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TESI DI "PHILOSOPHIAE DOCTOR"

LIMIT ANALYSIS OF SOME STOCHASTIC VARIATIONAL PROBLEMS

Settore: Analisi Funzionale ed applicazioni

Supervisore: Prof. Gianni Dal Maso

Candidato: Michele Balzano

Anno Accademico: 1987/1988

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Contents

INTRODUCTION.

During the past several years the asymptotic analysis of boundary value problems in highly perturbed domains, as well as of the Schroedinger equations with rapidly varying potentials, has been strongly developed.

A motivation of this development resides in the increasing interest in physics for the potential and scattering theory in domains with many obstacles, and in engineering for the behaviour of the so called *composite materials* (electric conduction, heat propagation, transmissions through thin isolating layers, etc. in multiphase media). For an initial approach to the mathematical aspects of the subject matter see, for istance, [10], [48] and [13].

Among the various problems which can be attacked, the Dirichlet problems in domains with random boundary and the Schroedinger equation with oscillating stochastic potentials, play an important role. As a matter of fact, it is enough to stress that when we describe, for instance, the structure of a composite material, the best approximation is the random one.

The main object of this thesis is just to provide a new general setting for investigating limit problems of this kind.

More precisely, given a bounded open region D of $\mathbf{R}^d$, $d \geq 2$, and a function $f \in L^2(\mathbf{R}^d)$, we focus our attention on the asymptotic behaviour, as $h \to +\infty$, of the solutions u_h of the following equations:

(I.1) **Dirichlet problems in domains with random holes.**

$$- \Delta u_h = f \qquad \text{in } D \setminus E_h$$

$$u_h \in H^1_0(D \setminus E_h) \ ,$$

where (E_h) is a sequence of random (closed) subsets of D;

(I.2) **Stationary Schroedinger equation with stochastic potentials.**

$$- \Delta u_h + q_h \, u_h = f \qquad \text{in } D$$

$$u_h \in H^1_0(D) \ ,$$

where (q_h) is a sequence of (wildly varying) random potentials;

(I.3) **Dirichlet problems in domains surrounded by thin layers with random thickness.**

$$-\Delta u_h = f \qquad \text{in } D$$

$$-\varepsilon_h \, \Delta u_h = f \qquad \text{in } A_h \setminus D$$

$$u_h \in H^1_o(A_h) \ ,$$

where the natural transmission conditions on ∂D are satisfied, (ε_h) is a sequence of real numbers such that $\varepsilon_h \to 0$, as $h \to +\infty$, and, for every $h \in \mathbb{N}$, A_h is a random (open) set such that $A_h \supseteq D$ and

$$\sup_{x \in A_h} \text{dist}(x,D) < \varepsilon_h \ .$$

The first significant result on the subject (I.1), which has been obtained by using the theory of Wiener sausage, can be found in the pioneering paper [38] of Kac. Successively, similar problems have been studied by many authors, who have contributed to produce a lot of methods. For instance, Brownian Motion techniques are used in the works [46], [47], [7] and, more recently, for problems on Riemannian manifolds, in [18], [19], [20], while Green function methods have been employed in [44], [45], [37].

There has been much work also on the corresponding deterministic case. In particular, we mention [39], [40], [41], [42]. Further interesting results can be found in [21], [22], [23], [24]. For the nonlinear case see the recent papers [28], [27].

To our knowledge no specific reference is avalaible for the full probabilistic problems (I.2) and (I.3).

A limit analysis of the problem (I.2) in the case of highly oscillating periodic potentials has been discussed in [2], [3], [12].

The corresponding deterministic cases of (I.3) are known in literature as *reinforcement problems* . In fact, they represent the mathematical model of an elastic rod in torsion reinforced by a thin coating of increasingly strong material.

Recent results on these topics have been obtained by several authors (see [17], [11], [16], [15]). In particular, nonlinear reinforcement problems have been attacked in [1].

In our research the problems (I.1), (I.2) and (I.3) have been tackled with some tools of the Calculus of Variations, known as *variational convergences* . They are especially useful for the limit analysis of sequences of variational problems. More specifically, Γ - convergence methods (see [2], [33], [34], [35]) have been applied for investigating (I.1) and (I.2); while, the notion of Mosco - convergence (see [2], [43]) has been used for dealing with (I.3).

In Chapter 1 of the thesis work we describe a general variational framework for studying both the Dirichlet problems in domains with randomly distributed small holes (I.1), and the stationary Schroedinger equation with rapidly oscillating random potentials (I.2).

Such problems can be regarded, in a unified context, as particular cases of the so called *relaxed Dirichlet problems* (see [14], [25], [30], [31], [32]), formally written as

$$\text{(I.4)} \qquad \begin{cases} -\Delta u_h + \mu_h u_h = f & \text{in } D \\ u_h = 0 & \text{on } \partial D \ , \end{cases}$$

where (μ_h) is a sequence of non negative Borel measures on D, which must vanish on sets of zero (harmonic) capacity, but may assume the value $+\infty$ on some subset of positive capacity.

According to [31], we have denoted by $\mathcal{M}_0$ the class of all Borel measures of this type.

Problem (I.1) can be written in the form (I.4) by taking $\mu_h = \infty_{E_h}$ for every $h \in \mathbb{N}$, where ∞_{E_h} is the Borel measure on D defined as

$$\infty_{E_h}(B) = \begin{cases} 0 & \text{if } \operatorname{cap}(B \cap E_h) = 0 \\ +\infty & \text{if } \operatorname{cap}(B \cap E_h) \neq 0 \end{cases}$$

while, problem (I.2) can get the form (I.4) by assuming that

$$\mu_h(B) = \int_B q_h(x)\, dx$$

At this stage of the analysis, the basic tool has been the variational μ - capacity defined as (see [31])

$$\text{(I.5)} \qquad C(\mu,B) = \inf \left\{ \int_D |Du|^2\, dx + \int_B (u-1)^2\, d\mu \ ; \ u \in H^1_0(D) \right\}$$

for every $\mu \in \mathcal{M}_0$ and for every Borel set $B \subseteq D$.

In fact, by endowing $\mathcal{M}_0$ with the minimal σ - algebra for which the maps $C(\cdot,K)$ are measurable for every compact subset K of D, the concrete probabilistic problem to attack can be formulated as follows.

Let (Ω,Σ,P) be a probability space. We are interested in studying the asymptotic behaviour, as $h \to +\infty$, of the solutions u_h of the relaxed Dirichlet problems of the form (I.4), where μ_h, for every $h \in \mathbb{N}$, has to be regarded as a random measure, namely, a measurable map from Ω into $\mathcal{M}_0$

We have found necessary and sufficient conditions (Theorem 4.1 of Chap.1) on (μ_h) for the convergence in probability of the sequence (u_h) toward the solution of a deterministic relaxed Dirichlet problem of the form

$$\text{(I.6)} \qquad \begin{cases} -\Delta u + \nu u = f & \text{in } D \\ u = 0 & \text{on } \partial D \ , \end{cases}$$

where ν is a suitable Radon measure of the class $\mathcal{M}_0$.

These conditions have been given in terms of the asymptotic behaviour of the expectations of the random variables $C(\mu_h,B)$ and of the covariances of the random variables $C(\mu_h,A)$ and $C(\mu_h,B)$ for disjoint subsets A and B of D.

The result has been obtained as consequence of a study concerning sequences of probability measures on $\mathcal{M}_0$ equipped with the topology attached to a notion of convergence for measures in $\mathcal{M}_0$, called γ - convergence (see [31], [25]), defined by means of the Γ - convergence of the functionals

$$\int_D |Du|^2 \, dx + \int_D u^2 \, d\mu$$

Note that, with this topology, $\mathcal{M}_0$ becomes a compact metric space (see [31]).

Similar problems of convergence of measures on spaces endowed with topology related to Γ - convergence have been studied in [26], [29].

Our study has also led to a meaningful characterization of the limit measure ν appearing in (I.6). We have shown, under our assumptions, that the expectations of the capacities converge weakly (in the sense of [36]) to a countably subadditive increasing set function $\alpha(B)$ (which turns out to be equal to $C(\nu,B)$) and ν is the least measure greater than or equal to α. This generalizes a result proven in [9].

Chapter 2 of the thesis has been devoted to explore further the notion of the least Borel measure, μ_α, greater than or equal to a given nonnegative countably subadditive set function α. This notion has been also employed in several recent papers (see, for istance, [25], [27]), concerning the asymptotic behaviour of relaxed Dirichlet problems.

The aim has been to seek a relation between the measure μ_α and the set function α. In the case in which α is the set function $C(\mu,\cdot)$ defined in (I.5) and μ is finite , it has been proved in [14] that $\mu_\alpha = \mu$ and the measure μ can be reconstructed by derivation of α with respect to a given Radon measure λ. This result is based heavily on the properties of the μ - capacitiy and the associated Euler differential equations.

Our main result (Theorem 1.1 of Chap.2) consists in showing that the same derivation theorem holds for an arbitrary nonnegative countably subadditive set function α. The new proof is based on general measure theoretic techniques, in particular on a refined version of the Vitali covering theorem.

The limit analysis of the problem (I.3) has been tackled (Chapter 3) using a method slightly different from that employed for investigating (I.1) and (I.2).

First, for every $h \in \mathbf{N}$, we have considered the quadratic form on $L^2(\mathbf{R}^d)$ associated with (I.3) and defined by

$$F_h(u) = \begin{cases} \displaystyle\int_D |Du|^2\, dx + \varepsilon_h \int_{A_h \setminus D} |Du|^2\, dx & \text{if } u \in H^1_0(A_h) \\ +\infty & \text{otherwise} \end{cases}$$

It must be stressed that the solution u_h of (I.3) coincides with the solution of the minimum problem

$$\min \left\{ F_h(u) - 2\int_{A_h} fu\, dx \ : u \in L^2(\mathbf{R}^d) \right\}$$

Next, we have introduced the class $\mathcal{E}$ of all convex semicontinuous functions from $L^2(\mathbf{R}^d)$ into $\mathbf{R}$. By endowing $\mathcal{E}$ with the topology attached to the $L^2(\mathbf{R}^d)$ - Mosco - convergence (a notion of variational convergence especially useful to study sequences of convex functionals), it becomes a separable complete metric space (see [2]).

Finally, by associating with the problems (I.3) a sequence of random functionals, namely, measurable maps $\omega \to F_h(\omega)$ from a probability space Ω into $\mathcal{E}$, the problem can be reduced to analyze the asymptotic behaviour of the sequence (F_h).

The first outcome of the research on this subject has been a compactness theorem for sequences of random functionals (Theorem 4.1 and Remark 5.5 of Chap.3).

It has been deduced from an abstract compactness result for sequences of probability measures on a complete metric space (Theorem 2.3 of Chap.3).

We have shown that, under suitable assumptions on the sequence (F_h), there exists a subsequence $(F_{\sigma(h)})$ converging in probability to a constant random functional F. Moreover, this functional turns out to be associated with the equation formally written as

$$
\text{(I.7)} \qquad \begin{cases} -\Delta u = f & \text{in } D \\[2ex] \dfrac{\partial u}{\partial n} + \mu u = 0 & \text{on } \partial D \end{cases}
$$

where μ is a measure of the class $\mathcal{M}_0$ supported by ∂D and n denotes the outer unit normal to D.

The assumptions of our theorem are expressed in terms of the asymptotic behaviour of the expectations and the covariances of suitable random capacities associated with the random functionals F_h.

We conclude this Chapter by proving a result which permits, under our hypotheses, to compute the limit measure μ appearing in (I.7), hence to determine the limit functional F.

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Chapter 1 :

Random relaxed Dirichlet problems.

RANDOM RELAXED DIRICHLET PROBLEMS

INTRODUCTION.

In this paper we provide a general framework to study both the classical Dirichlet problem in domains with randomly distributed small holes and the stationary Schroedinger equation with rapidly oscillating random potentials.

More precisely, given a bounded open region D of R^d, $d \geq 2$, and a function $f \in L^2(D)$, we deal with problems of the form:

(0.1)
$$\begin{cases} -\Delta u = f & \text{in } D \setminus F \\ u \in H_0^1(D \setminus F) \end{cases}$$

where F is a random subset of D, and of the form:

(0.2)
$$\begin{cases} -\Delta u + q(x) u = f & \text{in } D \\ u \in H_0^1(D) \end{cases}$$

where q is a random potential.

Problems (0.1) and (0.2) can be considered as particular cases of the so called relaxed Dirichlet problems (see [5], [8], [20], [21], [22]) formally written as:

$$(0.3)\qquad \begin{cases} -\Delta u + \mu u = f & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

where μ is a non negative Borel measure on D, which must vanish on sets of (harmonic) capacity zero, but may assume the value $+\infty$ on some subset of positive capacity.

Following [20] we denote by $\mathcal{M}_0$ the class of all Borel measure of this type.

Problem (0.1) can be written in the form (0.3) by taking $\mu = \infty_F$, where ∞_F is the Borel measure on D defined as:

$$\infty_F(B) = \begin{cases} 0 & \text{if } cap(B \cap F) = 0 \\ +\infty & \text{if } cap(B \cap F) \neq 0. \end{cases}$$

Problem (0.2) can be written in the form (0.3) by taking

$$\mu(B) = \int_B q(x)\, dx$$

In this paper we give a variational method for investigating sequences of Problems of the form (0.3), where μ are random measures of the class $\mathcal{M}_0$.

The basic tool in our analysis will be the variational μ- capacity defined as:

$$C(\mu, B) = \inf\left\{ \int_D |Du|^2 dx + \int_B (u-1)^2 d\mu \ ; \ u \in H_0^1(D) \right\}$$

for every $\mu \in \mathcal{M}_0$ and for every Borel set $B \subseteq D$.

The probabilistic problem we shall consider can be rigorously stated as follows. Let (Ω, Σ, P) be a probabilistic space. We consider a sequence M_h of random measures, i.e. of measurable maps between (Ω, Σ) and $\mathcal{M}_0$, endowed with the minimal σ- algebra $\mathcal{B}(\mathcal{M}_0)$ for which the maps $C(\cdot, K)$ are measurable for every compact subset K of D.

The problem is to analyze the asymptotic behaviour, as $h \to \infty$, of the solutions U_h of the random relaxed Dirichlet problems:

$$\begin{cases} -\Delta U_\varepsilon + M_\varepsilon U_\varepsilon = f & \text{on } D \\ U_\varepsilon = 0 & \text{on } \partial D \end{cases}$$

We find necessary and sufficient conditions on M_ε for the convergence in probability of the sequence U_ε toward the solution of a deterministic relaxed Dirichlet problem of the form:

$$\begin{cases} -\Delta U + \nu U = f & \text{in } D \\ U \in H_0^1(D) \end{cases} \tag{0.4}$$

where ν is a suitable Radon measure of the class $\mathcal{M}_0$. These conditions are given in terms of the asymptotic behaviour of the expectations of the random variables $C(M_\varepsilon, B)$ and of the covariances of the random variables $C(M_\varepsilon, A)$ and $C(M_\varepsilon, B)$ for disjoint subsets A and B of D.

When these conditions are satisfied, we obtain also a meaningful characterization of the limit measure ν. In fact, in this case, the expectations of the capacities $C(M_\varepsilon, B)$ converge weakly (in the sense of [26] to a countably subadditive increasing set function $\alpha(B)$ (which turns out to be equal to $C(\nu, B)$) and ν is the least measure such that $\nu \geq \alpha$. This generalizes a result proved in [6].

As a first application of our results we consider the asymptotic behaviour of a sequence of Dirichlet problems

$$\begin{cases} -\Delta u_\varepsilon = f & \text{on } D \setminus F_\varepsilon \\ u_\varepsilon \in H_0^1(D \setminus F_\varepsilon) \end{cases} \tag{0.5}$$

in which the random sets F_ε have the form:

$$F_\varepsilon = \bigcup_{i=1}^{\varepsilon} (x_i^\varepsilon + t_\varepsilon K) \tag{0.6}$$

where $(x_i^\varepsilon)_{1 \leq i \leq \varepsilon}$ is a family of independent identically distributed random variables in D with distribution law β given by:

$$\beta(B) = \int_B \varrho(x)\,dx \qquad (\varrho \in L^2(D)) \quad ,$$

K is an arbitrary compact subset contained in the unit ball and (τ_ℓ) is a sequence of positive real numbers such that

$$\lim_{\ell\to\infty} \ell\,\tau_\ell^{d-2} = l < +\infty$$

We prove that in this case the solutions U_ℓ of the random equation (0.5) converge in probability to the solution U of the deterministic equation (0.4) with $\nu = c\,\beta$, where $c = l\,C(K, \mathbb{R}^d)$, and

$$C(K,\mathbb{R}^d) = \min\left\{ \int_{\mathbb{R}^d} |Du|^2 dx \,;\, u \in H^1(\mathbb{R}^d),\, u \geq 1 \text{ q.e. on } K \right\}$$

Problems of this kind have been investigated in [4], [32], [38], [40], by Brownian motion methods and in [36], [37] by Green function methods. Recently the fluctuations around the solution of the limit problem have been investigated in [29] .

The corrisponding deterministic case has been studied in [30] by an orthogonal projection method, and in [31], [35] by a capacitary method. Other results on this argument can be found in [34], [13], [14], [15], [16]. Moreover, similar problems on Riemannian manifolds have been studied in [9, Chapter IX], [10], [11].

The second application of our abstract theorem concerns the asymptotic behaviour of a sequence of stationary Schroedinger equations with random potentials of the form

$$\begin{cases} -\Delta U_\ell + q_\ell U_\ell = f & \text{in } D \\ U_\ell \in H_0^1(D) & \end{cases}$$

where q_ℓ is given by

$$q_\ell(x) = \begin{cases} K_\ell & \text{if } x \in F_\ell \\ 0 & \text{otherwise,} \end{cases}$$

F_ℓ are the sets defined in (0.6) with K equal to the closed unit ball, and (K_ℓ) is a sequence of real numbers.

We prove that, in dimension $d = 3$, if $\lim_{\ell\to\infty} \sqrt{K_\ell}\,\tau_\ell = +\infty$, then the solutions U_ℓ

of the random equations converge to the solution of the deterministic equation (0.4), with $\nu = c\beta$, where $c = \ell\, C(B_1, \mathbb{R}^d)$.

Problems of this kind have been studied in the deterministic case in [2], [3] and [7].

I would like to thank Prof. G. Dal Maso, for suggesting me this research work, with the precious aid of his advice.

1. NOTATION AND PRELIMINARIES

Troughout the paper we denote by D a fixed bounded open subset of $\mathbb{R}^d$ with $d \geq 2$. Moreover, we denote by $\mathcal{U}$ the family of all open sets $U \subseteq D$ and by $\mathcal{K}$ the family of all compact sets $K \subseteq D$.

Let us recall some well-known definitions which will be often used in the sequel.

DEFINITION 1.1. For every compact set $K \in \mathcal{K}$ we define the capacity of K respect to D by:

$$C(K,D) = \inf\left\{ \int_D |D\varphi|^2,\ \varphi \in C_0^\infty(D),\ \varphi \geq 1 \quad \text{on } K \right\}$$

The definition is extended to the sets $U \in \mathcal{U}$ by:

$$C(U,D) = \sup\left\{ C(K);\ K \subseteq U,\ K \in \mathcal{K} \right\}$$

and to arbitrary sets $E \subseteq D$ by:

$$C(E,D) = \inf\left\{ C(U);\ U \supseteq E,\ U \in \mathcal{U} \right\}$$

When no confusion can arise, we will simply write $C(E)$ instead of $C(E,D)$.

Let E be any subset of D. When a property $P(x)$ is satisfied for all $x \in E$ except for a subset $N \subseteq E$ such that $C(N) = 0$, then we say that $P(x)$ holds quasi everywhere on E (q.e. on E).

A set $A \subseteq D$ is said to be *quasi open* (resp. *quasi closed*, *quasi compact*) in D if for every $\varepsilon > 0$ there exists an open (resp. closed, compact) set $U \subseteq D$ such that $C(A \Delta U) < \varepsilon$, where Δ denotes the symmetric difference (the topological notions are in the relative topology of D).

We say that a function $f : D \to \overline{\mathbb{R}}$ is *quasi continuous* in D if for every $\varepsilon > 0$ there exists a set $E \subseteq D$ such that $C(D-E) < \varepsilon$ and the restriction of f to E is continuous.

We denote by $H^1(D)$ the Sobolev space of all functions in $L^2(D)$ whose first weak derivatives belong to $L^2(D)$, and by $H_0^1(D)$ the closure of $C_0^\infty(D)$ in $H^1(D)$.

For every $x \in \mathbb{R}^d$ and every $t > 0$ we denote by

$$B_t(x) = \left\{ y \in \mathbb{R}^d : |y-x| < t \right\}$$

the open ball centered at x with radius r .

By the symbol $|B_r(x)|$ we mean the Lebesgue measure of the ball. By B_r we denote the ball of radius r centered at the origin.

Let $u \in H^1(D)$. It is well-known that the limit

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\,dy$$

exists and is finite for quasi every $x \in D$.

In the sequel we always require that for every $x \in D$

$$\liminf_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\,dy \le u(x) \le \limsup_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\,dy$$

Thus, the pointwise value $u(x)$ is determined quasi everywhere in D , and the function u is quasi continuous in D .

It can be shown that

$$C(E) = \min \left\{ \int_D |Du|^2 dx \; ; \; u \in H_0^1(D) \, , \, u \ge 1 \text{ q.e. on } E \right\}$$

for every subset E of D .

For these properties of the capacity and of the function of $H^1(D)$ see [28] .

We denote by $\mathcal{B}$ the σ-field of all Borel subsets of D . A nonnegative countable additive set function defined on $\mathcal{B}$ and with value in [0, +∞] is called a *Borel measure* on D . A Borel measure which assigns finite value to every compact subset of D is called *Radon measure.*

In our paper we deal with a peculiar class of Borel measures, defined as follows:

DEFINITION 1.2. $\mathcal{M}_0^*$ is the class of all Borel measures μ on D such that:

a) $\mu(B) = 0$ for every $B \in \mathcal{B}$ with $C(B) = 0$;

b) $\mu(B) = \inf \{ \mu(A) : A \text{ quasi open}, B \subseteq A \}$ for every $B \in \mathcal{B}$.

An easy example of measure belonging to $\mathcal{M}_0^*$ is the following:

$$\mu(B) = \int_B f\,dx$$

where $f \in L^1_{loc}(D)$. More generally, every Radon measure μ on D which satisfies a) belongs to $\mathcal{M}_0^*$.

We remark that the masures belonging to $\mathcal{M}_0^*$ are not required to be regular nor σ- finite. For istance, the measures introduced in the Definition below belong to the class $\mathcal{M}_0^*$. (see [17], Remark 3.3).

DEFINITION 1.3. For every quasi closed set F of D we denote by ∞_F the Borel measure defined by:

$$\infty_F(B) = \begin{cases} 0 & \text{if } \ C(F \cap B) = 0 \\ +\infty & \text{if } \ C(F \cap B) \neq 0 \end{cases}$$

for every $B \in \mathcal{B}$.

Other examples are given in [21].

Now, we give the definition of the variational μ- capacity associated with any measure $\mu \in \mathcal{M}_0^*$. This will be the basic tool in our investigation.

DEFINITION 1.4. Let $\mu \in \mathcal{M}_0^*$. For every $B \in \mathcal{B}$ we define the μ -capacity of B as:

$$C(\mu, B, D) = \inf \left\{ \int_D |Du|^2 dx + \int_B (u-1)^2 d\mu \ ; \ u \in H_0^1(D) \right\}$$

When no confusion can arise, we will simply write $C(\mu, B)$ instead of $C(\mu, B, D)$.

Since the functional is lower semicontinuous in the weak topology of $H_0^1(D)$, the minimum is achieved.

REMARK 1.1. It is easy to see that if μ is the measure ∞_F of the Definition 1.3 with F quasi closed in D , then $C(\mu, B) = C(B \cap F)$ for every $B \in \mathcal{B}$.

The main properties of the μ -capacity can be summarized in the next Proposition:

PROPOSITION 1.1. For every $\mu \in \mathcal{M}_0^*$ the set function $C(\mu,\cdot)$ satisfies the following properties:

a) $C(\mu,\phi) = 0$

b) if $B_1, B_2 \in \mathcal{B}$ and $B_1 \subseteq B_2$, then $C(\mu,B_1) \leq C(\mu,B_2)$

c) if (B_k) is an increasing sequence in $\mathcal{B}$ and $\bigcup_k B_k = B$, then

$$C(\mu,B) = \sup_k C(\mu,B_k)$$

d) if (B_k) is a sequence in $\mathcal{B}$ and $B \subseteq \bigcup_k B_k$, then

$$C(\mu,B) \leq \sum_k C(\mu,B_k)$$

e) $C(\mu,B_1 \cup B_2) + C(\mu,B_1 \cap B_2) \leq C(\mu,B_1) + C(\mu,B_2)$

for every $B_1, B_2 \in \mathcal{B}$.

f) $C(\mu,B) \leq C(B)$ for every $B \in \mathcal{B}$

g) $C(\mu,B) \leq \mu(B)$ for every $B \in \mathcal{B}$

h) $C(\mu,K) = \inf\{C(\mu,U);\ K \subseteq U,\ U \in \mathcal{U}\}$

for every $K \in \mathcal{K}$

i) $C(\mu,B) = \sup\{C(\mu,K);\ K \subseteq B,\ K \in \mathcal{K}\}$

for every $B \in \mathcal{B}$

For a proof we refer to ([17], Theorem 2.9 - Theorem 3.5 - Theorem 3.7).

The previous properties allow to show an explicit formula to reconstruct a measure $\mu \in \mathcal{M}_0^*$ from the corresponding μ - capacity (see [17], Theorem 4.5).

THEOREM 1.1. Let $\mu \in \mathcal{M}_0^*$. Then for every $B \in \mathcal{B}$ we have

$$\mu(B) = \lim_{h \to \infty} \sum_{i \in \mathbb{Z}^d} C(\mu, B \cap R_h^i)$$

where R_h^i denotes the cube:

$$R_h^i = \prod_{k=1}^{d} \left] \frac{i_k}{2^h}, \frac{i_{k+1}}{2^h} \right]$$

for every $h \in \mathbb{N}$ and for every $i = (i_1, \dots, i_d) \in \mathbb{Z}^d$.

In our paper we are interested in studing a class of equations formally written as:

(1.1) $$\Delta u + \mu u = f \quad \text{in } D$$

(1.2) $$u = g \quad \text{on } \partial D$$

where $g \in H^1(D)$, $f \in L^2(D)$ and $\mu \in \mathcal{M}_0^*$.

Following [20] we shall call the equation (1.1) a relaxed Dirichlet problem in D.

In order to give an appropriate sense to the equation (1.1), we need the following definitions.

DEFINTION 1.5. A function $u \in H^1_{loc}(D) \cap L^2_{loc}(D, \mu)$ is said to be a *local weak solution* of the equation (1.1) if

$$\int_D Du\, Dv\, dx + \int_D u v\, d\mu = \int_D f\, dx$$

for every $v \in H^1(D) \cap L^2(\mu, D)$ with compact support in D.

DEFINITION 1.6. A local weak solution of (1.1) is said to satisfy the boundary condition (1.2) if, in addition, $u - g \in H^1_0(D)$.

The non trivial relationships between the definitions above and the definitions in the sense of distributions are discussed extensively in [21].

REMARK 1.2. It can be proven (see [20]) that if $g \in H^1(D)$ is given in such a way that there exists some $w \in H^1(D) \cap L^2(D,\mu)$ with $w - g \in H_0^1(D)$, then there exists a unique weak solution of problem (1.1)-(1.2), this solution belongs to $H^1(D) \cap L^2(D,\mu)$ and coincides with the unique minimum point of the functional:

$$F(v) = \int_D |Dv|^2 dx + \int_D v^2 d\mu - 2\int_D fv\, dx$$

on the set $\{v : v \in H^1(D),\ v - g \in H_0^1(D)\}$.

In what follows we give two examples of relaxed Dirichlet problems which will be essential in the applications of our main theorems.

EXAMPLE 1.1. Dirichlet problems in domains with holes.

Let $K \in \mathcal{K}$. Let ∞_K be the measure introduced in Definition 1.3. If $\mu = \infty_K$ and $g = 0$ then the problem (1.1)-(1.2) becomes :

$$\text{(1.3)} \qquad \begin{cases} -\Delta u + \infty_K u = f & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

It can be seen in [21] that a function $u \in H^1_{loc}(D) \cap L^2_{loc}(D,\mu)$ is a local weak solution of equation (1.3) if and only if $u|_{D\setminus K}$ is a solution in the usual sense of the boundary value problem:

$$-\Delta u = f \qquad \text{in } D \setminus K$$
$$u \in H_0^1(D \setminus K)$$

and $u|_K = 0$ q.e. on K.

EXAMPLE 1.2. Schroedinger equation.

Let $q \in L^1_{loc}(D)$ with $q \geq 0$. If $\mu(B) = \int_B q(x)\,dx$

then the problem (1.1)-(1.2) becomes

$$(1.4) \qquad \begin{cases} -\Delta u + q(x)\, u = f & \text{in } D \\ u \in H_0^1(D) & \end{cases}$$

We shall also study the following relaxed Dirichlet problem:

$$\begin{cases} -\Delta u + (\mu + \lambda m)\, u = f & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

where $\mu \in \mathcal{M}_0^*$, $f \in L^2(D)$, m denotes the Lebesgue measure on $\mathbf{R}^d$ and $\lambda \geq 0$.

In view of Remark 1.2 we can define a family of operators from $L^2(D)$ into $L^2(D)$ which are called resolvent operators.

DEFINITION 1.7. For every $\lambda \geq 0$ and for every $\mu \in \mathcal{M}_0^*$, the *resolvent operator* R_μ^λ is the mapping which associates with every $f \in L^2(D)$ the unique weak solution $u \in H_0^1(D) \cap L^2(D, \mu) \subseteq L^2(D)$ of the problem (1.4).

REMARK 1.3. R_μ^λ is a linear continuous operator between $L^2(D)$ and $L^2(D)$ (see [5], Definition 2.3).

2. γ - CONVERGENCE

In this section we introduce a variational notion of convergence for sequences (μ_h) in $\mathcal{M}_0^*$ which will be useful to study the perturbations of the relaxed Dirichlet problem (1.2) - (1.3).

With every $\mu \in \mathcal{M}_0^*$ we associate the following functional F_μ defined on $L^2(D)$

$$F_\mu(u) = \begin{cases} \int_D |Du|^2 + \int_D u^2 d\mu & \text{if } u \in H_0^1(D) \\ +\infty & \text{if } u \in L^2(D),\ u \notin H_0^1(D) \end{cases}$$

Since $\mu(B) = 0$ for every $B \in \mathcal{B}$ with $C(B) = 0$, the functional F_μ is lower semicontinuous in $L^2(D)$

The following definition of γ-convergence for sequences of measures (μ_h) belonging to $\mathcal{M}_0^*$ is given in terms of the Γ-convergence of the corresponding functionals F_{μ_h}. For the definition of Γ-convergence and its applications to the study of perturbation problems in calculus of variations, we refer to [2], [23], [24],[25].

DEFINITION 2.1. Let (μ_h) be a sequence in $\mathcal{M}_0^*$ and let $\mu \in \mathcal{M}_0^*$; we say that (μ_h) γ-converges to μ if the following conditions are satisfied:

a) for every $u \in H_0^1(D)$ and for every sequence (u_h) in $H_0^1(D)$ converging to u in $L^2(D)$ we have:

$$F_\mu(u) \le \liminf_{h \to \infty} F_{\mu_h}(u_h)$$

b) for every $u \in H_0^1(D)$, there exists a sequence (u_h) in $H_0^1(D)$ converging to u in $L^2(D)$ such that:

$$F_\mu(u) \ge \limsup_{h \to \infty} F_{\mu_h}(u_h)$$

REMARK 2.1. There exists a unique metrizable topology on $\mathcal{M}_0^*$ which induces the γ-convergence, which will be called the *topology of γ - convergence*. All topological notions we shall consider on $\mathcal{M}_0^*$ are relative to this topology, with respect to which $\mathcal{M}_0^*$ is compact ([17] - Remark 5.4).

A relevant aspect of Definition 1.7 for our purpose is contained in the following Proposition (see [5], Theorem 2.1).

PROPOSITION 2.1. Let (μ_h) be a sequence of measures in $\mathcal{M}_0^*$ and let $\mu \in \mathcal{M}_0^*$. Given $\lambda \geq 0$, let $R^\lambda_{\mu_h}$ be a sequence of resolvent operators associated with the measures μ_h and R^λ_μ the resolvent operator associated with μ. The following statements are equivalent:

a) (μ_h) γ - converges to μ.

b) $(R^\lambda_{\mu_h})$ converges to R^λ_μ strongly in $L^2(D)$.

The following Proposition states the relationships between the γ - convergence of a sequence of measures (μ_h) and the behaviour of the corrisponding μ - capacities, (see [17], Theorem 6.3 and Theorem 5.9).

PROPOSITION 2.2. Let (μ_h) a sequence in $\mathcal{M}_0^*$ and $\mu \in \mathcal{M}_0^*$. Then (μ_h) γ - converges to μ in $\mathcal{M}_0^*$ if and only if the inequalities

a) $$C(\mu, U) \leq \liminf_{h \to \infty} C(\mu_h, U)$$

and

b) $$C(\mu, K) \geq \limsup_{h \to \infty} C(\mu_h, K)$$

hold for every $K \in \mathcal{K}$ and for every $U \in \mathcal{U}$.

REMARK 2.2. In view of Proposition 2.2 a sub-base for the topology induced by γ - convergence on $\mathcal{M}_0^*$ is given by the set of the form $\{\mu \in \mathcal{M}_0^* : C(\mu, U) > t\}$ and $\{\mu \in \mathcal{M}_0^* : C(\mu, K) < s\}$ with $t, s \in R^+$, $U \in \mathcal{U}$ and $K \in \mathcal{K}$.

We denote by $\mathcal{B}(\mathcal{M}_0^*)$ the borel σ - field of $\mathcal{M}_0^*$ endowed with the topology of γ - convergence.

PROPOSITION 2.3. $\mathcal{B}(\mathcal{M}_0^*)$ is the smallest σ-field in $\mathcal{M}_0^*$ for which the functions $C(\cdot, U)$ from $\mathcal{M}_0^*$ into $\mathbb{R}$ are measurable for every $U \in \mathcal{U}$ (respectively the functions $C(\cdot, K)$ are measurable for every $K \in \mathcal{K}$).

Proof: Denote by Σ_1 the smallest σ-field in $\mathcal{M}_0^*$ for which all functions $C(\cdot, U)$, $U \in \mathcal{U}$, are measurable, and by Σ_2 the smallest σ-field in $\mathcal{M}_0^*$ for which all functions $C(\cdot, K)$, $K \in \mathcal{K}$, are measurable.

First, let us show that $\Sigma_1 = \Sigma_2$.
It is enough to prove that

a) any function $C(\cdot, K)$, $K \in \mathcal{K}$ is Σ_1-measurable;

and

b) any function $C(\cdot, U)$, $U \in \mathcal{U}$ is Σ_2-measurable.

Let us prove a). For every $K \in \mathcal{K}$, consider the decreasing sequence of open set:

$$U_h = \{x \in D : d(x, K) < 1/h\}$$

We remark that $U_h \searrow K$. By (h) of Proposition 1.1 we have

$$C(\mu, K) = \inf_{h \in \mathbb{N}} C(\mu, U_h)$$

for every $\mu \in \mathcal{M}_0^*$, which proves a).

Assertion b) can be proved in the same way, by choosing, for every $U \in \mathcal{U}$, an increasing sequence (K_h) in $\mathcal{K}$ such that $K_h \nearrow U$ and by using Proposition 1.1, (i).

The proof of the Proposition is complete if we show that $\mathcal{B}(\mathcal{M}_0^*) = \Sigma_1$. The inclusion $\Sigma_1 \subseteq \mathcal{B}(\mathcal{M}_0^*)$ is trivial because $C(\cdot, U)$, $U \in \mathcal{U}$ is lower semicontinuous on $\mathcal{M}_0^*$ by Proposition 2.2 (a). In order to show that $\mathcal{B}(\mathcal{M}_0^*) \subseteq \Sigma_1$, we have only to observe that the sub-base for the topology of the γ-convergence given in Remark 2.2 is contained in Σ_1 (because $\Sigma_1 = \Sigma_2$) and that $\mathcal{M}_0^*$ admits a countable basis for the open sets. ■

The next Corollary follows directly from the previous proposition.

COROLLARY 2.1. Let (Ω, Σ, P) be a measure space. Let M be a function from Ω into $\mathcal{M}_0^*$. The following statements are equivalent:

a) M is Σ - $\mathcal{B}(\mathcal{M}_0^*)$ measurable;

b) $C(M(\cdot), U)$ is Σ - measurable for every $U \in \mathcal{U}$;
c) $C(M(\cdot), K)$ is Σ - measurable for every $K \in \mathcal{K}$.

We need also some result about the measurability of the function $C(\cdot, B)$ for every $B \in \mathcal{B}$. Let us denote by $\widehat{\mathcal{B}}(\mathcal{M}_0^*)$ the σ - algebra of all subset of $\mathcal{M}_0^*$ which are universally measurable with respect to $\mathcal{B}(\mathcal{M}_0^*)$ (i.e. Q- measurable for every probability measure Q on $(\mathcal{M}_0^*, \mathcal{B}(\mathcal{M}_0^*))$.

PROPOSITION 2.4 For every $B \in \mathcal{B}$ the function $C(\cdot, B)$ is $\widehat{\mathcal{B}}(\mathcal{M}_0^*)$ - measurable.

Proof: Let Q be a probability measure on $\mathcal{B}(\mathcal{M}_0^*)$. For every $B \in \mathcal{U} \cup \mathcal{K}$ we set

$$\alpha(B) = \int_{\mathcal{M}_0^*} C(\mu, B)\, dQ$$

By properties (h), (i) and (e) of $C(\mu, \cdot)$ in Proposition 1.1 we have that:

$$\alpha(K) = \inf\{\alpha(U) \,;\, U \supseteq K \,,\, U \in \mathcal{U}\} \tag{2.1}$$

for every $K \in \mathcal{K}$,

$$\alpha(U) = \sup\{\alpha(K) \,;\, K \subseteq U \,,\, K \in \mathcal{K}\} \tag{2.2}$$

for every $U \in \mathcal{U}$, and

$$\alpha(K_1 \cup K_2) + \alpha(K_1 \cap K_2) \leq \alpha(K_1) + \alpha(K_2) \tag{2.3}$$

for every $K_1, K_2 \in \mathcal{K}$.

We can extend the definition of α by

$$\alpha(B) = \inf\{\alpha(U) \,;\, U \supseteq B \,,\, U \in \mathcal{U}\} \tag{2.4}$$

for every $B \in \mathcal{B}$. We infer from (2.1), (2.2), (2.3). (2.4) that α is a Choquet capacity on B (see [27], Theorem 1.5). Applying the capacitabily Theorem (see [12]) we get

(2.5) $$\alpha(B) = \sup\{\alpha(K);\ K \subseteq B,\ K \in \mathcal{K}\}$$

for every $B \in \mathcal{B}$. Now, fix $B \in \mathcal{B}$. By (2.4) it follows that for every $\varepsilon > 0$ there exists $U \in \mathcal{U}$, $U \supseteq B$ such that:

(2.6) $$\alpha(B) + \varepsilon/2 > \alpha(U)$$

Moreover, by (2.5) we also get that for every $\varepsilon > 0$ there exists a $K \in \mathcal{K}$, $K \subseteq B$ such that:

(2.7) $$\alpha(B) - \varepsilon/2 < \alpha(K)$$

By (2.6) and (2.7) we get that for every $\varepsilon > 0$

(2.8) $$\int_{\mathcal{M}_0^*} [C(\mu, U) - C(\mu, K)]\, dQ < \varepsilon$$

Since $C(\cdot, K) \leq C(\cdot, B) \leq C(\cdot, U)$, (2.8) gives the measurability of $C(\cdot, B)$ respect to the σ- field of all subsets Q - measurable.Finally, the assertion follows noting that Q is an arbitrary probability measure on $\mathcal{B}(\mathcal{M}_0^*)$. ■

At the end of this Section we recall some probabilistic notions which we use in the sequel.

By $\mathcal{P}(\mathcal{M}_0^*)$ we mean the space of all probability measures defind on $\mathcal{B}(\mathcal{M}_0^*)$, i.e. an element $Q \in \mathcal{P}(\mathcal{M}_0^*)$ is a non negative countably additive set function defined on $\mathcal{B}(\mathcal{M}_0^*)$ with $Q(\mathcal{M}_0^*) = 1$.

We recall the concept of the weak convergence for a sequence (Q_h) of measures belonging to $\mathcal{P}(\mathcal{M}_0^*)$.

DEFINITION 2.2. We say that a sequence (Q_h) of measures in $\mathcal{P}(\mathcal{M}_0^*)$ converges *weakly* to a measure Q in $\mathcal{P}(\mathcal{M}_0^*)$ if:

$$\lim_{h \to \infty} \int_{\mathcal{M}_0^*} f\, dQ_h = \int_{\mathcal{M}_0^*} f\, dQ$$

for every continuous function $f : \mathcal{M}_0^* \longrightarrow R$.

Simillar problems of weak convergence of measures on spaces endowed with topology related to Γ -convergece have been studied in [18] and [19].

The two results that we give in the following hold for a generic compact metric space. For the proofs we refer respectively to [1], Theorem 4.5.1 and to [39], Theorem 6.4.

PROPOSITION 2.5. Let (Q_h) be a sequence of probability measures in $\mathcal{P}(\mathcal{M}_0^*)$ and let $Q \in \mathcal{P}(\mathcal{M}_0^*)$. The following statement are equivalent:

a) (Q_h) converges weakly to Q in $\mathcal{P}(\mathcal{M}_0^*)$.

b) $\lim_{h\to\infty} \int_{\mathcal{M}_0^*} f \, dQ_h = \int_{\mathcal{M}_0^*} f \, dQ$

for every function $f : \mathcal{M}_0^* \longrightarrow \mathbf{R}$ such that

$$Q\{\mu \in \mathcal{M}_0^* : f \text{ is continuous at } \mu\} = 1$$

PROPOSITION 2.6. For every sequence (Q_h) of measures in $\mathcal{P}(\mathcal{M}_0^*)$ there exists a sub-sequence (Q_{h_k}) weakly convergent in $\mathcal{P}(\mathcal{M}_0^*)$.

We conclude with some definitions:

DEFINITION 2.3. For every $\mathcal{B}(\mathcal{M}_0^*)$ - measurable function X we denote by $E_Q[X]$ the *expectation* of X in the probability space $(\mathcal{M}_0^*, \mathcal{B}(\mathcal{M}_0^*), Q)$, defined by

$$E_Q[X] = \int_{\mathcal{M}_0^*} X(\mu) \, dQ(\mu)$$

and by $Var_Q[X]$ the *variance* of X in the probability space $(\mathcal{M}_0^*, \mathcal{B}(\mathcal{M}_0^*), Q)$, defined by

$$Var_Q[X] = E_Q[(X - E_Q[X])^2]$$

DEFINITION 2.4. For every $X, Y \in L^2(\mathcal{M}_0^*, \mathcal{B}(\mathcal{M}_0^*), Q)$ we denote by $Cov_Q[X, Y]$ the *covariance* of X and Y in the probability space $(\mathcal{M}_0^*, \mathcal{B}(\mathcal{M}_0^*), Q)$ defined by

$$Cov_Q[X, Y] = E_Q[XY] - E_Q[X] E_Q[Y] .$$

3. THE MAIN RESULT

In this section we prove the main result of this paper: a necessary and sufficient condition for the convergence of a sequence (Q_k) of measures on $\mathcal{M}_0^*$ of the class $\mathcal{P}(\mathcal{M}_0^*)$ to a measure $\delta_\nu \in \mathcal{P}(\mathcal{M}_0^*)$ of the form:

$$\delta_\nu(\xi) = \begin{cases} 0 & \text{if } \nu \notin \xi \\ 1 & \text{if } \nu \in \xi \end{cases} \tag{3.1}$$

for every $\xi \in \mathcal{B}(\mathcal{M}_0^*)$
where ν is a finite Borel measure on D of the class $\mathcal{M}_0^*$. This condition is expressed in terms of the asymptotic behaviour, as $k \to \infty$, of the functions $C(\cdot, B)$, $B \in \mathcal{B}$, considered as a random variables on the probability spaces $(\mathcal{M}_0^*, \mathcal{B}(\mathcal{M}_0^*), Q_k)$.

We begin with some definitions. Let (Q_k) be a sequence in $\mathcal{P}(\mathcal{M}_0^*)$.
First, for every $U \in \mathcal{U}$ we define:

$$\alpha'(U) = \liminf_{k \to \infty} E_{Q_k}[C(\cdot, U)]$$

and

$$\alpha''(U) = \limsup_{k \to \infty} E_{Q_k}[C(\cdot, U)]$$

where E_{Q_k} denotes the expectation in the probability space $(\mathcal{M}_0^*, \mathcal{B}(\mathcal{M}_0^*), Q_k)$

Next we consider the inner regularizations α'_- and α''_- of α' and α'' defined for every $U \in \mathcal{U}$ by:

$$\alpha'_-(U) = \sup\{\alpha'(V);\ V \in \mathcal{U},\ \overline{V} \subset U\} \tag{3.2}$$

and

$$\alpha''_-(U) = \sup\{\alpha''(V);\ V \in \mathcal{U},\ \overline{V} \subset U\} \tag{3.3}$$

Then we extend the definitions of α'_- and α''_- to the arbitrary Borel sets $B \subseteq D$ by:

$$(3.4) \qquad \alpha'_-(B) = \inf\{\alpha'_-(U);\ U \in \mathcal{U},\ U \supseteq B\}$$

and

$$(3.5) \qquad \alpha''_-(B) = \inf\{\alpha''_-(U);\ U \in \mathcal{U},\ U \supseteq B\}$$

for every $B \in \mathcal{B}$.

Finally, we denote by ν' and ν'' the least superadditive set functions on $\mathcal{B}$ greater than or equal to α'_- and α''_- respetively.

We are now in a position to state our main result.

THEOREM 3.1. Let (Q_ε) be a sequence of measures on $\mathcal{M}_0^*$ of the class $\mathcal{P}(\mathcal{M}_0^*)$. Assume that

i) $\nu'(B) = \nu''(B) < +\infty$ for every $B \in \mathcal{B}$

and denote by $\nu(B)$ the common value of $\nu'(B)$ and $\nu''(B)$ for every $B \in \mathcal{B}$

.

Suppose in addition that

ii) there exist a constant $\varepsilon > 0$, an increasing continuous function

$$\xi : R \times R \longrightarrow R$$

with $\xi(0,0) = 0$ and a Radon measure β on $\mathcal{B}$ such that

$$\limsup_{h \to \infty} |\mathrm{cov}_{Q_\varepsilon}[C(\cdot,U), C(\cdot,V)]| \leq \xi(\mathrm{diam}\,U, \mathrm{diam}\,V)\, \beta(U)\, \beta(V)$$

for every pair $U, V \in \mathcal{U}$ such that $U \cap V = \emptyset$ with diam $U < \varepsilon$ and diam $V < \varepsilon$.

Then

a) ν is a finite Borel measure on $\mathcal{B}$ of the class $\mathcal{M}_0^*$;

b) (Q_ε) converges weakly to the probability measure δ_ν defined by:

$$\delta_\nu(\Xi) = \begin{cases} 0 & \text{if } \nu \notin \Xi \\ 1 & \text{if } \nu \in \Xi \end{cases}$$

for every $\Xi \in \mathcal{B}(\mathcal{M}_0^*)$;

c) $\alpha'_-(B) = \alpha''_-(B) = C(\nu, B)$ for every $B \in \mathcal{B}$.

REMARK 3.1. Let $\alpha_\varepsilon : \mathcal{U} \longrightarrow R$ be an increasing set function defined by:

$$\alpha_k(U) = E_{Q_k}[C(\cdot, U)]$$

and let $\alpha : \mathcal{U} \longrightarrow \mathbf{R}$ be an increasing set function defined by

$$\alpha(U) = C(\nu, U)$$

Then the condition c) of Theorem 3.1 is equivalent to say that (α_k) converges weakly to α in the sense of [26] (with respect to the pair $(\mathcal{U}, \mathcal{K})$).

For the proof of Theorem 3.1 we need some preliminay results. We begin with a general probabilistic Lemma.

LEMMA 3.1. Let (Ω, Σ, P) be a probability space. Consider a sequence (x_k) of non negative random variables on (Ω, Σ, P).
Suppose that

i) $x_k \in L^2(\Omega, P)$ for every $k \in \mathbb{N}$.

ii) x_k converges to x for P-almost every $\omega \in \Omega$.

iii) $\lim_{k \to \infty} var(x_k) = 0$

Then there exists a constant x_0 such that $x(\omega) = x_0$ for P-almost every $\omega \in \Omega$.

Proof: Choose a non negative sequence ε_k such that

$$\lim_{k \to \infty} \varepsilon_k = 0 \quad \text{and} \quad \lim_{k \to \infty} \frac{var(x_k)}{\varepsilon_k^2} = 0$$

Set

$$t_k = \frac{var(x_k)}{\varepsilon_k^2}$$

Then there exists a subsequence of t_k, still denoted by t_k, such that $\sum_{k \in \mathbb{N}} t_k < +\infty$.
Consider the sets

$$B_k = \{\omega \in \Omega : |x_k - E[x_k]| \geq \varepsilon_k\}$$

By Chebychev's inequality we have $P(B_\ell) < t_\ell$ for every ℓ and by Borel-Cantelli's Lemma it follows that

$$P(\limsup_{\ell \to \infty} B_\ell) = 0$$

Consequently, if ω_1, ω_2 are two elements in $\Omega \setminus \limsup_{\ell \to \infty} B_\ell$, we obtain

$$|X_\ell(\omega_1) - X_\ell(\omega_2)| < 2\varepsilon_\ell$$

for ℓ large enough. Passing to the limit, as $\ell \to \infty$, we get the proof of the assertion.

■

In the next Lemma we prove a result concerning increasing set functions, i.e. functions $\alpha : \mathcal{B} \to \overline{\mathbb{R}}$ such that $\alpha(A) \le \alpha(B)$ whenever $A, B \in \mathcal{B}$ and $A \subseteq B$.
First we need some elementary definitions.

DEFINITIONS 3.1. A subset $\mathcal{D}$ of $\mathcal{U}$ is said to be *dense* if for every pair $U, V \in \mathcal{U}$ such that $\overline{U} \subset V$, there exists a set $W \in \mathcal{D}$ such that $\overline{U} \subset W \subset \overline{W} \subset V$.

LEMMA 3.2. Let $\alpha : \mathcal{B} \to \overline{\mathbb{R}}$ be any increasing set function. Then the set:

$$\mathcal{D} = \{ W \in \mathcal{U} : \overline{W} \subset D,\ \alpha(W) = \alpha(\overline{W})$$

is dense in $\mathcal{U}$.

Proof: The Lemma is an immediate consequence of Proposition 4.7 of [26]. For the readers convenience we repeat here the proof in our particular case.

Let U, V be in $\mathcal{U}$ such that $\overline{U} \subset V$. By Uryshon's Lemma there exists a function $f \in C_0^\circ(V)$ such that $0 \le f(x) \le 1$ for every $x \in V$ and $f = 1$ on U. For every $t \in]0, 1[\equiv T$ we consider the open set:

$$U_t = \{ x \in V : f(x) > t \}$$

Let $g : T \to \mathbb{R}$ be the function defined in the following way:

$$g(t) = \alpha(U_t)$$

Then g is a decreasing function and for every $t \in T$ we have:

$$\inf_{s<t} g(s) \ge \alpha(\overline{U}_t) \ge \alpha(U_t) \ge \sup_{s>t} g(s)$$

Since the function g has at most a countable set of discontinuity points in T, there exists $t \in T$ such that $\alpha(\bar{U}_t) = \alpha(U_t)$ and this proves the Lemma. ■

In the following we give sufficient conditions in order to have that a probability measure $Q \in \mathcal{P}(\mathcal{M}_0^*)$ be equal to the measure δ_ν defined in (3.1). The conditions are given in terms of the functions $C(\cdot, B)$, $B \in \mathcal{B}$ considered as random variables on $(\mathcal{M}_0^*, \mathcal{B}(\mathcal{M}_0^*), Q)$.

LEMMA 3.3 Let Q be a probability measure on $\mathcal{M}_0^*$ of the class $\mathcal{P}(\mathcal{M}_0^*)$.
Define $\alpha(U) = E_Q[C(\cdot, U)]$ for every $U \in \mathcal{U}$, and

$$\alpha(B) = \inf\{\alpha(U);\ U \supseteq B,\ U \in \mathcal{U}\}$$

for every $B \in \mathcal{B}$. Assume that:

(i) There exists a Radon measure β_1 on $\mathcal{B}$ such that $\beta_1 \geq \alpha$ on $\mathcal{B}$;

(ii) There exist a constant $\varepsilon > 0$, a Radon measure β_2 on $\mathcal{B}$ and an increasing continuous function $\xi : \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$ with $\xi(0,0) = 0$ s.t.:

$$|\mathrm{Cov}_Q[C(\cdot, U), C(\cdot, V)]| \leq \xi(\mathrm{diam}\, U, \mathrm{diam}\, V)\, \beta_2(U)\, \beta_2(V) \tag{3.6}$$

for every pair $U, V \in \mathcal{U}$ such that $\bar{U} \cap \bar{V} = \emptyset$, with diam $U < \varepsilon$ and diam $V < \varepsilon$.

Let ν be the least superadditive set function on $\mathcal{B}$ such that $\nu \geq \alpha$ on $\mathcal{B}$. Then ν is a measure on $\mathcal{B}$ of the class $\mathcal{M}_0^*$ and

$$Q = \delta_\nu$$

Proof: The function α is countably subadditive on $\mathcal{U}$ (hence on $\mathcal{B}$) by the countable subadditivity of $C(\mu, \cdot)$ (Proposition 1.1, (d)). Therefore ν is a measure by Lemma 4.1 of [17]. We observe that the measure ν is in $\mathcal{M}_0^*$ because it is a Radon measure and $\nu(B) = 0$ whenever $C(B) = 0$ by Proposition 1.1, (f). By properties (h) and (i) of Proposition 1.1 we can extend the relation (3.6) to each pair of disjoint sets $A, B \in \mathcal{B}$ and check that:

$$\alpha(B) = E_Q[c(\cdot, B)]$$

for every $B \in \mathcal{B}$.

Let us denote by $z(\cdot, B)$ the random variable on the probability space $(\mathcal{M}_0^d, \mathcal{B}(\mathcal{M}_0^d), Q)$ defined by:

$$z(\mu, B) = \mu(B)$$

for every $B \in \mathcal{B}$.

By Theorem 1.1 we have that:

$$z(B) = \lim_{\ell \to \infty} \sum_{i \in \mathbb{Z}^d} c(\cdot, B \cap R_\ell^i)$$

for every $B \in \mathcal{B}$, where R_ℓ^i denotes the cube defined in Theorem 1.1. We apply now Lemma 3.1 to show that $z(\cdot, B)$ is a costant random variable. Therefore, we have only to prove that:

$$\lim_{\ell \to \infty} \mathrm{Var}_Q \Big[\sum_{i \in \mathbb{Z}^d} c(\cdot, B \cap R_\ell^i)\Big] = 0$$

Now, let us fix $B \in \mathcal{B}$ with $\overline{B} \subseteq D$. For every $\ell \in \mathbb{N}$, we have:

(3.7) $$\sum_{i \in \mathbb{Z}^d} \mathrm{Var}_Q[c(\cdot, B \cap R_\ell^i)] =$$

$$\sum_{i \in \mathbb{Z}^d} \Big\{E_Q[c(\cdot, B \cap R_\ell^i)^2] - (E_Q[c(\cdot, B \cap R_\ell^i)])^2\Big\} \leq$$

$$\sum_{i \in \mathbb{Z}^d} E_Q[c(\cdot, B \cap R_\ell^i)^2] \leq$$

$$\sum_{i \in \mathbb{Z}^d} c(B \cap R_\ell^i)\, E_Q[c(\cdot, B \cap R_\ell^i)] \leq$$

$$\sup_{i \in \mathbb{Z}^d} c(B \cap R_\ell^i) \sum_{i \in \mathbb{Z}^d} \alpha(B \cap R_\ell^i) \leq s_\ell\, \beta_1(B)$$

where we have set $s_h = \sup_{i \in \mathbb{Z}^d} C(B \cap R_h^i)$.

We observe that $s_h \to 0$ as $h \to \infty$ because the dimension d is greater than or equal to 2 and $\overline{B}$ is compact in D . On the other hand, by hypotheses there exists $h_0 \in \mathbb{N}$ such that, for every $h \geq h_0$,

(3.8) $$\Big| \sum_{\substack{i,j \in \mathbb{Z}^d \\ i \neq j}} \mathrm{cov}_Q [C(\cdot, B \cap R_h^i), C(\cdot, B \cap R_h^j)] \Big| \leq$$

$$\sum_{\substack{i,j \in \mathbb{Z}^d \\ i \neq j}} \xi(\mathrm{diam}(B \cap R_h^i), \mathrm{diam}(B \cap R_h^j)) \beta_2(B \cap R_h^i) \beta_2(B \cap R_h^j) \leq$$

$$\xi(\mathrm{diam}\, R_h^0, \mathrm{diam}\, R_h^0) \sum_{\substack{i,j \in \mathbb{Z}^d \\ i \neq j}} \beta_2(B \cap R_h^i) \beta_2(B \cap R_h^j) \leq$$

$$\xi(\mathrm{diam}\, R_h^0, \mathrm{diam}\, R_h^0) [\beta_2(B)]^2$$

By (3.7), (3.8) and by hypothesis we get

$$\lim_{h \to \infty} \mathrm{var}_Q [\sum_{i \in \mathbb{Z}^d} C(\cdot, B \cap R_h^i)] \leq$$

$$\lim_{h \to \infty} \Big\{ \sum_{i \in \mathbb{Z}^d} \mathrm{var}_Q [C(\cdot, B \cap R_h^i)] + \sum_{\substack{i,j \in \mathbb{Z}^d \\ i \neq j}} \mathrm{cov}_Q [C(\cdot, B \cap R_h^i), C(\cdot, B \cap R_h^j)] \Big\} \leq$$

$$\lim_{h \to \infty} \{ s_h \beta_1(B) + \xi(\mathrm{diam}\, R_h^0, \mathrm{diam}\, R_h^0) [\beta_2(B)]^2 \} = 0$$

Therefore Lemma 3.2 implies that for every Borel set $\Xi(\cdot, B)$ is a costant radom variable. Now, let us compute the expectation of $\Xi(\cdot, B)$. Since the sequence $(\sum_{i \in \mathbb{Z}^d} C(\cdot, B \cap R_h^i))_{h \in \mathbb{N}}$ is increasing, we get

$$E_Q[z(\cdot, B)] = \lim_{h\to\infty} E_Q\Big[\sum_{i\in\mathbb{Z}^d} C(\cdot, B\cap R_h^i)\Big] =$$

$$\lim_{h\to\infty} \sum_{i\in\mathbb{Z}^d} \alpha(B\cap R_h^i) = \nu(B)$$

for every $B\in\mathcal{B}$, where the last equality is proved in [17], Lemma 4.2.
Hence, for every $B\in\mathcal{B}$ there exists a subset $\mathcal{M}_B$ of $\mathcal{M}_0^*$ with $Q(\mathcal{M}_B = 1)$ such that $z(\mu,B) = \nu(B)$ for every $\mu\in\mathcal{M}_B$. Let $\mathcal{D}$ be a countable dense set in $\mathcal{U}$ and let us consider:

$$\mathcal{M} = \bigcap_{U\in\mathcal{D}} \mathcal{M}_U$$

We obtain that $z(\mu,U) = \nu(U)$ for every $\mu\in\mathcal{M}$ and $Q(\mathcal{M})=1$. This implies that $z(\mu,\cdot)$ is a Radon measure on $\mathcal{B}$ for every $\mu\in\mathcal{M}$, and since $z(\mu,\cdot)$ coincides with ν on a dense set $\mathcal{D}$ in $\mathcal{U}$, we can deduce that $z(\mu,B) = \nu(B)$ for every $B\in\mathcal{B}$ and for every $\mu\in\mathcal{M}$. This concludes the proof of the Lemma. ■

Proof of Theorem 3.1: The set function α'' is subadditive on $\mathcal{U}$, being the upper limit of a sequence of subadditive set functions on $\mathcal{U}$. Therefore its inner regularization α''_- is countably subadditive on $\mathcal{U}$ by Theorem 5.6 of [26]. It is now easy to see that α''_- is countably subadditive on $\mathcal{B}$, so that ν'' is a measure by Lemma 4.1 of [17]. Moreover, $\nu''(B) = 0$ whenever $C(B) = 0$ by Proposition 1.1(f). This proves assertion (a).
Since $\mathcal{P}(\mathcal{M}_0^*)$ is sequentially compact space and ν' and ν'' do not change by passing to a subsequence, in order to prove (b) we can assume that Q_h converges weakly to a probability measure $Q\in\mathcal{P}(\mathcal{M}_0^*)$ and we have only to prove that $Q = \delta_\nu$.
By Lemma 3.2 the set:

$$\mathcal{D} = \{U\in\mathcal{U} : E_Q[C(\cdot,\bar{U})] = E_Q[C(\cdot,U)]\}$$

is dense in $\mathcal{U}$.
Consequently, for every $U\in\mathcal{D}$, the equality $C(\mu,U) = C(\mu,\bar{U})$

holds for $\mathcal{Q}$ - almost all $\mu \in \mathcal{M}_o^*$. Therefore, by Proposition 2.2:

$$\mathcal{Q}\{\mu \in \mathcal{M}_o^* : C(\cdot, U) \text{ is } \gamma\text{-continuous at } \mu\} = 1$$

for every $U \in \mathcal{D}$. Then, by Proposition 2.5 we have:

$$\lim_{h \to \infty} E_{\mathcal{Q}_h}[C(\cdot, U)] = E_{\mathcal{Q}}[C(\cdot, U)] = \alpha'(U) = \alpha''(U) \tag{3.9}$$

for every $U \in \mathcal{D}$,and

$$\lim_{h \to \infty} E_{\mathcal{Q}_h}[C(\cdot, U) C(\cdot, V)] = E_{\mathcal{Q}}[C(\cdot, U) C(\cdot, V)] \tag{3.10}$$

for every $U, V \in \mathcal{D}$.

By (3.9), (3.10) by hypothesis (ii) and by the properties of the μ-capacity (Proposition 1.1, (h) and (i)) we get that

$$E_{\mathcal{Q}}[C(\cdot, U)] = \alpha'_-(U) = \alpha''_-(U) \tag{3.11}$$

for every $U \in \mathcal{U}$, and

$$|Cov_{\mathcal{Q}}[C(\cdot, U), C(\cdot\, V)]| \leq \xi(\operatorname{diam} U, \operatorname{diam} V)\, \beta(U)\, \beta(V)$$

for every pair $U, V \in \mathcal{U}$ with diam $U < \varepsilon$ and diam $V < \varepsilon$ such that $\bar{U} \cap \bar{V} = \phi$.

Assertion (b) follows now from Lemma 3.3.

Assertion (c) can be obtained from (b) and (3.11) by using (3.4), (3.5) and the property of $C(\mu, \cdot)$ stated in Proposition 1.1, (h) and (i). ∎

REMARK 3.2 Conditions (i) and (ii) of Theorem 3.1 are also necessary. In fact, if $\mathcal{Q}_h$ converges weakly to a probability measure of the form δ_ν (see (3.1)), where ν is a finite Borel measure on $\mathcal{B}$ of the class $\mathcal{M}_o^*$, then (3.9) and (3.10) imply that there exists a family $\mathcal{D}$ dense in $\mathcal{U}$ such that

$$\alpha'(U) = \alpha''(U) = C(\nu, U) \tag{3.12}$$

for every $U \in \mathcal{D}$ and

(3.13) $$\lim_{h\to\infty} |\operatorname{cov}_{Q_h}[C(\cdot, U), C(\cdot, V)]| = 0$$

for every $U, V \in \mathcal{U}$ with $\overline{U} \cap \overline{V} = \phi$. By the properties of the capacities $C(\mu, \cdot)$ (Proposition 1.1, (h), (i)), (3.12) implies that

(3.14) $$\alpha'_{-}(B) = \alpha''_{-}(B) = C(\nu, B)$$

for every $B \in \mathcal{B}$ and (3.13) implies condition (ii) of Theorem 3.1. The condition (i) follows now from (3.14) and from the characterization of ν as the least superadditive set function greater than or equal to $C(\nu, \cdot)$, (see[17], Theorem 4.3)

4. DIRICHLET PROBLEMS IN DOMAINS WITH RANDOM SMALL HOLES.

In this section we consider an application of our results to a Dirichlet problem in a domain with small holes. In order to semplify the computations we assume $d \geq 3$.

Let (Ω, Σ, P) be a probability space. We shall denote by E and by Cov respectively the expectation and the covariance of a random variable, with respect to the measure P.

DEFINTION 4.1. A measurable function $M : \Omega \longrightarrow \mathcal{M}_0^*$ will be called *random measure.*

We recall that necessary and sufficient conditions for the measurability of a function $M : \Omega \longrightarrow \mathcal{M}_0^*$ are given in Corollary 2.1.

Let M be a random measure.

DEFINITION 4.2. The probability measure in $\mathcal{P}(\mathcal{M}_0^*)$ defined by

$$Q(\mathcal{E}) = P\{M^{-1}(\mathcal{E})\} \qquad \text{for any } \mathcal{E} \in \mathcal{B}(\mathcal{M}_0^*)$$

will be called *the distribution law* of the random measure M.

Let M_ε be a sequence of random measures and M a random measure. Let Q_ε be the sequence of the distribution laws of M_ε and let Q be the distribution law of M.

DEFINTION 4.3. We say that M_ε *converges in law* to the random measure M if and only if the distribution laws Q_ε converges weakly in $\mathcal{P}(\mathcal{M}_0^*)$ to the distribution law Q.

Let Q be the distribution of a random measure M. It is easy to see that:

$$\text{(4.1)} \quad E_Q[C(\cdot, v)] = E[C(M(\cdot), v)] \qquad \text{for any } v \in \mathcal{U}$$

(4.2) $$\mathrm{Cov}_Q[C(\cdot, U)C(\cdot, V)] = E[C(M(\cdot), U)C(M(\cdot), V)] - E[C(M(\cdot), U)]E[C(M(\cdot), V)] = \mathrm{Cov}[C(M(\cdot), U)C(M(\cdot), V)]$$

for any pair $U, V \in \mathcal{U}$.

Let M_ε be a sequence of random measures and let Q_ε be the corresponding sequence of distribution laws.

Let us define the set functions:

(4.3) $$\alpha'(U) = \liminf_{\varepsilon \to \infty} E[C(M_\varepsilon(\cdot), U)]$$

(4.4) $$\alpha''(U) = \limsup_{\varepsilon \to \infty} E[C(M_\varepsilon(\cdot), U)]$$

for every $U \in \mathcal{U}$.

In the sequel we will denote by α'_- and α''_- respesctively the inner regularization of α' and α'' as defined in (3.2) and (3.3).

The functions ν' and ν'' will be the least superadditive set function on $\mathcal{B}$ greater than or equal to α'_- and α''_- , respectively.

REMARK 4.1. Equalities (4.1), (4.2), (4.3), (4.4) allow to reformulate the hypotheses of Theorem 3.1 in terms of the expectations and covariances of the random variables $C(M(\cdot), U)$. By definition 4.3 the theses of Theorem 3.1 can be reformulated saying that the sequence M_ε converges in law to a random measure M such that $M(\omega) = \nu$ for P -almost every $\omega \in \Omega$ (i.e. to the constant random measure $M = \nu$).

REMARK 4.2. It is well known that, whenever M is a constant random measure, the convergence in law and the convergence in probability toward M of the sequence M_ε of random measures are equivalent. Thus, by Remark 4.1, we can deduce that, if the assumptions of Theorem 3.1 hold, then the sequence M_h converges in probability to the measure ν in $\mathcal{M}_0^*$, that is: for every $\varepsilon > 0$

$$\lim_{h \to \infty} P\{\omega \in \Omega : d_\gamma(M_h(\omega), \nu) > \varepsilon\} = 0$$

where d_γ is any metric on $\mathcal{M}_0^*$ which induces γ-convergence (Remark 2.1).

We wish to study the following sequence of random relaxed Dirichlet problems:

$$\begin{cases} -\Delta u_h + (M_h + \lambda m) u_h = f & \text{in } D \\ u_h = 0 & \text{on } \partial D \end{cases}$$

where $\lambda \geq 0$, $f \in L^2(D)$, m denotes the Lebesgue measure on R^d.

Let $\nu \in \mathcal{M}_0^*$ and let R^λ be the resolvent operator associated with ν . The next Theorem states a relationship between the previous results and the convergence of the resolvent operators R_h^λ associated with the random measures M_h

THEOREM 4.1. Let M_h be a sequence of random measures. Let α' and α'' be the functions defined in (4.3) and (4.4) and let ν' and ν'' be the least superadditive set functions on $\mathcal{B}$ greater than or equal to α'_- and α''_- respectively.

Assume that:

(i) $\nu'(B) = \nu''(B) < +\infty$ for every $B \in \mathcal{B}$

and denote by $\nu(B)$ the common value of $\nu'(B)$ and $\nu''(B)$ for every $B \in \mathcal{B}$. Suppose, in addition, that:

(ii) There exist a constant $\varepsilon > 0$, an increasing continuous function $\xi : R \times R \to R$ with $\xi(0,0) = 0$ and a Radon measure β on $\mathcal{B}$ such that:

$$\limsup_{h \to \infty} |\operatorname{cov}[C(M_h(\cdot), U)\, C(M_h(\cdot), V)]| \leq$$

$$\xi(\operatorname{diam} U, \operatorname{diam} V)\, \beta(U)\, \beta(V)$$

for every pair $U, V \in \mathcal{U}$ such that $\bar{U} \cap \bar{V} = \phi$ and with diam $U < \varepsilon$, diam $V < \varepsilon$.

Then, for every $\lambda \geq 0$, R^λ_ε converges strongly in probability to R^λ, i.e.

$$\lim_{\varepsilon \to \infty} P\{\omega \in \Omega : \| R^\lambda_\varepsilon(\omega)[\varphi] - R^\lambda[\varphi] \|_{L^2(D)} > \varepsilon\} = 0$$

for every $\varepsilon > 0$, and for any $\varphi \in L^2(D)$.

Proof: By Remark 4.2 we have that the sequence M_ε converges in probability to ν in $\mathcal{M}_0^*$. To get the assertion it is enough to recall that, by Proposition 2.1, for every $\omega \in \Omega$ the sequence of measures M_ε γ-converges to ν if and only if the resolvent operators $R^\lambda_\varepsilon(\omega)$ converge to R^λ strongly in $L^2(D)$. ■

Next, we wish to consider a particular sequence M_ε of random measures related with Dirichlet problems in domains with random holes.

Let $\mathcal{F}(D)$ be the family of all closed sets contained in D.

DEFINITION 4.4. A function $F : \Omega \to \mathcal{F}(D)$ is called a *random set* if the function $M : \Omega \to \mathcal{M}_0^*$ defined by $M(\omega) = \infty_{F(\omega)}$ for each $\omega \in \Omega$ is Σ - measurable, where ∞_F is the measure in $\mathcal{M}_0^*$ as in Definition 1.3.

REMARK 4.3. Let $F : \Omega \to \mathcal{F}(D)$ be a function. By Corollary 2.1 and by the equality $C(\infty_E, B) = C(E \cap B)$ the following conditions are equivalent:

a) F a random set.

b) $C(F(\cdot) \cap U)$ is Σ -measurable for every $U \in \mathcal{U}$.

c) $C(F(\cdot) \cap K)$ is Σ -measurable for every $K \in \mathcal{K}$.

Let us take a sequence F_ε of random sets. Let M_ε be the sequence of random measures so defined:

$$M_\varepsilon(\omega) = \infty_{F_\varepsilon}(\omega) \qquad \text{for each } \omega \in \Omega$$

Let $f \in L^2(D)$ and $\lambda \geq 0$ a real parameter. We shall consider the weak solutions u_ε of the following Dirichlet problems on random domains:

$$(4.5) \qquad \begin{cases} -\Delta u_\varepsilon + \lambda u_\varepsilon = f & \text{on} \quad D \setminus F_\varepsilon \\ u_\varepsilon \in H_0^1(D \setminus F_\varepsilon) \end{cases}$$

In view of the example 1.1, setting $u_\varepsilon = 0$ on the set F_ε, we have that is the local weak solution of the relaxed Dirichlet problem :

$$\begin{cases} -\Delta u_\varepsilon + (\infty_{F_\varepsilon} + \lambda m) u_\varepsilon = f & \text{in } D \\ u_\varepsilon = 0 & \text{on } \partial D \end{cases}$$

where m denotes the Lebesgue measure in $\mathbb{R}^d$.

We are interested in the behaviour of the sequence u_ε as $\varepsilon \to \infty$. More specifically, we will study the convergence of the resolvent operators R_ε^λ associated with the measures ∞_{F_ε}, which are related to the resolvents operators $\hat{R}_\varepsilon^\lambda$ of the Dirichlet problems (4.5) by

$$R_\varepsilon^\lambda(f) = \begin{cases} \hat{R}_\varepsilon^\lambda(f) & \text{on} \quad D \setminus F_\varepsilon \\ 0 & \text{on} \quad F_\varepsilon \end{cases}$$

(see example 1.1).

To do that we consider the distribution law Q_ε of the random measures $M_\varepsilon = \infty_{F_\varepsilon}$, defined by

(4.6) $\quad Q_\varepsilon(\mathcal{E}) = P(\infty_{F_\varepsilon} \in \mathcal{E}) \qquad$ for any $\mathcal{E} \in \mathcal{B}(\mathcal{M}_0^*)$

It is easy to check that:

$$E_{Q_\varepsilon}[C(\cdot, U)] = E[C(F_\varepsilon(\cdot), U)] \qquad \text{for any } U \in \mathcal{U}$$

and

$$Cov_{Q_\varepsilon}[C(\cdot, U) C(\cdot, V)] = Cov[C(F_\varepsilon(\cdot) \cap U), C(F_\varepsilon(\cdot) \cap V)]$$

for any pair $U, V \in \mathcal{U}$.
In this case the functions α', α'' defined in (4.3) and (4.4), take the following form:

(4.7) $$\alpha'(U) = \liminf_{h \to \infty} E\left[C(F_h(\cdot) \cap U)\right]$$

(4.8) $$\alpha''(U) = \limsup_{h \to \infty} E\left[C(F_h(\cdot) \cap U)\right]$$

for every $U \in \mathcal{U}$.

An interesting case occours when the probability distribution of the random set is specified. We will assume the following <u>general hypotheses</u>:

(i_1) Let β be a probability law on D of the form

$$\beta(B) = \int_B f\,dx$$

for every $B \in \mathcal{B}$, where $f \in L^2(D)$.

(i_2) For every $h \in \mathbb{N}$ we set $I_h = \{1, \dots, h\}$ and we consider h measurable functions $x_i^h : \Omega \to D$, $i \in I_h$, such that $(x_i^h)_{i \in I_h}$ is a family of indipendent identically distributed random variables with probability distribution β.

(i_3) Let r_h be a sequence of strictly positive numbers such that:

$$\lim_{h \to \infty} r_h^{d-2}\, h = \ell$$

for some constant $\ell < +\infty$

Let $x \in R^d$. Let F be a closed set of R^d. We define the set $x+F$ by:

$$x + F = \{y \in \mathbb{R}^d : x - y \in F\}$$

The next Lemma will be useful to identify a class of random sets.

LEMMA 4.1 For every compact set K of R^d the function

$$(x_1,\dots,x_\ell) \longrightarrow C\big[\bigcup_{i=1}^{\ell}(x_i+F)\cap K\big] \qquad \text{from } (R^d)^h \text{ into } R$$

is upper semicontinuous in R^d.

Proof: For each $n\in\mathbb{N}$ we define the set:

$$F_n=\{x\in\mathbb{R}^d : dist\ (x,F)<\tfrac{1}{n}\}$$

Set $\bar{x}=(x_1,\dots,x_\ell)$. Let $(\bar{x}_k)_{k\in\mathbb{N}}$ be a sequence in $(R^d)^h$ converging to $\bar{x}$ in $(R^d)^h$. Then, for every $n\in\mathbb{N}$ there exists $k_0\in\mathbb{N}$ such that

$$(\bar{x}_k)_i+F \subseteq x_i+F_n$$

for every $k\geq k_0$ and for every $i\in\{1,\dots,\ell\}$.
Hence, for every $n\in\mathbb{N}$ and for every compact set K of R^d, we obtain:

$$C\Big(\big(\bigcup_{i=1}^{\ell}x_i+F_n\big)\cap K\Big)\geq \limsup_{k\to\infty} C\Big(\big(\bigcup_{i=1}^{\ell}(\bar{x}_k)_i+F\big)\cap K\Big)$$

Since:

$$\bigcap_{n\in\mathbb{N}}\Big[\big(\bigcup_{i=1}^{\ell}x_i+F_n\big)\cap K\Big]=\big(\bigcup_{i=1}^{\ell}x_i+F\big)\cap K$$

by property (h) of Proposition 1.1 we get that:

$$C\Big(\big(\bigcup_{i=1}^{\ell}x_i+F\big)\cap K\Big)\geq \limsup_{k\to\infty} C\Big(\big(\bigcup_{i=1}^{\ell}(\bar{x}_k)_i+F\big)\cap K\Big)$$

which proves the Lemma. ■

Let K be a compact set of R^d such that $K\subseteq B_1$. For any $\ell\in\mathbb{N}$, we denote by K^ℓ the following set:

$$K^\ell=\{x\in\mathbb{R}^d : \tfrac{x}{r_\ell}\in K\}$$

and by K_i^ℓ the random sets

$$K_i^\ell=\{x\in D : \tfrac{1}{r_\ell}(x-x_i^\ell)\in K\}$$

we note that $K_i^\ell\subseteq B_{r_\ell}(x_i^\ell)$. Finally, we denote by F_ℓ the random sets:

$$F_\ell = \bigcup_{i=1}^{\ell} K_i^\ell \qquad \ell \in \mathbb{N} \tag{4.9}$$

REMARK 4.4 By Lemma 4.1 and Remark 4.3 the sets F_ℓ , are actually random sets in according to Definition 4.4.

We will prove the following theorems:

THEOREM 4.2 Let F_ℓ be the sequence of random sets defined in (4.9). If the general hypotheses (i_1), (i_2) and (i_3) hold then the sequence Q_ℓ of distribution laws defined in (4.6) converges weakly to the distribution law σ_ν defined by:

$$\sigma_\nu(\Xi) = \begin{cases} 1 & \text{if } \nu \in \Xi \\ 0 & \text{otherwise} \end{cases}$$

for any $\Xi \in \mathcal{B}(\mathcal{M}_0^*)$, where $\nu = c\beta$, $c = \ell C(K, \mathbb{R}^d)$, and

$$C(K, \mathbb{R}^d) = \inf\left\{ \int_{\mathbb{R}^d} |Du|^2 ; u \in H^1(\mathbb{R}^d), u \geq 1 \text{ q.e. on } K \right\}$$

THEOREM 4.3 Let F_ℓ be the sequence of random sets defined in (4.9). Assume the general hypotheses (i_1), (i_2) and (i_3). Then, for any $f \in L^2(D)$ and for every $\varepsilon > 0$

$$\lim_{\ell \to \infty} P\{\omega \in \Omega : \| R_\ell^\lambda(\omega)[f] - R^\lambda[f] \|_{L^2(D)} > \varepsilon\} = 0$$

where $R_\ell^\lambda(\omega)$ is the sequence of resolvent operators associated with the random measures ∞_{F_ℓ} and R^λ is the resolvent operator associated with the measure ν .

Both the theorems will be consequences of the next Proposition 4.1. More specifically, Theorem 4.2 will follow by applying Theorem 3.1 and Proposition 4.1; while the proof of Theorem 4.3 will be obtained by Theorem 4.1. and propotion 4.1.

PROPOSITION 4.1 Let F_ℓ be the sequence of random sets defined in (4.9). Let α' , α'' be the set functions as defined in (4.7), (4.8) respectively. Then, if the general hypotheses (i_1), (i_2) and (i_3) hold we have:

(t_1) $\nu'(B) = \nu''(B) = c\beta(B)$ for every $B \in \mathcal{B}$;

(t_2) there exist a constant $\varepsilon > 0$, an increasing continuous funcion $\xi : R \times R \to R$ with $\xi(0,0) = 0$ and a Radon measure β_1, such that

$$\limsup_{h \to \infty} |\operatorname{cov}[C(F_h(\cdot) \cap U), C(F_h(\cdot) \cap V)]| \leq$$

$$\xi(\operatorname{diam} U, \operatorname{diam} V)\, \beta_1(U)\, \beta_2(V)$$

for any $U, V \in \mathcal{U}$ such that $\bar{U} \cap \bar{V} = \phi$ with diam $U < \varepsilon$ and diam $V < \varepsilon$

For the proof of Proposition 4.1 we need some preliminary results. First, we give a result which allows us to estimate from below the capacity of the union of a family of sets $(E_i)_{i \in I}$ by means of the sum of capacities of the sets E_i .

LEMMA 4.2 Let $(E_i)_{i \in I}$ be a family of subsets of D and let $E = \bigcup_{i \in I} E_i$
Assume that:
there exist a finite family $(x_i)_{i \in I}$ of points in D and two positive real numbers τ, R such that:

(i) $0 < \tau < R$;

(ii) $E_i \subseteq B_\tau(x_i) \subseteq B_R(x_i) \subseteq D$ for $i \in I$;

(iii) $B_R(x_i) \cap B_R(x_j) = \phi$ for $i, j \in I$, $i \neq j$.

Let us set:

$$\delta = \delta(E) = 4^{d+1} \frac{\tau^{d-2}}{R^d} (R \vee \operatorname{diam} E)^2$$

Then, if $\delta < 1$ we have:

$$C(E) \geq (1-\delta)^2 \sum_{i \in I} C(E_i, B_R(x_i))$$

Proof: Let $u \in H_0^1(D)$ be such that:

$$C(E) = \int_D |Du|^2 \, dx$$

and $u \geq 1$ q.e. on E .

It is well known that u is the unique solution of the variational inequality

$$u \in K_E \quad : \int_D Du \, D(v-u) \, dx \geq 0 \qquad \text{for } v \in K_E$$

where $K_E = \{ v \in H_0^1(D) \; ; \; v \geq 1 \text{ q.e. on } E \}$

Assume that:

(4.10) $u \leq \delta$ q.e. on $\partial B_R(x_i)$ for every $i \in I$.

We prove that the assertion follows. Let us define the function:

$$v = \frac{(u-\delta)^+}{1-\delta}$$

It is easy to see that: $v \in H_0^1(D)$, $v \geq 1$ q.e. on E and $v = 0$ q.e. on $\partial B_R(x_i)$ for each $i \in I$. Since (ii) holds, we have:

$$c(E_i, B_R(x_i)) \leq \int_{B_R(x_i)} |Dv|^2 dx$$

for any $i \in I$. Hence,

(4.11) $$\int_D |Dv|^2 dx \geq \sum_{i \in I} \int_{B_R(x_i)} |Dv|^2 dx \geq$$

$$\sum_{i \in I} c((E_i), B_R(x_i))$$

On the other hand, by definition of v we also have:

$$(4.12)\qquad \int_D |Dw|^2 dx = \frac{1}{(1-\bar{\sigma})^2}\int_D |D(u-\bar{\sigma})^+|^2\, dx \leq$$

$$\frac{1}{(1-\bar{\sigma})^2}\int_D |Dv|^2 dx = \frac{1}{(1-\bar{\sigma})^2}\, C(E)$$

By (4.11) and (4.12) we obtain the assertion.

Let us verify (4.10). For every $i \in I$ we consider the function u_i defined by:

$$u_i(x) = \left[\frac{r^{d-2}}{|x-x_i|^{d-2}} \wedge 1\right] \qquad x \in \mathbb{R}^d$$

It is not diffucult to check that $u_i \in H^1_{loc}(\mathbb{R}^d)$ and that

$$(4.13)\qquad \begin{cases} -\Delta u_i \geq 0 & \text{on } \mathbb{R}^d \\ u_i = 1 & \text{on } B_r(x_i) \end{cases}$$

for any $i \in I$. Let us set

$$(4.14)\qquad z(x) = \sum_{i \in I} u_i(x) \qquad x \in \mathbb{R}^d$$

We see that $z \in H^1_{loc}(\mathbb{R}^d)$ and it satisfies the following conditions:

$$(4.15)\qquad \begin{cases} -\Delta z \geq 0 & \text{in } D \\ z \geq 1 & \text{q.e. on } E \\ z \geq 0 & \text{on } \partial D \end{cases}$$

By a classical comparison Theorem ([33], Chapter II, Theorem 6.4), we can get, by (4.13) and (4.15), that:

$$(4.16)\qquad u \leq z \quad \text{q.e. on } D \ .$$

Let $y \in \partial B_R(x_i)$ for $i \in I$ fixed. We wish to estimate $z(y)$. By (4.14) we have:

(4.17) $$z(y) \leq \sum_{j \in I} \frac{r^{d-2}}{|x_j - y|^{d-2}}$$

To estimate the right-hand side we introduce the following sets:

$$C_k(y) = \left\{ x \in \mathbb{R}^d : kR \leq |x-y| \ (k+1)R \right\} \quad k = 0, 1, \ldots$$

Moreover, let

$$I_k(y) = \{ i \in I : x_i \in C_k(y) \}$$

and let $N_k(y)$ be the number of elements of $I_k(y)$. Since $|x_j - y| \geq R$ for each $j \in I$, it is easy to see that:

(4.18) $$\sum_{j \in I} \frac{1}{|x_j - y|^{d-2}} \leq \sum_{k=1}^{[\frac{1}{R} \operatorname{diam} E]+1} \frac{1}{(kR)^{d-2}} N_k(y)$$

where [a.] denotes the integer part of a.
Let us estimate $N_k(y)$. Since, for k fixed,

$$\bigcup_{i \in I_k(y)} B_R(x_i) \subseteq \left\{ x \in \mathbb{R}^d : (k-1)R \leq |x-y| \leq (k+2)R \right\}$$

we have:

$$\operatorname{meas}\left[\bigcup_{i \in I_k(y)} B_R(x_i) \right] \leq \omega_d R^d \left[(k+2)^d - (k-1)^d \right]$$

hence, using (iii),

(4.19) $$N_k(y) \leq (k+2)^d - (k-1)^d \leq 4^d k^{d-1}$$

By (4.17), (4.18), (4.19), we obtain:

$$z(y) \leq \frac{r^{d-2}}{R^{d-2}} 4^d \sum_{k=1}^{[\frac{1}{R}\operatorname{diam} E]+1} k \leq$$

$$4^d \frac{r^{d-2}}{R^{d-2}} \left(\left[\frac{\operatorname{diam} E}{R} \right] + 1 \right)^2 \leq$$

$$4^d \frac{r^{d-2}}{R^{d-2}} \left\{ 2 \frac{(R \vee \operatorname{diam} E)}{R} \right\}^2 = 4^{d+1} \frac{r^{d-2}}{R^d} (R \vee \operatorname{diam} E)^2$$

This inequality, toghether with (4.16), shows that assumption(4.10) is always satisfied and this completes the proof of the Lemma. ■

For each subset $Z \subseteq D$ we define the random set of indices:

$$I_\varepsilon(Z) = \{ i \in I_\varepsilon : x_i^\varepsilon \in Z \}$$

and the random variable:

(4.19) $N_\varepsilon(Z)$ = number of elements of $I_\varepsilon(Z)$.

For each $h \in \mathbb{N}$, let $R_\varepsilon^s = \left(\frac{s}{\varepsilon}\right)^{1/d}$ where s is a positive real number (we note that by (i_3) $r_\varepsilon < R_\varepsilon^s$ for ε large enough). For s fixed we also consider:

$$I_\varepsilon^s(Z) = \{ i \in I_\varepsilon(Z) : \exists j \in I_\varepsilon ,\ i \neq j \text{ such that } |x_i^\varepsilon - x_j^\varepsilon| < R_\varepsilon^s \}.$$

and

(4.20) $N_\varepsilon^s(Z)$ = number of elements of $I_\varepsilon^s(Z)$.

The following estimate is crucial for our result

LEMMA 4.3. If (i_1) and (i_2) hold then:

$$\limsup_{h \to \infty} \frac{E[N_\varepsilon^s(U)]}{\varepsilon} \leq \omega_d\, s \int_U f^2\, dx .$$

for any $U \in \mathcal{U}$, where ω_d is the volume of the unit ball.

Proof: Fix $U \in \mathcal{U}$. It is easy to check that $i \in I_\varepsilon^s(U)$ if and only if

$$\sum_{\substack{j=1\\ j\neq i}}^{h} \chi_{B_{R_h^s}(x_j^h)\cap U}(x_i^h) \geq 1$$

Therefore, we see that:

$$N_h^s(U) \leq \sum_{i=1}^{h} \sum_{\substack{j=1\\ j\neq i}}^{h} \chi_{B_{R_h^s}(x_j^h)\cap U}(x_i^h) \tag{4.21}$$

By (4.21) and the assumptions (i_1), (i_2) we obtain:

$$E[N_h^s(U)] \leq \sum_{i=1}^{h} \sum_{\substack{j=1\\ j\neq i}}^{h} \int_\Omega \chi_{B_{R_h^s}(x_j^h(\omega))\cap U}(x_i^h(\omega))\, dP(\omega) = \tag{4.22}$$

$$\sum_{i=1}^{h} \sum_{\substack{j=1\\ j\neq i}}^{h} \int_D \left[\int_D \chi_{B_{R_h^s}(y)\cap U}(x)\, d\beta(x) \right] d\beta(y)$$

$$\sum_{i=1}^{h} \sum_{\substack{j=1\\ j\neq i}}^{h} \int_D \beta(B_{R_h^s}(y)\cap U)\, d\beta(y)$$

$$h(h-1) \int_D \beta(B_{R_h^s}(y)\cap U)\, d\beta(y)$$

Finally, by (4.22) we get:

$$\limsup_{h\to\infty} \frac{E[N_h^s(U)]}{h} \leq$$

$$s \limsup_{h\to\infty} \left[\frac{h}{s} \int_D \beta(B_{R_h^s}(y)\cap U)\, d\beta(y) \right] =$$

$$s \limsup_{h\to\infty} \int_D \left[\frac{\omega_d}{|B_{R_h^s}(y)|} \int_{B_{R_h^s}(y)\cap U} f(x)\, dx \right] f(y)\, dy =$$

$$s\, \omega_d \int_U f^2(y)\, dy$$

by Lebesgue Theorem. ∎

Proof of the Proposition 4.1:
For any $U \in \mathcal{U}$,let

(4.23) $$U_\varepsilon' = \{x \in U : dist(x, \partial U) > R_\varepsilon^s\}$$

and
(4.24) $$U_\varepsilon'' = \{x \in D : dist(x, U) < R_\varepsilon^s\}$$

We observe that: $U_\varepsilon' \subseteq U \subseteq U_\varepsilon''$
Moreover, we note that:

(4.25) $$J_\varepsilon(U_\varepsilon') = I_\varepsilon(U_\varepsilon') \setminus I_\varepsilon^s(U)$$

is the set of all elements $i \in I_\varepsilon$ which satisfy the following conditions:

(a_1) $x_i^\varepsilon \in U$

(a_2) $B_{R_\varepsilon^s}(x_i^\varepsilon) \subseteq U$

(a_3) $|x_i^\varepsilon - x_j^\varepsilon| \geq R_\varepsilon^s$ for any $j \in I_\varepsilon$ with $i \neq j$

Denote by F_ε' the random set:

$$F_\varepsilon' = \bigcup_{i \in J_\varepsilon(U_\varepsilon')} K_\varepsilon^i$$

We have:

(b_1) $K_i^\varepsilon \subseteq B_{r_\varepsilon}(x_i^\varepsilon) \subseteq B_{R_\varepsilon^s}(x_i^\varepsilon)$

(b_2) $B_{R_\varepsilon^s}(x_i^\varepsilon) \cap B_{R_\varepsilon^s}(x_j^\varepsilon) = \phi$ for $i, j \in J_\varepsilon(U_\varepsilon')$ with $i \neq j$.

Let us set:

(4.26) $\quad \sigma(U,h) = 4^{d+1} \dfrac{r_h^{d-2}}{(R_h^s)^d} (\mathrm{diam}\, U)^2$

Choosing $\varepsilon = \sqrt{s/C_0}$, where $C_0 = 4^{d+1} \ell$, by assumption (i_3), we see that $\sigma(U,h)$ will be less than 1 for h large enough and diam $U < \varepsilon$,

Thus, by Lemma 4.2 we obtain that, for each $\omega \in \Omega$:

(4.27) $C(F_h(\omega) \cap U) \geq C(F_h'(\omega)) \geq (1-\sigma(U,h))^2 \sum_{i \in J_h(U_h')} C(K_i^h, B_{R_h^s}(x_i^h)) \geq$

$$(1-\sigma(U,h))^2 \left[N_h(U_h') - N_h^s(U)\right] C(K^h, B_{R_h^s}) =$$

$$(1-\sigma(U,h))^2 \left[\frac{N_h(U_h')}{h} - \frac{N_h^s(U)}{h}\right] h\, r_h^{d-2}\, C(K, B_{R_h^s/r_h})$$

whenever h is sufficiently large and diam $U < \varepsilon$. On the other hand, by using the elementary properties of the capacity, we immediately get that

(4.28) $\quad C(F_h \cap U) \leq \sum_{i \in I_h(U_h'')} C(K_i^h, B_{R_h^s}(x_i^h)) =$

$$\frac{N_h(U_h'')}{h}\, h\, r_h^{d-2}\, C(K, B_{R_h^s/r_h})$$

for every $U \in \mathcal{U}$.

Now we are in position to prove (t_1) and (t_2) of the Proposition 4.1.

Proof of (t_1): First, we observe that by the Law of Large Numbers we have:

(4.29) $\quad \lim_{h \to \infty} \dfrac{E[N_h(U_h')]}{h} = \lim_{h \to \infty} \dfrac{E[N_h(U_h'')]}{h} = \beta(U)$

for every $U \in \mathcal{U}$ with $\beta(\partial U) = 0$.

Moreover, by (i_3) and (4.26) we obtain:

(4.30) $\quad \lim_{h \to \infty} \sigma(U,h) = \sigma(U) = \dfrac{C_0}{s} (\mathrm{diam}\, U)^2$

where $c_0 = 4^{d+1} e$

Next, we observe that for every compact subset $K \subseteq B_R$

(4.31) $$\lim_{R \to \infty} C(K, B_R) = C(K, \mathbb{R}^d)$$

By Lemma 4.3, (4.27), (4.28), (4.29), (4.30) and (4.31) we deduce that:

(4.32) $$\alpha''_-(B) \leq c\beta(B)$$

for every $B \in \mathcal{B}$, and

(4.33) $$\alpha'_-(B) \geq (1 - \frac{c_0}{s}(\operatorname{diam} B)^2)^2 \, c \left[\beta(B) - \omega_d s \int_B \varphi^2(y)\,dy\right]$$

for every $B \in \mathcal{B}$ with sufficiently small diameter. By (4.32) we have that:

$$\nu''(B) \leq c\beta(B)$$

for every $B \in \mathcal{B}$.

Therefore, we have only to prove that:

(4.34) $$\nu'(B) \geq c\beta(B) \qquad \text{for every } B \in \mathcal{B} .$$

Let us fix $B \in \mathcal{B}$. Next, for arbitrary $\eta > 0$ choose a partition $(B_i)_{i \in I}$ of B such that $B_i \in \mathcal{B}$ and diam $B_i < \eta$ for every $i \in I$. Then, by (4.33) applied with $s = \eta$, we get:

(4.35) $$\nu'(B) = \sum_{i \in I} \nu'(B_i) \geq \sum_{i \in I} \alpha'_-(B_i) \geq$$

$$(1 - c_0\eta)^2 \, c \left[\beta(B) - \omega_d \eta \int_B \varphi^2(y)\,dy\right]$$

Since η is arbitrary, (4.34) follows from (4.35).

Proof of (t_2): Preliminary, we note that by the Strong Law of Large Numbers we have:

$$(4.36) \qquad \frac{N_\ell(U'_\ell)}{\ell} \xrightarrow[\ell \to \infty]{} \beta(U) \qquad \text{a.e. } \omega \in \Omega$$

and

$$(4.37) \qquad \frac{N_\ell(U'_\ell)}{\ell} \xrightarrow[\ell \to \infty]{} \beta(U) \qquad \text{in } L^1(\Omega)$$

for any $U \in \mathcal{U}$. Moreover, since $\frac{N_\ell(U'_\ell)}{\ell}$ is an equibounded sequence of random variables we also have:

$$(4.38) \qquad \frac{N_\ell(U'_\ell)}{\ell} \xrightarrow[\ell \to \infty]{} \beta(U) \qquad \text{in } L^2(\Omega).$$

for any $U \in \mathcal{U}$. We observe that: (4.36), (4.37) and (4.38) hold also with U'_ℓ replaced by U''_ℓ, provided $\beta(\partial U) = 0$.
By (4.27), (4.31), (4.30) we have:

$$(4.39) \qquad \liminf_{\ell \to \infty} E\left[C(F_\ell(\cdot) \cap U)\, C(F_\ell(\cdot) \cap V)\right] \geq$$

$$(1-\sigma(U))^2 (1-\sigma(V))^2 c^2 \times$$

$$\times \limsup_{\ell \to \infty} \left\{ E\left[\frac{N_\ell(U'_\ell)}{\ell} \cdot \frac{N_\ell(V'_\ell)}{\ell}\right] - E\left[\frac{N_\ell(U'_\ell)}{\ell} \cdot \frac{N^s_\ell(V)}{\ell}\right] - E\left[\frac{N_\ell(V'_\ell)}{\ell} \cdot \frac{N^s_\ell(U)}{\ell}\right] \right\}$$

for any pair $U, V \in \mathcal{U}$ such that $\bar{U} \cap \bar{V} = \phi$, diam $U < \varepsilon$, diam $V < \varepsilon$ with $\varepsilon = \sqrt{5}/c_0$

By (4.38) we have:

$$(4.40) \qquad \lim_{\ell \to \infty} E\left[\frac{N_\ell(U'_\ell)}{\ell} \cdot \frac{N_\ell(V'_\ell)}{\ell}\right] = \beta(U)\,\beta(V)$$

Moreover, by Lemma 4.3 and (4.36) it follows:

(4.41) $$\limsup_{h\to\infty} E\left[\frac{N_h(U_h')}{h}\cdot\frac{N_h^s(V)}{h}\right] \leq \omega_d\,\beta(U)\,s\int_V \varphi^2\,dx$$

and

(4.42) $$\limsup_{h\to\infty} E\left[\frac{N_h(V_h')}{h}\cdot\frac{N_h^s(U)}{h}\right] \leq \omega_d\,\beta(V)\,s\int_U \varphi^2\,dx$$

for any $U, V \in \mathcal{U}$

Then, (4.39), (4.40), (4.41) and (4.42) give:

(4.43) $$\liminf_{h\to\infty} E\left[C(F_h(\cdot)\cap U)\,C(F_h(\cdot)\cap V)\right] \geq$$

$$(1-2\delta(U)-2\delta(V))\,c^2\,\times$$

$$\times\left[\beta(U)\beta(V)-\beta(U)\omega_d\,s\int_V\varphi^2 dx-\beta(V)\omega_d\,s\int_U\varphi^2 dx\right]$$

for every $U, V \in \mathcal{U}$, such that $\bar{U}\cap\bar{V}=\phi$ with diam $U<\varepsilon$, diam $V<\varepsilon$

By (4.28) and (4.38) (applied with U_h'' instead of U_h') we also deduce:

(4.44) $$\limsup_{h\to\infty} E\left[C(F_h(\cdot)\cap U)\,C(F_h(\cdot)\cap V)\right] \leq c^2\,\beta(U)\,\beta(V)$$

for any $U, V \in \mathcal{U}$ with $\beta(\partial U)=\beta(\partial V)=0$.
Estimates like (4.43) and (4.44) for the upper and lower limit of the sequence $E[C(F_h(\cdot)\cap U)]\cdot E[C(F_h(\cdot)\cap V)]$ can be obtained in the same way.
Therefore, we deduce that:

(4.45) $$\limsup_{h\to\infty}\left|\mathrm{Cov}\left[C(F_h(\cdot)\cap U),\,C(F_h(\cdot)\cap V)\right]\right| \leq$$

$$c^2\beta(U)\beta(V)-[1-2\delta(U)-2\delta(V)]\,c^2\left[\beta(U)\beta(V)-\beta(U)\,\omega_d\,s\int_V\varphi^2 dx-\right.$$

$$\left.-\,\beta(V)\,\omega_d\,s\int_U\varphi^2 dx\right] \leq$$

$$c^2\{\beta(U)\,\omega_d\, s\int_V \varphi^2 dx + \beta(V)\,\omega_d\, s\int_U \varphi^2 dx + 2\,[\sigma(U)+\sigma(V)]\,\beta(U)\beta(V)\}$$

for every $U, V \in \mathcal{U}$ such that $\overline{U} \cap \overline{V} = \phi$ with diam $U < \varepsilon$, diam $V < \varepsilon$.
Taking $s = \max\{\text{diam } U, \text{diam } V\}$, by (4.30), formula (4.45) becomes:

$$(4.46)\quad \limsup_{h\to\infty} |\operatorname{cov}[C(F_h(\cdot)\cap U)\,C(F_h(\cdot)\cap V)]| \le$$

$$c^2\{\beta(U)\,\omega_d\, s\int_V \varphi^2 dx + \beta(V)\,\omega_d\, s\int_U \varphi^2 dx + 2c_0 s\,\beta(U)\beta(V)\} \le$$

$$c_1 s\{\beta(U)\int_V \varphi^2 dx + \beta(V)\int_U \varphi^2 dx + \beta(U)\beta(V)\}$$

for every $U, V \in \mathcal{U}$ such that $\overline{U} \cap \overline{V} = \phi$, with diam $U < \varepsilon$ and diam $V < \varepsilon$.
In the last inequality we have set $c_1 = c^2 \max\{\omega_d, 2c_0\}$. The assertion ($t_2$) follows by (4.46) taking $\beta_1(U) = \beta(U) + \int_U \varphi^2\, dx$ for every $U \in \mathcal{U}$ and $\xi(x,y) = \max\{x, y\}$. ∎

5. SCHROEDINGER EQUATION WITH RANDOM POTENTIALS.

In this section we consider another application of our main Theorem. We study a problem concerning the stationary Schroedinger equation in R^3 with particular random potentials.

We still denote by (Ω, Σ, P) a probability space. Moreover, for every $\ell \in \mathbb{N}$ we consider a family $(x_i^\ell)_{i \in I_\ell}$ of random variables satisfying the <u>general hypotheses</u> (i_1), (i_2), (i_3) in the previous section.

Denote by F_ℓ, $\ell \in \mathbb{N}$ the following random sets:

$$F_\ell = \bigcup_{i=1}^{\ell} B_{r_\ell}(x_i^\ell)$$

Let (k_ℓ) be a sequence of positive real numbers.
For each $\ell \in \mathbb{N}$ we define the random function:

$$q_\ell(x) = \begin{cases} k_\ell & \text{if } x \in F_\ell \\ 0 & \text{otherwise.} \end{cases}$$

We will study the equations:

$$(5.1) \qquad \begin{cases} -\Delta u_\ell + q_\ell(x)\, u_\ell + \lambda u_\ell = \varphi \text{ in } D \\ u_\ell \in H_0^1(D) \end{cases}$$

where $\lambda \geq 0$ is a real number and $\varphi \in L^2(D)$.
To use the theory developed in section 3 we consider the sequence μ_ℓ of random measures defined by

$$(5.2) \qquad \mu_\ell(B) = \int_B q_\ell(x)\, dx$$

for any $B \in \mathcal{B}$.

REMARK 5.1. For every $U \in \mathcal{U}$ the functions $C(\mu_\ell(\cdot), U)$ are Σ-measurable, each of them being the infimum of a sequence of measurable functions. To see this, it is enough to use the variational definition of $C(\mu_\ell(\cdot), U)$ and the fact that the functions q_ε are bounded so that

$$C(M_\varepsilon(\cdot),U) = \inf_{v\in H}\left\{\int_D |Dv|^2 dx + \int_D (v-1)^2 q_\varepsilon(\cdot)\,dx\right\}$$

where H is a countable dense subset of $H_0^1(D)$. Therefore the maps $M_\varepsilon : \Omega \to \mathcal{M}_0^*$ are actually random measures by Corollary 1.1.

The problems (5.1) are equivalent to the following relaxed Dirichlet problems:

$$\begin{cases} -\Delta u_\varepsilon + (M_\varepsilon(\omega) + \lambda m)\, u_\varepsilon = f & \text{in } D \\ u_\varepsilon = 0 & \text{on } \partial D \ . \end{cases}$$

We shall prove the following theorems:

THEOREM 5.1. Let Q_ε be the sequence of distribution laws on $\mathcal{M}_0^*$ associated with the sequence of random measures M_ε defined in (5.2). Assume that the general hypotheses (i_1), (i_2), (i_3) hold. Moreover, we suppose also that:

(i_4) $$\lim_{\varepsilon\to 0} \sqrt{K_\varepsilon}\, r_\varepsilon = +\infty$$

Then, Q_ε converges weakly to the distribution law δ_ν defined by:

$$\delta_\nu(\mathcal{E}) = \begin{cases} 1 & \text{if } \nu \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

for any $\mathcal{E} \in \mathcal{B}(\mathcal{M}_0^*)$, where $\nu = c\beta$, $c = \ell\, C(B_1, \mathbb{R}^3)$, and $C(B_1, \mathbb{R}^3)$ is defined as in Theorem 4.2 .

THEOREM 5.2. Let $\mathcal{M}_h$ be the sequence of random measures defined in (5.2). Assume that the general hypotheses (i_1), (i_2), (i_3) hold. Suppose also that:

(i_4)
$$\lim_{h\to\infty} \sqrt{k_h}\, \tau_h = +\infty$$

Then, for any $f \in L^2(D)$ and for every $\varepsilon > 0$

$$\lim_{h\to\infty} P\{\omega \in \Omega : \| R_h^\lambda(\omega)[f] - R^\lambda[f] \|_{L^2(D)} > \varepsilon\} = 0$$

where R_h^λ is the sequence of resolvent operators associated with the random potentials q_h (i.e. with the random measures $\mathcal{M}_h$) and R^λ is the resolvent operator associated with the constant potential c (i.e. with the measure $c\beta$).

The proofs of these theorems will depend on the next Proposition 5.1. In particular, the proof of theorem5.1 will be obtained by applying Theorem 3.1 and Proposition 5.1; the Theorem 5.2 will follow from Theorem 4.1 and Proposition 5.1.

PROPOSITION 5.1. Let $\mathcal{M}_h$ be the sequence of random measures defined in (5.2). Let α' and α'' be the set functions as defined respectively in (4.3) and (4.4). Assume the general hypotheses (i_1), (i_2), (i_3). In addition, suppose that:

(i_4)
$$\lim_{h\to\infty} \sqrt{k_h}\, \tau_h = +\infty$$

Then, the following assertions hold:

(t_1') $\quad \nu'(B) = \nu''(B) = c\,\beta(B) \quad$ for every $B \in \mathcal{B}$

(t_2') $\quad$ there exist a constant $\varepsilon > 0$, an increasing continuos function $\xi : R \times R \longrightarrow R$ with $\xi(0,0) = 0$ and a Radon measure β_1 such that:

$$\limsup_{h\to\infty} |\mathrm{cov}[C(\mathcal{M}_h(\cdot), U), C(\mathcal{M}_h(\cdot), V)]| \leq \xi(\mathrm{diam}\, U, \mathrm{diam}\, V)\, \beta_1(U), \beta_1(V$$

for any $U, V \subset \mathcal{U}$ such that $\overline{U} \cap \overline{V} = \phi$ with diam $U < \varepsilon$, diam $V < \varepsilon$

The proof will be based on the following two lemmas.

LEMMA 5.1. Let $\mu \in \mathcal{M}_0^*$. Then, the Lemma 4.2 holds if we replace $C(E)$ by $C(\mu, E)$.

Proof: It is enough to replace the function u used in the proof of Lemma 4.2 with the μ -capacitary potential of E in D, defined as the unique function $w \in H_0^1(D)$ such that

$$C(\mu, E) = \int_D |Dw|^2 dx + \int_E (w-1)^2 d\mu$$

and to use the comparison Theorem for relaxed Dirichlet problems ([20], Theorem 2.10) instead of the classical comparison Theorem for variational inequalities. ∎

We now compute the μ - capacitary potential of a ball with respect to a concentric ball, when μ is the Lebesgue measure (multiplied by a constant).

LEMMA 5.2. Let r, R be two positive real numbers such that $r < R$. Moreover, let μ be the Borel measure in $\mathcal{M}_0^*$ defined by:

$$\mu(B) = K \int_B dx$$

for any $B \in \mathcal{B}$, where K is constant.
Then, the μ -capacitary potential associated with $C(\mu, B_r, B_R)$ is the function:

$$(5.3) \qquad w(|x|) = \begin{cases} 1 - (a|x|^{-1} + b) & r \le |x| \le R \\ 1 - \dfrac{c \sinh \sqrt{K} x}{|x|} & 0 < |x| \le r \end{cases}$$

for $x \in B_R$, where

$$a = R - \frac{\sqrt{K} R^2 \cosh(\sqrt{K} r)}{\sinh(\sqrt{K} r) + (\sqrt{K} R - \sqrt{K} r) \cosh(\sqrt{K} r)}$$

$$b = \frac{\sqrt{K} R \cosh(\sqrt{K} r)}{\sinh(\sqrt{K} r) + (\sqrt{K} R - \sqrt{K} r) \cosh(\sqrt{K} r)}$$

$$c = \frac{R}{\sin h(\sqrt{k}\, r) + (\sqrt{k}R - \sqrt{k}\, r)\cos h(\sqrt{k}\, r)}$$

Moreover, setting $d = w(r)$ we have

(5.4) $(1-d)^2 C(B_r, B_R) \leq C(\mu, B_r, B_R) \leq C(B_r, B_R)$

Proof: The proof of (5.3) is obtained solving explicitly the Euler equation of the functional

$$F(u) = \int_{B_R} |Du|^2 dx + k\int_{B_r} u^2 dx$$

with the boundary condition $u - 1 \in H_0^1(B_R)$. In order to proof (5.4) we note that the relation $C(\mu, B_r, B_R) \leq C(B_r, B_R)$ follows by the property (f) of Proposition 1.1; moreover let us define

$$u = \frac{(w-d)^+}{1-d}$$

It is easy to see that $u \in H_0^1(B_R)$ and $u \geq 1$ q.e. on B_r.
Hence,

$$C(B_r, B_R) \leq \int_{B_R} \frac{|D(w-d)^+|^2}{(1-d)^2} \leq \frac{1}{(1-d)^2}\int_{B_R} |Dw|^2 dx = \frac{1}{(1-d)^2} C(\mu, B_r, B_R)$$

which proves (5.4). ■

Proof of Proposition 5.1.

For each $h \in \mathbb{N}$ let us define a sequence μ_h of Borel measures in the following way:

$$\mu_h(B) = k_h \int_B dx$$

for any $B \in \mathcal{B}$.

Let $U \in \mathcal{U}$. Let U_e' and U_e'' be the sets defined in (4.23) and (4.24) respectively. By $J_e(U_e')$ we denote the set of indices defined in (4.25). Furthermore let $\sigma(U, e)$ be as defined in (4.26) , $N_e(U)$ as in (4.19) and $N_e^s(U)$ as in (4.20). By hypothesis (i_3) , by Lemma 5.1 and Lemma 5.2 we can get that for each $\omega \in \Omega$:

(5.5) $$C(M_e, U) \geq (1 - \sigma(U, e))^2 \sum_{i \in J_e(U_e')} C(\mu_e, B_{r_e}(x_i^e), B_{R_e}(x_i^e)) =$$

$$(1 - \sigma(U, e))^2 [N_e(U_e') - N_e^s(U)] C(\mu_e, B_{r_e}, B_{R_e}) \geq$$

$$(1 - \sigma(U, e))^2 [N_e(U_e') - N_e^s(U)] (1 - d_e)^2 C(B_{r_e}, B_{R_e}) =$$

$$(1 - \sigma(U, e))^2 (1 - d_e)^2 \left[\frac{N_e(U_e')}{e} - \frac{N_e^s(U)}{e} \right] e r_e C(B_1, B_{\frac{R_e}{r_e}})$$

whenever e is sufficiently large and diam $U < \varepsilon$, with $\varepsilon = \sqrt{\frac{s}{C_0}}$
By (5.3) we have that for each $e \in \mathbb{N}$:

$$d_e = \frac{1}{\frac{r_e}{R_e} + \left[\sqrt{k_e} R_e \left(1 - \frac{r_e}{R_e}\right) \right] \frac{r_e}{R_e} \operatorname{cotg}(\sqrt{k_e} r_e)}$$

So, by hypothesis (i_4) it follows that .

On the other hand we have by the properties of the μ - capacity

(5.6) $$C(M_e, U) \leq \sum_{i \in I_e(U_e'')} C(\mu_e, B_{r_e}(x_i^e), B_{R_e}(x_i^e)) =$$

$$N_e(U_e'') C(\mu_e, B_{r_e}, B_{R_e}) \leq$$

$$N_e(U_e'') C(B_{r_e}, B_{R_e}) = \frac{N_e(U_e'')}{e} e r_e C(B_1, B_{\frac{r_e}{R_e}})$$

By repeating the same steps made in the proof of the assertions (t_1) and (t_2) of Proposition 4.1, we get by (5.5) and (5.6) immediately the equivalent assertion in this case. ■

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Chapter 2 :

A derivation theorem for countably subadditive set functions.

A DERIVATION THEOREM FOR COUNTABLY SUBADDITIVE SET FUNCTIONS.

1. INTRODUCTION AND STATEMENT OF THE RESULT .

Let α be a non negative countably subadditive set function defined on the σ - field $\mathcal{B}(\Omega)$ of all Borel subsets of an open set $\Omega \subseteq \mathbf{R}^n$.

A well - known result of measure theory states that there exists a minimal element μ_α in the class of all Borel measures μ on Ω such that $\alpha \leq \mu$ on $\mathcal{B}(\Omega)$.

The measure μ_α is used in [2] to define a "limiting capacity measure" for a sequence of closed sets in Ω, in order to study the solutions of the diffusion equation in regions with many small holes .

The notion of the least measure μ_α is also employed in several recent papers ([4] , [5] , [1]) concerning the study of asymptotic Dirichlet problems with boundary conditions on varying domains from the point of view of Γ - convergence .

The aim of this paper is to investigate the relations between the measure μ_α and the set function α .

An explicit formula which allows to represent the measure μ_α by means of α is the following (see , for istance [4] , Lemma 4.1) : for every $B \in \mathcal{B}(\Omega)$

$$\mu_\alpha (B) = \sup \sum_{i \in I} \alpha(B_i)$$

where the supremum is taken over all finite Borel partition $(B_i)_{i\in I}$ of B.

Our main result , stated in Theorem 1.1 , shows how to identify the least measure μ_α by means of the derivation of the function α with respect to a given Radon measure λ on Ω.

THEOREM 1.1. *Let* $\alpha : \mathcal{B}(\Omega) \to [0,+\infty]$ *be a countably subadditive set function such that* $\alpha(\emptyset) = 0$, *let* $\lambda : \mathcal{B}(\Omega) \to [0,+\infty]$ *be a Radon measure on* Ω , *and let f be the function defined for every* $x\in\Omega$ *by* :

$$f(x) = \liminf_{\rho\to 0^+} \frac{\alpha(B_\rho(x))}{\lambda(B_\rho(x))} \tag{1.1}$$

where $B_\rho(x)$ *is the closed ball centered at x with radius* ρ .

Assume that :

(i) $f(x) < +\infty$ $\quad(\lambda + \alpha)$ - a.e. $x\in\Omega$

(ii) $f\in L^1_{loc}(\Omega,\lambda)$

Then

$$\mu_\alpha(B) = \int_B f \, d\lambda$$

for any Borel subset of Ω *; moreover, the liminf in (1.1) is a limit* λ - a.e. $x\in\Omega$.

When the set function α is just the μ - capacity considered in [6] , this result has been obtained in [3] .

Theorem 1.1 will be deduced from Theorem 3.1 in which we prove a differentiation result with respect to a more general "Vitali system" of sets .

Section 2 of the paper is devoted to recall the definitions and the results of measure theory that we need .

In the section 3 we give the proof of the main result and discuss some its consequences .

I wish to thank Prof. G. Dal Maso for having introduced me to the subject and for his many suggestions .

2. BASIC DEFINITIONS AND PRELIMINARIES.

In the sequel we collect some definitions and results of measure theory . We shall denote by Ω any open subset of $\mathbf{R}^n$ and by $\mathcal{B}(\Omega)$ the σ-algebra of all Borel subsets of Ω .

A non-negative countable additive set function μ defined on $\mathcal{B}(\Omega)$ with values in $[0,+\infty]$ such that $\mu(\emptyset)=0$ will be called a *Borel measure* on Ω .

A Borel measure on Ω which assigns finite value to every compact subset of Ω will be called a *Radon measure* on Ω .

DEFINITION 2.1. Let $x \in \Omega$. A family $\mathcal{F} \subseteq \mathcal{B}(\Omega)$ of non empty sets is said to be *fine at* x if and only if for any $\rho > 0$ there is a set $F \in \mathcal{F}$ such that $x \in F$ and diam $F < \rho$.

Let μ be a Radon measure on Ω .

DEFINITION 2.2. A *Vitali system* for μ is a family $\mathcal{V} \subseteq \mathcal{B}(\Omega)$ which has the following properties :

(a) $\mathcal{V}$ is fine at each point $x \in \Omega$;

(b) if E is any subset of Ω and $\mathcal{F}$ is a subfamily of $\mathcal{V}$ which is fine at each point $x \in E$, then there are countably many disjoint sets $V_1, V_2, \ldots, V_h, \ldots$ in $\mathcal{F}$ such that :

$$\mu^*(E - \bigcup_{h \in \mathbf{N}} V_h) = 0$$

where μ^* is the outer measure associated to μ, i.e. for any subset A of Ω

$$\mu^*(A) = \inf \{ \mu(B) : B \in \mathcal{B}(\Omega) , B \supseteq A\}$$

Some examples of Vitali systems for every Radon measure μ are the following :

(i) the family of all closed cubes contained in Ω with sides parallel to the axes (see, for istance, [7] Theorem 3.2.) ;

(ii) the family of all closed balls contained in Ω (see, for istance, [7] Remark 3 of Theorem 3.2.) .

The examples (i) and (ii) are particular cases of the following Theorem due to A.P. Morse [10, Theorem 5.13] :

THEOREM 2.3. *For every* $x \in \Omega$ *let* $\mathcal{C}(x)$ *be a family of convex subsets of* Ω *with the following properties:*

(a) $x \in C$ *for every* $C \in \mathcal{C}(x)$;

(b) $\inf\{ \operatorname{diam} C \,;\, C \in \mathcal{C}(x) \} = 0$;

(c) *there exists a constant* $M > 0$ *such that for any* $x \in \Omega$ *and for any* $C \in \mathcal{C}(x)$ *there are two positive members* r_1 *and* r_2 *with* $r_2 / r_1 < M$ *for which*

$$B_{r_1}(x) \subseteq C \subseteq B_{r_2}(x)$$

Then the family $\mathcal{C} = \bigcup_{x \in \Omega} \mathcal{C}(x)$ *is a Vitali system for every Radon measure* μ *.*

Let $\mathcal{V}$ be a Vitali system for a Radon measure μ on Ω and let F be an $\mathbf{R}$ valued set function on $\mathcal{B}(\Omega)$.

DEFINITION 2.4. For any $x \in \Omega$ we define:

(a) $\liminf\limits_{\mathcal{V} \ni V \to x} F(V) = \sup\limits_{\rho > 0} \inf\{ F(V) : V \in \mathcal{V},\, x \in V ,\, \operatorname{diam} V < \rho \}$

(b) $\limsup\limits_{\mathcal{V} \ni V \to x} F(V) = \inf\limits_{\rho > 0} \sup\{ F(V) : V \in \mathcal{V},\, x \in V ,\, \operatorname{diam} V < \rho \}$

Moreover, we write:

(c) $\lim\limits_{\mathcal{V}\ni V\to x} F(V) = q$ if and only if $q = \liminf\limits_{\mathcal{V}\ni V\to x} F(V) = \limsup\limits_{\mathcal{V}\ni V\to x} F(V)$

We remark that (c) holds with $q \in \mathbf{R}$ if and only if for every $\varepsilon > 0$ there exists $\rho > 0$ such that $| F(V) - q | < \varepsilon$ for all sets $V \in \mathcal{V}$ for which $x \in V$ and diam $V < \rho$.

The following basic theorem on differentiation with respect to a Vitali system holds (see [8] , Theorem 2.9.5 , Lemma 2.9.6 , Theorem 2.9.7 .) .

THEOREM 2.5. *Let* μ, ν *be two Radon measures on* Ω, *and let* $\mathcal{V}$ *be a Vitali system for* μ *; then the limit*

$$g(x) = \lim_{\mathcal{V}\ni V\to x} \frac{\nu(V)}{\mu(V)}$$

exists and is finite for μ *- almost every* $x \in \Omega$, *and the function g is* μ *- measurable on* Ω *.Moreover, if* ν *is absolutely continuous with respect to* μ, *then*

$$\nu(B) = \int_B g \, d\mu$$

for every $B \in \mathcal{B}(\Omega)$.

Next, with the notation above we define the "derivative" of a Radon measure ν with respect to another Radon measure μ.

DEFINITION 2.6. Let μ,ν be two Radon measures on Ω with ν absolutely continuous with respect to μ, let $\mathcal{V}$ be a Vitali system for μ, and let $x \in \Omega$. Then the quantity

$$\frac{d\nu}{d\mu}(x) = \lim_{\mathcal{V} \ni V \to x} \frac{\nu(V)}{\mu(V)}$$

will be the *derivative* of ν with respect to μ at the point x relative to the Vitali system $\mathcal{V}$.

Let $\alpha : \mathcal{B}(\Omega) \to [0,+\infty]$ be an increasing countably subadditive set function such that $\alpha(\emptyset) = 0$. The existence of the least Borel measure on Ω greater than or equal to α on $\mathcal{B}(\Omega)$ is guaranted by the following result (for a proof see , for istance , [4] , Lemma 4.1) .

PROPOSITION 2.7. *Let μ be the least superadditive set function on $\mathcal{B}(\Omega)$ such that $\mu \geq \alpha$ on $\mathcal{B}(\Omega)$. Then for every $B \in \mathcal{B}(\Omega)$ we have*

$$\mu(B) = \sup \sum_{i \in I} \alpha(B_i)$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i\in I}$ of B . Moreover, μ is a Borel measure on Ω .

DEFINITION 2.8. We denote by μ_α the least Borel measure on Ω such that $\mu_\alpha \geq \alpha$ on $\mathcal{B}(\Omega)$.

Later we also need the following proposition (for a proof see [9] , Theorem 17.2.2) .

PROPOSITION 2.9. *Let λ be a Radon measure , and let $\mathcal{V}$ be a Vitali system for λ . Then the function defined for every $x \in \Omega$ by*

$$f(x) = \liminf_{\mathcal{V} \ni V \to x} \frac{\alpha(V)}{\lambda(V)}$$

is λ-measurable .

3 . PROOF OF THE RESULT .

We will deduce our main result from the following theorem :

THEOREM 3.1. *Let λ a Radon measure on Ω , and let $\alpha : \mathcal{B}(\Omega)\to[0,+\infty]$ be an increasing countably subadditive set function such that $\alpha(\emptyset) = 0$. Consider the Borel measure μ_α introduced in Definition 2.8, and assume that $\mu_\alpha(\Omega) < +\infty$. Let $\mathcal{V}$ be a Vitali system for $\lambda+\mu_\alpha$. For every $x\in\Omega$ define*

$$(3.1) \qquad h(x) = \liminf_{\mathcal{V}\ni V\to x} \frac{\alpha(V)}{\lambda(V)}$$

Suppose that

(i) $h(x) < +\infty$ for $(\lambda+\mu_\alpha)$- *almost every* $x\in\Omega$,

(ii) $h\in L^1_{loc}(\Omega,\lambda)$

.

Then

$$\mu_\alpha(B) = \int_B h\, d\lambda$$

for every $B\in\mathcal{B}(\Omega)$. Moreover the liminf in (3.1) is a limit for λ-almost every $x\in\Omega$.

REMARK 3.2. The proof of Theorem 1.1 is an obvious consequence of Theorem 3.1; infact, it is enough to note that, whenever we choose as Vitali system the family of all closed balls contained in Ω, the liminf in (1.1) is greater than or equal to the liminf in (3.1); and that, by Proposition 2.7, $\mu_\alpha(B) = 0$ for every $B\in \mathcal{B}(\Omega)$ such that $\alpha(B) = 0$.

The proof of Theorem 3.1 is essentially based on the next lemma :

LEMMA 3.3. *Let* α, μ_α *be as in Theorem 3.1. Let* $\mathcal{V}$ *be a Vitali system for* μ_α *. Then*

$$\lim_{\mathcal{V}\ni V\to x} \frac{\alpha(V)}{\mu_\alpha(V)} = 1$$

for μ_α*- almost every* $x\in\Omega$.

PROOF: For any $x\in\Omega$ we set

$$\liminf_{\mathcal{V}\ni V\to x} \frac{\alpha(V)}{\mu_\alpha(V)} = f(x) \tag{3.2}$$

To get the assertion it is enough to show that $f(x) \geq 1$ for μ_α - almost every $x\in\Omega$.

For any $t\in]0,1[$ we define the set :

$$E_t = \{ x\in\Omega : f(x) < t \} .$$

By Proposition 2.9, for each $t\in]0,1[$ the set E_t is μ_α - measurable . The proof is accomplished if we show that $\mu_\alpha(E_t) = 0$ for every $t\in]0,1[$. Preliminarily, we prove that for every $B\in \mathcal{B}(\Omega)$ such that $B\subseteq E_t$ we have :

$$\alpha(B) \leq t\, \mu_\alpha(B) . \tag{3.3}$$

Fix $t\in]0,1[$ and $B\in \mathcal{B}(\Omega)$ with $B\subseteq E_t$. For every $\eta > 0$ there exists an open set U , $U\supseteq B$ such that $\mu_\alpha(U) < \mu_\alpha(B) + \eta$. By the definition of E_t , for every $\rho > 0$ and for every $x\in B$

there exists $V\in\mathcal{V}$ such that $x\in V$, $V\subseteq U$, diam $V<\rho$ and $\dfrac{\alpha(V)}{\mu_\alpha(V)}<t$.

Let $\mathcal{F}$ be the subfamily of $\mathcal{V}$ so defined :

$$\mathcal{F}=\{\, V\in\mathcal{V}: V\subseteq U, \frac{\alpha(V)}{\mu_\alpha(V)}<t, B\cap V=\emptyset \,\}$$

By the previous remark $\mathcal{F}$ is fine at each point $x\in B$; thus, by condition (b) of Definition 2.2, there are countably many disjoint sets $V_1, V_2, \ldots\ldots, V_h, \ldots\ldots$ in $\mathcal{F}$ and a set $N\in\mathcal{B}(\Omega)$ such that $\mu_\alpha(N)=0$ and $B\subseteq \bigcup_{h\in\mathbf{N}} V_h \cup N$.

Therefore we have :

$$\alpha(B)\le\sum_{h\in\mathbf{N}}\alpha(V_h)+\alpha(N)\le$$

$$\le\sum_{h\in\mathbf{N}}\mu_\alpha(V_h)+\mu_\alpha(N)\le$$

$$\le t\,\mu_\alpha(U)\le t\,\mu_\alpha(B)+t\,\eta\,.$$

Since η is arbitrary the inequality (3.3) follows . Next , consider any set $B\in\mathcal{B}(\Omega)$. We see that :

$$\alpha(B)\le\alpha(B\cap E_t)+\alpha(B-E_t)\le$$

$$\le t\;\mu_\alpha(B\cap E_t)+\;\mu_\alpha(B-E_t)\,.$$

For any $B\in\mathcal{B}(\Omega)$ define :

$$\mu_t(B)=t\;\mu_\alpha(B\cap E_t)+\;\mu_\alpha(B-E_t)\,.$$

It is easy to see that μ_t is a Borel measure on Ω . Since $\alpha\le\mu_t$ on $\mathcal{B}(\Omega)$, by the definition of μ_α we have also $\mu_\alpha \leq \mu_t$ on $\mathcal{B}(\Omega)$. Hence ,

$$\mu_\alpha(E_t) \leq \mu_t(E_t) = t\,\mu_\alpha(E_t)$$

which proves that $\mu_\alpha(E_t) = 0$, and this concludes the proof of the lemma . ∎

PROOF OF THE THEOREM 3.1. Since $(\lambda + \mu_\alpha)$ is a Radon measure , by Definition 2.6 and by Lemma 3.3 , we obtain that there is a set E with $\mu_\alpha(E) = 0$ such that :

$$\text{(3.4)} \qquad \frac{d(\mu_\alpha)}{d(\lambda+\mu_\alpha)}(x) = \lim_{\mathcal{V}\ni V\to x} \frac{\mu_\alpha(V)}{(\lambda+\mu_\alpha)(V)} \lim_{\mathcal{V}\ni V\to x} \frac{\alpha(V)}{\mu_\alpha(V)} =$$

$$= \lim_{\mathcal{V}\ni V\to x} \frac{\alpha(V)}{(\lambda+\mu_\alpha)(V)} = \liminf_{\mathcal{V}\ni V\to x} \frac{\alpha(V)}{\lambda(V)} \lim_{\mathcal{V}\ni V\to x} \frac{\lambda(V)}{(\lambda+\mu_\alpha)(V)} =$$

$$= h(x) \frac{d\lambda}{d(\lambda+\mu_\alpha)}(x)$$

for any $x\in\Omega - E$.

We remark that , by hypothesis (i) , the equality (3.4) holds even if $\dfrac{d\lambda}{d(\lambda+\mu_\alpha)} = 0$ at x .

Moreover , noting that

$$\lim_{\mathcal{V}\ni V\to x} \frac{\mu_\alpha(V)}{\lambda(V)} = 0$$

for $(\lambda + \mu_\alpha)$ - almost every $x\in E$, and that

$$h(x) = \liminf_{\mathcal{V} \ni V \to x} \frac{\alpha(V)}{\lambda(V)} \leq \lim_{\mathcal{V} \ni V \to x} \frac{\mu_\alpha(V)}{\lambda(V)}$$

for λ - almost every $x \in \Omega$, we also have that the equality (3.4) holds for $(\lambda + \mu_\alpha)$ - almost every $x \in \Omega$. This implies that :

$$(3.5) \qquad \int_B \frac{d\mu_\alpha}{d(\lambda+\mu_\alpha)} \, d(\lambda+\mu_\alpha) = \int_B h \, \frac{d\lambda}{d(\lambda+\mu_\alpha)} \, d(\lambda+\mu_\alpha)$$

for every $B \in \mathcal{B}(\Omega)$. By hypothesis (ii) and by (3.5) we obtain

$$\int_B d\mu = \int_B h \, d\lambda$$

for any $B \in \mathcal{B}(\Omega)$. To prove that the liminf in (3.1) is a limit , it is enough to observe that , by Lemma 3.3 ,

$$h(x) = \liminf_{\mathcal{V} \ni V \to x} \frac{\alpha(V)}{\lambda(V)} \leq \limsup_{\mathcal{V} \ni V \to x} \frac{\alpha(V)}{\lambda(V)} =$$

$$= \lim_{\mathcal{V} \ni V \to x} \frac{\alpha(V)}{\mu_\alpha(V)} \lim_{\mathcal{V} \ni V \to x} \frac{\mu_\alpha(V)}{\lambda(V)} = h(x)$$

for λ - almost every $x \in \Omega$, which completes the proof . ∎

An interesting corollary , whose proof is deduced trivially from Theorem 3.1 , is the following :

COROLLARY 3.4. *Let* α *be as in Theorem 3.1. Let* λ *be a Radon measure on* Ω. *Suppose that there is a constant* $c > 0$ *such that* $\alpha \leq c\lambda$ *on* $\mathcal{B}(\Omega)$. *Let* $\mathcal{V}$ *be a Vitali system for* λ. *Then the limit*

$$f(x) = \lim_{\mathcal{V} \ni V \to x} \frac{\alpha(V)}{\lambda(V)} \tag{3.6}$$

exists for λ *- almost every* $x \in \Omega$ *and*

$$\mu_\alpha(B) = \int_B f \, d\lambda$$

for any $B \in \mathcal{B}(\Omega)$. ∎

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Chapter 3 :

Dirichlet problems in domains bounded by thin layers with random thickness

In this part of the thesis we present some results obtained in collaboration with G. Paderni.

DIRICHLET PROBLEMS IN DOMAINS BOUNDED BY THIN LAYERS WITH RANDOM THICKNESS.

INTRODUCTION

Recently G. Buttazzo, G. Dal Maso, and U. Mosco have proposed in [6] a new capacity method to investigate the asymptotic behaviour for Dirichlet problems in domains bounded by thin layers. In this paper, taking inspiration from that method and from some variational techniques developed in [3], we provide a setting to analyze the cases in which the domains are surrounded by thin layers with random thickness.

Let us describe more closely the problem we deal with.

Let D be a bounded Lipschitz domain of $\mathbf{R}^n$, $n \geq 2$, and let (ε_h) be a sequence of real numbers such that $\varepsilon_h \to 0$ as $h \to +\infty$. For every $h \in \mathbf{N}$ let us consider a class $\mathcal{A}_h$ of subsets of $\mathbf{R}^n$ defined by

$$\mathcal{A}_h = \{ A \subseteq \mathbf{R}^n ,\ A \supseteq \overline{D} : \sup_{x \in A} \operatorname{dist}(x,D) < \varepsilon_h \} .$$

Given $f \in L^2(\mathbf{R}^n)$ we are interested in the solutions of the equations

$$-\Delta u_h = f \ \text{ in } D \qquad -\varepsilon_h \Delta u_h = f \ \text{ in } A_h \setminus D \tag{0.1}$$

where A_h is a random set of the class $\mathcal{A}_h$, $u_h = 0$ on ∂A_h and the natural transmission conditions on ∂D are satisfied.

Let F_h be the quadratic form on $L^2(\mathbf{R}^n)$ defined by

$$F_h(u) = \begin{cases} \displaystyle\int_D |\nabla u|^2 dx + \varepsilon_h \int_{A \setminus D} |\nabla u|^2 dx & \text{if } u \in H^1_o(A_h) \\ + \infty & \text{otherwise} \end{cases} .$$

The solution u_h of (0.1) coincides with the solution of the minimum problem

$$\min \{ F_h(u) - 2 \int_A f u \, dx \ : \ u \in L^2(\mathbf{R}^n) \} .$$

Our aim is to characterize the behaviour of the sequence (u_h) in the limit as $h \to +\infty$. First, we introduce the class $\mathcal{E}$ of all convex, semicontinuous functions from $L^2(\mathbf{R}^n)$ into $\overline{\mathbf{R}}$. We equip $\mathcal{E}$ with a topological structure ($L^2(\mathbf{R}^n)$ - Mosco - convergence) so that it becomes a complete metric space. Then, we associate with the problems (0.1) a sequence (F_h) of "random functionals", that is measurable maps $\omega \to F_h(\omega)$ from a probability space Ω into $\mathcal{E}$. In this way the problem consists in analyzing the asymptotic behaviour, as $h \to +\infty$, of sequence of random functionals (F_h).

The first result we prove is a compactness theorem for sequences of random functionals

(Theorem 4.1 and Remark 5.5). It is deduced from an abstract compactness result for sequences of probability measures on a complete metric space (Theorem 2.3). We show that, under suitable assumptions on the sequence (F_h), there exists a subsequence $(F_{\sigma(h)})$ converging in probability to a constant random functional F. Moreover the functional F turns out to be associated with the equation, formally written as

$$(0.2) \qquad \begin{cases} -\Delta u = f & \text{in } D \\ \dfrac{\partial u}{\partial n} + \mu u = 0 & \text{on } \partial D \ , \end{cases}$$

where μ is a Borel measure on ∂D that vanishes on any set of zero (harmonic) capacity, but may assume the value $+\infty$ on some subset of positive capacity and n is the outer unit normal to D.

The second result we obtain is a characterization of the limit functional (hence of the measure μ that appears in (0.2)). For both results the assumptions are made in terms of the asymptotic behaviour of the expectations and of the covariances of suitable random capacities associated with the random functionals F_h .

In the deterministic case problems of the type (0.1) are known as "reinforcement problems". They have been investigated in the last years by several authors (see for istance [1], [5], [7], [8]).

To our knowledge no specific reference for the stochastic cases is available. We only mention the paper [11] which provides a general framework for the study of probabilistic problems in calculus of variations.

Our paper is organized as follows.

1. Notation and preliminaries.
2. Some abstract probabilistic results.
3. Mosco - convergence and random capacities.
4. Main results.
5. Dirichlet problems in domains surrounded by thin layers with random thickness.
6. An example.

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1. NOTATION AND PRELIMINARIES.

1.1 Let n be an integer with $n \geq 2$. We denote by $\mathcal{U}$ (resp. $\mathcal{K}$, $\mathcal{B}$) the family of all bounded open (resp. compact, Borel) subsets of $\mathbf{R}^n$. We recall some definitions which will be often used in the sequel. For every $U \in \mathcal{U}$ and for every $K \in \mathcal{K}$ such that $K \subseteq U$, we define the *capacity* of K with respect to U by

$$\operatorname{cap}(K,U) = \inf \left\{ \int_U |D\phi|^2 dx \ : \phi \in C_0^\infty(U), \phi \geq 1 \text{ on } K \right\} \quad ;$$

the definition is extended to the sets $V \in \mathcal{U}$ with $V \subseteq U$ by

$$\operatorname{cap}(V,U) = \sup \left\{ \operatorname{cap}(K,U) \ : K \in \mathcal{K}, K \subseteq V \right\} \ ;$$

and to the sets $B \in \mathcal{B}$ with $B \subseteq U$ by

$$\operatorname{cap}(B,U) = \inf \left\{ \operatorname{cap}(V,U) \ : V \in \mathcal{U}, V \supseteq B \right\} .$$

We say that a Borel set B of $\mathbf{R}^n$ has *capacity zero* if $\operatorname{cap}(B \cap U,U) = 0$ for every $U \in \mathcal{U}$. When a property P(x) is satisfied for all $x \in B$, except for a subset $N \subseteq B$ with zero capacity, then we say that P(x) holds *quasi everywhere* on B (q.e. on B). We say that a function $f : B \to \mathbf{R}$ is *quasi continuous* on B if for every $U \in \mathcal{U}$ and for every $\varepsilon > 0$ there exists $V \in \mathcal{U}$, $V \subseteq U$, with $\operatorname{cap}(V,U) < \varepsilon$ such that the restriction of f to $(B \cap U) \setminus V$ is continuous. A subset A of $\mathbf{R}^n$ is said to be *quasi open* (resp. *quasi closed*, *quasi compact*) if for every $\varepsilon > 0$ and for every $U \in \mathcal{U}$, there exists an open (resp. closed, compact) set $V \subseteq U$ such that $\operatorname{cap}((A \cap U) \Delta V,U) < \varepsilon$, where Δ denotes the symmetric difference between sets. We recall that a bounded set $B \subseteq \mathbf{R}^n$ has zero capacity (resp. B is quasi open or f is quasi continuous on B) if and only if the above conditions are satisfied for some $U \in \mathcal{U}$ with $B \subseteq U$.

1.2. For every open set $U \subseteq \mathbf{R}^n$ we denote by $H^1(U)$ the Sobolev space of all functions in $L^2(U)$ whose first weak derivatives belong to $L^2(U)$, and by $H_0^1(U)$ the closure of $C_0^\infty(U)$ in $H^1(U)$. For every $x \in \mathbf{R}^n$ and for every $r > 0$ we set

$$B_r(x) = \{ y \in \mathbf{R}^n : |x - y| < r \}$$

and for every Borel set $B \subseteq \mathbf{R}^n$ we denote by $|B|$ its Lebesgue measure. Let $U \in \mathcal{U}$. For every $u \in H^1(U)$ the limit

$$\tilde{u}(x) = \lim_{r \to 0} \frac{1}{|U \cap B_r(x)|} \int_{U \cap B_r(x)} u(y)\, dy \tag{1.1}$$

exists and is finite q.e. on U. The function $\tilde{u}$ defined q.e. by (1.1) is quasi continuous on U. Moreover, it can be shown that for every $B \in \mathcal{B}$, with $B \subset U$

$$\operatorname{cap}(B,U) = \min \left\{ \int_U |Du|^2 dx : u \in H_0^1(U) \ , \tilde{u} \geq 1 \text{ q.e. on } B \right\}.$$

For a proof of these properties of the capacity and of the functions of $H^1(U)$ we refer to [18].

1.3. A non negative countably additive set function defined on the Borel σ-algebra of $\mathbf{R}^n$ with values in $[0,+\infty]$ is called a *Borel measure*. A Borel measure which assigns finite values to every $K\in\mathcal{K}$ is called a *Radon measure*. In our paper we deal with a peculiar class of Borel measures.

Following [10] we denote by $\mathcal{M}_0^*$ the class of all Borel measures μ such that:

(a) $\mu(B)=0$ for every Borel set $B\subseteq\mathbf{R}^n$ with capacity zero;

(b) $\mu(B)=\inf\{\mu(A)$: A quasi open, $B\subseteq A\}$ for every Borel set $B\subseteq\mathbf{R}^n$.

An easy example of measure belonging to $\mathcal{M}_0^*$ is the measure μ defined by

$$\mu(B)=\int_B f dx$$

for every Borel set $B\subseteq\mathbf{R}^n$, where $f\in L^1_{loc}(\mathbf{R}^n)$. More generally, every Radon measure μ satisfying (a) belongs to $\mathcal{M}_0^*$. We remark that the measures belonging to $\mathcal{M}_0^*$ are not required to be regular nor σ-finite. For instance, the measures introduced in the definition below belong to the class $\mathcal{M}_0^*$ (see [10], Remark 3.3).

Definition 1.1. For every quasi closed set $F\subseteq\mathbf{R}^n$ we denote by ∞_F the Borel measure defined by

$$\infty_F(B) = \begin{cases} 0 & \text{if } F\cap B \text{ has capacity zero} \\ +\infty & \text{otherwise} \end{cases}$$

for every Borel set $B\subseteq\mathbf{R}^n$.

Other examples of measures in M_0^* are given in [13].

1.4. Troughout we denote by D a fixed set of $\mathcal{U}$ with a Lipschitz boundary and by L a fixed elliptic operator of the form

$$Lu = -\sum_{i,j=1}^{n} D_i(a_{i,j}(x)D_j u),$$

where $a_{i,j}=a_{j,i}\in L^\infty(\mathbf{R}^n)$ and for almost every $x\in\mathbf{R}^n$ and for every $\xi\in\mathbf{R}^n$ we have

$$\Lambda_1|\xi|^2 \le \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \le \Lambda_2|\xi|^2,$$

where $\Lambda_1,\Lambda_2\in\mathbf{R}$ with $0<\Lambda_1\le\Lambda_2<+\infty$. Let us fix a sequence (ε_h) of positive real numbers such that $\varepsilon_h\to 0$ as $h\to+\infty$. For every $h\in\mathbf{N}$ we consider the operator

$$L^{(h)}u = -\sum_{i,j=1}^{n} D_i(a^{(h)}_{i,j}(x)D_j u),$$

where

$$a^{(h)}_{i,j}(x) = \begin{cases} a_{i,j}(x) & \text{if } x\in D \\ \varepsilon_h a_{i,j}(x) & \text{if } x\in \mathbf{R}^n - D. \end{cases}$$

We denote by $a^{(h)}(x,\xi)$ and $a^{0}(x,\xi)$ the quadratic forms associated with the matrices $(a^{(h)}_{i,j})$ and

$$a^{0}_{i,j}(x) = \begin{cases} a_{i,j}(x) & \text{if } x\in D \\ 0 & \text{if } x\in \mathbf{R}^n - D, \end{cases}$$

more precicely,

$$a^{(h)}(x,\xi) = \sum_{i,j=1}^{n} a^{(h)}_{i,j}(x)\xi_i\xi_j , \tag{1.2}$$

$$a^{0}(x,\xi) = \sum_{i,j=1}^{n} a^{0}_{i,j}(x)\xi_i\xi_j .$$

Let (η_h) be another sequence of positive numbers such that $\eta_h \to 0$ as $h\to+\infty$.

Definition 1.2. For every $h\in \mathbf{N}$ we consider the class of sets

$$\mathcal{A}_h = \{ A\in \mathcal{U} : A\supseteq \bar{D},\ \sup_{x\in A} \operatorname{dist}(x,D) < \eta_h \}$$

and we denote by $\mathcal{F}_h$ the class of all functionals $F : L^2(\mathbf{R}^n)\to[0,+\infty]$ defined by

$$F(u) = \begin{cases} \int_{\mathbf{R}^n} a^{(h)}(x,Du)dx + \int_{\mathbf{R}^n} \tilde{u}^2 d\infty_{\partial A} & u\in H^1(\mathbf{R}^n) \\ +\infty & \text{otherwise} \end{cases}$$

where A is an open set belonging to the class $\mathcal{A}_h$ and $\infty_{\partial A}$ is the measure of $\mathcal{M}^*_0$ defined in Definition 1.1.

Remark 1.3. By definition 1.2. we have $\mathcal{F}_h \cap \mathcal{F}_k = \emptyset$ for $h \neq k$.

Definition 1.4. The class of measure $\mu\in \mathcal{M}^*_0$ such that spt $\mu \subseteq \partial D$ will be denoted by $\mathcal{M}^*_0(\partial D)$ and the class of all functionals $F : L^2(\mathbf{R}^n)\to[0,+\infty]$ defined by

$$F(u) = \begin{cases} \int_{D} a^{0}(x,Du)dx + \int_{\partial D} \tilde{u}^2 d\mu & \text{if } u_{|D}\in H^1(D) \\ +\infty & \text{otherwise} \end{cases}$$

where $\mu \in \mathcal{M}_0^*(\partial D)$, will be indicated by $\mathcal{F}_0$.

Remark 1.5. It can be seen that there is a one to one correspondance between the functionals of the class $\mathcal{F}_h$ and the measures $\infty_{\partial A}$ with $A \in \mathcal{A}_h$, and between the functionals of the class $\mathcal{F}_0$ and the measure $\mu \in \mathcal{M}_0^*(\partial D)$.

With every functional $F \in \mathcal{F}_h$ we associate a Dirichlet problem of the form

$$(1.3) \qquad \begin{cases} L^{(h)}u_h + \lambda u_h = g & \text{in } A \\ u_h \in H_0^1(A) \end{cases}$$

where $\lambda \geq 0$, $A \in \mathcal{A}_h$, and $g \in L^2(\mathbf{R}^n)$. Let u_h be the unique weak solution of (1.3). Let us consider the function

$$w_h = \begin{cases} u_h & \text{on } A \\ 0 & \text{on } \mathbf{R}^n - A. \end{cases}$$

Definition 1.6. Given $F \in \mathcal{F}_h$, for $\lambda \geq 0$ we define the resolvent operator $R^h(\lambda)$: $L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ associated with F by setting

$$R_h(\lambda)[g] = w_h.$$

With every functional $F \in \mathcal{F}_0$ we also associate a problem formally written as

$$(1.4) \qquad \begin{cases} Lu + \lambda u = g & \text{in } D \\ \dfrac{\partial u}{\partial n} + \mu u = 0 & \text{on } \partial D \end{cases}$$

where $\lambda > 0$, $g \in L^2(\mathbf{R}^n)$, $\mu \in \mathcal{M}_0^*(\partial D)$, and n is the outer unit normal to D.

A variational solution of the problem (1.4) is a function u such that $u \in H^1(D)$, $\tilde{u} \in L^2(\partial D, \mu)$ and

$$\int_D \left(\sum_{i,j=1}^n a_{i,j} D_i u D_j v \right) dx + \int_{\partial D} \tilde{u}\tilde{v} \, d\mu + \lambda \int_D uv \, dx = \int_D gv \, dx$$

for every $v \in H^1(D)$ with $\tilde{v} \in L^2(\partial D, \mu)$. Let u be a variational solution of the problem (1.4); u is the unique solution of the minimum problem

$$(1.5) \qquad \min\{ \int_D a(x, Dw) \, dx + \int_{\partial D} \tilde{w}^2 d\mu + \lambda \int_D w^2 \, dx - 2 \int_D gw \, dx \; ; \; w \in H^1(D) \}.$$

Let w be the function

$$w = \begin{cases} u & \text{on } D \\ 0 & \text{on } \mathbf{R}^n - D . \end{cases}$$

Definition 1.7. Given $F_0 \in \mathcal{F}_0$, for $\lambda > 0$ we define the resolvent operator $R^0(\lambda) : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ associated to F_0 by setting

(1.6) $$R^0(\lambda)[g] = w.$$

Remark 1.8. It can be shown that (1.5) and (1.6) hold also in the case $\lambda = 0$ when there exists a constant $c > 0$ such that for every $h \in \mathbf{N}$ $\eta_h = c\varepsilon_h$, i.e. if the following relation is satisfied

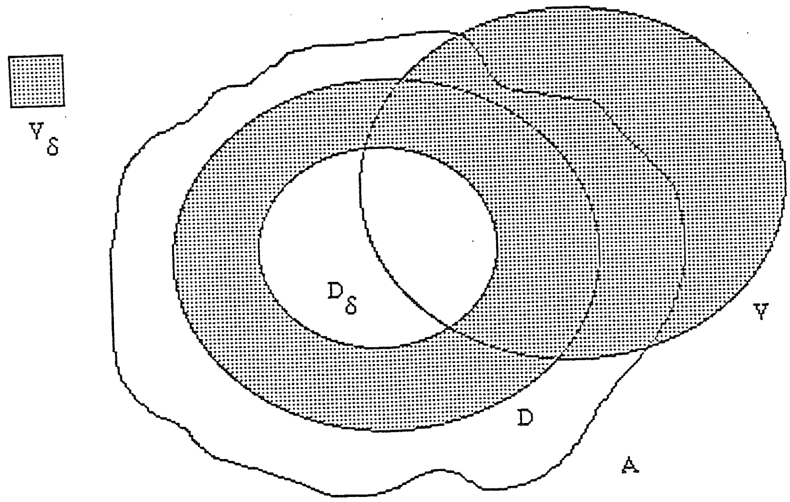
(1.7) $$A \subseteq \{ x \in \mathbf{R}^n : \mathrm{dist}(x,D) < c\varepsilon_h \}.$$

Let us set

(1.8) $$\mathcal{F} = \mathcal{F}_0 \cup \Big(\bigcup_{h \in \mathbf{N}} \mathcal{F}_h \Big).$$

In the following we define a set function of capacity type associated with any functional $F \in \mathcal{F}$. It will be the basic tool in our investigation.

Let $(D_\delta)_{\delta>0}$ be the family of the open subsets of D with Lipschitz boundary ∂D_δ such that $D_{\delta_2} \subset\subset D_{\delta_1}$ for $\delta_1 > \delta_2$ and $D = \bigcup_{\delta>0} D_\delta$. For every $V \in \mathcal{U}$ we set $V_\delta = (V \cup D) \setminus \bar{D}_\delta$.



Definition 1.9. Given $F \in \mathcal{F}$, for every $U \in \mathcal{U}$ and for every $\delta > 0$ we define the following set function

(1.9) $$b_\delta(F,V) = \min \Big\{ \int_{V_\delta} a^{(F)}(x,Du)\,dx + \int_{V_\delta} \tilde{u}^2 d\mu_F \ : \ u \in H^1(V_\delta),\ u=1 \text{ on } \partial D_\delta \Big\}$$

where

$$a^{(F)}(x,\xi) = \begin{cases} a^{(h)}(x,\xi) & \text{if } F\in \mathcal{F}_h \\ a^{0}(x,\xi) & \text{if } F\in \mathcal{F}_0 \end{cases}$$

and

$$\mu_F = \begin{cases} \infty_{\partial A} \ \text{ with } A\in \mathcal{A}_h & \text{if } F\in \mathcal{F}_h \\ \mu \quad \text{ with } \mu\in \mathcal{M}_0^*(\partial D) & \text{if } F\in \mathcal{F}_0 \end{cases}$$

For $F\in \mathcal{F}_0$ we extend the definition of b_δ to the Borel sets $B\in \mathcal{B}$ by

$$b_\delta(F,B) = \inf \{ b_\delta(F,V) \ : \ V\in \mathcal{U},\ V\supseteq B \}$$

The minimum in (1.9) is achieved by the lower semicontinuity and the coerciveness of the functional.

Remark 1.10. In order to study Dirichlet problems in domains bounded by thin layers, in [6] the authors introduce two set functions, depending on the choice of a pair $V,U\in \mathcal{U}$ such that $V\subseteq U$. By Lemma 3.5(c) in [6], it can be seen that, for $F\in \mathcal{F}_0$, the set function b_δ defined in (1.9) is equivalent to the set function b_δ^{o} defined by (3.11) in [6], i.e. for every $B\in \mathcal{B}$ $b_\delta(F,B) = b_\delta^{o}(\mu_F,B,U)$ where U is an arbitrary open set of U containing the region D. Moreover, for $F\in \mathcal{F}_h$, the set function b_δ is equivalent to the set function $b_\delta^{\epsilon h}$ defined by (6.7) in [6], i.e. $b_\delta(F,V)=b_\delta^{\epsilon h}(V,U)$ with $A=\Omega^{\epsilon h}$ and $U,V\in \mathcal{U}$ such that $V\subseteq U$.

If $F\in \mathcal{F}_0$ the main properties of the set function $b_\delta(F,\bullet)$ can be summarized in the next proposition.

Proposition 1.11. *For every* $F\in \mathcal{F}_0$ *and for every* $\delta > 0$, *the function* $b_\delta(F,\bullet)$ *satisfies the following properties:*

(a) $b_\delta(F,\emptyset) = 0$;

(b) *if* $B_1,B_2\in \mathcal{B}$, *with* $B_1\subseteq B_2$, *then* $b_\delta(F,B_1) \le b_\delta(F,B_2)$;

(c) *if* (B_h) *is an increasing sequence of sets in* $\mathcal{B}$ *and* $B = (\cup B_h)$, *then*

$$b_\delta(F,B) = \sup\{b_\delta(F,B_h) : \ h\in \mathbf{N} \};$$

(d) *if* (B_h) *is a sequence of sets in* $\mathcal{B}$ *and* B *is a Borel subset of* $(\cup B_h)$, *then*

$$b_\delta(F,B) \le \sum b_\delta(F,B_h);$$

(e) *if* $B_1,B_2\in \mathcal{B}$ *then* $b_\delta(F,B_1\cup B_2) + b_\delta(F,B_1\cap B_2) \le b_\delta(F,B_1) + b_\delta(F,B_2)$;

(f) *if* $B\in \mathcal{B}$ *and* $V\in \mathcal{U}$ *with* $(B\cap\partial D) \subseteq V$ *and* $V\cap\partial D_\delta=\emptyset$ *then* $b_\delta(F,B) \le \Lambda_2 \mathrm{cap}(B\cap\partial D,V)$;

(g) *if* $B\in \mathcal{B}$ *then* $b_\delta(F,B) = b_\delta(F,B\cap\partial D) \le \mu(B\cap\partial D)$, *where* μ *is the measure in* $\mathcal{M}_0^*$ *associated to the functional* F;

(h) *for every* $K\in \mathcal{K}$, $b_\delta(F,K) = \inf\{b_\delta(F,U) : U\in \mathcal{U},\ K\subseteq U\}$;

(i) *if* $B_1,B_2\in \mathcal{B}$ *and* $\mathrm{dist}(B_1,B_2)=\sigma>0$, *then for every* $\eta\in\,]0,1[$
$b_\delta(F,B_1) + b_\delta(F,B_2) \le (1-\eta)^{-1} b_\delta(F,B_1\cup B_2) + 4\Lambda_2\eta^{-1}\sigma^{2}|D-D_\delta|$;

(j) *for every* $B\in\mathcal{B}$, $b_\delta(F,B)=\sup\{b_\delta(F,K) : K\in\mathcal{K}, K\subseteq B\}$;

(k) *if* $\mu\in\mathcal{M}_0^*(\partial D)$ *is the measure associated to* F, *then* $\mu(B\cap\partial D)=\sup\{b_\delta(F,B):\delta>0\}$ *for every* $B\in\mathcal{B}$.

Proof. The properties (a), (b), (c), (d) can be deduced by standard capacity theory arguments. In view of Remark 1.10 the properties (e),(f),(g),(h),(i) follow directly from (3.12), (3.13), (3.14) and Lemma 3.5 in [6]. The property (j) is an easy consequence of properties (h),(c),(b), and the Choquet capacitability theorem (see [9]). The property (k) follows from Theorem 3.6 in [6]. ∎

Finally we recall the definition of capacity relative to the operator $L^{(h)}$.

Definition 1.12. For every $h\in\mathbf{N}$ we define

$$\mathrm{cap}^{(h)}(B)=\min\{\int_{\mathbf{R}^n}a^{(h)}(x,Du)\,dx+\int_{\mathbf{R}^n}u^2\,dx\ :\ u\in H^1(\mathbf{R}^n),\ \tilde{u}\geq 1 \text{ q.e. on } B\}$$

for every Borel set $B\subseteq\mathbf{R}^n$.

2. SOME ABSTRACT PROBABILISTIC RESULTS.

In this section we set up the probabilistic picture of our paper and give some results which will have a crucial role in the proofs of the main theorems in section 4.

Troughout we deal with the following abstract framework.

(2.1) (X,d) is a complete metric space;

(2.2) X_0 is a compact subset of X;

(2.3) (X_h) is a sequence of subsets of X satisfying the following proprerty : if (x_h) is a sequence of elements of X and $(X_{\sigma(h)})$ is any subsequence of (X_h) such that $x_h\in X_{\sigma(h)}$ then there exist a subsequence $x_{m(h)}$ of x_h and an element $x\in X_0$ such that $x_{m(h)}$ converges to x in X.

Remark 2.1. From (2.1), (2.2) and (2.3) it is immediate to deduce that for every open neighbourhood U of X_0, there exists $h_0\in\mathbf{N}$ such $X_h\subseteq U$ for every $h\geq h_0$.

We denote by $\mathcal{B}(X)$ the Borel σ-field of X. A probability measure Q on $(X,\mathcal{B}(X))$ is a non negative countably additive set function defined on $\mathcal{B}(X)$ with Q(X)=1.By $\mathcal{P}(X)$ we mean the space of all probabilty measure defined on $\mathcal{B}(X)$. On $\mathcal{P}(X)$ we consider the following definition of weak convergence.

Definition 2.2. We say that a sequence (Q_h) of measures in $\mathcal{P}(X)$ converges weakly to $Q \in \mathcal{P}(X)$ if

$$\lim_{h \to +\infty} \int_X f \, dQ_h = \int_X f \, dQ$$

for every $f \in C_b^o(X)$, where $C_b^o(X)$ denotes the class of all bounded continuous functions $f : X \to \mathbf{R}$.

Let $Q \in \mathcal{P}(X)$. For every $\mathcal{B}(X)$ - measurable real valued function f we define the expectation of f in the probability space $(X, \mathcal{B}(X), Q)$ by

$$E_Q[f] = \int_X f \, dQ \,.$$

Let f,g be two real valued functions in $L^2(X,Q)$. Then the covariance of f and g is defined by

$$\mathrm{Cov}_Q[f,g] = E_Q[fg] - E_Q[f] \, E_Q[g]$$

The variance of f is defined by

$$\mathrm{Var}_Q[f] = \mathrm{Cov}_Q[f,f]$$

For every $h \in \mathbf{N}$ the Borel σ-field of X_h equipped with the induced topology is denoted by $\mathcal{B}(X_h)$. Let Q_h be any probability measure on $(X_h, \mathcal{B}(X_h))$. We associate Q_h with the probability measure $\hat{Q}_h$ in $\mathcal{P}(X)$ defined by

$$\hat{Q}_h(B) = Q_h(B \cap X_h) \tag{2.4}$$

for every $B \in \mathcal{B}(X)$.

In what follows we consider sequences $(\hat{Q}_h)$ of probability measure in P(X) with $\hat{Q}_h$ defined by (2.4). We note that a probability measure P_h on $(X, \mathcal{B}(X))$ can be written in the form $\hat{Q}_h$ given by (2.4) if and only if $P_h^*(X_h) = 1$,where P_h^* denotes the outer measure associate with P_h. Infact, if $P_h^*(X_h) = 1$, then $P_h = \hat{Q}_h$ with Q_h defined by $Q_h(B) = P_h^*(B)$ for every $B \in \mathcal{B}(X_h)$.

We can state the following compacteness result.

Theorem 2.3. *For every sequence* $(\hat{Q}_h)$ *in* $\mathcal{P}(X)$ *of the form* (2.4), *there exist a subsequence* $(\hat{Q}_{\sigma(h)})$ *and a measure* $\hat{Q}$ *in* $\mathcal{P}(X)$ *such that* $\hat{Q}(X_o) = 1$ *and* $(\hat{Q}_{\sigma(h)})$ *converges weakly to* $\hat{Q}$ *in* $\mathcal{P}(X)$.

The proof of theorem (2.3) needs the next lemma.

Lemma 2.4. *Let* $(\widehat{Q}_h)$ *be a sequence in* $\mathcal{P}(X)$ *of the form* (2.4). *Let* $f,g \in C_b^{\circ}(X)$. *Assume that there exists* $\eta > 0$ *such that*

$$|f(x) - g(x)| < \eta$$

for every $x \in X_0$.*Then*

$$\limsup_{h \to +\infty} E_{\widehat{Q}_h}[f - g] < \eta .$$

Proof. Since the set $U = \{ x \in X : |f(x) - g(x)| < \eta \}$ is an open neighbourhood of X_0 by Remark 2.1 we have $X_h \subseteq U$ for h sufficiently large. Thus we obtain

$$\limsup_{h \to +\infty} E_{\widehat{Q}_h}[f - g] = \limsup_{h \to +\infty} \int_U |f - g| \, d\widehat{Q}_h < \eta$$

and the proof is complete. ∎

Proof of Theorem 2.3. Let $(\widehat{Q}_h)$ be a sequence in $\mathcal{P}(X)$ of the form (2.4). The proof is articuleted in two steps. In the first step we show that there exists a subsequence $(\widehat{Q}_{\sigma(h)})$ of $(\widehat{Q}_h)$ such that the limit

$$\lim_{h \to +\infty} E_{\widehat{Q}_{\sigma(h)}}[f] \tag{2.5}$$

exists for every $f \in C_b^{\circ}(X)$. In the second one we prove that there exists a measure $\widehat{Q} \in \mathcal{P}(X)$, with $\widehat{Q}(X_0) = 1$ such that the limit (2.5) is equal to $E_{\widehat{Q}}[f]$ for every $f \in C_b^{\circ}(X)$.

Step 1. Let $\mathcal{G} = (g_i)_{i \in I}$ be a countably set which is dense in $C^{\circ}(X_0)$. For every $i \in I$, let $f_i \in C_b^{\circ}(X)$ such that $f_i|_{X_0} = g_i$. By a diagonal procedure we can find a subsequence $(\widehat{Q}_{\sigma(h)})$ of $(\widehat{Q}_h)$ such that

$$\lim_{h \to +\infty} E_{\widehat{Q}_{\sigma(h)}}[f_i]$$

exists, for every $i \in I$. Denote by $I(f_i)$ this limt. In order to prove that the limit (2.5) exists we show that for every $f \in C_b^{\circ}(X)$ the sequence $(E_{\widehat{Q}_{\sigma(h)}}[f])$ is a Cauchy sequence. Let $f \in C_b^{\circ}(X)$. For every $\varepsilon > 0$ let us take $g_i \in \mathcal{G}$ such that

$$\sup_{x \in X_0} |f(x) - g_i(x)| < \frac{\varepsilon}{4} \tag{2.6}$$

Then, by (2.6), and by Lemma 2.4 we obtain that there exists $k_0 \in \mathbb{N}$ such that

$$|E_{\widehat{Q}_{\sigma(k)}}[f] - E_{\widehat{Q}_{\sigma(l)}}[f]| \leq$$

$$\leq |E_{\widehat{Q}_{\sigma(k)}}[f] - E_{\widehat{Q}_{\sigma(k)}}[f_i]| + |E_{\widehat{Q}_{\sigma(k)}}[f_i] - E_{\widehat{Q}_{\sigma(l)}}[f_i]| + |E_{\widehat{Q}_{\sigma(l)}}[f_i] - E_{\widehat{Q}_{\sigma(l)}}[f]| <$$

$$< \frac{\varepsilon}{2} + |E_{\widehat{Q}_{\sigma(k)}}[f_i] - E_{\widehat{Q}_{\sigma(l)}}[f_i]|$$

for every $k,l \geq k_0$. Since $(E_{\widehat{Q}_{\sigma(h)}}[f_i])$ is a Cauchy sequence, we get the first assertion.

Step 2. Let us denote by T any extention operator from $C^{\circ}(X_0)$ into $C_b^{\circ}(X)$ and let us introduce the following maps:

(a) $I : C_b^{\circ}(X) \to \mathbf{R}$ defined by

$$I(f) = \lim_{h \to +\infty} E_{\hat{Q}_{\sigma(h)}}[f]$$

(b) $J : C^{\circ}(X_0) \to \mathbf{R}$ defined by $J(f) = (I \circ T)(f)$.

Noting that $J(1) = 1$, by the classical Rietz Theorem's (see for example [20], Theorem 2.14) it follows that there exists a probability measure Q_0 on X_0 such that

$$J(g) = \int_{X_0} g\, dQ_0$$

for any $g \in C^{\circ}(X_0)$.

Let $\hat{Q}$ be the measure in $\mathcal{P}(X)$ defined by $\hat{Q}(B) = Q_0(B \cap X_0)$ for any $B \in \mathcal{B}(X)$. Then, by Lemma 2.4 we get

$$I(f) = I(Tf_{|X_0}) = J(f_{|X_0}) = \int_{X_0} f\, dQ_0 = E_{\hat{Q}}[f]$$

for every $f \in C_b^{\circ}(X)$. This accomplishes the proof. ∎

Let us set

$$Y = X_0 \cup \Big(\bigcup_{h \in N} X_h \Big)$$

We conclude this section with a basic result for our purposes.

Lemma 2.5. *Let* $(\hat{Q}_h)$ *be a sequence in* $\mathcal{P}(X)$ *of the form* (2.4). *Let* $\hat{Q} \in \mathcal{P}(X)$ *such that* $\hat{Q}(X_0) = 1$. *Suppose that* $(\hat{Q}_h)$ *converges weakly to* $\hat{Q}$ *in* $\mathcal{P}(X)$. *Let* $g : Y \to \mathbf{R}$ *be a function bounded from below. Assume that*

(i) $g|X_0$ *is lower semicontinuous;*

(ii) *let* $(\sigma(h))$ *be any sequence of natural numbers such that* $\sigma(h) \to +\infty$ *as* $h \to +\infty$, *then*

$$g(x) \le \liminf_{h \to +\infty} g(x_h)$$

for every sequence (x_h) *converging to* $x \in X_0$ *in* X *and such that* $x_h \in X_{\sigma(h)}$ *for every* $h \in N$;

(iii) $g|X_h$ *is* $\mathcal{B}(X_h)$ *- measurable for every* $h \in N$.

Then

$$E_Q[g] \le \liminf_{h \to +\infty} E_{Q_h}[g] \tag{2.7}$$

where $Q = \hat{Q}|X_0$ *and, for every* $h \in N$, Q_h *is the probability measure on* $(X_h, \mathcal{B}(X_h))$ *associated with* $\hat{Q}_h$ *by* (2.4).

Proof. Let $f\in C^{o}(X_0)$ such that $f \le g$ on X_0. To get the assertion it is enough to show that

$$E_Q[f] \le \liminf_{h\to+\infty} E_{Q_h}[g] \,. \tag{2.8}$$

Infact, since X_0 is compact, there exists an increasing sequence (f_k) of functions in $C^{o}(X_0)$ such that $f_k(x) \to g(x)$ for every $x\in X_0$, as $k \to +\infty$; then, the inequality (2.7) follows from (2.8) and the monotone convergence theorem. Let us prove (2.8). Let $h\in C^{o}_{b}(X)$ be such that $f = h\,|_{X_0}$. Preliminarly, we show that

$$\limsup_{h\to+\infty} \sup_{x\in X_{\sigma(h)}} (h(x) - g(x)) = l \le 0 \tag{2.9}$$

Suppose by contraddition $l > 0$; then there exist a subsequence $(X_{\sigma(\tau(h))})$ of $(X_{\sigma(h)})$ and a constant $c > 0$ such that

$$\sup_{x\in X_{\sigma(\tau(h))}} (h(x) - g(x)) > c \,.$$

Hence,there exists a sequence (x_h) in X such that $x_h\in X_{\sigma(\tau(h))}$ and $h(x_h) > g(x_h) + c$. By passing to a subsequence, by property (2.3) the sequence (x_h) converges in X to $x\in X_0$. Moreover, by (ii) and by continuity of h we obtain $f(x) = h(x) > g(x) + c$, which is in contraddiction with the assumption on the function f. This proves (2.9). Finally, the proof of (2.8) is obtained by noting that, if (2.9) holds then there exists a sequence η_h of positive real numbers such that $\eta_h \to 0$ and $h(x) \le g(x) + \eta_h$ for every $x\in X_0$ and $h\in \mathbf{N}$. ∎

3. MOSCO CONVERGENCE AND RANDOM CAPACITIES.

In this section we define a variational notion of convergence, introduced by U. Mosco in [19], for sequences of convex functions and discuss some its useful implications for the study of Dirichlet problems in domains surrounded by thin layers.

Definition 3.1. Let (X,τ) be a topological space. Let (F_h) be a sequence of functions from X into $\mathbf{R}$. We say that a function $F : X\to\bar{\mathbf{R}}$ is the sequential Γ-limit of (F_h) and we write

$$F = \Gamma_{seq}(\tau) \lim_{h\to\infty} F_h$$

if

(a) for every $x\in X$ and for every sequence (x_h) converging to x in X we have

$$F(x) \le \liminf_{h\to\infty} F_h(x_h) \,;$$

(b) for every $x\in X$ there exists a sequence (x_h) converging to x in X such that

$$F(x) \ge \limsup_{h\to\infty} F_h(x_h) \,.$$

For a general definition of Γ-convergence and for its applications in calculus of variation we refer to [14],[15],[2]. Let X be a Banach space, we consider on X both the weak and the strong topology, denoted by w and s, respectively.

Definition 3.2. A sequence (F_h) of function from X into $\bar{\mathbf{R}}$ is said to be Mosco convergent to F if

$$F = \Gamma_{seq}(w) \lim_{h \to +\infty} F_h = \Gamma_{seq}(s) \lim_{h \to +\infty} F_h .$$

In other words the sequence F_h Mosco converges to F if
(a) for every $x \in X$ and for every sequence (x_h) converging weakly to x in X we have

$$F(x) \le \liminf_{h \to \infty} F_h(x_h) ;$$

(b) for every $x \in X$ there exists a sequence converging strongly to x in X such that

$$F(x) \ge \limsup_{h \to \infty} F_h(x_h) .$$

Definition 3.3. We denote by $\mathcal{E}$ the class of convex, lower semicontinuous, proper functions from $L^2(\mathbf{R}^n)$ into $\bar{\mathbf{R}}$.

We note that the class $\mathcal{F}$, defined in (1.8), is contained in $\mathcal{E}$. On $\mathcal{E}$ the Mosco convergence is attached to a metrizable topology (see [2], Section 3.5), which will be called *the topology of the Mosco convergence* and denoted by τ_M . For our purpose the relevant topological aspects of the Mosco convergence are contained in the following theorem (see[2], Theorem 3.36).

Theorem 3.4. *There exists a metric on $\mathcal{E}$ which induces the Mosco convergence topology and which is complete and separable.*

If we consider $\mathcal{F}$ endowed with the topology induced by τ_M, the following compactness results can be obtained by adapting the proofs of Theorem 4.1, Lemma 5.2, and Lemma 6.2 of [6].

Proposition 3.5. (a) *$\mathcal{F}_0$ is compact in $\mathcal{F}$*; (b) *let $(\mathcal{F}_{m(h)})$ be any subsequence of $(\mathcal{F}_h)$, then for every sequence (F_h) in $\mathcal{F}$ such that $F_h \in \mathcal{F}_{m(h)}$ there exist a subsequence $(F_{\sigma(h)})$ and a functional $F_0 \in \mathcal{F}_0$ such that $(F_{\sigma(h)})$ Mosco converges to F_0 .*

For any sequence m(h) of natural numbers such that $m(h) \to +\infty$ as $h \to +\infty$, let (F_h) be a sequence in $\mathcal{F}$ such that $F_h \in \mathcal{F}_{m(h)}$ and let $F \in \mathcal{F}_0$. Given $\lambda > 0$, let $R_h(\lambda)$ and $R(\lambda)$ be the resolvent operators introduced in Definition 1.6 and Definition 1.7.

The next result is an easy consequence of Theorem 5.5 and Lemma 5.2 in [6].

Proposition 3.6. *For every* $\lambda > 0$ *the following statements are equivalent:*
(a) (F_h) *Mosco converges to* F;
(b) $(R_h(\lambda))$ *converges to* $R(\lambda)$ *strongly in* $L^2(\mathbf{R}^n)$.
The same result holds also for $\lambda = 0$ *if* $\eta_h = c\varepsilon_h$ *(see Remark 1.8).*

The following propositions show the connection between Mosco - convergence of a sequence of functionals in $\mathcal{F}$ and the behaviour of the corresponding functions b_δ introduced in Definition 1.9.

Proposition 3.7. *Let* (F_h) *be a sequence in* $\mathcal{F}$ *and let* $F \in \mathcal{F}$. *Suppose that* (F_h) *Mosco converges to* F *and one of the following assumptions holds* :
(i) $F_h, F \in \mathcal{F}_0$;
(ii) $F_h, F \in \mathcal{F}_m$, *where* m *is a fixed natural number* ;
(iii) $F_h \in \mathcal{F}_{m(h)}$ *for every* $h \in \mathbf{N}$ *and* $F \in \mathcal{F}_0$, *where* $(m(h))$ *is any sequence of natural numbers such that* $m(h) \to +\infty$ *as* $h \to +\infty$. *Then the inequalities*

$$b_\delta(F,U) \leq \liminf_{h\to+\infty} b_\delta(F_h,U) \tag{3.1}$$

$$b_\delta(F,U) \geq \limsup_{h\to+\infty} b_\delta(F_h,U') \tag{3.2}$$

are satisfied for every $\delta > 0$ *and for every pair* $U, U' \in \mathcal{U}$ *such that* $U' \subset\subset U$.

Proof. The case (i) and (iii) require minor changes in the proof of Lemmas 6.3,6.4,6.5,6.6,5.2 in [6]. In the case (ii) we can adapt Lemmas 5.5,5.6 and Proposition 5.7 in [10] ∎

Proposition 3.8. *If* (F_h) *is a sequence in* $\mathcal{F}_0$, $F \in \mathcal{F}_0$, *and* (F_h) *Mosco converges to* F *in* $\mathcal{F}_0$ *then the inequality*

$$b_\delta(F,K) \geq \limsup_{h\to+\infty} b_\delta(F_h,K)$$

holds for every $K \in \mathcal{K}$ *and for every* $\delta > 0$.

Proof. It is enough to use (3.2) and the property (h) in Proposition 1.11. ∎

Let us indicate by τ_0 the topology on $\mathcal{F}_0$ induced by τ_M, by $\mathcal{B}(\tau_M)$ the Borel σ-field of $\mathcal{E}$ equipped with τ_M, and by $\mathcal{B}(\tau_0)$ the Borel σ-field of $\mathcal{F}_0$ endowed with τ_0. As a consequence of the Propositions 3.7 and 3.8 we have that for every $\delta > 0$ the functions $b_\delta(\cdot,U)$, $U \in \mathcal{U}$, and $b_\delta(\cdot,K)$, $K \in \mathcal{K}$ from $\mathcal{F}_0$ into $\mathbf{R}$, are $\mathcal{B}(\tau_0)$ measurable. We have also to say something about measurability of the function $b_\delta(\bullet,B)$, $B\in\mathcal{B}$, from $\mathcal{F}_0$ into $\mathbf{R}$. Let us denote by $\widehat{\mathcal{B}}(\tau_0)$ the σ-field of all subsets of $\mathcal{F}_0$ which are Q measurable for every probability measure Q on $(\mathcal{F}_0,\widehat{\mathcal{B}}(\tau_0))$. The following result holds.

Proposition 3.9. *For every* $B\in\mathcal{B}$ *and for every* $\delta>0$ *the function* $b_\delta(\bullet,B)$ *from* $\mathcal{F}_0$ *into* $\mathbf{R}$ *is* $\mathcal{B}(\tau_0)$ *- measurable .*

Proof. The assertion can be obtained by suitable minor changes in the proof of Proposition 2.4 in [3]. ∎

For $h\in\mathbf{N}$ fixed, let $\mathrm{cap}^{(h)}$ be the set function of Definition 1.12. We recall the following result (see [10], Theorem 6.3, Theorem 5.9, and [4], Lemma 2.2).

Proposition 3.10. *Let* (F_j) *be a sequence in* $\mathcal{F}_h$ *and let* $F\in\mathcal{F}_h$. *Then* (F_j) *Mosco converges to* F *in* $\mathcal{F}_h$ *if and only if the inequalities*

(a) $$\mathrm{cap}^{(h)}(U\cap\partial A_F)\ \le\ \liminf_{j\to+\infty}\mathrm{cap}^{(h)}(U\cap\partial A_{F_j}),$$

(b) $$\mathrm{cap}^{(h)}(K\cap\partial A_F)\ \ge\ \limsup_{j\to+\infty}\mathrm{cap}^{(h)}(K\cap\partial A_{F_j})$$

hold for every $U\in\mathcal{U}$ *and for every* $K\in\mathcal{K}$, *where* A_F, $A_{F_j}\in\mathcal{A}_h$ *are , respectively , the open sets associated with* F *and* F_j *(see Remark 1.5) .*

For each $h\in\mathbf{N}$ let us denote by τ_h the topology induced on the class $\mathcal{F}_h$ by τ_M.

Remark 3.11. From Proposition 3.10 we deduce that a sub-base for the topology τ_h is given by the sets of the form $\{F\in\mathcal{F}_h : \mathrm{cap}^{(h)}(U\cap\partial A_F)>t\}$ and $\{F\in\mathcal{F}_h : \mathrm{cap}^{(h)}(K\cap\partial A_F)<s\}$, with $t,s\in\mathbf{R}^+$, $U\in\mathcal{U}$, $K\in\mathcal{K}$, where $A_F\in\mathcal{A}_h$ is the set associated with $F\in\mathcal{F}_h$.

We indicate by $\mathcal{B}(\tau_h)$ the Borel σ-field of $\mathcal{F}_h$ endowed with the topology τ_h.

Proposition 3.12. $\mathcal{B}(\tau_h)$ *is the smallest* σ*-field in* $\mathcal{F}_h$ *for which the functions* $F\to\mathrm{cap}^{(h)}(U\cap\partial A_F)$ *from* $\mathcal{F}_h$ *into* $\mathbf{R}$ *are measurable for every* $U\in\mathcal{U}$ *(respectively, the functions* $F\to\mathrm{cap}^{(h)}(K\cap\partial A_F)$ *are measurable for every* $K\in\mathcal{K}$*) .*

Proof. Denote by Σ_h' the smallest σ-field in $\mathcal{F}_h$ for which all functions $F\to\mathrm{cap}^{(h)}(U\cap\partial A_F)$, $U\in\mathcal{U}$, are measurable and by Σ_h'' the smallest σ-field in $\mathcal{F}_h$ for which all functions $F\to\mathrm{cap}^{(h)}(K\cap\partial A_F)$, $K\in\mathcal{K}$, are measurable. Let us show that $\Sigma_h'=\Sigma_h''$. It is enough to prove that

(a) any function $F \to \mathrm{cap}^{(h)}(K \cap \partial A_F)$, $K \in \mathcal{K}$ is Σ'_h measurable;
(b) any function $F \to \mathrm{cap}^{(h)}(U \cap \partial A_F)$, $U \in \mathcal{U}$ is Σ''_h measurable.
Let us prove (a). For every $K \in \mathcal{K}$ consider the decreasing sequence of open sets

$$U_h = \{x \in \mathbf{R}^n : d(x,K) < 1\backslash h\}.$$

We remark that $U_h \downarrow K$. From the well-known properties of $\mathrm{cap}^{(h)}$ we have

$$\mathrm{cap}^{(h)}(K \cap \partial A_F) = \inf_{h \in N} \mathrm{cap}^{(h)}(U_h \cap \partial A_F)$$

for every $F \in \mathcal{F}_h$, which proves (a). Assertion (b) can be proven in the same way, by choosing, for every $U \in \mathcal{U}$, an increasing sequence (K_h) in $\mathcal{K}$ such that $K_h \uparrow U$. The proof of the proposition is complete if we show that $\mathcal{B}(\tau_h) = \Sigma'_h$. The inclusion $\Sigma'_h \subseteq \mathcal{B}(\tau_h)$ is trivial because,by Proposition 3.10, $\mathrm{cap}^{(h)}(K \cap \partial A_F)$, $K \in \mathcal{K}$ and $\mathrm{cap}^{(h)}(U \cap \partial A_F)$, $U \in \mathcal{U}$, are respectively upper and lower semicontinuous on $\mathcal{F}_h$. On the other hand, noting that the sub-base for the topology τ_h, given in Remark 3.11, is contained in Σ'_h and that $\mathcal{F}_h$ admits a countable base for the topology τ_h, we obtain the inclusion $\mathcal{B}(\tau_h) \subseteq \Sigma'_h$. ∎

The next corollary is a direct consequence of the previous proposition.

Corollary 3.13. *Let* (Ω,Σ) *be a measure space. Let* F *be a function from* Ω *into* $\mathcal{F}_h$. *The following statements are equivalent:*
(a) F *is* Σ-$\mathcal{B}(\tau_h)$ *measurable;*
(b) $\mathrm{cap}^{(h)}(U \cap \partial A_{F(\bullet)})$ *is* Σ*-measurable for each* $U \in \mathcal{U}$;
(c) $\mathrm{cap}^{(h)}(K \cap \partial A_{F(\bullet)})$ *is* Σ*-measurable for each* $K \in \mathcal{K}$.

We conclude this section with some results on the functions $b_\delta(\bullet,B)$, $B \in \mathcal{B}$, $\delta > 0$, considered as random variables on the space $\mathcal{E}$. We shall deal with weak convergence of measures on the space $\mathcal{E}$. Similar problems of weak convergence of measures on spaces endowed with topology related to Γ-convergence have been studied in [11],[12], and[3].

Lemma 3.14. *Let* Q *be a probability measure on* $(\mathcal{F}_0, \mathcal{B}(\tau_0))$. *Then the following relations*

$$E_Q[b_\delta(\bullet,B)] = \sup\{ E_Q[b_\delta(\bullet,K)] : K \in \mathcal{K}, K \subseteq B \}, \tag{3.3}$$

$$E_Q[b_\delta(\bullet,B_1)b_\delta(\bullet,B_2)] = \sup\{E_Q[b_\delta(\bullet,K_1)b_\delta(\bullet,K_2)] : K_1,K_2 \in \mathcal{K}, K_1 \subseteq B_1, K_2 \subseteq B_2\} \tag{3.4}$$

hold for every $\delta > 0$ *and for every* $B,B_1,B_2 \in \mathcal{B}$.
Proof. We only prove (3.4) since (3.3) can be proven with similar arguments. Fix $\delta > 0$ and $B_2 \in \mathcal{B}$. For every $E \in (\mathcal{U} \cup \mathcal{K})$ we define

$$\beta(E) = E_Q[b_\delta(\bullet,E)b_\delta(\bullet,B_2)].$$

By properties (e),(h),and(j) of Proposition 1.11 we have that

(3.5) $$\beta(K_1 \cup K_2) + \beta(K_1 \cap K_2) \leq \beta(K_1) + \beta(K_2)$$

for every $K_1, K_2 \in \mathcal{K}$

(3.6) $$\beta(K) = \inf \{ \beta(U) : U \in \mathcal{U}, U \supseteq K \}$$

for every $K \in \mathcal{K}$ and

(3.7) $$\beta(U) = \sup \{ \beta(K) : K \in \mathcal{K}, K \subseteq U \}$$

for every $U \in \mathcal{U}$. Moreover, we can extend the definition of β by

(3.8) $$\beta(B) = \inf \{ \beta(U) : U \in \mathcal{U}, U \supseteq B \}$$

for every $B \in \mathcal{B}$. We deduce from (3.5),(3.6),(3.7),and (3.8) that β is a Choquet capacity (see[16], Theorem 1.5). Applying the capacitability theorem (see [9]) we get

(3.9) $$\begin{aligned} \beta(B_1) &= \sup \{ \beta(K_1) : K_1 \in \mathcal{K}, K_1 \subseteq B_1 \} = \\ &= \sup \{ E_Q[b_\delta(\cdot,K_1) b_\delta(\cdot,B_2)] : K_1 \in \mathcal{K}, K_1 \subseteq B_1 \} \leq \\ &\leq E_Q[b_\delta(\cdot,B_1) b_\delta(\cdot,B_2)] \leq \inf \{ E_Q[b_\delta(\cdot,U_1) b_\delta(\cdot,U_2)] : U_1 \in \mathcal{U}, U_1 \supseteq B_1 \} = \\ &= \inf \{ \beta(U_1) : U_1 \in \mathcal{U}, U_1 \supseteq B_1 \} = \beta(B_1). \end{aligned}$$

for every $B_1 \in \mathcal{B}$ and $B_2 \in \mathcal{B}$ fixed. From (3.9) and from the formula we can obtain exchanging in (3.9) the roles of B_1 and B_2, we get (3.4). ∎

For every $h \in \mathbf{N}$ let Q_h be a probability measure on $(\mathcal{F}_h, \mathcal{B}(\tau_h))$. From now on we consider sequences $(\hat{Q}_h)$ of measures in $\mathcal{P}(\mathcal{E})$ with $\hat{Q}_h$ defined by

(3.10) $$\hat{Q}_h(B) = Q_h(B \cap \mathcal{F}_h)$$

for every $B \in \mathcal{B}(\tau_M)$.

Lemma 3.15. *Let* $(\hat{Q}_h)$ *be a sequence in* $\mathcal{P}(\mathcal{E})$ *of the form (3.10), and let* $\hat{Q}$ *be a measure in* $\mathcal{P}(\mathcal{E})$ *such that* $\hat{Q}(\mathcal{F}_0)=1$. *Suppose that* $(\hat{Q}_h)$ *converges weakly in* $\mathcal{P}(\mathcal{E})$ *to* $\hat{Q}$. *Then, for every* $\delta > 0$ *and* $U, U' \in \mathcal{U}$ *with* $U' \subset\subset U$, *we have*

(3.11) $$E_Q[b_\delta(\cdot,U)] \leq \liminf_{h \to +\infty} E_{Q_h}[b_\delta(\cdot,U)] ,$$

(3.12) $$E_Q[b_\delta(\cdot,U)] \geq \limsup_{h \to +\infty} E_{Q_h}[b_\delta(\cdot,U')] ,$$

where $Q = \hat{Q}|_{X_0}$ *and* Q_h *is the probability measure on* $(F_h, \mathcal{B}(\tau_h))$ *associated with* $\hat{Q}_h$ *by* (3.10).

Proof. By Proposition 3.7 and by applying Lemma 2.5 with $g(F) = b_\delta(F,U)$ for $\delta > 0$ and $U \in \mathcal{U}$ fixed, we get the inequality (3.11). Let us prove (3.12). For every $F \in \mathcal{F}$ and for $\delta > 0$ fixed we define

$$b^*(F,U) = \inf \{ b_\delta(F,U') : U' \in \mathcal{U} , U \subset\subset U' \}$$

for every $U \in \mathcal{U}$. Preliminarly, we show that, for every $U \in \mathcal{U}$,

(a) $b^*(\cdot,U) |_{\mathcal{F}_0}$ is upper semicontinuous;

(b) let $(m(h))$ be any sequence of natural numbers such that $m(h) \to +\infty$ as $h \to +\infty$, then

$$b^*(F,U) \geq \limsup_{h\to+\infty} b^*(F_h,U)$$

for every sequence F_h, with $F_h \in \mathcal{F}_{m(h)}$, which Mosco converges to $F \in \mathcal{F}_0$;

(c) $b^*(\cdot,U) |_{\mathcal{F}_m}$ is upper semicontinuous, where m is a fixed natural number.

We prove (a). Properties (b) and (c) can be obtained by repeating the proof of (a) with suitable changes. Let (F_h) be a sequence in $\mathcal{F}_0$ Mosco converging to $F \in \mathcal{F}_0$ and let $U \in \mathcal{U}$. For every $t > b^*(F,U)$ there exists $U' \in \mathcal{U}$, with $U \subset\subset U'$ such that $t > b_\delta(F,U')$. Let $U'' \in \mathcal{U}$ be such that $U \subset\subset U'' \subset\subset U'$.Then by (3.2) it follows that

$$t > b_\delta(F,U') \geq \limsup_{h\to+\infty} b_\delta(F,U'') \geq \limsup_{h\to+\infty} b^*(F,U)$$

which proves (a). Now, by applying Lemma 2.5 to the function $g(F) = -b^*(F,U)$ with $U \in \mathcal{U}$ fixed we have

$$E_Q[b^*(\cdot,U)] \geq \limsup_{h\to+\infty} E_{Q_h}[b^*(\cdot,U)].$$

Let $U' \in \mathcal{U}$ such that $U' \subset\subset U$. Then

$$E_Q[b_\delta(\cdot,U)] \geq E_Q[b^*(\cdot,U')] \geq$$

$$\geq \limsup_{h\to+\infty} E_{Q_h}[b^*(\cdot,U')] \geq \limsup_{h\to+\infty} E_{Q_h}[b_\delta(\cdot,U')]$$

which proves (3.12). ∎

Lemma 3.16. *Let* $(\hat{Q}_h)$ *be a sequence in* $\mathcal{P}(\mathcal{E})$ *and let* $\hat{Q} \in \mathcal{P}(\mathcal{E})$ *as in the Lemma* 3.15. *Then for every* $\delta > 0$

(3.13) $$E_Q[b_\delta(\cdot,U_1)b_\delta(\cdot,U_2)] \leq \liminf_{h\to+\infty} E_{Q_h}[b_\delta(\cdot,U_1)b_\delta(\cdot,U_2)]$$

(3.14) $$E_Q[b_\delta(\cdot,U_1)b_\delta(\cdot,U_2)] \geq \limsup_{h\to+\infty} E_{Q_h}[b_\delta(\cdot,U_1')b_\delta(\cdot,U_2')]$$

for every $U_1,U_2,U_1',U_2' \in \mathcal{U}$ *with* $U_1' \subset\subset U_1$ *and* $U_2' \subset\subset U_2$, *where* $Q = \hat{Q}|_{X_0}$ *and* Q_h *is the probability measure on* $(F_h, \mathcal{B}(\tau_h))$ *associated with* $\hat{Q}_h$ *by* (3.10).

Proof. Let us fix $\delta > 0$; we set $\mathcal{N}(F,U_1,U_2) = b_\delta(F,U_1)b_\delta(F,U_2)$.

Let (F_h) be a sequence in $\mathcal{F}$ and let $F \in \mathcal{F}$. By Proposition 3.7, if (F_h) Mosco converges to F and one of the assumptions considered there is satisfied, it follows that

(3.15) $$\mathcal{N}(F,U_1,U_2) \leq \liminf_{h\to+\infty} \mathcal{N}(F_h,U_1,U_2)$$

(3.16) $$\mathcal{N}(F,U_1,U_2) \geq \limsup_{h\to+\infty} \mathcal{N}(F_h,U_1',U_2')$$

for every $U_1,U_2,U_1',U_2'\in \mathcal{U}$ with $U_1'\subset\subset U_1$ and $U_2'\subset\subset U_2$. From (3.15), by applying Lemma 2.5 with $g(F)=\mathcal{N}(F,U_1,U_2)$ for $U_1,U_2\in \mathcal{U}$ fixed, we obtain (3.13). Let us prove (3.14). For every $F\in \mathcal{F}$ and for every $U_1,U_2\in \mathcal{U}$ we define

$$\mathcal{N}^*(F,U_1,U_2) = \inf \{ \mathcal{N}(F,U_1',U_2') \,,\, U_1',U_2'\in \mathcal{U} \,,\, U_1\subset\subset U_1' \,,\, U_2\subset\subset U_2' \}$$

Preliminarly we show that for every $U_1,U_2\in \mathcal{U}$

(a) $\mathcal{N}^*(F,U_1,U_2)|_{\mathcal{F}_0}$ is upper semicontinuous;

(b) let m(h) be any sequence of natural numbers such that $m(h)\to+\infty$ as $h\to+\infty$, then

$$\mathcal{N}^*(F,U_1,U_2) \geq \limsup_{h\to+\infty} \mathcal{N}^*(F_h,U_1,U_2)$$

for every sequence (F_h) in $\mathcal{F}$, with $F_h\in \mathcal{F}_{m(h)}$, which Mosco converges to $F\in \mathcal{F}_0$;

(c) $\mathcal{N}^*(\cdot,U_1,U_2)|_{\mathcal{F}_m}$ is upper semicontinuous, where m is a fixed natural number.

We prove (b). Properties (a) and (c) can be obtained by adapting with minor changes the proof of (b). Let (F_h) be a sequence in $\mathcal{F}$, with $F_h\in F_{m(h)}$, which Mosco converges to $F\in \mathcal{F}_0$. For every $t > \mathcal{N}^*(F,U_1,U_2)$ there exist $U_1',U_2'\in \mathcal{U}$ with $U_1\subset\subset U_1'$ and $U_2\subset\subset U_2'$ such that $t > \mathcal{N}(F,U_1',U_2')$. Let $U_1'',U_2''\in \mathcal{U}$ be such that $U_1\subset\subset U_1''\subset\subset U_1'$ and $U_2\subset\subset U_2''\subset\subset U_2'$. Then from (3.16) it follows that

$$t > \mathcal{N}(F,U_1',U_2') \geq$$
$$\geq \limsup_{h\to+\infty} \mathcal{N}(F_h,U_1'',U_2'') \geq \limsup_{h\to+\infty} \mathcal{N}^*(F_h,U_1,U_2)$$

which proves (b). Properties (a), (b), and (c) allow to apply Lemma 2.5 to the function $g(F) = -\mathcal{N}^*(F,U_1,U_2)$ with $U_1,U_2\in \mathcal{U}$ fixed. Thus, we obtain

$$E_Q[\mathcal{N}^*(\cdot,U_1,U_2)] \geq \limsup_{h\to+\infty} E_{Q_h}[\mathcal{N}^*(\cdot,U_1,U_2)]$$

for every $U_1,U_2\in \mathcal{U}$. Finally, by taking $U_1',U_2',U_1,U_2\in \mathcal{U}$ such that $U_1'\subset\subset U_1$ and $U_2'\subset\subset U_2$, we have

$$E_Q[\mathcal{N}(\cdot,U_1,U_2)] \geq E_Q[\mathcal{N}^*(\cdot,U_1',U_2')] \geq$$
$$\geq \limsup_{h\to+\infty} E_{Q_h}[\mathcal{N}^*(\cdot,U_1',U_2')] \geq \limsup_{h\to+\infty} E_{Q_h}[\mathcal{N}(\cdot,U_1',U_2')]$$

which proves (3.14) and the proof is accomplished. ∎

Lemma 3.17. *Let* $(\hat{Q}_h)$ *a sequence in* $\mathcal{P}(\mathcal{E})$ *and* $\hat{Q}\in \mathcal{P}(\mathcal{E})$ *as in the Lemma* 3.15, *and let* $\delta > 0$. *If we assume that*

$$\lim_{h\to+\infty} \mathrm{Cov}_{Q_h} [b_\delta(\cdot,U_1)b_\delta(\cdot,U_2)] = 0$$

for each pair $U_1,U_2\in \mathcal{U}$ *such that* $\bar{U}_1\cap\bar{U}_2=\emptyset$, *we have*

$$\mathrm{Cov}_Q [b_\delta(\cdot,U_1)b_\delta(\cdot,U_2)] = 0$$

for any $U_1, U_2 \in \mathcal{U}$, *with* $\bar{U}_1 \cap \bar{U}_2 = \emptyset$.

Proof. Let $U_1, U_2, U_1', U_2' \in \mathcal{U}$ such that $U_1' \subset\subset U_1$, $U_2' \subset\subset U_2$ and $U_1 \cap U_2 = \emptyset$. By (3.13) and (3.12) it follows that

$$(3.17) \qquad E_Q[b_\delta(\bullet, U_1')b_\delta(\bullet, U_2')] \le \liminf_{h \to +\infty} E_{Q_h}[b_\delta(\bullet, U_1')b_\delta(\bullet, U_2')]$$

$$(3.18) \qquad E_Q[b_\delta(\bullet, U_1)]E_Q[b_\delta(\bullet, U_2)] \ge \limsup_{h \to +\infty} E_{Q_h}[b_\delta(\bullet, U_1')]E_{Q_h}[b_\delta(\bullet, U_2')]$$

By subtracting (3.18) from (3.17) we obtain

$$(3.19) \qquad E_Q[b_\delta(\bullet, U_1')b_\delta(\bullet, U_2')] - E_Q[b_\delta(\bullet, U_1)]E_Q[b_\delta(\bullet, U_2)] \le$$
$$\le \liminf_{h \to +\infty} \mathrm{Cov}_{Q_h}[b_\delta(\bullet, U_1'), b_\delta(\bullet, U_2')] = 0 .$$

By (3.14) and (3.11) we deduce that

$$(3.20) \qquad E_Q[b_\delta(\bullet, U_1)b_\delta(\bullet, U_2)] \ge \limsup_{h \to +\infty} E_{Q_h}[b_\delta(\bullet, U_1')b_\delta(\bullet, U_2')]$$

$$(3.21) \qquad E_Q[b_\delta(\bullet, U_1')]E_Q[b_\delta(\bullet, U_2')] \le \liminf_{h \to +\infty} E_{Q_h}[b_\delta(\bullet, U_1')]E_{Q_h}[b_\delta(\bullet, U_2')]$$

By subtracting (3.21) from (3.20) we have

$$(3.22) \qquad E_Q[b_\delta(\bullet, U_1)b_\delta(\bullet, U_2)] - E_Q[b_\delta(\bullet, U_1')]E_Q[b_\delta(\bullet, U_2')] \ge$$
$$\ge \limsup_{h \to +\infty} \mathrm{Cov}_{Q_h}[b_\delta(\bullet, U_1'), b_\delta(\bullet, U_2')] = 0 .$$

By (3.19), (3.22), and Lemma 3.14 we get the assertion. ∎

4. THE MAIN RESULTS.

This section is devoted to state and to prove the main theorems of this paper. They give full answer to the following questions.

(a) For every $h \in \mathbf{N}$, let Q_h be a probability measure on $(\mathcal{F}_h, \mathcal{B}(\tau_h))$.
Under which conditions a sequence of measures $(\widehat{Q}_h)$ in $\mathcal{P}(\mathcal{E})$ of the form (3.10) has a subsequence $(\widehat{Q}_{\sigma(h)})$ which converges in $\mathcal{P}(\mathcal{E})$ to a Dirac measure $\widehat{Q} = \delta_{F_0}$ with $F_0 \in \mathcal{F}_0$?

(b) How can this limit be characterize ?

We will show that both the answers depend on the asymptotic behaviour, as $h \to +\infty$, of the functions $b_\delta(\bullet, U)$, considered as random variables on the probability spaces $(\mathcal{F}_h, \mathcal{B}(\tau_h), Q_h)$.

Before to state our main results we put some definitions. For every $U \in \mathcal{U}$ we define

$$\alpha'_\delta(U) = \liminf_{h \to +\infty} E_{Q_h}[b_\delta(\cdot, U)]$$

$$\alpha''_\delta(U) = \limsup_{h \to +\infty} E_{Q_h}[b_\delta(\cdot, U)]$$

where E_{Q_h} denotes the expectation in the probability space $(\mathcal{F}_h, \mathcal{B}(\tau_h), Q_h)$. Next, we consider the inner regularizations β'_δ and β''_δ of the set functions α'_δ and α''_δ, defined for every $U \in \mathcal{U}$ by

(4.1) $$\beta'_\delta(U) = \sup\{\, \alpha'_\delta(V) \; : V \in \mathcal{U}, V \subset\subset U \,\}$$

(4.2) $$\beta''_\delta(U) = \sup\{\, \alpha''_\delta(V) \; : V \in \mathcal{U}, V \subset\subset U \,\}$$

We extend the definitions of β'_δ and β''_δ to the Borel sets $B \in \mathcal{B}$ by

$$\beta'_\delta(B) = \inf\{\, \beta'_\delta(U) : U \in \mathcal{U}, U \supseteq B \,\}$$

$$\beta''_\delta(B) = \inf\{\, \beta''_\delta(U) : U \in \mathcal{U}, U \supseteq B \,\}.$$

Finally, we define

(4.3) $$\nu'(B) = \sup\{\, \beta'_\delta(B) : B \in \mathcal{B}, \delta > 0 \,\}$$

(4.4) $$\nu''(B) = \sup\{\, \beta''_\delta(B) : B \in \mathcal{B}, \delta > 0 \,\}$$

We are now able to state our results.

Theorem 4.1.(Compacteness Theorem). *Let* (Q_h) *be a sequence of probability measure on* $(\mathcal{F}_h, \mathcal{B}(\tau_h))$; *for every* $h \in \mathbf{N}$ *let* $\hat{Q}_h$ *the measure in* $\mathcal{P}(\mathcal{E})$ *associated with* Q_h *by* (3.10). *Assume that there exists a Radon measure* β *with* $\operatorname{spt} \beta \subseteq \partial D$ *such that*

(4.5) $$\limsup_{h \to +\infty} E_{Q_h}[b_\delta(\cdot, U)] \le \beta(\bar{U})$$

for every $U \in \mathcal{U}$ *and* $\delta > 0$.

Moreover, suppose that for every $U_1, U_2 \in \mathcal{U}$, *with* $\bar{U}_1 \cap \bar{U}_2 = \emptyset$,

(4.6) $$\lim_{h \to +\infty} \operatorname{Cov}_{Q_h}[b_\delta(\cdot, U_1), b_\delta(\cdot, U_2)] = 0$$

Then, there exists a subsequence $(\hat{Q}_{\sigma(h)})$ *of* $(\hat{Q}_h)$ *and a functional* $F_0 \in \mathcal{F}_0$ *such that* $(\hat{Q}_{\sigma(h)})$ *converges weakly on* $\mathcal{P}(\mathcal{E})$ *to the Dirac measure* $\delta_{F_0} \in \mathcal{P}(\mathcal{E})$ *defined by*

(4.7) $$\delta_{F_0}(A) = \begin{cases} 0 & \text{if } F_0 \notin A \\ 1 & \text{if } F_0 \in A \end{cases}$$

for every $A \in \mathcal{B}(\tau_M)$.

The limit functional F_0 is determined by the next theorem.

Theorem 4.2. *Let* (Q_h) *be a sequence of probability measures as in Theorem 4.1. Assume that there exists a Radon measure* γ *such that*

(4.8) $$\nu'(B) = \nu''(B) \le \gamma(B)$$

for every $B \in \mathcal{B}$ *and call* $\nu(B)$ *the common value of* $\nu'(B)$ *and* $\nu''(B)$.

Suppose that (4.6) holds. Then,

(t_1) ν *is a Borel measure of the class* $\mathcal{M}_0^*(\partial D)$;

(t_2) $(\widehat{Q}_h)$ *converges in* $\mathcal{P}(\mathcal{E})$ *to the Dirac measure* $\delta_{F_0} \in \mathcal{P}(\mathcal{E})$ *defined in (4.7) where* F_0 *is the functional in* $\mathcal{F}_0$ *associated to the measure* ν *according to Remark 1.5 .*

We now show some preliminary results which allow to get the proofs of Theorem 4.1 and Theorem 4.2. The next lemma gives a peculiar representation of a measure $\mu \in \mathcal{M}_0^*(\partial D)$.

Lemma 4.3. *Let* $\mu \in \mathcal{M}_0^*(\partial D)$ *and let* F *be the corresponding functional in* $\mathcal{F}_0$. *Then, for every* $B \in \mathcal{B}$ *we have*

$$\mu(B) = \lim_{\delta \to 0} \sum_{i \in I_\delta} b_\delta(F, B_i^\delta).$$

where, for every $\delta > 0$, $(B_i^\delta)_{i \in I_\delta}$ *is any finite Borel partition of B.*

Proof. Let $B \in \mathcal{B}$. For every $\delta > 0$ fixed, denote by $(B_i^\delta)_{i \in I_\delta}$ any finite partition of B. Then, by (g) in Proposition 1.11, we have

$$\mu(B) = \sum_{i \in I_\delta} \mu(B_i^\delta) \geq \sum_{i \in I_\delta} b_\delta(F, B_i^\delta)$$

Hence

$$\mu(B) \geq \limsup_{\delta \to 0} \sum_{i \in I_\delta} b_\delta(F, B_i^\delta) \tag{4.9}$$

On the other hand, by (k) and (e) of Proposition 1.11, for every real number $t < \mu(B)$, there exists $\delta_0 > 0$ such that

$$t < b_\delta(F,B) \leq \sum_{i \in I_\delta} b_\delta(F, B_i^\delta)$$

for every $\delta < \delta_0$. Thus we have

$$t < \liminf_{\delta \to 0} \sum_{i \in I_\delta} b_\delta(F, B_i^\delta)$$

Hence

$$\mu(B) \leq \liminf_{\delta \to 0} \sum_{i \in I_\delta} b_\delta(F, B_i^\delta) \tag{4.10}$$

The inequalities (4.9) and (4.10) give the assertion. ∎

The following proposition provides a sufficient condition in order that a probability measure Q on $(\mathcal{F}_0,\mathcal{B}(\tau_0))$ be equal to a Dirac measure δ_{F_0}.

Proposition 4.4. *For every* $\delta > 0$ *we define*

$$\alpha_\delta(U) = E_Q[b_\delta(\bullet,U)]$$

for every $U\in \mathcal{U}$, *and*

$$\alpha_\delta(B) = \inf\{\ \alpha_\delta(U) : U\in \mathcal{U},\ U \supseteq B\}$$

for every $B\in \mathcal{B}$.

Moreover, let us set

$$\nu(B) = \sup_{\delta > 0} \alpha_\delta(B)$$

for every $B\in \mathcal{B}$.

Let us assume that

(i) *there exists a Radon measure* β *such that* $\nu(B) \le \beta(B)$ *on* $\mathcal{B}$;

(ii) $$\mathrm{Cov}_Q\,[b_\delta(\bullet,U_1),b_\delta(\bullet,U_2)] = 0$$

for every $\delta > 0$ *and for every pair* U_1,U_2 *of sets in* $\mathcal{U}$ *such that* $\bar{U}_1\cap\bar{U}_2 = \emptyset$.

Then ,

(t_1) ν *is a Borel measure of the class* $\mathcal{M}_0^*(\partial D)$;

(t_2) $Q = \delta_{F_0}$, *where* F_0 *is the functional in* $\mathcal{F}_0$ *associated with* ν.

Proof. From (b) and (d) of Proposition 1.11 we deduce that the function ν is increasing and countably subadditive on $\mathcal{B}$. In order to prove that ν is a Borel measure we first note that, by (h) of Proposition 1.11 and by (3.3), we have

$$\alpha_\delta(B) = E_Q[b_\delta(\bullet,B)] \tag{4.11}$$

for every $\delta > 0$ and $B\in \mathcal{B}$.

Thus, from (i) of Proposition 1.11 we deduce that

$$\alpha_\delta(B_1) + \alpha_\delta(B_2) \le (1-\eta)^{-1}\ \alpha_\delta(B_1\cup B_2) + 4\Lambda_2\,\eta^{-1}\,\sigma^2\,|D\text{-}D_\delta| \tag{4.12}$$

for every η, $\delta > 0$ and for every pair B_1, $B_2\in \mathcal{B}$ such that dist $(B_1,B_2) = \sigma > 0$.

By taking first the supremum over all $\delta > 0$ and then the limit as η goes to zero in (4.12), we get

$$\nu(B_1) + \nu(B_2) \le \nu(B_1\cup B_2)$$

for every B_1, $B_2\in \mathcal{B}$ such that dist $(B_1,B_2) > 0$.

Applying the Caratheodory criterion (see[17], 2.3.2(9)) we obtain that ν is Borel measure. Finally, the hypothesis (i) and Proposition 1.11 ((f),(g)) infer that $\nu\in \mathcal{M}_0^*(\partial D)$ and this completes the proof of (t_1). Let us prove (t_2). Let us denote by $Z(\bullet,B)$ the random variable on the probability space $(\mathcal{F}_0,\mathcal{B}(\tau_0),Q)$ defined for every Borel set B of ∂D by

$$Z(F,B) = \mu(B)$$

where μ is the measure in $\mathcal{M}_0^*(\partial D)$ associated with $F \in \mathcal{F}_0$.
We note that, by Lemma 4.3, for every $F \in \mathcal{F}_0$ and for every Borel set B of ∂D,

$$Z(F,B) = \lim_{\delta \to 0} \sum_{i \in I_\delta} b_\delta(F, B_i^\delta)$$

where, for each $\delta > 0$, $(B_i^\delta)_{i \in I_\delta}$ is any finite partition of B. Our aim is to show that $Z(\bullet,B)$ is a constant random variable. In view of Lemma 3.1 in [3], we have only to prove that

$$\text{(4.13)} \qquad \lim_{\delta \to 0} \mathrm{Var}_Q[\sum_{i \in I_\delta} b_\delta(\bullet, B_i^\delta)\ ] = 0$$

By (h) of Proposition 1.11 and by (3.3) and (3.4), we can extend the relation (ii) to each pair of disjoint sets B_1, $B_2 \in \mathcal{B}$. Therefore, to get (4.13) it is enough to prove

$$\text{(4.14)} \qquad \lim_{\delta \to 0} \sum_{i \in I_\delta} \mathrm{Var}_Q[b_\delta(\bullet, B_i^\delta)] = 0$$

Let B be a Borel set of ∂D and let $(r_\delta)_{\delta>0}$ and $(R_\delta)_{\delta>0}$ be two sequences of positive numbers such that : (a) $r_\delta < R_\delta$ for every $\delta > 0$;

(b) $s_\delta = \mathrm{cap}\,(B_{r_\delta}(0), B_{R_\delta}(0)) \to 0$ as $\delta \to 0$;

(c) for every $x \in \partial D$, $B_{R_\delta}(x) \cap \partial D_\delta = \emptyset$.

For every $\delta > 0$, let us choose a finite partition $(B_i^\delta)_{i \in I_\delta}$ of B such that

$$\sup_{i \in I_\delta} (\mathrm{diam}\, B_i^\delta) < \frac{r_\delta}{2}$$

moreover, for every $i \in I_\delta$ let us fix x_i^δ such that $B_i^\delta \subseteq B_{r_\delta}(x_i^\delta) \subseteq B_{R_\delta}(x_i^\delta)$.
Since $B_{R_\delta}(x_i^\delta) \cap \partial D_\delta = \emptyset$, by (f) of Proposition 1.11 we have

$$b_\delta(F, B_i^\delta) \le \Lambda_2\, \mathrm{cap}\,(B_i^\delta; B_{R_\delta}(x_i^\delta)\ ).$$

Then, for every $\delta > 0$ we get

$$\text{(4.15)} \qquad \sum_{i \in I_\delta} \mathrm{Var}_Q[b_\delta(\bullet, B_i^\delta)] =$$

$$= \sum_{i \in I_\delta} \left\{ E_Q[b_\delta(\bullet, B_i^\delta)^2] - \left(E_Q[b_\delta(\bullet, B_i^\delta)]\right)^2 \right\} \le$$

$$\le \sum_{i \in I_\delta} E_Q[b_\delta(\bullet, B_i^\delta)^2] \le \Lambda_2 \sum_{i \in I_\delta} \mathrm{cap}\,(B_i^\delta, B_{R_\delta}(x_i^\delta))\ \ E_Q[b_\delta(\bullet, B_i^\delta)] \le$$

$$\le \Lambda_2 \sup_{i \in I_\delta} \left\{ \mathrm{cap}\,(B_{r_\delta}(x_i^\delta), B_{R_\delta}(x_i^\delta)) \right\} \sum_{i \in I_\delta} E_Q[b_\delta(\bullet, B_i^\delta)] \le$$

$$\le \Lambda_2\, \mathrm{cap}\,(B_{r_\delta}(0), B_{R_\delta}(0)) \sum_{i \in I_\delta} \alpha_\delta(B_i^\delta) \le \Lambda_2\, s_\delta \sum_{i \in I_\delta} \beta(B_i^\delta) = \Lambda_2\, s_\delta\, \beta(B)$$

By taking the limit as $\delta \to 0$ in (4.15) we get (4.14) and this proves that $Z(\bullet,B)$ is a constant random variable. Now, let us compute the expectation of $Z(\bullet,B)$. By taking in account that the function $\delta \to b_\delta(F,B)$ is decreasing and by applying Lemma 4.3 with $B_i^\delta = B$ for every $i \in I_\delta$ and for every $\delta > 0$, we obtain

$$E_Q[Z(\bullet,B)] = \sup_{\delta > 0} E_Q[b_\delta(\bullet,B)] = \nu(B)$$

where in the last equality we have used (4.11).

Therefore, for every Borel set B of ∂D there exists a subset $\mathcal{F}_B$ of $\mathcal{F}_0$ with $Q(\mathcal{F}_B) = 1$ such that $Z(F,B) = \nu(B)$ for every $F \in \mathcal{F}_B$.

Finally, by means standard density arguments (see for istance the proof of Lemma 3.3 in [3]) we can deduce that there exists a subset $\tilde{\mathcal{F}}$ of $\mathcal{E}$ such that $Q(\tilde{\mathcal{F}}) = 1$ and $Z(F,B) = \nu(B)$ for every $F \in \tilde{\mathcal{F}}$ and for every Borel set B of ∂D. This completes the proof of (t_2). ∎

Proof of Theorem 4.1. By Theorem 2.2 there exists a subsequence of $(\hat{Q}_h)$ converging weakly to a measure $\hat{Q}$ in $\mathcal{P}(\mathcal{E})$ such that $\hat{Q}(\mathcal{F}_0) = 1$. By (4.5) and by Lemma 3.15 we obtain

$$E_Q[b_\delta(\bullet,U)] \le \beta(\bar{U})$$

for every $\delta > 0$ and $U \in \mathcal{U}$, where $Q = \hat{Q}|_{\mathcal{F}_0}$.

It is easy to see that also the relation $E_Q[b_\delta(\bullet,U)] \le \beta(U)$ holds.

Hypothesis (4.6) and Lemma 3.17 yield

$$\mathrm{Cov}_Q[b_\delta(\bullet,U1)\, b_\delta(\bullet,U2)] = 0$$

for every $\delta > 0$ and for every pair $U_1, U_2 \in \mathcal{U}$ with $\bar{U}_1 \cap \bar{U}_2 = \emptyset$.

The thesis is obtained easily from Proposition 4.4. ∎

Proof of Theorem 4.2. By Theorem 4.1 and by (4.8) we can assume that (Q_h) converges weakly to a Dirac measure $\delta_{F_0} \in \mathcal{P}(\mathcal{E})$ for some $F_0 \in \mathcal{F}_0$. By Lemma 3.15 it follows that

(4.16) $$E_{\delta_{F_0}}[b_\delta(\bullet,U)] = b_\delta(F_0,U) = \beta_\delta'(U) = \beta_\delta''(U)$$

for every $U \in \mathcal{U}$. By extending (4.16) to an arbitrary Borel set in D we have

$$b_\delta(F_0,B) = \beta_\delta'(B) = \beta_\delta''(B)$$

for every $B \in \mathcal{B}$, which gives

$$\nu(B) = \sup_{\delta > 0} b_\delta(F,B)$$

Property (k) in Proposition 1.11 implies that ν is just the measure in $\mathcal{M}_0^*(\partial D)$ associated with the functional F_0. This concludes the proof of the theorem. ∎

5. DIRICHLET PROBLEMS IN DOMAINS SURROUNDED BY THIN LAYERS WITH RANDOM THICKNESS.

In this section we apply the main results proved in the previous section to Dirichlet problems in domains surrounded by thin layers with random thickness.

From now on (Ω,Σ,P) will denote a probability space, that is, Ω is a set, Σ is a σ-field of subsets of Ω, and P is a probability measure on Σ.

Definition 5.1. (a) For every $h\in N$ a *random functional of the class* $\mathcal{F}_h$ is any measurable function $F_h : \Omega\to\mathcal{F}_h$, where $\mathcal{F}_h$ is equipped with the Borel σ-field $\mathcal{B}(\tau_h)$ generated by the topology τ_h induced by τ_M (topology of Mosco convergence); (b) a *random functional of the class* $\mathcal{F}_0$ is any measurable function $F_0 : \Omega\to\mathcal{F}_0$ where $\mathcal{F}_0$ is endowed with the Borel σ-field $\mathcal{B}(\tau_0)$ generated by the topology τ_0 induced by τ_M.

Remark 5.2. We recall that necessary and sufficient conditions for the measurability of a function $F_h : \Omega\to\mathcal{F}_h$ are given in Corollary 3.13.

Let F_h be a random functional of the class $\mathcal{F}_h$ and let Q_h be the probability measure on $(\mathcal{F}_h,\mathcal{B}(\tau_h))$ defined by

$$Q_h(A) = P\{F_h^{-1}(A)\}$$

for any $A\in \mathcal{B}(\tau_h)$. Q_h is called the *distribution law of* F_h. In the same way, given a random functional F_0 of the class $\mathcal{F}_0$ we can define the distribution law Q of F_0.

For every $h\in N$ let Q_h be the distribution law of a random functional F_h of the class $\mathcal{F}_h$ and let $\widehat{Q}_h$ be the measure in $\mathcal{P}(\mathcal{E})$ associated to Q_h by (3.10). Moreover, let Q be the distribution law of a random functional F_0 of the class $\mathcal{F}_0$ and let $\widehat{Q}$ be the measure in $\mathcal{P}(\mathcal{E})$ defined by

$$\widehat{Q}(B) = Q(B\cap\mathcal{F}_0)$$

for every $B\in \mathcal{B}(\tau_M)$.

Definition 5.3. We say that (F_h) *converges in law* to F_0 if $(\widehat{Q}_h)$ converges weakly in $\mathcal{P}(\mathcal{E})$ to $\widehat{Q}$.

We denote by E and by Cov respectively the expectation and the covariance of a random variable on Ω, with respect the measure P. It is easy to see that, for $\delta>0$ and for every $h\in N$,

$$E_{Q_h}[b_\delta(\cdot,U)] = E[b_\delta(F_h(\cdot),U)] \tag{5.1}$$

for any $U\in \mathcal{U}$ and

$$\mathrm{Cov}_{Q_h}[b_\delta(\cdot,U_1),b_\delta(\cdot,U_2)] = \mathrm{Cov}_{Q_h}[b_\delta(F_h(\cdot),U_1)b_\delta(F_h(\cdot),U_2)] \tag{5.2}$$

for any $U_1,U_2\in \mathcal{U}$.

Remark 5.4. Equalities (5.1) and (5.2) allow to reformulate the hypotheses of the compactness theorem in terms of the expectations and covariances of the real random variables $b_\delta(F_h(\bullet),U)$, $\delta > 0$. By Definition 5.3 the thesis of Theorem 4.1 can be restated by saying that the sequence of random functionals (F_h) has a subsequence $F_{(\sigma(h))}$ which converges in law to a random functional F_0 on $\mathcal{F}_0$ such that $F_0(\omega)=F_0$ for P-almost every $\omega\in\Omega$ (i.e. to the constant random functional F_0 on $\mathcal{F}_0$).

Remark 5.5. Since $\mathcal{E}$ is a metric space (let d_M be the metric) the convergence in law of the sequence (F_h) toward the random constant functional F_0 is equivalent to the convergence in probability. By Remark 5.4 we can deduce that if the assumtions of Theorem 4.1 on the random variables $b_\delta(F_h(\bullet),U)$, $\delta > 0$, $U\in \mathcal{U}$, hold, then the sequence (F_h) has a subsequence $(F_{\sigma(h)})$ which converges in probability to the constant functional $F_0\in \mathcal{F}_0$, that is,for every $\varepsilon > 0$

$$\lim_{h\to+\infty} P\{\ \omega\in\Omega : d_M(F_{\sigma(h)},F) > \varepsilon\ \} = 0 .$$

For every $h\in \mathbf{N}$, let A_h be a function from the set Ω into the class of sets $\mathcal{A}_h$ (see Definition 1.2). For every $\omega\in\Omega$ let $F_h(\omega)$ be the functional in $\mathcal{F}_h$ associated with $A_h(\omega)$ (see Remark 1.5).

Definition 5.6. We say that the function $A_h : \Omega\to\mathcal{A}_h$ is a *Random set of the class* $\mathcal{A}_h$ if the function $F_h : \Omega\to\mathcal{F}_h$ is a random functional of the class $\mathcal{F}_h$.

Remark 5.7. Necessary and sufficient conditions in order that a map $A_h : \Omega\to\mathcal{A}_h$ be a random set, can be deduced by Corollary 3.13.

We are interested in the study of the following sequence of random Dirichlet problems associated with a sequence of random sets, that is, for every $\omega\in\Omega$

$$(5.3)\qquad \begin{cases} L^{(h)}u_h + \lambda u_h = g & \text{in } A_h(\omega) \\ u_h\in H^1_0(A_h(\omega)) \end{cases}$$

where $\lambda \geq 0$, $g\in L^2(\mathbf{R}^n)$.

For each $\omega\in\Omega$ and $\lambda \geq 0$, let $(R_h(\lambda)[\omega])$ be the sequence of resolvent operators associated with the sequence $(F_h(\omega))$ (see Definition 1.6). We are now able to state a new version of the compactness Theorem 4.1.

Theorem 5.8. *Let* (A_h) *be a sequence of random sets and let* (F_h) *be the corresponding sequence of random functionals. Assume that there exists a Radon measure* γ *with* $\operatorname{spt}\gamma \subseteq \partial D$ *such that*

(5.4) $$\limsup_{h\to+\infty} E[b_\delta(F_h(\cdot),U)] \le \gamma(\bar{U})$$

for every $\delta > 0$ *and for every* $U \in \mathcal{U}$. *Moreover, suppose that for every* $U_1, U_2 \in \mathcal{U}$ *with* $\bar{U}_1 \cap \bar{U}_2 = \emptyset$ *we have*

(5.5) $$\limsup_{h\to+\infty} \mathrm{Cov}\,[b_\delta(F_h(\cdot),U_1) b_\delta(F_h(\cdot),U_2)] = 0 .$$

Then there exist a subsequence $(R_{\sigma(h)}(\lambda))$ *of* $(R_h(\lambda))$ *and a functional* $F_0 \in \mathcal{F}_0$ *such that, for every* $\lambda > 0$, $(R_{\sigma(h)}(\lambda))$ *converges strongly in probability to the resolvent operator* $R_0(\lambda)$ *associated to* F_0 *(see Definition 1.7), that is*

$$\lim_{h\to+\infty} P\{\, \omega\in\Omega : \|R_{\sigma(h)}(\lambda)[g] - R_0(\lambda)[g]\|_{L^2(\mathbb{R}^n)} > \varepsilon \,\} = 0$$

for every $\varepsilon > 0$ *and for every* $g \in L^2(\mathbb{R}^n)$. *If* $\eta_h = c\varepsilon_h$ *then the same result holds for* $\lambda = 0$ *(see Remark 1.8).*

Proof. By Remark 5.5 we have that there exists a subsequence $(F_{\sigma(h)})$ of (F_h) which converges in probability to some $F \in \mathcal{F}_0$. So the assertion is obtained easily by Proposition 3.6. ∎

For every $h \in \mathbb{N}$ let F_h be a random functional of the class $\mathcal{F}_h$. Given the sequence (F_h), let us define for every $\delta > 0$

$$\alpha'_\delta(U) = \liminf_{h\to+\infty} E[b_\delta(F_h(\cdot),U)] ,$$

$$\alpha''_\delta(U) = \limsup_{h\to+\infty} E[b_\delta(F_h(\cdot),U)] .$$

We denote by β'_δ, β''_δ respectively the inner regularization of α'_δ, α''_δ as definited in (4.1), (4.2) and by ν', ν'' the set functions as defined in (4.3), (4.4). It is easy to see that by (5.1), (5.2), Definition 5.3, Remark 5.4 and Remark 5.5, Theorem 4.2 can be restated in the following way.

Theorem 5.9. *Given a sequence of random sets* (A_h), *let* (F_h) *be the corresponding sequence of random functionals. Assume that there exists a Radon measure* γ *such that*

$$\nu'(B) = \nu''(B) \le \gamma(B)$$

for every $B \in \mathcal{B}$ *and call* $\nu(B)$ *the common value of* $\nu'(B)$ *and* $\nu''(B)$. *Suppose that (5.5) holds, then, for every* $\lambda > 0$

$$\lim_{h\to+\infty} P\{\, \omega\in\Omega : \|R_h(\lambda)[f] - R_0(\lambda)[f]\|_{L^2(\mathbb{R}^n)} > \varepsilon \,\} = 0$$

for every $\varepsilon > 0$ *and for any* $f \in L^2(\mathbb{R}^n)$, *where* $R_0(\lambda)$ *is the resolvent associated to the functional* $F_0 \in \mathcal{F}_0$, *which corresponds to the measure* $\nu \in \mathcal{M}_0^*(\partial\Omega)$. *If* $\eta_h = c\varepsilon_h$ *then the same result holds for* $\lambda = 0$ *(see Remark 1.8)* .

6. AN EXAMPLE.

In what follows we assume that the domain D of $\mathbf{R}^n$ has a C^2 boundary and that, for every $h\in\mathbf{N}$, $\eta_h = c\varepsilon_h$ (see Remark 1.8) .

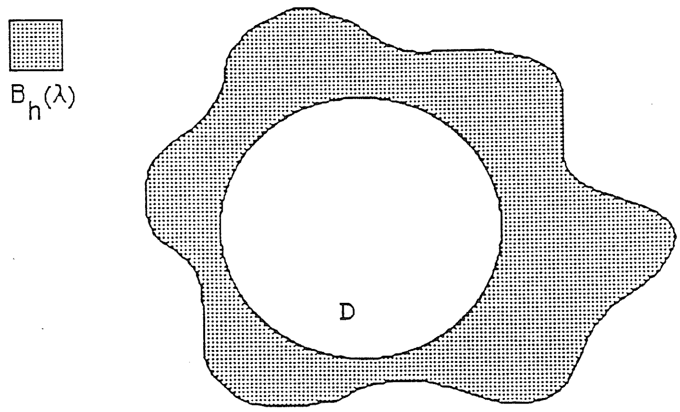
By $(Q_i^h)_{i\in I_h}$ we denote a finite open cover of ∂D such that

$$\max_{i\in I_h} \operatorname{diam} Q_i^h \to 0$$

as $h \to +\infty$. Let $(\phi_i^h)_{i\in I_h}$ be a partition of unity on ∂D, subordinate to the cover $(Q_i^h)_{i\in I_h}$ and let $(x_i^h)_{i\in I_h}$ be a family of indipendent random variables defined on the same probabilistic space (Ω,Σ,P) with values in the interval $[c,1]$, where c is a positive constant. We regard the family $(x_i^h)_{i\in I_h}$ as a vector random variable $\xi^{(h)}$ from Ω into $[c,1]^{I_h}$. For every $\lambda=(\lambda_i)_{i\in I_h}$ in $[c,1]^{I_h}$, we define the set (see fig.2)

$$B_h(\lambda) = \bigcup_{i\in I_h} \{\, x\in\mathbf{R}^n : x=\sigma+t\,n(\sigma)\,,\ \sigma\in\partial D\,,\ 0<t<\varepsilon_h\lambda_i\phi_h^i(\sigma) \,\}$$

where n is the outer unit normal to ∂D. Let us set $A_h(\lambda)=\bar{D}\cup B_h(\lambda)$.



We stress that the assumption on ∂D ensure that the mapping $(\sigma,t)\to\sigma+t\,n(\sigma)$ is invertible on $B_h(\lambda)$ if h is sufficiently large so that the boundary of the set $A_h(\lambda)$ is given by

$$\partial A_h(\lambda) = \bigcup_{i\in I_h} \{\, x\in\mathbf{R}^n : x=\sigma+\varepsilon_h\,\lambda_i\,\phi_i^h(\sigma)\,n(\sigma)\,,\ \sigma\in\partial D \,\}.$$

We note that for every $\lambda\in[c,1]^{I_h}$ the following inclusions hold

(6.1) $$D^{(h)} = \{\, x\in\mathbf{R}^n : d(x,D) < c\varepsilon_h \,\} \subseteq A_h(\lambda) \subseteq \{\, x\in\mathbf{R}^n : d(x,D) < \varepsilon_h \,\}$$

Furthermore, we associate with every $\lambda\in[c,1]^{I_h}$ the functional $F_h(\lambda):L^2(\mathbf{R}^n)\to\mathbf{R}$ defined by

$$F_h(\lambda)(u) = \begin{cases} \displaystyle\int_{\mathbf{R}^n} a^{(h)}(x,Du)\,dx + \int_{\mathbf{R}^n} \tilde{u}^{\,2}\, d\infty_{\partial A_h(\lambda)} & u\in H^1(\mathbf{R}^n) \\ +\infty & \text{otherwise} \end{cases}$$

where $a^{(h)}$ is the quadratic form defined in (1.2). Our aim is to show that the composit function $\omega\to F_h(\xi^{(h)}(\omega))$ from Ω into $\mathcal{F}_h$ is Σ–$B(\tau_h)$ measurable, i.e. is a random functional of the class $\mathcal{F}_h$. To get this we need the following lemma.

Lemma 6.1. *For every* $K\in\mathcal{K}$ *the function* $\lambda\to cap^{(h)}(\partial A_h(\lambda)\cap K)$ *from* $[c,1]^{I_h}$ *into* $\mathbf{R}$, *is upper semicontinuous in* $[c,1]^{I_h}$.

Proof. The lemma is similar to Lemma 4.1 in [3]. For the reader convenience we adapt the proof in our particular case. Let $(\lambda_i)_{i\in\mathbf{N}}$ be a sequence in $[c,1]^{I_h}$ converging to λ in $[c,1]^{I_h}$. For every $j\in\mathbf{N}$ we define the set

$$E_h^j(\lambda) = \{\, x\in\mathbf{R}^n : dist(x,\partial A_h(\lambda)) < 1/j \,\}.$$

By definition of $\partial A_h(\lambda)$ we have that for every $j\in\mathbf{N}$ there exists $i_0\in\mathbf{N}$ such that $E_h^j(\lambda)\supseteq\partial A_h(\lambda_i)$ for every $i\geq i_0$. Hence, for every $j\in\mathbf{N}$ and $K\in\mathcal{K}$ we obtain

$$cap^{(h)}(E_h^j(\lambda)\cap K) \geq \limsup_{i\to+\infty} cap^{(h)}(\partial A_h(\lambda_i)\cap K)\,.$$

Since

$$\bigcap_{j\in\mathbf{N}} (E_h^j(\lambda)\cap K) = \partial A_h(\lambda)\cap K$$

by the well-known properties of the capacity $cap^{(h)}$ we get

$$cap^{(h)}(\partial A_h(\lambda)\cap K) \geq \limsup_{i\to+\infty} cap^{(h)}(\partial A_h(\lambda_i)\cap K)\,,$$

which proves the lemma. ∎

Remark 6.2. Lemma 6.1 and Corollary 3.13 imply that the function $\omega\to F_h(\xi^{(h)}(\omega))$ is a random functional of the class $\mathcal{F}_h$ or equivalently, that the function $\omega\to A_h(\xi^{(h)}(\omega))$ is a random set.

Let us set $F_h(\omega)=F_h(\xi^{(h)}(\omega))$ for every $\omega\in\Omega$. In the following we want to show that the sequence (F_h) satisfies the assumptions of Theorem 5.8.

For every $x\in\mathbf{R}^n$, let us define the function

$$u_h(x) = \left(1 - \frac{1}{c\varepsilon_h}\, d(x,D)\right) \vee 0\,.$$

By (6.1) it is easy to see that for every $U \in \mathcal{U}$, $\delta > 0$, and $\lambda \in [c,1]^{I_h}$, the function u_h has the following properties

$$u_h = 0 \quad \text{q.e. on } U \cap \partial A_h(\lambda),$$
$$u_h = 1 \quad \text{q.e. on } \partial D_\delta.$$

Thus, for every $\delta > 0$, $\lambda \in [c,1]^{I_h}$, and $U \in \mathcal{U}$, we have

$$b_\delta(F_h(\lambda),U) \le \varepsilon_h \int_{(D^{(h)}/D)\cap U} a(x,Du)\, dx \le \tag{6.2}$$

$$\le \Lambda_2\, \varepsilon_h \int_{(D^{(h)}/D)\cap U} |Du_h|^2\, dx = \Lambda_2\, \varepsilon_h \int_{(D^{(h)}/D)\cap U} \frac{1}{\varepsilon_h^2 c^2}\, dx =$$

$$= \frac{\Lambda^2}{c^2} \frac{1}{\varepsilon_h} \, |(D^{(h)}/D)\cap U| \,.$$

By (6.2) it follows that for every $\delta > 0$ and $U \in \mathcal{U}$

$$E[b_\delta(F_h(\bullet),U] \le \frac{\Lambda_2}{c^2} \frac{1}{\varepsilon_h} \, |(D^{(h)}/D)\cap U| ,$$

hence, for every $U \in \mathcal{U}$, and $\delta > 0$, we have

$$\limsup_{h\to+\infty} E[b_\delta(F_h(\bullet),U)] \le \frac{\Lambda_2}{c^2} H^{n-1}(\partial D \cap U) \tag{6.3}$$

which proves the assumption (5.4) of Theorem 5.8.

Now, let I be any subset of I_h. We denote by Π_I the projection of $[c,1]^{I_h}$ on $[c,1]^I$. For every $U \in \mathcal{U}$ we set $I(U)=\{i \in I_h : Q_i^h \cap U \neq \emptyset\}$. By Definition 1.9 it is easy to see that, for any $\delta > 0$ fixed, the function $\lambda \to b_\delta(F_h(\lambda),U)$ from $[c,1]^{I_h}$ into **R** is actually a function of the variable $\lambda' = \Pi_{I(U)}(\lambda)$. So if we consider two sets $U_1, U_2 \in \mathcal{U}$ such that $\bar{U}_1 \cap \bar{U}_2 = \emptyset$, we find two disjoint sets I_1, I_2 of I_h such that, for any $\delta > 0$ and $\lambda \in [c,1]^{I_h}$,

$$b_\delta(F_h(\lambda),U_1) = \psi_1(\lambda') \qquad \text{and} \qquad b_\delta(F_h(\lambda),U_2) = \psi_2(\lambda'')$$

with $\lambda' = \Pi_{I_1}(\lambda)$ and $\lambda'' = \Pi_{I_2}(\lambda)$.

As the random vectors

$$\xi_1^{(h)} = (x_i^h)_{i\in I_1} \qquad \text{and} \qquad \xi_2^{(h)} = (x_i^h)_{i\in I_2}$$

are indipendent, it follows that the random variables

$$\omega \to \psi_1(\xi_1^{(h)})(\omega) \qquad \text{and} \qquad \omega \to \psi_2(\xi_2^{(h)})(\omega)$$

are indipendent too. This proves the assumption (5.5) of Theorem 5.8.

Finally, we point out that, by (6.3), the measure ν of Theorem 5.9 turns out to be absolutely continuous with respect to the (n−1)-dimensional Hausdorff measure H^{n-1}. Therefore, by

Radon-Nikodym theorem we obtain that there is a unique function $h \in L^1(H^{n-1})$ such that

$$v(B) = \int_B h \, dH^{n-1}$$

for every Borel set of ∂D.

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