_1 \dots q_n, \Gamma_1 \dots \Gamma_m))$$

To make this somewhat abstract setting more concrete let us give some explicit examples of lattice models satisfying the above requirements.

Examples 5.2

A) ϕ^4 with +bd conditions d=2

In this case the field ϕ has support on sites and is real valued; the measure $d\nu_\Lambda(\phi)$ is given by

$$d\nu_\Lambda(\phi) = \prod_{x\in\Lambda} d\phi_x\, e^{-\lambda(\phi_x^2-\rho_0^2)^2} \prod_{x\in\partial\Lambda} \delta(\phi_x-\rho_0)$$

where $d\phi_x$ is the Lesbesgue measure on $\mathbb{R}$, ρ_0, $\lambda \in \mathbb{R}_+$; and the function S_Λ is given by

$$S_\Lambda(d\phi) = \sum_{\langle xy\rangle\in\Lambda} \frac{1}{2}(d\phi)^2_{\langle xy\rangle} = \sum_{\langle xy\rangle\in\Lambda} \frac{1}{2}(\phi_x-\phi_y)^2$$

The measure $d\nu_\Lambda$ is invariant under $\phi_x \to -\phi_x$, $x\in\Lambda\backslash\partial\Lambda$ so $\mathcal{G} = \mathbb{Z}_2$.
If $\omega_{\langle xy\rangle} \in \mathbb{Z}_2 \simeq \{0,1\}$ denotes the associated gauge field, the covariant derivative is defined by $(\nabla_\omega\phi)_{\langle xy\rangle} = \phi_y - e^{i\pi\omega_{\langle xy\rangle}}\phi_x$

so that
$$D(\omega) = \prod_{\langle xy\rangle\in\operatorname{supp}\omega} \exp\left[\phi_y\left(e^{i\pi\omega_{\langle xy\rangle}}-1\right)\phi_x\right]$$

As an example of a $\mathbb{Z}$ gauge invariant field we can take e.g.

$$\rho_x = |\phi_x| \qquad \text{or} \qquad \phi_x \prod_{\langle xy\rangle' \in \Gamma_{xy}} e^{i\pi\,\omega_{\langle xy\rangle'}}\,\phi_y$$

where Γ_{xy} is a curve connecting x to y.
Using cluster expansion techniques, one can see that also the field $\psi_\Lambda(x) = \phi_x \prod_{\langle xy\rangle' \in \Gamma_x(\Lambda)} e^{i\pi\,\omega_{\langle xy\rangle'}}$ is allowed.
This plays a role similar to that of $\psi_\Lambda(x)$ in $(\text{Ising})_2$ in distinguishing between + and - conditions.

B) <u>Higgs model</u>.
In this case the measure $d\nu_\Lambda$ depends on a gauge field defined on links A and on a field defined on sites ϕ, the Higgs field.
We discuss two models:
i) <u>U(1) Higgs with gaussian gauge action</u>.
Here A is real, ϕ is $\mathbb{C}$ valued and

$$d\nu_\Lambda(\phi, A) = \prod_{x\in\Lambda} d\phi_x\, e^{-\lambda(\phi_x^2 - \rho_0^2)^2} \prod_{x\in\partial\Lambda} \delta(\phi_x - \rho_0)\cdot$$

$$\prod_{\langle xy\rangle\in\Lambda} dA_{\langle xy\rangle}\, e^{-\frac{1}{2}|\phi_x - \exp(ieA_{\langle xy\rangle})\,\phi_y|^2} \qquad F(A)$$

$$S_\Lambda(dA) = \frac{1}{2}\sum_{p\in\Lambda} (dA_p)^2\,,$$

and $F(A)$ is the necessary gauge fixing term, $e \in \mathbb{R}_+$.
We can take e.g. the Landau gauge fixing.

$$F(A) = \prod_{x\in\Lambda} e^{-\frac{1}{2\alpha}\,({}^*d{}^*A)_x^2 + C}$$

with α, C suitable constants; (see e.g. [23]). The measure $d\nu_\Lambda(\phi, A)$ is invariant under

$$A_{\langle xy\rangle} \longrightarrow A_{\langle xy\rangle} + \frac{2\pi}{e}\, n_{\langle xy\rangle}\;; \quad n_{\langle xy\rangle} \in \mathbb{Z}.$$

So if we consider the 2nd-rank field ω with value in $\frac{2\pi}{e}\mathbb{Z}$,

$$D(\omega) = \prod_{p\in \operatorname{supp}\omega} \exp\left\{-\frac{1}{2}\left[(dA_p + \omega_p)^2 - (dA)_p^2\right]\right\}$$

As an example of $\mathcal{Z} = \frac{2\pi}{e}\mathbb{Z}$ gauge invariant fields we can take e.g.

$$\exp ie(dA)_p$$

or its non local version, the Wilson loop:

$$W(C) = \prod_{\langle xy\rangle \in C} \exp ie A_{\langle xy\rangle}$$

where C is a loop, and the string variable

$$\phi_x \Big(\prod_{\langle xy\rangle' \in \Gamma_{xy}} \exp ie A_{\langle xy\rangle'} \Big) \phi_y$$

ii) Higgs Model SU(N) d=3

In this case the gauge field g takes values in the SU(N) group and the Higgs field ϕ in a Hilbert space V, with scalar product $(\cdot,\cdot)$ carrying a unitary representation U of $SU(N)/\mathbb{Z}_N$ (for simplicity we assume $(\phi,\phi)=1$)

$$d\nu_\Lambda(g,\phi) = \prod_{\langle xy\rangle} dg_{\langle xy\rangle} \prod_{x\in\Lambda} d\phi_x \prod_{\langle xy\rangle\in\Lambda} e^{\lambda \,\mathrm{Re}\,[(\phi_x, U(g_{\langle xy\rangle})\phi_y)-1]}$$

$$S_\Lambda(dg) = \frac{1}{e^2} \sum_{p\in\Lambda} \mathrm{Re}\,(\chi(g_{\partial p})-1)$$

with $dg_{\langle xy\rangle}$ the Haar measure on $SU(N)$, χ a faithful character of $SU(N)$, $e \in \mathbb{R}_+$, $d\phi_x$ a measure on V.

This $d\nu_\Lambda$ measure is clearly invariant under

$$g_{\langle xy\rangle} \longrightarrow \zeta_{\langle xy\rangle}\, g_{\langle xy\rangle} \quad , \quad \zeta_{\langle xy\rangle} \in \mathbb{Z}_N \subset SU(N)$$

Hence if ω is a rank 2 field, $\mathbb{Z}_N$ valued,

$$D(\omega) = \prod_{p \in \mathrm{supp}\,\omega} \exp\Big\{ -\frac{1}{e^2} \mathrm{Re}\, [\, \chi(\omega_p g_{\partial p}) - \chi(g_{\partial p}) \,]$$

C) <u>Rank k-Stückelberg models d=K+2</u>

Let A_k be a real valued field of rank k, and B_{k-1} a $U(1)$ valued field of rank-k-1 (for $k=0$ we set $B_{-1}=0$) and take

$$d\nu_\Lambda(A_k, B_{k-1}) = \prod_{c_k \in \Lambda} dA_k(c_k) \prod_{c_{k-1} \in \Lambda} dB_{k-1}(c_{k-1})$$

$$\cdot \prod_{c_k \in \Lambda} \exp\{ z[\cos(eA_k(c_k) + dB_{k-1}(c_k)) - 1] \}$$

$$S_\Lambda(dA_k) = \frac{1}{2} \sum_{c_{k+1} \in \Lambda} (dA_k(c_{k+1}))^2$$

where $dA_k(c_k)$ is the Lesbesgue measure on $\mathbb{R}$ and $dB_{k-1}(c_{k-1})$ is the Haar measure on $U(1)$; $z, e \in \mathbb{R}_+$.
The model described by the thermodynamic limit of

$$\langle(\cdot)\rangle_\Lambda = \frac{1}{Z_\Lambda}\int d\gamma_\Lambda(A_k, B_{k-1})\, e^{-S_\Lambda(dA_k)}(\cdot)\, F(A_k)$$

where $F(A_k)$ is a gauge fixing term (and $F(A_0) \equiv 1$), is sometimes called rank-k Stükelberg model.
For $k=0$ it reduces to the $\cos\phi$ theory.
The gauge fixing $F(A)$ is necessary if $k>0$ since $S_\Lambda(dA_k)$ is invariant under

$$A_k \longrightarrow A_k + d\lambda_{k-1} \equiv A_k^{\lambda_{k-1}}$$

$$\lambda_{k-1}(c_{k-1}) \in \mathbb{R}$$

so that the function $\exp - S_\Lambda(dA_k)$ is not $d\gamma_\Lambda$ integrable.
The gauge fixing term can be chosen as

$$F(A_k) = e^{-\frac{1}{2K}\sum [(*d*A)(c_{k-1})]^2 + c}$$

where c is fixed in such a way that

$$\int F(A_k^{\lambda_{k-1}}) \prod_{c_{k-1}\in \Lambda\setminus T_{k-1}} d\lambda_{k-1}(c_{k-1}) = 1$$

with T_{k-1} a maximal $k-1$ tree. (This is a generalization to k>1 of the gauge fixing discussed in [23]).
In this case $\chi = \frac{2\pi}{e}\mathbb{Z}$, corresponding to the invariance of $d\gamma$ under

$$A_k(c_k) \longrightarrow A_k(c_k) + \frac{2\pi}{e} n(c_k), \quad n(c_k) \in \mathbb{Z}$$

and the external gauge field ω has support on k+1 cells, so that

$$D(\omega) = \prod_{c_{k+1}\in \operatorname{supp}\omega} \exp\left\{-\frac{1}{2}\left[(dA_k(c_{k+1}) + \omega(c_{k+1}))^2 - (dA_k(c_{k+1}))^2\right]\right\}$$

As χ gauge invariant field one can take e.g.

$$\exp i\,[e\,A_k(C_k) + d\,B_{k-1}(C_k)]$$

or, a choice which is more familiar [19] in the $\cos\phi$ model,

$$\cos\,[e\,A_k(C_k) + d\,B_{k-1}(C_k)]$$

Having in mind these specific examples, let us come back to the general discussion and prove the

Theorem 5.3 (Reconstruction Theorem)

If the state $\langle(\cdot)\rangle \equiv \lim_{\Lambda\uparrow\mathbb{Z}^d}\int d\mu_\Lambda(\cdot)$ defined in (5.1) is lattice translation invariant and O.S positive and the correlation functions $S_{n,m}$ defined in (5.5), (5.6) are uniformly bounded in their time coordinates, then the sequence

$$\{\, S_{n,m}(x_1 q_1 \ldots x_n q_n\, ;\, \Gamma_1 \ldots \Gamma_m)\,\}_{n,m=0}^{\infty}$$

determines a Lattice Quantum Mechanics ($\mathcal{H}$, Ω, T, U).

Remark 5.4 The non trivial statement of this theorem is that only the properties of the state $\langle\ \rangle$ of the lattice field theory are involved in the reconstruction theorem also for mixed correlation functions of ordinary fields and disorder fields.

Proof: The idea, of course, is to slightly modify the standard reconstruction theorem 1.1.

So let $\underline{S}^*$ be the set of all finite sequences $\underline{f}^* = \{f_n^*\}$ of complex functions f_n^* on $(\mathbb{Z}^d \times \mathcal{Z}\setminus\{0\})^{\times n}$ vanishing except at finitely many points.(+)

Turn $\underline{S}^*$ into an algebra, define lattice translation in the obvious way and define the Θ operation by

(+) Here the * is not the dual but just a symbol to distinguish quantities associated to the disorder fields.

$$\theta \underline{f}^* = \{ \bar{f}^*_{n,r} \}$$

$$f^*_{n,r}(x_1 q_1 \ldots x_n q_n) = f_n(r x_1 - q_1 \ldots r x_n - q_n)$$

Now let $\underline{S}_E$ be the set of sequences relative to the ψ-fields with support on $\Gamma \in \mathcal{Y}$ and set

$$\underline{\mathcal{S}} \equiv \underline{S}^* \otimes \underline{S}_E$$

Define the functional on the sequences $\underline{f}^* \otimes \underline{g}$:

$$S : \underline{f}^* \otimes \underline{g} \longrightarrow \sum_{n,m=0}^{\infty} S_{n,m}(f_n^*, g_m)$$

where

$$S_{n,m}(f_n^*, g_m) = \prod_{i=1}^{n} \sum_{\substack{x_i \in \mathbb{Z}^d \\ q_i \in \mathbb{Z}\setminus\{0\}}} \prod_{j=1}^{m} \sum_{\substack{y_j \in \mathbb{Z}^d_{\frac{1}{2}} \\ \Gamma^0_j \in E}} \cdot \tag{5.7}$$

$$\cdot S_{n,m}(x_1 q_1 \ldots x_n q_n ; y_1 \Gamma^0_1 \ldots y_m \Gamma^0_m) \cdot$$

$$\cdot f_n^*(x_1 q_1 \ldots x_n q_n) \, g_m(y_1 \Gamma^0_1 \ldots y_m \Gamma^0_m)$$

and extend S to $\underline{\mathcal{S}}$ by linearity.
Therefore, from the lattice translation invariance of the state $\langle\ \rangle$ follows the lattice translation invariance of the functional S in eq. (5.7). From the boundedness of the functions $S_{n,m}$ in their time coordinate, hypothesis 3) of theorem 1.1 follows.
So the proof is deferred to the following lemma.

<u>Lemma 5.4</u> If the state $\langle\ \ \rangle$ is O.S. positive then the functional S is O.S. positive.

<u>Proof</u>. We show it in the

$$S\left(\theta\left(\underline{f}_1^* \otimes \underline{1\!\!1}\right) \times \left(\underline{f}_1^* \otimes \underline{1\!\!1}\right)\right)$$

$$\left(\underline{f}_1^*\right)_n = 0 \qquad , n \neq 1$$

$$= f(x,q) \qquad , n = 1$$

case, which only involves the two point function

$$S_{2,0}\,(x q \;\; x' q') \quad .$$

The general case follows easily by the same method.
We have to prove that for every complex function $f(x,q)$ with support on $\mathbb{Z}^d_+ \times \mathbb{Z}\setminus\{0\}$
($\mathbb{Z}^d_+$ is the positive time dual lattice) which vanishes except at finitely many points,

$$\sum_{x\in\mathbb{Z}^d}\ \sum_{q\in\mathbb{Z}\setminus\{0\}}\ \sum_{x'\in\mathbb{Z}^d}\ \sum_{q'\in\mathbb{Z}\setminus\{0\}} f(x,q)\,\overline{f(x',q')}\; S_{2,0}(xq \;\; \tau x' - q') \geq 0$$

where the bar denotes the complex conjugation, and τ the reflection w.r.t. the $t=0$ plane.
To prove this inequality let y be a point in $\mathbb{Z}^d_+$ and consider the path

$$\gamma_y : \quad y = (t, y^1 \dots y^{d-1}) \longrightarrow (t, y^1 \dots y^{d-2}, 0) \longrightarrow$$

$$\longrightarrow \quad \dots \quad \longrightarrow (t, 0, \dots, 0) \longrightarrow (0, \dots, 0)$$

if y is in the strictly positive time lattice and

$$\gamma_y : \quad y = (0, y^1 \dots y^{d-1}) \longrightarrow (1, y^1 \dots y^{d-1}) \longrightarrow$$

$$\longrightarrow (1, y^1 \dots y^{d-2}, 0) \longrightarrow \quad \dots \quad \longrightarrow (1, 0 \dots, 0) \longrightarrow$$

$$\longrightarrow (0, \dots 0)$$

if y is in the $t=0$ plane. (See Fig. 5.1).
We now put a compensating charge $q'-q$ at $z=(0,\vec{z})$ and take as support for the "gauge" field ω the dual of the curve $\gamma_x \cup \tau\gamma_{x'} \cup \gamma_z \cup \tau\gamma_z$ (see fig. 5.1). With charge q on γ_x, $-q'$ on $\tau\gamma_{x'}$, $-q$ on γ_z, q' on $\tau\gamma_z$, then the field ω naturally decomposes into

$$\omega = \omega_+ + \omega_-$$

where $\omega_{+ (-)}$ has support on $\gamma_x \cup \gamma_z$ $(\tau\gamma_{x'} \cup \tau\gamma_z)$ and we have the factorization property

$$D(\omega) = D(\omega_+)\, D(\omega_-) =$$

$$= D(xq, z-q)\, D(\tau x'-q', z\, q')$$

Now

$$D(\tau x'-q', z\, q') = \Theta\, D(x\, q', z-q') \equiv \tau\, \overline{D(x q', z-q')}$$

and hence

$$\sum_{x\in\mathbb{Z}^d}\;\sum_{q\in\mathbb{Z}\setminus\{0\}}\;\sum_{x'\in\mathbb{Z}^d}\;\sum_{q'\in\mathbb{Z}\setminus\{0\}} f(x,q)\,\overline{f(x'q')}\,\cdot$$

$$\cdot\, S_{2,0}(xq,\, \tau x'-q') = \lim_{z\to\infty}\;\lim_{\Lambda\uparrow\mathbb{Z}^d} \langle \Big[\sum_x\sum_q f(x,q)\,\cdot$$

$$\cdot\, D(xq,\, z-q)\Big]\,\theta\Big[\sum_x\sum_q f(x,q)\,D(x,q,z-q)\Big]\rangle \geq 0$$

by O.S. positivity of the measure $d\mu_\Lambda$.

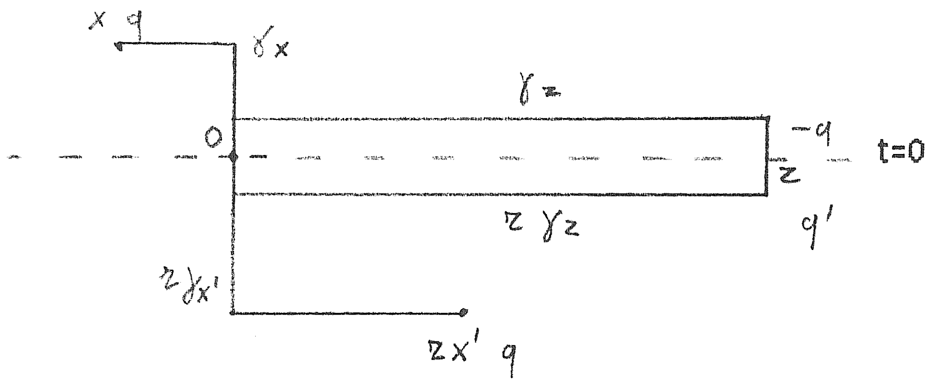


fig. 5.1

<u>Remark 5.5</u> On $[\underline{S}^{*+}\otimes\underline{S}_E^+]^\wedge$ operators

$$\phi(\underline{f}^*\otimes\underline{g})\,T(t)$$

are well defined, with $\underline{f}\otimes\underline{g}\in\underline{S}^{*+}\otimes\underline{S}_E^+$ having support in the time coordinate contained in the strip $[0,t]$.
They are defined by

$$\phi(\underline{f}^*\otimes\underline{g})\,T(t)\,[\underline{h}^*\otimes\underline{\ell}]^\wedge = [(\underline{f}^*\times R(t)\underline{h}^*)\otimes(\underline{g}\times R(t)\underline{\ell})]^\wedge$$

Clearly, for $t\in\mathbb{Z}_+$, $\vec{a}\in\mathbb{Z}^{d-1}$ and $a=(t,\vec{a})$

$$S(\theta(\underline{f}^{*}\otimes\underline{g})\times R(a)(\underline{h}^{*}\otimes\underline{\ell}))=$$
$$=\langle\phi(\underline{f}^{*}\otimes\underline{g})\Omega, T(t)\,U(\vec{a})\,\phi(\underline{h}^{*}\otimes\underline{\ell})\Omega\rangle$$

$\square$

We are now ready to construct the soliton sectors.

Theorem 5.6 (Construction of the local soliton sectors)
If the state $\langle\ \ \rangle$, defined in (5.1), has the cluster property also w.r.t. the support of the disorder field (this implies in particular, by (5.6), that all correlation functions with non-vanishing total charge are zero), then the Hilbert space $\mathcal{H}$ of Theorem 5.4 splits into orthogonal sectors $\mathcal{H}_q$ labelled by the elements $q\in\mathcal{Z}$ and the sectors corresponding to the non trivial elements of $\mathcal{Z}$ are soliton sectors.

Proof. The ideas of the proof are very simple:
1) By the cluster property, Ω is the unique translation invariant vector in $\mathcal{H}$.
2) Define the projection P_q of $\mathcal{H}$ onto a sector of fixed total charge q by

$$P_q:\ \mathcal{H}\longrightarrow\mathcal{H}_q$$
$$[\underline{f}^{*}\otimes\underline{g}]^{\wedge}\longmapsto[P_q\underline{f}^{*}\otimes\underline{g}]^{\wedge}$$

$$P_q\,\underline{f}^{*}=\{f_0^{*}\,\delta_{0q},\ f_1^{*}(x_1 q)\,\delta_{q_1 q},$$
$$f_2^{*}(x_1 q_1\, x_2 q_2)\,\delta_{q_1+q_2,q},\ \dots\ \}$$

From the vanishing of the correlation functions with non null total charge it follows that the sectors $\mathcal{H}_q$ are orthogonal and from

$$\sum_{q \in \mathbb{Z}} P_q = Id$$

it follows

$$\mathcal{H} = \bigoplus_{q \in \mathbb{Z}} \mathcal{H}_q$$

Each sector is invariant, by inspection, under T and U and since $\Omega = [\underline{1\!\!1}^* \otimes \underline{1\!\!1}]^\wedge$, by 1) all the sectors $\mathcal{H}_q, q \neq 0$ do not possess translation invariant vectors.
Finally, the field algebra

$$\{ \phi(\underline{1\!\!1}^* \otimes \underline{g}) T(t) \quad ; \quad \mathrm{supp}\, g_n \subset ([0,t] \times \mathbb{Z}^{d-1} \times E)^{\times n} \}$$

clearly leaves invariant each sector since:

$$\phi(\underline{1\!\!1}^* \otimes \underline{g}) T(t) [\underline{h}^* \otimes \underline{\ell}]^\wedge =$$

$$= [(\underline{1\!\!1}^* \times R(t) \underline{f}^*) \otimes (\underline{g} \times R(t) \underline{\ell})]^\wedge =$$

$$= [(R(t) \underline{f}^*) \otimes (\underline{g} \times R(t) \underline{\ell})]^\wedge$$

<u>Remark 5.7</u> Consider the operators labelled by the spatial coordinates $\vec{x} \in \mathbb{Z}^{d-1}$ and the charge $q \in \mathbb{Z} \setminus \{0\}$ defined by

$$S_q(\vec{x}) \; | x_1 q_1 \ldots x_n q_n \; ; \; \Gamma_1 \ldots \Gamma_m > =$$

$$= | (0,\vec{x}) q, x_1 q_1 \ldots x_n q_n \; ; \; \Gamma_1 \ldots \Gamma_m > \tag{5.8}$$

(In the notation used in the proof of Theorem 5.3

$$S_q(\vec{x}) = \phi(\underline{f}^*_{\vec{x},q} \otimes \underline{1\!\!1})$$

$$\left(\underline{f}^{*}_{\vec{x}q}\right)_n \begin{aligned} &= 0 && n \neq 1 \\ &= \delta(x_1, (0\vec{x})) \, \delta_{q_1 q} && n = 1 \end{aligned} \quad)$$

We notice that they are defined on $[\underline{S}^{*+} \otimes \underline{S}^{+}_{E}]^{1}$ and since $S_q^{+}(\vec{x}) = S_{-q}(\vec{x})$ (the adjoint $^{+}$ is taken w.r.t. the scalar product in $\mathcal{H}$) they are isometric and can be extended to the whole $\mathcal{H}$.

They are called soliton operators and, as in the $(\text{Ising})_2$ model, provide natural intertwiners between the $\mathcal{H}_q$ sectors.

<u>Examples 5.8</u>

<u>A)</u> For the ϕ^4 model we will show that the conditions of Theorem 5.6 are satisfied in d=2 in the two phase region (broken symmetry region). Therefore, the above construction gives us the soliton sectors of ϕ^4_2 , the so called kink sectors. The operator $S_q(\vec{x})$ introduced in (5.8) is the lattice analogue of the soliton intertwiner between vacuum and soliton sectors introduced in [19] by means of an algebraic construction. For fixed bd conditions the above considerations show that there is a unique vacuum sector $\mathcal{H}_+$ and a unique soliton sector $\mathcal{H}_S$ labelled by the non trivial element $1 \in \mathbb{Z}_2 \simeq \{0, 1\}$.

However, if we introduce the operator $P: \phi \to -\phi$ which turns +bd conditions into -bd conditions, we recover also an "anti" soliton sector $\mathcal{H}_{\bar{S}}$ besides the vacuum sector $\mathcal{H}_-$.

<u>B i)</u> For the $U(1)$ Higgs model we will show that the conditions of Theorem 5.6 are satisfied in d=3 in the superconducting (Higgsian) region and the above construction provides us with the $\mathbb{Z}$ vortex sectors.

<u>B ii)</u> For the $SU(N)$ Higgs model, we will show that the conditions of Theorem 5.6 are satisfied in d=3 in the Higgs phase and we obtain the $\mathbb{Z}_N$ vortex sectors.

<u>C)</u> For the rank k Stückelberg models $\mathbb{Z}$ soliton sectors exist in $d = k+2$ for large z and small e ; for $k=0$ we obtain the kink sectors of the sine-Gordon d=2 model. ($\equiv \cos\phi_2$)

<u>Remark 5.9</u> (Cohomological considerations)

In this remark we show how to extend the argument on the dependence on $d\omega$ of $\langle D(\omega)\rangle_\Lambda$ made for Λ convex to the case of a general lattice complex, making use of the concepts of cohomology.

The k-th cohomology group with coefficient in $\mathcal{Z}$ (discrete abelian group) of the lattice complex Λ, denoted by $H^k(\Lambda,\mathcal{Z})$ is defined to be the quotient of the closed $\mathcal{Z}$ valued local field of rank k, w.r.t. the exact $\mathcal{Z}$ valued local field of rank k, i.e. if ϕ_k denotes a $\mathcal{Z}$ valued k- rank field with support in Λ:

$$H^k(\Lambda,\mathcal{Z}) \equiv \frac{Z^{k+1}(\Lambda,\mathcal{Z})}{d\,C^k(\Lambda,\mathcal{Z})} \equiv \frac{\{\phi_k : d\phi_k = 0\}}{\{\phi_k : \exists\phi_{k-1} \text{ with } \phi_k = d\phi_{k-1}\}}$$

where the quotient is taken w.r.t. the addition.

The r-skeleton of Λ is defined to be the set of all $k \leq r$ dimensional cells in Λ and is denoted by Λ_r.

An open cell $\mathring{c}_k$ is defined by

$$\mathring{c}_k \equiv c_k \setminus \partial c_k$$

where ∂ denotes the boundary.

Then if Λ is a lattice complex, $\Lambda\setminus\{\mathring{c}_k\}$ is still a lattice complex and

$$\Lambda_r = (\Lambda\setminus\{\mathring{c}_k\})_r \qquad r < k \; .$$

Let now $\mathring{S}(d\omega)$ denote the set of all open cells dual to the support of the external "gauge" field ω.

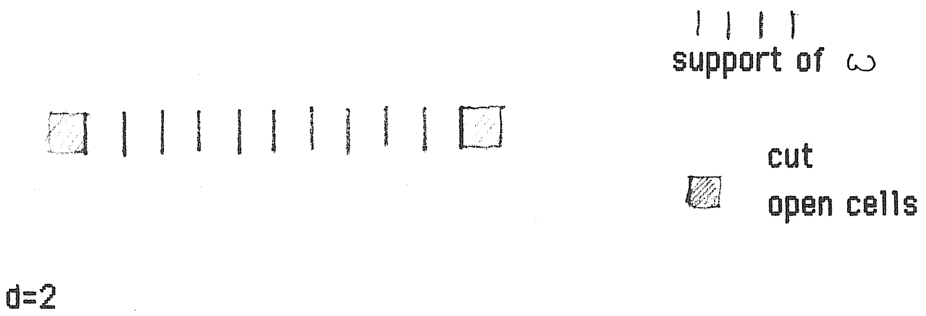


fig. 5.2

Then we have:

<u>Proposition 5.10</u> The expectation value $\langle D(\omega)\rangle_\Lambda$ depends only on the cohomology class $[\omega] \in H^{k+1}(\Lambda \setminus \mathring{S}(d\omega), \mathbb{Z})$.

<u>Proof.</u> We first remark that $(\Lambda \setminus \mathring{S}(d\omega))_r = \Lambda_r$ for $r = 1 \dots k+1$, i.e. cutting out $\mathring{S}(d\omega)$ does not change the set of cells of dimension less than $k+2$.
Therefore, substitution of Λ by $\Lambda \setminus \mathring{S}(d\omega)$ does not change the measure which does not involve fields of rank higher than $k+1$.
Hence,

$$\langle D(\omega)\rangle_\Lambda = \langle D(\omega)\rangle_{\Lambda \setminus \mathring{S}(d\omega)}$$

But on $\Lambda \setminus \mathring{S}(d\omega)$ we have $d\omega = 0$

i.e. $$\omega \in Z^{k+1}(\Lambda \setminus \mathring{S}(d\omega), \mathbb{Z}).$$

We already know, however, that $\langle D(\omega)\rangle_\Lambda$ is invariant under

$$\omega \longrightarrow \omega + d\xi$$

where ξ is a $\mathbb{Z}$ valued k-rank field defined on Λ and hence on $\Lambda \setminus \mathring{S}(d\omega)$. Therefore, $\langle D(\omega)\rangle_\Lambda$ depends only on $\omega \bmod d\xi =$
$= [\omega] \in H^{k+1}(\Lambda \setminus \mathring{S}(d\omega), \mathbb{Z})$
[We remember that $D(\omega)$ has been defined only for $\omega|_{\partial\Lambda} = 0$; if this condition is not satisfied we modify the bd conditions in the way suggested by subsect. 7.12 and the proof of the proposition now works in general as it stands]

This proposition shows that a natural classification of disorder fields on the lattice can be made by means of cohomology (at least for the present case), without any assumption of continuity needed in the the homotopic approach.
It is also worthwhile to remark that if Λ has non trivial homology (see remark 5.11), e.g. is homologous to a torus, then $\mathring{S}(d\omega) = \phi$ does not imply $H^{k+1}(\Lambda, \mathbb{Z}) = 0$, i.e. we can have $\langle D(\omega)\rangle_\Lambda$ non trivial, still with $d\omega = 0$ on the whole Λ. (See e.g. [15]).

Let us end this section with the answers to the following two questions. What happens if:

1) the dimension d is greater than $k+2$;

2) we do not choose $\mathbb{Z}$ gauge invariant ordinary fields to construct the mixed correlation functions.

If $d>k+2$, the support of $(d\omega)^*$ is given by a set of closed $d-k-2>0$ dimensional surfaces. Hence the expectation value $\langle D(\omega)\rangle_\Lambda$ will depend on these and on some set of $\mathbb{Z}$ charges. For $d=k+2$ given a set of points in the dual lattice and charges $\{y_i, q_i\}$, as we have seen, it is not guaranteed that we can find an ω contained in $\Lambda\backslash\partial\Lambda$ such that $supp\,(d\omega)^* = \{y_i\}$ and $d\omega(y_i^*) = q_i$. However, if we have to deal with closed surfaces of dimension $d-k-2>0$ and Λ is convex, as we assume here, those are always the boundary of a $d-k-1$ dimensional surface in Λ .

So, given a set of closed surface in the dual lattice $\{S_i\}$ and charges q_i , we can always find a field ω in $\Lambda\backslash\partial\Lambda$ such that $(d\omega)^*$ has support on $\{S_i\}$ and $d\omega(c_{k+2}) = q_i$ if $c_{k+2}^* \in S_i$.

Therefore, to introduce disorder fields in $d>k+2$ in the models discussed in this section cannot give rise to local soliton sectors, which require simultaneously the vanishing of correlation functions with ω of infinitely extended support, having $d\omega$ finite support. (See fig. 5.3).

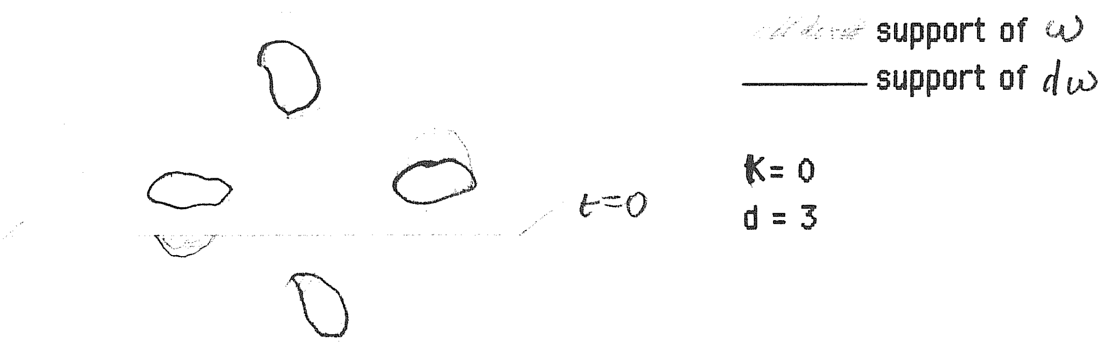


fig. 5.3

If we do not use $\mathbb{Z}$ gauge invariant ordinary field then the mixed correlation function will acquire an explicit dependence on the external gauge field ω .
However, this is essentially a dependence of topological (homological) nature.
To show this, let us give two examples: the spin in ϕ_2^4 and the Wilson loop with fractional charge in ($U(1)$ Higgs)$_3$.
In the ϕ_2^4 model

$$< D(\omega + d\xi) \, \phi_{x_1} \cdots \phi_{x_n} > =$$

$$= \; < D(\omega) \, e^{i\pi \xi_{x_1}} \phi_{x_1} \cdots e^{i\pi \xi_{x_n}} \phi_{x_n} > \tag{5.9}$$

where $\xi_{x_i} = 0$ if the support of $(d\xi)^*$ does not enclose point x_i .
Therefore the correlation function (59) is invariant under $\omega \to \omega + d\xi$ if the paths which are support of ω^* and of $(\omega + d\xi)^*$ are homotopic in $\mathbb{R}^2 \setminus \{x_1, \dots x_n\}$.
Analogously, for $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ in the (U(1) Higgs)$_3$ model consider

$$< D(\omega) \prod_{\langle xy \rangle \in C} e^{i \alpha e A_{\langle xy \rangle}} > \qquad \partial C = \phi \, . \tag{5.10}$$

This correlation function is invariant under $\omega \to \omega + d\xi$, $\xi_{\langle xy \rangle} \in \frac{2\pi}{e}\mathbb{Z}$ if the path corresponding to supp ω^* and supp $(\omega + d\xi)^*$ are homotopic in $\mathbb{R}^3 \setminus C$.

<u>Remark 5.11</u> (Homological considerations)
For lattice theories it is more appropriate to speak of homology instead of homotopy, since homotopy involves the concept of continuity which is not shared by lattice objects.

Let us just sketch how the previous arguments translate into homological language.
Given a $\mathcal{Z}$ valued field of rank k, ξ_k, we associate to it a $\mathcal{Z}$ valued chain ξ_* in Λ^* by

$$\xi_* \equiv \sum_{c_k^*} \xi_k^*(c_k^*)\; c_k^*$$

The k-th homology group with coefficient in $\mathcal{Z}$ of the lattice complex Λ, denoted $H_k(\Lambda,\mathcal{Z})$, is defined to be the quotient of the closed k chain w.r.t. the k boundaries. I.e. if $\mathcal{C}_k$ denotes a k-chain in Λ with coefficient in $\mathcal{Z}$

$$H_k(\Lambda,\mathcal{Z}) = \{\mathcal{C}_k : \partial\mathcal{C}_k = 0\} / \{\mathcal{C}_k : \exists\, \mathcal{C}_{k+1} \text{ with } \partial\mathcal{C}_{k+1} = \mathcal{C}_k\}$$

where the quotient is taken w.r.t. the addition.
In this language, we will say that the correlation function (5.9) depends on ξ through the homology class (see also remark 5.9) :

$$[(d\xi)_*] \in H_1(\mathbb{Z}^2_{\frac{1}{2}} \setminus \{(\mathring{x}_1^*), (\mathring{x}_n^*)\}, \mathbb{Z}_2)$$

and the correlation function (5.10) through:

$$[(d\xi)_*] \in H_1(\mathbb{Z}^3_{\frac{1}{2}} \setminus (\mathring{\mathcal{C}}^*), \mathbb{Z})$$

More details will appear elsewhere.

6. HIGH-LOW TEMPERATURE CLUSTER EXPANSION

In section 4, the particle analysis of the Ising model was based on a convergent low temperature expansion.
Similarly, for the models discussed in section 5, we need a cluster expansion which is convergent in the region of the parameters where soliton sectors can be constructed.
Although clearly this expansion changes from model to model, some characteristics are common to all, and in this section we outline this general construction.
The specific examples will be discussed in the next section together with the particle analysis of the soliton sector which is the main output in our context.
Let us start with some general considerations on the role of the group $\mathcal{Z}$.
The main difference between the models in section 5 and the $(\mathrm{Ising})_2$ model is that whereas the symmetry group $\mathcal{Z}$ is still a discrete group in the first one, as in the latter ones, the fields are continuous variables in ϕ^4, Higgs and Stuckelberg models.
Therefore, to pursue the analogy, we decompose the field ϕ in such a way to emphasize the role of the group $\mathcal{Z}$ and, as a consequence, the defects associated with it. This can be done in complete generality by associating, in a natural way, to the field ϕ in (5.1), taking value in a metric space W, the pair of fields (ρ, σ) taking values in $(W/\mathcal{Z}, \mathcal{Z})$.
To clarify this point and the next steps let us see what happens in the specific case of ϕ^4, the general case will follow easily.
The map defined above in this case reads

$$\phi_x \in \mathbb{R} \longrightarrow \left(\rho_x \equiv |\phi_x| \in \mathbb{R}_+ \,,\; \sigma_x \equiv \mathrm{sign}\, \phi_x \in \mathbb{Z}_2 \right)$$

This decomposition would hardly work in the continuum, but it is very useful on the lattice.

We take as field variables ρ_x and the analogue of the Ising gauge-invariant $v_{<xy>}$-variable:

$$v_{<xy>} = \sigma_y \, e^{i\pi \omega_{<xy>}} \sigma_x$$

We can do so because all correlation functions in ϕ_2^4 we discuss are $\mathbb{Z}_2$ gauge invariant (eventually ω -dependent) and will therefore depend on σ only through the field v.

The field v satisfies

$$dv = d\omega$$

and from the definitions (5.1)-(5.3) one has, for instance, the following representation for the v.e.v. of the disorder field:

$$\langle D(\omega) \rangle_\Lambda = \frac{Z_\Lambda(\omega)}{Z_\Lambda} = \frac{\displaystyle\sum_{\substack{v : dv = d\omega \\ v \subset \Lambda \setminus \partial\Lambda}} Z_\Lambda(v)}{\displaystyle\sum_{\substack{v : dv = 0 \\ v \subset \Lambda \setminus \partial\Lambda}} Z_\Lambda(v)} \tag{6.1}$$

with

$$Z_\Lambda(v) = \int \prod_{x \in \Lambda} d\rho_x \, e^{-\lambda(\rho_x^2 - \rho_0^2)^2} \prod_{<xy> \in \Lambda} e^{-\frac{1}{2}(\rho_x - \rho_y)^2} \cdot \tag{6.2}$$

$$\cdot \prod_{<xy> \in \Lambda} e^{\rho_x \rho_y (v_{<xy>} - 1)} \prod_{x \in \partial\Lambda} \delta(\rho_x - \rho_0)$$

where $d\rho_x$ is the Lesbesgue measure on $\mathbb{R}_+$. The analogy of (6.1) with (4.3)-(4.13), made explicit by the substitution

$$\prod_{<xy> \in \Lambda} e^{\beta(v_{<xy>} - 1)} \longleftrightarrow Z_\Lambda(v)$$

is immediate.
Since the $\mathbb{Z}_2$ valued v field in the numerator (resp denominator) satisfies $dv=d\omega$ (resp $dv=0$), it has support in the dual of a set of lines with boundary supp($d\omega$)*(resp vanishing boundary); moreover, a given set of such curves completely characterizes a v field configuration.
Therefore, supp $(v)^* = \{\gamma_1 \cdots \gamma_n, \gamma^\omega\}$,

where γ_i are maximally connected closed curves in the dual lattice and γ^ω is the union of all the maximally connected curves of the non-vanishing boundary. (See Fig. 6.1).

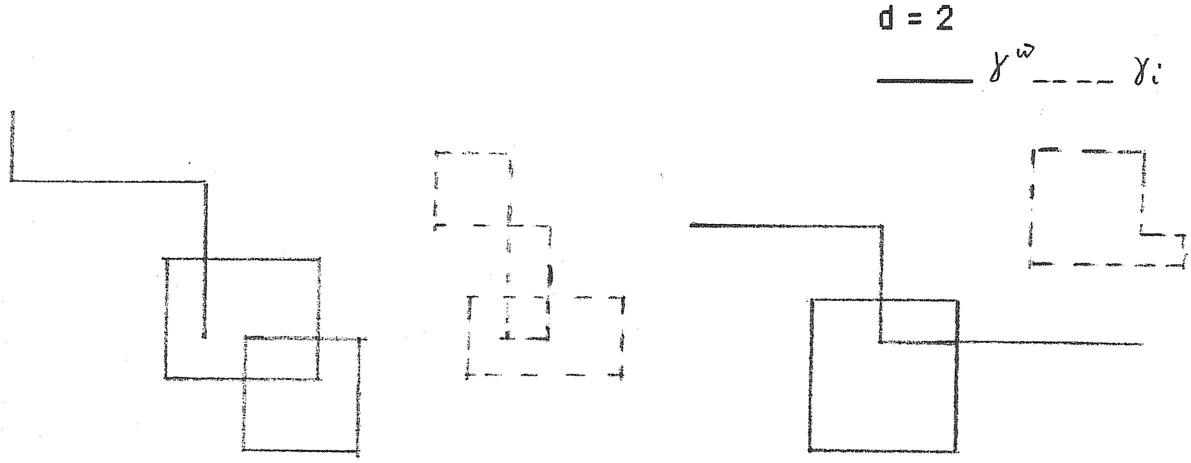


fig. 6.1

Hence we can rewrite, for a v satisfying (6.2),

$$Z_\Lambda(v) \equiv Z_\Lambda(\gamma_1 \ldots \gamma_n, \gamma^\omega)$$

and

$$\langle D(\omega) \rangle \equiv \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{\sum_{\{\gamma_1 \ldots \gamma_n, \gamma^\omega\}} Z_\Lambda(\gamma_1 \ldots \gamma_n, \gamma^\omega)}{\sum_{\{\gamma_1 \ldots \gamma_n\}} Z_\Lambda(\gamma_1 \ldots \gamma_n)} \tag{6.3}$$

The closed curves γ are nothing but the defects, called Peierls contours, of the $(\mathrm{Ising})_2$model underlying ϕ^4_2, corresponding to the discrete spin variable σ.
The open ended curve γ^{ω} is a distinguished defect associated to the introduction of the disorder field $D(\omega)$ [See e.g. 10].

Remark 6.1 From the above considerations, the following intuitive picture emerges.
To introduce a disorder field corresponds to "open" some Peierls contour. In a "particle" picture one can interpret the Peierls contours as the worldlines, in the euclidean section, of pairs of virtual solitons; therefore, to open a contour corresponds to create a soliton at one end and destroy it at the other end of the "open" contour.

Remark 6.2 In general, equation (6.1) is valid in the class of lattice field theories considered here, v being a $\mathbb{Z}$ valued field of rank $k+1$. In this case the support of v in $Z_\Lambda(v)$ in the numerator (resp denominator) is the dual of a set of $d-k-1$ dimensional surfaces with boundary supp $(d\omega)^*$(resp vanishing boundary). In general v is also characterized by some charges $q_i \in \mathbb{Z}\setminus\{0\}$ relative to these boundary surfaces.
The closed surfaces are the analogue of the Peierls contours. Also from this point of view, one can see that the soliton in this class of models can exist only in $d = k+2$, since this is the dimension in which the Peierls contours are unidimensional. Moreover, it is intuitive that soliton sectors can be constructed only in a phase where the contours do not condense.

These general preliminaries suggest that a convergent cluster expansion for $\langle D(\omega)\rangle$ in the region where a soliton sector can be constructed, should contain a contour expansion relative to v and could be seen as a generalization of a lattice version of the Glimm Jaffe Spencer [22] expansion for ϕ^4_2 (see also [23]).
The expansion we discuss here is, in fact, similar in spirit to the G.J.S. expansion, but, after the Peierls contour expansion, instead of using the

expansion around a gaussian, we use the simpler high temperature expansion.
This is possible because we are working with lattice field theory. Clearly this is not a good starting point for a continuum analysis. However, for lattice models it has the advantage of nicer decoupling properties between the clusters.

The main assumptions to set up the expansion are the following:
1) The presence of a contour gives rise to a strongly decreasing exponential factor, proportional to the length of the contour (and the square of the associated charge).
2) The measure $d\nu(\phi) \equiv d\nu(\rho)$ (since $d\nu(\phi)$ is $\mathbb{Z}$ gauge invariant it does not depend on ϑ) is strongly peaked around a value ρ_0 of ρ, and fluctuations around ρ_0 can be well approximated by gaussian fluctuations.

Point 2) implies that $S_{C_{k+1}}(\rho,\vartheta) \equiv S_{\Lambda=C_{k+1}}(\rho,\vartheta)$ should not, with high probability, differ from $S_{C_{k+1}}(\rho_0,\vartheta)$, and therefore

$$e^{-[S_{C_{k+1}}(\rho,\vartheta) - S_{C_{k+1}}(\rho_0,\vartheta)]}$$

should not be very different from 1 in large regions, so that we can apply to it the standard method of the high temperature expansion.
Moreover, in the convergence proof, we exploit the locality property of S_Λ and $d\nu_\Lambda(\rho)$ shared by all models discussed here, (i.e.

$$S_\Lambda = \sum_{C_{k+1} \in \Lambda} S_{C_{k+1}}$$

where $S_{C_{k+1}}$ just depends on the fields ρ and ϑ with support in C_{k+1} and

$$d\nu_\Lambda(\rho) = \prod_{C_k \in \Lambda} d\nu(\rho(C_k))$$

)

and the property

$$S_{c_{k+1}}(\rho_0, 0) = 0 \tag{6.4}$$

Let us start by considering the simpler case of the partition function. We can write (see also (6.1))

$$Z_\Lambda = \sum_{v : dv = 0} \prod_{c_{k+1} \in \Lambda} e^{-S_{c_{k+1}}(\rho_0, v)} \int d\nu_\Lambda(\rho)$$

$$\prod_{c_{k+1} \in \Lambda} \left\{ \left[e^{-[S_{c_{k+1}}(\rho, v) - S_{c_{k+1}}(\rho_0, v)]} - 1 \right] + 1 \right\} \tag{6.5}$$

Introducing the notations

$$e^{-[S_{c_{k+1}}(\rho, v) - S_{c_{k+1}}(\rho_0, v)]} - 1 \equiv P(c_{k+1}; \rho, v)$$

$$d\tilde{\nu}(\rho(c_k)) = \frac{d\nu(\rho(c_k))}{\int d\nu(\rho(c_k))} \tag{6.6}$$

$$\tilde{Z}(\Lambda) = \prod_{c_k \in \Lambda} \int d\nu(\rho(c_k)) ,$$

developing the sum in the integral and dividing by $\tilde{Z}(\Lambda)$, (6.5) gives

$$Z_\Lambda = \tilde{Z}(\Lambda) \left[\sum_{v : dv = 0} e^{-S_\Lambda(\rho_0, v)} \cdot \right.$$

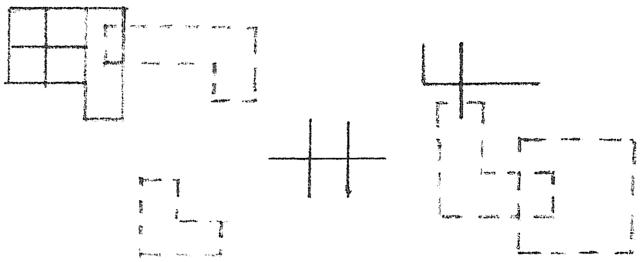
$$\left. \cdot \sum_{X \subset \Lambda} \int \prod_{c_k \in X} d\tilde{\nu}(\rho(c_k)) \prod_{c_{k+1} \in \Lambda} P(c_{k+1}; \rho, v) \right] \tag{6.7}$$

where X is a set of maximally connected clusters X_i of $k+1$ dimensional cells, $X = \{X_1 \dots X_n\}$.
Let $v = \{v_1 \dots v_m\}$, where the support of $(v_i)^*$ is maximally connected, then property (6.4) gives the factorization

$$\exp - S_\Lambda(\rho_0, v) = \prod_{i=1}^{m} \exp\Big\{-\sum_{c_{k+1} \in \operatorname{supp} v_i} S_{c_{k+1}}(\rho_0, v)\Big\}$$

______ X

------- supp v*



d = 2
k = 0

fig.6.2

Now call

$$Z(v_i) = \exp\Big\{-\sum_{c_{k+1} \in \operatorname{supp} v_i} S_{c_{k+1}}(\rho_0, v)\Big\}$$

$$Z(X_j) = \int \prod_{c_k \in X_j} d\tilde{\nu}(\rho(c_k)) \prod_{c_{k+1} \in X_j} P(c_{k+1}; \rho, v)$$

These can be considered as activities of two different types of polymers v-, X- and we can write

$$Z_\Lambda = \tilde{Z}(\Lambda) \sum_{X_1 \dots X_n} \sum_{v_1 \dots v_m} \prod_{j=1}^{n} z(X_j) \prod_{i=1}^{m} z(v_i) \qquad (6.8)$$

We now group together all the v-polymers and X-polymers into single "connected" clusters, as is done in [23].
A configuration $\{v_1 \dots v_m, X_1 \dots X_n\}$ is said to form a cluster on

$$C = \operatorname{supp}\Big(\bigcup_{j=1}^{n} X_j \; \bigcup_{i=1}^{m} v_i\Big)$$

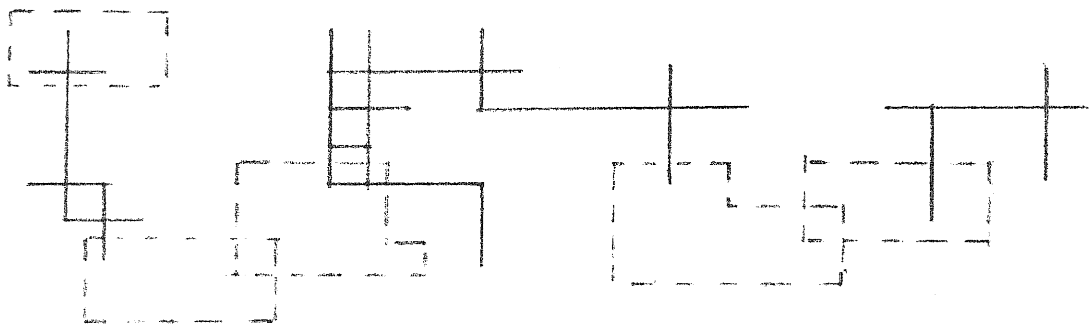
if

$$\mathrm{supp}\,(\bigcup_j X_j \bigcup_i (v_i)^*)$$

is connected.

One defines the activity of the cluster C by

$$Z(C) = \sum_{\{v_i X_j\} \text{ clusters on } C} \prod_i z(v_i) \prod_j z(X_j) \tag{6.9}$$



A configuration forming
a cluster on $C = X \cup \mathrm{supp}\, v$

d=2
k=0

——— X
- - - - $\mathrm{supp}(v)^*$

fig. 6.3

Now we can write

$$\frac{Z_\Lambda}{\tilde{Z}(\Lambda)} = \sum_{\substack{C_1 \ldots C_r \\ \text{disjoint}}} \prod_{\ell=1}^{r} z(C_\ell) \tag{6.10}$$

(clusters C_ℓ are then "maximally connected").

Eq. (6.10) is in the standard form of a cluster expansion, and we can formally take its logarithm by the well known formula

$$\log \frac{Z_\Lambda}{\tilde{Z}(\Lambda)} = \sum_{\underset{\sim}{C}} \prod_{C \in \underset{\sim}{C}} z(C) \frac{\varphi_T(\underset{\sim}{C})}{|\underset{\sim}{C}|!} \tag{6.11}$$

where

1) $\underset{\sim}{C}$ is a collection containing clusters repeated an arbitrary number of times;

2) $|\underset{\sim}{C}|! = \prod_{\mathrm{supp}\, C \,:\, C \in \underset{\sim}{C}} n(C)$

$n(C) = \{ \# \; C \in \underset{\sim}{C}$ with the same support, supp C$\}$

3) Let $\sum_G$ denote the sum over all connected graphs G, having as vertices the elements of $\underset{\sim}{C}$; let $<C_1, C_2>$ denote a link in the graph and

$$V(C_1, C_2) = \begin{cases} \infty & C_1 \cap C_2 \neq \phi \\ 0 & C_1 \cap C_2 = \phi \end{cases}$$

Then $$\varphi_T(\underset{\sim}{C}) \equiv \sum_G \prod_{<C_1,C_2> \in G} (e^{-V(C_1,C_2)} - 1)$$

(Notice that the combinatorial factor a_T , used in sect. 4, is given by

$$a_T(\underset{\sim}{\gamma}) = \frac{\varphi_T(\underset{\sim}{\gamma})}{|\underset{\sim}{\gamma}|!}$$)

Before giving a sketch of the arguments for the convergence proof of expansion (6.11), let us first discuss what happens to the correlation function $\langle D(\omega) \rangle_\Lambda$ when the thermodynamic limit gives the two point soliton function, i.e.

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \langle D(\omega) \rangle_\Lambda = S_{20}(x\,q, y-q).$$

Afterwards, mixed correlation functions can be worked out easily.
As previously discussed, the introduction of disorder field $D(\omega)$ introduces, in the sum over the field configurations v in the numerator of (6.1) and (6.3), a component v^{ω} such that $(v^{\omega})^{*}$ has support on a curve γ^{ω} with boundary $\{x, y\}$.
Therefore, from (6.1) and (6.8) we get

$$Z_{\Lambda}(\omega) = \tilde{Z}(\Lambda) \sum_{\substack{v_1 \dots v_m, v^{\omega} \\ dv_i = 0 \quad dv^{\omega} = d\omega}} \sum_{X_1 \dots X_n} \prod_{i=1}^{m} z(v_i) \cdot$$

$$\cdot \prod_{j=1}^{n} z(X_j) \, z(v^{\omega})$$

Again, between the connected component of $\bigcup_i (v_i)^* \bigcup_j X_j$ from which we form the clusters, there will be one which contains v^{ω}. The corresponding cluster , which necessarily contains also x, y , is denoted by C^{ω} .
Now, a configuration $\{ v_1 \dots v_m, v^{\omega}, X_1 \dots X_n \}$
(all v's are disjoint from each other; all X 's are disjoint from each other) is said to form a cluster on $C^{\omega} = \operatorname{supp} (\bigcup_i v_i \bigcup_j X_j \cup v^{\omega})$
if $\operatorname{supp} (\bigcup_i v_i^* \bigcup_j X_j \cup v^{\omega *})$ is connected.

Then we can write

$$\frac{Z_{\Lambda}(\omega)}{Z_{\Lambda}} = \sum_{\substack{\{C_1 \dots C_r, C^{\omega}\} \\ \text{disjoint}}} \prod_{l=1}^{r} z(C_l) \, z(C^{\omega})$$

We now use standard techniques of the cluster expansion (see e.g. [10]) to write

$$\langle D(\omega)\rangle_\Lambda = \frac{Z_\Lambda(\omega)/\tilde{Z}(\Lambda)}{Z_\Lambda/\tilde{Z}(\Lambda)} =$$

$$= \frac{d}{d\alpha}\log\left(Z_\Lambda + \alpha Z_\Lambda(\omega)\right)\Big|_{\alpha=0} = \tag{6.12}$$

$$= \sum_{\underset{\sim}{C}\, C^\omega} \frac{\varphi_{\Pi'}(\underset{\sim}{C}, C^\omega)}{|\underset{\sim}{C}|!} \prod_{C\in\underset{\sim}{C}} z(C)\, z(C^\omega)$$

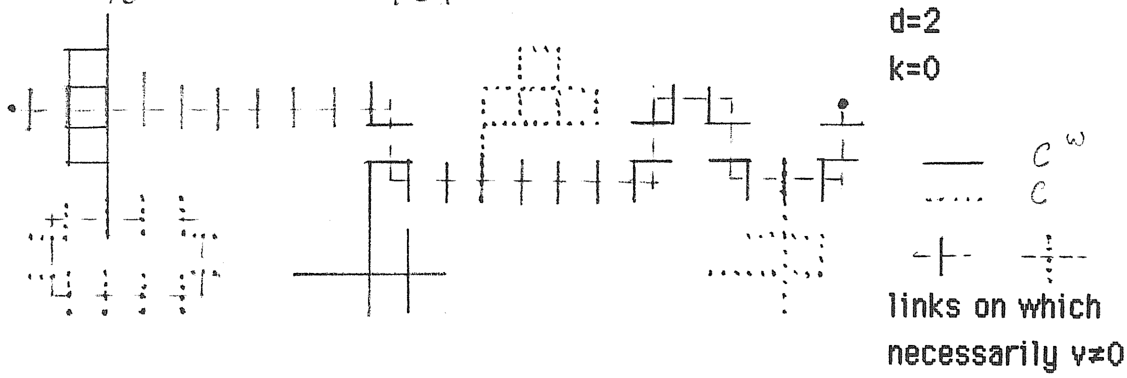


fig. 6.4

Let us come to the problem of convergence of (6.11), (6.12).
The convergence proof is based on the following steps:
1) obtain upper bounds on $z(v)$, $z(X)$ (and $z(v^\omega)$).
2) Using 1) and (6.9) obtain upper bounds on $z(C)$ (and $z(C^\omega)$).
3) Using 2) and (6.11) (or (6.12)) obtain the convergence of the expansion.
Let us first fix some notation.
We denote all constants in the bounds by $0 < c_i < \infty$; if one depends on the parameter of the model it will be denoted by $c_i(e)$, where e is intended to be small in the convergence region of the expansion.
The bound on $z(v)$ is typically trivial

$$|z(v)| \leq \prod_{c_{k+1}\in \operatorname{supp} v} e^{-c_0(e)\, q^2}, \quad c_0(e) \underset{e\searrow 0}{\nearrow} \infty \tag{6.13}$$

The bound on $Z(X)$ is typically obtained by the Holder inequality (see e.g. [10]):

$$|Z(X)| \leq \prod_{C_{k+1} \in X} \left[\int \prod_{C_k \in \partial C_{k+1}} d\tilde{\nu}(\rho(C_k)) |P(C_{k+1}; \rho, v)|^{r} \right]^{\frac{1}{r}}$$

where $r = \{$ number of C_{k+1} cells sharing a C_k cell$\}$.
We now approximate $d\tilde{\nu}(\rho(C_k))$ with a normalized gaussian measure $d\mu_G(\rho(C_k))$ (whose normalization factor we denote by Z_G) satisfying

$$\int \prod_{C_k \in \partial C_{k+1}} d\tilde{\nu}(\rho(C_k)) |P(C_{k+1})|^{r} \leq \left[\frac{Z_G}{\int d\nu(\rho(C_k))} \right]^{r} \int \prod_{C_k \in C_{k+1}} d\mu_G(\rho(C_k)) |P(C_{k+1})|^{r}$$

evaluate the last term by gaussian integration, and bound the first term, typically, by using the Jensen inequality.
One typically gets

$$|Z(X)| \leq c_1^{|X \cap \mathrm{supp}\, v|} e^{-c_2(e)|X \setminus (X \cap \mathrm{supp}\, v)|} \tag{6.14}$$

$$c_2(e) \underset{e \downarrow 0}{\nearrow} \infty \qquad c_0(e) >> c_2(e)$$

The bound on the cluster activity is obtained using the following

<u>Lemma 6.1</u> [23] Let Y and W denote sets of cells in Λ, f a function on the subsets of cells of Λ, and define

$$\|f\|_\kappa = \sup_{x \in \Lambda} \sum_{Y:\, Y \text{ touches } x} |f(Y)|\, e^{\kappa |Y|} \quad , \kappa \in \mathbb{R}_+$$

then

$$\sum_{\underset{\sim}{Y} :\ \text{each } Y \text{ touches } W} \frac{1}{|\underset{\sim}{Y}|!} \prod_{Y \in \underset{\sim}{Y}} |f(Y)| \leq e^{\|f\|_0 |W|}$$

To apply this lemma, if $\mathcal{Z} \neq \mathbb{Z}_2$, we first sum over all charges in $\mathcal{Z}\setminus\{0\}$, on every cell in the support of v, using

$$\sum_{q\in\mathcal{Z}\setminus\{0\}} e^{-c_0(e)\,q^2} \leq c_3\, e^{-c_0(e)}$$

(We omit in the following c_3, it will be adsorbed in $c_0(e)$).
Then, denoting by γ_i the support of v_i, we have

$$|Z(C)| \leq \sum_{\substack{\{\gamma_i X_j\}: \\ \{v_i X_j\} \text{ clusters on } C}} \prod_i e^{-[c_0(e)-c_2(e)]|\gamma_i|} \prod_j e^{-c_2(e)|X_j|} \tag{6.15i}$$

where we have used $c_0(e) >> c_2(e)$ to bound $c_1^{|X\cap \mathrm{supp}\, v|}$.
We now observe that

$$\bigcup_i |\gamma_i| \bigcup_j |X_j| \geq |C|$$

(since X's and γ's can overlap) and we extract an exponential factor decreasing as $|C|$. We then substitute $\{\gamma_i X_j\}:\{v_i X_j\}$ clusters on C with the collections $\{\underset{\sim}{\gamma}, \underset{\sim}{X}\}$ of γ's and X's just touching C (dividing by the number of identical permutations).
So, for $c_4 < c_2(e)$ large enough, we get

$$|Z(C)| \leq e^{-(c_2(e)-c_4)|C|} \sum_{\substack{\underset{\sim}{\gamma}\,\underset{\sim}{X} \\ \text{each } \gamma, X \text{ touches } C}} \frac{1}{|\underset{\sim}{\gamma}|!}\frac{1}{|\underset{\sim}{X}|!} \prod_{\gamma\in\underset{\sim}{\gamma}} e^{-c_4|\gamma|} \prod_{X\in\underset{\sim}{X}} e^{-c_4|X|} \tag{6.15ii}$$

and, applying lemma 6.1:

$$|Z(C)| \leq e^{-(c_2(e)-c_4)|C|}\, e^{c_5 \|e^{-c_4(\cdot)}\|_0 |C|} = e^{-(c_2(e)-c_6)|C|} \tag{6.16}$$

where

$$c_6 = c_5 \|e^{-c_4(\cdot)}\|_0 + c_4$$

If we have v^{ω} in the cluster, we cannot sum over all the charges on the support of v^{ω}, and in the collection $\underset{\sim}{\gamma}$ used in (6.15ii) we must start with a minimal γ, having the length $|\gamma_{xy}|$ of the shortest line connecting x and y.
Therefore, for the two point function of the charge q soliton we have:

$$|Z(C^{\omega})| \leq \sum_{\substack{\{\gamma_i X_j, \gamma^{\omega}\}: \\ \{v_i, X_j, v^{\omega}\} \text{ clusters on } C^{\omega}}} \prod_i e^{-(C_0(e)-C_2(e))|\gamma_i|} \cdot$$
$$\cdot\, e^{-[C_0(e)q^2 - C_2(e)]|\gamma^{\omega}|} \prod_j e^{-C_2(e)|X_j|} \leq \tag{6.17}$$
$$\leq e^{-[C_0(e)q^2 - C_2(e) - 2C_4]|\gamma_{xy}|}\, e^{-(C_2(e)-C_6')|C^{\omega}|}$$

Finally, to prove convergence of (6.11), we use the following

Theorem 6.4 [23] The cluster expansion (6.11) converges if $\exists\, \varepsilon, k > 0$, such that, for the activity of the cluster $z(C)$, the following bound holds:

$$\| z(\cdot) \|_{k+1+\varepsilon} \left(1 + \frac{1}{\varepsilon}\right) < 1$$

Using bound (6.16), is easy to see that

$$\| z(\cdot) \|_k \leq \| e^{-[C_2(e)-C_6]|\cdot|} \|_k \underset{C_2(e)\nearrow\infty}{\searrow} 0 \tag{6.18}$$

Now, for $z(C^{\omega})$, define

$$\| z^{\omega}(\cdot) \|_k = \sum_{C^{\omega}} |z(C^{\omega})| e^{k|C^{\omega}|}, \quad k \in \mathbb{R}_+$$

Notice that all C^ω must contain x, y , so

$$\| Z^\omega(\cdot)\|_K \leq \| e^{-[c_2(e) - c'_6]|\cdot|} \|_K \cdot$$

$$\cdot\, e^{-[c_0(e) q^2 - c_2(e) - 2c_4]|\gamma_{xy}|} \tag{6.19}$$

Theorem 6.5 Under the same hypothesis of theorem 6.4 and assuming $\| Z^\omega(\cdot) \|_{1+\varepsilon+K} < \infty$ the cluster expansion (6.12) converges and, moreover,

$$\left| \frac{Z_\Lambda(\omega)}{Z_\Lambda} \right| \leq \frac{1}{1 - \left(1 + \frac{1}{\varepsilon}\right) \| Z(\cdot) \|_{K+1+\varepsilon}} \frac{1}{\varepsilon} \| Z^\omega(\cdot) \|_{1+K+\varepsilon} \tag{6.20}$$

The proof of theorem (6.5) is a straightforward modification of the proof of theorem (6.4) and we will omit it.
Notice that, as a result of the above bounds (and analogous bounds for ordinary fields), if the expression sketched above converges for all mixed finite lattice correlation functions:

1) this proves the existence of the thermodynamic limit for the correlation functions by standard cluster techniques;

2) if $\sum_i q_i = q$, from (6.19) and (6.20)

$$| S_{n+1, m} (x_1 q_1 \ldots x_n q_n, z - q; \ldots)| \underset{|z| \to \infty}{\sim} e^{-C(e) |z| q^2}$$

where $C(e) = O(c_0(e))$.
Therefore, the conditions for the existence of the soliton sectors are satisfied.

7. LOCAL SOLITON: particle analysis

In this section, to study the particle structure of the soliton sectors in class A (local soliton), we will use the excitation expansion of sect. 3.
It turns out that the soliton state, corresponding to the $S_q(\vec{x})$ intertwiner operator introduced in (5.8), appears in these sectors as the lowest lying (massive) one-particle state with non vanishing upper gap.
Before giving a sketch of this adapted excitation expansion, and to apply it to specific models, let us first state a general theorem which shows that to know the mass gap we just need to analyze the two-point soliton function.

Theorem 7.1 If the measure $d\mu(\phi)$, defined as the (thermodynamic weak limit) of (5.1), is invariant under reflection w.r.t. the t=0 plane, then the mass gap m_q, in the q- soliton sector $\mathcal{H}_q$, $q \in \mathbb{Z}\setminus\{0\}$ of theorem 5.6, coincides with the soliton mass defined by

$$m_{S_q} = \lim_{t\to\infty} -\frac{1}{t} \ln \langle S_q(\vec{0})\Omega, T(t)\, S_q(\vec{0})\Omega\rangle =$$

$$= \lim_{t\to\infty} -\frac{1}{t} \ln S_{2,0}(0\, -q, (t,\vec{0})\, q) \tag{7.1}$$

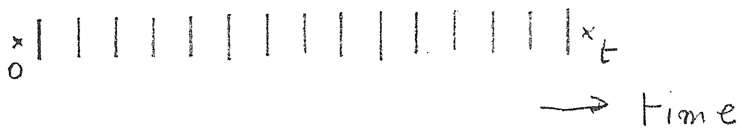
Proof: It is essentially similar to that of Theorem 5.6 in [24] and is given in Appendix 7A.

We then consider the two point function $S_2(0\, -q, (t,\vec{0})\, q)$.
In the previous section we have seen that it can be expressed in our models in the form

$$S_{2,0} = \lim_{\Lambda\uparrow\mathbb{Z}^d} \sum_{\underset{\sim}{C}\, C^\omega \subset\Lambda} \prod_{C\in\underset{\sim}{C}} z(C)\, z(C^\omega)\, \frac{\varphi_T(\underset{\sim}{C}, C^\omega)}{|\underset{\sim}{C}|!} \tag{7.2}$$

which appears as a sum on a "decorated" path from 0 to $(t,\vec{0})$ since $\varphi_{\eta} = 0$ unless all C touch each other.
Let us see which configuration $\underset{\sim}{C}$, C^{ω} produces the leading term in the sum.
Clearly, one must have $\underset{\sim}{C} = \phi$ and the support of C^{ω} must be the smallest possible: the dual of a straight line joining 0 and $t \equiv (t,\vec{0})$.
Let us call C_0^{ω} the corresponding cluster, it is easy to see that it is composed of t disjoint k+1 dimensional cells that are orthogonal to the time axis.

d = 2
k = 0



and the corresponding term is simply

$$\left[Z\, (C^{\omega} = c_{k+1}) \right]^{t}$$

Clearly, this is also the leading term in a similar expansion for

$$\sum_{\vec{y}} S_2 \,(0\, -q\, , (t,\vec{y})\, q) \tag{7.3}$$

where $(d\omega)^{*}$ has support on 0 and on a point in the hyperplane $x^0 = t$. (See also [17]).
Now, we identify regular parts and excitations in a configuration $\underset{\sim}{C}\ C^{\omega}$, as needed for the particle analysis.

<u>Definition 7.2</u> A k+1 cell $c_{k+1} \in \{\underset{\sim}{C}, C^{\omega}\}$ is called regular if:
1) it is orthogonal to the time axis,
2) it occurs only in C^{ω} ,
3) there are no other cells having the same projection on the time axis,

4) it is not connected to other cells.

As in section 3, the maximally connected set of regular cells are called straight lines.

The complements are sets of cells whose projections on the time axis are disjoint.

A maximally connected subset modulo vertical translation is called an excitation ε .

Define

$$\underset{\sim}{C}(\varepsilon) = \underset{\sim}{C} \cap \varepsilon \qquad C^{\omega}(\varepsilon) = C^{\omega} \cap \varepsilon$$

then, if a path $\{\underset{\sim}{C}, C^{\omega}\}$ (with $\varphi_{\pi}(\underset{\sim}{C}, C^{\omega}) \neq 0$) contains the excitations $\{\varepsilon_1 \ldots \varepsilon_{n}\}$ (compatible, allowed in $[0, t]$ along definition in sect. 3) by inspection we have the factorization property (see definition 3.) after eq. (6.11))

$$\varphi_{\pi}(\underset{\sim}{C}, C^{\omega}) = \prod_{i=1}^{n} \varphi_{\pi}(\underset{\sim}{C}(\varepsilon), C^{\omega}(\varepsilon))$$

Then, we see that every straight line of length τ gives, in the sum (7.1) (or the corresponding one in (7.2)), a contribution $[Z(C^{\omega} = c_{k+1})]^{\tau}$, and an excitation gives a contribution

$$\prod_{C(\varepsilon) \in \underset{\sim}{C}(\varepsilon)} Z(C(\varepsilon))\, Z(C^{\omega}(\varepsilon))\, \frac{\varphi_{\pi}(\underset{\sim}{C}(\varepsilon), C^{\omega}(\varepsilon))}{|\underset{\sim}{C}(\varepsilon)|!}$$

Therefore, if we define

$$\zeta(\varepsilon) = \prod_{C(\varepsilon) \in \underset{\sim}{C}(\varepsilon)} Z(C(\varepsilon))\, Z(C^{\omega}(\varepsilon))\, \frac{\varphi_{\pi}(\underset{\sim}{C}(\varepsilon), C^{\omega}(\varepsilon))}{|\underset{\sim}{C}(\varepsilon)|!} \left[Z(C^{\omega} = c_{k+1})\right]^{-|\pi(\varepsilon)|}$$

as activity for the excitation (see section 3), one can write

$$\sum_{\vec{y}} S_{2,0}(0 - q, (t\vec{y})\, q) = [Z(C^{\omega} = c_{k+1})]^{t} \cdot \sum_{\varepsilon_1 \ldots \varepsilon_n} \prod_{i=1}^{n} \zeta(\varepsilon_i)$$

Moreover, using the bounds of the previous section, the bound

$$|\zeta(\varepsilon)| \leq \left| \frac{\varphi_T(\underset{\sim}{C}(\varepsilon), C^w(\varepsilon))}{|\underset{\sim}{C}(\varepsilon)|!} \right| \prod_{C(\varepsilon)\in \underset{\sim}{C}(\varepsilon)} e^{-[c_2(e)-c_6]|C(\varepsilon)|} \cdot$$

$$\cdot\, e^{-[c_0(e)q^2-c_2(e)-2c_4]|\pi(\varepsilon)|} \, e^{-[c_2(e)-c_6']|C^w(\varepsilon)|} \, |Z(C^w=C_{k+1})|^{-|\pi(\varepsilon)|}$$

is obtained.
Now, using

$$\left| \frac{\varphi_T(\underset{\sim}{C}(\varepsilon), C^w(\varepsilon))}{|\underset{\sim}{C}(\varepsilon)|!} \right| \leq e^{c|\varepsilon|}$$

where c is a purely geometrical factor (see e.g. [17]), and if we are able to prove

$$c_Z\, e^{-c_0(e)q^2} \leq Z(C^w = C_{k+1}) \leq c_Z'\, e^{-c_0(e)q^2} \tag{7.4}$$

one gets

$$|\zeta(\varepsilon)| \leq \text{const}\; e^{-(c_2(e)-c_6'-c)|\varepsilon|} \tag{7.5}$$

Therefore, if we formally exponentiate:

$$\Big(\sum_{\varepsilon_1\ldots\varepsilon_n} \prod_{i=1}^{n} \zeta(\varepsilon_i)\Big) = \exp\Big\{ \sum_{\underset{\sim}{\varepsilon}} \prod_{\varepsilon\in\underset{\sim}{\varepsilon}} \zeta(\varepsilon)\, \frac{\varphi_T(\underset{\sim}{\varepsilon})}{|\underset{\sim}{\varepsilon}|!} \Big\} \tag{7.6}$$

the expansion in the exponential converges for $c_2(e)$ large enough.

Denote by $t^-(\underset{\sim}{\xi})$ $(t^+(\underset{\sim}{\xi}))$, respectively the lowest (highest) time coordinate in $\underset{\sim}{\xi}$. We split the sum over the clusters $\underset{\sim}{\xi}$ of the excitations allowed in $[0,t]$, in the right hand side of (7.6), into three terms:

1) $\underset{\sim}{\xi}$: $t^-(\underset{\sim}{\xi}) \in [0,t)$ $\qquad t^+(\underset{\sim}{\xi}) \in (0,t]$

2) $\underset{\sim}{\xi}$: $t^-(\underset{\sim}{\xi}) < 0$ $\qquad t^+(\underset{\sim}{\xi}) > t$

3) $\underset{\sim}{\xi}$: $t^-(\underset{\sim}{\xi}) > 0$ $\qquad t^+(\underset{\sim}{\xi}) > t$ $\qquad$ or
$\qquad t^-(\underset{\sim}{\xi}) < 0$ $\qquad t^+(\underset{\sim}{\xi}) < t$

We now add to the first term the clusters of excitations $\underset{\sim}{\xi}$ allowed in $(-\infty, +\infty)$ and, by adding them to the first term, we get
1') $\underset{\sim}{\xi}$ allowed in $(-\infty, +\infty)$: $t^-(\underset{\sim}{\xi}) \in [0,t)$.
2') is defined subtracting from 2) all the excitations present in 1'), such that $\Pi(\xi) \supset [0,t]$.
3') is defined subtracting from 3) all the excitations in 1') that are not in 1) and $\Pi(\xi) \not\supset [0,t]$

Therefore, using the translation invariance of the sum over excitations in 1'):

$$\sum_{\vec{y}} S_{2,0}(0-q\,;(t,\vec{y})\,q) = [Z(C^\omega = c_{K+1})]^t \cdot$$

$$\cdot \exp\left[\left(\sum_{\underset{\sim}{\xi}\in 1)'} + \sum_{\underset{\sim}{\xi}\in 2)'} + \sum_{\underset{\sim}{\xi}\in 3)'}\right)\left(\prod_{\xi\in\underset{\sim}{\xi}} \zeta(\xi)\,\frac{\varphi_T(\underset{\sim}{\xi})}{|\underset{\sim}{\xi}|!}\right)\right. = \tag{7.7}$$

$$= \exp\left[t \ln Z(C^\omega = c_{K+1}) + \left(t \sum_{\underset{\sim}{\xi}\in 1)' :\, t^-(\underset{\sim}{\xi})=0} + \right.\right.$$

$$\left.+ \sum_{\underset{\sim}{\xi}\in 2)'} + \sum_{\underset{\sim}{\xi}\in 3)'}\right)\left(\prod_{\xi\in\underset{\sim}{\xi}} \zeta(\xi)\,\frac{\varphi_T(\underset{\sim}{\xi})}{|\underset{\sim}{\xi}|!}\right)\Bigg]$$

From eq. (7.5) we obtain that, for large t, the sum $\sum_{\xi \in 2)'} (\cdot)$ behaves like a constant, and $\sum_{\xi \in 3)'} (\cdot)$ decrases exponentially in t, since every excitation in 3') has $\pi(\xi) \supset [0, t]$.
Hence $\exp \sum_{\xi \in 3)'} (\cdot) \underset{t\to\infty}{\sim} (1+e^{-\mu t})$ for some $\mu > 0$.
By the spectral theorem

$$< s_q(\vec{\sigma})\,\Omega \,,\, T(t)\,U(\vec{a})\,s_q(\vec{\sigma})\,\Omega > =$$

$$= \int d\rho_{s_q}(\lambda, \vec{k})\, e^{-\lambda t}\, e^{i \vec{k}\cdot\vec{a}}$$

where $d\rho_{s_q}$ is a positive measure and

$$\mathrm{spec}\ e^{-M} \supset \mathrm{supp}\ d\rho_{s_q}(\lambda, \vec{\sigma}) .$$

Using the above expansion we have shown that

$$\sum_{\vec{y}} < s_q(\vec{\sigma})\,\Omega \,,\, T(t)\,U(\vec{a})\,s_q(\vec{\sigma})\,\Omega > =$$

$$= \int d\rho_{s_q}(\lambda, \vec{\sigma})\, e^{\lambda t} \underset{t\to\infty}{\sim} e^{c}\, e^{-mt}\,(1+e^{-\mu t})$$

where m is the mass of the soliton (and, by theorem 7.1, the mass gap in the q - soliton sectors). From (7.7) we can identify

$$m_{s_q} = m = -\Big[\ln Z(c^{w} = c_{k+1}) + \sum_{\substack{\underline{\xi} \in 1)': \\ \varepsilon^{-}(\underline{\xi}) = 0}} \prod_{\varepsilon \in \underline{\xi}} \zeta(\varepsilon)\, \frac{\varphi_{T}(\underline{\xi})}{|\underline{\xi}|!}\Big] \tag{7.8}$$

To have more detailed information we now need an explicit analysis for the models which will be also helpful in clarifying the previous, somewhat abstract, setting.

<u>Example 7.3 ϕ^4_2 with +bd conditions.</u>
Using eq. (6.2), Z_Λ can be represented as

$$Z_\Lambda = \sum_{v:\, dv=0} \int \prod_{x\in\Lambda} d\rho_x \; e^{-\lambda(\rho_x^2-\rho_0^2)^2} \prod_{x\in\partial\Lambda} \delta(\rho_x-\rho_0) \; \cdot \tag{7.9}$$

$$\cdot \prod_{\langle xy\rangle \in\Lambda} \exp\left[-\tfrac{1}{2}(\rho_x-\rho_y)^2 + \rho_x\rho_y\,(v_{\langle xy\rangle}-1)\right]$$

One easily identifies

$$d\nu(\rho_x) = d\rho_x \, e^{-\lambda(\rho_x^2-\rho_0^2)^2} \begin{cases} \delta(\rho_x-\rho_0) & x\in\partial\Lambda \\ 1 & x\notin\partial\Lambda \end{cases}$$

This measure is clearly strongly peaked around ρ_0 if $\lambda\rho_0^2$ is large (requisite 2).

The function $S_{\langle xy\rangle}(\rho, v)$ is given by

$$\frac{1}{2}(\rho_x-\rho_y)^2 + \rho_x\rho_y\,(1-v_{\langle xy\rangle})$$

so that

$$S_{\langle xy\rangle}(\rho_0, v) = \rho_0^2\,(1-v_{\langle xy\rangle}) \tag{7.10}$$

Therefore Peierls contour (support of v^* $(v^*=-1)$) gives, in (7.9), a strongly decreasing exponential factor if ρ_0^2 is large (requisite 1).

The activity of v- and X-polymers are easily recognized to be

$$z(v) = e^{-2\rho_0^2|\gamma|} \qquad \gamma = \operatorname{supp} v^*$$

$$z(X) = \int \prod_{x\in X} d\tilde\nu(\rho_x) \prod_{\langle xy\rangle\in X} \left[e^{-\frac{1}{2}(\rho_x-\rho_y)^2+\rho_x\rho_y(v_{\langle xy\rangle}-1)} - 1\right] \equiv$$

$$\equiv \int \prod_{x\in X} d\tilde\nu(\rho_x) \prod_{\langle xy\rangle\in X} P(\langle xy\rangle;\rho, v)$$

where X is a connected cluster of links.

Let us now come to the upper bound for $z(X)$.

Define $d\mu_G(\rho_x)$ the normalized gaussian measure on $\mathbb{R}$ with covariance $C_\rho = \frac{1}{2\lambda\rho_0^2}$ and mean ρ_0, and Z_G its normalization factor.
By Jensen inequality, we have

$$\left| \frac{\int d\nu(\rho_x)}{Z_G} \right| = \int_{-\infty}^{+\infty} d\mu_G(\rho_x)\,[1-\chi(\rho_x \le 0)]\,\exp\{-\lambda[(\rho_x^2-\rho_0^2)^2 -$$

$$- 2\rho_0^2(\rho_x-\rho_0)^2]\} \geq e^{-\lambda \langle (\rho_x-\rho_0)^2(\rho_x-\rho_0)^2 - 2\rho_0^2(\rho_x-\rho_0)^2\rangle_G} - O(e^{-\lambda\rho_0^2}) \geq \bar{c}$$

with $\langle\ \ \rangle_G$ expectation value with respect to $d\mu_G$.
For large $\lambda\rho_0^2$ and $\lambda = O(1)$, $\bar{c} = O(1)$.
Let r = {# links sharing a site }, then using

$$e^{-\lambda(\rho_x^2-\rho_0^2)^2} \leq e^{-2\lambda\rho_0^2(\rho_x-\rho_0)^2}$$

and the Holder inequality:

$$|Z(X)| \leq \prod_{\langle xy\rangle\in X} \left[\int d\tilde{\nu}(\rho_x)\, d\tilde{\nu}(\rho_y)\, |P(\langle xy\rangle;\rho,v)|^r\right]^{\frac{1}{r}} \leq$$

$$\leq \frac{1}{\bar{c}} \prod_{\langle xy\rangle\in X} \left[\int d\mu_G(\rho_x)\, d\mu_G(\rho_y)\, |P(\langle xy\rangle;\rho,v)|^r\right]^{\frac{1}{r}}$$

If $v_{\langle xy\rangle} = -1$ we use the gaussian bound

$$\left[\int d\mu_G(\rho_x)\, d\mu_G(\rho_y)\, |P(\langle xy\rangle;\rho,v_{\langle xy\rangle}=-1)|^r\right]^{\frac{1}{r}} \leq O\left(\frac{\rho_0^2}{\lambda\rho_0^2}\right) = O\left(\frac{1}{\lambda}\right)$$

If $v_{\langle xy\rangle} = +1$, by using $|P| \leq 1$ and the Jensen inequality

$$\int d\mu_G(\rho_x)\, d\mu_G(\rho_y)\, |P(\langle xy\rangle;\rho,v_{\langle xy\rangle}=+1)|^r \leq$$

$$\int d\mu_G(\rho_x)\, d\mu_G(\rho_y)\, \left(1 - e^{-\frac{1}{2}(\rho_x - \rho_y)^2}\right) \leq$$

$$\leq 1 - e^{-\frac{1}{2}\langle(\rho_x-\rho_y)^2\rangle_G} = 1 - e^{-O(\frac{1}{\lambda\rho_0^2})} = O\left(\frac{1}{\lambda\rho_0^2}\right)$$

where $\langle\ \ \rangle_G$ is now the expectation value w.r.t. $d\mu_G(\rho_x)\, d\mu_G(\rho_y)$. Therefore, we obtain:

$$|Z(X)| \leq e^{-O(\ln\lambda\rho_0^2)|X \setminus X \cap \operatorname{supp} v|}\, O\left(\frac{1}{\lambda}\right)^{|X \cap \operatorname{supp} v|} \tag{7.11}$$

Making comparison with (6.13), (6.14), we get

$$\begin{aligned} c_0(e) &= 2\rho_0^2 \\ c_2(e) &= O(\ln\lambda\rho_0^2) \\ c_1 &= O\left(\frac{1}{\lambda}\right) \end{aligned} \tag{7.12}$$

It is now evident that, for $c_0(e) \gg c_2(e)$, the convergence condition for theorems 6.4 and 6.5. $\| e^{-(c_2(e)-c_0')|\cdot|} \|_K \searrow 0$

as $\rho_0 \nearrow \infty$, is satisfied. From eq. (7.11) we also get estimate (7.4) in the form

$$Z(C^w = \langle xy\rangle) = e^{-2\rho_0^2} + e^{-2\rho_0^2}\, Z(X = \langle xy\rangle : v_{\langle xy\rangle} = -1)$$

$$|Z(X = \langle xy\rangle : v_{\langle xy\rangle} = -1)| \leq O\left(\frac{1}{\lambda}\right) \tag{7.13}$$

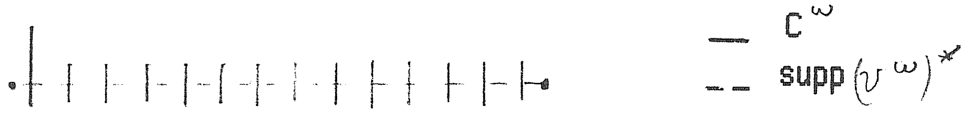
so that for large λ

$$\left(1 - O\left(\frac{1}{\lambda}\right)\right)e^{-2\rho_0^2} \leq Z(C^w = \langle xy\rangle) \leq \left(1 + O\left(\frac{1}{\lambda}\right)\right)e^{-2\rho_0^2} \tag{7.14}$$

(Note that the second term in (7.13) is due to the possibility that the $\langle xy\rangle$ link, support of v, is also one of the clusters X of the high temperature expansion).
From (7.12), (7.14) one immediately obtains the estimate for the excitation activity

$$|\zeta(\varepsilon)| \leqslant e^{-[O(\ln \lambda\rho_0^2) - c_6' - c]|\varepsilon|}$$

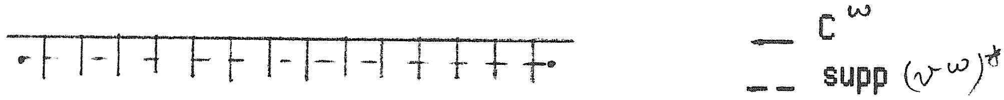
which, for large $\lambda\rho_0^2$, guarantees the convergence of the excitation expansion.
We are now ready for the estimate of the soliton mass and the upper gap in the two point soliton function.
The leading contribution of the excitations in the sum 1') is given by configurations like



which gives a contribution $e^{-O(\ln \lambda\rho_0^2)}$, so that from (7.8)

$$m_s = 2\rho_0^2 + e^{-O(\ln \lambda\rho_0^2)}$$

An estimate on the upper gap μ is obtained by considering the leading term in the sum 3'), corresponding to an excitation configuration like:



and hence gives a contribution of order $e^{-O(\ln \lambda\rho_0^2)t}$.
Therefore, $\mu = O(\ln \lambda\rho_0^2)$.
One can also easily prove the Ornstein Zernike behaviour.
We collect all these results in

Theorem 7.4 The ϕ_2^4 model defined in examples 5.2,A) for large ρ_0 and large $\lambda\rho_0^2$ ($\rho_0^2 >> \ln \lambda\rho_0^2$) (broken symmetry phase) possesses superselection sectors labelled by $\mathbb{Z}_2$; sector $\mathcal{H}_1$ corresponding to the non trivial element 1 of $\mathbb{Z}_2 \simeq \{0,1\}$ is a soliton sector with mass gap:

$$m = 2\rho_0^2 + e^{-O(\ln \lambda \rho_0^2)}$$

and the one-soliton state is the one-particle state with lowest mass in $\mathcal{H}_1$.

Examples 7.5 ($U(1)$ Higgs)$_3$ with gaussian gauge action.

We first rewrite the partition function of the model (see Example 5.2 B i) in term of the v-field. This has been done in [23] but it follows, quite naturally, also from the general procedure outlined in section 6.

The result is

$$Z_\Lambda = \sum_{v : dv = 0} \int \prod_{\langle xy \rangle \in \Lambda} dA_{\langle xy \rangle} \prod_{x \in \Lambda} d\rho_x \, e^{-\lambda(\rho_x^2 - \rho_0^2)^2}$$

$$\cdot \prod_{\langle xy \rangle \in \Lambda} e^{-\frac{1}{2}(\rho_x - \rho_y)^2 + \rho_x \rho_y (1 - \cos e A_{\langle xy \rangle})} \prod_{p \in \Lambda} e^{-\frac{1}{2}(dA_p + v_p)^2} \qquad (7.15)$$

where v_p is a field which takes values in $\frac{2\pi}{e}\mathbb{Z}$, $A_{\langle xy \rangle}$ is a field which takes values in $[-\frac{\pi}{e}, \frac{\pi}{e})$ and $dA_{\langle xy \rangle}$ is the corresponding Haar measure on $U(1) \simeq [-\frac{\pi}{e}, \frac{\pi}{e})$.

The field ρ_x takes values in $\mathbb{R}_+$ and $d\rho_x$ is the Lesbesgue measure on $\mathbb{R}_+$.

We now make a shift $\rho_x \to \rho_x + \rho_0$ so that ρ_x takes value in $[-\rho_0, +\infty)$.

As a formal "Higgs mechanism", this will produce a "mass term" for A in the form:

$$\rho_0^2 (1 - \cos e A) \underset{e \downarrow 0}{\sim} \rho_0^2 e^2 A^2$$

Now, defining

$$d\nu(\rho_x) = d\rho_x \exp\{-\lambda[\rho_x^4 + 4\rho_0\rho_x^3 + 4\rho_0^2\rho_x^2]\} \quad \text{i)}$$

$$d\nu(A_{\langle xy\rangle}) = dA_{\langle xy\rangle} \exp\{-\rho_0^2(1-\cos eA_{\langle xy\rangle})\} \quad \text{ii)} \qquad (7.16)$$

$$S_p(A,v) = \frac{1}{2}(dA_p + v_p)^2 \quad \text{iii)}$$

$$\mathcal{P}(\langle xy\rangle;\rho,A) = \exp\{-\frac{1}{2}(\rho_x-\rho_y)^2 + \rho_0(\rho_x+\rho_y)\cdot \quad \text{iv)}$$

$$\cdot(1-\cos eA_{\langle xy\rangle}) + \rho_x\rho_y(1-\cos eA_{\langle xy\rangle})\} - 1$$

$$\mathcal{P}(p;A,v) = \exp\{-\frac{1}{2}(dA_p+v_p)^2\} - 1 \quad \text{v)}$$

we can rewrite (7.15) as

$$Z_\Lambda = \sum_{v:dv=0} \int \prod_{\langle xy\rangle\in\Lambda} d\nu(A_{\langle xy\rangle}) \prod_{x\in\Lambda} d\nu(\rho_x)\cdot$$

$$\cdot \prod_{p\in\Lambda}[\mathcal{P}(p;A,v)+1] \prod_{\langle xy\rangle\in\Lambda}[\mathcal{P}(\langle xy\rangle;\rho,A)+1]$$

To have a large exponential decreasing factor from the Peierls contours associated to non vanishing Vortices v, one needs, (requisite 1), e to be small so that, from (7.16iii)), $\exp -\frac{1}{2}v_p^2 = \exp -2\pi^2\frac{n^2}{e^2}$ is large (n is the charge of v).

In order to converge the high temperature expansion in A, ρ, one needs, (requisite 2), large $\rho_0 e$ so is strongly peaked around $A=0$, and large $\lambda\rho_0^2$ to ensure $d\mu(\rho)$ strongly peaked around ρ_0.

Now let $v=\{v_1 \dots v_m\}$ where $\mathrm{supp}\, v_i^*$ is maximally connected.

Denote by X a maximally connected union of links and plaquettes. Then,

$$Z_\Lambda = \tilde{Z}(\Lambda) \sum_{\substack{v_1 \ldots v_m \\ \text{disjoint}}} \sum_{\substack{X_1 \ldots X_n \\ \text{disjoint}}} \prod_{i=1}^{m} z(v_i) \prod_{j=1}^{n} z(X_j)$$

$$z(v_i) = \exp\left\{ -\frac{1}{2} \sum_{p \in \operatorname{supp} v_i} v_p^2 \right\}$$

$$z(X_j) = \int \prod_{x \in X_j} d\tilde{\nu}(\rho_x) \prod_{\langle xy \rangle \in X_j} P(\langle xy \rangle; \rho, A) \cdot \tag{7.17}$$

$$\cdot \prod_{p \in X_j} P(p; A\, v)$$

$$\tilde{Z}(\Lambda) = \prod_{x \in \Lambda} \left[\int d\tilde{\nu}(\rho_x) \right] \prod_{\langle xy \rangle \in \Lambda} \left[\int d\nu(A_{\langle xy \rangle}) \right]$$

We now get an upper bound on $z(X)$, separating, at first, the contribution of $P(\langle xy \rangle)$ and $P(p)$ by means of a Schwartz inequality, and then using the Holder inequality:

$$|z(X)| \le \prod_{p \in X} \left[\int \prod_{\langle xy \rangle \in \partial p} d\tilde{\nu}(A_{\langle xy \rangle}) \, |P(p; A, v)|^{r_1} \right]^{\frac{1}{r_1}} \cdot$$

$$\cdot \prod_{\langle xy \rangle \in X} \left[\int d\tilde{\nu}(A_{\langle xy \rangle}) \, d\tilde{\nu}(\rho_x) \, d\tilde{\nu}(\rho_y) \, |P(\langle xy \rangle; A, \rho)|^{r_2} \right]^{\frac{1}{r_2}}$$

where

r_1 = {number of plaquettes sharing a link}

r_2 = {number of links sharing a site}

We now proceed as in the ϕ_2^4 treatment.
Using the two inequalities

$$\rho_0^2 (1 - \cos eA) \geq \frac{1}{2} \gamma_1 \rho_0^2 e^2 \qquad 0 < \gamma_1 < \frac{4}{\pi^2}$$

$$\rho_x^2 + 4 \rho_0 \rho_x^3 + 4 \rho_0^2 \rho_x^2 \geq \rho_0^2 \rho_x^2$$

and approximating $d\tilde{\gamma}(A_{<xy>})$ with a gaussian measure with mean zero and covariance $C_A = 1/\gamma_1 \rho_0^2 e^2$, and $d\tilde{\gamma}(\rho_x)$ with a gaussian measure with mean zero and covariance $C_\rho = 1/2\lambda\rho_{(0)}^2$ from gaussian evaluations we have the following bounds

$$\left[\int \prod_{<xy> \in \partial p} d\tilde{\gamma}(A_{<xy>}) \left| \mathbb{I}(p; A, v_p = 0) \right|^{2r_1} \right]^{\frac{1}{2r_1}} \leq O\left(\frac{1}{\rho_0^2 e^2}\right)$$

$$\left[\int \prod_{<xy> \in \partial p} d\tilde{\gamma}(A_{<xy>}) \left| \mathbb{I}(p; A, v_p = \frac{2\pi}{e} n \right|^{2r_1} \right]^{\frac{1}{2r_1}} \leq$$

$$\leq O\left(\frac{|n|}{\rho_0 e^2}\right) \left(1 + O\left(\frac{|n|}{\rho_0 e^2}\right)\right) e^{O\left(\frac{n^2}{\rho_0 e^3}\right)}$$

$$\left[\int d\tilde{\gamma}(A_{<xy>}) d\tilde{\gamma}(\rho_x) d\tilde{\gamma}(\rho_y) \left| \mathbb{I}(<xy>; \rho, A) \right|^{2r_2} \right]^{\frac{1}{2r_2}} \leq O\left(\frac{1}{\lambda \rho_0^2}\right)^{\frac{1}{2}}$$

Hence

$$|Z(X)| \leq \prod_{n \in \mathbb{Z}\setminus\{0\}} \prod_{p : v_p = \frac{2\pi}{e} n} e^{O\left(\frac{|n|}{\rho_0 e^2}\right) + O\left(\frac{n^2}{\rho_0 e^3}\right)} \cdot \tag{7.18}$$

$$\cdot \begin{cases} e^{-O(\ln \rho_0 e)|X \setminus X \cap \mathrm{supp}\, v|} & \lambda >> e^2 \\ e^{-O(\ln \lambda \rho_0^2)|X \setminus X \cap \mathrm{supp}\, v|} & \lambda << e^2 \end{cases}$$

Summing over all $n \in \mathbb{Z}\setminus\{0\}$ in each plaquette in the support of the vortices v, and for small e, and for large $\rho_0 e$, we get

$$\sum_{n \in \mathbb{Z}\setminus\{0\}} \exp\left[-\frac{2\pi^2}{e^2} n^2 + O\left(\frac{|n|}{\rho_0 e^2}\right) + O\left(\frac{n^2}{\rho_0 e^3}\right) \right] \leq$$

$$\leq 2 \exp\left\{-\frac{2\pi}{e^2}\left(1 - O\left(\frac{1}{\rho_0 e}\right)\right)\right\} \tag{7.19}$$

As before, a configuration $\{X_1 \ldots X_n, v_1 \ldots v_m\}$ is said to form a cluster on $C = \mathrm{supp} \bigcup_j X_j \bigcup_i v_i$ if $\mathrm{supp} \bigcup_j X_j \bigcup_i (v_i)^*$ is connected.
From (6.9), (7.17), (7.18), (7.19), denoting by γ_i the supports of $(v_i)^*$, we get

$$|Z(C)| \leq \sum_{\substack{\{\gamma_i, X_j\}: \\ \{v_i, X_j\} \text{ clusters on } C}} \prod_i e^{-\frac{2\pi}{e^2}\left(1 - O\left(\frac{1}{\rho_0 e}\right) - O(\ln \rho_0 e)\right)|\gamma_i|}$$

$$\cdot \prod_j e^{-O(\ln \rho_0 e)|X_j|} \qquad \text{if} \quad \lambda >> e^2$$

(otherwise substitute $O(\ln \rho_0 e)$ with $O(\ln \lambda \rho_0^2)$).

By comparison with (6.15), we get

$$c_0(e) = \frac{2\pi}{e^2}\left(1 - O\left(\frac{1}{\rho_0 e}\right)\right)$$

$$c_2(e) = O(\ln \rho_0 e) \qquad \lambda >> e^2$$

$$= O(\ln \lambda \rho_0^2) \qquad \lambda << e^2$$

From this bound we easily see that for large $\rho_0 \lambda$, large $\rho_0 e$ and small e the cluster expansion converges.
To get the convergence of the excitation expansion we also need $\rho_0 e^2$ to be large so that for a soliton of charge n in the formula

$$e^{-\frac{2\pi}{e^2}\left(1 - O\left(\frac{|n|}{\rho_0 e^2}\right)\right)} e^{O\left(\frac{|n|}{\rho_0 e^2}\right) + O\left(\frac{n^2}{\rho_0 e^3}\right)} \leq$$

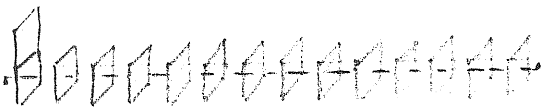
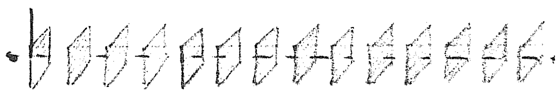
$$\leq Z(C^w = p) \leq e^{-\frac{2\pi}{e^2}}\left(1+O\left(\frac{|n|}{\rho_0 e^2}\right)\right)e^{O\left(\frac{|n|}{\rho_0 e^2}\right)+O\left(\frac{n^2}{\rho_0 e^3}\right)}$$

the term $O\left(\frac{|n|}{\rho_0 e^2}\right)$ is small compared with 1.
Then the first contribution from excitations in the sum 1') is of order

$e^{-O(\ln \rho_0 e)}$ if $\lambda >> e^2$

$e^{-O(\ln \lambda \rho_0^2)}$ if $\lambda << e^2$

and comes from configurations like

 or 

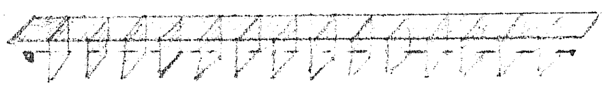
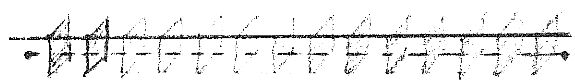
$\lambda >> e^2$ $\lambda << e^2$

The leading contribution in the sum 3') is of order

$e^{-O(\ln \rho_0 e)t}$ $\lambda >> e^2$

$e^{-O(\ln \lambda \rho_0^2)t}$ $\lambda << e^2$

and comes from configurations like

 or 

$\lambda >> e^2$ $\lambda << e^2$

Therefore, the mass of the soliton in the sector labelled by $n \in \mathbb{Z} \setminus \{0\}$ is given by

$$m_n = \frac{2\pi}{e^2} n^2\left(1 + O\left(\frac{1}{\rho_0 e}\right)\right) + \begin{cases} e^{-O(\ln \rho_0 e)} & \lambda >> e^2 \\ e^{-O(\ln \lambda \rho_0^2)} & \lambda << e^2 \end{cases}$$

and the upper gap in the two soliton point function is estimated by

$$\mu = \begin{cases} O(\ln \rho_0 e) & \lambda >> e^2 \\ O(\ln \lambda \rho_0^2) & \lambda << e^2 \end{cases}$$

We collect all these results in the

Theorem 7.6 The ($U(1)$ Higgs)$_3$ model (defined in example 5.2 B i) for a small e, large $\lambda \rho_0^2$, $\rho_0 e$ (superconducting phase) possesses superselection sectors labelled by $\mathbb{Z}$; sectors $\mathcal{H}_n$, $n \in \mathbb{Z}\setminus\{0\}$ are soliton sectors with mass gap

$$m_n = \frac{2\pi}{e^2} n^2 \left(1 + O\left(\frac{1}{\rho_0 e}\right)\right) + \begin{cases} e^{-O(\ln \rho_0 e)} & \lambda >> e \\ e^{-O(\ln \lambda \rho_0^2)} & \lambda << e \end{cases}$$

and the one-soliton state is the one-particle state with lowest mass in $\mathcal{H}_n$.

Remark 7.7 A similar theorem holds for the ($U(1)$ Higgs)$_3$ model with compact action for the gauge field, having a period q-times the period of the Higgs field. In this model one gets the $\mathbb{Z}_q$ soliton (vortex) sectors and similar mass gap estimate.

Example 7.8 SU(N) Higgs Model

To analyze this model we first go into the unitary gauge, i.e. we fix a $\phi_0 \in V$ with $(\phi_0, \phi_0) = 1$ and change variable $g_{\langle xy\rangle} \rightarrow g'_{\langle xy\rangle}$ in such a way that

$$(\phi_y, U(g_{\langle xy\rangle})\phi_x) = (\phi_0, U(g'_{\langle xy\rangle})\phi_0)$$

(we will henceforth omit the prime).

Then, we represent the partition function in terms of the v-fields as

$$Z_\Lambda = \sum_{v: dv=0} \int \prod_{\langle xy\rangle \in \Lambda} dg_{\langle xy\rangle} \exp\{-\lambda [Re(\phi_0, U(g_{\langle xy\rangle})\phi_0) - 1]\} \prod_{p\in\Lambda} \exp \frac{1}{e^2} [Re\, \chi(g_{\partial p}) \cos v_p - Im\, \chi(g_{\partial p}) \operatorname{sen} v_p + 1]$$

where $dg_{\langle xy\rangle}$ is the Haar measure on $SU(N)/\mathbb{Z}_N$ and v_p takes values in $\frac{2\pi}{e} \mathbb{Z}_N$; $\mathbb{Z}_N \simeq \{0, 1 \ldots N-1\}$.

We identify

$$S_p(g,v) = \frac{1}{e^2}\left[-\mathrm{Re}\,\chi(g_{\partial p})\cos v_p + \mathrm{Im}\,\chi(g_{\partial p})\sin v_p - 1\right]$$

so that

$v_p \neq 0$ gives a strongly decreasing factor if e is small (requisite 1).
Moreover, since $\mathrm{Re}(\phi_0, U(g_{<xy>})\phi_0) - 1 \geq 0$ and $= 0$ only for $g_{<xy>} = 1 \in SU(N)/\mathbb{Z}_N$ this term peaks the measure dg around $g = 1$ if λ is large (requisite 2).
However, as we shall see, this requirement is not strong enough for our purposes.
The analysis now proceeds as in the previous examples with only a slight modification in the estimate of $Z(X)$, which requires the previously mentioned stronger condition. Here X is a connected set of plaquettes

$$Z(X) = \int d\tilde{\nu}(g_{<xy>}) \prod_{p\in X} \mathbb{P}(p; g, v)$$

$$d\tilde{\nu}(g_{<xy>}) = dg_{<xy>}\, e^{-\lambda\left[\mathrm{Re}(\phi_0, U(g_{<xy>})\phi_0) - 1\right]}$$

$$\mathbb{P}(p; g, v) = \exp\left\{\frac{1}{e^2}\left[(\mathrm{Re}\,\chi(g_{\partial p}) - 1)\cos v_p - \right.\right.$$

$$\left.\left. - \mathrm{Im}\,\chi(g_{\partial p})\sin v_p\right]\right\} - 1$$

The main difference is that we do not approximate directly $d\tilde{\nu}(g_{<xy>})$ with a gaussian measure, but we split the integration over g in a neighbourhood $\mathcal{N}_1$ of $SU(N)/\mathbb{Z}_N$ identity and its complement $\mathcal{N}_1^c$; the gaussian approximation is used only in the first [25]. To be more precise, suppose in $\mathcal{N}_1$ is defined the exponential map $\exp: \mathbb{R}^d \longrightarrow G$ and let A^a $a = 1, \dots q$ = (dimension of $G \equiv SU(N)/\mathbb{Z}_N$) denote the associated coordinates in $\exp^{-1}\mathcal{N}_1$.
Then we define

$$\mathcal{N}_1 = \{ g : g = \exp i A^a \tau^a , |A^a| \le O(e) \}$$

where τ^a are the generators of the SU(N) algebra.

In $\mathcal{N}_1$

1)

$$dg_{\langle xy \rangle}(A^a_{\langle xy \rangle}) \sim (1 + O(A^2_{\langle xy \rangle}))\, d^a A^a_{\langle xy \rangle}$$

where $dA_{\langle xy \rangle}$ is the Lesbesgue measure on $\mathbb{R}$; $A = \sum \tau^a A^a$;

2) the following inequality holds

$$\frac{1}{2} A^2 - \frac{1}{4!} A^4 \le 1 - \cos A \le \frac{1}{2} A^2$$

So that, if f is a bounded function on G:

$$\int_{\mathcal{N}_1} d\nu(g_{\langle xy \rangle}) |f(g)| \le \int_{\exp^{-1} \mathcal{N}_1} d^a A^a_{\langle xy \rangle} (1 + A^2_{\langle xy \rangle})\, e^{-\lambda \sum_a c^a (A^a_{\langle xy \rangle})^2} |f(\exp^{-1} g)|$$

where $c^a = (\phi_0 , U((\tau^a)^2)\phi_0)$

In $\mathcal{N}_1$ we also have , roughly,

$$P(p; \exp^{-1} g, v) \simeq [1 - \cos(dA_p + O(e^2))] \cos v_p$$

$$+ [\mathrm{sen}\,(dA_p + O(e^2))]\, \mathrm{sen}\, v_p$$

The integration over $\mathcal{N}_1^c$ is, instead, roughly estimated by

$$(1 - (\phi_0 , U(g_{\langle xy \rangle})\, \phi_0)) \ge O(e) \quad \text{in } \mathcal{N}_1^c$$

so that

$$\int_{\mathcal{N}_1^c} d\nu(g) |f(g)| \le e^{-\lambda e}\, \mathrm{Max} |f(g)|$$

Moreover, in N_1^c

$$|\mathbb{P}(p;g,v)| \leq \exp O(\tfrac{1}{e^2})$$

so that $\mathbb{P}(p)$ is dominated by the exponential factor $e^{-\lambda e}$ if $1/\lambda e^2 << e$ (this is just due to a too rough estimate; large λe^2 should suffice).
Using these estimates it is not difficult to prove

$$|Z(X)| \leq \left[O\left(\frac{1}{e\sqrt{\lambda}}\right) + O(\sqrt{\lambda})\, e^{-\lambda e\left(O(1) - O\left(\frac{1}{\lambda e^3}\right)\right)}\right]^{|X \cap \operatorname{supp} v|}.$$

$$\cdot \left[O\left(\frac{1}{\lambda}\right) + O(\sqrt{\lambda})\, e^{-\lambda e\, O(1)}\right]^{|X \setminus X \cap \operatorname{supp} v|}$$

and hence the

Theorem 7.9
The SU(N) Higgs model (defined in example 5.2 B ii) for small e large λe^2 $\left(\frac{1}{e^2} >> \ln \lambda e^2\right)$ in $d=3$ possesses superselection sectors labelled by $\mathbb{Z}_N$; the $\mathcal{H}_n$ sectors corresponding to the non trivial elements $n \in \mathbb{Z}_N$ are soliton sectors with mass gap

$$m_n = \frac{1}{e^2}\left(1 - \cos\frac{2\pi n}{N}\right) + e^{-O(\ln \lambda e^2)}$$

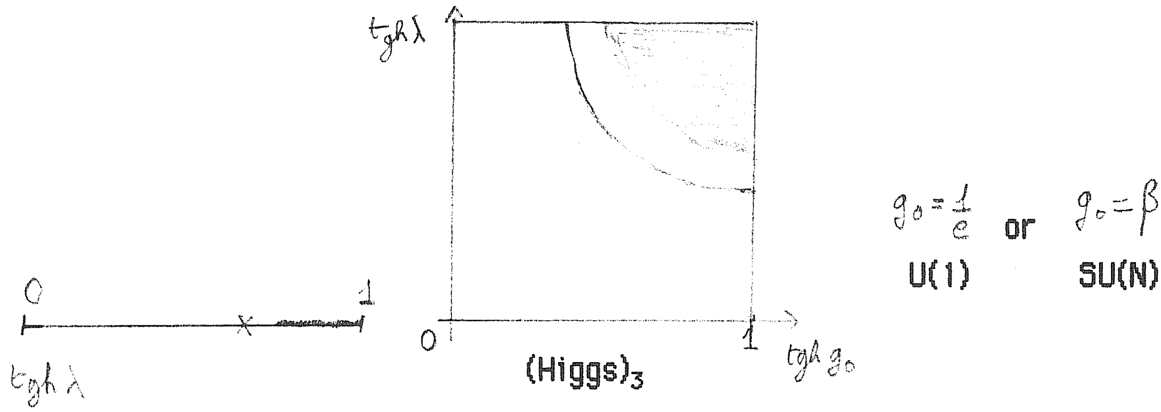
and the one-soliton state is the one-particle state with lowest mass in $\mathcal{H}_n$.

Remark 7.10 This theorem can be generalized to a G-Higgs model with G compact, having non trivial discrete center $\mathbb{Z}_G$, if the gauge action is given by characters in which $\mathbb{Z}_G$ is faithfully represented and the Higgs field takes values in a Hilbert space V carrying a unitary faithful representation of $G/\mathbb{Z}_G$.
For suitable values of the parameters we obtain $\mathbb{Z}_G$ soliton sectors in d=3. We simply state without proof (which is now trivial):

Theorem 7.11 The rank k-Stückelberg model (defined in example 5.2 C) for small e and large ze^2 $\left(\frac{1}{e^2} \gg \ln ze\right)$ in $d = k+2$ possesses superselection sectors labelled by $\mathbb{Z}$; the sectors $\mathcal{H}_n$ corresponding to the non trivial elements $n \in \mathbb{Z}$ are soliton sectors with mass gap

$$m_n = 2\pi^2 \frac{n^2}{e^2} + e^{-O(\ln ze^2)}$$

and the one-soliton state is the lowest one-particle state in $\mathcal{H}_n$.



Phase diagrams showing the parameter region where we have proven that soliton sectors exist. (Shaded region)

fig. 7.1

7.12 Soliton mass and surface tension

In this subsection we will show that the soliton mass coincides with the (corresponding) surface tension [26], in the convergence region of the previously discussed expansions.

Let us first define what the surface tension is in the simplest model, the $(\text{Ising})_2$, model and discuss its relation with the soliton mass in the low temperature region.

Let $\Lambda \equiv T \times L$ be the rectangular lattice, centered at the origin, with side length in the time (space) direction $T+1$ $(L+1)$.

Define $\partial\Lambda_{\pm} = \{ x \in \partial\Lambda : x^1 \gtrless 0 \}$, and put + bd conditions on $\partial\Lambda_+$ and - bd conditions on $\partial\Lambda_-$; then the partition function is given by

$$Z_{+-}(T \times L) = \sum_{\sigma_x = \pm 1,\, x \in \Lambda} \prod_{\langle xy \rangle \in \Lambda} e^{-\frac{\beta}{2}(\sigma_x - \sigma_y)^2} \prod_{x \in \partial\Lambda_+} \delta(\sigma_x - 1) \prod_{x \in \partial\Lambda_-} \delta(\sigma_x + 1)$$

Let $Z(T \times L)$ denote the partition function with +bd conditions, then the surface tension is defined by

$$\tau_- = \lim_{T \nearrow \infty} \lim_{L \nearrow \infty} -\frac{1}{T} \ln \frac{Z_{+-}(T \times L)}{Z_+(T \times L)}$$

The above defined ±bd conditions force an open Peierls contour, called Bloch wall(whose boundary is given by the points (see fig. 7.2.)

$$\left\{ \left(\frac{T}{2} + 1, 0\right), \left(-\frac{T}{2} - 1, 0\right) \right\} \qquad)$$

in the lattice Λ.

If the surface tension is not zero, the intrinsic thickness of the Bloch wall is non zero, even in the thermodynamic limit.

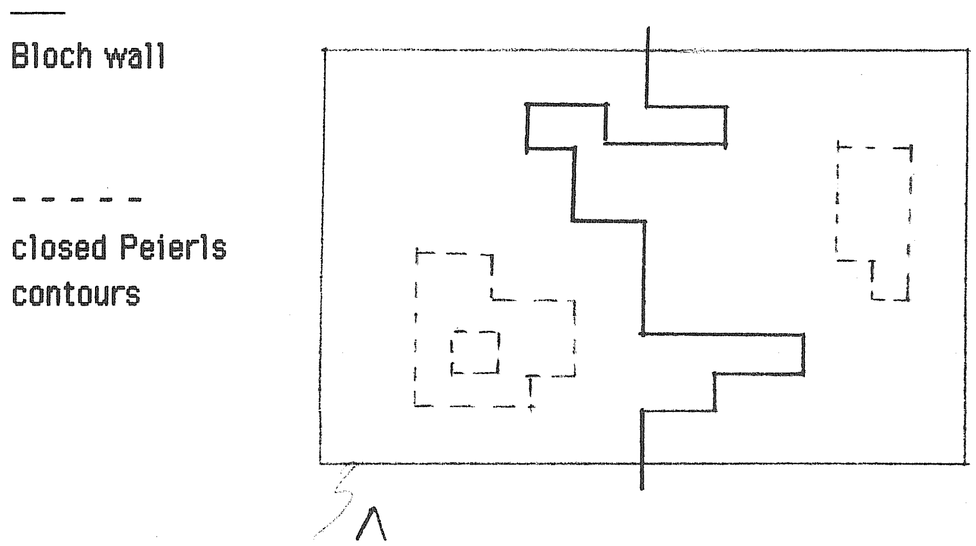


fig. 7.2

The situation for the contours is quite similar to that of a disorder field with support given by two points; it is, therefore, natural to investigate whether there is a relation.

An immediate observation is that if

$$\mathrm{supp}\; d\omega = \left\{ x_1 \equiv \left(-\frac{T}{2}, 0\right),\ x_2 \equiv \left(\frac{T}{2}, 0\right) \right\}$$

using a local gauge transformation, we have

$$\frac{Z_{+-}(T \times L)}{Z_{+}(T \times L)} = \langle D(\omega) \rangle_{\Lambda = T \times L}$$

In fact, let γ be the dual of the support of ω, then, making the change $\sigma_x \to -\sigma_x$ for all the x on the left of γ in the shadowed region in fig. 7.3, we have

$$\langle D(\omega)\rangle_{TXL} = \frac{1}{Z_+(TXL)} \sum_{\sigma_x=\pm 1,\, x\in\Lambda} \prod_{\langle xy\rangle\in\Lambda\setminus\gamma^*} e^{-\frac{\beta}{2}(\sigma_x-\sigma_y)^2}$$

$$\cdot \prod_{\langle xy\rangle\in\gamma^*} e^{-\frac{\beta}{2}(\sigma_x+\sigma_y)^2} \prod_{x\in\partial\Lambda} \delta(\sigma_x-1) =$$

$$= \frac{1}{Z_+(TXL)} \sum_{\sigma_x=\pm 1,\, x\in\Lambda} \prod_{\langle xy\rangle\in\Lambda\setminus\partial\Lambda_-} e^{-\frac{\beta}{2}(\sigma_x-\sigma_y)^2} \prod_{x\in\partial\Lambda_+} \delta(\sigma_x-1)\,\cdot$$

$$\cdot \prod_{x\in\partial\Lambda_-} \delta(\sigma_x+1) = \frac{Z_{+-}(TXL)}{Z_+(TXL)}$$

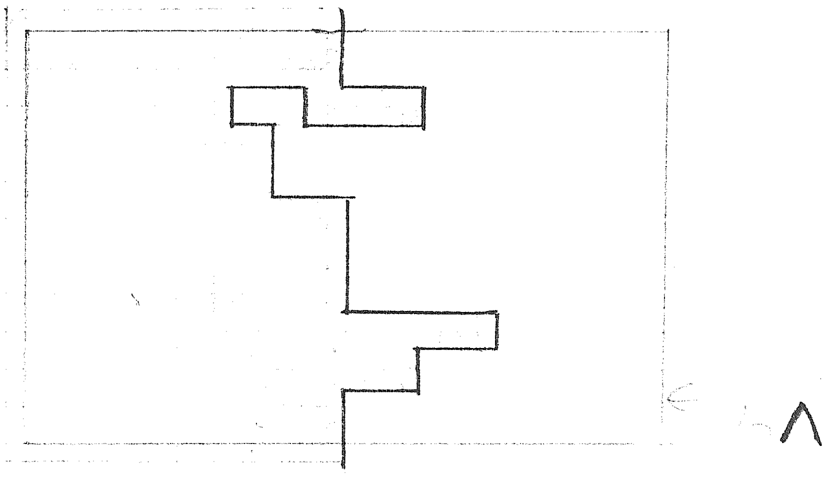


fig. 7.3

The soliton mass is defined by

$$m_S = \lim_{t\nearrow\infty} \lim_{L,T\nearrow\infty} -\frac{1}{t} \ln \langle D_t(\omega)\rangle_{LXT} =$$

$$= \lim_{t\nearrow\infty} -\frac{1}{t} \ln \langle D_t(\omega)\rangle$$

where $D_t(\omega)$ denote a disorder field with

$$\operatorname{supp} d\omega = \left\{ \left(\tfrac{t}{2}, 0\right), \left(-\tfrac{t}{2}, 0\right) \right\}$$

Therefore, we have

$$m_s - \tau_- = \lim_{T \uparrow \infty} -\frac{1}{T} \ln \frac{\langle D_T(\omega) \rangle}{\langle D_T(\omega) \rangle_{T \times \mathbb{Z}}}$$

We now use the excitation expansion to obtain

$$\langle D_T(\omega) \rangle = e^{-2\beta T} \exp\left\{ \sum_{\underset{\sim}{\xi} \subset \mathbb{Z}^2} \prod_{\varepsilon \in \underset{\sim}{\xi}} \zeta(\varepsilon) \frac{\varphi_T(\underset{\sim}{\xi})}{|\underset{\sim}{\xi}|!} \right\}$$

$$\langle D_T(\omega) \rangle_{T \times \mathbb{Z}} = e^{-2\beta T} \exp\left\{ \sum_{\underset{\sim}{\xi} \subset T \times \mathbb{Z}} \prod_{\varepsilon \in \underset{\sim}{\xi}} \zeta(\varepsilon) \frac{\varphi_T(\underset{\sim}{\xi})}{|\underset{\sim}{\xi}|!} \right\}$$

so that

$$\ln \frac{\langle D_T(\omega) \rangle}{\langle D_T(\omega) \rangle} = \sum_{\substack{\underset{\sim}{\xi} \subset \mathbb{Z}^2 \\ \underset{\sim}{\xi} \not\subset T \times \mathbb{Z}}} \prod_{\varepsilon \in \underset{\sim}{\xi}} \zeta(\varepsilon) \frac{\varphi_T(\underset{\sim}{\xi})}{|\underset{\sim}{\xi}|!}$$

We now give a bound on this sum, using the fact that all the contributing families of excitations $\underset{\sim}{\xi}$ must connect the boundary of γ to the boundary of $T \times \mathbb{Z}$ $\left(\partial(T \times \mathbb{Z}) = \left\{ t = \frac{T+1}{2}, t = -\frac{T-1}{2} \right\}\right)$

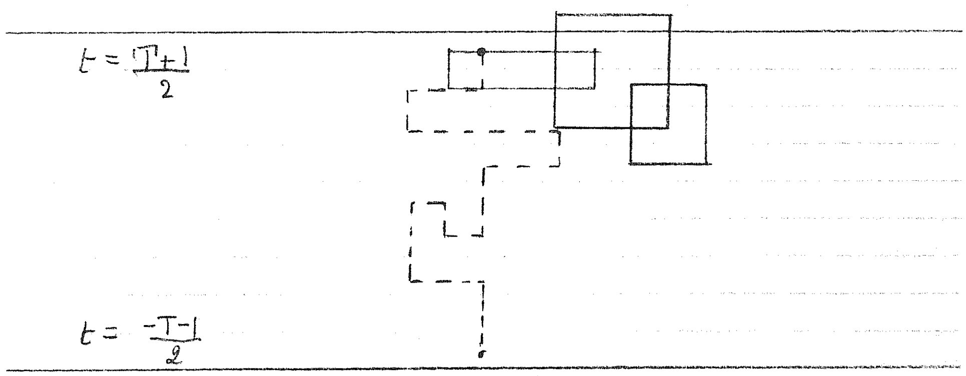


fig. 7.4

Using the symmetry for reflection w.r.t. the $t=0$ plane and the obvious bound

$$\left| \frac{\varphi_{T}(\underset{\sim}{\xi}) \prod_{\xi \in \underset{\sim}{\xi}} \zeta(\xi)}{|\underset{\sim}{\xi}|!} \right| \leq \prod_{\xi \in \underset{\sim}{\xi}} e^{-(2\beta-c)|\xi|} \equiv e^{-(2\beta-c)|\underset{\sim}{\xi}|}$$

we obtain

$$\left| \sum_{\substack{\underset{\sim}{\xi} \subset \mathbb{Z}^2 \\ \underset{\sim}{\xi} \not\subset T\times\mathbb{Z}}} \prod_{\xi \in \underset{\sim}{\xi}} \zeta(\xi) \frac{\varphi_{T}(\underset{\sim}{\xi})}{|\underset{\sim}{\xi}|!} \right| \leq$$

$$\leq 2 \Bigg[\sum_{x \in \{t = \frac{T+1}{2}\} \simeq \mathbb{Z}} \sum_{\substack{\underset{\sim}{\xi}: \\ \mathrm{supp}\, \underset{\sim}{\xi} \supset \{x\, x_1\}}} e^{-(2\beta-c)|\underset{\sim}{\xi}|}$$

$$+ \sum_{x \in \{t = \frac{-T-1}{2}\} \simeq \mathbb{Z}} \sum_{\substack{\underset{\sim}{\xi}: \\ \mathrm{supp}\, \underset{\sim}{\xi} \supset \{x\, x_1\}}} e^{-(2\beta-c)|\underset{\sim}{\xi}|} \Bigg] \leq$$

$$\leq 2 \Big(\sum_{\underset{\sim}{\xi} \subset \mathbb{Z}^2} e^{-\frac{(2\beta-c)}{2}|\underset{\sim}{\xi}|} \Big) \Big[\sum_{n \in \mathbb{Z}} e^{-\left(\frac{2\beta-c}{2}\right)n} +$$

$$+ e^{-\frac{T}{2}(2\beta-c)} \Big] \leq$$

$$\leq \mathrm{const} \left(\frac{1}{1 - e^{-\left(\frac{2\beta-c}{2}\right)}} + e^{-T\left(\frac{2\beta-c}{2}\right)} \right) \tag{7.20}$$

Therefore,

$$|m_s - \tau_-| = \lim_{T\nearrow\infty} \frac{1}{T} \left| \sum_{\substack{\underset{\sim}{\varepsilon} \subset \mathbb{Z}^2 \\ \underset{\sim}{\varepsilon} \subset T\times\mathbb{Z}}} \prod_{\varepsilon \in \underset{\sim}{\varepsilon}} \zeta(\varepsilon) \frac{\varphi_T(\varepsilon)}{|\underset{\sim}{\varepsilon}|} \right| \leq$$

$$\leq \lim_{T\nearrow\infty} \frac{const}{T} \left(\frac{1}{1 - e^{-\frac{(2\beta - c)}{2}}} - e^{-\frac{T}{2}(2\beta - c)} \right) = 0$$

i.e.

$$m_s = \tau_- \quad .$$

This is a general argument in the context of the models discussed here. Let in fact $\phi \to (\rho, \sigma)$ be the decomposition discussed in sect. 6 and, moreover, let $\Lambda = T\times L^{d-1}$ denote the dimensional lattice centered at the origin, with length of the side in the time direction $T+1$ and length of the sides in the space direction $L+1$.

Consider $\partial\Lambda$ as a d-1 dimensional lattice, and let Γ be a curve in its dual (which is well defined since $\partial(\partial\Lambda)=\phi$) joining

$$\bar{x}_1 = \left(-\frac{T-1}{2}, 0 \ldots 0 \right) \quad to \quad \bar{x}_2 = \left(\frac{T+1}{2}, 0 \ldots 0 \right)$$

The dual of Γ is a set of d-2 dimensional cells in $\partial\Lambda$ and the dual of $\partial\Gamma$ are two d-1 dimensional cells in $\partial\Lambda$. Since the soliton arises in dimension d=k+2, the σ field has support exactly on d-2 dimensional cells.

So we can impose to all σ, with support on the dual of Γ, the condition $\delta(\sigma - q)$, where $q \in \mathbb{Z}$ is the element labelling the soliton sector we consider, and $\delta(\sigma - 0)$ for all the other σ in $\partial\Lambda$.

The partition function with such "q"-bd conditions is denoted by $Z_q(T\times L^{d-1})$, and the related surface tension is defined by

$$\tau_q \equiv \lim_{T\nearrow\infty} \lim_{L\nearrow\infty} -\frac{1}{T} \ln \frac{Z_q(T\times L^{d-1})}{Z_0(T\times L^{d-1})}$$

Using the $\mathbb{Z}$ gauge invariance of

$$\langle D(\omega) \rangle_{T\times L^{d-1}}$$

one can easily show again that

$$\langle D_T(\omega) \rangle_{T\times L^{d-1}} = \frac{Z_q(T\times L^{d-1})}{Z_0(T\times L^{d-1})}$$

with obvious meaning of the symbols.
Then the argument previously discussed applies in a straightforward way, with an easy modification of inequality (7.20) if $d \neq 2$.

Therefore, we have the general

Theorem 7.13 For all the models discussed here in the region of the parameters where the expansion in excitation converges, the soliton mass equals the corresponding surface tension: $m_q = \tau_q$.
Since the mass m_q is analytic in the coupling parameters, this equality extends to their whole analiticity domain.

Appendix 7.A Proof of theorem 7.1

First, we show that for a generic element

$$[\underline{f}^* \otimes \underline{g}]^\Lambda \in P_q [\underline{S}^{*+} \otimes \underline{S}_E^+]^\Lambda$$

we have

$$\langle [\underline{f}^* \otimes \underline{g}]^\Lambda, T(t) [\underline{f}^* \otimes \underline{g}]^\Lambda \rangle \leq$$

$$\leq C_{\underline{f}^* \otimes \underline{g}} \langle S_q(\vec{0}') \Omega, T(2t) S_q(\vec{0}') \Omega \rangle^{\frac{1}{2}} \quad (7.A1)$$

$C_{\underline{f}^* \otimes \underline{g}} = \text{const}$

Instead of proving this inequality in full generality, let us consider the case

$$\phi(\underline{f}^* \otimes \underline{g}) \Omega = | x_1 q_1 \ldots x_n q_n ; C_1 \ldots C_m \rangle$$

where C_i refers to the Wilson loops $W(C_i)$.
Then,

$$S_{-q}(\vec{0})\,|x_1 q_1 \ldots x_n q_n ; C_1 \ldots C_m\rangle = |0 - q\ x_1 q_1 \ldots x_n q_n ; C_1 \ldots C_m\rangle$$

and we have (let $x+t \equiv (x^0+t, \vec{x})$)

$$\langle x_1 q_1 \ldots C_m |\, T(t)\, | x_1 q_1 \ldots C_m\rangle =$$

$$= \langle 0-q\ x_1 q_1 \ldots C_m |\, S_{-q}(\vec{0})\, T(t)\, S_q(\vec{0})\, |0-q\ x_1 \ldots C_m\rangle =$$

$$= \langle \theta [D(0-q\ x_1 \ldots q_n) \prod_{j=1}^{m} W(C_j)] [D(0-q, x_1+t \ldots q_n) \cdot \tag{7.A2}$$

$$\cdot \prod_{j=1}^{m} W(C_j+t)]\rangle$$

$$= \langle \theta [D(-\tfrac{t}{2}-q, x_1+\tfrac{t}{2} \ldots q_n) \prod_{j=1}^{m} W(C_j+\tfrac{t}{2})\, D^{\frac{1}{2}}(-\tfrac{t}{2}-q\ \tfrac{t}{2} q)] \cdot$$

$$\cdot [D(\tfrac{t}{2}-q, x_1+\tfrac{t}{2} \ldots q_n) \prod_{j=1}^{m} W(C_j+\tfrac{t}{2})\, D^{\frac{1}{2}}(-\tfrac{t}{2}-q\ \tfrac{t}{2} q)]\rangle$$

where we have used the invariance of $\langle\ \ \rangle$ w.r.t. the time translations. Using the Schwartz inequality, w.r.t. $L_2(d\mu)$ and time translation invariance, again we obtain

$$(7A2) \le \langle \theta\, |D(0-q \ldots q_n) \prod_{j=1}^{m} W(C_j)|^2 \cdot$$

$$\cdot D(0-q\ tq)\rangle^{\frac{1}{2}} \langle |D(0-q \ldots q_n) \prod_j W(C_j)|^2 \cdot \tag{7.A.3}$$

$$\cdot D(0-q\ tq)\rangle^{\frac{1}{2}}$$

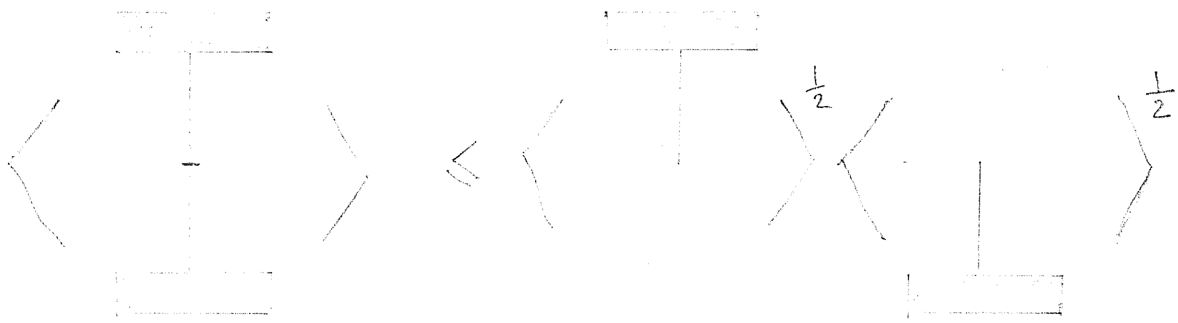
Now, the two terms are equal by the invariance of $\langle\ \rangle$, w.r.t. reflection in the t=0 plane, and applying the Schwartz inequality w.r.t. $\langle\ ,\Theta\ \rangle$, we immediately get

$$(7A3) \leq \langle D(0-q\ t\,q)\ \Theta\ D(0-q\ t\,q)\rangle^{\frac{1}{2}} \cdot$$

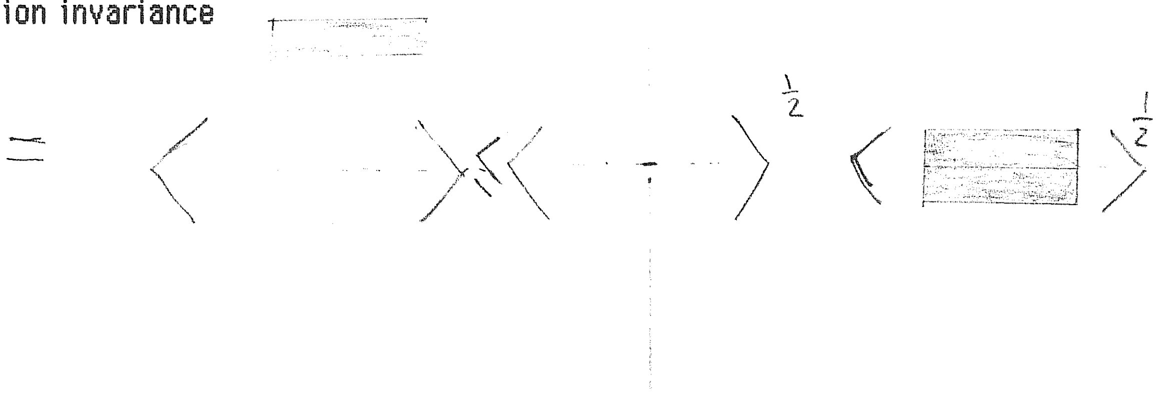
$$\cdot \langle |D(0-q \ldots q_n) \prod_j W(C_j)|^2\ \Theta\ |D(0-q \ldots q_n) \prod_j W(C_j)|^2 \rangle$$

$$= c_{\underline{f}^* \otimes \underline{g}} \langle S_q(\vec{\sigma})\Omega\ ,\ T(2t)\ S_q(\vec{\sigma})\Omega \rangle^{\frac{1}{2}}$$

Pictorially,



reflection invariance



Now define

$$\hat{T} \equiv T\, e^{m_q} \qquad \text{so that} \qquad 0 \leq \hat{T} \restriction \mathcal{H}_q \leq 1$$

Since by construction $P_q[\Sigma^{*+}\otimes\Sigma_\varepsilon^{+}]^\wedge$ is dense in $\mathcal{H}_q$, $\forall\varepsilon>0$ $\exists\ [\sum_i c_i \underline{f}_i^{+}\otimes \underline{g}_i]^\wedge \in P_q[\Sigma^{*+}\otimes\Sigma_\varepsilon^{+}]$ such that

$$\langle [\sum_i c_i \underline{f}_i^{+}\otimes \underline{g}_i]^\wedge, T(t)\, [\sum_i c_i \underline{f}_i^{+}\otimes \underline{g}_i]^\wedge \rangle \geq e^{-\varepsilon t}$$

But then, by inequality (7.A1) and using the spectral representation:

$$\varepsilon \geq -\lim_{t\to\infty} \frac{1}{t} \ln \langle [\sum_i c_i \underline{f}_i^{+}\otimes \underline{g}_i]^\wedge,$$

$$\cdot\ \hat{T}(t)\, [\sum_i c_i \underline{f}_i^{+}\otimes \underline{g}_i]^\wedge \rangle =$$

$$= \inf \operatorname{supp} d\rho_{\sum_i c_i \underline{f}_i^{+}\otimes g_i}(\lambda,\vec{0}) \geq$$

$$\geq -\lim_{t\to\infty} \frac{1}{t} \ln \langle S_q(\vec{0})\Omega, \hat{T}(2t) S_q(\vec{0})\Omega\rangle^{\frac{1}{2}} = m_s - m_q$$

By arbitrariness of ε we conclude

$$m_s = m_q$$

8. STRING-LIKE SOLITON

In this section we discuss how to construct soliton sectors with solitons of infinitely extended (euclidean) time-like support.
These sectors arise in (gauge+Higgs) models with fields taking values in a compact discrete group in d=3, as remarked in [15].
To be precise let n be a rank one and m a rank 0 fields, with value in $\mathbb{Z}_N \simeq \{0, \dots N-1\}$.
Consider the lattice field theory defined by the state $\langle\ \ \rangle$ given by the thermodynamic limit of

$$\langle (\cdot) \rangle_\Lambda = \frac{1}{Z_\Lambda} \, Tr_n \, Tr_m \prod_{p\in\Lambda} e^{\beta\left\{\cos\frac{2\pi}{N}(dn)_p - 1\right\}} , \tag{8.1}$$

$$\cdot \prod_{\langle xy\rangle} e^{\lambda \cos\frac{2\pi}{N}\left(dm_{\langle xy\rangle} - n_{\langle xy\rangle}\right)} \, (\cdot)$$

where

$$Tr_n = \sum_{n_{\langle xy\rangle}\in\mathbb{Z}_N,\ \langle xy\rangle\in\Lambda} \qquad\qquad Tr_m = \sum_{m_x\in\mathbb{Z}_N\ ,\ x\in\Lambda}$$

In d=3 the phase diagram for such models is likely to be

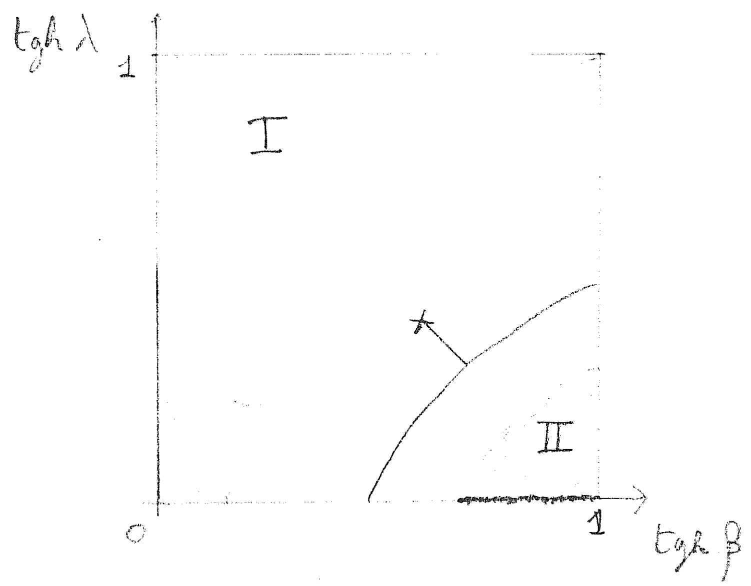


fig. 8.1

where the shaded regions denote the convergence domain for a high (I) and a low (II) temperature, respectively, expansion and the solid lines are the expected phase transitions.

Using the techniques of the previous section, one easily sees that in the marked region of the $\lambda = 0$ line, the pure gauge $\mathbb{Z}_N$ model possesses soliton sectors for each $q \in \mathbb{Z}_N \setminus \{0\}$.

The state Tr_n is clearly invariant under

$$n_{<xy>} \longrightarrow n_{<xy>} + \rho_{<xy>} , \qquad \rho_{<xy>} \in \mathbb{Z}_N , \tag{8.2}$$

In the region $\lambda \neq 0$, however, the measure

$$Tr_n \, Tr_m \prod_{<xy> \in \Lambda} e^{-\lambda \cos \frac{2\pi}{N} (dm_{<xy>} + n_{<xy>})} \; (\cdot)$$

is no longer invariant under the transformation (8.2), so the construction of the soliton sectors carried out in the previous sections breaks down, since it is based on the invariance under (8.2).

However, if λ is small in the II region, one should not expect dramatic changes, so it is natural to think that in some way the soliton should survive.

One point, however, is clear: due to the non invariant λ-term, the correlation functions with the ω external field introduced to construct the solitons will now depend on the whole support of ω, which is a stringlike region and not just on $d\omega$.

This is the characteristic feature of the models in class B of the introduction.

Whereas in the class A models the creation of the string between the support of the charges of the external field ω does not cost "energy" (really action); in the class B models, which we discuss here, it costs an "energy" growing with its length.

Therefore, in order to construct soliton states of finite energy we need both to arrange the support of ω in a suitable way and to normalize the correlation functions.

These requirements will lead to a slightly weaker form of the reconstruction theorem.

To make all these statements more explicit, let us refer to a concrete example: the simplest case of the Ising gauge model with $\mathbb{Z}_2$ Higgs fields in d=3 ($\equiv \mathbb{Z}_2$ Higgs)$_3$. It is common to discuss this model in terms of the multiplicative $\mathbb{Z}_2$ fields

$$\tau_{\langle xy\rangle} = e^{i\pi n_{\langle xy\rangle}} \qquad\qquad \sigma_x = e^{i\pi m_x}$$

so that (8.1) reads

$$\langle (\cdot) \rangle_\Lambda = \frac{1}{Z_\Lambda} \sum_{\substack{\tau_{\langle xy\rangle} = \pm 1,\ \langle xy\rangle\in\Lambda \\ \sigma_x = \pm 1,\ x\in\Lambda}} \prod_{p\in\Lambda} e^{\beta(\tau_{\partial p}-1)} \cdot \prod_{\langle xy\rangle\in\Lambda} e^{\lambda \sigma_x \tau_{\langle xy\rangle} \sigma_y} (\cdot)$$

To simplify the problem, let us first consider how to construct a state describing a soliton in the t=0 plane.

The ($\mathbb{Z}_2$ Higgs)$_3$ model is well known [27] to be self dual and the duality transformation maps the Wilson loop into the disorder field. (This point is sketched in a somewhat different situation in the appendix of [32]).

Now Fredenhagen and Marcu [28] constructed a charged state in ($\mathbb{Z}_2$ Higgs)$_3$ essentially making use of an increasing sequence of Wilson loops, so one expects that the soliton state is the dual of this charged state.

Translated into our language the dual of the Fredenhagen Marcu construction defines a soliton state $\langle S(x)\Omega, (\cdot)\, S(x)\Omega\rangle$ on the operator field algebra $\mathfrak{A}$ corresponding to the $\mathcal{Z}$ gauge invariant fields as follows.

Let us fix a spatial direction in the lattice, say x^1 and consider the path γ_x made by straight lines on the dual lattice

$$X = (0, x^1, x^2) \longrightarrow (\tau, x^1, x^2) \longrightarrow$$

$$\longrightarrow (\tau, x^1+2\tau, x^2) \longrightarrow (0, x^1+2\tau, x^2)$$

see Fig. 8.2, and let ω_p be a $\mathbb{Z}_2$ valued (external) field defined by

$$\omega_p = \begin{cases} 1 & p \notin (\gamma_{\vec{x}})^* \\ -1 & p \in (\gamma_{\vec{x}})^* \end{cases}$$

Then define the field

$$d_\tau^{FM}(X) = \prod_{p \in \Lambda} e^{\beta\, \Im_{\partial p}(\omega_p - 1)} = \prod_{p \in \operatorname{supp} \omega} e^{-2\beta\, \Im_{\partial p}}$$

$$d_\tau^{FM}(0, \vec{x}) \equiv d_\tau^{FM}(\vec{x})$$

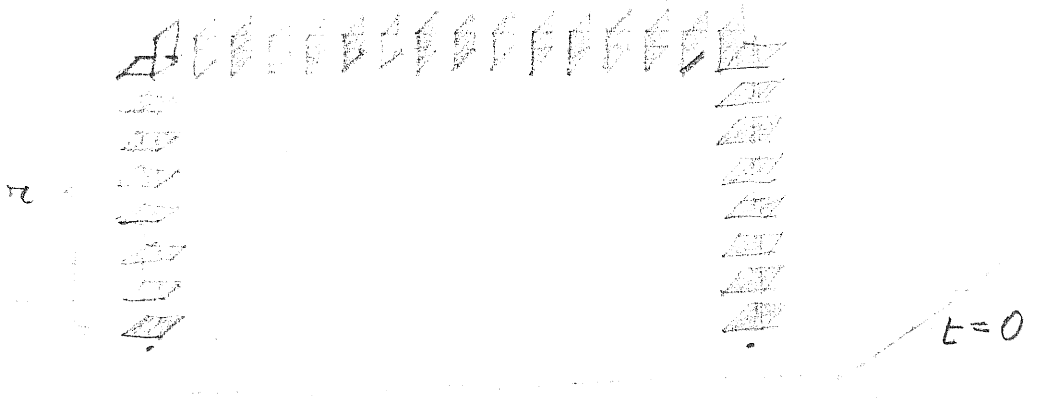


fig. 8.2

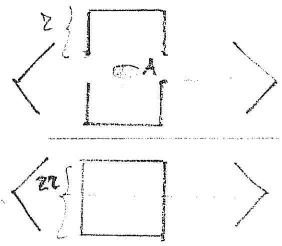
Now let $A(\Im_{\partial p}\omega_p,\ \sigma_y \Im_{\langle xy \rangle} \sigma_x \ ;\ p, \langle xy \rangle \subset [0,t] \times \mathbb{Z}^2)$ be a $\mathcal{I}$ gauge invariant field, let $\mathcal{A}$ denote the corresponding field operator, and let $T(t)$ be the time translation, then following F.M. define

$$\langle S(\vec{x})\Omega, A\, T(t)\, S(\vec{x})\Omega\rangle =$$

$$= \lim_{r\nearrow\infty} \lim_{\Lambda\uparrow\mathbb{Z}^3} \frac{\langle d_r^{FM}(t,\vec{x})\, A\, \theta\, d_r^{FM}(0,\vec{x})\rangle_\Lambda}{\langle d_r^{FM}(\vec{x})\, \theta\, d_r^{FM}(\vec{x})\rangle_\Lambda}$$

Pictorially,

$$\langle S(\vec{x})\Omega, A\, T(t)\, S(\vec{x})\Omega\rangle = \lim_{r\to\infty}$$



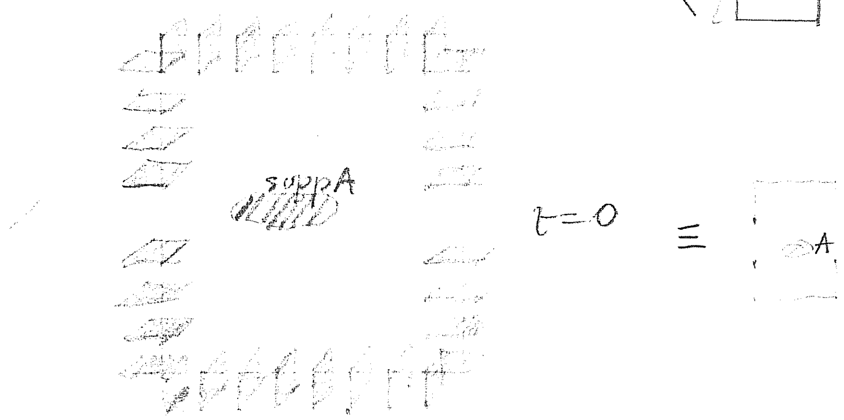


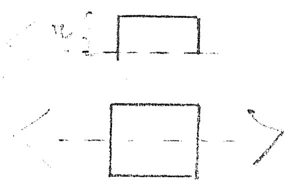
fig. 8.3

To show that $S(\vec{x})\Omega$ is a soliton state we first need to show that it is orthogonal to the vacuum, i.e.

$$\langle \Omega, S(\vec{x})\Omega\rangle = \lim_{r\nearrow\infty} \lim_{\Lambda\uparrow\mathbb{Z}^3} \frac{\langle d_r^{FM}(\vec{x})\rangle_\Lambda}{\langle d_r^{FM}(\vec{x})\, \theta\, d_r^{FM}(\vec{x})\rangle_\Lambda} = \tag{8.3}$$

$$= 0$$

The denominator in (8.3) has perimeter decay [10] for large   both in high (I) and low (II) temperature regions of the phase diagram (fig. 8.1). However, the numerator has quite a different behaviour in the two regions, more precisely



$$\underset{r\nearrow\infty}{\sim}\begin{cases} \dfrac{e^{-cr}}{(e^{-2cr})^{\frac{1}{2}}} & \text{region I} \\[2ex] \dfrac{e^{-cr}}{(r)^{\frac{d-1}{2}}\,(e^{-2cr})^{\frac{1}{2}}} & \text{region II} \end{cases}$$

So only in region II $\langle \Omega, S(\vec{x})\Omega \rangle = 0$.
The fact that the state $\langle S(\vec{x})\Omega, (\cdot)\, S(\vec{x})\Omega \rangle$ really gives a soliton sector in the region II can be shown by duality using the results of [28].
It is however convenient for the reconstruction theorem and the particle structure analysis to modify slightly the above construction.
Let us consider the model with free db conditions and define now the disorder fields

$$d_\Lambda(x,\pm) = \prod_{p\in\Lambda} e^{\beta\, T_{\partial p}(\omega_p - 1)} \;;\quad d_\Lambda((0,\vec{x}),\pm) \equiv d_\Lambda(\vec{x},\pm)$$

with $\omega_p = -1$ if p is in the dual of the straight line $\gamma^+(x)$ $(\gamma^-(x))$ in the time direction connecting x to the boundary $\partial\Lambda$, directed upwards (γ^+), for $d_\Lambda(x+)$, downwards (γ^-) for $d_\Lambda(x,-)$, and $\omega_p = +1$ otherwise.
More generally, for a set of points

$$(x_1 \dots x_{n_1}, y_1 \dots y_{n_2}) \quad \text{with} \quad x_i \geq y_j \quad \forall\, (i,j)$$

let

$$\begin{aligned} \omega_p &= -1 \qquad & p \in (\gamma^+(x_1)\Delta\gamma^+(x_2)\dots\Delta\gamma^-(y_{n_2})) \\ &= +1 & \text{otherwise} \quad (\Delta \equiv \text{symmetric difference}) \end{aligned}$$

and

$$d_\Lambda(x_1+ \quad x_{n_1}+,\ y_1- \dots y_{n_2}-) = \prod_{p\in\Lambda} e^{\beta\, \tau_{\partial p}(\omega_p - 1)}$$

Then we define for even $n = n_1 + n_2$; $m = \ell + s$:

$$S_{n,m}(x_1+ \dots x_{n_1}+,\ y_1- \dots y_{n_2}- \ ;\ p_1 \dots p_\ell,\ \langle xy\rangle_1 \dots \langle xy\rangle_s) \equiv$$

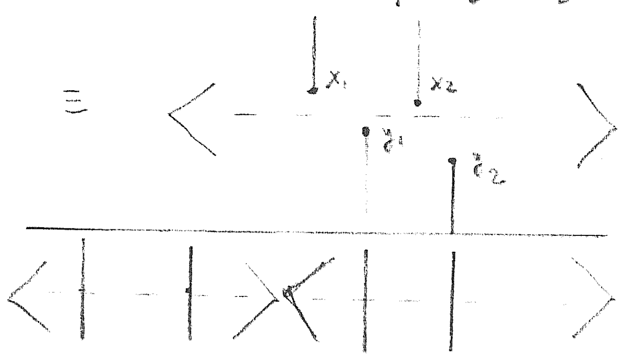
$$\equiv \lim_{\Lambda\uparrow\mathbb{Z}^3} \frac{\left\langle d_\Lambda(x_1+ \dots y_{n_2}-) \prod_{i=1}^{\ell} \tau_{\partial p_i}\omega_{p_i} \prod_{j=1}^{s} \sigma_{y_j}\, \tau_{\langle xy\rangle_j}\, \sigma_{x_j} \right\rangle_\Lambda}{\left\langle d_\Lambda(\vec{x}_1+ \dots \vec{x}_{n_1}+)\, \theta\, d_\Lambda(\vec{x}_1+ \dots \vec{x}_{n_1}+)\right\rangle_\Lambda^{1/2} \left\langle d_\Lambda(\vec{y}_1- \dots y_{n_2}-)\, \theta\, d_\Lambda(\vec{y}_1- \dots \vec{y}_{n_2}-)\right\rangle_\Lambda^{1/2}} \tag{8.4}$$

Later on we will show that this limit exists.
Odd correlation functions are defined by the limits

$$\lim_{z\to\infty} S_{n,m}(x_1+ \dots x_{n_1}+,\ z+,\ y_1 \dots \quad)$$

$$\lim_{z\to\infty} S_{n,m}(x_1+ \dots x_{n}+,\ z-,\ y_1 \dots \quad) \tag{8.5}$$

and correlation functions with $x_i < y_j$ are set equal to zero.
Pictorially, e.g. $S_n(x_1+\ x_2+\ \ y_1-\ y_2-) \equiv$



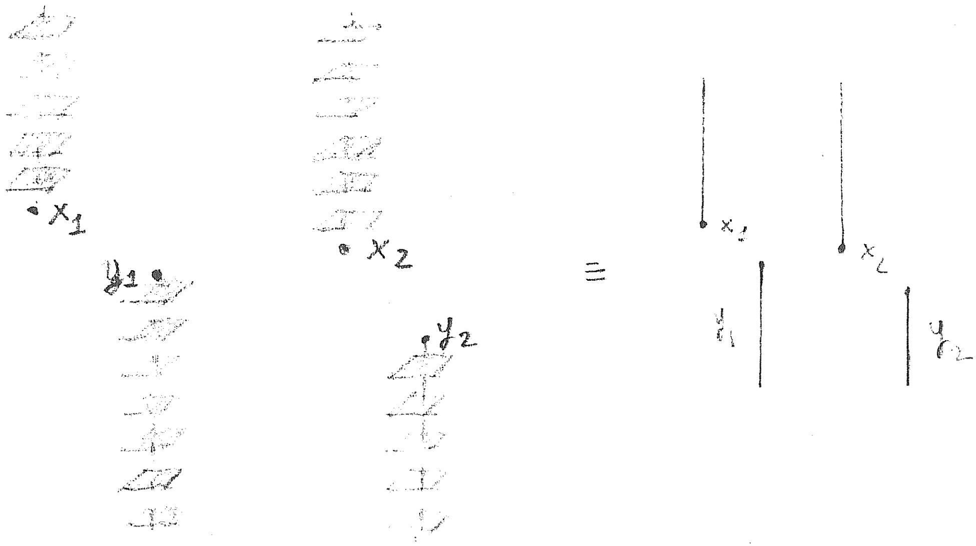


fig. 8.4

The program is now clear: we wish to apply to the sequences $S_{n,m}$ the reconstruction theorem.

Two observations are useful:

1) $S_{n,m}$ are translation invariant;

2) if Λ is symmetric w.r.t. the t=0 plane, θ denotes the reflection w.r.t. this plane r and the complex conjugation (as before), $x_1 \dots x_n$ are in the positive time lattice and $y_1 \dots y_m$ are in the negative time lattice

$$d_\Lambda(x_1+ \dots x_n+, y_1- \dots y_m-) =$$

$$= d_\Lambda(x_1+ \dots x_n+) \;\; \theta \;\; d_\Lambda(r y_1+ \dots r y_m+)$$

Let $E^* = \{\pm\}$ and define $\underline{S}^*$ as the space of all finite sequences $\underline{f}^*$ of complex functions f_n^* on $(\mathbb{Z}^3 \times E^+)^{\times n}$ vanishing except at finitely many points. Moreover, let $\underline{S}_E$ denote the space of all finite sequences $\underline{g}$ of functions on links and plaquettes.

The sequences of correlation functions $\{S_{n,m}\}_{n,m=0}^{\infty}$ define a functional S on $\underline{S}^* \otimes \underline{S}_E$ through

$$S(\underline{f}^{*} \otimes \underline{g}) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} S_{n,m}(f_n^{*}, g_m)$$

$$S_{n, m=m_1+m_2}(f_n^{*}, g_m) =$$

$$= \prod_{i=1}^{m} \sum_{\substack{x_i \in \mathbb{Z}^3 \\ \alpha_i \in E^{*}}} \prod_{j=1}^{m_1} \sum_{p_j \in \mathbb{Z}^3} \prod_{l=1}^{m_2} \sum_{\langle xy \rangle_l} S_{n,m}(x_1 \alpha_1, \ldots$$

$$\ldots x_n \alpha_n ; p_1 \ldots p_{m_1}, \langle xy \rangle_1 \ldots \langle xy \rangle_{m_2}) \cdot$$

$$\cdot f_n^{*}(x_1 \alpha_1 \ldots x_n \alpha_n)\, g_m(p_1 \ldots p_{m_1} \langle xy \rangle_1 \ldots \langle xy \rangle_{m_2})$$

Take $\underline{S}^{*+} = \{ \underline{f}^{*} : \operatorname{supp} f_n^{*} \subset (\Lambda + \chi\{+\})^{\times n} \}$
and then, thanks to the previous observations, it is clear that

the Reconstruction Theorem applies to the sequence of correlation functions $\{ S_{n,m} \}_{n,m=0}^{\infty}$ defined in (8.4) (8.5) and we obtain a Lattice Quantum Mechanics ($\mathcal{H}$, Ω , T , U).

Moreover, if the cluster property holds w.r.t. the χ 's, which in particular implies that by definition all odd correlation functions vanish, then
1) the vacuum Ω is the unique lattice translation invariant vector in $\mathcal{H}$;
2) $\mathcal{H}$ splits into two orthogonal sectors $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ corresponding to even and odd numbers of χ 's.
One then easily verifies that $\mathcal{H}_1$ is a soliton sector w.r.t. the field algebra corresponding to the ordinary fields $\tau_{\partial p} \omega_p$ and $\sigma_y \tau_{\langle xy \rangle} \sigma_x$.

As in Sect. 5, one can also introduce a soliton operator s. However, due to the timelike infinitely extended support of the soliton operator, we do not know whether Range $(s \upharpoonright \mathcal{H}_0)$ is dense in $\mathcal{H}_1$ because there may exist $\psi \in \mathcal{H}_0$ with $\|\psi\| = 0$ such that the vector corresponding to the formal expression $s\psi$ is a non-zero vector in $\mathcal{H}_1$. We feel that this cannot happen, but the proof is still missing.

Therefore, in order to prove that a soliton sector can be constructed in the way sketched above, we must show that

1) the correlation functions defined by the thermodynamic limit (8.4) exist. (This is not completely trivial since numerator and denominator in the right hand side of (8.4) tend to zero in the $\Lambda \nearrow \mathbb{Z}^3$ limit);

2) the cluster property holds w.r.t. the X coordinates and, in particular, all odd correlation functions vanish (by (8.5)).

We prove now that 1) and 2) hold in the region II for small λ and large β.

This is done by applying the excitation expansion to exponentiate the low temperature cluster expansion of Marra and Miracle-Solé [27].

Let us directly discuss a modified partition function

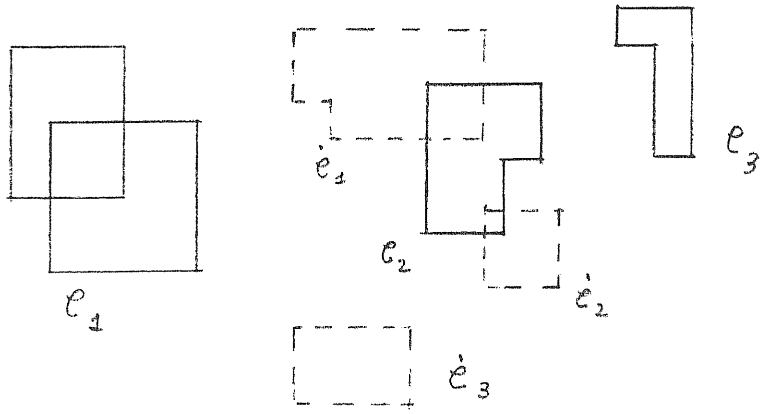
$$Z_\Lambda(\omega) = \sum_{\substack{\sigma_x = \pm 1,\ x \in \Lambda \\ \tau_{\langle xy \rangle} = \pm 1,\ \langle xy \rangle \in \Lambda}} \prod_{p \in \Lambda} e^{\beta(\tau_{\partial p}\omega_p - 1)} \prod_{\langle xy \rangle \in \Lambda} e^{\lambda \sigma_y \tau_{\langle xy \rangle} \sigma_x}$$

Denote by $\dot{\mathcal{C}}$ the curve (in general not connected) in the dual lattice which is the support of $\omega_p \tau_{\partial p}$, i.e. such that for $p \in (\dot{\mathcal{C}})^*$, $\omega_p \tau_{\partial p} = -1$. Clearly $\partial \dot{\mathcal{C}} = \operatorname{supp}(d\omega)^*$.

Now decompose $\dot{\mathcal{C}}$ in a set of connected disjoint curves $\dot{\mathcal{C}}_i$ with $\partial \dot{\mathcal{C}}_i = \phi$ and a curve $\dot{\mathcal{C}}^\omega$ disjoint from $\dot{\mathcal{C}}_i$ not necessarily connected but which has non vanishing boundary in each connected component. So

$$\dot{\mathcal{C}} = \{ \dot{\mathcal{C}}_i \ \ i = 1 \ldots m,\ \dot{\mathcal{C}}^\omega \}$$

Analogously, let $\mathcal{C}_j$, $j=1\ldots n$ denote a set of connected disjoint curves in the lattice, $\mathcal{C}=\{\mathcal{C}_1\ldots\mathcal{C}_n\}$.



(one dimension is suppressed)

fig. 8.5

Then we have

$$[\cosh\lambda]^{|\Lambda|}\, Z_\Lambda(\omega) = \sum_{\{\dot{\mathcal{C}}_1\ldots\dot{\mathcal{C}}_n\,:\,\partial\dot{\mathcal{C}}_i=\phi,\ \dot{\mathcal{C}}^\omega\,:\,\partial\dot{\mathcal{C}}^\omega=\mathrm{supp}(d\omega^*)\}} \prod e^{-2\beta|\dot{\mathcal{C}}_i|}\, e^{-2\beta|\dot{\mathcal{C}}^\omega|}$$

$$\sum_{\{\mathcal{C}_1\ldots\mathcal{C}_n\,:\,\partial\mathcal{C}_j=\phi\}} \prod_j (tgh\lambda)^{|\mathcal{C}_j|} \prod_{\langle xy\rangle\in\mathcal{C}} \tau_{\langle xy\rangle}$$

where the condition $\partial\mathcal{C}_j=\phi$ is due to the gauge invariance of the model and arises summing over the field σ.
One clearly sees that the constraint $\tau_{\partial p}\omega_p=-1$ on $\dot{\mathcal{C}}$ gives

$$\prod_{\langle xy\rangle\in\mathcal{C}} \tau_{\langle xy\rangle} = (-1)^{n(\mathcal{C},\,\dot{\mathcal{C}}\Delta\omega^*)}$$

where n is the number of intersections of $\mathcal{C}$ and $\dot{\mathcal{C}}\Delta\omega$.
Let us now decompose all these curves into clusters $C=\{\dot{\mathcal{C}}_1\ldots\dot{\mathcal{C}}_m,\ \mathcal{C}_1\ldots\mathcal{C}_n\}$ such that $n(C,C')=0$
i.e. they do not intersect.

Besides the clusters C there is also a distinguished cluster C^{ω} which contains $\dot{\mathcal{E}}^{\omega}$.
The term $(-1)^{n(\mathcal{E},\dot{\mathcal{E}}\Delta\omega^*)}$ then factorizes into

$$\prod_e (-1)^{n(C_e\cap\mathcal{E},\, C_e\cap\dot{\mathcal{E}})} \cdot (-1)^{n(C^{\omega}\cap\mathcal{E},\, C^{\omega}\Delta\omega^*\cap\dot{\mathcal{E}}^{\omega}\Delta\omega^*)}$$

Defining the cluster activity by

$$Z(C) = \sum_{\{\dot{\mathcal{E}}_i,\mathcal{E}_j\}\in C} \prod_i e^{-2\beta|\dot{\mathcal{E}}_i|} \prod_j (tgh\,\lambda)^{|\mathcal{E}_j|} (-1)^{n(C\cap\mathcal{E},\, C\cap\dot{\mathcal{E}})}$$

$$Z(C^{\omega}) = \sum_{\{\dot{\mathcal{E}}_i,\,\mathcal{E}_j,\,\mathcal{E}^{\omega}\}\in C^{\omega}} \prod_i e^{-2\beta|\dot{\mathcal{E}}_i|} \prod_j (tgh\,\lambda)^{|\mathcal{E}_j|} \cdot e^{-2\beta|\dot{\mathcal{E}}^{\omega}|} \cdot (-1)^{n(C^{\omega}\cap\mathcal{E},\, C^{\omega}\Delta\omega^*\cap\dot{\mathcal{E}}^{\omega}\Delta\omega^*)}$$

we have the expansion

$$Z_{\Lambda}(\omega) = \sum_{\substack{\{C_e,\, C^{\omega}\}\\ \text{disjoint}}} \prod_e Z(C_e)\; Z(C^{\omega}) \tag{8.6}$$

and for $\beta \gg \ln(tgh\,\lambda)^{-1}$

$$|Z(C)| \leq O(tgh\,\lambda)^{|C|}$$

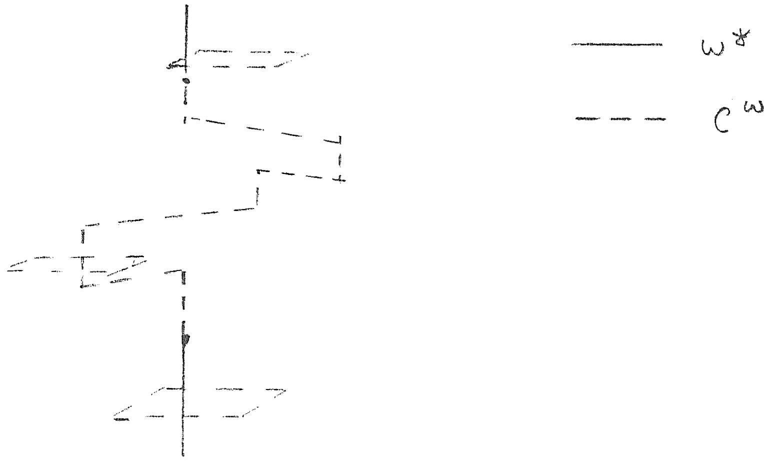
$$|Z(C^{\omega})| \leq O(tgh\,\lambda)^{|C^{\omega}|}\, e^{-[2\beta + O(\ln tgh\,\lambda)]|\dot{\mathcal{E}}^{\omega}|}$$

For ω of finite support the convergence of the expansion

$$\frac{Z_{\Lambda}(\omega)}{Z_{\Lambda}} = \sum_{\underset{\sim}{C},\, C^{\omega}\subset\Lambda} \prod_{C\in\underset{\sim}{C}} Z(C)\, Z(C^{\omega})\, \frac{\varphi_T(\underset{\sim}{C}, C^{\omega})}{|\underset{\sim}{C}|!} \tag{8.7}$$

follows.

However, this is not enough for our purposes, since in the thermodynamic limit, our ω will acquire an infinitely extended support. It is hard then to see, with this expansion (8.7) applied to numerator and denominator of (8.4), cancellations between numerator and denominator, which both tend to zero, since there is an infinite number of curves of arbitrary length linked to $\omega^* \triangle \mathscr{C}^\omega$ in the thermodynamic limit.



To improve the situation, the idea is to use an exponentiation in terms of excitations which exhibits explicitly this cancellation.

To make things easier, we discuss just the simple case of $S_{2,0}(x+0-)$ but the general proof of the existence of the thermodynamic limit of correlation functions and clustering would then easily follow.

As a result, we also obtain an estimate in the mass of the soliton and the upper gap in the two point soliton function. Since this last is larger than zero, one has shown that $S(x)\Omega$ is a one particle state, i.e. the soliton is a lattice particle although of infinitely extended support.

This is in agreement with the discussion of Buchholz and Fredenhangen [3] on the locality property of a charged one-particle state in a massive theory: it must be localized at most on a string like region.

So let us start by considering (8.5) with

ω_1^* having support on $\gamma^+(x) \cup \gamma^-(0)$, $x=(t,\vec{x})$

and ω_2^* having support on $\gamma^+(0) \cup \gamma^-(0)$

Then,

$$S_2(x+0-) = \lim_{\Lambda \uparrow Z^3} \frac{\sum_{\underset{\sim}{\mathcal{C}}\, C^{\omega_1}} \prod_{C \in \underset{\sim}{\mathcal{C}}} Z(C)\, Z(C^{\omega_1})\, \frac{\varphi_T(\underset{\sim}{\mathcal{C}}, C^{\omega_1})}{|\underset{\sim}{\mathcal{C}}|!}}{\sum_{\underset{\sim}{\mathcal{C}}\, C^{\omega_2}} \prod_{C \in \underset{\sim}{\mathcal{C}}} Z(C)\, Z(C^{\omega_2})\, \frac{\varphi_T(\underset{\sim}{\mathcal{C}}, C^{\omega_2})}{|\underset{\sim}{\mathcal{C}}|!}} \tag{8.8i}$$

Let us now identify in $\{\underset{\sim}{\mathcal{C}}, C^{\omega}\}$ the regular bonds as those in the dual lattice which are parallel to the time axis, and such that there exist no other links in $\underset{\sim}{\mathcal{C}} \cup C^{\omega} \cup \omega^*$ having time projection on them.
If we omit all regular bonds, we are left with a set of disjoint excitations which we divide into two classes

1) particle excitations $\mathcal{E}$: $\quad \pi(\mathcal{E}) \cap [0,t] \neq \phi$

2) string excitations $\mathcal{E}^s$: $\quad \pi(\mathcal{E}^s) \subset \pi(\omega^*)$

Only string excitations appear in the denominator.
We now assign to the excitations an activity defined by

$$\zeta(\mathcal{E}, \vec{K}) = e^{2\beta\,\pi(\mathcal{E})}\, \frac{\varphi_T(\underset{\sim}{\mathcal{C}}(\mathcal{E}), C^{\omega}(\mathcal{E}))}{|\underset{\sim}{\mathcal{C}}(\mathcal{E})|!}\,.$$

$$\cdot \prod_{C \in \underset{\sim}{\mathcal{C}}(\mathcal{E})} Z(C)\quad Z(C^{\omega}(\mathcal{E}))\quad e^{i\vec{K}\cdot\vec{\eta}(\mathcal{E})}$$

and

$$\zeta(\mathcal{E}^s) = \frac{\varphi_T(\underset{\sim}{\mathcal{C}}(\mathcal{E}^s), C^{\omega}(\mathcal{E}^s))}{|\underset{\sim}{\mathcal{C}}(\mathcal{E}^s)|!} \prod_{C \in \underset{\sim}{\mathcal{C}}(\mathcal{E}^s)} Z(C)\, Z(C^{\omega}(\mathcal{E}^s))$$

and using the techniques of [17] we can write

$$S_2(x+\ 0-) = e^{-2\beta t}\ \frac{1}{(2\pi)^2}\int d^2k\ e^{i\vec{k}\cdot\vec{x}}\ .$$

$$\frac{\displaystyle\sum_{\substack{\{\varepsilon_1 \dots \varepsilon_n,\ \varepsilon_1^s \dots \varepsilon_m^s\}\\ \text{relative to } \omega_1}} \prod_{i=1}^{n} \zeta(\varepsilon_i, \vec{k}) \prod_{j=1}^{m} \zeta(\varepsilon_j^s)}{\displaystyle\sum_{\substack{\{\varepsilon_1 \dots \varepsilon_m^s\}\\ \text{relative to } \omega_2}} \prod_{j=1}^{m} \zeta(\varepsilon_j^s)} \tag{8.8ii}$$

From the estimate (8.6) we see that the expansions in (8.8i) converge for large $\beta >> \ln(g\lambda)^{-1}$ and for finite lattice, so that we can exponentiate obtaining

$$S_2(x+\ 0-) = \lim_{\Lambda \uparrow \mathbb{Z}^3} \frac{1}{(2\pi)^2}\int d^2k\ e^{i\vec{k}\cdot\vec{x}}\ .$$

$$\exp\left\{\left[\sum_{\substack{\underset{\sim}{\varepsilon},\ \underset{\sim}{\varepsilon}^s \subset \Lambda\\ \text{relative to } \omega_1}} \frac{\varphi_T(\underset{\sim}{\varepsilon}, \underset{\sim}{\varepsilon}^s)}{|\underset{\sim}{\varepsilon}|!\ |\underset{\sim}{\varepsilon}^s|!} \prod_{\varepsilon \in \underset{\sim}{\varepsilon}} \zeta(\varepsilon, \vec{k}) \prod_{\varepsilon \in \underset{\sim}{\varepsilon}^s} \zeta(\varepsilon^s)\right] - \right. \tag{8.9}$$

$$\left. - \left[\sum_{\substack{\underset{\sim}{\varepsilon}^s \subset \Lambda\\ \text{relative to } \omega_2}} \frac{\varphi_T(\underset{\sim}{\varepsilon}^s)}{|\underset{\sim}{\varepsilon}^s|!} \prod_{\varepsilon^s \in \underset{\sim}{\varepsilon}^s} \zeta(\varepsilon^s)\right]\right\}$$

We now split the first sum in the exponential into four terms:

1) $\underset{\sim}{\varepsilon}\ :\quad t^-(\underset{\sim}{\varepsilon}) \in [0,t) \qquad t^+(\underset{\sim}{\varepsilon}) \in (0,t]$

$\underset{\sim}{\varepsilon}^s = \phi$

2) $\underset{\sim}{\varepsilon}, \underset{\sim}{\varepsilon}^s :\quad t^-(\underset{\sim}{\varepsilon} \cup \underset{\sim}{\varepsilon}^s) \le 0 \qquad t^+(\underset{\sim}{\varepsilon} \cup \underset{\sim}{\varepsilon}^s) \ge t$

3) $\underset{\sim}{\varepsilon}, \underset{\sim}{\varepsilon}^s :\quad t^-(\underset{\sim}{\varepsilon} \cup \underset{\sim}{\varepsilon}^s) > 0 \qquad t^+(\underset{\sim}{\varepsilon} \cup \underset{\sim}{\varepsilon}^s) > t$

$\qquad\qquad t^-(\underset{\sim}{\varepsilon} \cup \underset{\sim}{\varepsilon}^s) < 0 \qquad t^+(\underset{\sim}{\varepsilon} \cup \underset{\sim}{\varepsilon}^s) < t$

4) $\underset{\sim}{\mathcal{E}}^{\circ}, \underset{\sim}{\mathcal{E}} = \phi$

It is now rather clear that the sum of the fourth term and the term relative to ω_2 cancel all the contributions from clusters $\underset{\sim}{\mathcal{E}}$ such that do not touch $[0,t]$.

Then, the existence of the thermodynamic limit follows easily. In fact in the sum over excitations, all the contributions which come from an enlargement of the lattice involve now clusters of excitations $\underset{\sim}{\mathcal{E}}$, which connect the boundary of the lattice to the line $0 \to (t,\vec{x})$, so that by bounds (8.6) they decrease exponentially fast as Λ increases.

Clustering in time direction follows easily since as $t \nearrow \infty$ the term $e^{-2\beta t}$ in eq. (8.9) tends to zero (and the excitation expansion converges and, therefore, is uniformly bounded).

Clustering in the space direction follows from the fact that the sum of the heights of excitations, $\vec{h}(\varepsilon)$, must be equal to $\vec{x}$, but again from expansion convergence we can extract for large $|\vec{x}|$ a term $(tg h \lambda)^{|\vec{x}|}$ which gives the exponential clustering.

Finally, the particle analysis method still applies since we have on the $\mathcal{H}_1$ soliton sector a contractive representation of positive time translations and a unitary representation of the space translation. Therefore from

$$\sum_{\vec{x}} \langle S(\vec{0})\Omega, T(t)U(\vec{x})S(\vec{0})\Omega \rangle =$$

$$= \sum_{\vec{x}} S_{2,0}((t,\vec{x})+0-) \underset{t\to\infty}{\sim} e^{c} e^{-mt}(1+e^{-\mu t})$$

where m is the mass of the soliton and μ the upper gap in the two point function, we obtain

$$m = 2\beta + O(\ln tgh \lambda)$$

$$\mu = O(\ln tgh \lambda)$$

Let us now come back to the general ($\mathbb{Z}_N$ Higgs)$_3$ models and state

Theorem 8.1 The $\mathbb{Z}_N$ gauge model with $\mathbb{Z}_N$ Higgs field defined in eq. (8.1) in d=3 for large β and small λ possesses superselection sectors $\mathcal{H}_q$ labelled by $q \in \mathbb{Z}_N \simeq \{0 \ldots N-1\}$ and sectors $\mathcal{H}_q$, $q \neq 0$ are soliton sectors.
The soliton with charge q has support (for $\lambda \neq 0$) on an infinitely extended timelike string. It is a massive (lattice) particle with mass

$$m_q = \beta\left(1 - \cos\frac{2\pi}{N}q\right) + O(\ln\lambda)$$

Proof The proof of this theorem can be obtained using the low temperature expansion of Theorem 3.19 in [10].

9. SOLITON WITH COULOMB FIELD

The models discussed up to now have all a massive spectrum.
There is, however, a well known class of topological solitons associated to a massless theory: the monopoles.
Due to the absence of the mass gap in the spectrum, they are associated to a Coulomb (-like) field which, in a sense, makes them even less local than the stringlike solitons of the previous section.
There are two main examples of monopoles.
1) The Dirac monopole in the $U(1)_4$ gauge model in the Q.E.D. phase;
2) the t'Hooft-Polyakov monopole [29] in the $(SU(2)$ Georgi-Glashow$)_4$ model in the QED phase.
The last model can be described on the lattice by means of a SU(2) gauge field $g_{\langle xy \rangle}$ and a Higgs field ϕ_x in the spin 1 representation of SU(2) by the action

$$S(g,\phi) = \beta \sum_p \mathrm{Re}\, \chi(g_{\partial p}) + \lambda \sum_{\langle xy \rangle} (\phi_x, U_1(g_{\langle xy \rangle}) \phi_y)$$

where χ is the fundamental character of SU(2), and U_1 is the spin 1 representation of SU(2).
There are two (related) reasons to believe that an understanding of the Dirac monopole gives an essential intuition also for the t'Hooft-Polyakov one:
1) the large distance behaviour of the gauge field strength configurations of both monopoles coincides;
2) the $U(1)_4$ gauge model can be obtained as $\lambda \nearrow \infty$ limit of the Georgi-Glashow model, as shown in the expected phase diagram:

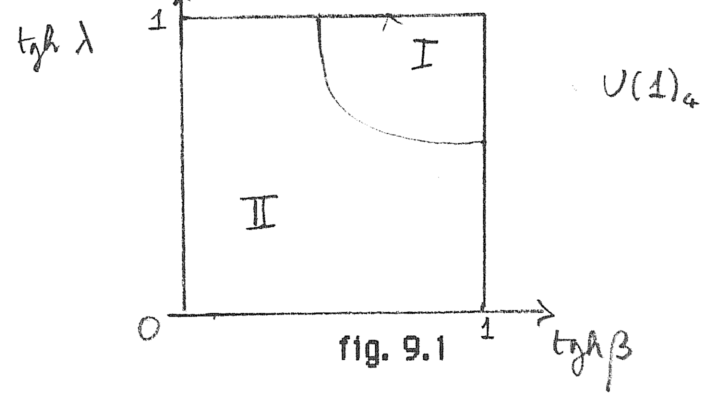


fig. 9.1

where in region I we have QED phase.

In this section we explicitly discuss the $U(1)_4$ model. Also in this simpler model, however, the particle structure analysis of the monopole sector cannot be completed, since we still lack a cluster expansion for $U(1)_4$ in the QED phase.

To analyze the $U(1)_4$ gauge model we only have the expansion in (renormalized) magnetic loops of Fröhlich and Spencer [30,31] and we must extract information from it.

To understand how to introduce the monopole in the theory, let us briefly recall two representations of the partition function of $U(1)_4$in the Villain form which explicitly show the monopole loops "analogue" of the Peierls contours in ϕ^4_2.

The first method is essentially based on the intuition that the monopole arises from the constraint on a $\mathbb{R}$ -valued gauge field, which turns it into a U(1) valued field.

The second method exploits the duality transformation which maps $U(1)_4$ into a $\mathbb{Z}$ -gauge field theory, then the monopole arises from the constraint which imposes to the dual $\mathbb{R}$ gauge field to be integer valued.

The $U(1)_4$model in the Villain form is described by the thermodynamic limit of the state

$$\langle (\cdot) \rangle_\Lambda = \frac{\int \prod_{\langle xy \rangle \in \Lambda} d\theta_{\langle xy \rangle} \sum_{n} \prod_{p \in \Lambda} e^{-\frac{\beta}{2}(d\theta_p + 2\pi n_p)^2} (\cdot)}{\int \prod_{\langle xy \rangle \in \Lambda} d\theta_{\langle xy \rangle} \sum_{n} \prod_{p \in \Lambda} e^{-\frac{\beta}{2}(d\theta + 2\pi n_p)^2}} \quad (9.1)$$

where $d\theta_{\langle xy \rangle}$ is the Haar measure on U(1), $n = \{n_p \in \mathbb{Z} : p \in \Lambda\}$

We now rewrite $\sum_n$ as $\sum_{m : dm = 0} \; \sum_{n' \, dn = m}$

and, using the techniques of [33, 34], we obtain

$$Z_\Lambda = \sum_{\substack{m:dm=0\\ m\subset\Lambda}} Z(A)\, e^{-2\pi^2\beta\,(m\,\Delta_\Lambda^{-1} m)} \tag{9.2}$$

$$Z(A) = \int_{-\infty}^{+\infty} \prod_{\langle xy\rangle\in\Lambda} dA_{\langle xy\rangle} \prod_{p\subset\Lambda} e^{-\frac{1}{2\beta}(dA_p)^2}$$

where Δ_Λ is the lattice laplacian with free bd conditions on $\partial\Lambda$ and $dA_{\langle xy\rangle}$ is the Lesbesgue measure on $\mathbb{R}$.(+)
The third rank field m , satisfying $dm=0$, has support in the dual of a set of loops: these are the monopole loops we are searching for.
From eq. (9.2) one obtains that the monopole loops in the thermodynamic limit interact via a Coulomb-like potential $\Delta^{-1}\equiv(\lim_{\Lambda\uparrow\mathbb{Z}^4}\Delta_\Lambda^{-1})$ so it is natural to expect that if one such closed contours is "opened" , at the end of the broken contour, for the conservation of the magnetic flux, we will have a Coulomb-like field.
On this line one is tempted to say that the two-point function, for a monopole $S_2(x,y)$, could be written, by analogy with (6.1), introducing a suitable external 3-field $\tilde\omega$ with

$$\mathrm{supp}\,(d\tilde\omega)^* = \{x,y\}$$

as (see also Fig. 9.2)

(+) If α , β are rank k fields with support in Λ

$$(\alpha,\beta)\equiv\sum_{c_k\in\Lambda}\alpha(c_k)\,\beta(c_k)$$

In particular, if η is a k+1 rank field, so that $\delta\eta = {}^*d^*\eta$ is a k field,

$$(\delta\eta,\beta)=(\eta,d\beta)$$

$$\frac{\displaystyle\sum_{m:\,dm=-d\tilde{\omega}} e^{-2\pi^2\beta(m+\tilde{\omega},\,\Delta^{-1}m+\tilde{\omega})}}{\displaystyle\sum_{m:\,dm=0} e^{-2\pi^2\beta(m,\,\Delta^{-1}m)}} \tag{9.3}$$

where $\tilde{\omega}$ takes care of the Coulomb field at the boundary of the connected open component of supp$(m)^*$ in the numerator.

If we leave this magnetic flux to spread out in the whole $\mathbb{Z}^4$, we hardly obtain O.S. positivity, so a natural choice is to leave it to spread only in the hyperplane of fixed time coordinate corresponding to x^0 and y^0.

Although such external $\tilde{\omega}$ field could be directly introduced in the (9.1) state, it is easier to understand its form by a duality argument.

By duality, in fact, a magnetic monopole with the Coulomb-like magnetic flux in the $U(1)_4$ model should behave like a charge in the dual model with the associated electric Coulomb field.

Therefore, one is naturally led to consider the second method of representing the U(1) model.

It is well known that the dual of the $U(1)_4$ model in the Villain form is a theory described by a $\mathbb{Z}$-valued gauge field α with the measure

$$\frac{1}{Z_\Lambda}\sum_{[\alpha]}\prod_{p\in\Lambda^*} e^{-\frac{1}{2\beta}(d\alpha_p)^2}(\cdot) \tag{9.4}$$

where $[\alpha]$ denotes the equivalence class

$$[\alpha]=\{\alpha' : d\alpha' = d\alpha\}$$

This model can be obtained as the $\lambda\nearrow\infty$ limit of the U(1) Higgs model described by the action

$$S(A,\phi)=\sum_{p\in\Lambda^*}\frac{1}{2}(dA_p)^2+\lambda\sum_{\langle xy\rangle\in\Lambda^*}\cos 2\pi\left(A_{\langle xy\rangle}+d\phi_{\langle xy\rangle}\right)$$

where $A_{\langle xy\rangle} \in \mathbb{R}$ and $\phi_x \in U(1) \simeq \mathbb{R}/\mathbb{Z}$.
This limit exists by Griffith inequalities and imposes the constraint that A is integer valued.
More precisely, the relation between the Villain U(1) gauge model and the U(1) Higgs model goes as follows.
Let $d\mu_\Lambda(A)$ denote the gaussian measure on the space of equivalence classes

$$[A] = \{A' : dA' = dA\}$$

determined by

$$\int d\mu_\Lambda(A)\, e^{i(A,\mu)} = e^{-\frac{\beta}{2}(\mu,\, \Delta_\Lambda^{-1}\mu)} \quad ; \delta\mu = 0$$

where Δ_Λ is the lattice laplacian with 0-Dirichlet condition on the outer boundary of Λ^*.
Using the Fourier expansion, write

$$\frac{e^{\lambda \cos 2\pi (A+d\phi)}}{\int_0^1 e^{\lambda \cos 2\pi A}\, dA} = \sum_{n=-\infty}^{+\infty} \zeta(n,\lambda)\, e^{i 2\pi n (A+d\phi)}$$

$$\zeta(n,\lambda) \underset{\lambda \nearrow \infty}{\searrow} 1$$

Now, one can express the expectation value on the Gibbs state of $U(1)_4$ as

$$\langle (\cdot) \rangle = \lim_{\Lambda \nearrow \mathbb{Z}^3} \frac{1}{Z_\Lambda} \sum_{\substack{n_{\langle xy\rangle} \in \mathbb{Z} \\ \langle xy\rangle \in \Lambda^*}} \int \prod_{x \in \Lambda^*} d\phi_x \,\cdot$$

$$\cdot\, d\mu_\Lambda(A) \quad e^{i 2\pi n_{\langle xy\rangle}(A_{\langle xy\rangle} + d\phi_{\langle xy\rangle})} \quad (\cdot) \tag{9.5}$$

In particular, for the partition function we obtain

$$Z_\Lambda \sim \sum_{n_{\langle xy\rangle}:\langle xy\rangle\in\Lambda^*} \int \prod_{x\in\Lambda^*} d\phi_x \, d\mu_\Lambda(A) \prod_{\langle xy\rangle\in\Lambda^*} e^{i\,2\pi\, n_{\langle xy\rangle}(A_{\langle xy\rangle}+d\phi_{\langle xy\rangle})} =$$

$$= \sum_{n;\,\delta n=0} \int d\mu_\Lambda(A)\, e^{i\,2\pi\,(n,A)} = \sum_{[\alpha]} \prod_{p\in\Lambda^*} e^{-\frac{1}{2\beta}(d\alpha_p)^2}$$

and one easily recognizes in the n field the monopole loops $(\delta n=0)$.
It is now easy to introduce in the dual model a charged field ϕ with its Coulomb tail spreading in a fixed time plane. Via eq. (9.5) its two point function will give in the $U(1)_4$ model the desired two point-function for the monopole.
Denote by $\Delta_\Lambda(t)$ the restriction of the lattice laplacian Δ_Λ to the fixed time t sublattice and consider the two points in the dual lattice $x_i=(t_i,\vec{x}_i)$, $i=1,2$.
An electric Coulomb field associated to a charge $q_1=+1$ in x_1 and a charge $q_2=-1$ in x_2 and spreading in the t_1 and t_2 plane, respectively, is described by

$$E_\Lambda(x_1 q_1, x_2 q_2) = d\left(\Delta_\Lambda^{-1}(t_1)\,\delta_{x_1} - \Delta_\Lambda^{-1}(t_2)\,\delta_{x_2}\right) \tag{9.6}$$

where d is here the 3-dimensional lattice differential, and δ_x is a rank 0 field which is 1 in x and zero otherwise.
Therefore, the corresponding charged field is described by

$$\psi_\Lambda(x_1 q_1 x_2 q_2) = e^{i\,2\pi\,\phi(x_1)}\, e^{i\,2\pi(A,E_\Lambda)}\, e^{-i\,2\pi\,\phi(x_2)} \tag{9.7}$$

which is clearly gauge invariant, since

$$\phi(x) \longrightarrow \phi(x) - \lambda(x)$$
$$A_{\langle xy \rangle} \longrightarrow A_{\langle xy \rangle} + d\lambda_{\langle xy \rangle}$$

implies

$$(A, E_\Lambda) \longrightarrow (A, E_\Lambda) + (\lambda, \delta E_\Lambda) = (A, E_\Lambda) + \lambda(x_1) - \lambda(x_2) \,.$$

Let now γ be a directed curve from x_2 to x_1, then (still denoting by γ the rank 1 field $\gamma_{\langle xy \rangle} = 1$
$= 0$ otherwise)

$$\delta(E_\Lambda - \gamma) = 0$$
$$(d\phi, \gamma) = \phi(x_1) - \phi(x_2) \tag{9.8}$$

Inserting (9.7) and (9.8) in the right hand side of (9.5) we obtain

$$\langle \psi(x_1 q_1\, x_2 q_2) \rangle = \lim_{\Lambda \uparrow \mathbb{Z}^4} \frac{1}{Z_\Lambda} \sum_n \int \prod_{x \in \Lambda} d\phi_x \, d\mu_\Lambda(A)$$
$$\cdot\, e^{i 2\pi [(n + E_\Lambda, A) + (n + \gamma, d\phi)]} =$$
$$= \lim_{\Lambda \uparrow \mathbb{Z}^4} \frac{\sum_{n;\, \delta n = \delta\gamma} \int d\mu_\Lambda(A)\; e^{i 2\pi (n + E_\Lambda, A)}}{\sum_{n;\, \delta n = 0} \int d\mu_\Lambda(A)\; e^{i 2\pi (n, A)}} =$$
$$= \lim_{\Lambda \uparrow \mathbb{Z}^4} \frac{\sum_{n;\, \delta n = -\delta E} e^{-2\pi^2 \beta (n + E_\Lambda, \Delta_\Lambda^{-1} n + E_\Lambda)}}{\sum_{n;\, \delta n = 0} e^{-2\pi^2 \beta (n, \Delta_\Lambda^{-1} n)}} \,.$$

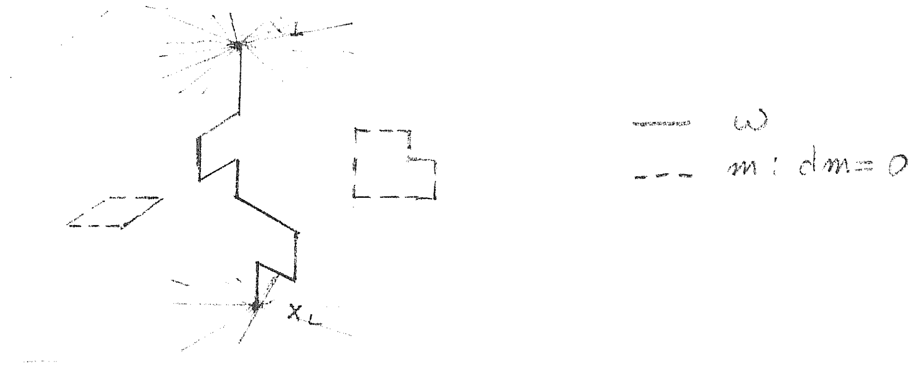
By comparison with (9.3) we see that we have found our external field $\tilde{\omega}$:

formally $\quad \tilde{\omega} \equiv \tilde{\omega}(x_1 q_1\, x_2 q_2) = E^*(x_1 q_1 x_2 q_2) \equiv E^*$

$$\tilde{\omega} = \lim_{\Lambda \uparrow \mathbb{Z}^4} \tilde{\omega}_\Lambda \qquad \tilde{\omega}_\Lambda = (E_\Lambda)^* \tag{9.9}$$

and the two-point function of the monopole should be

$$S_2(x_1 q_1\ x_2 q_2) = \lim_{\Lambda \nearrow \mathbb{Z}^4} \frac{\sum\limits_{m: dm = -d\tilde{\omega}} e^{-2\pi^2\beta (m+\tilde{\omega}_\Lambda, \Delta_\Lambda^{-1} m+\tilde{\omega}_\Lambda)}}{\sum\limits_{m: dm=0} e^{-2\pi^2\beta (m, \Delta_\Lambda^{-1} m)}} \tag{9.10}$$



(one dimension is suppressed)

fig. 9.2

Let us call

$$\omega_\Lambda = \omega_\Lambda(\gamma) = (\xi_\Lambda - \gamma)^* = \tilde{\omega}_\Lambda - \gamma^* \tag{9.11}$$

then it is easy to see that we can rewrite (9.10) as

$$S_2(x_1 q_1\ x_2 q_2) = \lim_{\Lambda \nearrow \mathbb{Z}^4} \frac{\sum\limits_{m: dm=0} e^{-2\pi^2\beta (m+\omega_\Lambda, \Delta_\Lambda^{-1} m+\omega_\Lambda)}}{\sum\limits_{m: dm=0} e^{-2\pi^2\beta (m, \Delta_\Lambda^{-1} m)}}$$

We check now that it is non vanishing. In fact by Jensen inequality

$$S_2(x_1 q_1\ x_2 q_2) \geq e^{-2\pi^2\beta (\omega, \Delta^{-1}\omega)} e^{-4\pi^2\beta \langle m, \Delta^{-1}\omega \rangle_M}$$

where

$$\langle (\cdot) \rangle_M = \frac{\sum\limits_{m: dm=0} e^{-2\pi^2\beta (m, \Delta^{-1} m)} (\cdot)}{\sum\limits_{m: dm=0} e^{-2\pi^2\beta (m, \Delta^{-1} m)}}$$

Since the measure $\langle\ \rangle_\mu$ is even in m, $\langle(m,\Delta^{-1}\omega)\rangle_\mu=0$ and, moreover, it is easy to show (see Appendix 9.A) that (choose for simplicity $\mathrm{supp}\,\tilde\omega\cap\gamma=$ $=\{x_1\ x_2\}$)

$$(\omega,\Delta^{-1}\omega)=(\tilde\omega,\Delta^{-1}\tilde\omega)+(\gamma^\dagger\Delta^{-1}\gamma^\dagger)=(E,\Delta^{-1}E)+(\gamma,\Delta^{-1}\gamma)<\infty$$

The fact that it is, really, a good two-point function for the monopole will be proved later.
Now let us remark that, since E represents the Coulomb electric field associated to a charge and, by (9.10), $\tilde\omega$ represents the magnetic flux associated to the monopole, eq. (9.9) is a form of the well known duality between magnetic and electric fields.
The oriented curve introduced in (9.8) can be seen in the U(1) model as a representation of the Dirac string of integer magnetic flux, whose introduction gives a conservation of the magnetic flux; $d\omega=d(\tilde\omega-\gamma^\dagger)=0$.
Let us now see how the field ω appears in the original Villain form of $U(1)_4$.
Using the fact that $(E-\gamma)$ satisfies condition (9.8), by the Harmonic decomposition

$$E-\gamma=\delta\Delta_\Lambda^{-1}d(E-\gamma)$$

so that

$$(A,E-\gamma)=(dA,\Delta_\Lambda^{-1}d(E-\gamma))$$

Then, using also (9.11), we have

$$\sum_n\int\prod_{x\in\Lambda^*}d\phi_x\ d\mu_\Lambda(A)\ e^{i2\pi(E+n,A)}\,e^{i2\pi(n+\gamma,d\phi)}=$$

$$=\sum_{n;\delta n=0}\int d\mu_\Lambda(A)\ e^{i2\pi(E-\gamma,A)}\ e^{i2\pi(n,A)}= \qquad (9.12)$$

$$=\sum_{n;\delta n=0}\int d\mu_\Lambda(A)\ e^{i2\pi(\Delta_\Lambda^{-1}d(E-\gamma),dA)}\ e^{i2\pi(n,A)}\sim$$

$$\sim\sum_{[\alpha]}\prod_{p\in\Lambda^*}e^{-\frac{1}{2\beta}(d\alpha)_p^2}\ e^{i2\pi(\Delta_\Lambda^{-1}d(E-\gamma),d\alpha)}\sim$$

$$\sim\sum_n\int\prod_{\langle xy\rangle\in\Lambda}d\theta_{\langle xy\rangle}\prod_{p\in\Lambda}e^{-\frac{\beta}{2}(d\theta+2\pi n+2\pi\Delta_\Lambda^{-1}\delta\omega)}$$

It is evident that this expression is invariant under

$$\omega \longrightarrow \omega + dn \qquad , \quad n_p \in \mathbb{Z} \tag{9.13i}$$

by the Harmonic decomposition:

$$\begin{cases} \Delta_\Lambda^{-1} \delta\omega \longrightarrow \Delta_\Lambda^{-1} \delta\omega + \Delta_\Lambda^{-1} \delta d n = \\ \qquad\qquad = \Delta_\Lambda^{-1} \delta\omega + n - d\Delta_\Lambda^{-1} \delta n \\ \theta \longrightarrow \theta + 2\pi \Delta_\Lambda^{-1} \delta n \end{cases} \tag{9.13ii}$$

leaves (9.12) unchanged.

This invariance corresponds to the possibility of freely moving the Dirac string γ^* of the monopole [as one can see from (9.11)] and it is similar to the invariance under the choice of the external field ω, for fixed $d\omega$, in the case of local solitons. What is really changed here is that, besides the source term, $d\gamma^*$, we have now a Coulomb like field $\tilde{\omega}$, as anticipated in the introduction.

Like in the theories with local solitons, we can introduce ordinary fields together with the monopole, preserving the invariance of the expectation value w.r.t. the shifts of the Dirac strings.

For example, we can take the ω dependent field

$$\psi_p(\omega) = e^{ie\theta_{\partial p} + (\delta\Delta_\Lambda^{-1}\omega)_p} \tag{9.14}$$

or its non-local version, corresponding to the Wilson loop

$$W_S(C) = \prod_{\langle xy\rangle \in C} e^{i\theta_{\langle xy\rangle}} \prod_{p\in S:\, \partial S = C} e^{i2\pi(\delta\Delta^{-1}\omega)_p} \tag{9.15}$$

To joint correlation functions of such ordinary fields and monopoles we apply the Reconstruction Theorem.
Let us start by noticing that the electric field, generated in the dual (U(1) Higgs)$_4$ model by n-charges $\{x_1 q_1 \dots x_n q_n\}$, is clearly given by

$$E(x_1 q_1 \dots x_n q_n) = \sum_{i=1}^{n} E(x_i q_i) \tag{9.16}$$

Equation (9.16) expresses the superposition principle.
If $\sum_i q_i = 0$, then there exists a $\mathbb{Z}$-valued rank 1 field z such that

$$\delta\left(E(x_1 q_1 \dots x_n q_n) - z\right) = 0 \ . \tag{9.17}$$

Let

$$\omega_\Lambda = \omega_\Lambda(x_1 q_1 \dots x_n q_n) = \left(E_\Lambda(x_1 q_1 \dots x_n q_n) - \gamma\right)^*$$

Define the correlation functions $S_{n,m}$ of monopoles and ψ_p fields [introduced in eq. (9.14)] by

$$S_{n,m}(x_1 q_1 \dots x_n q_n ; p_1 \dots p_m) =$$

$$= \lim_{\Lambda \uparrow \mathbb{Z}^4} \frac{1}{Z_\Lambda} \sum_n \int \prod_{\langle xy \rangle \in \Lambda} d\theta_{\langle xy \rangle} \prod_{p \in \Lambda} e^{-\frac{\beta}{2}(d\theta + 2\pi n + \delta \Delta_\Lambda^{-1} \omega_\Lambda)_p^2} \prod_{j=1} \psi_{p_j}(\omega_\Lambda) \ . \tag{9.18}$$

From the invariance under the (9.13) transformations, it follows that these correlation functions do not depend on the choice of z satisfying (9.17).

Finally, for $\sum_i q_i = q = 0$ we define

$$S_{n,m}(x_1 q_1 \dots x_n q_n \,;\, p_1 \dots p_m) =$$

$$= \lim_{y \to \infty} S_{n+1,m}(x_1 q_1 \dots x_n q_n \; y \, -q \,;\, p_1 \dots p_m) \tag{9.19}$$

In the dual representation in terms of A and ϕ the Ostewalder Schrader positivity (where $\theta: (x,q) \to (\tau x, -q)$) of the correlation functions (9.18) and (9.19) is evident since the measure in (A, ϕ) is O.S. positive, and the E field spreads out in fixed time planes.
Since $[\Delta^{-1}(t)](x,y)$ (resp $\Delta^{-1}(x,y)$) vanishes when $y \to \infty$ as $|x-y|^{-1}$ (resp $|x-y|^{-2}$), and the $U(1)_4$ measure is translation invariant, the translation invariance of the correlation functions follows.

Therefore, the Reconstruction Theorem applies to the correlation functions $\{S_{n,m}\}_{n,m=0}^{\infty}$ defined by eq. (9.18), (9.19) and we obtain a Lattice Quantum Mechanics ($\mathcal{H}$, Ω, T, U).
Now if the cluster property holds also w.r.t. the x 's (this implies in particular from (9.19) that all correlation with non null total charge vanish), then $\mathcal{H}$ decomposes into $\mathbb{Z}$ sectors labelled by the total monopole charge q and the sectors with $q \neq 0$ are the monopole (soliton) sectors.

Using the Kennedy-King techniques [42] applied to the dual of the two-point function in terms of A and ϕ, one easily finds, by duality:
For small enough β,

$$S_2(x\,q \;\; y\,-q) \geq \text{const} > 0 \;, \; \forall x, y \in \mathbb{Z}^4$$

and no monopole sectors can be constructed.
This can be heuristically understood from the fact that, for small enough β and large λ in the dual (U(1) Higgs)$_4$ model, A is massive (Higgs phase) with a mass $m_A \sim \lambda\beta$.
Therefore,

$$S_2(xq\ \ y-q) \sim e^{-\frac{1}{2\beta}(E,(\Delta+m_A)^{-1}E)} \xrightarrow[\lambda\uparrow\infty]{} 1$$

In Appendix 9.B we show that:
For large enough β, the cluster property for the $S_{n,m}$ correlation functions holds. In particular, correlation functions with total charge $q\neq 0$ vanish.
This will be done using the method of [30, 32], which consists essentially in a renormalization group procedure applied to the gas of monopole loops. One renormalizes their activity in such a way that the partition function can be written as a <u>convex</u> combination of positive measures on the monopole loops, which are dilute.
This renormalization allows us to extract in $S_2(xq, y-q)$ a factor $e^{-O(\beta)q^2|x-y|}$ from the component of the monopole paths which has non vanishing boundary, given by $\{x, y\}$.
Hence, we obtain

$$|S_2(xq\ \ y-q)| \leq e^{-O(\beta)q^2|x-y|} \tag{9.20}$$

This argument can be generalized to all the $S_{n,m}$ correlation functions, and gives an upper bound of the type

$$\exp -O(\beta)|z|^2_{min}$$

where $|z|^2_{min} \equiv min\{|z|^2;\ z \text{ satisfies } (9.17)\}$.
This ensures that for large enough β, monopole sectors do exist.
Moreover, since the support of the monopole with its magnetic flux is local in time, one can construct a $s_q(\vec{x})$ monopole operator with dense domain in $\mathcal{H}$, formally defined by

$$S_q(x)\ T(x^0)\,|q_1\ x_1 \ldots ;\ p_1 \ldots p_m\rangle =$$
$$= |q\ x\ \ q_1\ \ x_1+x^0 \ldots ;\ p_1+x^0 \ldots p_m+x^0\rangle$$

For $x^0 = 0$, this operator

$$S_q(x) = S_q(\vec{x})$$

is unitary in $\mathcal{H}$ and corresponds to the creation of a monopole with its magnetic flux in the time zero plane.
It provides, as for the local solitons, an intertwiner between sectors $\mathcal{H}_{q'}$ and $\mathcal{H}_{q+q'}$. Since for $x = (0, \vec{x})$, $y = (t, \vec{y})$

$$S_2(x q \; y -q) = \langle S_q(\vec{x})\Omega, T(t)\, U(\vec{y}-\vec{x})\, S_q(\vec{x})\Omega \rangle$$

equation (9.20) shows that:
The mass of the monopole of charge q is

$$m_q = q^2 O(\beta) .$$

We collect all these results into the

Theorem 9.1 The $U(1)_4$ model for large β possesses super-selection sectors $\mathcal{H}_q$, labelled by $q \in \mathbb{Z}$, and the sectors $\mathcal{H}_q$, $q \neq 0$ are monopole (soliton) sectors.
The soliton is associated to a Coulomb-like magnetic flux which extends to infinity in a fixed time plane. The monopole with charge q has mass

$$m_q = q^2 O(\beta) .$$

Remark 9.2 Using the above defined field E directly in the U(1) Higgs model with gaussian gauge action, one can construct the charged sectors (in class 2) of the introduction) in the QED phase by means of an O.S. like reconstruction theorem.
The sequences to which one applies the theorem are, e.g.

$$S_{n,m}(x_1 q_1 \ldots x_n q_n ; C_1 \ldots C_m) =$$

$$= \langle \prod_{i=1}^{n} W(C_i) \prod_{j=1}^{m} \phi_{x_j} e^{i E(q_j x_j)} \rangle \quad \text{for} \quad \sum_j q_j = 0, \quad x_j \in \Lambda, \quad q_j \in \mathbb{Z} \setminus \{0\}$$

and

$$S_{n,m}(x_1 q_1 \ldots x_n q_n ; C_1 \ldots C_m) =$$
$$= \lim_{z \to \infty} S_{n+1,m}(x_1 q_1 \ldots x_n q_n \; z \; -q ; C_1 \ldots C_m) \quad \text{for } \sum_{j=1}^{n} q_j = q \neq 0 \; .$$

By the same arguments of Appendices 9.A and 9.B one shows that in the QED phase the reconstructed Hilbert space decomposes into orthogonal sectors labelled by the total charge $q \in \mathbb{Z}$. For $q \neq 0$ the sectors are charged sectors.
More details will appear elsewhere [43].

Remark 9.3 It is clear that the whole construction of the monopole sectors applies, if we take $\tilde{\omega}$ given by a 3-form lying in the fixed time lattice (corresponding to the time coordinate of the monopole) satisfying

$$(d\tilde{\omega})^* = \{x_i \, q_i\}$$

and $(\tilde{\omega}, \Delta^{-1}\tilde{\omega}) < \infty$.
The choice we have made for $\tilde{\omega}$ corresponds to a symmetric Coulomb-like field around the monopole, which minimizes the energy.
There are two other interesting choices.
One is to take an $\tilde{\omega}$ with support in a cone of strictly positive opening angle. The corresponding soliton operators S is then localized inside the cone.
The dual of the state $S\Omega$ in the QED phase of the U(1) Higgs model is nothing but a lattice version of the state localised in a spacelike cone, discussed in an axiomatic framework, by Buchholz, in QED theories [35].
Another interesting choice could be to take as $(\tilde{\omega})^*$ the electric field E constructed as follows.
Consider a charge moving in the continuum in a lattice direction with velocity $\vec{v} = (v, 0, 0)$ which at the time t=0 is in the position $\vec{x} = (x^1 x^2 x^3)$, $x^i \in \mathbb{Z}$.

An electric field $\vec{\mathcal{E}}(v, t)$ is associated to that charge and we take its lattice approximation at t=0 as our $E (\equiv E(v) \equiv E(x\, q, \vec{v})$.

The Fourier components are ($\vec{e}_i$ unit vector in the i-th direction)

$$E^{(v)}_{\langle x, x+\vec{e}_i \rangle}(\vec{k}) = \begin{cases} i=1 & \dfrac{(1-e^{ik_1})(1-v^2)}{[1-\cos k_1](1-v^2)+[1-\cos k_2]+[1-\cos k_3]} \\[2ex] i=2 & \dfrac{(1-e^{ik_2})}{[1-\cos k_1](1-v^2)+[1-\cos k_2]+[1-\cos k_3]} \\[2ex] i=3 & \dfrac{(1-e^{ik_3})}{[1-\cos k_1](1-v^2)+[1-\cos k_2]+[1-\cos k_3]} \end{cases}$$

An interesting fact is that the monopole sectors constructed with

$$\tilde{\omega} = E(xq, \vec{v})^{*}$$

are orthogonal to those constructed with $\tilde{\omega} = E(xq, \,)^{*}$.
In fact, whereas $(E(xq, \vec{v}; y-q, \vec{v}), \Delta^{-1} E(xq, \vec{v}; y-q, \vec{v})) < \infty$

we have that

$$(E(xq, \vec{v}; y-q, \vec{0}), \Delta^{-1} E(xq, \vec{v}; y-q, \vec{0}))$$

diverges.
This could be viewed as a lattice analogue of the superselection sectors labelled by $\vec{v} \in [-1,1]^3$ in the continuum discussed in [6] (for more general sectors see also [35]).

<u>Remark 9.4</u> It is well known that the U(1) hypergauge theories, i.e. theories with a U(1) valued rank $k > 1$ gauge field θ with e.g. Villain action, possess in $d \geq k+3$ a decoupling transition for large β [30].

With the above methods one can easily prove that in the deconfined region, for large β , the theory possesses $\mathbb{Z}$ "monopole"-like soliton sectors in $d=k+3$.

Remark 9.5 Also for monopole disorder fields, a proposition similar to proposition 5.10 can be proved.
In fact, if $D(\omega)$ denotes the disorder field associated to the monopole, then, in a general lattice,

$$\langle D(\omega)\rangle_\Lambda = \Big\langle \prod_{p\in \mathrm{supp}\,\omega} e^{-\beta\left[(d\theta+2\pi n+2\pi\delta\Delta^{-1}\omega)_p^2-(d\theta+2\pi n)_p^2\right]}\Big\rangle_\Lambda$$

depends on $\tilde{\omega}$ and $[\omega-\tilde{\omega}]\in H^{k+2}(\Lambda\setminus \mathring{S}(d(\omega-\tilde{\omega})),\mathbb{Z})$.
Here one can also remark that if Λ is of non-trivial homology, e.g. $\Lambda\simeq T^4$, the expectation value $\langle D(\omega)\rangle_\Lambda$ can be non-trivial even if $\tilde{\omega}=0$, $d\omega=0$.
This would be the case for a "large" monopole loop encyrcling the torus.

Appendix 9.A

Lemma $(\Xi, \Delta^{-1} E) < \infty$

Proof. Let $E = E(x q\, y{-}q)$, and, for simplicity, assume $\vec{x}^{\,0} = \vec{y}$, $q = 1$.

The general case requires only a few, but straightforward, calculations.
Let $\vec{e}_i$ denote the unit vector in the i-space direction , define

$$\vec{E}(z) = \left(E_{\langle z, z+\vec{e}_1 \rangle}, E_{\langle z, z+\vec{e}_2 \rangle}, E_{\langle z, z+\vec{e}_3 \rangle} \right) ; \quad E_{\langle z, z+\vec{e}_0 \rangle} = 0,$$

then

$$\vec{E}(z) = \frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} d^3k \, \frac{(e^{i\vec{k}} - 1)\, e^{i\vec{k}(\vec{z}-\vec{x})} \left[\delta(z^0 - x^0) - \delta(z^0 - y^0)\right]}{\sum_{i=1}^{3} [1 - \cos k_i]}$$

$$(E, \Delta^{-1} E) = \sum_{z} \sum_{z'} \int_{-\pi}^{\pi} d^4p \left\{ \frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} d^3k \, (e^{-i\vec{k}} - 1) \cdot \right.$$

$$\left. \frac{e^{-i\vec{k}(\vec{z}-\vec{x})} \left[\delta(z^0 - x^0) - \delta(z^0 - y^0)\right]}{\sum_{i=1}^{3} [1 - \cos k_i]} \right\} \cdot \left\{ \frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} d^3k' \right.$$

$$(e^{-i\vec{k}'} - 1) \left. \frac{e^{i\vec{k}'(\vec{z}'-\vec{x})} \left[\delta(z'^0 - x^0) - \delta(z'^0 - y^0)\right]}{\sum_{i=1}^{3} [1 - \cos k_i]} \right\} \frac{e^{ip(z-z')}}{\sum_{i=0} [1 - \cos p_i]}$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} dp_0 \, d^3p \, \frac{1 - \cos p_0 (x^0 - y^0)}{\left\{ \sum_{i=0}^{3} [1 - \cos p_i] \right\} \left\{ \sum_{j=1}^{3} [1 - \cos p_j] \right\}}$$

From

$$1 - \cos p_i \geq \frac{2}{\pi^2} p_i^2 \quad , p \in [-\pi, \pi]$$

we finally obtain

$$(E, \Delta E) \leq \frac{1}{2}\left(\frac{\pi}{2}\right)^2 \int_{-\pi}^{\pi} dp_0 \,(1 - \cos p_0(x^0 - y^0)).$$

$$\cdot \int_{-\pi}^{\pi} d^3p \; \frac{1}{(\bar{p}^0)^2(\bar{p}^{0\,2} + p_0^2)} \leq$$

$$\leq \left(\frac{\pi}{2}\right)^2 \int_{-\infty}^{+\infty} dp_0 \; d^3p \; \frac{1}{(\bar{p}^0)^2(p_0^2 + \bar{p}^{0\,2})} < \infty$$

Appendix 9.B

Proposition For large enough β :

$$|S_2(xq\ y-q)| \leq e^{-O(\beta)q^2|x-y|}$$

Proof. Define a current density ρ as a mapping from the links in Λ^* into $2\pi\mathbb{Z}$, and let $\mathcal{E}$ denote a set of current densities with dist $(\rho,\rho') \geq 2^{\frac{1}{2}}$ if $\rho, \rho' \in \mathcal{E}$.
$\mathcal{E}$ is called a 1-ensemble.
We now quote[30] the

Lemma:

$$\prod_{\langle xy\rangle \in \Lambda^*} \sum_{n_{\langle xy\rangle}=-\infty}^{+\infty} e^{i2\pi n_{\langle xy\rangle} A_{\langle xy\rangle}} =$$

$$= \sum_{\sigma} d_\sigma \prod_{\rho \in \mathcal{E}_\sigma} [1 + K(\rho)\cos(A,\rho)] \quad (9.B1)$$

where σ ranges over some finite index sets, each $\mathcal{E}_\sigma$ is a 1-ensemble and

$$d_\sigma > 0$$

$$0 < K(\rho) \leq e^{c|\rho|^2}$$

$$|\rho|^2 = \sum_{\langle xy\rangle \in \operatorname{supp}\rho} \rho_{\langle xy\rangle}^2$$

with c some β independent constant.

We also notice that

$$Z_\Lambda = \lim_{\zeta \to 0} \int d\mu_\Lambda(A)\, e^{i(A,\, \zeta(E-\gamma))} \prod_{\langle xy\rangle \in \Lambda^*} \sum_{n_{\langle xy\rangle}=-\infty}^{\infty} e^{i 2\pi n_{\langle xy\rangle} A_{\langle xy\rangle}} \tag{9.B2}$$

Inserting (9.B1) in the definition of S_2 , by duality we obtain

$$S_2(xq\; y\, -q) = \lim_{\Lambda \uparrow \mathbb{Z}^4} \frac{1}{Z_\Lambda} \int d\mu_\Lambda(A)\, e^{i(A,\, E-\gamma)} \cdot$$

$$\cdot \prod_{\langle xy\rangle \in \Lambda^*} \sum_{n_{\langle xy\rangle}=-\infty}^{+\infty} e^{i 2\pi n_{\langle xy\rangle} A_{\langle xy\rangle}} = \tag{9.B3}$$

$$= \lim_{\Lambda \uparrow \mathbb{Z}^4} \int d\mu_\Lambda(A) \cos(A, E-\gamma) \sum_{\sigma} d\sigma \prod_{\substack{\rho \in \mathcal{E}_\sigma \\ \delta\rho = 0}} [1 + K(\rho)\cos(A, \rho)]$$

where we have used the fact that $d\mu_\Lambda(A)$ is even and

$$\int d\mu_\Lambda(A)\, e^{i(A,\mu)} = 0 \qquad \text{unless } \delta\mu = 0.$$

Given a set of links $N \subset \Lambda^*$ define

$$N' = \{ \langle xy\rangle \in \Lambda^* : \operatorname{dist}(N, \langle xy\rangle) \le 1 \}$$

then, using the techniques of [32], one obtains the identity

$$\cos(A, E-\gamma) \prod_{\substack{\rho : \\ \rho \cap (\mathrm{supp}\, E-\gamma)' \neq \phi}} [1 + K(\rho) \cos(A, \rho)] =$$

$$= \sum_{\tau} c_\tau \left[\cos(A, E-\gamma) + K(\rho^\tau) \cos(A, E-\gamma+\rho^\tau) \right] \tag{9.B.4}$$

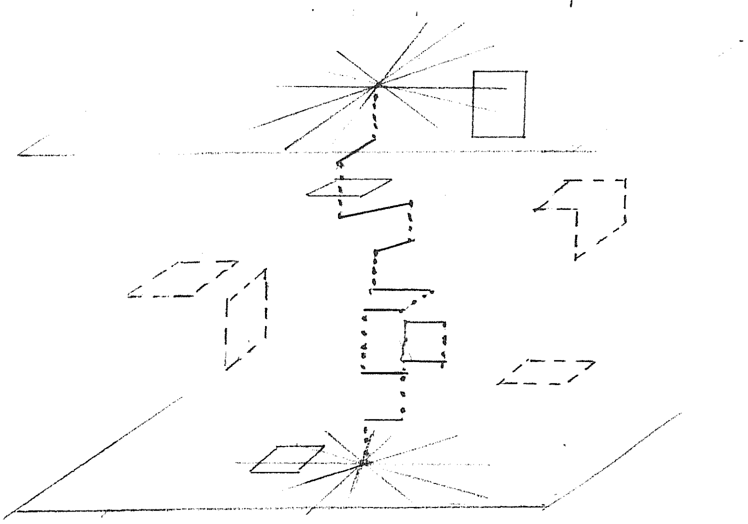
where ρ^τ is a current satisfying $\delta\rho^\tau = 0$, τ ranges over a finite index set, $c_\tau > 0$ and $|K(\rho^\tau)| \leq \exp c|\rho^\tau|^2$.

Inserting (9.B4) into (9.B3), and using (9.B2), we obtain that the expression under the limit in (9.B3) can be written

$$\Big\{ \int d\mu_\Lambda(A) \sum_\tau d_\tau \big[\cos(A, E-\gamma) + K(\rho^\tau)\cos(A, E-\gamma+\rho^\tau)\big] \cdot \prod_{\substack{\rho\in\mathcal{E}_\tau \\ \delta\rho=0}} \big[1+K(\rho)\cos(A,\rho)\big]\Big\} \cdot \Big\{ \int d\mu_\Lambda(A) \sum_\tau d_\tau \cdot \big[1+K(\rho)\cos(A,\rho^\tau)\big] \prod_{\rho\in\mathcal{E}_\tau\,:\,\delta\rho=0} \big[1+K(\rho)\cos(A,\rho)\big]\Big\}^{-1} \tag{9.B5}$$

Let now $\mathcal{B}_\rho$ ($\mathcal{B}_{\rho^\tau}$) be a set of links in supp $\rho\in\mathcal{E}_\tau$ (in supp $\rho^\tau-\gamma$), such that two different links in $\mathcal{B}_\rho$ ($\mathcal{B}_{\rho^\tau}$) do not belong to a common plaquette, and

$$\sum_{\langle xy\rangle\in\mathcal{B}_\rho} |\rho_{\langle xy\rangle}|^2 \geq c'|\rho|^2 \qquad \Big(\sum_{\langle xy\rangle\in\mathcal{B}_{\rho^\tau}} |\rho_{\langle xy\rangle}-\gamma_{\langle xy\rangle}|^2 \geq c'|\rho-\gamma|^2\Big)$$



(one dimension is suppressed)

fig. 9.B1

Define $n_{\langle xy\rangle} = \{\# p \in \Lambda^*; \langle xy\rangle \in \partial p\}$, then, using the renormalization transformation of [30] (and choosing $\gamma \cap E = \phi$), we obtain

$$\int d\mu_\Lambda(A) \; [\cos(A, E-\gamma) + K(p^T)\cos(A, E-\gamma+p^T)] \cdot$$

$$\cdot \prod_{\substack{p \in \mathcal{E}_T \\ \delta p = 0}} [1 + K(p)\cos(A, p)] =$$

$$= \int d\mu_\Lambda(A) \left\{ Z(\beta) \cos(A, E-\bar\gamma) + Z(\beta, p^T) \right.$$

$$\left. \cos\left(A, \overline{E-\gamma+p^T}\right) \right\} \prod_{\substack{p \in \mathcal{E}_T \\ \delta p = 0}} [1 + z'(\beta, p)\cos(A, \bar p)]$$

where

$$\bar\gamma_{\langle xy\rangle} = \gamma_{\langle xy\rangle} - \frac{(\delta d\gamma)_{\langle xy\rangle}}{n_{\langle xy\rangle}}$$

$$\overline{(E-\gamma+p^T)}_{\langle xy\rangle} = \begin{cases} (E-\gamma+p^T)_{\langle xy\rangle} , & \langle xy\rangle \notin \mathcal{B}_{p^T} \\ (E-\gamma+p^T)_{\langle xy\rangle} - \dfrac{(\delta d(E-\gamma+p^T))_{\langle xy\rangle}}{n_{\langle xy\rangle}} , & \langle xy\rangle \in \mathcal{B}_{p^T} \end{cases}$$

(9.B6i)

$$\bar p_{\langle xy\rangle} = \begin{cases} p_{\langle xy\rangle} , & \langle xy\rangle \notin \mathcal{B}_p \\ p_{\langle xy\rangle} - \dfrac{(\delta dp)_{\langle xy\rangle}}{n_{\langle xy\rangle}} , & \langle xy\rangle \in \mathcal{B}_p \end{cases}$$

$$Z(\beta) = \prod_{\langle xy\rangle \in \gamma} e^{-\frac{\beta}{2} \frac{\gamma^2_{\langle xy\rangle}}{n_{\langle xy\rangle}}}$$

$$Z(\beta, \rho^T) = \prod_{\langle xy\rangle \in \mathcal{B}_{\rho^T}} e^{-\beta \frac{(E-\gamma+\rho_T)^2_{\langle xy\rangle}}{n_{\langle xy\rangle}}} K(\rho^T)$$

$$Z(\beta, \rho) = \prod_{\langle xy\rangle \in \mathcal{B}_\rho} e^{-\beta \frac{\rho^2_{\langle xy\rangle}}{n_{\langle xy\rangle}}} K(\rho)$$

Using the bounds on $K(\rho)$, $K(\rho^T)$, for some constant c_1 independent on β, we obtain

$$Z(\beta, \rho^T) \le e^{-c_1(\beta - c)|\rho^T - \gamma|^2} \prod_{\langle xy\rangle \in \mathcal{B}_{\rho^T}} e^{-\frac{\beta}{2} E^2_{\langle xy\rangle}} \cdot \tag{9.B6ii}$$

$$\cdot \prod_{\langle xy\rangle \in \mathcal{B}_{\rho_T}} e^{\beta |E_{\langle xy\rangle}| |(\rho-\gamma)_{\langle xy\rangle}|}$$

$$Z(\beta, \rho) \le e^{-c_1(\beta - c)|\rho|^2}$$

For large enough is therefore less than 1.
To bound $Z(\beta,\rho^T)$ we have to take care that the E field is very large on the links near the boundary of γ.
Hence define

$$\mathcal{N}_{x_1 x_2} = \left\{ \langle xy\rangle : \ |E_{\langle xy\rangle}| \ge \frac{1}{2} \right\}$$

Then, for some $c_2 < c_1$

$$Z(\beta, \rho^T) \le e^{-c_2(\beta-c)|\rho^T-\gamma|^2} \prod_{\langle xy\rangle \in \mathcal{B}_{\rho^T} \cap \mathcal{N}_{x_1 x_2}} e^{\frac{\beta}{2}\left\{|E_{\langle xy\rangle}| |\rho^T_{\langle xy\rangle} - \gamma_{\langle xy\rangle}| - |E_{\langle xy\rangle}|^2\right\}}$$

so that

$$Z(\beta, \rho^T)\Big|_{E=0} \le e^{-c_2(\beta-c)|\rho^T-\gamma|^2}$$

which for large β is less than 1.

Therefore, from $|z\cos\alpha| \le |z|$ we finally obtain:

$$(9.B5) \le \max_{\tau} \left(z(\beta) + z(\beta, \rho^{\tau})\right) \cdot$$

$$\cdot \frac{\int d\mu_{\Lambda}(A) \sum_{T} d_{\tau} \prod_{\rho:\ \rho \cap (E-\gamma)' = \phi,\ \rho \in \mathcal{E}_{\tau}} [1 + z(\beta,\rho)\cos(A,\bar{\rho})]}{\int d\mu_{\Lambda}(A) \sum_{T} d_{\tau} \prod_{\rho \in \mathcal{E}_{\tau}} [1 + z(\beta,\rho)\cos(A,\bar{\rho})][1 + z(\beta,\rho^{\tau})\cos(A,\bar{\rho}^{\tau})]\Big|_{E=0}}$$

Now one uses the inequality

$$\int d\mu_{\Lambda}(A) \prod_{\rho \in \mathcal{E}_{\tau}} [1 + z(\beta,\rho)\cos(A,\bar{\rho})] \le$$

$$\le \int d\mu_{\Lambda}(A) \prod_{\rho \in \mathcal{E}_{\tau}} [1 + z(\beta,\rho)\cos(A,\bar{\rho})][1 + z(\beta,\rho^{\tau})\cos(A,\bar{\rho}^{\tau})]\Big|_{E=0}$$

to obtain

$$|S_2(xq\ y-q)| \le \max_{\tau:\ \mathcal{E}_{\tau} \subset \mathbb{Z}^4} \left(z(\beta) + z(\beta, \rho^{\tau})\right)$$

Since $|\rho^{\tau} - \gamma|^2 \le q^2 |x-y|$, from (9.B6i,ii) we finally have

$$|S_2(xq\ y-q)| \le e^{-O(\beta)\, q^2 |x-y|}$$

10. OPEN PROBLEMS

In this final short section we just want to mention some open problems suggested by the previous sections.

1) The first and fundamental one is the continuum limit of the lattice soliton sectors. In particular: do they survive when the lattice spacing go to zero? The answer is certainly positive for the ϕ^4_2 or $(\cos\phi)_2$ sectors, which have been explicitly constructed in [19] and [36] with algebraic tools.
We also believe that the continuum vortex sectors in the three dimensional Higgs models and the t'Hooft-Polyakov monopole sectors in the Georgi-Glashow model do exist. On the contrary, there are indications [15,31] that the $U(1)_4$ monopole sectors will not survive the continuum limit.
From a constructive point of view, the euclidean construction of the local solitons in $d=2$ should not be very difficult. For example, in the ϕ^4_2 model one should obtain the expectation value of the disorder field at finite volume, substituting in the gaussian measure the covariance $(\Delta + m^2)^{-1}(x,y)$ with the covariance $(\Delta_{\mathbb{Z}_2}(\omega) + m^2)^{-1}(x,y)$ where $\Delta_{\mathbb{Z}_2}(\omega)$ is the $\mathbb{Z}_2$ covariant laplacian in $\mathbb{R}^2 \setminus supp(d\omega)^*$.
The thermodynamic limit should then be taken by means of a sort of Glimm-Jaffe-Spencer expansion [22], which probably allows also a particle structure analysis with the excitation approach.
2) A second natural problem is the extension of the techniques developed here to other similar cases. In particular:
a) construction of the t'Hooft-Polyakov monopole sectors;
b) construction of the instanton [37] on sectors in d=5 in SU(N) gauge theories.
In this dimension, in fact, one believes that SU(N) gauge theories have a deconfining region for large β and the instanton, a line defect in $d=5$, should give rise to soliton sectors in class C.
3) To construct an excitation expansion also for sectors in class C, both to analyze their particle structure, and to extend to these models the equality between soliton mass and surface tension.

4) To extend the soliton sectors analysis to models which include the fermions.
In this context, the interesting new phenomenon of sectors labelled by fractional charges should appear [38], together with intriguing aspects on the relation between spin and statistics [39]. A simple model which should be possible to analyze in this context is the $(\phi^4 + \overline{\Psi}\gamma_5 \phi \psi)$ which, on general grounds, is known to possess soliton sectors in d=2 [40].

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INDEX