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TESI DI "PHILOSOPHIAE DOCTOR"

ASYMPTOTIC ANALYSIS FOR SOME VARIATIONAL PROBLEMS WITH CONSTRAINTS

Settore: Analisi Funzionale ed applicazioni

Supervisore: Prof. Gianni Dal Maso

Candidato: Gabriella Paderni

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Il presente lavoro costituisce la tesi presentata dalla Dott. Gabriella Paderni sotto la direzione del Prof. G. Dal Maso, in vista di ottenere l'attestato di ricerca postuniversitaria "Doctor Philosophiae", settore di Analisi Funzionale e Applicazioni.

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CONTENTS

INTRODUCTION

In this thesis work we investigate the asymptotic behaviour of the solutions of two classes of variational problems with constraints.

In the first part, we consider a sequence of minimum problems for a second order functional with bilateral obstacles of the form

$$(P_h) \qquad \min \{ \int_D |\Delta u|^2 \, dx + \int_D fu \, dx \; : \; u \in H^2_0(D) \; , \; \phi_h \le u \le \psi_h \text{ on } A \} ,$$

where A is an open set included in a bounded open region D of $\mathbf{R}^n$, $f \in L^2(D)$, and (ϕ_h) , (ψ_h) are sequences of functions from D into $\overline{\mathbf{R}}$.

In the second part, we analize a sequence of Dirichlet problems in domains bounded by thin layers with random thickness, where the role of the constraint is played by the Dirichlet condition on the outer boundary of the layer. More precisely, given a bounded open region D of $\mathbf{R}^n$, $n \ge 2$, for every $h \in \mathbf{N}$, we consider a random set A_h such that

$$A_h \supseteq \bar{D} \text{ and } \sup_{x \in A_h} \operatorname{dist}(x, D) < \varepsilon_h$$

and the corresponding quadratic form F_h , defined on $L^2(\mathbf{R}^n)$ by

$$(0.1) \qquad F_h(u) = \begin{cases} \int_D |Du|^2 \, dx + \varepsilon_h \int_{A_h \setminus D} |Du|^2 \, dx & \text{if } u \in H^1_0(A_h) \\ +\infty & \text{otherwise} , \end{cases}$$

where (ε_h) is a given sequence of real numbers such that $\varepsilon_h \to 0$ as $h \to +\infty$. Our aim is to study the sequence of random minimum problems

$$(Q_h) \qquad \min \{ F_h(u) - \int_{A_h} fu\,dx \ : \ u \in L^2(\mathbf{R}^n) \} .$$

with $f \in L^2(\mathbf{R}^n)$.

Our investigation of the asymptotic behaviour of the solutions u_h of (P_h) and (Q_h) , consists in determining a minimum problem whose solution is the limit , in a suitable sense , of the sequence (u_h). For each of the cases introduced above , we find that , under some assumptions , such limit problem is no longer a minimum problem with constraints, but it consists in minimizing an integral functional different from that one of the approximating problems (P_h) and (Q_h).

Both results rely on a compactness property of a class of functionals, suitably choosen according to the problem, with respect on a notion of variational convergence of functionals : Γ-convergence in the first case (see, for example [11],[2]) and Mosco-convergence in the second one (see , for example [2],[15]) .

In the first part of the thesis we prove the following result of convergence for the solutions to problems (P_h). For every $f \in L^2(D)$, and for every $h \in \mathbf{N}$, let us denote by (u_h) the solution of (P_h). Then, we show that there exists a subsequence $(u_{\sigma(h)})$ of (u_h) which converges in $H^1(D)$ to the solution u of a minimum problem that we can write in the form

$$(P) \qquad \min \{ \int_D |\Delta u|^2\,dx + G(u , A) + \int_D fu\,dx \ : \ u \in H_0^2(D) \} ,$$

where $G(\bullet , A)$ is , for every A, a convex and semicontinuous functional from $H^2(D)$ into $\overline{\mathbf{R}}$, independent of f .

Without any assumption of strong convergence on the sequences (ϕ_h) and (ψ_h) , the functional G in (P) , in general, fails to be an obstacle functional of the form

$$(0.2) \qquad G(u, A) = \begin{cases} 0 & \text{if } \phi \le u \le \psi \text{ on } A \\ +\infty & \text{otherwise} \end{cases}$$

with ϕ and ψ function from D into $\bar{\mathbf{R}}$. In some interesting situations, it turns out that G is finite everywhere , and in this case it can be represented in the following integral form

$$(0.3) \qquad G(u, A) = \int_A g(x, u)\, d\mu + \nu(A) ,$$

where g is a non negative Borel function , convex in u , μ and ν are non negative Radon measures on D , and μ is absolutely continuous with respect to H^2-capacity.

To apply the Γ-convergence theory to the sequence of problems (P_h), we introduce a class $\mathcal{G}$ of functionals .

Let $\mathcal{A}$ denote the family of all subsets of D. We say that a functional $G : H^2(D) \times \mathcal{A} \to \bar{\mathbf{R}}$ belongs to $\mathcal{G}$ if G is local, $G(\bullet , A)$ is lower semicontinuous in $H^2(D)$, $G(u , \bullet)$ is a measure on D, and G verifies the following convexity condition ; if $u,v \in H^2(D)$ and $\alpha , \beta \in \mathbf{R}$ with $0 \le \alpha \le \beta \le 1$, then

$$G(\phi u + (1-\phi)v , A) \le \beta\, G(u, A) + (1-\alpha)G(v , A)$$

for every $\phi \in C^\infty(D)$ such that $\alpha \le \phi(x) \le \beta$ for all $x \in A$.

It is not too difficult to verify that both the obstacle functionals like (0.2) and the integral functionals like (0.3) , belong to $\mathcal{G}$.

In chapter 1 (Theorem 3.3) we prove that $\mathcal{G}$ has a compactness property with respect to Γ-convergence. That is, we show that, given a sequence (G_h) in $\mathcal{G}$, there exist a subsequence

$(G_{\sigma(h)})$ of (G_h) and a functional $G \in \mathcal{G}$ such that

$$\int_D |\Delta u|^2 \, dx + G_{\sigma(h)}(u, A)$$

Γ-converge to

$$\int_D |\Delta u|^2 \, dx + G(u, A) .$$

From this compactness property of $\mathcal{G}$, by a well-known theorem ([11] , Corollary 2.4) about the equivalence between Γ-convergence of a sequence of functionals and the convergence of the corresponding minimum points, we can deduce the convergence of the sequence of the solutions u_h of (P_h) to the solution u of (P) .

Then Chapter 2 is devoted to find an integral representation for a functional of the class $\mathcal{G}$. We prove (Theorem 2.5) that if $G \in \mathcal{G}$ and G never takes the value $+\infty$, then G admits an integral representation formula like (0.3).

In two examples (Chapter 1 , Sections 5 and 6) we show significant cases in which the hypothesis $G < +\infty$ is verified . In these examples we assume $n > 4$ and $\phi_h = -\psi_h = -\infty$ everywhere , except for a set E_h which is the union of an increasing number of small balls , whose radii tend to zero with a critical size . For this problem we find an explicit formula for the limit functional in the form (0.3) . The same problem , in the case of unilateral obstacles ($\psi_h = +\infty$ everywhere) has been studied by C. Picard in [16] . The main difficulty of the bilateral case , compared with the unilateral one , is the absence of monotonicity of G , which is essential in the proof of [16].

The above results can be applied to the study of sequences of minimum problems in varying open sets with Dirichlet boundary conditions of the form

$$(P'_h) \qquad \min \{ \int_D |\Delta u|^2 \, dx + \int_D fu \, dx \; : \; u \in H^2_0(D \setminus E_h) \} ,$$

where , for every $h \in N$, E_h is an open subset of D and $f \in L^2(D)$. Problem (P'_h) can be regarded as a particular case of (P_h) if we set

$$\phi_h = -\psi_h = \begin{cases} 0 & \text{in } E_h \\ -\infty & \text{elsewhere .} \end{cases}$$

Among the results on this subject , obtained with different methods , we mention [12],[13],[14],[8].

The basic result of the second part of the thesis is the characterization of the limit behaviour of the sequence of random minimum problems (Q_h).

For every $h \in N$, let us denote by u_h the solution of (Q_h). We prove that there exists a subsequence $(u_{\sigma(h)})$ of (u_h) which converges in probability, as $h \to +\infty$, to the solution of the following deterministic minimum problem

$$(Q) \qquad \min \{ \int_D |Du|^2 \, dx + \int_D fu \, dx + \int_{\partial D} u^2 \, d\mu \; : \; u \in H^1(D) \} ,$$

where μ is a non negative Borel measure , supported by ∂D, which vanishes on sets of zero harmonic capacity.

Note that both the subsequence and the measure μ do not depend on $f \in L^2(D)$.

This result is the probabilistic version of a result obtained by G.Buttazzo, G.Dal Maso, and U. Mosco in [5]. In the deterministic case , the asymptotic behaviour of the solutions of Dirichlet problems in domains surrounded by variable thin layers , known as reinforcement problem , has been investigated , by different thecniques, by several authors (see, for example, [1],[4],[6],[7]).

We illustrate briefly the main ideas of the method used in the proof. First, we consider the class $\mathcal{E}$ of all convex, semicontinuous functions from $L^2(\mathbf{R}^n)$ into $\overline{\mathbf{R}}$ and we equip $\mathcal{E}$ with the topology of Mosco-convergence. This topology makes the space $\mathcal{E}$ a separable complete metric space (see [2]).

Then we regard the functionals F_h, defined in (0.1), as random functionals, i.e. measurable maps $\omega \rightarrow F_h(\omega)$ from a probability space Ω into $\mathcal{E}$.

Next, from an abstract compactness result for sequences of probability measures on a complete metric space, we deduce the convergence in probability (at least for a subsequence) of the distribution laws of the random functionals F_h.

Hence, we prove, under suitable hypotheses, that the limit distribution law is concentrated on a unique functional F : that is the sequence of random functionals F_h converges in probability to a deterministic functional F. Such hypotheses are made in terms of the asymptotic behaviour of the expectations and the covariances of opportune random capacities associated with the random functionals F_h.

Finally, using the variational meaning of the Mosco-convergence, we interpret this result as convergence property of the sequence of the minimum points of the functionals in (Q_h).

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Part 1, Chapter 1 :

Variational inequalities for the biharmonic operator with variable obstacles

VARIATIONAL INEQUALITIES FOR THE BIHARMONIC OPERATOR WITH VARIABLE OBSTACLES.

Summary. The asymptotic behaviour of the solutions of a sequence of variational inequalities for the biharmonic operator with variable two-sided obstacles is investigated by describing the form of the limit problem, which is computed explicitly in two meaningful examples.

INTRODUCTION

Let φ_h and ψ_h be two sequences of functions defined on a bounded open subset Ω of $\mathbf{R}^n$, let $f \in L^2(\Omega)$, and let A be an open subset of Ω. In this paper we study the limit behaviour, as $h \to +\infty$, of the sequence u_h of the solutions of the following variational inequalities for the biharmonic operator:

$$(0.1)\qquad \begin{cases} u_h \in H^2_0(\Omega)\,, \quad \varphi_h \le u_h \le \psi_h \quad \text{on } A\,, \\ \int_\Omega \Delta u_h\, \Delta(v-u_h)\,dx \ge \int_\Omega f(v-u_h)\,dx \\ \text{for every } v \in H^2_0(\Omega) \text{ such that } \varphi_h \le v \le \psi_h \text{ on } A\,. \end{cases}$$

By using Γ-convergence techniques (Theorem 3.3) we prove the following compactness result (Theorem 4.2): if there exists a sequence w_h bounded in $H^2_{loc}(\Omega)$

such that $\varphi_h \leq w_h \leq \psi_h$ on Ω, then there exist an increasing sequence of integers $\sigma(h)$ and a function $G(u,A)$ such that, for every $f \in L^2(\Omega)$ and for a generic open set $A \subset\subset \Omega$, the sequence $u_{\sigma(h)}$ of the solutions of (0.1) converges in $H^1(\Omega)$ to the unique solution u of the variational inequality

$$
(0.2) \qquad \begin{cases} u \in H_0^2(\Omega), \quad G(u,A) < +\infty \\ \int_\Omega \Delta u \Delta(v-u)dx + G(v,A) - G(u,A) \geq \int_\Omega f(v-u)dx \\ \text{for every } v \in H_0^2(\Omega) \text{ with } G(v,A) < +\infty. \end{cases}
$$

Moreover

$$
\lim_{h\to\infty} \int_\Omega |\Delta u_{\sigma(h)}|^2 dx = \int_\Omega |\Delta u|^2 dx + G(u,A) .
$$

In the most common situation the functional G has the form

$$
(0.3) \qquad G(u,A) = \begin{cases} 0 & \text{if } \varphi \leq u \leq \psi \text{ on } A, \\ +\infty & \text{otherwise,} \end{cases}
$$

where φ and ψ are suitable functions defined on Ω, so the limit problem (0.2) is still an obstacle problem .

However, in dimension $n \geq 4$, one can find in the literature many examples where G can not be given the form (0.3) (see [8], [9], [10], [2], [12], [14]).

In the general case, the functional G which occurs in (0.2) is local, $G(\cdot,A)$ is lower semicontinuous on $H^2(\Omega)$ for every open set $A \subseteq \Omega$, and $G(u,\cdot)$ is a measure for every $u \in H^2(\Omega)$ (see [14]) . In addition, we prove that for every open set $A \subseteq \Omega$ the function $G(\cdot,A)$ satisfies the following convexity condition: if $u, v \in H^2(\Omega)$ and α,β are two real numbers with $0 \le \alpha \le \beta \le 1$, then

$$G(\varphi u + (1-\varphi)v,A) \le \beta G(u,A) + (1-\alpha)G(v,A)$$

for every $\varphi \in C^\infty(\Omega)$ such that $\alpha \le \varphi(x) \le \beta$ for all $x \in A$.

By relying on our previous paper [5] , this last property allows us to prove that, if $G(u,\Omega) < +\infty$ for every $u \in H^2(\Omega)$, then G can be written in the form

$$G(u,A) = \int_A g(x,u)\, d\mu + \nu(A) ,$$

where μ and ν are non-negative Radon measures on Ω , μ is absolutely continuous with respect to H^2-capacity, and g is a non-negative Borel function, convex in u .

We use this integral representation theorem to compute explicitly the functional G in two meaningful examples.

The first example concerns the limit of a sequence of Dirichlet problems for the biharmonic operator Δ^2 in domains with periodically distributed small spherical holes. The same problem has been studied by different methods in [8] , [9] , [10] , [2].

In the second example we have

$$\psi_h = -\varphi_h = \begin{cases} 1 & \text{on } E_h , \\ +\infty & \text{elsewhere} , \end{cases}$$

where E_h is the union of an increasing number of small balls whose radii tend to zero with a critical size. In spite of the fact that the lower obstacles φ_h and the upper obstacles ψ_h are well separated , the limit functional G can not be computed by considering separately the effect of the lower obstacles φ_h and the upper obstacle ψ_h. This points out the main difference between the problem considered in this paper and the analogous problems for the second-order operators considered in [3] and [1].

1. PRELIMINARIES

1.1. Capacities and quasi-continuous functions.

For every open set $\Omega \subseteq \mathbf{R}^n$ we denote by $H^2(\Omega)$ the space of all functions $u \in L^2(\Omega)$ such that $D^\alpha u \in L^2(\Omega)$ for every multi index α with $|\alpha| \leq 2$. The norm in $H^2(\Omega)$ is defined by

$$\|u\|_{H^2(\Omega)} = \{ \sum_{|\alpha|\leq 2} \|D^\alpha u\|^2_{L^2(\Omega)} \}^{1/2}.$$

For every compact set $K \subseteq \mathbf{R}^n$ we define the (2,2)-capacity of K by

$$C(K) = \inf \{ \|\varphi\|^2_{H^2(\mathbf{R}^n)} : \varphi \in C_0^\infty(\mathbf{R}^n) , \varphi \geq 1 \text{ on } K \}.$$

The definition is extended to every open set $A \subseteq \mathbf{R}^n$ by

$$C(A) = \sup \{ C(K) : K \subseteq A , K \text{ compact} \}$$

and to arbitrary sets $E \subseteq \mathbf{R}^n$ by

$$C(E) = \inf \{ C(A) : A \supseteq E , A \text{ open} \} .$$

We say that a property $P(x)$ holds quasi everywhere (q.e.) on a set $E \subseteq \mathbf{R}^n$ if $P(x)$ holds for all $x \in E$ except for a subset of E with (2,2)-capacity zero.

A function $f:\mathbf{R}^n \to \mathbf{R}$ is said to be quasi continuous if for every $\varepsilon > 0$ there exists a open set A, with $C(A) < \varepsilon$ such that $f|_{A^c}$ is continuous on $A^c = \mathbf{R}^n - A$.

It is well known that every $u \in H^2(\mathbf{R}^n)$ has a quasi continuous representative which is uniquely determined quasi everywhere (see [11]). We shall identify every function $u \in H^2(\mathbf{R}^n)$ with its quasi continuous representative.

1.2. Γ-convergence.

To study the convergence of the solutions of the variational inequalities considered in the introduction, we use the notion of Γ-convergence.

Let (X,d) be a metric space and let f_h be a sequence of functions from X into $\overline{\mathbf{R}}$. For every $u \in X$ we set

$$\Gamma(d)\liminf_{\substack{h\to\infty \\ v\to u}} f_h(v) = \min \{ \liminf_{h\to\infty} f_h(u_h) : u_h \to u \}$$

and

$$\Gamma(d)\limsup_{\substack{h\to\infty \\ v\to u}} f_h(v) = \min \{ \limsup_{h\to\infty} f_h(u_h) : u_h \to u \} .$$

We say that f_h Γ-converges to a function $f:X \to \overline{\mathbf{R}}$ at a point $u \in X$ if and only if

$$f(u) = \Gamma(d)\liminf_{\substack{h\to\infty \\ v\to u}} f_h(v) = \Gamma(d)\limsup_{\substack{h\to\infty \\ v\to u}} f_h(v) .$$

In this case we write

$$f(u) = \Gamma(d)\lim_{\substack{h\to\infty \\ v\to u}} f_h(v) .$$

It is clear from the above definition that the Γ-convergence is preserved under the addition of a continuous term. The variational meaning of this convergence is given by the following statement (see [6] , Corollary 2.4) .

THEOREM 1.1. *Let* f_h *be a sequence in* X *which* Γ*-converges to* f *in* X. *Let* g *be a continuous function from* X *into* $\mathbf{R}$. *Suppose there exists a compact set* $K \subseteq X$ *such that*

$$\inf_{v \in X} (f_h + g)(v) = \inf_{v \in K} (f_h + g)(v)$$

for every $h \in \mathbf{N}$. *Then we have*

$$\min_{v \in X} (f + g)(v) = \min_{v \in K} (f+g)(v) = \lim_{h \to \infty} \inf_{v \in X} (f+g)(v)$$

Moreover, if $f_h + g$ *has a minimum point* u_h *in* X *and* $f+g$ *has a unique minimum point* u *in* X, *then* u_h *converges to* u *in* X.

2. A CLASS OF LOCAL FUNCTIONALS

In this section we introduce a class of local functionals which contains the obstacle functionals of the form

$$(2.1) \qquad G(u,B) = \begin{cases} 0 & \text{if } \varphi \le u \le \psi \text{ q.e. on } B, \\ +\infty & \text{otherwise}, \end{cases}$$

where φ and ψ are arbitrary functions from $\mathbf{R}^n$ into $\overline{\mathbf{R}}$. Let Ω a bounded open set in

$\mathbf{R}^n$. By $\mathcal{B}$ we denote the family of the Borel subsets of Ω. By $\mathcal{A}$ we denote the family of all open subsets A of Ω such that $A \subset\subset \Omega$ (i.e. A is compact and $A \subseteq \Omega$).

DEFINITION 2.1. We denote by $\mathcal{G}$ the class of all functionals $G:H^2(\Omega)\times\mathcal{B} \to [0,+\infty]$ with the following properties:

(a) for every $A\in\mathcal{A}$, the function $u \to G(u,A)$ is lower semicontinuous in $H^2(\Omega)$;

(b) for every $u\in H^2(\Omega)$, the function $B \to G(u,B)$ is a Borel measure on Ω ;

(c) if $u,v\in H^2(\Omega)$, $A\in\mathcal{A}$, and $u_{|A} = v_{|A}$, then $G(u,A) = G(v,A)$;

(d) if $u,v\in H^2(\Omega)$, $A\in\mathcal{A}$, $\alpha,\beta\in\mathbf{R}$ with $0 \le \alpha \le \beta \le 1$, then

$$G(\varphi u+(1-\varphi)v,A) \le \beta G(u,A) + (1-\alpha)G(v,A)$$

for every $\varphi\in C^\infty(\Omega)$ such that $\alpha \le \varphi(x) \le \beta$ for all $x\in A$.

EXAMPLE 2.2. Let $\varphi , \psi : \Omega\to\overline{\mathbf{R}}$ be two functions such that $\varphi \le \psi$ q.e. in Ω . Then the functional $G:H^2(\Omega)\times\mathcal{B} \to [0,+\infty]$ defined by (2.1) belongs to the class $\mathcal{G}$.

DEFINITION 2.3. We denote by $\mathcal{G}_0$ the class of functionals $G\in\mathcal{G}$ such that $G(t,A) < +\infty$ for every $t\in\mathbf{R}$ and for every $A\in\mathcal{A}$.

Every functional in $\mathcal{G}_0$ admits an integral representation , as the following theorem shows .

THEOREM 2.4. *Let* G *be a functional of the class* $\mathcal{G}_0$ *. Then there exist a Borel function* $g:\Omega\times\mathbf{R}\to[0,+\infty[$ *and two Radon measures on* Ω , μ *and* ν *, with the following* *properties:*

(a) for every $x \in \Omega$ *the function* $t \to g(x,t)$ *is continuous and convex on* $\mathbf{R}$;

(b) $\mu(B) = 0$ *for every Borel set* $B \subseteq \Omega$ *with* $C(B) = 0$;

(c) for every $t \in \mathbf{R}$ *and for every* $A \in \mathcal{A}$

$$G(t,A) = \int_A g(x,t)\, d\mu(x) + \nu(A) \,. \tag{2.2}$$

If, in addition,

$$\int_A g(x,u(x))\, d\mu(x) + \nu(A) < +\infty \tag{2.3}$$

for every $A \in \mathcal{A}$ *and for every* $u \in H^2(\Omega)$ *, then*

$$G(u,A) = \int_A g(x,u(x))\, d\mu(x) + \nu(A) \tag{2.4}$$

for every $A \in \mathcal{A}$ *and for every* $u \in H^2(\Omega)$.

PROOF. By the integral representation theorem for the class $\mathcal{G}_o$ ([5], Theorem 2.5) there exists a triple (g_0,μ_o,ν_o) satisfying (a) and (b) such that

$$G(u,A) = \int_A g_0(x,u(x))\, d\mu_0(x) + \nu_0(A) \tag{2.5}$$

for every $A \in \mathcal{A}$ and for every $u \in H^2(\Omega) \cap L^\infty(A)$.

Let now (g,μ,ν) be another triple which satisfies (a),(b) , and (c). We want to prove that

$$G(u,A) = \int_A g(x,u(x))\, d\mu(x) + \nu(A) \tag{2.6}$$

for every $A \in \mathcal{A}$ and for every $u \in H^2(\Omega) \cap L^\infty(A)$.

Note that, in general, the equalities $g = g_o$, $\mu = \mu_o$, $\nu = \nu_o$ do not hold. Nevertheless we shall prove that

$$\int_A g(x,u(x))\, d\mu(x) + \nu(A) = \int_A g_0(x,u(x))\, d\mu_0(x) + \nu(A) \tag{2.7}$$

for every $A \in \mathcal{A}$ and for every $u \in H^2(\Omega) \cap L^\infty(A)$, which yields (2.6). Let λ be the Radon measure defined by $\lambda = \mu + \nu + \mu_o + \nu_o$. From (2.2) and (2.5) it follows that

$$\int_A [g(x,t) \frac{d\mu}{d\lambda}(x) + \frac{d\nu}{d\lambda}(x)]\, d\lambda(x) = \int_A [g_0(x,t) \frac{d\mu_0}{d\lambda}(x) + \frac{d\nu_0}{d\lambda}(x)]\, d\lambda(x)$$

for every $t \in \mathbf{R}$ and for every $A \in \mathcal{A}$. Therefore , using the continuity of $g(x,t)$ and $g_o(x,t)$ with respect to t , we can prove that there exists a Borel set $N \subseteq \Omega$, with $\lambda(N) = 0$, such that

$$g(x,t) \frac{d\mu}{d\lambda}(x) + \frac{d\nu}{d\lambda}(x) = g_0(x,t) \frac{d\mu_0}{d\lambda}(x) + \frac{d\nu_0}{d\lambda}(x) \tag{2.8}$$

for every $t \in \mathbf{R}$ and for every $x \in \Omega - N$. From (2.8) it follows that

$$(2.9) \qquad g(x,u(x))\frac{d\mu}{d\lambda}(x) + \frac{d\nu}{d\lambda}(x) = g_0(x,u(x))\frac{d\mu_0}{d\lambda}(x) + \frac{d\nu_0}{d\lambda}(x)$$

for every $x \in \Omega - N$ and for every $u \in H^2(\Omega) \cap L^\infty(A)$. Integrating (2.9) with respect to λ we obtain (2.7), hence (2.6) .

Finally, if (2.3) holds, then (2.4) follows from (2.6) by Theorem 2.6 of [5] . ∎

3. COMPACTNESS THEOREM

In this section we prove that the class $\mathcal{G}$ is compact with respect to Γ-convergence. To this aim we introduce a larger class of functionals , whose compactness properties have been studied in [14] .

DEFINITION 3.1. We denote by $\mathcal{G}_1$ the class of all functionals $G : H^2(\Omega) \times \mathcal{B} \to [0,+\infty]$ with fulfil conditions (a), (b), (c) of Definition 2.1 and, in addition, have the following property

(d') if $u,v \in H^2(\Omega)$ and $A \in \mathcal{A}$, then

$$G(\varphi u + (1-\varphi)v\, , A) \le G(u\, , A) + G(v\, , A)$$

for every $\varphi \in C^\infty(\Omega)$ such that $0 \le \varphi(x) \le 1$ for all $x \in A$.

To state the compactness theorem for $\mathcal{G}$ we need the following definition.

DEFINITION 3.2. A subset $\mathcal{R}$ of $\mathcal{A}$ is rich in $\mathcal{A}$ if and only if , for every family $(A_t)_{t\in]0,1[}$ of elements of $\mathcal{A}$ such that $A_s \subset\subset A_t$ for $s<t$, the set $\{t\in]0,1[: A_t \notin \mathcal{R}\}$ is at most countable.

THEOREM 3.3. *Let* G_h *be a sequence of functionals in* $\mathcal{G}$ *. Then there exist a subsequence* $G_{\sigma(h)}$ *of* G_h *, a functional* $G\in\mathcal{G}$ *, and a rich family* $\mathcal{R}$ *of open subsets of* Ω *such that*

$$(3.1) \qquad \int_\Omega |\Delta u|^2\,dx + G(u,A) = \Gamma(H^1(\Omega))\lim_{\substack{h\to\infty\\ v\to u}} \Big[\int_\Omega |\Delta v|^2\,dx + G_{\sigma(h)}(v,A) \Big]$$

for every $u\in H^2(\Omega)$ *and for every* $A\in\mathcal{R}$.

PROOF. By Theorem 1 of [14] (with minor changes in the proof, for which we refer to [13]) there exist a subsequence $G_{\sigma(h)}$ of G_h , a functional $G\in\mathcal{G}_1$, and a rich family $\mathcal{R}$ of open subsets of Ω for which (3.1) holds for every $u\in H^2(\Omega)$ and $A\in\mathcal{A}$. Note that the boundedness hypothesis (H_b) of [14] is not necessary because the functional G is not required to be proper.

To conclude the proof we have only to verify property (d) of Definition 2.1. Let $u,v\in H^2(\Omega)$, let $A\in\mathcal{A}$, let $\alpha,\beta\in\mathbf{R}$, and let $\varphi\in C^\infty(\Omega)$ be such that $0\le\alpha\le\varphi(x)\le\beta\le1$ for every $x\in A$. We have to prove that

$$(3.2) \qquad G(\varphi u + (1-\varphi)v,A) \le \beta G(u,A) + (1-\alpha)G(v,A)\,.$$

Clearly we may suppose $G(u,A) < +\infty$, $G(v,A) < +\infty$, and $A\in\mathcal{R}$. By (3.1) there exist two sequences u_h and v_h in $H^2(\Omega)$ which converge respectively to u and v in

$H^1(\Omega)$, such that

(3.3) $$G(u\,,A) = \lim_{h\to\infty} [\int_\Omega |\Delta u_h|^2\,dx + G_{\sigma(h)}(u_h\,,A)] - \int_\Omega |\Delta u|^2\,dx < +\infty\ ,$$

(3.4) $$G(v\,,A) = \lim_{h\to\infty} [\int_\Omega |\Delta v_h|^2\,dx + G_{\sigma(h)}(v_h\,,A)] - \int_\Omega |\Delta v|^2\,dx < +\infty\ .$$

It follows that Δu_h and Δv_h converge weakly in $L^2(\Omega)$ to Δu and Δv respectively. Since $\varphi u_h+(1-\varphi)v_h$ converges to $\varphi u+(1-\varphi)v$ in $H^1(\Omega)$, by the definition of Γ-limit we have

$$G(\varphi u + (1-\varphi)v\,,A) \le \limsup_{h\to\infty} [\int_\Omega |\Delta(\varphi u_h+(1-\varphi)v_h)|^2\,dx -$$

$$- \int_\Omega |\Delta(\varphi u +(1-\varphi)v)|^2\,dx + G_{\sigma(h)}(\varphi u_h+(1-\varphi)v_h\,,A)] \le$$

$$\le \limsup_{h\to\infty} [\int_\Omega |\Delta[\varphi(u_h-u) + (1-\varphi)(v_h-v)]|^2 dx + G_{\sigma(h)}(\varphi u_h + (1-\varphi)v_h\,,A)] \le$$

$$\le \limsup_{h\to\infty} [\int_\Omega |\varphi\Delta(u_h-u) + (1-\varphi)\Delta(v_h-v)|^2\,dx + G_{\sigma(h)}(\varphi u_h + (1-\varphi)v_h\,,A)] \le$$

$$\le \limsup_{h\to\infty} [\int_\Omega \varphi|\Delta(u_h-u)|^2+(1-\varphi)|\Delta(v_h-v)|^2\,dx + G_{\sigma(h)}(\varphi u_h + (1-\varphi)v_h\,,A)] \le$$

$$\leq \limsup_{h\to\infty} \Big[\int_\Omega \beta|\Delta(u_h-u)|^2+(1-\alpha)|\Delta(v_h-v)|^2\, dx + \beta G_{\sigma(h)}(u_h, A) + (1-\alpha)G_{\sigma(h)}(v_h, A)\Big] \leq$$

$$\leq \beta \limsup_{h\to\infty} \Big[\int_\Omega |\Delta u_h|^2\, dx - \int_\Omega |\Delta u|^2\, dx + G_{\sigma(h)}(u_h, A)\Big] +$$

$$+ (1-\alpha)\limsup_{h\to\infty} \Big[\int_\Omega |\Delta v_h|^2\, dx - \int_\Omega |\Delta v|^2 dx + G_{\sigma(h)}(v_h, A)\Big] =$$

$$= \beta G(u, A) + (1-\alpha)G(v, A),$$

which concludes the proof of the theorem. ∎

4. BOUNDARY CONDITIONS AND CONVERGENCE OF THE MINIMA

Let G and G_h $(h\in \mathbf{N})$ be functionals of the class $\mathcal{G}$ and let $A\in\mathcal{A}$. Throughout this section we assume that

$$(4.1) \qquad \int_\Omega |\Delta u|^2\, dx + G(u,A) = \Gamma(H^1(\Omega))\lim_{\substack{h\to\infty \\ v\to u}} \Big[\int_\Omega |\Delta v|^2\, dx + G_h(v,A)\Big]$$

for every $u\in H^2(\Omega)$. Moreover we assume that there exists a sequence w_h, bounded in $H^2(\Omega)$, such that

$$(4.2) \qquad \sup_{h\in\mathbf{N}} G_h(w_h,A) < +\infty .$$

This implies easily that $G(\bullet,A)$ is proper (i.e. not identically $+\infty$) . We shall prove that for every $w\in H^2(\Omega)$ and for every $f\in L^2(\Omega)$ the sequence of the solutions u_h of the minimum problems

$$\min_{v-w\in H_0^2(\Omega)} \left[\int_\Omega |\Delta v|^2\,dx + G_h(v\,,A) + \int_\Omega fv\,dx\right] \tag{4.3}$$

converges in $H^1(\Omega)$ to the solutions u of the minimum problem

$$\min_{v-w\in H_0^2(\Omega)} \left[\int_\Omega |\Delta v|^2\,dx + G(v\,,A) + \int_\Omega fv\,dx\right] \tag{4.4}$$

To this aim, for every $w\in H^2(\Omega)$ we consider the functionals $F,F_h : H^2(\Omega)\to[0,+\infty]$ defined by

$$F^w(u) = \begin{cases} \int_\Omega |\Delta u|^2\,dx + G(u,A) & \text{if } u-w\in H_0^2(\Omega) \\ +\infty & \text{otherwise ,} \end{cases}$$

$$F_h^w(u) = \begin{cases} \int_\Omega |\Delta u|^2\,dx + G_h(u,A) & \text{if } u-w\in H_0^2(\Omega) \\ +\infty & \text{otherwise ,} \end{cases}$$

We are now in a position to prove the following theorem.

THEOREM 4.1. *Under the assumption (4.1), we have*

$$F^{w}(u)=\Gamma(H^{1}(\Omega))\lim_{\substack{h\to\infty\\ v\to u}} F_h^{w}(v)$$

for every $u,w\in H^2(\Omega)$.

PROOF. Let us fix $u,w\in H^2(\Omega)$. We have to prove that
(a) for every sequence u_h in $H^2(\Omega)$ which converges to u in $H^1(\Omega)$

$$F^{w}(u)\le\liminf_{h\to\infty} F_h^{w}(u_h)\ ;$$

(b) for every $\varepsilon>0$ there exists a sequence u_h in $H^2(\Omega)$ which converges to u in $H^1(\Omega)$ such that

$$F^{w}(u)+\varepsilon\ge\limsup_{h\to\infty} F_h^{w}(u_h)\ .$$

(a) Let u_h be a sequence which converges to u in $H^1(\Omega)$.

We may suppose that $\liminf_{h\to\infty} F_h^{w}(u_h)<+\infty$, which implies that there exists a subsequence of u_h , still denoted by u_h , such that

$$\lim_{h\to\infty} F_h^{w}(u_h)<+\infty \tag{4.5}$$

From (4.5) it follows that $u_h-w\in H_o^2(\Omega)$ and $\sup_h\int_\Omega|\Delta u_h|^2\,dx<+\infty$.

Therefore a subsequence of u_h-w converges weakly in $H^2(\Omega)$. Since u_h-w converges to $u-w$ strongly in $H^1(\Omega)$, it follows that $u-w\in H_o^2(\Omega)$. By the definitions of

F_h^w and F^w and by (4.1) we have

$$F^w(u) = \int_\Omega |\Delta u|^2\,dx + G(u,A) \le$$

$$\le \liminf_{h\to\infty} \Big[\int_\Omega |\Delta u_h|^2\,dx + G_h(u_h,A) \Big] \le \liminf_{h\to\infty} F_h^w(u_h)\,.$$

(b) We suppose $u-w \in H_o^2(\Omega)$ (otherwise we can choose $u_h = u$) . Then

$$F^w(u) = \int_\Omega |\Delta u|^2\,dx + G(u,A)$$

and by (4.1) there exists a sequence v_h in $H^2(\Omega)$, converging to u in $H^1(\Omega)$, such that

$$F^w(u) = \lim_{h\to\infty} \Big[\int_\Omega |\Delta v_h|^2\,dx + G_h(v_h,A) \Big] \tag{4.6}$$

Now we modify the functions v_h in neighbourhood of $\partial\Omega$ to obtain a sequence of functions u_h which still fulfils (4.6) and satisfies, in addition, the boundary condition $u_h - w \in H_0^2(\Omega)$. Given $\varepsilon>0$, we fix $\Omega' \in \mathcal{A}$ such that $A \subset\subset \Omega' \subset\subset \Omega$ and $\int_{\Omega-\Omega'} |\Delta u|^2\,dx < \varepsilon$.

Let $\varphi \in C_o^\infty(\Omega)$ with $0 \le \varphi \le 1$, $\varphi = 1$ on Ω' , and let $u_h = \varphi v_h + (1-\varphi)u$. Then $u_h \in H^2(\Omega)$, $u_h - w \in H_0^2(\Omega)$, and $u_h = v_h$ on A . Moreover for every $\varepsilon > 0$

$$\int_\Omega |\Delta u_h|^2\,dx \le \frac{1}{1-\varepsilon}\int_\Omega |\varphi\Delta v_h + (1-\varphi)\Delta u|^2\,dx + \frac{1}{\varepsilon}\int_\Omega |2D\varphi(Dv_h - Du) + \Delta\varphi(v_h - u)|^2\,dx \le$$

$$\le \frac{1}{1-\varepsilon}\int_\Omega |\Delta v_h|^2\,dx + \frac{1}{1-\varepsilon}\int_\Omega |\Delta u|^2 + C_h \quad ,$$

where $\lim_{h\to\infty} C_h = 0$ since v_h converges to u in $H^1(\Omega)$. Thus we have

$$\limsup_{h\to\infty} F_h(u_h,A) = \limsup_{h\to\infty} \Big[\int_\Omega |\Delta u_h|^2\,dx + G_h(u_h,A)\Big] \le$$

$$\le \limsup_{h\to\infty} \frac{1}{1-\varepsilon}\Big[\int_\Omega |\Delta v_h|^2\,dx + G_h(v_h,A)\Big] + \frac{1}{1-\varepsilon}\int_{\Omega-\Omega'} |\Delta u|^2\,dx =$$

$$= \frac{1}{1-\varepsilon} F^w(u) + \frac{\varepsilon}{1-\varepsilon} \;.$$

Since $\varepsilon>0$ is arbitrary, we obtain (b) and the theorem is proved. ∎

Given $w\in H^2(\Omega)$ and $f\in L^2(\Omega)$, we now study the minimum problem

(4.7) $$\min_{v-w\in H^2_0(\Omega)} \Big[\int_\Omega |\Delta v|^2\,dx + G(v,A) + \int_\Omega fv\,dx\Big] .$$

Since $G(\bullet,A)$ is proper (by (4.2)) and there exists a constant K such that

(4.8) $$\|u\|_{H^2(\Omega)} \leq K\,\|\Delta u\|_{L^2(\Omega)}$$

for every $u \in H_0^2(\Omega)$ (see, for instance, [7], Lemma 5.17), by using the direct method of the calculus of variations it is not difficult to prove that problem (4.7) has one and only one solution u, which can be characterized as the unique solution u of the variational inequality

$$\begin{cases} u \in H^2(\Omega), \quad u-w \in H_0^2(\Omega), \quad G(u,A) < +\infty, \\ 2\int_\Omega \Delta u \Delta(v-u)\,dx + G(v,A) - G(u,A) + \int_\Omega f(v-u)\,dx \geq 0 \\ \forall v \in H^2(\Omega): \quad v-w \in H_0^2(\Omega), \quad G(v,A) < +\infty \end{cases}$$

An analogous result can be obtained for the functionals G_h.

We are now in a position to prove the convergence of the solution of problems (4.3) to the solution of problem (4.4).

THEOREM 4.2. *Assume (4.1) and (4.2). Then for every* $w \in H^2(\Omega)$ *and for every* $f \in L^2(\Omega)$ *the sequence of the minimum values of problem (4.3) converges to the minimum value of problem (4.4). Moreover, if* u_h *and* u *are the minimum points of (4.3) and (4.4) respectively, then* u_h *converges to* u *strongly in* $H^1(\Omega)$.

PROOF. First we observe that the minimum problems (4.3) and (4.4) are equivalent to the minimum problems

(4.9)
$$\min_{u\in H^2(\Omega)} \left[F_h^w(u) + \int_\Omega fu\,dx \right]$$

and

(4.10)
$$\min_{u\in H^2(\Omega)} \left[F^w(u) + \int_\Omega fu\,dx \right],$$

in the sense that (4.3) (resp. (4.4)) and (4.9) (resp (4.10)) have the same minimum values and the same minimum points .

By (4.8) for every $\varepsilon\in]0,1[$ and for every $u\in H^2(\Omega)$ with $u\text{-}w\in H_0^2(\Omega)$ we have

$$\|u\|^2_{H^2(\Omega)} \le \frac{1}{1-\varepsilon}\|u-w\|^2_{H^2(\Omega)} + \frac{1}{\varepsilon}\|w\|^2_{H^2(\Omega)} \le$$

$$\le \frac{1}{1-\varepsilon} K\|\Delta u\|^2_{L^2(\Omega)} + \frac{1}{1-\varepsilon} K\|\Delta w\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\|w\|^2_{H^2(\Omega)}$$

so there exist two constants α , $\beta > 0$ such that

(4.11)
$$F_h^w(u) + \int_\Omega fu\,dx \ge \alpha\|u\|^2_{H^2(\Omega)} - \beta$$

for every $h\in\mathbf{N}$ and for every $u\in H^2(\Omega)$.

By (4.2) and (4.11) the sequence u_h of the minimum points of (4.9) is bounded in $H^2(\Omega)$, hence it is relatively compact in $H^1(\Omega)$. Since F_h^w Γ-converges to F^w in $H^1(\Omega)$

and $\int_\Omega fu\,dx$ is continuous in $H^1(\Omega)$ for every $f\in L^2(\Omega)$, the result follows immediately from Theorem 1.4. ∎

5. DIRICHLET PROBLEMS IN DOMAINS WITH HOLES

In this section we study the Γ-limit of a sequence of functionals related to Dirichlet problems for the biharmonic operator Δ^2 in domains with many small holes .

Assume $n > 4$. For every $h\in\mathbf{N}$ let $0 < r_h < R_h = 2^{-h}$. Let us divide $\mathbf{R}^n$ into a family $(Q_h^i)_{i\in\mathbf{Z}^n}$ of cubes with side $2R_h$. More precisely for $i=(i_1,\dots,i_n)\in\mathbf{Z}^n$ we set

$$Q_h^i = \prod_{k=1}^{n} \left] \frac{i_k}{2^{h-1}} , \frac{i_k+1}{2^{h-1}} \right[.$$

Let us denote by x_h^i the center of Q_h^i and let $B_h^i = B(x_h^i , r_h)$ and $D_h^i = B(x_h^i , R_h)$ be the open balls of center x_h^i and radius r_h and R_h respectively . Let

$$E_h = \bigcup_{i\in\mathbf{Z}^n} B_h^i .$$

Let $G_h : H^2(\Omega)\times\mathcal{A}\to[0,+\infty]$ be the functional defined by

$$G_h(u,A) = \begin{cases} 0 & \text{if } u=0 \text{ q.e. on } E_h\cap A \\ +\infty & \text{otherwise} \end{cases}$$

for every $u\in H^2(\Omega)$ and for every $A\in\mathcal{A}(\Omega)$.

THEOREM 5.1. *Assume that*

$$\lim_{h\to\infty} (2R_h)^{-n} r_h^{n-4} = a$$

and that $0 < a < +\infty$. *Then for every* $A \in \mathcal{A}$ *with* $\text{meas}(\partial A) = 0$ *and for every* $u \in H^2(\Omega)$

$$\Gamma(H^1(\Omega)) \lim_{\substack{h\to\infty \\ v\to u}} \left[\int_\Omega |\Delta v|^2 \, dx + G_h(v,A) \right] = \int_\Omega |\Delta u|^2 \, dx + aC^* \int_A u^2 \, dx \tag{5.1}$$

where

$$C^* = \min \left\{ \int_{\mathbf{R}^n} |\Delta u|^2 \, dx : u \in H^2(\mathbf{R}^n) , u=1 \text{ on } B_n \right\} = (n-4)(n-2)^2 S_n \,, \tag{5.2}$$

B_n *is the unit ball in* $\mathbf{R}^n$ *and* S_n *is the (n–1)dimensional measure of* ∂B_n.

PROOF. By the compactness theorem (Theorem 3.3) for every subsequence $G_{\sigma(h)}$ of G_h there exist a subsequence $G_{\rho(h)}$ of $G_{\sigma(h)}$, a functional $G \in \mathcal{G}$ and a rich family $\mathcal{R}$ of open sets in Ω such that

$$\int_\Omega |\Delta u|^2 \, dx + G(u,A) = \Gamma(H^1(\Omega)) \lim_{\substack{h\to\infty \\ v\to u}} \left[\int_\Omega |\Delta v|^2 \, dx + G_{\rho(h)}(v,A) \right]$$

for every $u \in H^2(\Omega)$ and for every $A \in \mathcal{R}$. We shall denote this subsequence G_h.

We shall prove that

$$G(u,A) = aC^* \int_A u^2\, dx \tag{5.3}$$

for every $u \in H^2(\Omega)$ and for every $A \in \mathcal{A}$.

Moreover , in Lemma 5.2 we shall prove that every open set $A \subset\subset \Omega$ with $\text{meas}(\partial A)=0$ belongs to the family $\mathcal{R}$. Since the limit does not depend on the subsequence , we conclude that (5.1) holds for every $A \in \mathcal{A}$ with $\text{meas}(\partial A)=0$, (see , for example , [4] , Proposition 1.15) .

By Theorem 2.4 to prove (5.3) it is enough to show that

$$G(t,A) = aC^* t^2 \,\text{meas}(A) \tag{5.4}$$

for every $t \in \mathbf{R}$ and for every $A \in \mathcal{A}$.

It is clear that it is sufficient to prove (5.4) when A is the union of a finite number of open rectangles whose vertices have dyadic coordinates . We recall that a dyadic number is a real number t of the form $t=r2^s$, with $r,s \in \mathbf{Z}$.

Let us fix such an open set A and let $I_h(A) = \{i \in \mathbf{Z}^n : Q_h^i \cap A \neq \emptyset\}$. Note that by the particular choice of A there exists $k \in \mathbf{N}$ such that $Q_h^i \subseteq A$ for every $h \geq k$ and for every $i \in I_h(A)$. Therefore

$$A \cap E_h = \bigcup_{i \in I_h(A)} B_h^i \tag{5.5}$$

for every $h \geq k$.

To obtain (5.4) we construct an auxiliary sequence of function w_h . For every $\rho > 0$ we denote by $B(\rho)$ the open ball centered in the origin with radius ρ . For every $h \in \mathbf{N}$ let w_h be the solution to the following minimum problem

$$(5.6) \qquad \min\{ \int_{B(R_h)} |\Delta u|^2\, dx : u \in H^2_0(B(R_h)) \,,\ u=1 \text{ on } B(r_h) \}$$

We extend w_h by setting $w_h = 0$ on $\mathbf{R}^n - B(R_h)$. For every $i \in \mathbf{Z}^n$ and for every $x \in \mathbf{R}^n$ let $w_h^i(x) = w_h(x - x_h^i)$ and let

$$(5.7) \qquad v_h = \sum_{i \in I_h(A)} w_h^i \,.$$

Then v_h belongs to $H^2(\Omega)$ and $v_h = 1$ on $\bigcup_{i \in I_h(A)} B_h^i$ (which is equal to $A \cap E_h$ for h large enough) . Moreover v_h converges to zero in $H^1(\Omega)$ (this can be proved , for istance , by using the explicit representation of w_h we shall give in (5.11)).

Let us fix $t \in \mathbf{R}$ and let $u_h = t - tv_h$. Then $u_h \in H^2(\Omega)$, $u_h=0$ on $E_h \cap A$ (for h large enough) and u_h converges to t in $H^1(\Omega)$. By the definition of Γ-limit we have

$$(5.8) \qquad G(t,A) \le \liminf_{h \to \infty} \int_\Omega |\Delta u_h|^2\, dx \,.$$

In the sequel we shall give an explicit representation of w_h from which it follows that

(5.9) $$\lim_{h\to\infty} \int_\Omega |\Delta u_h|^2\, dx = aC^*\, t^2 \,\mathrm{meas}(A)\,.$$

Moreover we shall prove that

(5.10) $$\liminf_{h\to\infty} \int_\Omega |\Delta z_h|^2\, dx \geq \lim_{h\to\infty} \int_\Omega |\Delta u_h|^2\, dx$$

for every sequence z_h in $H^2(\Omega)$ converging to t in $H^1(\Omega)$ such that $z_h=0$ on $E_h \cap A$. From (5.8),(5.9),(5.10) we obtain (5.4) .

To conclude the proof of the theorem it remains to prove (5.9) and (5.10) .

The solution w_h to the minimum problem (5.6) coincide on $B(R_h)-B(r_h)$ with the solution of the following boundary problem

$$\Delta^2 w_h = 0 \qquad \text{in } B(R_h)-B(r_h)\,,$$

$$w_h = \frac{\partial w_h}{\partial n} = 0 \qquad \text{on } \partial B(R_h)\,,$$

$$w_h = 1\,,\ \frac{\partial w_h}{\partial n} = 0 \qquad \text{on } \partial B(r_h)\,.$$

The function w_h can be written explicitely . Using polar coordinates we have

(5.11) $$w_h(\rho) = a_h\rho^{2-n} + b_h\rho^{4-n} + c_h + d_h\rho^2$$

where

$$a_h = 2n(n-4)\,\frac{r_h^{n-2} - R_h^{2-n}\,r_h^{2n-4}}{D_h}$$

$$b_h = 2n(n-2)\,\frac{R_h^{-n}\,r_h^{2n-4} - r_h^{n-4}}{D_h}$$

$$c_h = -n^2(n-4)\,\frac{R_h^{2-n}\,r_h^{n-2}}{D_h} + n(n-2)^2\,\frac{R_h^{4-n}\,r_h^{n-4}}{D_h} - 4n\,\frac{R_h^{4-2n}\,r_h^{2n-4}}{D_h}$$

$$d_h = n(n-2)(n-4)\,\frac{R_h^{-n}\,r_h^{n-2} - R_h^{2-n}\,r_h^{n-4}}{D_h}$$

with

$$D_h = -4n\,[\,1 + (\frac{r_h}{R_h})^{2n-4}\,] - 2n^2(n-4)(\frac{r_h}{R_h})^{n-2} + n(n-2)^2\,[\,(\frac{r_h}{R_h})^{n} + (\frac{r_h}{R_h})^{n-4}\,]$$

Let us prove (5.9) . For h large enough we have

$$\int_\Omega |\Delta u_h|^2\,dx = t^2 \int_\Omega |\Delta v_h|^2\,dx = t^2 \sum_{i\in I_h(A)} \int_{D_h^i} |\Delta v_h|^2\,dx =$$

$$= t^2\,\mathrm{card}(I_h(A)) \int_{B(R_h)} |\Delta w_h|^2\,dx = t^2\,\frac{\mathrm{meas}(A)}{(2R_h)^n} \int_{B(R_h)} |\Delta w_h|^2\,dx\ .$$

By computing explicitely the last integral we obtain (5.9) .

To prove (5.10) we fix a sequence z_h in $H^2(\Omega)$ converging to t in $H^1(\Omega)$ such that $z_h=0$ q.e. on $A\cap E_h$. It is not restrictive to assume that z_h converge to t weakly in $H^2_{loc}(\Omega)$ Since

$$\int_\Omega |\Delta z_h|^2\,dx - \int_\Omega |\Delta u_h|^2\,dx \geq 2\int_\Omega \Delta u_h \Delta(z_h-u_h)\,dx = -2t\int_\Omega \Delta v_h \Delta(z_h-u_h)\,dx\ ,$$

we have only to prove that

$$\limsup_{h\to\infty} \int_\Omega \Delta v_h \Delta(z_h-u_h)\,dx \leq 0$$

The distribution $\Delta^2 v_h$ can be decomposed in the following way

$$\Delta^2 v_h = \mu_h + \nu_h\,,$$

where

$$\operatorname{supp}\mu_h \subseteq \bigcup_{i\in I_h(A)} \partial D_h^i ,$$

$$\operatorname{supp}\nu_h \subseteq \bigcup_{i\in I_h(A)} \partial B_h^i .$$

Since $z_h-u_h=0$ and $\partial(z_h-u_h)/\partial n = 0$ on $\bigcup_{i\in I_h(A)} \partial B_h^i$ (see (5.5)) we have $<\nu_h,z_h-u_h> = 0$.

Therefore it is enough to prove that

(5.12) $$\lim_{h\to\infty} <\mu_h,z_h-u_h> = 0$$

Since supp $\mu_h \subseteq A \subset\subset \Omega$ and z_h-u_h converge to 0 weakly in $H^2_{loc}(\Omega)$, to obtain (5.12) it is enough to show that μ_h converge in $H^{-2}(\Omega)$.

To this aim we introduce the distributions $\delta_{i,h}$ and $\delta^*_{i,h}$ defined by

(5.13) $$< \delta_{i,h} , \varphi > = \int_{\partial D_h^i} \varphi \, dS$$

and

(5.14) $$< \delta^*_{i,h} ,\varphi > = \int_{\partial D_h^i} \frac{\partial \varphi}{\partial n} \, dS$$

Since

$$<\mu_h,\varphi> = \sum_{i\in I_h(A)} \Big[\int_{\partial D_h^i} \Delta v_h \frac{\partial \varphi}{\partial n} dS - \int_{\partial D_h^i} \frac{\partial \Delta v_h}{\partial n} \varphi \, dS \Big] ,$$

from the explicit expression of w_h we obtain

$$\mu_h = \sum_{i \in I_h(A)} [\, s_h R_h \delta_{i,h} + t_h \delta^*_{i,h} \,]$$

where

$$s_h = \frac{4n(n-2)^2(n-4)}{D_h} R_h^{-n} r_h^{n-4} [\, 1 - (\frac{r_h}{R_h})^n \,]$$

and

$$t_h = \frac{2n(n-2)(n-4)}{D_h} R_h^{-n} r_h^{n-4} (-2R_h^{2-n} r_h^n + n r_h^2 - (n-2) R_h^2) .$$

Note that

$$\lim_{h\to\infty} s_h = -2^n (n-4)(n-2)^2 a \qquad \text{and} \qquad \lim_{h\to\infty} t_h = 0 .$$

Since

(5.15)
$$\sum_{i \in I_h(A)} R_h \delta_{i,h} \to \frac{S_n}{2^n} 1_A \quad \text{in } H^{-1}(\Omega)$$

(see for example [2] lemma (2.3)) and

(5.16)
$$\sum_{i \in I_h(A)} \delta^*_{i,h} \to \Delta (\frac{1}{n} \frac{S^n}{2^n} 1_A) \ \text{ weakly in } H^{-2}(\Omega) ,$$

it is easy to see that μ_h converges strongly in $H^{-2}(\Omega)$ to the function $(n-4)(n-2)^2 a S_n 1_A$.

This proves (5.12) and concludes the proof of the theorem . ∎

We now prove the lemma used at the beginning of the proof of Theorem 5.1 .

LEMMA 5.2. *Let* $G_{\rho(h)}$ *be a subsequence of* G_h . *Suppose that there exist a rich family* $\mathcal{R}$ *of open subsets of* Ω *such that*

(5.17) $$\Gamma(H^1(\Omega)) \lim_{\substack{h\to\infty \\ v\to u}} [\int_\Omega |\Delta v|^2 dx + G_{\rho(h)}(v,A)] = \int_\Omega |\Delta u|^2 dx + aC^* \int_A u^2 dx$$

for every $u \in H^2(\Omega)$ *and for every* $A \in \mathcal{R}$. *Then (5.17) holds for every* $A \in \mathcal{A}$ *with* meas$(\partial A)=0$.

PROOF . Let $G', G'' : H^2(\Omega) \times \mathcal{A} \to [0,+\infty]$ be the functionals defined by

$$\int_\Omega |\Delta u|^2 dx + G'(u,A) = \Gamma(H^1(\Omega)) \liminf_{\substack{h\to\infty \\ v\to u}} [\int_\Omega |\Delta v|^2 dx + G_{\rho(h)}(v,A) ,$$

$$\int_\Omega |\Delta u|^2 dx + G''(u,A) = \Gamma(H^1(\Omega)) \limsup_{\substack{h\to\infty \\ v\to u}} [\int_\Omega |\Delta v|^2 dx + G_{\rho(h)}(v,A) .$$

Then for every $A\in\mathcal{R}$ and for every $u\in H^2(\Omega)$

(5.18) $$aC^* \int_A u^2\,dx = G'(u,A) = G''(u,A)\ .$$

By inner regular approximation , this implies that

(5.19) $$aC^* \int_A u^2\,dx \le G'(u,A) \le G''(u,A)\ .$$

for every $u\in H^2(\Omega)$ and for every $A\in\mathcal{A}$.

Since $G''(u,\bullet)$ is increasing , by (5.18) we have

(5.20) $$G''(u,A) \le \inf\{\, aC^* \int_{A'} u^2\,dx \ :\ A'\supset\supset A\ ,\ A'\in\mathcal{R}\} = aC^* \int_A u^2\,dx$$

So (5.19) and (5.20) imply that

$$aC^* \int_A u^2\,dx \le G'(u,A) \le G''(u,A) \le aC^* \int_A u^2\,dx\ ,$$

which yields (5.18) for every $A\in\mathcal{A}$ with $\mathrm{meas}(\partial A)=0$. ∎

6. EXPLICIT DETERMINATION OF THE LIMIT OF A SEQUENCE OF OBSTACLE PROBLEMS

In this section we study an example of a sequence G_h of obstacle functionals of the form (2.1) where the corresponding obstacles φ_h and ψ_h satisfy the inequality $\varphi_h \le -1$ and $\psi_h \ge 1$. In spite of the fact that the obstacles are well separated, we shall see that there is a strong interaction between lower and upper obstacles in the determination of the limit functional.

Assume $n > 4$. For every $h \in \mathbf{N}$ and for every $i \in \mathbf{Z}^n$ let r_h , R_h , Q_h^i , B_h^i , D_h^i , E_h be as in Section 5. Let $G_h : H^2(\Omega) x \mathcal{A} \to [0,+\infty]$ be the functional defined by

$$G_h(u,A) = \begin{cases} 0 & \text{if } \; -1 \le u \le 1 \; \text{ q.e. on } E_h \cap A , \\ +\infty & \text{otherwise .} \end{cases}$$

Note that G_h is the functional corresponding to the obstacle φ_h and ψ_h defined by

$$\psi_h = -\varphi_h = \begin{cases} 1 & \text{on } E_h \\ +\infty & \text{elsewhere .} \end{cases}$$

To state the convergence theorem , for every $t > 1$ we define

$$C(t) = \min \{ \int_{\mathbf{R}^n} |\Delta u|^2 \, dx \; : \; u \in H^2(\mathbf{R}^n) , \; 1 \le u \le \frac{t+1}{t-1} \; \text{ on } B_n \}$$

where B_n is the unit ball in $\mathbf{R}^n$. By explicit calculation we obtain

$$C(t) = S_n \frac{n}{2} (n-2)(n-4)$$

if $1 < t \leq (n+4)/(n-4)$, and

$$C(t) = S_n\, 4n(n-2)(n-4) \frac{2-n\lambda(t)^{n-2}-(2-n)\lambda(t)^n}{(4-n\lambda(t)^{n-2}+(n-4)\lambda(t)^n)^2}$$

if $t \geq (n+4)/(n-4)$, where $\lambda(t)$ is the unique solution $\lambda \in [0,1[$ of the equation

$$\frac{n(1-\lambda^2)}{4-n\lambda^{n-2}+(n-4)\lambda^n} = \frac{t+1}{t-1} .$$

THEOREM 6.1. *Assume that*

$$\lim_{h\to\infty} (2R_h)^{-n} r_h^{n-4} = a$$

and that $0 < a < +\infty$. *Then for every* $A \in \mathcal{A}$ *with* $\mathrm{meas}(\partial A) = 0$ *and for every* $u \in H^2(\Omega)$

(6.1) $$\Gamma(H^1(\Omega)) \lim_{\substack{h\to\infty \\ v\to u}} [\int_\Omega |\Delta v|^2\, dx + G_h(v,A)] = \int_\Omega |\Delta u|^2\, dx + G(u,A)$$

where

(6.2) $$G(u,A) = \int_A g(u)\, dx$$

and

$$g(t) = \begin{cases} 0 & \text{if } |t| \leq 1 \\ aC(|t|)(|t|-1)^2 & \text{if } |t| > 1 \end{cases}$$

for every $t \in \mathbf{R}$.

REMARK 6.2. For $|t| \leq 1$ the obstacles φ_h and ψ_h have no influence on the limit functional. In fact $g(t) = 0$.

For $1 \leq t \leq (n+4)/(n-4)$ only the effect of the upper obstacles ψ_h is present. Indeed in this case $g(t)$ coincides with the function $g^1(t)$ corresponding to the obstacles $\varphi_h^1 = -\infty$ and $\psi_h^1 = \psi_h$ (see [14] , Theorem 5) .

For $t \geq (n+4)/(n-4)$ the effect of both obstacles φ_h and ψ_h is present and the influence of φ_h in the determination of $g(t)$ increases as t increases.

Note that

$$\lim_{t \to \infty} \lambda(t) = 1,$$

hence

$$\lim_{t \to \infty} C(t) = C^*$$

where C^* is the constant , defined in (5.2) , which occurs in the problem considered in Section 5. This means that for very large t the functions $g(t)$ behaves as the function $g^2(t)$ which corresponds to the sequence of functionals

$$G_h^2(u,A) = \begin{cases} 0 & \text{if } u=1 \text{ q.e. in } E_h \cap A \\ +\infty & \text{otherwise .} \end{cases}$$

PROOF OF THEOREM 6.1. As in the proof of Theorem 5.1, we may assume (6.1) and we have only to prove (6.2). By Theorem 2.4 it is enough to show that

$$(6.4) \qquad G(t,A) = g(t)\,\text{meas}(A)$$

for every $t \in \mathbf{R}$ and for every open set $A \subset\subset \Omega$ which can be espressed as the union of a finite number of open rectangles whose vertices have dyadic coordinates.

Let us fix t and A as required. If $|t| \leq 1$, then (6.4) follows from the fact that $G_h(t,A) = 0$ for every $h \in \mathbf{N}$.

Let us consider the case $t > 1$. To obtain (6.4) we construct an auxiliary sequence of functions w_h . Let $\tau = (t+1)/(t-1)$. For every $\rho > 0$ we denote by $B(\rho)$ the open ball centred in the origin with radius ρ . For every $h \in \mathbf{N}$ let w_h be the solution to the following minimum problem

$$(6.5) \qquad \min \{ \int_{B(R_h)} |\Delta u|^2 \, dx : u \in H_0^2(B(R_h)) \,,\ 1 \leq u \leq \tau \text{ on } B(r_h) \}$$

Starting from w_h we construct the sequence v_h as in (5.7) . The sequence u_h is now defined by $u_h = t-(t-1)v_h$. Then $u_h \in H^2(\Omega)$, $-1 \le u_h \le 1$ on $E_h \cap A$ (for h large enough) and u_h converges to t in $H^1(\Omega)$.

In the sequel we shall give an explicit representation of w_h from which it follows that

(6.6) $$\lim_{h\to\infty} \int_\Omega |\Delta u_h|^2\, dx = aC(t)(t-1)^2 \,\mathrm{meas}(A) .$$

Moreover we shall prove that

(6.7) $$\liminf_{h\to\infty} \int_\Omega |\Delta z_h|^2\, dx \ge \lim_{h\to\infty} \int_\Omega |\Delta u_h|^2\, dx$$

for every sequence z_h in $H^2(\Omega)$ converging to t in $H^1(\Omega)$ such that $-1 \le z_h \le 1$ on $E_h \cap A$. As in Theorem 5.1, from (6.6) and (6.7) we obtain (6.4).

To conclude the proof of the theorem it remains to prove (6.6) and (6.7) .

For every $h \in \mathbf{N}$ there exists $\lambda_h = \lambda_h(\tau) \in [0,1[$ and two spherically symmetric functions w_h^1 and w_h^2 , defined respectively on $B(R_h)-B(r_h)$ and $B(r_h)-B(\lambda_h r_h)$, such that

$$w_h = \begin{cases} w_h^1 & \text{on } B(R_h)\text{-}B(r_h) \\ w_h^2 & \text{on } B(r_h)\text{-}B(\lambda_h r_h) \\ \tau & \text{on } B(\lambda_h r_h) \end{cases}$$

and the triple $(\lambda_h , w_h^1 , w_h^2)$ is the unique solution of the following system

$$
(6.8)\qquad
\begin{aligned}
&\Delta^2 w_h^1 = 0 \quad \text{on} \quad B(R_h)-B(r_h)\,, \qquad \Delta^2 w_h^2 = 0 \quad \text{on} \quad B(r_h)-B(\lambda_h r_h)\\
&w_h^1 = \frac{\partial w_h^1}{\partial n} = 0 \quad \text{on} \quad \partial B(R_h)\\
&w_h^1 = w_h^2 = 1\;, \quad \frac{\partial w_h^1}{\partial n} = \frac{\partial w_h^2}{\partial n}\;, \quad \Delta w_h^1 = \Delta w_h^2 \quad \text{on} \quad \partial B(r_h)\\
&w_h^2 = \tau\;, \quad \frac{\partial w_h^2}{\partial n} = \Delta w_h^2 = 0 \quad \text{on} \quad \partial B(\lambda_h r_h)\,.
\end{aligned}
$$

The dependence of λ_h on τ can not be given explicitly. On the contrary, given a constant $\lambda \in]0,1[$ there exists a unique $\tau = \tau_h(\lambda)$ such that the system (6.8) has a solution with $\lambda_h(\tau) = \lambda$, and this number τ as well as the corresponding functions w_h^1 and w_h^2 can be computed explicitly in terms of λ . More precisely we obtain

$$
(6.9)\qquad
\begin{aligned}
\tau_h = \tau_h(\lambda) = \frac{1}{D_h(\lambda)}\Big\{&\big[n(n-4)(1-\lambda^n) + n^2(\lambda^2-\lambda^{n-2})\big]\Big(\frac{r_h}{R_h}\Big)^{n-2} +\\
&+ 2n(\lambda^{n-2}-\lambda^n)\Big(\frac{r_h}{R_h}\Big)^{2n-4} + n(n-2)(\lambda^{n-2}-1)\Big(\frac{r_h}{R_h}\Big)^{n-4} +\\
&+ n(n-2)(\lambda^n-\lambda^2)\Big(\frac{r_h}{R_h}\Big)^{n} + 2n(1-\lambda^2)\Big\}
\end{aligned}
$$

where

$$D_h(\lambda) = 2\left[4-n\lambda^{n-2}+(n-4)\lambda^n\right] + 2n(n-4)(1-\lambda^n)\left(\frac{r_h}{R_h}\right)^{n-2} + 2n(\lambda^{n-2}-\lambda^n)\left(\frac{r_h}{R_h}\right)^{2n-4} +$$
$$+ n(n-2)(\lambda^{n-2}-1)\left(\frac{r_h}{R_h}\right)^{n-4} + (n-2)\left[(4-n) - n\lambda^{n-2} + 2(n-2)\lambda^n\right]\left(\frac{r_h}{R_h}\right)^n.$$

Let us define

$$\tau_h^{max} = \tau_h(0) = \frac{-n(n-4)\left(\frac{r_h}{R_h}\right)^{n-2} + n(n-2)\left(\frac{r_h}{R_h}\right)^{n-4} - 2n}{(n-2)(n-4)\left(\frac{r_h}{R_h}\right)^{n} - 2n(n-4)\left(\frac{r_h}{R_h}\right)^{n-2} + n(n-2)\left(\frac{r_h}{R_h}\right)^{n-4} - 8} .$$

By (6.9) we have

$$\tau_h(\lambda) < \tau_h^{max}$$

for every $\lambda \in]0,1[$. So for $\tau \geq \tau_h^{max}$ we have $\lambda_h(\tau) = 0$ and the last line in (6.8) disappears.

Using polar coordinates, the expression of w_h^1 and w_h^2 in terms of $\lambda = \lambda_h(\tau)$ is given by

$$\begin{cases} w_h^1(\rho) = \dfrac{a_h(\lambda)}{D_h(\lambda)}\rho^{2-n} + \dfrac{b_h(\lambda)}{D_h(\lambda)}\rho^{4-n} + \dfrac{c_h(\lambda)}{D_h(\lambda)} + \dfrac{d_h(\lambda)}{D_h(\lambda)}\rho^2 \\ w_h^2(\rho) = \dfrac{\alpha_h(\lambda)}{D_h(\lambda)}\rho^{2-n} + \dfrac{\beta_h(\lambda)}{D_h(\lambda)}\rho^{4-n} + \dfrac{\gamma_h(\lambda)}{D_h(\lambda)} + \dfrac{\delta_h(\lambda)}{D_h(\lambda)}\rho^2 \end{cases}$$

where

$$a_h(\lambda) = n(n-4)(\lambda^{n-2}-\lambda^n)R_h^{2-n}r_h^{2n-4} - 2(n-4)(1-\lambda^n)r_h^{n-2}$$

$$b_h(\lambda) = n(n-2)(\lambda^n-\lambda^{n-2})R_h^{-n}r_h^{2n-4} + 2n(1-\lambda^{n-2})r_h^{n-4}$$

$$c_h(\lambda) = n(n-4)(1-\lambda^n)(\frac{r_h}{R_h})^{n-2} + 2n(\lambda^{n-2}-\lambda^n)(\frac{r_h}{R_h})^{2n-4} + n(n-2)(\lambda^{n-2}-1)(\frac{r_h}{R_h})^{n-4}$$

$$d_h(\lambda) = (n-2)(n-4)(\lambda^n-1)R_h^{-n}r_h^{n-2} + n(n-4)(1-\lambda^{n-2})R_h^{2-n}r_h^{n-4}$$

$$\alpha_h(\lambda) = 2(n-4)\lambda^n r_h^{n-2} - n(n-4)\lambda^n R_h^{2-n}r_h^{2n-4} + (n-2)(n-4)\lambda^n R_h^{-n}r_h^{2n-2}$$

$$\beta_h(\lambda) = n^2\lambda^{n-2}R_h^{2-n}r_h^{2n-6} - 2n\lambda^{n-2}r_h^{n-4} - n(n-2)\lambda^{n-2}R_h^{-n}r_h^{2n-4}$$

$$\gamma_h(\lambda) = [n(n-4)(1-\lambda^n) - n^2\lambda^{n-2}](\frac{r_h}{R_h})^{n-2} + 2n(\lambda^{n-2}-\lambda^n)(\frac{r_h}{R_h})^{4-2n} +$$

$$+ n(n-2)(\lambda^{n-2}-1)(\frac{r_h}{R_h})^{n-4} + n(n-2)\lambda^n(\frac{r_h}{R_h})^n + 2n$$

$$\delta_h(\lambda) = n(n-4)R_h^{2-n}r_h^{n-4} - 2(n-4)r_h^{-2} - (n-2)(n-4)R_h^{-n}r_h^{n-2} .$$

To prove (6.7) we fix a sequence z_h in $H^2(\Omega)$ converging to t in $H^1(\Omega)$ such that $-1 \le z_h \le 1$ q.e. on $A \cap E_h$. It is not restrictive to assume that z_h converges to t weakly in $H^2_{loc}(\Omega)$.

Since

$$\int_\Omega |\Delta z_h|^2\, dx - \int_\Omega |\Delta u_h|^2\, dx \geq 2\int_\Omega \Delta u_h \Delta(z_h - u_h)\, dx = -2(t-1)\int_\Omega \Delta v_h \Delta(z_h - u_h)\, dx$$

we have only to prove that

$$\limsup_{h\to\infty} <\Delta^2 v_h, z_h - u_h> \leq 0 \ .$$

The distribution $\Delta^2 v_h$ can be decomposed in the following way:

$$\Delta^2 v_h = \mu_h + \nu_h + \eta_h \ ,$$

where

$$\operatorname{supp} \mu_h \subseteq \bigcup_{i \in I_h(A)} \partial B(x_h^i, R_h) \ ,$$

$$\operatorname{supp} \nu_h \subseteq \bigcup_{i \in I_h(A)} \partial B(x_h^i, r_h) \ ,$$

$$\operatorname{supp} \eta_h \subseteq \bigcup_{i \in I_h(A)} \partial B(x_h^i, \lambda_h r_h) \ .$$

By the minimum property of w_h we have $\nu_h \geq 0$ and $\eta_h \leq 0$. Since $z_h - u_h \leq 0$ on supp ν_h and $z_h - u_h \geq 0$ on supp η_h we have

$$<\nu_h + \eta_h, z_h - u_h> \leq 0$$

for every $h \in \mathbf{N}$. Therefore it is enough to prove that

$$\lim_{h\to\infty} <\mu_h, z_h - u_h> = 0 \tag{6.10}$$

Since $\operatorname{supp} \mu_h \subseteq A \subset\subset \Omega$ and $z_h - u_h$ converges to 0 weakly in $H^2_{loc}(\Omega)$, to obtain (6.10) it is enough to show that μ_h converges in $H^{-2}(\Omega)$.

Since

$$<\mu_h, \varphi> = \sum_{i \in I_h(A)} \Big[\int_{\partial D_h^i} \Delta v_h \frac{\partial \varphi}{\partial n} dS - \int_{\partial D_h^i} \frac{\partial \Delta v_h}{\partial n} \varphi \, dS \Big], \tag{6.11}$$

the distribution μ_h can be expressed in terms of the distributions $\delta_{i,h}$ and $\delta^*_{i,h}$ defined by (5.13) and (5.14). In fact, from (6.11) and from the explicit expression of w_h we obtain

$$\mu_h = \sum_{i \in I_h(A)} [s_h(\lambda_h) R_h \delta_{i,h} + t_h(\lambda_h) \delta^*_{i,h}],$$

where

$$s_h(\lambda) = \frac{1}{D_h} R_h^{-n} r_h^{n-4} [2n(n-2)^2(n-4)(\lambda^{n-2} - \lambda^n)(\frac{r_h}{R_h})^n - 4n(n-2)(n-4)(1-\lambda^{n-2})]$$

and

$$t_h(\lambda) = \frac{1}{D_h} R_h^{-n} r_h^{n-4} [2n(n-2)(n-4)(\lambda^{n-2}-\lambda^n) R_h^{2-n} r_h^n + 2n(n-2)(n-4)(1-\lambda^{n-2}) R_h^2 +$$
$$+ 2n(n-2)(n-4)(\lambda^n - 1) r_h^2] .$$

For every $\lambda \in]0,1[$ we define

(6.12)
$$\tau(\lambda) = \frac{n(1-\lambda^2)}{4-n\lambda^{n-2}+(n-4)\lambda^n} = \lim_{h\to\infty} \tau_h(\lambda) .$$

Let $\lambda(\tau)$ be the inverse function of $\tau(\lambda)$ on the interval $]1,n/4[$, and let $\lambda(\tau) = 0$ for $\tau \geq n/4$. By (6.9) and (6.12) the functions $\tau_h(\lambda)$ and $\tau(\lambda)$ are continuous and invertible on $]0,1[$ so the function $\lambda_h(\tau)$ and $\lambda(\tau)$ are continuous on $]1,\tau_h^{max}[$ and $]1,n/4[$ respectively. Since

$$\lim_{h\to\infty} \tau_h^{max} = n/4 ,$$

it follows from (6.12) that

(6.13)
$$\lambda(\tau) = \lim_{h\to\infty} \lambda_h(\tau)$$

for every $1 < \tau < n/4$. It is easy to see that (6.13) holds also for $\tau \geq n/4$. On the other hand if λ_h converges to $\lambda \in]0,1[$ then

(6.14)
$$\lim_{h\to\infty} s_h(\lambda_h) = s(\lambda) = \frac{2n(n-2)(n-4)(\lambda^{n-2}-1)}{4-n\lambda^{n-2}+(n-4)\lambda^n} \; 2^n a$$

and

(6.15)
$$\lim_{h\to\infty} t_h(\lambda_h) = 0$$

By (5.15), (5.16), (6.13), (6.14), (6.15) it follows that μ_h converges strongly in $H^{-2}(\Omega)$ to $\mu = s(\lambda(\tau))2^{-n} S_n 1_A$ and this concludes the proof of (6.7) .

Equality (6.6) can now be obtained by direct calculation as in Section 5 , by using (6.12) , (6.13) , and the explicit expression of w_h . ∎

and

(6.15) $$\lim_{h\to\infty} t_h(\lambda_h) = 0$$

By (5.15), (5.16), (6.13), (6.14), (6.15) it follows that μ_h converges strongly in $H^{-2}(\Omega)$ to $\mu = s(\lambda(\tau))2^{-n}\, S_n\, 1_A$ and this concludes the proof of (6.7) .

Equality (6.6) can now be obtained by direct calculation as in Section 5 , by using (6.12) , (6.13) , and the explicit expression of w_h . ∎

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Part 1 , Chapter 2 :

Integral representation of some convex local functionals

INTEGRAL REPRESENTATION OF SOME CONVEX LOCAL FUNCTIONALS

SUMMARY. In this paper we prove an integral representation theorem for a class of convex local functionals related to the study of limits of solutions to minimum problems for higher order functionals with bilateral obstacles.

INTRODUCTION

The purpose of this paper is to prove an integral representation theorem for a class of local functionals which arise in the study of the asymptotic behaviour of minimum problems for higher order functionals with bilateral obstacles (see[14]).

Given a bounded open set $\Omega \subseteq \mathbf{R}^n$, we consider a non-negative real valued functional $G(u, B)$ defined for every $u \in W^{m,p}(\Omega)$ and for every Borel set $B \subseteq \Omega$. We assume that G is local, in the sense that $G(u, A) = G(v, A)$ whenever $u = v$ on an open set $A \subseteq \Omega$. Moreover we suppose that $G(\cdot, A)$ is lower semicontinuous on $W^{m,p}(\Omega)$ for every open set $A \subseteq \Omega$ and that $G(u, \cdot)$ is a Borel measure on Ω for every $u \in W^{m,p}(\Omega)$. Finally we assume that for every open set $A \subseteq \Omega$ the function $G(\cdot, A)$ satisfies the following convexity assumption : if $u, v \in W^{m,p}(\Omega)$ and α, β are two real numbers with $0 \leq \alpha \leq \beta \leq 1$, then

$$G(\varphi u + (1-\varphi)v, A) \leq \beta G(u, A) + (1-\alpha)G(v, A) \tag{0.1}$$

for every $\varphi \in C^\infty(\Omega)$ such that $\alpha \leq \varphi(x) \leq \beta$ for all $x \in A$.

We prove that, under these assumptions, the functional G can be written in the form

$$G(u, B) = \int_B g(x, u(x))d\mu(x) + \nu(B),$$

where $g : \Omega \times \mathbf{R} \to [0, +\infty[$ is a non-negative Borel function convex in the last variable, μ is a non negative Radon measure on Ω which vanishes on all Borel subsets of Ω with (m,p)-capacity zero, and ν is a non-negative Radon measure on Ω.

In a forthcoming paper we shall use this integral representation theorem to prove some new results about limits of minimum problems for the functional

$$\int_A |\Delta u|^2 dx$$

with bilateral obstacles .

Integral representation theorems of this kind were proved in [9] , [8] , [4] for functionals defined on $W^{1,2}(\Omega)$ and in [2] for functionals defined on $W^{1,p}(\Omega)$. A similar theorem for functionals defined on $W^{m,p}(\Omega)$ was proved in [7] , where the convexity condition (0.1) is replaced by a monotonicity assumption, which is the natural one in the study of limits of minimum problems with unilateral obstacles (see[2] , [5] , [6] , [14]).

The new assumption (0.1) leads to a deep change in the proof of the integral representation theorem, which is completely different from the proofs of the quoted papers.

1. SOME PROPERTIES OF A CLASS OF LOCAL FUNCTIONALS

In this section we examine the properties of a class $\mathcal{G}$ of local functionals on $W^{m,p}$, related to the study of limits of obstacle problems for higher order functionals.

Let us fix two integers $m, n \geq 1$ and a real number p with $1 < p < +\infty$. For every open set A in $\mathbf{R}^n$ we denote by $W^{m,p}(A)$ the space of all functions $u \in L^p(A)$ such that $D^\alpha u \in L^p(A)$ for every multi-index α with $|\alpha| \leq m$ with the norm

$$||u||_{W^{m,p}} = \{ \sum_{|\alpha| \leq m} ||D^\alpha u||_p^p \}^{\frac{1}{p}}$$

For every compact set $K \subseteq \mathbf{R}^n$ we define the (m,p)-capacity of K by

$$C_{m,p}(K) = \inf \{ ||\varphi||^p_{W^{m,p}(\mathbf{R}^n)} : \varphi \in C_0^\infty(\mathbf{R}^n), \varphi \geq 1 \text{ on } K \}$$

The definition is extended to every open set $A \subseteq \mathbf{R}^n$ by

$$C_{m,p}(A) = \sup \{ C_{m,p}(K) : K \subseteq A , K \text{ compact} \}$$

and to arbitrary sets $E \subseteq \mathbf{R}^n$ by

$$C_{m,p}(E) = \inf \{ C_{m,p}(A) : A \supseteq E , A \text{ open} \} .$$

We say that a property $P(x)$ holds (m,p)-quasi everywhere ((m,p)-q.e.) on a set $E \subseteq \mathbf{R}^n$, if $P(x)$ holds for all $x \in E$ exept for a subset of E with (m,p)-capacity zero.

A function $f : \mathbf{R}^n \to \mathbf{R}$ is said to be (m,p)-quasi continuous if for every $\varepsilon > 0$ there exists a open set A, with $C_{m,p}(A) < \varepsilon$, such that $f_{|A^c}$ is continuous on $A^c = \mathbf{R}^n - A$.

It is well known that every $u \in W^{m,p}(\mathbf{R}^n)$ has an (m,p)-quasi continuous representative which is uniquely determined (m,p)-quasi everywhere (see [13]). We shall identify every function $u \in W^{m,p}(\mathbf{R}^n)$ with its (m,p)-quasi continuous representative.

We say that a non-negative Borel measure μ on $\mathbf{R}^n$ is (m,p)-absolutely continuous if $\mu(B) = 0$ for every Borel set $B \subseteq \mathbf{R}^n$ with $C_{m,p}(B) = 0$.

Let us fix a bounded open set $\Omega \subseteq \mathbf{R}^n$. We denote by $\mathcal{A}$ (resp. $\mathcal{B}$) the family of all open (resp. Borel) subsets of Ω .

DEFINITION 1.1. We denote by $\mathcal{G}$ the class of all functionals $G : W^{m,p}(\Omega) \times \mathcal{B} \to [0, +\infty]$ with the following properties

(a) for every $A \in \mathcal{A}$, the function $u \to G(u, A)$ is lower semicontinuous in $W^{m,p}(\Omega)$;

(b) for every $u \in W^{m,p}(\Omega)$, the function $B \to G(u, B)$ is a Borel measure on Ω ;

(c) if $u, v \in W^{m,p}(\Omega)$, $A \in \mathcal{A}$, and $u_{|A} = v_{|A}$, then $G(u, A) = G(v, A)$;

(d) if $u, v \in W^{m,p}(\Omega)$, $A \in \mathcal{A}$, $\alpha, \beta \in \mathbf{R}$ with $0 \leq \alpha \leq \beta \leq 1$, then

$$G(\varphi u + (1-\varphi)v, A) < \beta G(u, A) + (1-\alpha)G(v, A)$$

for every $\varphi \in C^\infty(\Omega)$ such that $\alpha \leq \varphi(x) \leq \beta$ for all $x \in A$.

We give now some examples of functionals belonging to the class $\mathcal{G}$.

EXAMPLE 1.2 (Obstacle functionals).

Let $\varphi, \psi : \Omega \to \overline{\mathbf{R}}$ be two functions such that $\varphi \leq \psi$ (m,p)-q.e. in Ω and let $G : W^{m,p}(\Omega) \times \mathcal{B} \to [0,+\infty]$ be the functional defined by :

$$G(u, B) = \begin{cases} 0 & \text{if } \varphi \leq u \leq \psi \quad \text{(m,p)-q.e. on } B \\ \\ +\infty & \text{otherwise} . \end{cases}$$

Then G belongs to the class $\mathcal{G}$.

EXAMPLE 1.3 (Integral functionals).

Let $g : \Omega \times \mathbf{R} \to [0,+\infty]$ be a Borel function and let μ, ν be two positive Radon measures on Ω, such that μ is (m,p) absolutely consinuous. We denote by G the functional defined by

$$(1.1) \qquad G(u, B) = \int_B g(x, u(x))\, d\mu(x) + \nu(B)$$

for every $B \in \mathcal{B}$ and for every $u \in W^{m,p}(\Omega)$. If the function $t \to g(x,t)$ is convex and lower semicontinuous on $\mathbf{R}$ for every $x \in \Omega$, then G belongs to the class $\mathcal{G}$.

EXAMPLE 1.4 (Functionals depending on u and grad u).

Let $m = p = 2$, $n = 1$, and $\Omega =]-1, 1[$. Let $F, G : W^{2,2}(\Omega) \times \mathcal{B} \to [0, +\infty]$ be the functionals defined by

$$F(u, B) = \begin{cases} 0 & \text{if } 0 \notin B, \\ (u'(0))^2 & \text{if } 0 \in B \text{ and } u(0) = 0, \\ +\infty & \text{if } 0 \in B \text{ and } u(0) \neq 0, \end{cases}$$

$$G(u, B) = \begin{cases} 0 & \text{if } 0 \notin B, \\ 0 & \text{if } 0 \in B,\ u(0) = 0 \text{ and } |u'(0)| \leq 1, \\ +\infty & \text{if } 0 \in B,\ u(0) \neq 0 \text{ and} \\ +\infty & \text{if } 0 \in B,\ u(0) = 0 \text{ and } |u'(0)| > 1. \end{cases}$$

It is easy to verify that F and G belong to the class $\mathcal{G}$.

EXAMPLE 1.5 (Γ-limits of obstacles for second order quadratic functionals).

Let $p = m = 2$ and let $\varphi_h , \psi_h : \Omega \to \overline{\mathbf{R}}$ be two sequences of functions such that $\varphi_h \leq \psi_h$ (2,2)-q.e. on Ω for every $h \in \mathbf{N}$. We consider the sequence of functionals G_h of the class $\mathcal{G}$ defined by

$$G_h(u,B) = \begin{cases} 0 & \text{if } \varphi_h \leq u \leq \psi_h \quad \text{(2,2)-q.e. on } B \\ +\infty & \text{otherwise} \end{cases}$$

Let $\mathcal{A}'$ be a family of open subsets of Ω with the property that for every $A_1 , A_2 \in \mathcal{A}$ with $A_1 \subset\subset A_2$ there exists $A' \in \mathcal{A}'$ such that $A_1 \subset\subset A' \subset\subset A_2$. Suppose that for every $A' \in \mathcal{A}'$ the sequence of functionals

$$\int_{A'} |\Delta u|^2 dx + G_h (u,A')$$

Γ-converges in $W^{1,2}(\Omega)$ to the functional

$$\text{(1.2)} \qquad \int_{A'} |\Delta u|^2 dx + G (u,A')$$

where $G : W^{2,2}(\Omega) \times \mathcal{B} \to [0, +\infty]$ satisfies

$$G(u , A) = \sup\{G(u , A') : A' \in \mathcal{A}' , A' \subset\subset A\} .$$

Then G belongs to the class $\mathcal{G}$, as we shall prove in a forthcoming paper. More generally, the same result holds if we assume only that each functional G_h belongs to the class $\mathcal{G}$. In other words, the class of all functionals of the form (1.2), with $G \in \mathcal{G}$, is closed for the Γ-convergence in $W^{1,2}(\Omega)$.

DEFINITION 1.6. We denote by $\mathcal{G}_0$ the class of all functionals $G \in \mathcal{G}$ such that $G(k, A) < +\infty$ for every $A \in \mathcal{A}$ and for every $k \in \mathbf{R}$.

REMARK 1.7. Properties (b) and (c) of Definition 1.1 imply that, if $G \in \mathcal{G}_0$, then $G(u, B) = G(v, B)$ for every $B \in \mathcal{B}$ and for every pair u, v of functions of $W^{m,p}(\Omega)$ which coincide on a neighbourhood of B.

We shall prove that every functional G of the class $\mathcal{G}_0$ admits an integral representation like (1.1). The same result does not hold in the larger class $\mathcal{G}$, as Example 1.4 shows. We observe that the functional G of Example 1.4 can be obtained as Γ-limit of a sequence of obstacle functionals. In fact, if in Example 1.5 we take $n = 1$, $\Omega =]-1, 1[$, and

$$\psi_h(x) = -\varphi_h(x) = \begin{cases} |x| & \text{if } |x| \leq \frac{1}{h}, \\ +\infty & \text{oterwise}, \end{cases}$$

then the Γ-limit in $W^{1,2}(\Omega)$ of the sequence

$$\int_A |u''|^2 dx + G_h(u,A)$$

is given by

$$\int_A |u''|^2 dx + G(u,A)$$

for every $A \in \mathcal{A}$ such that $0 \notin \partial A$, where G is the functional defined in Example 1.4. This proves that the integral representation (1.1) does not hold for all functionals G which belong to the Γ-closure of the obstacle functionals described in Example 1.5.

Before proving the integral representation theorem, we study some properties of the functionals of the class $\mathcal{G}_0$. Let us fix $G \in \mathcal{G}_0$.

PROPOSITION 1.8. Let $A \in \mathcal{A}$. Then $G(u, A) < +\infty$ for every $u \in W^{m,p}(\Omega) \cap L^\infty(A)$.

PROOF. We fix $u \in W^{m,p}(\Omega) \cap L^\infty(A)$. Then there exist two constants k', k'' such that $k' \leq u(x) \leq k''$ for every $x \in A$. For every $A' \in \mathcal{A}$, with $A' \subset\subset A$, there exists a sequence u_h of functions in $W^{m,p}(\Omega) \cap C^\infty(\Omega) \cap L^\infty(A')$ converging to u in $W^{m,p}(\Omega)$, such that $k' \leq u_h(x) \leq k''$ for every $x \in A'$ and $h \in \mathbf{N}$. Therefore there exists a sequence φ_h in $C^\infty(\Omega)$, with $0 \leq \varphi_h(x) \leq 1$ for every $x \in A'$ such that $u_h = \varphi_h k' + (1 - \varphi_h)k''$. From the properties (b) and (d) of the functional G it follows that

$$G(u, A') \leq \liminf_{h \to \infty} G(u_h, A') \leq G(k', A) + G(k'', A')$$

for every $A' \in \mathcal{A}$ with $A' \subset\subset A$. Taking the limit as $A' \uparrow A$ we obtain

$$G(u,A) \leq G(k',A) + G(k'', A) < +\infty,$$

which proves the proposition. ∎

PROPOSITION 1.9. Let $A \in \mathcal{A}$ and let $u, v \in W^{m,p}(\Omega) \cap L^\infty(A)$. Then

$$G(\varphi u + (1-\varphi)v, A) \leq \int_A \varphi(x)\, G(u, dx) + \int_A (1-\varphi(x))\, G(v, dx)$$

for every function $\varphi \in C^\infty(\Omega)$ with $0 \leq \varphi \leq 1$ on A.

PROOF. Since $G(\varphi u + (1-\varphi)v, A) < +\infty$, for every $\varepsilon > 0$ it is possible to find $n + 1$

real numbers t_i, with $0 = t_o < t_1 < ... < t_n = 1$ and $|t_i - t_{i-1}| < \varepsilon$ for every $i = 1, ..., n$, such that the sets $E_i = \{x \in A : \varphi(x) = t_i\}$ satisfy $G(\varphi u + (1 - \varphi)v , E_i) = 0$ for every $i = 1,2, ..., n-1$. We set

$$A_1 = \{ x \in A : t_1 > \varphi(x) \}$$
$$A_i = \{ x \in A : t_{i-1} < \varphi(x) < t_i \} \qquad i = 2,...,n-1$$
$$A_n = \{ x \in A : \varphi(x) > t_{n-1} \}.$$

Then

$$A = (\bigcup_{i=1}^{n} A_i) \cup N$$

where $G(\varphi u + (1 - \varphi)v , N) = 0$. From the properties (b) and (d) of the functional G it follows that

$$G(\varphi u + (1-\varphi)v , A) = \sum_{i=1}^{n} G(\varphi u + (1-\varphi)v , A_i) \leq$$

$$\sum_{i=1}^{n} [t_i G(u , A_i) + (1-t_{i-1})G(v , A_i)] =$$

$$\sum_{i=1}^{n} [\int_{A_i} t_{i-1}G(u , dx) + (1-t_i)G(v , dx)] + \sum_{i=1}^{n} (t_i-t_{i-1}) [G(u , A_i) + G(v , A_i)] \leq$$

$$\int_A \varphi(x)G(u , dx) + \int_A (1-\varphi(x))G(v , dx) + \varepsilon [G(u , A) + G(v , A)] .$$

Since $\varepsilon > 0$ is arbitrary, the proposition is proved. ∎

Let us consider the set function $\nu : \mathcal{B} \to [0 , +\infty[$ defined by

(1.3) $$\nu (B) = \inf \{ G(u ,B) : u \in W^{m,p}(\Omega) \cap L^\infty(\Omega) \}$$

for every $B \in \mathcal{B}$.

PROPOSITION 1.10. The set function ν is a Radon measure on Ω.

PROOF. Since $\nu(B) = \inf\{ \nu(A) : A \supseteq B , A \in \mathcal{A} \}$, we have only to prove that ν is subadditive, superadditive , and inner regular on $\mathcal{A}$ (see[10] , Theorem 5.6). The last two properties are proved in a similar context in [4] (Lemma 3.5). So we have only to verify that ν is subadditive. Let A_1, A_2 , $A_2 \in \mathcal{A}$, with $A_2 \subset\subset A_2$. For every $\varepsilon > 0$ there exist two functions u_1 , u_2 in $W^{m,p}(\Omega) \cap L^\infty(\Omega)$ such that :

$$\nu(A_1) + \frac{\varepsilon}{2} > G(u_1 , A_1) \qquad \text{and} \qquad \nu(A_2) + \frac{\varepsilon}{2} > G(u_2, A_2)$$

Let $\varphi \in C^\infty(A_2)$, with $0 \le \varphi \le 1$ and $\varphi = 1$ in a neighbourhood of A_2 . We set $u = (1 - \varphi)u_1 + \varphi u_2$. Then from properties (b), (c), and (d) of Definition 1.1 and from Remark 1.7 it follows that

$$\nu(A_1 \cup A_2') \le G(u , A_1 \cup A_2') \le$$

$$G(u_1 , A_1 - A_2) + G(u_2 , \overline{A}_2') + G((1-\varphi)u_1 + \varphi u_2 , (A_2 - \overline{A}_2') \cap A_1) \le$$

$$G(u_1 , A_1 - A_2) + G(u_2 , \overline{A}_2') + G(u_1 , (A_2 - \overline{A}_2') \cap A_1) + G(u_2 , (A_2 - \overline{A}_2') \cap A_1) \le$$

$$G(u_1 , A_1) + G(u_2 , A_2) < \nu(A_1) + \nu(A_2) + \varepsilon.$$

As $\varepsilon \downarrow 0$ and $A_2 \uparrow A_2$ we obtain

$$\nu(A_1 \cup A_2) \le \nu(A_1) + \nu(A_2)$$

which proves that ν is subadditive on $\mathcal{A}$. ∎

THEOREM 1.11. For every $u \in W^{m,p}(\Omega) \cap L^\infty(\Omega)$ the measure $G(u,\cdot) - \nu(\cdot)$ is (m,p)-absolutely continuous.

To prove Proposition 1.11 we need the following lemmas.

LEMMA 1.12. Let $A\in\mathcal{A}$, let $u,v\in W^{m,p}(\Omega)\cap L^\infty(A)$, and let $k\in\mathbf{R}$ be such that $v\le u\le v+k$ (m,p)-q.e. on A. Then

$$(1.4)\qquad G(u, B) - \nu(B) \le [\, G(v, B) - \nu(B)\,] + [\, G(v+k, B) - \nu(B)\,]$$

for every Borel set $B\subseteq A$.

PROOF. It is enough to prove the lemma for $B = A$. Let $w = u - v$. Then $w\in W^{m,p}(\Omega)\cap L^\infty(A)$ and $0\le w\le k$ (m,p)-q.e. on A. Therefore for every $A'\in\mathcal{A}$, with $A'\subset\subset A$, there exists a sequence w_h of functions in $C^\infty(\Omega)\cap W^{m,p}(\Omega)\cap L^\infty(A')$, converging to w in $W^{m,p}(\Omega)$, such that $0\le w_h\le k$ (m,p)-q.e. on A' for every $h\in\mathbf{N}$. If u_h, φ_h are the functions defined by $u_h = v + w_h$ and $\varphi_h = (k - w_h)/k$, then $\varphi_h\in C^\infty(\Omega)$, $0\le\varphi_h\le 1$ on A', $u_h = \varphi_h v + (1-\varphi_h)(v+k)$, and u_h converges to u in $W^{m,p}(\Omega)$. From Proposition 1.9 it follows that

$$G(u_h, A') - \nu(A') \le$$

$$\int_{A'} \varphi_h(x)\,[\, G(v, dx) - \nu(dx)\,] + \int_{A'} [1-\varphi_h(x)]\,[\, G(v+k, dx) - \nu(dx)\,] \le$$

$$[\, G(v, A') - \nu(A')\,] + [\, G(v+k, A') - \nu(A')\,].$$

Therefore, by the lower semicontinuity of $G(\cdot,A')$ in $W^{m,p}(\Omega)$ we have

$$G(u, A') - \nu(A') \le \liminf_{h\to\infty} [\, G(u_h, A') - \nu(A')\,] \le$$

$$[\, G(v, A') - \nu(A')\,] + [\, G(v+k, A') - \nu(A')\,].$$

Taking the limit as $A' \uparrow A$ we obtain (1.4) for $B = A$. ∎

LEMMA 1.13. Let $B \in \mathcal{B}$ with $C_{m,p}(B) = 0$. Then

$$G(v+k, B) - \nu(B) \leq G(v, \Omega) - \nu(\Omega)$$

for every $k \in \mathbf{R}$ and for every $v \in W^{m,p}(\Omega) \cap L^\infty(\Omega)$.

PROOF. It is enough to prove the lemma when $B = K$ is compact subset of Ω. Let $C_{m,p}(K) = \inf ||\varphi||_{W^{m,p}(\mathbf{R}^n)}$, where the infimum is over all $\varphi \in C_0^\infty(\mathbf{R}^n)$ such that $0 \leq \varphi \leq 1$ on $\mathbf{R}^n$ and $\varphi = 1$ in a neighbourhood of K. Since $C_{m,p}(K) = 0$, we have also $C_{m,p}(K) = 0$ (see[1], Theorem A, and [12], Section 9.3.2). Therefore for every $h \in \mathbf{N}$ there exists a function $\varphi_h \in C^\infty(\mathbf{R}^n)$ and an open set $A_h \subseteq \mathbf{R}^n$ such that $A_h \supseteq K$, $\varphi_h = 1$ on A_h, $0 \leq \varphi_h \leq 1$ on $\mathbf{R}^n$, and $||\varphi_h||_{W^{m,p}(\mathbf{R}^n)} < 1/h$. Let

$$A_h = (\bigcap_{i=1}^{h} A_i') \cap \Omega ,$$

so that $A_h \supseteq A_{h+1}$ for every $h \in \mathbf{N}$. We may assume that

$$K = \bigcap_{h=1}^{\infty} A_h . \tag{1.5}$$

We fix now $v \in W^{m,p}(\Omega) \cap L^\infty(\Omega)$ and $k \in \mathbf{R}$. Let $w_h = v + k(1 - \varphi_h)$; then $w_h = v$ (m,p)-q.e. on A_h, the sequence w_h converges to $v+k$ in $W^{m,p}(\Omega)$, $v \leq w_h \leq v+k$ on Ω for $k > 0$, and $v + k \leq w_h \leq v$ on Ω for $k < 0$. Using Lemma 1.12 and the properties (b) and (c) of the functional G we obtain :

$$\begin{aligned} G(w_h, \Omega) - \nu(\Omega) = {} & G(w_h, \Omega - A_h) - \nu(\Omega - A_h) + G(v, A_h) - \nu(A_h) \leq \\ & G(w_h, \Omega - K) - \nu(\Omega - K) + G(v, A_h) - \nu(A_h) \leq \end{aligned} \tag{1.6}$$

$$G(v\,,\Omega\!-\!K)-\nu(\Omega\!-\!K)+G(v\!+\!k\,,\Omega\!-\!K)-\nu(\Omega\!-\!K)+G(v\,,A_h)-\nu(A_h)$$

On the other hand the functional $G(\bullet,\Omega)$ is lower semicontinuous on $W^{m,p}(\Omega)$, thus we have

(1.7) $$G(v+k\,,\Omega-K)-\nu(\Omega-K)+G(v+k\,,K)-\nu(K)\leq \liminf_{h\to\infty} G(w_h\,,\Omega)-\nu(\Omega)\,.$$

Since $G(v,K)-\nu(K)=\lim_{h\to\infty} G(v,A_h)-\nu(A_h)$ by (1.5) , from (1.16) and (1.17) it follows that

$$G(v+k\,,K)-\nu(K)\leq G(v\,,\Omega)-\nu(\Omega)\,,$$

which concludes the proof of the lemma . ∎

PROOF OF THEOREM 1.11. We fix $u\in W^{m,p}(\Omega)\cap L^{\infty}(\Omega)$. Let $B\in\mathcal{B}$ with $C_{m,p}(B)=0$. Since

$$\inf\{\,G(u\,,\Omega)-\nu(\Omega):u\in W^{m,p}(\Omega)\cap L^{\infty}(\Omega)\,\}=0\,,$$

for each $\varepsilon>0$ there exists a function $v_\varepsilon\in W^{m,p}(\Omega)\cap L^{\infty}(\Omega)$ such that

$G(v_\varepsilon\,,\Omega)-\nu(\Omega)<\varepsilon/2$. On the other hand there exists two constants $k'\,,k''\in\mathbf{R}$ such that

$$v_\varepsilon+k'\leq u\leq v_\varepsilon+k'' \qquad (m,p)\text{-q.e. on }\Omega\,.$$

Using Lemmas 1.12 and 1.13 we obtain

$$G(u\,,B)-\nu(B)\leq$$

$$[\,G(v_\varepsilon+k'\,,B)-\nu(B)\,]+[\,G(v_\varepsilon+k''\,,B)-\nu(B)\,]\leq$$

$$2\,[\,G(v_\varepsilon\,,\Omega)-\nu(\Omega)\,]<\varepsilon\,.$$

The theorem follows now by taking the limit as ε goes to 0 . ∎

2. AN INTEGRAL REPRESENTATION THEOREM

In this section we shall prove that every functional G of the class $\mathcal{G}_0$ can be represented in the form

$$G(u,B) = \int_B g(x,u(x))\,d\mu(x) + \nu(B) \tag{2.1}$$

where g , μ satisfy the hypothesis of Example 1.3 and ν is the Radon measure defined in (1.3) .

Let us fix a functional G of the class $\mathcal{G}_0$.

The measure μ, occurring in (2.1) will be constructed by means of the sequence of measures μ_k defined by

$$\mu_k(B) = [\,G(k\,,B) - \nu(B)\,] + [\,G(-k\,,B) - \nu(B)\,]$$

for every $B\in\mathcal{B}$.

REMARK 2.1. For every $k\in\mathbf{N}$ the measure μ_k is (m,p)-absolutely continuous by Proposition 1.11. Moreover we deduce easily from Lemma 1.12 that

$$G(u\,,A) - \nu(A) \le \mu_k(A)$$

for every $A\in\mathcal{A}$ and $u\in W^{m,p}(\Omega)\cap L^\infty(A)$ with $\|u\|_{L^\infty(A)} \le k$.

In the following lemma we prove that an integral representation formula like (2.1) holds in the class of all constant functions.

LEMMA 2.2. For every $k\in\mathbf{N}$ there exists a Borel function $g_k : \Omega\times]-k\,,k[\to[0\,,1]$ such that :

(a) for every $B\in\mathcal{B}$ and for every $t\in\,]-k\,,k[$

$$G(t\,,B)=\int_B g_k(x\,,t)\,d\mu_k(x)+\nu(B)\,;$$

(b) for every $x\in\Omega$ the function $t\to g_k(x\,,t)$ is continuous and convex on $]-k\,,k[$.

PROOF. We fix $k\in\mathbf{N}$. As we observed in Remark 2.1 , if $t\in\,]-k\,,k[$ then $G(t\,,A)-\nu(A)\le\mu_k(A)$ for every $A\in\mathcal{A}$. By the Radon-Nikodym theorem there exists a Borel function $f_k(\cdot,t):\Omega\to[0\,,1]$ such that

$$G(t\,,B)=\int_B f_k(x\,,t)\,d\mu_k(x)+\nu(B) \tag{2.2}$$

for every $B\in\mathcal{B}$.

We shall prove now that there exists a set $N\in\mathcal{B}$, with $\mu_k(N)=0$, such that the function $t\to f_k(x\,,t)$ is convex on $]-k\,,k[\cap Q$.

Let $t_1\,,t_2\in\,]-k\,,k[\cap Q$, $\lambda\in[0\,,1]\cap Q$, and $t=\lambda t_1+(1-\lambda)t_2$. Since G is a convex functional there exists a set $N(\lambda\,,t_1\,,t_2)\in\mathcal{B}$, with $\mu_k(N(\lambda\,,t_1\,,t_2))=0$, such that

$$f_k(x\,,t)\le\lambda f_k(x\,,t_1)+(1-\lambda)f_k(x\,,t_2)$$

for every $x\in\Omega-N(\lambda\,,t_1\,,t_2)$. Let N be the union of all sets $N(\lambda\,,t_1\,,t_2)$ for $t_1\,,t_2\in\,]-k\,,k[\cap Q$ and $\lambda\in[0\,,1]\cap Q$. Then $\mu_k(N)=0$ and $f_k(x\,,\cdot)$ is convex on $]-k\,,k[\cap Q$ for every $x\in\Omega-N$.

Since f_k is bounded for every $x\in\Omega-N$, the function $f_k(x\,,\cdot)$ is locally Lipschitz on $]-k\,,k[\cap Q$ and this property guarantees the existence of the limit

$$\lim_{\substack{s\to t\\ s\in Q}} f_k(x\,,s)$$

for every $x \in \Omega - N$ and for every $t \in]-k, k[$.

Let $g_k : \Omega x]-k, k[\to [0, 1]$ be the function defined by

$$g_k(x, t) = \begin{cases} \lim\limits_{s \in Q} f_k(x, s) & \text{if } x \in \Omega - N \\ 0 & \text{if } x \in N \end{cases}$$

It is easy to check that $g_k(x, t) = f_k(x, t)$ for every $x \in \Omega - N$ and $t \in]-k, k[\cap Q$ and that the function g is convex and continuous in t for every $x \in \Omega$.

Since for every $A \in \mathcal{A}$ the function $t \to G(t, A)$ is finite and convex (hence continuous) on $]-k, k[$, we can extend the representation formula (2.2) by continuity. Thus we obtain

$$G(t, A) = \int_A g_k(x,t)\, d\mu_k(x) + \nu(A)$$

for every $A \in \mathcal{A}$ and $t \in]-k, k[$. The extension of this equality to all Borel subsets of Ω is trivial. ∎

The following lemmas provide some continuity properties of the functional G which will be essential in the proof of the integral representation theorem for arbitrary functions in $W^{m,p}(\Omega) \cap L^\infty(\Omega)$.

First we give an estimate of the modulus of continuity of G in $L^\infty(A)$ for every $A \in \mathcal{A}$.

LEMMA 2.3. Let $A \in \mathcal{A}$ and let $\eta > 0$. Then

$$| G(u, A) - G(v, A) | \leq \frac{1}{\eta} \mu_k(A) \, ||u-v||_{L^\infty(A)}$$

for every $u, v \in W^{m,p}(\Omega) \cap L^\infty(A)$ such that $||u||_{L^\infty(A)} \leq k - \eta$, $||v||_{L^\infty(A)} \leq k - \eta$.

PROOF. We consider the space $W^{m,p}(\Omega) \cap L^\infty(A)$ with the seminorm $||\cdot||_{L^\infty(A)}$. Since $G(\cdot, A)$ is convex and

$$0 \leq G(u, A) - \nu(A) \leq \mu_k(A)$$

for every $u \in W^{m,p}(\Omega) \cap L^\infty(A)$ with $||u||_{L^\infty(A)} \leq k$, the thesis of the lemma follows easily from the well known estimate of the Lipschitz constant of a convex function on a normed space (see, for instance, [11], Chapter I, Corollary 2.4). ∎

We now prove that $G(\cdot, A)$ is continuous in $L^1(A, \mu_{4k})$.

LEMMA 2.4. Let $A \in \mathcal{A}$. Then

$$|G(u, A) - G(v, A)| \leq \int_A |u - v| \, d\mu_{4k} \tag{2.3}$$

for every $u, v \in W^{m,p}(\Omega) \cap L^\infty(A)$ with $||u||_{L^\infty(A)} \leq k$ and $||v||_{L^\infty(A)} \leq k$.

PROOF. Let $u, v \in W^{m,p}(\Omega) \cap L^\infty(A)$ be such that $u - v \in C^\infty(\Omega)$ and $||u||_{L^\infty(A)} \leq 3k$.

For each $\varepsilon > 0$ we take $n+1$ points t_i in $\mathbf{R}$, with $0 = t_o < t_1 < \ldots < t_n = 6k$ and $|t_i - t_{i-1}| < \varepsilon$ for $i = 1, 2, \ldots, n$, such that $\mu_k(\{x \in A : |u(x) - v(x)| = t_i\}) = 0$ for $i = 1, 2, \ldots, n-1$. Let $A_1, \ldots, A_n$ be the open sets defined by

$$A_1 = \{x \in A : |u(x) - v(x)| < t_1\}$$

$$A_i = \{x \in A : t_{i-1} < |u(x) - v(x)| < t_i\} \qquad i = 2, \ldots, n-1$$

$$A_n = \{x \in A : |u(x) - v(x)| > t_{n-1}\}.$$

Then, using Lemma 2.3 with $\eta = k$, we obtain

$$| G(u , A) - G(v , A) | \leq \sum_{i=1}^{n} | G(u , A_i) - G(v , A_i) | \leq$$

$$\frac{1}{k} \sum_{i=1}^{n} \mu_{4k}(A_i)\, t_i \leq \sum_{i=1}^{n} \mu_{4k}(A_i)(t_i - t_{i-1}) + \sum_{i=1}^{n} \mu_{4k}(A_i) t_{i-1} <$$

$$\varepsilon\, \mu_{4k}(A) + \sum_{i=1}^{n} \int_{A_i} |u-v|\, d\mu_{4k} = \varepsilon\, \mu_{4k}(A) + \int_{A} |u-v|\, d\mu_{4k} .$$

Since $\varepsilon > 0$ is arbitrary, (2.3) holds when $u - v \in C^\infty(\Omega)$ and $||u||_{L^\infty(A)} \leq 3k$, $||v||_{L^\infty(A)} \leq 3k$.

Let now $u, v \in W^{m,p}(\Omega) \cap L^\infty(\Omega)$ be such that $||u||_{L^\infty(A)} \leq k$ and $||v||_{L^\infty(A)} \leq k$. We set $w = u - v$, and for every $A' \in \mathcal{A}$, with $A' \subset\subset A$, we consider a sequence w_h in $C^\infty(\Omega) \cap W^{m,p}(\Omega)$, converging to w in $W^{m,p}(\Omega)$, such that $||w_h||_{L^\infty(A')} \leq 2k$ for every $h \in \mathbf{N}$. Let $u_h = w_h - v$; then $u_h - v \in C^\infty(\Omega)$ and $||u_h||_{L^\infty(A')} \leq 3k$ for every $h \in \mathbf{N}$. Moreover u_h converges to u in $W^{m,p}(\Omega)$, thus by passing to a subsequence, we may also assume that u_h converges to u (m,p)-q.e. on Ω (see [13]), hence μ_{4k}-q.e. on Ω.

From the previous step of the proof it follows that

$$G(u_h , A') \leq G(v , A') + \int_{A'} |u_h - v|\, d\mu_{4k} .$$

Therefore, by the lower semicontinuity of G and by the dominated convergence theorem we have

$$G(u , A') \leq \liminf_{h\to\infty} G(u_h , A') \leq G(v , A') + \int_{A'} |u-v|\, d\mu_{4k} .$$

Taking the limit as $A' \uparrow A$ we obtain

$$G(u, A) \leq G(v, A) + \int_A |u - v| \, d\mu_{4k}.$$

By exchanging the roles of u and v we obtain (2.3) . ∎

THEOREM 2.5. Let G be a functional of the class $\mathcal{G}_0$ and let ν be the Radon measure defined in (1.3). Then there exist an (m,p)-absolutely continuous Radon measure μ on Ω and a Borel function $g : \Omega \times \mathbf{R} \to [0, +\infty[$, such that :

(a) $$G(u, A) = \int_A g(x, u(x)) \, d\mu(x) + \nu(A)$$

for every $A \in \mathcal{A}$ and for every $u \in W^{m,p}(\Omega) \cap L^\infty(A)$;

(b) for every $x \in \Omega$ the function $t \to g(x,t)$ is continuous and convex on $\mathbf{R}$.

PROOF. Step 1. Let $A \in \mathcal{A}$, $k \in \mathbf{N}$, and $u \in C^\infty(\Omega) \cap W^{m,p}(\Omega) \cap L^\infty(A)$ such that $||u||_{L^\infty(A)} < k$ and let $\eta = k - ||u||_{L^\infty(A)}$. For each $\varepsilon > 0$ let t_0, t_n such that $-||u||_{L^\infty(A)} = t_0 < t_1 < \ldots < t_n = ||u||_{L^\infty(A)}$, $|t_i - t_{i-1}| < \varepsilon$ for every $i = 1, \ldots, n$, and $\mu_k(\{x \in A : u(x) = t_i\}) = 0$ for every $i = 1, \ldots, n-1$. Let $A_1, \ldots A_n$ be the open sets defined by

$$A_1 = \{x \in A : u(x) < t_1\}$$

$$A_i = \{x \in A : t_{i-1} < u(x) < t_i\} \qquad i = 2, \ldots, n-1$$

$$A_n = \{x \in A : u(x) > t_{n-1}\}.$$

Then $A = (\bigcup_{i=1} A_i) \cup N$ where $\mu_k(N) = 0$. Since $||u||_{L^\infty(A)} < k$ we have also $G(u, N) - \nu(N) = 0$ and from Lemmas 2.2 and 2.3 it follows that

(2.4) $$G(u, A) - \nu(A) = \sum_{i=1}^{n} [G(u, A_i) - \nu(A_i)] <$$

$$< \sum_{i=1}^{n} [G(t_i, A_i) - \nu(A_i) + \frac{\varepsilon}{\eta} \mu_k(A_i)] =$$

$$= \sum_{i=1}^{n} [\int_{A_i} g_k(x, t_i) d\mu_k(x) + \frac{\varepsilon}{\eta} \mu_k(A_i)$$

Since $g_k(x, \cdot)$ is a convex function and $0 \le g_k(x, t) \le 1$ for $|t| < k$, we have $|g_k(x, s) - g_k(x, t)| \le_\eta |s - t|$ for every $s, t \in \mathbf{R}$ such that $|t| \le ||u||_{L^\infty(A)}$, $|s| \le ||u||_{L^\infty(A)}$. Then (2.4) hields

(2.5) $$G(u, A) - \nu(A) < \int_A g_k(x, u(x)) d\mu_k(x) + \frac{2\varepsilon}{\eta} \mu_k(A)$$

Since (2.5) holds for every $\varepsilon > 0$ we have

$$G(u, A) = \int_A g_k(x, u(x)) d\mu_k(x) + \nu(A)$$

We can now exchange the roles of G and of the integral to obtain the inverse inequality.

Hence

(2.6) $$G(u, A) = \int_A g_k(x, u(x)) d\mu_k(x) + \nu(A)$$

for every $u \in C^\infty(\Omega) \cap W^{m,p}(\Omega) \cap L^\infty(A)$ with $||u||_{L^\infty(A)} < k$.

Step 2. Let $A \in \mathcal{A}$, let $k \in \mathbf{N}$, and let $u \in W^{m,p}(\Omega) \cap L^\infty(A)$, with $||u||_{L^\infty(A)} < k$. For every $A' \in \mathcal{A}$, with $A' \subset\subset A$, there exists a sequence u_h in $C^\infty(\Omega) \cap W^{m,p}(\Omega) \cap L^\infty(A')$ such that u_h converges to u in $W^{m,p}(\Omega)$ and $||u_h||_{L^\infty(A')} < k$ for every $h \in \mathbf{N}$. Moreover we can assume that u_h converges to u (m,p)-q.e. (see[13]) and therefore, by the (m,p) - absolute continuity of μ_{4k} and by the dominated convergence theorem, sequence u_h converges to u in $L^1(A', \mu_{4k})$. Thus, by lemma 2.4 we have

$$G(u, A') = \lim_{k\to\infty} G(u_h, A').$$

We use now the results of step 1 and the dominated convergence theorem to prove that

$$\int_{A'} g_k(x, u(x))d\mu_k(x) - \nu(A')$$

Taking the limit as $A' \uparrow A$ we obtain (2.6) for every $u \in W^{m,p}(\Omega) \cap L^\infty(A)$, with $||u||_{L^\infty(A)} < k$.

Step 3. Let μ be the Radon measure defined by

$$\mu(B) = \sum_{k=1}^{\infty} 2^{-k} \frac{\mu_k(B)}{\mu_k(\Omega)}$$

for every $B \in \mathcal{B}$. Since each measure μ_k is (m,p)-absolutely continuous (Remark 2.1), the measure μ is (m,p)-absolutely continuous. For every $k \in \mathbf{N}$ we denote by $\frac{d\mu_k}{d\mu}$ the Radon-Nikodym derivative of μ_k with respect to μ and we define the function f_k : $\Omega x]-k, k[\to[0, +\infty[$ by $f_k(x,t) = g_k(x, t)\frac{d\mu_k}{d\mu}(x)$.

Then

$$(2.7) \qquad G(u, A) = \int_A f_k(x, u(x))d\mu(x) + \nu(A)$$

for every $A \in \mathcal{A}$ and for every $u \in W^{m,p}(\Omega) \cap L^\infty(A)$ with $||u||_{L^\infty(A)} < k$. Moreover for every $x \in W$ the function $f_k(x, \cdot)$ is continuous and convex on $]-k, k[$.

Let $h, k \in \mathbf{N}$ and $t \in A$, with $|t| < h \leq k$. By (2.7) it follows that

$$\int_A f_h(x\,,t)d\mu(x)\ +\nu(A)\ \ =\ \ G(t\,,A)\ \ =\ \ \int_A f_h(x\,,t)d\mu(x)\ +\ \nu(A)$$

for every $A\in\mathcal{A}$. Therefore there exists a set $N(h\,,k\,,t)\in\mathcal{B}$ with $m(N(h\,,k\,,t))=0$ such that

$$f_h(x\,,t)\ =\ f_k(x\,,t) \tag{2.8}$$

for every $x\in\Omega-N(h\,,k\,;t)$. Let N be the union of the sets $N(h\,,k\,,t)$ for $h\,,k\in\mathbf{N}$ and $t\in\mathcal{A}$ with $|t|<h\le k$. Then $\mu(N)=0$ and (2.8) holds for every $x\in\Omega-N$ and for every h, $k\in\mathbf{N}$ and $t\in\mathcal{A}$ with $|t|<h\le k$. Since $f_h(x\,,\cdot)$ and $f_k(x\,,\cdot)$ are continuous the equality (2.8) continuos to hold for every $t\in\mathbf{R}$ with $|t|<h\le k$.

Let $g:\Omega x\mathbf{R}\to[0\,,+\infty[$ be the function defined by

$$g(x\,,t)\ =\ \begin{cases} 0 & \text{if } x\in N \\ f_k(x\,,t) & \text{if } x\in\Omega-N \text{ and } |t|<k \end{cases}$$

It is now easy to see that the function g is continuous and convex in the second variable and that

$$G(u\,,A)\ \ =\ \int_A g(x\,,u(x))d\mu(x)\ +\ \nu(A)$$

for every $u\in W^{m,p}(\Omega)\cap L^{\infty}(A)$. ∎

We conclude this section by proving an integral representation theorem on $W^{m,p}(\Omega)$.

THEOREM 2.6. Let G be a functional of the class $\mathcal{G}_o$, let ν be the Radon measure defined in (1.3), and let g and μ be respectively the function and the measure given by theorem 2.5. The following conditions are equivalent :

(a) $G(u, \Omega) < +\infty$ for every $u \in W^{m,p}(\Omega)$;

(b) $\int_\Omega g(x, u(x))d\mu(x) + \nu(\Omega) < +\infty$ for every $u \in W^{m,p}(\Omega)$.

Each of the previous conditions implies that

$$G(u, A) = \int_A g(x, u(x))d\mu(x) + \nu(A) \tag{2.9}$$

for every $u \in W^{m,p}(\Omega)$ and for every $A \in \mathcal{A}$.

PROOF. We first prove that

$$G(u, A) \leq \int_{A'} g(x, u(x))d\mu(x) + \nu(A) \tag{2.10}$$

for every $A \in \mathcal{A}$ and $u \in W^{m,p}(\Omega)$. Suppose

$$\int_{A'} g(x, u(x))d\mu(x) + \nu(A') < +\infty$$

Then for every $A' \in \mathcal{A}$, with $A' \subset\subset A$ there exist a sequence u_h in $W^{m,p}(\Omega) \cap L^\infty(\Omega)$ which converges to u in $W^{m,p}(A')$, such that $|u_h| \leq |u|$ and $u_h u \geq 0$ (m,p)-q.e. on A' for every $h \in \mathbf{N}$ (see[3] Theorem 2). We may also assume that u_h converges to u (m,p)-q.e. on A' (see[13]), hence m-q.e. on A'. Since g is a convex function, we have $g(x, u_h(x)) \leq g(x, u(x)) + g(x, 0)$ for every $x \in A'$ and $h \in \mathbf{N}$. By Theorem 2.5 we have

$$G(u_h, A') = \int_{A'} g(x, u_h(x))d\mu(x) + \nu(A')$$

for every $h \in \mathbf{N}$. Thus the lower semicontinuity of G and the dominated convergence theorem imply

$$G(u\,,A') \;\le\; \lim G(u_h\,,A') \;=\; \int_{A'} G(x\,,u(x))d\mu(x) \;+\; \nu(A')$$

Taking the limit as $A' \uparrow A$ we obtain (2.10) , which proves that (b) implies (a) .

Let us prove that (a) implies (b). Assume (a). For every $A \in A$ the function $G(\cdot\,,A)$ is convex, lower semicontinuous, and everywhere finite on $W^{m,p}(\Omega)$, thus it is continuous on $W^{m,p}(\Omega)$ (see, for instance [11] Chapter I, Corollary 2.5). Let $u \in W^{m,p}(\Omega)$ and, for every $A' \in A$, with $A' \subset\subset A$, let u_h be the sequence considered in the first part of the proof. Since m is (m,p)-absolutely continuous, from the Fatou lemma and from Theorem 2.5 it follows that

$$\int_A g(x\,,u(x))d\mu(x) + \nu(A) \;\le\; \liminf \int_{A'} g(x\,,u_h(x))d\mu(x) + \nu(A') \;=$$

$$=\; \lim G(u_h\,,A') \;=\; G(u\,,A') \;<\; +\infty$$

Taking the limit as $A' \uparrow A$, we obtain

$$\int_A g(x\,,u(x))d\mu(x) + \nu(A) \;\le\; G(u\,,A) \;<\; +\infty$$

for every $u \in W^{m,p}(\Omega)$ and for every $A \in \mathcal{A}$. This proves (b) and, together with (2.10) , yields (2.9) . ∎

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Part 2 :

Dirichlet problems in domains bounded by thin layers with random thickness

In this part of the thesis we present some results obtained in collaboration with M. Balzano.

DIRICHLET PROBLEMS IN DOMAINS BOUNDED BY THIN LAYERS WITH RANDOM THICKNESS.

INTRODUCTION

Recently G. Buttazzo, G. Dal Maso, and U. Mosco have proposed in [6] a new capacity method to investigate the asymptotic behaviour for Dirichlet problems in domains bounded by thin layers. In this paper, taking inspiration from that method and from some variational techniques developed in [3], we provide a setting to analyze the cases in which the domains are surrounded by thin layers with random thickness.

Let us describe more closely the problem we deal with.

Let D be a bounded Lipschitz domain of $\mathbf{R}^n$, $n \geq 2$, and let (ε_h) be a sequence of real numbers such that $\varepsilon_h \to 0$ as $h \to +\infty$. For every $h \in \mathbf{N}$ let us consider a class $\mathcal{A}_h$ of subsets of $\mathbf{R}^n$ defined by

$$\mathcal{A}_h = \{ A \subseteq \mathbf{R}^n ,\ A \supseteq \overline{D} : \sup_{x \in A} \operatorname{dist}(x,D) < \varepsilon_h \} .$$

Given $f \in L^2(\mathbf{R}^n)$ we are interested in the solutions of the equations

$$(0.1) \qquad -\Delta u_h = f \ \text{ in } D \qquad -\varepsilon_h \Delta u_h = f \ \text{ in } A_h \setminus D$$

where A_h is a random set of the class $\mathcal{A}_h$, $u_h = 0$ on ∂A_h and the natural transmission conditions on ∂D are satisfied.

Let F_h be the quadratic form on $L^2(\mathbf{R}^n)$ defined by

$$F_h(u) = \begin{cases} \displaystyle\int_D |\nabla u|^2 dx + \varepsilon_h \int_{A \setminus D} |\nabla u|^2 dx & \text{if } u \in H_0^1(A_h) \\ +\infty & \text{otherwise} \end{cases} .$$

The solution u_h of (0.1) coincides with the solution of the minimum problem

$$\min \{ F_h(u) - 2 \int_A f u \, dx \ : \ u \in L^2(\mathbf{R}^n) \} .$$

Our aim is to characterize the behaviour of the sequence (u_h) in the limit as $h \to +\infty$. First, we introduce the class $\mathcal{E}$ of all convex, semicontinuous functions from $L^2(\mathbf{R}^n)$ into $\overline{\mathbf{R}}$. We equip $\mathcal{E}$ with a topological structure ($L^2(\mathbf{R}^n)$ - Mosco - convergence) so that it becomes a complete metric space. Then, we associate with the problems (0.1) a sequence (F_h) of "random functionals", that is measurable maps $\omega \to F_h(\omega)$ from a probability space Ω into $\mathcal{E}$. In this way the problem consists in analyzing the asymptotic behaviour, as $h \to +\infty$, of sequence of random functionals (F_h).

The first result we prove is a compactness theorem for sequences of random functionals (Theorem 4.1 and Remark 5.5). It is deduced from an abstract compactness result for sequences of probability measures on a complete metric space (Theorem 2.3). We show that, under suitable assumptions on the sequence (F_h), there exists a subsequence $(F_{\sigma(h)})$ converging in probability to a constant random functional F. Moreover the functional F turns out to be associated with the equation, formally written as

$$
(0.2) \qquad \begin{cases} -\Delta u = f & \text{in } D \\ \dfrac{\partial u}{\partial n} + \mu u = 0 & \text{on } \partial D \ , \end{cases}
$$

where μ is a Borel measure on ∂D that vanishes on any set of zero (harmonic) capacity, but may assume the value $+\infty$ on some subset of positive capacity and n is the outer unit normal to D.

The second result we obtain is a characterization of the limit functional (hence of the measure μ that appears in (0.2)). For both results the assumptions are made in terms of the asymptotic behaviour of the expectations and of the covariances of suitable random capacities associated with the random functionals F_h .

In the deterministic case problems of the type (0.1) are known as "reinforcement problems". They have been investigated in the last years by several authors (see for istance [1], [5], [7], [8]).

To our knowledge no specific reference for the stochastic cases is available. We only mention the paper [11] which provides a general framework for the study of probabilistic problems in calculus of variations.

Our paper is organized as follows.

1. Notation and preliminaries.
2. Some abstract probabilistic results.
3. Mosco - convergence and random capacities.
4. Main results.
5. Dirichlet problems in domains surrounded by thin layers with random thickness.
6. An example.

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1. NOTATION AND PRELIMINARIES.

1.1 Let n be an integer with $n \geq 2$. We denote by $\mathcal{U}$ (resp. $\mathcal{K}$, $\mathcal{B}$) the family of all bounded open (resp. compact, Borel) subsets of $\mathbf{R}^n$. We recall some definitions which will be often used in the sequel. For every $U \in \mathcal{U}$ and for every $K \in \mathcal{K}$ such that $K \subseteq U$, we define the *capacity* of K with respect to U by

$$\text{cap}\,(K,U) = \inf \left\{ \int_U |D\phi|^2 dx \; : \phi \in C_0^\infty(U),\ \phi \geq 1 \text{ on } K \right\} \quad ;$$

the definition is extended to the sets $V \in \mathcal{U}$ with $V \subseteq U$ by

$$\text{cap}\,(V,U) = \sup \left\{ \text{cap}\,(K,U) \; : K \in \mathcal{K},\ K \subseteq V \right\} \quad ;$$

and to the sets $B \in \mathcal{B}$ with $B \subseteq U$ by

$$\text{cap}\,(B,U) = \inf \left\{ \text{cap}\,(V,U) \; : V \in \mathcal{U},\ V \supseteq B \right\} .$$

We say that a Borel set B of $\mathbf{R}^n$ has *capacity zero* if $\text{cap}\,(B \cap U, U) = 0$ for every $U \in \mathcal{U}$. When a property P(x) is satisfied for all $x \in B$, except for a subset $N \subseteq B$ with zero capacity, then we say that P(x) holds *quasi everywhere* on B (q.e. on B). We say that a function $f : B \to \mathbf{R}$ is *quasi continuous* on B if for every $U \in \mathcal{U}$ and for every $\varepsilon > 0$ there exists $V \in \mathcal{U}$, $V \subseteq U$, with $\text{cap}\,(V,U) < \varepsilon$ such that the restriction of f to $(B \cap U) \setminus V$ is continuous. A subset A of $\mathbf{R}^n$ is said to be *quasi open* (resp. *quasi closed*, *quasi compact*) if for every $\varepsilon > 0$ and for every $U \in \mathcal{U}$, there exists an open (resp. closed, compact) set $V \subseteq U$ such that $\text{cap}\,((A \cap U) \Delta V, U) < \varepsilon$, where Δ denotes the symmetric difference between sets. We recall that a bounded set $B \subseteq \mathbf{R}^n$ has zero capacity (resp. B is quasi open or f is quasi continuous on B) if and only if the above conditions are satisfied for some $U \in \mathcal{U}$ with $B \subseteq U$.

1.2. For every open set $U \subseteq \mathbf{R}^n$ we denote by $H^1(U)$ the Sobolev space of all functions in $L^2(U)$ whose first weak derivatives belong to $L^2(U)$, and by $H_0^1(U)$ the closure of $C_0^\infty(U)$ in $H^1(U)$. For every $x \in \mathbf{R}^n$ and for every $r > 0$ we set

$$B_r(x) = \{ y \in \mathbf{R}^n : |x - y| < r \}$$

and for every Borel set $B \subseteq \mathbf{R}^n$ we denote by $|B|$ its Lebesgue measure. Let $U \in \mathcal{U}$. For every $u \in H^1(U)$ the limit

$$\tilde{u}(x) = \lim_{r \to 0} \frac{1}{|U \cap B_r(x)|} \int_{U \cap B_r(x)} u(y)\, dy \qquad (1.1)$$

exists and is finite q.e. on U. The function $\tilde{u}$ defined q.e. by (1.1) is quasi continuous on U. Moreover, it can be shown that for every $B \in \mathcal{B}$, with $B \subset U$

$$\mathrm{cap}(B,U) = \min \{ \int_U |Du|^2 dx : u \in H_0^1(U) \, , \, \tilde{u} \geq 1 \text{ q.e. on } B \ \}.$$

For a proof of these properties of the capacity and of the functions of $H^1(U)$ we refer to [18].

1.3. A non negative countably additive set function defined on the Borel σ-algebra of $\mathbf{R}^n$ with values in $[0,+\infty]$ is called a *Borel measure*. A Borel measure which assigns finite values to every $K \in \mathcal{K}$ is called a *Radon measure*. In our paper we deal with a peculiar class of Borel measures.

Following [10] we denote by $\mathcal{M}_0^*$ the class of all Borel measures μ such that:

(a) $\mu(B)=0$ for every Borel set $B \subseteq \mathbf{R}^n$ with capacity zero;

(b) $\mu(B)=\inf\{\mu(A)$: A quasi open, $B \subseteq A\}$ for every Borel set $B \subseteq \mathbf{R}^n$.

An easy example of measure belonging to $\mathcal{M}_0^*$ is the measure μ defined by

$$\mu(B)=\int_B f dx$$

for every Borel set $B \subseteq \mathbf{R}^n$, where $f \in L^1_{loc}(\mathbf{R}^n)$. More generally, every Radon measure μ satisfying (a) belongs to $\mathcal{M}_0^*$. We remark that the measures belonging to $\mathcal{M}_0^*$ are not required to be regular nor σ-finite. For instance, the measures introduced in the definition below belong to the class $\mathcal{M}_0^*$ (see [10], Remark 3.3).

Definition 1.1. For every quasi closed set $F \subseteq \mathbf{R}^n$ we denote by ∞_F the Borel measure defined by

$$\infty_F(B) = \begin{cases} 0 & \text{if } F \cap B \text{ has capacity zero} \\ +\infty & \text{otherwise} \end{cases}$$

for every Borel set $B \subseteq \mathbf{R}^n$.

Other examples of measures in M_0^* are given in [13].

1.4. Troughout we denote by D a fixed set of $\mathcal{U}$ with a Lipschitz boundary and by L a fixed elliptic operator of the form

$$Lu = -\sum_{i,j=1}^{n} D_i(a_{i,j}(x)D_j u),$$

where $a_{i,j}=a_{j,i} \in L^\infty(\mathbf{R}^n)$ and for almost every $x \in \mathbf{R}^n$ and for every $\xi \in \mathbf{R}^n$ we have

$$\Lambda_1 |\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \leq \Lambda_2 |\xi|^2,$$

where $\Lambda_1, \Lambda_2 \in \mathbf{R}$ with $0<\Lambda_1 \leq \Lambda_2 <+\infty$. Let us fix a sequence (ε_h) of positive real numbers such that $\varepsilon_h \to 0$ as $h \to +\infty$. For every $h \in \mathbf{N}$ we consider the operator

$$L^{(h)}u = -\sum_{i,j=1}^{n} D_i(a_{i,j}^{(h)}(x)D_j u),$$

where

$$a_{i,j}^{(h)}(x) = \begin{cases} a_{i,j}(x) & \text{if } x \in D \\ \varepsilon_h a_{i,j}(x) & \text{if } x \in \mathbf{R}^n - D. \end{cases}$$

We denote by $a^{(h)}(x,\xi)$ and $a^0(x,\xi)$ the quadratic forms associated with the matrices $(a_{i,j}^{(h)})$ and

$$a_{i,j}^0(x) = \begin{cases} a_{i,j}(x) & \text{if } x \in D \\ 0 & \text{if } x \in \mathbf{R}^n - D, \end{cases}$$

more precicely,

(1.2)
$$a^{(h)}(x,\xi) = \sum_{i,j=1}^{n} a_{i,j}^{(h)}(x)\xi_i\xi_j,$$

$$a^0(x,\xi) = \sum_{i,j=1}^{n} a_{i,j}^0(x)\xi_i\xi_j.$$

Let (η_h) be another sequence of positive numbers such that $\eta_h \to 0$ as $h \to +\infty$.

Definition 1.2. For every $h \in \mathbf{N}$ we consider the class of sets

$$\mathcal{A}_h = \{ A \in \mathcal{U} : A \supseteq \bar{D}, \sup_{x \in A} \text{dist}(x,D) < \eta_h \}$$

and we denote by $\mathcal{F}_h$ the class of all functionals $F : L^2(\mathbf{R}^n) \to [0,+\infty]$ defined by

$$F(u) = \begin{cases} \int_{\mathbf{R}^n} a^{(h)}(x,Du)dx + \int_{\mathbf{R}^n} \tilde{u}^2 d\infty_{\partial A} & u \in H^1(\mathbf{R}^n) \\ +\infty & \text{otherwise} \end{cases}$$

where A is an open set belonging to the class $\mathcal{A}_h$ and $\infty_{\partial A}$ is the measure of $\mathcal{M}_0^*$ defined in Definition 1.1.

Remark 1.3. By definition 1.2. we have $\mathcal{F}_h \cap \mathcal{F}_k = \emptyset$ for $h \neq k$.

Definition 1.4. The class of measure $\mu \in \mathcal{M}_0^*$ such that spt $\mu \subseteq \partial D$ will be denoted by $\mathcal{M}_0^*(\partial D)$ and the class of all functionals $F : L^2(\mathbf{R}^n) \to [0,+\infty]$ defined by

$$F(u) = \begin{cases} \int_D a^0(x,Du)dx + \int_{\partial D} \tilde{u}^2 d\mu & \text{if } u_{|D} \in H^1(D) \\ +\infty & \text{otherwise} \end{cases}$$

where $\mu \in \mathcal{M}_0^*(\partial D)$, will be indicated by $\mathcal{F}_o$.

Remark 1.5. It can be seen that there is a one to one correspondance between the functionals of the class $\mathcal{F}_h$ and the measures $\infty_{\partial A}$ with $A \in \mathcal{A}_h$, and between the functionals of the class $\mathcal{F}_o$ and the measure $\mu \in \mathcal{M}_0^*(\partial D)$.

With every functional $F \in \mathcal{F}_h$ we associate a Dirichlet problem of the form

$$(1.3) \qquad \begin{cases} L^{(h)}u_h + \lambda u_h = g & \text{in } A \\ u_h \in H_0^1(A) \end{cases}$$

where $\lambda \geq 0$, $A \in \mathcal{A}_h$, and $g \in L^2(\mathbf{R}^n)$. Let u_h be the unique weak solution of (1.3). Let us consider the function

$$w_h = \begin{cases} u_h & \text{on } A \\ 0 & \text{on } \mathbf{R}^n - A. \end{cases}$$

Definition 1.6. Given $F \in \mathcal{F}_h$, for $\lambda \geq 0$ we define the resolvent operator $R^h(\lambda)$: $L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ associated with F by setting

$$R_h(\lambda)[g] = w_h.\,.$$

With every functional $F \in \mathcal{F}_o$ we also associate a problem formally written as

$$(1.4) \qquad \begin{cases} Lu + \lambda u = g & \text{in } D \\ \dfrac{\partial u}{\partial n} + \mu u = 0 & \text{on } \partial D \end{cases}$$

where $\lambda > 0$, $g \in L^2(\mathbf{R}^n)$, $\mu \in \mathcal{M}_o^*(\partial D)$, and n is the outer unit normal to D.

A variational solution of the problem (1.4) is a function u such that $u \in H^1(D)$, $\tilde{u} \in L^2(\partial D, \mu)$ and

$$\int_D \left(\sum_{i,j=1}^n a_{i,j} D_i u D_j v \right) dx + \int_{\partial D} \tilde{u}\tilde{v}\, d\mu + \lambda \int_D uv\, dx = \int_D gv\, dx$$

for every $v \in H^1(D)$ with $\tilde{v} \in L^2(\partial D, \mu)$. Let u be a variational solution of the problem (1.4); u is the unique solution of the minimum problem

(1.5) $$\min\{ \int_D a(x,Dw)\,dx + \int_{\partial D} \tilde{w}^2 d\mu + \lambda \int_D w^2\,dx - 2\int_D gw\,dx\ ;\ w \in H^1(D)\}.$$

Let w be the function

$$w = \begin{cases} u \text{ on } D \\ 0 \text{ on } \mathbf{R}^n - D . \end{cases}$$

Definition 1.7. Given $F_0 \in \mathcal{F}_0$, for $\lambda > 0$ we define the resolvent operator $R^o(\lambda) : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ associated to F_0 by setting

(1.6) $$R^o(\lambda)[g] = w.$$

Remark 1.8. It can be shown that (1.5) and (1.6) hold also in the case $\lambda = 0$ when there exists a constant $c > 0$ such that for every $h \in \mathbf{N}$ $\eta_h = c\varepsilon_h$, i.e. if the following relation is satisfied

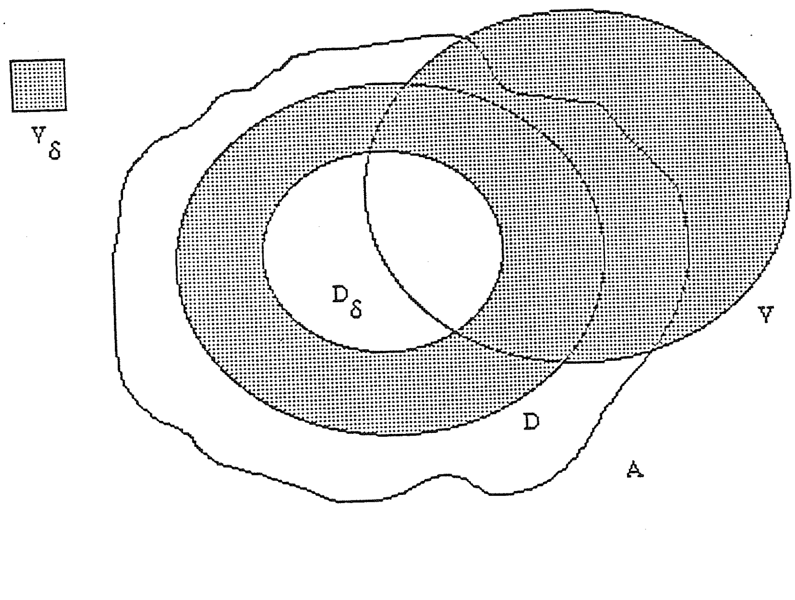
(1.7) $$A \subseteq \{ x \in \mathbf{R}^n : \text{dist}(x,D) < c\varepsilon_h \}.$$

Let us set

(1.8) $$\mathcal{F} = \mathcal{F}_0 \cup (\bigcup_{h \in \mathbf{N}} \mathcal{F}_h).$$

In the following we define a set function of capacity type associated with any functional $F \in \mathcal{F}$. It will be the basic tool in our investigation.

Let $(D_\delta)_{\delta>0}$ be the family of the open subsets of D with Lipschitz boundary ∂D_δ such that $D_{\delta_2} \subset\subset D_{\delta_1}$ for $\delta_1 > \delta_2$ and $D = \bigcup_{\delta>0} D_\delta$. For every $V \in \mathcal{U}$ we set $V_\delta = (V \cup D) \setminus \bar{D}_\delta$.



Definition 1.9. Given $F\in\mathcal{F}$, for every $U\in\mathcal{U}$ and for every $\delta>0$ we define the following set function

(1.9) $$b_\delta(F,V)=\min\{\int_{V_\delta} a^{(F)}(x,Du)\,dx+\int_{V_\delta}\tilde{u}^2 d\mu_F \;:\; u\in H^1(V_\delta)\,,\ u=1 \text{ on } \partial D_\delta\}$$

where

$$a^{(F)}(x,\xi)=\begin{cases} a^{(h)}(x,\xi) & \text{if } F\in\mathcal{F}_h \\ a^{0}(x,\xi) & \text{if } F\in\mathcal{F}_0\end{cases}$$

and

$$\mu_F=\begin{cases} \infty_{\partial A} \text{ with } A\in\mathcal{A}_h & \text{if } F\in\mathcal{F}_h \\ \mu\ , \text{ with } \mu\in\mathcal{M}_0^*(\partial D) & \text{if } F\in\mathcal{F}_0\end{cases}$$

For $F\in\mathcal{F}_0$ we extend the definition of b_δ to the Borel sets $B\in\mathcal{B}$ by

$$b_\delta(F,B)=\inf\{\ b_\delta(F,V)\ :\ V\in\mathcal{U}\,,\ V\supseteq B\ \}$$

The minimum in (1.9) is achieved by the lower semicontinuity and the coerciveness of the functional.

Remark 1.10. In order to study Dirichlet problems in domains bounded by thin layers, in [6] the authors introduce two set functions, depending on the choice of a pair $V,U\in\mathcal{U}$ such that $V\subseteq U$. By Lemma 3.5(c) in [6], it can be seen that, for $F\in\mathcal{F}_0$, the set function b_δ defined in (1.9) is equivalent to the set function b_δ^{o} defined by (3.11) in [6], i.e. for every $B\in\mathcal{B}$ $b_\delta(F,B)=b_\delta^{o}(\mu_F,B,U)$ where U is an arbitrary open set of U containing the region D. Moreover, for $F\in\mathcal{F}_h$, the set function b_δ is equivalent to the set function $b_\delta^{\epsilon_h}$ defined by (6.7) in [6], i.e. $b_\delta(F,V)=b_\delta^{\epsilon_h}(V,U)$ with $A=\Omega^{\epsilon_h}$ and $U,V\in\mathcal{U}$ such that $V\subseteq U$.

If $F\in\mathcal{F}_0$ the main properties of the set function $b_\delta(F,\bullet)$ can be summarized in the next proposition.

Proposition 1.11. *For every* $F\in\mathcal{F}_0$ *and for every* $\delta>0$, *the function* $b_\delta(F,\bullet)$ *satisfies the following properties:*

(a) $b_\delta(F,\emptyset)=0$;

(b) *if* $B_1,B_2\in\mathcal{B}$, *with* $B_1\subseteq B_2$, *then* $b_\delta(F,B_1)\le b_\delta(F,B_2)$;

(c) *if* (B_h) *is an increasing sequence of sets in* $\mathcal{B}$ *and* $B=(\cup B_h)$, *then*

$$b_\delta(F,B)=\sup\{b_\delta(F,B_h)\ :\ h\in\mathbb{N}\ \};$$

(d) *if* (B_h) *is a sequence of sets in* $\mathcal{B}$ *and* B *is a Borel subset of* $(\cup B_h)$, *then*

$$b_\delta(F,B)\le\Sigma\, b_\delta(F,B_h);$$

(e) *if* $B_1,B_2\in\mathcal{B}$ *then* $b_\delta(F,B_1\cup B_2)+b_\delta(F,B_1\cap B_2)\le b_\delta(F,B_1)+b_\delta(F,B_2)$;

(f) *if* $B\in\mathcal{B}$ *and* $V\in\mathcal{U}$ *with* $(B\cap\partial D)\subseteq V$ *and* $V\cap\partial D_\delta=\emptyset$ *then* $b_\delta(F,B)\le\Lambda_2\text{cap}(B\cap\partial D,V)$;

(g) *if* $B\in\mathcal{B}$ *then* $b_\delta(F,B) = b_\delta(F,B\cap\partial D) \le \mu(B\cap\partial D)$, *where* μ *is the measure in* $\mathcal{M}_0^*$ *associated to the functional* F;

(h) *for every* $K\in\mathcal{K}$, $b_\delta(F,K) = \inf\{b_\delta(F,U) : U\in\mathcal{U}, K\subseteq U\}$;

(i) *if* $B_1, B_2 \in \mathcal{B}$ *and* $\mathrm{dist}(B_1,B_2)=\sigma>0$, *then for every* $\eta\in\,]0,1[$
$$b_\delta(F,B_1) + b_\delta(F,B_2) \le (1-\eta)^{-1}b_\delta(F,B_1\cup B_2) + 4\Lambda_2\eta^{-1}\sigma^{-2}|D-D_\delta|;$$

(j) *for every* $B\in\mathcal{B}$, $b_\delta(F,B)=\sup\{b_\delta(F,K) : K\in\mathcal{K}, K\subseteq B\}$;

(k) *if* $\mu\in\mathcal{M}_0^*(\partial D)$ *is the measure associated to* F, *then* $\mu(B\cap\partial D) = \sup\{b_\delta(F,B) : \delta>0\}$ *for every* $B\in\mathcal{B}$.

Proof. The properties (a), (b), (c), (d) can be deduced by standard capacity theory arguments. In view of Remark 1.10 the properties (e),(f),(g),(h),(i) follow directly from (3.12), (3.13), (3.14) and Lemma 3.5 in [6]. The property (j) is an easy consequence of properties (h),(c),(b), and the Choquet capacitability theorem (see [9]). The property (k) follows from Theorem 3.6 in [6]. ∎

Finally we recall the definition of capacity relative to the operator $L^{(h)}$.

Definition 1.12. For every $h\in\mathbf{N}$ we define
$$\mathrm{cap}^{(h)}(B) = \min\left\{ \int_{\mathbf{R}^n} a^{(h)}(x,Du)\,dx + \int_{\mathbf{R}^n} u^2\,dx \;:\; u\in H^1(\mathbf{R}^n),\ \tilde{u}\ge 1 \text{ q.e. on } B \right\}$$
for every Borel set $B\subseteq\mathbf{R}^n$.

2. SOME ABSTRACT PROBABILISTIC RESULTS.

In this section we set up the probabilistic picture of our paper and give some results which will have a crucial role in the proofs of the main theorems in section 4.

Troughout we deal with the following abstract framework.

(2.1) (X,d) is a complete metric space;

(2.2) X_0 is a compact subset of X;

(2.3) (X_h) is a sequence of subsets of X satisfying the following proprerty :
if (x_h) is a sequence of elements of X and $(X_{\sigma(h)})$ is any subsequence of (X_h) such that $x_h\in X_{\sigma(h)}$ then there exist a subsequence $x_{m(h)}$ of x_h and an element $x\in X_0$ such that $x_{m(h)}$ converges to x in X.

Remark 2.1. From (2.1), (2.2) and (2.3) it is immediate to deduce that for every open neighbourhood U of X_0, there exists $h_0 \in N$ such $X_h \subseteq U$ for every $h \geq h_0$.

We denote by $\mathcal{B}(X)$ the Borel σ-field of X. A probability measure Q on $(X,\mathcal{B}(X))$ is a non negative countably additive set function defined on $\mathcal{B}(X)$ with Q(X)=1.By $\mathcal{P}(X)$ we mean the space of all probabilty measure defined on $\mathcal{B}(X)$. On $\mathcal{P}(X)$ we consider the following definition of weak convergence.

Definition 2.2. We say that a sequence (Q_h) of measures in $\mathcal{P}(X)$ converges weakly to $Q \in \mathcal{P}(X)$ if

$$\lim_{h \to +\infty} \int_X f \, dQ_h = \int_X f \, dQ$$

for every $f \in C_b^0(X)$, where $C_b^0(X)$ denotes the class of all bounded continuous functions $f : X \to \mathbf{R}$.

Let $Q \in \mathcal{P}(X)$. For every $\mathcal{B}(X)$ - measurable real valued function f we define the expectation of f in the probability space $(X,\mathcal{B}(X),Q)$ by

$$E_Q[f] = \int_X f \, dQ \, .$$

Let f,g be two real valued functions in $L^2(X,Q)$. Then the covariance of f and g is defined by

$$\mathrm{Cov}_Q[f,g] = E_Q[fg] - E_Q[f]\, E_Q[g]$$

The variance of f is defined by

$$\mathrm{Var}_Q[f] = \mathrm{Cov}_Q[f,f]$$

For every $h \in N$ the Borel σ-field of X_h equipped with the induced topology is denoted by $\mathcal{B}(X_h)$. Let Q_h be any probability measure on $(X_h,\mathcal{B}(X_h))$. We associate Q_h with the probability measure $\widehat{Q}_h$ in $\mathcal{P}(X)$ defined by

(2.4) $$\widehat{Q}_h(B) = Q_h(B \cap X_h)$$

for every $B \in \mathcal{B}(X)$.

In what follows we consider sequences $(\widehat{Q}_h)$ of probability measure in P(X) with $\widehat{Q}_h$ defined by (2.4). We note that a probability measure P_h on $(X,\mathcal{B}(X))$ can be written in the form $\widehat{Q}_h$ given by (2.4) if and only if $P_h^*(X_h) = 1$,where P_h^* denotes the outer measure associate with P_h. Infact, if $P_h^*(X_h) = 1$, then $P_h = \widehat{Q}_h$ with Q_h defined by $Q_h(B) = P_h^*(B)$ for every $B \in \mathcal{B}(X_h)$.

We can state the following compacteness result.

Theorem 2.3. *For every sequence* $(\widehat{Q}_h)$ *in* $\mathcal{P}(X)$ *of the form* (2.4), *there exist a subsequence* $(\widehat{Q}_{\sigma(h)})$ *and a measure* $\widehat{Q}$ *in* $\mathcal{P}(X)$ *such that* $\widehat{Q}(X_0) = 1$ *and* $(\widehat{Q}_{\sigma(h)})$ *converges* *weakly to* $\hat{Q}$ *in* $\mathcal{P}(X)$.

The proof of theorem (2.3) needs the next lemma.

Lemma 2.4. *Let* $(\hat{Q}_h)$ *be a sequence in* $\mathcal{P}(X)$ *of the form* (2.4). *Let* $f,g \in C_b^o(X)$. *Assume that there exists* $\eta > 0$ *such that*

$$|f(x) - g(x)| < \eta$$

for every $x \in X_0$. *Then*

$$\limsup_{h \to +\infty} E_{\hat{Q}_h}[f - g] < \eta .$$

Proof. Since the set $U = \{ x \in X : |f(x) - g(x)| < \eta \}$ is an open neighbourhood of X_0 by Remark 2.1 we have $X_h \subseteq U$ for h sufficiently large. Thus we obtain

$$\limsup_{h \to +\infty} E_{\hat{Q}_h}[f - g] = \limsup_{h \to +\infty} \int_U |f - g| \, d\hat{Q}_h < \eta$$

and the proof is complete. ∎

Proof of Theorem 2.3. Let $(\hat{Q}_h)$ be a sequence in $\mathcal{P}(X)$ of the form (2.4). The proof is articuleted in two steps. In the first step we show that there exists a subsequence $(\hat{Q}_{\sigma(h)})$ of $(\hat{Q}_h)$ such that the limit

$$\lim_{h \to +\infty} E_{\hat{Q}_{\sigma(h)}}[f] \tag{2.5}$$

exists for every $f \in C_b^o(X)$. In the second one we prove that there exists a measure $\hat{Q} \in \mathcal{P}(X)$, with $\hat{Q}(X_0) = 1$ such that the limit (2.5) is equal to $E_Q[f]$ for every $f \in C_b^o(X)$.

Step 1. Let $\mathcal{G} = (g_i)_{i \in I}$ be a countably set which is dense in $C^o(X_0)$. For every $i \in I$, let $f_i \in C_b^o(X)$ such that $f_i|X_0 = g_i$. By a diagonal procedure we can find a subsequence $(\hat{Q}_{\sigma(h)})$ of $(\hat{Q}_h)$ such that

$$\lim_{h \to +\infty} E_{\hat{Q}_{\sigma(h)}}[f_i]$$

exists, for every $i \in I$. Denote by $I(f_i)$ this limt. In order to prove that the limit (2.5) exists we show that for every $f \in C_b^o(X)$ the sequence $(E_{\hat{Q}_{\sigma(h)}}[f])$ is a Cauchy sequence. Let $f \in C_b^o(X)$. For every $\varepsilon > 0$ let us take $g_i \in \mathcal{G}$ such that

$$\sup_{x \in X_0} |f(x) - g_i(x)| < \frac{\varepsilon}{4} \tag{2.6}$$

Then, by (2.6), and by Lemma 2.4 we obtain that there exists $k_0 \in N$ such that

$$|E_{\hat{Q}_{\sigma(k)}}[f] - E_{\hat{Q}_{\sigma(l)}}[f]| \le$$

$$\le |E_{\hat{Q}_{\sigma(k)}}[f] - E_{\hat{Q}_{\sigma(k)}}[f_i]| + |E_{\hat{Q}_{\sigma(k)}}[f_i] - E_{\hat{Q}_{\sigma(l)}}[f_i]| + |E_{\hat{Q}_{\sigma(l)}}[f_i] - E_{\hat{Q}_{\sigma(l)}}[f]| <$$

$$< \frac{\varepsilon}{2} + |E_{\hat{Q}_{\sigma(k)}}[f_i] - E_{\hat{Q}_{\sigma(l)}}[f_i]|$$

for every $k,l \ge k_0$. Since $(E_{\hat{Q}_{\sigma(h)}}[f_i])$ is a Cauchy sequence, we get the first assertion.

Step 2. Let us denote by T any extention operator from $C^o(X_0)$ into $C^o_b(X)$ and let us introduce the following maps:

(a) $I : C^o_b(X) \to \mathbf{R}$ defined by

$$I(f) = \lim_{h \to +\infty} E_{\hat{Q}_{\sigma(h)}}[f]$$

(b) $J : C^o(X_0) \to \mathbf{R}$ defined by $J(f) = (I \circ T)(f)$.

Noting that $J(1) = 1$, by the classical Rietz Theorem's (see for example [20], Theorem 2.14) it follows that there exists a probability measure Q_0 on X_0 such that

$$J(g) = \int_{X_0} g \, dQ_0$$

for any $g \in C^o(X_0)$.

Let $\hat{Q}$ be the measure in $\mathcal{P}(X)$ defined by $\hat{Q}(B) = Q_0(B \cap X_0)$ for any $B \in \mathcal{B}(X)$. Then, by Lemma 2.4 we get

$$I(f) = I(Tf_{|X_0}) = J(f_{|X_0}) = \int_{X_0} f \, dQ_0 = E_{\hat{Q}}[f]$$

for every $f \in C^o_b(X)$. This accomplishes the proof. ∎

Let us set

$$Y = X_0 \cup \left(\bigcup_{h \in N} X_h \right)$$

We conclude this section with a basic result for our purposes.

Lemma 2.5. *Let* $(\hat{Q}_h)$ *be a sequence in* $\mathcal{P}(X)$ *of the form* (2.4). *Let* $\hat{Q} \in \mathcal{P}(X)$ *such that* $\hat{Q}(X_0) = 1$. *Suppose that* $(\hat{Q}_h)$ *converges weakly to* $\hat{Q}$ *in* $\mathcal{P}(X)$. *Let* $g : Y \to \mathbf{R}$ *be a function bounded from below. Assume that*

(i) $g|_{X_0}$ *is lower semicontinuous*;

(ii) *let* $(\sigma(h))$ *be any sequence of natural numbers such that* $\sigma(h) \to +\infty$ *as* $h \to +\infty$, *then*

$$g(x) \le \liminf_{h \to +\infty} g(x_h)$$

for every sequence (x_h) *converging to* $x \in X_0$ *in* X *and such that* $x_h \in X_{\sigma(h)}$ *for every* $h \in \mathbf{N}$;
(iii) $g|X_h$ *is* $\mathcal{B}(X_h)$ *- measurable for every* $h \in \mathbf{N}$.
Then

(2.7) $$E_Q[g] \leq \liminf_{h \to +\infty} E_{Q_h}[g]$$

where $Q = \widehat{Q}|X_0$ *and, for every* $h \in \mathbf{N}$, Q_h *is the probability measure on* $(X_h, \mathcal{B}(X_h))$ *associated with* $\widehat{Q}_h$ *by* (2.4).

Proof. Let $f \in C^\circ(X_0)$ such that $f \leq g$ on X_0. To get the assertion it is enough to show that

(2.8) $$E_Q[f] \leq \liminf_{h \to +\infty} E_{Q_h}[g] .$$

Infact, since X_0 is compact, there exists an increasing sequence (f_k) of functions in $C^\circ(X_0)$ such that $f_k(x) \to g(x)$ for every $x \in X_0$, as $k \to +\infty$; then, the inequality (2.7) follows from (2.8) and the monotone convergence theorem. Let us prove (2.8). Let $h \in C_b^\circ(X)$ be such that $f = h|_{X_0}$. Preliminarly, we show that

(2.9) $$\limsup_{h \to +\infty} \sup_{x \in X_{\sigma(h)}} (h(x) - g(x)) = l \leq 0$$

Suppose by contraddition $l > 0$; then there exist a subsequence $(X_{\sigma(\tau(h))})$ of $(X_{\sigma(h)})$ and a constant $c > 0$ such that

$$\sup_{x \in X_{\sigma(\tau(h))}} (h(x) - g(x)) > c .$$

Hence,there exists a sequence (x_h) in X such that $x_h \in X_{\sigma(\tau(h))}$ and $h(x_h) > g(x_h) + c$. By passing to a subsequence, by property (2.3) the sequence (x_h) converges in X to $x \in X_0$. Moreover, by (ii) and by continuity of h we obtain $f(x) = h(x) > g(x) + c$, which is in contraddiction with the assumption on the function f. This proves (2.9). Finally, the proof of (2.8) is obtained by noting that, if (2.9) holds then there exists a sequence η_h of positive real numbers such that $\eta_h \to 0$ and $h(x) \leq g(x) + \eta_h$ for every $x \in X_0$ and $h \in \mathbf{N}$. ∎

3. MOSCO CONVERGENCE AND RANDOM CAPACITIES.

In this section we define a variational notion of convergence, introduced by U. Mosco in [19], for sequences of convex functions and discuss some its useful implications for the study of Dirichlet problems in domains surrounded by thin layers.

Definition 3.1. Let (X,τ) be a topological space. Let (F_h) be a sequence of functions from X into **R**. We say that a function $F : X \to \overline{\mathbf{R}}$ is the sequential Γ-limit of (F_h) and we write

$$F = \Gamma_{seq}(\tau) \lim_{h\to\infty} F_h$$

if

(a) for every $x \in X$ and for every sequence (x_h) converging to x in X we have

$$F(x) \leq \liminf_{h\to\infty} F_h(x_h) ;$$

(b) for every $x \in X$ there exists a sequence (x_h) converging to x in X such that

$$F(x) \geq \limsup_{h\to\infty} F_h(x_h) .$$

For a general definition of Γ-convergence and for its applications in calculus of variation we refer to [14],[15],[2]. Let X be a Banach space, we consider on X both the weak and the strong topology, denoted by w and s, respectively.

Definition 3.2. A sequence (F_h) of function from X into $\overline{\mathbf{R}}$ is said to be Mosco convergent to F if

$$F = \Gamma_{seq}(w) \lim_{h\to+\infty} F_h = \Gamma_{seq}(s) \lim_{h\to+\infty} F_h .$$

In other words the sequence F_h Mosco converges to F if

(a) for every $x \in X$ and for every sequence (x_h) converging weakly to x in X we have

$$F(x) \leq \liminf_{h\to\infty} F_h(x_h) ;$$

(b) for every $x \in X$ there exists a sequence converging strongly to x in X such that

$$F(x) \geq \limsup_{h\to\infty} F_h(x_h) .$$

Definition 3.3. We denote by $\mathcal{E}$ the class of convex, lower semicontinuous, proper functions from $L^2(\mathbf{R}^n)$ into $\overline{\mathbf{R}}$.

We note that the class $\mathcal{F}$, defined in (1.8), is contained in $\mathcal{E}$. On $\mathcal{E}$ the Mosco convergence is attached to a metrizable topology (see [2], Section 3.5), which will be called *the topology of the Mosco convergence* and denoted by τ_M . For our purpose the relevant topological aspects of the Mosco convergence are contained in the following theorem (see[2], Theorem 3.36).

Theorem 3.4. *There exists a metric on $\mathcal{E}$ which induces the Mosco convergence topology and which is complete and separable.*

If we consider $\mathcal{F}$ endowed with the topology induced by τ_M, the following compactness results can be obtained by adapting the proofs of Theorem 4.1, Lemma 5.2, and Lemma 6.2 of [6].

Proposition 3.5. (a) *$\mathcal{F}_0$ is compact in $\mathcal{F}$*; (b) *let $(\mathcal{F}_{m(h)})$ be any subsequence of $(\mathcal{F}_h)$, then for every sequence (F_h) in $\mathcal{F}$ such that $F_h \in \mathcal{F}_{m(h)}$ there exist a subsequence $(F_{\sigma(h)})$ and a functional $F_0 \in \mathcal{F}_0$ such that $(F_{\sigma(h)})$ Mosco converges to F_0.*

For any sequence m(h) of natural numbers such that $m(h) \to +\infty$ as $h \to +\infty$, let (F_h) be a sequence in $\mathcal{F}$ such that $F_h \in \mathcal{F}_{m(h)}$ and let $F \in \mathcal{F}_0$. Given $\lambda > 0$, let $R_h(\lambda)$ and $R(\lambda)$ be the resolvent operators introduced in Definition 1.6 and Definition 1.7.

The next result is an easy consequence of Theorem 5.5 and Lemma 5.2 in [6].

Proposition 3.6. *For every $\lambda > 0$ the following statements are equivalent:*
(a) (F_h) *Mosco converges to* F;
(b) $(R_h(\lambda))$ *converges to* $R(\lambda)$ *strongly in* $L^2(\mathbf{R}^n)$.
The same result holds also for $\lambda = 0$ if $\eta_h = c\varepsilon_h$ (see Remark 1.8).

The following propositions show the connection between Mosco - convergence of a sequence of functionals in $\mathcal{F}$ and the behaviour of the corresponding functions b_δ introduced in Definition 1.9.

Proposition 3.7. *Let (F_h) be a sequence in $\mathcal{F}$ and let $F \in \mathcal{F}$. Suppose that (F_h) Mosco converges to F and one of the following assumptions holds :*
(i) $F_h, F \in \mathcal{F}_0$;
(ii) $F_h, F \in \mathcal{F}_m$, *where* m *is a fixed natural number* ;
(iii) $F_h \in \mathcal{F}_{m(h)}$ *for every* $h \in \mathbf{N}$ *and* $F \in \mathcal{F}_0$, *where* $(m(h))$ *is any sequence of natural numbers such that* $m(h) \to +\infty$ *as* $h \to +\infty$. *Then the inequalities*

$$b_\delta(F,U) \leq \liminf_{h \to +\infty} b_\delta(F_h,U) \tag{3.1}$$

$$b_\delta(F,U) \geq \limsup_{h \to +\infty} b_\delta(F_h,U') \tag{3.2}$$

are satisfied for every $\delta > 0$ and for every pair $U,U' \in \mathcal{U}$ such that $U' \subset\subset U$.

Proof. The case (i) and (iii) require minor changes in the proof of Lemmas 6.3,6.4,6.5,6.6,5.2 in [6]. In the case (ii) we can adapt Lemmas 5.5,5.6 and Proposition 5.7 in [10] ∎

Proposition 3.8. *If* (F_h) *is a sequence in* $\mathcal{F}_0$, $F\in\mathcal{F}_0$, *and* (F_h) *Mosco converges to* F *in* $\mathcal{F}_0$ *then the inequality*

$$b_\delta(F,K) \geq \limsup_{h\to+\infty} b_\delta(F_h,K)$$

holds for every $K\in\mathcal{K}$ *and for every* $\delta > 0$.

Proof. It is enough to use (3.2) and the property (h) in Proposition 1.11. ∎

Let us indicate by τ_0 the topology on $\mathcal{F}_0$ induced by τ_M, by $\mathcal{B}(\tau_M)$ the Borel σ-field of $\mathcal{E}$ equipped with τ_M, and by $\mathcal{B}(\tau_0)$ the Borel σ-field of $\mathcal{F}_0$ endowed with τ_0. As a consequence of the Propositions 3.7 and 3.8 we have that for every $\delta > 0$ the functions $b_\delta(\bullet,U)$, $U\in\mathcal{U}$, and $b_\delta(\bullet,K)$, $K\in\mathcal{K}$ from $\mathcal{F}_0$ into **R**, are $\mathcal{B}(\tau_0)$ measurable. We have also to say something about measurability of the function $b_\delta(\bullet,B)$, $B\in\mathcal{B}$, from $\mathcal{F}_0$ into **R**. Let us denote by $\widehat{\mathcal{B}}(\tau_0)$ the σ-field of all subsets of $\mathcal{F}_0$ which are Q measurable for every probability measure Q on $(\mathcal{F}_0,\widehat{\mathcal{B}}(\tau_0))$. The following result holds.

Proposition 3.9. *For every* $B\in\mathcal{B}$ *and for every* $\delta > 0$ *the function* $b_\delta(\bullet,B)$ *from* $\mathcal{F}_0$ *into* **R** *is* $\mathcal{B}(\tau_0)$ *- measurable .*

Proof. The assertion can be obtained by suitable minor changes in the proof of Proposition 2.4 in [3]. ∎

For $h\in\mathbf{N}$ fixed, let $cap^{(h)}$ be the set function of Definition 1.12. We recall the following result (see [10], Theorem 6.3, Theorem 5.9, and [4], Lemma 2.2).

Proposition 3.10. *Let* (F_j) *be a sequence in* $\mathcal{F}_h$ *and let* $F\in\mathcal{F}_h$. *Then* (F_j) *Mosco converges to* F *in* $\mathcal{F}_h$ *if and only if the inequalities*

(a) $$cap^{(h)}(U\cap\partial A_F) \leq \liminf_{j\to+\infty} cap^{(h)}(U\cap\partial A_{F_j}),$$

(b) $$cap^{(h)}(K\cap\partial A_F) \geq \limsup_{j\to+\infty} cap^{(h)}(K\cap\partial A_{F_j})$$

hold for every $U\in\mathcal{U}$ *and for every* $K\in\mathcal{K}$ *where* A_F, $A_{F_j}\in\mathcal{A}_h$ *are , respectively , the open sets associated with* F *and* F_j *(see Remark 1.5) .*

For each $h\in\mathbf{N}$ let us denote by τ_h the topology induced on the class $\mathcal{F}_h$ by τ_M.

Remark 3.11. From Proposition 3.10 we deduce that a sub-base for the topology τ_h is given by the sets of the form $\{F\in\mathcal{F}_h : cap^{(h)}(U\cap\partial A_F) > t\}$ and $\{F\in\mathcal{F}_h : cap^{(h)}(K\cap\partial A_F) < s\}$, with

$t,s\in \mathbf{R}^+$, $U\in \mathcal{U}$, $K\in \mathcal{K}$, where $A_F\in \mathcal{A}_h$ is the set associated with $F\in \mathcal{F}_h$.

We indicate by $\mathcal{B}(\tau_h)$ the Borel σ-field of $\mathcal{F}_h$ endowed with the topology τ_h.

Proposition 3.12. *$\mathcal{B}(\tau_h)$ is the smallest σ-field in $\mathcal{F}_h$ for which the functions $F\to cap^{(h)}(U\cap\partial A_F)$ from $\mathcal{F}_h$ into $\mathbf{R}$ are measurable for every $U\in \mathcal{U}$ (respectively, the functions $F\to cap^{(h)}(K\cap\partial A_F)$ are measurable for every $K\in \mathcal{K}$).*

Proof. Denote by Σ_h' the smallest σ-field in $\mathcal{F}_h$ for which all functions $F\to cap^{(h)}(U\cap\partial A_F)$, $U\in \mathcal{U}$, are measurable and by Σ_h'' the smallest σ-field in $\mathcal{F}_h$ for which all functions $F\to cap^{(h)}(K\cap\partial A_F)$, $K\in \mathcal{K}$ are measurable. Let us show that $\Sigma_h' = \Sigma_h''$. It is enough to prove that
(a) any function $F\to cap^{(h)}(K\cap\partial A_F)$, $K\in \mathcal{K}$ is Σ_h' measurable;
(b) any function $F\to cap^{(h)}(U\cap\partial A_F)$, $U\in \mathcal{U}$ is Σ_h'' measurable.
Let us prove (a). For every $K\in \mathcal{K}$ consider the decreasing sequence of open sets

$$U_h = \{x\in \mathbf{R}^n : d(x,K) < 1\backslash h\}.$$

We remark that $U_h\downarrow K$. From the well-known properties of $cap^{(h)}$ we have

$$cap^{(h)}(K\cap\partial A_F) = \inf_{h\in N} cap^{(h)}(U_h\cap\partial A_F)$$

for every $F\in \mathcal{F}_h$, which proves (a). Assertion (b) can be proven in the same way, by choosing, for every $U\in \mathcal{U}$, an increasing sequence (K_h) in $\mathcal{K}$ such that $K_h\uparrow U$. The proof of the proposition is complete if we show that $\mathcal{B}(\tau_h)=\Sigma_h'$. The inclusion $\Sigma_h' \subseteq \mathcal{B}(\tau_h)$ is trivial because,by Proposition 3.10, $cap^{(h)}(K\cap\partial A_F)$, $K\in \mathcal{K}$ and $cap^{(h)}(U\cap\partial A_F)$, $U\in \mathcal{U}$, are respectively upper and lower semicontinuous on $\mathcal{F}_h$. On the other hand, noting that the sub-base for the topology τ_h, given in Remark 3.11, is contained in Σ_h' and that $\mathcal{F}_h$ admits a countable base for the topology τ_h, we obtain the inclusion $\mathcal{B}(\tau_h) \subseteq \Sigma_h'$. ∎

The next corollary is a direct consequence of the previous proposition.

Corollary 3.13. *Let (Ω,Σ) be a measure space. Let F be a function from Ω into $\mathcal{F}_h$. The following statements are equivalent:*
(a) F is Σ-$\mathcal{B}(\tau_h)$ measurable;
(b) $cap^{(h)}(U\cap\partial A_{F(\bullet)})$ is Σ-measurable for each $U\in \mathcal{U}$;
(c) $cap^{(h)}(K\cap\partial A_{F(\bullet)})$ is Σ-measurable for each $K\in \mathcal{K}$.

We conclude this section with some results on the functions $b_\delta(\bullet,B)$, $B\in \mathcal{B}$, $\delta > 0$, considered as random variables on the space $\mathcal{E}$. We shall deal with weak convergence of measures on the space $\mathcal{E}$. Similar problems of weak convergence of measures on spaces endowed with topology related to Γ-convergence have been studied in [11],[12], and[3].

Lemma 3.14. *Let* Q *be a probability measure on* $(\mathcal{F}_0,\mathcal{B}(\tau_0))$. *Then the following relations*

(3.3) $$E_Q[b_\delta(\bullet,B)] = \sup\{ E_Q[b_\delta(\bullet,K)] : K\in \mathcal{K}, K \subseteq B \},$$

(3.4) $$E_Q[b_\delta(\bullet,B_1)b_\delta(\bullet,B_2)] = \sup\{E_Q[b_\delta(\bullet,K_1)b_\delta(\bullet,K_2)] : K_1,K_2\in \mathcal{K}, K_1 \subseteq B_1, K_2 \subseteq B_2\}$$

hold for every $\delta > 0$ *and for every* $B,B_1,B_2\in \mathcal{B}$.

Proof. We only prove (3.4) since (3.3) can be proven with similar arguments. Fix $\delta > 0$ and $B_2\in \mathcal{B}$. For every $E\in (\mathcal{U}\cup\mathcal{K})$ we define

$$\beta(E) = E_Q[b_\delta(\bullet,E)b_\delta(\bullet,B_2)] .$$

By properties (e),(h),and(j) of Proposition 1.11 we have that

(3.5) $$\beta(K_1\cup K_2) + \beta(K_1\cap K_2) \leq \beta(K_1) + \beta(K_2)$$

for every $K_1,K_2\in \mathcal{K}$

(3.6) $$\beta(K) = \inf\{ \beta(U) : U\in \mathcal{U}, U \supseteq K \}$$

for every $K\in \mathcal{K}$ and

(3.7) $$\beta(U) = \sup\{\beta(K) : K\in \mathcal{K}, K \subseteq U \}$$

for every $U\in \mathcal{U}$. Moreover, we can extend the definition of β by

(3.8) $$\beta(B) = \inf\{\beta(U) : U\in \mathcal{U} , U \supseteq B \}$$

for every $B\in \mathcal{B}$. We deduce from (3.5),(3.6),(3.7),and (3.8) that β is a Choquet capacity (see[16], Theorem 1.5). Applying the capacitability theorem (see [9]) we get

(3.9) $$\beta(B_1) = \sup\{ \beta(K_1) : K_1\in \mathcal{K}, K_1 \subseteq B_1 \} =$$

$$= \sup\{ E_Q[b_\delta(\bullet,K_1)b_\delta(\bullet,B_2)] : K_1\in \mathcal{K}, K_1 \subseteq B_1 \} \leq$$

$$\leq E_Q[b_\delta(\bullet,B_1)b_\delta(\bullet,B_2)] \leq \inf\{ E_Q[b_\delta(\bullet,U_1)b_\delta(\bullet,U_2)] : U_1\in \mathcal{U}, U_1 \supseteq B_1 \} =$$

$$= \inf\{ \beta(U_1) : U_1\in \mathcal{U} , U_1 \supseteq B_1 \} = \beta(B_1).$$

for every $B_1\in \mathcal{B}$ and $B_2\in \mathcal{B}$ fixed. From (3.9) and from the formula we can obtain exchanging in (3.9) the roles of B_1 and B_2, we get (3.4). ∎

For every $h\in \mathbf{N}$ let Q_h be a probability measure on $(\mathcal{F}_h,\mathcal{B}(\tau_h))$. From now on we consider sequences $(\widehat{Q}_h)$ of measures in $\mathcal{P}(\mathcal{E})$ with $\widehat{Q}_h$ defined by

(3.10) $$\widehat{Q}_h(B) = Q_h(B\cap\mathcal{F}_h)$$

for every $B\in \mathcal{B}(\tau_M)$.

Lemma 3.15. *Let* $(\widehat{Q}_h)$ *be a sequence in* $\mathcal{P}(\mathcal{E})$ *of the form (3.10), and let* $\widehat{Q}$ *be a measure in* $\mathcal{P}(\mathcal{E})$ *such that* $\widehat{Q}(\mathcal{F}_0)=1$. *Suppose that* $(\widehat{Q}_h)$ *converges weakly in* $\mathcal{P}(\mathcal{E})$ *to* $\widehat{Q}$. *Then, for every* $\delta > 0$ *and* $U,U'\in \mathcal{U}$ *with* $U'\subset\subset U$, *we have*

(3.11) $$E_Q[b_\delta(\bullet,U)] \le \liminf_{h\to+\infty} E_{Q_h}[b_\delta(\bullet,U)]\,,$$

(3.12) $$E_Q[b_\delta(\bullet,U)] \ge \limsup_{h\to+\infty} E_{Q_h}[b_\delta(\bullet,U')]\,,$$

where $Q=\widehat{Q}|_{X_0}$ *and* Q_h *is the probability measure on* $(F_h,\mathcal{B}(\tau_h))$ *associated with* $\widehat{Q}_h$ *by* (3.10).

Proof. By Proposition 3.7 and by applying Lemma 2.5 with $g(F) = b_\delta(F,U)$ for $\delta > 0$ and $U\in \mathcal{U}$ fixed, we get the inequality (3.11). Let us prove (3.12). For every $F\in \mathcal{F}$ and for $\delta > 0$ fixed we define

$$b^*(F,U) = \inf\{\, b_\delta(F,U') : U'\in \mathcal{U}\,,\ U\subset\subset U' \,\}$$

for every $U\in \mathcal{U}$. Preliminarily, we show that, for every $U\in \mathcal{U}$,

(a) $b^*(\bullet,U)|_{\mathcal{F}_0}$ is upper semicontinuous;

(b) let $(m(h))$ be any sequence of natural numbers such that $m(h)\to+\infty$ as $h\to+\infty$, then

$$b^*(F,U) \ \ge\ \limsup_{h\to+\infty} b^*(F_h,U)$$

for every sequence F_h, with $F_h\in \mathcal{F}_{m(h)}$, which Mosco converges to $F\in \mathcal{F}_0$;

(c) $b^*(\bullet,U)|_{\mathcal{F}_m}$ is upper semicontinuous, where m is a fixed natural number.

We prove (a). Properties (b) and (c) can be obtained by repeating the proof of (a) with suitable changes. Let (F_h) be a sequence in $\mathcal{F}_0$ Mosco converging to $F\in \mathcal{F}_0$ and let $U\in \mathcal{U}$. For every $t > b^*(F,U)$ there exists $U'\in \mathcal{U}$, with $U\subset\subset U'$ such that $t > b_\delta(F,U')$. Let $U''\in \mathcal{U}$ be such that $U\subset\subset U''\subset\subset U'$. Then by (3.2) it follows that

$$t > b_\delta(F,U') \ge \limsup_{h\to+\infty} b_\delta(F,U'') \ge \limsup_{h\to+\infty} b^*(F,U)$$

which proves (a). Now, by applying Lemma 2.5 to the function $g(F)= -b^*(F,U)$ with $U\in \mathcal{U}$ fixed we have

$$E_Q[b^*(\bullet,U)] \ \ge\ \limsup_{h\to+\infty} E_{Q_h}[b^*(\bullet,U)].$$

Let $U'\in \mathcal{U}$ such that $U'\subset\subset U$. Then

$$E_Q[b_\delta(\bullet,U)] \ge E_Q[b^*(\bullet,U')] \ge$$

$$\ge \limsup_{h\to+\infty} E_{Q_h}[b^*(\bullet,U')] \ \ge\ \limsup_{h\to+\infty} E_{Q_h}[b_\delta(\bullet,U')]$$

which proves (3.12). ∎

Lemma 3.16. *Let* $(\hat{Q}_h)$ *be a sequence in* $\mathcal{P}(\mathcal{E})$ *and let* $\hat{Q}\in\mathcal{P}(\mathcal{E})$ *as in the Lemma* 3.15. *Then for every* $\delta>0$

(3.13) $$E_Q[b_\delta(\bullet,U_1)b_\delta(\bullet,U_2)] \le \liminf_{h\to+\infty} E_{Q_h}[b_\delta(\bullet,U_1)b_\delta(\bullet,U_2)]$$

(3.14) $$E_Q[b_\delta(\bullet,U_1)b_\delta(\bullet,U_2)] \ge \limsup_{h\to+\infty} E_{Q_h}[b_\delta(\bullet,U_1')b_\delta(\bullet,U_2')]$$

for every $U_1,U_2,U_1',U_2'\in\mathcal{U}$ *with* $U_1'\subset\subset U_1$ *and* $U_2'\subset\subset U_2$, *where* $Q=\hat{Q}|_{X_0}$ *and* Q_h *is the probability measure on* $(F_h,\mathcal{B}(\tau_h))$ *associated with* $\hat{Q}_h$ *by* (3.10).

Proof. Let us fix $\delta>0$; we set $\mathcal{N}(F,U_1,U_2)=b_\delta(F,U_1)b_\delta(F,U_2)$.

Let (F_h) be a sequence in $\mathcal{F}$ and let $F\in\mathcal{F}$. By Proposition 3.7, if (F_h) Mosco converges to F and one of the assumptions considered there is satisfied, it follows that

(3.15) $$\mathcal{N}(F,U_1,U_2) \le \liminf_{h\to+\infty} \mathcal{N}(F_h,U_1,U_2)$$

(3.16) $$\mathcal{N}(F,U_1,U_2) \ge \limsup_{h\to+\infty} \mathcal{N}(F_h,U_1',U_2')$$

for every $U_1,U_2,U_1',U_2'\in\mathcal{U}$ with $U_1'\subset\subset U_1$ and $U_2'\subset\subset U_2$. From (3.15), by applying Lemma 2.5 with $g(F)=\mathcal{N}(F,U_1,U_2)$ for $U_1,U_2\in\mathcal{U}$ fixed, we obtain (3.13). Let us prove (3.14). For every $F\in\mathcal{F}$ and for every $U_1,U_2\in\mathcal{U}$ we define

$$\mathcal{N}^*(F,U_1,U_2)=\inf\{\,\mathcal{N}(F,U_1',U_2')\,,\ U_1',U_2'\in\mathcal{U}\,,\ U_1\subset\subset U_1'\,,\ U_2\subset\subset U_2'\,\}$$

Preliminarly we show that for every $U_1,U_2\in\mathcal{U}$

(a) $\mathcal{N}^*(F,U_1,U_2)|_{\mathcal{F}_0}$ is upper semicontinuous;

(b) let m(h) be any sequence of natural numbers such that $m(h)\to+\infty$ as $h\to+\infty$, then

$$\mathcal{N}^*(F,U_1,U_2) \ge \limsup_{h\to+\infty} \mathcal{N}^*(F_h,U_1,U_2)$$

for every sequence (F_h) in $\mathcal{F}$, with $F_h\in\mathcal{F}_{m(h)}$, which Mosco converges to $F\in\mathcal{F}_0$;

(c) $\mathcal{N}^*(\bullet,U_1,U_2)|_{\mathcal{F}_m}$ is upper semicontinuous, where m is a fixed natural number.

We prove (b). Properties (a) and (c) can be obtained by adapting with minor changes the proof of (b). Let (F_h) be a sequence in $\mathcal{F}$, with $F_h\in F_{m(h)}$, which Mosco converges to $F\in\mathcal{F}_0$. For every $t>\mathcal{N}^*(F,U_1,U_2)$ there exist $U_1',U_2'\in\mathcal{U}$ with $U_1\subset\subset U_1'$ and $U_2\subset\subset U_2'$ such that $t>\mathcal{N}(F,U_1',U_2')$. Let $U_1'',U_2''\in\mathcal{U}$ be such that $U_1\subset\subset U_1''\subset\subset U_1'$ and $U_2\subset\subset U_2''\subset\subset U_2'$. Then from (3.16) it follows that

$$t>\mathcal{N}(F,U_1',U_2')\ge$$
$$\ge \limsup_{h\to+\infty} \mathcal{N}(F_h,U_1'',U_2'') \ge \limsup_{h\to+\infty} \mathcal{N}^*(F_h,U_1,U_2)$$

which proves (b). Properties (a), (b), and (c) allow to apply Lemma 2.5 to the function $g(F)=-\mathcal{N}^*(F,U_1,U_2)$ with $U_1,U_2\in\mathcal{U}$ fixed. Thus, we obtain

$$E_Q[\mathcal{N}^*(\bullet,U_1,U_2)] \geq \limsup_{h\to+\infty} E_{Q_h}[\mathcal{N}^*(\bullet,U_1,U_2)]$$

for every $U_1,U_2 \in \mathcal{U}$. Finally, by taking $U_1',U_2',U_1,U_2 \in \mathcal{U}$ such that $U_1' \subset\subset U_1$ and $U_2' \subset\subset U_2$, we have

$$E_Q[\mathcal{N}(\bullet,U_1,U_2)] \geq E_Q[\mathcal{N}^*(\bullet,U_1',U_2')] \geq$$
$$\geq \limsup_{h\to+\infty} E_{Q_h}[\mathcal{N}^*(\bullet,U_1',U_2')] \geq \limsup_{h\to+\infty} E_{Q_h}[\mathcal{N}(\bullet,U_1',U_2')]$$

which proves (3.14) and the proof is accomplished. ∎

Lemma 3.17. *Let* $(\hat{Q}_h)$ *a sequence in* $\mathcal{P}(\mathcal{E})$ *and* $\hat{Q} \in \mathcal{P}(\mathcal{E})$ *as in the Lemma* 3.15, *and let* $\delta > 0$. *If we assume that*

$$\lim_{h\to+\infty} \mathrm{Cov}_{Q_h}[b_\delta(\bullet,U_1)b_\delta(\bullet,U_2)] = 0$$

for each pair $U_1,U_2 \in \mathcal{U}$ *such that* $\bar{U}_1 \cap \bar{U}_2 = \emptyset$, *we have*

$$\mathrm{Cov}_Q[b_\delta(\bullet,U_1)b_\delta(\bullet,U_2)] = 0$$

for any $U_1,U_2 \in \mathcal{U}$, *with* $\bar{U}_1 \cap \bar{U}_2 = \emptyset$.

Proof. Let $U_1,U_2,U_1',U_2' \in \mathcal{U}$ such that $U_1' \subset\subset U_1$, $U_2' \subset\subset U_2$ and $U_1 \cap U_2 = \emptyset$. By (3.13) and (3.12) it follows that

(3.17) $$E_Q[b_\delta(\bullet,U_1')b_\delta(\bullet,U_2')] \leq \liminf_{h\to+\infty} E_{Q_h}[b_\delta(\bullet,U_1')b_\delta(\bullet,U_2')]$$

(3.18) $$E_Q[b_\delta(\bullet,U_1)]E_Q[b_\delta(\bullet,U_2)] \geq \limsup_{h\to+\infty} E_{Q_h}[b_\delta(\bullet,U_1')]E_{Q_h}[b_\delta(\bullet,U_2')]$$

By subtracting (3.18) from (3.17) we obtain

(3.19) $$E_Q[b_\delta(\bullet,U_1')b_\delta(\bullet,U_2')] - E_Q[b_\delta(\bullet,U_1)]E_Q[b_\delta(\bullet,U_2)] \leq$$
$$\leq \liminf_{h\to+\infty} \mathrm{Cov}_{Q_h}[b_\delta(\bullet,U_1'),b_\delta(\bullet,U_2')] = 0 .$$

By (3.14) and (3.11) we deduce that

(3.20) $$E_Q[b_\delta(\bullet,U_1)b_\delta(\bullet,U_2)] \geq \limsup_{h\to+\infty} E_{Q_h}[b_\delta(\bullet,U_1')b_\delta(\bullet,U_2')]$$

(3.21) $$E_Q[b_\delta(\bullet,U_1')]E_Q[b_\delta(\bullet,U_2')] \leq \liminf_{h\to+\infty} E_{Q_h}[b_\delta(\bullet,U_1')]E_{Q_h}[b_\delta(\bullet,U_2')]$$

By subtracting (3.21) from (3.20) we have

(3.22) $$E_Q[b_\delta(\bullet,U_1)b_\delta(\bullet,U_2)] - E_Q[b_\delta(\bullet,U_1')]E_Q[b_\delta(\bullet,U_2')] \geq$$
$$\geq \limsup_{h\to+\infty} \mathrm{Cov}_{Q_h}[b_\delta(\bullet,U_1'),b_\delta(\bullet,U_2')] = 0 .$$

By (3.19), (3.22), and Lemma 3.14 we get the assertion. ∎

4. THE MAIN RESULTS.

This section is devoted to state and to prove the main theorems of this paper. They give full answer to the following questions.

(a) For every $h \in \mathbb{N}$, let Q_h be a probabality measure on $(\mathcal{F}_h, \mathcal{B}(\tau_h))$. Under which conditions a sequence of measures $(\hat{Q}_h)$ in $\mathcal{P}(\mathcal{E})$ of the form (3.10) has a subsequence $(\hat{Q}_{\sigma(h)})$ which converges in $\mathcal{P}(\mathcal{E})$ to a Dirac measure $\hat{Q} = \delta_{F_0}$ with $F_0 \in \mathcal{F}_0$?

(b) How can this limit be characterizé ?

We will show that both the answers depend on the asymptotic behaviour, as $h \to +\infty$, of the functions $b_\delta(\bullet, U)$, considered as random variables on the probability spaces $(\mathcal{F}_h, \mathcal{B}(\tau_h), Q_h)$.

Before to state our main results we put some definitions. For every $U \in \mathcal{U}$ we define

$$\alpha'_\delta(U) = \liminf_{h \to +\infty} E_{Q_h}[b_\delta(\bullet, U)]$$

$$\alpha''_\delta(U) = \limsup_{h \to +\infty} E_{Q_h}[b_\delta(\bullet, U)]$$

where E_{Q_h} denotes the expectation in the probability space $(\mathcal{F}_h, \mathcal{B}(\tau_h), Q_h)$. Next, we consider the inner regularizations β'_δ and β''_δ of the set functions α'_δ and α''_δ, defined for every $U \in \mathcal{U}$ by

(4.1) $$\beta'_\delta(U) = \sup\{\alpha'_\delta(V) : V \in \mathcal{U}, V \subset\subset U\}$$

(4.2) $$\beta''_\delta(U) = \sup\{\alpha''_\delta(V) : V \in \mathcal{U}, V \subset\subset U\}$$

We extend the definitions of β'_δ and β''_δ to the Borel sets $B \in \mathcal{B}$ by

$$\beta'_\delta(B) = \inf\{\beta'_\delta(U) : U \in \mathcal{U}, U \supseteq B\}$$

$$\beta''_\delta(B) = \inf\{\beta''_\delta(U) : U \in \mathcal{U}, U \supseteq B\}.$$

Finally, we define

(4.3) $$\nu'(B) = \sup\{\beta'_\delta(B) : B \in \mathcal{B}, \delta > 0\}$$

(4.4) $$\nu''(B) = \sup\{\beta''_\delta(B) : B \in \mathcal{B}, \delta > 0\}$$

We are now able to state our results.

Theorem 4.1.(Compacteness Theorem). *Let* (Q_h) *be a sequence of probability measure on* $(\mathcal{F}_h, \mathcal{B}(\tau_h))$; *for every* $h \in \mathbb{N}$ *let* $\hat{Q}_h$ *the measure in* $\mathcal{P}(\mathcal{E})$ *associated with* Q_h *by* (3.10). *Assume that there exists a Radon measure* β *with* $\operatorname{spt}\beta \subseteq \partial D$ *such that*

(4.5) $$\limsup_{h\to+\infty} E_{Q_h}[b_\delta(\bullet,U)] \le \beta(\bar{U})$$

for every $U \in \mathcal{U}$ *and* $\delta > 0$.

Moreover, suppose that for every $U_1, U_2 \in \mathcal{U}$, *with* $\bar{U}_1 \cap \bar{U}_2 = \emptyset$,

(4.6) $$\lim_{h\to+\infty} \mathrm{Cov}_{Q_h}[b_\delta(\bullet,U_1), b_\delta(\bullet,U_2)] = 0$$

Then, there exists a subsequence $(\widehat{Q}_{\sigma(h)})$ *of* $(\widehat{Q}_h)$ *and a functional* $F_0 \in \mathcal{F}_0$ *such that* $(\widehat{Q}_{\sigma(h)})$ *converges weakly on* $\mathcal{P}(\mathcal{E})$ *to the Dirac measure* $\delta_{F_0} \in \mathcal{P}(\mathcal{E})$ *defined by*

(4.7) $$\delta_{F_0}(A) = \begin{cases} 0 & \text{if } F_0 \notin A \\ 1 & \text{if } F_0 \in A \end{cases}$$

for every $A \in \mathcal{B}(\tau_M)$.

The limit functional F_0 is determined by the next theorem.

Theorem 4.2. *Let* (Q_h) *be a sequence of probability measures as in Theorem 4.1. Assume that there exists a Radon measure* γ *such that*

(4.8) $$\nu'(B) = \nu''(B) \le \gamma(B)$$

for every $B \in \mathcal{B}$ *and call* $\nu(B)$ *the common value of* $\nu'(B)$ *and* $\nu''(B)$.

Suppose that (4.6) holds. Then,

(t_1) ν *is a Borel measure of the class* $\mathcal{M}_0^*(\partial D)$;

(t_2) $(\widehat{Q}_h)$ *converges in* $\mathcal{P}(\mathcal{E})$ *to the Dirac measure* $\delta_{F_0} \in \mathcal{P}(\mathcal{E})$ *defined in (4.7) where* F_0 *is the functional in* $\mathcal{F}_0$ *associated to the measure* ν *according to Remark 1.5* .

We now show some preliminary results which allow to get the proofs of Theorem 4.1 and Theorem 4.2. The next lemma gives a peculiar representation of a measure $\mu \in \mathcal{M}_0^*(\partial D)$.

Lemma 4.3. *Let* $\mu \in \mathcal{M}_0^*(\partial D)$ *and let* F *be the corresponding functional in* $\mathcal{F}_0$. *Then, for every* $B \in \mathcal{B}$ *we have*

$$\mu(B) = \lim_{\delta\to 0} \sum_{i\in I_\delta} b_\delta(F, B_i^\delta)$$

where, for every $\delta > 0$, $(B_i^\delta)_{i\in I_\delta}$ *is any finite Borel partition of* B.

Proof. Let $B \in \mathcal{B}$. For every $\delta > 0$ fixed, denote by $(B_i^\delta)_{i\in I_\delta}$ any finite partition of B. Then, by (g) in Proposition 1.11, we have

$$\mu(B) = \sum_{i\in I_\delta} \mu(B_i^\delta) \ge \sum_{i\in I_\delta} b_\delta(F, B_i^\delta)$$

Hence

(4.9) $$\mu(B) \geq \limsup_{\delta \to 0} \sum_{i \in I_\delta} b_\delta(F, B_i^\delta)$$

On the other hand, by (k) and (e) of Proposition 1.11, for every real number $t < \mu(B)$, there exists $\delta_0 > 0$ such that

$$t < b_\delta(F,B) \leq \sum_{i \in I_\delta} b_\delta(F, B_i^\delta)$$

for every $\delta < \delta_0$. Thus we have

$$t < \liminf_{\delta \to 0} \sum_{i \in I_\delta} b_\delta(F, B_i^\delta)$$

Hence

(4.10) $$\mu(B) \leq \liminf_{\delta \to 0} \sum_{i \in I_\delta} b_\delta(F, B_i^\delta)$$

The inequalities (4.9) and (4.10) give the assertion. ∎

The following proposition provides a sufficient condition in order that a probability measure Q on $(\mathcal{F}_0, \mathcal{B}(\tau_0))$ be equal to a Dirac measure δ_{F_0}.

Proposition 4.4. *For every* $\delta > 0$ *we define*

$$\alpha_\delta(U) = E_Q[b_\delta(\bullet, U)]$$

for every $U \in \mathcal{U}$, *and*

$$\alpha_\delta(B) = \inf\{\alpha_\delta(U) : U \in \mathcal{U}, U \supseteq B\}$$

for every $B \in \mathcal{B}$.

Moreover, let us set

$$\nu(B) = \sup_{\delta > 0} \alpha_\delta(B)$$

for every $B \in \mathcal{B}$.

Let us assume that

(i) *there exists a Radon measure* β *such that* $\nu(B) \leq \beta(B)$ *on* $\mathcal{B}$;

(ii) $$\mathrm{Cov}_Q[b_\delta(\bullet, U_1), b_\delta(\bullet, U_2)] = 0$$

for every $\delta > 0$ *and for every pair* U_1, U_2 *of sets in* $\mathcal{U}$ *such that* $\bar{U}_1 \cap \bar{U}_2 = \emptyset$.

Then ,

(t_1) ν *is a Borel measure of the class* $\mathcal{M}_0^*(\partial D)$;

(t_2) $Q = \delta_{F_0}$, *where* F_0 *is the functional in* $\mathcal{F}_0$ *associated with* ν.

Proof. From (b) and (d) of Proposition 1.11 we deduce that the function ν is increasing and countably subadditive on $\mathcal{B}$. In order to prove that ν is a Borel measure we first note that, by (h) of Proposition 1.11 and by (3.3), we have

(4.11) $$\alpha_\delta(B) = E_Q[b_\delta(\bullet,B)]$$

for every $\delta > 0$ and $B \in \mathcal{B}$.

Thus, from (i) of Proposition 1.11 we deduce that

(4.12) $$\alpha_\delta(B_1) + \alpha_\delta(B_2) \le (1-\eta)^{-1}\ \alpha_\delta(B_1 \cup B_2) + 4\Lambda_2\,\eta^{-1}\,\sigma^2\,|D-D_\delta|$$

for every η, $\delta > 0$ and for every pair $B_1, B_2 \in \mathcal{B}$ such that dist $(B_1,B_2) = \sigma > 0$.

By taking first the supremum over all $\delta > 0$ and then the limit as η goes to zero in (4.12), we get

$$\nu(B_1) + \nu(B_2) \le \nu(B_1 \cup B_2)$$

for every $B_1, B_2 \in \mathcal{B}$ such that dist $(B_1,B_2) > 0$.

Applying the Caratheodory criterion (see[17], 2.3.2(9)) we obtain that ν is Borel measure. Finally, the hypothesis (i) and Proposition 1.11 ((f),(g)) infer that $\nu \in \mathcal{M}_0^*(\partial D)$ and this completes the proof of (t_1). Let us prove (t_2). Let us denote by $Z(\bullet,B)$ the random variable on the probability space $(\mathcal{F}_0,\mathcal{B}(\tau_0),Q)$ defined for every Borel set B of ∂D by

$$Z(F,B) = \mu(B)$$

where μ is the measure in $\mathcal{M}_0^*(\partial D)$ associated with $F \in \mathcal{F}_0$.

We note that, by Lemma 4.3, for every $F \in \mathcal{F}_0$ and for every Borel set B of ∂D,

$$Z(F,B) = \lim_{\delta \to 0} \sum_{i \in I_\delta} b_\delta(F,B_i^\delta)$$

where, for each $\delta > 0$, $(B_i^\delta)_{i \in I_\delta}$ is any finite partition of B. Our aim is to show that $Z(\bullet,B)$ is a constant random variable. In view of Lemma 3.1 in [3], we have only to prove that

(4.13) $$\lim_{\delta \to 0} \mathrm{Var}_Q[\sum_{i \in I_\delta} b_\delta(\bullet,B_i^\delta)\,] = 0$$

By (h) of Proposition 1.11 and by (3.3) and (3.4), we can extend the relation (ii) to each pair of disjoint sets $B_1, B_2 \in \mathcal{B}$. Therefore, to get (4.13) it is enough to prove

(4.14) $$\lim_{\delta \to 0} \sum_{i \in I_\delta} \mathrm{Var}_Q[b_\delta(\bullet,B_i^\delta)] = 0$$

Let B be a Borel set of ∂D and let $(r_\delta)_{\delta>0}$ and $(R_\delta)_{\delta>0}$ be two sequences of positive numbers such that : (a) $r_\delta < R_\delta$ for every $\delta > 0$;

(b) $s_\delta = \mathrm{cap}\,(B_{r_\delta}(0),B_{R_\delta}(0)) \to 0$ as $\delta \to 0$;

(c) for every $x \in \partial D$, $B_{R_\delta}(x) \cap \partial D_\delta = \emptyset$.

For every $\delta > 0$, let us choose a finite partition $(B_i^\delta)_{i \in I_\delta}$ of B such that

$$\sup_{i \in I_\delta} (\mathrm{diam}\ B_i^\delta) < \frac{r_\delta}{2}$$

moreover, for every $i \in I_\delta$ let us fix x_i^δ such that $B_i^\delta \subseteq B_{r_\delta}(x_i^\delta) \subseteq B_{R_\delta}(x_i^\delta)$.
Since $B_{R_\delta}(x_i^\delta) \cap \partial D_\delta = \emptyset$, by (f) of Proposition 1.11 we have

$$b_\delta(F,B_i^\delta) \le \Lambda_2 \operatorname{cap}(B_i^\delta; B_{R_\delta}(x_i^\delta)).$$

Then, for every $\delta > 0$ we get

$$\sum_{i \in I_\delta} \mathrm{Var}_Q[b_\delta(\bullet,B_i^\delta)] = \tag{4.15}$$

$$= \sum_{i \in I_\delta} \left\{ E_Q[b_\delta(\bullet,B_i^\delta)^2] - \left(E_Q[b_\delta(\bullet,B_i^\delta)]\right)^2 \right\} \le$$

$$\le \sum_{i \in I_\delta} E_Q[b_\delta(\bullet,B_i^\delta)^2] \le \Lambda_2 \sum_{i \in I_\delta} \operatorname{cap}(B_i^\delta, B_{R_\delta}(x_i^\delta))\; E_Q[b_\delta(\bullet,B_i^\delta)] \le$$

$$\le \Lambda_2 \sup_{i \in I_\delta} \left\{ \operatorname{cap}(B_{r_\delta}(x_i^\delta), B_{R_\delta}(x_i^\delta)) \right\} \sum_{i \in I_\delta} E_Q[b_\delta(\bullet,B_i^\delta)] \le$$

$$\le \Lambda_2 \operatorname{cap}(B_{r_\delta}(0), B_{R_\delta}(0)) \sum_{i \in I_\delta} \alpha_\delta(B_i^\delta) \le \Lambda_2\, s_\delta \sum_{i \in I_\delta} \beta(B_i^\delta) = \Lambda_2\, s_\delta\, \beta(B)$$

By taking the limit as $\delta \to 0$ in (4.15) we get (4.14) and this proves that $Z(\bullet,B)$ is a constant random variable. Now, let us compute the expectation of $Z(\bullet,B)$. By taking in account that the function $\delta \to b_\delta(F,B)$ is decreasing and by applying Lemma 4.3 with $B_i^\delta = B$ for every $i \in I_\delta$ and for every $\delta > 0$, we obtain

$$E_Q[Z(\bullet,B)] = \sup_{\delta > 0} E_Q[b_\delta(\bullet,B)] = \nu(B)$$

where in the last equality we have used (4.11).
Therefore, for every Borel set B of ∂D there exists a subset $\mathcal{F}_B$ of $\mathcal{F}_0$ with $Q(\mathcal{F}_B) = 1$ such that $Z(F,B) = \nu(B)$ for every $F \in \mathcal{F}_B$.
Finally, by means standard density arguments (see for istance the proof of Lemma 3.3 in [3]) we can deduce that there exists a subset $\mathcal{F}$ of $\mathcal{E}$ such that $Q(\mathcal{F}) = 1$ and $Z(F,B) = \nu(B)$ for every $F \in \mathcal{F}$ and for every Borel set B of ∂D. This completes the proof of (t_2). ∎

Proof of Theorem 4.1. By Theorem 2.2 there exists a subsequence of $(\hat{Q}_h)$ converging weakly to a measure $\hat{Q}$ in $\mathcal{P}(\mathcal{E})$ such that $\hat{Q}(\mathcal{F}_0) = 1$. By (4.5) and by Lemma 3.15 we obtain

$$E_Q[b_\delta(\bullet,U)] \le \beta(\bar{U})$$

for every $\delta > 0$ and $U \in \mathcal{U}$, where $Q = Q_{|\mathcal{F}_0}$.
It is easy to see that also the relation $E_Q[b_\delta(\bullet,U)] \le \beta(U)$ holds.
Hypothesis (4.6) and Lemma 3.17 yield

$$\mathrm{Cov}_Q[b_\delta(\bullet,U1)\; b_\delta(\bullet,U2)] = 0$$

for every $\delta > 0$ and for every pair $U_1, U_2 \in \mathcal{U}$ with $\bar{U}_1 \cap \bar{U}_2 = \emptyset$.

The thesis is obtained easily from Proposition 4.4. ∎

Proof of Theorem 4.2. By Theorem 4.1 and by (4.8) we can assume that (Q_h) converges weakly to a Dirac measure $\delta_{F_0} \in \mathcal{P}(\mathcal{E})$ for some $F_0 \in \mathcal{F}_0$. By Lemma 3.15 it follows that

(4.16) $$E_{\delta_{F_0}}[b_\delta(\cdot,U)] = b_\delta(F_0,U) = \beta'_\delta(U) = \beta''_\delta(U)$$

for every $U \in \mathcal{U}$. By extending (4.16) to an arbitrary Borel set in D we have

$$b_\delta(F_0,B) = \beta'_\delta(B) = \beta''_\delta(B)$$

for every $B \in \mathcal{B}$, which gives

$$\nu(B) = \sup_{\delta > 0} b_\delta(F,B)$$

Property (k) in Proposition 1.11 implies that ν is just the measure in $\mathcal{M}_0^*(\partial D)$ associated with the functional F_0. This concludes the proof of the theorem. ∎

5. DIRICHLET PROBLEMS IN DOMAINS SURROUNDED BY THIN LAYERS WITH RANDOM THICKNESS.

In this section we apply the main results proved in the previous section to Dirichlet problems in domains surrounded by thin layers with random thickness.

From now on (Ω,Σ,P) will denote a probability space, that is, Ω is a set, Σ is a σ-field of subsets of Ω, and P is a probability measure on Σ.

Definition 5.1. (a) For every $h \in N$ a *random functional of the class* $\mathcal{F}_h$ is any measurable function $F_h : \Omega \to \mathcal{F}_h$, where $\mathcal{F}_h$ is equipped with the Borel σ-field $\mathcal{B}(\tau_h)$ generated by the topology τ_h induced by τ_M (topology of Mosco convergence); (b) a *random functional of the class* $\mathcal{F}_0$ is any measurable function $F_0 : \Omega \to \mathcal{F}_0$ where $\mathcal{F}_0$ is endowed with the Borel σ-field $\mathcal{B}(\tau_0)$ generated by the topology τ_0 induced by τ_M.

Remark 5.2. We recall that necessary and sufficient conditions for the measurability of a function $F_h : \Omega \to \mathcal{F}_h$ are given in Corollary 3.13.

Let F_h be a random functional of the class $\mathcal{F}_h$ and let Q_h be the probability measure on $(\mathcal{F}_h,\mathcal{B}(\tau_h))$ defined by

$$Q_h(A) = P\{F_h^{-1}(A)\}$$

for any $A \in \mathcal{B}(\tau_h)$. Q_h is called the *distribution law of* F_h. In the same way, given a random functional F_0 of the class $\mathcal{F}_0$ we can define the distribution law Q of F_0.

For every $h \in \mathbf{N}$ let Q_h be the distribution law of a random functional F_h of the class $\mathcal{F}_h$ and let $\widehat{Q}_h$ be the measure in $\mathcal{P}(\mathcal{E})$ associated to Q_h by (3.10). Moreover, let Q be the distribution law of a random functional F_0 of the class $\mathcal{F}_0$ and let $\widehat{Q}$ be the measure in $\mathcal{P}(\mathcal{E})$ defined by

$$\widehat{Q}(B) = Q(B \cap \mathcal{F}_0)$$

for every $B \in \mathcal{B}(\tau_M)$.

Definition 5.3. We say that (F_h) *converges in law* to F_0 if $(\widehat{Q}_h)$ converges weakly in $\mathcal{P}(\mathcal{E})$ to $\widehat{Q}$.

We denote by E and by Cov respectively the expectation and the covariance of a random variable on Ω, with respect the measure P. It is easy to see that, for $\delta > 0$ and for every $h \in \mathbf{N}$,

(5.1) $$E_{Q_h}[b_\delta(\bullet,U)] = E[b_\delta(F_h(\bullet),U)]$$

for any $U \in \mathcal{U}$ and

(5.2) $$\mathrm{Cov}_{Q_h}[b_\delta(\bullet,U_1),b_\delta(\bullet,U_2)] = \mathrm{Cov}_{Q_h}[b_\delta(F_h(\bullet),U_1)b_\delta(F_h(\bullet),U_2)]$$

for any $U_1, U_2 \in \mathcal{U}$.

Remark 5.4. Equalities (5.1) and (5.2) allow to reformulate the hypotheses of the compactness theorem in terms of the expectations and covariances of the real random variables $b_\delta(F_h(\bullet),U)$, $\delta > 0$. By Definition 5.3 the thesis of Theorem 4.1 can be restated by saying that the sequence of random functionals (F_h) has a subsequence $F_{(\sigma(h))}$ which converges in law to a random functional F_0 on $\mathcal{F}_0$ such that $F_0(\omega)=F_0$ for P-almost every $\omega \in \Omega$ (i.e. to the constant random functional F_0 on $\mathcal{F}_0$).

Remark 5.5. Since $\mathcal{E}$ is a metric space (let d_M be the metric) the convergence in law of the sequence (F_h) toward the random constant functional F_0 is equivalent to the convergence in probability. By Remark 5.4 we can deduce that if the assumtions of Theorem 4.1 on the random variables $b_\delta(F_h(\bullet),U)$, $\delta > 0$, $U \in \mathcal{U}$, hold, then the sequence (F_h) has a subsequence $(F_{\sigma(h)})$ which converges in probability to the constant functional $F_0 \in \mathcal{F}_0$, that is,for every $\varepsilon > 0$

$$\lim_{h \to +\infty} P\{ \omega \in \Omega : d_M(F_{\sigma(h)},F) > \varepsilon \} = 0 .$$

For every $h \in \mathbf{N}$, let A_h be a function from the set Ω into the class of sets $\mathcal{A}_h$ (see Definition 1.2). For every $\omega \in \Omega$ let $F_h(\omega)$ be the functional in $\mathcal{F}_h$ associated with $A_h(\omega)$ (see Remark 1.5).

Definition 5.6. We say that the function $A_h : \Omega \to \mathcal{A}_h$ is a *Random set of the class* $\mathcal{A}_h$ if the function $F_h : \Omega \to \mathcal{F}_h$ is a random functional of the class $\mathcal{F}_h$.

Remark 5.7. Necessary and sufficient conditions in order that a map $A_h : \Omega \to \mathcal{A}_h$ be a random set, can be deduced by Corollary 3.13.

We are interested in the study of the following sequence of random Dirichlet problems associated with a sequence of random sets, that is, for every $\omega \in \Omega$

$$(5.3) \qquad \begin{cases} L^{(h)} u_h + \lambda u_h = g & \text{in } A_h(\omega) \\ u_h \in H_0^1(A_h(\omega)) \end{cases}$$

where $\lambda \geq 0$, $g \in L^2(\mathbf{R}^n)$.

For each $\omega \in \Omega$ and $\lambda \geq 0$, let $(R_h(\lambda)[\omega])$ be the sequence of resolvent operators associated with the sequence $(F_h(\omega))$ (see Definition 1.6). We are now able to state a new version of the compactness Theorem 4.1.

Theorem 5.8. *Let* (A_h) *be a sequence of random sets and let* (F_h) *be the corresponding sequence of random functionals. Assume that there exists a Radon measure* γ *with* $\operatorname{spt} \gamma \subseteq \partial D$ *such that*

$$(5.4) \qquad \limsup_{h \to +\infty} E[b_\delta(F_h(\bullet), U)] \leq \gamma(\bar{U})$$

for every $\delta > 0$ *and for every* $U \in \mathcal{U}$. *Moreover, suppose that for every* $U_1, U_2 \in \mathcal{U}$ *with* $\bar{U}_1 \cap \bar{U}_2 = \emptyset$ *we have*

$$(5.5) \qquad \limsup_{h \to +\infty} \operatorname{Cov}[b_\delta(F_h(\bullet), U_1) b_\delta(F_h(\bullet), U_2)] = 0 .$$

Then there exist a subsequence $(R_{\sigma(h)}(\lambda))$ *of* $(R_h(\lambda))$ *and a functional* $F_0 \in \mathcal{F}_0$ *such that, for every* $\lambda > 0$, $(R_{\sigma(h)}(\lambda))$ *converges strongly in probability to the resolvent operator* $R_0(\lambda)$ *associated to* F_0 *(see Definition 1.7), that is*

$$\lim_{h \to +\infty} P\{\omega \in \Omega : \| R_{\sigma(h)}(\lambda)[g] - R_0(\lambda)[g] \|_{L^2(\mathbf{R}^n)} > \varepsilon\} = 0$$

for every $\varepsilon > 0$ *and for every* $g \in L^2(\mathbf{R}^n)$. *If* $\eta_h = c\varepsilon_h$ *then the same result holds for* $\lambda = 0$ *(see Remark 1.8).*

Proof. By Remark 5.5 we have that there exists a subsequence $(F_{\sigma(h)})$ of (F_h) which converges in probability to some $F \in \mathcal{F}_0$. So the assertion is obtained easily by Proposition 3.6. ∎

For every $h \in \mathbf{N}$ let F_h be a random functional of the class $\mathcal{F}_h$. Given the sequence (F_h), let us define for every $\delta > 0$

$$\alpha'_\delta(U) = \liminf_{h\to+\infty} \; E[b_\delta(F_h(\cdot),U)] ,$$

$$\alpha''_\delta(U) = \limsup_{h\to+\infty} E[b_\delta(F_h(\cdot),U)] .$$

We denote by β'_δ, β''_δ respectively the inner regularization of α'_δ, α''_δ as definited in (4.1), (4.2) and by ν', ν'' the set functions as defined in (4.3), (4.4). It is easy to see that by (5.1), (5.2), Definition 5.3, Remark 5.4 and Remark 5.5, Theorem 4.2 can be restated in the following way.

Theorem 5.9. *Given a sequence of random sets* (A_h), *let* (F_h) *be the corresponding sequence of random functionals. Assume that there exists a Radon measure* γ *such that*

$$\nu'(B) = \nu''(B) \le \gamma(B)$$

for every $B \in \mathcal{B}$ *and call* $\nu(B)$ *the common value of* $\nu'(B)$ *and* $\nu''(B)$. *Suppose that (5.5) holds, then, for every* $\lambda > 0$

$$\lim_{h\to+\infty} P\{\, \omega\in\Omega : ||R_h(\lambda)[f] - R_0(\lambda)[f]||_{L^2(\mathbf{R}^n)} > \varepsilon \,\} = 0$$

for every $\varepsilon > 0$ *and for any* $f\in L^2(\mathbf{R}^n)$, *where* $R_0(\lambda)$ *is the resolvent associated to the functional* $F_0 \in \mathcal{F}_0$, *which corresponds to the measure* $\nu \in \mathcal{M}^*_0(\partial\Omega)$. *If* $\eta_h = c\varepsilon_h$ *then the same result holds for* $\lambda = 0$ *(see Remark 1.8)* .

6. AN EXAMPLE.

In what follows we assume that the domain D of $\mathbf{R}^n$ has a C^2 boundary and that, for every $h\in N$, $\eta_h = c\varepsilon_h$ (see Remark 1.8) .

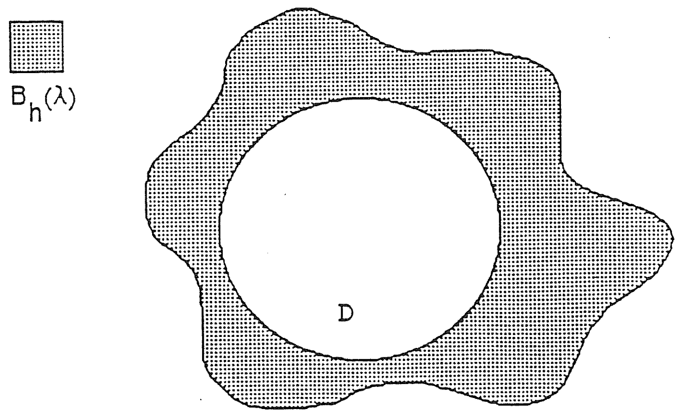
By $(Q_i^h)_{i\in I_h}$ we denote a finite open cover of ∂D such that

$$\max_{i\in I_h} \operatorname{diam} Q_i^h \to 0$$

as $h \to +\infty$. Let $(\phi_i^h)_{i\in I_h}$ be a partition of unity on ∂D, subordinate to the cover $(Q_i^h)_{i\in I_h}$ and let $(x_i^h)_{i\in I_h}$ be a family of indipendent random variables defined on the same probabilistic space (Ω,Σ,P) with values in the interval $[c,1]$, where c is a positive constant. We regard the family $(x_i^h)_{i\in I_h}$ as a vector random variable $\xi^{(h)}$ from Ω into $[c,1]^{I_h}$. For every $\lambda=(\lambda_i)_{i\in I_h}$ in $[c,1]^{I_h}$, we define the set (see fig.2)

$$B_h(\lambda) = \bigcup_{i\in I_h} \{\, x\in\mathbf{R}^n : x = \sigma + t\, n(\sigma) \,,\, \sigma\in\partial D \,,\, 0<t<\varepsilon_h\lambda_i\phi_h^i(\sigma) \,\}$$

where n is the outer unit normal to ∂D. Let us set $A_h(\lambda)=\bar{D}\cup B_h(\lambda)$.



We stress that the assumption on ∂D ensure that the mapping $(\sigma,t)\to\sigma+t\,n(\sigma)$ is invertible on $B_h(\lambda)$ if h is sufficiently large so that the boundary of the set $A_h(\lambda)$ is given by

$$\partial A_h(\lambda) = \bigcup_{i\in I_h} \{\, x\in \mathbf{R}^n : x=\sigma+\varepsilon_h\,\lambda_i\,\phi_i^h(\sigma)\,n(\sigma)\,,\ \sigma\in\partial D\,\}.$$

We note that for every $\lambda\in[c,1]^{I_h}$ the following inclusions hold

$$\text{(6.1)}\qquad D^{(h)} = \{\, x\in \mathbf{R}^n : d(x,D) < c\varepsilon_h \,\} \subseteq A_h(\lambda) \subseteq \{\, x\in \mathbf{R}^n : d(x,D) < \varepsilon_h \,\}$$

Furthermore, we associate with every $\lambda\in[c,1]^{I_h}$ the functional $F_h(\lambda):L^2(\mathbf{R}^n)\to\mathbf{R}$ defined by

$$F_h(\lambda)(u) = \begin{cases} \displaystyle\int_{\mathbf{R}^n} a^{(h)}(x,Du)\,dx + \int_{\mathbf{R}^n} \tilde{u}^{\,2}\, d\infty_{\partial A_h(\lambda)} & u\in H^1(\mathbf{R}^n) \\ +\infty & \text{otherwise} \end{cases}$$

where $a^{(h)}$ is the quadratic form defined in (1.2). Our aim is to show that the composit function $\omega\to F_h(\xi^{(h)}(\omega))$ from Ω into $\mathcal{F}_h$ is Σ–$B(\tau_h)$ measurable, i.e. is a random functional of the class $\mathcal{F}_h$. To get this we need the following lemma.

Lemma 6.1. *For every* $K\in\mathcal{K}$ *the function* $\lambda\to\mathrm{cap}^{(h)}(\partial A_h(\lambda)\cap K)$ *from* $[c,1]^{I_h}$ *into* $\mathbf{R}$, *is upper semicontinuous in* $[c,1]^{I_h}$.

Proof. The lemma is similar to Lemma 4.1 in [3]. For the reader convenience we adapt the proof in our particular case. Let $(\lambda_i)_{i\in\mathbf{N}}$ be a sequence in $[c,1]^{I_h}$ converging to λ in $[c,1]^{I_h}$. For every $j\in\mathbf{N}$ we define the set

$$E_h^j(\lambda) = \{\, x\in\mathbf{R}^n : \mathrm{dist}(x,\partial A_h(\lambda)) < 1/j \,\}.$$

By definition of $\partial A_h(\lambda)$ we have that for every $j\in\mathbf{N}$ there exists $i_0\in\mathbf{N}$ such that $E_h^j(\lambda) \supseteq \partial A_h(\lambda_i)$

for every $i \geq i_0$. Hence, for every $j \in \mathbf{N}$ and $K \in \mathcal{K}$ we obtain

$$\mathrm{cap}^{(h)}(E_h^j(\lambda) \cap K) \geq \limsup_{i \to +\infty} \mathrm{cap}^{(h)}(\partial A_h(\lambda_i) \cap K) .$$

Since

$$\bigcap_{j \in \mathbf{N}} (E_h^j(\lambda) \cap K) = \partial A_h(\lambda) \cap K$$

by the well-known properties of the capacity $\mathrm{cap}^{(h)}$ we get

$$\mathrm{cap}^{(h)}(\partial A_h(\lambda) \cap K) \geq \limsup_{i \to +\infty} \mathrm{cap}^{(h)}(\partial A_h(\lambda_i) \cap K) ,$$

which proves the lemma. ∎

Remark 6.2. Lemma 6.1 and Corollary 3.13 imply that the function $\omega \to F_h(\xi^{(h)}(\omega))$ is a random functional of the class $\mathcal{F}_h$ or equivalently, that the function $\omega \to A_h(\xi^{(h)}(\omega))$ is a random set.

Let us set $F_h(\omega) = F_h(\xi^{(h)}(\omega))$ for every $\omega \in \Omega$. In the following we want to show that the sequence (F_h) satisfies the assumptions of Theorem 5.8.

For every $x \in \mathbf{R}^n$, let us define the function

$$u_h(x) = \left(1 - \frac{1}{c\varepsilon_h} d(x,D)\right) \vee 0 .$$

By (6.1) it is easy to see that for every $U \in \mathcal{U}$, $\delta > 0$, and $\lambda \in [c,1]^{I_h}$, the function u_h has the following properties

$$u_h = 0 \quad \text{q.e. on } U \cap \partial A_h(\lambda),$$
$$u_h = 1 \quad \text{q.e. on } \partial D_\delta .$$

Thus, for every $\delta > 0$, $\lambda \in [c,1]^{I_h}$, and $U \in \mathcal{U}$, we have

(6.2) $$b_\delta(F_h(\lambda),U) \leq \varepsilon_h \int_{(D^{(h)}/D) \cap U} a(x,Du)\, dx \leq$$

$$\leq \Lambda_2 \varepsilon_h \int_{(D^{(h)}/D) \cap U} |Du_h|^2 dx = \Lambda_2 \varepsilon_h \int_{(D^{(h)}/D) \cap U} \frac{1}{\varepsilon_h^2 c^2} dx =$$

$$= \frac{\Lambda^2}{c^2} \frac{1}{\varepsilon_h} |(D^{(h)}/D) \cap U| .$$

By (6.2) it follows that for every $\delta > 0$ and $U \in \mathcal{U}$

$$E[b_\delta(F_h(\bullet),U] \le \frac{\Lambda_2}{c^2}\,\frac{1}{\varepsilon_h}\;|(D^{(h)}/D)\cap U|,$$

hence, for every $U\in\mathcal{U}$, and $\delta > 0$, we have

$$\limsup_{h\to+\infty}\; E[b_\delta(F_h(\bullet),U)] \le \frac{\Lambda_2}{c^2}\, H^{n-1}(\partial D\cap U) \tag{6.3}$$

which proves the assumption (5.4) of Theorem 5.8.

Now, let I be any subset of I_h. We denote by Π_I the projection of $[c,1]^{I_h}$ on $[c,1]^I$. For every $U\in\mathcal{U}$ we set $I(U)=\{i\in I_h : Q_i^h\cap U\neq\emptyset\}$. By Definition 1.9 it is easy to see that, for any $\delta > 0$ fixed, the function $\lambda\to b_\delta(F_h(\lambda),U)$ from $[c,1]^{I_h}$ into **R** is actually a function of the variable $\lambda'=\Pi_{I(U)}(\lambda)$. So if we consider two sets $U_1,U_2\in\mathcal{U}$ such that $\bar{U}_1\cap\bar{U}_2=\emptyset$, we find two disjoint sets I_1,I_2 of I_h such that, for any $\delta > 0$ and $\lambda\in[c,1]^{I_h}$,

$$b_\delta(F_h(\lambda),U_1) = \psi_1(\lambda') \qquad\text{and}\qquad b_\delta(F_h(\lambda),U_2) = \psi_2(\lambda'')$$

with $\lambda'=\Pi_{I_1}(\lambda)$ and $\lambda''=\Pi_{I_2}(\lambda)$.

As the random vectors

$$\xi_1^{(h)} = (x_i^h)_{i\in I_1} \qquad\text{and}\qquad \xi_2^{(h)} = (x_i^h)_{i\in I_2}$$

are indipendent, it follows that the random variables

$$\omega\to\psi_1(\xi_1^{(h)})(\omega) \qquad\text{and}\qquad \omega\to\psi_2(\xi_2^{(h)})(\omega)$$

are indipendent too. This proves the assumption (5.5) of Theorem 5.8.

Finally, we point out that, by (6.3), the measure ν of Theorem 5.9 turns out to be absolutely continuous with respect to the (n−1)-dimensional Hausdorff measure H^{n-1}. Therefore, by Radon-Nikodym theorem we obtain that there is a unique function $h\in L^1(H^{n-1})$ such that

$$\nu(B) = \int_B h\, dH^{n-1}$$

for every Borel set of ∂D.

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