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HIGHER-ORDER FIELD THEORY AND GENERAL RELATIVITY

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Part One. Introduction

From a predictive point of view the best theory we have in order to describe the macroscopic effects of the gravitational field is General Relativity. The same is not true from a field theoretical point of view. The main problem is that concerning with the energetic content of the theory: just to start, with the standard definitions, the so called canonical energy-momentum "tensor" is not a tensor; secondly, the canonical definition must be supplemented with surface terms. It is our belief that most of this kind of problems in General Relativity arise due to a misunderstanding of the foundations and an inadequate application of field theory.

The Lagrangian for General Relativity contains second-order derivatives of the fields, the components of the metric or any other equivalent set. The previous fact has been considered as a defect of General Relativity. People argue that due to the presence of second-order derivatives in the Lagrangian a Hamiltonian formalism is lacking. People then have recourse to

i. the Palatini first-order formalism in which g and Γ are considered as independent fields; or
ii. to the $\Gamma\Gamma$ Lagrangian, obtained by discarding second-order derivatives through a divergence.

The first procedure has the disadvantage of not being a general procedure, it is just a "lucky strike" that it works for General Relativity, as do it for any metric field theory. Concerning the second one, it is not true that the $\Gamma\Gamma$ Lagrangian gives exactly General Relativity since the boundary behavior of the fields is changed.

As we already mentioned it, a further problem is related with surface terms. In order to have the correct asymptotic behavior of the energy, the linear and the angular momentum, one must supplement the canonical definitions with surface terms. They were first introduced by Regge and Teitelboim (1974).

All previous criticisms to the field theoretical aspects of General Relativity can be overcome if one applies field theory in its most orthodox way and avoid any *ad hoc* construction for

General Relativity. Summarising, in order to correctly address the field theoretical aspects of General Relativity, we must get acquainted of higher-order field theory and how to correctly deal with it. That is the subject of this work.

Apart from the previous motivation, there has been recently much interest in the formulation of field theories governed by Lagrangians depending on higher-order derivatives of the fields, both for purely mathematical reasons and in view of their possible applications to concrete physical theories, e.g., in the domain of alternative theories of gravitation; see, e.g. (Ferraris and Francaviglia, 1987) and references quoted therein.

A historical review of the role of higher-order Lagrangians in physics it is not out of place here. The first suggestions of higher-order Lagrangians date back to the early days of General Relativity; in fact, just after the arrival of General Relativity it was suggested that the Einstein equations for the gravitational field be replaced by others involving derivatives of higher than second order. Early suggestions (Weyl, 1921; Eddington, 1924) concerned an attempt to include the electromagnetic field in a unified geometrical framework, but this line of approach proved unfruitful and was eventually abandoned (Pais and Uhlenbeck, 1950). Later suggestions were given in (Lanczos, 1938); Buchdahl, 1948).

The first known theory using second-order Lagrangians is the Podolski electrodynamics (Bopp, 1940; Podolski, 1942; Podolski and Kikuchi, 1944, 1945; Chang, 1946, 1947; Kanai and Tagaki, 1946; Montgomery, 1946; Green, 1947, 1948, 1949; Podolski and Schwed, 1948; Matthews, 1949). The argument to introduce higher-order derivatives is the fact that they are important for high frequencies (ultraviolet), but they are not well described by the usual second-order field equations. The quantisation of this generalised electrodynamics gave finite energies.

The previous facts motivated a deeper study of the generalised Hamiltonian formalism. The first contributions were given by DeWet (1948). However, his formulation did not contain classical mechanics as a particular case. Next Chang (1948) tried to rewrite the higher-order Lagrangians in a first-order form. Further developments were given by (Thielheim, 1967; Coelho de

Souza and Rodrigues, 1969).

The success of the previous results lead to Pais and Uhlembeck (1950) to study higher-derivative theories in the general context of quantum field theory and to consider the possibility of cancelling the divergent features of field theories using Lagrangians containing higher-order derivatives. However, they were not able to reconcile the convergence, the positive definiteness of the free field energy and the absolutely causal behaviour of the wave function for a physical system. Another attempts to put farther the Podolski electrodynamics are due to Katayama (1953) and Taniuti (1955, 1956).

Later on, for the same reasons as in electrodynamics, to control the high frequencies behavior and in order to permit renormalisation of divergences in the quantum corrections to the interactions of matter fields, quantum field theory also gave rise to suggestions that higher-derivative terms be included in the gravitational Lagrangian for quantum gravity (Utiyama and DeWitt, 1962; DeWitt, 1965; Sakharov, 1967). More recent use of higher derivatives to regularise the stress tensor is reviewed in (DeWitt, 1975). For a review see the book by Christensen (1984) and references therein.

Most recently there have been proposals of adding a term proportional to the extrinsic curvature, a second- order term, to the Nambu-Goto Lagrangian in order to describe the fine structure of strings (Polyakov, 1986; Alonso and Espriu, 1987).

In spite of the previous interest on the subject the progresses done from a more mathematical and formal point of view are not well known. The purpose of this work is to introduce the most recent results, most of them original, concerning higher-order field theories. The philosophy of the work can be well summarised with the alternative title "How to deal with higher-order field theories?"

The work is organised as follows. In sec.I.1 we start by introducing the fundamentals of General Relativity and showing explicitly the defects we talk about at the very beginning. Next, sec.I.2 we consider some elements of differential geometry. In fact, one of the most natural mathematical setting for dealing with higher-order field theories is with the Poincaré-Cartan form over fibered manifolds. This formalism has however the disadvantage of being not known to the physics community. That the reason why on parallel lines we present the formalism for dealing with higher-order field theories as a physicist will do it.

Next we consider field theory from a local point of view. There a physical system is described by a set of fields ϕ over a four-dimensional space-time. We start by deriving the fields equations from a variational principle.

We show that a Lagrangian density of the form $\mathcal{L}=d_\mu \Lambda^\mu$, where $d_\mu = d/dx^\mu$ is the total, or formal, derivative with respect to x^μ, yields identically vanishing field equations. Accordingly, two Lagrangians locally differing by a divergence yield the same field equations; this property will be called *d-equivalence*. The previous is a sufficient condition, and an important problem is to show that the condition is also necessary, i.e. to show that $\delta_A(\mathcal{L})\equiv 0$, where δ_A is the variational derivative, locally implies $\mathcal{L}=d_\mu \Lambda^\mu$. This can be easily done in Classical Mechanics, as well as for its higher-order generalisations, and also for first-order field theory; see, e.g. (Hojman, 1983). A classical constructive method of proof, which, e.g., can be found in (Hojman, 1983), consists into explicitly writing the total derivatives involved in the field equations. The terms containing derivatives of each different order must be independently zero. This means that the factors multiplying them must be zero, giving differential conditions which might be integrated to obtain the explicit structure of the Lagrangian.

We then construct the local canonical energy-momentum tensor and consider the local conservation laws to which it gives rise. In this connection we show then that the energy-momentum tensor is not invariant under the addition of a divergence to the Lagrangian, i.e., it is not d-invariant. It seems that the only way to solve this *impasse* is by selecting a representative for the d-equivalent Lagrangians. Suitable ways of generalising these results is currently under investigation.

One of the main problems one must face in field theory is the increase of the base space dimension from one (classical mechanics) to some dimension greater than one, two or four in the physically relevant cases. There are two alternatives. The first alternative is canonical field theory which is constructed as a classical mechanics with infinitely many degrees of freedom. The second alternative is covariant field theory (Tapia, 1988b) which, on the contrary, is constructed as a multitime classical mechanics.

We show how canonical field theory can be constructed still on the same lines as classical mechanics. We then construct the canonical Hamiltonian formalism. The canonical Hamiltonian is defined as the time-time component of the canonical energy-momentum tensor. When considering the time evolution of an arbitrary functional one realises that the canonical Hamiltonian is the correct time evolution generator. Furthermore, it induces a symplectic structure over the phase space. Finally we consider the conservation laws induced by the canonical energy-momentum tensor.

The boundary conditions appearing to guarantee the stationarity of the action must be considered as constraints for the formalism. This can be avoided (Tapia, 1987a) if one consider them as sources at the boundary for the field equations.

We construct then a covariant field theory in which all four directions of space-time are dealt with on the same footing (Tapia, 1988b). For this aim an extra term must be added to the canonical energy-momentum tensor. Contact with the canonical field theory is made be considering the time-time component of this modified energy-momentum tensor. The additional term turns to be a surface term. The modified Hamiltonian has the same properties as the canonical one, viz., it is the time evolution generator and induces a symplectic structure over the phase space. The energy, the linear and angular momentum, all acquire an extra surface term. Similar conclusions, in the sense that the physically relevant quantities are the canonical ones supplemented by surface terms, were reached by Ferraris and Francaviglia (1988) using the Poincaré-Cartan form as extended to higher-order derivatives field theories.

We next go to local higher-order field theory, i.e., one in which the Lagrangian depends on higher-order derivatives of the fields. The Lagrangian formalism can be developed on the same lines as for the first-order case with the exception that momenta must be defined in a generalised way depending on the order of the

Lagrangian.

We then address the problem of characterising Lagrangian densities (of an arbitrary order) yielding identically vanishing field equations. The Hojman method of proof complicates very much for orders larger than one. Some partial results for second-order field theory were obtained by Shadwick (1982); more general results for the second-order case were also recently obtained by the author (Tapia, 1987b). However, for higher-order Lagrangians this method of proof becomes practically unmanageable, and a new method of proof must be looked for. In any case, an implicit proof that $\delta_A(\mathcal{L})\bar{=}0$ is equivalent to the Lagrangian being a divergence was given by Krupka (1982), in the context of the theory of Lepagean equivalence. This proof, however, amounts only to an existence theorem, and it is still interesting to derive a method for constructing out in general the divergences which generate identically satisfied field equations.

Inspired by some previous results by Vainberg (1964), Atherton and Homsy (1975) and Engels (1975, 1978), Tapia et al. (1988) derived, in the framework of higher-order field theory, a remarkable identity for the Lagrangian, from which one can immediately infer that the necessary and sufficient condition for obtaining identically vanishing field equations is that the Lagrangian is a divergence. Furthermore, it enables us to remove from field theory the ambiguities related to the non-invariance of the energy-momentum tensor under the addition of a divergence to the Lagrangian.

We construct the canonical energy-momentum tensor. This can be done on the same lines as for the first-order case. The conservation laws for the energy, the linear and the angular momenta, are constructed in the same way as for the first-order case.

We then construct a d-invariant field theory, i.e., one for which the energy-momentum tensor is independent of the divergences which can be added to the Lagrangian. This must be done by selecting a representative for d-equivalent Lagrangians. Once again the previous identity helps us in this task (Tapia et al., 1988); in fact, it allows us to select a local representative for the d- equivalent Lagrangians from which the divergence part has been removed.

We then specialise the previous formalism to the second-order case, i.e., a field theory in which the Lagrangian density depends on up to the second-order derivatives of the fields. In analogy with what was done for second-order classical mechanics (Tapia, 1985) the momenta are defined as the terms multiplying the variation of the fields in the boundary term of the variation of the action (Tapia, 1988a). The main difference with respect to the first-order case appears in the definition of the momenta, functional rather than ordinary derivatives must be used.

We then construct the canonical Hamiltonian formalism. Lastly we consider the surface terms for second-order field theories.

To finish we apply all the previous results to General Relativity. The conserved quantities are just surface terms.

Aldo the theory is written in a local form it can be conveniently applied to asymptotically flat spaces where outside the compact domain one can choose a single coordinate patch which cover all of the infinity.

I.1. General Relativity

Here we introduce the field theoretical aspects of General Relativity. We start by considering some fundamentals of Riemannian geometry at the manner a relativistic will do it. Later on we consider the equations describing the gravitational field, the Einstein equations. Then we look for the variational principles from which they follow.

I.1.1. Riemannian Geometry

A Riemannian manifold is intrinsically characterised by the metric tensor $g_{\mu\nu}$. From it we can define the Christoffel symbols

$$\Gamma_{\lambda\mu\nu} = \frac{1}{2} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \tag{I.1.1}$$

and the contractions

$$\Gamma_\lambda = g^{\mu\nu} \Gamma_{\lambda\mu\nu} , \tag{I.1.2a}$$

$$\tilde{\Gamma}_\lambda = g^{\mu\nu} \Gamma_{\mu\nu\lambda} , \tag{I.1.2b}$$

where $g^{\mu\nu}$ is the contravariant metric tensor satisfying $g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu$. Indices are raised and lowered with the metric g even for the Christoffel symbol which is not a tensor. The following derivatives will be useful in what follows

$$\frac{\partial \Gamma_{\lambda,\mu\nu}}{\partial \partial_\alpha g_{\beta\gamma}} = \frac{1}{2} (\delta^\alpha_\mu \, \delta^{(\beta\gamma)}_{(\nu\lambda)} + \delta^\alpha_\nu \, \delta^{(\beta\gamma)}_{(\mu\lambda)} - \delta^\alpha_\lambda \, \delta^{(\beta\gamma)}_{(\mu\nu)}) , \tag{I.1.3}$$

and the contractions

$$\frac{\partial \Gamma_\lambda}{\partial \partial_\alpha g_{\beta\gamma}} = g^{\mu\nu} \frac{\partial \Gamma_{\lambda,\mu\nu}}{\partial \partial_\alpha g_{\beta\gamma}} = \frac{1}{2} (g^{\alpha\beta} \delta^\gamma_\lambda + g^{\alpha\gamma} \delta^\beta_\lambda - g^{\beta\gamma} \delta^\alpha_\lambda) ,$$
$$\frac{\partial \tilde{\Gamma}_\lambda}{\partial \partial_\alpha g_{\beta\gamma}} = g^{\mu\nu} \frac{\partial \Gamma_{\mu,\nu\lambda}}{\partial \partial_\alpha g_{\beta\gamma}} = \frac{1}{2} \delta^\alpha_\lambda \, g^{\beta\gamma} . \tag{I.1.4}$$

The Riemann tensor is defined as

$$R_{\lambda\mu\nu\rho} = \frac{1}{2} (\partial_{\lambda\rho} g_{\mu\nu} + \partial_{\mu\nu} g_{\lambda\rho} - \partial_{\lambda\nu} g_{\mu\rho} - \partial_{\mu\rho} g_{\nu\lambda})$$

$$+ g^{\sigma\tau} (\Gamma_{\sigma\mu\nu} \Gamma_{\tau\lambda\rho} - \Gamma_{\sigma\lambda\nu} \Gamma_{\tau\mu\rho}) \ . \tag{I.1.5}$$

The Ricci tensor is

$$R_{\nu\rho} = \frac{1}{2} g^{\lambda\sigma} (\partial_{\lambda\rho} g_{\nu\sigma} + \partial_{\nu\sigma} g_{\lambda\rho} - \partial_{\lambda\sigma} g_{\nu\rho} - \partial_{\nu\rho} g_{\lambda\sigma})$$

$$+ \Gamma^{\sigma}{}_{\lambda\rho} \Gamma_{\sigma}{}^{\lambda}{}_{\nu} - \tilde{\Gamma}^{\sigma} \Gamma_{\sigma\nu\rho} \ , \tag{I.1.6}$$

and the scalar curvature is given by

$$R = g^{\lambda\nu} g^{\mu\rho} R_{\lambda\mu\nu\rho}$$

$$= - (G^{\lambda\nu\mu\rho} \partial_{\lambda\nu} g_{\mu\rho} + G^{\lambda\nu\mu\rho} g^{\sigma\tau} \Gamma_{\sigma\lambda\nu} \Gamma_{\tau\mu\rho}) \ . \tag{I.1.7}$$

where

$$G^{\mu\nu\lambda\rho} = g^{\mu\nu} g^{\lambda\rho} - \frac{1}{2} (g^{\mu\lambda} g^{\nu\rho} + g^{\mu\rho} g^{\nu\lambda}) \ , \tag{I.1.8}$$

is the DeWitt metric. The Einstein tensor is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \ . \tag{I.1.9}$$

The Riemann tensor satisfies the Bianchi identities

$$R_{\alpha\beta\lambda\mu\nu} + R_{\alpha\beta\mu\nu\lambda} + R_{\alpha\beta\nu\lambda\mu} = 0 \ . \tag{I.1.10}$$

A double contraction of them leads to

$$G^{\mu\nu}{}_{;\nu} = 0 \ . \tag{I.1.11}$$

I.1.2. Variational Principles for General Relativity

According to General Relativity the field equations describing the gravitational field are the Einstein equations

$$G_{\mu\nu} = 0 \ . \tag{I.1.12}$$

They are second-order equations, therefore one hopes they follow from a first-order Lagrangian. However, a first- order Lagrangian cannot be constructed since it can be put equal to zero by means of a convenient coordinate transformation. This *impasse* was solved by Hilbert who noticed that if the second-order derivatives appear linearly in the Lagrangian and only involved in a total divergence one can equally well obtain Einstein field equations. In this case, furthermore, one can construct an invariant Lagrangian linear in the second-order derivatives. The solution is

$$\mathcal{L}_H = \sqrt{-g}\, R = \mathcal{L}_{\Gamma\Gamma} + d_\mu \mathcal{V}^\mu \ , \tag{I.1.13}$$

$$\mathcal{L}_{\Gamma\Gamma} = \sqrt{-g}\, (\Gamma^{\nu\mu\lambda}\, \Gamma_{\lambda\mu\nu} - \tilde{\Gamma}^\mu\, \Gamma_\mu) \ , \tag{I.1.14}$$

$$\mathcal{V}^\mu = \sqrt{-g}\, (\Gamma^\mu - \tilde{\Gamma}^\mu) \ . \tag{I.1.15}$$

I.1.3. The Formulation a là Palatini

The purpose of this formalism is to get rid of second-order derivatives, i.e. to obtain a canonical form for Einstein equations, that is to put them in the form $\dot{q}=\partial H/\partial p$, $\dot{p}=-\partial H/\partial q$. As a preliminary step, we will rewrite the Lagrangian so that the equations of motion have two of the properties of canonical equations:

1. they are first-order equations; and
2. they are solved explicitly for the time derivatives.

The second property will be obtained by a 3+1 dimensional breakup of the original four-dimensional quantities, as will be discussed below. The first property is insured by using a Lagrangian linear in first derivatives. In General Relativity, this is called the Palatini Lagrangian, and consists in regarding the Christoffel symbols $\Gamma^\alpha_{\mu\nu}$ as independent quantities in the variational principle. Thus, one may rewrite (I.1.13) as

$$S = \int g^{\mu\nu}\, R_{\mu\nu}(\Gamma)\, d^4x \ , \tag{I.1.16}$$

$$R_{\mu\nu}(\Gamma) = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\alpha\mu,\nu} + \Gamma^\alpha_{\mu\nu}\, \Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\mu\beta}\, \Gamma^\beta_{\nu\alpha} \ . \tag{I.1.17}$$

Note that these covariant components $R_{\mu\nu}$ of the Ricci tensor do not involve the metric but only the affinity $\Gamma^\alpha_{\mu\nu}$. Thus, by varying $g^{\mu\nu}$, one obtains directly the Einstein field equations

$$G_{\mu\nu} = R_{\mu\nu}(\Gamma) - \tfrac{1}{2}\, g^{\alpha\beta}\, R_{\alpha\beta}(\Gamma)\, g_{\mu\nu} = 0 \ . \tag{I.1.18}$$

These equations no longer express the full content of the theory, since the relation between the now independent quantities $\Gamma^{\alpha}{}_{\mu\nu}$ and $g_{\mu\nu}$ is still required. This is obtained as a field equation by varying $\Gamma^{\alpha}{}_{\mu\nu}$. One then finds

$$g^{\mu\nu}{}_{;\alpha} = g^{\mu\nu}{}_{,\alpha} + g^{\mu\beta}\,\Gamma^{\nu}{}_{\alpha\beta} + g^{\nu\beta}\,\Gamma^{\mu}{}_{\alpha\beta} - g^{\mu\nu}\,\Gamma^{\beta}{}_{\alpha\beta} = 0 \ , \qquad \text{(I.1.19)}$$

which can be solved for $\Gamma^{\alpha}{}_{\mu\nu}$ to give the usual relation

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}\, g^{\alpha\beta}\,(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}) \ . \qquad \text{(I.1.20)}$$

I.1.4. The ΓΓ Approach

Most of later developments of General Relativity have been based on the ΓΓ Lagrangian. In this case the Lagrangian is a first-order one.

The corresponding momenta are

$$\Pi^{(\alpha\beta)\mu}(\mathcal{L}_{\Gamma\Gamma}) = \sqrt{-g}\ (\Gamma^{\mu\alpha\beta} - \tilde{\Gamma}^{\sigma}\, g^{\mu\tau}\, \delta^{(\alpha\beta)}_{(\sigma\tau)} + \frac{1}{2}\,\tilde{\Gamma}^{\mu}\, g^{\alpha\beta} - \frac{1}{2}\,\Gamma^{\mu}\, g^{\alpha\beta}) \ . \qquad \text{(I.1.21)}$$

The corresponding energy-momentum tensor is

$$\begin{aligned}\mathcal{H}^{\mu}{}_{\nu}(\mathcal{L}_{\Gamma\Gamma}) &= \partial_{\nu} g_{\alpha\beta}\ \Pi^{(\alpha\beta)\mu}(\mathcal{L}_{\Gamma\Gamma}) - \delta^{\mu}_{\nu}\, \mathcal{L}_{\Gamma\Gamma} \\ &= -\frac{\sqrt{-g}}{2}\,(2\,\tilde{\Gamma}_{\nu}\,\Gamma^{\mu} - 4\,\Gamma^{\mu\alpha\beta}\,\Gamma_{\alpha\beta\nu} - 2\,\tilde{\Gamma}_{\nu}\,\tilde{\Gamma}^{\mu} \\ &\qquad + 2\,\Gamma^{\mu}{}_{\beta\nu}\,\tilde{\Gamma}^{\beta} + \tilde{\Gamma}^{\beta}\,\Gamma_{\beta}{}^{\mu}{}_{\nu}) \ . \end{aligned} \qquad \text{(I.1.22)}$$

The formalism based on the ΓΓ Lagrangian it is not the correct one; first of all, the energy-momentum "tensor" (I.1.22) is not a tensor.

I.1.5. The ADM Formalism

One of the common procedures to formulate a Hamiltonian theory of gravitation is based on a slice dependent formalism which was developed around 1960 by Arnowitt, Deser and Misner, and is commonly known as *ADM formalism* (Arnowitt et al., 1962). In its classical formulation it relies first on the choice of an appropriate first-order Lagrangian equivalent to the Hilbert Lagrangian obtained by pushing second-order derivatives of the metric into a total space-time divergence. As a second step, one then chooses a foliation of the four-dimensional space-time into a continuous family of three-dimensional (space-like) hypersurfaces, onto which the four-dimensional dynamical variables can be projected. This helps to select a set of three-dimensional canonical variables, whose Hamiltonian evolution in time allows to suitably reproduce Einstein field equations, see, e.g. (Isenberg and Nester, 1980).

It was soon realised that, since General Relativity is a constrained system, in the sense of Dirac (1950, 1964), the standard Hamiltonian vanishes on the constraints and has therefore to be properly corrected by the addition of suitable boundary integrals, i.e., contributions at infinity, representing the asymptotic mass and angular momentum, see (Regge and Teitelboim, 1974).

The classical formulation of this formalism depends heavily on the choice of coordinates or frames adapted to the foliation, so that several problems arise when trying to restore the full four-dimensional covariance of Einstein's theory. In recent years, various authors have therefore attempted to construct a fully covariant approach to ADM formalism for first-order theories, generally by relying on symplectic or multisymplectic techniques, or on particular geometric structures implied by the choice of a class of Lagrangians. Important work in this direction can be found in (Isenberg and Nester, 1980; Sniaticki, 1985; Kijowski, 1985) and references quoted therein; we mention in particular some papers by Szczyrba (1978, 1981, 1987), which contain an important discussion of ADM formalism for first-order covariant field theories, together with an extension to cover the case of General Relativity. These last papers are based on the extensive use of the Poincaré-Cartan form as the basic tool to define the Hamiltonian, or energy-momentum, of the given theory.

It has been often argued in the literature that the classical ADM formalism applies to General Relativity only because this theory behaves essentially as a Lagrangian field theory of order one. Moreover, doubts have been raised, see (Szczyrba, 1987), about the existence of a similar general method for arbitrary higher-order field theories. Nevertheless, also in view of recent investigations concerning the physical interest of truly second-order Lagrangians for relativistic theories of gravitation, see (Magnano et al., 1987; Stelle, 1978; Whitt, 1984) and references quoted therein, it is interesting, at least on the theoretical level, to reconsider the problem and to provide a fully covariant approach directly applicable to higher-order field theories, i.e., to Lagrangians depending on higher-order derivatives, no matter whether constrained or not.

ADM start by considering the usual action for General Relativity

$$S = \int \mathcal{L}\, d^4x = \int R \sqrt{-g}\, d^4x \ , \tag{I.1.23}$$

which yields the Einstein field equations when one considers variations in the metric (e.g., $g_{\mu\nu}$ or the density $\mathfrak{g}^{\mu\nu}=\sqrt{-g}g^{\mu\nu}$). These Lagrange equations of motion are then second-order differential equations.

The three-dimensional quantities appropriate for the Einstein field are

$$g_{ij} = {}^4g_{ij} \ , \tag{I.1.24a}$$

$$N = (-{}^4g^{00})^{-1/2} \ , \tag{I.1.24b}$$

$$N_i = {}^4g_{0i} \ , \tag{I.1.24c}$$

$$\pi^{ij} = (-{}^4g^{00})^{1/2}\, ({}^4\Gamma^0{}_{pq} - g_{pq}\, {}^4\Gamma^0{}_{rs}\, g^{rs})\, g^{ip}\, g^{jq}. \tag{I.1.24d}$$

Here and subsequently we mark every four-dimensional quantity with the prefix ^{4}, so that all unmarked quantities are understood as three- dimensional. In particular, g^{ij} in (I.1.24d), is the reciprocal matrix to g_{ij}. The full metric ${}^4g_{\mu\nu}$ and ${}^4g^{\mu\nu}$ may, with (I.1.24), be written as

$${}^4g_{00} = N^2 + N^i\, N_i \ , \tag{I.1.25a}$$

where $N^i=g^{ij}N_j$, and

$$ {}^4g^{0i} = - N^i/N^2 \ , \tag{I.1.25b} $$

$$ {}^4g^{00} = 1/N^2 \ , \tag{I.1.25c} $$

$$ {}^4g^{ij} = g^{ij} + N^i N^j/N^2 \ . \tag{I.1.25d} $$

One further useful relation is

$$ \sqrt{-{}^4g} = N \sqrt{-g} \ . \tag{I.1.26} $$

In terms of the basic quantities (I.1.24), the Lagrangian for General Relativity becomes

$$ \mathcal{L} = \sqrt{-{}^4g}\ {}^4R = - g_{ij}\ \partial_t \pi^{ij} - N \mathcal{H}^0 - N_i \mathcal{H}^i $$

$$ - 2 \left(\pi^{ij} N_j - \tfrac{1}{2} \pi N^i + N^{|i} \sqrt{-g}\right)_{,i} \tag{I.1.27} $$

where

$$ \mathcal{H}^0 = - \sqrt{-g}\ [{}^3R + g^{-1} (\tfrac{1}{2} \pi^2 - \pi^{ij} \pi_{ij})] \ , \tag{I.1.28a} $$

$$ \mathcal{H}^i = - 2\ \pi^{ij}{}_{|j} \ . \tag{I.1.28b} $$

The quantity ${}^3R$ is the curvature scalar formed from the spatial metric g_{ij}, $|$ indicates the covariant derivative using this metric, and spatial indices are raised and lowered using g^{ij} and g_{ij}. Similarly, $\pi = \pi^i{}_i$. We have allowed second-order space derivatives to appear by elliminating such quantities as $\Gamma^k{}_{ij}$ in terms of $\partial_k g_{ij}$.

One may verify directly that the first-order Lagrangian (I.1.27) correctly gives rise to Einstein equations. One obtains

$$ \partial_t g_{ij} = 2\ N\ g^{-1/2} (\pi_{ij} - \tfrac{1}{2} \pi\ g_{ij}) + N_{i|j} + N_{j|i} \ , \tag{I.1.29a} $$

$$ \partial_t \pi^{ij} = - N \sqrt{-g}\ ({}^3R^{ij} - \tfrac{1}{2}\ {}^3R\ g^{ij}) $$

$$ + \tfrac{1}{2} N g^{-1/2} g^{ij} (\pi^{mn} \pi_{mn} - \tfrac{1}{2} \pi^2) $$

$$ - 2 N g^{-1/2} (\pi^{im} \pi_m{}^j - \tfrac{1}{2} \pi\ \pi^{ij}) $$

$$ + \sqrt{-g}\ (N^{i|j} - g^{ij} N^{|m}{}_{|m}) + (\pi^{ij} N^m)_{|m} $$

$$- N^i{}_{|m}\, \pi^{mj} - N^j{}_{|m}\, \pi^{mi} \ , \qquad \text{(I.1.29b)}$$

$$\mathscr{H}^{\mu}(g_{ij}, \pi^{ij}) = 0 \ . \qquad \text{(I.1.29c)}$$

Equation (I.1.29a), which results from varying π^{ij}, would be viewed as the defining equation for π^{ij} in a second-order formalism. Variation of N and N_i yields equations (I.1.29c), which are the ${}^4G^0{}_{\mu}={}^4R^0{}_{\mu}-\frac{1}{2}{}^4R\delta^0{}_{\mu}=0$ equations, while equations (I.1.29b) are linear combinations of these equations and the remaining six Einstein equations (${}^4G_{ij}=0$).

I.1.6. Surface Terms

It has been recognised long ago that there are drastic differences in the Hamiltonian formulation of General Relativity depending upon whether three-space is open or closed. DeWitt (1967) noticed that in the case of an asymptotically flat space-time the usual Hamiltonian

$$H_0 = \int [N(x)\, \mathscr{H}^0(x) + N_i(x)\, \mathscr{H}^i(x)]\, d\Sigma \qquad \text{(I.1.30)}$$

for General Relativity must be supplemented by the addition of a surface integral at infinity (Arnowitt et al., 1962)

$$E[g_{ij}] = \int (g_{ik,i} - g_{ii,k})\, dS_k \qquad \text{(I.1.31)}$$

in order for the modified Hamiltonian

$$H = H_0 + E[g_{ij}] \qquad \text{(I.1.32)}$$

to coincide asymptotically with the usual expression of the linearised theory. However Dewitt stated at the same time that: "...although such a partial integration leaves the dynamical equations unaffected it does change the definition of energy...." This point of view seems to be commonly accepted; Dirac (1959), for example, also states that: "The removal of this surface integral does not disturb the validity of H_{main} [our H of eq.(I.1.32) with N=1 and N_i=0] for giving equations of motion, but it results in H_{main} not vanishing weakly...."

We would like to point out that with a proper definition of the functional space of the gravitational fields the inclusion of the surface integral (I.1.31) in the Hamiltonian is not a matter of choice and that it is not necessary either to justify it by making appeal to the linearised theory. Rather, in our case, the usual Hamiltonian H_0 not only does not give the correct equations of motion (Einstein's equations) but, worse than that, it (H_0) gives no well-defined set of equations of motion at all. It will be in fact shown that if the phase space of the system is to include the trajectories representing the solutions of the equations of motion, which is a necessary and basic consistency requirement for any Hamiltonian theory, then the Hamiltonian for General Relativity must be expression (I.1.32) which includes the surface integral (I.1.31). If one would use H_0 given by (I.1.30) instead of H he would find that Hamilton's principle

$$\delta \int L \, dt = 0 \tag{I.1.33}$$

has no solutions.

Once one sees that the surface integral (I.1.31) must be included in the Hamiltonian to start with by a fundamental reason and not by some *ad hoc* considerations, the logical procedure leading to the identification of the energy in General Relativity becomes considerably simpler: the energy is just the (non-zero!) numerical value of the Hamiltonian which, on account of the initial value equations

$$\mathcal{H}^0 = 0 \; , \tag{I.1.34a}$$

$$\mathcal{H}^i = 0 \; , \tag{I.1.34b}$$

is precisely the surface integral (I.1.31).

In much the same way as the energy is identified as the conserved quantity (the Hamiltonian) associated to invariance of the action under time displacements at infinity one can identify the momentum

$$P^i = - 2 \int \pi^{ij} \, d^2 s_j \; , \tag{I.1.35}$$

and the angular momentum

$$L_i = 2 \int \epsilon_{ijk} \, \pi^{lj} \, x^k \, d^2 s_l \; , \tag{I.1.36}$$

as the conserved quantities associated to translations and rotations at infinity.

What we are trying to emphasize at this point is that the identification of energy, momentum and angular momentum can be achieved without solving (even in principle) the constraint equations and, consequantly, before fixing the space-time coordinates and finding the corresponding reduced Hamiltonian of the theory ("true Hamiltonian" in the terminology of ADM). On the contrary the reduced Hamiltonian corresponding to a particular fixation of the coordinates is obtained by inserting the solutions of the constraint equations (I.1.34) into the surface integral (I.1.31). It becomes then trivial to see that any fixation of the coordinates leads to the same value for the energy. Similar remarks apply to the other conserved quantities. ADM worked in the opposite direction, reducing first the Hamiltonian and then identifying the value of the reduced Hamiltonian with the energy. It is then not obvious at all that different reductions will lead to the same value for the energy (Arnowitt et al., 1961).

The understanding of the necessity of including the surface integral $E[g_{ij}]$ in the Hamiltonian to start with clarifies also a difficulty found by ADM, who discarded the surface integral at the beginning but found that if they would do so also at the end (i.e., after coordinate conditions are imposed) they would be left with no Hamiltonian. To bypass the difficulty they stated (Arnowitt et al., 1960, 1962) that "It should be emphasized that, while the energy and momentum are indeed divergences, the integrands in the Hamiltonian and in the space translation generators are not divergences when expressed as functions of the canonical variables." What ADM seemed to have in mind is that before solving the constraints one can assume that δg_{ij} and $\delta\pi^{ij}$ vanish sufficiently fast at infinity so that $\delta E[g_{ij}]=0$, but that, on the other hand, one cannot make such an assuption after solving the constraints, because at that moment the bahavior of some of the canonical variables is determined by the behavior of the others. As we will see, in our approach the asymptotic behavior of δg_{ij} and $\delta\pi^{ij}$ is determined from general considerations having little to do with imposing coordinate conditions. Such an asymptotic behavior stays the same before and after solving the constraints. Consequently a surface integral that can be eliminated at the beginning (as the surface terms other than $E[g_{ij}]$ in (I.1.32)) can also be elimimated at the end. On the other hand, a surface integral having a non-zero variation at the beginning ($E[g_{ij}]$) will also have a non-zero variation at the end and consequently cannot ever be discarded.

Another bonus of the present treatment is a clear cut proof of the correctness of Dirac's result (Dirac, 1959) in the reduction of the Hamiltonian for maximal slicing of space-time. (It should be emphasized that although we find the result to be correct we are not satisfied with the original reasoning of Dirac. In this aspect we agree with ADM (Arnowitt et al., 1960) who found the original Dirac's procedure "logically incomplete.") For the sake of completeness we also show how the reduction of the Hamiltonian by ADM coordinate conditions is achieved in our approach.

Finally the problem of invariance of the theory under Poincaré transformations at infinity is considered. To begin with, the previous results are generalised so as to allow for asymptotic Poincaré transformations among the permissible deformations of the hypersurface. In that case nine surface integrals besides (I.1.32) contribute to the Hamiltonian. These integrals are the six momentum and angular momentum expression (I.1.35), (I.1.36) and three other quantities related to the "center-of-mass" motion. If one fixes the coordinates in the "interior" but still leaves open the possibility of making Poincaré transformations at infinity then the ten generators of the Poincaré group are obtained by inserting the solutions of the constraints into the surface integrals in question, in total analogy with the procedure for reducing the (less general) Hamiltonian (I.1.32). The steps described above are not however the end in achieving a Hamiltonian formalism which is manifestly covariant under Poincaré transformations at infinity. To reach this goal one must introduce, according to the general method of Dirac (1951, 1964), the ten variables describing the asymptotic location of the surface (together with their conjugate momenta) as extra canonical variables besides, and in the same footing with, the (g_{ij},π^{ij}). In doing so one acquires ten new constraints which enter into the

Hamiltonian in the same footing with eqs.(I.1.34). The new Hamiltonian obtained in this way vanishes then weakly and is this quantity which is the analog of H_0 given by (I.1.30) for compact spaces.

The true phase space of the gravitational field for asymptotically flat spaces is therefore the space of the (g_{ij}, π^{ij}) "completed by the introduction of boundary conditions as canonical variables." We might mention, incidentally, that in this enlarged phase space the canonical transformations involving inverse Laplacians used by ADM (Arnowitt et al., 1962) become one to one and time independent, two properties which they did not simultaneously possess in the original space of g_{ij}, π^{ij}.

I.2. Field Theory a la Poincaré-Cartan

Here we introduce the fundamentals of field theory formulated as a variational principle over a fibered manifold. One of the principal bases for any reasonable physical field theory consists of the principle of general invariance (or also general covariance or relativistic invariance). Many formulations of this fundamental principle may be found in the literature, but all of them esentially amount to requiring that the differential equations governing the dynamics of fields (i.e., the field equations) have the same form for all the observers, i.e., for all frames of reference in space-time. Differential calculus over manifolds tells us that this happens if and only if field equations have tensorial character, a property which was known since the formulation of Einstein's theory of General Relativity. What seems to be relatively less understood is that general invariance does not require all fields to be tensor fields over space-time, but merely the much less stringent requirement that fields may be Lie-dragged along the flow of any vectore field in space-time. Again, differential calculus over manifolds tells us that this happens only if the fields are fields of geometric objects, which, roughly speaking, amounts to requiring that changes of coordinates in space-time define uniquely the transformation laws of the object themselves. Lagrangian field theories depending on geometric objects and having generally invariant field equations will be called here geometric field theories; space-time will be generally denoted by M.

Geometric field theories are important for several reasons. To our understanding, the main feature of this class of theories consits in the fact that they are the natural framework for defining and investigating the physically fundamental concept of energy. In fact, for any generally invariant Lagrangian $\mathcal{L}$ and any vector field X on M we can uniquely define a vector density $E^i(\mathcal{L};X)$, called the energy flow of $\mathcal{L}$ along X, such that its divergence vanishes along all solutions to the relevant field equations (namely, the Euler-Lagrange equations of $\mathcal{L}$). According to this property, Stokes' theorem implies that the integral of E^i

is zero over any closed (i.e., compact without boundary) hypersurface of M; this justifies the physical interpretation of E^i itself as the flux of energy associated with all fields. The vanishing of the divergence of E^i expresses then the conservation of energy.

As is well known, the physically most significant theories are those which describe the interaction of several fields, or, even better, the dynamics of a single field which unifies more than one elementary field. Acoording to a standard viewpoint, when dealing with theories of interacting fields one tends to select some of of the fields involved and to interpret them as the basic fields, considering the remaining ones as sources for the basic fields. Widely known examples are provided by the relativistic theories of gravitation (see, e.g., Misner et al., 1973) and by gauge theories of elementary particles interacting with fundamental forces (see, e.g., Carmeli, 1982). More generally, one may envisage situatons in which the fields are split into more than two groups.

In all these cases, it is either assumed *a priori* or obtained by some trick (such as, e.g., partial Legendre transformation, spontaneous symmetry breaking, etc.), that the Lagrangian governing the theory splits into a suitable number of partial Lagrangians $\mathcal{L}_{(\alpha)}$. Each partial Lagrangian $\mathcal{L}_{(\alpha)}$ is then interpreted as the basic Lagrangian for one or more of the fields. Physically meaningful interpretation of these partial Lagrangians are in fact available only if some additional requirement is made on the splitting, like, for example, the minimal coupling conditions.

When dealing with interacting fields, it is a classical procedure to describe the details of their interaction through suitable tensorial or pseudotensorial objects, called stress tensors or stress pseudotensors, which roughly speaking express the response of some of the fields when the remaining ones are subjected to deformations induced by changes of coordinates in space-time. Again, we see, just from this naive definition, that geometric field theories constitute the natural framework for defining and discussing the notion of stress tensors. In this chapter, we shall deal with geometric theories of interacting fields, with the aim of providing a general framework suited to investigate energy, conservation laws, and stress tensors for any Lagrangian theory of geometric fields, no matter how many fields are involved and how many of their derivatives enter the Lagrangian.

We start by discussing some generalities concerning the global structure of higher-order calculus of variations. We shall assume that the reader is familiar with the main concepts from the theory of fibered manifolds, jet prolongations, and bundles of geometric objects. All manifolds, mappings, and objects considered here are assumed to be smooth (in the C^∞ sense). Further discussions of these general concepts may be found, for example, in (Choquet-Bruhat et al., 1977; Pommaret, 1978; Schouten and Haantjes, 1936; Haantjes and Laman, 1953a, b; Kuiper and Yano, 1955; Salvioli, 1972).

Next we consider geometric field theories and natural conservation laws, i.e., Lagrangian field theories which are based on the jet prolongations of bundles of geometric objects. This, in fact, is the natural framework for introducing the concept of natural conservation laws.

I.2.1. Fibered Manifolds and Jet Prolongations

Let M be a differentiable manifold, with dim(M)=m. The following standard notation will be used throughout: diff(M) denotes the set of local diffeomorphisms of M into M; T(M) denotes the tangent bundle of M; and $A^p_q(M)$, respectively $S^p_q(M)$, is the vector bundle of p-contravariant and q-covariant skew-symmetric, respectively symmetric, tensors over M. In particular, the bundle $A^0_q(M)$ coincides with the qth exterior power $\Lambda^q[T^*(M)]$ of the cotangent bundle of M.

Let $B=(B,M,\beta)$ be a fibered manifold over the manifold M. We shall adopt the following notation: aut(B) is the set of all local automorphisms of the fibered manifold B; $C^\infty(B)$ is the set of local sections of class C^∞ of B; and $V(B)=(V(B),B,\nu_B)$ denotes the vertical bundle of B. V(B) is a vector bundle over B, whose sections are called vertical vector fields over B.

The kth-order jet prolongation of a fibered manifold B, where k is any positive integer, will be denoted by $J^k(B)=(J^k(B),M,\beta^k)$; we agree that the zeroth-order prolongation of B coincides with B itself. We recall also that for any $h\geq k$ there is a canonical projection β^h_k from $J^h(B)$ onto $J^k(B)$, such that $J^h_k(B)=(J^h(B),J^k(B),\beta^h_k)$ is a fiber bundle; in particular, we know that $J^{k+1}_k(B)$ is an affine bundle. Furthermore, if $Z=(Z,J^k(B),\zeta)$ is a fibered manifold and h is any integer larger than k, the pullback $(\beta^h_k)^*(Z)$ is a bundle over $J^h(B)$; accordingly, any fibered morphism $f:J^h(B)\rightarrow Z$ over the projection β^h_k may be canonically identified to a section of the pullback bundle $(\beta^h_k)^*(Z)$ over $J^h(B)$.

We consider now the following family of vector bundles over $J^k(B)$

$$A^0_q(M) \otimes \Lambda^p\{V^*[J^k(B)]\} \ , \qquad \text{(I.2.1)}$$

where (q,p,k) are three non-negative integers, which will often enter our nexts considerations on higher-order calculus of variations. Given any integer $h\geq k$, one can use the canonical projections β^h_k and β^h to identify, by pullback, the bundle above with a vector subbundle of $A^0_{p+q}[J^h(B)]$ which will be denoted by $\Phi^p_q(\beta^h_k)$. Accordingly, any fibered morphism $f:J^h(B)\rightarrow A^0_q(M)\otimes\Lambda^p\{V^*[J^k(B)]\}$, over the projection β^h_k, may be canonically identified to a section of the bundle $\Phi^p_q(\beta^h_k)$, i.e., to a suitable (p+q) form on $J^h(Q)$. Clearly, this applies also to local fibered morphisms, which give rise to local (p+q)-forms.

Let us now give some local coordinate notations. In any local chart $(U;x^i)$ of the manifold M, we define the following, local, forms

$$ds(x) = dx^1\wedge\ldots\wedge dx^m \ , \qquad \text{(I.2.2)}$$

$$ds_{h_1\ldots h_r}(x) = \partial_{h_r}\rfloor\ldots\rfloor\partial_{h_1}\rfloor ds(x) \ . \qquad \text{(I.2.3)}$$

The m-form ds(x) defines a basis of the vector bundle $A^0_m(U)$, while the (m-1) forms $(ds_1(x),\ldots,ds_m(x))$ constitute a basis for the vector bundle $A^0_{m-1}(U)$.

Now, let B be a fibered manifold over M. In the sequel, we agree to consider only charts $(W;z^\alpha)$ of B, with $\alpha=1,\ldots,\dim(B)$, which are fibered over the charts of and atlas of M; these charts will be shortly called fibered charts of B. The corresponding coordinates, called fibered coordinates, will be generally denoted by (x^i, y^A), with $i=1,\dots,m$, and $A=1,\dots,\dim(B)-m$, where (x^i) stands for any system of local coordinates in the open subset $\beta(W)\subseteq M$.

For any fibered chart $(W;x^i,y^A)$ of B there exists an induced fibered chart $(V(W);x^i,y^A,\upsilon^A)$ of V(B), where $V(W)=(\nu_B)^{-1}(W)$, which is called the natural fibered chart of V(B), induced by the fibered chart W of B. Moreover, any fibered chart $(W;x^i,y^A)$ of B induces, uniquely, natural fibered charts in each one of the bundles $J^k(B)$, $V[J^k(B)]$ and $J^s[T(M)]$. The domains of these charts will be denoted by $J^k(W)$, $V[J^k(W)]$ and $J^s[T(W)]$, respectively, while the corresponding local coordinates will be denoted, respectively, by

$$(x^i;\ y^A,\ y^A_{j_1},\ \dots,\ y^A_{j_1\dots j_k})\ ,$$

$$(x^i;\ y^A,\ y^A_{j_1},\ \dots,\ y^A_{j_1\dots j_k};\ \upsilon^A,\ \upsilon^A_j,\ \dots,\ \upsilon^A_{j_1\dots j_k})\ ,$$

$$(x^i;\ X^i,\ \dots,\ X^i_{j_1\dots j_s})\ . \tag{I.2.4}$$

We finally recall that for any fibered manifold B and any integer k, $k\geq 0$, there exists a canonical isomorphism $i^k:J^k[V(B)]\to V[J^k(B)]$. For any local differentiable function $f:J^k(W)\to R$, where $W\subseteq B$ is the domain of a fibered chart, we shall denote by $d_i(f):J^{k+1}(W)\to R$ the formal partial derivative of f with respect to the coordinate x^i.

I.2.2. Linear Connections and Tensorisation Procedures

We shall recall here the procedure which allows us to replace the partial derivatives of the components of a vector field, having non-tensorial character, with a suitable set of tensors constructed using the symmetrised covariant derivatives of these components with respect to an arbitrary, linear, connection.

Let C be any linear connection on a differentiable manifold M. The connection C induces a linear isomorphism

$$\psi_C : J^r[T(M)] \to \mathop{\oplus}_{p=0}^{r} S^1_p(M) \ , \tag{I.2.5}$$

having the following local representation

$$(x^h,\ X^h,\ X^h_{j_1},\ \ldots,\ X^h_{j_1\ldots j_r})$$

$$\to (x^h,\ X^h,\ \nabla_{j_1} X^h,\ \ldots,\ \nabla_{(j_1}\ldots\nabla_{j_r)}X^h) \ , \tag{I.2.6}$$

where ∇ denotes the covariant derivative with respect to C. Notice that we can also define a dual isomorphism for any pair of non-negative integers (q,r)

$$\bar{\psi}_C : [A^0_{m-q}(M)]\otimes\{J^r[T(M)]\}^* \to [A^0_{m-q}(M)]\otimes[\mathop{\oplus}_{p=0}^{r} S^p_1(M)] \ , \tag{I.2.7}$$

by setting

$$<\zeta|j^r(X)> = <\bar{\psi}_C(\zeta)|\psi_C[j^r(X)]> \ , \tag{I.2.8}$$

for any point ζ in $[A^0_{m-q}(M)] \otimes \{J^r[T(M)]\}^*$ and any vector field X.

Let

$$(x^i,\ z_i^{h_1\ldots h_q},\ z_i^{h_1\ldots h_q,j_1},\ \ldots,\ z_i^{h_1\ldots h_q,j_1\ldots j_r}) \ ,$$

$$(x^i,\ Z_i^{h_1\ldots h_q},\ Z_i^{h_1\ldots h_q,j_1},\ \ldots,\ Z_i^{h_1\ldots h_q,j_1\ldots j_r}) \ , \tag{I.2.9}$$

now, respectively, be natural fibered coordinates in the two vector bundles $[A^0_{m-q}(M)]\otimes\{J^r[T(M)]\}^*$ and $[A^0_{m-q}(M)]\otimes[\mathop{\oplus}_{p=0}^{r} S^p_1(M)]$. In these local coordinates we can describe as follows the action of the dual morphism $\bar{\psi}_C$. We first represent the point ζ and its image $\bar{\psi}_C(\zeta)$ as follows

$$<\zeta|j^r(X)> = \frac{1}{q!}\ ds_{h_1\ldots h_q}(x)\ <z^{h_1\ldots h_q}|j^r(X)> \ ,$$

$$<\bar{\psi}_C(\zeta)|\psi_C[j^r(X)]> = \frac{1}{q!}\ ds_{h_1\ldots h_q}(x)$$

$$\times <Z^{h_1\ldots h_q}|\psi_C[j^r(X)]> \ , \tag{I.2.10}$$

where $<z^{h_1\ldots h_q}|j^r(X)>$ and $<Z^{h_1\ldots h_q}|\psi_C[j^r(X)]>$ are the, skew-symmetric, tensor densities defined by the following expansions

$$<z^{h_1\ldots h_q}|j^r(X)> = z^{h_1\ldots h_q}_i X^i + z^{h_1\ldots h_q,j_1}_i X^i_{j_1}$$

$$+ \dots + z^{h_1 \dots h_q, j_1 \dots j_r} X^i_{j_1 \dots j_r} \ , \qquad \text{(I.2.11)}$$

and

$$\langle Z^{h_1 \dots h_q} | \psi_C [j^r(X)] \rangle = Z^{h_1 \dots h_q} X^i + Z^{h_1 \dots h_q, j_1} \nabla_{j_1} X^i$$

$$+ \dots + Z^{h_1 \dots h_q, j_1 \dots j_r} \nabla_{(j_1} \dots \nabla_{j_r)} X^i \ . \qquad \text{(I.2.12)}$$

The explicit relations between the coefficients z and Z may be thus obtained by applying (I.2.8), i.e., by equating the right-hand sides of expressions (I.2.11) and (I.2.12).

I.2.3. Bundles of Geometric Objects and Lie Derivatives

The correct setting for dealing with generally invariant field theories is the framework of bundles of geometric objects, of finite order, also known as natural bundles. In fact, in the formulation of physical field theories it is unavoidable to introduce non-tensorial entities, like, for example, jet prolongations of tensor fields and linear connections. We shall here assume that the reader is familiar with the notion of bundles of geometric objects and we shall limit ourselves to recall its role in the definition of Lie derivatives. Further details and references may be found in (Schouten and Haantjes, 1936; Haantjes and Laman, 1953a, b; Kuiper and Yano, 1955; Salvioli, 1972; Ferraris, Francaviglia and Reina, 1983a, b).

Let $B=(B,M,\beta)$ be a bundle of geometric objects, of finite order, over a manifold M. A functorial mapping is then defined as

$$(\ )_B : \mathrm{diff}(M) \to \mathrm{aut}(B) \ , \qquad \text{(I.2.13)}$$

which lifts any local diffeomorphism φ of the basis M into a, unique, local automorphism φ_B, over φ, of the bundle B; this automorphism is called the natural lift of φ. Examples of bundles of geometric objects over a manifold M are the following: the tangent bundle T(M), with the lift given by $\varphi \mapsto T(\varphi)$; all tensor bundles over M, with the natural lift defined by the push forward of tensors; and the bundle C(M) of linear connection over M, with the lift defined by the natural action of diff(M) on connections, etc. Recall also that for any bundle of geometric objects $B=(B,M,\beta)$ the tensor bundles $(T^r_s(B),M,\beta\circ\tau^r_s)$, the bundle $(V(B),M,\beta\circ\nu_B)$, and all the jet prolongations $(J^k(B), M, \beta^k)$ are bundles of geometric objects over M.

Let $X:M\to T(M)$ be a vector field on M. Using $()_B$ we can associate to X a unique vector field $X_B:B\to T(B)$, which is called the natural lift of X to the bundle B. The vector field X_B is defined as follows: for any $b\in B$ one sets

$$X_B(b) = d_t[(\varphi_t)_B(b)]\Big|_{t=0} , \tag{I.2.14}$$

where φ_t denotes the local flow in M generated by X. The mapping

$$(\ )_B\colon C^\infty[T(M)] \to C^\infty[T(B)] , \tag{I.2.15}$$

defined by $X\to X_B$, is linear and satisfies the following properties:

(i) for any vector field X over M one has $T(\beta)\circ X_B=X\circ\beta$; and

(ii) for any pair (X,Y) of vector fields on M, one has $([X,Y])_B=[X_B,Y_B]$.

Now, let σ be a local section of B, i.e., $\sigma:U\to B$, and X be a vector field over U. We can define a local section $L_X(\sigma):U\to V(B)$ of the bundle $(V(B),M,\beta\circ\nu_B)$ by setting

$$L_X(\sigma) = T(\sigma)\circ X - X_B\circ\sigma . \tag{I.2.16}$$

The local section $L_X(\sigma)$ is called the Lie derivative of σ along the vector field X and it satisfies the following property: $\nu_B\circ L_X(\sigma)=\sigma$. We now recall that for any positive integer $k\geq 0$ there exists a canonical isomorphism from $V[J^k(B)]$ to $J^k[V(B)]$. Then, it can be shown that the following holds

$$L_X[j^k(\sigma)] = j^k[L_X(\sigma)] , \tag{I.2.17}$$

for any local section σ of the bundle B and for any local vector field X over the basis manifold M. Moreover, the following properties hold:

(i) For any vector field X over M, the mapping $\sigma\to L_X(\sigma)$ is a first-order quasilinear differential operator.

(ii) For any local section σ of B, the mapping $X\to L_X(\sigma)$ is a linear differential operator, having as order the order s of B as a bundle of geometric objects.

From (i) and (ii) above it follows that the local representations of $L_X(\sigma)$ have necessarily the following form

$$x^i \circ L_X(\sigma) = x^i \circ \sigma \ , \qquad y^A \circ L_X(\sigma) = y^A \circ \sigma \ ,$$

$$\upsilon^A \circ L_X(\sigma) = X^i \ y^A_{\ i} \circ [j^1(\sigma)] + b^A_{\ i}(x, y\circ\sigma) \ X^i + b^{Aj_1}_{\ \ \ i}(x, y\circ\sigma) \ X^i_{\ j_1}$$

$$+ \ldots + b^{Aj_1 \ldots j_s}_{\ \ \ \ \ \ \ \ i}(x, y\circ\sigma) \ X^i_{\ j_1 \ldots j_s} \ , \qquad \text{(I.2.19)}$$

where $(x^i, y^A, \upsilon^A, y^A_{\ i})$ are natural fibered coordinates and the coefficients $b^A_{\ i}(x,y)$ and $b^{Aj_1 \ldots j_p}_{\ \ \ \ \ \ \ \ i}(x,y)$, with $1 \le p \le s$, are functions of (x^i, y^A) which depend on the choice of the fibered chart in B but do not depend on the particular section σ chosen.

We finally recall from (Yano, 1955) that a bundle of geometric objects is said to be of differentiable type if it admits, at least, one natural atlas whose fibered charts (x^i, y^A) are such that the coefficients $b^A_{\ i}(x,y)$ vanish identically, while the coefficients $b^{Aj_1 \ldots j_p}_{\ \ \ \ \ \ \ \ i}(x,y)$, $1 \le p \le s$, depend only on the fiber coordinates y^A. We remark that, to our knowledge, all the bundles of geometric objects which enter the formulation of physical field theories are precisely of this type. Accordingly, in the following we shall restrict our attention only to this class of bundles.

I.2.4. Calculus of Variations on Fibered Manifolds

We now have to recall some concepts we need from the geometric formulation of the calculus of variations on fibered manifolds, as it was developed in (Krupka, 1973, 1975a, b, 1982; Ferraris and Francaviglia, 1982). In these papers the reader may find more details and further references, also concerning alternative viewpoints.

According to Krupka (1973, 1975a, b) a variational problem of order k is defined by assigning the following:

(i) a fibered manifold $Q=(Q,M,\pi)$ over a differentiable manifold M of dimension m, and

(ii) a morphism $\mathcal{L}: J^k(Q) \to A^0_m(M)$ of fibered manifolds over M.

The fibered manifold Q is called the configuration space and its local sections represent the physical fields. The fibered morphism $\mathcal{L}$ is called the Lagrangian density of the variational problem; it defines the action functionals $\mathcal{L}_D : C^\infty \to R$ by

$$\mathcal{L}_D(\sigma) = \int_D \mathcal{L} \circ j^k(\sigma) \ , \tag{I.2.20}$$

where $D \subseteq M$ is any compact domain. Solving the variational problem consists thence in finding the critical sections of the action functionals, i.e., those local sections $\sigma \in C^\infty(Q)$ which make stationary all functionals above when D ranges through all compact domains of M.

We remark that there exists the following canonical isomorphism of vector bundles over $J^K(Q)$

$$(\pi^k)^*[A^0_m(M)] \simeq A^0_m(M) \otimes \Lambda^0\{V^*[J^k(Q)]\} \ . \tag{I.2.21}$$

Therefore, according to the remarks of sec.I.2.1, the Lagrangian $\mathcal{L}$ can be canonically identified to a global section $\Phi(\mathcal{L})$ of the vector bundle $\Phi^0_m[J^k(Q)]$ i.e. to a global m-form over $J^k(Q)$.

Then, let $\mathcal{L} : J^k(Q) \to A^0_m(M)$ be a Lagrangian, of order k, over the configuration space $Q=(Q,M,\pi)$ and let $\mathcal{L}=ds(x) \otimes L$ be its local representation with respect to any system of natural fibered coordinates. Restricting $T(\mathcal{L})$ to the vector subbundle $V[J^k(Q)]$ of $T[J^k(Q)]$ and taking into account the linearity of the tangent map, we may define uniquely a fibered morphism over the identity of M,

$$\hat{p}(\mathcal{L}) : J^k(Q) \to A^0_m(M) \otimes V^*[J^k(Q)] \ . \tag{I.2.22}$$

According to sec.I.2.1, the morphism $\hat{p}(\mathcal{L})$ can also be interpreted as a global section $\Phi[\hat{p}(\mathcal{L})]$ of the vector bundle $\Phi^1_m[J^k(B)]$ over $J^k(B)$, i.e., as a global (m+1)-form over $J^k(B)$. It can be seen that the (m+1)-form $\Phi[\hat{p}(\mathcal{L})]$ so defined is in fact the exterior differential $d\Phi(\mathcal{L})$ of the m-form $\Phi(\mathcal{L})$ which is canonically associated to the Lagrangian $\mathcal{L}$ itself. The action over $V[J^k(Q)]$ of the morphism $\hat{p}(\mathcal{L})$ may be represented as follows for any system of natural fibered coordinates

$$\begin{aligned} \langle \hat{p}(\mathcal{L}) | \upsilon \rangle &= ds(x)\ \langle p(L) | \upsilon \rangle \\ &= ds(x)\ [p_A(L)\ \upsilon^A + p_A^{\ j_1}(L)\ \upsilon^A_{\ j_1} \\ &\qquad + \ldots + p_A^{\ j_1 \ldots j_k}(L)\ \upsilon^A_{\ j_1 \ldots j_k}] \ , \end{aligned} \tag{I.2.23}$$

where we have set

$$p_A(L) = \partial_A L \; ,$$

$$p_A^{\,j_1 \ldots j_h}(L) = \partial_A^{\,j_1 \ldots j_h} L \; , \qquad 1 \le h \le k \; . \qquad (I.2.24)$$

The critical sections σ of our variational problem satisfy the Euler-Lagrange equations, which, although being generally written in coordinates, are globally and intrinsically well-defined over M, see e.g. (Krupka, 1974). In fact, these equations may be represented as follows

$$\hat{e}(\mathcal{L}) \circ j^{2k}(\sigma) = 0 \; , \quad \hat{e}(\mathcal{L}) : J^{2k}(Q) \to A^0_m(M) \otimes V^*(Q) \; , \qquad (I.2.25)$$

a global morphism of fibered manifolds over M. The action of $\hat{e}(\mathcal{L})$ over V(Q) is defined, for any natural fibered chart, by the following local expressions

$$<\hat{e}(\mathcal{L}) | \upsilon> = ds(x) <e(L)|\upsilon> = ds(x) \; [e_A(L) \; \upsilon^A] \; ,$$

$$e_A(L) = p_A(L) - d_{j_1} [p_A^{\,j_1}(L)]$$

$$+ \ldots + (-1)^k \, d_{j_1} \ldots d_{j_k} [p_A^{\,j_1 \ldots j_k}(L)] \; . \qquad (I.2.26)$$

This morphism is known as the Euler-Lagrange operator associated to the Lagrangian $\mathcal{L}$, see e.g. (Krupka, 1974, 1982). We remark that it can be canonically identified to a global section $\Phi[\hat{e}(\mathcal{L})]$ of the vector bundle $\Phi^1_m[J^{2k}(Q)]$, i.e., to a global (m+1)-form over $J^{2k}(Q)$.

I.2.5. First Variation Formula and the Poincaré-Cartan Forms

This section is a short discussion about the role which the Poincaré-Cartan form plays in the formulation of the first variation formula in higher-order variational problems. The results developed here will be of use for a precise definition of stress tensors and energy-momentum tensors, which will be investigated in secs.I.2.6 and onwards.

Let $\mathcal{L}: J^k(Q) \to A^0_m(M)$ be a Lagrangian, of order k, over the configuration space $Q=(Q,M,\pi)$. It is possible to show that there exists, at least, one global morphism $\hat{f}(\mathcal{L})$ of fiber bundles over

$J^{k-1}(Q)$

$$\hat{f}(\mathcal{L}) : J^{2k-1}(Q) \to A^{0}_{m-1}(M) \otimes V^{*}[J^{k-1}(Q)] \ , \qquad (I.2.27)$$

such that the following holds

$$\langle \hat{p}(\mathcal{L}) \circ j^{k}(\sigma) | j^{k}(\upsilon)\rangle = \langle \hat{e}(\mathcal{L}) \circ j^{2k}(\sigma) | \upsilon\rangle$$

$$+ \, d\langle \hat{f}(\mathcal{L}) \circ j^{2k-1}(\sigma) | j^{k-1}(\upsilon)\rangle \ , \qquad (I.2.28)$$

where $\upsilon : U \to V(Q)$ is any local section and $\sigma : U \to Q$ is the local section of Q defined by $\sigma = \nu_{Q} \circ \upsilon$. Integrating (I.2.28) over any compact domain $D \subseteq U$, we recover the first variation formula, over D, for the Lagrangian $\mathcal{L}$. According to the results of sec.I.2.1, also in this case the global morphism $\hat{f}(\mathcal{L})$ can be canonically identified to a global section $\Phi[\hat{f}(\mathcal{L})]$ of the vector bundle $\Phi^{1}_{m-1}[J^{2k-1}(Q)]$, i.e., to a global m-form on $J^{2k-1}(Q)$. The action of the global morphism $\hat{f}(\mathcal{L})$ on the vector bundle $V[J^{k-1}(Q)]$ is defined, in any natural fibered chart by the following local expression

$$\langle \hat{f}(\mathcal{L}) | \upsilon\rangle = ds_{i}(x) \ \langle f^{i}(L) | \upsilon\rangle$$

$$= ds_{i}(x) \ [f^{i}{}_{A}(L) \ \upsilon^{A} + f^{ij_{1}}{}_{A}(L) \ \upsilon^{A}{}_{j_{1}}$$

$$+ \ldots + f^{ij_{1}\ldots j_{k-1}}{}_{A}(L) \ \upsilon^{A}{}_{j_{1}\ldots j_{k-1}} \ ] \ . \qquad (I.2.29)$$

Moreover, rewriting (I.2.28) in a natural fibered chart we find the following local expression

$$\langle p(L) \circ j^{k}(\sigma) | j^{k}(\upsilon)\rangle = \langle e(L) \circ j^{2k}(\sigma) | \upsilon\rangle$$

$$+ \, d_{i}\langle f^{i}(L) \circ j^{2k-1}(\sigma) | j^{k-1}(\upsilon)\rangle \ , \qquad (I.2.30)$$

which is more significant than (I.2.28) itself, since, roughly speaking, it amounts to decomposing the first variation of $\mathcal{L}$ into field equations plus a formal total divergence. From the physical viewpoint, the divergence appearing in (I.2.30) will allow us to define the Noether's conserved current, see secs.I.2.6 and onwards.

As far as the uniqueness of the morphisms $\hat{e}(\mathcal{L})$ and $\hat{f}(\mathcal{L})$ is concerned, we remark the following:

(i) In general, there exists a whole family of global morphisms $\hat{f}(\mathcal{L})$ which satisfy the properties above. In any case, no matter which one of them is chosen, the morphism $\hat{e}(\mathcal{L})$ such that (I.2.28) holds is uniquely defined. This property is known as the unicity of the Euler-Lagrange operator, see e.g. (Krupka, 1974).

(ii) The morphism $\hat{f}(\mathcal{L})$ is uniquely defined if the order of the Lagrangian is 1, no matter which is the dimension m of the basis M, i.e., $m\geq 1$, $k=1$ (Goldschmidt and Sternberg, 1973). It is also uniquely defined if the basis manifold is one dimensional, no matter which is the order k of the Lagrangian, i.e., $m=1$, $k\geq 1$, see (Sternberg, 1977).

(iii) However, when $m\geq 2$ and also $k\geq 2$, the morphism $\hat{f}(\mathcal{L})$ is globally but not uniquely defined. In fact, let $\hat{f}_1(\mathcal{L})$ be a global morphism which satisfies the required properties; then, if we consider any global fibered morphism over $J^{k-2}(Q)$

$$\hat{h}: J^{2k-2}(Q) \to A^{0}_{m-2}(M) \otimes V^{*}[J^{k-2}(Q)] , \tag{I.2.31}$$

and we denote by

$$\mathrm{div}(\hat{h}): J^{2k-1}(Q) \to A^{0}_{m-1}(M) \otimes V^{*}[J^{k-1}(Q)] \tag{I.2.32}$$

its formal divergence, we see that the global morphism $\hat{f}_2(\mathcal{L},\hat{h})=\hat{f}_1(\mathcal{L})+\mathrm{div}(\hat{h})$ satisfies the required properties, too, see (Krupka, 1982). For example, as was shown in (Garcia and Muñoz, 1982; Ferraris, 1983; Kolar, 1983), from any such morphism one can generate a whole family which depends on a connection.

(iv) We finally remark that in the particular case $m\geq 2$ and $k=2$, there exists a canonical global morphism $\hat{f}(\mathcal{L})$ satisfying the required properties, which is defined, in any natural fibered chart, by the following local expressions

$$f^{i}{}_{A}(L) = p^{i}{}_{A}(L) - d_{j}p^{ij}{}_{A}(L) , \tag{I.2.33}$$

$$f^{ij}{}_{A}(L) = p^{ij}{}_{A}(L) . \tag{I.2.34}$$

Let us now remark that, taking into account the methods discussed in sec.I.2.1, to any kth-order Lagrangian and to each global morphism $\hat{F}:J^{2k-1}(Q)\to A^{0}_{m-1}(M)\otimes V^{*}[J^{k-1}(Q)]$ there corresponds a global m-form $\theta(\mathcal{L},\hat{F})$ over $J^{2k-1}(Q)$, defined by the following prescription

$$\theta(\mathcal{L}, \hat{F}) = (\pi_k^{2k-1})^* \Phi(\mathcal{L}) + \Phi(\hat{F}) \ . \qquad \text{(I.2.35)}$$

According to the terminology used by Krupka (1982) when the morphism $\hat{F}$ is any one of the morphisms $\hat{f}(\mathcal{L})$ which satisfy eq.(I.2.30), the corresponding m-form $\theta[\mathcal{L}, \hat{f}(\mathcal{L})]$ is said to be a global Lepagean equivalent Lagrangian $\mathcal{L}$. Any such m-form can be assumed as a Poincaré-Cartan form associated to the Lagrangian $\mathcal{L}$. According to our remarks above, we see thus that the following holds:

(i) the Poincaré-Cartan form is uniquely defined if the order of the Lagrangian is 1, no matter which is the dimension m of the basis M, i.e., $m \geq 1$, $k=1$, as well as if the basis manifold M is one dimensional, no matter which is the order of the Lagrangian, i.e., $m=1$, $k \geq 1$;

(ii) when $m \geq 2$ and also $k \geq 2$, there exists a whole family of global Poincaré-Cartan forms, which locally differ by a formal divergence, for example, one can generate a family of global Poincaré-Cartan forms which depend on a couple of connections; and (iii) in the particular case $m \geq 2$ and $k=2$, there exists however a canonical Poincaré-Cartan form.

I.2.6. General Invariance, Geometric Field Theories, and Energy Flow

One of the fundamental requirements of physical field theories over space-time is the invariance of their field equations with respect to any change of (local) coordinates in the space-time manifold itself; this requirement, which has been generally accepted since the early developments of Einsten's theory of General Relativity, is commonly known as general invariance (or relativistic invariance, or also as general covariance). However, when dealing with Lagrangian field theories one usually makes a stronger assumption; namely, one requires the general invariance of the Lagrangian itself. Requiring the general invariance of the Lagrangian allows us then to apply the second Noether's theorem and to generate, as a consequence, a whole family of natural conservation laws, i.e., those conservation laws which are naturally associated with the dragging of physical fields along the flows generated by vector fields over the basis. We shall assume here that the reader is familiar with the fundamental concepts and ideas of this theory; for a detailed and comprehensive account of them we refer to (Trautman, 1962; Krupka and Trautman, 1974; Krupka, 1974, 1976, 1978) and references quoted therein.

We are now in position to give the following definition: A *k*th-order Lagrangian theory is called a geometric field theory if the following two conditions are satisfied.

(i) The configuration space $Q=(Q,M,\pi)$ is a bundle of geometric objects of order s.

(ii) The Lagrangian $\mathcal{L}$ governing the dynamics of fields is generally invariant (Krupka and Trautman, 1974).

The order s of the bundle Q is the geometric order of the theory, while the sum $r=k+s-1$ is the differential degree of the theory. This terminology will be clarified later.

According to standard results (Krupka and Trautman, 1974), requiring the general invariance of the Lagrangian $\mathcal{L}$ in a geometric field theory implies that $\mathcal{L}$ should satisfy the following relation

$$T(\mathcal{L})\circ j^k[L_X(\sigma)] = L_X[\mathcal{L}\circ j^k(\sigma)] \ , \qquad (I.2.36)$$

where X is any vector field over the basis manifold M and σ is any (local) section of the bundle Q. Since $\mathcal{L}\circ j^k(\sigma)$ is a scalar density, the following holds

$$L_X[\mathcal{L}\circ j^k(\sigma)] = d_i\{X^i[L\circ j^k(\sigma)]\}\ ds(x) \ . \qquad (I.2.37)$$

As a consequence, relation (I.2.xx) can be rewritten as follows

$$T(L)\circ j^k[L_X(\sigma)] = d_i\{X^i[L\circ j^k(\sigma)]\} \ . \qquad (I.2.38)$$

We can now apply the general relation (I.2.30) to obtain the following result: for any vector field X over M and any local section σ of Q we have

$$d_i[E^i(L;X,\sigma)] = -\langle e(L)\circ j^{2k}(\sigma)|L_X(\sigma)\rangle \ , \qquad (I.2.39)$$

where the vector density $E^i(L;X,\sigma)$ is defined by

$$E^i(L;X,\sigma) = \langle f^i(L)\circ j^{2k-1}(\sigma) | j^{k-1}[L_X(\sigma)]\rangle$$

$$- X^i[L\circ j^k(\sigma)] \ . \qquad \text{(I.2.40)}$$

If we restrict the above relation (I.2.39) to any local section σ of Q which satisfies the Euler-Lagrange equations $e(L)\circ j^{2k}(\sigma)=0$, we find

$$d_i[E^i(L;X,\sigma)] = 0, \ \forall X\in C^\infty[T(M)] \ . \qquad \text{(I.2.41)}$$

Relations (I.2.41) express the (weak) natural conservation laws associated with the Lagrangian $\mathcal{L}$ (Trautman, 1962, 1967).

The following remarks are in order here

(i) We can also define a (local) (m-1) form $E(\mathcal{L};X,\sigma)$ on the basis manifold M by setting

$$E(\mathcal{L};X,\sigma) = E^i(L;X,\sigma)\ ds_i(x) \ . \qquad \text{(I.2.42)}$$

This form is the energy flow of the Lagrangian $\mathcal{L}$ along the vector field X and the (local) section σ of Q. Using the energy flow $E(\mathcal{L};X,\sigma)$, the natural conservation laws (I.2.41) are turned into the equivalent expression

$$d[E(\mathcal{L};X,\sigma)] = 0 \ . \qquad \text{(I.2.43)}$$

Using the Stokes theorem, eq.(I.2.43) implies in turn the following relation

$$\int_{\partial D} E(\mathcal{L};X,\sigma) = 0 \ , \qquad \text{(I.2.44)}$$

where ∂D denotes the (m-1)-dimensional boundary of any regular domain $D\subseteq dom(\sigma)\subseteq M$. This last relation expresses the natural conservation laws in their integral form.

(ii) If one eliminates the arbitrary vector field X and the arbitrary section σ from the relation (I.2.36) [or, equivalently, from (I.2.39)], one obtains a set of first-order differential equations in the unknown Lagrangian $\mathcal{L}$. These equations characterise the whole family of generally invariant kth-order Lagrangians over Q, i.e., depending on the given fields together with their derivatives up to the order k. See (Krupka, 1974, 1976, 1978; Ferraris and Francaviglia, 1983) for examples of application.

I.2.7. Energy Momentum Tensors Associated to a Generally Invariant Lagrangian

In this section we shall investigate in detail the energy flow defined above. Recalling that the geometric order of the theory is s, from eq.(I.2.40) we see that the energy flow $E(\mathcal{L};X,\sigma)$ satisfies the following properties.

(i) For any (local) section σ of the configuration space Q, the mapping $X \to E(\mathcal{L};X,\sigma)$ is a linear differential operator of order equal to the differential degree $r=k+s-1$.

(ii) For any vector field X on M, the mapping $\sigma \to E(\mathcal{L};X,\sigma)$ is a (generally nonlinear) differential operator of order $2k-1$.

From these two properties we infer the existence of a morphism of fibered manifolds

$$\hat{E}(\mathcal{L}) : J^{2k-1}(Q) \to [A^0_{m-1}(M)] \otimes \{J^r[T(M)]\}^* , \tag{I.2.45}$$

such that the following holds

$$\begin{aligned} E^i(L;X,\sigma) &= \langle E^i(L) \circ j^{2k-1}(\sigma) \,|\, j^r(X) \rangle \\ &= \langle E^i(L) \,|\, j^r(X) \rangle \circ j^{2k-1}(\sigma) , \end{aligned} \tag{I.2.46}$$

where σ is any (local) section of Q, X is any vector field over M and $E^i(L)$ denotes the ith component of $\hat{E}(\mathcal{L})$, i.e., $\hat{E}(\mathcal{L}) = ds_i(x) \otimes E^i(L)$. Representing the relation (I.2.46) in any natural fibered chart we obtain the following expansion

$$\begin{aligned} E^i(L;X,\sigma) = {} & [e^i{}_h(L) \circ j^{2k-1}(\sigma)]\, X^h + [e^{ij_1}{}_h(L) \circ j^{2k-1}(\sigma)]\, X^h{}_{j_1} \\ & + \dots + [e^{ij_1 \dots j_r}{}_h(L) \circ j^{2k-1}(\sigma)]\, X^h{}_{j_1 \dots j_r} , \end{aligned} \tag{I.2.47}$$

where $(x^h, X^h, X^h{}_{j_1}, \dots, X^h{}_{j_1 \dots j_r})$ and $(x^h, e^i{}_h, e^{ij_1}{}_h, \dots, e^{ij_1 \dots j_r}{}_h)$ denote, respectively, the natural fibered coordinates in the vector bundles $J^r[T(M)]$ and $[A^0_{m-1}(M)] \otimes \{J^r[T(M)]\}^*$. Let us now remark that the sections of these vector bundles are fields of geometric objects over the basis manifold M, but in general they do not have tensorial character. Therefore, also the coefficients $(e^i{}_h, e^{ij_1}{}_h, \dots, e^{ij_1 \dots j_r}{}_h)$, which aresymmetric woith respect to the upper indices j, are not tensors over M; they are the energy-momentum pseudotensors associated with the Lagrangian $\mathcal{L}$. This clarifies the meaning of the differential degree r: it is in fact the maximum order of derivatives of X which appear in the expansion above. In other words, for any theory of differential degree r there are exactly $r+1$ energy pseudotensors.

Using the general procedure described in sec.I.1.2, we can reexpand the energy flow $E(\mathcal{L};X,\sigma)$ by means of tensorial coefficients, rather than with the pseudotensors above. Let us then consider any linear connection C over the basis manifold M, together with the linear isomorphism

$$\psi_C : J^r\{T(M)\} \to \bigoplus_{p=0}^{r} S^1_p(M) , \qquad (I.2.48)$$

and the dual morphism

$$\bar{\psi}_C : [A^0_{m-1}(M)]\otimes\{J^r[T(M)]\}^* \to [A^0_{m-1}(M)]\otimes[\bigoplus_{p=o}^{r} S^p_1(M)] , \qquad (I.2.49)$$

(defined above). Equation (I.2.8) then gives

$$<\hat{E}(\mathcal{L})\,|\,j^r(X)> = <\bar{\psi}_C[\hat{E}(\mathcal{L})\,|\,\bar{\psi}_C[j^r(X)]> , \qquad (I.2.50)$$

from which follows immediately the equivalent expansion

$$<E^i(L)\,|\,j^r(X)> = E^i{}_h(L;C)\; X^h + E^{ij_1}{}_h(L;C)\; \nabla_{j_1} X^h$$

$$+ \dots + E^{ij_1\dots j_r}(L;C)\; \nabla_{(j_1}\dots\nabla_{j_r)} X^h \; . \qquad (I.2.51)$$

The coefficients $(E^i{}_h(L;C), E^{ij_1}{}_h(L;C),\dots,E^{ij_1\dots j_r}(L;C))$, which appear in (I.2.51), are tensor densities, symmetric with respect to their upper indices j. They are the energy-momentum tensor (of the Lagrangian $\mathcal{L}$) associated with the connection C.

We can now insert the explicit expansion (I.2.47) into the natural conservation laws (I.2.41) and eliminate the arbitrary vector field X from the resulting expression. Owing to the linearity of the differential operator $X \to E(\mathcal{L};X,\sigma)$ and to the Leibniz rule for the formal derivative d_i, it turns out also that the quantity $d_i[E^i(L;X,\sigma)]$ may be expanded as a linear combination of the quantities X^h and $X^h_{j_1\dots j_p}$, with $1 \le p \le r+s=k+s$. Therefore, eliminating X amounts to setting all the coefficients in this expansion equal to zero, which gives rise to the following set of (first-order) linear differential equations in the pseudotensors

$e^i_h(L)$ and $e^{ij_1\dots j_p}_h(L)$

$$e^{i(j_1\dots j_r)}_h(L) = 0 ,$$

$$e^{(j_1 j_2\dots j_r)}_h(L) + d_i[e^{(ij_1 j_2\dots j_r)}_h(L)] = 0 , \quad (1\le p\le r) ,$$

$$d_i[e^i_h(L)] = 0 . \qquad (I.2.52)$$

These equations, which are completely equivalent to the single equation (I.2.41), take often the name of conservation laws for the energy-momentum pseudotensors.

It is clear that an equivalent set of natural conservationlaws may be obtained also for the energy-momentum tensors (associated with any connection C). To obtain these equations, which are rather more complicated than eq.(I.2.52) (and therefore will not be written here explicitly), there are at least two possible ways. A first method consists in inserting directly into eq.(I.2.52) the explicit expressions of the energy-momentum pseudotensors $(e^i_h, e^{ij_1}_h, \dots, e^{ij_1\dots j_r}_h)$ in terms of the tensors $(E^i_h, E^{ij_1}_h, \dots, E^{ij_1\dots j_r}_h)$ and the appropriate jet prolongation of the connection C itself; the resulting equations may be further simplified by taking suitable linear combinations which are directly suggested by their very structure.

The second approach, on the contrary, does not require us to express explicitly the pseudotensors in terms of the corresponding tensors (which, as we said above, is not always a simple matter). The method consists first in reexpressing the natural conservation laws (I.2.41) by means of the formal covariant derivative with respect to the connection C chosen; one finds

$$d_i[E^i(L;X,\sigma)] = \nabla_i[E^i(L;X,\sigma)] + T^a{}_{ai}(C)[E^i(L;X,\sigma)] , \qquad (I.2.53)$$

where $T^a{}_{bi}(C)$ denotes the torsion of C. Then the expansion (I.2.51) should be inserted into eq.(I.2.53) and the resulting expression should be rewritten as a linear combination of the components X^h and their symetrised covariant derivatives $\nabla_{(j_1}\dots\nabla_{j_p)}X^h$, with $1\le p\le r$ [this is, of course, possible, in virtue of the linearity of the differential operator $X\to E(\mathcal{L};X,\sigma)$ and of the bundle morphism ψ_C]. Finally, the appropriate set of first-order differential equations for the energy- momentum tensors $(E^{i}_{h}, E^{ij_1}_{h}, \ldots, E^{ij_1 \ldots j_r}_{h})$ is obtained by setting equal to zero all the coefficients of the resulting linear combination. We remark that, owing to the commutation rules for iterated covariant derivatives, the equations so found will contain explicitly the Riemann curvature tensor and the torsion of the connection C, together with their covariant derivatives up to the order at most r-2. These equations may be further simplified by using Bianchi identities and all the existing symmetries of the Riemann tensor.

I.2.8. The Notion and the Existence of Superpotentials

Superpotentials (Benn, 1982) are of primary importance, since they provide a method for generating the energy content of relativistic field theories. An example is represented by the theory of General Relativity itself, which is still source of debate since there is yet no agreement on the concept of gravitational energy. There are however indications, based on the general theoretical framework described in sections I.2.5 and I.2.6 above, that the gravitational energy may be described through a slight generalisation of Komar's superpotentials (Komar, 1959, 1962), as it was first suggested in (Kijowski, 1978). In recent investigations (Robutti, 1984), we have thus constructed a general framework for defining superpotentials, as it will be shortly discussed below, since it appeared that a consistent and general theory was still missing, in spite of the very many known examples.

As we remarked in section I.2.5 and I.2.6, expanding $E(L,\gamma;\xi)$ as a linear combination of symmetrised γ-covariant derivatives of ξ does not provide completely symmetric coefficients $E_{\alpha}^{\lambda\rho_1\ldots\rho_p}$. However, the following result was proved in (Robutti, 1984) and announced in (Ferraris et al., 1986a, b): For any pair of connections $(\gamma,\tilde{\gamma})$ thre exist at least a global (n-2)-form $U(L,\gamma,\tilde{\gamma};\xi)$ over $J^{2k-1}B$, such that

(i) the following holds

$$E(L,\gamma;\xi) = \tilde{E}(L,\gamma,\tilde{\gamma};\xi) + \mathrm{div}[U(L,\gamma,\tilde{\gamma};\xi)] \; ; \qquad (I.2.54)$$

(ii) the energy-momentum tensors of $\tilde{E}$ with respect to $\tilde{\gamma}$ are totally symmetric.

The form $\tilde{E}$ is the *reduced energy density* (with respect to the connection $\tilde{\gamma}$). We can now take any local section $\sigma \in \Gamma(\pi)$ defined in the neighbourhood of a compact hypersurface $\Sigma \subset M$ with smooth boundary $\partial\Sigma$ and recall that for any p-form $\omega \in \Omega^p(J^{2k-1}B)$ it is by definition

$$(j^{2k-1}\sigma)^*(\mathrm{div}\omega) = d[j^{2k-1}\sigma)^*\omega] \ . \qquad \text{(I.2.55)}$$

Accordingly, if we pull back (I.2.54) via $(j^{2k-1}\sigma)$, by integration over the domain Σ and an application of Stokes' theorem we find

$$\int_\Sigma E(\sigma) = \int_\Sigma \tilde{E}(\sigma) + \int_{\partial\Sigma} U(\sigma) \ , \qquad \text{(I.2.56)}$$

with $E(\sigma)=(j^{2k-1}\sigma)^*E$, $\tilde{E}(\sigma)=(j^{2k-1}\sigma)^*\tilde{E}$ and $U(\sigma)=(j^{2k-1}\sigma)^*U$. However, for any free theory, the reduced energy density $\tilde{E}(\sigma)$ has to vanish along any critical section, as it may be immediately seen by relying on the general covariance of L and on field equations. Therefore, for any critical section σ we have

$$\int_\Sigma E(\sigma) = \int_{\partial\Sigma} U(\sigma) \ , \qquad \text{(I.2.57)}$$

so that the whole energetic content of a geometric (free) field theory along its critical sections is generated by the (n-2)-form $U(L,\gamma,\tilde{\gamma},;\xi)$. For this reason this form is called a superpotential. Setting locally

$$U(\sigma) = \frac{1}{2} U^{\lambda\rho} \, ds_{\lambda\rho} \ , \qquad \text{(I.2.58)}$$

(in the natural basis $ds_{\lambda\rho}$ of $\Omega^{n-2}(M)$), eq.(I.2.54) will read as follows

$$E^\lambda = \tilde{E}^\lambda + d_\mu U^{\mu\lambda} \ . \qquad \text{(I.2.59)}$$

According to (I.2.54), also $U(\sigma)$ is a linear partial differential operator of order r-2 in the vector field ξ. Thence, the skew-symmetric coefficients $U^{\mu\lambda}$ may be expanded as follows

$$U^{\lambda\mu} = u_\alpha{}^{\lambda\mu} \xi^\alpha + u_\alpha{}^{\lambda\mu\rho} (\partial_\rho \xi^\alpha)$$
$$+ \ldots + u_\alpha{}^{\lambda\mu\rho_1\ldots\rho_{r-2}} (\partial_{\rho_{r-2}} \ldots \partial_{\rho_1} \xi^\alpha) \ , \qquad \text{(I.2.60)}$$

or equivalently

$$U^{\lambda\mu} = U_{\alpha}{}^{\lambda\mu}\, \xi^{\alpha} + U_{\alpha}{}^{\lambda\mu\rho}\, (\nabla_{\rho}\xi^{\alpha})$$

$$+ \ldots + U_{\alpha}{}^{\lambda\mu\rho_1\ldots\rho_{r-2}}\, (\nabla_{\rho_{r-2}} \ldots \nabla_{\rho_1}\xi^{\alpha}) \ . \qquad (I.2.61)$$

The coefficients $u_{\alpha}{}^{\lambda\mu\rho_1\ldots\rho_p}$, respectively $U_{\alpha}{}^{\lambda\mu\rho_1\ldots\rho_p}$, which are skew-symmetric in $\lambda\mu$ and symmetric in ρ_k, are the superpotential pseudo-tensors, respectively the superpotential canonical tensors. Here canonical refers to the choice of the already given connection γ to expand U as a linear combination of symmetrised covariant derivatives of the vectorfield ξ; ,one might, of course, choose a further connection $\hat{\gamma}$ to expand U, but the procedure would be, of course, rather artificial.

I.2.9. Superpotentials at Order Three

Calculating explicitly the energy-momentum tensors and their superpotentials is in principle possible at all orders r, although the process of symmetrising higher-order covariant derivatives, which requires one to use higher-order commutation relations, becomes increasingly more complicated at large orders. On the other hand, for the purpose of application to relativistic geometric theories of gravitation it is of course enough to investigate in detail only the case r=3. In fact, we have either k=2 and s=1 for purely metric theories, or k=1 and s=2 for purely affine and metric-affine theories.

Superpotentials at order three were calculated explicitly in (Robutti, 1984; Ferraris et al., 1986a, b), obtaining the following results. Starting from the explicit decomposition

$$E^{\lambda} = e_{\alpha}{}^{\lambda}\, \xi^{\alpha} + e_{\alpha}{}^{\lambda\rho}\, (\partial_{\rho}\xi^{\alpha}) + \ldots + e_{\alpha}{}^{\lambda\rho\sigma}\, (\partial_{\rho}\partial_{\sigma}\xi^{\alpha}) \ , \qquad (I.2.62)$$

it is found that for any linear connection γ the following explicit relations hold among the energy-momentum pseudo-tensors and tensors associated to the connection γ

$$E_{\alpha}{}^{\lambda\rho\sigma} = e_{\alpha}{}^{\lambda\rho\sigma} \ ,$$

$$E_{\alpha}{}^{\lambda\rho} = e_{\alpha}{}^{\lambda\rho} + \gamma^{\rho}{}_{\sigma\omega}\, e_{\alpha}{}^{\lambda\omega\sigma} - 2\, \gamma^{\kappa}{}_{\alpha\sigma}\, e_{\kappa}{}^{\lambda\rho\sigma} \ ,$$

$$E_\alpha{}^\lambda = e_\alpha{}^\lambda - \gamma^\omega{}_{\alpha\rho}\, e_\omega{}^{\lambda\rho} + \gamma^\kappa{}_{\omega\sigma}\, \gamma^\omega{}_{\alpha\rho}\, e_\kappa{}^{\lambda\rho\sigma}$$

$$- \gamma^\kappa{}_{\alpha\sigma,\rho}\, e_\kappa{}^{\lambda\rho\sigma} \,. \tag{I.2.63}$$

In order to calculate the superpotentials, it turns out to be convenient to introoduce the auxiliary covariant derivative $\overset{*}{\nabla}_\lambda=\nabla_\lambda+T^\alpha{}_{\alpha\lambda}$, where $T^\alpha{}_{\beta\lambda}$ is the torsion of γ. Integrating by parts the tensorial analogue of (I.2.62) one finds that the covariant expansion of the superpotential $U=\tilde{U}$ admits the following coefficients

$$U_\alpha{}^{\lambda\mu\rho} = \frac{4}{3} E_\alpha{}^{[\lambda\mu]\rho} \,,$$

$$U_\alpha{}^{\lambda\mu} = E_\alpha{}^{[\lambda\mu]} + \frac{2}{3} \overset{*}{\nabla}_\rho E_\alpha{}^{[\lambda\mu]\rho} + \frac{1}{3} T^{[\lambda}{}_{\rho\sigma}\, E_\alpha{}^{|\rho\sigma|\mu]} \,, \tag{I.2.64}$$

while the corresponding expansion in terms of pseudo-tensors is given by

$$u_\alpha{}^{\lambda\mu\rho} = \frac{4}{3} e_\alpha{}^{[\lambda\mu]\rho} \,,$$

$$u_\alpha{}^{\lambda\mu} = e_\alpha{}^{[\lambda\mu]} + \partial_\rho e_\alpha{}^{\rho[\lambda\mu]} + \phi_\alpha{}^{\rho\sigma[\lambda}\, \gamma^{\mu]}{}_{\rho\sigma} \,, \tag{I.2.65}$$

where we set

$$\phi_\alpha{}^{\rho\sigma\lambda} = e_\alpha{}^{(\rho\sigma\lambda)} + \frac{1}{2} u_\alpha{}^{\rho\sigma\lambda} \,. \tag{I.2.66}$$

Accordingly, the reduced energy flow $\tilde{E}$ admits the following expansion in terms of energy-momentum pseudo-tensors

$$\tilde{e}_\alpha{}^{\lambda\rho\sigma} = e_\alpha{}^{(\lambda\rho\sigma)} \,,$$

$$\tilde{e}_\alpha{}^{\lambda\rho} = e_\alpha{}^{(\lambda\rho)} + \partial_\sigma e_\alpha{}^{\sigma(\lambda\rho)} + \phi_\alpha{}^{\omega\sigma[\lambda}\, \gamma^{\rho]}{}_{\omega\sigma} \,,$$

$$\tilde{e}_\alpha{}^{\lambda} = e_\alpha{}^{\lambda} - \partial_\mu [e_\alpha{}^{[\lambda\mu]} + \partial_\rho e_\alpha{}^{\rho[\lambda\mu]} + \phi_\alpha{}^{\rho\sigma[\lambda}\, \gamma^{\mu]}{}_{\rho\sigma}] \,. \tag{I.2.67}$$

I.2.10. Application to General Relativity

In this section we will discuss a basic example of application to relativistic field theories, by considering the generally covariant Lagrangian density describing the standard gravitational field.

Let us now consider the generally covariant Lagrangian density describing the standard Einstein theory

$$\mathcal{L}(j^2g) = \mathcal{L}_H(j^2g) = \sqrt{-g}\, R \,, \qquad \text{(I.2.68)}$$

where $R_{\alpha\beta}=R_{\alpha\beta}(j^2g)$ is the Ricci tensor of the metric $g_{\mu\nu}$.

Using directly (I.2.40) and taking into account the linearity of the partial differential operator $E^\lambda(\mathcal{L},\gamma;\xi)$ with respect to $\mathcal{L}$, one finds explicitly

$$E^\lambda(\mathcal{L},\{\ \}_g;\xi) = E^\lambda(\mathcal{L}_H,\{\ \}_g;\xi)$$

$$= [\mathrm{g}^{\alpha\beta}\, L_\xi\{^\lambda{}_{\alpha\beta}\}_g - \mathrm{g}^{\lambda\sigma}\, L_\xi\{^\alpha{}_{\alpha\sigma}\}_g] - \mathcal{L}_H\, \xi^\lambda \,, \qquad \text{(I.2.69)}$$

where $\{\ \}_g$ denotes the Christoffel symbol of $g_{\mu\nu}$, with $\mathrm{g}^{\alpha\beta}=\sqrt{-g}g^{\alpha\beta}$.

We insert now the explicit expressions of the Lie-derivatives into (I.2.69), and we apply the result of section I.2.9. By a covariant integration by parts with respect to the Levi-Civita connection $\{\ \}_g$, we obtain the following results

$$\begin{aligned} E^\lambda(\mathcal{L}_H,\{\ \}_g;\xi) &= 2\sqrt{-g}\, G^\lambda{}_\omega\, \xi^\omega \\ &\quad - \partial_\mu[(\mathrm{g}^{\lambda\rho}\, \nabla_\rho\xi^\mu - \mathrm{g}^{\rho\mu}\, \nabla_\rho\xi^\lambda)] \\ &= 2\sqrt{-g}\, G^\lambda{}_\nu\, \xi^\nu \\ &\quad - \partial_\mu[\sqrt{-g}(\Gamma^{\mu\lambda}{}_\nu - \Gamma^{\lambda\mu}{}_\nu)]\, \xi^\nu \\ &\quad - [\partial_\nu \mathrm{g}^{\lambda\rho} - \partial_\mu \mathrm{g}^{\mu\rho}\, \delta^\lambda_\nu + \sqrt{-g}\,(\Gamma^{\rho\lambda}{}_\nu - \Gamma^{\lambda\rho}{}_\nu)]\, \partial_\rho\xi^\nu \\ &\quad - (\mathrm{g}^{\lambda\rho}\, \delta^\mu_\nu - \mathrm{g}^{\mu\rho}\, \delta^\lambda_\nu)\, \partial_\mu\partial_\rho\xi^\nu \,. \end{aligned} \qquad \text{(I.2.70)}$$

Accordingly, the reduced energy flow for the Einstein theory is the following

$$\tilde{E}^\lambda(\mathcal{L}_H,\{\ \}_g;\xi) = 2\sqrt{-g}\, G^\lambda{}_\omega\, \xi^\omega \,, \qquad \text{(I.2.71)}$$

while the superpotential has the following expression

$$U^{\lambda\mu}(\mathcal{L}_H,\{\ \}_g;\xi) = -\,(\mathrm{g}^{\lambda\rho}\, \nabla_\rho\xi^\mu - \mathrm{g}^{\rho\mu}\, \nabla_\rho\xi^\lambda) \,. \qquad \text{(I.2.72)}$$

It must be remarked that, owing to the field equations

$$G^{\lambda}_{\omega} = 0 \ , \tag{I.2.73}$$

the reduced energy flow $\tilde{E}^{\lambda}(\mathcal{L}_H, \{\ \}_g; \xi)$ vanishes identically "on shell." On the contrary, the superpotential $U^{\lambda\mu}(\mathcal{L}_H, \{\ \}_g; \xi)$ does not vanish "on shell," but reduces to the following

$$U^{\lambda\mu}(\mathcal{L}_H, \{\ \}_g; \xi) = - (g^{\lambda\rho} \nabla_{\rho} \xi^{\mu} - g^{\rho\mu} \nabla_{\rho} \xi^{\lambda}) \ , \tag{I.2.74}$$

which is non-zero for a generic vector field ξ. Thus, according to our view point, the energy of the Einstein theory turns out to be generated by the superpotential (I.2.74). This result is in agreement with the expression found by Komar (1959, 1962) for time-like Killing vectors and for Einstein's theory, generalising it to an arbitrary vectorfield and to theories other than Einstein's theory of gravitation. Moreover, the above results agree also with those obtained by considering the equivalent purely affine picture, which in (Ferraris, 1985) was shown to generate a coherent unified theory of gravitation and electromagnetism. We finally remark that superpotentials bearing some analogy with (I.2.74) have been heuristically considered in (Benn, 1982).

II. Part Two. Local First-Order Field Theory

Field theory is the study of dynamical systems in which the dimension of the base space is greater than one. With an eye on physical applications we will concentrate our efforts in a hyperbolic four-dimensional base space. Currently people restricts the considerations to Minkowski space but it is advisable to admit some more general geometries and topologies, e.g., de Sitter spaces.

In field theory a dynamical system is described by the fields $\phi^A(x^\mu)$, where A=1,...,m, and m is the number of fields; we assume they are sections of suitable vector bundles over the base space. x^μ, μ=0,...,n-1, are local coordinates in the n-dimensional base space; most of the times we will we thinking of n=3, such that x^μ, with μ=0,1,2,3, are local coordinates in the four-dimensional space-time; i=1,2,3, has the usual meaning of a space-like index.

We assume the base space to be a hyperbolic Riemannian space. We restric our considerations to Minkowski space; the generalisation to other geometries and topologies is straightforward. We assume that the region Ω of the space-time over which the theory is formulated is simply connected. We then can locally write $\Omega=\Sigma\otimes[t_1,t_2]$, i.e., the space-time can be conveniently splitted in space and time with the introduction of a system of simply connected space-like surfaces Σ and a transverse time-like vector field over Ω; this is equivalent to the splitting $x^\mu=(t,x^i)$. The region Ω is then limited in the time-like direction by the space-like surfaces Σ_1 and Σ_2 at times t_1 and t_2, respectively. In the space-like direction is limited by $\partial\Sigma$. For a closed without boundary Σ no additional assumptions are introduced. If Σ is closed with a boundary $\partial\Sigma$ we assume $\Sigma=I\otimes S$, where I=[0,a] and S has the topology of $\partial\Sigma$. For an open space we assume S to have the topology of a two-sphere, S^2, and $I=[0,\infty)$; in this case we formally put the boundary at infinity, $a\to\infty$. In the last two cases we call r the coordinate on I and r=a the boundary.

We start by introducing the fundamentals of field theory. We introduce the Lagrangian formalism. We consider identically vanishing field equations. We construct the canonical energy-momentum tensor. We show that field theory is not d-invariant.

In classical mechanics the time plays a quite preferential role since it is the only coordinate of the base space. When going to field theory one must face an increase of the base space dimension from one to four. There are two possibilities. Firstly, one can consider field theory as classical mechanics with an infinite number of degrees of freedom, the canonical theory, or, secondly, one can consider it as multi-time classical mechanics, the covariant theory.

II.1. General Aspects of Field Theory

Here we consider the foundations of field theory.

II.1.1. The Lagrangian Formalism

As in classical mechanics the dynamical information of a physical system is contained in the action S which is given as an integral over a region Ω of the space-time of the Lagrangian density

$$\mathcal{L} = \mathcal{L}(\phi^A, \phi^A_\mu) \ , \tag{II.1.1}$$

$$S[\mathcal{L}] = \int_\Omega \mathcal{L} \, d^4x \ , \tag{II.1.2}$$

where $\phi^A_\mu = \partial\phi^A/\partial x^\mu$. For simplicity we restrict our considerations to the case in which the Lagrangian density does not depend explicitly on the coordinates x^μ of the base space; the generalisation to the explicitly dependent case is straightforward.

The variation of the action is

$$\delta S[\mathcal{L}] = \int_\Omega \delta_A \mathcal{L} \, \delta\phi^A \, d^4x + \int_{\partial\Omega} \partial_A{}^\mu \mathcal{L} \, \delta\phi^A \, d\Sigma_\mu \ , \tag{II.1.3}$$

where

$$\delta_A \mathcal{L} = \partial_A \mathcal{L} - d_\mu \partial_A{}^\mu \mathcal{L} \ . \tag{II.1.4}$$

$d_\mu = d/dx^\mu$ is the total, or formal derivative with respect to x^μ. The last term has been obtained by partial integration using the fact that

$$\delta\partial_\mu\phi^A = \partial_\mu\delta\phi^A \ . \tag{II.1.5}$$

As for the classical mechanics of discrete systems the equations of motion are obtained by requiring $\delta S[\mathcal{L}]=0$. The first integral in (II.1.3) depends on the values of $\delta\phi^A$ in the interior Ω while the second one does it at $\partial\Omega$. Therefore both terms must be independently zero.

For the second integral it must be

$$\delta\phi^A\Big|_{\partial\Omega} = 0 \ , \tag{II.1.6}$$

but $\delta\phi^A$ is unrestricted and arbitrary in the rest.

Since the variations $\delta\phi^A$ in the first integral are arbitrary what must be zero are the factors multiplying $\delta\phi^A$. The sufficient condition is

$$\delta_A \mathcal{L} = 0 \ . \tag{II.1.7}$$

Equations (II.1.7) then imply that in order to integrate the field equations (II.1.3) a complete set of Cauchy data is provided by the functions $\phi^A|_{\partial\Omega}$.

II.1.2. Identically Vanishing Field Equations

The operator δ_A acting on a four-divergence of the form $\mathcal{L}=d_\mu\Lambda^\mu(\phi)$ is identically zero

$$\delta_A(d_\mu\Lambda^\mu(\phi)) \doteq 0 \ . \tag{II.1.8}$$

The previous is a sufficient condition. The necessary condition can be obtained following Hojman (1983). The general solution to this problem will be provided in the context of higher-order field theory.

Here we will prove that, in spite of what is usually believed to be correct (Hill, 1951; Courant and Hilbert, 1953), the converse statement may be stated in more general fashion. In other words, two Lagrangians may differ by the divergence of $\omega^\mu(\phi,\partial\phi)$ whose dependence in $\partial\phi$ will be explicitly found in what follows. We will prove that even if ω^μ depends on ϕ and $\partial\phi$, Λ will still be a function of ϕ and $\partial\phi$ only, and the two Lagrangians will have exactly the same Euler-Lagrange derivatives.

Now for the proof. Consider Λ and assume it has identically vanishing Euler-Lagrange derivatives

$$\delta_A\Lambda = \partial_A\Lambda - \partial_B\partial_A{}^\mu\mathcal{L}\ \phi^B{}_\mu - \partial_B{}^\nu\partial_A{}^\mu\mathcal{L}\ \phi^B{}_{\mu\nu} \doteq 0 \ . \tag{II.1.9}$$

The problem consists in finding the most general function Λ which will satisfy identity (II.1.9).

Due to the fact that eq.(II.1.9) has to be identically satisfied and the second derivatives of the field appear only in the first term, one must have that

$$\partial_B{}^{\nu}\partial_A{}^{\mu}\mathcal{L}\ \phi^B{}_{\mu\nu} \doteq 0 \ , \tag{II.1.10}$$

$$\partial_A\Lambda - \partial_B\partial_A{}^{\mu}\mathcal{L}\ \phi^B{}_{\mu} \doteq 0 \ , \tag{II.1.11}$$

holds simultaneously.

Define

$$\Psi^{\alpha\beta}_{AB} = \partial_A{}^{\alpha}\partial_B{}^{\beta}\mathcal{L} = \Psi^{\beta\alpha}_{BA} \ , \tag{II.1.12}$$

i.e., by definition, Ψ is symmetric under the simultaneous interchange of capital Latin and Greek indices.

The most general solution to identity (II.1.10) is

$$\Psi^{\beta\alpha}_{AB} = - \Psi^{\alpha\beta}_{AB} \tag{II.1.13}$$

because

$$\phi^B{}_{\mu\nu} = \phi^B{}_{\nu\mu} \tag{II.1.14}$$

and Ψ does not depend on second derivatives of the field ϕ.

Therefore

$$\Psi^{\alpha\beta}_{BA} = - \Psi^{\alpha\beta}_{AB} \tag{II.1.15}$$

due to eqs.(II.1.12) and (II.1.13).

Consider now the third derivatives of Λ

$$\Psi^{\alpha\beta\gamma}_{ABC} = \partial_A{}^{\alpha}\partial_B{}^{\beta}\partial_C{}^{\gamma}\mathcal{L} \ . \tag{II.1.16}$$

Then

$$\Psi^{\alpha\beta\gamma}_{ABC} = \partial_A{}^{\alpha}\Psi^{\beta\gamma}_{BC} = \partial_B{}^{\beta}\Psi^{\alpha\gamma}_{AC} = \partial_C{}^{\gamma}\Psi^{\alpha\beta}_{AB} \tag{II.1.17}$$

and it is straighforward to see that $\Psi^{\alpha\beta\gamma}_{ABC}$ is antisymmetric under the exchange of any pair of (Latin or Greek) indices. Similarly, the kth derivative of Λ,

$$\Psi^{\alpha\beta\gamma\ldots\kappa}_{ABC\ldots K} = \partial_A{}^{\alpha}\partial_B{}^{\beta}\partial_C{}^{\gamma}\ldots\partial_K{}^{\kappa}\Lambda \ , \tag{II.1.18}$$

is completely antisymmetric under the exchange of any pair of indices.

Define l by

$$l = \min(m,\ n)\ . \tag{II.1.19}$$

It is clear that the $(l+1)$-th derivative of Λ vanishes (because it is a completely antisymmetric object with at least two indices with the same value). This fact implies that Λ is a polynomial in the derivatives of the field of degree l at most, i.e.,

$$\Lambda = f(\phi) + \sum_{k=1}^{l} F^{\alpha_1 \ldots \alpha_k}_{A_1 \ldots A_k}(\phi)\ \phi^{A_1}_{\alpha_1} \ldots \phi^{A_k}_{\alpha_k}\ , \tag{II.1.20}$$

where the F's are arbitrary functions of the fields ϕ, completely antisymmetric under the interchange of any pair of (Latin or Greek) indices. The expression (II.1.20) is the most general solution to identity (II.1.10).

Consider now identity (II.1.11). It is clear from the structure of identity (II.1.11) that the terms of Λ with different powers of derivatives of the fields do not mix. Therefore, identity (II.1.11) must be satisfied separately by each term of Λ.

For the zeroth-order term one gets

$$f(\phi) = f = \text{constant}\ . \tag{II.1.21}$$

For the *k*th term one gets from identity (II.1.11) (the square brackets mean complete antisymmetrisation)

$$\partial_{[A_{k+1}} F^{\alpha_1 \ldots \alpha_k}_{A_1 \ldots A_k]}(\phi) \doteq 0\ , \tag{II.1.22}$$

i.e., the "curl" of F vanishes meaning that F is a "gradient,"

$$F^{\alpha_1 \ldots \alpha_k}_{A_1 \ldots A_k}(\phi) = \partial_{[A_k} F^{\alpha_1 \ldots \alpha_{k-1} \alpha_k}_{A_1 \ldots A_{k-1}]}(\phi)\ , \tag{II.1.23}$$

where $F^{\alpha_1 \ldots \alpha_{k-1} \alpha_k}_{A_1 \ldots A_{k-1}}(\phi)$ is an arbitrary function of the fields ϕ completely antisymmetric under the interchange of (Latin or Greek) indices.

The same result can be stated in the language of differential forms saying that eq.(II.1.22) implies that the k-form F is closed and, therefore, locally exact, which is the statement equivalent to eq.(II.1.23) (Spivak, 1965). The most general solution Λ to identities (II.1.10) and (II.1.11) is

$$\Lambda = f + \sum_{k=1}^{l} \partial_{[A_k} F^{\alpha_1 \ldots \alpha_{k-1} \alpha_k}_{A_1 \ldots A_{k-1}]}(\phi)\ \phi^{A_1}_{\alpha_1} \ldots \phi^{A_k}_{\alpha_k}\ , \tag{II.1.24}$$

where f is a constant and the F's are arbitrary functions of the fields ϕ completely antisymmetric in its indices.

It is straightforward to see that

$$\omega^{\mu} = \frac{fx^{\mu}}{n} + \sum_{k=1}^{l} F^{\alpha_1 \ldots \alpha_{k-1}\mu}_{A_1 \ldots A_{k-1}}(\phi)\, \phi^{A_1}_{\alpha_1} \ldots \phi^{A_{k-1}}_{\alpha_{k-1}} \quad , \qquad \text{(II.1.25)}$$

is such that

$$\Lambda = d_{\mu}\omega^{\mu} \ . \qquad \text{(II.1.26)}$$

Of course, ω^{μ} is not unique, but the point is to show that the most general Λ can be written as a divergence of ω^{μ} which depends on the first derivatives of the field ϕ. Proving that Λ defined by eq.(II.1.24) [or by eqs.(II.1.25) and (II.1.26)] satisfies identity (II.1.9) is straigthforward, and it is not necessary to go through the steps in detail.

It is now perhaps convenient to comment on why this result remained unnoticed. The already classic review paper by Hill on symmetries and conservation laws (Hill, 1951) wrongly states that the necessary and sufficient condition for two Lagrangians densities to have exactly the same Euler-Lagrange derivatives is that their difference Λ may be written as a total divergence of ω^{μ} which may depends on the fields only. This claim is not proved in (Hill, 1951) but rather the readers are referred to the also classic book on mathematical physics by Courant and Hilbert (1953). As a matter of fact, the proof in (Courant and Hilbert, 1953) is correct, but it is done only for the case of *one* field. From the arguments given previously, it is clear that when either m=1 (one field) or n=1 (one independent variable, i.e., classical mechanics), l=1, and therefore, the "standard" result holds [see eqs.(II.1.24) and (II.1.25)]. Whenever m>1 and n>1, a more general situation arises. It is fair to say then that the statement made in (Courant and Hilbert, 1953) is correct but misleading, while the one in (Hill, 1951) is wrong in general.

The result presented previously then makes it possible to understand why two Lagrangians which do nor differ by the total derivative of $\omega^{\mu}(\phi)$ have the same Euler-Lagrange derivatives. Finally it should be noted that the determinant of any (square) matrix obtained by suppressing rows and/or columns of ϕ^{A}_{α} will have Euler-Lagrange derivatives which are identically zero. Also, the product of such determinants by an arbitrary function of the fields which appear in the square matrix considered will have the same property.

II.1.3. The Canonical Energy-Momentum Tensor

In field theory the conservation laws are written as covariant continuity equations

$$d_\mu Q_a{}^\mu = 0 \ , \tag{II.1.27}$$

where a indexes them. Examples of conservation laws are provided by the energy-momentum tensor.

Contracting the field equations (II.1.7) with $\phi^A{}_\nu$ one obtains that the energy-momentum tensor

$$\mathcal{H}^\mu{}_\nu(\mathcal{L}) = \phi^A{}_\nu \ \partial_A{}^\mu \mathcal{L} - \delta^\mu_\nu \ \mathcal{L} \ , \tag{II.1.28}$$

is a covariantly conserved quantity

$$d_\mu \mathcal{H}^\mu{}_\nu(\mathcal{L}) = 0 \ . \tag{II.1.29}$$

The conservation equation (II.1.29) does not determine uniquely the energy-momentum tensor since they are not altered under the replacement

$$\mathcal{H}^\mu{}_\nu(\mathcal{L}) \to \mathcal{H}^\mu{}_\nu(\mathcal{L}) + d_\lambda \mathcal{U}^{\lambda\mu}{}_\nu \ , \tag{II.1.30}$$

with $\mathcal{U}^{\lambda\mu}{}_\nu$ antisymmetric in λ and μ. $\mathcal{U}$ is called a superpotential, however it must not be confused with the superpotentials introduced in sec.I.2.8. Its existence has been used to satisfy different requirements for the energy-momentum tensor. For example, when there is a metric available, the canonical energy-momentum tensor is not symmetric. Then, it is possible to choose $\mathcal{U}^{\lambda\mu}{}_\nu$ such that the resulting energy-momentum tensor is symmetric (Belinfante, 1939; Rosenfeld, 1940).

When the energy-momentum tensor is symmetric the quantity

$$\mathcal{M}^{\lambda\mu\nu}(\mathcal{L}) = T^{\lambda\mu}(\mathcal{L}) \ x^\nu - T^{\lambda\nu}(\mathcal{L}) \ x^\mu \ , \tag{II.1.31}$$

is covariantly conserved

$$D_\lambda \mathcal{M}^{\lambda\mu\nu}(\mathcal{L}) = 0 \ . \tag{II.1.32}$$

This corresponds to the angular momentum of the field.

For isolated systems the quantities derived from the energy-momentum tensor show the right asymptotic behaviour only if their canonical definitions are supplied with convenient surface terms. This aspect of the problem will we considered in sec.II.3.

II.1.4. Field Theory is not D-Invariant

A nice property of classical mechanics is d-invariance. This property is however not possessed by field theory. Let us remember that the operator δ_A acting on a four-divergence is identically zero. This does not mean however that one can add a four-divergence to the Lagrangian density without changing the field equations. The previous fact is due to the appearance of surface terms and therefore changing the boundary behaviour of the fields. Furthermore, the energy-momentum tensor is not invariant under the addition to the Lagrangian density of a divergence.

Let us now consider two first-order Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ differing by a divergence, which for simplicity we choose of the form $d_\mu \Lambda^\mu(\phi) = \partial_A \Lambda^\mu \phi^A{}_\mu$

$$\mathcal{L}' = \mathcal{L} + d_\mu \Lambda^\mu(\phi) = \mathcal{L} + \partial_A \Lambda^\mu \, \phi^A{}_\mu \, . \tag{II.1.33}$$

Since

$$\delta_A \mathcal{L}' = \delta_A \mathcal{L} \, , \tag{II.1.34}$$

the field equations to which they give rise are the same. For the canonical energy-momentum tensor we obtain instead

$$\mathcal{H}^\mu{}_\nu(\mathcal{L}') = \mathcal{H}^\mu{}_\nu(\mathcal{L}) + \phi^A{}_\nu \, \partial_A \Lambda^\mu - \delta^\mu_\nu \, \partial_A \Lambda^\lambda \, \phi^A{}_\lambda \, . \tag{II.1.35}$$

This expresses the non d-invariance of the energy-momentum tensor we mentioned above.

The construction of a d-invariant field theory and that of identically vanishing field equations will be postponed to higher-order field theory, where we will show the existence of a convenient substitute of the energy-momentum tensor which is independent of the pure divergences added to the Lagrangian.

II.2. Canonical Field Theory

In the classical mechanics of discrete systems the time plays a quite preferential role since it is the only coordinate of the base space. The Hamiltonian is at the same time the generator of the time evolution and a conserved quantity.

Canonical field theory mimics the Lagrangian mechanics of discrete systems considering the system as a mechanical one but with an infinite number of degrees of freedom, the values of the field components at the various points of the space-like surfaces Σ for fixed t. This canonical formulation of field theory was first given by Heisenberg and Pauli (1929). The first step is to individuate an evolution parameter playing the role of the time. The 1+3 splitting of the space-time provides the answer to this. The time is given the role of the evolution parameter. The second step is to consider the system as a mechanical one but with an infinite number of degrees of freedom. These infinite degrees of freedom are the values of the field components at the various points of the space-like surfaces Σ for fixed t. Then the discrete label i in q^i (of classical mechanics) becomes a continuous label x^i plus additional discrete labels A; in this way q^i is replaced by the fields $\phi^A(x^i)$. The sum over the discrete label i becomes an integration over the continuous label x^i over all Σ plus a sum over the discrete label A, in such a way that the Lagrangian is

$$L[\phi^A(x^i)] = \int_{\Sigma(t)} \mathcal{L}(\phi^A(x^i),\dot{\phi}^A(x^i),\phi^A{}_{,j}(x^i))\, d\Sigma \ , \qquad \text{(II.2.1)}$$

where $\mathcal{L}$ is the Lagrangian density. The derivatives of the field components with respect to the time, $\dot{\phi}^A(x^i)$, are defined as the velocities. In what follows we will suppres the continuous label x^i; this cannot give rise to any confusion.

The equations of motion can be considered as those from classsical mechanics but with an infinite number of degrees of freedom. In this case one must face an increase of the configuration space dimension from n to infinity. In this transition partial derivatives become partial functional (Lagrangian) derivatives. The way in which we will obtain canonical field theory is by considering a 1+3 splittong of the results of the previous section.

II.2.1. The Lagrangian Formalism

The action is now written as

$$S[\mathcal{L}] = \int_{t_1}^{t_2} L\ dt = \int_{t_1}^{t_2} \int_{\Sigma(t)} \mathcal{L}\ d\Sigma\ dt\ . \tag{II.2.2}$$

Under arbitrary variations of the fields the variation of the action is given by

$$\delta S[\mathcal{L}] = \int_{t_2}^{t_1} \left[\int_{\Sigma} \delta_A \mathcal{L}\ \delta\phi^A\ d\Sigma + \int_{\partial\Sigma} \partial_A{}^P \mathcal{L}\ \delta\phi^A\ dS \right] dt$$
$$+ \int_{\Sigma} \dot{\partial}_A \mathcal{L}\ \delta\phi^A\ d\Sigma \Big|_{t_2}^{t_1} \tag{II.2.3}$$

$\partial_A{}^P\mathcal{L}=n_i\partial_A{}^i\mathcal{L}$, with n_i the outer normal to $\partial\Sigma$; $d\Sigma$ and dS are the volume and surface elements on Σ and $\partial\Sigma$, respectively; $d\Sigma$=drdS; $\dot{\phi}^A=d\phi^A/dt$.

The functions multiplying the variations $\delta\phi^A$ in the second integral are defined as the momenta $\Pi_A(\mathcal{L})$ canonically conjugated to the fields ϕ^A

$$\Pi_A(\mathcal{L}) = \dot{\partial}_A \mathcal{L}\ . \tag{II.2.4}$$

As for the classical mechanics of discrete systems the equations of motion are obtained by requiring $\delta S[\mathcal{L}]=0$. The first integral in (II.2.3) depends on the values of $\delta\phi^A$ in the interior of the interval $[t_1,t_2]$ while the second one does it at t_1 and t_2. Therefore both terms must be independently zero.

For the second integral it must be

$$\delta\phi^A(t_1) = \delta\phi^A(t_2) = 0\ , \tag{II.2.5}$$

but $\delta\phi^A$ is unrestricted and arbitrary in the rest.

For the first integral there exist drastic differences depending upon whether the three-space Σ is open or closed with or without boundary. For a closed without boundary space one is certain that no complications could possibly arise since then $\partial\Sigma=0$ and the surface term does not appear in (II.2.3). For an open or closed with boundary space the presence of the surface integral is unavoidable.

Since the variations $\delta\phi^A$ in the first integral are arbitrary what must be zero are the factors multiplying $\delta\phi^A$. The sufficient condition is

$$\delta_A \mathcal{L} = 0 \tag{II.2.6}$$

plus, in the case of an open or closed with boundary space, a condition on the boundary behaviour of the fields

$$\partial_A{}^p \mathcal{L} \Big|_{\partial\Sigma} = 0 \ . \tag{II.2.7}$$

Equations (II.2.5) then imply that in order to integrate the field equations (II.2.6) a complete set of Cauchy data is provided by the functions $\phi^A(t_1)$ and $\phi^A(t_2)$ restricted only to satisfy (II.2.7).

II.2.2. The Canonical Hamiltonian Formalism

A Hamiltonian mechanics for field theory may be set up still in parallel with classical mechanics. The canonical Hamiltonian is defined as the time-time component of the canonical energy-momentum tensor integrated over the space-like sections, i.e.

$$H_c[\mathcal{L}] = \int_\Sigma \mathcal{H}^0{}_0(\mathcal{L}) \ d\Sigma = \int_\Sigma \left(\dot\phi^A \ \Pi_A(\mathcal{L}) - \mathcal{L}\right) \ d\Sigma \ . \tag{II.2.8}$$

This allows to define the canonical Hamiltonian density

$$\mathcal{H}_c(\mathcal{L}) = \mathcal{H}^0{}_0(\mathcal{L}) = \dot\phi^A \ \Pi_A(\mathcal{L}) - \mathcal{L} \ . \tag{II.2.9}$$

The variation of $\mathcal{H}_c(\mathcal{L})$ is

$$\begin{aligned} \delta\mathcal{H}_c(\mathcal{L}) &= - \partial_A \mathcal{L} \ \delta\phi^A - \partial_A{}^i \mathcal{L} \ \delta\phi^A{}_i + \dot\phi^A \ \delta\Pi_A(\mathcal{L}) \\ &= - \left(\partial_A \mathcal{L} - d_i \partial_A{}^i \mathcal{L}\right) \delta\phi^A + \dot\phi^A \ \delta\Pi_A(\mathcal{L}) - d_i(\partial_A{}^i \mathcal{L} \ \delta\phi^A) \ , \end{aligned} \tag{II.2.10}$$

which shows that $\mathcal{H}_c(\mathcal{L})$ has the dependence

$$\mathcal{H}_c(\mathcal{L}) = \mathcal{H}_c(\phi^A, \dot\phi^A, \phi^A{}_i) \ . \tag{II.2.11}$$

The energy is defined as the numerical value of the canonical Hamiltonian, i.e.

$$E[\mathcal{L}] = \int_\Sigma \left(\dot\phi^A \ \Pi_A(\mathcal{L}) - \mathcal{L}\right) \ d\Sigma \ . \tag{II.2.12}$$

It is furthermore a conserved quantity

$$\dot{E}[\mathcal{L}] = -\int_{\partial\Sigma} \dot{\phi}^A \, \partial_A{}^p\mathcal{L} \, dS = 0 \ . \tag{II.2.13}$$

The variation of the Hamiltonian is

$$\delta H_c[\mathcal{L}] = \int_\Sigma \delta\mathcal{H}_c(\mathcal{L}) \, d\Sigma$$
$$= \int_\Sigma \left[\dot{\phi}^A \, \delta\Pi_A(\mathcal{L}) - \left(\partial_A\mathcal{L} - D_i\partial_A{}^i\mathcal{L}\right) \delta\phi^A\right] d\Sigma$$
$$- \int_{\partial\Sigma} \partial_A{}^p\mathcal{L} \, \delta\phi^A \, dS \ . \tag{II.2.14}$$

From here one obtains

$$\frac{\delta\mathcal{H}_c}{\delta\Pi_A} = \dot{\phi}^A \ , \tag{II.2.15a}$$

$$\frac{\delta\mathcal{H}_c}{\delta\phi^A} = -\left(\partial_A\mathcal{L} - d_i\partial_A{}^i\mathcal{L}\right) . \tag{II.2.15b'}$$

When the field equations (II.2.6) hold eq.(II.15b') reduces to

$$\frac{\delta\mathcal{H}_c}{\delta\phi^A} = -\dot{\Pi}_A(\mathcal{L}) \ . \tag{II.2.15b}$$

The time derivative of a functional

$$F = \int_\Sigma \mathcal{F} \, d\Sigma \ , \tag{II.2.16}$$

is

$$\dot{F} = \{F, H_c\} + \int_{\partial\Sigma} \partial_A{}^p\mathcal{F} \, \dot{\phi}^A \, dS \ , \tag{II.2.17}$$

where

$$\{F, G\} = \int_\Sigma \left(\frac{\delta^x\mathcal{F}(z)}{\delta\phi^A(z)} \frac{\delta^x\mathcal{G}(z)}{\delta\Pi_A(z)} - (\mathcal{F}\longleftrightarrow\mathcal{G})\right) d\Sigma(z) \ , \tag{II.2.18}$$

is the Poisson bracket; δ^x is defined as δ except in that the range of Greek indices is restricted to Latin indices.

At this point a further restriction is unavoidable. In order to induce a symplectic-like structure on the phase space we must require

$$\partial_A{}^p\mathcal{F}\Big|_{\partial\Sigma} = 0 \ . \tag{II.2.19}$$

In this way eq.(II.2.17) reduces to

$$\dot{F} = \{F, H_c\} \ . \tag{II.2.20}$$

The canonical variables can be written as functionals with a delta function as kernel

$$\phi^A(\mathbf{x}) = \int_\Sigma \phi^A(\mathbf{z})\ \delta^{(3)}(\mathbf{x} - \mathbf{z})\ \delta\Sigma(\mathbf{z})\ , \tag{II.2.21a}$$

$$\Pi_A(\mathbf{x}) = \int_\Sigma \Pi_A(\mathbf{z})\ \delta^{(3)}(\mathbf{x} - \mathbf{z})\ \delta\Sigma(\mathbf{z})\ . \tag{II.2.21b}$$

where boldface letters stand by space-like coordinates. This allows to define the densities

$$\varphi^A(\mathbf{x};\ \mathbf{z}) = \phi^A(\mathbf{z})\ \delta^{(3)}(\mathbf{x} - \mathbf{z})\ , \tag{II.2.22a}$$

$$\prod_A(\mathbf{x};\ \mathbf{z}) = \Pi_A(\mathbf{z})\ \delta^{(3)}(\mathbf{x} - \mathbf{z})\ . \tag{II.2.22b}$$

Then, one obtains the Lagrangian derivatives

$$\frac{\delta\varphi^A(\mathbf{x};\mathbf{z})}{\delta\phi^B(\mathbf{z})} = \delta^A_B\ \delta^{(3)}(\mathbf{x} - \mathbf{z})\ ,\quad \frac{\delta\phi^A(\mathbf{x};\mathbf{z})}{\delta\prod_B(\mathbf{z})} = 0\ ,$$
$$\frac{\delta\prod_A(\mathbf{x};\mathbf{z})}{\delta\phi^B(\mathbf{z})} = 0\ ,\quad \frac{\delta\prod_A(\mathbf{x};\mathbf{z})}{\delta\prod_B(\mathbf{z})} = \delta^B_A\ \delta^{(3)}(\mathbf{x}-\mathbf{z})\ . \tag{II.2.23}$$

Therefore, for the canonical variables one obtains

$$\{\phi^A(\mathbf{x}),\ \Pi_B(\mathbf{y})\} = \delta^A_B\ \delta^{(3)}(\mathbf{x} - \mathbf{y})\ . \tag{II.2.24}$$

The Hamilton equations are obtained by putting F equal to the canonical variables in (II.2.20). Commutation relations are also obtained for ϕ^A_i

$$\{\phi^A_i(\mathbf{x}),\ \Pi_B(\mathbf{y})\} = -\ \delta^A_B\ d^x_i\delta^{(3)}(\mathbf{x} - \mathbf{y})\ . \tag{II.2.25}$$

II.2.3. Conservation Laws

In canonical field theory the conservation laws are obtained by rewritting the covariant continuity equations (II.1.27) in a 1+3 form

$$d_\mu Q_a^{\ \mu} = d_0 Q_a^{\ 0} + d_i Q_a^{\ i} = 0\ , \tag{II.2.26}$$

where a indexes them. The conserved quantities, called the *charges*, are the time-like components of $Q_a^{\ \mu}$ integrated over the space-like sections, i.e.

$$q_a = \int_\Sigma Q_a^{\ 0}\ d\Sigma\ . \tag{II.2.27}$$

Examples of conservation laws are provided by the energy-momentum tensor. The canonical energy is one example. Analogously one can obtain other conservation laws. For example

$$P_i(\mathcal{L}) = \int_\Sigma T^0{}_i(\mathcal{L})\ d\Sigma\ , \qquad \text{(II.2.28)}$$

then

$$\dot{P}_i(\mathcal{L}) = \delta^p_i \int_{\partial\Sigma} \mathcal{L}\ dS\ . \qquad \text{(II.2.29)}$$

This quantity measures the flux of momentum at the boundary.

Superpotentials are transformed in surface terms. In fact, let us consider the energy, then the modified energy is

$$E_{mod} = E_c + \int d_i \mathcal{U}^{i0}{}_0\ d\Sigma\ , \qquad \text{(II.2.30)}$$

due to the antisymmetry of the superpotential. Finally, then

$$E_{mod} = E_c + \int \mathcal{U}^{p0}{}_0\ dS\ . \qquad \text{(II.2.31)}$$

Further conserved quantities are obtained analogously for the angular momentum

$$M^{\mu\nu}(\mathcal{L}) = \int_\Sigma \mathcal{M}^{0\mu\nu}(\mathcal{L})\ d\Sigma\ , \qquad \text{(II.2.32)}$$

In particular

$$M^{ij}(\mathcal{L}) = \int_\Sigma \mathcal{M}^{0ij}(\mathcal{L})\ d\Sigma\ , \text{ etc.} \qquad \text{(II.2.33)}$$

For isolated systems the quantities derived from the energy-momentum tensor show the right asymptotic behaviour only if their canonical definitions are supplied with convenient surface terms, i.e., a convenient superpotential. This aspect of the problem will we considered later on.

The non d-invariance of field theory also manifests in the canonical formalism. In the example given in sec.I.4 the two first-order Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ differing by a divergence $d_\mu\Lambda^\mu(\phi)=\partial_A\Lambda^\mu\phi^A_\mu$ gave rise to canonical energy-momentum tensors related by

$$\mathcal{H}^\mu{}_\nu(\mathcal{L}') = \mathcal{H}^\mu{}_\nu(\mathcal{L}) + \phi^A_\nu\ \partial_A\Lambda^\mu - \delta^\mu_\nu\ \partial_A\Lambda^\lambda\ \phi^A_\lambda\ . \qquad \text{(II.2.34)}$$

The canonical energy is now given by

$$E[\mathcal{L}'] = \int_\Sigma \mathcal{H}^0{}_0(\mathcal{L}')\ d\Sigma = \int_\Sigma (\mathcal{H}^0{}_0(\mathcal{L}) - d_i\Lambda^i)\ d\Sigma$$

$$= E[\mathcal{L}] - \int_{\partial\Sigma} \Lambda^p \, dS \ . \qquad \text{(II.2.35)}$$

where Λ^p is the component of Λ^i normal to the boundary $\partial\Sigma$ of Σ; dS is the (n-2)-dimensional volume element of $\partial\Sigma$. The canonical energy can, therefore, be given an arbitrary value, unless boundary conditions are chosen so to satisfy $\Lambda^{\perp}\doteq 0$ on $\partial\Sigma$, but in this case the arbitrariness is shifted to the boundary behaviour of the fields. The energy can, therefore, be given an arbitrary value. This expresses the non d-invariance of the energy-momentum tensor in the canonical formalism.

The construction of a d-invariant field theory and that of identically vanishing field equations will be postponed to higher-order field theory, where we will show the existence of a convenient substitute of the energy-momentum tensor which is independent of the pure divergences added to the Lagrangian.

II.2.4. Field Theory with Surface Terms

There are some problems with the previous procedure. In fact, for an open or closed with boundary space the Cauchy data, evolved with (II.2.6), must satisfy (II.2.7) they cannot be given all independently. This means that the system is a constrained one. The Hamiltonian formalism is then constructed as for a constrained system with (II.2.7) as constraint.

The energy is defined as the numerical value of the function generating the time evolution. For a closed without boundary space this quantity is directly provided by the canonical Hamiltonian. For an open or closed with boundary space one is dealing with a constrained system and H_c does not give the correct field equations. In this case the function generating the time evolution is the canonical Hamiltonian plus a linear combination of the constraints

$$H_1 = H_c + \int_{\partial\Sigma} \partial_A{}^p\mathcal{L} \, \lambda^A \, dS \ , \qquad \text{(II.2.36)}$$

with λ^A some Lagrange multipliers. Therefore, the Hamiltonian is the canonical one plus a surface integral.

For a closed without boundary space H_c is the correct

Hamiltonian. For an open or closed with boundary space the correct hamiltonian is H_1. The energy is then the numerical value of H_1. The surface energy term has been long known in general relativity (Dirac, 1959; Arnowitt et al., 1962; DeWitt, 1967; Regge and Teitelboim, 1974).

II.2.5. Field Theory without Surface Terms

It is shown that the surface terms appearing in the variation of the action can always be given account by means of an extra term in the Lagrangian, which does not modify the action, in such a way that there is no need for impossing conditions on the boundary or asymptotic behaviour of the fields for the stationarity of the action.

In field theory the surface terms appearing in the variation of the action are currently taken away by fixing the boundary or asymptotic behaviour of the fields. The previous procedure involves a sufficient condition for the stationarity of the action. The necessary condition leads to a source term at the boundary in the field equations. In this case it is not necessary to fix the boundary behaviour of the fields. This is the correct field theory following from the given action. The conditions at the boundary diminish the number of Cauchy data which can be given independently, therefore both approaches lead to different theories.

The field equations with the source term can be obtained from a new Lagrangian constructed in terms of the original one, which does not modifies the action, but having no surface terms and such that the action remains the same. The Hamiltonian formulation of the theory is then straightforward.

The necessary condition in order that the first integral in (II.2.3) be zero can be obtained by rewriting it as

$$\int_{t_1}^{t_2} \int_{\Sigma} \left[\delta_A \mathcal{L} + 2\, \partial_A^{\,P}\mathcal{L}\, \delta(r-a)\right] \delta\phi^A \, dr\, dS \quad dt \,. \qquad \text{(II.2.37)}$$

The field equations are now

$$\delta_A \mathcal{L} + 2\, \partial_A^{\,P}\mathcal{L}\, \delta(r-a) = 0 \,. \qquad \text{(II.2.38)}$$

The extra term can be considered as a source or force at the boundary depending however on the fields.

As in classical mechanics, when a generalised force is present, one would like to find a Lagrangian $\mathcal{L}'$ giving account of the extra term in (II.2.38). Difficulties appear when trying to formulate the Hamiltonian formalism since it is not known how to deal with the surface term appearing in (II.2.14), unless of proceeding as in (II.2.36), i.e., by impossing $\partial_A{}^p\mathcal{L}\big|_{\partial\Sigma}=0$. Therefore, nothing is gained unless the new Lagrangian gives no rise to surface terms.

The solution is

$$\mathcal{L}' = \mathcal{L} - 2\,\partial_A{}^p\mathcal{L}\Big|_{\partial\Sigma}\,\theta(r-a)\,\phi^A{}_p \tag{II.2.39}$$

where $\phi^A{}_p=n^i\partial_i\phi^A$, with n^i normal to $\partial\Sigma$ and satisfying $n^in_i=1$; $\theta(x)$ is the Heavyside step function

$$\theta(x) = \begin{cases} 1 & \text{for } x>0 \\ 1/2 & \text{for } x=0\ . \\ 0 & \text{for } x<0 \end{cases} \tag{II.2.40}$$

Let us observe first of all that the action remains the same

$$S' = \int_\Omega \mathcal{L}'\,d^4x = \int_\Omega \mathcal{L}\,d^4x = S\ . \tag{II.2.41}$$

The variation of the primed Lagrangian is

$$\begin{aligned} \delta\mathcal{L}' &= \partial_A\mathcal{L}\,\delta\phi^A + \partial_A{}^0\mathcal{L}\,\delta\phi^A{}_0 + \partial_A{}^i\mathcal{L}\,\delta\phi^A{}_i \\ &\quad - 2\,\delta\Big(\partial_A{}^p\mathcal{L}\Big|_{\partial\Sigma}\Big)\,\theta(r-a)\,\phi^A{}_p - 2\,\partial_A{}^p\mathcal{L}\Big|_{\partial\Sigma}\,\theta(r-a)\,\delta\phi^A{}_p \\ &= \delta_A\mathcal{L}\,\delta\phi^A + D_0\big(\pi_A\,\delta\phi^A\big)\;D_i\big(\partial_A{}^i\mathcal{L}\,\delta\phi^A\big) \\ &\quad - 2\,\delta\Big(\partial_A{}^p\mathcal{L}\Big|_{\partial\Sigma}\Big)\,\theta(r-a)\,\phi^A{}_p + 2\,\partial_A{}^p\mathcal{L}\Big|_{\partial\Sigma}\,\partial_r\theta(r-a)\,\delta\phi^A \\ &\quad - 2\,\partial_r\Big(\partial_A{}^p\mathcal{L}\Big|_{\partial\Sigma}\,\theta(r-a)\,\delta\phi^A\Big)\ . \end{aligned} \tag{II.2.42}$$

The previous variation must now be integrated over the region Ω. The first and fifth terms add to give

$$\Big[\delta_A\mathcal{L} + 2\,\partial_A{}^p\mathcal{L}\Big|_{\partial\Sigma}\,\delta(r-a)\Big]\,\delta\phi^A \tag{II.2.43}$$

The second one is the same appearing before and represents no problem. The third and sixth terms cancel exactly. The fourth term gives no contribution.

A comment is relevant here concerning the unequal treatment of the last two terms in (II.2.42). The last one was partially transformed by partial integration to a boundary term. The same was not done for the first term since it involves the variation of a funtion, not a functional. One can from here conclude that $\partial_A{}^p\mathcal{L}\big|_{\partial\Sigma}$, a funtion, behaves as a constant with respect to funtional derivation. Therefore, in the second term of (II.2.42) derivatives are taken only with respect to r in θ and ϕ^A_p. This is assumed in what follows.

The variation of the action is then

$$\delta S' = \int_\Omega \left[\delta_A\mathcal{L} + 2\ \partial_A{}^p\mathcal{L}\ \delta(r-a)\right]\ \delta\phi^A\ d^4x$$

$$+ \int_\Sigma \pi_A(\mathcal{L})\ \delta\phi^A\ d\Sigma\Big|_{t_1}^{t_2} , \qquad \text{(II.2.44)}$$

which is equivalent to (II.2.3). It must be noticed furthermore that

$$\partial_A{}^p\mathcal{L}'\Big|_{\partial\Sigma} \doteq 0\ . \qquad \text{(II.2.45)}$$

Therefore, one can give account of the extra term in the field equations by means of an extra term in the Lagrangian without however modify the action.

The energy density is

$$\epsilon' = \phi^A{}_0\ \partial_A{}^o\mathcal{L}' - \mathcal{L}' = \epsilon + \partial_A{}^p\mathcal{L}\Big|_{\partial\Sigma}\ \theta(r-a)\ \phi^A{}_p\ , \qquad \text{(II.2.46)}$$

such that the total energy is

$$E' = \int_\Sigma \epsilon'\ d\Sigma = E\ . \qquad \text{(II.2.47)}$$

Furthermore, it is a conserved quantity

$$\frac{dE}{dt} = - \int_{\partial\Sigma} \phi^A{}_0\ \partial_A{}^p\mathcal{L}'\ dS_2 = 0\ , \qquad \text{(II.2.48)}$$

due to (II.2.45).

Now for the formulation of the Hamiltonian formalism. We first observe that the canonical momenta are the same

$$\Pi_A(\mathcal{L}') = \partial_A{}^0\mathcal{L}' = \Pi_A(\mathcal{L})\ . \qquad \text{(II.2.49)}$$

The canonical Hamiltonian is the function appearing at the right-hand side in the definition of the energy, such that the energy is the numerical value of the Hamiltonian. Due to (II.2.49) the equation (II.2.47) rewrites as

$$H'_c = H_c \ . \qquad \text{(II.2.50)}$$

The fact that H'_c and H_1 in (II.2.36) differ is not strange since when introducing conditions at the boundary one is not dealing with the same theory. The possibility for H'_c of reproducing the standard results, e.g., the behaviour of the energy for asymptotically flat spaces in General Relativity, is not excluded since it is evaluated on solutions of (II.2.38) while H_1 does it on solutions of (II.2.6) subjected to (II.2.7).

To conclude, for any Lagrangian there exists a new Lagrangian equivalent to the original one in the sense of giving the same field equations but lacking of surface terms and not changing the action. Therefore, we have been able to free field theory from the surface terms or, equivalently, from the conditions on the boundary behaviour of the fields.

II.3. Four-Dimensional Hamiltonian Formalism for Field Theory and Surface Terms

We develop a formalism for field theory in which all the directions of the space-time are treated in an equivalent way. Consequently four velocities and four momenta are defined for each component of the fields. The role of the Hamiltonian is played by the energy-momentum tensor modified by an (unique) identically conserved tensor. This gives the covariant version of Hamilton equations, Poisson brackets and of the Jacobi identity. The surface terms of the canonical field theory, appearing for the energy, for the linear and angular momentum, found a natural explanation within this formalism.

The deadly sin of canonical field theory is to give a preferential role to the time over the space. Many arguments have been given in favour of the canonical approach and people feel that, no matter how arbitrary the distinction between space and time may be, the conventional language is both necessary and appropriate. Something which is not under discussion is the fact that covariance has not been respected. At this point a covariant formalism is not only desirable but conceptually strictly necessary.

The first steps towards a covariant formulation of field theory were given by Born (1934), Weyl (1934) and de Donder (1935) in the thirties. Unfortunately they failed in their attempts. Recently other attempts has been done by Marsden et al. (1986) and by Crnkovic and Witten (1986).

In a covariant field theory all the four space-time directions are treated in an equivalent way. Then all the coordinates of the space-time take the place that the time alone takes in classical mechanics. The four space-time derivatives of a field component are treated as four independent velocities and four momenta are defined as the partial derivatives of the Lagrangian density with respect to the four velocities. The fields and the momenta defined in this way span a covariant phase space. The canonical energy-momentum tensor, even when being a conserved quantity, is not the generator of space-time displacements. The only possible modification is the addition of an identically conserved tensor. The requirement of a symplectic-like structure on the covariant phase space fixes uniquely this term. The covariant versions of Hamilton equations, Poisson brackets and of the Jacobi identity naturally follow.

Contact with the canonical formalism is made when considering the time-time component of the modified energy-momentum tensor integrated over the space-like sections. This quantity turns out to be the canonical Hamiltonian plus a surface term. The same considerations are extended to the linear and angular momentum.

II.3.1. The Modified Energy-Momentum Tensor

Let us start by considering the canonical Hamiltonian tensor density

$$\mathcal{H}_{c\ \mu}^{\ \nu}(\mathcal{L}) = \Pi_{A}^{\ \nu}(\mathcal{L})\ \phi^{A}_{\ \mu} - \delta^{\nu}_{\mu}\ \mathcal{L}\ . \tag{II.3.1}$$

The variation of $\mathcal{H}_{c\ \mu}^{\ \nu}(\mathcal{L})$ is

$$\delta\mathcal{H}_{c\ \mu}^{\ \nu}(\mathcal{L}) = -\ \delta^{\nu}_{\mu}\ \partial_{A}\mathcal{L}\ \delta\phi^{A} + \phi^{A}_{\ \mu}\ \delta\Pi_{A}^{\ \nu}(\mathcal{L})$$
$$+ \left(\delta^{\lambda}_{\mu}\ \Pi_{A}^{\ \nu}(\mathcal{L}) - \delta^{\nu}_{\mu}\ \partial_{A}^{\ \lambda}\mathcal{L}\right)\ \delta\phi^{A}_{\ \lambda}\ . \tag{II.3.2}$$

The first unpleasant aspect of this variation is the fact that the term multiplying the variations $\delta\phi^{A}_{\ \lambda}$ does not cancel by virtue of the definition of the momenta as happens in classical mechanics.

This impasse is overcome by observing that $\mathcal{H}_{c\ \mu}^{\ \nu}(\mathcal{L})$, as derived from (II.1.29), is defined up to an additive identically conserved quantity $\mathcal{F}^{\nu}_{\ \mu}(\mathcal{L})$ of the form

$$\mathcal{F}^{\nu}_{\ \mu}(\mathcal{L}) = d_{\lambda}\mathcal{F}^{\lambda\nu}_{\ \ \mu}(\mathcal{L}) \tag{II.3.3}$$

with $\mathcal{F}^{\lambda\nu}_{\ \ \mu}(\mathcal{L})$ antisymmetric in λ and ν. The modified Hamiltonian is then defined as

$$\mathcal{H}_{\mathrm{mod}\ \mu}^{\ \ \nu}(\mathcal{L}) = \mathcal{H}_{c\ \mu}^{\ \nu}(\mathcal{L}) + \mathcal{F}^{\nu}_{\ \mu}(\mathcal{L})\ . \tag{II.3.4}$$

Then, eq.(II.1.29) is correspondingly written as

$$d_{\nu}\mathcal{H}_{\mathrm{mod}\ \mu}^{\ \ \nu}(\mathcal{L}) = 0\ . \tag{II.3.5}$$

The canonical variables $(\phi^A, \Pi_A^{\ \mu}(\mathcal{L}))$ span a covariant phase space. In classical mechanics a symplectic structure is induced on the phase space when considering the time- derivative of the observables. Here we would like a symplectic structure to be induced on the covariant phase space when considering the derivative with respect to x^μ of a generic observable $F=F(\phi^A, \Pi_A^{\ \mu}(\mathcal{L}))$

$$d_\mu F = \frac{\partial F}{\partial \phi^A} \phi^A_{\ \mu} + \frac{\partial F}{\partial \Pi_A^{\ \nu}} d_\mu \Pi_A^{\ \nu} . \tag{II.3.6}$$

We would like to write the previous equation as something like

$$d_\mu F = \alpha \, \{F, \mathcal{H}_{\text{mod}}{}^{\nu}{}_{\mu}(\mathcal{L})\}_\nu + \beta \, \{F, \mathcal{H}_{\text{mod}}{}^{\nu}{}_{\nu}(\mathcal{L})\}_\mu \tag{II.3.7}$$

where

$$\{F, G\}_\mu = \frac{\partial F}{\partial \phi^A} \frac{\partial G}{\partial \Pi_A^{\ \mu}} - \frac{\partial G}{\partial \phi^A} \frac{\partial F}{\partial \Pi_A^{\ \mu}} \tag{II.3.8}$$

is a covariant Poisson bracket. Then, it must be

$$\phi^A_{\ \mu} = \alpha \frac{\partial \mathcal{H}_{\text{mod}}{}^{\lambda}{}_{\mu}}{\partial \Pi_A^{\ \lambda}} + \beta \frac{\partial \mathcal{H}_{\text{mod}}{}^{\lambda}{}_{\lambda}}{\partial \Pi_A^{\ \mu}} , \tag{II.3.9a}$$

$$d_\mu \Pi_A^{\ \nu} = - \alpha \frac{\partial \mathcal{H}_{\text{mod}}{}^{\nu}{}_{\mu}}{\partial \phi^A} - \beta \frac{\partial \mathcal{H}_{\text{mod}}{}^{\lambda}{}_{\lambda}}{\partial \phi^A} \delta^\nu_\mu , \tag{II.3.9b}$$

with α and β numerical constants.

Equations (II.3.7) and the requirement of not having terms $\delta\phi^A_{\ \lambda}$ in the variation of $\mathcal{H}_{\text{mod}}{}^{\nu}{}_{\mu}(\mathcal{L})$ determine uniquely $\mathcal{F}^\nu{}_\mu(\mathcal{L})$. The solution is

$$\mathcal{F}^{\lambda\nu}{}_\mu(\mathcal{L}) = - \phi^A \left(\delta^\lambda_\mu \, \Pi_A^{\ \nu}(\mathcal{L}) - \delta^\nu_\mu \, \Pi_A^{\ \lambda}(\mathcal{L})\right) , \tag{II.3.10a}$$

$$\mathcal{F}^\nu{}_\mu(\mathcal{L}) = - d_\lambda\left[\phi^A \left(\delta^\lambda_\mu \, \Pi_A^{\ \nu}(\mathcal{L}) - \delta^\nu_\mu \, \Pi_A^{\ \lambda}(\mathcal{L})\right)\right] . \tag{II.3.10b}$$

In spite of the resemblance, this term has nothing to do with that obtained by Belinfante (1939) and Rosenfeld (1940) in order to symmetrise the energy-momentum tensor and to guarantee the conservation of the angular momentum.

The modified Hamiltonian is

$$\mathcal{H}_{\text{mod}}{}^{\nu}{}_{\mu}(\mathcal{L}) = - \phi^A \, d_\mu \Pi_A^{\ \nu}(\mathcal{L}) - \delta^\nu_\mu \left(\mathcal{L} - \phi^A_{\ \lambda} \, \Pi_A^{\ \lambda}(\mathcal{L}) - \phi^A \, d_\lambda \Pi_A^{\ \lambda}(\mathcal{L})\right) . \tag{II.3.11}$$

Its variation is given by

$$\delta\mathcal{H}_{\mathrm{mod}}{}^{\nu}{}_{\mu}(\mathcal{L}) = - \left[d_{\mu}\Pi_{A}{}^{\nu}(\mathcal{L}) + \delta^{\nu}_{\mu} \left(\partial_{A}\mathcal{L} - d_{\lambda}\Pi_{A}{}^{\lambda}(\mathcal{L})\right)\right] \delta\phi^{A}$$

$$+ \delta^{\nu}_{\mu}\, \phi^{A}{}_{\lambda}\, \delta\Pi_{A}{}^{\lambda}(\mathcal{L}) - \delta^{\nu}_{\mu} \left(\partial_{A}{}^{\lambda}\mathcal{L} - \Pi_{A}{}^{\lambda}(\mathcal{L})\right) \delta\phi^{A}{}_{\lambda}$$

$$- \phi^{A} \left(\delta^{\alpha}_{\mu}\, \delta^{\nu}_{\beta} - \delta^{\alpha}_{\beta}\, \delta^{\nu}_{\mu}\right) \delta d_{\alpha}\Pi_{A}{}^{\beta}(\mathcal{L}) \ . \qquad \text{(II.3.12)}$$

From the definition of the momenta, it follows that $\mathcal{H}_{\mathrm{mod}}{}^{\nu}{}_{\mu}(\mathcal{L})$ depends only on ϕ^{A}, $\Pi_{A}{}^{\mu}(\mathcal{L})$ and $d_{\nu}\Pi_{A}{}^{\mu}(\mathcal{L})$. The price paid in order to have the correct dependence on ϕ^{A} and $\Pi_{A}{}^{\mu}(\mathcal{L})$ is the additional dependence on $d_{\nu}\Pi_{A}{}^{\mu}(\mathcal{L})$ coming from $\mathcal{F}^{\nu}{}_{\mu}(\mathcal{L})$.

From (II.3.12) one can write

$$\frac{\partial\mathcal{H}_{\mathrm{mod}}{}^{\nu}{}_{\mu}(\mathcal{L})}{\partial\phi^{A}_{\nu}} = - d_{\mu}\Pi_{A}{}^{\nu}(\mathcal{L}) - \delta^{\nu}_{\mu}\, \delta_{A}\mathcal{L} \ , \qquad \text{(II.3.13a)}$$

$$\frac{\partial\mathcal{H}_{\mathrm{mod}}{}^{\nu}{}_{\mu}(\mathcal{L})}{\partial\Pi_{A}{}^{\lambda}(\mathcal{L})} = \delta^{\nu}_{\mu}\, \phi^{A}{}_{\lambda} \ , \qquad \text{(II.3.13b)}$$

$$\frac{\partial\mathcal{H}_{\mathrm{mod}}{}^{\nu}{}_{\mu}(\mathcal{L})}{\partial D_{\alpha}\Pi_{A}{}^{\beta}(\mathcal{L})} = - \phi^{A} \left(\delta^{\alpha}_{\mu}\, \delta^{\nu}_{\beta} - \delta^{\nu}_{\mu}\, \delta^{\alpha}_{\beta}\right) \ . \qquad \text{(II.3.13c)}$$

At a first sight eqs.(II.3.13) are more than those necessary to establish the equivalence with the original field equations. The point is that many of them, as can be seen by using (II.3.11), are identities. We can restrict our considerations to the following ones. If the field equations (II.2.7) hold then eq.(II.3.13a) reduces to

$$\frac{\partial\mathcal{H}_{\mathrm{mod}}{}^{\nu}{}_{\mu}(\mathcal{L})}{\partial\phi^{A}} = - d_{\mu}\Pi_{A}{}^{\nu}(\mathcal{L}) \ . \qquad \text{(II.3.14a)}$$

From a contraction of eq.(II.3.13b) it follows that

$$\frac{\partial\mathcal{H}_{\mathrm{mod}}{}^{\nu}{}_{\mu}(\mathcal{L})}{\partial\Pi_{A}{}^{\nu}(\mathcal{L})} = \phi^{A}{}_{\mu} \ , \qquad \text{(II.3.14b)}$$

Equations (II.3.14) are the covariant Hamilton equations.

Comparison of eqs.(II.3.14) and (II.3.9) gives $\alpha=1$, $\beta=0$, in such a way that eq.(II.3.7) can be written as

$$d_{\mu}F = \{F, \mathcal{H}_{\mathrm{mod}}{}^{\nu}{}_{\mu}(\mathcal{L})\}_{\nu} \ . \qquad \text{(II.3.15)}$$

This shows that the modified Hamiltonian is the generator of space-time displacements.

For the canonical variables

$$\{\phi^A,\ \Pi_B^{\ \nu}(\mathcal{L})\}_\mu = \delta_B^A\ \delta_\mu^\nu \qquad \text{(II.3.16)}$$

and all other brackets equal to zero. The covariant Poisson bracket induces a covariant symplectic structure on the covariant phase space. The Jacobi identity is

$$\{F,\{G,H\}_{(\mu}\}_{\nu)} + \{G,\{H,F\}_{(\mu}\}_{\nu)} + \{H,\{F,G\}_{(\mu}\}_{\nu)} = 0 \qquad \text{(II.3.17)}$$

where $(\cdot)$ denotes symmetrisation.

II.3.2. Surface Terms

The meaning of $\mathcal{F}^\nu_{\ \mu}(\mathcal{L})$ is made clear when looking at the canonical field theory. In the same way in which the time-time component of the canonical Hamiltonian, $\mathcal{H}_{c\ 0}^{\ 0}(\mathcal{L})$, is identified with the Hamiltonian density of the canonical formulation, the time-time component of the modified Hamiltonian is identified with the Hamiltonian density of a modified theory. Let us consider

$$\mathcal{H}_{\text{mod}\ 0}^{\ \ 0}(\mathcal{L}) = \mathcal{H}_{c\ 0}^{\ 0}(\mathcal{L}) + D_i\mathcal{F}^{i0}_{\ \ 0}(\mathcal{L}) \ . \qquad \text{(II.3.18)}$$

Integration of eq.(23) over Σ gives

$$H_{\text{mod}}(\mathcal{L}) = H_c(\mathcal{L}) + \int_{\partial\Sigma} \phi^A\ \Pi_A^{\ p}(\mathcal{L})\ dS \qquad \text{(II.3.19)}$$

where dS is the surface element of $\partial\Sigma$ and $v^p = n_i v^i$ with n_i the outer normal to dS. Therefore, the extra term corresponds to a surface term in the definition of the energy. This surface term has been dealt with from another point of view in (Tapia, 1987). Additionally $\mathcal{F}^\nu_{\ \mu}(\mathcal{L})$ gives, without awkward calculations, the Lagrange multiplier appearing in eq.(6) of (Tapia, 1987), the answer being $\lambda^A = \phi^A$.

The same results can be extended to other physically relevant quantities. If $\mathcal{H}_c^{\ \nu\lambda} = \mathcal{H}_{c\ \mu}^{\ \nu}\eta^{\mu\lambda}$, with $\eta^{\mu\lambda}$ the metric of the Minkowski space, is symmetric we have a further conservation law

$$d_\alpha \mathcal{M}_c^{\ \alpha\mu\nu}(\mathcal{L}) = 0 \ , \qquad \text{(II.3.20)}$$

where

$$\mathcal{M}_c^{\ \alpha\mu\nu}(\mathcal{L}) = \mathcal{H}_c^{\ \alpha\mu}(\mathcal{L})\ x^\nu - \mathcal{H}_c^{\ \alpha\nu}(\mathcal{L})\ x^\mu \ , \qquad \text{(II.3.21)}$$

is the angular momentum tensor density. As for the energy-momentum tensor, (II.3.20) is defined up to an additive identically conserved tensor. We would like to have $\mathcal{M}$ constructed in terms of $\mathcal{H}_{mod}$ rather than in terms of $\mathcal{H}_c$. Then, the solution is

$$\mathcal{M}_{mod}{}^{\alpha\mu\nu}(\mathcal{L}) = \mathcal{M}_c{}^{\alpha\mu\nu}(\mathcal{L}) + \left\{\mathcal{F}^{\alpha\mu}(\mathcal{L})\ x^\nu - \mathcal{F}^{\alpha\nu}(\mathcal{L})\ x^\mu - \mathcal{F}^{\mu\nu\alpha}(\mathcal{L})\right\} ,$$

$$= \mathcal{H}_{mod}{}^{\alpha\mu}(\mathcal{L})\ x^\nu - \mathcal{H}_{mod}{}^{\alpha\nu}(\mathcal{L})\ x^\mu - \mathcal{F}^{\mu\nu\alpha}(\mathcal{L}) \ . \quad \text{(II.3.22)}$$

The second term in (II.3.22) is identically conserved due to the remarkable identity

$$D_\alpha \mathcal{F}^{\mu\nu\alpha}(\mathcal{L}) \doteq \mathcal{F}^{\nu\mu}(\mathcal{L}) - \mathcal{F}^{\mu\nu}(\mathcal{L}) \ . \quad \text{(II.3.23)}$$

We can therefore rewrite (II.3.20) as

$$D_\alpha \mathcal{M}_{mod}{}^{\alpha\mu\nu}(\mathcal{L}) = 0 \ . \quad \text{(II.3.24)}$$

From the continuity equations (II.3.5) and (II.3.20) we obtain: the energy

$$E_{mod}(\mathcal{L}) = \int_\Sigma \mathcal{H}_{mod}{}^0{}_0(\mathcal{L})\ d\Sigma$$

$$= E_c(\mathcal{L}) + \int_{\partial\Sigma} \phi^A\ \Pi_A{}^p(\mathcal{L})\ dS \ , \quad \text{(II.3.25a)}$$

the linear momentum

$$P^{mod}{}_i(\mathcal{L}) = \int_\Sigma \mathcal{H}_{mod}{}^0{}_i(\mathcal{L})\ d\Sigma$$

$$= P^c{}_i(\mathcal{L}) - \delta^p_i \int_{\partial\Sigma} \phi^A\ \Pi_A{}^0(\mathcal{L})\ dS \ , \quad \text{(II.3.25b)}$$

and the angular momentum

$$M_{mod}{}^{\mu\nu}(\mathcal{L}) = \int_\Sigma \mathcal{M}_{mod}{}^{0\mu\nu}(\mathcal{L})\ d\Sigma$$

$$= M_c{}^{\mu\nu}(\mathcal{L}) + \int_{\partial\Sigma} \left[\ \mathcal{F}^{p0\mu}(\mathcal{L})\ x^\nu - \mathcal{F}^{p0\nu}(\mathcal{L})\ x^\mu\ \right] dS \ . \quad \text{(II.3.25c)}$$

Concluding, $\mathcal{H}_{mod}{}^\nu{}_\mu(\mathcal{L})$ together with the covariant Poisson bracket (II.3.8) provides a correct covariant approach to field theory. As a by-product the surface terms of the canonical field theory appearing for the energy, for the linear and angular momentum are easily calculated by means of eqs. (II.3.25). It must be furthermore remarked that no asymptotic or boundary conditions on the behaviour of the fields are needed in order to obtain the surface terms. In general relativity the previous surface terms have a long history (Dirac,1959; Arnowitt et al., 1962; DeWitt, 1963; Regge and Teitelboim, 1974); however, they are introduced in a quite *ad hoc* and not systematic way. Conclusions similar to ours, in the sense that the physically relevant quantities are the canonical ones supplemented by surface terms, were reached in another context by Ferraris and Francaviglia (1987) using the Poincaré- Cartan form as extended to higher-order derivatives field theories.

To finish it must be observed that the modified Hamiltonian gives rise to the same dynamics as the canonical one

$$\dot{F} = \{F, H_{mod}\} \ . \tag{II.3.26}$$

Part Three. Local Higher-Order Field Theory

We consider higher-order field theory, i.e., Lagrangians depending on higher-order derivatives of the fields. We start introducing the fundamentals of higher-order field theory. The momenta are defined as the terms multiplying the variation of the fields in the boundary term of the variation of the action. The main difference with respect to the first-order case appears in the definition of the momenta, functional rather than ordinary derivatives must be used.

We establish a remarkable identity for the Lagrangian. Then we study the problem of identically vanishing field equations. The previously established identity helps us to prove that the necessary and sufficient condition for identically vanishing field equations is the Lagrangian being a divergence. Then we construct the canonical energy-momentum tensor. As was mentioned previously the canonical energy-momentum tensor is not d-invariant. Once again the previous identity helps us to solve this problem. In fact, with it we are able to select a representative for d-equivalent Lagrangians such that when the Lagrangian is a divergence the representative is identically zero, therefore it has also an identically null canonical energy-momentum tensor.

We study then second-order field theory. Some developments are done only for the second-order case, i.e., Lagrangians depending on up to the second-order derivatives of the fields. Where not explicitly written the notation is the same as that introduced in Part Two.

III.1. Higher-Order Field Theory

III.1.1. The Lagrangian Formalism

The dynamical information of the physical system is contained in the action S

$$S[\mathcal{L}] = \int_\Omega \mathcal{L}\, d^4x \ , \tag{III.1.1}$$

where now the Lagrangian density is of the following type

$$\mathcal{L} = \mathcal{L}(\phi^A, \phi^A_\mu, \ldots, \phi^A_{\mu_1 \ldots \mu_s}, \ldots) \ . \tag{III.1.2}$$

$\phi^A_{\mu_1 \ldots \mu_s}$ denote the s-th order partial derivative of the fields with respect to the local coordinates, which appear into $\mathcal{L}$ up to a maximum order k, which is called the *order of the Lagrangian*. The following notation is also used: $\partial_A^{\mu_1 \ldots \mu_s} = \partial / \partial \phi^A_{\mu_1 \ldots \mu_s}$.

The action can then be rewritten as

$$S = \int_{t_1}^{t_2} \int_{\Sigma(t)} \mathcal{L}\, d\Sigma\, dt \ . \tag{III.1.3}$$

Under arbitrary variations of the fields the variation of the action is

$$\begin{aligned} \delta S = & \int_{t_1}^{t_2} \int_{\Sigma(t)} \delta_A \mathcal{L}\, \delta\phi^A\, d\Sigma\, dt \\ & + \sum_{n=1}^{k} \int \delta_A^{\mu_1 \ldots \mu_n} \mathcal{L}\, \delta\phi^A_{\mu_1 \ldots \mu_{n-1}}\, d\Sigma_{\mu_n} \ , \end{aligned} \tag{III.1.4}$$

where

$$\delta_A \mathcal{L} = \partial_A \mathcal{L} - d_\mu \delta_A^\mu \mathcal{L} \ , \tag{III.1.5}$$

with the operator δ_A^μ recursively defined at all orders by

$$\delta_A^\mu \doteq \partial_A^\mu - d_\nu \delta_A^{\nu\mu} \ ,$$

$$\delta_A^{\mu\nu} \doteq \partial_A^{\mu\nu} - d_\lambda \delta_A^{\lambda\mu\nu} \ , \text{ etc.} \tag{III.1.6}$$

is the Lagrangian functional derivative. The last terms have been obtained by integration by parts using the fact that δ and d_μ conmute.

The functions multiplying the variations $\delta\phi^A$, $\delta\phi^A{}_\mu$, ..., $\delta\phi^A{}_{\mu_1\ldots\mu_s}$, etc., in the boundary term are defined as the momenta $\Pi_A{}^\mu(\mathcal{L})$, $\Pi_A{}^{\mu\nu}(\mathcal{L})$, ..., $\Pi_A{}^{\mu_1\ldots\mu_s}(\mathcal{L})$, etc., canonically conjugated to the fields ϕ^A, $\phi^A{}_\mu$, ..., $\phi^A{}_{\mu_1\ldots\mu_s}$, etc.

$$\Pi_A{}^\mu(\mathcal{L}) = \delta_A{}^\mu\mathcal{L} \ ,$$

$$\Pi_A{}^{\mu\nu}(\mathcal{L}) = \delta_A{}^{\mu\nu}\mathcal{L} \ , \text{ etc.} \tag{III.1.7}$$

The field equations are obtained by requiring $\delta S=0$. The first integral in (III.1.4) depends on the values of $\delta\phi^A$ in the interior of the region Ω while the boundary term depends only on the values of $\delta\phi^A$ on $\partial\Omega$. Therefore, both terms must be independently zero. Since the variations $\delta\phi^A$ are arbitrary it must be

$$\partial_A\mathcal{L} - d_\mu\delta_A{}^\mu\mathcal{L} = 0 \ . \tag{III.1.8}$$

For the boundary term it must be $\delta_A{}^{\mu_1\ldots\mu_n}\mathcal{L}\ \delta\phi^A{}_{\mu_1\ldots\mu_{n-1}}\Big|_{\partial\Omega}=0$.

Contracting the field equations (III.1.8) with $\phi^A{}_\mu$ shows that the *canonical energy-momentum tensor*, defined by

$$\mathcal{H}^\mu{}_\nu(\mathcal{L}) = \phi^A{}_\nu\ \delta_A{}^\mu(\mathcal{L}) + \phi^A{}_{\nu\lambda}\ \delta_A{}^{\lambda\mu}(\mathcal{L}) + \ldots - \delta^\mu_\nu\ \mathcal{L} \ , \tag{III.1.9}$$

is a (formally) conserved quantity, i.e. it satisfies

$$d_\mu\mathcal{H}^\mu{}_\nu(\mathcal{L}) = 0 \ . \tag{III.1.10}$$

The corresponding conservation laws are constructed as for a first-order field theory.

III.1.2. A Remarkable Identity for The Lagrangian

Inspired by some previous results by Vainberg (1964), Atherton and Homsy (1975) and Engels (1975, 1978), Tapia et al. (1988b) derived a remarkable identity for the Lagrangian. Its importance is due to the fact that from it one can immediately infer that a Lagrangian leading to identically vanishing field equations must be a divergence.

Let us consider a k-th order Lagrangian $\mathcal{L}$ and define $f^\mu(\mathcal{L})$ by

$$f^{\mu}(\mathcal{L}) = \int_0^1 [\phi^A(\delta_A{}^{\mu}(\mathcal{L}))^\wedge + \phi^A{}_{\nu}(\delta_A{}^{\nu\mu}(\mathcal{L}))^\wedge + \ldots]\, d\tau, \quad \text{(III.1.11)}$$

where $(\cdot)^\wedge$ means that in the corresponding expression all variables, ϕ, $\partial\phi$, etc, have been scaled by a real factor $\tau\in[0,1]$ (i.e., $\hat{\phi}=\tau\phi$, etc). Then we have

$$d_{\mu}f^{\mu}(\mathcal{L}) = -\phi^A \int_0^1 (\delta_A(\mathcal{L}))^\wedge\, d\tau + \mathcal{L} - \mathcal{L}(0) \ , \quad \text{(III.1.12)}$$

where $\mathcal{L}(0)=\mathcal{L}(\tau)\big|_{\tau=0}$. Since $\mathcal{L}(0)$ is an irrelevant constant we can simply write

$$\mathcal{L} \,\bar{=}\, \phi^A \int_0^1 (\delta_A(\mathcal{L}))^\wedge\, d\tau + d_{\mu}f^{\mu}(\mathcal{L}) \ . \quad \text{(III.1.13)}$$

This is the announced identity.

III.1.3. Identically Vanishing Field Equations

First we investigate the problem of characterising Lagrangian densities (of an arbitrary order) yielding identically vanishing field equations. A k-th order Lagrangian of the form $\mathcal{L}=d_{\mu}\Lambda^{\mu}$, where $\Lambda^{\mu}=\Lambda^{\mu}(\phi^A,\phi^A{}_{\mu},\ldots,\phi^A{}_{\mu_1\ldots\mu_s},\ldots)$ is any arbitrary function of fields and their derivatives at least up to the order k-1 included, $d_{\mu}=d/dx^{\mu}$ is the total, or formal, derivative with respect to x^{μ}, yields identically vanishing field equations. This can be immediately verified just by direct replacement of $d_{\mu}\Lambda^{\mu}$ into eq.(III.1.8). Accordingly, two Lagrangians differing by a divergence yield the same field equations; this property will be called *d-equivalence*. The previous fact involves a sufficient condition for a k-th order Lagrangian to have identically vanishing field equations.

An important problem is to show that the condition is also necessary, i.e. to show that $\delta_A(\mathcal{L})\bar{=}0$, implies $\mathcal{L}=d_{\mu}\Lambda^{\mu}$. For the necessary condition one must prove that $\delta_A(\mathcal{L})\bar{=}0$ implies $\mathcal{L}=d_{\mu}\Lambda^{\mu}$, where the functions Λ^{μ} may however depend also on k-th order derivatives. In fact, known examples show that second-order Lagrangians yielding identically vanishing field equations may not be divergences of first-order functions, while they can be expressed as divergences of functions containing also second-order derivatives of the fields; see (Shadwick, 1982; Tapia, 1987b).

The necessary condition can be easily obtained in Classical Mechanics, as well as for its higher-order generalisations, and also for first-order field theory; see, e.g. (Hojman, 1983). A classical constructive method of proof, which, e.g., can be found in (Hojman, 1983), consists into explicitly writing the total derivatives involved in the field equations. The terms containing derivatives of each different order must be independently zero. This means that the factors multiplying them must be zero, giving differential conditions which might be integrated to obtain the explicit structure of the Lagrangian.

The problem, however, complicates very much for orders larger than one. Some partial results for second-order field theory were obtained by Shadwick (1982); more general results for the second-order case were also recently obtained by the author (Tapia, 1987a). However, for higher-order Lagrangians this constructive method of proof becomes practically unmanageable, cf. sec.III.2.3, and a new method of proof must be looked for. In any case, an implicit proof that $\delta_A(\mathcal{L}) \bar{=} 0$ is equivalent to the Lagrangian being a divergence was given by Krupka (1982), in the context of the theory of Lepagean equivalence. This proof, however, amounts only to an existence theorem, and it is still interesting to derive a method for constructing out in general the divergences which generate identically satisfied field equations.

Using the previous identity it is straightforward to prove that $\delta_A(\mathcal{L}) \bar{=} 0$ implies $\mathcal{L}=d_\mu f^\mu$, which is the result we aimed to establish. Furthermore, this method provides a constructive way to find, through eq.(III.1.11), the function f^μ such that the field equations are identically vanishing. Summarising, we have provided a constructive proof at any order of the fact that $\delta_A(\mathcal{L}) \bar{=} 0$ is equivalent to $\mathcal{L}=d_\mu \Lambda^\mu$.

III.1.4. D-Invariant Field Theory

We investigate the fact that, in Field Theory, the energy-momentum tensor is not invariant under the addition of a divergence to the Lagrangian. As it was mentioned in section xx in field theory the energy-momentum tensor is not invariant under the addition of a divergence to the Lagrangian. Therefore, one must select a representative for d-equivalent Lagrangians. Once again the previous identity helps us in this task; in fact, it allows us to select a representative for the d-equivalent Lagrangians from which the divergence part has been removed. We can then construct a "d-invariant energy-momentum tensor," i.e. an energy-momentum tensor independent on the divergences which can be added to the Lagrangian.

In order to obtain a d-invariant energy-momentum tensor we must find a method to pick up a representative for the d-equivalent Lagrangians, in such a way that the energy-momentum tensor corresponding to a pure divergence is identically zero.

We must therefore look for a linear operator f acting on Lagrangians by

$$\tilde{\mathcal{L}} \doteq f(\mathcal{L}) = \mathcal{L} + d_\mu g^\mu \ , \qquad \text{(III.1.14)}$$

with g^μ some function to determine, such that the following holds

$$\mathcal{H}^\mu{}_\nu((d_\lambda \Lambda^\lambda)\sim) \doteq 0 \ . \qquad \text{(III.1.15)}$$

From the expression (II.1.9) of the canonical energy-momentum tensor we conclude that to satisfy (II.1.15) it is enough to require that f is identically zero when applied to a pure divergence $d_\lambda \Lambda^\lambda$, i.e.

$$f(d_\lambda \Lambda^\lambda) \doteq 0 \ . \qquad \text{(III.1.16)}$$

The operator f must furthermore be a projector, i.e., it must satisfy

$$f^2 = f \ . \qquad \text{(III.1.17)}$$

An obvious solution is thence given by formula (III.1.13)

$$\tilde{\mathcal{L}} \doteq f(\mathcal{L}) \doteq \mathcal{L} - d_\mu f^\mu(\mathcal{L}) = \phi^A \int_0^1 (\delta_A(\mathcal{L}))\hat{}\ d\tau \ . \qquad \text{(III.1.18)}$$

This is in fact identically zero when applied to a pure divergence Lagrangian $\mathcal{L} \doteq d_\lambda \Lambda^\lambda$. It is furthermore a projector, since

$$\begin{aligned} \mathcal{L}^{\approx} &\doteq f^2(\mathcal{L}) \doteq f(\tilde{\mathcal{L}}) = \phi^A \int_0^1 (\delta_A(\tilde{\mathcal{L}}))\hat{}\ d\tau \\ &= \phi^A \int_0^1 (\delta_A(\mathcal{L}))\hat{}\ d\tau = f(\mathcal{L}) \ . \end{aligned} \qquad \text{(III.1.19)}$$

We have been able to free Field Theory from the ambiguities existing under the addition of a pure divergence term to the Lagrangian.

III.2. Second-Order Field Theory

We consider Lagrangian densities of the type

$$\mathcal{L} = \mathcal{L}(\phi^A, \phi^A_\mu, \phi^A_{\mu\nu}) \ . \tag{III.2.1}$$

The corresponding field equations are given by

$$\delta_A \mathcal{L} = \partial_A \mathcal{L} - d_\mu \partial_A^{\ \mu} \mathcal{L} + d_\mu d_\nu \partial_A^{\ \mu\nu} \mathcal{L} = 0 \ . \tag{III.2.2}$$

These equations are of fourth-order in the derivatives of the fields and can be written in a more extended form displaying the fourth-order derivatives as

$$\begin{aligned} &\partial_A \mathcal{L} - \tilde{d}_\mu \partial_A^{\ \mu} \mathcal{L} + \tilde{d}_\mu \tilde{d}_\nu \partial_A^{\ \mu\nu} \mathcal{L} \\ &+ \left[\partial_A^{\ \mu\nu} \partial_B^{\ \lambda} \mathcal{L} - \partial_B^{\ \mu\nu} \partial_A^{\ \lambda} \mathcal{L} + 2\, \tilde{d}_\rho W^{\rho\mu\nu\lambda}_{AB}\right] \phi^B_{\mu\nu\lambda} \\ &+ \partial_A^{\ \mu\nu} \partial_B^{\ \lambda\rho} \partial_C^{\ \sigma\tau} \mathcal{L}\ \phi^B_{\mu\lambda\rho}\ \phi^C_{\nu\sigma\tau} + W^{\mu\nu\lambda\rho}_{AB}\ \phi^B_{\mu\nu\lambda\rho} = 0 \ , \end{aligned} \tag{III.2.3}$$

where

$$\tilde{d}_\mu = \phi^A_\mu\ \partial_A + \phi^A_{\mu\nu}\ \partial_A^{\ \nu} \ , \tag{III.2.4}$$

and

$$W^{\mu\nu\lambda\rho}_{AB} = \partial_A^{\ \mu\nu} \partial_B^{\ \lambda\rho} \mathcal{L} \ , \tag{III.2.5}$$

is a generalised Hessian matrix.

From the definition (III.2.5) follows that W is symmetric in the first and second pair of Greek indices and that

$$W^{\mu\nu\lambda\rho}_{AB} = W^{\lambda\rho\mu\nu}_{BA} \ . \tag{III.2.6}$$

We are going next to consider the conditions under which the Lagrangian field equations (III.2.2) reduce their order. This can be considered as the extension to second-order of the results by Hojman (1983) presented in sec.II.1.2. Some partial results on this line has been obtained by (Shadwick, 1982). When the field equations reduce their order there are some constraints on the system. We are not going to consider this aspect of the problem since it can be developed on the same lines as for classical mechanics, cf. (Tapia, 1985; 1988a).

The notation of differentiation with respect to $\phi^A_{\mu\nu}$ has a symbolic character. The derivative $\partial_A^{\mu\nu}\mathcal{L}$ must be understood as the factor appearing in $d\mathcal{L}=\partial_A^{\mu\nu}\mathcal{L}\ d\phi^A_{\mu\nu}$. However, in the sum $d_\mu d_\nu \partial_A^{\mu\nu}\mathcal{L}$ the terms with $\mu\neq\nu$ appear twice, therefore, when differentiating the real expression for $\mathcal{L}$ with respect to some $\phi^A_{\mu\nu}$, with $\mu\neq\nu$, one obtains a result double from that denoted by $\partial_A^{\mu\nu}\mathcal{L}$. In order to keep this observation in mind when assigning definite values to μ and ν in formulae involving derivatives with respect to $\phi^A_{\mu\nu}$. The following practical rule shows to be useful

$$\partial_A^{\mu\nu}\mathcal{L} \text{ really means } \begin{cases} \partial_A^{\mu\nu}\mathcal{L} \text{ for } \mu=\nu, \\ \frac{1}{2}\partial_A^{\mu\nu}\mathcal{L} \text{ for } \mu\neq\nu. \end{cases} \tag{III.2.7}$$

III.2.1 The Lagrangian Formalism

In this case the variation of the action reduces to

$$\begin{aligned} \delta S = & \int_{t_1}^{t_2}\int_{\Sigma(t)} \delta_A\mathcal{L}\ \delta\phi^A\ d\Sigma\ dt \\ & + \int \delta_A^{\mu}\mathcal{L}\ \delta\phi^A\ d\Sigma_\mu + \int \delta_A^{\mu\nu}\mathcal{L}\ \delta\phi^A_\nu\ d\Sigma_\mu \\ = & \int_{t_1}^{t_2}\int_{\Sigma(t)} \delta_A\mathcal{L}\ \delta\phi^A\ d\Sigma\ dt \\ & + \int_\Sigma \dot{\delta}_A\mathcal{L}\ \delta\phi^A\ d\Sigma\ \Big|_{t_1}^{t_2} + \int_{\partial\Sigma} \delta_A^{p}\mathcal{L}\ \delta\phi^A\ dS\ dt \\ & + \int_\Sigma \ddot{\delta}_A\mathcal{L}\ \delta\dot{\phi}^A\ d\Sigma\ \Big|_{t_1}^{t_2} + \frac{1}{2}\int_{\partial\Sigma} \dot{\delta}_A^{p}\mathcal{L}\ \delta\dot{\phi}^A\ dS\ dt \\ & + \frac{1}{2}\int_\Sigma \dot{\delta}_A^{i}\mathcal{L}\ \delta\phi^A_i\ d\Sigma\ \Big|_{t_1}^{t_2} + \int_{\partial\Sigma} \delta_A^{ip}\mathcal{L}\ \delta\phi^A_i\ dS\ dt\ . \end{aligned} \tag{III.2.8}$$

The one half factors appearing in the fifth and sixth integrals are due to the rule (III.2.7).

Let us now introduce the following definitions

$$\delta_A\mathcal{L} = \delta_A^x\mathcal{L} - \dot{\Pi}_A(\mathcal{L}),$$

$$\delta_A^x\mathcal{L} = \partial_A\mathcal{L} - d_i\partial_A^{i}\mathcal{L} + d_i d_j \partial_A^{ij}\mathcal{L}\ ,$$

$$\Pi_A(\mathcal{L}) = \dot{\delta}^x_A \mathcal{L} - \dot{N}_A(\mathcal{L}) \ ,$$

$$\dot{\delta}^x_A \mathcal{L} = \dot{\partial}_A \mathcal{L} - d_i \dot{\partial}_A{}^i \mathcal{L} \ ,$$

$$N_A(\mathcal{L}) = \ddot{\delta}^x_A \mathcal{L} = \ddot{\partial}_A \mathcal{L} \ . \qquad \text{(III.2.9)}$$

From here one sees that functional rather than ordinary derivatives must be used in the definition of the momenta.

From second-order classical mechanics (Tapia, 1985) one learns that coordinates, q's, and velocities, $\dot{q}$'s, must be treated on the same footing as generalised coordinates. Analogously, in second-order field theory (Tapia, 1988a), the fields ϕ^A, its time- and space-like derivatives, $\dot{\phi}^A$ and $\phi^A{}_i$, must be treated as independent fields. Therefore, the functions multiplying the variations $\delta\phi^A$, $\delta\dot{\phi}^A$ and $\delta\phi^A{}_i$ in the boundary terms in (III.2.2) are defined as independent momenta $\Pi_A(\mathcal{L})$, $N_A(\mathcal{L})$ and $N_A{}^i(\mathcal{L})$ canonically conjugated to the fields ϕ^A, $\dot{\phi}^A$ and $\phi^A{}_i$.

The dynamical behaviour of the physical system is obtained by requiring $\delta S=0$. The first integral depends on the values of $\delta\phi^A$ inside the region Ω while the other ones do it at the boundary. Therefore, each term must be independently zero.

In order to cancel the surface terms in (III.2.2) we assume

$$\delta_A{}^p \mathcal{L} = 0 \ , \qquad \text{(III.2.10a)}$$

$$\dot{\partial}_A{}^p \mathcal{L} = 0 \ , \qquad \text{(III.2.10b)}$$

$$\partial_A{}^{pi} \mathcal{L} = 0 \ , \qquad \text{(III.2.10c)}$$

on $\partial\Sigma$. The previous procedure involves a sufficient condition, the necessary condition, which considers the boundary conditions (III.2.10) as sources at the boundary for the field equations, can be obtained on lines similar as for first-order Lagrangians (Tapia, 1987a).

The dynamical behaviour of the physical system is then given by

$$\delta_A \mathcal{L} = 0 \qquad \text{(III.2.11)}$$

restricted to the boundary conditions (III.2.10). For the volume integrals it must be $\delta\phi^A(t_1)=\delta\phi^A(t_2)=0$, $\delta\dot\phi^A(t_1)=\delta\dot\phi^A(t_2)=0$, $\delta\phi^A{}_i(t_1)=\delta\phi^A{}_i(t_2)=0$. Therefore, a complete set of Cauchy data to integrate eqs.(III.2.6) is provided by the space functions $\phi^A(t_1)$, $\phi^A(t_2)$, $\dot\phi^A(t_1)$, $\dot\phi^A(t_2)$, $\phi^A{}_i(t_1)$, $\phi^A{}_i(t_2)$.

Contraction of the field equations (III.2.11) with $\phi^A{}_\mu$ shows that the *canonical energy-momentum tensor*, defined by

$$\mathcal{H}^\mu{}_\nu(\mathcal{L}) = \phi^A{}_\nu\ \delta_A{}^\mu(\mathcal{L}) + \phi^A{}_{\nu\lambda}\ \delta_A{}^{\lambda\mu}(\mathcal{L}) - \delta^\mu_\nu\ \mathcal{L}\ , \tag{III.2.12}$$

is a (formally) conserved quantity, i.e. it satisfies

$$d_\mu \mathcal{H}^\mu{}_\nu(\mathcal{L}) = 0\ . \tag{III.2.13}$$

The corresponding conservation laws are constructed as for a first-order field theory.

III.2.2. The Hamiltonian Formalism

The canonical Hamiltonian density is defined as the zero-zero component of the energy-momentum tensor which turns to be

$$\mathcal{H}_c(\mathcal{L}) = \dot\phi^A\ \Pi_A(\mathcal{L}) + \ddot\phi^A\ N_A(\mathcal{L}) - \mathcal{L}$$
$$+ \frac{1}{2}\ d_i(\dot\phi^A\ N_A{}^i(\mathcal{L}))\ , \tag{III.2.14}$$

and it cooresponds to the canonical energy density. The scalar canonical Hamiltonian, corresponding to the canonical energy, is therefore

$$H_c[\mathcal{L}] = \int_\Sigma \left(\dot\phi^A\ \Pi_A(\mathcal{L}) + \ddot\phi^A\ N_A(\mathcal{L}) - \mathcal{L}\right)\ d\Sigma$$
$$+ \int_{\partial\Sigma} \frac{1}{2}\ d_i(\dot\phi^A\ N_A{}^i(\mathcal{L}))\ dS\ , \tag{III.2.15}$$

and, in virtue of the field equations (III.2.11) subjected to the boundary conditions (III.2.10), it is a conserved quantity, $\dot{E}=0$.

From the space-like variations of the canonical Hamiltonian one obtains

$$\frac{\delta^x\mathcal{H}_c}{\delta\Pi_A} = \dot\phi^A\ , \qquad \frac{\delta^x\mathcal{H}_c}{\delta\phi^A} = -\ \dot\Pi_A(\mathcal{L})\ ,$$
$$\frac{\delta^x\mathcal{H}_c}{\delta N_A} = \ddot\phi^A\ , \qquad \frac{\delta^x\mathcal{H}_c}{\delta\dot\phi^A} = -\ \dot N_A(\mathcal{L})\ ,$$

$$\frac{\delta^x \mathcal{H}_c}{\delta N_A{}^i} = 0 \ , \qquad \frac{\delta^x \mathcal{H}_c}{\delta \phi^A{}_i} = - \dot{N}_A{}^i(\mathcal{L}) \ . \tag{III.2.16}$$

The canonical variables $(\phi^A, \dot{\phi}^A, \phi^A{}_i, \Pi_A(\mathcal{L}), N_A(\mathcal{L}), N_A{}^i(\mathcal{L}))$ span an infinite-dimensional phase space. In first-order field theory a symplectic structure is induced on the phase space when considering the time derivative of a functional. In fact, let us consider the family of functionals of the form

$$F = \int_\Sigma \mathcal{F}\big(\phi^A, \phi^A{}_i, \phi^A{}_{ij}, \dot{\phi}^A, \dot{\phi}^A{}_i, \Pi_A(\mathcal{L}), N_A(\mathcal{L})\big) \ d\Sigma \ . \tag{III.2.17}$$

Its time derivative is given by

$$\dot{F} \neq \{F, H_c(\mathcal{L})\} \ , \tag{III.2.18}$$

where, in analogy with first-order field theory we define the generalised Poisson bracket as

$$\{F, G\} = \int_{\Sigma(t)} \Bigg[\frac{\delta^x \mathcal{F}(z)}{\delta \phi^A(z)} \frac{\delta^x \mathcal{G}(z)}{\delta \Pi_A(z)} + \frac{\delta^x \mathcal{F}(z)}{\delta \dot{\phi}^A(z)} \frac{\delta^x \mathcal{G}(z)}{\delta N_A(z)}$$

$$+ \frac{\delta^x \mathcal{F}(z)}{\delta \phi^A{}_i(z)} \frac{\delta^x \mathcal{G}(z)}{\delta N_A{}^i(z)} - (\mathcal{F} \leftrightarrow \mathcal{G})\Bigg] \ d\Sigma(z) \ . \tag{III.2.19}$$

The canonical variables can be written as functionals with a delta function as kernel, e.g.

$$\phi^A(x) = \int_{\Sigma(t)} \phi^A(y) \ \delta^{(3)}(x - y) \ d\Sigma(y) \ . \tag{III.2.20}$$

Then, for the canonical variables one has

$$\{\phi^A(x), \ \Pi_B(y)\} = \delta^A_B \ \delta^{(3)}(x - y) \ , \tag{III.2.21a}$$

$$\{\dot{\phi}^A(x), \ N_B(y)\} = \delta^A_B \ \delta^{(3)}(x - y) \ , \tag{III.2.21b}$$

$$\{\dot{\phi}^A{}_i(x), \ N_B{}^j(y)\} = \delta^A_B \ \delta^j_i \ \delta^{(3)}(x - y) \ , \tag{III.2.21c}$$

and all other brackets equal to zero.

The generalisation to second-order of the covariant formalism developed on sec.II.3 gives the modified energy-momentum tensor

$$\mathcal{T}^{\lambda\mu}{}_\nu(\mathcal{L}) = - \phi^A \ (\delta^\lambda_\nu \ \pi_A{}^\mu(\mathcal{L}) - \delta^\mu_\nu \ \pi_A{}^\lambda(\mathcal{L}))$$

$$- \phi^A{}_\rho \ (\delta^\lambda_\nu \ \pi_A{}^{\mu\rho}(\mathcal{L}) - \delta^\mu_\nu \ \pi_A{}^{\lambda\rho}(\mathcal{L})) \ . \tag{III.2.22}$$

For the modified Hamiltonian we obtain

$$\mathcal{H}_{mod}(\mathcal{L}) = \mathcal{H}_c(\mathcal{L}) + d_i \mathcal{U}^i(\mathcal{L}) \quad , \qquad \text{(III.2.23)}$$

with

$$\mathcal{U}^i(\mathcal{L}) = \phi^A \, \Pi_A{}^i(\mathcal{L}) + \dot{\phi}^A \, N_A{}^i(\mathcal{L}) + \phi^A{}_j \, N_A{}^{ij}(\mathcal{L}) \quad , \qquad \text{(III.2.24)}$$

Now we can equally write

$$\dot{F} = \{F, H_{mod}(\mathcal{L})\} \quad , \qquad \text{(III.2.25)}$$

such that

$$\dot{H}_{mod}(\mathcal{L}) = 0 \ . \qquad \text{(III.2.26)}$$

III.2.2. Non-Standard Lagrangians

As we saw at the beginning of this section a second-order Lagrangian density yields fourth-order field equations. Here we will consider the necessary and sufficient conditions for a second-order Lagrangian density to yield third-order and second-order field equations, which we will call non-standard Lagrangians. With the previous conditions one is making the Lagrangian a constrained one. However, we will not be interested in this aspect since it is a straightforward generalisation of what was done in (Tapia, 1985) for classical mechanics. By abuse of language we will call velocities and accelerations the first-order and second-order derivatives of the fields, respectively.

The importance of characterising non-standard Lagrangians lies on the direct applicability it has to some current field theories, General Relativity at the first place. Furthermore, thre has been some recent proposals in some alternative theories of gravity of considering Lagrangians polynomial in the curvature. For example, the Lovelock Lagrangian is defined as the most general metric dependent Lagrangian yielding second-order field equations.

A sth-order Lagrangian yields 2sth-order equations of motion. One can introduce a hierarchical classification of the sth-order Lagrangians by considering the necessary and sufficient conditions under which these Lagrangians yield rth-order, r<2s, equations of motion. In classical mechanics for $s=1$ this classification is trivial; there are only two hierarchies: standard and non-standard Lagrangians. Non-standard Lagrangians are those linear in the velocities. The case $s=2$ was studied in (Tapia, 1985). Its extension to $s>2$ is straightforward. In field theory the case $s=1$ was studied by Hojman (1983). Here we will consider the case $s=2$. Once again the extension to $s>2$ is straightforward, but with an eye on physical applications we restrict our considerations to the second-order case.

In classical mechanics the necessary and sufficient condition for a second-order Lagrangian to yield second-order equations of motion is to differ from a first-order Lagrangian by the total time-derivative of an arbitrary first-order function. The total time-derivative does not contribute to the equations of motion and therefore one usually restricts the considerations to the first-order Lagrangian. Furthermore, since a sth-order Lagrangian yields 2sth-order equations of motion, in order to obtain second-order equations of motion one needs to consider only first-order Lagrangians. The two previous facts have lead to the usual (wrong) statement in field theory that the necessary and sufficient condition to have second-order field equations is that the Lagrangian density be of first-order.

In classical mechanics all the previous arguments involve a necessary and sufficient condition. In field theory only a sufficient one. The purpose of this section is to look for the necessary conditions.

For a second-order Lagrangian density yielding third-order field equations, case I, the Lagrangian density turns to be a polynomial in the accelerations, its order being given in terms of the dimension of the base space (the space-time) and of the number of fields. The field equations are linear in the third-order derivatives of the fields.

We find that a non-standard Lagrangian density yielding second-order field equations, case II, must be a polynomial in the accelerations. The order of this polynomial still depends on the dimension of the base space (the space-time) and on the number of fields. These conditions do not restrict the order of the polynomial but act only as integrability conditions on the coefficients of the polynomial. We do not solve this integrability conditions since they depend on the particular Lagrangian under consideration. Instead we provide an explicit example of a Lagrangian of this kind.

Finally, we consider some explicit examples: firstly, the scalar field and, secondly, the two-dimensional case. The form of the Lagrangian can be determined quite explicitly.

Case I

By non-standard Lagrangians of the case I here we mean those yielding third-order field equations. In this case it must be required, cf. eq.(III.2.3)

$$W^{(\mu\nu\lambda\rho)}_{AB} \equiv 0 \ , \tag{III.2.27}$$

where $(\cdot)$ means complete antisymmetrisation.

It is convenient to decompose $W^{\mu\nu\lambda\rho}_{AB}$ as

$$W^{\mu\nu\lambda\rho}_{AB} = S^{\mu\nu\lambda\rho}_{AB} + A^{\mu\nu\lambda\rho}_{AB} \ , \tag{III.2.28}$$

where S and A are the symmetric and antisymmetric parts of W with respect to the first and second pair of Greek indices or, given the eq.(III.2.6), with respect to the Latin indices, respectively. Then, it can be verified that the condition (III.2.27) acts only on the symmetric part of W

$$S^{\mu\{\nu\lambda\rho\}}_{AB} = \partial_{(A}{}^{\mu\{\nu}\partial_{B)}{}^{\lambda\rho\}}\mathcal{L} \equiv 0 \ , \tag{III.2.29}$$

where $\{\cdot\}$ denotes cyclic permutation.

Equations (III.2.29) are a set of differential equations for $\mathcal{L}$ with respect to the accelerations. Unfortunately, it seems not possible to obtain the general form of $\mathcal{L}$ without integrating eqs. (III.2.29) in a quite pedestrian way. We are not going to do that but instead we will obtain some general but not less important results.

The first step is to rewrite eqs. (III.2.29) as

$$\partial_A{}^{\mu\mu}\partial_B{}^{\mu\mu}\mathcal{L} \doteq 0 \ , \tag{III.2.30a}$$

$$\partial_A^{\mu\mu}\partial_B^{\mu\nu}\mathcal{L} + \partial_B^{\mu\mu}\partial_A^{\mu\nu}\mathcal{L} \doteq 0 \ , \qquad \text{(III.2.30b)}$$

$$\partial_A^{\mu\mu}\partial_B^{\nu\nu}\mathcal{L} + \partial_B^{\mu\mu}\partial_A^{\nu\nu}\mathcal{L} + \partial_A^{\mu\nu}\partial_B^{\mu\nu}\mathcal{L} \doteq 0 \ , \qquad \text{(III.2.30c)}$$

$$\partial_{(A}^{\ \mu\mu}\partial_{B)}^{\ \nu\lambda}\mathcal{L} + \partial_{(A}^{\ \mu\nu}\partial_{B)}^{\ \mu\lambda}\mathcal{L} \doteq 0 \ , \qquad \text{(III.2.30d)}$$

$$\partial_{(A}^{\ \mu\nu}\partial_{B)}^{\ \lambda\rho}\mathcal{L} + \partial_{(A}^{\ \mu\lambda}\partial_{B)}^{\ \rho\nu}\mathcal{L} + \partial_{(A}^{\ \mu\rho}\partial_{B)}^{\ \nu\lambda}\mathcal{L} \doteq 0 \ . \qquad \text{(III.2.30e)}$$

In these equations μ,ν,λ and ρ are all different, there is no summation over repeated indices and use has already been made of the practical rule (III.2.7).

Equation (III.2.30a) implies that the accelerations with equal indices, which we will call A-couples, appear at most linearly. Deriving eq.(III.2.30c) with respect to $\phi^C_{\mu\nu}$, $\mu\neq\nu$, and using (III.2.30b) one concludes that the accelerations with unequal indices, which we will call B-couples, appear at most quadratically.

From (III.2.30a) and (III.2.30c) we see that

$$\partial_A^{\mu\mu}\partial_B^{\mu\nu}\partial_C^{\mu\nu}\mathcal{L} \doteq 0 \ . \qquad \text{(III.2.31)}$$

Therefore, the B-couples involving at least one of the indices already contained in one of the A-couples can appear at most linearly.

Theorem. If equations (III.2.30) are satisfied then $\mathcal{L}$ is a polynomial in the accelerations of order at most $n(n+1)/2$.

Proof 1. Let us consider the term containing k, $0\leq k\leq n$, A-couples. Then there are $k[n-(k+1)/2]$ B-couples involving at least one of the indices already contained in the A-couples, therefore they can appear at most linearly. From the other $(n-k)(n-k-1)/2$ B-couples, due to (III.2.31), only a number $\mathrm{Int}[(n-k)/2]$, where Int denotes the integer part of, of them can appear at most quadratically. Therefore we have

$$N_k = k + \frac{n(n-1)}{2} + \mathrm{int}\left(\frac{n-k}{2}\right) \ . \qquad \text{(III.2.32)}$$

Maximisation with respect to k is obtained for k=n

$$N_{max} = N = n(n+1)/2 \ . \qquad \text{(III.2.33)}$$

A second proof is by *reductio ad absurdum*.

Proof 2. Let us assume that $\mathcal{L}$ is a polynomial in the accelerations of order $N_{max}=N+m$, $m\geq 0$. Let us consider the term containing k, $0\leq k\leq n$, A-couples. The B-couples will then appear $l=N+m-k\geq n(n-1)/2$ times. Then, due to (III.2.31), there will be $l-n(n-1)/2$ B-couples appearing cubically. The indices involved in such B-couples cannot appear in a A-couple. Therefore

$$k \leq n - 2\,[l - n(n-1)/2]\ , \tag{III.2.34}$$

which implies

$$k \geq n + 2\,m\ . \tag{III.2.35}$$

The only form of satisfying this is through the equality with m=0, i.e., $N_{max}=N$.

Proof 3. Equation (III.2.30a) implies that the Lagrangian is of the form

$$\mathcal{L} = \sum_{i=0}^{n} \mathcal{L}^{(\mu_1\mu_1\ldots\mu_i\mu_i)_1}_{A_1\ldots A_i}(\phi,\partial\phi,(\partial^2\phi)')$$
$$\times\ \phi^{A_1}_{\mu_1\mu_1}\ldots\phi^{A_2}_{\mu_2\mu_2}\ , \tag{III.2.36}$$

where $(.)_1$ means that the couples of indices $\mu_i\mu_i$ are not repeated and that they appear in some standard order. $(\partial^2\phi)'$ stands by those accelerations $\phi^A_{\mu\nu}$ with $\mu\neq\nu$.

Equation (III.2.30b) can be rewritten as

$$\partial^{\mu\nu}_{A}\mathcal{L}^{(\mu_1\mu_1\ldots\mu\mu\ldots\mu_i\mu_i)_1}_{A_1\ldots B\ldots A_i}$$
$$+\ \partial^{\mu\nu}_{B}\mathcal{L}^{(\mu_1\mu_1\ldots\mu\mu\ldots\mu_i\mu_i)_1}_{A_1\ldots A\ldots A_i} = 0\ . \tag{III.2.37}$$

From here one deduces

$$\partial^{\mu\nu}_{A}\partial^{\mu\nu}_{B}\mathcal{L}^{(\mu_1\mu_1\ldots\mu\mu\ldots\mu_i\mu_i)_1}_{A_1\ldots C\ldots A_i} = 0\ . \tag{III.2.38}$$

Therefore $\mathcal{L}^{(.)_1}$ is linear in those accelerations $(\partial^2\phi)'$ involving at least one index of those already contained in $(.)_1$

$$\mathcal{L}^{(\mu_1\mu_1\ldots\mu_i\mu_i)_1}_{A_1\ldots A_i}$$
$$= \sum_{j=0}^{j_i} \mathcal{L}^{(\mu_1\mu_1\ldots\mu_i\mu_i)_1(\nu_1\lambda_1\ldots\nu_j\lambda_j)_2}_{A_1\ldots A_i\ B_1\ldots B_j}(\phi,\partial\phi,(\partial^2\phi)'')$$
$$\times\ \phi^{B_1}_{\nu_1\lambda_1}\ldots\phi^{B_j}_{\nu_j\lambda_j}\ , \tag{III.2.39}$$

where

$$j_i = i(n-i) + i(i-1)/2 = i(2n-i-1)/2 \ . \qquad \text{(III.2.40)}$$

$(.)_2$ means that the couples of indices $\nu_j\lambda_j$, $\nu_j\neq\lambda_j$, with ν_j or λ_j already contained in $(.)_1$, are not repeated and that they appear in some standard order. $(\partial^2\phi)$" stands by those accelerations $\phi^A_{\mu\nu}$ with $\mu\neq\nu$ and neither μ nor ν contained in $(.)_1$. Equation (III.2.xx) implies that the coefficients $\mathcal{L}^{(.)_1(.)_2}_{\cdot\ \ \cdot}$ are antisymmetric in those AB indices which have at least one common Greek index.

Equation (III.2.30c) can be written as

$$\mathcal{L}^{(\mu_1\mu_1\ldots\mu_i\mu_i\mu\mu\nu\nu)_1}_{A_1\ldots A_i\ (AB)} + \partial_A^{\ \mu\nu}\partial_B^{\ \mu\nu}\mathcal{L}^{(\mu_1\mu_1\ldots\mu_i\mu_i)_1}_{A_1\ldots A_i} = 0, \qquad \text{(III.2.41)}$$

where (AB)=AB+BA. Then, using eq.(III.2.xx) one obtains

$$\partial_A^{\ \mu\nu}\partial_B^{\ \mu\nu}\partial_C^{\ \mu\nu}\mathcal{L}^{(\mu_1\mu_1\ldots\mu_i\mu_i)}_{A_1\ldots A_i} = 0 \ . \qquad \text{(III.2.42)}$$

Therefore, $\mathcal{L}^{(.)_1}_{\cdot}$ depends at most quadratically on the accelerations $(\partial^2\phi)$". Then one can write

$$\mathcal{L}^{(\mu_1\mu_1\ldots\mu_i\mu_i)_1(\nu_1\lambda_1\ldots\nu_j\lambda_j)}_{A_1\ldots A_i\ \ B_1\ldots B_j}$$
$$= \sum_{k=0}^{k_i}\mathcal{L}^{(\mu_1\mu_1\ldots\mu_i\mu_i)_1(\nu_1\lambda_1\ldots\nu_j\lambda_j)_2(\rho_1\sigma_1\ldots\rho_k\sigma_k)_3}_{A_1\ldots A_i\ \ B_1\ldots B_j\ \ C_1\ldots C_k}(\phi,\partial\phi)$$
$$\times\ \phi^{C_1}_{\rho_1\sigma_1}\ldots\phi^{C_k}_{\rho_k\sigma_k}, \qquad \text{(III.2.43)}$$

with $0\leq j\leq j_i$, and $k_i=(n-i)(n-i-1)$, where $(.)_3$ means that the couples of indices $\rho_k\sigma_k$, $\rho_k\neq\sigma_k$, with ρ_k and σ_k not contained neither in $(.)_1$ nor in $(.)_2$, are not repeated and that they appear in some standard order.

The value of k_i is restricted when considering eq.(III.2.30d) which can be written as

$$\partial_{(A}^{\ \ \nu\lambda}\mathcal{L}^{(\mu\mu\mu_1\mu_1\ldots\mu_i\mu_i)_1}_{B)\ A_1\ldots A_i}$$
$$+\ \partial_{(A}^{\ \ \mu\lambda}\partial_{B)}^{\ \ \mu\nu}\mathcal{L}^{(\mu_1\mu_1\ldots\mu_i\mu_i)_1}_{A_1\ldots A_i} = 0 \ . \qquad \text{(III.2.44)}$$

From here one obtains

$$\partial_A^{\ \mu\nu}\partial_B^{\ \mu\nu}\partial_C^{\ \mu\lambda}\partial_D^{\ \mu\lambda}\mathcal{L}^{(\mu_1\mu_1\ldots\mu_i\mu_i)_1}_{A_1\ldots A_i} = 0 \ . \qquad \text{(III.2.45)}$$

This restrict the range of k_i to

$$k'_i = (n-i)(n-i-1)/2 + \mathrm{Int}[(n-i)/2] \ , \qquad \text{(III.2.46)}$$

where Int[] stands by the integer part of. Equation (III.2.30e) does not restrict more the range of k'_i, but only imposes some symmetry conditions on the coefficients of the polynomial. The maximum power, for fixed i, is given by

$$N_i = i + n(n-1)/2 + \mathrm{Int}[(n-i)/2] \ . \qquad \text{(III.2.47)}$$

Maximisation with respect to i obtained for i=n

$$N_{max} = N = n(n+1)/2 \ . \qquad \text{(III.2.48)}$$

Therefore, $\mathcal{L}$ is a polynomial in the accelerations of maximum order equal to $N=n(n+1)/2$.

This is not the end of the story because there are some other conditions to be satisfied. We have not yet considered the additional restrictions imposed by the symmetries in the Latin indices, deriving from (III.2.30b) when A-couples and B-couples have common Greek indices. This extra restriction is well illustrated by the term containing N powers of the accelerations. Then the last term in eqs.(III.2.30c) does not appear and one can conclude from (III.2.30b) and (III.2.30c) that this term is of the form

$$F_{[A_1 \dots A_N]}(\phi,\partial\phi) \ \underbrace{\in \cdot \cdot \cdot \in}_{n+1 \text{ times}} \ \partial^2\phi^{A_1} \dots \partial^2\phi^{A_n} \ , \qquad \text{(III.2.49)}$$

with $[\cdot]$ denoting complete antisymmetrisation. $\in$ are n-dimensional Levi-Civita tensors, their Greek indices being contracted with those of the derivatives of the fields (there is only one possible non-trivial contraction). Since $F_{[A_1 \dots A_N]}$ is completely antisymmetric with respect to the Latin indices this imposes an additional restriction on the order of $\mathcal{L}$. Then, for $m<N$, $N_{max} \le N-1$. The main point is that $\mathcal{L}$ is a polynomial of finite order in the accelerations.

A Lagrangian satisfying eqs.(III.2.30) will yield third-order field equations. By looking at (III.2.3) it seems that these equations will be quadratic in the third-order derivatives of the fields. However, the coefficient multiplyng this term is, due to (III.2.31)

$$\partial_{(C}{}^{(\sigma\tau}\partial_{|A|}{}^{\mu)(\nu}\partial_{B)}{}^{\lambda\rho)}\mathcal{L} = \partial_{(C}{}^{\{\sigma\tau}A^{\mu\}\{\nu\lambda\rho\}}{}_{|A|B)} \quad , \qquad \text{(III.2.50)}$$

where $|\cdot|$ means exclusion from the symmetrisation. An analysis of the symmetries of this object shows that it is identically zero

$$\underbrace{C\,\widehat{A\;B}} = BAC = - BCA = - ACB = ABC = CBA = - CAB \quad , \qquad \text{(III.2.51)}$$

where $\cup$ and $\wedge$ mean symmetric and antisymmetric with respect to those indices, respectively.

Case II

By non-standard Lagrangians of the case II here we mean those yielding second-order field equations. This means to require

$$\partial_A{}^{(\mu\nu}\partial_B{}^{\lambda)}\mathcal{L} - \partial_B{}^{(\mu\nu}\partial_A{}^{\lambda)}\mathcal{L} + 2\,\tilde{d}_\rho W^{\rho(\mu\nu\lambda)}{}_{AB} \doteq 0 \ . \qquad \text{(III.2.52)}$$

These additional restrictions on $\mathcal{L}$ are only on the coefficients of the polynomial expression for $\mathcal{L}$ and do not restrict its order. A look at eq.(III.2.3) shows that the resulting field equations are polynomial in the accelerations of order N+1.

A concrete example of this kind of Lagrangians is provided by the Hilbert Lagrangian for general relativity when written in term of the fields describing the embedding of the four-dimensional space-time into a ten-dimensional flat pseudo-Euclidean space (Regge and Teitelboim, 1976; Tapia, 1988c). In this formalism the Ricci tensor is of the form

$$R_{\mu\nu} = M_{\mu\nu ab}{}^{\lambda\rho\sigma\tau}(\partial\phi)\ \phi^a{}_{\lambda\rho}\ \phi^b{}_{\sigma\tau} \ , \qquad \text{(III.2.53)}$$

with ϕ^a, a=1,...,6, the fields describing the embedding. The metric tensor depends on only up to the velocities, therefore the Lagrangian $\mathcal{L}=\sqrt{-g}R$ is quadratic in the accelerations. The field equations are

$$\left(R^{\mu\nu} - (R/2)\ g^{\mu\nu}\right)\ \phi^a{}_{\mu\nu} = 0 \ , \qquad \text{(III.2.54)}$$

and therefore are cubic in the accelerations.

The Scalar Field

Condition (III.2.33) is identically satisfied when m=1, a scalar field. In this case eqs.(III.2.11) simplify considerably

$$\partial^{\mu\mu}\partial^{\mu\mu}\mathcal{L} \doteq 0 \ , \tag{III.2.55a}$$

$$\partial^{\mu\mu}\partial^{\mu\nu}\mathcal{L} \doteq 0 \ , \tag{III.2.55b}$$

$$2\ \partial^{\mu\mu}\partial^{\nu\nu}\mathcal{L} + \partial^{\mu\nu}\partial^{\mu\nu}\mathcal{L} \doteq 0 \ , \tag{III.2.55c}$$

$$\partial^{\mu\mu}\partial^{\nu\lambda}\mathcal{L} + \partial^{\mu\nu}\partial^{\mu\lambda}\mathcal{L} \doteq 0 \ , \tag{III.2.55d}$$

$$\partial^{\mu\nu}\partial^{\lambda\rho}\mathcal{L} + \partial^{\mu\lambda}\partial^{\rho\nu}\mathcal{L} + \partial^{\mu\rho}\partial^{\nu\lambda}\mathcal{L} \doteq 0 \ . \tag{III.2.55e}$$

In this case the field equations will be automatically of second-order since condition (III.2.52) is identically satisfied.

The general solution is

$$\begin{aligned} \mathcal{L} &= b \in^{\mu_1 \ldots \mu_n} \in^{\nu_1 \ldots \nu_n} \phi_{\mu_1\nu_1} \ldots \phi_{\mu_n\nu_n} \\ &+ b_{\mu_1\nu_1} \in^{\mu_1 \ldots \mu_n} \in^{\nu_1 \ldots \nu_n} \phi_{\mu_2\nu_2} \ldots \phi_{\mu_n\nu_n} \\ &+ b_{\mu_1\mu_2\nu_1\nu_2} \in^{\mu_1 \ldots \mu_n} \in^{\nu_1 \ldots \nu_n} \phi_{\mu_3\nu_3} \ldots \phi_{\mu_n\nu_n} \\ &+ \ldots + B \ , \end{aligned} \tag{III.2.56}$$

where the functions b and B depend only on ϕ and its first order derivatives. The number of independent components is given by 1, $n(n+1)/2$, $n^2(n^2-1)/12$, etc.

Particularly interesting is the two-dimensional case. In this case the Lagrangian density is

$$\begin{aligned} \mathcal{L} &= a + b^{\mu\nu}\ \phi_{\mu\nu} + \tfrac{c}{2} \in^{\mu\lambda} \in^{\nu\rho}\ \phi_{\mu\nu}\ \phi_{\lambda\rho} \\ &= a + b^{00}\ \phi_{00} + 2\ b^{01}\ \phi_{01} + b^{11}\ \phi_{11} \\ &+ c\ \left(\phi_{00}\ \phi_{11} - (\phi_{01})^2\right) \ . \end{aligned} \tag{III.2.57}$$

Since the condition (III.2.52) is identically satisfied the field equations are of second-order.

Two interesting particular cases of the previous Lagrangian are

$$\mathcal{L} = c_0 \left(\phi_{00}\, \phi_{11} - (\phi_{01})^2\right) , \qquad \text{(III.2.58)}$$

with c_0=const. Then one can write

$$\mathcal{L} = c_0\, d_\mu\left(\in^{\mu\lambda} \in^{\nu\rho}\, \phi_\nu\, \phi_{\lambda\rho}\right) = d_\mu \omega^\mu . \qquad \text{(III.2.59)}$$

The field equations are identically vanishing. This is due to the fact that the Lagrangian is the divergence of a vector ω^μ which depends, however, on the accelerations.

The situation is however different from that in classical mechanics where the only component of ω^μ can depend on only up to the velocities. Concrete examples of this kind are the topological invariants for even dimensional, n≥4, Riemannian manifolds.

The second example is given by

$$\mathcal{L} = c_0\, \phi \left(\phi_{00}\, \phi_{11} - (\phi_{01})^2\right) , \qquad \text{(III.2.60)}$$

with c_0=const. The field equations are

$$\phi_{00}\, \phi_{11} - (\phi_{01})^2 = 0 . \qquad \text{(III.2.61)}$$

This provides the simplest concrete example of a Lagrangian depending non-trivially on the accelerations yielding second-order field equations.

The Two-Dimensional Case

Besides being one of the simplest cases to start with the two-dimensional case has direct applications in the determination of the fine structure of strings (Polyakov, 1986; Alonso and Espriu, 1987) where the fields ϕ are the functions describing the embedding of the two-dimensional world-sheet in a D-dimensional, D≥2, flat space.

In this case eqs.(III.2.30) reduce to

$$\partial_A^{\mu\mu} \partial_B^{\mu\mu} \mathcal{L} \doteq 0 , \qquad \text{(III.2.62a)}$$

$$\partial_A^{\mu\mu} \partial_B^{\mu\nu} \mathcal{L} + \partial_B^{\mu\mu} \partial_A^{\mu\nu} \mathcal{L} \doteq 0 , \qquad \text{(III.2.62b)}$$

$$\partial_A^{\mu\mu}\partial_B^{\nu\nu}\mathcal{L} + \partial_B^{\mu\mu}\partial_A^{\nu\nu}\mathcal{L} + \partial_A^{\mu\nu}\partial_B^{\mu\nu}\mathcal{L} \doteq 0 \ , \qquad \text{(III.2.62c)}$$

where $\mu\nu$=0,1, A,B=1,...,m. L is then given by

$$\mathcal{L} = a + b_A^{\mu\nu}\ \phi^A_{\ \mu\nu} + c_{AB} \in^{\mu\lambda} \in^{\nu\rho} \phi^A_{\ \mu\nu}\ \phi^B_{\ \lambda\rho}$$

$$+ d^{\alpha\beta}_{AB} \in^{\lambda\gamma} \phi^A_{\ \alpha\lambda}\ \phi^B_{\ \beta\gamma}$$

$$+ d_{ABC} \in^{\mu\alpha} \in^{\nu\beta} \in^{\lambda\gamma} \phi^A_{\ \mu\nu}\ \phi^B_{\ \beta\lambda}\ \phi^C_{\ \alpha\gamma} \ . \qquad \text{(III.2.63)}$$

The functions a, b, c, d and e depend on only up to the velocities, c_{AB} is symmetric, d_{ABC} is completely antisymmetric; $\in^{\mu\nu}$ are two-dimensional Levi-Civita tensors.

The previous Lagrangian yields third-order field equations. In order to obtain second-order ones one must still satisfy the condition (III.2.52).

IV. Part Four. Applications: the Klein-Gordon Field, the Electromagnetic Field and General Relativity

Here we present applications of our formalism to the Klein-Gordon field, to the electromagnetic field and to general relativity.

IV.1. The Klein-Gordon Field

Commonly the Klein-Gordon field is described by the Lagrangian

$$\mathcal{L}_{KG} = \frac{1}{2} (\phi_\mu \phi^\mu + m^2 \phi^2) , \tag{IV.1.1}$$

where indices are raised and lowered with the Minkowski metric $\eta_{\mu\nu}$=diag(+---).

The canonical momenta is

$$\pi(\mathcal{L}_{KG}) = \dot{\phi} \tag{IV.1.2}$$

The canonical Hamiltonian is

$$\mathcal{H}_c(\mathcal{L}_{KG}) = \frac{1}{2} (\dot{\phi}^2 - \phi_i \phi^i - m^2 \phi^2) . \tag{IV.1.3}$$

The surface term for the energy is

$$\int n_i \phi \phi^i d^2x . \tag{IV.1.4}$$

The d-invariant Lagrangian is

$$\tilde{\mathcal{L}}_{KG} = - \frac{1}{2} \phi (\Box\phi - m^2 \phi) = \mathcal{L}_{KG} + d_\mu(\phi \phi^\mu) , \tag{IV.1.5}$$

where $\Box = \partial_\mu \partial^\mu$. Since it differs for a total divergence from the standard Lagrangian yields the same field equations. However, the energy-momentum tensor must be constructed using the second-order formalism, i.e., by introducing conjugated momenta of second-order. These momenta are given by

$$\pi^\mu(\tilde{\mathcal{L}}_{KG}) = \frac{\partial\tilde{\mathcal{L}}}{\partial\phi_\mu} - d_\nu(\frac{\partial\tilde{\mathcal{L}}}{\partial\phi_{\mu\nu}}) = \frac{1}{2} \phi^\mu , \tag{IV.1.6a}$$

$$\tilde{\pi}^{\mu\nu}(\tilde{\mathcal{L}}_{KG}) = \frac{\partial\tilde{\mathcal{L}}}{\partial\phi_{\mu\nu}} = - \frac{1}{2} \phi \eta^{\mu\nu} . \tag{IV.1.6b}$$

The corresponding energy-momentum tensor is given by

$$\mathcal{H}^{\mu}{}_{\nu}(\tilde{\mathcal{L}}_{KG}) = \frac{1}{2}(\phi_{\nu}\ \phi^{\mu} - \phi\ \phi^{\mu}{}_{\nu}) - \delta^{\mu}_{\nu}\ \tilde{\mathcal{L}}_{KG} \quad , \tag{IV.1.7}$$

where we must put $\tilde{\mathcal{L}}_{KG}=0$. The canonical Hamiltonian is

$$\mathcal{H}_{c}(\tilde{\mathcal{L}}) = \mathcal{H}_{c}(\mathcal{L}) + \frac{1}{2}\ d_{i}(\phi\ \phi^{i}) \ . \tag{IV.1.8}$$

IV.2. The Electromagnetic Field

Commonly electrodynamics is described by the Lagrangian

$$\mathcal{L}_{EM} = \frac{1}{4}\ F_{\mu\nu}\ F^{\mu\nu}$$

$$= \frac{1}{2}\ (\eta^{\mu\lambda}\ \eta^{\nu\rho} - \eta^{\mu\rho}\ \eta^{\nu\lambda})\ \partial_{\mu}A_{\nu}\ \partial_{\lambda}A_{\rho}$$

$$= \frac{1}{2}\ (E_{i}\ E^{i} - B_{i}\ B^{i}) \ , \tag{IV.2.1}$$

where

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \ , \tag{IV.2.2}$$

$$E_{i} = \partial_{0}A_{i} - \partial_{i}A_{0} \ , \tag{IV.2.3a}$$

$$B_{i} = \in_{ijk}\ \partial^{j}B^{k} \ . \tag{IV.2.3b}$$

The canonical momenta are

$$\pi^{0}(\mathcal{L}_{EM}) = 0 \ , \tag{IV.2.4a}$$

$$\pi^{i}(\mathcal{L}_{EM}) = E^{i} \ . \tag{IV.2.4b}$$

The canonical Hamiltonian is

$$\mathcal{H}_{c}(\mathcal{L}_{EM}) = \frac{1}{2}\ (E_{i}\ E^{i} - B_{i}\ B^{i}) + \partial_{i}(A_{0}\ E^{i})$$

$$= -\frac{1}{2}\ (E^{2} + B^{2}) + \partial_{i}(A_{0}\ E^{i}) \ . \tag{IV.2.5}$$

The surface term for the energy is

$$-\int n_{i}\ (A_{0}\ E^{i} - \in^{ijk}\ A_{j}\ B_{k})\ d^{2}x \ . \tag{IV.2.6}$$

The d-invariant Lagrangian is

$$\tilde{\mathcal{L}}_{EM} = -\frac{1}{2} A^{\mu} [\Box A_{\mu} - \partial_{\mu}(\partial_{\lambda} A^{\lambda})]$$

$$= -\frac{1}{2} A^{\mu} \eta^{\lambda\rho} (\partial_{\lambda}\partial_{\rho} A_{\mu} - \partial_{\mu}\partial_{\lambda} A_{\rho})$$

$$= \mathcal{L}_{EM} - \frac{1}{2} d_{\lambda}(A_{\mu} F^{\lambda\mu}) \; . \tag{IV.2.7}$$

Since it differs for a total divergence from the standard Lagrangian yields the same field equations. However, the energy-momentum tensor must be constructed using the second-order formalism, i.e., by introducing conjugated momenta of second-order. These momenta are given by

$$\pi^{\alpha,\mu}(\tilde{\mathcal{L}}_{EM}) = \frac{\partial \tilde{\mathcal{L}}_{EM}}{\partial A_{\alpha,\mu}} - d_{\nu}(\frac{\partial \tilde{\mathcal{L}}_{EM}}{\partial A_{\alpha,\mu\nu}})$$

$$= \frac{1}{4} (2\, \partial^{\mu} A^{\alpha} - \partial^{\alpha} A^{\mu} - \eta^{\alpha\mu} \partial_{\lambda} A^{\lambda}) \; , \tag{IV.2.8a}$$

$$\pi^{\alpha,\mu\nu}(\tilde{\mathcal{L}}_{EM}) = \frac{\partial \tilde{\mathcal{L}}_{EM}}{\partial A_{\alpha,\mu\nu}}$$

$$= -\frac{1}{4} (2\, A^{\alpha} \eta^{\mu\nu} - \eta^{\alpha\mu} A^{\nu} - \eta^{\alpha\nu} A^{\mu}) \; . \tag{IV.2.8b}$$

The corresponding energy-momentum tensor is given by

$$\mathcal{H}^{\mu}_{\ \nu}(\tilde{\mathcal{L}}_{EM}) = \frac{1}{4} \partial_{\nu} A_{\alpha} (2\, \partial^{\mu} A^{\alpha} - \partial^{\alpha} A^{\mu} - \eta^{\alpha\mu} \partial_{\lambda} A^{\lambda})$$

$$- \frac{1}{4} \partial_{\nu}\partial_{\lambda} A_{\alpha} (2\, A^{\alpha} \eta^{\mu\lambda} - \eta^{\alpha\mu} A^{\lambda} - \eta^{\alpha\lambda} A^{\mu})$$

$$- \delta^{\mu}_{\nu} \tilde{\mathcal{L}}_{EM} \; , \tag{IV.2.9}$$

where we must put $\tilde{\mathcal{L}}_{EM}=0$. The canonical Hamiltonian is

$$\mathcal{H}_{c}(\tilde{\mathcal{L}}_{EM}) = \mathcal{H}_{c}(\mathcal{L}_{EM}) + \frac{1}{2} d_{i}(A_{j} F^{ji} - A^{i} \partial_{0} A^{0} - A_{0} E^{i}) . \tag{IV.2.10}$$

IV.3. General Relativity

The solution is partially provided by our d-invariant formalism. From there it is obvious that

$$\tilde{\mathcal{L}} = g^{\mu\nu} \int_0^1 (\sqrt{-g}\, G_{\mu\nu})^{\sim} d\tau = \sqrt{-g}\, R = \mathcal{L}_{H} \; . \tag{IV.3.1}$$

Therefore the Hilbert Lagrangian is the correct d-invariant

Lagrangian to start with. The Hilbert Lagrangian is, explicitly

$$\mathcal{L}_{H} = - \sqrt{-g}\ (G^{\mu\nu\lambda\rho}\ \partial_{\mu\nu} g_{\lambda\rho} + G^{\mu\nu\lambda\rho}\ g^{\sigma\tau}\ \Gamma_{\sigma\mu\nu}\ \Gamma_{\tau\lambda\rho}) \qquad \text{(IV.3.2)}$$

Some useful derivatives are

$$\frac{\partial \mathcal{L}_{H}}{\partial \partial_{\lambda} g_{\mu\nu}} = - \sqrt{-g}\ (G^{\lambda\nu\sigma\tau}\ \Gamma^{\mu}{}_{\sigma\tau} + G^{\lambda\mu\sigma\tau}\ \Gamma^{\nu}{}_{\sigma\tau} - G^{\mu\nu\sigma\tau}\ \Gamma^{\lambda}{}_{\sigma\tau})\ , \qquad \text{(IV.3.3)}$$

$$d_{\mu}(\sqrt{-g}\ G^{\alpha\beta\mu\nu}) = \frac{1}{2}\ \sqrt{-g}\ (G^{\beta\nu\mu\tau}\ \Gamma^{\alpha}{}_{\mu\tau} + G^{\alpha\nu\mu\tau}\ \Gamma^{\beta}{}_{\mu\tau} - 2\ G^{\alpha\beta\mu\tau}\ \Gamma^{\nu}{}_{\mu\tau})\ . \qquad \text{(IV.3.4)}$$

The previous formulae is obtained by using

$$\nabla_{\rho} G^{\mu\nu\lambda\rho} = \partial_{\rho} G^{\mu\nu\lambda\rho} + G^{\alpha\nu\lambda\rho}\ \Gamma^{\mu}{}_{\alpha\rho} + G^{\mu\alpha\lambda\rho}\ \Gamma^{\nu}{}_{\alpha\rho} + G^{\mu\nu\alpha\rho}\ \Gamma^{\lambda}{}_{\alpha\rho} + G^{\mu\nu\lambda\alpha}\ \Gamma_{\alpha} = 0\ . \qquad \text{(IV.3.5)}$$

The corresponding momenta are

$$\Pi^{(\mu\nu)\lambda}(\mathcal{L}_{H}) = - \frac{1}{2}\ \sqrt{-g}\ (G^{\lambda\mu\sigma\tau}\ \Gamma^{\nu}{}_{\sigma\tau} + G^{\lambda\nu\sigma\tau}\ \Gamma^{\mu}{}_{\sigma\tau})\ , \qquad \text{(IV.3.6)}$$

$$\Pi^{(\mu\nu)\lambda\rho}(\mathcal{L}_{H}) = - \sqrt{-g}\ G^{\mu\nu\lambda\rho}\ . \qquad \text{(IV.3.7)}$$

The corresponding energy-momentum tensor is

$$\mathcal{H}^{\mu}{}_{\nu}(\mathcal{L}_{H}) = 2\ \sqrt{-g}\ G^{\mu}{}_{\nu} - d_{\lambda}[\sqrt{-g}\ (\Gamma^{\lambda\mu}{}_{\nu} - \Gamma^{\mu\lambda}{}_{\nu})]\ . \qquad \text{(IV.3.8)}$$

It can be easily verified that the second term, due to its particular form, it is antisymmetric in λ and μ, is an identically conserved quantity. However, it contributes to the conserved quantities with a surface term.

Application of formula (III.2.17) to the Hilbert Lagrangian gives

$$\mathcal{F}^{\lambda\mu}{}_{\nu}(\mathcal{L}_{H}) = 0\ , \qquad \text{(IV.3.9)}$$

such that

$$\mathcal{U}^{i}(\mathcal{L}) = 0\ . \qquad \text{(IV.3.10)}$$

The energy for the gravitational field reduces then to the expression

$$E(\mathcal{L}_H) = - \int d_i [\sqrt{-g}\ (\Gamma^{i0}{}_0 - \Gamma^{0i}{}_0)]\ d^3x$$

$$= - \int \sqrt{-g}\ (\Gamma^{\perp 0}{}_0 - \Gamma^{0\perp}{}_0)]\ dS\ . \qquad \text{(IV.3.11)}$$

For a diagonal spherically symmetric metric the previous formula reduces to

$$E(\mathcal{L}_H) = - 4\pi \int_{r_-}^{r_+} d_r [\sqrt{-g}\ g^{00}\ g^{11}\ \partial_r g_{00}]\ dr$$

$$= - 4\pi \sqrt{-g}\ g^{00}\ g^{11}\ \partial_r g_{00} \Big|_{r_-}^{r_+} . \qquad \text{(IV.3.12)}$$

It is now easy to verify that for a Schwarzschild field E=0 in: satandard coordinates

$$ds^2 = (1 - 2m/r)\ dt^2 - (1 - 2m/r)^{-1}\ dr^2 - r^2\ d\Omega^2\ , \qquad \text{(IV.3.13a)}$$

isotropic coordinates

$$ds^2 = [(2r - m)/(2r + m)]^2\ dt^2$$

$$- (1 + m/2r)^4\ (dr^2 + r^2\ d\Omega^2)\ , \qquad \text{(IV.3.13b)}$$

and Fock harmonic coordinates

$$ds^2 = [(r - m)/(r + m)]\ dt^2 - [(r + m)/(r - m)]\ dr^2$$

$$- (1 + m/r)^2\ r^2\ d\Omega^2\ , \qquad \text{(IV.3.13c)}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta\ d\varphi^2\ . \qquad \text{(IV.3.14)}$$

V. Concluding Remarks

We have considered the local aspects of higher-order field theory. We have seen that in order to preserve d-invariance in field theory one must unavoidably select a representative for the d-equivalent Lagrangians. This is proportional to the field equations and therefore, in general, it will be a second-order one. Instead of dealing with this kind of Lagrangians by substracting a total divergence containing the second-order derivatives we must deal with the full Lagrangian, in order also to avoid the d-invariance problem. After applying field theory in the correct way it must be done we arrive to physically meaningful conservation laws. In these the role of the surface terms is clear. What is lacking now is the globalisation of the previous results.

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