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GEOMETRICAL ASPECTS OF NON-LINEAR σ -MODEL AND STRING THEORY

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بسم الله الرحمن الرحيم

« وعلّمَكَ ما لَمْ تَعْلَمْ وكانَ فضلُ الله عليْكَ عظيما »

صدق الله العظيم

Dedicated to the memory of my mother.

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The chief role of mathematics in physics consists not in its being an instrument (e.g. computations) but in being the language of physics.

Eugene Wigner.

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I. Introduction:

It is said that Plato wrote on the door of his Academy in Athens: "If you are not a geometer, don't enter this place". This illustrates the very important role geometry played in Greek thinking. The rigorous deductive nature of this science became for them a model for all branches of knowledge, to follow. Physics and geometry in the works of the later Greek philosophers were more or less indistinguishable. The universe in their opinion, was spherical in shape, because the sphere is the most perfect of geometrical figures. Plato thought that every thing consisted of various proportions of atoms having uniform geometrical shapes (equilateral triangles, tetrahedron, cubes, ets.). Even such a modern philosopher as Spinoza wrote his famous book "Ethics" in a form similar to that of the books of Euclead, with axioms, lemmas, theorems and corollaries.

Physics in the last few centuries has became a quantitative science, and hence had to depend on mathematics to a large extenet. Newton had to invent calculus in order to formulate his mechanics. Since then, physics has started to drift away from geometry, and to depend more and more on the analytical methods of algebra and calculus even the mathematicians themselves neglected geometry somewhat, and we can see this from what R. Courant wrote in the introduction to his book "Differential and Integral Calculus", when he says that the difference between modern mathematics and ancient mathematics is the close association of the latter with geometry.

However, things have changed, and mathematicians have gone back to study geometry intensively. New branches of geometry such as differential and algebraic geometry[1] have appeared, and mathematics now is as closeley associated with geometry as it ever has been. Physics also has come closer to geometry. This trend started with Einstein's general theory of relativity (1916), which is basically a geometrical theory. For a long time after that physicists were interested in the quantum theory[2] of matter and radiation, which did not appear initially to have any geometrical significance, and for many years only general relativists had anything to do wiht geometry, while mainstream theoretical physicists, to quote a famous saying by one of them "neede only a knowledge of the Latin and Greek alphabet". This deplorable state of affairs has changed in the last twenty years when theoretical physicists have started to consider theories such as Yang-Mills [3], Kaluza-Klien[4] and supersymmetry theories [5] that turned out to have direct geometrical interpretations, although this took some time to be fully understood[6]. The discovery that Yang-Mill's theories have a variety of classical instanton and monopole solutions[7,8] lead physicists to see such theories as they really are: as theories of connections[9] on principal fibre bundles, objects which mathematicians have already studied intensively. A period of new fruitful interaction between physics and mathematics have thus begun, with benefits to both sides. Geometrical ideas helped physicists to achieve a better understanding of their theories, and the problem they faced gave an impetus for much new mathematical research. Powerful arguments from Yang-Mills theory have changed the picture of the theory of smooth four-dimensional manifolds (Donaldson),[10] and led to a great progress in the theory of complex geometry and Kähler manifolds, and very recently the work of E.Witten on topological quantum field theory[11].

Supersymmetry theory also led to a new area of mutual interaction.

E.Witten[12], using supersymmetry arguments, could prove many of the theorems in Morse theory, thus opening the way for more research in both mathematics and physics. Also anomalies have became a favourite topic for research among both physicists [13] and mathematicians[14]. The non-linear σ-models proved to be closely related at least at the classical level to the mathematical theory of harmonic maps[15].

The recent interest in string theories[16] has made physics and geometry come even closer. One way to get four-dimensional string theories is to compactify the ten-dimensional superstring on a Calabi-Yau[17] manifold (a Ricci-flat, Kähler manifold), thus introducing a new application of differential and algebraic geometry in physics[18]. Treating string multiloop diagrams leads naturally to Riemann surfaces and their moduli spaces, with Grassmanian manifolds[19] as a natural context for studying these spaces. The attempts to construct a covariant string field theory[20] have led to consider non commutative geometries. Studying the string symmetries has led to study infinite-dimensional differential geometry[21]. All these problems have been considered by both physicists and mathematicians, and made them work even closer.

No one knows the full extent to which this trend will go, but perhaps in the near future we will see the application of such mathematical structures as the Hopf algebras[22] (quantum groups) and many others to physics.

In this thesis, we consider some mathematical mostly geometrical aspects of non-linear σ-model and string theories. The topics we shall deal with include bosonization and supersymmetric sigma models.

The origin of bosonization (fermi-bose equivalence) goes back to the work of Skyrme[23], who demonstrated that solitons in some two and four dimensional model may have half-integer spin, and hence behave as fermions.

The two dimensional fermi-bose equivalence was made explicit by S.Colman and S.Mandelstam[24,25] by relating the massive Sine-Gordon theory to the massive Thirring model, i.e., to a two-dimensional self-coupled fermi-field with vector interaction.

The Sine-Gordon field satisfies the following equation;

$$\frac{\partial^2\Phi(x,t)}{\partial t^2} - \frac{\partial^2\Phi(x,t)}{\partial x^2} + (\mu^2/\beta) :\sin[\beta\Phi(x,t)]: = 0 \tag{1}$$

where β is a real parameter, and μ is a mass parameter. Eq.(1) and the corresponding action are invariant under the transformation,

$$\Phi \longrightarrow \Phi + 2\pi n\beta^{-1} \tag{2}$$

where $n \in Z$. This means that the vacuum posseses a discrete degeneracy characterized by the index n.

Solitons are solutions of eq.(2) in which the vacuum well to the left of the disturbance is different from the vacuum to the right. The boundary conditions of solitons and antisolitons are defined by

$$\lim_{x \longrightarrow +\infty} \Phi(x) \longrightarrow 0$$

$$\lim_{x \longrightarrow -\infty} \Phi(x) \longrightarrow \mp 2\pi\beta^{-1} \tag{3}$$

where -(+) corresponds to solitons (antisolitons).

Coleman[24] showed that if $\beta^2 > 8\pi$ the energy density is unbounded below; if $\beta^2 = 4\pi$ the zero-soliton sector of the Sine-Gordon theory is equivalent to the zero-charge sector of the theory of the free massive fermi field; for other values of β the theory is equivalent to the zero-charge sector of the massive Thirring model.

Almost at the same time Mandelstam[25] constructed operators for the creation and annhilation of quantum Sine-Gordon soliton which are given by;

$$\Psi_1(x) = (c\mu/2\pi)^{1/2} e^{\mu/8\epsilon} :\exp[-2\pi i\beta^{-1}\int_{-\infty}^{+x} d\xi\dot{\Phi}(\xi) - 1/2 i\beta\Phi(x)]: \quad (4.a)$$

$$\Psi_2(x) = -i(c\mu/2\pi)^{1/2} e^{\mu/8\epsilon} :\exp[-2\pi i\beta^{-1}\int_{-\infty}^{+x} d\xi\dot{\Phi}(\xi) + 1/2 i\beta\Phi(x)]: \quad (4.b)$$

where $c=1/2i\beta$. These operators satisfy anticommutation relations; and the field equations of the massive Thirring model. The field Φ above has to satisfy the following boundary conditions;

$$\lim_{x\to+\infty} \langle\Phi(x)\rangle = 0$$

$$\lim_{x\to-\infty} \langle\Phi(x)\rangle = n\sqrt{\pi} \qquad ,n\in Z \qquad . \qquad (5)$$

Note that in the free case there are no topologically non-trivial static soliton solutions unlike the Sine-Gordon field. However, if in the free case the field Φ lives on a two-dimensional manifold with spacelike has non-trivial homotopy group, and has values in a manifold with non-trivial first homotopy group, there can be a static topological non-trivial solitons. This is the case for the free bosonic string[26] (living on a Riemann surface) $(X_\mu(\xi))$ when some spatial dimentions are compactified.

A two-dimensional free scaler field $\Phi(x)$ and its conjugate momentum $\pi(x)=\partial_o\Phi(x)$ can be written in terms of harmonic oscillators and with Fourier coefficients satisfying the usual commutation relations. The corresponding two-dimensional fermion Ψ has a Lagrangian density

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - M)\Psi \qquad .$$

We shall take $\gamma^o=\sigma_1, \gamma^1=i\sigma_2$ and $\gamma_5=-\sigma_3$, where $\sigma_1,\sigma_2,\sigma_3$ are the usual Pauli-matrices.

The two quantum theories are equivalent if one makes the following correspondence .

$$\text{fermion bilinear-}1/2\ Z\bar{\Psi}(1\pm\gamma_5)\Psi \longleftrightarrow 1/2\ :e^{\pm i\sqrt{4\pi}\,\Phi}: \tag{5.a}$$

$$\text{current } J^{\mu} =: \bar{\Psi}\gamma^{\mu}\Psi : \longleftrightarrow -\sqrt{\pi}\ \epsilon^{\mu}\partial_{\gamma}\Phi \quad . \tag{5.b}$$

Note that the current $\sqrt{\pi}\,\epsilon^{\mu}\partial_{\gamma}\Phi$ is topological, since the corresponding charge Q is

$$Q = \oint dx\ J_{o} = \oint \partial\Phi$$

which is a topological invariant not affected by a local changes in Φ, since such charges lead to representatives of the same cohomology class as the original field $(\pi_1(\Sigma))=H^1(\Sigma,R)$.

In string theory bosonization[26] (fermionization) was shown to hold for any loop, (i.e. for any number $g\geq 1$), by finding the spin structure dependence of the ζ-function regulated determinant of the Dirac operator D. It was shown that by combining the Quillen theorem[27] and algebraic geometry[28] that the explicit expression for the Dirac determinant is;

$$\det D^{\dagger}D = \text{const}\left|\theta\left[{}^{\epsilon_1}_{\epsilon_2}\right](0|\Omega)\right|^2 \tag{6}$$

where $\theta\left[{}^{\epsilon_1}_{\epsilon_2}\right](0|\Omega)$ is the theta divisor with chracteristics ϵ_1 and ϵ_2 which correspond to spin structures. Physically the spin structures correspond to assigning a ± 1 multiplicative factor to the fermion field Ψ as we go arround non-contractible loops. Also in ref.(26) it was shown that by summing over all instanton (soliton) sectors, the partition function of a single boson on a compact Riemann surface Σ of genus g with values in U(1), is given by

$$\left(\frac{\det'-\nabla^2}{\int_{\Sigma}\sqrt{g}\ \det\mathrm{Im}\Omega}\right)^{-1/2}\sum_{\epsilon_1\epsilon_2\in(1/2\ Z/Z)^{2g}}\left|\theta\left[{}^{\epsilon_1}_{\epsilon_2}\right](0|\Omega)\right|^2 \tag{7}$$

Therefore one concludes that the constant in eq.(6) is equal to the first factor in eq.(7), which also can be checked by using the family index theorem[30] or the relative Grothendieck Riemann Roch (G.R.R.) theorem[28]. The bosonization formula on a torus obtained in ref.(26) also can be obtained using the analytic torsion of Ray and Singer[31] which we will describe later on in this thesis.

The connection between the determinant or the volume form and the theta divisor was first made in a non-explicit way by G.Falting[32].

In string theory bosonization was shown to be very powerfull. For example, via bosonization, one can show the equivalence of the Green-Schwarz and Neveu-Schwarz-Ramond superstring in the light cone gauge[33]. Also understanding gauge and supersymmetry of the Hetrotic string[34] and the construction of the covariant fermionic vertex in superstring[35] was possible through bosonization.

Following the work of Alvarez-Gaumé et al.[26] in this thesis we give an explicit computation for the bose-fermi equivalence[36] in the case of D bosons on a Riemann surface Σ of genus g, with values in a d-dimensional general torus $T^D=R^D/\Lambda_D$, where Λ_D is a lattice in R^D. We assume that the symmetric matrix $Q_{k\upsilon}=\sum \delta_{\mu\lambda}P^{\mu}_{k}P^{\lambda}_{\upsilon}$ is rational, where P^{μ}_{k}, μ,k=1,...,D are the generators of the lattice. We will show that when the matrix Q is the identity, then we recover the extended bosonization formula (fermionization) obtained in ref.(26) for D bosons:

$$Z_{bose}=(1/2)^{3/2gD}\left(\frac{\det'-\nabla^2}{\int_\Sigma\sqrt{g}\ \det\mathrm{Im}\Omega}\right)^{-D/2}\sum_{(\epsilon_1,\epsilon_2)\in(Z_2^{2g})^D}\left|\theta\left[\begin{smallmatrix}\epsilon_1\\ \epsilon_2\end{smallmatrix}\right](0|\Omega)\right|^{2D}$$

$$.\exp(4\pi i D\epsilon_1.\epsilon_2) \qquad . \qquad (8)$$

When Q is orthogonal in the above bosonization formula (eq.8), we obtain the theta function associated with the quadratic form, introduced by Mumford[37] and denoted by θ^Q. It is the generalization of the product of D-theta functions and is given by;

$$\theta^Q = \sum_{N\in Z^{(g,D)}} e^{i\pi tr^TN.\Omega.N.Q+2\pi i tr^TN.Z}$$

where

$$Z = (z_1, \ldots, z_D) \text{ a } g\times D \text{ matrix}$$

$$N = (n_1, \ldots, n_D) \text{ a } g\times D \text{ matrix} \quad .$$

So in this case the bosonization formula reads;

$$Z_{bose} = \sum_{(\epsilon_1,\epsilon_2)\in(Z_2^{2g})^D} (1/2)^{3/2gD}(\det Q)^{-g/2}\left(\frac{\det'-\nabla^2}{\int_\Sigma\sqrt{g}\ \det\mathrm{Im}\Omega}\right)^{-D/2}$$
$$\left|\theta^Q\left[\begin{smallmatrix}\epsilon_1\\ \epsilon_2\end{smallmatrix}\right](0|\Omega)\right|^2 .\exp(4\pi i\mathrm{tr}\epsilon_1.\epsilon_2) \quad . \tag{9}$$

Now we turn to the case in which the matrix Q is general. We will show that in this case we get rational conformal field theory[38] in the sence that the partition function is written as a finite sum of terms each has the form of a product of holomorphic function $f_i(\Omega)$ and an antiholomorphic function $\overline{g_i(\Omega)}$.

From these results we see that, when we compactify D bosons on a generic (rational) torus, we get a generalization of the usual formulas involving different type of theta functions. Therefore it is natural to look for conditions on a twisted spin bundle L_E, which ensure that our partition functions may rise from some generalized bosonization formulas. For that we will use the G.R.R. theorem.

Also, in this thesis, we discuss some mathematical aspects of the non-linear σ-model and bosonic string theory whose classical solutions correspond to harmonic maps[15].

A mathematical background for the theory of harmonic maps is given so that this part will be self contained (section II). Also in this section we will show that the second variation of the harmonic maps is nothing but the second variation of the action of the bosonic non-linear σ-model using normal coordinate expansions which will be discussed also in this thesis.

Finally, we give the explicit expression for the classical energy momentum tensor T_f, where f corresponds to the imbedding or X_μ in string theory. We use the language of harmonic maps and show that $T_f=0$ if and only if the dimension in which the field is defined (the world sheet in string theory) is 2.

As an application we look at the supersymmetric non-linear σ-model[39] which consists of the ordinary bosonic non-linear σ-model coupled to fermions in a supersymmetric way with one, two or four supersymmetries.

We consider the ultraviolet behaviour of softly broken N=1,d=2 supersymmetric non-linear σ-model (soft breaking means the divergences that the breaking terms induce are at most logarithmic). First we give an extension of the supergraph methods[40] to include possible breaking terms in two dimensional N=1 supersymmetry. Once this is done, the method of supergraphs is used to analyse the structure of the divergences[41] induced by the breaking terms. we will show that the proposed breaking term;

$$1/4\int d^2x d^2\theta\ \theta^2\lambda\ R_{ijkl}(\Phi)(D^\alpha\Phi^i)(D_\alpha\Phi^l)(D^\beta\Phi^j)(D_\beta\Phi^k)$$

yields a non-vanishing contribution to the three loop metric tensor β-function even when the target manifold is Ricci-flat.

The thesis is organized as follows:

Section II is divided into two parts, the first part, II.A containes some concepts from differential geometry required for understanding the theory of harmonic map to which II.B is devoted. There we give the definition of a harmonic map, and its relation to Euler-Lagrange equations, and give the first variation formula, and the derivation of the second variation formula and the energy momentum tensor.

Section III is also divided into two parts. In III.A we review the theory of Riemann surfaces, and in III.B we give a brief account of the theory of moduli space and study the line bundles that can be defined on the moduli space.

In section IV we give the geometrical definition of the theta functions, Dirac determinants and our work on the explicit computations of the bose-fermi equivalence on Riemann surfaces of genus g.

Finally in section V we consider the two dimensional supersymmetry and supersymmetric non-linear σ-model, including our works on a two dimensional N=1 supergrphs andexplicitly broken supersymmetries, and the ultraviolet behaviour of softly broken N=1, D=2 supersymmetric non-linear σ-models.

II. DIFFERENTIAL GEOMETRICAL ASPECTS OF HARMONIC MAPS AND APPLICATIONS.

In this section, we shall briefly review some of the concepts of differential geometry such as vector bundles, connection, curvatures, etc., mainly to establish our notations, and as a quick reminder to the reader.

A. DIFFERENTIAL GEOMETRICAL BACKGROUND

1. Vector bundles:

Definition: A real (resp. complex) vector bundle E over a C^{∞}-manifold M is a C^{∞}-mapping $\pi:E\longrightarrow M$, such that $\pi^{-1}(x)=R^k$ (resp.C^k) $\forall x \in M$, and for every $x_o \in M$ ther is a neighborhood U of x_o in M and a diffeomorphism $\varphi_u:\pi^{-1}(U)\longrightarrow U\times R^k$ (resp.$U\times C^k$). This means that E is locally a direct product (trivial).

The C^{∞}-mapping π is called the projection, φ_u a trivialisation over U, $E_x=\pi^{-1}(x)$ the fibre at x and k is the rank of E. If rank k=1, E is called a line bundle. A vector bundle E can be viewed as a family of vector space E_x parametrized by a manifold M such that it is locally trivial.

If φ_u and φ_v are two trivialisation with $U\cap V\neq\emptyset$, the map $g_{uv}:U\cap V\longrightarrow GL(k)$ defined by

$$g_{uv}(x) = (\varphi_u\circ\varphi_v^{-1})_{X\times R^k}(\text{resp.}X\times C^k)^{,}$$

is a C^{∞}-map and is called the transition function which satisfies the following conditions;

$$g_{uu}(x) = \text{identity in } GL(k)$$

$$g_{uv}(x)g_{vu}(x) = \text{identity} \quad ,\text{for } U\cap V\neq\emptyset$$

$$g_{uv}(x)g_{vw}(x)g_{wu}(x) = \text{identity} \quad ,\text{for } U\cap V\cap W\neq\emptyset$$

Given an open covering $\{U_\alpha\}$ of M and a C^∞-map $g_{\alpha\beta}:U_\alpha\cap U_\beta \longrightarrow GL(k)$ satisfying the above conditions, then there is a unique (up to isomorphisms) real (resp.copmlex) vector bundle $E\longrightarrow M$ with transition functions $\{g_{\alpha\beta}\}$.

A section of a vector bundle E is a map $S:M\longrightarrow E$ such that $\pi\circ S$=identity. Two sections can be added to give a section and multiplying a section by a real (resp. complex) valued function is still remaining a section, therefore the set of all sections of E form a module denoted by $\Gamma(E)$.

<u>The dual bundle</u>: If $E\longrightarrow M$ is a vector bundle with transition function $\{g_{\alpha\beta}(x)\}$, then the dual bundle $E^*\longrightarrow M$ has $\{{}^T g_{\alpha\beta}(x)^{-1}\}$ as transition functions.

<u>The tensor product bundle</u>: If E and F are two vector bundles of ranks k and l and defined by the transition functions $\{g_{\alpha\beta}\}$ resp.$\{h_{\alpha\beta}\}$ on the same covering $\{U_\alpha\}$ of M, then the tensor product $E\otimes F$ is given by the transition functions

$$G_{\alpha\beta}(x) = \{g_{\alpha\beta}\}\otimes\{h_{\alpha\beta}\} \quad \in GL(R^k\otimes R^l) \quad \text{resp.} GL(C^k\otimes C^l) \quad .$$

If we consider the r^{th} exterior power of $E,\Lambda^r E$ then the corresponding transition functions are given by

$$G_{\alpha\beta}(x) = \Lambda^r g_{\alpha\beta}(x)$$

where $\Lambda^r g(x)$ denotes the r^{th} antisymmetrized tensor product. For r=k, $\Lambda^r E$ is a line bundle, called the determinant bundle of E, det(E) its transition functions are given by:

$$G_{\alpha\beta}(x) = \det g_{\alpha\beta}(x) \quad \in GL(1,R) \text{ resp.} GL(1,C) \qquad .$$

<u>The pull back bundle</u>: Let $f:M \longrightarrow N$ be a differentiable map and E a vector bundle over N. The pull back bundle, $f^{-1}E \longrightarrow M$, its fibre at point $x \in M$, is defined by $(f^{-1}E)_x = E_{f(x)}$ and its transition function at that point is $(f^{-1}g_{\alpha\beta})_x = g_{\alpha\beta}[f(x)]$.

<u>Examples</u>:

(i) The tangent bundle TM, which is the set of all tangent vectors X at all points $x \in M$, in other words the union $\cup_{x\in M} T_x M$ of the tangent spaces to M.

(ii) The tangent bundle T^*M, the space of all covariant vectors at all points $x \in M$ (also called the space of differentials of M).

(iii) The bundle of tensors of type (r,s); $T^r_s(M)$, in particular the tangent bundle $TM = T^1_0(M)$, is of type $(1,0)$ and the cotangent bundle $T^*M = T^1_0(M)$ is of type $(0,1)$.

A tensor field T of type (r,s) is a function $T:U \longrightarrow T^r_s M$, where the domain U of T is such that of M, such that for every $x \in U$ we have $T(x) \in (T^r_s M)_x$. If $r=1$, $s=0$, the tensor field of type $(1,0)$ is a vector field. If $r=s=0$, T assigns a scalar to each $x \in U$. If f is a C^∞-function on $U \subset M$ then for every $x \in U$, $df_x \in (TM^*)_x = (T^0_1 M)_x$, thus the differential of f, $df:U \longrightarrow T^0_1 M$, is a tensor field of type $(0,1)$.

2. <u>Connections on vector bundles:</u>

Let $\pi:M \longrightarrow N$ be a C^∞-vector bundle and let $\Gamma(E)$, $\mathcal{E}(TM)$, and $\mathcal{E}(M)$ be the set of smooth sections, the set of smooth vector fields and the set of smooth functions on M, respectively. A connection on E is bilinear map ∇ on smooth sections:

$$\nabla : \mathcal{C}(TM)\times\Gamma(E) \longrightarrow \Gamma(E)$$

$$\nabla(x,s) = \nabla_x(s)$$

such that the following properties hold;

$$\text{(i)}\ \nabla_x(fs) = x(f)+f\nabla_x(s)$$

$$\text{(ii)}\ \nabla_{fx}(s) = f\nabla_x(s) \qquad , \forall X \in \mathcal{C}(TM),\ s\in\Gamma(E),\ f\in\mathcal{C}(M)$$

∇_x is a kind of directional derivative which differenciates smooth sections of E in the X-direction. $\nabla_x s$ is called the covariant derivative of s in the direction of X. When E=TM, the tangent of M;∇ is called a linear connection on M.

<u>Connection on the dual bundle E^*</u>: Given a connection ∇ on E, the connection on the dual bundle E^* is obtained by requiring that ∇ commute with the contraction $E_x^*\otimes E_x \longrightarrow R$ in each fibre. In particular, for $s^*\in\Gamma(E^*)$,$\nabla_x s^*$ is defined by the formula;

$$X(s^*(s)) = (\nabla_x s^*)(s)+s^*(\nabla_x s)$$

where $X\in\mathcal{C}(TM)$, $s\in\Gamma(E)$.

<u>The tensor product connection</u>: Given two vector bundle $E\longrightarrow M$ and $E'\longrightarrow M$, the tensor product on $E\otimes E'$ is defined by the linear extension of ∇,

$$\nabla_x(s\otimes\sigma) = (\nabla_x^E s)\otimes\sigma+s\otimes(\nabla_x^{E'}\sigma)$$

where $s\in\Gamma(E)$ and $\sigma\in\Gamma(E')$.

<u>The pull back connection</u>: Given a smooth map $f:M\longrightarrow N$ and a vector bundle $E\longrightarrow N$ with connection ∇^E, the pull back connection on $f^{-1}E$ is defined as a unique connection on $f^{-1}E$ such that;$\forall X\in M$, with $y=f(x)\in N$, $X\in T_xM$, $s\in\Gamma(E)$, we have

$$\nabla_x^{f^{-1}E}(f^{-1}s) = f^{-1}\nabla_{f*x}^{E}(s)$$

where $f_*:T_xM \longrightarrow T_yN$ is the differential map between M and N also denoted by df, and $f^{-1}s=s\circ f$ a section in $\Gamma(f^{-1}E)$.

If we set X(f)=df, then condition (i) in the definition of a connection can be written as

$$\nabla_x(fs) = (df)s+f\nabla_x s \qquad (i') \quad .$$

Now (fs) can be thought as a 0.form on M with values in E, since $s\in\Gamma(E)$. Therefore A connectin ∇ can be thought of as a linear map that takes 0.forms with values in E into 1.forms with values in E, i.e.

$$\nabla:A^o(E) \longrightarrow A'(E)$$

is a "generalization" of exterior derivative.

3. Local description of a connection:

Let U be a neighborhood of point p in M and take in it a frame field,i.e., K-sections, $e=(e_1,\ldots,e_K)$ which are linearly independent. Then the matrix ω of 1-forms is defined by

$$\nabla e_\alpha = \omega_\alpha^\beta e_\beta$$

the matrix of 1-forms $\omega=(\omega_\alpha^\beta)$ is called the connection matrix. The frame e and the matrix connection determine the connection ∇, since for $s\in\Gamma(E)|_u=A^o(E)|_u$ we have

$$s = s^\alpha e_\alpha$$

and

$$\nabla s = ds^\alpha e_\alpha + s^\alpha \nabla e_\alpha = (ds^\alpha + s^\beta \omega^\alpha_\beta) e_\alpha \quad .$$

Therefore

$$\nabla_X s = (Xs^\alpha + s^\beta \omega^\alpha_\beta(X)) e_\alpha$$

If (x^i) are the local coordinates on U, then the vector field **X** is given by $X = X^i \partial/\partial x^i$ and the covariant derivative in the direction of X, of a section s is

$$\nabla_x s = (X^i \frac{\partial}{\partial x^i} s^\alpha + s^\beta \omega^\alpha_\beta (\frac{\partial}{\partial x^i})) e_\alpha \quad .$$

Let

$$\nabla_{\frac{\partial}{\partial x^j}} e_\alpha = \omega^\beta_\alpha (\frac{\partial}{\partial x^i}) e_\beta = \Gamma^\beta_{i\alpha} e_\beta$$

where $\Gamma^\beta_{i\alpha}$ are the coefficients of the connection with respect to e_α, therefore,

$$\nabla_x s = X^i (\frac{\partial}{\partial x^i} s^\alpha + s^\beta \Gamma^\alpha_{i\beta}) e_\alpha \quad .$$

Under the change frame fields in $E|_u : e'_\alpha = g^\beta_\alpha e_\beta$. The coefficient of the connection with respect to the new frame field are given by

$$\nabla e'_\alpha = \omega'^\gamma_\alpha e'_\gamma = \omega'_\alpha g^\beta_\alpha e_\beta$$

but

$$\nabla e'_\alpha = dg^\beta_\alpha e_\beta + g^\gamma_\alpha \omega^\beta_\gamma e_\beta$$

Therefore,

$$\omega'^\gamma_\alpha = dg^\beta_\alpha (g^{-1})^\gamma_\beta + g^\delta_\alpha \omega^\beta_\delta (g^{-1})^\gamma_\beta$$

or in the matrix form, $\omega' = dgg^{-1} + g\omega g^{-1}$, i.e., the connection form transforms like a gauge potential in field theory.

4. <u>The curvature of a connection:</u>

Let $\pi: E \longrightarrow M$ be a vector bundle over a manifold M where the connection is denoted by ∇, then the curvature of E is defined by;

$$R(X,Y)s = \nabla_x \nabla_y s - \nabla_y \nabla_x s - \nabla_{[x,y]} s$$

where $X, Y \in C^{\infty}(TM)$, $s \in \Gamma(E)$. The right hand side of eq.(   ) tells us that $R(X,Y)s$ is a 2.forms with values in $\Gamma(E)$, i.e., the curvature is a section of the tensor bundle $C^{\infty}(\Lambda^2(T^*M)) \otimes \mathrm{Hom}(E,E)$; where T^*M is the cotangent bundle of M and Hom(E,E) is the set of linear maps sending sections of E into itself. Locally, set $X = X^j \partial/\partial x^j$ and $s = s^{\alpha} e_{\alpha}$, then by using the linearity of the curvature operator R(X,Y) we obtain

$$R(X,Y)s = R^{\alpha}_{\beta ij} X^i X^j s^{\beta} e_{\alpha}$$

where $R^{\alpha}_{\beta ij}$ is the curvature matrix.

If E=TM, there is a second important tensor associated to E, the torsion T and is defined by

$$T(X,Y) = \nabla_x Y - \nabla_y X - \nabla_{[x,y]} \quad .$$

Consider the case in which E is equiped with a fibre inner product; i.e., an invertable section $g \in C^{\infty}(E^* \otimes E^*)$, where for each $x \in M$, g is a positive definite inner product on each fibre E_x. Then the connection ∇ on E is called compatible with g if $\nabla g = 0$, that is to say

$$Xg(s,\sigma) = g(\nabla_x s, \sigma) + g(s, \nabla_x \sigma) \quad , \forall X \in C^{\infty}(TM), s, \sigma \in \Gamma(E) \quad .$$

When E=TM, g is a Riemannian inner product structure, M is called a Riemannian manifold.

One can show that there exists a unique torsion-free Riemannian connection called the levi-civita connection on any Riemannian manifold.

<u>The pull back curvature</u>: If $f:M \longrightarrow N$ is a smooth map between two manifolds M and N, and E is a ve8
∇ and curvature $R(X,Y)$, then the pull back curvature R^f via f is given by

$$R^f(X_x,Y_x)S_x = R(f_*(X_x),f_*(Y_x))\sigma_{f(x)}$$

where $X_x, Y_x \in T_x(M)$ are the vector field at x, and $s=\sigma \circ f \in \Gamma(f^{-1}E)$ is the pull back section of E.

Now if the curvature tensor in N is $R^\alpha_{\delta ij}$ then the pull back curvature M is given by

$$(R^f_{ij})^\alpha_\beta = \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} R^\alpha_\beta$$

where the pull back connection is

$$\Gamma^{f\alpha}_{i\beta}(x) = \frac{\partial f}{\partial x^i}\Gamma^\alpha_\beta \quad .$$

B. Harmonic maps and its applications

1. Definition:

Let (M,g) and (N,h) be 2-Riemannian manifolds with dimension m,n respectively. M is assumed to be compact and without a boundary.

A smooth map $f:M \longrightarrow N$ (functional class can be extended to $\mathcal{L}^1_2(M,N)$ induces a linear map $f_*:TM \longrightarrow TM_{f(p)}$, $p \in M$. This is also denoted by df, and can be viewed as an element of tensor product $T^*_p(M)\otimes f^{-1}(TN)$, where $f^{-1}(TN)$ is the pullback of the tangent bundle $TN \longrightarrow N$. Locally the differential f_* is given by

$$f_* = \frac{\partial f^\alpha}{\partial x^i}\, dx^i \frac{\partial}{\partial u^\alpha}\bigg|_{f(p)} \tag{1}$$

where α, $\beta,\ldots=1,\ldots,n$, i, $j,\ldots=1,\ldots,m$, x^i,u^α are local coordinates on M, and N respectively.

The energy of a smooth map f is the functional[15] defined by

$$E(f) = \frac{1}{2}\int_M \|f_*\|^2_{T^*M\otimes f^{-1}TN} * 1 \tag{2}$$

$*1$ is the volume element of M, which is locally given by: $*1=[\det(g_{ij})]^{1/2}dx^1\wedge\ldots\wedge dx^n$. The norm $\|f_*\|$, is taken in $T^*M\otimes f^{-1}TN$ and is locally given by

$$\|f_*\|^2 = g^{ij}\left\langle\frac{\partial f^\alpha}{\partial x^i},\frac{\partial f^\beta}{\partial x^j}\right\rangle_{f^{-1}TN} \tag{3}$$

i.e., $\|f_*\|^2$ is the trace of the pullback via f of the metric tensor of N. Therefore, the energy functional is independent of the local coordinates.

Note that the energy functional E(f) is finite since M is assumed to be compact and f is smooth. f is called harmonic if it is a critical point of E(f), i.e., satisfying the Euler-Lagrange equations:

$$\Delta f^{\alpha}+{}^{N}\Gamma^{\alpha}_{\beta\gamma}\frac{\partial f^{\beta}}{\partial x^{i}}\;\frac{\partial f^{\gamma}}{\partial x^{j}}\;g^{ij}=0 \tag{4}$$

where Δf^{α} is the Laplace-Beltrami operator on M;

$$\Delta f^{\alpha}=\frac{1}{\sqrt{g}}\;\frac{\partial}{\partial x^{i}}\left(\sqrt{g}\;g^{ij}\frac{\partial}{\partial x^{j}}\;f^{\alpha}\right) \quad . \tag{5}$$

Next, we would like to give an intrinsic definition of harmonic maps and show that they are equivalent to above Euler-Lagrange eguations.

Let ∇^{M},∇^{N} be the linear connections on M and N respectively, and ∇^{f} be the induced connection on $f^{-1}TN$. If we denote by ∇ the connection on the tensor product $T^{*}M\otimes f^{-1}TN$, then the second fundamental form of f is given by

$$\beta(f)=\nabla f_{*} \quad . \tag{6}$$

The map f is harmonic if the trace of $\beta(f)$ with respect to g (the Riemannian metric on M) vanishes,i.e.

$$\tau(f)=:\operatorname{tr}(\nabla f_{*})=0 \tag{7}$$

In the litrature $\tau(f)$ is called the tension field of f. Having given the intrinsic definition of the harmonic map, let us now show the equivalence between $\tau(f)$=0 and the Euler-Lagrange equations. The easiest way to see that is to go over to local coordinates; by using eq.(1) and the definition of the Christoffel symbols in M and N. Then;

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x^{i}}}(f_{*}) &= \nabla_{\frac{\partial}{\partial x^{i}}}\left(\frac{\partial f^{\alpha}}{\partial x^{j}}dx^{j}\frac{\partial}{\partial u^{\alpha}}\right)\\ &= \frac{\partial}{\partial x^{i}}\left(\frac{\partial f^{\alpha}}{\partial x^{j}}\right)dx^{j}\frac{\partial}{\partial u^{\alpha}}+\left(\nabla^{f}_{\frac{\partial}{\partial x^{i}}}\;\frac{\partial}{\partial u^{\alpha}}\right)\frac{\partial f^{\alpha}}{\partial x^{j}}\;dx^{j}\\ &\quad+\left(\nabla^{T^{*}M}_{\frac{\partial}{\partial x^{i}}}\;dx^{j}\right)\frac{\partial f^{\alpha}}{\partial x^{j}}\;\frac{\partial}{\partial u^{\alpha}}\\ &= \frac{\partial^{2}f^{\alpha}}{\partial x^{i}\partial x^{j}}\;dx^{j}\frac{\partial}{\partial u^{\alpha}}+{}^{N}\Gamma^{\gamma}_{\beta\alpha}\frac{\partial}{\partial u^{\gamma}}\;\frac{\partial f^{\beta}}{\partial x^{i}}\;\frac{\partial f^{\alpha}}{\partial x^{j}}\;dx^{j}\end{aligned}$$

$$-{}^{M}\Gamma^{j}_{ik}\, dx^k \frac{\partial f^\alpha}{\partial x^j} \frac{\partial}{\partial u^\alpha}$$

Taking the trace with respect to g, of the α^{th} component of $\tau(f)$

$$(\tau(f))_{ij} = \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - {}^{M}\Gamma^{k}_{ij} \frac{\partial f^\alpha}{\partial x^k} + {}^{N}\Gamma^{\alpha}_{\beta\gamma} \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\gamma}{\partial x^j}$$

we get

$$\tau^\alpha(f) = g^{ij} \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - g^{ij}\, {}^{M}\Gamma^{k}_{ij} \frac{\partial f^\alpha}{\partial x^k} + g^{ij}\, {}^{N}\Gamma^{\alpha}_{\beta\gamma} \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\gamma}{\partial x^j}$$

$$\tau^\alpha(f) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\; g^{ij} \frac{\partial}{\partial x^j} f^\alpha\right) + g^{ij}\, {}^{N}\Gamma^{\alpha}_{\beta\gamma} \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\gamma}{\partial x^j}$$

$$= \Delta f^\alpha + g^{ij}\, {}^{N}\Gamma^{\alpha}_{\beta\gamma} \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\gamma}{\partial x^j} \qquad (8)$$

so the equation $\tau^\alpha(f)=0$ is equivalent to the Euler-Lagrange equations. From eq.(8), we see that harmonic maps are generalization of the solution, of the Laplace equation $\Delta f=0$. The relation between the vanishing of the tension field $\tau(f)$ and the criticality of the energy functional $E(f)$ can be seen through the so called "the first variation formula" which we represent next.

<u>The first variation formula</u>: For a given field v along f (a section of $f^{-1}TN$), consider a family of maps f_t such that $f_o=f$ and $\partial f_t/\partial t|_{t=0}=v$.

If one takes $f_t(x)=\exp_{f(x)}(tv)$, then the derivative of E in the direction of v, $D_vE(f)$ is equal to $dE(f_t)/dt|_{t=0}$. Following ref.(15) the first variation reads;

$$\frac{d}{dt}E(f_t)\Big|_{t=0} = -\int_M \langle v, \tau(f)\rangle * 1 \qquad . \qquad (9)$$

Therefore, if the map f is harmonic, then the first vanishes since $\tau(f)=0$.

Next we will give an explicit computation for "the second variation formula" the Hessian of a harmonic map.

2. "The second variation formula" (the Hessian):

A harmonic map $f:M \longrightarrow M$ is a stable, if for every smooth homotopy f_t, such that $f_o=f$. The second variation

$$\frac{d^2}{dt^2}E(f_t)\Big|_{t=0} \geq 0 . \tag{1}$$

The expresion for the Hessian of a harmonic map is given by the following proposition;

Proposition: The Hessian of a harmonic map is given by;

$$\frac{d^2}{dt^2}E(f_t)\Big|_{t=0} = H_f(v,v) = \int[<\nabla^f v,\nabla^f v>-trR^N(df,v)df,v>*1]$$

$$= \int_M <\nabla^f v-trR^N(df,v)df,v>*1 \tag{2}$$

where $\Delta^f = d^* d = -tr\nabla\nabla^f$ is the Laplacian with respect to the connection ∇^f of the pullback tangent bundle $f^{-1}TN$. R^N is the curvature of the manifold N, and v is a section in $f^{-1}TN$, i.e., $v \in C(f^{-1}TN)$ which can also be, viewed as a zero.form with values in $f^{-1}TN$ denoted by $A^o(f^{-1}TN)$.

Proof: In the second variation formula we have to concider a 2-parameter variation $\Phi(.,s,t)=f_{s,t}$ corresponding to two vector fields v,w along f, i.e. $v,w \in C(f^{-1}TN)$. Just like for "the first variation formula" $\Phi(.,s,t)$ is constructed such that

$$f_{o,o} = f$$

$$v = \frac{\partial f_{s,t}}{\partial s}\Big|_{s,t=0} , \quad w = \frac{\partial f_{s,t}}{\partial t}\Big|_{s,t=0} \tag{3}$$

"The second variation" or the Hessian is a symmetric bilinear form on $C(f^{-1}TN)$ defined by $H_f(v,w)=\partial^2 E/\partial t\partial s|_{s,t=0}$.

Let ∇^{Φ} be the connection in $\Phi^{-1}TN$, consider the first variation formula (II.B.1 eq.9)

$$\frac{\partial}{\partial s}\langle df_{s,t},df_{s,t}\rangle = \langle\nabla^{\Phi}\frac{\partial\Phi}{\partial s},df_{s,t}\rangle \quad . \tag{4}$$

Acting now with the vector field $\partial/\partial t$, we obtain

$$\frac{\partial^2}{\partial t\partial s}\langle df_{s,t},df_{s,t}\rangle = \frac{\partial}{\partial t}\langle\nabla^{\Phi}\frac{\partial\Phi}{\partial s},df_{s,t}\rangle$$

$$= \langle\nabla_{\partial/\partial t}\nabla^{\Phi}\frac{\partial\Phi}{\partial s},df_{s,t}\rangle+\langle\nabla^{\Phi}\frac{\partial\Phi}{\partial s},\nabla_{\partial/\partial t}df_{s,t}\rangle. \tag{5}$$

But $\nabla_{\partial/\partial t}df_{s,t}=\nabla^{\Phi}\partial/\partial t$, which can be shown as follows:

For $X\in TM$, we have

$$\nabla^{\Phi}_{\partial/\partial t}(df_{s,t}.X) = (\nabla_{\partial/\partial t}df_{s,t})X+df_{s,t}.\nabla^{T(MxR)}_{\partial/\partial t}X$$

$$= (\nabla_{\partial/\partial t}df_{s,t})X \quad , \tag{6}$$

since $[\partial/\partial t,X]=0$.

Using the fact that $\nabla^{\Phi}_{\partial/\partial t}(df_{s,t}.X)=\nabla^{\Phi}_{\partial/\partial t}(d\Phi.X)$, and the torsion free condition;

$$0 = -\nabla_{\partial/\partial t}X+\nabla_{X}\partial/\partial t+[X,\partial/\partial t]$$

and hence one obtains $\nabla^{\Phi}_{\partial/\partial t}(d\Phi.X)=\nabla^{\Phi}_{X}\partial\Phi/\partial t$,

$$(\nabla_{\partial/\partial t}df_{s,t}) = \nabla^{\Phi}\frac{\partial\Phi}{\partial t} \quad . \tag{7}$$

Therefore the second term on the right hand side of $\partial^2/\partial t\partial s\langle\ ,\ \rangle$ can be written as

$$\langle\nabla^{\Phi}\frac{\partial\Phi}{\partial s},\nabla_{\partial/\partial t}df_{s,t}\rangle\Big|_{s,t=0} = \langle\nabla^{\Phi}v,\nabla^{\Phi}w\rangle \tag{8}$$

with

$$v = \frac{\partial\Phi}{\partial s}\Big|_{s,t=0} \quad , \quad w = \frac{\partial\Phi}{\partial t}\Big|_{s,t=0} \quad .$$

To evaluate the first term in $\partial^2/\partial t\partial s\langle\ ,\ \rangle$ we use the relation between the curvature and the covariant derivative in the X and $\partial/\partial t$ directions, namely

$$R^{\Phi}(X,\partial/\partial t) = -\nabla^{\Phi}_{X}\nabla^{\Phi}_{\partial/\partial t}+\nabla^{\Phi}_{\partial/\partial t}\nabla^{\Phi}_{X}+[X,\partial/\partial t] = 0 \qquad . \tag{9}$$

But the pullback curvature is

$$R^{\Phi}(X,\partial/\partial t) = R^{N}(d\Phi.X,d\Phi.\partial/\partial t) = R^{N}(d\Phi.X,\partial\Phi/\partial t) \qquad ,$$

thus the first term is given by

$$\langle\nabla_{\partial/\partial t}\nabla^{\Phi}\frac{\partial\Phi}{\partial s},df_{s,t}\rangle = \langle\nabla^{\Phi}\nabla^{\Phi}_{\partial/\partial t}\frac{\partial\Phi}{\partial s},df_{s,t}\rangle$$

$$+\langle R^{N}(df_{s,t}.,\frac{\partial\Phi}{\partial t})\frac{\partial\Phi}{\partial s},df_{s,t}.\rangle \qquad , \tag{10}$$

Now ∇^{Φ} and d coinside on $C(\Phi^{-1}TN)=A^{o}(\Phi^{-1}TN)$, i.e.

$$\nabla^{\Phi}\nabla^{\Phi}_{\partial/\partial t}\frac{\partial\Phi}{\partial s} = d\nabla^{\Phi}_{\partial/\partial t}\frac{\partial\Phi}{\partial s}$$

since $\nabla^{\Phi}_{\partial/\partial t}\frac{\partial\Phi}{\partial s}$ can be viewed as an element of $C(\Phi^{-1}TN)$ and since the map f is harmonic. Therefore

$$\int\langle\nabla^{\Phi}_{\partial/\partial t}\frac{\partial\Phi}{\partial s},d^{*}d\Phi_{s,t}\rangle\Big|_{s,t=0}*1 = 0$$

Finally, the second variation formula can be written as

$$\frac{\partial^2}{\partial t\partial s}E(f_{s,t})\Big|_{s,t=0} = \int\langle\nabla^{\Phi}v,\nabla^{\Phi}w\rangle*1-\int_{M}\langle trR^{N}(df,v)df,w\rangle*1$$

$$= \int_{M}\langle\nabla^{\Phi}v\ \ trR^{N}(df,v)df,w\rangle*1 \qquad . \tag{11}$$

We will give another proof of this formula, in section V.B.2, where we discuss the "background field expansion", by using normal coordinates.

Before deriving the stress-energy tensor (energy-momentum tensor) of a harmonic map, we would like to make the following remarks concerning the relationship between harmonic maps and non-linear

σ-models.

The energy functional $E(f)=1/2\int_M \|df\|^2 *1$ in component-form is $E(f)=\int_M g^{ij}\partial f^\beta/\partial x^i \partial f^\beta/\partial x^j h_{\alpha\beta}$, where g^{ij} is the metric of the compact manifold M, and $h_{\alpha\beta}$ is the metric of the tangent manifold N. E(f) in this form is exactly the action for a (classical) non-linear σ-model where M is the space-time, and f the field defined on M with values in N. Also E(f) can be thought of as the action for the bosonic string where M is the world-sheet (for instance a compact Riemann surface) and N is the space-time.

In σ-model theory, choosing a critical point of the energy functional (a harmonic map) corresponds to choosing one "vacuum" say f_o.This vacuum will be stable to first order if the Hessian of E(f) is positive definite. In perturbative quantum field theory, one studies "perturbations" around a chosen vacuum, f_o.

From the classical point of view, quantum fluctuations ζ's correspond to perturbations around f_o, which are in $T\mathscr{C}(M,N)|_{f_o} \simeq \Gamma(f^{-1}TN)$,i.e.forgetting about their operator nature) quantum fluctuations are section of the pullback tangent bundle at f_o.

3. The stress energy tensor[15]:

If g(t) is a smooth 1-parameter family of metrics, g(0)=g and $\delta g=\partial g/\partial t|_{t=0}$, then for a fixed map f, the variation of E(f) with respect to t at t=0 is given by

$$\left.\frac{dE(f)}{dt}\right|_{t=0} = \int_M \langle T_f, \delta g\rangle *1 \tag{1}$$

where T_f is the stress-energy tensor which is a symmetric 2-form and is given by;

$$T_f = e(f).g - f^*h \qquad , \qquad (2)$$

where

$$e(f) = 1/2 g^{ij} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} h_{\alpha\beta} \qquad ,$$

and $$f^*h = \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} h_{\alpha\beta} \qquad ,$$

is the pull back metric on M. To prove the above relation for T_f, let us first suppose that E(f) has the form $E(f)=\int \mathcal{L}(f)*1$, where $\mathcal{L}(f)$ is a Lagrangian for f. Then in this case, $T_f = \partial\mathcal{L}/\partial g + 1/2 g.\mathcal{L}$, which we can see as follows;

$$\left(\frac{dE(f)}{dt}\bigg|_{t=0}\right) = \int_M \frac{\partial \mathcal{L}}{\partial g_{kl}} \delta g*1 + \int_M \mathcal{L} \frac{\partial}{\partial g_{kl}} (*1) g$$

$$\frac{\partial}{\partial g_{kl}}(*1) = \frac{\partial}{\partial g_{kl}} \left[(\det g)^{-1/2} dx^1 \Lambda \ldots \Lambda dx^m\right]$$

$$= \frac{1}{2}\left[(\det g)^{-1/2} (\text{cofactor of } g_{kl}) dx^1 \Lambda \ldots \Lambda dx^m\right]$$

$$= \frac{1}{2} (\det g)^{-1} (\text{cofactor of } g_{kl}) (\det g)^{1/2} dx^1 \Lambda \ldots \Lambda dx^m$$

$$= \frac{1}{2} g^{kl} * 1 \qquad ,$$

where m is the dimension of M and g^{kl} is the inverse matrix of g_{ks}. Therefore the stress-energy tensor is

$$T_f = \frac{\partial \mathcal{L}}{\partial g} + \frac{1}{2} g.\mathcal{L} \qquad . \qquad (3)$$

In our case

$$E(f) = \frac{1}{2} g^{ij} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} \qquad ,$$

then we have

$$\frac{\partial \mathcal{L}}{\partial g_{kl}} = \left(\frac{\partial g^{ij}}{\partial g_{kl}}\right)\frac{\partial f^{\alpha}}{\partial x^{i}}\;\frac{\partial f^{\beta}}{\partial x^{j}}h_{\alpha\beta}$$

Using the fact that $g^{ij}g_{jm}=\delta^{l}_{m}$, we have

$$\frac{\partial g^{ij}}{\partial g_{kl}} = -g^{ik}g^{jl} \quad ,$$

and

$$\frac{\partial \mathcal{L}}{\partial g_{kl}} = -g^{ik}g^{jl}\frac{\partial f^{\alpha}}{\partial x^{i}}\;\frac{\partial f^{\beta}}{\partial x^{j}}h_{\alpha\beta} \quad .$$

Thus

$$\frac{\partial \mathcal{L}}{\partial g} = -f^{*}h$$

Then the stress- energy tensor T_f, with $\mathcal{L}=\|df\|^2$ becomes

$$T_f = e.g-f^{*}h$$

it was shown in ref.(15) that

$$(\mathrm{div}T_f) = -<\tau,df> \tag{4}$$

where $(\mathrm{div}\ T_f)_i=g^{jk}\nabla_{\partial/\partial x^j}T_{ki}$, (the tension field $\tau=0$). Therefore, if the map f is harmonic, the stress-energy tensor is conserved; $\mathrm{div}T_f=0$. this corresponds to the fact that the energy momentum tensor is conserved when we have translation symmetry, by Noether theorem only when equation of motion are satisfied.

<u>Proposition</u>: Let $f:(M,g)\longrightarrow(N,h)$ be a non-constant map, then $T_f=0\longleftrightarrow\dim M=2$ and the map f is conformal (f is conformal if $f^{*}h=\mu g$, where μ is a positive C^{∞}-function).

<u>Proof</u>: Set $m=\dim M$, and recall that

$$e=1/2\|df\|^2=1/2\,\mathrm{tr}f^{*}h$$

and

$$T_f=0 \Longrightarrow f^*h = e.g.$$

Then

$$\mathrm{tr}T_f = e.\mathrm{tr}g - \mathrm{tr}f^*h = 0$$

$$= e.m-2e = 0$$

i.e., $m=2=\dim M$. Conversly, suppose that $f^*h=\mu g$, then $e=m\mu/2$ and $T_f=((m-2)/2)\mu g$. Thus for $m=2$, $T_f=0$.

Note that this proposition is applicable to the classical bosonic string theory[16], since $\dim M=2$. In particular, T is traceless, because of the invariance of the string action (the energy functional) under a local rescaling of the metric.

III. RIEMANN SURFACES AND THE MODULI SPACE

A. RIEMANN SURFACES

Let us first recall the definition of complex manifolds[43] and then give the definition of Riemann surfaces[44] as a special case of a complex manifold.

1. <u>Definitions</u>: A Hausdorff topological space M is called a complex manifold of complex dimension n if there are given open covering $\{U_i\}_{i\in I}$ and a family $\{\Phi_i\}_{i\in I}$ of homomorphism of U_i onto C^n (n-dimensional complex space) such that in the overlap $U_i \cap U_j$, the mapping $\Phi_i \circ \Phi_j^{-1}$. $\Phi_j(U_i\cap U_j) \longrightarrow \Phi_i(U_i \cap U_j)$ is biholomorphic (i.e., holomorphic homeomorphism). M is called the underlying topological space of this complex manifold, and we say that $\{U_i,\Phi_i\}_{i\in I}$ define a complex structure on M.

A Riemann surface Σ is a one dimensional complex connected manifold.

<u>Example</u>: The sphere S^2 can be given a complex structure. S^2 can be thought of as the one point compactification of the complex plane C, i.e. $S^2 = C\cup\{\infty\}$, where C is the z-plane. Set

$$U_1 = C\ , \qquad \Phi_1 : U_1 \longrightarrow C, \qquad \Phi_1(z) = z$$

$$U_2 = (C\cup\{\infty\}) - \{0\}, \qquad \Phi_2 : U_2 \longrightarrow C, \qquad \Phi_2(z) = \begin{cases} 1/z & ,\ z\in C-\{0\} \\ 0 & ,\ \text{for } z=\infty \end{cases}$$

Thus $\{U_i,\Phi_i\}$, $i=1,2$, makes $C\cup\{\infty\}$ a Riemann surface called the Riemann sphere.

Topologically, a compact Riemann surface Σ is characterized by its "genus" g, the number of handles that must be fastened to the sphere in order to obtain Σ . Given two Riemann surfaces, Σ_1 , Σ_2 and a continuous map f between them; $f:\Sigma_1 \longrightarrow \Sigma_2$, then f is called holomorphic if for each point $z \in \Sigma_1$ and any coordinate system Φ and ψ around z and f(z), respectively, the function $\psi[f^{-1}(\Phi)]$ defined in the complex plane C, is holomorphic. Σ_1, Σ_2 are said to be conformally equivalent if the continuous map is biholomorphic.

<u>Harmonic and holomorphic forms on a Riemann surface</u>: Since Riemann surfaces are considered as real 2-dimensional differentiable manifolds of class C^{∞}, then the forms that we can construct on a Riemann surface are of order only up to 2.

<u>Definition</u>: Let U be an open set of a Riemann surface Σ. A differentiable 1-form ω on U is said to be an abelian differential of the first kind . If it is of the form $\omega = f dz$, (a (1,0) type), where f is holomorphic, i.e., $\partial f / \partial \bar{z} = 0$.

<u>The Hodge operator, *</u>: Let ω be a complex-valued one-form and set $\omega = \omega_1 + \omega_2$, where ω_1 is of type (1,0) and ω_2 is of type (0,1). The hodge operator * on ω is defined by $*\omega = i(\bar{\omega}_1 - \bar{\omega}_2)$. The complex valued 1-form ω is called harmonic if $d\omega = d{*\omega} = 0$.

Another equivalent definition of the hodge or the conjugate * operator is given by writting a 1-form on a Riemann surface as $\alpha = U(x,y) + V(x,y)dy$, then the conjugate operator *, is defined by:

$$*\alpha = -Vdx + Udy \quad , \tag{1}$$

hence

$$*dx = dy \quad ; \quad *dy = -dx \quad . \tag{2}$$

Other important properties of the * operator are the following:

(1) If f is a function, then $*(f\alpha) = f(*\alpha)$.

(2) $\alpha\Lambda*\beta = \beta\Lambda*\alpha$, where α, and β are 1-forms.

The product $\alpha\Lambda*\bar{\beta}$ is a (1,1) form, i.e., can be integrated over a Riemann surface Σ to give a number;

$$(\alpha,\beta) = \int_{\Sigma} \alpha\Lambda*\bar{\beta} \quad . \tag{3}$$

This new product (α,β) defines a scalar product. this can be checked as follows;

$$\|\alpha\| = \int_{\Sigma} \alpha\Lambda*\bar{\alpha} = \int_{\Sigma}(|U|^2+|V|^2)dx\Lambda dy \quad ,$$

since $(|U|^2+|V|^2)>0$ unless $\alpha=0$, i.e., $\|\alpha\|=(\alpha,\alpha)=0\Leftarrow \Rightarrow\alpha=0$.

The $*$-operator on Abelian differentials: Abelian differentials of the form $\omega=f(z)dz$ can be written as: $\omega=\alpha+i*\alpha$, where $f(z)=U+iV$ and $dz=dx+idy$. Using the properties of the $*$-operator we obtain the following identities;

$$*\omega = -i\omega \quad , \quad *\bar{\omega} = i\bar{\omega} \quad . \tag{4}$$

The genus g of a Riemann surface Σ is equal to half the number of closed curves needed to generate the first homology group; $H_1(\Sigma,Z)$, i.e., there are 2g-closed curves $\gamma_1,\ldots\gamma_{2g}$ such that every closed curve in Σ is "homologous" to a unique integral linear combination $\Sigma n_i\gamma_i$. Recall that two 1-cycles $C,C'\in Z_1(\Sigma)$ are homologous if $\int_C\omega = \int_{C'}\omega$ for every closed 1-form ω.

Also, one can show that 2g is the number of linearly independent real holomorphic differential forms of degree 1, i.e., g is the dimension of the space of holomorphic 1-forms (the dimension of the cohomology group $H^1(\Sigma,R)$).

Let us now consider a Riemann surface Σ with a simplectic basis $(a_1,\ldots a_g;b_1,\ldots b_g)$ of $H_1(\Sigma,Z)$,which means that we have on this homology group a quadratic form J represented by the intersection matrix:

$$\begin{pmatrix} a.a & a.b \\ b.a & a.a \end{pmatrix} = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} .$$

This particular basis is called the canonical homology basis. A Riemann surface with a canonical homology basis is called a marked Riemann surface. The cup product[45] of two cohomology classes represented by the 1-forms α and β is given by the Riemann identity;

$$\int_\Sigma \alpha\Lambda\beta = \sum_{i=1}^{g} \int_{a_i} \alpha \int_{b_i} \beta - \int_{a_i} \beta \int_{b_i} \alpha \tag{5}$$

If $\omega_1, \ldots \omega_g$ is a basis of the complex-vector space of holomorphic differentials on Σ, then the complex g×2g matrix;

$$\tilde{\Omega} = [\int_{a_i} \omega_j, \int_{b_i} \omega_j] \tag{6}$$

is called the period matrix of Σ.

Riemann observed that it is possible to choose the basis $\omega_1, \ldots \omega_g$ such that $\int_a \omega_j = \delta_{ij}$ and $\int_b \omega_j = \Omega_{ij}$, i.e., the period matrix becomes;

$$\tilde{\Omega} = (1, \Omega) \tag{7}$$

Ω in eq.(7) is called the canonical period matrix of Σ. Using the Riemann identity, one can show that ${}^T\Omega = \Omega$ and $\mathrm{Im}\Omega > 0$. A set of Ω's having this property is called the Siegel upper-half plane, and is denoted by $\mathcal{H}_g$.

The period matrix Ω determines the complex analytic structure according to theorem of R.Torelli, which says that given a marked Riemann surface Σ, then two complex structures on Σ having the same period matrix Ω, are isomorphic.

<u>The Jacobian of a Riemann surface</u>: Let C be a compact Riemann surface of genus g with first homology basis $\gamma_1, \ldots \gamma_{2g}$, and with

$\omega_1, \ldots \omega_g$ its corresponding dual in $H^1(C,Z)$. The vector integral

$$\int_{\gamma_i} \vec{\omega} = \left(\int_{\gamma_1} \omega_1, \ldots, \int_{\gamma_g} \omega_g \right) ,$$

is called the period.

Given a fixed base point p_o on a Riemann surface C the following vector integral

$$\int_{p_o}^{p} \vec{\omega} = \left(\int_{p_o}^{p} \omega_1, \ldots, \int_{p_o}^{p} \omega_g \right)$$

is well defined modulo periods. In this way, we obtain a mapping from the Riemann surface C into C^g/periods and is called the Jacobian of C, denoted by Jac(C).

2. Sheaves:

The essential idea in a sheaf theory[46] is to associate with each point x of a topological space X an algebraic structure $\mathcal{F}_x$ (such as a ring structure, group structure, module structure, etc.) called the stalk, the union of these structure $\bigcup_{x \in X} \mathcal{F}_x$ is treated as a topological space, and is called a sheaf.

After this brief intuitive idea of a sheaf, now we turn to the formal definitions of presheaves and sheaves:

Definition of a presheaf and sheaf: A presheaf $\mathcal{F}$ is determined if, given a space X, we associate with every set U of X an algebraic structure $\mathcal{F}(u)$ such that

(1) If U is empty, $\mathcal{F}(u)=0$ (i.e., consist of zero only).

(2) If $V \subset U$, there is a holomorphism ρ_u^v of $\mathcal{F}(u)$ into $\mathcal{F}(v)$.

(3) If $W \subset V \subset U$ then $\rho_U^W = \rho_V^W \circ \rho_W^V$ and ρ_u^u is the identical mapping.

A sheaf $\mathcal{F}$ over X is a presheaf such that whenever $V = \cup_i V_i$ the following are satisfied;

(4) If $s\in\mathcal{F}(V)$ and $s|_{V_i}=0$ for any i, then $s=0$.

(5) Given $s_i\in\mathcal{F}(V_i)$ such that $\forall i,j$ $s_i|_{V_i\cap V_j}=s_j|_{V_i\cap V_j}$, then there is $s\in\mathcal{F}(V)$ such that $s|_{V_i}=s_i$, by (4) such s is necessarily unique.

Examples:

(i) On a complex manifold X, the structure sheaf $\mathcal{O}_X$ is defined by $\mathcal{O}_X(u)=\{f|f:U\longrightarrow C,\text{where f is holomrphic}\}$ with the homomorphism ρ^u_V being the restriction functions.

In the mathematical literature, a topological space X with a structure sheaf $\mathcal{O}_X$ is called a ringed space (Espace Annalé)

(ii) Let $\mathcal{L}\xrightarrow{\pi}C$ be a holomorphic line bundle over a Riemann surface C. The sheaf $\underline{\mathcal{L}}_C$ of holomorphic sections of $\mathcal{L}$ is defined by $\underline{\mathcal{L}}_C(u)=\{\sigma|\sigma:U\longrightarrow\pi^{-1}(U),\text{where } \sigma \text{ is holomorphic}\}$ and ρ_{uv} being the restriction.

(iii) The canonical sheaf of Σ, is a sheaf of differential 1-forms, whose sections are locally given by $\sigma=f(z)dz$, and is usually denoted by ω_C or K_C. The j^{th} canonical sheaf has sections of the form $\sigma=g(z)(dz)^j$, and is denoted by ω^j_C. Physically this corresponds to a field of spin j.

(iv) For any abelian group G we define the constant sheaf G_X by

$$G_X(u) = G \qquad , \qquad \rho_{uv}=id_G$$

As an example we have Z_X, the sheaf of integer valued functions.

<u>Direct image sheaf $f_*\mathcal{F}$</u>: Let $f:X\longrightarrow Y$ be a continous map (in particular f is holomorphic whenevr X,Y are complex manifolds). For any sheaf $\mathcal{F}$ over X we define its direct image sheaf $f_*\mathcal{F}$ on Y by

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$$

<u>Homomorphism of sheaves</u>: Let $\mathcal{F},\mathcal{G}$ be sheaves on X, a homomorphism

$\varphi:\mathcal{F}\longrightarrow\mathcal{Y}$ of sheaves on X is a collection of homomorphisms $\varphi_u:\mathcal{F}(U)\longrightarrow\mathcal{Y}(U)$ which are compatible with restrictions in the sense that the diagram

$$\begin{array}{ccc}\mathcal{F}(U) & \longrightarrow & \mathcal{Y}(U)\\ \rho_{uv}\downarrow & & \downarrow \rho'_{uv}\\ \mathcal{F}(V) & \longrightarrow & \mathcal{Y}(V)\end{array}$$

comutes, i.e., $\varphi_u(s)|_u=\varphi_v(s)|_u$. φ is an isomorphism, whenever φ'_us are. A short exact sequence of sheaves is a sequence of homomorphism

$$0\longrightarrow\mathcal{F}\xrightarrow{\alpha}\mathcal{Y}\xrightarrow{\beta}\mathcal{H}\longrightarrow 0$$

such that $\mathcal{F}$=Kerβ and $\mathcal{H}$=coKerα.

<u>Examples</u>: on a Riemann surface C, let $Z,\mathcal{O},\mathcal{O}^*$ denote the constant sheaf of integers, the structure sheaf and the sheaf of non-vanishing holomorphic functions. Then we can form an exact sequence

$$0\longrightarrow Z\hookrightarrow\mathcal{O}\xrightarrow{f}\mathcal{O}^*\longrightarrow 0$$

where $Z\hookrightarrow\mathcal{O}$ is an inclusion map and f is the exponential map $e^{2\pi if}$.

<u>Cohomology of sheaves</u>: The cohomology group $H^q(X,\mathcal{F})$ of X with coefficients in $\mathcal{F}$ is defined through the Čech procedures which can be described as follows;

If $U=(U_i)$ is a finite open covering of X by open sets U_i, and $q\in\{0,1,2,\ldots\}\equiv I$, then a q-cochain with respect to U is a function f that assigns to each $(i_o,\ldots i_q)$ an element $f(i_o,\ldots i_q)\in\Gamma(U_o\cap\ldots\cap U_q,\mathcal{F})$, subject to the condition that f is an alternating function in the indices. The set of all q-cochain with respect to U is denoted by $C^q(\mathcal{U},\mathcal{F})$. Define the coboundary operator $\delta:C^q(\mathcal{U},\mathcal{F})\longrightarrow C^{q+1}(\mathcal{U},\mathcal{F})$, by

$$(\delta f)(i_o,\ldots i_{q+1}) = \sum_{j=0}^{q+1}(-1)^j f(i_o,\ldots,\hat{i}_j,\ldots i_{q+1})$$

here ^ denotes deletion. In particular if $f=(f_i)\in C^0(\mathcal{U},\mathcal{F})$

$$(\delta f)(i,j) = -f_i+f_j \quad ,$$

and if $f = \{f_{i,j}\}\in C'(\mathcal{U},\mathcal{F})$,

$$(\delta f)(i,j,k) = f_{ij}+f_{jk}-f_{ik}$$

A q-cochain f is called a cochain if $\delta f=0$, and it is called a coboundary if $f=\delta g$ for some $g\in C^{q-1}(\mathcal{U},\mathcal{F})$. A direct computation shows that $\delta\circ\delta=0$, i.e., a coboundary is a cocycle.

Let $Z^q(U,F)=\{f\in C^q(\mathcal{U},\mathcal{F}):\delta f=0\}$ and $B^q(U,F)=\delta(C^q(\mathcal{U},\mathcal{F}))$. Then the cohomology group with respect to the covering $\mathcal{U}$ and with values in $\mathcal{F}$ is defined by;

$$H^q(\mathcal{U},\mathcal{F}) = \frac{Z^q(\mathcal{U},\mathcal{F})}{B^q(\mathcal{U},\mathcal{F})} \quad .$$

The cohomology $H^q(\mathcal{U},\mathcal{F})$ depends on the covering $\mathcal{U}=(U_i)$ of X, but by passing to a finer covering, (going to the inductive limit) it can be made independent of the choice of the covering, and the result is the q^{th} cohomology group of X with values in $\mathcal{F}$, $H^q(X,\mathcal{F})$, and is called the Čech cohomology.

For q=0, one has $H^0(\mathcal{U},\mathcal{F})=Z^0(\mathcal{U},\mathcal{F})=\mathcal{F}(X)$, to see that first the 0^{th}-cohomology group vanishes, i.e., $H^0(\mathcal{U},\mathcal{F})=Z^0(\mathcal{U},\mathcal{F})$. But using the definition of the coboundary operator δ, one concludes that a 0-cochain $(f_i)\in C^0(\mathcal{U},\mathcal{F})$ belong to $Z^0(\mathcal{U},\mathcal{F})$ precisely if

$$f_i|_{u_i\cap v_j} = f_j|_{u_i\cap v_j} \quad \forall i,j\in(0,1,2,\ldots).$$

Using sheaf axiom (5), f_i fit together to give a global element $f\in\mathcal{F}(X)$, therefore, $H^0(\mathcal{U},\mathcal{F})=Z^0(\mathcal{U},\mathcal{F})\simeq\mathcal{F}(X)$.

<u>Examples</u>[47]:

(i) $H^1(X,G)$, G is a lie group. Let X be a complex manifold and U an open set in X. Let $\mathcal{U}=\{U_i\}_{i\in I}$ be a covering of X by open sets U_i and $I(\mathcal{U})$ be the set of indeces needed for covering.

A $\mathcal{U}$-cocycle is a function f that associates two ordered elements in $I(\mathcal{U})$ with an element f_{ij}, $f_{ij}\in\Gamma(U_i\cap U_j, G)$ such that

$$f_{ij}(x)f_{JK}(x) = f_{iK}(x) \quad \text{for} \quad x\in U_i\cap U_j\cap U_k, \quad i,j,k\in I(\mathcal{U}).$$

Note that the multiplication takes place in G.

In the space $\Gamma(\mathcal{U},G)$, of maps from U into G, the unit element, is given by f_{ii}=neutral element in $\Gamma(U_i,G)$ and the inverse element is $(f_{ij})^{-1}=f_{ji}$. If $V\subset U$, then we have a restriction map $\rho^u_V:H^1(U,g)\longrightarrow H^1(V,g)$. The inductive limit of the system $\{H^1(U,g),\rho^u_V\}$ is denoted by $H^1(X,g)$.

Two cocycles f and f′ are said to be equivalent if there exists an element $g_i\in\Gamma(U_i,G)$ for all $i\in I(\mathcal{U})$ such that

$$f'_{ij}(x) = g_i^{-1}(x)f_{ij}(x)g_i(x) \qquad \text{in } U_i\cap U_j \ .$$

The set of equivalence classes is denoted by $H^1(U,G)$.

(ii) The isomorphism classes of fibres bundles E with base B, fibre F and structure group G are in one to one correspondence with $H^1(B,G)$. The unit element of $H^1(B,G)$ corresponds to the trivial fibre bundle $E=B\times F$.

To see that, let $\gamma=\{g_{ij}\}$ and $\gamma'=\{g'_{ij}\}$ be two equivalent cocycles, i.e., $g'_{ij}=g_i^{-1}g_{ij}g_j$ in $U_i\cap U_j$ with $g_i\in\Gamma(U_i,g)$. For $x\in U_i$, the mapping $h:(x,f)\longrightarrow(x,g_j^{-1}(x)f)$ is a homeomorphism from $U_j\times F$ onto $U_j\times F$. For $x\in U_i\cap U_j$, (x,f) and $(x,g_{ij}(x)f)$ are identified in the fibre bundle E_γ. But from the definition of the homeomorphism h, we have

$$h(x,g_{ij}(x)f) = (x,g_i^{-1}(x)\ g_{ij}(x)f)$$

$$= (x, g'_i\ (x)\ g_{ij}^{-1}(x)f) \qquad ,$$

so $h(x,g_{ij}(x)f)$ and $(x,g_j^{-1}(x)f)$ are identified in the fibre bundle $E_{\gamma'}$, which means that h defines a homeomorphism $E_\gamma \longrightarrow E_{\gamma'}$. Hence, we see that the fibre bundle E is determined by the class of cohomology in $H^1(U,G)$ up to isomorphisms.

3. Divisors, line bundles and the Riemann Roch Theorem[28]:

A divisor D on a compact Riemann surface Σ (or an algebraic curve C) is a finite sum $D=\sum n_i p_i$ of points $p_i \in C$ with multiplicities n_i. The set of divisors on C form an abelian group denoted by Div(C). D is called effective if $n_i \geq 0$, for all i. A degree of a divisor D is defined by the following map;

$$\deg : \mathrm{Div}(D) \longrightarrow Z, \qquad \deg(D) = \sum_i n_i$$

Let U be an open subset of C, and f a non identically vanishing meromorphic function on any connected component of U, then f defines a divisor $(f)=\sum_{i\in C} \upsilon_i(f)p_i$, on C , where $\upsilon_i(f)$ corresponds to the order of f at p_i which is defined by

$$\mathrm{order}_i(f) = \begin{cases} k & \text{if f has a zero of order k at } p_i \\ -k & \text{if f has a pole of order k at } p_i \end{cases} .$$

Since a meromorphic function f on C has as many zeros as poles, therefore deg(f)=0.

A divisor D∈Div(C) is called principal if there exists a non-vanishing meromorphic function f such that D=(f), the set of principal divisor is denoted by $\mathrm{div}_p(C)$.

The set of divisors of degree zero on C is denoted by $\mathrm{div}_0(C)$,

$div_0(C) \subset div(C)$ and $div_p(C) \subset div_0(C)$, therefore one can form the following quotient: $Pic(C) \equiv Div(C)/Div_p(C)$ which is called the Picard group of C, also we can form its subgroup $Pic_0(C) \equiv Div_0(C)/Div_p(C)$.

An alternative definition of a divisor on a Riemann surface C is defined by assigning to each open set u_i of the covering $\{u_i\}$ of C, a meromorphic function on f_i such that $f_{ij}=f_i/f_j$ is a non-vanishing holomorphic function on $U_i \cap U_j$. Clearly $\{f_{ij}\}$ defines a 1-cocycle in the multiplicative sense, namely $f_{ij}.f_{jk}.f_{ki}=1$. Hence a divisor defines an element of $H^1(C,\mathcal{O}^*)$ where $\mathcal{O}^*$ is the multiplicative sheaf of the non-vanishing holomorphic functions. From example (ii) above, one concludes that $H^1(C,\mathcal{O}^*)$ corresponds to the set of all line bundles $\mathcal{L}$ over C which has a structure of a group, the group operation being the tensor product of line bundles, and the inverse is given by dual line bundle $\mathcal{L}^{-1}$. As a consequence of the alternative definition of divisors, $Pic(C)=H^1(C,\mathcal{O}^*)$.

From the following exact sequence;

$$0 \longrightarrow Z \xrightarrow{\hookrightarrow} \mathcal{O} \xrightarrow{e} \mathcal{O}^* \longrightarrow 0$$

where $e(f)=\exp 2\pi i f$, we have the associated exact cohomology sequence

$$H^1(c,Z) \longrightarrow H^1(c,\mathcal{O}) \longrightarrow H^1(c,\mathcal{O}^*) \longrightarrow H^2(c,Z) \longrightarrow H^2(c,\mathcal{O})$$

where δ is the connecting homomorphism;

$$\delta : H^1(c,\mathcal{O}^*) \longrightarrow H^2(c,Z) \quad .$$

But on a Riemann surface one can show that $H^2(C,\mathcal{O})=0$, therefore the above exact cohomology sequence becomes;

$$0 \longrightarrow H^1(c,\mathcal{O})/H^1(c,Z) \longrightarrow H^1(c,\mathcal{O}^*) \xrightarrow{\delta} H^2(c,Z) \longrightarrow 0 \ .$$

This shows that the homomorphism δ is in fact an isomorphism which takes a line bundle $\mathcal{L}\in H^1(c,\mathcal{O}^*)$ into its first Chern class, $C_1(L)=\delta(L)\in H^2(C,Z)$, since $H^2(C,Z)\simeq Z$, hence $C_1(L)$ is integer-valued. It can be shown that $C_1(L)=\deg(L)$.

<u>Riemann Roch theorem</u>: Let $\mathcal{L}$ be a line bundle on a Riemann space C, and set $h^i(\mathcal{L})=\dim_C H^i(C,\mathcal{L})$ then

$$h^o(\mathcal{L}) - h^1(\mathcal{L}) = \deg\mathcal{L} - g + 1 \quad .$$

From the Serre duality for line bundles on Riemann surfaces, one has $H^i(C,\mathcal{L})\simeq H^o(C,\mathcal{L}^{-1}\otimes k)$, where k is the canonical line bundle. Therefore the final form of the Riemann Roch theorem is

$$h^o(\mathcal{L}) - h^o(k\otimes\mathcal{L}^{-1}) = \deg\mathcal{L} - g + 1$$

since $h^o(k\otimes\mathcal{L}^{-1})$ is non-negative, we obtain the Riemann Roch inequality;

$$h^o(\mathcal{L}) \geq \deg\mathcal{L} - g + 1$$

In particular, if $\deg(\mathcal{L}) \geq g + 1$, then $h^o(C,\mathcal{L}) \geq 2$ and there exist at least two linearly independent sections s_1,s_2 of $\mathcal{L}$ over C such that $f=s_1/s_2$ is a non-constant meromorphic function on C. Thus the Riemann Roch theorem, for Riemann surfaces gives imformation concerning the existence of meromorphic functions with given divisor on C.

<u>**Example**</u>: If $\mathcal{L}$=K, $h^o(k)=g$, then by the Riemann Roch theorem, the degree of the canonical line bundle K is 2g-2, and as a consequence the spin bundle $L=K^{1/2}$ has degree g-1.

Finally, we would like to make the following remark concerning the relationship between the index of the Cauchy Riemann operation coupled to a holomorphic line bundle $\mathcal{L}$ on C and the Riemann Roch theorem. The index of $\bar{\partial}_{\mathcal{L}}$ is defined by

$$\text{index}(\bar{\partial}_{\mathcal{L}}) \equiv \dim\ker\bar{\partial}_{\mathcal{L}} - \dim\operatorname{coker}\bar{\partial}_{\mathcal{L}}$$

Now the vector space $\ker\bar{\partial}_{\mathcal{L}}$, is the space of holomorphic sections of $\mathcal{L}$, and $\operatorname{coker}\bar{\partial}=\ker\bar{\partial}^{*}$ is the space of antiholomorphic differential 1-forms, and is isomorphic to the space of holomorphic differential 1-form. So $\operatorname{coker}\bar{\partial}_{\mathcal{L}}=\ker\bar{\partial}^{*}_{\mathcal{L}\otimes k}\simeq\ker\bar{\partial}_{\mathcal{L}\otimes k}$

$$\therefore \operatorname{index}(\bar{\partial}_{\mathcal{L}}) = \dim \ker\bar{\partial}_{\mathcal{L}} - \dim \operatorname{coker}\bar{\partial}_{\mathcal{L}}$$

$$= h^{\circ}(\mathcal{L}) - h^{\circ}(K\otimes\mathcal{L}^{-1})$$

$$= \deg(\mathcal{L}) - g + 1$$

i.e., the Riemann Roch theorem in this case reduces to the index theorem for $\bar{\partial}_{\mathcal{L}}$.

B. THE MODULI SPACE[48]

1. Introduction:

The concept of moduli arieses in connection with classification problems in algebraic geometry. The basic set that enters in moduli theory is the collection of algebraic objects modulo the obvious equivalence relation (isomorphisms) of these objects. The main aim is to give, some algebraic-geometrical structure to this set.

The origin of the moduli space of curves, goes back to the theory of elliptic functions where one shows that there is a continuous family of such function whose parameter space is the field of complex numbers, C.

Riemann in his famous memoires (1857) gave a heuristic argument that the space of conformally inequivalent Riemann surfaces Σ of fixed genus $g\geq2$ has a complex dimension of 3g-3.

It was during the second world war that the German mathematician Teichmüller gave a rigourous proof for Riemann's conjecture. By using the Riemann-Roch theorem, he was able to identify the 3g-3 complex-parametrers as the dimension of the space of quadratic differentials. Actually, it turns out that the so called Teichmüller space T_g is a complex ball in C^{3g-3}. The moduli space $\mathcal{M}_g$ is the quotient $\mathcal{M}_g=T_g/\Gamma_g$, where Γ_g is the group of components of orientation preserving diffeomorphisms of Σ (the mapping class group).

2. Families, and deformation therory[49]:

In this section, we review briefly the concept of a family, and the Kodaira-Spencer-Kuranishi deformation theory in which one studies complex structures "close" to a given one.

let χ and S be complex spaces, and let $\pi:\chi \longrightarrow S$ be a surjective proper holomorphic map. A holomorphic family of compact complex manifolds consistes of the triplet (χ,π,S) such that

(i) Every fibre $\chi_s=\pi^{-1}(s)$, $s\in S$, is a connected complex manifold.

(ii) π is a simple (smooth) holomorphic map, i.e., for each point $x\in\chi$, there is a neighbourhood U, an open subset of $V\subset C^n$ and a biholomorphic map $\eta:U\longrightarrow\pi(U)\times V$ such that the diagram

$$\begin{tikzcd} U \arrow[rr, "\eta"] \arrow[dr] & & \pi(U)\times V \arrow[dl] \\ & \pi(U) & \end{tikzcd}$$

is commutative.

Sometimes, a family is denoted by $\{\chi_s\}_{s\in S}$ instead of (χ,π,S). If the parameterizing space S of the family $\{\chi_s\}_{s\in S}$ is connected, then we say that χ_t is a deformation of χ_s for any $s,t\in S$.

One can prove that all the fibres are diffeomorphic to each other and have the same real analytic structure. Therefore, one can regard the complex structures of the fibres to be complex structures on a given compact differentiable manifold that we denote by $\underline{X}$. If X is a complex structure on $\underline{X}$, then a holomorphic family (χ,s,π) is said to be a holomorphic deformation of X if there exists an isomorphism, $i:X\longrightarrow\chi_{s_o}$ where χ_{s_o} is the fibre at a reference point s_o in S. Such a deformation is denoted by (χ,s,π,s_o,X,i).

<u>Example</u>: Let T_g be the Teichmüller space of compact Riemann surfaces of genus $g\geq 2$. For $s\in T_g$, let X_s be the compact Riemann surface corresponding to t. Then $\{X_s\}_{s\in Tg}$ is a family of compact Riemann surfaces of genus g. This familly has the following properties:

(i) For any compact Riemann surface Σ of genus g, there is a point

$s \in T_g$ such that, Σ is biholomorphic to X_s.

(ii) For any reference point $\{p\} \in T_g$, there is an open neighbourhood U of $\{p\}$ in T_g such that X_s is not biholomorphic to X_p $\forall s \in U - \{p\}$.

<u>Construction of the Kodaira-Spencer map[49] and the dimension of the moduli-space of algebraic curves, (compact Riemann surfaces), $\mathcal{M}_g$</u>:
Let $(\chi, \pi, B) : \{C_s\}_{s \in B\lambda}$ be a family of compact Riemann surfaces parameterized by a ball $B_\lambda = \{s \in C^m, |s| < \lambda\}$. We can fix 0 as a reference point in B_λ, and set $\pi^{-1}(0) = C_o = i(C)$. From now on we will not distinguish between C_o and C. Let $\mathcal{U}_i = U_i \times B_\lambda$ be a covering of χ (where $\{U_i\}$ is an open covering of C) with coordinates (z_i, s^μ). On each intersection $\mathcal{U}_i \cap \mathcal{U}_k$, one obviously has $z_i = f_{ik}(t_k, s^\mu)$

Now given a tangent vector $v = v^r \partial / \partial s^r \in T_s B_\lambda$ one can construct the holomorphic vector-valued one-cochain $\theta_{ik}(s)$ on $C_s \cap \mathcal{U}_k$ by

$$\theta_{ik}(s) = v^r \frac{\partial}{\partial s^r} f^\alpha_{ik}(z_k, s) \frac{\partial}{\partial z^\alpha_i}$$

From the cocycle condition $f^\alpha_{ik}(z_k, s) = f^\alpha_{ij}([f_{jk}(z_k), s], s)$, one gets a cocycle condition

$$\theta_{ik} = \theta_{ij} + \theta_{jk} \qquad ;$$

on the θ_i's. Finally, we can associate to v the cohomology class $[\theta^1_{ik}(s)]$ in $H^1(C_s, \Theta_s)$ where Θ is the sheaf of germs of holomorphic vector fields. We get in this way a map: $P_s : T_s B_\lambda \longrightarrow H^1(C_s, \Theta_s)$. Called the Kodaira-Spensor homomorphism. The deformation $(\chi, B, \pi, C, \imath)$ is complete

iff P_s surjective

(uni)versal iff P_s is an isomorphism.

Now to compute the dimension of $\mathcal{M}_g$ it is enough to compute the dimension of the base of a versal deformation, i.e. $\dim_C H^1(C_s, \Theta_s)$.

We can now compute the dimension of the moduli space $\mathcal{M}_g$ of algebraic curves which amounts to compute the dimension of tangent bundle to $\mathcal{M}_g$. Now by Serre duality $H^1(C,\Theta)=H^1(C,\omega_c^{-1})=H^o(C,\omega_c^2)$ where ω_c is the canonical bundle (the tangent bundle). Riemann Rock theorem for a line bundle, L on a curve C reads;

$$\dim H^o(C,L) - \dim H^o(C,\omega_c \otimes L^{-1}) = \deg L - g + 1$$

for $L=\omega_c^2$ the term $\dim H^o(C,\omega_c \otimes L^{-1})=0$ by the Kodaira vanishing theorem, therefore

$$\dim \mathcal{M}_g = H^o(C,\omega_c^2) = 3g-3$$

3. Line bundle over the moduli space[50,48]:

Let $\mathcal{M}_g^o$ denotes the moduli space of smooth algebraic curves without automorphisms. One can show that it is a complex manifold. On $\mathcal{M}_g^o$ one has a universal family of automorphism-free curves $\pi : X_g^o \longrightarrow \mathcal{M}_g^o$ whose fibre at $[c] \in \mathcal{M}_g^o$ is a representative curve of the set of equivalent classes of Riemann surfaces [c].

The hodge bundle: The differential $d\pi$ maps TX_g^o onto $T\mathcal{M}_g^o$, and its kernel is called tha relative tangent bundle T_x, i.e., we have an exact sequence

$$0 \longrightarrow T_{X^o/\mathcal{M}^o} \longrightarrow T\mathcal{M}^o \longrightarrow 0$$

the dual $\omega_{X^o/\mathcal{M}^o} = \mathcal{O}(T_{X^o/\mathcal{M}^o})^{V}$ is the relative canonical sheaf.

The hodge sheaf $R^o\pi_*\omega_{X^o/\mathcal{M}^o}$ is by definition the sheaf associated with the presheaf on $\mathcal{M}_g^o$ given by

$$\mathcal{M}_g^o \supset U \longrightarrow H^o(\pi^{-1}(u), \omega_{X^o/m^o})$$

clearly $R^o\pi_*\omega$ is the locally free sheaf of rank g, which may be thought of as the sheaf of sections of the vector bundle $\bigcup_{[m]\in mg^o} H^o(C_n, \omega_c)$ whose fibre at C_m is the space of abelian differentials on C_m itself.

Having given the definition of the Hodge bundle, we next consider the universal family of smooth curves, $f: X_g^o \longrightarrow \mathcal{M}_g^o$ and find what the natural line bundle on $\mathcal{M}_g^o$ will be. The fibres here are smooth curves, therefore the dualizing sheaf ω_c and the sheaf of holomorphic 1-forms on C are isomorphic: $\omega_c \approx \Omega_c^1$. The hodge bundle E on $\mathcal{M}_g^o$ has $H^o(C,\omega_c)$ as a fibre of the space of differentials, and because of the Serre duality $H^o(C,\omega_c) \simeq H^1(C,\mathcal{O})$. $H^o(C,\omega_c)$ is a g-dimensional vector space, therefore the natural line bundle L on $\mathcal{M}_g^o$ would be the determinant of this Hodge bundle, i.e., $L = \bigcup_{[c]} \Lambda^g H^o(C,\omega_c)$.

By combining Harer theorem with that of Arbarello and Cornalba[51,52] the Hodge bundle L over $\mathcal{M}_g^o$ has the following non-trivial properties:

(i) L is non-trivial on $\mathcal{M}_p^o$, i.e., its first Chern class $C_1(L) = \lambda \neq 0$.

(ii) λ generates the group $H^2(\mathcal{M}_g^o, Z) = Z$, so that the first Chern class of any other line bundle L′ on $\mathcal{M}_g^o$ is an integral multiple of λ, i.e., $L' = L^n$ for some integer n.

On χ_g^o we could consider another sheaves as the n^{th} power of the canonical sheaf, i.e., $H^o(C,\omega_c^{\otimes n})$. For any such line bundle ($n \geq 1$) over the universal family X_g^o, Riemann-Roch theorem tells us that $H^o(C,\mathcal{L}|_{Cm}) \simeq C^n$, i.e., independent of $m \in \mathcal{M}_g^o$. So the collection $\bigcup_{[c]} H^o(\pi^{-1}(c),\mathcal{L})$ over $\mathcal{M}_g^o$ is a vector bundle denoted by $\pi_!\mathcal{L}$, and called the push-forward of the relative line bundle $\tilde{\mathcal{L}}$ on X_g^o. The corresponding determinant line bundle on $\mathcal{M}_g^o$ would be $\bigcup_{[c]} \Lambda^n H^o(\pi^{-1}(c),\mathcal{L})$, and is denoted by $\det \pi_! \tilde{\mathcal{L}}$.

An interesting example that comes up in string theory is the construction of the push-forward of the relative dualizing sheaf squared, $\pi_!(\tilde{\omega}_c^{\otimes 2})$. Since the fibre of $\pi_!(\tilde{\omega}_c^{\otimes 2})$ at $[c]\in\mathcal{M}_g^o$ is given by $H^o(C,\omega_c^{\otimes 2})$ which is the holomorphic cotangent space to $\mathcal{M}_g^o$, therefore $\pi_!(\tilde{\omega}_c^{\otimes 2})$ corresponds to the cotangent bundle to $\mathcal{M}_g^o$, and $\det(\pi_!\tilde{\omega}_c^{\otimes 2})$ is the canonical line bundle of the moduli space $K_{\mathcal{M}_g^o}$ (i.e. locally generated by $\underset{\sim}{\ }\ \psi^1\Lambda\ldots\Lambda\psi^{3g-3}$) where ψ^i are local basis in $T^*\mathcal{M}_g^o$. From the properties of the hodge bundle L, we have $K_{\mathcal{M}_g^o}=L^m$ for some m.

To compute the first Chern class we apply relative G.R.R. theorem to our case. If $\tilde{E}$ is the relative line bundle to X_g^o and $\tilde{T}$ is the relative tangent bundle to X_g^o, the the relative G.R.R. theorem reads as follows;

$$Ch((\pi,\tilde{E}) = \pi_*(Ch(\tilde{E}).Td(\tilde{T})) \quad .$$

Using the fact that $\tilde{T}=\tilde{\omega}^{-1}$ then

$$Td(\tilde{T}) = 1-1/2C_1(\tilde{\omega})+1/12C_1^2(\tilde{\omega})+\ldots..$$

$$Ch(\tilde{E}) = 1+C_1(\tilde{E})+1/2C_1^2(\tilde{E})+\ldots.. \quad .$$

If we think of π_* as an integration along the fibre of π , then keeping only four forms on the right hand side, then

$$C_1(\pi_1\tilde{E}) = \pi_*(1/2C_1(\tilde{E})-1/2C_1(\tilde{\omega}).C_1(\tilde{E})+1/12C_1^2(\tilde{\omega})) \quad .$$

Now we apply this formula to the hodge bundle $L=\pi_!\tilde{\omega}$ and tothe canonical line bundle of the moduli space, $K_{\mathcal{M}_g^o}$. The first Chern class of the hodge line bundle is $\lambda=C_1(L)=\pi_*(1/12C_1^2(\tilde{\omega}))$. Using the fact that $C_1(K_{\mathcal{M}_g^o})=C_1(\det\pi_!\tilde{\omega}^{\otimes 2})=C_1(\pi_!\tilde{\omega}^{\otimes 2})$, with $C_1(\tilde{\omega}^{\otimes 2})=2C_1(\tilde{\omega})$, we get

$$C_1(K_{\mathcal{M}_g^o}) = C_1(\pi_!\tilde{\omega}^{\otimes 2}) = \pi_*(13/12C_1^2(\tilde{\omega}))=13\lambda$$

4. The determinant line bundles and the Quillen metric[53]:

Let E be a complex vector space of dimension n, and P be an endomorphism of E, then the determinant of the operator P is the linear map is defined by

$$\det P(e_1 \wedge \ldots \wedge e_n) = C\, e_1 \wedge \ldots \wedge e_n$$

where $\{e_i\}$ is a basis in E, and C is a complex number called the determinant of P. If e_i is an eigenvector basis for P, then C is the product of the eigenvalues. Note that $\det P: \mathrm{DET}E \longrightarrow \mathrm{DET}E$ where $\mathrm{DET}E = \Lambda^n E$, isthe highest exterior power of E. If the operator P is not an endomorphism but a linear map, $P: E \longrightarrow F$, where F is another complex vector space with dimE=dimF, then we have an induced map, the determinant map $\det P: \mathrm{DET}E \longrightarrow \mathrm{DET}F$, in this case detP is not a number. Also we cannot define an eigenvalue problem for P, since the operator P is in Hom(E,F), and $\det P \in (\mathrm{DET}E, \mathrm{DET}F) = (\mathrm{DET}E)^* \otimes (\mathrm{DET}F)$. So $\det P = C_p (f'_1 \wedge \ldots f'_n) \otimes (f^*_1 \wedge \ldots f^*_n)$, where $\{f'_i\}$ is basis and $\{e^*_i\}$ is the dual basis to $\{e_i\}$.

Consider now the case in wich E and F are vector bundles over a parameter space X, and assume the general case in which E and F do not have the same rank. Suppose that the operator $P_x: E_x \longrightarrow F_x$ varies smothly over $x \in X$. The operator P_x is not invertible, hence, the above construction fails. However, if we write $E_x = (\ker P_x) + E'_x$, $F_x = (\mathrm{coker} P_x) + F'_x$ where E'_x and F'_x have the same dimension, then the above construction for the operator P_x restricted to these spaces holds. It turns out that in this case $(\mathrm{DET}E_x)^* \otimes (\mathrm{DET}F_x) \simeq (\mathrm{DET}\ \ker P_x)^* (\mathrm{DET}\ \mathrm{coker} P_x)$; i.e., $\det P_x$ is a point in $(\mathrm{DET}\ \ker P_x) \overset{*}{\otimes} (\mathrm{DET}\ \mathrm{coker} P_x)$.

Now $\underset{x}{\cup}(\mathrm{DET}\ \ker P_x)^{*}\otimes(\mathrm{DET}\ \mathrm{coker} P_x)\equiv \mathrm{DET}P$ is the "determinant" line bundle of the family P whose section is detP. For example, let us consider the determinant line bundle of the Cauchy Riemann operator $\bar{\partial}$ over a family of Riemann surfaces. For that let $\pi:X\longrightarrow S$ be a holomorphic family of compact Riemann surfaces, and let E be a holomorphic vector bundle over X. Then $F=E\otimes\tilde{\bar{\omega}}$, where $\tilde{\omega}$ is the relative conjugate cotangent bundle, sometimes denoted by $\bar{\omega}_{X|S}$. At the fibre level, the Cauchy Riemann operator $\bar{\partial}$ with values in E is defined by;

$$\bar{\partial}_{E,S} : C^{\infty}(\pi^{-1}(s),E)\longrightarrow C^{\infty}(\pi^{-1}(s),E\otimes\tilde{\bar{\omega}}) \quad .$$

We denote the family of these operator by $\bar{\partial}_E$. Therefore,

$$\mathrm{DET}\bar{\partial}_E \simeq \underset{s}{\cup}(\mathrm{DET}\ \ker\bar{\partial}_s)^{*}\otimes(\mathrm{DET}\ \mathrm{coker}\bar{\partial}_s)$$

$$\simeq (\Lambda^{h^o}E_o)^{*}\otimes(\Lambda^{h^1}E_1)$$

where

$$h^{o} = \dim \ker\bar{\partial}_{E,S} = \dim H^{o}(\pi^{-1}(s),E)$$

$$h^{1} = \dim \mathrm{coker}\bar{\partial}_{E,S} \simeq \dim H^{1}(\pi^{-1}(s),E)$$

The Quillen metric on the determinant line bundle, $\mathrm{DET}\bar{\partial}_E$: As before let $\pi:X\longrightarrow S$ be a holomorphic family of compact Riemann surfaces, and E a holomorphic vector bundle over X, Let $\|\ \|_T$ be a C^{∞} hermitian metric on the relative tangent bundle $\tilde{T}$ on X and let $\|\ \|_E$ be a C^{∞} hermitian metric on E. Then on the space of differential forms $\Omega^{o,q}(E)$ with values in E , we have an inner product which in turn implies that we can define the formal adjoint Cauchy Riemann operator $\bar{\partial}_E^{\dagger}$ associated to $\bar{\partial}_E$ and its Laplacian, $\Delta_E=\bar{\partial}_E^{\dagger}.\bar{\partial}_E$.

The Quillen metric on $\mathrm{DET}\bar{\partial}_E=(\Lambda^{\max}\ker\bar{\partial}_E)\otimes(\Lambda^{\max}\ker\bar{\partial}_E^{\dagger})$, is obtained by multiplying the L_2-metric $\| \ \|_{L^2}$ on $\mathrm{DET}\bar\partial_E$ by the regularized zeta determinant $\det'\Delta_E=\exp(-\zeta'(0))$, i.e., $\| \ \|_Q^2=(\det\bar\partial_E^*\bar\partial_E)\| \ \|_{L^2}^2$. In particular we can consider the Quillen metric associated with determinant line bundles $\mathrm{DET}\bar\partial_n=(\Lambda^{\max}\ker\bar\partial_n)^{-1}\otimes(\Lambda^{\max}\ker\bar\partial_n^\dagger)$, where

$$\bar\partial_n:\Gamma(K^{\otimes n})\longrightarrow\Gamma(K^{\otimes(n-1)}) \qquad ,$$

and $K^\otimes$ is the n^{th} canonical bundle on a Riemann surface in this case, $\| \ \|_Q^2=(\det\bar\partial_n^*\bar\partial_n)_N\| \ \|_{L^2}$. Recalling that $\ker\bar\partial_o=\{$global holomorphic section on a Riemann surface$\}\approx C^g$, and $\mathrm{coker}\bar\partial_o=\{$holomorphic 1-form$\}$. Thus the Quillen metric on $(\Lambda^{\max}\ker\bar\partial_o)^{-1}\otimes(\Lambda^{\max}\ker\bar\partial_o^\dagger)$ is given by

$$\|(\omega_1\Lambda\ldots.\Lambda\omega_g)^{-1}\otimes 1\|_Q^2 = \frac{\det'\Delta_o}{\det(\omega_i,\omega_j)\int_\Sigma\sqrt{g}}$$

where $\{\omega_i\}$ is a basis of holomorphic one-forms, and 1 is a constant section, $(1,1)_g=\|1\|_g=\int_\Sigma\sqrt{g}$.

As an application let us consider the Polyakov bosonic string (54) which is described by the following partition function for genus, $g\geq 2$

$$Z_g = \int\prod_{i=1}^{3g-3} d\Phi_i\Lambda d\bar\Phi_i\left|\det(\omega_i,\omega_j)\right|^{-d/2}.\left(\frac{\det'\Delta_o}{\det(\omega_i,\omega_j)\int_\Sigma\sqrt{g}}\right)$$

$$.\frac{\det'_\zeta\Delta_o}{\det(\Phi_i,\Phi_j)}$$

where $\{\Phi_i\}$, $i=1,\ldots 3g-3$ is basis for $H^o(\Sigma,\omega^2)$, and $\{\omega_i\}$ $i=1,\ldots g$ is an orthonormal basis for $H^o(\Sigma,\omega)$, and d is the dimension for the target space which is assumed to be R^d.

By using the Quillen metric, the partition function can be rewritten as;

$$Z_g =\int\prod_{i=1}^{3g-3} d\Phi_i\Lambda d\bar\Phi_i\left|\det(\omega_i,\omega_j)\right|^{-d/2}.(\|\det\bar\partial_o\|_Q^{-d}.\|\det\partial_2\|_Q^2) \qquad .$$

The quantized bosonic string described by Z_g will not suffer from conformal anomalies if the integrand in Z_g is a holomorphic function rather than a section. This will happen when

$$C_1[((\mathrm{DET}\bar\partial_o)^{\otimes(-d)},\| \ \|_Q)\otimes((\mathrm{DET}\bar\partial_2)^{\otimes 2},\| \ \|_Q)] = 0$$

$$-(d-26)(\lambda) = 0 \qquad .$$

when d is equal to 26, the partition function is

$$Z_g = \int \prod_{i=1}^{3g-3} d\Phi_i \Lambda d\bar\Phi_i \left|\det(\omega_i,\omega_j)\right|^{-13}\left|F(\Phi)\right|^2 \qquad ,$$

where $F(\Phi)=((\omega_1\Lambda\ldots\Lambda\omega_g)^{-13}.(\Phi_1\Lambda\ldots\Lambda\Phi_{3g-3})^{-1})$ is a global holomorphic section of $(\mathrm{DET}\bar\partial_2)\otimes(\mathrm{DET}\bar\partial_o)^{\otimes(-13)}=K\otimes\lambda^{-13}$. The form $F(\Phi)$ is called the Mumford form, so the bosonic string measure in d=26 is the square modulus of the global holomorphic section Φ of $K\otimes\lambda^{-13}$.

Next we consider the fermionic string(55) whose action is given by

$$S = \int \partial X\Lambda\bar\partial X + \bar\psi\partial\psi \qquad ,$$

where $\bar\partial_n:K^n \longrightarrow K^n\otimes\Lambda^{0,1}=K^{n-1}\in Z$, $\partial:\bar\partial_{1/2}:K^{1/2}\longrightarrow K^{-1/2}$. The differential operator ∂ depends on the choice of $\sqrt{K}$ (i.e. on the choice among the 2^{2g} spin structures). The partition function for the fermionic string is given by

$$W = (\det\bar\partial_o)^{-d/2}(\det\bar\partial_{1/2})(\det\bar\partial_2)(\det\bar\partial_{3/2})^{-1}$$

To obtain the critical dimension for the fermionic string we use the identity $C_1(\mathrm{DET}\ \bar\partial_n)=(6n^2-6n+1)\lambda$. the partition function W is a section of

$$(\mathrm{DET}\bar\partial_o)^{\otimes -d/2}(\mathrm{DET}\bar\partial_{1/2})(\mathrm{DET}\bar\partial_2)(\mathrm{DET}\bar\partial_{3/2})^{\otimes -1} \qquad ,$$

so W has no conformal anomalies if

$$C_1[(\mathrm{DET}\bar\partial_o)^{\otimes -d/2}(\mathrm{DET}\bar\partial_{1/2})(\mathrm{DET}\bar\partial_2)(\mathrm{DET}\bar\partial_{3/2})^{\otimes -1}] = 0$$

$$0 = (-d/2-d/4+13-11/2)c_1(\lambda)$$

i.e. $d=10$.

N.B.: Note that in this case all determinant line bundles should be considered on the moduli space of Riemann surfaces with spin structure. As it is well known that these give a finite covering space of moduli spaces. Accordingly the vanishing of conformal anomalies is a priori not sufficient to guarantee absence of the mapping class group anomalies.

IV. Theta functions, Dirac determinants and bosonization

1. The geometrical meaning of the theta function[28]:

Before giving definitions of theta functions and their properties, we first review some geometrical aspects of the theta function[28].

Let $T=C^n/\Lambda$ be an n-dimensional complex torus with period lattice Λ, i.e., $\Lambda=Z\gamma_1+\ldots+Z\gamma_{2n}$, where the γ_i's, $i=1,\ldots 2m$, $2m=n$ are the generators of the lattice. A holomorphic function $f:C^n \longrightarrow C$ is called a theta function with respect to Λ, if it satisfies the condition

$$\text{(i)} \quad f(z+g) = \mu g(z)f(z) \qquad \forall z\in C^n, g\in\Lambda.$$

where the "theta factors" $\mu g:C^n \longrightarrow C^*$ are functions of the form

$$\text{(ii)} \quad \mu g(z) = e^{L_g(z)},$$

and $L_g(z)=\sum_{k=1}^{n} a_{gk} z_k + b_g$ is an affine linear function. From (i), for any two elements $g\ g'\in\Lambda$, $f(z+g+g')$ is given by

$$f(z+g+g') = \mu_{g+g'}(z)f(z) = \mu_g(z+g')f(z+g')$$

$$= \mu_g(z+g')\mu_{g'}(z)f(z) \quad .$$

Therefore, if $f(z)\neq 0$ then the μ_g's satisfy the relations

$$\mu_{g+g'}(z) = \mu_g(z+g')\mu_{g'}(z) \qquad \text{for all } z\in C^n, g, g'\in\Lambda.$$

Example: Take n=1, so $\Lambda\subset C^*$, and can be written as $\Lambda=Z+Z\tau$, $\mathrm{Im}\tau>0$. The series $\theta(z)=\sum_{n\in Z} e^{i\pi n\tau} e^{2\pi i n z}$ converges very rapidly with respect to Λ, and satisfies the above condition.

Given two theta functions f_1 and f_2 with $f_2 \neq 0$, then the function $F=f_1/f_2$ is periodic with respect to Λ, and therefore defines a meromorphic function on $T=C^n/\Lambda$. Usually $\mu_g \neq 0$, then a theta function is not periodic with respect to the lattice Λ, hence does not define a function in the usual sence. However, one can identify the theta function with a holomorphic section of a line bundle on the complex torus C^n/Λ.

To see that, first let us recall that a line bundle L is a vector bundle of rank 1 with fibres L_x isomorphic to the complex numbers, C. However, the isomorphism $L_x \simeq C$ is not uniquely determined, but is determined only up to a factor $\lambda \in C^*$. Thus a holomorphic section $s:T \longrightarrow L$ cannot be interpreted as a holomorphic function on $T=C^n/\Lambda$.

Now given two holomorphic sections $s_1, s_2 : T \longrightarrow L$ with $s_2 \neq 0$, the quotient $f=s_1/s_2$ is a well defined meromorphic function on T. In this case the indeteminacy cancels out.

Let $\mu_g : C^n \longrightarrow C^*, g \in \Lambda$ be a system of theta factors. we would like to associate to this system a line bundle L on $T=C^n/\Lambda$. For that, let us define on $C^n \times C$ an equivalence relation $(z,v) \simeq (z',v')$ if there exists an element $g \in \Lambda$ such that $z'=z+g$, $v'=\mu_g(z)v$. Set $L_\mu = C^n \times C/\sim$. The projection map $pr_1 : C^n \times C \longrightarrow C^n$ induces a map $\pi : L_\mu \longrightarrow T$, with all fibres isomorphic to C, and L_μ becomes a line bundle over T. By construction the holomorphic sections of L are in one-to-one correspondance with holomorphic functions $f:C^n \longrightarrow C^*$ satisying the theta function condition. With the geometrical meaning given above to the theta functions as sections of a holomorphic line bundle on an n-dimensionel complex torus, now we give the definition and the properties of the theta function associated with the period matrix $\Omega_{ij} = \int_{b_i} \omega_j$, where $\{\omega_i\}$ i=1,...g is a basis of C-vector-spaces of holomorphic differentials on a Riemann surface Σ.

2. Theta functions and the Dirac deteminants:

The theta function is given by series expansion:

$$\theta(z|\Omega) = \sum_{n\in Z^g} \exp(i\pi {}^T n\Omega n+2\pi i n.z) \quad , \tag{1}$$

where $z=(z_1,\ldots z_g)$ and $\Omega=(\Omega_{ij})$ are coordinates on C^g and the siegel upper half-plane $\mathcal{H}_g$ respectively. Given a point $\Omega\in\mathcal{H}_g$, consider the lattice $\Lambda\Omega\subset C^g$ generated by $(1,\Omega)$, i.e., $\Lambda_\Omega=Z^g+Z^g\Omega$. So we can form the complex torus $Jac(\Sigma)=C^g/\Lambda_\Omega$. The theta function on it is holomorphic in $C^g\times\mathcal{H}^g$, and satisfies the quasi periodicity property:

$$\theta(z+n+{}^T m.\Omega,\Omega) = \exp(\pi i {}^T m.\Omega.m-2\pi i {}^T m.z)\theta(z,\Omega) \tag{2}$$

for $n,m\in Z^g$. This means that when we shift the argument z by an element in the lattice Λ_Ω, the theta function is mulitplied by a simple factor. However, when we shift by half the lattice, namely z goes into $z+n$, $n\in Z^g$, the theta function is periodic. So if $f(z)$ is an entire function such that

$$f(z+m) = f(z) \tag{3}$$

$$f(z+\Omega m) = \exp(-\pi i {}^T m\Omega m-2\pi i {}^T m.z)f(z) \tag{4}$$

then one can show that $f(z)$[37] is a theta function up to a costant, i.e. $f(z)=\text{const.}\theta(z,\Omega)$. Later on we will need this fact when we come to the Dirac determinant.

Another important property of the theta function is its evenness with respect to z, $\theta(-z|\Omega)=\theta(z|\Omega)$, which can be proved easily by reindexing the summation.

A theta function with characteristics: Given a, $b\in R^g$ a theta function with characteristics a, b is defined by;

$$\theta\begin{bmatrix}a\\b\end{bmatrix} = \sum_{n\in Z^g} \exp\{i\pi^T(n+a).\Omega(n+b)+2i\pi(n+a).(z+b)\}$$

$$= e^{i\pi a\Omega a+2i\pi a(z+b)}\theta(z+\Omega a+b|\Omega) \qquad (5)$$

So the theta functions with characteristics are nothing but the translation of $\theta(z|\Omega)=\theta\begin{bmatrix}0\\0\end{bmatrix}(z|\Omega)$ by an element in the lattice L_Ω times an elementary exponential factor. For $\epsilon_1^i,\epsilon_2^j\in\{0,1/2\}$ we define

$$\epsilon_1 = \begin{pmatrix}\epsilon_1^1\\ \vdots\\ \epsilon_1^g\end{pmatrix} \quad , \epsilon_2 = \begin{pmatrix}\epsilon_2^1\\ \vdots\\ \epsilon_2^g\end{pmatrix}$$

where $i,j=1,\ldots,g$. Then one can show that

$$\theta\begin{bmatrix}\epsilon_1\\ \epsilon_2\end{bmatrix}(-z|\Omega) = (-1)^{4\epsilon_1\epsilon_2}\,\theta\begin{bmatrix}\epsilon_1\\ \epsilon_2\end{bmatrix}(z|\Omega)$$

This means that $\theta\begin{bmatrix}\epsilon_1\\ \epsilon\end{bmatrix}(z|\Omega)$ are even or odd according to whether $4.\epsilon_1\epsilon_2=0$ (mod2) or 1(mod2), respectively.

Dirac determinants: It was shown explicitly in ref.(26) that by combining the Quillen theorem, gauge invariance and the vanishing Riemann theorem, the ζ-function regulated Dirac determinant can be expressed in terms of the theta divisor Θ.

The Quillen theorem[27] shows that, when a given differential operator D_x on a Riemann surface varies holomorphically with x, where x is a point in the complex parametrizing manifold, then the determinant line bundle: DETD$\longrightarrow$X has a holomorphic structure, and if D_x has no index, then the ζ-function regulated determinant satisfies:

$$\det D_x^\dagger D_x = e^{-q(x,x)}|\det D_x|^2 \qquad , \qquad (6)$$

where $\det D_x$ is a holomorphic function on X, and q is the quadratic function defined by choosing an origin in family of differential operators;

$$q(D)=\|D-D_o\|^2 = i/2\pi\int_\Sigma tr(D-D_o)^\dagger\Lambda(D-D_o) \quad . \tag{7}$$

To express the spin structure-dependence of the determinants of the Dirac operators, one needs to know how spin bundles (line bundles of degree g-1) are characterized. Holomorphic line bundles of degree zero are parametrized by unitary equivalence classes of flat connections, where the transition functions are constant. This corresponds to the first cohomology group with values in $U(1):H^1(\Sigma,U(1))=Pic_o(\Sigma)=R^{2g}/Z^{2g}$. The section s along the a_i cycle is identified with $e^{-2\pi i\Phi i}$ times s along a_i^{-1} and s along b_i with $e^{2\pi i\Theta}$ times s along b_i^{-1}.

Therefore a trivial holomorphic line bundle corresponds to a point on $Pic_o(\Sigma)$. So any holomorphic line bundle $\mathcal{L}$ is completely chracterized by any integer (its first Chern class or degree), and a point on $Pic_o(\Sigma)$. In particular all line bundles $\mathcal{L}$ of degree g-1 are given by $L_\alpha\otimes V(\Theta,\Phi)$, where L_α is a fixed bundle for some chosen spin structure α, and $V(\Theta,\Phi)$ is a flat holomorphic bundle. Since the dual spin bundle L_β^{-1} has degree 1-g, the square of the tensor bundle $L_\alpha\otimes L_\beta^{-1}$ is trivial. So spin structures are in one-to-one corresponence with half-points in $Pic_o(\Sigma)$, i.e.$(\Theta,\Phi)\in(1/2Z)/Z$.

The spin-structure dependence of the Dirac determinant in ref.(26) has been found by coupling the chiral Dirac operator on L_α^{-1} to a flat U(1) gauge field A, constructed in terms of the harmonic 1-forms:

$$A = 2\pi i\left(\sum_{i=1}^{g}\Theta^i\alpha_i - \sum_{i=1}^{g}\Phi^i\beta_i\right) \quad . \tag{8}$$

Writing the 1-forms α_i,β_i in terms of the Abelian differentials ω and their congugate $\bar{\omega}$, one can show that eq.(8) becomes;

$$A = 2\pi i(\Phi+\Omega\Theta).(\Omega.\bar{\Omega})^{-1}.\bar{\omega} + h.c. \tag{9}$$

Therefore, the chiral Dirac operators coupled to A takes the form,

$$D(z) = \nabla_z^{-1} + 2\pi i(\Phi+\Omega\Theta).(\Omega.\bar{\Omega})^{-1}.\bar{\omega} \quad , \qquad (10)$$

where $z=\Phi+\Omega\Theta$. If $(\Theta,\Phi)\in(1/2Z/Z)^{2g}$ Then the chiral operator $D(z)$ is explicitly spin-structure dependent.

Applying the Quillen theorem to $D(z)$, and using the identities

$$\int_{b_i} \omega_j = \Omega_{ij} \quad , \qquad 1/2i\int \bar{\omega} \wedge \omega = \mathrm{Im}\,\Omega \quad ,$$

one can show that

$$\det_\alpha D^\dagger(z)D(z) = e^{i\pi(z-z)(\Omega-\bar{\Omega})^{-1}(z-z)}|g(z)|^2 \quad , \qquad (11)$$

where $g(z)$ is holomorphic function of z. Shifting z by $z+n+\Omega m$ corresponds to well-defined U(1) gauge transformations on Σ given by

$$U(P) = \exp[-2\pi i(m\int_{P_o}^{P}\alpha - n\int_{P_o}^{P}\beta)]$$

Under this shift, the regulated determinant eq.(11) transforms into

$$e^{i\pi(z-z)(\Omega-\bar{\Omega})+i\pi m(\Omega-\bar{\Omega})}\left|g(z+n+\Omega m)\right|^2 \quad . \qquad (12)$$

But the whole expresion must be gauge invariant, since the ζ-function regularization is gauge invariant. Therfore to cancel the non-gauge invariant prefactor exponential, the holomorphic function $g(z)$ must transform as follows

$$g(z+n) = e^{2\pi i n.a}g(z) \quad ,$$

$$g(z+\Omega m) = e^{-2\pi i m.b}e^{-i\pi m\Omega m - 2\pi i m.z}g(z) \quad , \qquad (13)$$

where $e^{2\pi i n.a}$, $e^{-2\pi i m.b}$ are the unitary representation of Z^g. The behaviour of $g(z)$ is the same as the entire function $f(z)$ discussed above (see eq. 3. and 4.). Therefore, we conclude that

$$g(z) = \text{const.}\,\theta\begin{bmatrix} a \\ b \end{bmatrix}(z|\Omega) \tag{14}$$

where **a**,**b** are real characteristics to be determined.

The chiral Dirac operator D(z) is the same as Cauchy-Riemann operator $\bar{\partial}$ acting on sections of the line bundle $\mathcal{L}(z)$ parametrized by the Jacobian of Σ. Therefore to determine the holomorphic section of $\mathcal{L}(z)$ is equivalent to determining its divisor, i.e., the zeros of g(z). By using Riemann vanishing theorem, one can show that to any spin structure one can associate a symmetric translate of the Θ-divisor which are translated by points of order 2 and in particular some spin structures corresponds to the Θ-divisor itself, i.e., without a translate. Therefore, if we denot this particular spin structure by D_o, then

$$\det_{D_o} D^{\dagger}D = \text{const.}\left|\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0|\Omega)\right|^2 \quad , \tag{15}$$

and for other spin structures which correspond to the translate of the Θ-divisor, we have

$$\det_{D_\alpha} D^{\dagger}D = \text{const.}\left|\theta\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(0|\Omega)\right|^2 \tag{16}$$

by combinig the bosonization formula[26] and the G.R.R. theorem[28] one can show that the constant is given by

$$\text{const.} = \left(\frac{\det' - \nabla^2}{\int_\Sigma \sqrt{g} \quad \det\text{Im}\Omega}\right)^{-1/2} .$$

Therefore the fermionic partition function for a given spin structure α is;

$$\det_{D_\alpha} D^{\dagger}D = \left(\frac{\det' - \nabla^2}{\int_\Sigma \sqrt{g} \quad \det\text{Im}\Omega}\right)^{-1/2} \left|\theta\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(0|\Omega)\right|^2 \tag{17}$$

Note that on a torus T, the expresion for the Dirac determinant (fermionic partition function) can be obtained directly using the analytic torsion theorem on elliptic curves Ray and Singer[31].

The analytic torsion on a Riemann surface, for a non-trivial representation of $\pi_1(C)$ with hermitian metric is denoted by $T(C,\chi)$ and is defined as the positive root of $\mathrm{Log}T_p(C,\chi)=-1/2\zeta'(0)$. In this theorem the expression for the torsion was shown to be given by;

$$T_p(T,\chi) = \left|\frac{e^{\pi i v^2\tau}\theta_1(u-\tau v,\tau)}{\eta(\tau)}\right| \tag{18}$$

for p=0,1, where $\theta_1(v,\tau)=\theta(1/2-v,\tau)$, $\eta(\tau)=e^{\pi i\tau/12}\prod_{k=1}^{\infty}(1-e^{2\pi ik\tau})$ is the Dedekind function, and χ is a non-trivial character given by $\chi(m\tau+n)=e^{2\pi i(mu+nv)}$.

The determinant for the scalar Laplacian in terms of the ζ-function is;

$$\det{}'_{\zeta}\Delta = \det{}'_{\zeta}\bar{\partial}^{*}\bar{\partial} = e^{-\zeta'(0)} \quad . \tag{19}$$

The spin bundle on a torus is topologically trivial, since its degree is zero. Therefore the scalar Laplacian determinant and the spinor Laplacian on elliptic curves coincides, i.e., we have bosonization.

The explicit expresion for the fermionic partition function can be obtained by using the definition of a torsion on a Riemann surface

$$\mathrm{Log}T_p(T,\chi) = -\frac{1}{2}\zeta'(0)$$

thus

$$(\det{}'_{\zeta}\bar{\partial}^{*}\bar{\partial})_{\mathrm{fermionic}} = e^{-\zeta'(0)}$$

$$= \left|\frac{e^{\pi i v^2\tau}\theta_1(u-\tau v,\tau)}{\eta(\tau)}\right|^2 \tag{20}$$

using, $\theta_1(u-\tau v,\tau)=\theta(\tau v+1/2-u,\tau)$ and

$$\theta\left[{v \atop 1/2-u}\right](0|\tau) = e^{\pi i v^2\tau+2\pi i v(1/2-u)}.\theta(\tau v+1/2-u,\tau)$$

then eq.(20) can be written interms of a theta function with characteristics v and (1/2-u). However, if we want to make contact with the bosonization formula for g=1 obtained in ref.(26), we set Θ=v, and Φ=1/2-u, then one obtains;

$$(\det{}'_{\zeta}\bar{\partial}^{*}\bar{\partial}) = \left|\frac{\theta\left[\begin{smallmatrix}\theta\\ \Phi\end{smallmatrix}\right](0|\tau)}{\eta(\tau)}\right|^{2} \tag{21}$$

3. Explicit computation of the bose-fermi equivalence on Riemann surfaces of genus g:

3.1. Introduction:

Alvarez-Gaumè, Moore and Vafa[26] have shown that by summing over all instanton (soliton) sectors, the partition function of a single boson on a compact Riemann surface Σ of genus g with values in the circle U(1), (i.e., compactified on S^1), is equivalent to the partition of a single Dirac fermion with spin structures summed over. More explicitly, it was shown that

$$Z_{bose} = \sum_{(inst.)=m,n\,\in\, Z^{g}} Z(m,n)$$

$$= \sum_{\text{spin structures}} Z_{fermi} \quad \text{with}$$

$$Z(m,n) = \left(\frac{\det - \nabla^{2}}{\int_{\Sigma}\sqrt{g}}\right)^{-1/2} \exp(-E_{n,m}(\phi))$$

$$= Z_{(Quantum)} \cdot Z_{(inst)} \tag{1.1}$$

The factorization of Z_{bose} into the Quantum partition function Z_{Quant} and the instanton partition function Z_{inst} can be seen as a consequence of the fact that the Hessian[15] of the energy functional $E(\phi,g)$ does not depend on the instanton sectors. Therefore evaluating the partition function at one loop in the non-linear σ-model one obtains eq.(1.1). If, for simplicity, there are no twists ($\Theta=0$, $\Phi=0$) around the homology basis a_i, b_i in $H_1(\Sigma,Z)$, $i=1,...,g$, then the contribution from summing over all instantons in the bosonic partition function is given in terms of the theta functions with characteristics

$$\sum_{\epsilon_1,\epsilon_2\in Z_2^g} |\theta\, [^{\epsilon_1}_{\epsilon_2}]\ (0|\Omega)|^2 \exp(4\pi i\, \epsilon_1\epsilon_2) \ ,$$

where ϵ_1, ϵ_2 are the half-point on the Jacobian of the Riemann surface Σ which corespond to the spin structures and Ω is the period matrix.

The aim of this work is to give an explicit computation for the instanton summation for D bosons compactified on a D-dimensional torus, $T^D=R^D/\Lambda_D$, where Λ_D is a lattice with generators P^{μ}_k, $\mu,k=1,...,D$. Notice that the matrix $\delta_{\mu\lambda}P^{\mu}_k\cdot P^{\lambda}_{\upsilon}=P_k\cdot P_{\upsilon}=Q_{k\upsilon}$, is ofcourse a D×D symmetric and positive definite matrix. we assume it is also rational in order to relate the instanton sum to the theta function θ^Q associated to the quadratic form Q as introduced by Mumford[37].

In section 2 of this work, we will show that when the matrix Q is the identity matrix, we recover what one expects[26], namely the product of eq.(1.1) D-times, and we recover the usual bosonization formula. When Q is assumed to be an orthogonal matrix, the instanton sum gives rise to the quadratic theta function θ^Q with characteristics. However, when Q is neither the identity nor an orthogonal matrix the result is still given in terms of theta functions but with twisted characteristics. In this generic case we will show that we get rational conformal field theory,[38] in the sence that the partition function is written as a finite sum of a holomorphic function $f_i(\Omega)$ times an antiholomorphic function $\overline{g_i(\Omega)}$.

Finally, in section 3 we consider a fermion multiplet, i.e. a section of a spin bundle L, twisted by a vector bundle E and we look for necessary conditions on E such that the partition functions considered above may arise from bosonizing such a fermionic system. We will find that the relative Grothendieck-Riemann-Roch theorem (GRR) theorem[28] puts conditions on this sort of non ablian bosonization.

3.2 The instanton Sum:

Let Σ be a Riemann surface of genus g and with a fixed homology basis a_i, b_i in $H_1(\Sigma,Z)$, $i=1,\dots,g$; and let $T^D=R^D/\Lambda_D$ be a D-dimentional torus, $_D$ is its lattice with generators denoted by P^μ_k, $\mu,k=1,\dots,D$, such that the symmetric D×D matrix $Q_{\upsilon k}=\sum\delta_{\mu\lambda}P^\lambda_\upsilon\cdot P^\lambda_k=P_\upsilon\cdot P_k$ is rational and positive definite.

Following the work of reference (26), the energy functional (the action) $E(\phi)$ for the instantons ϕ (harmonic maps) on the Riemann surface Σ with values in T^D, can be written as

$$E(\phi) = 2\pi\sum_{\mu=1}^{D}\int d\phi^\mu \wedge * d\phi^\mu \quad , \tag{2.1}$$

if we assume that there are no twists around the cycles a_i (resp. b_i) then the boundary conditions as we go around those cycles are respectively given by

$$\phi^\mu = \phi^\mu + \sum_\upsilon n^\upsilon_i P^\mu_\upsilon \quad ; \quad \phi^\mu = \phi^\mu + \sum_\upsilon m^\upsilon_i P^\mu_\upsilon \tag{2.2}$$

where n^υ_i and m^υ_i are the winding numbers around the cycle a_i (resp.b_i). The instantons (solitons) are nothing but harmonic maps, therefore the instanton solutions satisfy the following equation

$$d\phi^\mu = \sum_{\upsilon=1}^{D}\sum_{i=1}^{g}(n^\upsilon_i P^\mu_\upsilon \alpha^i + m^\upsilon_i P^\mu_\upsilon \beta^i) \tag{2.3}$$

where α_i, β_i in $H^1(\Sigma,Z)$ are harmonic forms dual to a_i, b_i cycles.

The energy functional for the instantons, $E(\phi)$ therefore reads

$$E(\phi) = 2\pi \sum_{\mu} \int d\phi^{\mu} \wedge * d\phi^{\mu} = \sum_{\mu\lambda} \delta_{\mu\lambda} \int d\phi^{\mu} \wedge * d\phi^{\lambda}$$

$$= 2\pi \sum_{\upsilon,k} Q_{\upsilon k} \, (n_i^{\upsilon} A_{ij} n_j^{k} + m_i^{\upsilon} B_{ij} m_j^{k}$$

$$+ n_i^{\upsilon} C_{ij} m_j^{k} + m_i^{\upsilon} C_{ij} m^{k}) \qquad (2.4)$$

where

$$A_{ij} = \int \alpha^i \wedge * \alpha^j \; , \qquad B_{ij} = \int \beta^i \wedge * \beta^j \; ,$$

$$C_{ij} = \int \alpha^i \wedge * \beta^j = \int \beta^i \wedge * \alpha^j \quad . \qquad (2.5)$$

In the formalism of Mumford[37] the energy functional, eq.(2.4), can be written in a more compact form, namely

$$E(\phi) = 2\pi \, \mathrm{tr}[Q\cdot({}^{T}N\cdot A\cdot N + {}^{T}M\cdot B\cdot M + {}^{T}N\cdot C\cdot M + {}^{T}M\cdot C\cdot N)] \qquad (2.6)$$

where the $N=(n_1,\ldots,n_D)$, $M=(m_1,\ldots,m_D)$ are $g\times D$ matrices, i.e., $N,M\in Z^{(g,D)}$, the matrices A, B and C are $g\times g$ matrices and Q was defined before.

Next we compute explicitly the matrices A, B and C which will be needed later on. The dual homology basis α^i, β^i in terms of the Abelian differentials ω and their complex conjugate can be written as

$$\alpha_i = \lambda_{ij}\omega_j + c\cdot c \quad , \qquad \beta_i = \lambda_{ij} w_j + c\cdot c \quad , \qquad (2.7)$$

using the standard relations,

$$\int_{a_i} \alpha_i = \delta_{ij} \quad , \qquad \int_{a_i} \beta_j = 0 \; ,$$

$$\int_{b_i} \alpha_j = 0 \quad , \qquad \int_{b_i} \beta_j = \delta_{ij} \ ,$$

$$\int_{a_i} \omega_j = \delta_{ij} \quad , \qquad \int_{b_i} \omega_j = \Omega_{ij} \quad , \tag{2.8}$$

we get

$$\alpha = \left(\frac{-\bar{\Omega}}{\Omega - \bar{\Omega}} \right)_{ij} \omega_j + \text{c.c} \ ,$$

$$\beta = \left(\frac{-1}{\Omega - \bar{\Omega}} \right)_{ij} \omega_j + \text{c.c} \quad , \tag{2.9}$$

To obtain the matrices A, B and C we need the Hodge $*$-operator properties on Abelian differentials and the Riemann bilinear identity,[44] namely

$$* \omega = - i \omega \quad , \qquad * \bar{\omega} = - i \bar{\omega} \ ,$$

$$\int_\Sigma \theta \wedge \eta = \sum_{i=1}^{g} \left(\int_{a_i} \theta \int_{b_i} \eta - \int_{b_i} \theta \int_{a_i} \eta \right) ,$$

from which

$$\int_\Sigma \bar{\omega} \wedge * \bar{\omega} = \int_\Sigma \omega \wedge * \omega = 0$$

$$\int_\Sigma \bar{\omega} \wedge * \omega = \int_\Sigma \omega \wedge * \bar{\omega} = i \ [\ \bar{\Omega} - \Omega \ ] \ .$$

Therefore the matrices A, B and C in terms of the Riemann period matrix Ω read ;

$$A = \frac{2i \ \bar{\Omega} \ \Omega}{\Omega - \bar{\Omega}} \ ; \ B = \frac{2 \ i}{\Omega - \bar{\Omega}} \ , \ \text{and} \ C = \frac{i(\bar{\Omega} + \Omega)}{\Omega - \bar{\Omega}} \ . \tag{2.10}$$

The partition function for all the instanton takes then the form

$$Z_{inst.} = \sum_{N,M \in Z^{(g,D)}} e^{-2\pi tr[Q\cdot({}^T N\cdot A\cdot N + {}^T M\cdot B\cdot M + {}^T N\cdot C\cdot M + {}^T M\cdot C\cdot N)]} \ .$$

Next, we want to apply the Poisson summation formula to the sum over M, keeping N fixed. In the g-dimensional case, the Poisson summation formula applied to a test function f, on R^g is $\sum_{n\in Z^g} f(n)=\sum_{n\in Z^g}\hat{f}(n)$ where $\hat{f}$ denotes the Fourier transform of f. Thus in analogy with the usual case, the Poisson summation formula in our case would be

$$\sum_{M\in Z^{(g,D)}} f(M) = \sum_{M\in Z^{(g,D)}} \hat{f}(M) .$$

Now the natural Fourier transform definition of a test function f which is of the form of eq.(2.11) would be

$$\hat{f}(K) = \int_{R^{(g,D)}} f(X) \exp(2 \pi i \operatorname{tr} ({}^{T}K\cdot X))\, dX \tag{2.12}$$

where the measure dX is on $R^{(g,D)}$, a g×D variable matrix, the matrix variable k is also a g×D matrix.

Let us first consider the following Fourier transformation

$$\int_{R^{(g,D)}} \exp[-\pi \operatorname{tr}({}^{T}X\cdot B\cdot X)\cdot Q + 2\pi i \operatorname{tr}({}^{T}K\cdot X)]\, dX \quad , \tag{2.13}$$

where X and K can be thought of as D columns vectors, each of which is a g×1 matrix, i.e., $X=(x_1,\ldots,x_D)$, $K=(k_1,\ldots,k_D)$. Since the matrices B and Q are symmetric and positive definite of rank g (resp. D) it follows from linear algebra that there exist non-singular symmetric matrices F, G which are g×g (resp. D×D) such that $B=F^2$, $Q=G^2$. Changing variables by setting $Y=FX^{T}G$, eq.(2.13) becomes

$$(\det F)^{-D/2}(\det Q)^{-g/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\pi \operatorname{tr}({}^{T}Y.Y)}\cdot e^{-2\pi i \operatorname{tr}(KF^{-1}KG^{-1})}dY$$

$$=(\det F)^{-D/2}(\det Q)^{-g/2}\int\cdots\int e^{-\pi \operatorname{tr}{}^{T}(Y+iZ)(Y+iZ)}\cdot e^{-\pi \operatorname{tr}(G^{-1}KF^{-2}KG^{-1})}dY$$

$$= (\det F)^{-D/2}\cdot(\det Q)^{-g/2}e^{-\pi \operatorname{tr}(KB^{-1}KQ^{-1})} \tag{2.14}$$

where the second step is obtained by setting $Z=F^{-1}KG^{-1}$. Now we have to apply the Poisson summation formula to the expression

$$\sum_{M\in Z^{(g,D)}} e^{-2\pi \mathrm{tr}\left[M\left(\frac{2\ i}{\Omega-\bar{\Omega}}\right)M+N\cdot i\left(\frac{\Omega+\bar{\Omega}}{\Omega-\bar{\Omega}}\right)M+M\cdot i\left(\frac{\Omega+\bar{\Omega}}{\Omega-\bar{\Omega}}\right)\cdot N\right]\cdot Q} \tag{2.15}$$

Note that the terms linear in M can be considered as shifts in Fourier transforming the quadratic term in M. Therefore, by using eq.(2.14), with $B=1/\mathrm{Im}\Omega$, the Poisson summation formula applied to eq.(2.15), finally gives the following contribution

$$(1/2)^{gD/2}\ (\det(\mathrm{Im}\ \Omega\ ))^{D/2}\ (\det\ Q)^{-g/2}$$

$$\cdot \sum_{M\in Z^{(g,D)}} \exp\left[\frac{i\pi}{4}\mathrm{tr}\left(M + 2N\left(\frac{\Omega+\bar{\Omega}}{\Omega-\bar{\Omega}}\right).Q\right)\left(\Omega-\bar{\Omega}\right)\left(M + 2N\left(\frac{\Omega+\bar{\Omega}}{\Omega-\bar{\Omega}}\right).Q\right).Q^{-1}\right] \tag{2.16}$$

by combining the terms in the exponential of eq.(2.16) with the quadratic term in N of eq.(2.11) the instanton partition function becomes

$$Z_{(\mathrm{inst.})} = (1/2)^{gD/2}\ (\det(\mathrm{Im}\Omega))^{D/2}\ (\det Q)^{-g/2}$$

$$\cdot \sum_{N,M\ \in\ Z^{(g,D)}} \exp\Big[i\pi\ \mathrm{tr}({}^{T}N(\bar{\Omega}-\Omega)N.Q)\ +\ i\pi/2\,\mathrm{tr}({}^{T}N(\bar{\Omega}+\Omega)M)$$

$$+\ i\pi/2\ \mathrm{tr}({}^{T}N(\Omega+\bar{\Omega})N)\ + i\pi/4\ \mathrm{tr}({}^{T}M(\Omega-\bar{\Omega})M.Q)\Big]$$

$$= (1/2)^{gD/2}\ (\det(\mathrm{Im}\Omega))^{D/2}\ (\det Q)^{-g/2}$$

$$\cdot \sum_{N,M\in Z^{(g,D)}} e^{A^{Q}_{N,M}}\ e^{\bar{A}^{Q}_{N,M}} \tag{2.17}$$

where

$$A^{Q}_{N,M} = i\pi\ \mathrm{tr}(N + M/2\ Q^{-1})\ \Omega\ (N + M/2\ Q^{-1})\cdot Q$$

$$\bar{A}^{Q}_{N,M} = -i\pi\ \mathrm{tr}(N - M/2\ Q^{-1})\ \bar{\Omega}\ (N - M/2\ Q^{-1})\cdot Q \quad . \tag{2.18}$$

Not that in obtaining eq.(2.17) we did not use eq.(6.5) of ref.(1). Having obtained the expression for the instantons partition function, next we consider the following three cases:

(i) $Q = I$,

(ii) ${}^{T}Q = Q^{-1}$, an orthogonal matrix,

(iii) the case in which Q is generic (rational and positive definite).

Thus in the first case, Q=I, $A^{Q}_{N,M}$ and $A^{Q}_{N,M}$ are given by

$$A^{1}_{N,M} = i\pi \operatorname{tr} (N + M/2) \Omega (N + M/2)$$

$$\bar{A}^{1}_{N,M} = - i\pi \operatorname{tr} (N - M/2) \bar{\Omega} (N - M/2) \qquad . \qquad (2.19)$$

To relate the instanton partition function $Z_{(inst)}$ to the theta function with characteristics, we have to decouple eq.(2.17) with Q=I, into two independent summation in which each one gives rise to the theta function with characteristics. To achieve that, we follow the trick used in ref.(56), and set let $N=(A+B)/2$, $M=(B-A)/2+\underline{\epsilon}_1$, where $A,B\in Z^{(g,D)}$ and $\underline{\epsilon}_1\in(Z^{g}_{2})^{D}$. Now the eveness of $A\pm B$ is necessary so that N, M are in $Z^{(g,D)}$, therefore if we were to sum over A, B in $Z^{(g,D)}$ keeping the eveness, we should include in our sum the following Dirac δ function mod (2) given by

$$\delta(A-B(\mathrm{mod}(2))) = (1/2)^{gD} \sum_{\underline{\epsilon}_2\in(Z^{g}_{2})^{D}} \exp(2\pi i \operatorname{tr}(A-B)\cdot\underline{\epsilon}_2) \qquad . \qquad (2.20)$$

Thus with these new variables; the instanton partition function becomes;

$$Z_{(inst.)}=(1/2)^{3gD/2}(\det(\mathrm{Im}\Omega)^{D/2} \cdot \sum_{\underline{\epsilon}_1,\underline{\epsilon}_2\in(Z^{g}_{2})^{D}} \left|\theta\left[\begin{smallmatrix}\underline{\epsilon}_1\\ \underline{\epsilon}_2\end{smallmatrix}\right](0|\Omega)\right|^2 . e^{4\pi i \operatorname{tr}\underline{\epsilon}_1\cdot\underline{\epsilon}_2}$$

and therefore the full partition function is ;

$$Z_{(bose)}=(1/2)^{3gD/2}\left(\frac{\det' - \nabla^2}{\int_\Sigma \sqrt{g}\ \det(\mathrm{Im}\Omega)}\right)^{-D/2}$$

$$\cdot \sum_{\underline{\epsilon}_1,\underline{\epsilon}_2\in(Z_2)^{gD}} \left| \theta \left[{\underline{\epsilon}_1 \atop \underline{\epsilon}_2}\right](0|\Omega) \right|^2 \exp(4\pi i \mathrm{tr}\,\underline{\epsilon}_1\underline{\epsilon}_2) \quad . \qquad (2.21)$$

Following again Mumford's notation, this coresponds to

$$Z_{(bose)}=(1/2)^{3gD/2}\left(\frac{\det' - \nabla^2}{\int_\Sigma \sqrt{g}\ \det(\mathrm{Im}\Omega)}\right)^{-D/2}$$

$$\sum_{\substack{\epsilon_1\in Z_2^g \\ \epsilon_2\in Z_2^g}} \left| \theta \left[{\epsilon_1 \atop \epsilon_2}\right](0|\Omega) \right|^{2D} \exp(4\pi i D\ \epsilon_1\ \epsilon_2) \qquad (2.22)$$

where we denote by $\underline{\epsilon}_1=(\epsilon_1,\epsilon_1,\dots\epsilon_1),\underline{\epsilon}_2=(\epsilon_2,\epsilon_2,\dots\epsilon_2)$ two D-vectors with costant entries.

Thus summing over all instanton sectors in the partition function of D-bosons on a Riemann surface with values in a D-dimensional torus, $T^D=R^D/\Lambda^D$ corresponds to a patition function for D-Dirac fermions with spin structures, (ϵ_1,ϵ_2) summed over.

Now we come to the case when the matrix Q is orthogonal. Since the square root of the Q and its inverse are then equal, one can write $A^Q_{N,M}$ and $\bar{A}^Q_{N,M}$ as

$$A^Q_{N,M} = i\pi\mathrm{tr}\ (N + M/2)\Omega(N + M/2)\cdot Q$$

$$\bar{A}^Q_{N,M} = -i\pi\mathrm{tr}\ (N - M/2)\bar{\Omega}(N - M/2)\cdot Q \quad . \qquad (2.23)$$

Following the same procedure as in the first case one can write the coresponding full partition function as ;

$$Z^{(Q)}_{(bose)} = (1/2)^{3gD/2}(\det Q\ )^{-g/2}\left(\frac{\det' - \nabla^2}{\int_\Sigma \sqrt{g}\ \det(\mathrm{Im}\Omega)}\right)^{-D/2}$$

$$\cdot \sum_{\substack{A,B\in Z^{(g,D)} \\ \underline{\epsilon}_1,\underline{\epsilon}_2\in (Z_2^g)^D}} \exp\{(i\pi \mathrm{tr}(B+\underline{\epsilon}_1)\Omega(B+\underline{\epsilon}_1)\cdot Q\}$$

$$\cdot \quad \exp\{-i\pi \mathrm{tr}(A-\underline{\epsilon}_1)\bar{\Omega}(A-\underline{\epsilon}_1)\cdot Q\}\exp\{2\pi i \mathrm{tr}(A-B)\underline{\epsilon}_2\}$$

$$= \sum_{\underline{\epsilon}_1,\underline{\epsilon}_2\in(Z_2^g)^D} (1/2)^{3gD/2}(\det Q)^{-g/2}\left(\frac{\det' - \nabla^2}{\int_\Sigma \sqrt{g}\, \det(\mathrm{Im}\Omega)}\right)^{-D/2} \cdot$$

$$| \,\theta^Q \, [\begin{smallmatrix}\underline{\epsilon}_1 \\ \underline{\epsilon}_2\end{smallmatrix}](0|\Omega)\, |^2 . \exp(4\pi i \mathrm{tr}\, \underline{\epsilon}_1 \underline{\epsilon}_2) \qquad (2.24)$$

where θ^Q is of course the theta function associated to the quadratic form Q. In the pesent case we can express $Z^{(Q)}_{(\mathrm{bose})}$ in terms of the theta lattice[57] by writing eq.(2.23) as

$$A_{N',M'} = i\pi\, (N'+ M/2)\, \Omega\, (N' + M'/2)$$

$$\bar{A}_{N',M'} = -i\pi\, (N' - M'/2)\, \bar{\Omega}\, (N' - M'/2) \qquad (2.25)$$

where in this case, the lattice Λ_D and its dual, $\tilde{\Lambda}_D$ are equal, i.e., $N',M'\in Z^g\otimes\Lambda_D=L$. As before, the partition function can be writen as

$$Z^{(Q)}_{(\mathrm{bose})} = (1/2)^{3gD/2}(\det Q)^{-g/2}\left(\frac{\det' - \nabla^2}{\int_\Sigma \sqrt{g}\quad \det(\mathrm{Im}\Omega)}\right)^{-D/2}$$

$$\sum_{\tilde{\epsilon}_1,\tilde{\epsilon}_2\in 1/2L/L} \left| \,\theta_{\Lambda_D}\left[\begin{smallmatrix}\tilde{\epsilon}_1 \\ \tilde{\epsilon}_2\end{smallmatrix}(0|\Omega)\right] \right|^2 . \exp(4\pi\, \tilde{\epsilon}_1\tilde{\epsilon}_2) \qquad (2.26)$$

where

$$\theta_{\Lambda_D}\left[\begin{smallmatrix}\epsilon_1 \\ \epsilon_2\end{smallmatrix}(0|\Omega)\right] = \sum_{N\in Z^g\otimes\,\Lambda_D} \exp(i\pi(N+\tilde{\epsilon}_1).\Omega(N+\tilde{\epsilon}_1)+2\pi i(N+\tilde{\epsilon}_1).\tilde{\epsilon}_2$$

Finally we consider the case in which the rational matrix Q is generic, the procedure for obtaining the θ-function from the instanton sum follows basically the same strategy as before the difference, however, being that the lattice, Λ_D and its dual, $\tilde{\Lambda}_D$ are not assumed to be equal and $\tilde{\Lambda}_D$ does not contain Λ_D as a whole. Therefore the variables N',M' in eq.(2.25) are in $\Lambda=Z^g\otimes\Lambda_D$ (resp. $\tilde{\Lambda}=Z^g\otimes\tilde{\Lambda}_D$). Now if λ is the intersection lattice between Λ and $\tilde{\Lambda}$, then due to the rationality of Q, it is a non empty D-dimensional lattice, making the following change of variables

$$N' = \alpha + \beta \qquad \alpha \in \lambda \ , \ \beta \in \Lambda/\lambda$$

$$M' = \alpha' + \beta' \qquad \alpha' \in \lambda \ , \ \beta' \in \tilde{\Lambda}/\lambda \ .$$

the expressions for $A_{N',M'}$, and $\bar{A}_{N',M'}$ become

$$A = i\pi \ (\alpha + \alpha'/2 + \beta + \beta'/2) \ \Omega \ (\alpha + \alpha'/2 + \beta + \beta'/2)$$

$$\bar{A} = -i\pi \ (\alpha - \alpha'/2 + \beta - \beta'/2) \ \bar{\Omega} \ (\alpha - \alpha'/2 + \beta - \beta'/2)$$

Now A and $\bar{A}$ are still dependent on each other, and to decouple them we let $\alpha=(\gamma+\delta)/2$, $\alpha'/2=(\delta-\gamma)/2+\vartheta$ where $\gamma,\delta\in\lambda$ and $\theta\in(\lambda/2)/\lambda$. The partition function then becomes

$$Z^{(Q)}_{(bose)} = (1/2)^{3gD/2}(\det Q)^{-g/2}\left(\frac{\det' - \nabla^2}{\int_\Sigma \sqrt{g} \quad \det(\mathrm{Im}\Omega)}\right)^{-D/2} \cdot$$

$$\sum_{\substack{\vartheta,\eta\in 1/2\lambda/\lambda \\ \beta,\beta' \in /\lambda \ ,\tilde{}/\lambda}} \theta\begin{bmatrix}\vartheta+\beta+\beta'/2 \\ -\eta\end{bmatrix}(0|\Omega)\,\overline{\theta\begin{bmatrix}-\vartheta+\beta-\beta'/2 \\ -\eta\end{bmatrix}(0|\Omega)}\cdot\exp\{4\pi i(\vartheta+\beta/2)\cdot\eta\} \qquad . \ (2.27)$$

Due to the fact that Λ/λ (resp. $\tilde{\Lambda}/\lambda$) are finite sets, the summation above is finite, we see that when Q is rational we get rational conformal field theory.

3.3. Bosonization formula for a twisted spin bundle L_E:

As we saw, when we compactify D bosons on a generic (rational) torus, we get generalization of the usual formulas involving different type of θ-functions. It is natural to ask wheather some kind of bosonization formula holds even in this more general setup. One possibility comes from non abelian bosonization.

To examine this case, we couple a fermion, that is a section of a spin bundle L on a Riemann surface (L^2=K, K being the canonical bundle) to a vector bundle E and ask when we may have bosonization formula of the form,

$$|| \det \bar{\partial}_{L\otimes E} ||^2 = \left(\frac{\det - \nabla '}{\int_\Sigma \sqrt{g} \quad \det(\mathrm{Im}\Omega)} \right)^{-D/2} |F(\Omega)|^2 \tag{3.1}$$

where $F(\Omega)$ is holomorphic function on moduli space $\mathcal{M}_g$. The equality in eq.(3.1) implies conditions on the vector bundle E. The easiest to get, come from the relative GRR theorem[28,50]. When applied to the determinant line bundle, Det $\bar{\partial}_{L\otimes E}$ whose section is $\det\bar{\partial}_{L\otimes E}$ this theorem reads

$$C_1(f_!(\mathrm{DET}\ \bar{\partial}_{L\otimes E})^{\otimes 2}) = 2f_*(\mathrm{Ch}(L\otimes E).\mathrm{Td}\mathcal{T}_f)_4 \tag{3.2}$$

where f is the projection map of the universal family of compact Riemann surfaces; $f:X\longrightarrow\mathcal{M}_g$ and $\mathcal{T}_f$ is the relative tangent bundle. By recalling that:

$$\mathrm{Ch}(E) = \mathrm{rank}(E) + C_1(E) + (C_1^{\,2}(E) - 2\ C_2(E))/2 + \dots ,$$

$$\mathrm{Td}(\mathcal{T}) = 1 + C_1(\mathcal{T})/2 + (C_1^2(\mathcal{T}) + C_2(\mathcal{T}))/12 + \dots ,$$

and that

$$Ch(L\otimes E) = (ChL).(Ch(E)) ,$$

we have

$$Ch(L\otimes E) = rank(E) + C_1(E) + C_1{}^2(E)/2 - C_2(E)$$

$$+ rank[C_1(L)] + C_1(L)\cdot C_1(E)/2$$

$$+ rank[C_1^2(L)]/2 + \ldots \quad , \tag{3.3}$$

where we have kept only terms up to 4 forms.

The relative tangent bundle $\mathcal{T}$ and the spin bundle L are related to the canonical K, by $\mathcal{T}=K^{-1}$ (resp. $L=K^{1/2}$). Therefore eq.(3.2) after some algebra becomes

$$C_1(f_!(DET\ \bar{\partial}_{L\otimes E})^{\otimes 2}) = 2f_*\Big[C_1^2(E)/2 - C_2(E)$$

$$+ rank\{C_1^2(K)/12 - C_1(L).C_1(K)/2 + C_1^2(L)/2\}\Big] .$$

By substituting L for $K^{1/2}$ and using the definition of the first Chern class of the Hodge bundle, $\lambda=f_*(1/12C_1^2(K))$ one obtains

$$C_1(f_!(DET\ \bar{\partial}_{L\otimes E})^{\otimes 2})=-rank(E)\lambda_1+2f_*(1/2C_1^2(E)-C_2(E)) \tag{3.4}$$

on the other hand, the first Chern class of the determinant of the boson Laplacian is twice the Chern class of the Hodge bundle λ, therefore in order that eq.(3.1) holds the rank of the vector bundle E must be D and $f_*(1/2C_1^2(E)-C_2(E))=0$.

It may be nice to relate this condition on E to the non-Abelian bosonization conditions one gets in representation theory of the Kac-Moody algebra[58].

V. TWO DIMENSIONAL SUPERSYMMETRY AND SUPERSYMMETRIC NON-LINEAR σ-MODEL

A. TWO DIMENSIONAL SUPERSYMMETRY

1. Supersymmetric Algebra and Superspace[59]:

A supersymmetry (SUSY) algebra is a graded version of the Poincare algebra, i.e., its composition rule contains both commutators and anticommutators. The structure of the algebra is

$$[P,P] = 0 \qquad [M,M] \sim M \qquad [P,M] \sim P$$

$$[P,Q] = 0 \qquad [M,Q] \sim Q \qquad \{Q,Q\} \sim P \tag{1}$$

where P, M are the generators of the translation, and the lorentz rotations respectively, and Q is the super generator.

Superspace: The N=1 - D=2 superspace has its points labeled by the coordinates $Z^A = (x^{\mu\upsilon};\ \theta^{\mu})$, where $x_{\mu\upsilon}=(\gamma_\alpha)_{\mu\upsilon}x^\alpha$ is the space-time coordinate with $\{\gamma_\alpha,\gamma_\beta\}=2g_{\alpha\beta}$ and θ^μ is a Majorana spinor. The representations of SUSY generators are differential operators acting on superfields and are given by

$$P_{\mu\upsilon} = i\partial_{\mu\upsilon} \qquad , \qquad Q_\mu = i(\partial_\mu - \theta^\upsilon i\partial_{\upsilon\mu}) \tag{2}$$

and

$$M_{\mu\upsilon} = -i/2(x^\lambda{}_{(\mu}\partial_{\upsilon)\lambda} + \theta_{(\mu}\partial_{\upsilon)}) \tag{3}$$

Note that the spinorial derivative ∂_μ (the anticommuting partial

θ-derivative) satisfies $\partial_\mu \theta^\upsilon = \delta^\upsilon_\mu$.

Under the supercoordinates transformation

$$x'^{\mu\upsilon} = x^{\mu\upsilon} + \xi^{\mu\upsilon} - i/2\epsilon^{(\mu}\theta^{\upsilon)} \qquad (4)$$

$$\theta' = \theta^\mu + \epsilon^\mu \qquad (5)$$

where ξ, ϵ are Grassman parameters, the superfield $\psi(x^{\mu\upsilon}, \theta^\mu)$ goes into

$$\psi(x^{\mu\upsilon}+\xi^{\mu\upsilon}-i/2\epsilon^{(\mu}\theta^{\upsilon)}, \theta^\mu+\epsilon^\mu) = \exp[-i(\xi^{\lambda\ell}P_{\lambda\ell}+\epsilon^\lambda Q_\lambda)]\ \psi(x^{\mu\upsilon}, \theta^\mu) \qquad (6)$$

i.e., supersymmetry transformations are realized as rotations and transformations in superspace.

Note that the above transformation is also shared by the space-time derivarive of the superfield $\partial_{\mu\upsilon}\psi$. However, the spinorial derivative $\partial_\mu\psi$, do not transform covariantly. The spinorial derivative that anticommutes with Q_μ is given by

$$D_\mu = \partial_\mu + i\theta^\lambda\partial_{\mu\lambda} \qquad . \qquad (7)$$

This is not to be confused with the covariant derivative (there is no connection). The D algebra is isomorphic to the algebra of Q

$$\{D_\mu, D_\upsilon\} = 2i\partial_{\mu\upsilon} \qquad . \qquad (8)$$

Superfields are in general reducible representations of the SUSY algebra, however, by imposing constraints on the superfields using the covariant spinorial derivative, one can construct irreducible representaions.

Integration in superspace: Integrating over a single Grassman variable γ is defined such that the integral is invariant under translation, i.e., one is forced to require $\int d\gamma = 0$, $\int d\gamma\ \gamma$ =constant normalized to 1 which are both translation invariant. A function of γ, $f(\gamma)$, in Taylor series is $f(\gamma)=f(0)+\gamma f'(0)$ since $\gamma^2=0$, the γ's anticommute among themselves, therefore $\int df(\gamma)=f'(0)$, i.e., integration is equivalent to differentiation.

Applying this to the spinorial coordinates we have

$$\int d\theta_\alpha = \partial_\alpha \tag{9}$$

and hence,

$$\int d\theta_\alpha \theta^\beta = \partial_\alpha \theta^\beta = \delta^\beta_\alpha \tag{10}$$

$$\int d^2\theta\ \theta^2 = \frac{1}{4}\ \partial^\beta \partial_\beta (\theta^2 \theta_\alpha) = -1 \tag{11}$$

therefore the 2-dimesional δ fuction $\delta^2(\theta)$ can be defined by $\delta^2(\theta)=-\theta^2=-\frac{1}{2}\ \theta^\alpha\theta_\alpha$ these techniques can be used in the superspace integration and one gets

$$\int d^2x\ d\theta^2(\qquad) = \int d^2x\ D^2(\qquad)|_{\theta=0} \quad . \tag{12}$$

Note that we are using the spinorial derivative instead of ∂_α, but after the calculation is carried out, θ is put equal to zero.

2. N=1 scalar superfield:

The N=1 real scaler field ϕ is an irreducible representation of N=1 SUSY. To make contact with ordinary space-time, the superfield ψ is expanded in a terminating Taylor series in θ , with the component fields appearing as the cooficients of the different powers of θ;

$$\psi(x,\theta) = A(x) + \theta^{\alpha}\Psi_{\alpha} - \theta^{2}F(x) \tag{1}$$

where

$$A(x) = \phi(x,\theta)|_{\theta=0} ,$$

$$\psi(x) = D_{\alpha}\phi(x,\theta)|_{\theta=0}$$

$$F(x) = D^{2}\phi(x,\theta)|_{\theta=0} \tag{2}$$

where ψ is a two component magorana spinor, the component fields A, F are real and bosonic, F is called the auxialiary field, it is a non-propogating field. The presence of F is to make the SUSY algebra off-shell closed which also means that the fermionic degrees of freedom=bosonic degrees of freedom off shell.

Using the above definitions of the projection one then can find the component action from their superspace counterparts. The kinetic action for a scalar N=1 superfield is given by

$$S = \frac{1}{4}\int d^{2}x\ d^{2}\theta\ D^{\alpha}\phi D_{\alpha}\phi = \frac{1}{4}\int d^{2}x\ \frac{1}{2}\ D^{\beta}D_{\beta}(D^{\alpha}\phi D_{\alpha}\phi) \tag{3}$$

$$S = \frac{1}{4}\int d^{2}x\ \left[-D^{\beta}D^{\alpha}\phi D_{\beta}D_{\alpha}\phi + 2D^{\alpha}\phi D^{2}D_{\alpha}\phi\right]\Big|_{\theta=0} \tag{4}$$

to get the scalar A, the spin ψ and the auxialiary field F we use the following identities,

$$D_\alpha D_\beta = i\partial_{\alpha\beta} + C_{\beta\alpha} D^2$$

$$(D^2)^2 = \Box \qquad , \qquad D^2 D_\alpha = -i\partial_{\alpha\beta} D^\beta \tag{5}$$

then S takes the form

$$S = \int d^2x \left[-\frac{1}{2} A\Box A + \frac{1}{2} F^2 - i\psi^\alpha \partial_{\alpha\beta} \psi^\beta \right] \tag{6}$$

This is all we need to know about the matter superfields to discuss later the supersymmetric non-linear σ-models. Now we should like to present a general description of the formulation of the N=1 -D=2 supersymmetric gauge theories.

3. N=1, D=2 gauge superfield[59]:

In case of the gauge superfields, we use the gauge spinor potential Γ_α which contains the vector two component gauge potential.

Abelian case: In the abelian case, the gauge transformation is given by $\partial\Gamma = -iDX$, where X is a real scalar superfield. The vector potential is constructed as

$$\Gamma_{\alpha\beta} = \frac{i}{2} D_{(\alpha} \Gamma_{\beta)} \tag{1}$$

and transforming as

$$\partial\Gamma_{\alpha\beta} = -i\partial_{\alpha\beta} X \tag{2}$$

the spinor field strength F_α is given by

$$F_\alpha = iD^\beta D_\alpha \Gamma_\beta \ , \quad D^\alpha F_\alpha = 0 \quad \text{(divergence free)} \tag{3}$$

the vector field strength $F_{\alpha\beta}$ is related to F_α through

$$F_{\alpha\beta} = iD_{(\alpha}F_{\beta)} \tag{4}$$

just like the components of ϕ, the different components of F_α are obtained by projections of F_α onto the θ-independent sectors

$$\lambda_\alpha = F_\alpha|_{\theta=0} \quad , \quad f_{\alpha\beta} = iD_{(\alpha}F_{\beta)}|_{\theta=0} = F_{\alpha\beta}|_{\theta=0} \tag{5}$$

since the spinor strength F_α is divergence free, therfore, $D^2F|_{\theta=0}$ is not independent $D^2F_\alpha|_{\theta=0} = 2i\partial_\alpha{}^\beta\lambda_\beta$. Γ_α can be thought of as the spinor component of a connection super-1 form

$$\Gamma = \Gamma_\alpha d\theta^\alpha + \Gamma_{\alpha\beta}dx^{\alpha\beta} \tag{6}$$

where $\Gamma_{\alpha\beta}$, Γ_α are superfields, having defined the spinor connection, and the vector potential thus one can form the different covariant derivatives namely;

$$\nabla_\alpha = D_\alpha + \Gamma_\alpha \qquad , \qquad \nabla_{\alpha\beta} = \partial_{\alpha\beta} + \Gamma_{\alpha\beta} \tag{7}$$

and think of them as components of the gauge covariant derivative $\nabla_A = (\nabla_\alpha, \nabla_{\alpha\beta})$ which satisfy the algebra

$$\{\nabla_\alpha, \nabla_\beta\} = 2i\nabla_{\alpha\beta} \quad , \quad [\nabla_\alpha, \nabla_{\beta\alpha}] = -\frac{1}{2}C_{\alpha(}F_{\beta\gamma)} \tag{8}$$

$$[\nabla_{\alpha\beta}, \nabla_{\gamma\delta}] = \frac{1}{2}(C_{\alpha(\gamma}F_{\delta)\beta} + C_{\beta(\delta}F_{\gamma)\alpha}) \tag{9}$$

where the curvature F_α, $F_{\alpha\beta}$ are the spinor and vector field defined above. The non-abelian case is defined by eq.(7) and eq.(8) where the connections and curvatures are lie algebra-valued.

4. The Feynman supergraph rules:

The formulation of supesymmetric field theories in terms of component fields is not the most suitable one, and if one wishes to do perturbation theory, the numerous propogators and interaction vertices lead to a considerable number of graphs to be calculated. Once a supersymmetric theory is written down in terms of superfields (constrained or non-constrained), the best way to be followed is the so-called quantum superspace: Feynman supergraph rules can be written down by means of a direct inspection of the superfield action.

The Feynman ruels for the scalar superfield can be read directly from the $\mathcal{L}$agrangian. By considering the generating functional for a massive scalar superfield $\phi(x,\theta)$ with an arbitrary self-interaction $f(\phi)$,

$$Z[J] = \int \mathcal{D}\phi \; e^{\int d^2x d^2\theta [\frac{1}{2}\phi D^2\phi + \frac{1}{L} m\phi^2 + f(\phi) + J\phi]}$$

$$= e^{\int d^2x d^2y f(\frac{\partial}{\partial J})} \int \mathcal{D}\phi \; e^{\int d^2x d^2\theta [\frac{1}{L}\phi(D^2+m)\phi + J\phi]} \tag{1}$$

we complete the square in the usual fashion, do the Gaussian functional integral over ϕ and obtain:

$$Z[J] = e^{\int d^2x d^2\theta f(\frac{\partial}{\partial J})} \; e^{-\int d^2x d^2\theta \frac{1}{L} J \frac{1}{D^2+m} J} \tag{2}$$

By using the identity;

$$\frac{1}{D^2+m} = \frac{D^2-m}{\Box-m^2} \quad , \tag{3}$$

we can write

$$<T(\phi(1)\phi(2))> = \frac{D_1^2 - m}{k^2 + m^2}\, \delta^2(\theta_1 - \theta_2) \quad , \tag{4}$$

where

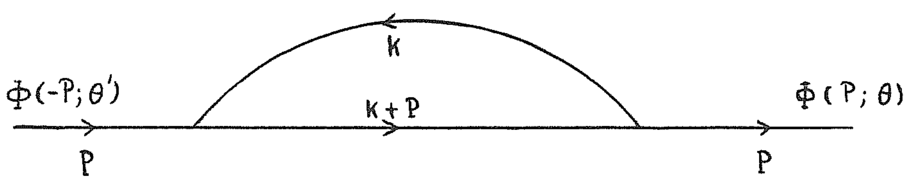
$$D_{1\alpha}(k) \equiv \frac{\partial}{\partial \theta_1{}^\alpha} + \theta_1{}^\beta k_{\alpha\beta} \quad . \tag{5}$$

remembering that

$$\delta^2(\theta_1 - \theta_2) D_1{}^2 \delta^2(\theta_1 - \theta_2) = \delta^2(\theta_1 - \theta_2) \quad . \tag{6}$$

This is all what we need to carry out superfield loop corrections.

We illustrate what was explained above by calculating a self-energy correction to a massless model with a $\lambda\phi^3$-interaction. It is represented by the graph drawn below,



the effective action contribution of such a graph is:

$$\Gamma_2 = \lambda^2 \int \frac{d^2p}{(2\pi)^2} \int d^2\theta d^2\theta' \;\; \Phi(-p;\theta')\Phi(p;\theta).$$

$$\times \int \frac{d^2k}{(2\pi)^2} \; \frac{D^2\delta^2(\theta-\theta')}{k^2} \; \frac{D^2\delta^2(\theta'-\theta)}{(k+p)^2} \quad . \tag{7}$$

The terms in θ can be reduced upon superspace partial integration:

$$D^2\delta^2(\theta-\theta')D^2\delta(\theta'-\theta)\Phi(p;\theta) = \delta^2(\theta-\theta')\big[D^2D^2\delta^2(\theta'-\theta).\Phi(p;\theta)$$

$$+D^{\alpha}D^2\delta^2(\theta'-\theta).D_{\alpha}\Phi(p;\theta)+D^2\delta^2(\theta'-\theta).D^2\Phi(p;\theta)] \qquad (8)$$

however, using the identity

$$(D^2) = -k^2 \qquad ,$$

and

$$D^{\alpha}D^2 = K^{\alpha\beta}D_{\beta} \qquad ,$$

we can reduce our superspace loop computation to

$$\Gamma_2 = \lambda^2 \int\frac{d^2p}{(2\pi)^2}\int d^2\theta\ \Phi(-p;\theta')D^2\Phi(p;\theta)\int\frac{d^2k}{(2\pi)^2}\ \frac{1}{k^2(k+p)^2} \quad . \qquad (9)$$

This is a finite correction to the Φ-kinetic term.

Now, in calculating loop corrections to supersymmetric non-linear σ-models, we have basically to deal with the Feynman rules derived above, as only scalar superfields are involved in their formulation.

5. Two-dimentional N=1 supergraphs and explicitly broken supersymmetries[40]:

There is a renewed interest in two-dimensional supersymmetric theories (especially non-linear σ-models) in connection with superstring theory. Here, we propose to analyse the effect that explicit breakings of N=1, d=2 supersymmetry might have on models constructed in terms of scalar and gauge superfields. The former are indeed of major relevance in connection with N=1 supersymmetric non-linear σ-models[60,39,61].

Our main purpose here is first to set up modified Feynman rules to perform our analysis while working in superspace, even though supersymmetry has been broken. Instead of treating the breaking terms as spurionic vertex insertions into propagators, we shall adopt the technique of shifted superpropagators[62]. This will permit us to account for all powers of the breaking parameters to a given order in perturbation theory.

Once this step has been accomplished, we shall present the results of our superspace calculations, and discuss the effects of the breakings on the infinity structure of the effective action. This is the initial step towards a broader programme concerninig the analysis of the breaking terms for the more interesting situation of the two-dimensional supersymmetric σ-models.

Notice that a complete analysis of all possible soft breakings of N=1 d=4 supersymmetry has been made in ref.(63); however, dimensional reduction of these results would be cosistent with breakings of N=2, d=2 supersymmetry, and not the N=1, d=2, one which intend to consider here.

The following (anti-)commutation relations for N=1 supersymmetry in two dimensions will be of use in the course of our algebraic manipulations:

$$\{D_\alpha, \theta_\beta\} = C_{\alpha\beta} \quad , \quad [D^2, \theta_\alpha] = D_\alpha \quad , \quad [D_\alpha, \theta^2\} = \theta_\alpha$$

$$[D^2, \theta^2] = -1 + \theta^\alpha D_\alpha \quad , \quad D_\alpha \equiv \partial_\alpha + i\theta^\alpha \partial_\alpha \quad , \tag{1}$$

where θ is a Majorana spinor and C is the charge conjugation matrix. We shall adobt here the notation and convention of ref.(59).

Scalar superfields $\Phi(x,\theta)$ can be defined by the following projections:

$$A(x) = \Phi(x;\theta)|_{\theta=0} \quad , \quad \psi_\alpha(x) = D_\alpha\Phi(x;\theta)|_{\theta=0} \; ,$$

$$F(x) = D^2\Phi(x;\theta)|_{\theta=0} \quad . \tag{2}$$

To consider the case of gauge theories, we need to specify the superfield wich accommodates the massless gauge vector field, v, and its fermionic partner, λ. It is a real spinor superfield $\Gamma_\alpha(x;\theta)$ (64) whose components are defined by

$$\Gamma_\alpha|_{\theta=0} = \zeta_\alpha \quad , \quad D^\alpha\Gamma_\alpha|_{\theta=0} = B \; ,$$

$$D_{(\beta}\Gamma_{\alpha)}|_{\theta=0} = v_{\alpha\beta} \qquad \text{(gauge field plus gauge-invariant physical scalar)}$$

$$D^\beta D_\alpha\Gamma_\beta|_{\theta=0} = \lambda_\alpha \qquad \text{(gaugino)}. \tag{3}$$

Considering now a set of scalar superfields $\Phi^i(x;\theta)(i=1, \ldots, M)$, we add to the quadratic supersymmetric action

$$\mathcal{L}_{(Q)} = -\frac{1}{4}\int d^2\theta \; (D^\alpha\Phi^i)(D_\alpha\Phi^i) + \frac{1}{2}\int d^2\theta \; M_{ij}\Phi^i\Phi^j \tag{4}$$

terms which explicitly break N=1 supersymmetry; their net effect is to shift the masses of the physical scalar and spinor fields contained in $\Phi^i(x;\theta)$. They are collected into the breaking lagrangian, $\mathcal{L}_B$, given by

$$\mathcal{L}_B = \frac{1}{2}\int d^2\theta \; \theta^2[m^2_{ij}\Phi^i\Phi^j + \mu_{ij}(D^\alpha\Phi^i)(D_\alpha\Phi^j)] \quad , \tag{5}$$

where m^2 and μ are symmetric M×M mass matrices.

Further on, other explicit breaking terms will be discussed in considering possible couplings for the superfield $\Phi^i(x;\theta)$. The most interesting ones are of the kind $\int d^2\theta\ \theta^2\Phi^3$, $\int d^2\theta\ \theta^2\Phi(D^\alpha\Phi)(D_\alpha\Phi)$, $\int d^2\theta\ \theta^2\Phi(D^2\Phi)(D^2\Phi)$, $\int d^2\theta\ \theta^2(D^2\Phi)(D^\alpha\Phi)(D_\alpha\Phi)$.

Concentrating now on the terms contained in $\mathcal{L}_o+\mathcal{L}_B$, the first step will consist in the derivation of the superpropagator $<T(\Phi^i\Phi^j)>$ with the breaking parameters m^2 and μ taken into account to all orders.

By coupling the superfield $\Phi^i(x;\theta)$ of Lagrangians eq.'s(4) and (5) to external sources, it follows that the most general superpropogator has in principle the following form:

$$P(x_1,\theta_2;x_2,\theta_2)= -\left[1+\sum_{n=1}^{12} X_n A_n(x_1,\theta_1)\right](D_1^2+M)^{-1}\delta(x_1-x_2)\delta^2(\theta_1-\theta_2), \tag{6}$$

where the coefficients X_n are c-number valued M×M matrices to be determined, the operators A_n are defined below, and D_1^2 stands for $\frac{1}{2}D^\alpha(x_1,\theta)D_\alpha(x_1,\theta_1)$:

$$A_1\equiv D^2 \quad , \quad A_2\equiv \theta^\alpha D_\alpha \quad , \quad A_3\equiv D^\alpha\theta_\alpha \quad , \quad A_4\equiv \theta^2 D^2,$$

$$A_5\equiv D^2\theta^2 \quad , \quad A_6\equiv D^\alpha\theta^2 D_\alpha \quad , \quad A_7\equiv \partial_{\alpha\beta}D^\alpha\theta^2 D^\beta \quad , \quad A_8\equiv \partial_{\alpha\beta}\theta^\alpha D^\beta,$$

$$A_9\equiv \partial_{\alpha\beta}D^\alpha D^\beta \quad , \quad A_{10}\equiv D^2D^\alpha\theta^2 D_\alpha \quad , \quad A_{11}\equiv D^\alpha\theta^2 D_\alpha D^2 \quad , \quad A_{12}\equiv \theta^2. \tag{7}$$

Because of the (anti-)commutation relations among the D's and θ's given in eq.(1), the truly independent operators can be shown to be just A_1, A_2, A_4, A_8, and A_{12}: all the others can be written as linear combinations of them. Moreover, A_1, A_2, A_4, A_8, and A_{12} form a closed set under multiplication.

This observation considerably simplifies the task of obtaining the superpropagator of expression of eq.(6), which reduces to

$$\mathcal{P}(x_1,\theta_1;x_2;\theta_2) = -(1+XA_2+YA_4+ZA_8+WA_{12})(D_1^2+M)^{-1}\delta^2(x_1-x_2)\delta^2(\theta_1-\theta_2) \quad , \tag{8}$$

with the matrices X, Y, Z and W being determined by

$$(1+A)X+\Box CZ = -A \quad , \quad CX+(1+A)Z = -C \quad ,$$

$$2(A-B)X-(1+2A-B))Y+2\Box CZ = B \quad ,$$

$$2i\Box CX-2i\Box(A-B)Z+(1+2A-B)W = D \quad . \tag{9}$$

A, B, and C are matrices given in terms of the masses of the theory:

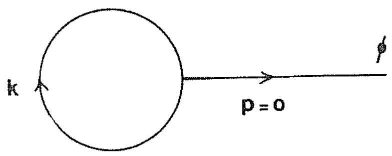


Fig.1. Tadpole graph for the real scalar superfield.

$$A = \frac{1}{\Box-M^2-m^2}(M\mu+m^2) \quad , \quad B = \frac{1}{\Box-M^2-m^2}(2M\mu+m^2)$$

$$C = i\frac{1}{\Box-M^2-m^2}\mu \quad , \quad D = \frac{1}{\Box-M^2-m^2}(Mm^2+2\mu\Box) \quad . \tag{10}$$

After some algebra, our superpropagator reads

$$P(k;\theta_1,\theta_2) \equiv \langle T(\Phi(1)\Phi(2))\rangle = \alpha(k^2)(D_1^2-M)\delta^2(\theta_{12})$$

$$-\beta(k^2)\theta^{\alpha}D_{1\alpha}\delta^2(\theta_{12})+\gamma(k^2)\theta_1^2D_1^2\delta^2(\theta_{12})$$

$$+i\eta(k^2)k^{\alpha\beta}\theta_{1\alpha}D_{1\beta}\delta^2(\theta_{12})-\epsilon(k^2)\theta_1^2\delta^2(\theta_{12}) \quad , \tag{11}$$

where

$$\alpha(k^2) \equiv \frac{1}{k^2-M^2-m^2} \quad , \qquad \beta(k^2) \equiv X\alpha(k^2)M+iZk^2\alpha(k^2) \quad ,$$

$$\gamma(k^2) \equiv Y\alpha(k^2)M-W\alpha(k^2) \quad , \quad \eta(k^2) \equiv iX\alpha(k^2)+Z\alpha(k^2)M$$

$$\epsilon(k^2) \equiv Yk^2\alpha(k^2)M+W\alpha(k^2)M \qquad . \tag{12}$$

Having the complete expressions for the superpropagators acconting for the breaking terms, we next present the results of our supergraph evaluation of different supersymmetric breaking couplings. In doing so, we can have a first estimate of how the breakings affect the structure of the divergences of the effective action of the supersymmetric model.

Case(i): the coupling is $(\lambda/3!)\int d^2\theta\Phi^3$. In this case, the supergraph of fig. 1 contributes the following term to the effective action:

$$\lambda\int\frac{d^2k}{(2\pi)^2}\ \alpha(k^2)\int d^2\theta\ \Phi+\lambda\int\frac{d^2k}{(2\pi)^2}\ \gamma(k^2)\int d^2\theta\ \theta^2\Phi \ . \tag{13}$$

The diagram of fig. 2 contributes the term

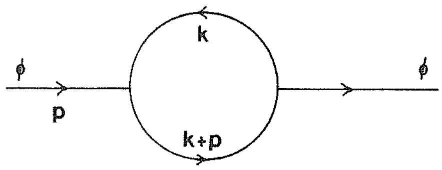


Fig. 2. One-loop self-energy for the real scalar superfield.

$$\lambda^2\int\frac{d^2k}{(2\pi)^2}\{2\alpha(k^2)\gamma[(k+p)^2]-2M\alpha[(k^2)]-M\alpha(k^2)\gamma[(k+p)^2]\}.$$

$$\int d^2\theta\Phi\Phi\ +\lambda^2\int\frac{d^2k}{(2\pi)^2}\{2\alpha(k^2)\epsilon[(k+p)^2]+2\beta(k^2)\beta[(k+p)^2]$$

$$+4\beta(k^2)\gamma[(k+p)^2]-\gamma(k^2)\gamma[(k+p)^2]+2\eta(k^2)\eta[(k+p)^2]k^2\}.$$

$$\int d^2\theta\ \theta^2\Phi\Phi\ +\lambda^2\int\frac{d^2k}{(2\pi)^2}\alpha(k^2)\alpha[(k+p)^2]\int d^2\theta(D^2\Phi)\Phi$$

$$-\lambda^2\int\frac{d^2k}{(2\pi)^2}\alpha(k^2)\beta[(k+p)^2]\int d^2\theta D^\alpha[\theta_\alpha\Phi]\Phi$$

$$-2\lambda^2\int\frac{d^2k}{(2\pi)^2}\alpha(k^2)\gamma[(k+p)^2]\int d^2\theta[(\theta^2D^2+\theta^\alpha D_\alpha)\Phi]\Phi. \qquad (14)$$

In this case, the mass breaking terms do not affect the ultraviolet behaviour of the exact model, as already expected by a power-counting arguments. The only logarithmic divergence is the one arising from the supersymmetric piece of eq.(11) [the term proportional to $\alpha(k^2)$], and it can be eliminated by a field redefinition of the physical scalar A.

case(ii): the coupling is $(\lambda/2)\int d^2\theta\Phi(D^\alpha\Phi)(D_\alpha\Phi)$. The result for the tadpole of fig. 1 is

$$2\lambda^2\int\frac{d^2k}{(2\pi)^2}[\alpha(k^2)M+\beta(k^2)]\int d^2\theta\Phi$$

$$+\lambda\int\frac{d^2k}{(2\pi)^2}[\gamma(k^2)-\beta(k^2)]\int d^2\theta\theta^\alpha D_\alpha\Phi-2\lambda\int\frac{d^2k}{(2\pi)^2}\epsilon(k^2)\int d^2\theta\theta^2\Phi. \qquad (15)$$

We would like to remark that the first (supersummetric) piece of the above contributon contains a term coming exclusively from the breaking, namely, the term proportional to $\beta(k^2)$. It can be shown to vanish in the case $\mu=0$, but if $\mu\neq0$ this term introduces a new logarithmic contribution to the effective action. It is also worthwhile to notice that if either m or μ is non-vanishing, the term proportional to $\epsilon(k^2)$ in eq.(15) is also logarithmically divrgent, however, its contribution has an explicit θ-dependence and hence is non-supersymmetric.

case(iii): the additional coupling term is $,(\lambda/3!)\int d^2\theta\theta^2\Phi^3$, which *explicitly breaks* the supersymmetry, introdusing a three-scalar vertex. The contribution of the tadpole graph to the effective action is

$$\lambda\int\frac{d^2k}{(2\pi)^2}\alpha(k^2)\int d^2\theta\theta^2\Phi \quad , \tag{16}$$

which is a logarithmically divergent F-term.

case(iv): the additional coupling term, $(\lambda/2)\int d^2\theta\theta^2\Phi(D^\alpha\Phi)(D_\alpha\Phi)$, which *breaks* supersymmetry. The contribution of the tadpole graph is

$$\lambda\int\frac{d^2k}{(2\pi)^2}\,[M\alpha(k^2)-\beta(k^2)]\int d^2\theta\;\theta^2\Phi \quad . \tag{17}$$

In this case there is no supersymmetric contribution and the logarithmic divergences arise from the supersymmetric and supersymmetry breaking parts of the propagator eq.(11).
The latter, however, shows up only in the case $\mu\neq 0$.

case(v): the additional coupling term, $\lambda\int d^2\theta\theta^2\Phi(D^2\Phi)(D^2\Phi)$, which *breaks* supersymmetry and whose coupling constant is dimensionless, generating an *on-shell* non-polynomial interaction in the physical scalar A.

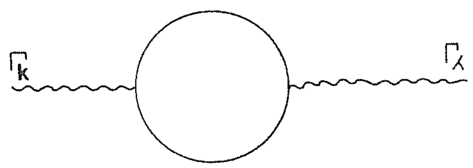


Fig. 3. Gaugino mass one-loop correction.

In this case the tadpole graph contributes

$$\frac{\lambda}{2}\int\frac{d^2k}{(2\pi)^2}[M\alpha(k^2)-2\beta(k^2)+\gamma(k^2)]\int d^2\theta\Phi \quad . \tag{18}$$

So, the whole contribution is linear in F and hence supersymmetric.

This breaking introduces a logarithmic divergence only in the case $\mu\neq 0$.

case(vi): the coupling *breaks* supersymmetry and the coupling constant is dimensionless, $(\lambda/2)\int d^2\theta\theta^2(D^2\Phi)(D^\alpha\Phi)(D_\alpha\Phi)$, generating an *on-shell* a quaric four-fermion coupling, $(\psi^\alpha\psi_\alpha)^2$.

In this case the tadpole graph contributes

$$\lambda\int\frac{d^2k}{(2\pi)^2}\,[M\alpha(k^2)-\beta(k^2)]\int d^2\theta\Phi \quad , \tag{19}$$

which is supersymmetric and diverges logarithmically only in the case $\mu\neq 0$.

Let us finally contemplate the interesting case where the matter superfields $\Phi^i(x;\theta)$ are coupled minimally to the gauge multiplet $\Gamma_\alpha(x;\theta)$[59,64].

The gauge piece of the action is

$$S = \frac{1}{8}\int d^2xd^2\theta(D^\kappa D^\alpha\Gamma_\kappa)(D^\lambda D_\alpha\Gamma_\lambda)-\frac{1}{4\alpha}\int d^2xd^2\theta(D^\alpha\Gamma_\alpha)D^2(D^\beta\Gamma_\beta)$$

$$+\frac{1}{2}\int d^2xd^2\theta\;\theta^2m(D^\kappa D^\alpha\Gamma_\kappa)(D^\lambda D_\alpha\Gamma_\lambda) \quad , \tag{20}$$

where α is the gauge-fixing parameter, and the last term is the explicit breaking which gives the guagino a mass m, while keeping the gauge vector boson massless.

In this case, by adopting the Feynman gauge (α=1), the operator we need to invert in order to get the vector superpropagator is

$$O^\beta_\alpha = (\Box-mA_1-miA_8)\delta^\beta_\alpha - m(1-A_2-2A_4)i\partial^\beta_\alpha \quad . \tag{21}$$

By using the algebra of the operators A_1, A_2, A_4, A_8 and A_{12}, we can work out the expression for $(O^{-1})^\gamma_\alpha$. Since it is very cumbersome, we do not report it here.

Finally, we would like to mention that with the superpropagator of eq.(11), it can be shown that the supergraph of fig.3 generates among other terms a finite contribution whose superspace dependence is of the form

$$\int d^2x d^2\theta \ \theta^2 (D^\kappa D^\alpha \Gamma_\kappa)(D^\lambda D_\alpha \Gamma_\lambda) \quad , \tag{22}$$

which is the term giving the gaugino a mass.

To summarize, we have in this section pursued an attempt to extend well-known techniques for N=1 d=2 supersymmetry in order to include explicit terms which may be of two types: quadratic terms shifting the mass of physical field in the same supermultiplet, and interation terms. We derived modified Feynman rules and applied them to analyse the effect of the breaking terms on the divergence structure of the effective action. *All* breaking terms presented here can be classified as soft breaking terms, in that the divergences they induce are at most logarithmic, and hence not worse than the divergences already present in the unbroken theory.

B. Supersymmetric Non-linear σ-model[59,60]

1. Introduction:

The action that describes a two dimensional Non-linear σ-model is given by

$$S = \int d^2x d^2\theta \; \mathcal{L} = \int d^2x d^2\theta \; g_{ab}(\Phi) D^\alpha \Phi^a D_\alpha \Phi^b \quad . \tag{1}$$

The superfields Φ^a are coordinates on a Riemannian manifold M, called the target manifold.

The Φ^a's, $a=1,\ldots\ldots \dim_R M$ are real N=1 superfields and g_{ab} is the Riemannian metric tensor of the target manifold M. The superfield can be written in the form

$$\Phi^a = A^a + \bar{\theta}\psi^a - \theta\bar{\psi}^a + \bar{\theta}\theta F^a \tag{2}$$

where A^a is a set of real coordinate, ψ and $\bar{\psi}$ take their values in the tensor product of two dimensional Weyl spinors and the Pullback tangent bundle, i.e. the space of tangent vectors at the point A^a of the target manifold.

The equation of motion associated with the action (1) is given by

$$\frac{\partial \mathcal{L}}{\partial \Phi^a} - D^\alpha \frac{\partial \mathcal{L}}{\partial D^\alpha \Phi^a} = 0 \quad . \tag{3}$$

Under an infinitesimal variational $\delta\Phi^a$ in the superfield Φ^a, the change in the lagrangian density $\delta\mathcal{L}$, is given by

$$\delta\mathcal{L} = D \left(\frac{\partial \mathcal{L}}{\partial D^\alpha \Phi^a} \delta\Phi^a \right)$$

By using eq.(1), the Euler lagrangian equation (eq.3) is

$$\nabla^{\alpha} D_{\alpha} \Phi^{a} = 0 \tag{4}$$

where ∇ is the covariant form of D with respect to the target manifold M;

$$\nabla^{\alpha} D_{\alpha} \Phi^{a} = D^{\alpha} D_{\alpha} \Phi^{a} + \Gamma^{a}_{bc} D^{\alpha} \Phi^{b} D_{\alpha} \Phi^{c} \tag{5}$$

Γ^{a}_{bc} is the Christoffel symbol in M. In mathematics, the field (superfield) Φ^{a} that satisfies eq.(5) is called a harmonic map ,i.e. a generalization of the harmonic equation.

$$D^{\alpha} D_{\alpha} \Phi^{a} = 0 \qquad (\Gamma^{a}_{bc}=0) \quad .$$

Under the ordinary supersymmetry transformation

$$\delta_{\alpha} \Phi^{a} = i(\zeta^{\alpha} Q_{\alpha}) \tag{6}$$

where ζ^{α} is a Grassman parameter, the action in eq.(1) is manifestly invariant.

2. Quantisation: The background field method, normal coordinate expansion[65,61].

The vacuum to vacuum amplitude in the presence of a source J corresponds to the generating functional of the full Green's function, i.e.

$$<0|0>_{J} = Z[J] \quad ; \tag{1}$$

$$Z[J] = N\int \mathcal{D}\phi \; e^{iS[\phi]+i\int d^4x J\phi} \tag{2}$$

The full Green's function $g^{(n)}(x_1,\ldots,x_n)$ is given by

$$g^{(n)}(x_1,\ldots,x_n) = \langle|T(\phi(x_1)\ldots\phi(x_n))|\rangle$$

$$= \frac{1}{i^n}\frac{\delta}{\delta J(x_1)}\cdots\cdots\frac{\delta}{\delta J(x_n)}Z[J]\Big|_{J=0} \quad . \tag{3}$$

The generating functional of the connected Green functions W[J] is defined by,

$$\langle 0|0\rangle_J = Z[J] = N\int \mathcal{D}\phi e^{iS[\phi]+i\int d^4xJ\Phi} = e^{iW[J]} \tag{4}$$

i.e., W[J] is defined such that W[J]=-ilnZ[J]. Correspondingly the connected Green's functions are given by

$$g^{(n)}(x^1,\ldots,x^n) = \langle|T(\phi(x_1)\ldots\phi(x_n))|\rangle_c$$

$$= \frac{1}{i^{n-1}}\frac{\delta}{\delta J(x_1)}\cdots\cdots\frac{\delta}{\delta J(x_n)}W[J]\Big|_{J=0} \tag{5}$$

The effective action is the generating functional of all 1-particle irreducible n-point Green's functions which is defined by;

$$\Gamma[\phi_{class}] \equiv W[J]-\int d^4x\, J\phi_{class} \tag{6}$$

(ϕ_{class} corresponds to the classical field, the background field)

$$J = -\frac{\delta\Gamma[\phi_{class}]}{\delta\phi_{class}}$$

and

$$\phi_{class} = \frac{\delta W[J]}{\delta J}$$

from which it follows that

$$<0|0>_J = Z[J] = N\int\mathcal{D}\phi\ e^{iS[\phi]+i\int d^4xJ\phi} = e^{i\Gamma[\phi_{class}]+i\int d^4xJ\phi_{class}}$$

hence

$$e^{i\Gamma[\phi_{class}]} = N\int\mathcal{D}\phi\ e^{iS[\phi]+i\int d^4x(\phi-\phi_{class})} \tag{7}$$

with $\phi=\phi_{class}+\pi$, where π is the quantum fluctuation.

Note that in the non-linear σ-model, where the field ϕ_{class} takes values on a manifold, the quantum fluctuation would correspond to a section in the pullback of the tangent bundle of the manifold (see section II.B.). Explicitly $e^{i\Gamma[\phi_{class}]}$ can be written as follows;

$$e^{i\Gamma[\phi_{class}]} = N\int\mathcal{D}\pi\ e^{iS[\phi_{class}+\pi]-i\int d^4x\frac{\delta\Gamma}{\delta\phi_{class}}.\pi} \quad . \tag{8}$$

From this equation, the total effective action may be split in the following form;

$$\Gamma[\phi_{class}] = S[\phi_{class}]+\Gamma^{loop}[\phi_{class}] \tag{9}$$

where $S[\phi_{class}]$ is just the classical action, and $\Gamma^{loop}[\phi_{class}]$ is the sum of all loop corrections to the effective action $\Gamma[\phi_{class}]$

$$e^{i\Gamma^{loop}[\phi_{class}]}= N\int\mathcal{D}\pi\ e^{iS[\phi_{class}+\pi]-iS[\phi_{class}]}e^{-i\int d^4x\frac{\delta\Gamma}{\delta\phi_{class}}.\pi} \tag{10}$$

This functional was defined by de Witt[65] and is denoted by $\Omega_B[\phi_{class}]$, and it generates all diagrames containing at least 1-loop and with amputed external legs.

The action of the bosonic non-linear σ-model is given by

$$S[\phi] = \frac{1}{2}\int d^2x\ g_{ij}(\phi)\partial^{\mu}\phi^i\partial_{\mu}\phi^j \tag{11}$$

where

$$\phi^i(x)=\phi^i_{class}(x)+\pi^i(x) \qquad . \qquad (12)$$

Since the quantum fluctuation $\pi^i(x)$ fields are a coordinate differences on the manifold, they are not covariant under reparameterization. Which in turn implies that the perturbation thoery developed in terms of $\pi^i(x)$ is not manifestly covariant.

To circumvent this problem, we can express the fluctuation $\pi^i(x)$ as a local power series in terms of new fields $\zeta^i(x)$ which transform as contrvariant vectors; these will be our quantum fields. To achieve this we study the geodesic line connecting the two points whose coordinates are $\phi^i_{class}(x)$ and $\phi^i_{class}(x)+\pi^i(x)$, and with tangent vector $\zeta^i(x)$ at the initial point $\phi^i_{class}(x)$. Such a geodesic can be written as [65]

$$\phi^i(s) = \phi^i_{class}(x)+\zeta^i s - \frac{1}{2}\Gamma^i_{jk}\zeta^j\zeta^k\Big|_{\phi_{class}} s^2 - \frac{1}{3}\Gamma^i_{jkl}\zeta^j\zeta^k\zeta^l\Big|_{\phi_{class}} s^3 + \ldots\ldots \qquad (13)$$

$$\text{for } s = 0 \qquad , \qquad \phi^i = \phi^i_{class}$$

$$\text{and for } s = 1 \qquad , \qquad \phi^i = \phi^i_{class}+\pi^i \qquad (14)$$

$$\pi^i(\zeta) = \zeta^i - \frac{1}{2}\Gamma^i_{jk}\Big|_{\phi_{class}} \zeta^i\zeta^k + \ldots\ldots \qquad (15)$$

<u>*Remark:*</u> the fact that $\pi^i(\zeta)$ can be written in terms of the contravariant vector $\zeta^i(x)$ is of importance because instead of expanding a tensor $T_{i1\ldots in}(\phi_{class}+\pi(\zeta))$ in terms of π^i, we can expand it in terms of $\zeta^i(x)$ and such expansion will be covariant.

$$T_{k_1\ldots k_m}(\phi_{class}+\pi(\varsigma)) = \mathcal{T}_{k_1\ldots k_m}(\varsigma) \tag{16}$$

$$\mathcal{T}_{k_1\ldots k_m}(\varsigma) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial}{\partial\varsigma^{i_1}}\cdots\cdots\frac{\partial}{\partial\varsigma^{i_n}}\mathcal{T}_{k_1\ldots k_n}(\varsigma)\Big|_{\varsigma=0}\varsigma^{i_1}\ldots\varsigma^{i_n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial}{\partial\varsigma^{i_1}}\cdots\cdots\frac{\partial}{\partial\varsigma^{i_n}}T_{k_1\ldots k_n}\Big|_{\phi_{class}}\varsigma^{i_1}\ldots\varsigma^{i_n} \tag{17}$$

the importance of this expansion is that it is completely written in terms of the covariant objects, namely the tensor T itself and the tangent vectors ς^i's. The coefficients of the power series can be shown to be expressible in terms of the covariant derivatives of T and the curvature tensor of the manifold.

Now the easiest way to obtain the manifest covariant form of the coefficients is to use the method of normal coordinates. In such a frame the derivatives of T with respect to ς^i can be easily expressed as covariant derivatives of T.

Normal coordinates: Suppose we have a geodesic passing through the origin where it has tangent vector ς^i. Its equation is thus given by[9]

$$X^i(s) = \varsigma^i s - \frac{1}{2}\Gamma^i_{jk}\Big|_0 \varsigma^j\varsigma^k s^2 - \frac{1}{3}\Gamma^i_{jkl}\Big|_0 \varsigma^j\varsigma^k\varsigma^l s^3 + \ldots\ldots \quad . \tag{18}$$

where

$$\Gamma^i_{jkl} = \partial_j\Gamma^i_{kl} - \Gamma^n_{kl}\Gamma^i_{nl} - \Gamma^n_{jl}\Gamma^i_{kn} \tag{19}$$

In this way each point of the geodesic (a fixed value of s) has its coordinates determined by a power series in ς^i. In the normal frame (coordinates $\overline{X}^i$), we have $\overline{\Gamma}^i_{jk}=0$ and $\overline{\Gamma}^i_{(jkl)}=0,\ldots.$ where (jkl) means that the indices are symmetrized, so that the geodesic equation becomes $\overline{X}^i(s)=\overline{\varsigma}^i s$. Then we can see that the points of the geodesic in this new coordinates system are given by the coordinates $\bar{\varsigma}^i$ of the tangent vector at the origin, up to a scalar parameter s.

So, that the tensor $T_{ij}(\phi_{class}+\pi(\varsigma)) \equiv \mathcal{T}_{ij}(\varsigma)$ in the normal coordinates becomes $\bar{\mathcal{T}}_{ij}(\bar{\varsigma})$;

$$\bar{\mathcal{T}}_{ij}(\bar{\varsigma}) = \bar{\mathcal{T}}_{ij}(0) + \frac{\partial}{\partial \bar{\varsigma}^m} \bar{\mathcal{T}}_{ij}\Big|_0 \bar{\varsigma}^m + \frac{1}{2} \frac{\partial}{\partial \bar{\varsigma}^m} \frac{\partial}{\partial \bar{\varsigma}^n} \bar{\mathcal{T}}_{ij}\Big|_0 \bar{\varsigma}^m \bar{\varsigma}^n$$

$$+ \frac{1}{2} \frac{\partial}{\partial \bar{\varsigma}^m} \frac{\partial}{\partial \bar{\varsigma}^n} \frac{\partial}{\partial \bar{\varsigma}^k} \bar{\mathcal{T}}_{ij}\Big|_0 \bar{\varsigma}^m \bar{\varsigma}^n \bar{\varsigma}^k + \ldots\ldots \quad . \qquad (20)$$

Our next step is to write the derivatives of $\bar{\mathcal{T}}_{ij}$ in terms of the covariant derivatives and curvature tensor. In normal coordinates we have

$$\frac{\partial}{\partial \bar{\varsigma}^m} \bar{\mathcal{T}}_{ij} = \nabla_m \bar{\mathcal{T}}_{ij} \qquad , \qquad \text{as } \bar{\Gamma}'_{jk}\Big|_0 = 0$$

$$\frac{\partial}{\partial \bar{\varsigma}^m} \frac{\partial}{\partial \bar{\varsigma}^n} \bar{\mathcal{T}}_{ij} = \bar{\nabla}_{(m} \bar{\nabla}_{n)} \bar{\mathcal{T}}_{ij} - \frac{1}{3} \bar{R}^l_{(min)} \bar{\mathcal{T}}_{lj} - \frac{1}{3} \bar{R}^l_{(mjn)} \bar{\mathcal{T}}_{ml}$$

where (min) and (mjn) denotes symmetric with respect to m and n only

$$\frac{\partial}{\partial \bar{\varsigma}^m} \frac{\partial}{\partial \bar{\varsigma}^n} \frac{\partial}{\partial \bar{\varsigma}^n} \bar{\mathcal{T}}_{ij} = \bar{\nabla}_{(m} \bar{\nabla}_n \bar{\nabla}_{p)} \bar{\mathcal{T}}_{ij} + \bar{R}^l_{(mni} \nabla_{p)} \bar{\mathcal{T}}_{lj}$$

$$+ \bar{R}^l_{(m\ j\ n} \nabla_{p)} \bar{\mathcal{T}}_{il} + \frac{1}{2} \nabla_{(m} \bar{R}^l_{np)i} \bar{\mathcal{T}}_{li} + \frac{1}{2} \nabla_{(m} \bar{R}^l_{np)j} \bar{\mathcal{T}}_{il}$$

Now, since we know all coefficients in the complete covariant form, we have in any frame

$$T_{ij}(\phi_{class}+\pi(\varsigma)) \equiv \mathcal{T}_{ij}(\varsigma) = \mathcal{T}_{ij}(\phi) + \frac{\partial}{\partial \varsigma^m} \mathcal{T}_{ij}(\phi)\Big|_\phi \varsigma^m$$

$$+ \frac{1}{2} \frac{\partial}{\partial \varsigma^m} \frac{\partial}{\partial \varsigma^n} \mathcal{T}_{ij}(\phi)\Big|_\phi \varsigma^m \varsigma^n$$

$$+\frac{1}{3}\frac{\partial}{\partial\zeta^{m}}\frac{\partial}{\partial\zeta^{n}}\frac{\partial}{\partial\zeta^{l}}\mathcal{T}_{ij}(\phi)\Big|_{\phi}\zeta^{m}\zeta^{n}\zeta^{l}+\dots \tag{21}$$

with

$$\frac{\partial}{\partial\zeta^{m}}\mathcal{T}_{ij}(\phi)\Big|_{\phi}=\nabla_{m}\mathcal{T}_{ij}$$

$$\frac{\partial}{\partial\zeta^{m}}\frac{\partial}{\partial\zeta^{n}}\mathcal{T}_{ij}(\phi)\Big|_{\phi}=\nabla_{(m}\nabla_{n)}\mathcal{T}_{ij}+\frac{1}{3}R^{l}_{(mn)i}\mathcal{T}_{li}+\frac{1}{3}R^{l}_{(mn)j}\mathcal{T}_{il}$$

and

$$\begin{aligned}\frac{\partial}{\partial\zeta^{m}}\frac{\partial}{\partial\zeta^{n}}\frac{\partial}{\partial\zeta^{l}}\mathcal{T}_{ij}(\phi)\Big|_{\phi}=&\nabla_{(m}\nabla_{n}\nabla_{p)}\mathcal{T}_{ij}+R^{l}_{(mn\ i}\nabla_{p)}\mathcal{T}_{li}\\&+R^{l}_{(mn\ j}\nabla_{p)}\mathcal{T}_{il}+\frac{1}{2}\nabla_{(m}R^{l}_{np)i}\mathcal{T}_{lj}\\&+\frac{1}{2}\nabla_{(m}R^{l}_{np)j}\mathcal{T}_{il}\end{aligned} \tag{22}$$

As for the non-linear σ-model, the tensor of our interest is $g_{ij}(\phi_{class}+\pi)$. In this case, the coefficient of the previous expansion simplifies, and g_{ij} is covariantly constant, so we can write

$$g_{ij}(\phi_{class}+\pi)=g_{ij}(\phi_{class})+\frac{1}{3}R_{imnj}\zeta^{m}\zeta^{n}+\frac{1}{3}\nabla_{m}R_{inpj}\zeta^{m}\zeta^{n}\zeta^{l} \tag{23}$$

Before writing the background field expansion of the bosonic σ-model action, one needs to know the expansion for $\partial_{\mu}(\phi^{i}+\pi(\zeta^{i}))$. By recalling that the definition of $\phi^{i}(s)\partial_{\mu}(\phi^{i}+\pi(\zeta^{i}))$ which is nothing but the derivative of $\phi^{i}(s)$ at $s=1$, that is

$$\partial_{\mu}(\phi^{i}+\pi(\zeta^{i}))=\partial_{\mu}\phi^{i}+\partial_{\mu}\zeta^{i}-(\frac{1}{2}\partial_{\underset{m}{l}}\Gamma^{i}_{jk}\zeta^{j}\zeta^{k}+\frac{1}{3!}\partial_{\underset{m}{l}}\Gamma^{i}_{jkl}\zeta^{j}\zeta^{k}\zeta^{l})+\dots \tag{24.a}$$

Now the curvature in the normal coordinates can be written as

$$\bar{R}^i_{jkl} = \partial_k \bar{\Gamma}^i_{jl} - \partial_l \bar{\Gamma}^i_{jk} ,$$

and by using $\partial_{(k}\bar{\Gamma}^i_{jl)}=0$ one obtains

$$\partial_k \bar{\Gamma}^i_{jl} = \frac{1}{3}(\bar{R}^i_{jkl}+\bar{R}^i_{lkj}) .$$

Therefore, the tensorial expansion for $\partial_\mu(\phi^i+\pi(\zeta^i))$ component up to the second order in ζ^i reads

$$\partial_\mu(\phi^i+\pi(\zeta^i)) = \partial_\mu\phi^i+D_\mu\zeta^i+ \frac{1}{R}R^i_{jk}\zeta^j\zeta^k+\ldots.. \tag{24.b}$$

where

$$D_\mu\zeta^i = \partial_\mu\zeta^i+\Gamma^i_{jk}\zeta^i\partial_\mu\phi^k .$$

By combining the expansion in eq.(23) and eq.(24.b), the background field expansion of the bosonic non-linear σ-model reads;

$$I_\beta[\phi+\pi(\zeta)] = \frac{1}{2}\int d^2x g_{ij}(\phi^k)\partial^\mu\phi^i\partial_\mu\phi^j+\int d^2x g_{ij}\partial_\mu\phi^j D_\mu\zeta^j$$

$$+ \frac{1}{2}\int d^2x\{g_{ij}D_\mu\zeta^i D^\mu\zeta^j+R_{iklj}\zeta^i\zeta^j\partial_\mu\phi^i\partial^\mu\phi^j\}+\ldots \tag{25}$$

Note that the second order in the bosonic non-linear σ-model is exactly the Hessian obtained in section II.B.2., where we considered a more general case, as the space-time in the non-linear σ-model need not be flat. Note the difference in notation, the latin indices here correspond to the Greek ones in the second-variation formula and vice-versa.

3. On the ultraviolet behaviour of softly broken N=1-D=2 supersymmetric non-linear σ-models[41]:

Two-dimensional supersymmetric non-linear σ-models consist of the ordinary non-linear σ-model coupled to fermions in such a way as to make the theory invariant under one, two or four supersymmetries. The restrictions on the nature of the target manifold of the supersymmetric model arising from the requirement of invariance under extended supersymmetries, provide a very appealing connection between supersymmetry and complex manifold theory[66,39].

More recently, the interest in these 2-dimensional models has increased since σ-models defined on a Riemann surface and taking values in an arbitrary D-dimensional Riemannian space have a close relationship with string theories[67]. Also very fascinating is the remarkable quantum behaviour of the supersymmetric σ-models in the ultraviolet limit. The N=1 and N=2 models are on-shell three-loop finite for Ricci-flat manifolds[66,68], whereas for N=4, finiteness holds through to all orders in perturbation theory .

In this section, it is our purpose to pursue a reassessment of the convergence properties of a general N=1 σ-model when supersymmetry is explicitly broken. The breaking terms we propose to study here are all soft breakings which modify the scaler-fermion couplings of the model. For particular choices of target spaces, there may also occur a shift in the fermionic masses. Performing supergraph computations, suitably modified to account for the explicit breaking terms[36], we shall investigate the divergence structure of the softly broken σ-model and discuss how the breakings may affect the β-function of the exact model.

We wish to add to the action of the N=1 supersymmetric non-linear

σ-model,

$$S = -\frac{1}{4}\int d^2x d^2\theta \; g_{ij}(\Phi)(D^\alpha\Phi^i)(D_\alpha\Phi^j) \tag{1}$$

new coupling terms which explicitly break supersymmetry but, contrary to the mass terms in section V.A.5 (eq.5), respect the diffeomorphisms invariance of the target manifold, F. The terms we propose to study here are:

$$\frac{1}{2}\int d^2x d^2\theta \; \theta^2 \mu g_{ij}(\Phi)(D^\alpha\Phi^i)(D_\alpha\Phi^j) \tag{2}$$

and

$$\frac{1}{2}\int d^2x d^2\theta \; \theta^2 \lambda R_{ijkl}(\Phi)(D^\alpha\Phi^i)(D_\alpha\Phi^l)(D^\beta\Phi^j)(D_\beta\Phi^k) \tag{3}$$

where μ has the dimension of the mass, and λ is dimensionless, and R_{ijkl} is the Riemann tensor of the target space F.

The breaking term (eq.2), besides modifying the coupling between the physical scalars and fermions of the σ-model, may also shift the masses of the fields Ψ^i_α. Indeed, when F is a homogeneous space like the n-sphere, for example, μ is nothing but the mass of the fermionic component fields. As for the breaking term (eq.3), it does not shift masses, but only affects the scalar-spinor couplings of the originally supersymmetric σ-model.

What is said above can be clearly seen if we rewrite the breaking terms(eq.2) and (eq.3) in terms of component fields. They respectively read:

$$-\frac{1}{2}\int d^2x \mu g_{ij}(A)\Psi^i\Psi^j \tag{4}$$

$$-\frac{1}{4}\int d^2x\ \lambda R_{ijkl}(A)(\Psi^i\Psi^l)(\Phi^j\Psi^k) \tag{5}$$

Before turning into our supergraph calculations, we would like to quote the normal coordinate expansions of the tensors appearing in eq.(2) and eq.(3). We present their respective expansions only up to third order in the field ζ^i relevant for our loop computations. The results are:

$$\begin{aligned} g_{ij}(\Phi+\pi(\zeta)) = {} & g_{ij}(\Phi)-\frac{1}{3}R_{ik_1jk_2}(\Phi)\zeta^{k1}\zeta^{k2} \\ & -\frac{1}{3}D_{k1}R_{ik_2jk_3}(\Phi)\zeta^{k1}\zeta^{k2}\zeta^{k3} \\ & +\frac{1}{5!}\Big[\frac{16}{3}R^m_{k_1jk_1}(\Phi)R_{k_3ik_4m}(\Phi)-6D_{k_1}D_{k_2}R_{ik_3jk_4}(\Phi)\Big]\cdot \\ & \zeta^{k1}\zeta^{k2}\zeta^{k3}\zeta^{k4}+O(\zeta^5) \end{aligned} \tag{6}$$

and

$$\begin{aligned} R_{ijkl}(\Phi+\pi(\zeta)) = {} & R_{ijkl}(\Phi)+D_mR_{ijkl}(\Phi)\zeta^m \\ & +\frac{1}{2}\Big[D_{m1}D_{m2}R_{ijkl}(\Phi)+\frac{1}{3}R^n_{m1m2i}(\Phi)R_{njkl}(\Phi) \\ & +\frac{1}{3}R^n_{m1m2j}(\Phi)R_{inkl}(\Phi)+\frac{1}{3}R^n_{m1m2k}(\Phi)R_{ijnl}(\Phi) \\ & +\frac{1}{3}R^n_{m1m2l}(\Phi)R_{ijkn}(\Phi)\Big]\zeta^{m1}\zeta^{m2}+O(\zeta^3) \quad , \end{aligned} \tag{7}$$

where Φ is taken as a background field, and $\pi(\zeta)$ is the quantum fluctuation expressed in terms of the true quantum field ζ^i[(69)].

We can obtain superpropagators correct to all orders in the breaking mass parameter μ using the results of eq.(11) in section V.B.5. Denoting by $\zeta^a(x;\theta)$ the quantum superfield with frame index, a, of the target space, our quantum propagator reads:

$$<T(\zeta^a(1)\zeta^b(2))> = \delta^{ab}\frac{1}{k^2}D_1^2\delta^2(\theta_{12})$$

$$+\delta^{ab}\frac{\mu}{k^2+\mu^2}(\theta_1^\alpha D_{1\alpha}+2\theta_1^2 D_1^2+\frac{\mu}{k^2}k_{\alpha\beta}\theta_1^\alpha D_1^\beta+2\mu\theta_1^2)\delta^2(\theta_{12}) \quad .(8)$$

However, through the quantum-background vertices arising from eq.(2), the parameter μ will still have to be taken into account when calculating graphs. As for the dimensionless coupling parameter λ, its effect can not be introduced into the quantum propagators, so it will always appear as coupling constant governing the quantum-background vertices stemming from eq.(3).

We are now ready to start presenting and discussing the results of our loop corrections.

Besides the well-known metric tensor renormalisation, which at one-loop is the same as in the case of the unbroken model, the only rôle of the one-loop supergraph of fig.1 is to renormalise the supersymmetry-breaking vertex of eq.(3) by means of the term:

$$\frac{1}{32\pi\epsilon}\int d^2\theta\ \theta^2\lambda(\frac{1}{2}D_m D^m R_{ijkl}+\frac{4}{3}R_{mi}R^m_{jkl})\ (D^\alpha\Phi^i)(D_\alpha\Phi^l)(D^\beta\Phi^j)(D_\beta\Phi^k)\ . \tag{9}$$

Notice however that such a renormalisation is required in the cases of locally symmetric and Ricci-flat target spaces. The mass-breaking parameter μ does not require an independent renormalisation: the metric tensor renormalisation automatically removes such an infinity. Moreover, it would be worthwhile to remark that, power-counting and reparameterisation invariance arguments, can be used to show that only the breaking parameter λ, and not μ, can induce supersymmetric (i.e. non-explicitly θ-dependent) higher-order corrections into the effective action.

Still at the one-loop level, we would like to consider the type of diagrams depicted in fig.2. They fall into three different categories: order zero, linear and quadratic in the breaking parameter λ. They all give finite one-loop contributions to the effective action, so that no new renormalisations are rquired; however, it is interesting to notice that they yeild supersymmetric corrections that arise exclusively from the terms which break supersymmetry at the tree level. These finite one-loop correction are:

$$-\frac{3\mu}{64}\int\frac{d^2k}{(2\pi)^2}\,\frac{1}{k^2}\,\frac{1}{k^2+\mu^2}\int d^2\theta\; R_{imnj}R_{k\;l}^{\;mn}\,(D^\alpha\Phi^i)(D_\alpha\Phi^j)(D^\beta\Phi^k)(D_\beta\Phi^l)\;, \tag{10}$$

and

$$\frac{\lambda}{128}\int\frac{d^2k}{(2\pi)^2}\,\frac{1}{(k^2+m^2_{reg.})}\int d^2\theta\; R_{i\;\;j}^{\;pq}(D_pD_qR_{klmn}$$

$$+\frac{1}{3}R^h_{pqk}R_{hlmn}+\frac{1}{3}R^h_{pql}R_{khmn}+\frac{1}{3}R^h_{pqm}R_{klhn}+\frac{1}{3}R^h_{pqn}R_{klmh})$$

$$.(D^\alpha\Phi^i)(D_\alpha\Phi^j)(D^\beta\Phi^k)(D_\beta\Phi^n)(D^\gamma\Phi^l)(D_\gamma\Phi^m)\;, \tag{11}$$

where $m_{reg.}$ denotes an infrared cutoff mass.

At order λ^2, no supersymmetric correction arises from the supergraph of fig.2.

At the two-loop approximation, no genuine divergence (i.e., a divergence in $1/\epsilon$) which appears alters the vanishing of the two-loop β-function of the exact model. Graphs exhibiting the topology drawn in fig.3 do contribute divergent two-loop corrections of order λ to the metric tensor renormalisation. However, such divergences are of the type $1/\epsilon^2$, and so they do not give any contribution to the β-function.

Finally, going to the three-loop approximation, the situation changes, as expected from power-counting arguments. Indeed, by considering the supergrphs whose topology is as shown in fig.4, where we have at the vertex 1 the quantum-background coupling following from the breaking term eq.(3), one can show that a genuine $1/\epsilon$ three-loop supersymmetric correction is induced which renormalises the metric tensor and is non-vanishing in the Ricci-flat case. Up to the numerical coefficients and $1/\epsilon$-factor coming from the momentum-space loop integrals, the tensorial form of this three-loop divergent contribution is simply:

$$\lambda\int d^2\theta\ R_{ilmn}R_j{}^{kmh}R^n{}_{kh}{}^l(D^\alpha\Phi^i)(D_\alpha\Phi^j) \qquad , \tag{12}$$

and, as it has been checked, such a divergent correction is not cancelled by any other three-loop contribution. This result clearly shows that the breaking term eq.(3) yields a non-zero three-loop contribution to the metric tensor β-function of the non-linear supersymmetric σ-model which persists even when the target manifold is chosen to be Ricci-flat. This is the lowest non-trivial contribution to β_{ij} induced by the breaking interaction term of eq.(3).

To conclude, we have studied two different ways of explicitly breaking N=1-D=2 supersymmetry in the framework of an arbitrary non-linear σ-model. The breaking terms we have proposed here are both soft breakings, respect the diffeormophism invariance of the target space, modify the couplings between the physical scalars and fermions of the σ-model, and also lead to a fermion mass shift. The net result of our analysis is that the breaking accomplished by the term eq.(3) yeilds a non-vanishing contribution to the three-loop metric tensor β-function.

There still remains however the investigation and justification for the ad-hoc breaking terms we propose in this section. Their origin in a superstring compactification context, and also a more detailed analysis of the two-, three- and four-loop supersymmetric corrections generated by the dimensionless coupling parameter λ, is now under investigation[70].

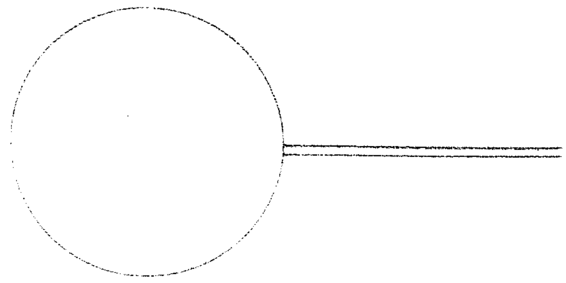


FIG.1

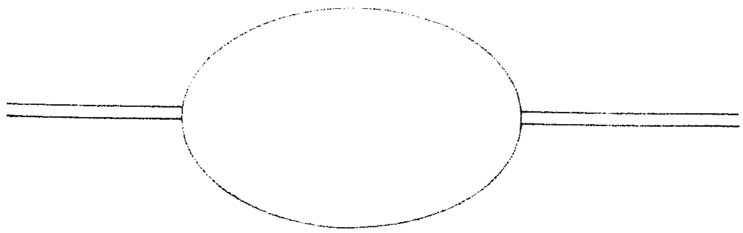


FIG.2

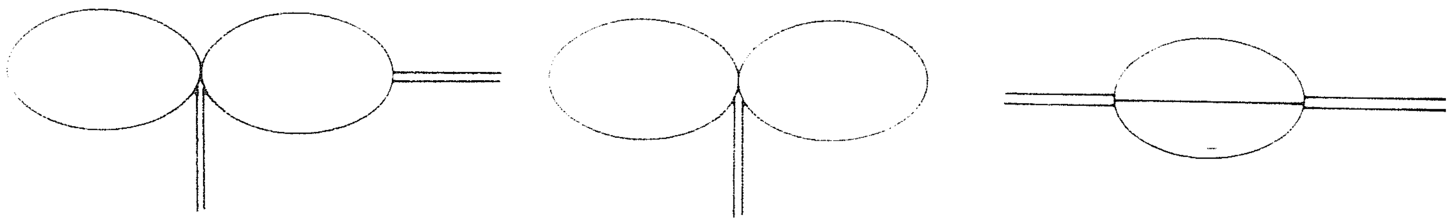


FIG.3

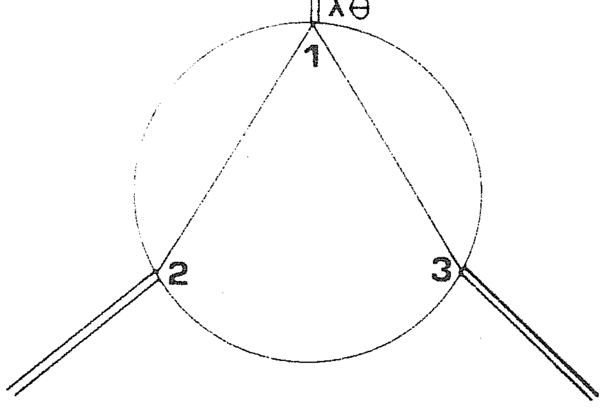


FIG.4

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