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EXISTENCE AND MULTIPLICITY RESULTS FOR NONLINEAR BOUNDARY VALUE PROBLEMS BY THE USE OF TOPOLOGICAL AND VARIATIONAL METHODS

presented by

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Preface

This thesis has been written with the purpose of obtaining a "Doctor Philosophiae" diploma at the International School of Advanced Studies in Trieste, Italy. It contains the research work I have been able to produce during my two years staying in Louvain - la - Neuve, Belgium. While there, I was sponsored by an Italian fellowship from the Consiglio Nazionale delle Ricerche, which I would like to greatly acknowledge.

The work is divided into seven chapters, each being a research paper to be published in some international journals. Chapter 1 was written in collaboration with Patrick Habets, Chapter 2 with Christian Fabry, Chapters 3 and 6 with Jean Mawhin, Chapter 4 with Jean Pierre Gossez, Chapter 5 with Daniela Lupo and Chapter 7 with Michel Willem. I would like to warmly thank them all for the very fruitful and interesting time spent while working together.

In particular, I would like to thank Jean Mawhin for having invited me at the Institute of Mathematics of Louvain - la - Neuve and for kindly supervising my research.

Alessandro Fonda

Contents

Introduction

The first systematic study of periodic solutions of nonlinear differential equations was started and developed by Henri Poincaré. His first work on the existence of periodic solutions appeared in 1883, and his great interest on this subject was to remain constant during all his life. From the very beginning Poincaré realized the importance of the theory of fixed points, and made explicit use of topological degree arguments.

The work of Fredholm on linear integral equations, the topological contributions of Birkoff and Kellogg, and the further developments of functional analysis, mainly by Banach, Schauder and the Polish school, brought to a new way of considering boundary value problems: the problem of the existence of solutions was replaced by that of finding the fixed points of an operator T, defined on a suitable space of functions, which in some cases is compact, i.e. has the property of transforming bounded sets into relatively compact ones.

In 1934, the Brouwer's finite dimensional theory of the topological degree was completely extended by Leray and Schauder [83] to operators of the form $I - T$, with T being a compact operator. This theory permits in many cases to establish the existence of solutions to a certain boundary value problem by simply proving that the set of all possible solutions is in some sense bounded. In other words, it is sufficient to prove some a priori estimates on the set of solutions in order to conclude that this set is nonempty. The theory developed by Leray and Schauder provides a very powerful tool in nonlinear analysis which has been extensively used for proving existence and multiplicity results for boundary value problems.

A different approach to the study of boundary value problems is possible when the problem has a variational structure. In this case, the existence of solutions can be reduced to the study of the critical points of a functional, defined on a suitable space of functions. In particular, when such a functional is bounded from below or from above, one can try to find out if a point of minimum or maximum exists. For instance, if the functional is coercive and satisfies a lower semi-continuity condition, it is easy to show that a minimum must exist. This relatively simple idea was already applied in some of Poincaré's works; it led Cinquini [30] in 1938 to an elegant proof of an existence result of Hammerstein [71] for a Dirichlet problem with nonlinearity "to the left" of the spectrum.

However, such a simple situation rarely occurs, and it can happen that the given functional is unbounded both from below and from above. In order to attack this kind of problems, Morse started to develop a general topological theory making wide use of algebraic topological tools. His book [116] on the calculus of variations in the large appeared in 1934, and it was again in the same year that Ljusternik and Schnirelmann [86] presented their topological theory for variational problems, proposing a different approach to the problem.

These theories have been extended by Palais [121,122,123], Smale [124,136] and Rothe [135] to infinite dimensional manifolds, thus providing new tools to the study of boundary value problems and inspiring a series of generalizations and new theorems for the applications. In particular, the Mountain Pass Theorem by Ambrosetti and Rabinowitz [10] and the Saddle Point Theorem by Rabinowitz [127] have been extensively used in proving existence and multiplicity results for boundary value problems.

In order to understand better the problems we are going to consider, let us begin with a simple example. We look for 2π - periodic solutions of the following linear equation:

$$x''(t) + \lambda x(t) = h(t) \quad , \tag{1}$$

where λ is a real parameter. It is well known that if $\lambda \notin \{n^2 : n \in \mathbb{N}\}$, there are 2π - periodic solutions to (1) for any forcing term h. On the other hand, if $\lambda = n^2$ for some $n \in \mathbb{N}$, (1) has a 2π - periodic solution if and only if h satisfies

$$\int_0^{2\pi} h(t) \sin nt \, dt = \int_0^{2\pi} h(t) \cos nt \, dt = 0 \ . \tag{2}$$

The above is a simple application of the Fredholm alternative to a selfadjoint differential problem. Notice that the homogeneous equation

$$x''(t) + \lambda x(t) = 0 \tag{3}$$

has nontrivial 2π - periodic solutions if and only if $\lambda = n^2$ for some $n \in \mathbb{N}$, in which case the set $\mathfrak{S}$ of solutions is made of functions which are linear combinations of $\sin nt$ and $\cos nt$. The real numbers n^2, $n \in \mathbb{N}$, are called the eigenvalues of the 2π - periodic problem associated to equation (3).

When one considers 2π - periodic solutions of a nonlinear equation of the type

$$x''(t) + g(x(t)) = h(t) \ , \tag{4}$$

one would like to extend the known results of the linear case to more general situations. Denoting by G a primitive of g, we remark that in the linear case when $g(x) = \lambda x$, one can write $\lambda = g'(x) = g(x)/x = 2G(x)/x^2$. Beginning with Dolph [44] (for the Dirichlet problem), many papers have been devoted to the study of the existence of solutions to boundary value problems where the asymptotic behavior either of $g'(x)$ or of the quotients $g(x)/x$ and $2G(x)/x^2$ has been compared to the spectrum of the differential operator. For the periodic case, see Loud [88].

According to whether solutions exist for any forcing term h or not, the problem is called nonresonant or resonant, respectively. For example, if the function g satisfies the following condition:

$$n^2 < \liminf_{|x|\to\infty} g(x)/x \leq \limsup_{|x|\to\infty} g(x)/x < (n+1)^2 , \qquad (5)$$

for some $n \in \mathbb{N}$, it can be proved that the problem

$$\begin{cases} x''(t) + g(x(t)) = h(t) , \\ x(.) \text{ is } 2\pi \text{ - periodic} \end{cases}$$

is nonresonant. On the other hand, if for example the first strict inequality in (5) is replaced by a non strict one, the problem is resonant with the eigenvalue n^2 , and some further condition has to be considered for a 2π - periodic solution of (4) to exist.

It seems to be unknown if a condition like (2) alone is sufficient in the above situation to assure this existence. Assuming the function $\tilde{g}(x) = g(x) - n^2x$ to be bounded, Landesman and Lazer [78] proposed their by now classical condition in order to prove the existence of a solution:

(LL) *if* v *is any function in the above defined space* $\mathfrak{S}$,

$$\int_0^{2\pi} hv > \liminf_{x\to+\infty} \tilde{g}(x) \int_{\{v>0\}} v - \limsup_{x\to-\infty} \tilde{g}(x) \int_{\{v<0\}} v .$$

In the same setting, Ahmad, Lazer and Paul [2] proposed the following condition, which can be shown to be more general than (LL):

(ALP)$_\pm$ *for* $v \in \mathfrak{S}$, $\displaystyle\lim_{\|v\|\to\infty} \int_0^{2\pi} [- \tilde{G}(v(t)) + h(t)\, v(t)]\, dt = \pm\infty,$

where $\tilde{G}$ *denotes a primitive of* $\tilde{g}$, *and* $\|.\|$ *is a norm in the two - dimensional space* $\mathfrak{S}$.

Dealing with the more general equation

$$x''(t) + g(t, x(t)) = h(t) , \qquad (6)$$

one can consider a situation like the following:

$$a(t) \leq \liminf_{|x|\to\infty} g(t,x)/x \leq \limsup_{|x|\to\infty} g(t,x)/x \leq b(t) \ , \qquad (7)$$

a and b being, for instance, bounded functions. It can then be proved that if for any choice of a function p such that $a(t) \leq p(t) \leq b(t)$ the equation

$$x''(t) + p(t)\, x(t) = 0$$

does not have any nontrivial 2π - periodic solutions, then there exists a 2π - periodic solution of equation (6), for any forcing term h . This is a particular case of a theorem by Lasota and Opial [79] on systems of ordinary differential equations. There is in the above fact a similarity with the homogeneous case, the eigenvalues being replaced by bounded functions.

Another generalization for equation (4) was given by Fučik [60,61] and Dancer [38,39], considering a nonlinearity g with a quotient $g(x)/x$ which, instead of verifying a condition like (5), has different behaviors at $+\infty$ and $-\infty$. Such kind of functions g were called "jumping nonlinearities". Generalizing the above results for equation (6), Habets and Metzen [69] were able to prove essentially the following: assume

$$a(t) \leq \liminf_{x\to+\infty} g(t,x)/x \leq \limsup_{x\to+\infty} g(t,x)/x \leq b(t) \ ,$$

$$c(t) \leq \liminf_{x\to-\infty} g(t,x)/x \leq \limsup_{x\to-\infty} g(t,x)/x \leq d(t) \ ,$$

for some bounded functions a, b, c, d, and suppose that, for any choice of functions p and q such that $a(t) \leq p(t) \leq b(t)$ and $c(t) \leq q(t) \leq d(t)$, the equation

$$x''(t) + p(t)\, x_+(t) - q(t)\, x_-(t) = 0 \ ,$$

where $x_+ = \max\{x,0\}$, $x_- = \max\{-x,0\}$, does not have any nontrivial 2π - periodic solutions. Then there exists a 2π - periodic solution of (6), for any forcing term h .

In Chapter 1 we will show that the above results can be extended to systems of differential equations having nonlinearities which are in some sense asymptotically positively homogeneous. We will moreover give some applications to second order differential equations of Liénard type:

$$x''(t) + f(x(t))x'(t) + g(t, x(t)) = h(t) \quad ,$$

and of Rayleigh type:

$$x''(t) + f(x'(t)) + g(t, x(t)) = h(t) \quad ,$$

and to third order differential equations of the form

$$x'''(t) + ax''(t) + bx'(t) + g(t, x(t)) = h(t) \quad ,$$

where the nonlinearities are of "jumping" type.

In Chapter 2 we will study the 2π - periodic solutions of equation (6) in the setting of assumption (7), admitting the possibility that either the equation

$$x''(t) + a(t)\, x(t) = 0$$

or the equation

$$x''(t) + b(t)\, x(t) = 0$$

or both, do have nontrivial 2π - periodic solutions. In order to overcome this double resonance situation, we will impose at both sides some Landesman - Lazer type conditions, thus generalizing for an equation like (6) the numerous results in the literature dealing with a one - sided resonance situation.

In Chapter 3 we will compare different approaches to the study of an equation like (6), and we will give a characterization of the Lasota - Opial condition in terms of quadratic forms, thus generalizing various existence results by e.g. Mawhin, Ward, Gossez (cf. [65,99,108,109]).

In Chapters 1, 2 and 3, the main tool used to prove our existence results is the theory of the topological degree as developed by Leray and Schauder. More precisely, we make use of the more general coincidence degree theory by Mawhin, of which we will recall the main features in Appendix 1. An abstract existence result related to those in Chapters 1 and 3 is also given in Appendix 1.

There exists a completely different way to study the periodic solutions of equation (6). It comes out in fact that this problem has a variational structure, and it is possible to define on a subspace of the Sobolev space $H^1(0,2\pi)$ a functional

$$f(x) = \int_0^{2\pi} [\ \tfrac{1}{2}|x'(t)|^2 - G(t, x(t)) + h(t)x(t)]\ dt$$

whose critical points correspond to the 2π - periodic solutions of (6). Using this approach it seems more natural to consider assumptions on G rather that on g, since the function g does not appear in the definition of the functional f. In order to find a critical point of the functional f, the first idea is, for example, to look if there exists a point of minimum. Using an argument of Mawhin, Ward and Willem [110], it can be shown that if G satisfies

$$\limsup_{|x|\to\infty}\ 2G(t,x)/x^2 \leq 0 \tag{8}$$

for any t, with strict inequality on a set of positive measure, then the functional f is coercive and has a minimum, which is a solution. Condition (8) means, roughly speaking, that the nonlinearity lies to the left of the spectrum, 0 being the first eigenvalue of our problem. The result is still true for Dirichlet problems associated to partial differential equations of elliptic type (cf. [110]). For these problems, de Figueiredo and Gossez [41] showed that even in case equality in (8) holds for all t, it is possible to prove the existence of a solution by imposing a density type condition on the quotient $2G(t,x)/x^2$. Different arguments were used by Berger [21], Mawhin [104,105] and Anane [11] to prove the existence of solutions under certain assumptions for which a minimum can be found for the functional f.

In Chapter 4 we give a general abstract theorem for the coercivity of a functional like f, and show how all the above mentioned results can be generalized in this setting. Different kinds of boundary value problems for ordinary or partial differential equations of elliptic type will be considered. Here, the Ahmad - Lazer - Paul condition plays a fundamental role. In fact, condition $(ALP)_+$ can be interpreted as a coercivity assumption on f along the space $\mathfrak{S}$. We show that, even if the $(ALP)_+$ condition is necessary for the coercivity of the functional f, it is not sufficient to guarantee the existence of a solution. It is our aim to show some way in which $(ALP)_+$ can be strengthened in order to assure this existence.

In Chapter 5 we will again deal with a periodic problem for an equation like (6), where the nonlinearity is in resonance with the first eigenvalue, but, roughly speaking, stays to the right of it. An Ahmad - Lazer - Paul type condition will be considered at the first eigenvalue, and a nonresonance condition at the second one. In this case the functional f is not any more bounded from below, and we will prove the existence of a saddle point by means of the theorem of Rabinowitz we mentioned above.

In Chapter 6 we will consider the periodic problem associated to conservative systems of second order ordinary differential equations, or to first order Hamiltonian systems, with a periodic nonlinearity. We will prove some multiplicity results by means of a theorem of Chang [27]. A survey on the algebraic topological concepts used in this chapter will be found in Appendix 2. Our results generalize previous theorems by Conley and Zehnder [34], Mawhin and Willem [111,113] and Mawhin [107]. As a very particular case, one can consider the pendulum equation

$$x''(t) + A \sin x(t) = f(t) , \tag{9}$$

where $f(t)$ is a forcing term with mean value zero, and recover the result that (9) has at least two periodic solutions.

In Chapter 7 we study equations like (9) with a periodic forcing term $f(t)$, and we study the existence of subharmonic oscillations, i.e. of periodic solutions having as minimal period a multiple of the period of f. Our main theorem uses Morse theory together with some iteration formulas for an index associated to the solutions of (9).

Chapter 1

PERIODIC SOLUTIONS OF ASYMPTOTICALLY POSITIVELY HOMOGENEOUS DIFFERENTIAL EQUATIONS

§1. INTRODUCTION.

First studies of periodic solutions for a differential equation

$$\ddot{x} + c\dot{x} + g(x) = e(t),$$

where g is asymptotically linear in some sense, are due to W.S. Loud [88] and A.C. Lazer [80]. This was the starting point of a vast literature on Liénard equation

$$\ddot{x} + f(x)\,\dot{x} + g(t,x) = e(t) \tag{1}$$

and its special case, the Duffing equation

$$\ddot{x} + c\dot{x} + g(t,x) = e(t). \tag{2}$$

One can mention for example the papers by R. Reissig [132], M. Martelli [95], J. Mawhin and J.R. Ward [108], J. Mawhin [99], C. Fabry [55] and the literature therein. In these papers, the asymptotic behaviour of the nonlinearity g is controlled through inequalities such as

$$a(t) \le \liminf_{|x|\to\infty} \frac{g(t,x)}{x} \le \limsup_{|x|\to\infty} \frac{g(t,x)}{x} \le b(t). \tag{3}$$

These tend to keep away the quotient $g(t,x)/x$ from the spectrum of the linear operator $Lx = -\ddot{x}$ as $|x| \to \infty$. Closely related results can be found in J. Mawhin [97], J. Mawhin and J.R. Ward [109], P. Omari and F. Zanolin [120]. Similar results for systems have been worked out in A.C. Lazer and D.A. Sanchez [82], P. Habets and M.N. Nkashama [70], for a Rayleigh equation in R. Reissig [133] and for third order equations in G. Villari [140], O.C. Ezeilo and M.N. Nkashama [54]. See also the references therein.

A major generalization was considered in E.N. Dancer [38], [39] and S. Fučik [60], [61]. There, existence of solutions for the equation

$$\ddot{x} + g(x) = e(t) \tag{4}$$

is investigated when the function g is asymptotically positively homogeneous, i.e.

$$\lim_{x \to +\infty} \frac{g(x)}{x} = \mu , \quad \lim_{x \to -\infty} \frac{g(x)}{x} = \nu . \tag{5}$$

Noticing that the quotient g(x)/x could vary from one eigenvalue of L as $x \to -\infty$ to the next one as $x \to +\infty$, or even could cross eigenvalues of L, S. Fučik called the function g a "jumping nonlinearity". These authors considered the positively homogeneous equation

$$\ddot{x} + \mu\, x_+ - \nu\, x_- = 0, \tag{6}$$

where $x_+ = \max(x,0)$ and $x_- = \max(-x,0)$ and introduced the set K, known as Fučik spectrum, of points $(\mu,\nu) \in \mathbb{R}^2$ such that (6) has a non zero periodic solution. Basically they proved that if $(\mu,\nu) \notin K$ and g satisfies (5), equation (4) has a periodic solution. Later, condition (5) has been generalized for a Duffing equation (2) using assumptions of the type (3) . In P. Habets and G. Metzen [69], the asymptotic values of the quotient g(t,x)/x are controlled by the inequalities

$$a(t) \le \liminf_{x \to +\infty} \frac{g(t,x)}{x} \le \limsup_{x \to +\infty} \frac{g(t,x)}{x} \le b(t),$$
$$c(t) \le \liminf_{x \to -\infty} \frac{g(t,x)}{x} \le \limsup_{x \to -\infty} \frac{g(t,x)}{x} \le d(t),$$

together with a condition called property P. This property replaces the assumption $(\mu,\nu) \notin K$ by imposing that zero is the only periodic solution of the positively homogeneous equation

$$\ddot{x} + c\dot{x} + p(t)x_+ - q(t)x_- = 0,$$

whenever $a(t) \le p(t) \le b(t)$, $c(t) \le q(t) \le d(t)$. Such a property P appears already more or less implicitly in A. Lasota and Z. Opial [79] and S. Invernizzi [76].

Recent results along these lines are in P. Drabek and S. Invernizzi [45], R. Iannacci, M.N. Nkashama, P. Omari and F. Zanolin [74]. In the case of one-sided growth restrictions, see also P. Omari, G. Villari and F. Zanolin [119] and L. Fernandes and F. Zanolin [57].

A similar phenomenon was observed by A. Fonda and F. Zanolin [59] for the Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = e(t). \tag{7}$$

Assuming (5) as well as

$$\lim_{x \to +\infty} f(x) = p, \quad \lim_{x \to -\infty} f(x) = q,$$

they indicate a set K in the (μ,ν,p,q) space which generalizes the Fučik spectrum and is such that if $(\mu,\nu,p,q) \notin K$, the equation (7) has at least one periodic solution.

The original motivation of our paper was to prove the existence of periodic solutions for (7) using a property P so as to weaken the above conditions on f and g. Our purpose was also to apply these ideas to other problems such as the Rayleigh equation

$$\ddot{x} + f(t,\dot{x}) + g(t,x) = e(t) \tag{8}$$

and the third order equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + g(t,x) = e(t). \tag{9}$$

The paper is organized as follows. In section 2, we consider a general 1st order equation in $\mathbb{R}^n$

$$\dot{x} = F(t,x). \tag{10}$$

We describe what we mean by F being asymptotically positively homogeneous and check this property in applications . Section 3 is devoted to property P and the main existence theorem for periodic solutions of (10). In section 4, we investigate property P for equations in $\mathbb{R}^2$ using phase plane methods. This applies to Liénard and Rayleigh equations. Section 5 studies property P for equations in $\mathbb{R}^3$ using L^2-estimates on the solutions and their derivatives. In section 6, we deduce

some existence theorems for Liénard equation (1), Rayleigh equation (8) and the third order equation (9).
These contain and generalize results in P. Drabek and S. Invernizzi [45], P. Habets and G. Metzen [69], A. Fonda and F. Zanolin [59] and O.C. Ezeilo and M.N. Nkashama [54].

§2. THE MAIN PROBLEM.

2.1. Consider the periodic boundary value problem

$$\begin{aligned} \dot{x} &= F(t,x) \\ x(0) &= x(2\pi) \ , \end{aligned} \tag{11}$$

where $F : [0,2\pi] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function.

The following assumption expresses the fact that F is asymptotically positively homogeneous.

ASSUMPTION H.

(i) *Let*

$$G(t,x,u) = G_0(t,x) + G_u(t,x)u, \qquad (t,x,u) \in [0,2\pi] \times \mathbb{R}^n \times \mathbb{R}^p,$$

be a continuous function which is positively homogeneous in x, *i.e.*

$$\forall (t,x,u) \in [0,2\pi] \times \mathbb{R}^n \times \mathbb{R}^p, \forall \lambda > 0, \ G(t,\lambda x,u) = \lambda G(t,x,u) \ ;$$

(ii) *let*

$$\alpha : [0,2\pi] \to \mathbb{R}^p \quad \text{and} \quad \beta : [0,2\pi] \to \mathbb{R}^p$$

be continuous functions and

(iii) *assume that for any* $\varepsilon > 0$, *there exist* $\gamma > 0$ *and a continuous function* $u(t,x)$ *such that for every* $(t,x) \in [0,2\pi] \times \mathbb{R}^n$ *one has*

$$u(t,x) \in [\alpha(t) - \varepsilon e , \beta(t) + \varepsilon e] ,$$

where $e \in \mathbb{R}^p$ *is the vector with all components equal to* 1, *and*

$$|G(t,x,u(t,x)) - F(t,x)| \leq \gamma.$$

This assumption holds true in several important applications.

2.2. **Application 1.** Consider the system of equations

$$\begin{aligned} \dot{x} &= y - f(t,x), \\ \dot{y} &= e(t) - g(t,x), \end{aligned} \tag{12}$$

where f,g and e are continuous functions defined for $t \in [0,2\pi]$, $x \in \mathbb{R}$. Recall that the Liénard equation

$$\ddot{x} + h(x)\dot{x} + g(t,x) = e(t)$$

can be written in such a form.

In this application, we assume the following.

ASSUMPTION A1. *There exist continuous functions* a,b,c,d,p,q,r,s *such that the following inequalities hold uniformly in* t :

$$a(t) \leq \liminf_{n\to+\infty} \frac{f(t,x)}{x} \leq \limsup_{n\to+\infty} \frac{f(t,x)}{x} \leq b(t),$$

$$c(t) \leq \liminf_{n\to-\infty} \frac{f(t,x)}{x} \leq \limsup_{n\to-\infty} \frac{f(t,x)}{x} \leq d(t),$$

$$p(t) \leq \liminf_{n\to+\infty} \frac{g(t,x)}{x} \leq \limsup_{n\to+\infty} \frac{g(t,x)}{x} \leq q(t),$$

$$r(t) \leq \liminf_{n\to-\infty} \frac{g(t,x)}{x} \leq \limsup_{n\to-\infty} \frac{g(t,x)}{x} \leq s(t).$$

Let us show that the function

$$F(t,x,y) = (y - f(t,x) \, , \, e(t) - g(t,x))$$

satisfies assumption H.

We shall first introduce the functions

$$
\begin{aligned}
\delta(a,x,b) &= a \quad, && \text{if} \quad x \leq a,\\
&= x \quad, && \text{if} \quad x \in (a,b),\\
&= b \quad, && \text{if} \quad x \geq b,
\end{aligned}
$$

and

$$
\begin{aligned}
\varphi(x) &= 0 \quad, && \text{if} \quad x \in [0,1]\ ,\\
&= x-1 \quad, && \text{if} \quad x \in (1,2],\\
&= 1 \quad, && \text{if} \quad x > 2.
\end{aligned}
$$

With these notations and for any $\varepsilon > 0$, we write (12) as

$$
\begin{aligned}
\dot{x} &= y - u_1(t,x)x_+ + u_2(t,x)x_- + h_1(t,x),\\
\dot{y} &= - u_3(t,x)x_+ + u_4(t,x)x_- + h_2(t,x),
\end{aligned}
$$

where

$$
\begin{aligned}
&x_+ = \max(x,0), \quad x_- = \max(-x,0),\\
&u_1(t,x) = \delta(a(t)-\varepsilon, \frac{f(t,x)}{x}\varphi(|x|), b(t)+\varepsilon),\\
&u_2(t,x) = \delta(c(t)-\varepsilon, \frac{f(t,x)}{x}\varphi(|x|), d(t)+\varepsilon),\\
&u_3(t,x) = \delta(p(t)-\varepsilon, \frac{g(t,x)}{x}\varphi(|x|), q(t)+\varepsilon),\\
&u_4(t,x) = \delta(r(t)-\varepsilon, \frac{g(t,x)}{x}\varphi(|x|), s(t)+\varepsilon).
\end{aligned}
$$

Notice that we can choose R large enough, so that if $x \geq R$ one has

$$
\begin{aligned}
a(t) - \varepsilon \leq \frac{f(t,x)}{x} \leq b(t) + \varepsilon,\\
p(t) - \varepsilon \leq \frac{g(t,x)}{x} \leq q(t) + \varepsilon.
\end{aligned}
$$

Similarly, if $x \leq -R$, one has

$$
\begin{aligned}
c(t) - \varepsilon \leq \frac{f(t,x)}{x} \leq d(t) + \varepsilon,\\
r(t) - \varepsilon \leq \frac{g(t,x)}{x} \leq s(t) + \varepsilon.
\end{aligned}
$$

If we define

$$u = (u_1,u_2,u_3,u_4),$$
$$G(x,y,u) = (y - u_1x_+ + u_2x_- \,,\ - u_3x_+ + u_4x_-),$$
$$\alpha(t) = (a(t),c(t),p(t),r(t)),$$
$$\beta(t) = (b(t),d(t),q(t),s(t)),$$

it is clear that

$$\alpha(t) - \varepsilon\, e \le u(t,x) \le \beta(t) + \varepsilon\, e$$

and that the function

$$\begin{aligned} h(t,x) &= F(t,x,y) - G(x,y,u(t,x)) \\ &= (-f + u_1x_+ - u_2x_- \,,\ e - g + u_3x_+ - u_4x_-) \end{aligned}$$

is bounded as it is continuous with compact support.

2.3. **Application 2.** The Rayleigh equation

$$\ddot{x} + f(t,\dot{x}) + g(t,x) = e(t)$$

can be written in vector form

$$\dot{x} = y,\ \ \dot{y} = e(t) - g(t,x) - f(t,y). \tag{13}$$

As in application 1, we assume that the functions f,g and e are continuous functions defined for $t \in [0,2\pi]$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$. We also assume that assumption A1 holds.

It is then easy to see that the function

$$F(t,x,y) = (y,\ e(t) - g(t,x) - f(t,y))$$

verifies assumption H with

$$G(x,y,u) = (y,\ -u_1y_+ + u_2y_- - u_3x_+ + u_4x_-),$$
$$u(t,x,y) = (u_1(t,y),u_2(t,y),u_3(t,x),u_4(t,x)),$$

where the functions u_i, α and β are defined as in application 1.

2.4. **Application 3.** The third order equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + g(t,x) = e(t,x,\dot{x},\ddot{x})$$

can be written as

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = e(t,x,y,z) - g(t,x) - by - az. \tag{14}$$

We assume that the functions g and e are continuous functions defined for $t \in [0,2\pi]$, $x \in \mathbb{R}$, $y \in \mathbb{R}$, $z \in \mathbb{R}$, and that the following condition holds.

ASSUMPTION A3. *There exist continuous functions* p,q,r,s *such that the following inequalities hold uniformly in* t

$$p(t) \le \liminf_{x\to+\infty} \frac{g(t,x)}{x} \le \limsup_{x\to+\infty} \frac{g(t,x)}{x} \le q(t),$$
$$r(t) \le \liminf_{x\to-\infty} \frac{g(t,x)}{x} \le \limsup_{x\to-\infty} \frac{g(t,x)}{x} \le s(t),$$

and there exist $\delta_0 > 0$, $\Delta_0 > 0$ *such that for any* $t \in [0,2\pi]$, $(x,y,z) \in \mathbb{R}^3$, *one has*

$$|e(t,x,y,z)| \le \delta_0 + \Delta_0 (|x| + |y| + |z|).$$

Let us prove that the function

$$F(t,x,y,z) = (y,\ z,\ e(t,x,y,z) - g(t,x) - by - az)$$

verifies assumption H.

For any $\varepsilon > 0$, we write (14) as

$$\dot{x} = y, \ \dot{y} = z,$$
$$\dot{z} = u_1(t,x,y,z)(|x| + |y| + |z|) - u_2(t,x)x_+ + u_3(t,x)x_- -$$
$$- by - az + h(t,x,y,z),$$

where

$$u_1(t,x,y,z) = e(t,x,y,z)\ \varphi(\frac{\varepsilon}{\delta_0}(|x|+|y|+|z|))/(|x|+|y|+|z|),$$
$$u_2(t,x) = \delta(p(t)-\varepsilon,\ \frac{g(t,x)}{x}\varphi(|x|),\ q(t)+\varepsilon),$$
$$u_3(t,x) = \delta(r(t)-\varepsilon,\ \frac{g(t,x)}{x}\varphi(|x|),\ s(t)+\varepsilon).$$

If we define

$$G(x,y,z,u) = (y,\ z,\ u_1(|x|+|y|+|z|) - u_2x_+ + u_3x_- - by - az),$$
$$\alpha(t) = (-\Delta_0,\ p(t),\ r(t)),$$
$$\beta(t) = (\Delta_0,\ q(t),\ s(t)),$$

it is clear that

$$\alpha(t) - \varepsilon\, e \le u(t,x,y,z) \le \beta(t) + \varepsilon\, e.$$

Moreover

$$\begin{aligned} h(t,x,y,z) &= F(t,x,y,z) - G(x,y,z,\ u(t,x,y,z)) \\ &= (0,0,\ h_1(t,x,y,z) + h_2(t,x)) \end{aligned}$$

is bounded since the functions

$$h_1(t,x,y,z) = e(t,x,y,z) - u_1(t,x,y,z)(|x|+|y|+|z|)$$
$$h_2(t,x) = -g(t,x) + u_2(t,x)x_+ - u_3(t,x)x_-$$

are continuous functions with compact support.

§3. PROPERTY P AND THE MAIN THEOREM.

3.1. **Definition.** Given functions $G(t,x,u)$, $\alpha(t)$ and $\beta(t)$ as in assumption H, we say that the triplet (G,α,β) has *property P* if for any $u \in L^2$ such that

$$\forall\, t \in [0,2\pi] \qquad u(t) \in [\alpha(t),\beta(t)]\ ,$$

zero is the only solution of the boundary value problem

$$\dot{x} = G(t,x,u(t))$$
$$x(0) = x(2\pi).$$

3.2. In order to prove our main theorem, we need the following two lemmas.

Lemma 1. *Let* G(t,x,u), $\alpha(t)$ *and* $\beta(t)$ *be as in assumption* H. *If* (G,α,β) *has property* P, *then there exists* $\varepsilon > 0$ *such that* $(G,\alpha-\varepsilon e,\beta+\varepsilon e)$ *has property* P.

Proof. Suppose the contrary is true. Then for any $n \in \mathbb{N}$, there exists $u_n \in L^2$ such that

$$\forall\, t \in [0,2\pi] \qquad u_n(t) \in [\alpha(t) - \frac{1}{n}e,\ \beta(t) + \frac{1}{n}e\,]$$

and $x_n \in H^1$, $x_n \neq 0$ such that

$$\dot{x}_n(t) = G(t,x_n(t),u_n(t)) \tag{15}$$

$$x_n(0) = x_n(2\pi). \tag{16}$$

The positive homogeneity of G in x allows us to choose x_n such that $\|x_n\|_{H^1} = 1$.

As $C \subset H^1$, the x_n are uniformly bounded in $\|.\|_\infty$. The u_n are also uniformly bounded. Hence from (15) it follows that the x_n are equicontinuous. Going to a subsequence, we can then suppose $x_n \xrightarrow{C} x$.

Likewise, since u_n is a bounded sequence in L^2 we can suppose $u_n \xrightarrow{L^2} u$, for some $u \in L^2$. It follows that

$$G(.,x_n,u_n) \xrightarrow{L^2} G(.,x,u).$$

Indeed, for any $\varphi \in L^2$ we have

$$\int_0^{2\pi} [G(t,x_n(t),u_n(t)) - G(t,x(t),u(t))]\varphi(t)dt =$$

$$= \int_0^{2\pi} [G_0(t,x_n(t)) - G_0(t,x(t))]\varphi(t)dt + \int_0^{2\pi} [G_u(t,x_n(t)) -$$

$$- G_u(t,x(t))]u_n(t)\varphi(t)dt + \int_0^{2\pi} G_u(t,x(t))(u_n(t) - u(t))\varphi(t)dt.$$

From Lebesgue dominated convergence theorem, the two first terms go to zero. As $u_n \xrightarrow{L^2} u$, the same holds true for the third one.

Taking the weak limit of (15) in L^2, and the limit in (16) we obtain

$$\begin{aligned} \dot{x}(t) &= G(t,x(t),u(t)) \\ x(0) &= x(2\pi). \end{aligned} \tag{17}$$

As $u_n \xrightarrow{L^2} u$, it is easy to see that for each $i = 1,\ldots,p$ and almost every $t \in [0,2\pi]$, one has

$$\alpha_i(t) \leq \liminf_{n\to\infty} u_{n_i}(t) \leq \limsup_{n\to\infty} u_{n_i}(t) \leq \beta_i(t).$$

Hence changing u on a set of measure zero, we can assume

$$\forall\, t \in [0,2\pi] \quad , \quad u(t) \in [\alpha(t),\beta(t)].$$

As (G,α,β) has Property P, we deduce from (17) that $x \equiv 0$.

On the other hand, from the positive homogeneity of G in x, we can find $K > 0$ such that

$$\forall (t,x) \in [0,2\pi] \times \mathbb{R}^n,\ \forall\, u \in [\alpha(t)\text{-}\mathbf{e},\beta(t)+\mathbf{e}],\ |G(t,x,u)| \leq K|x|.$$

Hence we can write

$$\begin{aligned} 1 \quad &= \|x_n\|^2_{H^1} = \|x_n\|^2_{L^2} + \|\dot{x}_n\|^2_{L^2} \\ &\leq 2\pi\|x_n\|^2_\infty + \int_0^{2\pi} G^2(t,x_n(t),u_n(t))dt \leq 2\pi(1+K)^2\|x_n\|^2_\infty \end{aligned}$$

which implies

$$x = \lim x_n \neq 0.$$

This is a contradiction. ■

3.3. **Lemma 2.** *Suppose* $F(t,x)$ *is such that assumption* H *holds and that the triplet* (G,α,β) *has property* P.

Then there exists a constant $c > 0$ *such that if* $x \in C^1$ *is a solution of* (11), *then* $\|x\|_\infty \leq c$.

Proof. Assume the lemma were false. Then there exists a sequence of solutions x_n of (1) such that $\forall\ n \in \mathbb{N}$, $\|x_n\|_\infty \geq n$. Let

$$v_n = \frac{x_n}{\|x_n\|_\infty}.$$

By Lemma 1, we can choose $\varepsilon > 0$ such that $(G,\alpha-\varepsilon e,\beta+\varepsilon e)$ has property P. Next, from assumption H, we can choose $\gamma > 0$ and

$$u(t,x) \in [\alpha(t)-\varepsilon e,\beta(t)+\varepsilon e]$$

such that

$$H(t,x) = F(t,x) - G(t,x,u(t,x))$$

verifies

$$|H(t,x)| \leq \gamma.$$

The functions v_n are solutions of

$$\dot{v}_n(t) = G(t,v_n(t),u(t,x_n(t))) + \frac{H(t,x_n(t))}{\|x_n\|_\infty}$$

$$v_n(0) = v_n(2\pi). \tag{18}$$

Clearly the $\dot{v}_n$ are uniformly bounded. Hence the v_n are bounded in H^1 and, going to subsequences, we can assume $v_n \xrightarrow{H^1} v \in H^1$, $v_n \xrightarrow{C} v \neq 0$. Likewise, as the functions u_n are uniformly bounded, we can assume $u_n \xrightarrow{L^2} u \in L^2$. Going to the limit in (18) we obtain

$$\dot{v}(t) = G(t,v(t),u(t))$$
$$v(0) = v(2\pi).$$

As in Lemma 1, it is clear that changing u on a set of measure zero, we have

$$\forall\, t \in [0,2\pi] \qquad u(t) \in [\alpha(t)-\varepsilon e, \beta(t)+\varepsilon e]$$

and, from property P, that $v = 0$, which contradicts $v_n \xrightarrow{C} v$. ■

3.4. To prove the existence of solutions of (11), we shall apply coincidence degree theory [98]. It is clear that Leray-Schauder's degree

[87] could be used at the expense of reformulating the problem as a fixed point problem.

Given functions F(t,x) and G(t,x,u) as in assumption H, and a continuous function $u_0 : [0,2\pi] \to \mathbb{R}^p$, we shall use the following notations :

$$\begin{aligned}
&\mathrm{Dom}\, L = \{x \in C^1 \mid x(0) = x(2\pi)\} \ ; \\
&L \ \ : \mathrm{Dom}\, L \to C, \ x \to x' \ ; \\
&N_1 \ : C \to C, \quad x \to F(.,x) \ ; \\
&N_0 \ : C \to C, \quad x \to G(.,x,u_0).
\end{aligned} \tag{19}$$

It is clear that N_0 and N_1 are L-compact on bounded subsets of C and that L is a linear Fredholm map of index zero.

Theorem 1. *Assume :*
(i) F *satisfies assumption* H ;
(ii) *the triplet* (G,α,β) *has property* P ;
(iii) *for some continuous function* u_0: $[0,2\pi] \to \mathbb{R}^p$ *such that*

$$\forall\, t \in [0,2\pi] \qquad u_0(t) \in [\alpha(t),\beta(t)],$$

one has

$$d\,(L-N_0, \Omega_0) \neq 0,$$

where N_0 *is defined as in* (19) *and* $\Omega_0 = \{x \in C \mid \|x\|_\infty < 1\}$.
Then the problem (11) *has at least one solution.*

<u>Proof.</u> We consider the homotopy

$$Lx - \lambda\, N_1 x - (1-\lambda) N_0 x = 0$$

which corresponds to the boundary value problem

$$\begin{aligned}
\dot{x} &= \lambda F(t,x) + (1-\lambda) G(t,x,u_0(t)), \\
x(0) &= x(2\pi).
\end{aligned} \tag{20}$$

For any $\lambda \in [0,1]$, the function

$$\Psi(t,x,\lambda) = \lambda F(t,x) + (1-\lambda)G(t,x,u_0(t))$$

verifies assumption H with the same functions G,α,β. Indeed, let us fix $\varepsilon > 0$. There exist $\gamma > 0$ and $u(t,x) \in [\alpha(t)-\varepsilon e, \beta(t)+\varepsilon e]$ such that the function

$$H(t,x) = F(t,x) - G(t,x,u(t,x))$$

verifies $|H(t,x)| \leq \gamma$. Further, we can write

$$\Psi(t,x,\lambda) = G(t,x,\lambda u(t,x)+(1-\lambda)u_0(t))+ \lambda H(t,x)$$

which is such that

$$\lambda u(t,x) + (1-\lambda)u_0(t) \in [\alpha(t)-\varepsilon\, e, \beta(t)+\varepsilon\, e]$$

and

$$|\lambda H(t,x)| \leq \gamma.$$

From Lemma 2, there is a constant c such that, for any λ and any solution x of (20),

$$\|x\|_\infty \leq c.$$

By invariance of the degree with respect to an homotopy and excision, one has

$$d(L-N_1,\Omega_1) = d(L-N_0,\Omega_1) = d(L-N_0,\Omega_0) \neq 0,$$

where

$$\Omega_1 = \{x \in C \mid \|x\|_\infty < c + 1\}.$$

Hence, there exists $x \in \overline{\Omega}_1$ such that $Lx = N_1x$, i.e. the problem (11) has at least one solution. ■

Corollary 1. *Assume :*
(i)F *satisfies assumption* H ;
(ii)*the triplet* (G,α,β) *has property* P ;
(iii)*for some continuous function* $u_0: [0,2\pi] \to \mathbb{R}^p$ *such that*

$$\forall\, t \in [0,2\pi] \qquad u_0(t) \in [\alpha(t),\beta(t)],$$

the function $G(t,x,u_0(t))$ *is linear in* x.
Then the problem (1) *has at least one solution.*

The proof follows from the observation that property P implies L - N_0 is one to one and therefore that $d(L - N_0, \Omega_0) \neq 0$.

§4. THE PROPERTY P FOR SECOND ORDER SYSTEMS.

4.1. Consider the equation

$$\dot{x} = G(t,x) \tag{21}$$

where the function $G : [0,2\pi] \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfies Caratheodory conditions and is positively homogeneous in x :

$$\forall (t,x) \in [0,2\pi] \times \mathbb{R}^2, \quad \forall \lambda > 0 : \; G(t,\lambda x) = \lambda\, G(t,x).$$

We will establish some conditions under which the only 2π-periodic solution of (21) is the trivial one. To this end, let us introduce polar coordinates

$$x = (r \cos \theta, r \sin \theta).$$

One computes

$$\begin{aligned} \dot{\theta} &= \mathcal{G}(t,\cos \theta,\sin \theta) = \\ &= \cos \theta\, G_2(t,\cos \theta,\sin\theta) - \sin \theta\, G_1(t,\cos \theta,\sin \theta). \end{aligned} \tag{22}$$

Consider also comparison systems

$$\dot{x} = A(t,x) \quad \text{and} \quad \dot{x} = B(t,x),$$

where the functions A and B are positively homogeneous in x and such that the functions

$$\mathcal{A}(t,x) = x_1 A_2(t,x) - x_2 A_1(t,x), \quad \mathbb{B}(t,x) = x_1 B_2(t,x) - x_2 B_1(t,x)$$

are continuous. Introducing polar coordinates, we have respectively

$$\dot{\theta} = \mathcal{A}(t, \cos \theta, \sin \theta), \tag{23}$$

$$\dot{\theta} = \mathbb{B}(t, \cos \theta, \sin \theta). \tag{24}$$

Proposition 1. *Assume*

$$\mathcal{A}(t,x) \le \mathcal{G}(t,x) \le \mathbf{B}(t,x) \tag{25}$$

and let θ *be a solution of* (22), φ *be a minimal solution of* (23) *and* ψ *be a maximal solution of* (24), *each of them defined on* $[0,2\pi]$ *and such that* $\theta(0) = \varphi(0) = \psi(0)$. *Then for any* $t \in [0,2\pi]$ *one has*

$$\varphi(t) \le \theta(t) \le \psi(t).$$

Proof : See P. Hartman [72] Theorem 4.1 p. 26. ∎

Corollary 2. *Assume* (25) *holds. For any* $\theta_0 \in [0,2\pi]$, *suppose that the functions* φ, *minimal solution of* (23) *such that* $\varphi(0) = \theta_0$, *and* ψ,*maximal solution of* (24) *such that* $\psi(0) = \theta_0$, *are such that*

$$[\varphi(2\pi),\psi(2\pi)] \cap (2\pi)\,\mathbb{Z} = \emptyset.$$

Then equation (21) *has no nontrivial* 2π*-periodic solution.*

Suppose now that A and B are independent of t and that for any x

$$\mathbf{B}(x) = x_1 B_2(x) - x_2\, B_1(x) < 0. \tag{26}$$

Then we know that $\varphi(t)$ and $\psi(t)$, solutions of (23) and (24), decrease. In this case, let t_φ and t_ψ be the time necessary for φ and ψ to decrease of 2π.

Corollary 3. *Assume* (25) *and* (26) *hold and* A, B *are independent of* t; φ *and* ψ *are defined as in Proposition 1.*

If $t_\varphi \ge \dfrac{2\pi}{n+1}$, *then any* 2π*-periodic solution* x *of* (21) *has at most* $2(n+1)$ *zeros in* $[0,2\pi[$ *and if* $t_\varphi > \dfrac{2\pi}{n+1}$ *then x has less than* $2(n+1)$ *zeros.*

If $t_\psi \le \dfrac{2\pi}{n}$, *then* x *has at least* 2n *zeros and if* $t_\psi < \dfrac{2\pi}{n}$, x *has more than* 2n *zeros.*

4.2. In order to compute t_φ and t_ψ in applications, we often have to investigate a comparison system which is piecewise linear

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -K & -L \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and corresponds to the scalar equation

$$\ddot{x} + L\dot{x} + Kx = 0.$$

The equivalent of (23) reads then

$$\dot{\theta} = -(\sin^2\theta + L\cos\theta\sin\theta + K\cos^2\theta), \tag{27}$$

the solution of which is decreasing if

$$L^2 - 4K < 0.$$

Let $t_1(L,K)$ be the smallest positive time such that (27) has a solution with

$$\theta(0) = \frac{\pi}{2}, \quad \theta(t_1) = 0.$$

One computes (see P.B. Bailey, L.F. Shampine and P.E. Waltman [16] p. 36)

$$t_1(L,K) = \int_0^{\pi/2} \frac{d\theta}{\sin^2\theta + L\cos\theta\sin\theta + K\cos^2\theta} = \frac{1}{\sqrt{K - L^2/4}} \cos^{-1}\frac{L}{2\sqrt{K}}. \tag{28}$$

It is easy to see also that if we define $t_i(L,K)$ $(i=1,\ldots,4)$ as the time necessary for a solution of (27) to go from $\theta = \pi - i\frac{\pi}{2}$ to $\theta = \frac{\pi}{2} - i\frac{\pi}{2}$, one has

$$t_1(L,K) = t_2(-L,K) = t_3(L,K) = t_4(-L,K).$$

4.3. **Application 1.** Considering (12), we have to investigate property P for the functions

$$\begin{aligned} G(x,y,u) &= (y - u_1x_+ + u_2x_-, -u_3x_+ + u_4x_-), \\ \alpha &= (a,c,p,r), \\ \beta &= (b,d,q,s). \end{aligned} \tag{29}$$

We shall assume for simplicity that the functions α and β are constant. Hence, we must prove that under appropriate conditions on α and β, the system

$$\begin{aligned} \dot{x} &= y - u_1(t)x_+ + u_2(t)x_- \\ \dot{y} &= - u_3(t)x_+ + u_4(t)x_- \end{aligned} \tag{30}$$

has no nontrivial periodic solution if $u \in C$ and

$$\alpha \le u(t) \le \beta.$$

Proposition 2. *Assume* $a \le b$, $c \le d$, $p \le q$, $r \le s$ *and*

$$a^2 - 4p < 0,\ b^2 - 4p < 0,\ c^2 - 4r < 0,\ d^2 - 4r < 0,$$

and that for some $n \in \mathbb{N}$

$$\begin{aligned} \frac{\pi}{n+1} &< \frac{1}{\sqrt{4q-a^2}} \cos^{-1}\left(-\frac{a}{2\sqrt{q}}\right) + \frac{1}{\sqrt{4s-d^2}} \cos^{-1}\left(\frac{d}{2\sqrt{s}}\right) \\ &+ \frac{1}{\sqrt{4s-c^2}} \cos^{-1}\left(-\frac{c}{2\sqrt{s}}\right) + \frac{1}{\sqrt{4q-b^2}} \cos^{-1}\left(\frac{b}{2\sqrt{q}}\right) \\ &\le \frac{1}{\sqrt{4p-b^2}} \cos^{-1}\left(-\frac{b}{2\sqrt{p}}\right) + \frac{1}{\sqrt{4r-c^2}} \cos^{-1}\left(\frac{c}{2\sqrt{r}}\right) \\ &+ \frac{1}{\sqrt{4r-d^2}} \cos^{-1}\left(-\frac{d}{2\sqrt{r}}\right) + \frac{1}{\sqrt{4p-a^2}} \cos^{-1}\left(\frac{a}{2\sqrt{p}}\right) < \frac{\pi}{n} \end{aligned} \tag{31}$$

then the triplet (G,α,β) *defined in* (29) *has property* P.

Remark. Notice that if $a = b = c = d = 0$, the assumption (31) reduces to

$$\frac{2}{n+1} < \frac{1}{\sqrt{q}} + \frac{1}{\sqrt{s}} \le \frac{1}{\sqrt{p}} + \frac{1}{\sqrt{r}} < \frac{2}{n}, \tag{32}$$

which is the usual condition imposing that the rectangle $[p,q] \times [r,s]$ keeps away from the Fučik's spectrum (see e.g. [69]).

<u>Proof.</u> Let u be a function such that $\alpha \le u(t) \le \beta$ and let (x,y) be a non trivial solution of (30). Consider the functions

$$
\begin{aligned}
A_1(x,y) &= y - ax_+ + dx_- \,;\ \text{if } y \geq 0,\\
&= y - bx_+ + cx_- \,,\ \text{if } y < 0,\\
A_2(x,y) &= -q\,x_+ + s\,x_-,\\
\\
B_1(x,y) &= y - bx_+ + cx_- \,,\ \text{if } y \geq 0,\\
&= y - ax_+ + dx_- \,,\ \text{if } y < 0,\\
B_2(x,y) &= -p\,x_+ + r\,x_- \,.
\end{aligned}
$$

One easily checks that (25) and (26) hold. Next one computes from (28)

$$
\begin{aligned}
t_\varphi &= t_1(-a,q) + t_1(d,s) + t_1(-c,s) + t_1(b,q)\\
&= \frac{2}{\sqrt{4q-a^2}}\cos^{-1}\Big(-\frac{a}{2\sqrt{q}}\Big) + \frac{2}{\sqrt{4s-d^2}}\cos^{-1}\Big(\frac{d}{2\sqrt{s}}\Big)\\
&+ \frac{2}{\sqrt{4s-c^2}}\cos^{-1}\Big(-\frac{c}{2\sqrt{s}}\Big) + \frac{2}{\sqrt{4q-b^2}}\cos^{-1}\Big(\frac{b}{2\sqrt{q}}\Big)\\
&> \frac{2\pi}{n+1}
\end{aligned}
$$

and

$$
\begin{aligned}
t_\psi &= t_1(-b,p) + t_1(c,r) + t_1(-d,r) + t_1(a,p)\\
&= \frac{2}{\sqrt{4p-b^2}}\cos^{-1}\Big(-\frac{b}{2\sqrt{p}}\Big) + \frac{2}{\sqrt{4r-c^2}}\cos^{-1}\Big(\frac{c}{2\sqrt{r}}\Big)\\
&+ \frac{2}{\sqrt{4r-d^2}}\cos^{-1}\Big(-\frac{d}{2\sqrt{r}}\Big) + \frac{2}{\sqrt{4p-a^2}}\cos^{-1}\Big(\frac{a}{2\sqrt{p}}\Big)\\
&< \frac{2\pi}{n}.
\end{aligned}
$$

From Corollary 3, it follows that the number N_0 of zeros of x on $[0,2\pi[$ is such that

$$2(n+1) > N_0 > 2n\,,$$

which is a contradiction. ∎

Corollary 4. *Assume* $a = b$, $c = d$, $p \leq q$, $r \leq s$ *and*

$$a^2 - 4p < 0, \qquad c^2 - 4r < 0$$

and that, for some $n \in \mathbb{N}$

$$\frac{1}{n+1} < \frac{1}{\sqrt{4q-a^2}} + \frac{1}{\sqrt{4s-c^2}} \leq \frac{1}{\sqrt{4p-a^2}} + \frac{1}{\sqrt{4r-c^2}} < \frac{1}{n},$$

then the triplet (G,α,β) *defined in* (29) *has property* P.

Proposition 3. *Assume* $a \leq b$, $c \leq d$, $p \leq q$, $r \leq s$ *and*

$$a^2 - 4q < 0, \qquad c^2 - 4s < 0.$$

Assume further that

$$\frac{1}{\sqrt{4q-a^2}} + \frac{1}{\sqrt{4s-c^2}} \geq 2.$$

Then the triplet (G,α,β) *defined in* (29) *has property* P.

Proof. To prove this result, one computes as in Proposition 2

$$t_\varphi > \frac{\pi}{\sqrt{4q-a^2}} + \frac{\pi}{\sqrt{4s-c^2}} \geq 2\pi.$$

From Proposition 1, it follows that the time necessary for θ to decrease of 2π is larger than $t_\varphi > 2\pi$. Hence, we have no nontrivial periodic solution. ■

Proposition 4. *Assume* $a = b$, $c = d$, $p \leq q$, $r \leq s$ *and*

$$a^2 - 4q < 0, \qquad c^2 - 4s < 0.$$

If further

$$\frac{1}{\sqrt{4q-a^2}} + \frac{1}{\sqrt{4s-c^2}} > 1,$$

then the triplet (G,α,β) *defined in* (29) *has property* P.

Proof. As above one computes

$$t_\varphi = \frac{2\pi}{\sqrt{4q-a^2}} + \frac{2\pi}{\sqrt{4s-c^2}} > 2\pi,$$

and the proof follows. ■

The following proposition gives a necessary and sufficient condition for the system with constant coefficients

$$\dot{x} = y - ax_+ + cx_-$$

$$\dot{y} = - px_+ + rx_-$$

to have only the trivial solution.

Proposition 5. *Assume* $a = b$, $c = d$, $p = q$, $r = s$. *Then the triplet* (G,α,β) *defined in* (29) *has property* P *if and only if one of the following does not hold* :

(i) $a^2 - 4p < 0$, $c^2 - 4r < 0$;

(ii) $\frac{c}{\sqrt{r}} + \frac{a}{\sqrt{p}} = 0$;

(iii) $\left(\frac{1}{\sqrt{4p-a^2}} + \frac{1}{\sqrt{4r-c^2}}\right)^{-1} \in \mathbb{N}$.

Proof. The proof follows from direct computation of the solutions (see A. Fonda and F. Zanolin [59] Lemma 1).

4.4. **Application 2.** To investigate periodic solutions of (13) we consider property P for the functions

$$\begin{aligned} G(x,y,u) &= (y, -u_1y_+ + u_2y_- - u_3x_+ + u_4x_-), \\ \alpha &= (a,c,p,r), \\ \beta &= (b,d,q,s), \end{aligned} \tag{33}$$

and we assume as above that α and β are constant.

Proposition 6. *Assume* $a \le b$, $c \le d$, $p \le q$, $r \le s$ *and*

$$a^2 - 4p < 0,\ d^2 - 4r < 0,\ \ c^2 - 4r < 0,\ b^2 - 4p < 0.$$

If further

$$\frac{\pi}{n+1} < \frac{1}{\sqrt{4q-b^2}}\cos^{-1}\left(\frac{b}{2\sqrt{q}}\right) + \frac{1}{\sqrt{4q-c^2}}\cos^{-1}\left(-\frac{c}{2\sqrt{q}}\right)$$
$$+ \frac{1}{\sqrt{4s-d^2}}\cos^{-1}\left(\frac{d}{2\sqrt{s}}\right) + \frac{1}{\sqrt{4s-a^2}}\cos^{-1}\left(-\frac{a}{2\sqrt{s}}\right)$$
$$\leq \frac{1}{\sqrt{4p-a^2}}\cos^{-1}\left(\frac{a}{2\sqrt{p}}\right) + \frac{1}{\sqrt{4p-d^2}}\cos^{-1}\left(-\frac{d}{2\sqrt{p}}\right)$$
$$+ \frac{1}{\sqrt{4r-c^2}}\cos^{-1}\left(\frac{c}{2\sqrt{r}}\right) + \frac{1}{\sqrt{4r-b^2}}\cos^{-1}\left(-\frac{b}{2\sqrt{r}}\right) < \frac{\pi}{n} \tag{34}$$

then the triplet (G,α,β) *defined in* (33) *has property* P.

Remark. As for Proposition 2, we notice that if $a = b = c = d = 0$, assumption (34) reduces to (32).

Proof. Let $u \in L^2$ be such that $\alpha \leq u(t) \leq \beta$ and (x,y) be a nontrivial solution of

$$\dot{x} = y\,,\quad \dot{y} = -u_1(t)y_+ + u_2(t)y_- - u_3(t)x_+ + u_4(t)x_-.$$

Consider the functions

$$\begin{aligned} A(x,y) &= (y, -by_+ + cy_- - qx), &&\text{if} \quad x \geq 0,\\ &= (y, -ay_+ + dy_- - sx), &&\text{if} \quad x < 0,\end{aligned}$$

$$\begin{aligned} B(x,y) &= (y, -ay_+ + dy_- - px), &&\text{if} \quad x \geq 0,\\ &= (y, -by_+ + cy_- - rx), &&\text{if} \quad x < 0.\end{aligned}$$

The proof follows then as the proof of Proposition 2. ■

Statements similar to Corollary 4, Propositions 3 and 4 are easy to obtain. For example we can write the following.

Proposition 7. *Assume* $a = b = c = d$, $p \leq q$, $r \leq s$,

$$a^2 - 4q < 0 \quad , \qquad a^2 - 4r < 0$$

and

$$\frac{1}{\sqrt{4q-a^2}} + \frac{1}{\sqrt{4r-a^2}} > 1.$$

Then the triplet (G,α,β) *defined in* (33) *has property* P.

The constant coefficient case can also be investigated and need some more care.

Proposition 8. Assume $a = b$, $c = d$, $p = q$, $r = s$ *and*

$$a + c \neq 0.$$

Then the triplet (G,α,β) *defined in* (14) *has property* P.

Proof. Property P refers to 2π-periodic solutions of the differential equation

$$\dot{x} = y,\ \dot{y} = -(ay_+ - cy_-) - (px_+ - rx_-). \tag{35}$$

Assume $a + c < 0$ and let $(x(t),y(t))$ be such a periodic solution. We can assume that for some $t_1 > 0$

$$y(0) = y(t_1) = 0 \quad \text{and} \quad \forall\, t \in (0,t_1),\ y(t) > 0.$$

Consider next the closed curve defined by the function

$$\gamma : [0,2t_1] \to \mathbb{R}^2, \qquad \begin{array}{l} t \leq t_1 \to \gamma_+(t) = (x(t),y(t)) \\ t > t_1 \to \gamma_-(t) = (x(2t_1-t),-y(2t_1-t)). \end{array}$$

It is easy to see that this curve encloses a bounded domain Γ in $\mathbb{R}^2$ which is negatively invariant for (35). The upper part γ_+ of the boundary is the orbit of a solution of (35). For any point $\gamma_-(t)$, $t \in (t_1,2t_1)$ on the lower part γ_- one computes

$$\gamma'(t) = (-\,y(2t_1-t),\ -\,a\ y(2t_1-t) - p\ x_+(2t_1-t) + r\ x_-(2t_1-t))$$

and

$$G(\gamma(t)) = (-\,y(2t_1-t),\ c\ y(2t_1-t) - p\ x_+(2t_1-t) + r\ x_-(2t_1-t)).$$

Since

$$cy - p\,x_+ + r\,x_- < - a\,y - p\,x_+ + r\,x_-,$$

it follows that the vector field G points outward. The orbit of $(x(t),y(t))$ leaves Γ at time t and cannot come back at the point $(x(0),y(0)) \in \partial\Gamma$. This contradicts the periodicity of (x,y). ■

More generally we can prove the following necessary and sufficient condition for the constant coefficient case to have property P.

Proposition 9. *Assume* $a = b$, $c = d$, $p = q$, $r = s$. *Then the triplet* (G,α,β) *defined in* (33) *has property* P *if and only if one of the following does not hold* :

(i) $a^2 - 4p < 0$, $a^2 - 4r < 0$;

(ii) $a + c = 0$;

(iii) $\pi \left[\frac{1}{\sqrt{4p-a^2}} \cos^{-1}\left(\frac{a}{2\sqrt{p}}\right) + \frac{1}{\sqrt{4r-a^2}} \cos^{-1}\left(\frac{a}{2\sqrt{r}}\right)\right] \in \mathbb{N}.$

The proof is by direct computation of the solutions and since it is rather involved, we prefer to omit it.

§5. Property P for 3D order systems.

5.1. Consider the equation

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = - u_2(t)x_+ + u_3(t)x_- - by - az \tag{36}$$

or, which is equivalent,

$$\dddot{x} + a\ddot{x} + b\dot{x} + u_2(t)x_+ - u_3(t)x_- = 0 \tag{37}$$

together with the conditions : $a \in \mathbb{R}$, $b \in \mathbb{R}$,

$$p(t) \le u_2(t) \le q(t)\,,$$
$$r(t) \le u_3(t) \le s(t)\,.$$

Proposition 10. *Let* $a \neq 0$ *and assume that for some* $n \in \mathbb{N}$

$$n^2 a \leq p(t) = r(t) \quad , \qquad q(t) = s(t) \leq (n+1)^2 a \, ,$$

both inequalities being strict on a subset of $[0,2\pi]$ *of positive measure.*

Then the triplet

$$G(x,y,z,u) = (y,\, z,\, -u_2 x_+ + u_3 x_- - by - az), \tag{38}$$

$$\alpha(t) = (p(t),\, p(t)) \, ,$$

$$\beta(t) = (q(t),\, q(t)) \, ,$$

has property P.

The proof follows from Lemma 1 in O.C. Ezeilo and M.N. Nkashama [54]. ■

We can extend this result to the somewhat more general equation

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = u_1(t)\,(|x|+|y|+|z|) - u_2(t)x_+ + u_3(t)x_- - by - az.$$

for which we have the following.

Corollary 5. *If the triplet* (G,α,β) *with* G *defined as in* (38) *and*

$$\alpha(t) = (p(t),r(t)) \, , \; \beta(t) = (q(t),s(t)),$$

has property P, *there exists* $\varepsilon_0 > 0$ *such that the triplet*

$$\hat{G}(x,y,z,u) = (y,\, z,\, u_1(|x|+|y|+|z|) - u_2 x_+ + u_3 x_- - by - az),$$

$$\hat{\alpha}(t) = (-\varepsilon_0, p(t), r(t)) \, ,$$

$$\hat{\beta}(t) = (\varepsilon_0, q(t), s(t))$$

has property P.

<u>Proof.</u> It is clear that the triplet

$$\hat{G} \, , \; \tilde{\alpha}(t) = (0,p(t),r(t)), \; \tilde{\beta}(t) = (0,q(t),s(t))$$

has property P. The proof follows then from Lemma 1. ■

5.2. **Proposition 11.** *Let* $\mu > 0, \nu > 0, \rho \geq 0$ *and define*

$$m = \min(\mu,\nu),\ M = \max(\mu,\nu),\ b_0 = \rho\left(\frac{a}{m-\rho}\right)^{1/2}.$$

Assume $m > \rho$ *and one of the following conditions holds* :

(i) $a \leq 0$;

(ii) $a > M + b_0$;

(iii) $b > 1 - b_0$;

(iv) $m > ab + b_0\,(a + \sqrt{b+b_0})$;

(v) $M < ab - b_0\,(a + \sqrt{b+b_0})$.

Then the triplet

$$G(x,y,z,u) = (y,\ z,\ -u_2x_+ + u_3x_- - by - az)\ ,$$
$$\alpha = (\mu-\rho,\ \nu-\rho)\ ,$$
$$\beta = (\mu+\rho,\ \nu+\rho)$$

has property P.

Proof. Let us suppose that x is a nontrivial 2π-periodic solution of (37) and let

$$f(t) = (\mu-u_2(t))\ x_+(t) - (\nu-u_3(t))\ x_-(t).$$

Equation (37) reads

$$\dddot{x} + a\ddot{x} + b\dot{x} + \mu\, x_+ - \nu\, x_- = f(t). \tag{39}$$

Multiplying (39) by x and integrating gives

$$m\ \|x\|^2_{L^2} - \|f\|_{L^2}\,\|x\|_{L^2} \leq a\|\dot{x}\|^2_{L^2}\ .$$

We notice further that $\|f\|_{L^2} \leq \rho\|x\|_{L^2}$, from which follows

$$0 < (m-\rho)\ \|x\|^2_{L^2} \leq a\ \|\dot{x}\|^2_{L^2} \tag{40}$$

i.e.

$$a > 0.$$

This contradicts (i).

Multiplying (39) by $\dot{x}$ and integrating, one gets

$$\left|\ \|\ddot{x}\|^2_{L^2} - b\|\dot{x}\|^2_{L^2}\right| \leq \|f\|_{L^2}\ \|\dot{x}\|_{L^2} \leq \rho\ \|x\|_{L^2}\ \ \|\dot{x}\|_{L^2}$$

and from (40)

$$\left|\ \|\ddot{x}\|^2_{L^2} - b\|\dot{x}\|^2_{L^2}\right| \leq \rho\left(\frac{a}{m-\rho}\right)^{1/2}\|\dot{x}\|^2_{L^2} = b_0\|\dot{x}\|^2_{L^2}$$

i.e.

$$(b - b_0)\ \|\dot{x}\|^2_{L^2} \leq \|\ddot{x}\|^2_{L^2} \leq (b + b_0)\ \|\dot{x}\|^2_{L^2}\ . \tag{41}$$

In particular, by Wirtinger inequality,

$$1 \leq b + b_0\ ,$$

which contradicts (iii).

Multiplying (39) by $\ddot{x}$ and integrating, one gets

$$m\ \|\dot{x}\|^2_{L^2} - \|f\|_{L^2}\|\ddot{x}\|_{L^2} \leq a\ \|\ddot{x}\|^2_{L^2} \leq M\ \|\dot{x}\|^2_{L^2} + \|f\|_{L^2}\|\ddot{x}\|_{L^2}\,. \tag{42}$$

From (40) and Wirtinger inequality we have

$$a \leq M + b_0\ ,$$

contradicting (ii).

From (42), (41) and (40), it follows

$$[m - b_0(b+b_0)^{1/2}]\ \|\dot{x}\|^2_{L^2} \leq a\ \|\ddot{x}\|^2_{L^2} \leq a\ (b+b_0)\ \|\dot{x}\|^2_{L^2}$$

and

$$m - b_0(b+b_0)^{1/2} \leq a\,(b+b_0)\,,$$

which contradicts (iv).

Similarly, it follows from (42), (41) and (40) that

$$a(b-b_0)\,\|\dot{x}\|^2_{L^2} \leq a\,\|\ddot{x}\|^2_{L^2} \leq \left[M + b_0(b+b_0)^{1/2}\right]\|\dot{x}\|^2_{L^2}$$

and

$$a(b-b_0) \leq M + b_0(b+b_0)^{1/2},$$

which contradicts (v). ■

5.3. Let us consider the constant coefficients case

$$\dddot{x} + a\ddot{x} + b\dot{x} + \mu\, x_+ - \nu\, x_- = 0\,, \tag{43}$$

with $(\mu,\nu) \in \mathbb{R}^2$. We define a subset $U(a,b)$ of $\mathbb{R}^2$ as follows.

1) If $a = 0$ or $b < 1$, set

$$U(a,b) = \{(\mu,\nu) : \mu.\nu > 0\}.$$

2) If $a \neq 0$, $b \geq 1$ and $n \in \mathbb{N}$ is such that $b \in [n^2,(n+1)^2[$, set

$$U(a,b) = \{(\mu,\nu) \mid \mu.\nu > 0,\ (\mu - ab)(\nu - ab) > 0\} \cup$$
$$\cup\ \{(\mu,\nu) \mid \mu\,,\nu \in\ ]\,an^2,\ a(n+1)^2\,[\}.$$

Proposition 12. *Assume* $(\mu,\nu) \in U(a,b)$. *Then the triplet*

$$\begin{aligned} G(x,y,z,u) &= (y,\ z,\ -u_2x_+ + u_3x_- - by - az) \\ \alpha(t) &= (\mu,\nu) \\ \beta(t) &= (\mu,\nu) \end{aligned}$$

has property P.

Proof. It is a consequence of Propositions 10 and 11, together with a symmetric formulation of Proposition 11 for the case $\mu < 0$, $\nu < 0$.

Remark. Proposition 12 gives sufficient conditions for a third order system with constant coefficients to have only the trivial solution. In case $\mu = \nu$, it is well known (see [54]) that a necessary and sufficient condition for property P to hold is

$$\mu \neq 0 \text{ and } \forall\, n \in \mathbb{N}^* \; [b \neq n^2 \text{ or } \mu \neq an^2].$$

One can check that such an assumption is equivalent to

$$(\mu, \mu) \in U(a,b).$$

Hence, Proposition 12 generalizes the linear case. Necessary and sufficient condition in the general case $\mu \neq \nu$ seem to be unknown.

§6. EXISTENCE OF PERIODIC SOLUTIONS IN APPLICATIONS.

6.1. Consider the boundary value problem

$$\begin{aligned} &\dot{x} = y - f(t,x) , \\ &\dot{y} = e(t) - g(t,x) , \\ &x(0) = x(2\pi), \; y(0) = y(2\pi). \end{aligned} \tag{44}$$

Theorem 2. *Assume* :

(i) *the functions* f, g *and* e *are continuous and defined for* $t \in [0,2\pi]$, $x \in \mathbb{R}$;

(ii) *assumption* A1 *holds* ;

(iii) *the triplet*

$$\begin{aligned} G(x,y,u) &= (y - u_1x_+ + u_2x_-, - u_3x_+ + u_4x_-), \\ \alpha(t) &= (a(t), c(t), p(t), r(t)), \\ \beta(t) &= (b(t), d(t), q(t), s(t)), \end{aligned}$$

has property P ;

(iv) *there exists some constant* $u^0 \in \mathbb{R}^4$ *such that*

$$\alpha(t) \leq u^0 \leq \beta(t).$$

Then the problem (44) *has at least one solution.*

Proof. We will apply Theorem 1. From paragraph 2.2 it is clear that assumption H holds.

Next we can find a path $u^\lambda = (u_1^\lambda, u_2^\lambda, u_3^\lambda, u_4^\lambda)$ in $\mathbb{R}^4$, $\lambda \in [0,1]$, that links u^0 to a point u^1 such that

$$u_1^1 = u_2^1 \, , \; u_3^1 = u_4^1$$

and for any $\lambda \in \,]0,1]$,

$$\frac{u_2^\lambda}{\sqrt{u_4^\lambda}} + \frac{u_1^\lambda}{\sqrt{u_3^\lambda}} \neq 0.$$

Hence, from Proposition 5, the differential equations

$$\begin{aligned} \dot{x} &= y - u_1^\lambda \, x_+ + u_2^\lambda \, x_- \\ \dot{y} &= - u_3^\lambda \, x_+ + u_4^\lambda \, x_- \end{aligned} \qquad , \lambda \in [0, 1] , \tag{45}$$

have no nontrivial 2π-periodic solutions. As for $\lambda = 1$, (44) reduces to a linear system, it follows that for $\Omega_0 = \{x \in C : \|x\|_\infty < 1\}$ and N_0 defined in (19),

$$d\,(L - N_0, \Omega_0) \neq 0.$$

The proof follows now from Theorem 1. ∎

Conditions for (iii) to hold are given in Proposition 2, 3, 4 and in Corollary 4. Other methods can be used as in P. Habets and G. Metzen [69]. Theorem 2 generalizes among others results from [69] and A. Fonda and F. Zanolin [59].

6.2. In our second application we consider the boundary value problem

$$\begin{aligned} \dot{x} = y, \quad \dot{y} &= e(t) - g(t,x) - f(t,y), \\ x(0) = x(2\pi), \qquad y(0) &= y(2\pi). \end{aligned} \tag{46}$$

Theorem 3. *Assume :*
(i)*the functions* f,g *and* e *are continuous and defined for* $t \in [0,2\pi]$, $x \in \mathbb{R}$, $y \in \mathbb{R}$;
(ii) *assumption* A1 *holds* ;
(iii) *the triplet*

$$\begin{aligned} G(x,y,u) &= (y,\ -u_1y_+ + u_2y_- - u_3x_+ + u_4x_-), \\ \alpha(t) &= (a(t),\ c(t),\ p(t),\ r(t)) , \\ \beta(t) &= (b(t),\ d(t),\ q(t),\ s(t)) \end{aligned}$$

has property P ;
(iv) *there exists some constant* $u^0 \in \mathbb{R}^4$ *such that*

$$\alpha(t) \leq u^0 \leq \beta(t).$$

Then the problem (46) *has at least one solution.*

The proof is identical to the proof of Theorem 2 but uses Proposition 8 instead of Proposition 5. Conditions ensuring (iii) are given in Propositions 6 and 7.

6.3. Consider the third order problem

$$\begin{aligned} &\dddot{x} + a\ddot{x} + b\dot{x} + g(t,x) = e(t,x,\dot{x},\ddot{x}) , \\ &x(0) = x(2\pi),\ \dot{x}(0) = \dot{x}(2\pi),\ \ddot{x}(0) = \ddot{x}(2\pi). \end{aligned} \tag{47}$$

Theorem 4. *Assume* :
(i) *the functions* g(t,x) and e(t,x,y,z) *are continuous functions defined for* $t \in [0,2\pi]$, $x \in \mathbb{R}$, $y \in \mathbb{R}$, $z \in \mathbb{R}$;
(ii) *assumption* A3 *holds* ;
(iii)*the triplet*

$$\begin{aligned} G(x,y,z,u) &= (y,\ z,\ -u_2x_+, + u_3x_- - by - az), \\ \alpha(t) &= (p(t),\ r(t)) , \\ \beta(t) &= (q(t),\ s(t)) \end{aligned}$$

has property P ;

(iv)*there exists some constant* $u^0 \in U(a,b)$, *where* $U(a,b)$ *is defined as in* 5.3, *such that*

$$\alpha(t) \leq u^0 \leq \beta(t).$$

Then there exists $\varepsilon_0 > 0$ *such that if* $\Delta_0 \leq \varepsilon_0$, *the problem* (47) *has a solution.*

The proof of this theorem goes as the proof of Theorem 2. One has only to notice that from Corollary 5, there exists $\varepsilon_0 > 0$ such that the triplet

$$\begin{aligned}
\hat{G}(x,y,z,u) &= (y,\, z,\, u(|x| + |y| + |z|) - u_2 x_+,\, + u_3 x_- - by - az),\\
\hat{\alpha}(t) &= (-\varepsilon_0,\, p(t),\, r(t)),\\
\hat{\beta}(t) &= (\varepsilon_0,\, q(t),\, s(t))
\end{aligned}$$

has property P.

Assumption (iii) can be obtained from Propositions 10 and 11. Theorem 4 generalizes then a result of O.C. Ezeilo and M.N. Nkashama [54].

6.4. Let us remark that in Theorems 2 and 3, assumption (iv) can be replaced by :

(iv') *there exists some functions*

$$u^0(t) = (u_1^0\,(t),\, u_1^0\,(t),\, u_3^0\,(t),\, u_3^0\,(t))$$

such that

$$\alpha(t) \leq u^0(t) \leq \beta(t).$$

In this case, the proof uses Corollary 1 instead of Theorem 1.

A similar statement holds for Theorem 4.

Chapter 2

PERIODIC SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS WITH DOUBLE RESONANCE.

1. INTRODUCTION

Consider the nonlinear periodic boundary value problem

$$x'' + g(t,x) = 0 \tag{1}$$
$$x(0) - x(T) = x'(0) - x'(T) = 0 \tag{2}$$

where $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and grows at most linearly. More precisely, letting $I = [0,T]$, we assume the existence of two functions $a,b \in L^\infty(I)$ such that

$$a(t) \leq \liminf_{|x|\to\infty} \frac{g(t,x)}{x} \leq \limsup_{|x|\to\infty} \frac{g(t,x)}{x} \leq b(t)\ , \tag{3}$$

uniformly for a.e. $t \in I$. We will suppose that $a(t) < b(t)$ on a subset of I of positive measure. The following property will be imposed on a,b :

(A) *Taken any function* $p \in L^\infty(I)$ *such that* $a(t) \leq p(t) \leq b(t)$ *a.e. on* I, *the inequalities being strict on sets of positive measure, the problem*

$$x'' + p(t)x = 0 \tag{4}$$
$$x(0) - x(T) = x'(0) - x'(T) = 0 \tag{2}$$

has only the trivial solution.

If the trivial solution is also the only solution of (4) (2) for $p \equiv a$ and for $p \equiv b$, the problem (1)(2) can be called nonresonant. Such problems have been extensively studied : let us mention, among many others, the works of Lasota and Opial [79] (where a property analogous to (A) does already appear), Mawhin and Ward [108], Mawhin [99], Habets and Metzen [69].

In this paper, we are interested mainly in the case where the problem (4)(2) has nontrivial solutions for $p \equiv a$ and for $p \equiv b$; this will be called a double resonance situation. This occurs, for instance, when, for some $n \in \mathbb{N}$, we have $a(t) \equiv n^2(2\pi/T)^2$ and $b(t) \equiv (n+1)^2 (2\pi/T)^2$. For such problems, we will assume that a pair of so-called "Landesman-Lazer conditions" is satisfied. Such conditions, introduced by Landesman and Lazer [78], appear in many studies of resonance problems : see, for instance, the papers of Brézis and Nirenberg [25], de Figueiredo [40], Iannacci and Nkashama [73] and references therein. However, these papers always consider situations in which (4)(2) is admitted to have nontrivial solutions for $p \equiv a$ or for $p \equiv b$, but not for both. For different type of conditions, we refer to Ding [42] and Omari and Zanolin [120].

The proofs of our results are based on coincidence degree arguments (see Mawhin [98] for the basic theory). The paper is organized as follows : in section 2, we associate to the functions a, b a positive semi-definite quadratic form which will play a major role in the proof of our main result. In section 3, preliminary lemmas are presented, which will be useful to produce a priori estimates for components of the solutions of (1)(2). The main theorem is then proved is section 4, whereas section 5 is devoted to the more specific case when 0 is the first eigenvalue of the operator $L : x \to -x'' - a(.)x$. Finally, in section 6, we describe the resonant case to the left of the first eigenvalue.

2. PROPERTY (A) AND POSITIVE DEFINITENESS OF A QUADRATIC FORM.

To the functions a,b appearing in (3), we associate the linear operators L_a, L_b : Dom L_a = Dom $L_b \subset L^2(I) \to L^2(I)$ defined by

$$\text{Dom } L_a = \text{Dom } L_b = \{x \in H^2(I) \mid x \text{ verifies } (2)\},$$
$$L_a x = -x'' - a(.)x,$$
$$L_b x = -x'' - b(.)x,$$

where $H^2(I) = \{x : I \to \mathbb{R} \mid x$ and x' are absolutely continuous on I and $x'' \in L^2(I)\}$.

It is well known (see, for instance, Eastham [47]) that L_a, L_b are self-adjoint operators with discrete spectrum whose eigenvalues, denoted respectively by α_i and β_i, are such that

$$\alpha_1 < \alpha_2 \leq \alpha_3 < \alpha_4 \leq \alpha_5 < \ldots \text{ and } \alpha_n \to \infty \text{ as } n \to \infty,$$
$$\beta_1 < \beta_2 \leq \beta_3 < \beta_4 \leq \beta_5 < \ldots \text{ and } \beta_n \to \infty \text{ as } n \to \infty,$$

any double eigenvalue appearing twice. Moreover, a comparison argument shows that

$$\beta_i \leq \alpha_i \qquad \text{for} \quad i = 1,2,\ldots$$

Using the spectra of L_a and L_b, we will relate property (A) to the positive semi-definiteness of a quadratic form, defined on the space

$$H_T^1 = \{x \in H^1(I) \mid x(0) = x(T)\},$$

where

$$H^1(I) = \{x : I \to \mathbb{R} \mid x \text{ is absolutely continuous on I and } x' \in L^2(I)\}.$$

Proposition 1. *The following conditions are equivalent* :

(i) the functions a,b *satisfy property* (A) ;

(ii) either $\beta_1 \geq 0$ *or, for some* $\bar{n} \geq 1$, *one has* $\alpha_{\bar{n}} \leq 0 \leq \beta_{\bar{n}+1}$;

(iii) the space $L^2(I)$ *can be decomposed into a direct sum* $H^- \oplus H^+$, *where either*

$$H^- = \{0\}, \quad H^+ = L^2(I),$$

in which case, for any $x \in H^1_T \setminus \operatorname{Ker} L_b$, *we have*

$$\int_I (x'^2 - bx^2) > 0$$

or, for some $\bar{n} \geq 1$,

$$H^- = \oplus_{j=1,\dots,\bar{n}} \operatorname{Ker}(L_a - \alpha_j I), \quad H^+ = \overline{\oplus_{j\geq\bar{n}+1} \operatorname{Ker}(L_b - \beta_j I)}, \tag{5}$$

in which case, for any $x = \bar{x} + \tilde{x} \in H^1_T$, *with* $\bar{x} \in H^-, \tilde{x} \in H^+$, *we have*

$$B_{a,b}(x) := \int_I (\tilde{x}'^2 - b\tilde{x}^2) - \int_I (\bar{x}'^2 - a\bar{x}^2) \geq 0, \tag{6}$$

the inequality being strict when $x \notin \operatorname{Ker} L_a \oplus \operatorname{Ker} L_b$.

<u>Proof</u> : a) <u>(i) ⇒ (ii)</u> Suppose (ii) is not true. Let n be the largest integer such that $\beta_n < 0$. Then, $\beta_{n+1} \geq 0$ and $\alpha_n > 0$. Defining $p_\lambda(t) = (1-\lambda)a(t) + \lambda b(t)$ and using the continuity of the eigenvalues with respect to λ, we conclude that there exists $\bar{\lambda} \in]0,1[$ such that the n^{th} eigenvalue of $L_{p_{\bar{\lambda}}}$ (defined as L_a and L_b) is equal to 0. This contradicts (i).

b) <u>(ii) ⇒ (iii).</u> In case $\beta_1 \geq 0$, take $H^- = \{0\}$, $H^+ = L^2(I)$. Then, for any $x \in (H^1_T \cap H^2(I)) \setminus \operatorname{Ker} L_b$, one has

$$\int_I (x'^2 - bx^2) = (L_b x, x)_{L^2} \geq \sigma \|x\|^2_{L^2}$$

for some $\sigma > 0$. Since $H^2(I)$ is dense in $H^1(I)$, it follows that, for any $x \in H^1_T \setminus \operatorname{Ker} L_b$,

$$\int_I (x'^2 - bx^2) \geq \sigma \|x\|^2_{L^2} > 0. \tag{7}$$

In case $\alpha_{\bar{n}} \leq 0 \leq \beta_{\bar{n}+1}$, take H^-, H^+ as in (5). Since the dimension of H^- is finite and equal to the codimension of H^+, we will have $L^2(I) = H^- \oplus H^+$ if we can prove that $H^- \cap H^+ = \{0\}$ (see Lazer [81]).
Take $x \in H^- \cap H^+$; we then have

$$\alpha_{\bar{n}} \|x\|^2_{L^2} \geq (L_a x, x)_{L^2} = \int_I (x'^2 - ax^2) \geq$$

$$\geq \int_I (x'^2 - bx^2) = (L_b x, x) \geq \beta_{\bar{n}+1} \|x\|^2_{L^2} \tag{8}$$

If $x \neq 0$, it then follows that $\alpha_{\bar{n}} = \beta_{\bar{n}+1} = 0$ and also that $(L_a x, x) = (L_b x, x) = 0$.
On H^-, the quadratic form $(L_a x, x)$ vanishes only if $x \in \operatorname{Ker} L_a$; similarly on $H^+ \cap H^2(I)$, the quadratic form $(L_b x, x)$ vanishes only if $x \in \operatorname{Ker} L_b$; since $x \in H^- \cap H^+$, we conclude that $x \in \operatorname{Ker} L_a \cap \operatorname{Ker} L_b$.
But, it is easy to show that $\operatorname{Ker} L_a \cap \operatorname{Ker} L_b = \{0\}$, because $a(t) < b(t)$ on a subset of positive measure. Hence, we have proved that $H^- \cap H^+ = \{0\}$, from which follows that $L^2(I) = H^- \oplus H^+$. Moreover, if $x \in H^1_T \cap H^2(I)$ is decomposed into $x = \bar{x} + \tilde{x}$ with $\bar{x} \in H^-$, $\tilde{x} \in H^+$, we have

$$B_{a,b}(x) = (L_b \tilde{x}, \tilde{x}) - (L_a \bar{x}, \bar{x}) \geq \beta_{\bar{n}+1} \|\tilde{x}\|^2_{L^2} - \alpha_{\bar{n}} \|\bar{x}\|^2_{L^2},$$

and

$$B_{a,b}(x) \geq \sigma \|x\|^2_{L^2} \tag{9}$$

for some $\sigma > 0$, unless $\tilde{x} \in \operatorname{Ker} L_b$ and $\bar{x} \in \operatorname{Ker} L_a$. By a density argument, (9) still holds for any $x \in H^1_T$, unless $\tilde{x} \in \operatorname{Ker} L_b$ and $\bar{x} \in \operatorname{Ker} L_a$.

c) <u>(iii) $\Rightarrow$ (i).</u> Take p as in property (A) and let x be a solution of (4)(2). Multiplying (4) par $(\bar{x} - \tilde{x})$ and integrating over I, one gets

$$0 = \int_I [\tilde{x}'^2 - \bar{x}'^2 + p\,(\bar{x}^2 - \tilde{x}^2)\,] \geq B_{a,b}(x),$$

the last inequality being strict unless $x \equiv 0$ (the functions $\bar{x}$ and $\tilde{x}$ cannot vanish simultaneously on a set of positive measure if x is a nontrivial solution of (4)). Condition (iii) then implies that $x \equiv 0$. ■

3. A PRIORI ESTIMATES.

The main result of this paper is prepared by the lemmas given below. One of the purposes of this section is to provide a priori estimates for components of any solution to (1)(2). We want to distinguish between the component in Ker L_a, the component in Ker L_b and a complementary component. Therefore, we introduce the subspace H* such that $L^2(I) = \text{Ker } L_a \oplus \text{Ker } L_b \oplus H^*$ and, for $x \in L^2(I)$, we will write $x = x^a + x^b + x^*$ with $x^a \in \text{Ker } L_a$, $x^b \in \text{Ker} L_b$, $x^* \in H^*$.

Lemma 1. *Assume that* a,b *satisfy property (A). Then there exists* a $\delta > 0$ *such that , for every* $x \in H^1_T$,

$$B_{a,b}(x) \geq \delta\, \|x^*\|^2_{H^1} \tag{10}$$

Proof. Notice first that $B_{a,b}(x) = B_{a,b}(x^*)$. Hence, if (10) does not hold, there exists a sequence (x_n) in H* such that $\|x_n\|_{H^1} = 1$ and $B_{a,b}(x_n) \to 0$ for $n \to \infty$. Taking a subsequence, we can suppose that (x_n) converges weakly to some x in $H^1_T \cap H^*$. Then, using the decomposition $x_n = \bar{x}_n + \tilde{x}_n$ of Proposition 1, we assert that $(\bar{x}_n)$ converges strongly to $\bar{x}$ in $H^1(I)$, since all norms are equivalent on a finite dimensional space, and $(\tilde{x}_n)$ converges uniformly to $\tilde{x}$, since $C_0(I)$ is compactly imbedded in $H^1(I)$.

Consequently,

$$\int_I (\tilde{x}'_n)^2 \to \int_I (b\tilde{x}^2 + (\bar{x}')^2 - a\bar{x}^2) \quad \text{for} \quad n \to \infty \tag{11}$$

and, by the weak lower semi-continuity of the L^2-norm, we obtain $B_{a,b}(x) \leq 0$. By Proposition 1, this implies that $B_{a,b}(x) = 0$ and that

$x \in \operatorname{Ker} L_a \oplus \operatorname{Ker} L_b$. As we also have $x \in H^*$, we conclude that $x = 0$ and, from (11), that $\int_I (\tilde{x}'_n)^2 \to 0$ for $n \to \infty$. This, combined with the uniform convergence of $(\tilde{x}_n)$ and the strong convergence in $H^1(I)$ of $(\bar{x}_n)$, shows that $\|x_n\|_{H^1} \to 0$ as $n \to \infty$, contradicting the fact that

$\|x_n\|_{H^1} = 1$. ■

The next lemma shows that, if $\operatorname{Ker} L_a = \{0\}$ (resp. $\operatorname{Ker} L_b = \{0\}$) property (A) is stable with respect to small perturbations of a (resp. b).

Lemma 2. *Let the functions* a,b *satisfy property* (A). *If* $\operatorname{Ker} L_a = \{0\}$, *there exists* $\varepsilon > 0$ *such that the functions* $a - \varepsilon$, b *still satisfy property* (A). *Symmetrically, if* $\operatorname{Ker} L_b = \{0\}$, *there is an* $\varepsilon > 0$ *such that the functions* a, $b + \varepsilon$ *satisfy* (A).

Proof. By Proposition 1, condition (ii) holds. If $\operatorname{Ker} L_a = \{0\}$, we have either $\alpha_1 > 0$ or $\alpha_{\bar{n}} < 0 \leq \beta_{\bar{n}+1}$. If $\alpha_1 > 0$, then $\beta_1 \geq 0$ by (ii) in Proposition 1. Hence, property (A) holds for any functions $\tilde{a}$,b with $\tilde{a} \leq$ b.
If $\alpha_{\bar{n}} < 0 \leq \beta_{\bar{n}+1}$, we deduce from (9) that $B_{a-\varepsilon,b}(x) \geq 0$, for any $0 < \varepsilon \leq |\alpha_{\bar{n}}|$ the inequality being strict, except when

$x \in \operatorname{Ker} L_{a-\varepsilon} \oplus \operatorname{Ker} L_b$. The conclusion then results again from Proposition 1. ■

In the sequel, it will be useful to decompose the function g in (1) as the sum of a "pseudo-linear" function and a bounded function. For this purpose, we introduce the following hypothesis :

(H) *If* $\operatorname{Ker} L_a \neq \{0\}$, *there exists a function* $H \in L^2(I)$ *such that, for every* $x \in \mathbb{R}$,

$$\operatorname{sgn} x \,[g(t,x) - a(t)x\,] \geq - H(t) \tag{12}$$

If $\operatorname{Ker} L_b \neq \{0\}$, *there exists a function* $K \in L^2(I)$ *such that, for every* $x \in \mathbb{R}$,

$$\operatorname{sgn} x \,[b(t)x - g(t,x)\,] \geq - K(t). \tag{13}$$

Lemma 3. *Let the Carathéodory function g verify* (3), *hypothesis* (H) *and be such that, for all* $R > 0$, *there exists* $h_R \in L^2(I)$ *such that*

$$|g(t,x)| \leq h_R(t) \text{ for } |x| \leq R. \tag{14}$$

Assume that the functions a,b *satisfy property* (A). *Then, there exist functions* $\bar{a},\bar{b} \in L^\infty(I)$, *with* $\bar{a}(t) \leq a(t)$, $b(t) \leq \bar{b}(t)$ *a.e. on* I, $\bar{a},\bar{b}$ *still satisfying* (A),(H) *and we can write*

$$g(t,x) = x\gamma(t,x) + h(t,x) \tag{15}$$

where

$$\bar{a}(t) \leq \gamma(t,x) \leq \bar{b}(t) \tag{16}$$

for a.e. $t \in I$, *all* $x \in \mathbb{R}$, *and* $h(.,.)$ *is a function satisfying Carathéodory conditions and such that*

$$|h(t,x)| \le \hat{h}(t) \tag{17}$$

for a.e. $t \in I$, *where* $\hat{h} \in L^2(I)$.

Proof. By (3) and (14), given $\varepsilon > 0$, we can find $R > 0$ such that, for $x \ge 0$,

$$g(t,x) \ge (a-\varepsilon)x - h_R(t) - |a-\varepsilon|\, R;$$

a similar inequality holding for $x \le 0$. By lemma 2, if Ker $L_a = \{0\}$, chosing $\varepsilon > 0$ small enough, the functions $a-\varepsilon, b$ satisfy property (A). In this case, we take $\bar{a} \equiv a - \varepsilon$; we notice that (12) holds with a replaced by $\bar{a}$ and H replaced by h_R.

If Ker $L_a \neq \{0\}$, we take $\bar{a} \equiv a$. The function $\bar{b}$ is defined in a similar way. The functions $\bar{a}, \bar{b}$ satisfy property (A) and we can now use, in all cases, the inequalities (12),(13) if we replace therein a by $\bar{a}$, b by $\bar{b}$. We introduce the function $\delta : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$\delta(u,x,v) = \begin{cases} \min\{u,v\} & \text{if } x < \min\{u,v\} \\ \max\{u,v\} & \text{if } x > \max\{u,v\} \\ x & \text{otherwise.} \end{cases}$$

Let us then define $\gamma(t,x)$ and $h(t,x)$ by

$$\gamma(t,x) = \begin{cases} \frac{1}{x}\,\delta(\bar{a}(t)x,\, g(t,x),\, \bar{b}(t)x) & \text{for } x \neq 0 \\ \bar{a}(t) & \text{for } x = 0 \end{cases},$$

$$h(t,x) = g(t,x) - x\,\gamma(t,x) ;$$

the function $\gamma(t,.)$ need not be continuous at 0, but the function $x \to x\,\gamma(t,x)$ will be (for a.e. $t \in I$) ; so will be the function $h(t,.)$.

It is clear from the definition of $\gamma(t,x)$ that (16) holds. On the other hand, (17) will result from (12) (13) (a,b having been replaced by $\bar{a},\bar{b}$) with $\hat{h}(t) = \max\{|H(t)|, |K(t)|\}$. ■

In the sequel, we can assume, without loss of generality, that $\bar{a} \equiv a$, $\bar{b} \equiv b$. Using Lemma3, we can obtain an estimate of the component in H^* of any solution of (1)(2).

Lemma 4. *Assume that* g,a,b *satisfy the hypotheses of Lemma 3. Then, there exists a constant* $C_1 > 0$ *such that, for every solution* x *of* (1) (2), *we have*

$$\|x^*\|^2_{H^1} \leq C_1 \|x\|_{L^2}, \tag{18}$$

x^* *being the component of* x *in* H^*.

Proof. We decompose g as in (15) ; multiplying (1) by $(\bar{x} - \tilde{x})$, where $\bar{x}$, $\tilde{x}$ are as in Proposition 1, and integrating, we get

$$0 = \int_I [(\tilde{x}')^2 - (\bar{x}')^2 + (\bar{x}^2 - \tilde{x}^2)\gamma(t,x) + (\bar{x} - \tilde{x})h(t,x)]$$

$$= B_{a,b}(x) + \int_I [\bar{x}^2(\gamma(t,x) - a(t)) + \tilde{x}^2(b(t) - \gamma(t,x)) + (\bar{x} - \tilde{x})h(t,x)].$$

By the remark preceding this lemma, we can assume, without loss of generality, that $a(t) \leq \gamma(t,x) \leq b(t)$. Hence, by Lemma 1 and inequality (17), we have

$$0 \geq \delta \|x^*\|^2_{H^1} - C \|x\|_{L^2} \|\hat{h}\|_{L^2},$$

for some constant C . The conclusion then follows immediately. ■

Consider, as in Lemma 1, the decomposition $L^2(I) = \mathrm{Ker}\, L_a \oplus \mathrm{Ker}\, L_b \oplus H^*$; as above, for $x \in L^2(I)$, we write

$x = x^a + x^b + x^*$. The next lemma will be useful to estimate x^b (or x^a) when x is a solution of (1)(2).

Lemma 5. *Let* a,b *satisfy property* (A) and (s_n) *be a sequence in* $L^2(I)$ *converging weakly to* a, *and such that* $a(t) \le s_n(t) \le b(t)$ *a.e. on* I. *Let* x_n *be a solution of*

$$x_n'' + s_n(t)\, x_n + h(t,x_n) = 0, \tag{19}$$

$$x_n(0) - x_n(T) = x_n'(0) - x_n'(T) = 0\ , \tag{2}$$

where h *is a Caratheodory function satisfying* (17). *Then, there exist constants* C_2, C_3 *such that, for any* $n \in \mathbb{N}$,

$$\|x_n^b\|^2 \le C_2\, \|x_n\|_{L^2} + C_3. \tag{20}$$

Remark. Any norm can be used for x_n^b, since Ker L_b is finite dimensional.

Proof. As in Lemma 4, we multiply (19) by $(\bar{x}_n - \tilde{x}_n)$ and integrate, which yields

$$0 = B_{a,b}(x_n) + \int_I [\bar{x}_n^2\,(s_n(t) - a(t)) + \tilde{x}_n^2\,(b(t) - s_n(t)) + (\bar{x}_n - \tilde{x}_n)\, h(t,x_n)]dt.$$

Since $B_{a,b}(x_n) \ge 0$ and $s_n(t) \ge a(t)$ a.e. on I, it follows that

$$\int_I \tilde{x}_n^2\,(b(t) - s_n(t))dt \le C\, \|x_n\|_{L^2}\, \|\hat{h}\|_{L^2}\ ,$$

for some constant C. Isolating the component of $\tilde{x}_n$ in Ker L_b, we obtain

$$\int_I (x_n^b)^2(b(t) - s_n(t))dt \le C \|x_n\|_{L^2} \|\hat{h}\|_{L^2} +$$

$$+ \|b - a\|_{L^\infty} \int_I [|\tilde{x}_n - x_n^b|^2 + 2\,|x_n^b|\,|\tilde{x}_n - x_n^b|\,]$$

$$\le C \|x_n\|_{L^2} \|\hat{h}\|_{L^2} + \|b - a\|_{L^\infty}[\|\tilde{x}_n - x_n^b\|^2_{L^2} + 2\|x_n^b\|_{L^2} \|\tilde{x}_n - x_n^b\|_{L^2} \quad (21)$$

We claim that there exists $\eta > 0$ such that, for n sufficiently large, we have

$$\int_I (x_n^b)^2(b(t) - s_n(t))dt \ge \eta\, \|x_n^b\|^2 \quad (22)$$

Indeed, if this would not be the case, we could find a subsequence, which will still be denoted by (x_n), such that

$$\int_I \left(\frac{x_n^b}{\|x_n^b\|}\right)^2 (b(t) - s_n(t))dt \to 0 \quad \text{as } n \to \infty. \quad (23)$$

Since Ker L_b is finite dimensional, we can assume, without loss of generality, that $x_n^b/\|x_n^b\|$ converges uniformly towards a non-zero function $y \in$ Ker L_b. By (23), it follows that

$$\int_I y^2(t)\,(b(t) - a(t))dt = 0\ ,$$

Since $a(t) < b(t)$ on a subset of positive measure, we conclude that y vanishes on such a subset. But since y belongs to Ker L_b, this would imply $y \equiv 0$, leading to a contradiction. Consequently, we deduce from (21) and (22) that, for n sufficiently large,

$$\eta\|x_n^b\|^2 \le C\|x_n\|_{L^2} \|\hat{h}\|_{L^2} + \|b - a\|_{L^\infty} [\,\|\tilde{x}_n - x_n^b\|^2_{L} + 2\,\|x_n^b\|_{L^2} \|\tilde{x}_n - x_n^b\|_{L^2}\,].$$

Since $\tilde{x}_n - x_n^b$ is a component of x_n^*, the result follows from Lemma 4 and easy computations. ■

A result similar to Lemma 5 can of course be proven when the sequence (s_n) converges weakly to b, with $a(t) \leq s_n(t) \leq b(t)$ a.e. on I. In that case, we obtain a relation of the form

$$\|x_n^a\|^2 \leq C_2'\|x_n\|_{L^2} + C_3'.$$

4. THE MAIN RESULT.

As explained above, we want to prove the existence of solutions for the problem (1)(2) in particular when a double resonance does occur, i.e. when the problem (4)(2) has non trivial solutions for $p \equiv a$ and for $p \equiv b$. For that purpose, we introduce the following "Landesman-Lazer conditions".

(LL) *For every* $u \in \operatorname{Ker} L_a \setminus \{0\}$,

$$0 < \int_{u>0} \liminf_{x\to+\infty} [g(t,x) - a(t)x\,]u(t)dt + \int_{u<0} \limsup_{x\to-\infty} [g(t,x) - a(t)x\,]u(t)dt$$

For every $v \in \operatorname{Ker} L_b \setminus \{0\}$,

$$0 < \int_{v>0} \liminf_{x\to+\infty} [b(t)x - g(t,x)\,]v(t)dt + \int_{v<0} \limsup_{x\to-\infty} [b(t)x - g(t,x)\,]v(t)dt.$$

Theorem 1. *Let the function g verify* (3), *hypothesis* (H) *and the Landerman-Lazer condition* (LL). *Moreover, assume that, for all* $R > 0$, *there exists* $h_R \in L^2(I)$ *such that*

$$|g(t,x)| \leq h_R(t) \quad \textit{for} \quad |x| \leq R.$$

Assume that the functions a,b *satisfy property* (A). *Then, problem* (1)(2) *has at least one solution.*

Proof. By classical arguments from the theory of the coincidence topological degree (see Mawhin [98]), the result will be proven if we can find an a priori bound for the solutions of the problems

$$x'' + \lambda p(t)x + (1-\lambda)g(t,x) = 0 \tag{24}$$
$$x(0) - x(T) = x'(0) - x'(T) = 0 \tag{2}$$

where $\lambda \in \,]\,0,1\,[$ and $p(t) = (a(t) + b(t))/2$ (notice that, with this definition of p, the problem (4)(2) has only the trivial solution, by property (A)).

As the function

$$g_\lambda(t,x) = \lambda p(t)x + (1-\lambda)g(t,x)$$

verifies the same hypotheses as the function g, with the same functions a,b in the inequalities corresponding to (3), the estimate (18) of Lemma 4 will hold for the solutions of (24)(2), with a constant C_1 independent of λ.

By contradiction, suppose that there exists sequences (x_n), (λ_n), with $\|x_n\|_{H^1} \to \infty$, such that x_n is a solution of (24)(2) with $\lambda = \lambda_n$. Set $u_n = x_n/\|x_n\|_{H^1}$; u_n then satisfies the equation

$$u_n'' + [\lambda_n\, p(t) + (1-\lambda_n)\gamma(t,x_n)]\, u_n + (1-\lambda_n)\, \frac{h(t,x_n)}{\|x_n\|_{H^1}} = 0, \tag{25}$$

where we have used the decomposition (15). It is easily shown that the sequence (u_n) is bounded in $H^2(I)$; therefore, passing to a subsequence, we can assume that (u_n) converges weakly in $H^2(I)$ and strongly in $C^1(I)$ to a certain map u. Moreover, letting

$$s_n(t) = \lambda_n p(t) + (1-\lambda_n)\gamma(t,x_n(t)),$$

we can suppose that (s_n) converges weakly in $L^2(I)$ to some function s and, by the weak closure of the set $\{\sigma \in L^2(I) : a(t) \le \sigma(t) \le b(t)$ a.e. on $I\}$, we have that $a(t) \le s(t) \le b(t)$ a.e. on I. Hence, passing to the weak limit in (25), we obtain

$$u'' + s(t)\,u = 0$$
$$u(0) - u(T) = u'(0) - u'(T) = 0$$

and, by property (A), we conclude that either $s \equiv a$, $s \equiv b$, or $u \equiv 0$. But this last possibility is excluded since $\|u_n\|_{H^1} = 1$ and u_n converges strongly to u in $C^1(I)$. So, we must have either $s \equiv a$ and Ker $L_a \neq \{0\}$ or $s \equiv b$ and Ker $L_b \neq \{0\}$; let us consider the first case, the other one being treated in a similar way. Multiplying (25) by $v_n = x_n^a/\|x_n\|_{H^1}$ and integrating we get, using the fact that $v_n'' = -a(t)v_n$,

$$\lambda_n \int_I (p(t) - a(t))u_n\, v_n\, dt + \\ + (1-\lambda_n) \int_I [(\gamma(t,x_n) - a(t))\, u_n\, v_n + \frac{h(t,x_n)}{\|x_n\|_{H^1}} v_n\,]dt = 0. \quad (26)$$

Since s_n converges weakly to a, we can apply Lemma 5 ; by the estimate (20) of Lemma 5 and the estimate (18) of Lemma 4, it is easy to show that $v_n \to u$ as $n \to \infty$ and, consequently,

$$\lim_{n\to\infty} \int_I (p(t) - a(t))u_n\, v_n\, dt = \int_I (p(t) - a(t))u^2\, dt.$$

So, for n large enough, the first term of the sum in (26) is positive and, therefore, we have the inequality

$$\int_I [(\gamma(t,x_n) - a(t))u_n\, v_n\, dt + \frac{h(t,x_n)}{\|x_n\|_{H^1}} v_n\,]dt \le 0,$$

which implies

$$\limsup_{n\to\infty} \int_I (g(t,x_n) - a(t)x_n)v_n\, dt \le 0 \quad (27)$$

We want to apply Fatou's Lemma ; for that purpose, we need to find a function $\gamma \in L^1(I)$ such that

$$(g(t,x_n(t)) - a(t)x_n(t))\, v_n(t) \geq \gamma(t) \quad \text{a.e. on I.}$$

This function will be obtained in two steps. First, since (v_n) is uniformly bounded and since h satisfies (17), there exists a function $\gamma_1 \in L^1(I)$ such that

$$h(t,x_n(t))v_n(t) \geq \gamma_1(t) \quad \text{a.e.} \quad \text{on I.} \tag{28}$$

Secondly, using the relations

$$2\, x_n\, x_n^a \geq - (x_n - x_n^a)^2 = - (x_n^* + x_n^b)^2 \geq - 2\, [(x_n^*)^2 + (x_n^b)^2\,]$$

and the fact that $\gamma(t,x_n(t)) \geq a(t)$ a.e. on I, we obtain

$$(\gamma(t,x_n(t)) - a(t))x_n(t)\, v_n(t) \geq - \frac{1}{\|x_n\|_{H^1}} (\gamma(t,x_n(t)) - a(t))\, [((x_n^*)^2 + (x_n^b)^2\,].$$

Using the estimates of Lemmas 4 and 5, it is then an easy matter to find a function $\gamma_2 \in L^1(I)$ such that

$$(\gamma(t,x_n(t)) - a(t))x_n(t)v_n(t) \geq \gamma_2(t) \quad \text{a.e. on I.} \tag{29}$$

Combining (28) and (29), we can now apply Fatou's Lemma to the integral in (27). The sequence (x_n) converges pointwise to $+\infty$ on the set $\{t \in I\!: u(t) > 0\}$, while it converges to $-\infty$ on the set $\{t \in I : u(t) < 0\}$. The application of Fatou's Lemma gives

$$\int_{u>0} \liminf_{n\to+\infty} [g(t,x_n) - a(t)x_n]u(t)dt \;+$$

$$+ \int_{u<0} \limsup_{n\to-\infty} [g(t,x_n) - a(t)x_n]u(t)dt \leq 0,$$

contradicting the first of the two Landesman-Lazer conditions in (LL). ■

5. RESONANCE AT THE FIRST TWO EIGENVALUES.

When $\alpha_1 = 0$, the first part of condition (LL) can be replaced by a different hypothesis, using the fact that Ker L_a is of dimension 1 and consists of functions of constant sign.

Theorem 2. *Assume that* α_1, *the first eigenvalue of* L_a, *is* 0. *Let* Ker $L_a = \{k\varphi : k \in \mathbb{R}\}$, *where* φ *is a positive function. Let the function* g *verify the same hypotheses as in Theorem* 1, *except that the first part of condition* (LL) *is replaced by the following assumption : there exist functions* $h^+, h^- \in L^2(I)$ *and a number* $d > 0$ *such that*

$$g(t,x) - a(t)x \geq h^+(t) \quad \text{for } x \geq d, \text{ a.e. } t \in I \tag{30}$$

$$g(t,x) - a(t)x \leq h^-(t) \quad \text{for } x \leq -d, \text{ a.e. } t \in I \tag{31}$$

$$\int_I h^+(t)\varphi(t)dt \geq 0, \tag{32}$$

$$\int_I h^-(t)\varphi(t)dt \leq 0. \tag{33}$$

Then, problem (1)(2) *has at least one solution.*

Proof. The proof begins as the proof of Theorem 1. Defining u_n and s_n in the same way, it is shown that (u_n) converges strongly in $C^1(I)$ to some $u \in \text{Ker } L_a \cup \text{Ker } L_b$ and that (s_n) converges weakly in $L^2(I)$ to some function s. If $a(t) < s(t)$ on a subset of positive measure, the proof is unchanged with respect to that of Theorem 1. The only case that needs to be treated here is the case where $s \equiv a$. In that case, multiply (25) by $\|x_n\|_{H^1} u$ and integrate over I. Since $u''(t) = -a(t)u$, this yields

$$\lambda_n \int_I (p(t) - a(t))x_n(t)u(t)dt + (1-\lambda_n) \int_I [g(t,x_n) - a(t)x_n]u(t)dt = 0 \, . \tag{34}$$

Since Ker $L_a \setminus \{0\}$ consists of functions of constant sign, let us assume, for instance, that $u(t) > 0$, $\forall\, t \in I$, the other case being treated in a similar way. We then have, for n sufficiently large, $u_n(t) > 0$ and $x_n(t) \geq d$, $\forall\, t \in I$. Consequently, using (30), (32), we have, for n sufficiently large,

$$\int_I [g(t,x_n) - a(t)x_n]\, u(t)dt \geq \int_I h^+(t)u(t)dt \geq 0.$$

On the other hand, since $p(t) > a(t)$ on a subset of positive measure, we have

$$\lim_{n\to\infty} \int_I (p(t) - a(t))u_n(t)u(t)dt = \int_I (p(t) - a(t))u^2(t)dt > 0.$$

Hence, the first term of the sum in (34) is positive for n large, while the second is non negative, leading to a contradiction. ■

Remarks.

1. If $h^+(t) = -e(t) + \Gamma$, $h^-(t) \equiv -e(t) + \gamma$, the conditions (30), (31) become

$$g(t,x) - a(t)x + e(t) \geq \Gamma \qquad \text{for } x \geq d, \text{ a.e. } t \in I,$$
$$g(t,x) - a(t)x + e(t) \leq \gamma \qquad \text{for } x \leq -d, \text{ a.e. } t \in I,$$

and (32)(33) write

$$\gamma \leq \frac{1}{T} \int_I e(t)\varphi(\tau) \leq \Gamma.$$

In the particular case where $a(t) \equiv 0$, this corresponds to a result of Mawhin and Ward [109], Ker L_a consisting then of constant functions.

2. When $\gamma = \Gamma = 0$, the above assumptions become

$$x\, [g(t,x) - a(t)x + e(t)\,] \geq 0 \quad \text{for } |x| \geq d, \text{ a.e. } t \in I, \tag{35}$$
$$\int_I e(t)\varphi(t)dt = 0. \tag{36}$$

Similar conditions have been proposed by Iannacci, Nkashama and Ward [75] for elliptic problems. Notice that this situation also covers the linear resonant case, when $g(t,x) = a(t)x - e(t)$.

3. A counterexample of Iannacci and Nkashama [73] shows that conditions (35)(36) are not sufficient to guarantee the existence of solutions when 0 is an eigenvalue of L_a other than the first one.

4. Conditions (30)(31) could be replaced by conditions of integral type : there exist functions $k^+, k^- \in L^2(I)$ and a number $d > 0$ such that

$$x\,[g(t,x) - a(t)x] \geq k^+(t) \qquad \text{for } x \geq d, \quad \text{a.e. } t \in I,$$
$$x\,[g(t,x) - a(t)x] \geq k^-(t) \qquad \text{for } x \leq -d, \text{ a.e. } t \in I,$$
$$\int_I k^+(t)dt > 0$$
$$\int_I k^-(t)dt > 0.$$

6. RESONANCE TO THE LEFT OF THE FIRST EIGENVALUE.

For the sake of completeness, we consider in this section a case of one-sided resonance, assuming only the existence of a function $b \in L^\infty(I)$ such that

$$\lim_{|x|\to\infty} \frac{g(t,x)}{x} \leq b(t), \tag{37}$$

uniformly for a.e. $t \in I$.

Let us introduce the following assumption, which is the analogous, in the setting of the above one-sided condition (37), of property (A).

(A') *Taken any function* $p \in L^\infty(I)$ *such that* $p(t) \leq b(t)$, *a.e. on* I, *the inequality being strict on a set of positive measure, the problem*

$$x'' + p(t)x = 0$$
$$x(0) - x(T) = x'(0) - x'(T) = 0$$

has only the trivial solution.

Notice that g is no longer required to grow at most linearly. As in Proposition 1 and Lemma 1, one can prove :

Proposition 2. *The following conditions are equivalent :*

(i) the function b *satisfy property* (A') *;*

(ii) $\beta_1 \geq 0$ *;*

(iii) for any $x \in H^1_T$*, one has*

$$B_b(x) := \int_I (x'^2 - bx^2) \geq 0,$$

the inequality being strict when $x \notin \operatorname{Ker} L_b$.

Lemma 6. *Assume that* b *satisfies property* (A'). *Let* H^* *be such that* $L^2(I) = \operatorname{Ker} L_b \oplus H^*$. *Then there exists a* $\delta > 0$ *such that, for every* $x = x^b + x^* \in H^1_T$, *with* $x^b \in \operatorname{Ker} L_b$, $x^* \in H^*$, *one has*

$$B_b(x) \geq \delta \, \|x^*\|^2_{H^1}. \tag{38}$$

We will also assume that the second part of condition (H) holds. For convenience, we rewrite it under the following form :

(H') *If* $\operatorname{Ker} L_b \neq \{0\}$, *there exist functions* $k^+, k^- \in L^2(I)$ *and a number* $d \geq 0$ *such that*

$$b(t)x - g(t,x) \geq k^+(t) \qquad \text{for } x \geq d, \text{ a.e. } t \in I, \tag{39}$$

$$b(t)x - g(t,x) \leq k^-(t) \qquad \text{for } x \leq -d, \text{ a.e. } t \in I. \tag{40}$$

A slight modification of the proof of Lemma 3 gives the following (see [40,Lemma 2]).

Lemma 7. *Let the Carathéodory function* g *verify* (37), *hypothesis* (H') *and be such that, for all* $R > 0$, *there exists* $h_R \in L^2(I)$ *such that*

$$|g(t,x)| \leq h_R(t) \qquad \text{for } |x| \leq R.$$

Assume that the function b *satisfies property* (A'). *Then, there exists* $\bar{b}$ $\in L^{\infty}(I)$, *with* $b(t) \leq \bar{b}(t)$ *a.e. on* I, $\bar{b}$ *still satisfying* (A'), (H') *and we can write*

$$g(t,x) = x\,\gamma(t,x) + h(t,x)$$

where

$$\gamma(t,x) \leq \bar{b}(t)$$

for a.e. $t \in I$, *all* $x \in \mathbb{R}$, *and* h(.,.) *is a function satisfying Carathéodory conditions and such that*

$$|h(t,x)| \leq \hat{h}(t)$$

for a.e. $t \in I$, *where* $\hat{h} \in L^2(I)$.

We are now able to prove the analogous of Theorems 1 and 2.

Theorem 3. *Let the function g verify* (37), *hypotheses* (H') *and the Landesman-Lazer condition*

(LL') *For every* $\psi \in \operatorname{Ker} L_b \setminus \{0\}$ *such that* $\psi(t) > 0$ *a.e. on* I,

$$\int_I \limsup_{x\to-\infty} [b(t)x - g(t,x)]\psi(t)dt < 0 < \int_I \liminf_{x\to+\infty} [b(t)x - g(t,x)]\psi(t)dt.$$

Moreover, assume that, for all $R > 0$, *there exists* $h_R \in L^2(I)$ *such that*

$$|g(t,x)| \leq h_R(t) \qquad \text{for } |x| \leq R.$$

If b *satisfies property* (A'), *then problem* (1) (2) *has at least one solution.*

Theorem 4. *Assume that* β_1, *the first eigenvalue of* L_b, is 0. *Let* Ker $L_b = \{k\psi : k \in \mathbb{R}\}$, *where* ψ *is a positive function. Let the function* g *verify the same hypotheses as in Theorem 3, except that the condition* (LL') *is replaced by*

$$\int_I k^-(t)\ \psi(t)dt \le 0 \le \int_I k^+(t)\ \psi(t)dt. \tag{41}$$

Then, problem (1) (2) *has at least one solution.*

Proofs. The proofs start in the same way as in the one of Theorem 1, with the only differences that one choses $p(t) = b(t) - 1$ and, by Lemma 6, where we assume without loss of generality that $b \equiv \bar{b}$,

$$\gamma(t,x) \le b(t)$$

for all $x \in \mathbb{R}$, a.e. $t \in I$.

Once we arrive at (25), multiplying by $(-u_n)$ and integrating one has

$$\int_I (u_n'^2 - s_n(t)u_n^2) = \int_I (1 - \lambda_n) \frac{h(t,x_n)}{\|x_n\|_{H^1}} u_n ,$$

which converges to zero as $n \to \infty$. This implies, by (38), that $u_n^* \overset{H^1}{\to} 0$. Passing to a subsequence, Ker L_b being finite dimensional, we can assume $u_n^b \to u \in$ Ker L_b, and hence $u_n \overset{H^1}{\to} u \ne 0$. Multiplying (25) by $\|x_n\|_{H^1}\, u$ and integrating, by the choice of p, one obtains

$$\lambda_n \int_I x_n(t)u(t)dt + (1 - \lambda_n) \int_I (b(t)x_n - g(t,x_n))u(t)dt = 0 \tag{42}$$

Since $u \in$ Ker $L_b \setminus \{0\}$ and β_1 is the first eigenvalue, u has constant sign. Let us suppose $u(t) < 0$ a.e. in I, the other case being treated similarly. By the strong converge $u_n \to u$, we have that, for n sufficiently large, $x_n(t) < -d$ for a.e. $t \in I$. By assumption (H'), a contradiction is obtained from (42) and (41), proving Theorem 4. On the other hand, from (42) one has

$$\limsup_{n\to\infty} \int_I (b(t)x_n - g(t,x_n))u(t)dt \le 0. \tag{43}$$

By assumption (H'), the Fatou's Lemma can be applied to (43), and this leads to a contradiction with (LL'), thus proving Theorem 3.

Chapter 3

QUADRATIC FORMS, WEIGHTED EIGENFUNCTIONS AND BOUNDARY VALUE PROBLEMS FOR NONLINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

1. INTRODUCTION

This paper is devoted to the study of various auxiliary tools employed when dealing with boundary value problems associated to second order differential equations of the form

$$x'' + f(t,x) = 0 . \tag{1}$$

Many existence conditions for (1) deal with the relation between the asymptotic behavior of the nonlinearity f and the spectrum of the differential operator. In particular, Mawhin and Ward (cf. [99, 108,109]) have introduced and used some quadratic forms associated to the eigenvalues and eigenfunctions of $-x''$. They were able to treat in this way many cases where $f(t,x)/x$ stays asymptotically between two consecutive eigenvalues or to the left of the spectrum.

On the other hand, Lasota and Opial [79] introduced a method of study of some boundary value problems for (1) which gives the existence of solutions in particular when $f(t,x)/x$ behaves asymptotically as a function $p(t)$ such that the equation

$$x''(t) + p(t)x(t) = 0 , \tag{2}$$

with the corresponding boundary conditions, has only the trivial solution. This approach was recently extended by Habets and Metzen [69] to the case of a jumping nonlinearity.

It is natural to study the relations between the two approaches. The aim of this paper is to provide a complete comparison as well as some abstract theorems generalizing the above mentioned results.

Section 2 emphasizes the fact that many known existence results for some boundary value problems associated to (1) when $f(t,x)/x$ stays asymptotically at the left of the spectrum of $-x''$ or between two consecutive eigenvalues still hold when we only assume that some quadratic forms, naturally associated to the asymptotic behavior of $f(t,x)/x$, are positive definite. This observation, combined with topological or variational approaches, provides some general existence theorems.

In section 3 we prove the equivalence of the approach based on the quadratic forms and the one based on the above property of the associated linear problem (2), in the case of the Dirichlet or the Neumann conditions. This is done in a straightforward way by using the weighted regular Sturm-Liouville theory as developed in [15,52,53]. The extension of this linear theory to the periodic case would make possible to prove the analogous of Theorems 5 and 6 under periodic boundary conditions.

2. EXISTENCE RESULTS BY THE USE OF QUADRATIC FORMS

We consider the following second order differential equation:

$$x'' + f(t,x) = 0 \quad ,$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ $(I = [0,T])$ is a Caratheodory function, i.e. $f(.,x)$ is measurable for every $x \in \mathbb{R}$ and $f(t,.)$ is continuous for almost every $t \in I$. Moreover, we assume the following condition.

(C) *For every* $R > 0$ *there is a* $k_R \in L^1(I)$ *such that*

$$|f(t,x)| \leq k_R(t)$$

for all $|x| \leq R$ *and almost every* $t \in I$.

Associated to equation (1) we consider one of the following boundary conditions:

the Dirichlet conditions:

$$x(0) = x(T) = 0 \quad ;$$

the Neumann conditions:

$$x'(0) = x'(T) = 0 \quad ;$$

the periodic conditions:

$$x(0) - x(T) = x'(0) - x'(T) = 0 \quad .$$

We will look for C^1-functions x with absolutely continuous derivatives (i.e. $x \in W^{2,1}(I)$) verifying (1) almost everywhere and one of the above boundary conditions. Such a function x will be called a solution of the considered boundary value problem.

According to which of the boundary conditions is considered, we will denote by H^1_* and W_* the following sets:

Dirichlet problem:

$$H^1_* = \{x \in H^1(I) \mid x(0) = x(T) = 0\} \ ,$$
$$W_* = \{x \in W^{2,1}(I) \mid x(0) = x(T) = 0\} \ ;$$

Neumann problem:

$$H^1_* = H^1(I) ,$$
$$W_* = \{x \in W^{2,1}(I) \mid x'(0) = x'(T) = 0\} ;$$

periodic problem:

$$H^1_* = \{x \in H^1(I) \mid x(0) = x(T)\} ,$$
$$W_* = \{x \in W^{2,1}(I) \mid x(0) - x(T) = x'(0) - x'(T) = 0\};$$

Our first existence result is the following generalization of a result in [99].

Theorem 1. *Let* $b \in L^1(I)$ *be such that*

(A1) *for every* $\varepsilon > 0$ *there exist* $\beta_\varepsilon, \gamma_\varepsilon \in L^1(I)$ *such that*

$$f(t,x)x \leq (b(t) + \varepsilon)x^2 + \beta_\varepsilon(t)\,|x| + \gamma_\varepsilon(t) ;$$

(B1) *for any* $x \in H^1_* \setminus \{0\}$ *one has* $\int_I((x')^2 - bx^2) > 0$.

Then (1) *has a solution in* W_*.

To prove Theorem 1 we need the following lemma, which is essentially proved in [99].

Lemma 1 . *Condition* (B1) *is equivalent to*

(B2) *there exists* $\bar{\varepsilon} > 0$ *such that for any* $x \in H^1_*$ *one has*

$$\int_I ((x')^2 - bx^2) \geq \bar{\varepsilon}\, ||x||^2_{H^1} .$$

Proof. It is clear that (B2) implies (B1). If (B2) is false, we can find a sequence (x_n) in H^1_* such that $||x_n||_{H^1} = 1$ and $\int_I((x'_n)^2 - bx_n^2) \to 0$. Taking a subsequence we can assume $x_n \rightharpoonup x$ in $H^1(I)$. Then (x_n) converges uniformly, and the weak lower semicontinuity of the L^2-norm of x'_n implies $\int_I((x')^2 - bx^2) \leq 0$. By (B1), $x = 0$, and the above implies $||x_n||_{H^1} \to 0$, which is impossible.

Proof of Theorem 1. Let us define the following operators.

$$\mathrm{dom}(\mathcal{L}) = W_*$$
$$\mathcal{L} : \mathrm{dom}(\mathcal{L}) \to L^1(I) \;,\; x \to x'' \;,$$
$$N : C(I) \to L^1(I) \;,\; x \to f(.\,, x(.)) \;.$$

It is well known (cf.[62,98]) that N is $\mathcal{L}$ - completely continuous, and the result will be proved if we find an a priori bound for the solutions in W_* of the equations

$$x'' + \lambda f(t,x) + (1 - \lambda)b(t)x = 0 \tag{3}$$

for every $\lambda \in [0,1[$. To this aim, fix $\varepsilon < \bar{\varepsilon}$, multiply (3) by $(- x)$ and integrate, to obtain

$$\begin{aligned} 0 &= \int_I (x')^2 - \lambda f(t,x)x - (1 - \lambda)b(t)x^2 \\ &\geq \int_I (x')^2 - \lambda[(b(t) + \varepsilon)x^2 + \beta_\varepsilon(t)|x| + \gamma_\varepsilon(t)] - (1 - \lambda)b(t)x^2 \\ &\geq (\bar{\varepsilon} - \varepsilon)\, ||x||^2_{H^1} - C\, ||\beta_\varepsilon||_{L^1} ||x||_{H^1} - ||\gamma_\varepsilon||_{L^1} . \end{aligned}$$

The a priori bound follows.

Let us define the function F by $F(t,x) = \int_0^x f(t,s)\, ds$. Then we have a result which is the analogue of Theorem 1 and extends Theorem 1.1 of [104].

Theorem 2. *Let* $b \in L^1(I)$ *be such that*

(A2) *for every* $\varepsilon > 0$ *there exist* $\beta_\varepsilon, \gamma_\varepsilon \in L^1(I)$ *such that*

$$2F(t,x) \leq (b(t) + \varepsilon)\, x^2 + \beta_\varepsilon(t)\, |x| + \gamma_\varepsilon(t)\ ,$$

and suppose (B1) *holds. Then* (1) *has a solution in* W_*.

Proof. We consider the functional associated to our problem, defined on H^1_*:

$$\rho(x) = \int_I [(x'(t))^2 - 2F(t,x(t))]\, dt\ . \tag{4}$$

One can easily see that ρ is weakly lower semicontinuous. Moreover, fix $\varepsilon < \bar{\varepsilon}$. By Lemma 1,

$$\begin{aligned} \rho(x) &\geq \int_I \{(x')^2 - [(b(t) + \varepsilon)x^2 + \beta_\varepsilon(t)|x| + \gamma_\varepsilon(t)]\} \\ &\geq (\varepsilon - \varepsilon)||x||^2_{H^1} - c||\beta_\varepsilon||_{L^1}||x||_{H^1} - ||\gamma_\varepsilon||_{L^1}\ . \end{aligned}$$

Hence ρ is coercive, and then it has a minimum, giving the solution we are looking for.

Remarks. 1) Theorems 1 and 2 can be extended in a straightforward way to the vector case. In Theorem 1 the products have to be replaced by scalar products in $\mathbb{R}^n$ and for Theorem 2 the system must be supposed to be in variational form, i.e. $f = D_xF$ for some $F : I \times \mathbb{R}^n \to \mathbb{R}$.

2) Using the hypothesis (C) it is easy to show that assumption (A1) is verified if, for some $b \in L^1(I)$, the following holds uniformly for almost every $t \in I$:

$$\limsup_{|x| \to \infty} f(t,x)/x \leq b(t) .$$

Analogously assumption (A2) is verified if, uniformly for almost every $t \in I$,

$$\limsup_{|x| \to \infty} 2F(t,x)/x^2 \leq b(t) .$$

3) Sufficient conditions for (B1) to hold can be found in [65,99].

We will now introduce a condition (B3) which generalizes (B1) and prove analogous existence results.

We begin with the following lemma, which is an immediate consequence of condition (C) .

Proposition 1. *Assume* $a,b \in L^1(I)$ *are such that*

(A3) $a(t) \leq \liminf_{|x| \to \infty} f(t,x)/x \leq \limsup_{|x| \to \infty} f(t,x)/x \leq b(t)$

.

Then the following holds:

(A4)*for every* $\varepsilon > 0$ *there exist* $g_\varepsilon, h_\varepsilon : I \times \mathbb{R} \to \mathbb{R}$, $\hbar_\varepsilon \in L^1(I)$ *such that*

$$f(t,x) = g_\varepsilon(t,x)x + h_\varepsilon(t,x)$$
$$a(t) - \varepsilon \leq g_\varepsilon(t,x) \leq b(t) + \varepsilon$$
$$|h_\varepsilon(t,x)| \leq \hbar_\varepsilon(t) .$$

The following theorem is a generalization of a result in [99].

Theorem 3. *Let* $a,b \in L^1(I)$ *satisfy* (A4) *and*

(B3) $L^2(I) = H^- \oplus H^+$, *where* H^- *is finite dimensional and contained in* H^1_*, *and for every* $x = \bar{x} + \breve{x} \in H^1_* \setminus \{0\}$, *with* $\bar{x} \in H^-$ *and* $\breve{x} \in H^+$, *one has*

$$B_{a,b}(x) := \int_I((\breve{x}')^2 - b\,\breve{x}^2) - \int_I((\bar{x}')^2 - a\,\bar{x}^2) > 0.$$

Then (1) *has a solution in* W_*.

To prove Theorem 3 we need the following lemma, which is essentially proved in [99].

Lemma 2. *If* (B3) *holds, then there exists* $\bar{\varepsilon} > 0$ *such that, for every* $x \in H^1_*$,

(B4) $B_{a,b}(x) \geq \bar{\varepsilon}\ \|x\|^2_{H^1}$.

Proof. If the conclusion is false, one can find a sequence (x_n) in H^1_* such that $\|x_n\|_{H^1} = 1$ and $B_{a,b}(x_n) \to 0$. Taking a subsequence, we can suppose $x_n \rightharpoonup x$ in $H^1(I)$. Then $\bar{x}_n \to \bar{x}$ in $H^1(I)$, all norms being equivalent in a finite dimensional space, and then also $\breve{x}_n \to \breve{x}$ uniformly. So,

$$\int_I(\breve{x}'_n)^2 \to \int_I(b\breve{x}^2 + (\bar{x}')^2 - a\bar{x}^2),$$

and by the weak lower semicontinuity of the L^2-norm of $\breve{x}'_n$ we get $B_{a,b}(x) \leq 0$. Then, by (B3), $x = 0$. It then follows from the above that $\|x_n\|_{H^1} \to 0$, which is a contradiction.

Proof of Theorem 3. Fix $\varepsilon < \bar{\varepsilon}$. By the arguments in the proof of Theorem 1, the proof will be complete if we find an a priori bound for the solutions of (3) in W_* for every $\lambda \in [0,1]$. Multiplying by $(\bar{x} - \breve{x})$ and integrating we get

$$0=\int_I[(\breve{x}')^2-(\bar{x}')^2+\lambda g_\varepsilon(t,x)(\bar{x}^2 - \breve{x}^2)+ \lambda h_\varepsilon(t,x)(\bar{x} - \breve{x}) + (1-\lambda)b(t)(\bar{x}^2 - \breve{x}^2)]$$
$$\geq \int_I [(\breve{x}')^2 - (\bar{x}')^2 + (a(t) - \varepsilon)\bar{x}^2 - (b(t) + \varepsilon)\breve{x}^2 + \lambda h_\varepsilon(t,x)(\bar{x} - \breve{x})]$$
$$\geq (\bar{\varepsilon} - \varepsilon) ||x||^2_{H^1} - c||\bar{h}_\varepsilon||_{L^1}||x||_{H^1} ,$$

and the a priori bound follows.

A similar result can be obtained by a variational method if assumption (A4) is replaced by an analogous one concerning the primitive of f . In this case we have to consider the functional ρ defined as in (3), whose critical points are precisely the solutions of the boundary value problem associated to (1). Let us recall the definition of the Palais - Smale condition.

Definition. The functional ρ satisfies (PS) if any sequence (x_n) in H^1_* such that $\rho(x_n)$ is bounded and $\rho'(x_n) \to 0$ has a convergent subsequence.

Proposition 2. Assume $a,b \in L^1(I)$ are such that

(A5) $a(t) \leq \liminf_{|x|\to\infty} 2F(t,x)/x^2 \leq \limsup_{|x|\to\infty} 2F(t,x)/x^2 \leq b(t)$

Then the following holds:

(A6) *for every* $\varepsilon > 0$ *there exist* $G_\varepsilon, H_\varepsilon : I \times \mathbb{R} \to \mathbb{R}$, $\mathcal{H}_\varepsilon \in L^1(I)$ *such that*

$$2F(t,x) = G_\varepsilon(t,x)x^2 + H_\varepsilon(t,x)$$
$$a(t) - \varepsilon \le G_\varepsilon(t,x) \le b(t) + \varepsilon$$
$$|H_\varepsilon(t,x)| \le \mathcal{H}_\varepsilon(t) \ .$$

Theorem 4. *Let* $a,b \in L^1(I)$ *satisfy* (B3) *and* (A6). *If moreover the functional* ρ *satisfies* (PS), *then* (1) *has a solution in* W_*.

Proof. Take in (A6) $\varepsilon = \bar{\varepsilon}/2$, with $\bar{\varepsilon}$ as in Lemma 2. If $\bar{x} \in H^-$, one has, by (A6) and (B4),

$$\begin{aligned}
\rho(\bar{x}) &= \int_I [(\bar{x}'(t))^2 - G_\varepsilon(t,\bar{x}(t))\bar{x}^2 - H_\varepsilon(t,\bar{x}(t))]\ dt \\
&\le \int_I [(\bar{x}')^2 - (a(t) - \varepsilon)\bar{x}^2] + ||\mathcal{H}_\varepsilon||_{L^1} \\
&= - B_{a,b}(\bar{x}) + \varepsilon||\bar{x}||^2_{L^2} + C \\
&\le - (\bar{\varepsilon}/2)\ ||\bar{x}||^2_{H^1} + C\ ,
\end{aligned}$$

where $C = ||\mathcal{H}_\varepsilon||_{L^1}$. Analogously, if $\check{x} \in H^+$, one proves that

$$\rho(\check{x}) \ge (\bar{\varepsilon}/2)\ ||\check{x}||^2_{H^1} - C\ .$$

The above implies that we are in the geometrical setting of the Saddle Point Theorem of Rabinowitz (cf. [129]), and since ρ satisfies (PS), the result follows.

Remark. Conditions which imply (B3) are given in [55, 65, 68, 99, 108,109].

3. A COERCIVE QUADRATIC FORM FOR SOME CLASSES OF NONCOERCIVE LINEAR PROBLEMS

In this section we will restrict ourself to Dirichlet or Neumann boundary conditions.

Theorem 5. *Given* $b \in L^1(I)$, *the following assertions are equivalent:*

(i) *assumption* (B1) *holds;*

(ii) *for each* $p \in L^1(I)$ *such that* $p(t) \leq b(t)$ *for almost every* $t \in I$, *the equation*

$$x'' + p(t)x = 0 \tag{5}$$

has only the trivial solution in W_*;

(iii) *for every* $\lambda \leq 0$ *the equation*

$$x'' + (b(t) + \lambda)x = 0 \tag{6}$$

has only the trivial solution in W_*.

Proof. To show that (i) implies (ii) , take p as in (ii) and let x be a solution of (5) in W_* . Multiply (5) by $(-x)$ and integrate, to obtain

$$0 = \int_I((x')^2 - px^2) \geq \int_I((x')^2 - bx^2)$$

It then follows from (B1) that $x = 0$.

It is clear that (ii) implies (iii). Let us then show that (iii) implies (i). By the theory of linear second order differential operators (see [15, 52, 53]), the eigenvalues of (6) with Dirichlet or Neumann boundary conditions form a sequence $\lambda_1 < \lambda_2 < \ldots$ which tends to $+\infty$, and the corresponding eigenfunctions $\phi_1 , \phi_2 , \ldots$ are an orthonormal base of $L^2(I)$. Hence, given any $x \in H^1_*$, we can write

$$x(t) = \sum_{i\geq 1} c_i \phi_i(t) \quad ,$$

and

$$\int_I((x')^2 - bx^2) = \sum_{i\geq 1} c_i^2 \int_I((\phi_i')^2 - b\phi_i^2)$$
$$= \sum_{i\geq 1} c_i^2 \lambda_i \int_I \phi_i^2$$
$$\geq \lambda_1 \int_I x^2 \ .$$

By (iii), $\lambda_1 > 0$, and (B1) follows.

An analogous result holds when (B3) is considered. Given $a,b \in L^1(I)$ as in (B3), we can suppose without restriction that $a(t) < b(t)$ for almost every $t \in I$. In fact, by Lemma 2, if a,b verify condition (B3), this is also true for $a - \bar{\varepsilon}/4$, $b + \bar{\varepsilon}/4$.

Theorem 6 . *Given* $a,b \in L^1(I)$ *such that* $a(t) < b(t)$ *for almost every* $t \in I$, *the following assertions are equivalent:*

(j) *assumption* (B3) *holds;*

(jj) *taken* $p \in L^1(I)$ *such that* $a(t) \leq p(t) \leq b(t)$ *for almost every* $t \in I$, *the equation*

$$x'' + p(t)x = 0 \tag{7}$$

has only the trivial solution in W_* ;

(jjj) *for every* $\mu \in [0,1]$, *the problem*

$$x'' + [(1 - \mu)a(t) + \mu b(t)]x = 0 \tag{8}$$

has only the trivial solution in W_*.

Proof. Assume (B3) holds. Multiplying (7) by $(\bar{x} - \check{x})$ and integrating, one has

$$0 = \int_I ((\check{x}')^2 - (\bar{x}')^2 + p(\bar{x}^2 - \check{x}^2) \geq B_{a,b}(x) \ .$$

Hence $x = 0$, so that (j) implies (jj).

It is clear that (jj) implies (jjj). Let us show that (jjj) implies (j). Notice that (7) can be written as

$$x'' + ax + \mu(b - a)x = 0 \ . \tag{9}$$

The theory developed in [15, 52, 53] can be applied: the eigenvalues of (9) with Dirichlet or Neumann boundary conditions form a sequence $\mu_1 < \mu_2 < \ldots$ which tends to $+\infty$. The corresponding eigenfunctions ψ_1 , ψ_2 , ... are an orthonormal base in the space $L^2_{b-a}(I)$ of measurable functions u such that

$$\int_I (b(t) - a(t))\, u(t)^2\, dt < + \infty ,$$

with scalar product given by

$$(u|v) = \int_I (b(t) - a(t))\, u(t)\, v(t)\, dt \ .$$

Hence, for $i \neq j$, one has

$$\int_I (b - a)\, \psi_i \psi_j = 0$$

and, as one can easily check,

$$\int_I (\psi_i' \psi_j' - a \psi_i \psi_j) = \int_I (\psi_i' \psi_j' - b \psi_i \psi_j) = 0 \ . \tag{10}$$

Given $x \in H^1_*$, we can write $x(t) = \sum_{i \geq 1} c_i \psi_i(t)$. By (jjj), there are no eigenvalues μ_n in the interval $[0,1]$. So, either $\mu_1 > 1$, or there exists a $n \geq 1$ such that $\mu_n < 0 < 1 < \mu_{n+1}$. If $\mu_1 > 1$, define $H^- = \{0\}$ and $H^+ = L^2_{b-a}(I)$. Then, by (10) one has:

$$\begin{aligned} B_{a,b}(x) &= \int_I ((x')^2 - bx^2) \\ &= \sum_{i\geq 1} c_i^2 \int_I ((\psi_i')^2 - b\,\psi_i^2) \\ &= \sum_{i\geq 1} c_i^2 (\mu_i - 1) \int_I ((b-a)\,\psi_i^2) \\ &\geq (\mu_1 - 1) \int_I ((b-a)\,x^2)\ , \end{aligned}$$

and (B3) follows. On the other hand, if $\mu_n < 0 < 1 < \mu_{n+1}$, set $H^- = \operatorname{span}\{\psi_1, \dots, \psi_n\}$ and $H^+ = \operatorname{span}\{\psi_{n+1}, \psi_{n+2}, \dots\}$. With analogous calculations, one has

$$\begin{aligned} &\int_I ((\breve{x}')^2 - b)\breve{x}^2) \geq (\mu_{n+1} - 1) \int_I ((b-a)\breve{x}^2)\ , \\ &\int_I ((\bar{x}')^2 - a\bar{x}^2) \leq \mu_n \int_I ((b-a)\bar{x}^2)\ , \end{aligned}$$

and (B3) follows in this case, as well.

Chapter 4

SEMICOERCIVE VARIATIONAL PROBLEMS AT RESONANCE : AN ABSTRACT APPROACH

1. INTRODUCTION

This paper is concerned with the existence of solutions to elliptic boundary value problems at resonance with the first eigenvalue. We consider the Dirichlet problem

$$\begin{aligned} -\Delta u - \lambda_1 u + g(x,u) &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega , \end{aligned} \tag{P}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, and λ_1 the first eigenvalue of $(-\Delta)$ on $H_0^1(\Omega)$. The Caratheodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is supposed to satisfy the usual growth condition

$$|g(x,u)| \leq a|u|^{q-1} + b(x)$$

where $q < \infty$ if $N = 2$, $q < 2^* = 2N/(N - 2)$ if $N \geq 3$, and where $b(x) \in L^{q'}(\Omega)$, with q' the Hölder conjugate exponent of q ; if $N = 1$, it suffices to assume that for any $r > 0$,

$$\sup_{|u| \leq r} |g(x,u)| \in L^1(\Omega) .$$

Under this condition, the associated functional

$$f(u) = \frac{1}{2}\int_\Omega [|\nabla u|^2 - \lambda_1|u|^2] + \int_\Omega G(x,u(x))\, dx ,$$

where $G(x,u) = \int_0^u g(x,s)\, ds$, is a weakly lower semicontinuous C^1 functional on H_0^1 whose critical points are the weak solutions of (P). It follows that if f is coercive (i.e. $f(u) \to +\infty$ as $\|u\| \to \infty$ in H_0^1), then f has a minimum and consequently (P) has a solution. We are mainly interested in this paper in the conditions which guarantee the coercivity of f .

The functional $f(u)$ can be written as the sum of a quadratic term

$$a(u) = \frac{1}{2}\int_\Omega [|\nabla u|^2 - \lambda_1 |u|^2]$$

coming from the linear part of the equation, and a term

$$b(u) = \int_\Omega G(x,u(x))\, dx$$

coming from the nonlinear part. Denoting by $\bar{H}$ the space spanned by the first (positive) eigenfunction ϕ_1 of $(-\Delta)$ on H_0^1 , we see that a necessary condition for f to be coercive on H_0^1 is that f be coercive on $\bar{H}$, i.e. that b be coercive on $\bar{H}$. A condition of this type was first considered by Ahmad, Lazer and Paul [2] in a slightly different situation. One can ask whether this condition is also sufficient for the coercivity of f , or at least for the existence of a solution to (P). The answer to this question is negative, as is shown by the following

Example 1. Consider the one - dimensional problem

$$\begin{aligned} u'' + u &= s(x)u + h(x) \quad \text{on }]0,\pi[\\ u(0) &= 0 = u(\pi) . \end{aligned} \qquad (Q)$$

The associated functional is

$$f(u) = \int_0^\pi [\tfrac{1}{2}(u')^2 - \tfrac{1}{2}(1 - s(x))\, u^2 + h(x)u] \,.$$

The coefficient $s(x)$ in (Q) is defined as follows: first choose $C > 0$ in such a way that the graphs of the two functions

$$u_1(x) = C(e^{2x} - e^{-2x}) \quad , \quad u_2(x) = - \sin(2x)$$

be tangent one to the other at a certain point $\bar{x} \in \,]\frac{\pi}{2},\pi\,[$; then take

$$s(x) = \begin{cases} 5 & \text{if } x \in [0,\bar{x}\,[\\ -3 & \text{if } x \in [\bar{x}\,,\pi] \,. \end{cases}$$

It follows that the corresponding homogeneous problem

$$\begin{aligned} &u'' + u = s(x)u \\ &u(0) = 0 = u(\pi) \end{aligned}$$

has a nontrivial solution given by

$$u_0(x) = \begin{cases} u_1(x) & \text{if } x \in [0,\bar{x}\,] \\ u_2(x) & \text{if } x \in [\bar{x},\pi] \,. \end{cases}$$

This implies that if $\int_0^\pi hu_0 \neq 0$ then problem (Q) has no solution. For such h 's the functional f can not be coercive; actually it is easily seen directly that f is not coercive on the line $\mathbb{R}u_0$. Nevertheless, f is coercive on $\bar{H}$. Indeed

$$f(r \sin x) = (r^2/2) \int_0^\pi s(x) \sin^2 x \, dx + r \int_0^\pi h(x) \sin x \, dx$$

and since

$$\int_0^\pi s(x) \sin^2 x \, dx = \int_0^{\bar{x}} 5 \sin^2 x \, dx + \int_{\bar{x}}^\pi - 3 \sin^2 x \, dx > 0 ,$$

one has $f(r \sin x) \to + \infty$ as $|r| \to \infty$.

In order to try to understand what can go wrong with the coercivity of f when b is coercive on $\bar{H}$, let us write the orthogonal decomposition $H_0^1 = \bar{H} \oplus \tilde{H}$, where $\tilde{H}$ is generated by the higher eigenfunctions of $(- \Delta)$ on H_0^1 , and for every $u \in H_0^1$, let $u = \bar{u} + \tilde{u}$, with $\bar{u} \in \bar{H}$, $\tilde{u} \in \tilde{H}$. Taking Fourier's expansions, it is easily seen that the functional a is coercive on $\tilde{H}$; more precisely, there exists $\delta > 0$ such that

$$a(u) \geq \delta \, \|\tilde{u}\|^2$$

for every $u \in H_0^1$. One reason why this semi - coercivity of a on H_0^1 together with a coercivity assumption of b on $\bar{H}$ do not necessarily imply the coercivity of f on H_0^1 is that b may decrease too rapidly outside $\bar{H}$. This phenomenon is apparent in the above example where one has $b(ru_0) \to - \infty$ with speed r^2 as $|r| \to \infty$ (since $b(ru_0) = - a(ru_0) + r \int_0^\pi hu_0$).

This suggests to try to impose some control on the decreasing speed of b outside $\bar{H}$. One can also reinforce the coercivity of b on $\bar{H}$ and look for a compromise with the decreasing speed of b outside $\bar{H}$. These ideas are the content of condition (iii) of our abstract theorem in section 2. A particular case of this theorem deals with the situation where b is Lipschitz continuous (see Remark 2). We provide in this way an abstract explanation for the use in [104] of a boundedness assumption on the nonlinearity g(x,u). This particular case can also be used to recover a theorem of Mawhin [105] relative to the situation where b is convex (see Corollary 1).

Beginning with Hammerstein [71], several papers have been concerned with the study of problem (P) under conditions on the asymptotic behaviour of the quotient $2G(x,u)/u^2$. In [110], Mawhin, Ward and Willem proved an existence result by assuming

$$\liminf_{|u|\to\infty} 2G(x,u)/u^2 \geq 0 \qquad (1)$$

for a.e. $x \in \Omega$, the inequality being strict on a set of positive measure. In [41], de Figueiredo and Gossez considered the case in which equality holds in (1) for a.e. $x \in \Omega$ and proposed a so called density condition in order to obtain existence. Conditions on the quotient $G(x,u)/|u|^p$ for $1 \leq p < 2$ were also considered by Anane in [11] (see also[12]).

In section 3, we show how these results can all be recovered and sometimes improved or generalized by using our abstract theorem. The present approach in addition provides a new insight to the role of some assumptions in the above mentioned results; this is particularly apparent for the linear growth restriction imposed in [41] on the nonlinearity g(x,u) .We also consider the limiting situation where $p = 0$ in the quotient $G(x,u)/|u|^p$, in which case the functional may not be coercive. Our result here is related to some recent work of Ramos and Sanchez [131] and provides an improvement of a theorem of Berger (cf. [21], [104]).

We finally observe that most of our results can be adapted to systems of differential equations or to other types of boundary conditions, like the Neumann conditions, or, for ODE's, the periodic conditions.

2. AN ABSTRACT THEOREM

In this section we study the coercivity of functionals of the form $f = a + b$, where a is semicoercive with respect to a subspace and b is coercive on a complementary subspace.

Theorem 1. *Let* H *be a normed space,* $H = \bar{H} \oplus \tilde{H}$, *and for any* $u \in H$, *write* $u = \bar{u} + \tilde{u}$, *with* $\bar{u} \in \bar{H}$, $\tilde{u} \in \tilde{H}$. *Let* $a,b : H \to \mathbb{R}$ *be two functionals satisfying the following properties:*

(i) *there exists* $\delta > 0$ *such that*
$$a(u) \geq \delta \, \|\tilde{u}\|^2$$
for every $u \in H$;

(ii) $\liminf\limits_{\|u\| \to \infty} \dfrac{b(u)}{\|u\|^2} \geq 0$;

(iii) *there exist a functional* $\hat{b} : H \to \mathbb{R}$, $\hat{b} \leq b$, *and* $\beta \geq 1$ *such that*
$$|\hat{b}(u) - \hat{b}(w)| \leq \|u - w\| \, [A(\|u\| + \|w\|)^{\beta-1} + B] \qquad (2)$$
for every $u,w \in H$, *and either*

$$\lim_{\substack{\|\bar{u}\|\to\infty \\ \bar{u}\in\bar{H}}} \frac{\hat{b}(\bar{u})}{\|\bar{u}\|^{2(\beta-1)}} = +\infty \tag{3}$$

or

$$\liminf_{\substack{\|\bar{u}\|\to\infty \\ \bar{u}\in\bar{H}}} \frac{\hat{b}(\bar{u})}{\|\bar{u}\|^{\beta}} > 0\ . \tag{4}$$

Then the functional $f = a + b$ *is coercive on* H .

Remark 1. When $1 \leq \beta < 2$, property (ii) is a consequence of (2); moreover one has $2(\beta - 1) < \beta$ and so (3) is a less restrictive requirement than (4). On the contrary when $\beta \geq 2$, (4) becomes less restrictive than (3). If $\hat{b}$ is differentiable, it is easily seen that (2) is equivalent to the following growth condition on $\hat{b}'$:

$$\|\hat{b}'(u)\| \leq A'\|u\|^{\beta-1} + B'\ . \tag{2'}$$

Remark 2. In the case $\beta = 1$, condition (iii) becomes:

(iii)' *there exists* $\hat{b} : H \to \mathbb{R}$ *such that* $\hat{b} \leq b$ on H , $\hat{b}$ *is coercive on* $\bar{H}$ *and Lipschitz continuous on* H .

Proof of Theorem 1. Assume by contradiction that there exists a sequence (u_n) in H and a real constant C such that $\|u_n\| \to \infty$ and $f(u_n) \leq C$ for every n . Then, by (i),

$$\delta \frac{\|\tilde{u}_n\|^2}{\|u_n\|^2} + \frac{b(u_n)}{\|u_n\|^2} \leq \frac{C}{\|u_n\|^2} \quad ,$$

which by (ii) implies that $\|\tilde{u}_n\| / \|u_n\| \to 0$. As a consequence,

$$\liminf \ \|\bar{u}_n\| / \|u_n\| > 0 \ , \tag{5}$$

and, in particular, $\|\bar{u}_n\| \to \infty$.

Assume that (3) holds. By (iii),

$$f(u_n) \geq \delta \ \|\tilde{u}_n\|^2 + \hat{b}(u_n) - \hat{b}(\bar{u}_n) + \hat{b}(\bar{u}_n)$$

$$\geq \delta \ \|\tilde{u}_n\|^2 - \|\tilde{u}_n\| \ [A(\|u_n\| + \|\bar{u}_n\|)^{\beta-1} + B] + \hat{b}(\bar{u}_n)$$

$$\geq \delta \ \|\tilde{u}_n\|^2 - B \ \|\tilde{u}_n\| - \frac{\delta}{2} \|\tilde{u}_n\|^2 - \frac{A^2}{2\delta} (\|u_n\| + \|\bar{u}_n\|)^{2(\beta-1)} + \hat{b}(\bar{u}_n)$$

$$\geq [\frac{\delta}{2} \|\tilde{u}_n\|^2 - B \ \|\tilde{u}_n\|] +$$

$$+ [\frac{\hat{b}(\bar{u}_n)}{\|\bar{u}_n\|^{2(\beta-1)}} - \frac{A^2}{2\delta} (\frac{\|u_n\|}{\|\bar{u}_n\|} + 1)^{2(\beta-1)}] \ \|\bar{u}_n\|^{2(\beta-1)} \ , \tag{6}$$

which by (5) and (3) goes to $+\infty$ as $n \to \infty$, in contradiction with our assumption.

Assume now that (4) holds. By (iii),

$$\left| \frac{\hat{b}(u_n)}{\|\bar{u}_n\|^{\beta}} - \frac{\hat{b}(\bar{u}_n)}{\|\bar{u}_n\|^{\beta}} \right| \leq \frac{\|\tilde{u}_n\|}{\|\bar{u}_n\|} \ [\ A \ (\frac{\|u_n\|}{\|\bar{u}_n\|} + 1)^{\beta-1} + \frac{B}{\|\bar{u}_n\|^{\beta-1}}] \ ,$$

which goes to 0, by (5). Consequently, by (4),

$$\liminf \frac{\hat{b}(u_n)}{\|\bar{u}_n\|^\beta} > 0 \ ,$$

which implies $\hat{b}(u_n) \to +\infty$ and so also $f(u_n) \to +\infty$, a contradiction. The theorem is thus proved.

Remark 3. When $\beta = 1$, condition (i) of Theorem 1 can be weakened to the following:

(i)' *there exists* $\alpha : \mathbb{R}_+ \to \mathbb{R}$, *bounded below, with* $\alpha(t) \to +\infty$ *as* $t \to +\infty$, *such that*

$$a(u) \geq \alpha(\|\tilde{u}\|)\, \|\tilde{u}\|$$

for every $u \in H$.

Indeed assume by contradiction that there exist a sequence (u_n) in H and a constant C such that $\|u_n\| \to \infty$ and $f(u_n) \leq C$ for every n . Then

$$\frac{\alpha(\|\tilde{u}_n\|)\, \|\tilde{u}_n\|}{\alpha(\|u_n\|)\, \|u_n\|} + \frac{\hat{b}(u_n)}{\alpha(\|u_n\|)\, \|u_n\|} \leq \frac{C}{\alpha(\|u_n\|)\, \|u_n\|} \ .$$

Since $\hat{b}$ is Lipschitz continuous, this implies

$$\frac{\alpha(\|\tilde{u}_n\|)\, \|\tilde{u}_n\|}{\alpha(\|u_n\|)\, \|u_n\|} \to 0 \ . \tag{7}$$

It is not restrictive to assume α increasing and concave (cf. [66]). As a consequence, $\|\bar{u}_n\| \to \infty$. Indeed, if this is not true, there exists a subsequence, still denoted (u_n) , such that $\|\bar{u}_n\|$ is bounded. Then

$$\liminf \frac{\|\tilde{u}_n\|}{\|u_n\|} \geq 1 \tag{8}$$

and so, for n large enough,

$$\liminf \frac{\alpha(\|\tilde{u}_n\|)}{\alpha(\|u_n\|)} \geq \liminf \frac{\alpha(\frac{1}{2}\|u_n\|)}{\alpha(\|u_n\|)} > 0 \tag{9}$$

since $\alpha(2t) \leq K\,\alpha(t)$ for a certain $K > 0$ and all t sufficiently large. A contradiction with (7) then follows from (8) and (9). Now, if ℓ is the Lipschitz constant of $\hat{b}$, one has

$$f(u_n) \geq \alpha(\|\tilde{u}_n\|)\|\tilde{u}_n\| + \hat{b}(u_n) - \hat{b}(\bar{u}_n) + \hat{b}(\bar{u}_n)$$

$$\geq [\alpha(\|\tilde{u}_n\|) - \ell]\, \|\tilde{u}_n\| + \hat{b}(\bar{u}_n) \ .$$

Since the first term in the above sum is bounded below and $\hat{b}(\bar{u}_n) \to +\infty$, we have that $f(u_n) \to +\infty$, a contradiction.

We deduce from Theorem 1 with $\beta = 1$ (actually from Remark 3) the following result which contains a theorem of Mawhin [105] obtained by convex analysis methods.

Corollary 1. *Let* H *be as in Theorem 1,* $a : H \to \mathbb{R}$ *a functional satisfying property* (i)' *and suppose that* $b : H \to \mathbb{R}$ *satisfies* $b \geq \hat{b}$, *with* $\hat{b} : H \to \mathbb{R}$ *a convex and continuous functional which is coercive on* $\bar{H}$. *Then* $f = a + b$ *is coercive on* H.

Proof. By Remark 3, it suffices to show that there exists $\hat{\hat{b}} : H \to \mathbb{R}$ with $\hat{\hat{b}} \leq \hat{b}$ on H, $\hat{\hat{b}}$ coercive on $\bar{H}$ and Lipschitz continuous on H. Since $\hat{b}$ is convex continuous and coercive on $\bar{H}$, it is easily seen that there exist constants C_1 and C_2 such that

$$\hat{b}(\bar{u}) \geq C_1 \|\bar{u}\| - C_2$$

for every $\bar{u} \in \bar{H}$. Moreover, adding a constant to $\hat{b}$, we can always assume $C_2 = 0$. For every $\bar{u} \in \bar{H}$, define a linear functional $L_{\bar{u}}$ on the one - dimensional space $\mathbb{R}\bar{u}$ as follows: $L_{\bar{u}}(t\bar{u}) = C_1\, t\, \|\bar{u}\|$. By the Hahn - Banach theorem and the continuity of $\hat{b}$, $L_{\bar{u}}$ can be extended on the whole space H into a continuous linear functional $L'_{\bar{u}}$ such that $L'_{\bar{u}}(w) \leq \hat{b}(w)$ for every $w \in H$. Define

$$\hat{\hat{b}}(w) = \sup_{\bar{u} \in \bar{H}} L'_{\bar{u}}(w) .$$

Clearly $\hat{\hat{b}} \leq \hat{b}$ on H ; moreover, $\hat{\hat{b}}(\bar{u}) \geq C_1 \|\bar{u}\|$ on $\bar{H}$, and so $\hat{\hat{b}}$ is coercive on $\bar{H}$. It remains to see that $\hat{\hat{b}}$ is Lipschitz continuous on H. The continuity of $\hat{b}$ implies that the linear functionals $L'_{\bar{u}}$, $\bar{u} \in \bar{H}$, remain bounded, say $\|L'_{\bar{u}}\| \leq M$. Given $v, w \in H$ and $\varepsilon > 0$, take $\bar{u} \in \bar{H}$ such that $\hat{\hat{b}}(v) - \varepsilon \leq L'_{\bar{u}}(v)$. Adding and substracting $L'_{\bar{u}}(w)$ and letting $\varepsilon \to 0$, one gets $\hat{\hat{b}}(v) - \hat{\hat{b}}(w) \leq M \|v - w\|$, which yields the conclusion.

Remark 4. If

$$|b(u) - \hat{b}(u)| \leq C$$

for some constant C and all $u \in \overline{H}$, then the coercivity requirement on $\hat{b}$ in the above results is equivalent to an analogous requirement on b.

3. APPLICATIONS TO PROBLEM (P)

As an immediate consequence of Remark 2, Corollary 1 and Remark 4, we have the following result (which could also be deduced from the arguments in [104]).

Theorem 2. *Assume that*

$$\lim_{|r| \to \infty} \int_\Omega G(x, r\phi_1(x))\, dx = +\infty . \tag{10}$$

If moreover there exists a Caratheodory function $\hat{G}(x,u)$ *which is either convex in* u *or Lipschitzian in* u *with a Lipschitz constant independent of* x, *and which satisfies*

$$|G(x,u) - \hat{G}(x,u)| \leq c(x)$$

for some function $c(x) \in L^1(\Omega)$, *then problem* (P) *has a solution.*

Theorem 2 is an application of Theorem 1 with the choice $\beta = 1$. The following result deals with a case where $\beta \in [1,2[$.

Theorem 3. *Let* $0 < p < 2$ *and* $c(x) \in L^{\infty}(\Omega)$ *be such that*

$$\liminf_{|u| \to \infty} G(x,u) / |u|^p \geq c(x) \tag{11}$$

uniformly for a.e. $x \in \Omega$. *If*

$$\int_{\Omega} c(x)\, \phi_1(x)^p\, dx > 0 \ , \tag{12}$$

then problem (P) has a solution.

Proof. Fix $\varepsilon > 0$ as follows:

$$\varepsilon < \int_{\Omega} c(x)\, \phi_1(x)^p\, dx \ \left(\int_{\Omega} \phi_1(x)^p\, dx \right)^{-1}.$$

There exists $k_{\varepsilon} \in L^1(\Omega)$ such that

$$G(x,u) \geq (c(x) - \varepsilon)|u|^p - k_{\varepsilon}(x) \ .$$

Suppose first $1 < p < 2$. Set in this case

$$\hat{b}(u) = \int_{\Omega} [(c(x) - \varepsilon)\, |u(x)|^p - k_{\varepsilon}(x)]\, dx \ .$$

Then

$$|\hat{b}'(u)v| \leq \int_{\Omega} |c(x) - \varepsilon|\, p\, |u(x)|^{p-1} |v(x)|\, dx$$

$$\leq C\, \|c(.) - \varepsilon\|_{\infty}\, \|u\|^{p-1} \|v\| \ ,$$

for a certain $C > 0$ and every $v \in H^1_0$, so that $\|\hat{b}'(u)\| \leq A\, \|u\|^{p-1}$, with $A = C\, \|c(.) - \varepsilon\|_{\infty}$. This implies (2) with $\beta = p$, as was observed in Remark 1. Moreover

$$\hat{b}(r\phi_1) \geq |r|^p \int_{\Omega} (c(x) - \varepsilon)\, \phi_1(x)^p\, dx - \|k_{\varepsilon}\|_{L^1} \ .$$

By the choice of ε , $\lim_{|r|\to\infty} \hat{b}(r\phi_1) / |r|^p > 0$, and consequently (4) is verified. So Theorem 1 with $\beta = p$ implies the result in this case.

Suppose now $0 < p \leq 1$. Define $\hat{G}(x,u)$ as follows:

$$\hat{G}(x,u) = \begin{cases} (c(x) - \varepsilon)\, |u|^p - |c(x) - \varepsilon| - k_\varepsilon(x) & \text{if } |u| \geq 1 \\ (c(x) - \varepsilon) - |c(x) - \varepsilon| - k_\varepsilon(x) & \text{if } |u| \leq 1 \,. \end{cases}$$

Then $\hat{G}$ is a Caratheodory function, which is Lipschitz continuous in u , with a Lipschitz constant independent of x . Moreover, $\hat{G} \leq G$, and defining $\hat{b}(u) = \int_\Omega \hat{G}(x,u(x))\, dx$, one has

$$\hat{b}(r\phi_1) \geq \int_\Omega [(c(x) - \varepsilon)\, |r\phi_1(x)|^p - 2\, |c(x) - \varepsilon| - k_\varepsilon(x)]\, dx$$

$$\geq |r|^p \int_\Omega (c(x) - \varepsilon)\, \phi_1(x)^p\, dx - 2\, \|c(.) - \varepsilon\|_{L^1} - \|k_\varepsilon\|_{L^1} \,.$$

So, $\hat{b}(r\phi_1) \to +\infty$ as $|r| \to \infty$, and the conclusion follows from Theorem 1 with $\beta = 1$.

A result analogous to Theorem 3 for $1 \leq p < 2$ was obtained in [11, 12]. The case $p = 0$ will be considered at the end of this section.

As can be seen from Example 1 given in the Introduction, Theorem 3 is not true when $p = 2$. In this case one has to reinforce condition (12). This is done in the following theorem. (The fact that the condition on $c(x)$ below implies (12) with $p = 2$ follows easily from the variational characterization of the first eigenvalue of an elliptic operator.)

Theorem 4. *Let* $c(x)$ *be such that*

$$\liminf_{|u|\to\infty} 2G(x,u) / u^2 \geq c(x) \tag{13}$$

uniformly for a.e. $x \in \Omega$ *, with* $c \in L^{N/2}(\Omega)$ *if* $N \geq 3$ *,* $c \in L^r(\Omega)$ *for some* $r > 1$ *if* $N = 2$ *,* $c \in L^1(\Omega)$ *if* $N = 1$ *. Suppose that the first eigenvalue of the operator* $Lu = -\Delta u - \lambda_1 u + c(.)u$ *, with Dirichlet boundary conditions, is positive. Then problem (P) has a solution.*

Proof. Let μ_1 be the first eigenvalue of L . Problem (P) can then be written as

$$-Lu + \mu_1 u = m(x,u) \quad ,$$

where $m(x,u) = g(x,u) + (\mu_1 - c(x))u$. Denoting by M the primitive of m with respect to u , we have

$$\liminf_{|u|\to\infty} 2M(x,u) / u^2 \geq \mu_1$$

uniformly for a.e. $x \in \Omega$. So, there is a $k \in L^1(\Omega)$ such that

$$M(x,u) \geq (\mu_1/4)u^2 - k(x) \ . \tag{14}$$

Let us now decompose H_0^1 according to the eigenfunctions of L (instead of those of $-\Delta$) . We set $\overline{H}$ to be the space spanned by the first eigenfunction of L and we define

$$a(u) = \frac{1}{2}[(Lu,u)_{L^2} - \mu_1 |u|_{L^2}^2] \ ,$$

$$b(u) = \int_\Omega M(x,u(x))\, dx \ .$$

The result then follows from Corollary 1 since, by (14), b is minorized by a convex continuous functional which is coercive on $\bar{H}$.

As a consequence of Theorem 4, we have the following

Corollary 2. (Mawhin, Ward and Willem [110]). *Let* $c \in L^\infty(\Omega)$ *be such that (13) holds uniformly for a.e.* $x \in \Omega$. *If* $c(x) \geq 0$ *for a.e.* $x \in \Omega$, *with strict inequality on a set of positive measure, then problem (P) has a solution.*

Proof. Let L be the operator defined in Theorem 4 and let μ_1 and ψ_1 be the first eigenvalue and the corresponding normalized eigenfunction of L . By the variational characterization of the first eigenvalue of an elliptic operator and Poincaré's inequality, we have:

$$\mu_1 = \int_\Omega [|\nabla\psi_1|^2 - (\lambda_1 - c(x))|\psi_1|^2] > \int_\Omega [|\nabla\psi_1|^2 - \lambda_1 |\psi_1|^2] \geq 0 \ .$$

Hence we are in the situation of Theorem 4.

The following example, which is a variation of Example 1, shows that a function $c(x)$ may satisfy the assumption of Theorem 4 without being nonnegative a.e.. Theorem 4 applies to problem (P) with $g(x,u) = c(x)u + h(x)$, while Corollary 2 does not.

Example 2. Choose $C > 0$ such that the graphs of the two functions

$$u_1(x) = C\,(e^{\sqrt{3}\,x} - e^{-\sqrt{3}\,x}) \ , \ u_2(x) = -\sin(\sqrt{5}\,(x - \pi))$$

be tangent one to the other at a certain point $\bar{x} \in [0,\pi]$. We define

$$c(x) = \begin{cases} 5 & \text{if } x \in [0,\bar{x}[\\ -3 & \text{if } x \in [\bar{x},\pi] . \end{cases}$$

Then the first eigenvalue of the operator $Lu = - u'' - u + c(x)u$ on $H_0^1(0,\pi)$ is 1 and the corresponding (positive) eigenfunction is

$$u_0(x) = \begin{cases} u_1(x) & \text{if } x \in [0,\bar{x}] \\ u_2(x) & \text{if } x \in [\bar{x},\pi] . \end{cases}$$

We will now apply Theorem 1 with β possibly greater than 2 to a situation where $c(x)$ in (13) may be identically equal to zero. Let us first recall the following

Definition 1. *Let* E *be a measurable subset of* $\mathbb{R}$ *and* $\nu \in [0,1[$. *We say that* E *has a positive* ν *-density at* $+\infty$ *if*

$$\liminf_{r\to+\infty} \frac{m_1(E \cap [\nu r,r])}{m_1([\nu r,r])} > 0 \ ,$$

where m_1 *is the Lebesgue measure on* $\mathbb{R}$. *An analogous definition can be given at* $-\infty$.

The following two theorems provide some improvement to results in [41].

Theorem 5. *Assume that there exist* $\beta \geq 1$, *with* $\beta \leq 2^*$ *if* $N \geq 3$, *and a Caratheodory function* $\hat{g} : \Omega \times \mathbb{R} \to \mathbb{R}$ *such that, if* $\hat{G}(x,u) = \int_0^u \hat{g}(x,s)\, ds$, *one has* $\hat{G} \leq G$ and

(a) $$\liminf_{|u|\to\infty} \frac{\hat{G}(x,u)}{|u|^\sigma} \geq 0$$

uniformly for a.e. $x \in \Omega$, *where* $\sigma = \min\{\beta, 2\}$;

(b) *there exist a constant* A *and a function* $B(x) \in L^{q'}(\Omega)$ *such that*

$$|\hat{g}(x,u)| \leq A|u|^{\beta-1} + B(x) \tag{15}$$

for a.e. $x \in \Omega$ *and all* $u \in \mathbb{R}$ (here q' is the Hölder conjugate exponent of the exponent q considered in the introduction; if $N = 1$, then $q' = 1$);

(c) *there exists a full subset* $\Omega' \subset \Omega$ *and* $\eta > 0$ *such that the set*

$$E_\eta = \bigcap_{x \in \Omega'} \{u \in \mathbb{R} : \hat{G}(x,u) \geq \eta |u|^\beta\}$$

has a positive 0 - density at both $+\infty$ *and* $-\infty$.

Then problem (P) has a solution.

Theorem 6. *The result of Theorem 6 remains true if condition* (c) *is replaced by the following:*

(c') *there exist* ω_+ , ω_- *open subsets of* Ω *and corresponding full subsets* $\omega_+' \subset \omega_+$ *and* $\omega_-' \subset \omega_-$ *with the following property: for every* $\nu \in [0,1[$ *there exists* $\eta > 0$ *such that the set*

$$E_\eta^+ = \bigcap_{x \in \omega_+'} \{u \in \mathbb{R} : \hat{G}(x,u) \geq \eta |u|^\beta\}$$

has a positive ν *- density at* $+\infty$ *, and the set*

$$E_\eta^- = \bigcap_{x \in \omega_-'} \{u \in \mathbb{R} : \hat{G}(x,u) \geq \eta |u|^\beta\}$$

has a positive ν *- density at* $-\infty$.

The proofs of these two theorems are based, as in [41], on the following lemma.

Lemma 1. (cf. [41]) *Let* Ω *be an open subset of* $\mathbb{R}^N$ *with finite Lebesgue measure* $m_N(\Omega)$. *There exists a constant* $c > 0$ *such that*

$$m_N(u^{-1}(B)) \geq c \, \frac{m_1(B)^N}{\ell(u)^N}$$

for any nonconstant Lipschitz continuous function u , *with Lipschitz constant* $\ell(u)$, *which vanishes on* $\partial\Omega$ *and any Borelian set* B *in the* *range of* u *. The same inequality holds for functions* u *which do not necessarily vanish on* $\partial\Omega$ *when* Ω *is an open parallelepiped.*

Proof of Theorem 5. We will apply Theorem 1 with β as above. Condition (ii) follows from (a). Set $\hat{b}(u) = \int_\Omega \hat{G}(x,u(x))\, dx$. From (15) one can easily deduce (2) (see Remark 1). In order to prove (4), suppose by contradiction that there exists a sequence (r_n) such that $|r_n| \to \infty$ and

$$\lim |r_n|^{-\beta} \int_\Omega \hat{G}(x,r_n\phi_1(x))\, dx \leq 0 . \tag{16}$$

Taking a subsequence if necessary, we have either $r_n \to +\infty$ or $r_n \to -\infty$. Let us consider the first case (the second case is treated similarly). By assumption (c), there exists a positive number γ such that, for n sufficiently large,

$$m_1 (E_\eta \cap [0, r_n \max \phi_1]) \geq \gamma\, r_n \max \phi_1 .$$

Setting $u = r_n\phi_1$ and $B = E_\eta \cap [0, r_n \max \phi_1]$, we have $u^{-1}(B) \cap \Omega' \subset \{x \in \Omega : \hat{G}(x, r_n\phi_1(x)) \geq \eta\, r_n^\beta\, \phi_1(x)^\beta\} := F_n$, and consequently, by Lemma 1,

$$m_N(F_n) \geq c \left(\frac{\gamma\, r_n \max\phi_1}{r_n\, L}\right)^N := k > 0$$

for n sufficiently large, where L is the Lipschitz constant of ϕ_1 . Now, by (16),

$$\lim \left(\int_{F_n} \frac{\hat{G}(x,r_n\phi_1(x))}{r_n^\beta\, \phi_1(x)^\beta} \phi_1(x)^\beta\, dx + \int_{\Omega\setminus F_n} \frac{\hat{G}(x,r_n\phi_1(x))}{r_n^\beta\, \phi_1(x)^\beta} \phi_1(x)^\beta\, dx \right) \leq 0,$$

and consequently, by Fatou's Lemma together with (a) and the definition of F_n ,

$$\eta \ \liminf \int_{F_n} \phi_1(x)^\beta \, dx \ \leq 0 \ .$$

This leads to a contradiction since $m_N(F_n) \geq k$ for every n .

Proof of Theorem 6. The proof goes as that of Theorem 5, except for showing that there exists $k > 0$ for which $m_N(F_n) \geq k$ for every n . Assuming without loss of generality that ω_+ is an open parallelepiped and setting $u = r_n\phi_1$, $B = E_\eta^+ \cap [r_n \inf_{\omega_+} \phi_1, r_n \sup_{\omega_+} \phi_1]$, we get here from Lemma 1 and Definition 1 with $\nu = \inf_{\omega_+} \phi_1 \, / \sup_{\omega_+} \phi_1$,

$$m_N(F_n) \geq c \ \Big(\frac{\gamma^+ \, r_n(\sup_{\omega_+} \phi_1 \ - \inf_{\omega_+} \phi_1)}{r_n \ L}\Big)^N \quad := k > 0$$

for a certain positive constant γ^+ and n sufficiently large.

Theorems 5 and 6 exhibit some kind of compromise between the growth condition on the nonlinearity and the density condition. This goes in the line of the main idea behind our abstract Theorem 1. The linear growth restriction on the nonlinearity imposed in [41] can be partially relaxed in this way, as illustrated by the following

Example 3. Take β as in Theorem 5 with in addition $1 < \beta < 2^*$ and define

$$G(x,u) = |u|^\beta \, (1 - \sin \log(1 + |u|)) + h(x) \, u$$

where, say, $h(x) \in L^{\infty}(\Omega)$. Direct computation shows that Theorem 5 applies (with $\hat{G} = G$).

To conclude, we show that Theorem 3 still holds when $p = 0$. In this case the functional f need not be coercive.

Theorem 7. *Assume that there exists an integrable function* $c(x)$ *such that*

$$\liminf_{|u|\to\infty} G(x,u) \geq c(x) \tag{17}$$

uniformly for a.e. $x \in \Omega$, *and*

$$\int_{\Omega} c(x)\, dx > 0 \ .$$

Then problem (P) has a solution.

While this paper was being completed, we learned of a slightly more general result by Ramos and Sanchez [131]. Their proof is based on the verification of the Palais - Smale condition at the level of the infimum of f. The proof below gives some insight of the geometry of the functional. It is based on the following simple lemma.

Lemma 2. *Let* H *be a reflexive Banach space and* $f : H \to \mathbb{R}$ *be a weakly lower semicontinuous and differentiable functional. Assume that there exists a* $R > 0$ *such that for every* u *with* $\|u\| = R$, *one has* $f(u) > f(0)$. *Then* f *has a critical point.*

Proof. The restriction of f to $B_R = \{u : \|u\| \leq R\}$ attains its minimum at some point $\bar{u} \in B_R$. By assumption, $\bar{u}$ must be in the interior of B_R, and is thus a local minimum for f on H, hence a critical point.

Proof of Theorem 7. Suppose by contradiction that f has no critical point. Then by Lemma 2 there exists a sequence (u_n) in H_0^1 such that $\|u_n\| = n$ and $f(u_n) \leq f(0)$. By (17), there exists $d \in L^1(\Omega)$ such that $G(x,u) \geq d(x)$ for a.e. x and all u. Since

$$f(0) \geq f(u_n) \geq \delta \, \|\tilde{u}_n\|^2 - \|d\|_{L^1} ,$$

$\|\tilde{u}_n\|$ is bounded. It follows that $\|\bar{u}_n\| \to \infty$ and that for a subsequence, $\tilde{u}_n(x) \to w(x)$ for a.e. x; this implies that $|u_n(x)| \to \infty$ a.e.. By Fatou's Lemma,

$$0 \geq \liminf \, (f(u_n) - f(0)) \geq \liminf \int_\Omega G(x,u_n(x)) \, dx$$

$$\geq \int_\Omega \liminf G(x,u_n(x)) \, dx \geq \int_\Omega c(x) \, dx > 0 ,$$

which gives the contradiction.

Remark 5. The uniformity in (17) is used in the proof above only to guarantee the existence of $d \in L^1(\Omega)$ such that $G(x,u) \geq d(x)$ for a.e. x and all u.

As an easy consequence of Theorem 7 and Remark 5, we have the following corollary which generalizes in several ways a result of Berger (cf. [21,104]).

Corollary 3. *Assume that there exist* $d \in L^1(\Omega)$ *such that*

$$G(x,u) \geq d(x)$$

for a.e. x *and all* u *, and a subset* Ω_0 *of* Ω *with positive measure such that for a.e.* $x \in \Omega_0$ *,*

$$G(x,u) \to +\infty \quad \text{as} \quad |u| \to \infty .$$

Then problem (P) has a solution.

Corollary 3 can also be derived from Theorem 1 with $\beta = 1$ (which shows that the corresponding functional is coercive). This is done by constructing a function $\hat{G}(x,u)$ with the following properties: $\hat{G}$ is Caratheodory, $\hat{G} \leq G$, $\hat{G}(x,u)$ is Lipschitz continuous in u with Lipschitz constant independent of x , $\hat{G}(x,u) \geq d(x)$ and, for a.e. $x \in \Omega_0$, $\hat{G}(x,u) \to +\infty$ as $|u| \to \infty$.

Chapter 5

Periodic Solutions of Second Order Ordinary Differential Equations at The First Two Eigenvalues

1. Introduction.

In this paper we study the existence of periodic solutions of second order differential equations of the form

$$- x''(t) = g(t,x(t)) \ ,$$

where the nonlinearity g interacts, roughly speaking, with the first two eigenvalues of the differential operator. More precisely, we impose at the first eigenvalue a resonance condition of the type introduced by Ahmad, Lazer and Paul [2], and a nonuniform nonresonance condition at the second eigenvalue.

We prove the existence of solutions using a variational approach. Our geometrical setting is the one of the Saddle Point Theorem of Rabinowitz [127]. It is obtained by splitting naturally the considered space into the eigenspace generated by the first eigenvalue and its orthogonal space.

The main technical problem lies in the verification of the Palais - Smale condition. This is done in the line of an argument introduced in [119] in order to find an a priori bound for the solutions of a Liénard type equation, and later developed in [57,74].

In order to obtain the Palais - Smale condition, we have to require $g(t,x)$ to be bounded below for x positive and bounded above for x negative.

Our theorem then gives a partial answer to a problem raised by Mawhin [102] . However, the problem of eliminating the boundedness assumptions on g remains still open.

2. THE MAIN RESULT.

We consider the following one dimensional periodic problem.

$$\begin{aligned} &x''(t) + g(t,x(t)) = 0 \\ &x(0) - x(T) = x'(0) - x'(T) = 0 \ . \end{aligned} \tag{1}$$

Being $T > 0$, we set $I = [0,T]$. The function $g : I \times \mathbb{R} \to \mathbb{R}$ is of Caratheodory type, i.e.

(a) $g(.,x)$ is measurable for every $x \in \mathbb{R}$,

(b) $g(t,.)$ is continuous for a.e. $t \in I$,

and

(c) for every $R > 0$ there exists a $k_R \in L^1(I)$ such that

$$|g(t,x)| \leq k_R(t)$$

for all $|x| \leq R$ and a.e. $t \in I$.

Statements concerning the variable t will always be intended to be true a.e..

We define the function G as follows:

$$G(t,x) = \int_0^x g(t,s)\ ds \ .$$

Theorem 1. *Assume the following conditions.*

(i) *There exists* $K \in L^1(I)$ *such that*

$$\begin{aligned} g(t,x) &\geq - K(t) \quad \text{for } x \geq 0 \ , \\ g(t,x) &\leq K(t) \quad \text{for } x \leq 0 \ . \end{aligned}$$

(ii) $\lim\limits_{|x|\to\infty} \int_I G(t,x)\, dt = +\infty$.

(iii) $\limsup\limits_{x\to\pm\infty} g(t,x)/x \leq \Gamma_\pm(t) \leq (2\pi/T)^2$,

where $\Gamma_+ , \Gamma_- \in L^1(I)$ *and the set*

$$S = \{t \in [0,T] : \Gamma_+(t) < (2\pi/T)^2 \text{ and } \Gamma_-(t) < (2\pi/T)^2\}$$

has positive measure.

Then problem (1) has a solution.

Remark 1. Condition (i) generalizes some assumptions which are frequently made in studying a problem like (1). It is in particular satisfied if one of the following holds.

(i)' $\exists\ \rho > 0 :\ g(t,x)x \geq 0 \qquad (|x| \geq \rho)$.

(i)" $\exists\ k \in L^1(I) :\ |g(t,x)| \leq k(t) \qquad (x \in \mathbb{R})$.

(i)''' $g(t,.)$ is increasing.

In the following, we will denote by H^1_T the Hilbert space of absolutely continuous functions $x : I \to \mathbb{R}$ such that $x(0) = x(T)$ and whose derivatives are square integrable, with the usual norm

$$\|x\| = \left\{ \int_I [|x(t)|^2 + |x'(t)|^2]\, dt \right\}^{1/2} .$$

Define the functional $f : H^1_T \to \mathbb{R}$, associated to problem (1), by

$$f(x) = \int_I [(1/2)|x'(t)|^2 - G(t,x(t))]\, dt \ .$$

It is well known that solving problem (1) is equivalent to finding a critical point of the functional f .

Remark 2. Condition (ii) imposes the functional f to be coercive on the space of constant functions, i.e. on the eigenspace associated to the first eigenvalue of the operator (-x") under periodic boundary conditions. An assumption of this type was first introduced by Ahmad, Lazer and Paul in [2] .

Condition (iii) means, roughly speaking, that the nonlinearity stays asymptotically below the second eigenvalue. Assumptions of this type are usually called "nonuniform nonresonance conditions", and were first introduced by Mawhin and Ward [99,108] .

Proof of Theorem 1. We will use the Saddle Point Theorem of Rabinowitz (cf. [129]). Let us write $H^1_T = H_0 \oplus H_1$, where H_0 is the space of constant functions, and H_1 is the space of functions having mean value zero. First of all, we will show that there exists an $R > 0$ such that, setting

$$S_0 = \{x \in H_0 : \|x\| = R\}$$

one has

$$\max_{S_0} f < \inf_{H_1} f \ . \tag{2}$$

Indeed, proceeding as in [99, Lemma 2.2], from (iii) it is possible to find a $\delta > 0$ such that, for every $x_1 \in H_1$,

$$\int_I [|x_1'(t)|^2 - \Gamma_+(t)|x_1^+(t)|^2 - \Gamma_-(t)|x_1^-(t)|^2]\, dt \geq 2\delta\|x_1\|^2 ,$$

where, as usual, $x^+ = \max\{x,0\}$ and $x^- = \max\{-x,0\}$.
Moreover, (iii) and (c) imply that there is a $K_\delta \in L^1(I)$ such that

$$G(t,x) \le (1/2)(\Gamma_+(t) + \delta)\, x^2 + K_\delta(t)$$

for every $x \ge 0$, and

$$G(t,x) \le (1/2)(\Gamma_-(t) + \delta)\, x^2 + K_\delta(t)$$

for every $x \le 0$.

Then, for $x_1 \in H_1$, we have

$$f(x_1) = \int_I (1/2)|x_1'(t)|^2 dt - \int_{\{x_1 \ge 0\}} G(t,x_1(t))\, dt - \int_{\{x_1 < 0\}} G(t,x_1(t))\, dt$$

$$\ge (1/2) \int_I \left[|x_1'(t)|^2 - (\Gamma_+(t) + \delta)|x_1^+(t)|^2 - (\Gamma_-(t) + \delta)|x_1^-(t)|^2 - 2K_\delta(t)\right] dt$$

$$\ge (1/2)\, [2\delta\, \|x_1\|^2 - \delta\, \|x_1\|_{L^2}^2] - \|K_\delta\|_{L^1}\ .$$

This implies

$$\inf_{H_1} f \ge - \|K_\delta\|_{L^1}\ .$$

On the other hand, if $x_0 \in H_0$,

$$f(x_0) = - \int_I G(t,x_0)\, dt\ .$$

By (ii), there exists an $R > 0$ such that, if $\|x_0\| = R$, then $f(x_0) \le - \|K_\delta\|_{L^1} - 1$. So (2) is satisfied.

To conclude the proof, we need to show that the functional f satisfies the Palais-Smale condition. By means of [129], it will be sufficient to show that, if (x_n) is a sequence in H^1_T such that $f(x_n)$ is bounded and $f'(x_n) \to 0$, then $\|x_n\|$ is bounded.

First of all we show that (x_n) is either uniformly bounded from above or from below. In fact, if this is not true, there must exist a subsequence, still denoted by (x_n), such that

$$m_n = \min x_n \to -\infty \tag{3}$$

and

$$M_n = \max x_n \to +\infty . \tag{4}$$

Then there surely exist α_n, β_n, γ_n and δ_n in $[0,2T]$ such that, extending by T-periodicity x_n over $[0,2T]$, one has

$$x_n(\alpha_n) = x_n(\beta_n) = 0$$
$$x_n(t) > 0 \quad \text{for } t \in \,]\alpha_n,\beta_n[$$
$$M_n = \max \{x_n(t) : t \in [\alpha_n,\beta_n]\}$$

and

$$x_n(\gamma_n) = x_n(\delta_n) = 0$$
$$x_n(t) < 0 \quad \text{for } t \in \,]\gamma_n,\delta_n[$$
$$m_n = \min \{x_n(t) : t \in [\gamma_n,\delta_n]\}.$$

Using a technical lemma proved in [74], we can say that, for a further subsequence (x_n), either there is a $\rho_- > 0$ such that

$$\int_{\gamma_n}^{\delta_n} [|x_n'(t)|^2 - \Gamma_-(t)|x_n(t)|^2]\, dt \geq (\rho_-/2) \int_{\gamma_n}^{\delta_n} |x_n'(t)|^2\, dt \tag{5}$$

or there is a $\rho_+ > 0$ such that

$$\int_{\alpha_n}^{\beta_n} [|x_n'(t)|^2 - \Gamma_+(t)|x_n(t)|^2]\, dt \geq (\rho_+/2) \int_{\alpha_n}^{\beta_n} |x_n'(t)|^2\, dt \tag{6}$$

for n sufficiently large.

Assume for example that (6) holds. Define the function y_n as follows:

$$\begin{aligned} y_n(t) &= x_n(t) \quad \text{when } t \in [\alpha_n, \beta_n] \\ &= 0 \qquad \text{otherwise}. \end{aligned}$$

It is possible to see that $y_n \in H_T^1$ (cf. [77]).
Since $f'(x_n) \to 0$, for some constant $C_1 > 0$ we have that

$$|\langle f'(x_n), v\rangle| \leq C_1 \|v\| \tag{7}$$

for every $v \in H_T^1$. Fix $\varepsilon < \rho_+\pi^2/2T^2$. From (iii) and (c), there exists a $\tilde{k}_\varepsilon \in L^1(I)$ such that

$$g(t,x)\, x \leq (\Gamma_+(t) + \varepsilon)\, x^2 + \tilde{k}_\varepsilon(t)$$

for all $x \geq 0$ and a.e. $t \in I$. Hence, by (7), the definition of y_n, and (6), one has

$$\begin{aligned} C_1 \|y_n\| &\geq \langle f'(x_n), y_n\rangle \\ &= \int_{\alpha_n}^{\beta_n} [|x_n'(t)|^2 - g(t,x_n(t))x_n(t)]\, dt \\ &\geq \int_{\alpha_n}^{\beta_n} [|x_n'(t)|^2 - (\Gamma_+(t) + \varepsilon)\, |x_n(t)|^2 - \tilde{k}_\varepsilon(t)]\, dt \end{aligned}$$

$$\geq (\rho_+/2) \int_{\alpha_n}^{\beta_n} |x_n'(t)|^2\, dt - \varepsilon \left(\frac{\beta_n - \alpha_n}{\pi}\right)^2 \int_{\alpha_n}^{\beta_n} |x_n'(t)|^2\, dt - \|\tilde{k}_\varepsilon\|_{L^1}$$

$$\geq C_2 \int_{\alpha_n}^{\beta_n} |x_n'(t)|^2\, dt - \|\tilde{k}_\varepsilon\|_{L^1} ,$$

where C_2 is a positive constant. It follows from the Poincaré inequality that

$$\int_{\alpha_n}^{\beta_n} |x_n'(t)|^2\, dt \leq C_3 ,$$

for a certain constant C_3. But this implies

$$M_n \leq (T\, C_3)^{1/2} ,$$

which is in contradiction with (4). Analogously, if (5) holds, we get a contradiction with (3).

We then proved that (x_n) is uniformly bounded either from above or from below. Let us suppose for example that (x_n) is uniformly bounded from above by a constant M. Taking $v \equiv 1$ in (7) we get

$$\left| \int_I g(t,x_n(t))\, dt \right| \leq C_1 \sqrt{T} . \tag{8}$$

On the other hand, since $x_n(t) \leq M$ for every n and a.e. $t \in I$, one has, by (i), the compactness of $[0,M]$ and (c), that

$$\int_{\{g\geq 0\}} g(t,x_n(t))\,dt = \int_{\{g\geq 0\}\cap\{0\leq x_n\leq M\}} g + \int_{\{g\geq 0\}\cap\{x_n<0\}} g \leq C_4 . \tag{9}$$

Combining (8) and (9) one easily gets

$$\int_I |g(t,x_n(t))|\,dt \leq C_5 . \tag{10}$$

The same conclusion (10) can be obtained when supposing (x_n) uniformly bounded from below.

Writing $x_n = x_0^n + x_1^n$, with $x_0^n \in H_0$ and $x_1^n \in H_1$, and recalling the Sobolev inequality

$$\|x_1^n\|_{L^\infty} \leq \sqrt{T/12}\ \|x_n'\|_{L^2} ,$$

from (7) and (10) we get

$$\begin{aligned} C_1\|x_1^n\| &\geq \langle f'(x_n), x_1^n\rangle \\ &= \int_I [|x_n'(t)|^2 - g(t,x_n(t))\,x_1^n(t)]\,dt \\ &\geq \|x_n'\|_{L^2}^2 - C_5\sqrt{T/12}\,\|x_n'\|_{L^2} . \end{aligned}$$

From the Wirtinger inequality we can conclude that

$$\|x_n'\|_{L^2} \leq C_6 . \tag{11}$$

Suppose by contradiction that $\|x_n\|$ is not bounded. By (11), this implies that there is a subsequence, still denoted (x_n), such that either

$$M_n = \max x_n \to -\infty \tag{12}$$

or

$$m_n = \min x_n \to +\infty \tag{13}$$

In case (12) holds, by the definition of G, hypotheses (i) and (11), for n large we have:

$$\int_I G(t,x_n(t))\,dt = \int_I [G(t,M_n) + \int_{M_n}^{x_n(t)} g(t,s)\,ds]\,dt$$

$$\geq \int_I G(t,M_n)\,dt - \int_I (M_n - x_n(t))K(t)\,dt$$

$$\geq \int_I G(t,M_n)\,dt - C_7, \tag{14}$$

which tends to $+\infty$ by hypotheses (ii).

Recall now that $f(x_n)$ is supposed to be bounded, i.e. there is a constant C_8 such that

$$\left| \int_I [(1/2)|x_n'(t)|^2 - G(t,x_n(t))]\,dt \right| \leq C_8 .$$

This is in contradiction with (11) and (14). An analogous contradiction is obtained when (13) holds. The Theorem of Rabinowitz can thus be applied, to achieve the proof.

Chapter 6

MULTIPLE PERIODIC SOLUTIONS OF CONSERVATIVE SYSTEMS WITH PERIODIC NONLINEARITY

1. INTRODUCTION

The first aim of this paper is to extend the results obtained by Mawhin [107] concerning the multiplicity of periodic solutions for a second order system of the form

$$(M(t)u')' + Au + D_uF(t,u) = h(t), \qquad (1.1)$$

where M(t) and A are symmetric matrices, M(t) being positive definite, F and D_uF are bounded and satisfy a periodicity condition along the directions of the null-space of A, and h belongs to a suitable subspace of L^1.

The special case $M(t) \equiv Id$ and $A = 0$ has been studied in [112]. The existence of two distinct solutions was proved, generalizing a result of [111] for the pendulum equation (see also [103], [113]).

In [107], Mawhin proved,whenever A is semi-negative definite, the existence of (m + 1) distinct periodic solutions of (1), where m is the dimension of the null-space of A. His result unifies and completes previous existence theorems for the satellite equation (cf. [106]), the linearly coupled pendulum (cf. [94]) and the Josephson multipoint system (cf. [84]). See also [46].

Here we will prove the existence of at least (m + 1) distinct periodic solutions of (1), without requiring A to be semi-negative definite. As already shown in [107], if all the solutions are non-degenerate, then there are at least 2^m of them. The proof consists in applying an abstract theorem of Chang [27], which is based on algebraic topology methods and Morse theory.

Our second objective is to give a multiplicity result for the periodic solutions of a Hamiltonian system of the form

$$J\dot{u} + Au + D_u H(t,u) = h(t), \tag{1.2}$$

where J is the standard symplectic matrix, A is a symmetric matrix, H, $D_u H$ and $D_{uu} H$ are bounded and satisfy a periodicity condition along the directions of the null-space of A, and h belongs to a suitable subspace of L^2.

The situation differs from the above one for the fact that the functional associated to (1.2) is strongly indefinite. A finite dimensional reduction will be used in order to overpass this difficulty.

The special case A = 0 has been studied by Conley and Zehnder [34, 35] and Chang [27]. Here we extend their results to the case $A \neq 0$, and prove a theorem which is the perfect analogous to the one we have for system (1.1). When the paper was written, we have received a preprint of Chang [29] which contains very close results.

The paper is organized as follows. In section 2 we recall some concepts of algebraic topology, two deformation lemmas and some results of Morse Theory.

In section 3 we present two abstract existence theorems by Chang [27,28]. The proofs are also carried out for the reader's convenience.

In sections 4 and 5 we apply the abstract results of section 3 to prove multiplicity results for periodic solutions of (1.1) and (1.2), respectively.

2. SOME PRELIMINARIES

Given a topological space X and a subspace $A \subset X$, we denote by $H_n(X,A)$ $[H^n(X,A)]$ the n^{th} singular homology [cohomology] vector space of X relative to A, with respect to a given field (e.g. $\mathbb{R}$).

We recall that the elements of $H_n(X,A)$ are equivalence classes of singular chains having zero boundary. These elements are invariant under every continuous deformation $\tau : X \to X$ such that $\tau_{|A} = id_A$, in the sense that τ induces an isomorphism τ_* of $H_n(X,A)$ into itself, and we identify each element of $H_n(X,A)$ with its image under τ_*. The analogous is true for cohomologies, as well.

We state the following two properties of homologies, which hold for cohomologies as well.

(a) If X' is a strong deformation retract of X and $A' \subset X'$ is a strong deformation retract of A, then $H_n(X,A) \approx H_n(X',A')$.

(b) (Künneth formula) For any topological space Y,

$$H_n(X \times Y, A \times Y) \approx \oplus_{p+q=n} [H_p(X,A) \otimes H_q(Y)].$$

Let us now briefly recall the concepts of cup and cap products. Suppose A and B are subspaces of X such that, for example, one of the following three situations is true :

$$A = \phi, \quad A = B, \qquad B = \phi.$$

Then there exist an operation, called the "cup product"

$$\begin{gathered} H^n(X,B) \times H^m(X,A) \to H^{n+m}(X,A \cup B) \\ (\omega,\rho) \mapsto \omega \cup \rho \end{gathered} \tag{2.1}$$

and an operation, called the "cap product"

$$H_{n+m}(X, A \cup B) \times H^n(X,B) \to H_m(X,A) \qquad (2.2)$$
$$(z,\omega) \mapsto z \cap \omega,$$

which are bilinear and invariant under continuous deformations leaving $A \cup B$ fixed (in the sense we saw above).

The cup and cap products are naturally induced by two operators, defined on the corresponding spaces of singular chains and cochains, which are denoted by the same symbols $\cup$ and $\cap$.

If $z \in H_{n+m}(X, A \cup B)$, $\omega \in H^n(X,B)$ and $\rho \in H^m(X,A)$, then for all $\eta \in z$, $c \in \omega$ and $d \in \rho$ one has

$$\langle \eta, c \cup d \rangle = \langle \eta \cap c, d \rangle, \qquad (2.3)$$

where $\langle .,. \rangle$ denotes the pairing between singular chains and cochains. Moreover, denoting by $|\eta|$ the support of the chain η, one has

$$|\eta \cap c| \subset |\eta|. \qquad (2.4)$$

We now consider a Riemannian manifold $\mathcal{M}$ of class C^2 and a C^1 functional $f : \mathcal{M} \to \mathbb{R}$. We will use the following notations.

$$f^a = \{x \in \mathcal{M} : f(x) \leq a\}$$
$$K_c = \{x \in \mathcal{M} : f(x) = c,\ df(x) = 0\}.$$

That is, K_c is the set of critical points at the level c.

The Palais-Smale condition, in short (PS), plays a funda-mental role in the following deformation lemmas. Recall that (PS) holds iff every sequence (x_n) in $\mathcal{M}$ such that $f(x_n)$ is bounded and $df(x_n) \to 0$ possesses a convergent subsequence.

Assume that at every point x of the boundary of $\mathcal{M}$, df(x) points inwards in $\mathcal{M}$. Then the following lemmas hold.

First Deformation Lemma. *Assume* (PS) *holds for* f. *Fix* $c \in \mathbb{R}$ *and let* N *be a closed neighborhood of* K_c. *Then there is a continuous map* $\tau : [0,1] \times \mathcal{M} \to \mathcal{M}$, *as well as numbers* $\bar{\varepsilon} > \varepsilon > 0$ *such that*

(1) $\eta(t,.)|_{\mathcal{M}\setminus f^{-1}([c-\bar{\varepsilon},\ c+\bar{\varepsilon}])} = \mathrm{id}$,

(2) $\eta(0,.) = \mathrm{id}$,

(3) $\eta(1, f^{c+\varepsilon}\setminus N) \subset f^{c-\varepsilon}$,

(4) $\forall\, t \in [0,1]$, $\eta(t,.)$ *is a homeomorphism.*

Second Deformation Lemma. *Assume* (PS) *holds for* f *and that* df *is locally Lipshitzian. If there are no critical values in the open interval* (a, b), *then* f^a *is a strong deformation retract of* $(f^b \setminus K_b)$.

For the proofs, cf. [113], [31], [123], [27]. Since $\mathcal{M}$ is a Riemannian manifold, we can consider the flow determined by the gradient of f. The deformations are then constructed along this flow.

Let a < b, and suppose that $x_1, \ldots, x_j$ are the only critical points of f in $f^{-1}([a,b])$. Let $C_n(f, x_i)$ denote the n^{th} critical group of x_i, and suppose that all these critical groups (which are vector spaces) are finite dimensional, and that they are trivial for n sufficiently large. Then we can define the Morse polynomial

$$M(t,f^b,f^a) = \sum_{n=0}^{\infty} \Big(\sum_{i=1}^{j} \dim C_n(f,x_i)\Big)\, t^n .$$

Moreover, the Poincaré polynomial

$$P(t,f^b,f^a) = \sum_{n=0}^{\infty} \dim H_n(f^b,f^a)t^n$$

is also well defined and we have that

$$M(t,f^b,f^a) = P(t,f^b,f^a) + (1+t)\, Q(t), \qquad (2.5)$$

where Q is a polynomial with nonnegative integer coefficients (cf. [113]).

As a consequence of (2.5), if for every $i \in \{1, \ldots, j\}$ we have that $\sum_{n=0}^{\infty} \dim C_n(f,x_i) \leq 1$, then

$$j \geq \sum_{n=0}^{\infty} \dim H_n(f^b,f^a). \tag{2.6}$$

This is the case if all the critical points $x_1, \ldots, x_j$ are nondegenerate. Indeed, in that case one has

$$C_n(f,x_i) = \delta_{n,m_i} \mathbb{R}$$

where m_i is the Morse index of the point x_i.

3. AN ABSTRACT MULTIPLICITY RESULT

In this section we expose some results of Chang [27].

Definition. *Let* X *be a topological space and* $A \subset X$. *Consider two non-zero singular homology classes*

$$z_1 \in H_m(X,A), \quad z_2 \in H_{n+m}(X,A).$$

We set $z_1 \prec z_2$ *whenever* $n > 0$ *and there exists* $\omega \in H^n(X)$ *such that*

$$z_1 = z_2 \cap \omega$$

(the cap product is as in (2.2), *with* $B = \phi$*).*

Theorem 1. *Assume* (PS) *holds for* f. *Let* $a < b$ *be two real numbers such that* f *has only a finite number of isolated critical points in* $f^{-1}[a,b]$. *If there are k non-zero singular homology classes* $z_1 \in H_{n_1}(f^b,f^a), \ldots, z_k \in H_{n_k}(f^b,f^a)$ *with* $z_1 \prec z_2 \prec \ldots \prec z_k$, *then* f *has at least* k *distinct critical values.*

Proof. Define the following quantities :

$$c_i = \inf_{\eta \in z_i} \sup_{x \in |\eta|} f(x) \qquad (i = 1, 2, \ldots, k).$$

The Minimax Principle tells us that whenever c_i is finite and the family of sets z_i is invariant under homeomorphisms, c_i is a critical value of f. Since $z_i \in H_{n_i}(f^b, f^a)$, for all $\eta \in z_i$ we have that $|\eta| \subset f^b$. Hence $c_i \le b$. Moreover, since z_i is supposed to be non-zero in $H_{n_i}(f^b, f^a)$, this means that for any $\eta \in z_i$ we have that $|\eta|$ has non empty intersection with $f^b \setminus f^a$. Hence $c_i \ge a$. So c_i is finite; z_i is invariant under homeomorphisms, as we said in section 2.

So the c_i's are critical values of f. We want to prove now that we have

$$c_1 < c_2 < \ldots < c_k.$$

Let us concentrate on c_1 and c_2, the reasoning being the same for the others. We know, since $z_1 \prec z_2$ that there exists $\omega \in H^{n_2 - n_1}(f^b)$, $n_2 - n_1 > 0$ such that $z_1 = z_2 \cap \omega$. This means that for any $\eta_2 \in z_2$ and any $c \in \omega$ we can define $\eta_1 = \eta_2 \cap c \in z_1$, and by (2.4) we have that $|\eta_1| \subset |\eta_2|$. Hence

$$\forall \eta_2 \in z_2 \, \exists \eta_1 \in z_1 : \sup_{x \in |\eta_1|} f(x) \le \sup_{x \in |\eta_2|} f(x).$$

This immediately implies $c_1 \le c_2$.

Suppose by contradiction that $c_1 = c_2$, and denote by c this common value. Then for every $\varepsilon > 0$ there must exist a $\eta_2 \in z_2$ such that $|\eta_2| \subset f^{c+\varepsilon}$. Since K_c is a set of isolated critical points, we choose two contractible neighborhoods of K_c in $f^b \setminus f^a$: $N \subset N'$. We can then write $\eta_2 = \eta_2' + \eta_2''$, where $|\eta_2'| \subset N'$ and $|\eta_2''| \subset f^{c+\varepsilon} \setminus N$.

We can consider η_2' as a cycle of f^b relative to $f^b \setminus N$. Hence $[\eta_2'] \in H_{n_2}(f^b, f^b \setminus N)$ and $[\eta_2'] \cap \omega \in H_{n_1}(f^b, f^b \setminus N)$. The cap product does not change if we shrink N', and hence η_2', to a point. Since $\omega \in$

$H^{n_2-n_1}(f^b)$ and $n_2-n_1 > 0$, there exists $c \in \omega$ which, applied to any chain having support in N', gives 0. In particular, by the definition of the cap product, $\eta_2' \cap c = 0$. Set $\eta_1 = \eta_2 \cap c \in z_1$; then

$$\eta_1 = \eta_2 \cap c = \eta_2' \cap c + \eta_2'' \cap c = \eta_2'' \cap c.$$

Hence $|\eta_1| \subset |\eta_2'' \cap c| \subset |\eta_2''| \subset f^{c+\varepsilon} \setminus N$.
Consider now the homeomorphism $\tau : f^{c+\varepsilon} \setminus N \to f^{c-\varepsilon}$ given by the First Deformation Lemma. We have that $\tau(|\eta_1|) \subset f^{c-\varepsilon}$. But $\tau \circ \eta_1 \in z_1$ because of the invariance under homeomorphisms, and this contradicts the definition of c_1 (= c).

Given a compact manifold $\mathcal{V}$, let us define cuplength ($\mathcal{V}$) as the greatest natural number ℓ such that

$$(\forall\, i \in \{1, \ldots, \ell\})\ (\exists\, k_i > 0)\ (\exists\, \omega_i \in H^{k_i}(\mathcal{V})) : \omega_1 \cup \ldots \cup \omega_\ell \neq 0.$$

It can be seen that for a compact manifold such a number indeed exists.

Theorem 2. *Let* L *be a bounded self-adjoint operator with a bounded inverse, defined on a Hilbert space* H. *Suppose that the negative space determined by* L *is finite dimensional. Let* $\mathcal{V}$ *be a* C^2*-compact manifold without boundary. Let* $g \in C^2(H \times \mathcal{V}, \mathbb{R})$ *be a function having bounded and compact differential* dg. *Then the function*

$$f(x, v) = \frac{1}{2}(Lx, x) + g(x, v)$$

has at least [cuplength ($\mathcal{V}$) + 1] *critical points.*

Moreover, if all the critical points of f *are non degenerate, there are at least* $\sum_{n=0}^{\infty} \dim H_n(\mathcal{V})$ *of them.*

<u>Proof</u>. 1) In order to apply the previous theorem, let us verify that (PS) holds for f on H x $\mathcal{V}$. Take a sequence (x_n, v_n) in H x $\mathcal{V}$ such that $(f(x_n, v_n))$ is bounded and

$$df(x_n,v_n) = Lx_n + dg(x_n,v_n) \to 0. \tag{3.1}$$

Since $(dg(x_n,v_n))$ is bounded by hypothesis, it follows that (Lx_n) is also bounded. We deduce that, L having a bounded inverse, (x_n) is bounded, and since $\mathcal{V}$ is compact, the sequence (x_n,v_n) is bounded. Since dg is supposed to be compact and $\mathcal{V}$ is compact, there exists a subsequence (x_{n_k},v_{n_k}) such that $(dg(x_{n_k},v_{n_k}))$ and (v_{n_k}) are convergent. From (3.1) and the boundedness of L^{-1}, it then follows that (x_{n_k},v_{n_k}) itself is convergent.

2) Let $H = H^+ \oplus H^-$, where H^+ and H^- are the invariant subspaces corresponding to the positive and negative spectrum of L, respectively. Accordingly, every $x \in H$ can be written as $x = x^+ + x^-$, with $x^+ \in H^+$ and $x^- \in H^-$. Let $\gamma = \dim H^-$. Set $\varepsilon_\pm = \inf \{\|Lx_\pm\| : \|x_\pm\| = 1\}$. These are positive numbers. If $\bar{m}$ is such that $\|dg(x,v)\| \leq \bar{m}$ $\forall$ $(x,v) \in H \times \mathcal{V}$, set $R_+ = (\bar{m} + 1) / \varepsilon_+$. From now on it will be convenient to work on the manifold $\mathcal{M} = (H^+ \cap B_{R_+}) \times H^- \times \mathcal{V}$.

In order to be sure that the Deformation Lemmas can be used, we have to check that -df points inward to $\mathcal{M}$ on each point of the boundary $\partial\mathcal{M}$, i.e. for every (x, v) such that $\|x^+\| = R_+$. Indeed in such a case we have

$$\begin{aligned}(-df(x,v), x^+) &= - (Lx^+,x^+) - (dg(x,v), x^+)\\ &\leq - \varepsilon_+ \|x^+\|^2 + \bar{m} \|x^+\|\\ &= - R_+ (\varepsilon_+R_+ - \bar{m}) = - R_+ < 0.\end{aligned}$$

Since (on $\mathcal{M}$)

$$f(x,v) \leq \frac{1}{2}\, \|L\|\, R_+^2 - \frac{1}{2}\, \varepsilon_- \|x^-\|^2 + \bar{m}\, (R_+ + \|x^-\| + \|v\|)$$

and since $\mathcal{V}$ is compact, we have that

$$f(x, v) \to -\infty \quad \text{as} \quad \|x^-\| \to \infty \quad \text{uniformly in } x^+ \text{ and } v. \qquad (3.2)$$

It is not restrictive to suppose there exist only a finite number of critical points, which are isolated and contained in $f^b \setminus f^a$, for certain fixed $a < b$. From (3.2) it follows that there exists an $R_1 > 0$ such that

$$(H^+ \cap B_{R_+}) \times (H^- \setminus B_{R_1}) \times \mathcal{V} \subset f^a.$$

Since

$$f(x,v) \geq \frac{1}{2}\, \varepsilon_+ R_+^2 - \frac{1}{2}\, \|L\|\, \|x^-\|^2 - \bar{m}(R_+ + \|x^-\| + \|v\|)$$

this implies f is bounded below on $(H^+ \cap B_{R_+}) \times (H^- \cap B_{R_1}) \times \mathcal{V}$, and hence there exists an $a' < a$ such that

$$f^{a'} \subset (H^+ \cap B_{R_+}) \times (H^- \setminus B_{R_1}) \times \mathcal{V}$$

and, again, there exists an $R_2 > R_1$ such that

$$(H^+ \cap B_{R_+}) \times (H^- \setminus B_{R_2}) \times \mathcal{V} \subset f^{a'}.$$

The Second Deformation Lemma gives us a strong deformation retraction $\tau_1 : f^a \to f^{a'}$. Moreover a strong deformation retraction

$$\tau_2 : (H^+ \cap B_{R_+}) \times (H^- \setminus B_{R_1}) \times \mathcal{V} \to (H^+ \cap B_{R_-}) \times (H^- \setminus B_{R_2}) \times \mathcal{V}$$

can be constructed by hand (see [6]). So $\tau = \tau_2 \circ \tau_1$ is a strong deformation retraction from f^a to $(H^+ \cap B_{R_+}) \times (H^- \setminus B_{R_2}) \times \mathcal{V}$, and we have

$$\begin{aligned}
H_n(f^b,f^a) &\approx H_n(\mathcal{M}, f^a) && \text{(deformation)}\\
&\approx H_n(\mathcal{M}, (H^+ \cap B_{R_+}) \times (H^- \setminus B_{R_2}) \times \mathcal{V}) && \text{(deformation)}\\
&\approx H_n\,(H^- \times \mathcal{V}, (H^- \setminus B_{R_2}) \times \mathcal{V}) && \text{(Künneth)}\\
&\approx \oplus_{p+q=n}\,[H_p\,(H^-, H^- \setminus B_{R_2}) \otimes H_q(\mathcal{V})] && \text{(Künneth)}\\
&\approx \oplus_{p+q=n}\,[H_p(B_{R_2}, \partial B_{R_2}) \otimes H_q(\mathcal{V})] && \text{(deformation)}\\
&\approx H_{n-\gamma}(\mathcal{V}).
\end{aligned}$$

Moreover,

$$\begin{aligned}
H^n(f^b) &\approx H^n(\mathcal{M})\\
&\approx H^n((H^+ \cap B_{R_+}) \times H^- \times \mathcal{V})\\
&\approx \oplus_{p+q=n}\,[H^p((H^+ \cap B_{R_+}) \times H^-) \otimes H^q(\mathcal{V})]\\
&\approx H^n(\mathcal{V}).
\end{aligned}$$

Let ℓ = cuplength $(\mathcal{V})$. Then, according to the above isomorphisms, for every $i \in \{1, \ldots, \ell\}$ there exists $\omega_i \in H^{k_i}(f^b)$, $k_i > 0$ such that $\omega_1 \cup \ldots \cup \omega_\ell \neq 0$. Hence there exists $z_1 \in H_{k_1+\ldots+k_\ell+\gamma}(f^b,f^a)$ such that

$$(\, z_1, \omega_1 \cup \ldots \cup \omega_\ell) \neq 0.$$

Then we can define recursively $z_{j+1} \in H_{k_{j+1}+\ldots+k_\ell+\gamma}(f^b,f^a)$ by

$$z_{j+1} = z_j \cap \omega_j,$$

$j = 1, \ldots, \ell$. We thus obtain $(\ell+1)$ non-zero homology classes such that $z_{\ell+1} \prec z_\ell \prec \ldots \prec z_1$. Theorem 1 then proves the first part of the theorem. As for the second part, it is immediate from (2.6) and the above isomorphisms.

4. PERIODIC SOLUTIONS OF SECOND ORDER SYSTEMS

In this section we will apply the abstract multiplicity results of section 3 to the following second order system.

$$\begin{aligned} &(M(t)u')' + Au + D_uF(t,u) = h(t) \\ &u(0) - u(T) = u'(0) - u'(T) = 0 \end{aligned} \tag{4.1}$$

We assume $T > 0$, $F = [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is continuous, bounded and such that

(i) $F(t + T,u) = F(t,u)$ for all $(t,u) \in [0,T] \times \mathbb{R}^n$.

(ii) D_uF exists, is continuous and bounded, and $D_{uu}F$ exists and is continuous.

Moreover, $S(\mathbb{R}^n,\mathbb{R}^n)$ being the space of symmetric real $(n \times n)$-matrices, we have $A \in S(\mathbb{R}^n,\mathbb{R}^n)$, $M : [0,T] \to S(\mathbb{R}^n,\mathbb{R}^n)$ is continuous and such that, for some $\mu > 0$ and all $(t,v) \in [0,T] \times \mathbb{R}^n$,

$$(M(t)v \mid v) \geq \mu \, |v|^2, \tag{4.2}$$

and $h \in L^1(0,T; \mathbb{R}^n)$.

Using Schauder's fixed point theorem, one can prove the existence of at least one solution to (4.1) whenever the "linearized" problem

$$\begin{aligned} &(M(t)u')' + Au = 0 \\ &u(0) - u(T) = u'(0) - u'(T) = 0 \end{aligned} \tag{4.3}$$

has only the zero solution.

We will consider the situation described by the following assumptions, where $N(A)$ denotes the null-space of A.

(A1) $N(A) = \text{span}\,\{a_1, \ldots, a_m\}$, $1 \leq m \leq n$, *and problem* (4.3) *has as solutions only the elements of* $N(A)$.

(A2) *For every* $v \in N(A)$,

$$\int_0^T (h(t) \mid v)\, dt = 0.$$

(A3) *There are positive numbers* $T_1, \ldots, T_m$ *such that*

$$F(t, u + T_j a_j) = F(t,u)$$

for every $(t,u) \in [0,T] \times \mathbb{R}^n$ *and* $j \in \{1, \ldots, m\}$.

Theorem 3. *Under the above assumptions, problem* (4.1) *has at least* $(m + 1)$ *geometrically distinct solutions.*

If moreover all the solutions of (4.1) *are nondegenerate, then there are at least* 2^m *of them.*

Remark. Theorem 3 generalizes previous results by Mawhin [107] where A was supposed to be semi-negative definite.

Proof. Let us consider the Hilbert space

$$H^1_T = \{u \in H^1(0,T; \mathbb{R}^n) : u(0) = u(T)\}$$

equipped with the inner product

$$\langle u \mid v \rangle = \int_0^T [(M(t)u'(t) \mid v'(t)) + (u(t) \mid v(t))]\, dt.$$

The corresponding norm $\|u\| = \langle u \mid u \rangle^{1/2}$ is by (4.2) equivalent to the classical norm of $H^1(0,T; \mathbb{R}^n)$.

Let us define the operator $L : H^1_T \to H^1_T$ such that

$$\langle Lu \mid v\rangle = \int_0^T [(M(t)u'(t) \mid v'(t)) - (Au(t) \mid v(t))]\, dt.$$

It can easily be seen that L is a self-adjoint operator on H^1_T. Because of the compact imbedding of H^1_T into $C([0,T], \mathbb{R}^n)$, $(I - L)^{-1}$ is compact. This implies that, writing $H^1_T = H^- \oplus H^\circ \oplus H^+$, where H^-, H° and H^+ are the invariant subspaces corresponding to the negative, zero and positive spectrum of L, respectively, the space H^- is finite dimensional. Moreover, by (A1), $H^\circ = N(A)$.

Let us consider the space $H = H^- \oplus H^+$. Then L can be considered as a bounded self-adjoint operator on H with a bounded inverse.

Let $T^m = \mathbb{R}^m/\mathbb{Z}^m$ be the m-fold torus, and define on $H \times T^m$ the following functional

$$g(u,(v_1, \ldots, v_m)) = \int_0^T (-F(t, u(t) + \sum_{i=1}^m v_i a_i) + (h(t) \mid u(t))]\, dt.$$

By (ii), g is of class C^2. Because of (A2), (A3) and classical arguments, the critical points of the C^2-functional f defined by

$$f(u,v) = \frac{1}{2}(Lu,u) + g(u,v)$$

correspond to geometrically distinct solutions of (4.1). It is easy to see that dg is bounded and compact, because of the compact imbedding of H^1_T into the space of continuous functions. So all the assumptions of Theorem 2 are satisfied, and the result follows from the following well known facts:

$$\text{cuplength}\,(T^m) = m + 1 \qquad (4.4)$$

$$\dim H_n\,(T^m) = \binom{m}{n}. \qquad (4.5)$$

5. PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS

In this section we will give a multiplicity result for the following Hamiltonian system

$$\begin{aligned} J\dot{u} + Au + D_uH(t,u) &= h(t) \\ u(0) &= u(T) \end{aligned} \tag{5.1}$$

We assume $T > 0$, $H = [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}$ is continuous, bounded and such that

(i) $H(t + T,u) = H(t,u)$ for all $(t,u) \in [0,T] \times \mathbb{R}^{2n}$;

(ii) D_uH and $D_{uu}H$ exist, they are continuous and bounded.

Moreover, A is a symmetric real $(2n \times 2n)$-matrix with null-space $N(A)$, $h \in L^2(0,T; \mathbb{R}^{2n})$ and $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ is the standard symplectic matrix.

It is not difficult to see, by Schauder's fixed point theorem, that if the "linearized" problem

$$\begin{aligned} J\dot{u} + Au &= 0 \\ u(0) &= u(T) \end{aligned} \tag{5.2}$$

has only the zero solution, then (5.1) has at least one solution.

We will consider the situation described by the following assumptions.

(A1) $N(A) = \text{span}\ \{a_1, \ldots, a_m\}$, $1 \leq m \leq 2n$, *and problem* (5.2) *has as solutions only the elements of* $N(A)$.

(A2) *For every* $v \in N(A)$,

$$\int_0^T (h(t) \mid v)\, dt = 0.$$

(A3) *There are positive numbers* $T_1, \ldots, T_m$ *such that*

$$H(t, u + T_j a_j) = H(t,u)$$

for every $(t,u) \in [0,T] \times \mathbb{R}^{2n}$ *and* $j \in \{1, \ldots, m\}$.

Theorem 4. *Under the above assumptions, problem* (5.1) *has at least* $(m + 1)$ *geometrically distinct solutions.*

If moreover all the solutions of (5.1) *are nondegenerate, then there are at least* 2^m *of them.*

Remark. Theorem 4 generalizes previous results obtained by Conley and Zehnder [34,35] and Chang [27] . They all consider the case $A = 0$ and hence $N(A) = \mathbb{R}^{2n}$. In [34] and [27] a conjecture of Arnold was proved (cf. [13], [14]).

Proof. We define a self-adjoint operator L on the Hilbert space $X = L^2(0,T; \mathbb{R}^{2n})$:

$$D(L) = \{u \in H^1(0,T; \mathbb{R}^{2n}) : u(0) = u(T)\}$$
$$Lu = J\dot{u} + Au.$$

It is well known that L is self-adjoint, has closed range and a discrete spectrum $\sigma(L) = \{\ldots < \lambda_{-1} < \lambda_0 < \lambda_1 < \ldots\}$ unbounded from below and from above, made of eigenvalues of finite multiplicity which do not accumulate at any finite point.

We define the operator $N : X \to X$ by

$$(Nu)(t) = - D_u H(t,u(t)) + h(t).$$

By (ii), N is Lipshitz continuous and possesses a symmetric Gateaux derivative given by

$$[N'(u)\eta](t) = - D_{uu} H(t,u(t))\eta(t) \tag{5.3}$$

for every $\eta \in X$.

Let α be the Lipshitz constant of N. By the properties of $\sigma(L)$ we can choose $\alpha' \geq \alpha$ and $\varepsilon > 0$ such that $[-(\alpha' + \varepsilon), -\alpha'] \cap \sigma(L)$ and $[\alpha', \alpha' + \varepsilon] \cap \sigma(L)$ are both empty.

By the above definitions of the operators L and N, it is clear that problem (5.1) is equivalent to the equation

$$Lu = Nu. \tag{5.4}$$

In order to be able to apply the abstract results of section 3 we need a reduction to a finite dimensional equation. To this aim, let us consider $\{E_\lambda : \lambda \in \mathbb{R}\}$, the spectral resolution of L, and define the following orthogonal projector in X :

$$P = \int_{-(\alpha'+\varepsilon)}^{\alpha'+\varepsilon} d\,E_\lambda.$$

For any $u \in D(L)$, we will write $u = v + w$, where $v = Pu$ and $w = (I - P)u$. Equation (5.4) is then equivalent to the following system.

$$Lw = (I - P)\, N\, (v + w) \tag{5.5}$$
$$Lv = P\, N\, (v + w). \tag{5.6}$$

Remark that, if V denotes the range of P and W the range of (I - P), then V is finite dimensional, and for every $u \in D(L)$, Pu can be expressed as a finite sum of terms in the spaces $\ker(\lambda I - L) \subset D(L)$, with $\lambda \in [-(\alpha' + \varepsilon), \alpha' + \varepsilon]$. Hence, if we pose $E = C([0,T], \mathbb{R}^{2n})$, we have that $V \subset D(L) \hookrightarrow E$. From (5.3) it follows moreover that $N'|_E$ is continuous.

We will now prove the following

Claim. *For each fixed* $v \in V$ *there exists a unique* $w \in W$ *which solves* (5.5). *Further,* $w \in E$ *and the map* ξ *associating* w *to* v *is of class* C^1 *from* V *to* E. *Moreover, for every* $v \in V$ *and* $j \in \{1, \dots, m\}$,

$$\xi(v + T_j a_j) = \xi(v). \qquad (5.7)$$

First of all, notice that, taking $\tau \in \,]-(\alpha' + \varepsilon), \alpha' + \varepsilon[\, \setminus \sigma(L)$, (5.5) becomes equivalent to the following fixed point problem :

$$w = (L - \tau I)^{-1} (I - P) [N(v + w) - \tau w] := T_v(w).$$

We want to show that, if τ is appropriately chosen, the map T_v is a contraction, for all v. Since

$$(L - \tau I)^{-1}(I - P) = \int_{-\infty}^{-(\alpha'+\varepsilon)} (\lambda - \tau)^{-1} \, d E_\lambda + \int_{\alpha'+\varepsilon}^{+\infty} (\lambda - \tau)^{-1} \, d E_\lambda,$$

we have that

$$|(L - \tau I)^{-1}(I - P)| \le (\alpha' + \varepsilon - |\tau|)^{-1}.$$

Hence, since N is Lipshitzian of constant α,

$$|T_v(w) - T_v(\tilde{w})| \le (\alpha' + \varepsilon - |\tau|)^{-1}(\alpha + |\tau|) \, |w - \tilde{w}|.$$

This shows that T_v is a contraction whenever $|\tau| < \frac{1}{2}(\alpha' + \varepsilon - \alpha)$, and such a choice is always possible because of the structure of $\sigma(L)$. Then T_v has a unique fixed point w, and we set

$$\xi(v) = w.$$

Hence we have that, for all $v \in V$,

$$\xi(v) = (L - \tau I)^{-1}(I - P) [N(v + \xi(v)) - \tau\xi(v)] . \qquad (5.8)$$

It can be shown (see [100]) that ξ is Lipshitz continuous from V to W. Moreover, for every $v \in H$,

$$[(\tau - L)^{-1}v](t) = \exp[tJ(A - \tau)]u_0 + \int_0^t \exp[(t - s)J(A - \tau)]\, Jv(s)\, ds$$

where

$$u_0 = [I_{2n} - \exp[TJ(A - \tau)]]^{-1} \int_0^T \exp[(T - s)J(A - \tau)]\, Jv(s)\, ds.$$

This implies

$$(\tau I - L)^{-1} \in \mathcal{L}(X,E).$$

Hence from (5.8) we have that ξ is continuous from V to E. Now consider the function

$$\varphi : V \times E \to E$$
$$\varphi(v,w) = w - T_v(w).$$

Recalling the fact that $N'|_E$ is continuous, since T_v is also a contraction as a map from E to E, we have that the implicit function theorem can be applied to φ. As a consequence, we have that ξ is of class C^1 from V to E. Finally, (5.7) holds since, by (A3), equation (5.5) does not change if we substitute v with $v + T_j a_j$. The Claim is then proved.

By the Claim proved above, we have that equation (5.4) is reduced to equation (5.6), with $w = \xi(v)$, i.e. to

$$Lv = PN(v + \xi(v)) \tag{5.9}$$

where v varies in the finite dimensional space V. By the spectral decomposition of L we can write $V = V^- \oplus V^0 \oplus V^+$, where V^-, V^0 and V^+ are the invariant subspaces of V corresponding to the negative, zero and positive spectrum of L. By (A1), $V^0 = N(A)$. Let us consider the Hilbert space $H = V^- \oplus V^+$. Then L can be considered as a bounded self-adjoint operator on H with a bounded inverse.

Let $T^m = \mathbb{R}^m/\mathbb{Z}^m$ be the m-fold torus, and define on H x T^m the following map

$$g(v, (\tau_1,\dots,\tau_m)) = \int_0^T [-H(t,v(t) + \xi(v)(t) + \sum_{i=1}^m \tau_i a_i) +$$
$$+ (h(t), v(t) + \xi(v)(t))] \, dt.$$

Since $\xi \in C^1(H,E)$, $H \hookrightarrow E$ and $N'|_E$ is continuous, g is of class C^2. Set

$$f(v,\tau) = \frac{1}{2}(Lv,v) + g(v,\tau).$$

By (A2), (A3) and classical arguments, the critical points of the functional f correspond to geometrically distinct solutions of (5.9).

It is easy to see that dg is bounded and continuous. Since g is defined on a finite dimensional space, this implies dg is also compact. So all the assumptions of Theorem 2 are satisfied, and the result follows from (4.4) and (4.5).

Chapter 7

SUBHARMONIC OSCILLATIONS OF FORCED PENDULUM - TYPE EQUATIONS

1. INTRODUCTION

In this paper we are concerned with the existence of subharmonic solutions of second order differential equations of the form

$$\ddot{x} + g(x) = f(t) ,$$

where f is periodic with minimal period T and mean value zero. We have in mind as a particular case the pendulum equation, where $g(x) = A \sin x$.

First results on the existence of subharmonic orbits in a neighborhood of a given periodic motion were obtained by Birkoff and Lewis (cf. [22] and [117) by perturbation - type techniques. Rabinowitz [128] was able to prove the existence of subharmonic solutions for Hamiltonian systems by the use of variational methods. His approach is not of local type like the one in [22], and permits to obtain a sequence of solutions whose minimal period tend towards infinity in the case when the Hamiltonian function has subquadratic or superquadratic growth. These results have been extended in various directions, cf. [19, 32, 36, 50, 114, 139, 141, 142]. Local results on subharmonics for the forced pendulum equation can be found in [143].

Hamiltonian systems with periodic nonlinearity were studied by Conley and Zehnder [36]. They proved the existence of subharmonic solutions under some assumptions on the nondegenerateness of the solutions, by the use of Morse - Conley theory.

In this paper we will prove the existence of subharmonic oscillations of a pendulum - type equation by the use of classical Morse theory together with an iteration formula for the index due to Bott [23] and developed in [49] and [17].

2. THE MAIN RESULT

Let T be a fixed positive number and $k \geq 2$ an integer. Assume $f : \mathbb{R} \to \mathbb{R}$ to be a continuous periodic function, with minimal period T, and such that

$$\int_0^T f(t)\, dt = 0 \,. \tag{1}$$

We consider the following equation:

$$\ddot{x}(t) + g(x(t)) = f(t) \,, \tag{2}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that, setting

$$G(x) = \int_0^x g(s)\, ds \,,$$

the function G is 2π - periodic.

We want to prove the existence of subharmonic solutions, i.e. we look for periodic solutions of (2) having kT as minimal period. The kT - periodic solutions of (2) correspond to the critical points of the functional ϕ_k , defined on the Hilbert space $H^1_{kT} = \{x \in H^1([0,kT]) : x(0) = x(kT)\}$ as follows:

$$\phi_k(x) = \int_0^{kT} [\, \tfrac{1}{2}(\dot{x}(t))^2 - G(x(t)) + f(t)x(t)]\, dt \,. \tag{3}$$

However, the critical points of ϕ_k do not necessarily correspond to periodic solutions of (2) with *minimal* period kT, as can be seen from the case $g \equiv 0$. In fact, in this case the kT - periodic solutions of (2) are of the form

$$x(t) = C_0 - t\frac{1}{kT} \int_0^{kT} \left(\int_0^s f(u)\, du \right) ds \; + \int_0^t \left(\int_0^s f(u)\, du \right) ds \,, \tag{4}$$

where $C_0 = x(0)$ can be chosen arbitrarily in $\mathbb{R}$. Because of (1),

$$\frac{1}{kT} \int_0^{kT} \left(\int_0^s f(u)\, du \right) ds \; = \frac{1}{T} \int_0^{T} \left(\int_0^s f(u)\, du \right) ds \,,$$

and then any $x(t)$ like in (4) has in fact period T .

It can be shown, cf. [103,111,113], that the functional ϕ_k is bounded from below and satisfies the Palais - Smale condition. So ϕ_k always has a minimum. If $g \equiv 0$, the minimum points of ϕ_k are like in (4), where C_0 is an arbitrary real number. In particular, they are not isolated.

Let x_0 be a T - periodic solution of equation (2). Define, for λ and t in $\mathbb{R}$, the matrix

$$A_\lambda(t) = \begin{pmatrix} 0 & -1 \\ \lambda + g'(x_0(t)) & 0 \end{pmatrix}$$

and consider the fundamental solution $X_\lambda(t)$ which satisfies

$$\dot{X}_\lambda(t) = A_\lambda(t)\, X_\lambda(t)$$
$$X_\lambda(0) = \mathrm{Id}.$$

It is well known (see e.g. [90]) that the eigenvalues $\sigma'_{\lambda,T}$ and $\sigma''_{\lambda,T}$ of $X_\lambda(T)$ have the following properties:

(i) either both $\sigma'_{\lambda,T}$ and $\sigma''_{\lambda,T}$ are in $\mathbb{R}$, or $\sigma'_{\lambda,T} = \bar{\sigma}''_{\lambda,T}$;

(ii) $\sigma'_{\lambda,T} \bullet \sigma''_{\lambda,T} = 1$;

(iii) there exists $\lambda_0 < \lambda_1$ such that the maps $\lambda \mapsto \sigma'_{\lambda,T}$ and $\lambda \mapsto \sigma''_{\lambda,T}$ are continuous and one to one if $\lambda \le \lambda_1$. Moreover,

$$0 < \sigma'_{\lambda,T} < 1 < \sigma''_{\lambda,T} \qquad (\lambda < \lambda_0)\,,$$
$$\sigma'_{\lambda,T} = \bar{\sigma}''_{\lambda,T} \in S^1 \qquad (\lambda_0 \le \lambda \le \lambda_1)\,.$$

The T - periodic solution x_0 is said to be nondegenerate if $1 \notin \{\sigma'_{0,T}, \sigma''_{0,T}\}$.

Given $\sigma \in S^1$, we define $J(x_0, T, \sigma)$ to be the number of negative λ 's for which $\sigma \in \{\sigma'_{\lambda,T}, \sigma''_{\lambda,T}\}$. The number $J(x_0, T, 1)$ is then the Morse index of the T - periodic solution x_0 .

We are now able to formulate our main result.

Theorem 1. *Assume the following conditions:*

(a) *the* T *- periodic solutions of equation (2) are isolated;*
(b) *every* T *- periodic solution of (2) having Morse index equal to zero is nondegenerate.*

Then there exists a $k_0 \geq 2$ *such that, for every prime integer* $k \geq k_0$, *there is a periodic solution of (2) with minimal period* kT .

Remarks. 1) We have seen above that in the case $g \equiv 0$ there are no subharmonic solutions of (2), and the T - periodic solutions are not isolated, and therefore degenerate. So, neither (a) nor (b) is verified in this case.

2) In [36], Conley and Zehnder proved the existence of subharmonic solutions for a system with Hamiltonian function periodic in each of its variables. They showed that when all the T - periodic solutions, together with their iterates, are nondegenerate, then there exists a periodic solution with minimal period kT if k is a sufficiently large prime number.

We don't need to assume, like in [36], that also the iterates of the T - periodic solutions of (2) are nondegenerate. Since for a T - periodic solution x_0 one has $\sigma'_{\lambda,kT} = (\sigma'_{\lambda,T})^k$ and $\sigma''_{\lambda,kT} = (\sigma''_{\lambda,T})^k$, it could then happen in principle that $1 \in \{\sigma'_{\lambda,kT}, \sigma''_{\lambda,kT}\}$ even if $1 \notin \{\sigma'_{\lambda,T}, \sigma''_{\lambda,T}\}$.

Proof of Theorem 1. Let us introduce the Hilbert space

$$\tilde{H}_k = \{ \tilde{x} \in H^1_{kT} : \int_0^{kT} \tilde{x}(t)\, dt = 0 \} .$$

By (1) and the 2π - periodicity of G , we have that

$$\phi_k(x + 2\pi) = \phi_k(x)$$

for every $x \in H^1_{kT}$. Set $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$. It is then equivalent to consider the functional ψ_k defined on $S^1 \times \tilde{H}_k$ by

$$\psi_k(x) = \phi_k(\bar{x} + \tilde{x}) ,$$

for every $x = (\bar{x}, \tilde{x}) \in S^1 \times \tilde{H}_k$. The functionals ψ_k are bounded from below and satisfy the Palais - Smale condition (cf. [103,111,113]). By assumption (a), the functional ψ_1 has only a finite number of cririral points $x_0, \dots, x_n$. It is clear that the functions x_i $(0 \le i \le n)$, extended by T - periodicity on $[0, kT]$ are also critical points of ψ_k for $k \ge 2$.

We now assert the following.

Claim. *There exists an integer* k_0 *such that, for* $k \ge k_0$ *and* $0 \le i \le n$, *either* $J(x_i, kT, 1) = 0$ *and* x_i *is nondegenerate, or* $J(x_i, kT, 1) \ge 2$.

Assume for the moment that the above Claim holds true. In case $k \ge k_0$ is a prime number, since f has minimal period T , the critical points of ψ_k have as minimal period either T or kT . Assume by contradiction that $x_0, \dots, x_n$ are the only critical points of ψ_k . Since the Poincaré polynomial of $S^1 \times \tilde{H}_k$ is $(1 + t)$, we have

$$\sum_{i=0}^{n} P_k(t,x_i) = (1 + t)\,[1 + Q(t)] , \tag{5}$$

where $Q(t)$ is a polynomial with nonnegative integer coefficients and $P_k(t,x_i) = \sum_j \dim C_j(\psi_k,x_i)\, t^j$ is the usual Morse polynomial of x_i (see e.g. [113]). By the Claim, if $J(x_i, kT, 1) = 0$, then $P_k(t,x_i) = 1$. Otherwise, if $J(x_i, kT, 1) \ge 2$, then $\dim C_j(\psi_k,x_i) = 0$ for $j = 0,1$. This implies that equation (5) can never be satisfied, and we have a contradiction.

To conclude the proof of the theorem we need then to prove the above Claim. In order to do so, let x_i be a critical point of ψ_1 and let $\lambda_0 < \lambda_1$ be as in property (iii). First of all, we claim that $\lambda_0 \neq 0$. Indeed, if on the contrary $\lambda_0 = 0$, we would have, for every negative λ, $0 < \sigma'_{\lambda,T} < 1 < \sigma''_{\lambda,T}$, which implies $J(x_i ,T,1) = 0$. On the other hand, by (iii), $\sigma'_{0,T} = 1 = \sigma''_{0,T}$, so that x_i would be a degenerate T - periodic solution with Morse index equal to zero, in contradiction with assumption (b).

Suppose $\lambda_0 > 0$. Then, for every $\lambda \leq 0$, we have $0 < \sigma'_{\lambda,T} < 1 < \sigma''_{\lambda,T}$ and hence $J(x_i ,T,\sigma) = 0$ for every $\sigma \in S^1$. By [23, Theorem 1] we have

$$J(x_i ,kT,1) = \sum_{\sigma^k=1} J(x_i ,T,\sigma) = 0 .$$

Moreover x_i , as a critical point of ψ_k , is also nondegenerate, since

$$0 < \sigma'_{0,kT} = (\sigma'_{0,T})^k < 1 < \sigma''_{0,kT} = (\sigma''_{0,T})^k .$$

Suppose now $\lambda_0 < 0$. Then for every $\lambda \in]\lambda_0 , \lambda_0 + \varepsilon[$, for $\varepsilon > 0$ small enough, we have $\sigma'_{\lambda,T} = \bar{\sigma}''_{\lambda,T} \in S^1$ and

$$J(x_i ,T,\sigma'_{\lambda,T}) = J(x_i ,T,\sigma''_{\lambda,T}) > 0 .$$

Hence, for k large enough, we have

$$J(x_i ,kT,1) = \sum_{\sigma^k=1} J(x_i ,T,\sigma) \geq 2 .$$

This proves the Claim, and completes the proof of Theorem 1.

Under a stronger assumption, in the following theorem we will obtain the existence of *two* subharmonic oscillations.

Theorem 2. *Suppose that the* kT *- periodic solutions of (2) are nondegenerate for* $k = 1$ *and for every prime integer* k *. Then there exists* $k_0 \geq 2$ *such that, for every prime integer* $k \geq k_0$ *, there are two geometrically distinct periodic solutions of (2) with minimal period* kT .

Proof. As a consequence of the assumption, for every prime number k , the number n_k of critical points of ψ_k is finite. Since the Morse polynomial of $S^1 \times \tilde{H}_k$ is $(1 + t)$, n_k must be even. It follows from Theorem 1 that, for $k \geq k_0$, $n_k \geq n_1 + 1$. Then, $n_k \geq n_1 + 2$, and the proof is complete.

Appendix 1

A SKETCH OF COINCIDENCE DEGREE THEORY AND AN ABSTRACT EXISTENCE RESULT

1. THE COINCIDENCE DEGREE

Let X and Z be real normed spaces, and $L : \mathrm{dom}\, L \subset X \to Z$ a linear Fredholm operator of index zero. By this we mean that the range of L is a closed set and its codimension is finite and equal to the dimension of $\ker L$.

It can be shown that there exist continuous projections $P : X \to X$ and $Q : Z \to Z$ such that

$$R(P) = \ker L \ , \ \ker Q = R(L) \ ,$$

and a continuous bijection $J : \ker L \to R(Q)$ such that $L + JP : \mathrm{dom}\, L \to Z$ is invertible.

Let Ω be an open bounded subset of X, and $N : \overline{\Omega} \to Z$ a map, generally non linear. We are interested in finding solutions of the equation

$$Lx = Nx \tag{1}$$

in the set $\mathrm{dom}\, L \cap \overline{\Omega}$. It is easily seen that this is equivalent to solving the equation

$$x = (L + JP)^{-1}(N + JP)\, x$$

in the set $\overline{\Omega}$. Let us introduce the following

Definition. *A map* $G : E \to Z$ *is said to be* L - compact *on the metric space* E *if the map* $(L + JP)^{-1} G : E \to X$ *is compact (i.e. continuous and such that the image of any bounded subset of* E *is relatively compact).* G *will be said to be* L - completely continuous *if it is L - compact on every bounded subset of* E .

It can be seen that the above definition is independent of the choice of P, Q and J .

We will suppose N to be L - compact on $\bar{\Omega}$; this can be seen to be equivalent to assuming that $QN : \bar{\Omega} \to Z$ is continuous with bounded image and, denoting by $K_{P,Q} = (L|_{\mathrm{dom}\, L \cap \ker P})^{-1}(I - Q)$ the right inverse of L , that $K_{P,Q}N : \bar{\Omega} \to X$ is compact.

If we moreover assume

$$0 \notin (L - N)(\mathrm{dom}\, L \cap \partial\Omega) ,$$

since $(L - JP)^{-1}JP$, having finite dimentional range, is compact, it is possible to define the coincidence degree as follows:

$$D_L(L - N, \Omega) = D_I(I - (L + JP)^{-1}(N + JP) , \Omega) ,$$

where D_I denotes the Leray - Schauder degree for compact perturbations of the identity. The notation is justified by the fact that $D_L = D_I$ when L = I . It can be shown that the above definition is independent of the choice of P, Q and J. Moreover, one has

$$I - (L + JP)^{-1}(N + JP) = I - P - J^{-1}QN - K_{P,Q}N .$$

Here are the main properties of the coincidence degree D_L:

1. Addition - excision: if Ω_1 and Ω_2 are disjoint open subsets of Ω and $0 \notin (L - N)[\text{dom } L \cap \bar{\Omega} \setminus (\Omega_1 \cup \Omega_2)]$, then

$$D_L(L - N, \Omega) = D_L(L - N, \Omega_1) + D_L(L - N, \Omega_2).$$

2. Homotopy invariance: if $H : (\text{dom } L \cap \bar{\Omega}) \times [0,1] \to Z$ is in the form $H(x,t) = Lx + N(x,t)$, where $N : \bar{\Omega} \times [0,1] \to Z$ is L - compact on $\bar{\Omega} \times [0,1]$, and if $0 \notin H((\text{dom } L \cap \partial\Omega) \times [0,1])$, then the map $t \to D_L(H(.,t), \Omega)$ is constant on $[0,1]$.

3. Normalization: if N is linear and $L - N$ is injective, then

$$D_L(L - N - z, \Omega) = \begin{cases} \pm 1 & \text{if } z \in (L - N)(\text{dom } L \cap \Omega) \\ 0 & \text{otherwise.} \end{cases}$$

4. Existence: if $D_L(L - N, \Omega) \neq 0$, then $0 \in (L - N)(\text{dom } L \cap \Omega)$.

The following existence theorem is a consequence of the above properties of the coincidence degree.

Theorem 1. *Let* $N, N' : \overline{\Omega} \to Z$ *be L - compact and such that*

(i) $D_L(L - N', \Omega) \neq 0$;

(ii) $Lx \neq (1 - t)\, N'x + t\, Nx$

for all $(x,t) \in (\mathrm{dom}\, L \cap \partial\Omega) \times \,]0,1[$.

Then equation (1) has a solution in $\mathrm{dom}\, L \cap \overline{\Omega}$.

Corollary. *Let* $N : \overline{\Omega} \to Z$ *be* L *- compact,* $N' : X \to Z$ *be linear and L - completely continuous, and* $z \in (L - N')(\mathrm{dom}\, L \cap \Omega)$. *Suppose that:*

(i) $L - N'$ *is injective*;

(ii) $Lx \neq (1 - t)\,(N'x - z) + t\, Nx$

for all $(x,t) \in (\mathrm{dom}\, L \cap \partial\Omega) \times \,]0,1[$.

Then equation (1) has a solution in $\mathrm{dom}\, L \cap \overline{\Omega}$.

2. AN ABSTRACT RESULT

In this section we give an abstract version of the existence result of Chapter 1, generalizing the concept of property P.

Let X and Z be real normed spaces, and $L : \operatorname{dom} L \subset X \to Z$ a linear Fredholm operator of index zero. Let $N : X \to Z$ a (possibly nonlinear) L - completely continuous map. We are concerned with the problem of the existence of solutions to the equation

$$Lx = Nx \ . \tag{1}$$

Let A be a weakly compact convex subset of a normed space Y, and consider the comparison function

$$M : X \times A \to Z$$

having the following properties:

(i) M is L - completely continuous on $X \times A$;

(ii) M is positively homogeneous in its first variable, i.e.
$$M(rx,u) = r\, M(x,u) \qquad (r \geq 0,\ x \in X,\ u \in A) \ ;$$

(iii) M is affine in its second variable, i.e.

$$M(x, tu_1 + (1 - t)u_2) = t\, M(x,u_1) + (1 - t)\, M(x,u_2)$$
$$(t \in [0,1],\ x \in X,\ u_i \in A) \ ;$$

(iv) for any $u \in A$, $x = 0$ is the only solution of the equation

$$Lx = M(x,u) \ ;$$

(v) there exists $c > 0$ and a function $u : \operatorname{dom} L \to A$ such that

$$\|Nx - M(x,u(x))\| \leq c \ , \tag{2}$$

for every $x \in \operatorname{dom} L$.

Theorem 2. *Under the above assumptions, if there exists* $\bar{u} \in A$ *such that, being* $\Omega = \{x \in X : \|x\| < 1\}$,

$$D_L(L - M(.,\bar{u}), \Omega) \neq 0 \quad ,$$

then equation (1) has a solution.

Proof. It will be sufficient to prove that there exists an a priori bound for the solutions of the equations

$$Lx = t\,Nx + (1 - t)\,M(x,\bar{u}) \;:= N_t x \quad , \tag{3}$$

for every $t \in \,]0,1[$. By contradiction, suppose that there exists a sequence (x_n,t_n) in $\operatorname{dom} L \times \,]0,1[$ such that $\|x_n\| \to \infty$ and

$$Lx_n = N_{t_n} x_n \quad .$$

Let $v_n = x_n / \|x_n\|$. From (3), (ii) and (iii), one obtains:

$$Lv_n = t_n \frac{Nx_n - M(x_n,u(x_n))}{\|x_n\|} + M(v_n\,, t_n\,u(x_n) + (1 - t_n)\,\bar{u}), \tag{4}$$

where $u(x_n)$ is defined as in (v). Set $u_n = t_n\,u(x_n) + (1 - t_n)\,\bar{u}$. Since A is convex, $u_n \in A$. Being A weakly compact, there exists a subsequence, still denoted by (u_n) , which converges weakly to some $u \in A$. Moreover, applying the (continuous) right inverse $K_{P,Q}$ to (4), one has

$$(I - P)v_n = K_{P,Q}\Big(t_n \frac{Nx_n - M(x_n,u(x_n))}{\|x_n\|} + M(v_n\,,u_n)\Big)\,. \tag{5}$$

It follows from (i), (v) and the fact that ker L is finite dimensional, that v_n has a subsequence which strongly converges to a certain $v \in \mathrm{dom}\, L$. Passing to the limit in (5), one has

$$(I - P)v = K_{P,Q}M(v,u) \quad ,$$

i.e.

$$Lv = M(v,u) \quad .$$

By (iv), $v = 0$, which is a contradiction since $\|v_n\| = 1$ for every n.

Appendix 2

SINGULAR HOMOLOGY AND COHOMOLOGY

1. SINGULAR CHAINS AND SINGULAR COCHAINS

We are given a topological space X, and we consider the set of real sequences $\mathbb{R}^\infty$.

Let $\Delta_n \subset \mathbb{R}^\infty$ be the convex hull of the following (n+1) points

$$E_0 = (0,0,...,0,...)$$

$$E_1 = (1,0,...,0,...)$$

$$\cdots$$

$$E_n = (0,0,...,1,...).$$

We define a n-singular simplex of X to be a continuous map

$$\sigma : \Delta_n \to X.$$

The image of a singular simplex σ is called its support and will be denoted by $|\sigma|$. The support of a 0, 1 or 2-singular simplex will thus be a point, a curve or a surface in X.

It is useful to be able to formally sum singular simplexes and multiply them by real numbers. This is why we will consider the set $S_n(X)$ whose elements are the formal finite sums of the form

$$\sum_{\sigma} r_{\sigma}\, \sigma\ ,$$

where the r_σ are real coefficients and the σ are n-singular simplices. This set $S_n(X)$ comes out to be a vector space over $\mathbb{R}$, and its elements are called n-singular chains of X.

The support of a chain $\eta = \sum_{\sigma} r_\sigma\, \sigma$ is defined to be the union of the supports of the σ's appearing in the sum, and is still denoted by $|\eta|$. So, for example, the support of a 0-chain is a finite number of points of X, etc.

A n-singular cochain of X is by definition a continuous linear map c from $S_n(X)$ to $\mathbb{R}$. These maps form a vector space that we denote by $S^n(X)$. We have a duality between $S_n(X)$ and $S^n(X)$:

$$<.,.> : S_n(X) \times S^n(X) \to \mathbb{R}$$

defined by $<\eta,c> = c(\eta)$. Clearly a n-singular cochain is determined, by the linearity, if we know how it acts on the n-singular simplexes of X.

2. CUP AND CAP PRODUCTS

There are two operations between singular chains and singular cochains that we would like to introduce. In order to do this, we need the following.

Fact. *Let* $p \leq n$ *and consider the face of* Δ_n *determined by the* $(p+1)$ *points* $E_{i_0}, E_{i_1}, \ldots, E_{i_p}$*.There exists an isomorphism between* Δ_p *and this face.*

We will denote this isomorphism by

$$\mathcal{L}(E_{i_0}, E_{i_1}, ..., E_{i_p}).$$

Given a n-singular simplex σ, we will say that the p-singular simplex $\sigma \circ \mathcal{L}(E_{i_0}, ..., E_{i_p})$ is a p-face of σ.

Now we are able to define an operation called the "cup product"

$$S^n(X) \times S^m(X) \to S^{n+m}(X)$$

$$(c,d) \to c \cup d$$

as follows : for each (n+m)-singular simplex σ we have

$$< \sigma, c \cup d > = < \sigma \circ \mathcal{L}(E_0, ..., E_n), c > \cdot < \sigma \circ \mathcal{L}(E_n, ..., E_{n+m}), d > . \qquad (1)$$

This means, intuitively, that $c \cup d$ acts on σ by letting c act on the "front" n-face of σ and d on the "back" m-face of σ, and then multiplying the two numbers thus obtained.

The r.h.s. of (1) is the product of two real numbers. Considering the first of these as a coefficient, we could write

$$< \sigma, c \cup d > = << \sigma \circ \mathcal{L}(E_0, ..., E_n), c > \sigma \circ \mathcal{L}(E_n, ..., E_{n+m}) , d >.$$

It is natural then to define an operation, called the "cap product"

$$S_{n+m}(X) \times S^n(X) \to S_m(X)$$

$$(\eta, c) \rightarrow \eta \frown c$$

which, for a (n+m)-singular simplex σ, is such that

$$\sigma \frown c = < \sigma \circ \mathcal{L}(E_0, \ldots, E_n), c > \sigma \circ \mathcal{L}(E_n, \ldots, E_{n+m})$$

and which extends to an arbitrary (n+m) singular chain η by linearity. Intuitively, $\sigma \frown c$ is the "back" m-face of σ with a coefficient given by the action of c on the "front" n-face of σ. Clearly the support of $\eta \frown c$ is contained in the support of η, i.e.

$$|\eta \frown c| \subset |\eta|.$$

Moreover one has :

$$< \eta, c \smile d > = < \eta \frown c, d >.$$

3. DEFINITION OF SINGULAR HOMOLOGY AND SINGULAR COHOMOLOGY

We define a "boundary operator" as a homomorphism

$$\partial_n : S_n(X) \rightarrow S_{n-1}(X)$$

such that for every n-singular simplex σ,

$$\partial_n \sigma = \sum_{i=1}^{n} (-1)^i \ \sigma \circ \mathcal{L}(E_0, \ldots, \hat{E}_i, \ldots, E_n),$$

where $\hat{E}_i$ means "omit E_i". So $\partial_n \sigma$ is the sum of all the (n-1)-faces of σ, taken with appropriate "orientation".

If we define the space of n-singular cycles by

$$Z_n(X) = \ker\ \partial_n$$

and the space of n-singular boundaries by

$$B_n(X) = \mathrm{Im}\ \partial_{n+1},$$

it is not difficult to see that all boundaries are cycles, i.e. $B_n(X) \subset Z_n(X)$. The quotient space

$$H_n(X) = Z_n(X)/B_n(X)$$

is called the n-singular homology vector space of X.

Dually, we can define the "coboundary operator"

$$\delta^n : S^n(X) \to S^{n+1}(X)$$

by

$$< \eta, \delta^n c > = < \partial_n \eta, c > \qquad (2)$$

for any (n+1)-singular chain η and any $c \in S^n(X)$.

We can analogously define the space of n-singular cocycles

$$Z^n(X) = \ker \delta^n,$$

and the space of n-singular coboundaries

$$B^n(X) = \mathrm{Im}\ \delta^{n-1}.$$

Since $B^n(X) \subset Z^n(X)$, we can define the quotient space

$$H^n(X) = Z^n(X)/B^n(X)$$

which is called the n-singular cohomology vector space of X.

The above notions can be generalized as follows. Let A be a subspace of X. Define the set of n-singular chains in X relative to A as

$$S_n(X,A) = S_n(X)/S_n(A).$$

Since the boundary operator $\delta_n : S_n(X) \to S_{n-1}(X)$ reduces to the boundary operator $\delta_n : S_n(A) \to S_{n-1}(A)$, it also induces a boundary operator

$$\delta_n : S_n(X,A) \to S_{n-1}(X,A) : \partial_n [\eta] = [\partial_n \eta]. \qquad (3)$$

So we can define as above the set of relative n-cycles $Z_n(X,A)$, the set of relative n-boundaries $B_n(X,A)$, and the relative n-singular homology

$$H_n(X,A) = Z_n(X,A)/B_n(X,A).$$

Analogously to what we did before, we define the set of relative n-cochains $S^n(X,A)$ as the set of continuous linear maps from $S_n(X,A)$ to $\mathbb{R}$. The coboundary operator

$$\delta_n : S^n(X,A) \to S^{n+1}(X,A)$$

is defined as in (2), where δ_n is as in (3). Finally we can define as above the set of relative n-cocycles $Z^n(X,A)$, the set of relative n-coboundaries $B^n(X,A)$, and the relative n-singular cohomology

$$H^n(X,A) = Z^n(X,A)/B^n(X,A).$$

If A and B are subspaces of X such that one of the following situations is true :

$$A = \phi, \qquad A = B, \qquad B = \phi$$

then we can define in a natural way the "cup product"

$$H^n(X,B) \times H^m(X,A) \to H^{n+m}(X, A \cup B)$$

by $[c] \cup [d] = [c \cup d]$.

Similarly we can define the "cap product"

$$H_{n+m}(X, A \cup B) \times H^n(X,B) \to H_m(X,A)$$

by $[z] \cap [c] = [z \cap c]$.

4. Some properties

We will present here some properties of homology and cohomology vector spaces which will be used later.

We are given the topological spaces $A \subset X$ and $A' \subset X'$, and a map $f : (X,A) \to (X',A')$ (i.e. a continuous function $f : X \to X'$ such that $f(A) \subset A'$). Then for every $n \in \mathbb{N}$, there is an induced homomorphism

$$H_n(f) : H^n(X,A) \to H_n(X',A'),$$

(whose explicit form is given by

$$H_n(f)\,[\sum_\sigma r_\sigma\, \sigma] = [\sum_\sigma r_\sigma\, (f \circ \sigma)])$$

which has the following properties.

(a) $H_n(id) = id$.

(b) $H_n(g \circ f) = H_n(g) \circ H_n(f)$.

(c) If f and g are homotopic, then $H_n(f) = H_n(g)$.

(d) Assume B to be an open subset of X whose closure is contained in the interior of A, and let $i : (X \setminus B , A \setminus B) \to (X,A)$ be the inclusion map. Then $H_n(i)$ is an isomorphism.

We will now point out some consequences of (a), (b) and (c). Let us first recall that A is said to be a strong deformation retract of X if there exists an $h \in C([0,1] \times X, X)$ with the following properties.

$$\begin{array}{llll} h(t,u) = u & \text{whenever } u \in A & \text{and } t \in [0,1] \\ h(0,u) = u & \text{and} \quad h(1,u) \in A & \text{for all } u \in X. \end{array}$$

Let $\tilde{A} \subset A \subset X$. The following properties hold.

If A is a strong deformation retract of X, then

$$H_n(X,\tilde{A}) \simeq H_n(A,\tilde{A}).$$

If $\tilde{A}$ is a strong deformation retract of A, then

$$H_n(X,A) \simeq H_n(X,\tilde{A}).$$

If $\tau = X \to X$ is a homeomorphism such that $\tau|_A = id_A$, then

$$H_n(\tau) : H_n(X,A) \to H_n(X,A) \text{ is an isomorphism.}$$

Accordingly, one can identify any $[z] \in H_n(X,A)$ with the corresponding $H_n(\tau)[z]$. Hence we will have that $\tau([z]) \subset [z]$.

Here are some other properties.

(e) Let $i : (A,\tilde{A}) \to (X,\tilde{A})$ and $j : (X,\tilde{A}) \to (X,A)$ be inclusion maps. Then for every $n \in \mathbb{N}$, there is a "boundary" homomorphism

$$\partial_n : H_n(X,A) \to H_n(A,\tilde{A})$$

such that the sequence

$$\dots \to H_{n+1}(X,A) \xrightarrow{\partial_{n+1}} H_n(A,\tilde{A}) \xrightarrow{H_n(i)} H_n(X,\tilde{A}) \xrightarrow{H_n(j)} H_n(X,A) \xrightarrow{\partial_n} \dots$$

is exact.

(f) Let $\tilde{A} \subset A \subset X$ and $\tilde{A}' \subset A' \subset X'$. If $f : X \to X'$ is a continuous map such that $f(A) \subset A'$ and $f(\tilde{A}) \subset \tilde{A}'$, then

$$\partial_n \circ H_n(f) = H_{n-1}(f|_A) \circ \partial_n.$$

(g) For any point x in X, $H_n(\{x\})$ is trivial if $n > 0$, while $H_0(\{x\}) = \mathbb{R}$.

(h) If $[z] \in H_n(X,A)$, there exist X_0 and A_0 compact such that $[z]$ is the image of the homomorphism $H_n(X_0,A_0) \to H_n(X,A)$ induced by inclusion.

Eilenberg and Steenrod proved that the properties (a) - (h) characterize homology theory.

Here are some consequences of (a) - (h).

If B^m is a closed ball in $\mathbb{R}^m$ and S^{m-1} is its boundary, then

$$H_n(B^m, S^{m-1})$$

is, for n = m, isomorphic to $\mathbb{R}$, and is trivial otherwise.

(Künneth formula). If X, Y are topological spaces and $A \subset X$, then

$$H_n(X \times Y, A \times Y) \simeq \oplus_{p+q=n} [H_p(X,A) \otimes H_q(Y)]$$

and the same formula holds for cohomologies, too.

The cup and the cap products are bilinear operations and they are invariant under continuous deformations.

Finally let us define the Poincaré polynomial as

$$P(t,X,A) = \sum_{n=0}^{\infty} \dim \; H_n(X,A) t^n$$

whenever for every $n \in \mathbb{N}$, dim $H_n(X,A)$ is finite and is equal to zero for all n sufficiently large.

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