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FOR ADVANCED STUDIES

TESI DI "PHILOSOPHIAE DOCTOR"

MORSE THEORY
AND
PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS

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TRIESTE

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Introduction.

In this tesis work, we have tried to describe, in an unified way, the results which are the outcome of a research undertaken under the supervision of Prof. A. Ambrosetti. Such a research had, as main objective, the study of dynamical systems obying to systems of differential equations of the form

$$\text{(H)} \qquad -Jz' = \nabla_z H(t,z),$$

where $z = (p,q) \in \mathbf{R}^{2N}$, $H \in C^1(\mathbf{R}\times\Omega,\mathbf{R})$, Ω open subset of $\mathbf{R}^{2N}$ and J, the symplectic matrix, is defined by $J(p,q) = (-q,p)$. Such a system of equations is called an Hamiltonian system. It describes the motion of a classical mechanical system, and it has been widely studied. See for example [10] for a detailed analysis.

The particular problem which we have studied is that of finding T-periodic solutions of (H) and of the so called classical Hamiltonian system

$$\text{(V)} \qquad -\ddot{y} = \nabla_y V(t,y),$$

where $y \in \mathbf{R}^N$, $V \in C^1(\mathbf{R}\times\Omega,\mathbf{R})$, Ω open subset of $\mathbf{R}^N$. We remark that (H) reduces to (V) for Hamiltonian functions $H(t,p,q) = (1/2)|p|^2 + V(t,q)$.

This problem has been tackled with the methods of variational nonlinear analyis, associating to (H) or (V) a functional f defined in a suitable Hilbert space, and finding the T-periodic solutions of the systems (H) and (V) as critical points of this functional. Such an approach has drawn a lot of attention in recent years and has generated a lot of interesting abstract results in nonlinear analysis (such as the so called "infinite dimensional linking-theorems" or the S^1 invariant indeces). We want in particular to recall here the pioneering paper of Rabinowitz [62], and the papers of [8, 41, 42]. For a review of the results, see [63].

More precisely, we have here found the critical points of the functional f by means of Morse theory, and we have tried to use such a theory to unify all the results we have obtained in our research.

The tesis is organized as follows: Chapter I is devoted to abstract Morse theory, with a particular emphasis on its connection with critical point theory. In particular in §2 we prove, using Morse theory, some well know theorems on existence of critical points, such as the Mountain Pass Theorem, as well as some theorems, such as Theorem 2.22, which are not known to us in this form. In §3 we give some results on Morse theory and perturbations, which are essentially contained in [7, 31, 58].

Chapter II is devoted to the application of the theory developed in Chapter I, to the search of

T-periodic solutions of (V) for potentials V bounded and such that $\nabla_y V(t,y) \to 0$ as $y \to \infty$. Such a problem, discussed in the papers [5, 33], is particularly interesting since the functional f associated to it does not satisfies the compactness condition of PS, so that the application of the usual existence theorem is problematic. Morse theory seems instead a very natural tool to deal with this kind of problems, see also [13]. For a different approach, see [14].

Chapter III is devoted to the study of singular potentials, i.e. potentials defined in an open subset Ω of $\mathbf{R}^N$ such that $V(y) \to \pm\infty$ as $y \to \partial\Omega$. Various cases are examined there; in particular §6,7 deals with potentials of the form $V(y) = \pm|y|^{-\alpha}$, $\alpha \geq 2$, and rely to arguments similar to those of Chapter II (i.e. Morse theory) while §8,9 deal with the case on Ω bounded and V convex (or concave). To study this last cases, we use the Dual Action Principle (see [27, 28]). The results of this chapter are contained in [4, 5, 6, 35, 36].

In the last chapter we have studied the perturbation problem, i.e., we suppose that (V) has one T-periodic solution and we look for T-periodic of

$$(H_\varepsilon) \qquad -Jz' = \nabla_z H(t,z) + \varepsilon h(t),$$

for ε small. In particular we have analyzed the problem of perturbing a T-periodic orbit of a time independent Hamiltonian by a small forcing term; such a problem is particularly interesting since, thanks to the natural invariance under the time-translations of time-independent Hamiltonians, one usually finds many T-solutions of the unperturbed problem. We find that, for every unperturbed solutions, one usually finds two solutions of the perturbed one. The results of this chapter are contained in [7, 31].

Chapter I: Morse theory and critical points.

In this chapter we want to recall the basic facts of Morse theory and show some of the ways in which it can be used to find critical points of functionals. We will employ here the "classical" Morse theory, as it has been developed by Morse in [56]. Standard references are [55] for the finite dimensional theory, [57, 59] for the infinite dimensional one, and [18, 19, 68] for the extension of the theory to deal with functions having nondegenerate critical manifolds and functions invariant under the action of some compact group G.

In recent years several authors have modified Morse theory wishing to extend its applications to a wider class of problems, in particular

(a) to deal with degenerate critical points [49, 58];

(b) to deal with functionals having less regularity [20];

(c) to deal with functionals having less "compactness" than it is usually required [13].

In the following, we will use some of these results. We will not make use, though, of some of the other interesting developements the theory has had, such as the Conley index theory [29, 30], the Morse-Conley index theory [16] or the equivariant Morse theory [19].

§1. Preliminaries.

Let E be an Hilbert space with scalar product $(\cdot,\cdot)$ and corresponding norm $\|\cdot\|$. Let $f \in C^1(E;\mathbf{R})$. We say that $u \in E$ is a critical point for f if $f'(u) \equiv \mathrm{grad} f(u) = 0$. We set

$$Z(f) \equiv \{u \in E : f'(u) = 0\},$$

and, for $c \in \mathbf{R}$,

$$Z_c(f) \equiv \{u \in Z(f) : f(u) = c\}.$$

If $Z_c(f) \neq \emptyset$ we say that c is a critical value for f. For $-\infty \leq a \leq b \leq +\infty$, we also set

$$\{a \leq f \leq b\} \equiv \{u \in E : a \leq f(u) \leq b\},$$
$$\{a < f \leq b\} \equiv \{u \in E : a < f(u) \leq b\},$$
$$f^b \equiv \{f \leq b\} \equiv \{u \in E : -\infty < f(u) \leq b\},$$

and so on.

1.1. Definition. We will say that $f \in C^1(E;\mathbf{R})$ satisfies the PS (Palais-Smale) condition in S, where S is a subset of E, if for every sequence $\{u_n\}$ in S such that $f(u_n)$ is bounded and $f'(u_n) \to 0$ there exists a subsequence converging to an element of S.

1.2. Definition. Let $f \in C^1(E;R)$. A vector field $V: E\setminus Z(f) \to E$, lipschitzian, such that

$$(V(x),f'(x)) \geq \|f'(x)\|^2$$
$$\|V(x)\| \leq 2\|f'(x)\|,$$

is said to be a pseudo gradient vector field for f.

1.3. Lemma. Let $f \in C^1(E;R)$. Then there exists a pseudo gradient vector field for f.

Proof. See [60, theorem 4.4].

1.4. Theorem. (deformation lemma) Let $-\infty < a \leq b \leq +\infty$. Suppose that $f \in C^1(E;R)$ satisfies PS in the set $\{a \leq f \leq b\}$ and that $\{a \leq f \leq b\} \cap Z(f) = \emptyset$. Then there exists a deformation $H \in C([0,1] \times f^b; f^b)$ such that

a) $H(0,x) = x \quad \forall\, x \in f^b$;

b) $H(t,x) = x \quad \forall\, (t,x) \in [0,1] \times f^a$;

c) $H(1,x) \in f^a \quad \forall\, x \in f^b$.

Proof. The proof is similar to that of theorem 5.9 in [60].
Let V be any pseudo gradient vector field for f. Since $\{a \leq f \leq b\} \cap Z(f) = \emptyset$, V is defined $\forall$ $x \in \{a \leq f \leq b\}$. Take any $x \in \{a \leq f \leq b\}$ and consider the solution $\alpha_x(t)$ of

$$(1.1) \qquad \begin{cases} \dfrac{d\alpha}{dt} = - V(\alpha) \\ \alpha_x(0) = x \end{cases}$$

Let $[0,ß[$ be the maximal interval of definition of α_x. Then there exists a unique $T=T(x) \in [0,ß[$ such that

$$f(\alpha_x(T(x))) = a.$$

In fact, by Theorem 5.4 of [60], either $ß = +\infty$ or $f(\alpha_x(t)) \to -\infty$ as $t \to ß$. So, if $ß < +\infty$, the existence of $T(x)$ follows from $f(\alpha_x(0)) = f(x) \geq a$ and the continuity of $f(\alpha_x(t))$ in t. So we can suppose $ß = +\infty$ and $f(\alpha_x(t)) > a$ $\forall$ t. Since

$$(1.2) \qquad \frac{d\,f(\alpha_x(t))}{dt} = -(V(\alpha_x(t)), f'(\alpha_x(t))) \leq -\|f'(\alpha_x(t))\|^2,$$

we deduce $f(\alpha_x(t)) \leq f(x) \leq b \ \forall\, t \geq 0$, hence $\alpha_x(t) \in \{a \leq f \leq f(x)\} \ \forall\, t \geq 0$. Since PS holds in $\{a \leq f \leq b\}$, we have that $\|f'(y)\| \geq \delta > 0 \ \forall\, y \in \{a \leq f \leq f(x)\}$, so we can deduce from (1.2) that $df(\alpha_x(t))/dt \leq -\delta^2$. It then follows

$$\begin{aligned} a &< f(\alpha_x(t)) \\ &= f(x) + \int_0^t (df(\alpha_x(s))/ds)\, ds \\ &\leq f(x) - \delta^2 t \end{aligned}$$

and we reach a contradiction taking $t = (f(x) - a)/\delta^2 \geq 0$.
So we have proved that $\forall\, x \in \{a \leq f \leq b\}$ there exists $T \in [0,\beta[$ such that $f(\alpha_x(T)) = a$. The uniqueness of such a T follows from the the fact that (1.2) implies that $f(\alpha_x(t))$ is strictly decreasing in t for $f(\alpha_x(t)) \in [a,b]$. We will denote such a T by $T(x)$. From the continuous dependence on the data of the solution of (1.1) and the strict monotonicity of $f(\alpha_x(t))$ in t, it follows that $T(x)$ is a continuous function of x. We set

$$H(t,x) = \begin{cases} x & \forall\, (t,x) \in [0,1] \times f^a \\ \alpha_x(T(x)) & \forall\, (t,x) \in [0,1] \times (f^b \setminus f^a). \end{cases}$$

Such an H clearly satisfies a), b) and c). It also follows, from the continuity of $T(x)$ in x and of $\alpha_x(t)$ in (t,x), that H is continuous in (t,x), hence the lemma is proved.

1.5. <u>Definition</u>. Given $f \in C^1(E;R)$, we say that a subset S of E is positively invariant under the steepest descent flow of f if for every pseudo gradient vector field V of f, and for every $x \in S \setminus Z(f)$

$$\alpha_x(t) \in S \quad \forall\, t \in [0,\beta[$$

where $\alpha_x(t)$ is the solution of (1.1).

1.6. <u>Remark</u>. It is easy to check that Theorem 1.4 can be generalized as follows:
Suppose $f \in C^1(E;R)$ satisfies PS in $S \cap \{a \leq f \leq b\}$, where S is a subset of E positively invariant under the steepest descent flow of f. Suppose, moreover, that $S \cap \{a \leq f \leq b\} \cap Z(f) = \emptyset$. Then there exists a deformation $H \in C([0,1] \times (f^b \cap S); (f^b \cap S))$ such that
a) $H(0,x) = x \quad \forall\, x \in f^b \cap S$;
b) $H(t,x) = x \quad \forall\, (t,x) \in [0,1] \times (f^a \cap S)$;

c) $H(1,x) \in f^a \cap S \ \ \forall x \in f^b \cap S$.

1.7. Let A, B be subset of an Hilbert space E, $A \supset B$. With

$$H_q(A,B;G)$$

we will denote the q-th homology group of the couple of topological spaces (A,B) relative to the coefficient group G. We set $H_q(A;G) \equiv H_q(A,\emptyset;G)$ and usually we will simply write $H_q(A,B)$, $H_q(A)$. With $\mathcal{H}_q(A,B;G)$ we will denote the q-th reduced homology group of the couple of topological spaces (A,B). See [40] for definitions and properties.

1.8. Definition. Given two subsets A,B of E, we say that A is a deformation retract of B (in symbols $A \approx B$) if $B \supset A$ and $\exists\, \sigma \in C([0,1] \times B; B)$ such that
i) $\sigma(0,b) = b \ \ \forall\, b \in B$;
ii) $\sigma(1,a) = a \ \ \forall\, a \in A$;
iii) $\sigma(1,b) \in A \ \ \forall\, b \in B$.

It is well known that

1.9. Proposition. $A \approx B$ implies $H_q(A) \cong H_q(B) \ \forall\, q$.

Proof. See [40].

1.10. Remark. It is clear that the hypothesis of theorem 1.4 imply $f^b \approx f^a$, hence $H_q(f^a) \cong H_q(f^b)$.

1.11. Proposition. Let $f \in C^2(E;R)$ satisfy the PS condition in the set $\{-\infty < a \leq f \leq b\}$, and suppose $\{a < f < b\} \cap Z(f) = \emptyset$ and that $Z_a(f)$ consists of finitely many critical points. Then

$$H_q(f^b \setminus Z_b(f)) \cong H_q(f^a) \qquad \forall\, q.$$

Proof. See [25,Thm 1.2].

1.12. Definition. Let $u \in Z(f)$, $f \in C^2(E;R)$. We can define the Hessian operator $H(f)_u : E \to E$ as $(H(f)_u v,w) = d^2f(u)[v,w]$. We will say that $u \in Z(f)$ has (Morse) index $= n$ if n is the dimension of the maximal subspace V of E on which $d^2f(u)$ is negative definite, and we will say that u has nullity m if m if the dimension of the largest subspace N of E on which $d^2f(u)$ is indefinite (we remark that $N = \ker H(f)_u$).

1.13. Definition. A submanifold V of E will be said to be a critical manifold for f if $\partial V = \emptyset$ and $Z(f) \supset V$. It is clear that $\forall\, u \in V$ T_uV (the tangent space to V in u) is contained in ker $H(f)_u$. It is then possible to define $H_1(f)_u$: $E/T_uV \to E/T_uV$ as $H_1(f)_u[v] = [H(f)_u v]$. If $H_1(f)_u$ is an isomorphism $\forall\, u \in V$, then V is said to be a nondegenerate critical manifold. In such a case every $u \in V$ has the same index; such an index is called index of the nondegenerate critical manifold V. We remark that, if V is nondegenerate, then $\ker H(f)_u = T_uV$ $\forall\, u \in V$.

1.14. Definition. Let $f \in C^2(E;R)$. f is said to be a Morse function if it satisfies PS and if $Z(f)$ is the union of nondegenerate critical points, and f is said to be a generalized Morse function if it satisfies PS and if $Z(f)$ is the union of nondegenerate critical manifold.

§2. Morse theory and existence of critical points.

One of the basic facts in critical point theory, which will be often used in the applications, is the following:

2.1. Theorem. Let $f \in C^1(E;R)$, and let Y be a subset of E positively invariant under the steepest descent flow of f. Suppose f satisfies the PS in $\{a \le f \le b\} \cap Y$, where $-\infty < a \le b \le +\infty$. If it exists $q \in \mathbf{Z}$ such that

$$H_q(f^b \cap Y) \ne H_q(f^a \cap Y), \tag{2.1}$$

then it exists at least one critical point u of f with $u \in \{a \le f \le b\} \cap Y$.

Proof. It is an immediate consequence of Remark 1.6 and Proposition 1.9. In fact, if $\{a \le f \le b\} \cap Y = \emptyset$, one deduces $H_q(f^b \cap Y) \cong H_q(f^a \cap Y)$ $\forall\, q \in \mathbf{Z}$.

We can deduce from theorem 2.1 two well known theorems on existence of critical points.

2.2. Theorem (existence of minimum). Suppose $f \in C^1(E;R)$ is bounded from below . Set $m = \inf f$, and suppose f satisfies PS in the set $\{f \le m+\varepsilon_0\}$ for some $\varepsilon_0 > 0$. Then it exists $u \in Z(f)$ such that $f(u) = m$.

Proof. $\forall\, \varepsilon \in\,]0, \varepsilon_0]$ we have that $f^{m+\varepsilon} \ne \emptyset$ and that PS holds in $f^{m+\varepsilon}$. Since $f^{m-\varepsilon} = \emptyset$, $H_0(f^{m-\varepsilon}) \ne H_0(f^{m+\varepsilon})$ $\forall\, \varepsilon \in\,]0, \varepsilon_0]$. Hence, applying Theorem 2.1 with $Y = E$ we deduce that $\forall\, \varepsilon \in\,]0, \varepsilon_0]$ it exists $u_\varepsilon \in Z(f) \cap \{m-\varepsilon \le f \le m+\varepsilon\}$. Since $Z(f) \cap \{m-\varepsilon \le f \le m+\varepsilon\}$ is compact (PS holds there), we can deduce that exists a subsequence u_i such that $u_i \to v$, $f(u_i) \to m$, $f'(u_i) = 0$; this implies $f(v) = m$, $f'(v) = 0$.

2.3. Theorem (Mountain Pass Theorem [9]). Suppose $f \in C^1(E;R)$, $f(0) = 0$ and, moreover, that exist $r,\delta > 0$ and $e \in E$ such that

(i) $f(u) \geq \delta > 0 \quad \forall\, u$ such that $\|u\| = r$;

(ii) $\|e\| > r$ and $f(e) \leq 0$.

Setting $\Gamma = \{\mu \in C([0,1];E) : \mu(0) = 0,\ \mu(1) = e\}$ and

$$c = \inf \{\max \{f(\mu(s)) :\ s \in [0,1]\} :\ \mu \in \Gamma\},$$

one has that $c \geq \delta > 0$ and, if PS holds in $\{c- \varepsilon_0 \leq f \leq c+ \varepsilon_0\}$ for some $\varepsilon_0 > 0$, then $Z_c(f) \neq \emptyset$.

Proof. $c \geq \delta > 0$ since, by continuity, every $\mu \in \Gamma$ has to cross $\{\|u\| = r\}$. Set $\varepsilon_1 = \min\{\varepsilon_0, c\}$. Take any $\varepsilon \in\,]0, \varepsilon_1[$ and $\mu_1 \in \Gamma$ such that $\max\{\mu_1(s) : s \in [0,1]\} \leq c+\varepsilon$. Let A be the connected component of $f^{c+\varepsilon}$ containing $\mu_1([0,1])$. It is well known that rank $H_0(A) = 1$, and that A is positively invariant under the steepest descent flow of f (in fact, for fixed t, $\alpha_x(t)$ sends $f^{c+\varepsilon}$ into itself; since it is continuous in (t,x) it also sends connected components into connected components). Consider now $B = f^{c-\varepsilon} \cap A$. First of all we remark that $0 \in B$, $e \in B$. In fact 0, $e \in A$, $f(e) \leq f(0) = 0 < c-\varepsilon$. Then we observe that B cannot be connected, otherwise it would exists $\mu_2 \in C([0,1],B)$ with $\mu_2(0) = 0$, $\mu_2(1) = e$, hence $\mu_2 \in \Gamma$, and $f(\mu_2(t)) \leq c-\varepsilon \quad \forall\, t \in [0,1]$. So B has at least two connected components and rank $H_0(B) \geq 2$. From this fact follows that $H_0(f^{c+\varepsilon} \cap A) \neq H_0(f^{c-\varepsilon} \cap A)$ $\forall\, \varepsilon \in\,]0, \varepsilon_1[$, hence, applying Theorem 2.1 we deduce that $\forall\, \varepsilon \in\,]0, \varepsilon_1[$ it exists $u_\varepsilon \in Z(f) \cap \{c-\varepsilon \leq f \leq c+\varepsilon\} \cap A$. Using PS, one can pass to a subsequence converging to $z \in Z_c(f)$.

2.4. Remarks. a) We have proved the Theorems 2.2 and 2.3 for functionals definited in an Hilbert space. Actually the proof can be carried over without any change also for functionals defined in a Banach space.

b) As we shall see later, the use of Morse theory permits to prove, for functionale of class C^2, additional properties of the critical point of "Mountain Pass type", in particular that its Morse index is 0 or 1 (1 if it is nondegenerate). On this argument see [1, 43, 50, 52, 25 §7].

c) Using the same tecniques of Theorems 2.2, 2.3 it would be possible to prove also many finite dimensional "linking" theorems.

d) In the applications we will, sometime, use directly Theorem 2.1. In such cases the problem will be to evaluate the homology of suitable sublevels f^c.

We have seen that whenever $Z(f) \cap \{a \leq f \leq b\} = \emptyset$ and PS holds, $H_q(f^b) \cong H_q(f^a)$.

Morse theory permits to study the change in homology in crossing a critical level c. More precisely

2.5. Theorem. Suppose $f \in C^2(E;\mathbf{R})$ satisfies PS in $\{a \le f \le b\}$. Moreover suppose that c is the only critical level in [a,b] and that all the critical points at level c are nondegenerate. If f has n_λ critical points of finite index λ in $Z_c(f)$, then

$$\operatorname{rank} H_q(f^b, f^a) = n_\lambda.$$

Proof. See [58, Theorem 3.1].

2.6. Remarks. a) Since every nondegenerate critical point is isolated and $Z_c(f)$ is compact (PS holds) we have that in the hypothesis of Theorem 2.5 $Z_c(f)$ consists of finitely many critical points, hence $\Sigma\, n_\lambda < +\infty$.

b) if $Z_c(f) = \{u\}$ and u is a nondegenerate critical point of infinite index, then

$$\operatorname{rank} H_q(f^b, f^a) = 0.$$

2.7. Using Theorem 2.5, Proposition 1.11 and the subadditivity of $R_q(a,b) \equiv \operatorname{rank} H_q(f^b, f^a)$ (i.e., the fact that $a < b < c$ implies $R_q(a,c) \le R_q(a,b) + R_q(b,c)$) one can deduce the Morse inequalities. Here we will present them following [19]. Suppose $f \in C^2(E;\mathbf{R})$ satisfies PS in $\{a \le f \le b\}$, with $-\infty \le a < b \le +\infty$ and that f has only nondegenerate critical points in $\{a \le f \le b\}$. We can then associate to such an f a (formal) power series

$$\mathcal{M}_t(f,a,b) = \Sigma_q\, N_q t^q \tag{2.2}$$

where $N_q \equiv \#\{u \in Z(f) \cap \{a \le f \le b\}: u \text{ has index } q\}$. We can also introduce another (formal) power series, the Poincarè polynomial of the couple (f^b, f^a), defined as

$$\mathcal{P}_t(f^b, f^a, G) = \Sigma_q\, t^q \operatorname{rank} H_q(f^b, f^a, G). \tag{2.3}$$

We will usually write just $\mathcal{P}_t(f^b, f^a)$.

We can now state

2.8. Theorem (Morse inequalities). Suppose $f \in C^2(E;\mathbf{R})$ satisfies PS in $\{a \le f \le b\}$, with $-\infty \le a < b \le +\infty$, that $Z_a(f) = Z_b(f) = \emptyset$ and that f has only nondegenrate critical points in $\{a \le f \le b\}$. Then it exists a (formal) power series $\mathcal{Q}_t(f) = q_0+q_1t+q_2t^2+\ldots$ with nonnegative coefficients $q_i \ge 0$ such that

(2.4) $\qquad \mathcal{M}_t(f,a,b) - \mathcal{P}_t(f^b, f^a, G) = (1+t)\, Q_t(f).$

Clearly this equality means that the coefficients of each power in left and right member of (2.4) must be equal.

2.9. Remark. If $-\infty < a < b < +\infty$ and the assumptions of theorem 2.8 hold, then follows from PS and the nondegeneracy of the critical points (which implies that the critical points are isolated) that $N_q < +\infty \ \forall q$, and that there exists a q_0 such that $N_q = 0 \ \forall q > q_0$. Using Morse inequalities one deduces

$$\text{rank } H_q(f^b, f^a) < +\infty \qquad \forall q,$$

$$\text{rank } H_q(f^b, f^a) = 0 \qquad \forall q > q_0.$$

We will see later (Remark 2.15) that this fact is true for every $f \in C^2(E;\mathbf{R})$ satisfying PS in $\{-\infty < a \leq f \leq b < +\infty\}$ and such that f' is Fredholm of index zero, even if we allow degenerate critical points.

2.10. Some consequences of Morse inequalities. First of all, equating the coefficients of the k-th powers in (2.4), one deduces

(2.5) $\qquad N_k - \text{rank } H_k(f^b, f^a) = q_k + q_{k+1}$

and, in particular

(2.6) $\qquad N_k \geq \text{rank } H_k(f^b, f^a).$

The peculiar form of the right member of (2.4) allows us to get other information. In particular one can see that the critical points on index λ interact with critical points of index $\lambda \pm 1$. To better illustrate this concept, suppose f is what is usually called a perfect Morse functional in $\{-\infty < a \leq f \leq b < +\infty\}$, i.e. a functional $f: E \to \mathbf{R}$ such that

$$N_k = \text{rank } H_k(f^b, f^a).$$

Let $N(f,a,b) = \Sigma_k N_k$. We claim that any other functional g satisfying the assumptions of Theorem 2.8 and such that $g^a = f^a$, $g^b = f^b$ will satisfy

(2.7) $N(g,a,b) = N(f,a,b) + 2m \quad$ for some $m \in \mathbf{N}$

and, moreover, if $N_k(g,a,b) > N_k(f,a,b)$, then also $N_{k+1}(g,a,b) > N_{k+1}(f,a,b)$ or $N_{k-1}(g,a,b) > N_{k-1}(f,a,b)$. In fact, from (2.5)

$$\begin{aligned} N(g,a,b) &= \Sigma_k \operatorname{rank} H_k(g^b, g^a) + \Sigma_k (q_k + q_{k+1}) \\ &= \Sigma_k \operatorname{rank} H_k(f^b, f^a) + 2\, \Sigma_k q_k \\ &= N(f,a,b) + 2m. \end{aligned}$$

(m is finite since N(g,a,b) is finite). If $N_k(g,a,b) > N_k(f,a,b)$ we have that

$$\begin{aligned} N_k(g,a,b) &= \operatorname{rank} H_k(f^b, f^a) + q_k + q_{k-1} \\ &= N_k(f,a,b) + q_k + q_{k-1}. \end{aligned}$$

hence $q_k + q_{k+1} = N_k(g,a,b) - N_k(f,a,b) \geq 1$ and either $q_k \geq 1$ or $q_{k-1} \geq 1$. If $q_{k-1} \geq 1$ we deduce from (2.5) $N_{k-1}(g,a,b) = N_{k-1}(f,a,b) + q_{k-1} + q_{k-2} > N_{k-1}(f,a,b)$ while if $q_k \geq 1$, always from (2.5) we get $N_{k+1}(g,a,b) = N_{k+1}(f,a,b) + q_{k+1} + q_k > N_{k+1}(f,a,b)$.

2.11. The Morse inequalities require f to have only nondegenrate critical points. In spite the fact that this is a generic property (at least in the finite dimensianal case - see [11, Chapter 2, §6, Proposition 8] and Theorem 2.14 below), this is a major drawback of the theory when one uses it to find critical points of an assigned functional. Moreover in some situations one knows a priori that all the critical points are degenerate - for example every time f is invariant under the action of some continuous Lie group G. To this purpose, let $f \in C^2(E;\mathbf{R})$ be a generalized Morse function in the sense of Definition 1.14. Let N be a nondegenerate critical manifold for f, and νN the normal disc bundle of N. The nondegeneracy of N implies that f restricted to each normal disc is nondegenerate; this gives a decomposition of νN into positive and negative part: $\nu N = \nu^+N \oplus \nu^-N$, where $\nu_p{}^+N$ and $\nu_p{}^-N$ are respectively spanned by the positive and negative eigendirections of the Hessian of f at p. The fiber dimension of ν^-N is the index λ_N of N. Setting θ^- = orientation bundle of ν^-N, one has

2.12. Theorem. Let $f \in C^2(E;\mathbf{R})$ be a Morse function in $\{a \leq f \leq b\}$. Then setting

(2.8) $\mathcal{M}_t(f,a,b) = \Sigma_{N \in Z(f) \cap \{a \leq f \leq b\}}\, t^{\lambda_N}\, \mathcal{P}_t(N;\theta^- \otimes G)$

where $\mathcal{P}_t(N, \theta^- \otimes G) = \Sigma_q t^q \operatorname{rank} H_q(N, \theta^- \otimes G)$, one has

(2.9) $\mathcal{M}_t(f,a,b) - \mathcal{P}_t(f^b, f^a, G) = (1+t)\, \mathcal{Q}_t(f)$, for some $Q_t(f) = \Sigma_k q_k t^k$, $q_k \geq 0$.

Proof. See [18,19].

We will now state some theorems which will allows us to deal with degenerate critical points of manifolds.

2.13. Theorem. Let E be an Hilbert space and $f \in C^2(E;\mathbf{R})$. Suppose that, given $-\infty \leq a \leq b \leq +\infty$, the following holds

1) $f'(x)$ is Fredholm of index zero in $Z(f) \cap \{a \leq f \leq b\}$;

2) $Z_a(f) = Z_b(f) = \emptyset$;

3) $Z(f) \cap \{a \leq f \leq b\}$ is compact.

Then $\forall\, \alpha, \delta > 0\ \exists\, g \in C^2(E;\mathbf{R})$ such that

i) $g(x) = f(x)$ if $d(x,Z(f)) \geq \delta$ or $x \notin \{a \leq f \leq b\}$;

ii) $|g(x) - f(x)| \leq \varepsilon$, $\|g'(x) - f'(x)\| \leq \varepsilon$, $\|g''(x) - f''(x)\| \leq \varepsilon$.

iii) g has only finitely many, non-degenerate critical points in $\{a \leq f \leq b\}$.

Moreover, if (PS) holds for f in $\{a \leq f \leq b\}$, g can be chosen satysfying the same property.

Proof. See [58, Lemma 2.4 and Theorem 2.2].

2.14. Remark. We can deduce from Theorem 2.13 that given any $f \in C^2(E;\mathbf{R})$ satisfying the hypothesis of Theorem 2.13 and PS, one can attach to it a series $\mathcal{M}^*_t(f,a,b) \equiv \mathcal{M}_t(g,a,b)$ for which the Morse inequalities must hold. Such a correspondence it is not unique, since g is not uniquely defined, but it is nontheless useful in critical point theory. Benci, in [16], has developed a theory which defines uniquely such a correspondence.

2.15. Remark. As already remarked in Remark 2.9, one has as an easy consequence of Theorem 2.13, that for any $f \in C^2(E;\mathbf{R})$ satisfying PS in $\{a \leq f \leq b\}$, with $-\infty < a \leq b < +\infty$, 1) of Theorem 2.13 and such that $Z_a(f) = Z_b(f) = \emptyset$

$$H_q(f^b, f^a) \cong H_q(g^b, g^a) \cong 0 \qquad \forall\, q \geq q_0$$

for a suitable $q_0 \geq 0$. Moreover, rank $H_q(f^b, f^a) < +\infty$ $\forall\, q$.

2.16. We want now to describe more in detail what happens of $H_k(f^c)$ in crossing a critical value c where one has only isolated critical points. Suppose $f \in C^2(E;\mathbf{R})$ satisfies PS in $\{a \leq f \leq b\}$. Let c be the only critical value in $[a,b]$, and suppose $Z_c(f) = \{z_1,\ldots,z_k\}$. Then, by

Proposition 1.11

$$H_q(f^b, f^a) \cong H_q(f^c, f^c\backslash\{z_1,\dots,z_k\}).$$

If U_j, $j=1,\dots,k$, are small neighborhoods of $z_1,\dots,z_k$ such that $U_j \cap U_i = \emptyset \quad \forall\ i \neq j$, then, by excision

$$H_q(f^c, f^c\backslash\{z_1,\dots,z_k\}) \cong \oplus_j H_q(f^c \cap U_j, U_j\backslash z_j).$$

So, to have a complete picture, we only have to to describe $H_q(f^c \cap U_j, U_j\backslash z_j) \cong C_q(f,p)$, the q-th critical group, which does not depend on the particular choice of the neighborhoods of z_j (see [25, §2]).

2.17. <u>Theorem</u>. If $f \in C^2(M;R)$, where M is an n-dimensional manifold, and p is an isolated cririal point of f, then, setting $c = f(p)$

(i) if p is a local minimum $C_0(f,p) = G$, $C_q(f,p) = 0 \ \forall\ q \neq 0$;

(ii) if p is a local maximum $C_n(f,p) = G$, $C_q(f,p) = 0 \ \forall\ q \neq n$;

(iii) if p is neither a local minimum nor a local maximum $C_0(f,p) = 0$, $C_q(f,p) = 0 \ \forall\ q \geq n$.

<u>Proof</u>. See [25, §1, Example 1 and 4].

2.18. <u>Theorem</u>. Let $f \in C^2(E;R)$ and suppose p is an isolated critical point for f. Let $c = f(p)$ and assume f satisfies PS in $\{c-\varepsilon \leq f \leq c+\varepsilon\}$ for some $\varepsilon \geq 0$. Moreover assume that 0 is either an isolated point of the spectrum of $d^2f(p)$ or does not belong to it and let $n = \dim \ker d^2f(p)$. If λ is the Morse index of p, we have that

$$C_q(f,p) = C_{q-\lambda}(h,p) \quad \forall\ q,$$

where h is a function defined on a n-dimensional manifold which has p has an isolated (completely degenerate) critical point.

<u>Proof</u>. See [25, §2] and [37].

2.19. <u>Remark</u>. From Theorem 2.17 and 2.18 it follows that $C_q(f,p) = 0 \quad \forall\ q < \lambda,\ \forall\ q > \lambda + n$ and that

a) $C_\lambda(f,p) \neq 0$ implies $C_q(f,p) = 0 \quad \forall\ q \neq \lambda$;

b) $C_{\lambda+n}(f,p) \neq 0$ implies $C_q(f,p) = 0 \quad \forall\ q \neq \lambda+n$.

We can reformulate the Morse inequalities in terms of the critical groups. Precisely

2.20. Theorem. Let $-\infty < a < b < +\infty$ and suppose $f \in C^2(E;\mathbb{R})$ satisfies PS in $\{a \leq f \leq b\}$, $Z_a(f) = Z_b(f) = \emptyset$. Assume f has only isolated critical values in $]a,b[$, each of them corresponding to a finite number of critical points and that f' is Fredholm of index 0. Then we have that

$$\mathcal{M}^{**}_t(f,a,b) - \mathcal{P}_t(f^b, f^a, G) = (1+t)\, \mathcal{Q}(t)$$

where $\mathcal{M}^{**}_t(f,a,b) = \Sigma_k \beta_k(a,b)\, t^k$, $\beta_k(a,b) = \Sigma_{p \in Z(f) \cap \{a \leq f \leq b\}} \operatorname{rank} C_k(f,p)$ and $\mathcal{Q}(t) = \Sigma\, q_k t^k$, $q_k \geq 0$.

Proof. See [25, Remark 1.4] and remark that each critical point has finite rank critical groups since Remark 2.15 holds.

As already observed in Remark 2.3 b), one can use Morse theory to prove additional properties of the critical points found in Theorem 2.2 and 2.3.

2.21. Proposition. (i) Suppose the assumptions of Thorem 2.2 are satisfied and that the point u of minimum (which exists by Theorem 2.2) is isolated. Then $C_0(f,u) = G$, $C_q(f,u) = 0$ $\forall\, q \neq 0$. Moreover, if $f \in C^2(E;\mathbb{R})$, the Morse index λ of u is 0.

(ii) Suppose the assumptions of Theorem 2.3 are satisfied. Suppose, moreover that c, which is a critical level for f by Theorem 3.2, is an isolated critical level for f. Then, if $f \in C^2(E;\mathbb{R})$, it exists a critical point at level c whose Morse index λ is 1 or 0.

Proof. (i) It is clear that $C_q(f,u) \cong H_q(f^c \cap U, f^c \cap U \setminus \{u\}) \cong H_q(\{u\}, \emptyset)$ and the first statement of (i) follows. It is then an immediate consequence of Remark 2.19 that $\lambda > 0$.
(ii) It is easy to show that (with the same notations of Theorem 3.2), $\operatorname{rank} H_0(A) = 1$ and $\operatorname{rank} H_0(B) \geq 2$ imply $\operatorname{rank} H_1(A,B) \geq 1$. Suppose f has only critical points of Morse index ≥ 2. Then also every g given by Theorem 2.13 sufficently close to f will have only nondegenerate critical points of index ≥ 2 in $\{c - \varepsilon \leq f \leq c + \varepsilon\} \cap A = \{c - \varepsilon \leq g \leq c + \varepsilon\} \cap A$. But then we reach a contradiction applying the Morse inequalities to g.

The following is an application of theorms 2.17 and 2.18 to critical point theory.

2.22. Theorem. Lert E be an Hilbert space and $f \in C^2(E;\mathbb{R})$. Suppose that, given $-\infty < a \leq b \leq +\infty$ one has that
(i) f satisfies PS in $\{a \leq f \leq b\}$ and $Z_a(f) = Z_b(f) = \emptyset$;

(ii) $\forall\, u \in Z(f) \cap \{a \le f \le b\}$ 0 is either an isolated critical point of the spectrum of $d^2f(p)$ or does not belong to it; moreover $\dim \ker d^2f(p) \le N$;
(iii) $\exists\, q,q' \in \mathbb{N}$, $|q-q'| \ge N$ such that $H_q(f^b, f^a) \ne 0$, $H_{q'}(f^b, f^a) \ne 0$.
Then f has at least two critical points in $\{a \le f \le b\}$.

Proof. The existence of at least one critical point u_1 follows from Theorem 2.1. Suppose

$$Z(f) \cap \{a \le f \le b\} = \{u_1\}.$$

Then, setting $c = f(u_1)$, it follows from 2.16 that

$$H_q(f^b, f^a) \cong C_q(f, u_1).$$

Let λ be the Morse index of u_1. Then from Theorem 2.18 follows

$$H_q(f^b, f^a) \cong C_{q-\lambda}(h, u_1) \quad \forall\, q$$

where h is defined on a n-dimensional manifold with $n \le N$. From Theorem 2.17 we deduce that only threee situations are possible
a) $C_{q-\lambda}(h,u_1) = 0 \quad \forall\, q \ne \lambda$;
b) $C_{q-\lambda}(h,u_1) = 0 \quad \forall\, q \ne \lambda + n$;
c) $C_{q-\lambda}(h,u_1) = 0 \quad \forall\, q \le \lambda \quad \forall\, q \ge \lambda + n$,
and each one of them is in contradiction with the assumption $H_q(f^b, f^a) \ne 0$, $H_{q'}(f^b, f^a) \ne 0$, $|q-q'| \ge N$.

On this argument, see also [Ber 1].

§3. Morse theory and perturbations.

In this section we will show how Morse theory can be employed in perturbation theory. Our discussion will mainly rely on the paper by Marino and Prodi [58] and on the papers [7,31].

The following Lemma, due to Marino and Prodi, is the fondamental tool.

3.1. Lemma. Let A, X, B, A', Y, B' be topological spaces such that

$$B' \supset Y \supset A' \supset B \supset X \supset A.$$

Suppose that $H_q(B,A) \cong 0$, $H_q(B',A') \cong 0$ $\forall$ q. Then the canonical homomorphism

$$h: H_q(A',A) \to H_q(Y,X)$$

is injective.

Proof. See [58, Lemma 4.1].

A direct consequence of such a Lemma is the following

3.2. Theorem. Let c be the only critical level for $f \in C^2(E;R)$ in $[c-\varepsilon,c+\varepsilon]$ for some $\varepsilon > 0$, and let $H_{q'}(f^{c+\varepsilon},f^{c-\varepsilon}) \neq 0$ for some q'. Suppose, moreover, that PS holds in $\{c - \varepsilon \leq f \leq c + \varepsilon\}$. Then, every $g \in C^2(E;R)$ satisfying the PS condition in $\{c - \varepsilon \leq g \leq c+ \varepsilon\}$ and such that $|f - g| \leq \varepsilon/3$ has at least one critical point in $\{c - \varepsilon \leq g \leq c+ \varepsilon\}$.

Proof. See [58, Theorem 4.1].

An application of Theorems 2.5 and 3.2 yields

3.3. Proposition. Let c be the only critical level for $f \in C^2(E;R)$ in $[c-\varepsilon,c+\varepsilon]$ for some $\varepsilon > 0$, and suppose f has only nondegenerate critical points at level c. Suppose, moreover, that PS holds in $\{c - \varepsilon \leq f \leq c + \varepsilon\}$. Then, every $g \in C^2(E;R)$ satisfying the PS condition in $\{c - \varepsilon \leq g \leq c+ \varepsilon\}$ and such that $|f - g| \leq \varepsilon/3$ has at least one critical point in $\{c - \varepsilon \leq g \leq c+ \varepsilon\}$.

Proof. From Theorem 2.5 follws that $H_{q'}(f^{c+\varepsilon},f^{c-\varepsilon}) \cong H_{q'}(f^c,f^c \setminus Z_c(f)) \neq 0$ for some q', and the Proposition follows from Theorem 3.2 (see also [7,§1]).

3.4. Remark. Theorem 3.2 and Propositon 3.3 only require f and g to be close in the C^0 norm and give an explicit extimate on the distance between f and g; these are the main advantages of this approach. On the other hand, one has no information on the closeness of the critical points of f and g (we remark that, if f and g are close in the C^2 norm, then a simple application of the local inversion Theorem gives existence of one critical point of g close to every nondegenerate critical point of f).

3.5. Remark. It often happens that our perturbed functional g is not close to f in the C^0 norm; this is the case when one adds a small forcing term to an Hamiltonian system (see §10,11). To this purpose we remark that the assumption $|f - g| \leq \varepsilon/3$ can be replaced by the more general

(3.1) $\qquad f^{c+\varepsilon} \supset g^{c+\varepsilon/2} \supset f^{c+\varepsilon/6} \supset f^{c-\varepsilon/6} \supset g^{c-\varepsilon/2} \supset f^{c-\varepsilon}$;

see [58,7].

Another way to deal with unbounded perturbations is to introduce the following "localization" assumption [31]. Let $f \in C^2(E;\mathbf{R})$. We will say that f satisfies the localization assumption if the following hold:

$$\exists\ c_1, c_2 \in \mathbf{R},\ R' > 0 \text{ such that } \forall\ a,b \in]c_1, c_2[\ \ H_{q'}(f^b,f^a) \text{ one has}$$
$$H_q(f^b,f^a) \cong H_q(f^b \cap B_{R'}, f^a \cap B_{R'})\ \forall\ q.$$

The following proposition estabilishes a sufficient condition under which the localization assumption holds

3.6. Proposition. Let $f \in C^2(E;\mathbf{R})$ be such that $\exists\ R > 0$, $\exists\ c_1, c_2$ such that
(i) $\forall\ a,b \in]c_1, c_2[\quad f^a \setminus \mathrm{cl}\, B_R$ is a deformation retract of $f^b \setminus \mathrm{cl}\, B_R$ (*);
(ii) $\forall\ a,b \in]c_1,c_2[\quad f^a \cap C(R;R+1)$ is a deformation retract of $f^b \cap C(R;R+1)$,
where $C(R;R+1) \equiv \{u \in E : R < \|u\| < R+1\}$.
Then the localization assumption holds for f with $R' = R + 1$, same c_1 and c_2 as in (i).

Proof. See [31, Proposition 2.1].

When the localization assumption holds, one can prove

3.7. Theorem. Let $f, g \in C^2(E;\mathbf{R})$ satisfy the PS and the localization assumption for some $R' \in]0,+\infty]$, c_1, c_2. Let $c \in]c_1,c_2[$ be the only critical level for f in $[c-\varepsilon, c+\varepsilon]$ $(\varepsilon > 0)$ and suppose f has only nondegenerate critical points at level c. Suppose, moreover, that $|f - g| \le \varepsilon/3\ \forall\ u \in B_{R'}$. Then g has at least one critical point in $\{c - \varepsilon \le g \le c + \varepsilon\}$.

Proof. See [31, Theorem 2.3].

3.8. In the applications we will be mainly interested in the case in which the umperturbed functional f has a nondegenerate critical manifold; in fact, as we have already remarked in 2.11, this is always the case for a functional invariant under the action of some continuous group, and the time-independent Hamiltonian systems have a natural S^1 invariance. In this case the perturbated functional will in general have more then one critical point. Quite a few authors have dealt with this

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(*) With $\mathrm{cl}\,\Omega$ we will denote the closure of the set Ω.

problem from an abstract point of view; we recall here [26, 38, 64, 66]. In all this papers the perturbed functional is close to the unperturbed one in the C^2 norm. More precisely, if we denote by cat(Z) the Lusternik-Schnirelman (LS) category of Z with respect to itself, namely the

least integer k such that Z is contained in $X_1 \cup \ldots \cup X_k$, each X_i closed and contractible to a point in Z, what one can prove is the following

3.9. Theorem. Let $f(\varepsilon,z)$ be smooth in $(\varepsilon,z) \in \mathbf{R} \times E$, E Hilbert space. Let Z be a compact, connected, nondegenerate critical manifold for f and suppose that $f''(\varepsilon,z)$ is a Fredholm operator of index zero $\forall\, z \in Z$. Then $\exists\, \varepsilon_0 > 0$ and a neighborhood U of Z such that $\forall\ \varepsilon \in]-\varepsilon_0,\varepsilon_0[$ $f(\varepsilon,\bullet)$ has at least cat(Z) critical ponts in U.

Proof. See [7, Theorem 2.1] The method of the proof consists of showing that the critical points of $f(\varepsilon,\bullet)$ can be found as critical points of the restriction $f(\varepsilon,\bullet)_{|Z(\varepsilon)}$, where $Z(\varepsilon)$ is a compact manifold diffeomorphic to Z.

3.10. Remark. As already stated, a C^2 regularity would be enough in proving the theorem, and the same proof works with minor modification also for a functional defined in a Banach space.

Even if it does not seem possible to obtain such a result by means of Morse theory, notheless such a theory is useful when one has less regularity. In particular one can prove

3.11. Theorem. Let $f, g \in C^1(E;R)$ satisfy PS and be bounded from below, with $c = \min_E f$, $\gamma = \min_E g$. Further, we suppose that

(3.2) $$\exists\, q^* > 0 \text{ such that } H_{q^*}(Z_c(f)) \neq 0$$

and that there exists $\varepsilon > 0$ such that

(3.3) $$Z_b(f) = \emptyset \quad \forall\, b \in]c,c+\varepsilon];$$

(3.4) $$f^{c+\varepsilon} \supset g^{c+\varepsilon/2} \supset f^c.$$

Then g has at least two critical points in $g^{c+\varepsilon/2}$.

Proof. ([7]) First of all we have that $f^c = Z_c(f)$ is a deformation retract of $f^{c+\varepsilon}$ because of (3.2) and PS holds (and remarking that c is the minimum of f, so that Theorem 1.4 still holds). Hence

(3.4) $$H_q(f^{c+\varepsilon}) \cong H_q(Z_c(f)).$$

Next, let $u^* \in Z_\gamma(g)$ $(\neq\emptyset)$. Remark that $\gamma \leq c + (1/2)\,\varepsilon$. Suppose, by contradiction, that $Z(g) \cap g^{c+\varepsilon/2} = \{u^*\}$. Then, by the same arguments recalled before, g^γ is a deformation retract of $g^{c+\varepsilon/2}$ and

(3.5) $$H_q(g^{c+\varepsilon/2}) \cong H_q(g^\gamma) \cong H_q(\{u^*\}).$$

From (3.3) we deduce, using Lemma 3.1 with $A = X = B = \emptyset$, $A' = f^c$, $Y = g^{c+\varepsilon/2}$, $B' = f^{c+\varepsilon}$, that $h\colon H_q(f^c) \to H_q(g^{c+\varepsilon/2})$ is an injection, so that $H_{q^*}(g^{c+\varepsilon/2}) \neq 0$, contradiction with 3.5.

Chapter II: Applications of Morse theory to the search of periodic solutions for second order Hamiltonian systems: bounded potentials.

In this chapter we will use Morse theory, as it has been developed in Chapter I, to deduce existence of T-periodic solutions of Hamiltonian systems of the form

$$\text{(V)} \qquad -\ddot{y} = \nabla_y V(t,y),$$

where $y \in \mathbf{R}^N$, $V: \mathbf{R} \times \Omega \to \mathbf{R}$, Ω is a subset of $\mathbf{R}^N$, $V(t+T,y) = V(t,y)$ $\forall\ (t,y) \in \mathbf{R} \times \Omega$. Such a system is a particular case of the more general Hamiltonian system

$$\text{(H)} \qquad -Jz' = \nabla_z H(t,z),$$

where $z = (p,q) \in \mathbf{R}^N \times \mathbf{R}^N$, J is the symplectic matrix defined by $J(p,q) = (-q,p)$, $H: \mathbf{R} \times \Omega \to \mathbf{R}$, Ω is a subset of $\mathbf{R}^{2N}$, $H(t+T,z) = H(t,z)$ $\forall\ (t,z) \in \mathbf{R} \times \Omega$. (H) reduces to (V) for Hamiltonian functions of the form $H(t,p,q) = (1/2)\ |p|^2 + V(t,q)$ (classical Hamiltonians).

In particular, we will deal in this chapter with bounded potentials. Most of these results are contained in [5, 33, 34]. The comparisons with related results will be done during the exposition.

§4. Potentials bounded from above.

In this section we will be concerned with potentials $V \in C^1(\mathbf{R} \times \mathbf{R}^N; \mathbf{R})$ such that

$$\text{(4.1)} \qquad \exists\, M \in \mathbf{R}\ \exists\, R > 0 \text{ such that } V(t,y) \leq M\ \ \forall\ (t,y) \in \mathbf{R} \times \mathbf{R}^N \text{ with } |y| \geq R.$$

The results of this section are well known; see for example [65]. We will simply introduce the notations and the methods, as well as some of the results which will be useful later.

4.1. Remarks. a) (4.1) implies that $V(t,y) \leq M'$ $\forall\ (t,y) \in \mathbf{R} \times \mathbf{R}^N$, with $M' \geq M$.
b) (4.1) is satisfied if $V(t,y) \to -\infty$ as $|y| \to +\infty$ uniformly in t.

Let $S^1 \equiv [0,T]/\{0,T\}$ and $E = H^1(S^1; \mathbf{R}^N)$ with scalar product

$$(x,y) = \int_0^T \langle x',y' \rangle dt + \int_0^T \langle x,y \rangle dt$$

(here $<\bullet,\bullet>$ denotes the scalar product in $\mathbf{R}^N$) and corresponding norm $\|y\|^2 = (y,y)$. It is known that $H^1(S^1;\mathbf{R}^N)$ is compactely embedded in $L^p(0,T;\mathbf{R}^N)$ $\forall\, p \geq 1$ and in $C(0,T;\mathbf{R}^N)$. We also have that $E = W \oplus \mathbf{R}^N$, where, $\forall\, y \in E$, $y = w_y + \xi_y$ (we will drop the the subscritp y when no confusion can arise), $\xi_y = (1/T)\int y$, $w_y = y - \xi_y$, $\int w_y = 0$ (from now on, unless explicitely stated, all the integrals will run from 0 to T). It is known that (denoting with $|\bullet|_p$ the $L^p(0,T;\mathbf{R}^N)$ norm)

(4.2) $$|w_y|^2 \leq (T^2/4\pi^2)|w_y'|_2^2 = (T^2/4\pi^2)|y'|_2^2,$$

so that

(4.3) $$|y|_2^2 = |w_y|_2^2 + |\xi_y|_2^2 \leq (T^2/4\pi^2)|y'|_2^2 + |\xi_y|_2^2 \leq \max(T^2/4\pi^2,1)\|y\|^2,$$

while

(4.4) $$|w_y|_\infty^2 \leq (T/6)\, |y'|_2^2.$$

We define $f\colon E \to \mathbf{R}^N$ as

(4.5) $$f(y) = (1/2)\int |y'|^2 - \int V(t,y).$$

It is well knonw that the critical points of f on E are T-periodic, classical (i.e. of class C^2) solutions of

(4.6) $$-\ddot{y} = \nabla_y V(t,y).$$

We start by proving

4.2. Lemma. f is bounded from below on E and satisfies PS in $\{f \leq -TM - \varepsilon\}$ $\forall\, \varepsilon > 0$.

Proof. From the Remark 4.1 a) it follows that

$$f(y) = (1/2)\int |y'|^2 - \int V(t,y) \geq -TM',$$

hence f is bounded from below. Let (y_n) be a sequence in E such that $f(y_n) \leq -TM - \varepsilon$, $f'(y_n) \to 0$. Then, from

$$(1/2)\int |y_n'|^2 - \int V(t,y_n) \geq (1/2)\int |y_n'|^2 - TM'$$

and $f(y_n) \leq -TM - \varepsilon$, we get

$$(1/2) \int |y_n'|^2 \leq T(M' - M) - \varepsilon,$$

hence

$$\int |y_n'|^2 \leq 2T(M' - M) - 2\varepsilon \equiv c_1 \tag{4.7}$$

and, from (4.4)

$$|w_n|_\infty^2 \leq (T/6)\, c_1. \tag{4.8}$$

Suppose $\|y_n\|$ is unbounded. Then, from (4.3) and (4.7) it follows that ξ_n is unbounded and

$$\begin{aligned} |y_n(t)| &= |w_n(t) + \xi_n| \\ &\geq |\xi_n| - |w_n(t)| \\ &\geq |\xi_n| - \{(T/6)\, c_1\}^{1/2}, \end{aligned}$$

so that $|y_n(t)| \to +\infty$ uniformly in t. Let n_0 be so large that $|y_n(t)| \geq R \;\; \forall\, t \in S^1 \;\; \forall\, n \geq n_0$. Then $V(t,y_n(t)) \leq M \;\; \forall\, t \in S^1$, hence

$$f(y_n) = (1/2) \int |y_n'|^2 - \int V(t,y_n) \geq -TM \quad \forall\, n \geq n_0,$$

in contradiction with $f \leq -TM - \varepsilon$. We deduce that $\|y_n\|$ must be bounded and so, up to a subsequence, y_n converges weakly in $H^1(S^1; \mathbf{R}^N)$ and strongly in $C(0,T; \mathbf{R}^N)$ to z. Then

$$\int \langle y_n', z' \rangle \to \int |z'|^2$$

$$\int \langle \nabla_y V(t,y), y_n - z \rangle \to 0.$$

Using $(f'(y_n), y_n - z) \to 0$ we then deduce

$$\begin{aligned} \int |y_n'|^2 - \int \langle y_n', z' \rangle &= \int \langle y_n', y_n' - z \rangle \\ &= (f'(y_n), y_n - z) + \int (\nabla_y V(t,y), y_n - z) \to 0, \end{aligned}$$

so that

$$\int |y_n'|^2 \to \int |z'|^2,$$

and $y_n \to z$ strongly in $H^1(S^1; \mathbf{R}^N)$. This proves the PS.

We are now in position to prove the following

4.3. Theorem. Let $V \in C^1(\mathbf{R} \times \mathbf{R}^N; \mathbf{R})$, $V(t+T,y) = V(t,y)$ $\forall (t,y) \in \mathbf{R} \times \mathbf{R}^N$. Suppose that (4.1) holds and that

(4.9) $\qquad \exists\, \xi \in \mathbf{R}^N$ such that $\int V(t,\xi) > TM.$

Then (V) has at least one T-periodic solution.

Proof. (see [65]). From assumption (4.9) it follows that

$$f(\xi) = -\int V(t,\xi) < -TM,$$

hence $m = \inf\{f(y) \colon y \in E\} < -TM$.
Since from Lemma 4.2 it follows that $m > -\infty$ and that PS holds in $\{f \leq -TM - \varepsilon_0\}$ for $\varepsilon_0 > 0$ sufficiently small, we can apply Theorem 2.2 to find a critical point of f, the minimum of f.

4.4. Remark. In the case in which V does not depend upon the time, a potential V satisfying the assumptions (4.1), (4.9) has always at least one critical point z, where $V(z) = \max V(\xi)$. Such a critical point z of V is a T-periodic solution of (V) $\forall\, T > 0$, hence it is always a critical point of f. So it could be that the critical point found via Theorem 4.3 is equal to z. This fact cannot, in general, be avoided. For example, if $V(\xi)$ is concave, it is easy to prove that $f(y)$ is convex and has only one critical point, the minimum of f.

§5. Bounded potentials.

We will deal, here, with potentials $V \in C^1(\mathbf{R} \times \mathbf{R}^N; \mathbf{R})$, $V(t+T,y) = V(t,y)$ $\forall (t,y) \in \mathbf{R} \times \mathbf{R}^N$ satisfying

(5.1) $\qquad 0 \leq V(t,y) \leq K \qquad \forall (t,y) \in \mathbf{R} \times \mathbf{R}^N.$

We will set

(5.2) $$m_* = \liminf_{|y|\to+\infty}(\inf_{t\in\mathbf{R}} V(t,y)), \quad m^* = \limsup_{|y|\to+\infty}(\sup_{t\in\mathbf{R}} V(t,y)).$$

From (5.1) follows that $0 \le m_* \le m^* \le K$. We remark that (4.1) is satisfied with $M = m^* + \varepsilon$ $\forall\, \varepsilon > 0$. We also set

(5.3) $$\alpha = \sup_{\xi\in\mathbf{R}^N} \int V(t,\xi).$$

If $\alpha > Tm^*$, then (4.9) is satisfied and we can apply Theorem 4.3 to find a T-periodic solution y of (V) such that $f(y) \le -\alpha$. We will now to investigate the situation $\alpha \le Tm^*$.

Quite a few results are known, in the literature, on bounded potentials. In particular, see [21, 53, 54, 69] for V even or periodic in y, and [24] for a case closer to our onesThe main problem, when one studies bounded potentials, is that the functional f defined in (4.5) does not satisfis the PS condition on the whole space E. In fact one can find diverging sequences of constants $\{\xi_n\}$ such that $f(\xi_n) \to 0$ and $f'(\xi_n) \to 0$. In particular the functional f, which is bounded from below, does not, in some situations, attains its infimum. The way which have been used to overcome such a problem are: a) to restrict the functional, whenever V is even in y, on the subspace of E of odd functions, where it can be shown to satisfy the PS condition, as in [21]; to show that, whenever V is periodic in x, the functional satisfies a weakened PS condition, as in [21, 54, 69]; to show that the PS condition holds in $\{f \ge c\}$ for some c and impose conditions on V such that f has a mini-max level greater than c, as in [24] and, applied to a different problem, in [14].

We start by studying the PS condition.

5.1. <u>Lemma</u>. Suppose $V \in C^1(\mathbf{R}\times\mathbf{R}^N; \mathbf{R})$, $V(t+T,y) = V(t,y)$ $\forall\,(t,y) \in \mathbf{R}\times\mathbf{R}^N$ and assume (5.1) holds for V. Then f is bounded from below. If

(5.4) $$\nabla_y V(t,y) \to 0 \text{ as } |y| \to +\infty, \text{ uniformly in } t,$$

then PS holds for f in $\{f \le -Tm^* - \varepsilon\} \cup \{f \ge -Tm_* + \varepsilon\}$ $\forall\, \varepsilon > 0$.

<u>Proof</u>. The facts that f is bounded and that PS holds for f in $\{f \le -Tm^* - \varepsilon\}$ follows from Lemma 4.2, since we have already remarked that (4.1) holds for V with $M = m^* + \varepsilon$ $\forall\, \varepsilon > 0$. So let us take (y_n) contained in E such that

$$c_2 \ge f(y_n) \ge -Tm_* + \varepsilon, \quad f'(y_n) \to 0.$$

From

$$(1/2)\int |y_n'|^2 - \int V(t,y_n) \le c_2$$

follows

$$(1/2)\int |y_n'|^2 \le c_2 + \int V(t,y_n) \le c_3,$$

hence, as in Lemma 4.2, we can deduce $|w_n(t)| \le c_4$. Suppose $\|y_n\| \to +\infty$. Since $\int |y_n'|^2 \le 2c_3$, we deduce that $|\xi_n| \to +\infty$, hence $|y_n(t)| \ge |\xi_n| - |w_n(t)| \to +\infty$ uniformly in t. Since

$$\int |y_n'|^2 = (f'(y_n),w_n) + \int (\nabla_y V(t,y_n(t)),w_n(t)).$$

From $f'(y_n) \to 0$, since $\|w_n\|$ is bounded and $|y_n(t)| \to +\infty$ uniformly, we deduce from (5.4) that

$$\int |y_n'|^2 \to 0.$$

Hence, using again the fact that $|y_n(t)| \to +\infty$ uniformly, we deduce that for n large enough

$$f(y_n) = (1/2)\int |y_n'|^2 - \int V(t,y_n)$$

$$\le \varepsilon' - Tm_* + \varepsilon',$$

with $\varepsilon' = \varepsilon'(n) \to 0$ as $n \to +\infty$, contradiction which proves the boundedness of $\|y_n\|$. Now one can proceed as in Lemma 4.2 to prove the existence of a converging subsequence.

We have already seen that, whenever $\{f \le -Tm^* - \varepsilon\} \ne \emptyset$, f attains its minimum and a solution exists. But this set could as well be empty, as it is the case whenever $m^* = K$. To find a solution in this case, we will make an assumption which will imply that $H_q(E,f^{-Tm_*+\varepsilon})$ is nontrivial for some $q \ge 0$.

5.2. Lemma. Suppose $V \in C^1(\mathbf{R} \times \mathbf{R}^N; \mathbf{R})$, $V(t+T,y) = V(t,y)$ $\forall$ $(t,y) \in \mathbf{R} \times \mathbf{R}^N$ and that V satisfies (5.1) and (5.4). Suppose, moreover

$$(5.5) \qquad \begin{cases} \exists\, \zeta \in \mathbf{R}^N \text{ and } \delta > 0 \text{ such that} \\ |\xi - \zeta| \le T\{(K-m_*)/3\}^{1/2} + \delta \quad \text{implies } V(t,\xi) < m_*. \end{cases}$$

Then $\exists\ \varepsilon_0 > 0$ such that $\forall\ \varepsilon \in [0,\varepsilon_0]$ $H_N(E, f^{-Tm_*+\varepsilon}) \neq 0$.

Proof. First of all we remark that $\forall\ \varepsilon > 0\ \exists\ R_1 > 0$ such that

$$A_1 \equiv \{\xi \in \mathbf{R}^N: |\xi| > R_1\}$$

is a subset of $f^{-Tm_*+\varepsilon}$. In fact $\forall\ \xi \in \mathbf{R}^N$

$$f(\xi) = -\int V(t, \xi)$$

and from the definition of m_* follows that exists R_1 such that $|\xi| \geq R_1$ implies $V(t, \xi) \geq m_* - \varepsilon/T$, so that

$$f(\xi) \leq -Tm_* + \varepsilon.$$

We will now show that $\exists\ \varepsilon_0, \rho > 0$ such that

$$f^{-Tm_*+\varepsilon_0} \text{ is a subset of } B_\rho \equiv \{y = w_y + \xi_y:\ |\xi_y - \zeta| \geq \rho\}$$

(ζ given by (5.5)). If not, $\forall\ \varepsilon_0 > 0$ it exists a sequence (y_n) in $f^{-Tm_*+\varepsilon_0}$ such that $(1/T)\int y_n = \xi_n$ with ξ_n converging to ζ. Then

$$(1/2)\int |y_n'|^2 - \int V(t,y_n) \leq -Tm_* + \varepsilon_0 \qquad (5.6)$$

implies

$$\int |y_n'|^2 \leq 2T(K - m_*) + 2\varepsilon_0$$

and, using (4.4)

$$|w_n(t)| \leq (2T/6)^{1/2}\{T(K - m_*) + \varepsilon_0\}^{1/2} \leq T\{(K - m_*)/3\}^{1/2} + (T\varepsilon_0/3)^{1/2}.$$

Then

$$\begin{aligned} |y_n(t) - \zeta| &= |w_n(t) + \xi_n - \zeta| \\ &\leq T\{(K - m_*)/3\}^{1/2} + (T\varepsilon_0/3)^{1/2} + |\xi_n - \zeta| \end{aligned}$$

taking ε_0 such that $(T\varepsilon_0/3)^{1/2} < \delta/2$ and n_0 large enough so that $|\xi_n - \zeta| < \delta/2 \ \forall\ n \geq n_0$, from (5.5) follows that

$$\sup_{t\in[0,T]} V(t,y_n(t)) \leq \mu < m_*$$

so that

$$f(y_n) = (1/2)\int |y_n'|^2 - \int V(t,y_n) \geq -T\mu > -Tm_*,$$

in contradiction with (5.6) for ε_0 eventually smaller.
Since $f^{-Tm_*+\varepsilon_0} \supseteq f^{-Tm_*+\varepsilon}\ \forall\ \varepsilon \in]0,\varepsilon_0[$, we have shown that $\forall\ \varepsilon \in]0,\varepsilon_0[\ \ \exists\ R_1, \rho > 0$ such that

$$B_\rho \supseteq f^{-Tm_*+\varepsilon} \supseteq A_1.$$

Since A_1 is a deformation retract of B_ρ, we deduce that A_1 is a retract of $f^{-Tm_*+\varepsilon}$, hence [40, formula (4.15) pg.37]

$$H_q(f^{-Tm_*+\varepsilon}) \cong H_q(f^{-Tm_*+\varepsilon}, A_1) \oplus H_q(A_1)$$

and, since $H_q(A_1) \cong H_q(S^{n-1})$, we deduce that

$$\mathcal{H}_N(f^{-Tm_*+\varepsilon}) \neq 0.$$

From the exact reduced homology sequence of the couple $(E, f^{-Tm_*+\varepsilon})$ (remark that $f^{-Tm_*+\varepsilon} \neq \emptyset$), we then deduce

$$\rightarrow \mathcal{H}_q(E) \rightarrow H_q(E,f^{-Tm_*+\varepsilon}) \rightarrow \mathcal{H}_{q-1}(f^{-Tm_*+\varepsilon}) \rightarrow \mathcal{H}_{q-1}(E) \rightarrow,$$

and using the well knonw fact that $\mathcal{H}_q(E) = 0\ \forall\ q$ we find that

$$0 \rightarrow H_q(E,f^{-Tm_*+\varepsilon}) \rightarrow \mathcal{H}_{q-1}(f^{-Tm_*+\varepsilon}) \rightarrow 0$$

from which we get

$$H_q(E,f^{-Tm_*+\varepsilon}) \cong \mathcal{H}_{q-1}(f^{-Tm_*+\varepsilon}),$$

so that

$$H_N(E, f^{-Tm_*+\varepsilon}) \neq 0.$$

We are now in position to prove

5.3. Theorem. Suppose $V \in C^1(\mathbf{R} \times \mathbf{R}^N; \mathbf{R})$, $V(t+T,y) = V(t,y)$ $\forall$ $(t,y) \in \mathbf{R} \times \mathbf{R}^N$ and that V satisfies (5.1), (5.4) and (5.5). Then the system of ordinary differential equations

$$\text{(V)} \qquad -\ddot{y} = \nabla_y V(t,y)$$

has at least one T-periodic solution. If, moreover $\exists$ $\xi \in \mathbf{R}^N$ such that $\int V(t, \xi) > m^*$, then (V) has at least two T-periodic solutions.

Proof. The Theorem follows directly from Theorem 2.1 and the remark preceeding Lemma 5.1, simply observing that the solution y_1 found via Theorem 2.1 is such that $f(y_1) > -Tm_*$ (actually $f(y_1) \geq -Tm_* + \varepsilon_0$, ε_0 given by Lemma 5.2).

Morse theory gives us some additional information on the solutions found in Theorem 5.3. This enables us to prove

5.4. Proposition. Suppose the assumptions of Theorem 5.3 are satisfied. Suppose, moreover, that $V \in C^2(\mathbf{R} \times \mathbf{R}^N; \mathbf{R})$ and

$$\text{(5.7)} \qquad V(t,0) = 0 < m_*, \ \nabla_y V(t,0) = 0, \ V''_{yy}(t,0) = \mu I \ \ \forall\, t \in [0,T], \ \text{where } \mu > (4\pi^2/T^2).$$

Then (V) has at least one non-trivial (i.e. not identically zero) T-periodic solution . If, moreover $\exists$ $\xi \in \mathbf{R}^N$ such that $\int V(t, \xi) > m^*$, then (V) has at least two non-trivial T-periodic solutions.

Proof. We know that $0 \in Z(f)$ and that $f(0) = 0 > -Tm_*$. Suppose $Z(f) \cap \{f \geq -Tm_* + \varepsilon_0\} \equiv \{0\}$. Then, by Proposition 1.11, we have that

$$H_q(E) \cong H_q(f^0) \quad \text{and} \quad H_q(f^0\backslash\{0\}) \cong H_q(f^{-Tm_*+\varepsilon}).$$

Since

$$0 \cong \mathcal{H}_q(f^0) \to H_q(f^0, f^0\backslash\{0\}) \to \mathcal{H}_{q-1}(f^0\backslash\{0\}) \to \mathcal{H}_q(f^0) \cong 0$$

we have that $\mathcal{H}_{q-1}(f^0/\{0\}) \cong H_q(f^0,f^0\backslash\{0\})$ (see proof of Lemma 5.2) and

$$H_q(f^0,f^0\backslash\{0\}) \cong \mathcal{H}_{q-1}(f^0/\{0\}) \cong \mathcal{H}_{q-1}(f^{-Tm_*+\varepsilon}).$$

The groups $H_q(f^0,f^0\backslash\{0\})$ are the critical groups $C_q(f,0)$ (see 2.16). From Lemma 5.2 and $H_q(f^{-Tm_*+\varepsilon}) \cong \mathcal{H}_q(f^{-Tm_*+\varepsilon})$ $\forall\, q \neq 0$ we deduce that

(5.8) $\qquad C_N(f,0) \neq 0.$

Let us now evaluate the Morse index of zero. We have that

$$d^2f(0)[v,v] = \int |v'|^2 - \mu\int |v|^2 \qquad \forall\ v \in E.$$

Setting, $\forall\ v \in E$, $v(t) = \Sigma_{k\in \mathbf{Z}}\, c_k e^{ikt}$, where $c_k \in \mathbf{C}^N$, c_{-k} = complex conjugate of c_k, we get

$$d^2f(0)[v,v] = T\, \Sigma_{k\in \mathbf{Z}}\ \{(4\pi^2/T^2)k^2 - \mu\}\ |c_k|^2$$

hence index(0) $\geq N + 2N = 3N$. Using Remark 2.19 we then have $C_q(f,p) = 0$ $\forall\ q < 3N$, in contradiction with (5.8). The second statement follows as in Theorem 5.5.

It is possible, under some additional assumptions, to evaluate exactly the homology groups $H_q(E,f^{-Tm_*+\varepsilon})$; this fact will allows us to prove existence of one more solution under suitable assumptions. We start by studying the sublevel $f^{-Tm_*+\varepsilon}$.

5.5. Lemma. Suppose $V\in C^1(\mathbf{R}\times\mathbf{R}^N;\mathbf{R})$, $V(t+T,y) = V(t,y)$ $\forall\ (t,y) \in \mathbf{R}\times\mathbf{R}^N$ and that V satisfies assumptions (5.1), (5.4). Suppose, moreover, that

(5.9) $\qquad \lim_{|y|\to+\infty} V(t,y) = m \quad \forall\, t \in [0,T],$

(5.10) $\qquad \exists\ R > 0$ such that $|y| > R$ implies that $V(t,\lambda y)$ is strictly increasing in $\lambda \geq 1$.

Then $\exists\ \varepsilon_0 > 0$ such that $\forall\ \varepsilon \in\,]0,\varepsilon_0]$

(5.11) $\qquad f^{-Tm_*+\varepsilon} = \Gamma_1^{\varepsilon} \cup \Gamma_2^{\varepsilon}$

with $\Gamma_2^{\varepsilon} \neq \emptyset$, $\mathrm{cl}\,(\Gamma_1^{\varepsilon}) \cap \mathrm{cl}\,(\Gamma_2^{\varepsilon}) = \emptyset$, Γ_i^{ε} positively invariant and closed. We also have that PS holds in Γ_1^{ε}. Moreover if for some $\varepsilon > 0$ $\Gamma_2^{\varepsilon} \cap Z(f) \cap \{f > m\} = \emptyset$, then

$$H_q(\Gamma_2{}^{\varepsilon}) \equiv H_q(S^{n-1}).$$

Proof. We show, first of all, that it exists $R_2 \geq R$ and $\varepsilon_0 > 0$ such that

$$f(y) \geq -Tm + \varepsilon_0 \quad \forall\, y \in E \text{ with } |\xi_y| = R_2. \tag{5.12}$$

Set $R_2 = R + T\,\{(k\text{-}m)/3\}^{1/2} + (T/3)^{1/2}$. Suppose $|\xi_y| = R_2$. Then, if

$$(1/2)\int |w'|^2 \geq T(K\text{-}m) + 1/2$$

we have that

$$\begin{aligned} f(y) &= (1/2)\int |w'|^2 - \int V(t,y) \\ &\geq T(K\text{-}m) + 1/2 - TK \\ &= -Tm + 1/2, \end{aligned}$$

while, if

$$(1/2)\int |w'|^2 < T(K - m) + 1/2$$

we have that (see proof of Lemma 5.2)

$$|w_y(t)| < T\{(K - m)/3\}^{1/2} + (T/3)^{1/2},$$

hence

$$\begin{aligned} |y(t)| &\geq |\xi_y| - |w_y(t)| \\ &> R + T\{(K - m)/3\}^{1/2} + (T/3)^{1/2} - T\{(K - m)/3\}^{1/2} - (T/3)^{1/2} \geq R, \end{aligned}$$

and since $V(t,\lambda y)$ is stricly decreasing in λ for $|y| \geq R$ and $\lambda \geq 1$, follows from (5.9), (5.10) that

$$m_1 \equiv \sup\{V(t,\xi)\colon\ t \in \mathbf{R},\, R \leq |\xi| \leq R_2 + T\{(K - m)/3\}^{1/2} + (T/3)^{1/2}\} < m$$

hence

$$f(y) = (1/2)\int |w'|^2 - \int V(t,y)$$

$$\geq -Tm_1 = -Tm + T(m - m_1),$$

and (5.12) holds with $\varepsilon_0 = \min(1/2, T(m - m_1))$.
Define now

$$\Gamma_1^{\varepsilon} = \{y = w + \xi \in f^{-Tm+\varepsilon} : |\xi| < R_2\}$$

$$\Gamma_2^{\varepsilon} = \{y = w + \xi \in f^{-Tm+\varepsilon} : |\xi| > R_2\}.$$

It is trivial to check that $f^{-Tm+\varepsilon} = \Gamma_1^{\varepsilon} \cup \Gamma_2^{\varepsilon}$, that $\Gamma_2^{\varepsilon} \neq \emptyset$ $\forall\, \varepsilon > 0$ (it follows from (5.9), that $\Gamma_1^{\varepsilon} \cap \Gamma_2^{\varepsilon} = \emptyset$ and that Γ_i^{ε} is closed for i = 1,2 $\forall\, \varepsilon < \varepsilon_0$. We also remark that

(5.13) $\qquad \forall\, \varepsilon \in \,]0,\varepsilon_0[,\ \forall\, y \in \Gamma_2^{\varepsilon},\ V(t,y(t)) < m \ \ \forall\, t \in \mathbb{R}.$

To prove that PS holds in $\Gamma_1^{\varepsilon_0}$ it is enough to remark that $\forall$ sequence $y_n \in \Gamma_1^{\varepsilon_0}$ one has that $|\xi_n| \leq R_2$ by definition. Then using the fact that $\Gamma_1^{\varepsilon_0}$ is closed, the same proof of Lemma 5.1 applies.

It only remains to show that $\Gamma_2^{\varepsilon} \cap Z(f) = \emptyset$ implies $H_q(\Gamma_2^{\varepsilon_0}) \equiv H_q(S^{n-1})$. To this purpose, we first remark that, $\forall\, n \geq 1$, $\Gamma_2^{\varepsilon_0/n}$ is a deformation retract of $\Gamma_2^{\varepsilon_0}$; in fact, since $\Gamma_2^{\varepsilon_0}$ is positively invariant and PS holds in $\{\varepsilon_0/n \leq f \leq \varepsilon_0\}$, we can apply Remark 1.6. Let η be the corresponding deformation.

We then define a second deformation

$$\Pi : [0,1] \times \Gamma_2^{\varepsilon_0/n} \to \Gamma_2^{\varepsilon_0}$$

as follows

$$\Pi(t,y) = \xi_y + (1 - t)w_y.$$

We claim that $\exists\, n_0$ such that $\forall\, n \geq n_0$

(5.14) $\qquad y \in \Gamma_2^{\varepsilon_0/n}$ implies $\Pi(t,y) \in \Gamma_2^{\varepsilon_0}$ $\forall\, t \in [0,1]$.

If not, $\forall\, n\ \exists\, y_n \in \Gamma_2^{\varepsilon_0/n}$, $t_n \in [0,1]$ such that

(5.15) $\qquad \Pi(t_n, y_n) \notin \Gamma_2^{\varepsilon_0}.$

Since $(1/T)\int\Pi(t,y(s))ds = \xi_y \ \forall\, y, \forall\, s \in [0,1]$ and since $\Gamma_2^{\varepsilon_0/n}$ is a subset of $\Gamma_2^{\varepsilon_0}$, we have that (5.15) implies

(5.16) $\qquad f(\Pi(t_n, y_n)) > -Tm + \varepsilon_0.$

From

$$f(y_n) = (1/2)\int |w_n'|^2 - \int V(t, y_n) \leq -Tm + \varepsilon_0/n,$$

taking into account (5.13), we get

(5.17) $\qquad (1/2)\int |w_n'|^2 \leq \varepsilon_0/n \quad$ and $\quad |w_n(t)|^2 \leq (\varepsilon_0 T)/(3n),$

so that $w_n \to 0$. Moreover, from

$$Tm \geq \int V(t,y_n) \geq Tm - \varepsilon_0/n + (1/2)\int |w_n'|^2,$$

it follows that

$$\int V(t,y_n) \to Tm,$$

which, together with (5.9), (5.13) implies $|\xi_n| \to +\infty$. Take now R_3 so large that $|\xi| > R_3$ implies $V(t,\xi) \geq Tm - \varepsilon_0/(2T)$. From our preeceding discussion follows that it exists n_0 such that $\forall\, n \geq n_0$ $|y_n(t)| > R_3$ $\forall\, t$, hence $V(t, y_n(t)) \geq m - \varepsilon_0/(2T)$ $\forall\, t, \forall\, n \geq n_0$. We finally get that $\forall\, n \geq \max(n_0,2)$

$$\begin{aligned} f(\Pi(t_n,y_n)) &= (1/2)\, t_n^2 \int |w_n'|^2 - \int V(t, y_n(t)) \\ &\leq (1/2)\int |w_n'|^2 - \int V(t, y_n(t)) \\ &\leq \varepsilon_0/n - Tm + \varepsilon_0/2 \\ &\leq -Tm + \varepsilon_0, \end{aligned}$$

in contradiction with (5.16).

We finally consider a third deformation, the projection P of $\mathbb{R}^N \backslash B_R$ on a sphere of radius $R_4 > R$, where R_4 is chosen in such a way that $f(|\xi| = R_4) \leq \varepsilon_0/n_0$. It is easy to check that the composition of the three deformations h, Π and P is a deformation $\sigma \in C([0,1] \times f^{-Tm+\varepsilon_0};$

$f^{-Tm+\varepsilon_0}$) such that

i) $\sigma(0,y) = y \quad \forall\, y \in f^{-Tm+\varepsilon_0}$;

ii) $\sigma(t,y) = y \quad \forall\, y \in \{\xi \in \mathbf{R}^N : |\xi| = R_4\}\ \forall\, t \in [0,1]$;

iii) $\sigma(1,y) \in \{\xi \in \mathbf{R}^N : |\xi| = R_4\} \quad \forall\, y \in f^{-Tm+\varepsilon_0}$.

This proves that $H_q(\Gamma_2^{\varepsilon_0}) \cong H_q(S^{n-1})$.

Using this Lemma we can prove

5.6. Theorem. Suppose $V \in C^2(\mathbf{R} \times \mathbf{R}^N; \mathbf{R})$, $V(t+T,y) = V(t,y)\ \forall\, (t,y) \in \mathbf{R} \times \mathbf{R}^N$, that V satisfies (5.1), (5.4), (5.9) and (5.10) and also assume

(5.18) $\quad V(t,0) = 0 < m,\quad \nabla_y V(t,0) = 0,\quad V''_{yy}(t,0) = \mu I \quad \forall\, t \in [0,T]$, where $\mu > (16\pi^2/T^2)$, $\mu \neq k^2\,(4\pi^2/T^2)$.

Then (V) has at least two non-trivial T-periodic solutions . If, moreover, $\exists\, \xi \in \mathbf{R}^N$ such that $\int V(t, \xi) > m$, then (V) has at least three non-trivial T-periodic solutions.

Proof. We first remark that assumptions (5.9), (5.10) imply that (5.5) holds. Then the existence of at least one nontrivial T-periodic solution y_1 for (V) follows from Proposition 5.4. We also remark that from the proof of such a Proposition follows that $f(y_1) > -Tm$. Suppose

(5.19) $\quad \{f > -Tm\} \cap Z(f) = \{0, y_1\}$.

Let $\varepsilon_1 = \min\{\varepsilon_0, 0, f(y_1)\}$ (ε_0 given by Lemma 5.5) and take $\varepsilon \in\,]0,\varepsilon_1[$. From Lemma 5.5 follows that $f^{-Tm+\varepsilon} = \Gamma_1^{\varepsilon} \cup \Gamma_2^{\varepsilon}$ and that these sets are positively invariant. Then

$$H_q(f^{-Tm+\varepsilon}) \cong H_q(\Gamma_1^{\varepsilon}) \oplus H_q(\Gamma_2^{\varepsilon})$$

$$\cong H_q(S^{n-1}) \oplus H_q(\Gamma_1^{\varepsilon}).$$

From this we deduce

$$H_q(E, f^{-Tm+\varepsilon}) \cong \begin{cases} \mathcal{H}_q(S^{n-1}) & \text{if } \Gamma_1^{\varepsilon} = \emptyset \\ H_q(S^{n-1}) \oplus \mathcal{H}_q(\Gamma_2^{\varepsilon_0}) & \text{if } \Gamma_2^{\varepsilon} \neq \emptyset. \end{cases}$$

Step 1. $\Gamma_2^{\varepsilon} = \emptyset$.

Suppose that $\Gamma_2^{\varepsilon} = \emptyset$. Then

$$H_q(E, f^{-Tm+\varepsilon}) \cong \mathcal{H}_q(S^{n-1}).$$

Reasoning as in Proposition 5.4 one can easily show that 0 is a nondegenerate critical point of index $i_0 = N + 2N[(T/2\pi)\,\mu^{1/2}]$, where $[\alpha]$ denotes the integer part of α. From our assumptions follows $i_0 \geq 5N$. Then $C_q(f,0) = G$ if $q = i_0$ and $C_q(f,p) = 0$ if $q \neq i_0$, while, if $\lambda = \mathrm{index}(y_1)$, we have that (see Remark 2.19)

$$C_q(f,y_1) = 0 \quad \forall\, q < \lambda,\ \forall\, q > \lambda + m,$$

where $m = \dim \ker d^2f(y_1)$. Since $u \in \ker d^2f(y_1)$ iff $-\ddot{u} = V''_{yy}(t,y_1(t))\,u$, $u \in E$, we have that $m \leq 2N$, hence

$$C_q(f,y_1) = 0 \quad \forall\, q < \lambda,\ \forall\, q > \lambda + 2N.$$

We can now apply the Morse inequalities (Proposition 2.20), to deduce

$$t^{i_0} + \Sigma_{i=0,2N}\, C_{\lambda+i}(f,y_1)t^{\lambda+i} = t^N + (1+t)\,\mathcal{Q}(t), \tag{5.20}$$

which can never be satisfied under our assumptions. In fact, we first deduce from $i_0 \geq 5N$ that $\lambda \leq N$, and then

$$\begin{aligned}\Sigma_{i=0,2N}\, C_{\lambda+i}(f,y_1)t^{\lambda+i} &= t^N + \Sigma_{i=0,2N}\,(q_i + q_{i+1})\, t^i \\ &= t^N + \Sigma_{i=0,2N+j}\,(q_i + q_{i-1})\, t^i \quad \forall\, 0 \leq j < i_0.\end{aligned}$$

This implies

$$\Sigma_{i=2N+1,2n+j}\,(q_i + q_{i+1})\, t^i = 0 \quad \forall\, 1 \leq j < i_0$$

and since $q_i \geq 0$ we deduce $q_i = 0$ $\forall\, 2N \leq i < i_0 - 1$. Equating the coefficients of t^{i_0} and t^{i_0+1} in (5.20), we then deduce

$$\begin{aligned}1 &= q_{i_0} + q_{i_0-1}\\ 0 &= q_{i_0+1} + q_{i_0}\end{aligned}$$

Since $q_{i_0-1} = 0$, $q_{i_0}=1$, we reach the desired contradiction.

Step 2. $\Gamma_1^{\varepsilon} \neq \emptyset$.

Suppose now $\Gamma_1^{\varepsilon} \neq \emptyset$ - remark that this is the case whenever $\exists\, \xi \in \mathbf{R}^N$ such that $\int V(t, \xi) > m$. In this situation one can find a solution y_i minimizing f in Γ_2^{ε}; in fact f is bounded from below in Γ_2^{ε} and PS holds there and this set is positively invariant. If y_1 is not the only critical point of f in Γ_1^{ε}, we are done (remark that $0 \notin \Gamma_1^{\varepsilon}$). If y_2 is the only critical point of f in Γ_1^{ε}, it is immediate to deduce from Proposition 1.11 and Theorem 2.17 that

$$H_q(\Gamma_1^{\varepsilon}) = C_q(f, y_1) = 0 \text{ if } q \neq 0, \quad G \text{ if } q = 0,$$

hence $\mathcal{H}_q(\Gamma_2^{\varepsilon}) = \{0\}$ and the proof of Step 1 applies to prove the existence of a third critical point.

5.7. Proposition. Suppose $V \in C^2(\mathbf{R} \times \mathbf{R}^N; \mathbf{R})$, $V(t+T,y) = V(t,y)\ \forall\, (t,y) \in \mathbf{R} \times \mathbf{R}^N$ and that V satisfies (5.1), (5.4), (5.19) and

(5.21) $\qquad \lim_{|y| \to +\infty} V(t,y) = K \quad \forall\, t \in [0,T],$ where $K = \sup_{\mathbf{R} \times \mathbf{R}^N} V(t,y),$

(5.22) $\qquad \exists\, R > 0$ such that $|y| > R$ implies $V(t,\lambda y)$ is nondecreasing in $\lambda \geq 1$.

If (5.7') holds with $m = K$, then (5.7) has at least two non-trivial T-periodic solutions.

Proof. It is easy to see that Lemma 5.5 continues to hold when assumptions (5.9), (5.10) are replaced by (5.21), (5.22); then the Proposition follows exactly as in Theorm 5.6.

Using Proposition 5.7 we can prove

5.8. Proposition. Let $V \in C^2(\mathbf{R} \times \mathbf{R}^N; \mathbf{R})$, $V(t+T,y) = V(t,y)\ \forall\, (t,y) \in \mathbf{R} \times \mathbf{R}^N$. Suppose $\exists$ Ω, subset of $\mathbf{R}^N$, Ω open, convex and diffeomorphic to a ball such that

$$0 \leq V(t,y) \leq m \quad \forall\, y \in \Omega;$$
$$V(t,y) = m \text{ and } \nabla_y V(t,y) = 0 \quad \forall\, t \in [0,T],\ \forall\, y \in \partial\Omega;$$
$$V(t,0) = 0 < m,\ \nabla_y V(t,0) = 0,\ V''_{yy}(t,0) = \alpha I \quad \forall\, t \in [0,T].$$

Then, if $\alpha > (4\pi^2)/T^2$, (V) has at least one T-periodic solution $y(t) \in \Omega\ \forall\, t$, while if $\alpha > (16\pi^2)/T^2$, $\alpha \neq k^2(4\pi^2)/T^2\ \forall\, k \in \mathbf{N}$, then (5.16) has at least two non-trivial T-periodic solutions $y_i(t) \in \Omega\ \forall\, t$.

Proof. Define $U(t,y)$ to be equal to $V(t,y)$ if $y \in \Omega$, and to m is $y \notin \Omega$. Such a U is of class $C^{1,1}$. Suppose $y(t)$ is a T-periodic solution of $-\ddot{y} = \nabla_y U(t,y)$. Suppose it exists $t' \in \mathbf{R}^N$ such that $y(t')$ does not belong to Ω. Then either $y(t) \notin \Omega\ \forall t$ or $\exists\, t_1$ such that $y(t_1) \in \partial\Omega$ and $y(t)$ is not constant. In the first case $-\ddot{y}(t) = 0\ \forall t$, hence (since y is T-periodic), $y(t) = \xi \notin \Omega$ and $f(\xi) = -Tm$. In the second case $u(t) \equiv x(t+t_1)$ solves the Cauchy problem

$$\begin{aligned} &-\ddot{u} = \nabla_y U(t,u) \\ &u(0) = y(t_1) \in \partial\Omega \\ &u'(0) = y'(t_1). \end{aligned}$$

If $y'(t_1) = 0$, the solution of such a problem is $y(t+t_1) = u(t) = y(t_1)\ \forall t$, so that $y(t) \in \partial\Omega\ \forall t$ and $f(y) = -Tm$. If $y'(t_1) \neq 0$ follows from the convexity of Ω that $y(t)$ must describe a straight line in the future or in the past, in contradiction with the periodicity of y. From the above discussion follows that a T-periodic solution y of the problem $-\ddot{y} = \nabla_y U(t,y)$ is either a constant ξ such that $f(\xi) = -Tm$ or is contained in Ω and it is a solution of $-\ddot{y} = \nabla_y V(t,y)$. From this follows that any critical point y of

$$g(y) = (1/2) \int |y'|^2 - \int U(t,y)$$

such that $g(y) > -Tm$ is a T-periodic solution of our original problem; moreover g is of class C^2 in a neighborhood of $Z(g) \cap \{g \geq -Tm + \varepsilon\}\ \forall \varepsilon > 0$. This allows us to use Theorem 5.3 and Proposition 5.7 to prove the Theorem. We only remark that in the case $\Gamma_2^{\varepsilon} \neq \emptyset$ one could have that $-Tm = \inf\{g(y)\colon y \in \Gamma_1^{\varepsilon}\}$, so that one of the critical points would be at level $-Tm$. But $g(y) = -Tm$ imply $(1/2)\int |y'|^2 = 0$, so that $y(t) = \xi$, while $y \in \Gamma_1^{\varepsilon}$ implies $\xi \in \Omega$, so that this is still a solution of our original problem.

5.9. Remark. The above Proposition covers the case of a potential $V(\xi) = p(|\xi|)$, where p is a τ-periodic function from $\mathbf{R}$ to $\mathbf{R}$.

Chapter III: Applications of Morse theory to the search of periodic solutions for second order Hamiltonian systems: singular potentials.

In this chapter we will study existence of T-periodic solutions for the equation

$$(V) \qquad -\ddot{y} = \nabla_y V(t,y)$$

for potentials $V \in C^2(\mathbf{R}\times\Omega,\mathbf{R})$, $V(t+T,y) = V(t,y)\ \forall\ (t,y) \in \mathbf{R}\times\Omega$, where Ω is and open subset of $\mathbf{R}^N$, $N \geq 2$, $\Omega = \mathbf{R}^N/C$ with C compact and $V \to \pm\infty$ as $y \to \partial\Omega$. The existence of one or more solutions for such an equation will be proved finding critical points of the functional f introduced in (4.5), the main deifference being now that f it is not defined on the whole Hilbert space E. In fact we will take f to be defined in the set $\Lambda \equiv \{u \in E\colon\ u(t) \in \Omega\ \forall\ t\}$, which is an open subset of E. In such a situation, one has to study the behaviour of f on $\partial\Omega$. We will show in Lemma 6.1, following Gordon [45], that, assuming the so called "strong force condition" on V (see [45]), one has that "$f(u) = \pm\infty$ on $\partial\Omega$". This fact will allow us to use the usual variational tecniques in every set $\{-\infty < b \leq f \leq a < +\infty\}$. The results discussed here are contained in the papers [3, 4, 5, 35, 36]. For other results on singular potentials, see [15, 22, 23, 45, 46, 47, 48, 51] and [6] for a review of such papers.

More precisely, in §6 we will prove existence of infinitely many T-periodic solutions in the case $V \to -\infty$ as $y \to \partial\Omega$, in §7 we will prove existence of at least one T-periodic solution in the case $V \to +\infty$ as $y \to \partial\Omega$. In both these sections we will assume the strong force condition.

The restrictions the strong force assumption pose on the potential V imply that, if $V(y) = \pm|y|^{-\alpha}$, then $\alpha \geq 2$, so that the Newton potential is not cover by our discussion. Little is known on the case in which the strong force assumption is violated (on this argument, see [46, 39]). In §8 we will discuss such a problem in the case in which Ω is convex and bounded and V is concave; this problem will be tackled using the Dual variational principle of Clarke and Ekeland [27, 28], while in §9 we will deal with the problem Ω convex and bounded and V convex, proving existence of a solution of minimal period T (on this problem see also [15] . On the problem of existence of solutions of minimal period see [2, 8, 28, 43, 44]).

§6. The case $V \to +\infty$ as $y \to \partial\Omega$.

Let $\Omega = \mathbf{R}^N \setminus C$, C compact, $N \geq 2$ and assume

$$(6.1) \qquad V \in C^2(\mathbf{R}\times\Omega,\mathbf{R}),\ V(t+T,y) = V(t,y)\ \forall\ (t,y) \in \mathbf{R}\times\Omega.$$

Define $\Lambda \equiv \{u \in E\colon\ u(t) \in \Omega\ \forall\ t\}$, and let $f\colon \Lambda \to \mathbf{R}$ be defined by (4.5). From (6.1) follows that $f \in C^2(\Lambda,\mathbf{R})$.

On the behaviour of V near C we will assume

(SF) there exist $\varepsilon' > 0$ and $U \in C^1(\Omega,\mathbf{R})$ such that

(6.2) $\quad U(y) \to -\infty$ as $y \to y^* \in C,\ y \in \Omega$;

(6.3) $\quad V(t,y) \le -|\nabla_y U(y)|^2 \quad \forall\, t \in \mathbf{R},\ \forall\, y \in C_{\varepsilon'} \equiv \{y \in \Omega:\ \mathrm{dist}\,(y,C) < \varepsilon'\}$.

Conditon (SF) (=Strong force) has been first introduced by Gordon [45]. The motivation of such a contition is the following

6.1. Lemma. If (SF) holds, then $\forall\,(u_n)$ in Λ, u_n converging weakly in E to $z \in \partial\Lambda$ one has that $f(u_n) \to +\infty$.

Proof. See [45, 47]. We give the proof for completeness. From u_n converging weakly in E, to z, we deduce $\|u_n\| \le$ const and $u_n \to z$ strongly in $C(0,T;\mathbf{R})$. Suppose $f(u_n)$ bounded. Let t^* be such that $z(t^*) \in \partial\Omega$. If $z(t) \in \Omega\ \forall\, t$, then, since u_n converges to z uniformly, we have that $V(t,u_n(t)) \to -\infty$ uniformly in t (follows from (SF)), hence, using the boundedness of $\|u_n\|$, we have $f(u_n) \to -\infty$. Then $\exists\, \delta > 0$ such that $z(t^*+\delta) \notin \partial\Omega$ and $z(t) \in C_{\varepsilon'}\ \forall\, t \in [t^*,t^*+\delta]$. Using (SF) we deduce

$$U(u_n(t^*+\delta)) - U(u_n(t^*)) = \int_{t^*}^{t^*+\delta} \frac{d}{ds}\, U(u_n(s))\, ds$$

$$= \int_{t^*}^{t^*+\delta} \langle \nabla_y U(u_n(s)),\ u_n'(s)\rangle\, ds$$

$$\le \Big\{\int_{t^*}^{t^*+\delta} |\nabla_y U(u_n(s))|^2 ds\Big\}^{1/2} \Big\{\int_{t^*}^{t^*+\delta} |u_n'(s)|^2 ds\Big\}^{1/2}$$

$$\le \Big\{\int_{t^*}^{t^*+\delta} -V(s,u_n(s))\, ds\Big\}^{1/2} \|u_n\|^2$$

Since $u_n(t^*+\delta) \to z(t^*+\delta) \notin \partial\Omega$, $U(u_n(t^*+\delta)$ is bounded, while $U(u_n(t^*)) \to -\infty$. Being $\|u_n\|$ bounded, we deduce that

$$\int_{t^*}^{t^*+\delta} V(s,u_n(s))\, ds \to -\infty \quad \text{as} \quad n \to +\infty.$$

Since V is bounded from above in a neighborhood of z(t) (V can only go to $-\infty$), we have that

$$\int_0^T V(s,u_n(s))\,ds = \int_{t^*}^{t^*+\delta} V(s,u_n(s))\,ds + \int_{\mathcal{B}} V(s,u_n(s))\,ds \qquad (\mathcal{B} \equiv [0,T]\setminus[t^*,t^*+\delta])$$

$$\leq \int_{t^*}^{t^*+\delta} V(s,u_n(s))\,ds + \text{const} \to -\infty \quad \text{as} \quad n \to +\infty.$$

and again we find that $f(u_n) \to +\infty$.

6.2. Remark. The above Lemma permits to use the usual variational methods in working with the functional f, regardless of the fact that it is defined in an open subset of E (for a different approach, see [15]).

In order to prove existence of T-periodic solutions for V in the present situation, some assumptions on the behaviour of V at infinity are in order. We will make here assumptions which are similar to the ones made for V bounded in §5, even if also different situations could be dealt with in an analogous way (see [47, 48] for some other kinds of behaviour).More precisely, we assume

(6.4) $\qquad V(t,y) \to 0$ as $|y| \to +\infty$ uniformly in t, and $\exists\, R_1 > 0$: $V(t,y) < 0 \;\forall\, |y| > R_1$;

(6.5) $\qquad \nabla_y V(t,y) \to 0$ as $|y| \to +\infty$ uniformly in t.

Using these assumptions, we can prove

6.3. Lemma. f is bounded from below. Moreover: (i) $\forall\, \varepsilon > 0$, PS holds in the set $\{f \geq \varepsilon\}$; (ii) (PS) holds in every closed set where $|\xi_u| \leq$ const.

Proof. [5]. The proof is essentially the same of that of Lemma 5.1; in fact the proof there, one only uses the boundedness from above of V and the boundedness at infinity of V and (6.5), so that from that Lemma follows that every sequence u_n such that $f(u_n)$ is bounded and $f'(u_n) \to 0$ converges weakly (up to a subsequence) to $z \in E$. From Lemma 6.1 we then deduce that $z \notin \partial\Lambda$, and the (i) follows. (ii) is immediate; it is enough to observe that the boundeness of $\|w_u\|$ is a direct consequence of the fact that V is bounded from above, and then proceed as in (i).

As in §5, we study the sublevels f^ε. We find, similarly to Lemma 5.5,

6.4. Lemma. It exists $R^* > 0$, $\varepsilon^* > 0$ such that $\forall\, \varepsilon \in\,]0,\varepsilon^*]$

$$f^\varepsilon = \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon$$

with

$$\Gamma_1^\varepsilon \equiv \{u \in f^\varepsilon : |\xi_u| < R^*\},$$

$$\Gamma_2^\varepsilon \equiv \{u \in f^\varepsilon : |\xi_u| > R^*\},$$

where $\mathrm{cl}\,\Gamma_1^\varepsilon \cap \mathrm{cl}\,\Gamma_2^\varepsilon = \emptyset$ and PS holds in Γ_1^ε. The sets $\Gamma_1^\varepsilon, \Gamma_2^\varepsilon$ are positively invariant under the steepeste descent flow of f. Moreover, if $Z(f) \cap \Gamma_2^\varepsilon = \emptyset$, then S^{N-1} is a deformation retract of Γ_2^ε.

Proof. See [5, Corollary 2.3, Lemma 2.8 and 5.1]. The proof is completely analogous to the proof of Lemma 5.5: in fact, there, only the boundedness from above of V has been used; then, taking into account the discussion in the proof of Lemma 6.3 and remarking (i) $m = 0$ in the present situation and (ii) replacing (5.9) and (5.10) with (6.4) one does not affect the validity of the arguments there (in fact (5.10) has been used only in proving $m_1 < m$), the Lemma follows.

To be able to apply the kind of reasoning used in Chapter II to prove the existence of T-periodic solutions for (V), it is necessary to study the topology of set Λ in which f is defined. To this purpose, we prove:

6.5. Lemma. If $\Omega = \mathbf{R}^N \setminus C$, with C compact, then

(6.6) $\quad H_q(\Lambda) \neq 0$ for infinitely many q.

Proof. [5, Lemma 6.2]. Let $p \in C$ and $R > 0$ be such that C is contained in B_R. Set $\Omega_1 \equiv \mathbf{R}^N \setminus B_R$, $\Omega_2 \equiv \mathbf{R}^N \setminus \{p\}$ and $\Lambda_1 = \{u \in E : u(t) \in \Omega_1 \ \forall t\}$, $\Lambda_2 = \{u \in E : u(t) \in \Omega_2 \ \forall t\}$. Clearly $\Lambda_2 \supset \Lambda \supset \Lambda_1$, and since Λ_1 is a deformation retract of Λ_2, Λ_1 is a retract of Λ. Then [40, (4.15)]

(6.7) $\quad H_q(\Lambda) \cong H_q(\Lambda_1) \oplus H_q(\Lambda, \Lambda_1)$.

Since it is well known [19, (3.10)] that (6.6) holds for Λ_1, then (6.7) implies that (6.6) holds for Λ as well.

We can now prove

6.6. Theorem. Suppose $\Omega \equiv \mathbf{R}^N \backslash C$, C compatto, and let V satisfy (6.1), (SF), (6.4) and (6.5). Then (V) has infinitely many T-periodic solutions.

Proof. [5, Theorem 6.3]. Suppose, by contradiction, that $Z(f) = \{u_1,\ldots,u_k\}$, $k < +\infty$. Take $\varepsilon \in]0,\varepsilon_0]$ such that $f^\varepsilon \cap Z(f) = \emptyset$ and $\Gamma_2^\varepsilon \cap Z(f) = \emptyset$. Set $M = \max_j f(u_j)$, $m = \min_\Lambda f(u)$. Then, by Proposition 1.11 and Lemma 6.1 follows

$$H_q(\Lambda, f^\varepsilon) \cong H_q(f^{M+1}, f^\varepsilon).$$

Since f is of the form identity - compact, it is Fredholm of index zero $\forall\ u \in \Lambda$. Then we can use Remark 2.15, together with Lemmas 6.3, to prove that $\exists\ q_0 \geq 0$ such that $H_q(\Lambda, f^\varepsilon) \cong 0\ \ \forall\ q \geq q_0$. This fact , together with the exact homology sequence of the couple (Λ, f^ε) (see proof of Lemma 5.2) permits us to prove that

$$\begin{aligned} H_q(\Lambda) &\cong H_q(f^{M+1}) & \forall\, q \\ &\cong H_q(f^\varepsilon) & \forall\, q \geq q_0 \end{aligned}$$

and, using Lemma 6.4,

$$\begin{aligned} H_q(\Lambda) &\cong H_q(\Gamma_1^\varepsilon) \oplus H_q(\Gamma_2^\varepsilon) & \forall\, q \geq q_0 \\ &\cong H_q(\Gamma_1^\varepsilon) \oplus H_q(S^{N-1}) & \forall\, q \geq q_0. \end{aligned}$$

Since PS holds in Γ_1^ε, we can apply again Remark 2.15 to prove that $H_q(\Gamma_1^\varepsilon) \cong 0\ \ \forall\ q \geq q_1$ for some q_1. Setting $q_2 = \max\{q_0, q_1, N-1\}$ we finally deduce

$$H_q(\Lambda) \cong 0 \qquad \forall\ q > q_1,$$

contradiction which proves the theorem.

It is also possible to prove

6.7. Propositon. Let the assumptions of Theorem 6.6 be satisfied. If, moreover, $\exists\, R, \delta > 0$ such that $|y| \geq R$, $|\xi| \leq \delta$ imply $\langle \nabla_y V(t, y+\xi), \xi \rangle > 0$, then it esists a sequence u_k of T-periodic solutions of (V) such that $f(u_k) \to +\infty$.

Proof. [5, Theorem 6.4]. We only remark that the assumption we are making ensures us that there are no critical points of f in Γ_2^ε for ε sufficiently small; this fact implies that we can reach a contradiction exactly as in the proof Theorem 6.3 assuming that $M = \sup\{f(u): u \in Z(f)\} < +\infty$.

In another situation it is possible to prove existence of infinitely many T-periodic solutions for (V): it is the case when V satisfies (6.1), (SF) and, instead of (6.5), satisfies

(6.8) $\quad V(t,y) \to 0$ as $|y| \to +\infty$, uniformly in t, and

$\exists\, R_2$: $<\nabla_y V(t,y),y> < 0 \quad \forall\, |y| \geq R_2, \ \forall\, t.$

The situation is now quite different from the situation seen above and in §5.In fact, while Lemma 6.3 continues to hold, different is the "topology" of the set f^ε. In fact, one can prove

6.8. Lemma. $\forall\, \varepsilon > 0$ esite $R > 0$ such that

(6.9) $\quad (f'(u),u) > 0 \quad \forall\, u \in f^\varepsilon, \ \|u\| > R.$

In particular $f^\varepsilon \cap B_R$ is positively invariant. Moreover $f^\varepsilon \cap B_R$ is a deformation retract of f^ε.

Proof. [5, Lemmas 3.2, 3.4 and 6.5]. We know that $u \in f^\varepsilon$ implies $|w_u(t)| \leq$ const. Then, If the first statement is not true, it must exists a sequence u_n such that $\|w_n\| \leq$ const, $|\xi_n| \to +\infty$ and $(f'(u_n),u_n) \leq 0$. Then $|u_n(t)| \to +\infty$ uniformly in t and, using (6.8), we get

$$\int |u_n|^2 = (f'(u_n),u_n) + \int <\nabla_y V(t,u_n(t)),u_n(t)> < 0 \ \text{ for n sufficiently large,}$$

contradction which proves the first statement. The other statements follow easily from the first one; in particular the deformation is just the projection on the ball $B_R \cap f^\varepsilon$ (it is decreasing thanks to (6.9).

Using this Lemma, we prove

6.9. Theorem. Suppose $\Omega \equiv R^N \backslash C$, C compatto, and let V satisfy (6.1), (SF), (6.5) and (6.8). Then (V) has infinitely many T-periodic solutions; moreover it exists a sequence u_n of solutions such that $f(u_n) \to +\infty$.

Proof. Suppose that $M = \sup\{f(u): u \in Z(f)\} < +\infty$. As in Theorem 6.4 we deduce that esite q_0 such that

$$H_q(\Lambda) \cong H_q(f^\varepsilon) \qquad \forall\, q \geq q_0$$

But now Lemma 6.5 implies that

$$H_q(\Lambda) \cong H_q(f^\varepsilon) \qquad \forall\, q \geq q_0$$

$$\cong H_q(f^\varepsilon \cap B_R) \qquad \forall\, q \geq q_0$$

and since PS holds in $f^\varepsilon \cap B_R$ ($|\xi_u|$ is bounded $\forall\ u \in B_R$), and $f^\varepsilon \cap B_R$ is positively invariant, again fron Remark 2.15 we deduce that $H_q(f^\varepsilon \cap B_R) \cong 0 \quad \forall\, q \geq q_1$, contradiction with Lemma 6.5.

If V does not depend on t, the problem becams to find non-constant periodic soluitons of

(6.10) $\qquad -\ddot{y} = \nabla_y V(y).$

In such a case one can prove, essentially with the same arguments used in Theorems 6.6, 6.7 and 6.9 , the following

6.10. Theorem. Suppose $\Omega \equiv R^N \setminus C$, C compatto, and let V satisfy (6.1), (SF), (6.4) and either (6.5) or (6.8). Suppose, moreover, that

$$Z(V) \text{ is a compact subset of } \Omega.$$

Then (V) has infinitely many T-periodic non constant geometrically distinct solutions.

Proof. See [5, Theorem 7.1].

§7. Effective-like potentials.

In this section we will study existence of T-periodic solutions for the equation

(V) $\qquad -\ddot{y} = \nabla_y V(t,y)$

for potentials $V \in C^2(R \times \Omega, R)$, $V(t+T,y) = V(t,y)\ \forall\ (t,y) \in R \times \Omega$, where $\Omega = R^N/\{0\}$ and $V \to +\infty$ as $y \to \partial\Omega$. The "model" potential is $V(y) = -|y|^{-1} + |x|^{-2}$, the Newton effective potential.

This case, even if can be dealt with the same tools as the cases studied in §6, it is quite different from that one for the fact that now the functional $f: \Lambda \to \mathbf{R}$ it is no more bounded from below. The results we will describe here are contained in [35].

More precisely, on V we will assume

(7.1) $V \in C^2(\mathbf{R}\times\Omega,\mathbf{R})$, $V(t+T,y) = V(t,y)\ \forall\,(t,y) \in \mathbf{R}\times\Omega$, where $\Omega = \mathbf{R}^N/\{0\}$;

(7.2) $\lim_{x\to 0} V(t,y) = +\infty$, uniformly in t and monotonically increasing along the rays as $|y|$ small;

(SF') $\exists\, U' \in C^1(\Omega,\mathbf{R})$, $\varepsilon' > 0$ such that
(i) $U'(y) \to +\infty$ as $y \to 0$;
(ii) $V(t,y) \geq |\nabla_y U'(y)|^2\ \ \forall\, y$ with $|y| \leq \varepsilon'$, $\forall\, t \in \mathbf{R}$.

(7.3) $V(t,y) \to 0$ as $|y| \to +\infty$, uniformly in t and monotonically increasing along the rays as $|y|$ small, and $\nabla_y V(t,y) \to 0$ as $|y| \to +\infty$, uniformly in t;

(7.4) $<\nabla_y V(t,y),y> \leq c_1\ \ \forall\,(t,y) \in \mathbf{R}\times\Omega$ for some $c_1 \geq 0$ (*).

7.1. <u>Remarks</u>. (i) if V satisfies (SF'), then $-V$ satisfies (SF) with $U = -U'$ (U' given by (SF')) and the same ε'.

(ii) (7.3) implies that $\exists\, m > 0$ such that

(7.5) $V(t,y) \geq -m\ \ \forall\,(t,y) \in \mathbf{R}\times\Omega$,

and that $\exists\, R > 0$ such that

(7.6) $-m \leq V(t,y) \leq 0\ \ \forall\,(t,y) \in \mathbf{R}\times\Omega$ with $|y| > R$.

(iii) (7.4) is a conditon at infinity; in fact (7.2) implies that (7.4) is satisfied in a neighborhood of the origin.

(iv) (7.2) implies that

(7.7) $\exists\, d > 0$ such that $V(t,y) \geq d$ implies $V(t,\beta y) \geq V(t,y)\ \ \forall\ \beta \in\ ]0,1]$.

Let Λ be defined as in §6, and let $f: \Lambda \to \mathbf{R}$ be given by (4.5). First of all, we have that

(*) In this section c_i ($i= 1, 2,\ldots$) will denote a nonnegative constant.

7.2. Lemma. If (SF') holds, then $\forall\ (u_n)$ in Λ, u_n converging weakly in E to $z \in \partial\Lambda$ one has that $f(u_n) \to -\infty$.

Proof. It follows from Lemma 6.1, taking into account Remark 7.1 (i).

Next, we investigate the PS condition.

7.3. Lemma. The PS condition holds: (i) in the sets $\{f \geq \varepsilon\}$ and $\{f \leq -\varepsilon\}$ $\forall\ \varepsilon > 0$; (ii) in every closed set A such that, for some nonnegative constants c_A, $c_{A'}$, $|\xi_u| \leq c_A\{\int |w_u'|^2\}^{1/2} + c_{A'}$, $\forall\ u \in A$.

Proof. See [35, Lemma3]. Even if the result is similar to that of Lemmas 5.1 and 6.3, the proof is sligthly different.

(i) Let (u_n) be a sequence in Λ such that

(7.8) $$f(u_n) \to c \neq 0 \ \text{ and } \ f'(u_n) \to 0.$$

Then

$$(f'(u_n),u_n) = \int |u_n'|^2 - \int <\nabla_y V(t,u_n),u_n>$$

implies

$$\int |u_n'|^2 \leq (f'(u_n),u_n) + Tc_1,$$

so that

$$\int |u_n'|^2 \leq \|u_n\| \, \|f'(u_n)\|_{E^*} + Tc_1$$

where $E^* \equiv H^{-1}(S^1,R^N)$. Using the fact that $f'(u_n) \to 0$, together with $\|u_n\| \leq c_2|w_n'|_2 + c_3\,|\xi_n|$ (see (4.2) and (4.3)), one then gets

(7.9) $$\int |u_n'|^2 \leq c_4\|f'(u_n)\|_{E^*}\,|\xi_n| + c_5 \qquad \forall\ \{u_n\} \text{ satysfing (7.8)..}$$

Using now (4.4), we find that

(7.10) $$|u_n(t)| \geq |\xi_n| - c_6\{c_4\|f'(u_n)\|_{E^*}\,|\xi_n| + c_5\}^{1/2} \quad \forall\ \{u_n\} \text{ satysfing (7.8).}$$

Suppose now, by contradiction, that $|\xi_n| \to +\infty$. Using (7.10), which implies that $|u_n(t)| \to +\infty$ uniformly in t, and (7.3) we have that

(7.11) $\qquad V(t,u_n(t)) \to 0$ uniformly in t as $n \to +\infty$,

(7.12) $\qquad \nabla_y V(t,u_n(t)) \to 0$ uniformly in t as $n \to +\infty$.

And using

$$(f'(u_n),w_n) = \int|u_n'|^2 - \int<\nabla_y V(t,u_n(t)),w_n>,$$

we deduce

$$\int|u_n'|^2 \leq \|f'(u_n)\|_{E*} \|w_n\| + |\nabla_y V(t,u_n(t)|_2 |w_n(t)|_2.$$

$$\leq c_7(\|f'(u_n)\|_{E*} + |\nabla_y V(t,u_n(t))|_2) \{\int|u_n'|^2\}^{1/2}.$$

Using (7.12) we finally deduce

(7.13) $\qquad \int|u_n'|^2 \to 0$ as $n \to +\infty$,

and from this and (7.11) we get that $f(u_n) \to 0$ as $n \to +\infty$, contradiction which proves the boundedness of $|\xi_n|$ and hence, using (7.9) the boundedness of $\{u_n\}$ in E. From now on the proof proceeds exactly as in Lemma 6.3.

(ii) It is enough to remark that, under this assumption, one immediately deduce from (7.9) the boundedness of w_n and, using again this assumption, also that of ξ_n.

We can now state our main Theorem of this section.

7.4. Theorem. Let V satisfy (7.1), (7.2), (7.3), (7.4) and SF'. Then (V) has at least one T-periodic solution.

Proof. See [35].

Let us suppose, by contradiction, that f has no critical points in Λ. The proof will be carried on in several steps.

Step 1. The homology groups of $f^{-\varepsilon}$, $\varepsilon > 0$.

First of all we prove that

(7.14) $\qquad A_\delta \equiv \{u \in \Lambda: \|u\| \le \delta\}$ is a deformation retract of Λ $\forall \delta > 0$.

This follows easily using as a deformation the Projection P on the ball B_δ of E defined by

$$F(s,u) = \begin{cases} u & \text{if } \|u\| \le \delta \\ \{1 - (1 - \delta/\|u\|) s & \text{if } \|u\| > \delta. \end{cases}$$

Such a deformation is well defined even when restricted to Λ, since it is just a multiplication by a nonnegative constant, so that if $u \in \Lambda$ (which is equivalent to $u \in E$ and $u(t) \neq 0$ $\forall$ t) also $P(s,u) \in \Lambda$ $\forall s \in [0,1]$, $\forall u \in \Lambda$.

Then we prove

(7.15) $\qquad \exists\, \delta_0 > 0$ such that $\forall \delta \in]0, \delta_0[$ A_δ is a deformation retract of $f^{-\varepsilon}$.

Since $u \in A_\delta$ implies $\|u\| < \delta$, we have that $(1/2)\int |u'|^2 \le \delta^2$ and $|u(t)| \le c_8\delta$. We deduce that $V(t,u(t)) \ge b(\delta)$ and $(1/2)\int |u'|^2 \le \delta^2$ $\forall u \in A_\delta$, where $b(\delta) \to +\infty$ as $\delta \to 0$. Then

$$f(u) = (1/2)\int |u'|^2 - \int V(t,u(t)) \le \delta^2/2 - b(\delta)T \to -\infty \quad \text{as} \quad \delta \to 0.$$

This implies that

(7.16) $\qquad \forall\, K > 0$, $\exists\, \delta_0 > 0$ such that $\forall\, \delta \in]0,\delta_0[$, A_δ is a subset of f^{-K}.

Take now $q \in N$ such that $q \ge 1 + T(d + m)/\varepsilon$, where d is given by (7.7) and m by (7.5). We have that

$$(1/2)\int |u'|^2 - \int V(t,u) \le -q\varepsilon \quad \forall\, u \in f^{-q\varepsilon},$$

which implies

$$\int V(t,u) \ge q\varepsilon + (1/2)\int |u'|^2 \quad \forall\, u \in f^{-q\varepsilon}.$$

Let $A(u) \equiv \{t \in [0,T]: V(t,u(t)) \ge d\}$, $B(u) \equiv \{t \in [0,T]: V(u(t)) < d\}$. Then

(7.17) $\qquad \int_A V(t,u(t)) \geq q\varepsilon + (1/2)\int |u'|^2 - \int_B V(t,u(t)) \geq q\varepsilon + (1/2)\int |u'|^2 - dT.$

Take now δ such that $f^{-q\varepsilon} \supset A_\delta$ and consider the deformation F(s,u) defined above. We have that from (7.7) follows that $A(F(s,u)) \supset A(u)$ $\forall\, s \in [0,1]$, and that, $\forall\, t \in A(u)$ $V(t,F(s,u)(t)) \geq V(t,u(t))$. Then

$$\int_{A(F(s,u))} V(t,F(s,u)) \geq \int_A V(t,u)$$

and, if $\|u\| > \delta$, using (7.17) one gets

$$\begin{aligned} f(F(s,u)) &= (1/2)\,\{1 - (1 - \delta/\|u\|)\}^2 - \int_{A(F(s,u))} V(F(s,u)) - \int_{B(F(s,u))} V(F(s,u)) \\ &\leq (1/2)\,\{1 - (1 - \delta/\|u\|)\}^2 - \int_A V(F(s,u)) + Tm \\ &\leq (1/2)\,\{1 - (1 - \delta/\|u\|)\}^2 - q\varepsilon - (1/2)\int |u'|^2 + Td + Tm \\ &\leq (1/2)\,\{(1 - \delta/\|u\|)\}^2 s^2 + 2(1 - \delta/\|u\|)\, s\}\ (1/2)\int |u'|^2 - q\varepsilon + T(d+m) \\ &\leq -q\varepsilon + T(d+m) \leq -\varepsilon, \end{aligned}$$

so that

(7.18) $\qquad f(F(s,u) \leq -\varepsilon \quad \forall\, s \in [0,1], \quad \forall\, u \in f^{-q\varepsilon}.$

Since in $f^{-\varepsilon}$ PS holds, and there are no critical point of f, there exists a deformation η from $f^{-\varepsilon}$ into $f^{-q\varepsilon}$. Since (7.18) holds, then the composition of the two deformations, η and F, shows that A_δ is a deformation retract of $f^{-\varepsilon}$, so that (7.15) holds with δ_0 given by (7.16), where K $= q\varepsilon$. Using (7.14) and (7.15), we deduce that

(7.19) $\qquad H_q(f^{-\varepsilon}) \cong H_q(\Lambda) \quad \forall\, \varepsilon > 0.$

<u>Step 2</u>. The homology groups of f^{ε}, $\varepsilon > 0$.

Let's prove, first of all, that

(7.20) $\qquad \exists\, \varepsilon_0 > 0,\ a \in\,]0,1[,\ R^* > 0$ such that $f(u) > \varepsilon_0 \quad \forall\, u \in S_{a,R^*}$,

where

(7.21) $$S_{a,R^*} \equiv \{u \in \Lambda:\ a^2 |\xi_u|^2 - (1/2)\int |u'|^2 = R^{*2}\}.$$

In fact, if $u \in S_{a,R^*}$, we deduce

$$|w_u(t)| \le c_9\{a^2 |\xi_u|^2 - R^{*2}\}^{1/2}$$

so that

$$|u(t)| \ge |\xi_u| - c_9\{a^2 |\xi_u|^2 - R^{*2}\}^{1/2}.$$

Taking $a < c_9^{-1/2}$, one can prove that

(7.22) $$(R^*/a)\{1 - (c_9 a)^2\}^{1/2} \le |u(t)| \le |\xi_u| + c_9\{a^2 |\xi_u|^2 - R^{*2}\}^{1/2} \quad \forall\, u \in S_{a,R^*}.$$

Let R be as in (7.6), and take $R^* > R$ such that $(R^*/a)\{1 - (c_9 a)^2\}^{1/2} \ge R$. Then, $\forall\, u \in S_{a,R^*}$, we have that

$$f(u) = (1/2)\int |u'|^2 - \int V(t,u)$$

$$= (1/2)(a^2 |\xi_u|^2 - R^{*2}) - \int V(t,u).$$

If $(1/2)(a^2 |\xi_u|^2 - R^{*2}) \ge 1$ from (7.22) follows $(V(t,u) \le 0)$ and $f(u) \ge 1$. If $(1/2)(a^2 |\xi_u|^2 - R^{*2}) < 1$, one deduces $|\xi_u|^2 < (R^{*2} + 2)/a^2$ and, from (7.22) follows $(R^*/a)\{1 - (c_9 a)^2\}^{1/2} \le |u(t)| \le \{R^{*2} + 2\}^{1/2}/a + c_9 2^{1/2}$.

Let

$$M = \sup\{V(\xi):\ (R^*/a)\{1 - (c_9 a)^2\}^{1/2} \le |\xi| \le \{R^{*2} + 2\}^{1/2}/a + c_9 2^{1/2}\} < 0.$$

Then

$$f(u) = (1/2)(a^2 |\xi_u|^2 - R^{*2}) - \int V(t,u)$$

$$\ge -\int V(t,u) \ge TM > 0.$$

It is then clear that, taking $\varepsilon_0 < 1$, $\varepsilon_0 < TM$, (7.20) follows. Set now

$$C_1 = \{u \in \Lambda:\ a^2 |\xi_u|^2 - (1/2)\int |u'|^2 \le R^{*2}\},$$

$$C_2 = \{u \in \Lambda:\ a^2 |\xi_u|^2 - (1/2)\int |u'|^2 \geq R^{*2}\}.$$

Both this sets have non empty intersection with f^ε (in fact C_1 contains the constant of small norm, while C_2 contains the constant of very large norm. Moreover, the PS condition holds in $C_1 \cap f^\varepsilon$ (see Lemma 7.3). From (7.20) it also follows that $C_1 \cap C_2 = \emptyset$, and that these two sets are closed.

Now, as already done in Lemmas 5.5, 6.4, one can prove that

(7.23) $$H_q(C_2) \cong H_q(S^{N-1}),$$

while, since PS holds in C_1, one finds

(7.24) $$H_q(C_1) \cong H_q(f^{-\varepsilon}).$$

Step 3. Proof of theorem compled.

Since PS holds in $\{f \geq \varepsilon\}$, taking into account (7.19), and (7.24), we deduce

$$\begin{aligned} H_q(\Lambda) &\cong H_q(f^\varepsilon) \\ &\cong H_q(C_1) \oplus H_q(C_2) \\ &\cong H_q(f^{-\varepsilon}) \oplus H_q(S^{N-1}) \\ &\cong H_q(\Lambda) \oplus H_q(S^{N-1}), \end{aligned}$$

which yields a contradiction for $q = 0,N$.

We state here, without proving it, a theorem on time-independent effective-like potentials.

7.5. Theorem. Let V be a time-independent potential satisfying (7.1), (7.2), (7.3), (7.4), SF' and

Z(V) consists only of finitely many nondegenerate critical points.

Then it exists a $T_0 > 0$ such that $\forall\ T \geq T_0$ (V) has at least one T-periodic, non-constant solution.

Proof. See [35, Theorem 1].

§8. Weak forces.

Let Ω be an open, bounded and convex subset of $\mathbf{R}^N$, with $0 \in \Omega$. Let $V \in C^1(\Omega; \mathbf{R})$, $h \in C^1(S^1; \mathbf{R}^N)$, where $S^1=[0,T]/\{0,T\}$. In this section we will study the existence of T-periodic solutions for the system of ordinary differential equations

$$\text{(V)} \qquad -\ddot{y} = \nabla_y V(t,y)$$

where $V(t,y) = V(y) + \langle h(t),y\rangle$, under the assumptions:

(8.1) $V(y) \to -\infty$ as $y \to \partial\Omega$, uniformly;

(8.2) it exists a $m > 0$ such that $-V(y) + (1/2)\, m\, |y|^2$ is strictly convex;

(8.3) there exists $\theta \in]0,1/2[$, $\varepsilon > 0$ such that $V(y) \geq \theta\langle y,\nabla V(y)\rangle$ for every $y \in (\partial\Omega)_\varepsilon \equiv \{y \in \Omega : \text{dist}\,(y,\partial\Omega) < \varepsilon\}$.

8.1. Remark. (8.2) is a condition on the behaviour of V near the boundary of Ω.

8.2. Remark. (8.3) is always satisfied if V is concave and radial. In particular, it is satisfied if $\Omega = \{y \in \mathbf{R}^N : |y| < 1\}$ and $V(y) = -\rho(y)\,(1 - |y|)^{-1}$, where ρ is of class C^1, radial, $\rho(0) = \rho'(0) = 0$, $\rho(y) = 1$ if $|y| > 1/2$ and is such that V is concave.

8.3. Remark. (8.3) implies that there exists a $c_1 \geq 0$ such that

$$\text{(8.4)} \qquad V(y) \geq \theta\langle y,\nabla V(y)\rangle - c_1 \qquad \forall\, y \in \Omega.$$

To study (V) under assumptions (8.1), (8.2) and (8.3) we will employ the Dual Action Principle. The same device will be used in §9, where we will deal with the case $V(y) \to +\infty$ as $y \to \partial\Omega$.

The content of this section is contained in [36]. This paper is strictly related to [4], a preceding paper which will be described in §9, and to which we will often refer to.

Setting $V_m(t,y) = -V(y) + (1/2)\, m\, |y|^2 - \langle h(t),y\rangle$, one has that (V) is equivalent to:

(8.5) $$-\ddot{y} - m\,y = -\nabla_y V_m(t,y).$$

We also remark that V_m is such that

(8.6) $V_m(t,y) \to +\infty$ as $y \to \partial\Omega$ uniformly in (t,y);

(8.7) $V_m(t,y)$ is strictly convex in y for every t;

(8.8) there exist $\theta' \in\,]0,1/2[$, $\varepsilon > 0$ such that $V_m(t,y) \le \theta'\langle y, \nabla_y V_m(t,y)\rangle$ for every $y \in (\partial\Omega)_\varepsilon$.

In fact (8.6) and (8.7) are direct consequences of (8.1), (8.2), while (8.8) follows from (8.1) and (8.3) taking $\theta' = \alpha\theta$ (θ given by (8.1))) with any $\alpha > 1$ such that $\alpha\theta' \in\,]0,1/2[$.

Let us now introduce the Legendre transform of the function $V_m^*(t,x)$, defined as

$$V_m^*(t,x) = \sup\, \{\langle x,y\rangle - V_m(t,y) :\ y \in \Omega\}.$$

From (8.6), (8.7) and (8.8) follows, in a standard way (see [4], and, for more general propertis of the Legendre transform [27, 28]), that

(8.9) $V_m^* \in C^1(S^1 \times \mathbf{R}^N; \mathbf{R})$;

(8.10) $c_2|x| - c_3 \le V_m^*(t,x) \le c_4|x| + c_5 \quad \forall\, (t,x) \in \mathbf{R}\times\mathbf{R}^N$ (*);

(8.11) $V_m^*(t,x) \ge (1 - \theta')\langle x, \nabla_x V_m^*(t,x)\rangle - c_6 \quad \forall\, (t,x) \in \mathbf{R}\times\mathbf{R}^N$;

(8.12) $|\nabla_x V_m^*(t,x)| \le c_7$.

We take now $X = L^1(0,T;\mathbf{R}^N)$ and define, for $m \ne k^2\omega^2$, $\omega \equiv 2\pi/T$, the operator $L_m : X \to X$ setting

(8.13) $$L_m v = u \quad \text{iff} \quad \ddot{u} + m\,u = v.$$

We remark that $L_m(X)$ is a subset of $W^{2,1}$.

(*) Here and in the following, c_i will denote non-negative constants.

We set, for every $v \in X$,

$$\Phi(v) = \int V_m^*(t,v) - (1/2) \int <v, L_m v>.$$

Φ is clearly of class C^1 on X. If $v \in X$ is such that $\Phi'(v) = 0$, then $L_m v = \nabla_x V_m^*(t,v)$. Set

(8.14) $$u \equiv L_m v = \nabla_x V_m^*(t,v).$$

From $u \equiv L_m v$ it follows that $u \in W^{2,1}$ and, in particular, that u' is an absolutely continuous function. Since (see [27, 28])

(8.15) $$x = \nabla_y V_m(t,y) \quad \text{iff} \quad y = \nabla_x V_m^*(t,x),$$

one has that $\nabla_x V_m^*(t,R^N) = \Omega$, hence, since v is a.e. bounded, one has $u(t) \in \Omega$ for almost all t and, since u is continuous, $u(t)$ belongs to the closure of Ω for all t's. Without loss of generality we can assume $u(0) \in W$. From (8.13), (8.14) and (8.15) we deduce

$$\ddot{u} + m\, u = \nabla_y V_m(t,u)$$

(in the $W^{2,1}$ sense), hence

(8.16) $$-\ddot{u}(t) = \nabla_x V(u(t)) + h(t) \qquad \text{in the } W^{2,1} \text{ sense.}$$

Let $I = [0,t^*]$ be an interval such that $u(t) \in W$ for every $t \in I$ (it exists since u is continuous, $u(0) \in \Omega$). Define, for $t \in I$,

(8.17) $$E(t) = (1/2)\, |u'(t)|^2 + V(u(t)).$$

For $t \in I$ we find that

$$E(t) = E(0) - \int_0^t <h(s),u'(s)>ds.$$

We note that:

(i) $E(0) = (1/2)\, |u'(0)|^2 + V(u(0))$ is bounded because $u(0) \in \Omega$ and u' is absolutely continuous;

(ii) $\quad |\int_0^t \langle h(s),u'(s)\rangle| \leq \|h\|_{L^1} \|u'\|_{L^\infty} \leq$ const.

From (i) and (ii) it follows that $|E(t)| \leq c_8$ for every $t \in I$, and c_8 is independent from I. This clearly implies that $E(t) \leq$ const. for every t, and from this follows that $u(t) \in \Omega$ for every t since $|u'(t)| \leq$ const for every t.

The above facts can be collected in the following

8.4. Lemma. If $v \in L^1$ is a critical point of Φ, then

$$u = L_m v \in W^{2,1}$$

is T-periodic solution of (V) such that $u(t) \in \Omega$ for every t.

8.5. Remark. One can use the usual regularity theorems to prove that the solution given by lemma 8.4 is actually of class C^2.

In order to find a critical point of Φ, we prove

8.6. Lemma. Φ satisfies the Palais Smale condition.

Proof. See [36, Lemmas 6 and ACz1, Lemma 2.4]. From

$$|\Phi(v_n)| = |\int V_m^*(t,v_n) - (1/2) \int \langle v_n, L_m v_n\rangle| \leq c_{10}$$

and

$$|\int \langle \nabla_x V_m^*(t,v_n),v_n\rangle - \int \langle v_n|L_m v_n\rangle| = |\langle \Phi'(v_n)|v_n\rangle| \leq \varepsilon_n \|v_n\|_{L^1},$$

using (8.10) and (8.11), we deduce

$$\|v_n\|_{L^1} \leq \text{const.}$$

Hence (up to a subsequence), $L_m v_n \to z$ (recall that L_m is compact) and $z_n = \nabla_x V_m^*(t,v_n) = L_m v_n - \Phi'(v_n) \to z$ in L^∞; moreover $z(t)$ belongs to the closure of Ω for every t. As in Lemma 8.4, one can actually prove that $z(t) \in \Omega$ for every t. Then

$$v_n = \nabla_y V_m(t,z_n) \to \nabla_y V_m(t,z) \quad \text{in } L^\infty,$$

and the lemma follows.

Define

$$X_1 = \text{span}\ \{ e^{ik\omega t},\ \omega^2 k^2 < m\},$$

$$X_2 = \text{span}\ \{ e^{ik\omega t},\ \omega^2 k^2 > m\}.$$

It is well known that $X = X_1 \oplus X_2$. Moreover

$$(8.18) \qquad \int \langle v, L_m v\rangle \leq 0 \quad \forall\ v \in X_2,$$

since it is true for all the finite combinations. Thus

$$(8.19) \qquad \Phi(v) \geq \int V_m^*(t,v) \leq c_2 \|v\|_{L^1} - c_{11} \qquad \forall\ v \in X_2$$

and Φ is coercive on X_2. On X_1 one has $(v = \Sigma\, v_k e^{ik\omega t})$

$$\Phi(v) \leq c_4 \|v\|_{L^1} - \sum \frac{|v_k|^2}{(m-\omega^2 k^2)} - c_{12} \qquad \forall\ v \in X_1$$

$$\leq c_4 \|v\|_{L^1} - c_{13} \Sigma\, |v_k|^2 - c_{12} \qquad \forall\ v \in X_1.$$

Since X_1 is finite dimensional, all the norms are equivalent, hence

$$(8.20) \qquad \Phi(v) \leq c_4 \|v\|_{L^1} - c_{14} (\|v_n\|_{L^1})^2 - c_{12} \qquad \forall\ v \in X_1,$$

and $-\Phi$ is coercive on X_1. (8.19) and (8.20) allows us to apply a well known linking theorem [61] which implies

8.7. <u>Theorem</u>. Let Ω be an open, bounded and convex subset of $\mathbf{R}^N$. Let $V \in C^1(\Omega; \mathbf{R})$ satisfy (8.1), (8.2) and (8.3). Then, for every $h \in C^1(\mathbf{R};\mathbf{R}^N)$, h T-periodic, it exists at least one T-periodic solution $y(t)$ of (V) such that $y(t) \in \Omega$ for every t.

8.8. <u>Remark</u>. The interesting point of this theorem is that no strong force condition is required on the behaviour of V near the singularity set. As we have seen in Lemma 8.4, one can rule out the solutions crossing the singularity thanks to the additional regularity of the solutions corresponding to the critical points of Φ. Let us point out that this additional regularity it is not shared by all solutions of (V); in fact there can exist solutions of (V) which are of class $H^{1,2}$ but not of class $W^{2,1}$. For example, in one dimension (but the example easily extend to any dimension), consider the Kepler's problem

$$-\ddot{y} = \nabla(-|y|^{-1}).$$

A particular (classical) solution of such an equation is given by

$$E^{-1}[y(1+Ey)]^{1/2} + (2E\sqrt{-E})^{-1}\mathit{arcsin}(2Ey+1) = t + \pi/(4E\sqrt{-E}), \ \forall t \neq 0, \ y > 0, \tag{8.21}$$

where E is a negative constant, the energy of the motion:

$$(1/2)|y'(t)|^2 - y(t)^{-1} = E \qquad \forall\, t \neq 0.$$

Such a solution has the following properties:

(a) $y(t) \approx t^{2/3}$ as $t \to 0$.
(b) $|y(t)| \leq -E^{-1}$ $\forall\, t \neq 0$;
(c) $y'(\tau/2)=0$, where $\tau=\pi/(2|E|^3)^{1/2}$.

Take now any potential $V:]0,1[\to\mathbf{R}$ satisfying the hypothesis of theorem 8.7 (in particular the potential of Remark 8.2)

$$V(y) = -y^{-1} \qquad \forall y \in\,]0,\varepsilon[.$$

Suppose any $T>0$ is given and choose $E<0$ such that

$$-E^{-1} < \varepsilon;$$
$$T(2|E|^3)^{1/2}/\pi = k \in \mathbf{N}.$$

It is easy to check that (8.21) defines a $H^{1,2}$ $T=k\tau$-periodic function which is a weak (in the $H^{1,2}$ sense) solution of equation (V). Such a function, though, is not, by (a), a $W^{2,1}$ function.

The above argument rules out the possibility to prove that <u>any</u> weak solution of (V) is of class $W^{2,1}$.

§9. Singular potentials and minimal period.

In this section we will describe some of the results contained in the paper [4]. The method employed here is based on the use of the Dual Action Principle (see Lemma 8.4), and on the connection between critical points of Mountain Pass type (see Theorem 2.3 and Remark 2.4 b)) and solutions of minimal period of Hamiltonian systems. Such a connection has been estabilished by Ekeland and Hofer (see [43, 44]) and is based on some geometrical properties of the critical points of Mountain Pass type (see [50]).

The setting will be very similar to that of §8, and, since the metheds we are going to use are quite different from those used up to here, we will skip most of the proofs.

Let Ω be an open, bounded and convex subset of $\mathbf{R}^N$, with $0 \in \Omega$. In this section we will study the existence of T-periodic solutions for the system of ordinary differential equations

(V) $$-\ddot{y} = \nabla V(y)$$

under the assumptions:

(9.1) $V \in C^2(\Omega; \mathbf{R})$;

(9.2) $V(0) = 0 = \min_\Omega V$;

(9.3) $V(y) \to +\infty$ as $y \to \partial\Omega$, uniformly;

(9.4) $\exists\ \theta \in]0,1/2[$, $\varepsilon > 0$ such that

$$V(y) \le \theta<y,\nabla V(y)> \quad \forall\ y \in (\partial\Omega)_\varepsilon \equiv \{y \in \Omega : \text{dist}\,(y,\partial\Omega) < \varepsilon\}.$$

Let us point out that (9.4) is the usual assumption of "superquadraticity" (near $\partial\Omega$). If U is radial and convex, (9.4) follows from (9.3), and that no "strong force" condition is required. See Remark 8.2.

We will prove

9.1. Theorem. Suppose V satisfies (9.1), (9.2), (9.3) and (9.4) and

(9.5) $$\exists\, k > 0 \text{ such that } (V''(y)x,x) \ge k|x|^2 \quad \forall\, y \in \Omega,\ \forall\, x \in \mathbf{R}^N.$$

Let ω_N be the gratest eigenvalue of $V''(0)$, and $T_0 \equiv (2/\omega_N)^{1/2}$. Then, $\forall\, T \in \,]0,T_0[$, (V) has at least one T-periodic solution $u \neq 0$ having T as minimal period.

9.2. Remark . Existence of at least one T-periodic solution for a similar problem has been proved by Benci in [15].

Proof. The proof will be carried on in the following steps:
Step 1: use of the dual action principle, as in §8, to transform (V) in a critical point problem for a functional Φ in a Banach space E.
Step 2: application of the Mountain Pass theorem to Φ.
Step 3. use Ekeland-Hofer's argument (see [43]) to show that the Mountain Pass solution has minimal period.

Step 1: (dual action principle).

As in §8, one finds that the Legendre transform of V, defined as

$$V^*(x) = \sup \{\langle x,y\rangle - V(y) : \; y \in \Omega\},$$

is such that (see §8 and [4, Lemma 2.2 and Remark 2.3])

(9.6) $V^* \in C^2(\mathbf{R}^N; \mathbf{R})$ and it is strictly convex;

(9.7) $c_1|y| - c_2 \leq V^*(y) \leq c_3|y| + c_4 \quad \forall\, y \in \mathbf{R}^N$;

(9.8) $V^*(y) \geq (1-\theta)\langle y,\nabla V^*(y)\rangle - c_5 \quad \forall\, y \in \mathbf{R}^N$;

(9.9) $|\nabla V^*(y)| \leq c_6 \quad \forall\, y \in \mathbf{R}^N$.

We now introduce

$$E = \{v \in L^1(0,T; \mathbf{R}^N) \colon \textstyle\int v = 0\},$$

and define, $\forall\, u \in E$, the operator L setting

$$Lv \equiv u \qquad \text{iff} \qquad -\ddot{u} = v.$$

Then, we set

$$\Phi(v) = \int [V^*(v) - (1/2)<v,Lv>].$$

Φ is clearly of class C^1 on X. If $v \in X$ is such that $\Phi'(v) = 0$, then $\exists\, \xi \in \mathbf{R}^N$ such that $Lv = \nabla V^*(v) + \xi$. Set

$$u \equiv Lv - \xi = \nabla V^*(v). \tag{9.10}$$

Then $-\ddot{u} = v$ and, from the properties of the Legendre transform (see (8.15), $-\ddot{u} = \nabla V(u)$. This concludes step 1.

Step 2. (application of the Mountain Pass theorem)

One starts by proving that Φ satisfies the PS condition (see [4, Lemma2.4]). The proof is similar to that of Lemma 8.6 and will be omitted here. Then one investigates the bahaviour of Φ at $v = 0$ and at infinity. One proves (see [4]) that
(i) esite r, $a > 0$ such that $\Phi(v) \geq a$ $\forall$ $\|u\|_1 = r$;
(ii) $\exists$ $v^* \in E$, $\|v^*\|_1 > r$ such that $\Phi(v^*) \leq 0$.
Then it is possible to apply the Mountain Pass theorem (see Theorem 2.3) to Φ, thus finding a critical point $u \in E$ of Φ. Such a u, according to step1, gives rise to a T-periodic solution of (V).

Step 3. (minimality of period)

To show the solution of Mountain Pass found in step2 has minimal period T, one repeats the arguments of [43]. The proof is quite technical, since one has to use a finite dimensional reduction to show that our critical point has "index 0 or 1" (we remark that our functional is not of class C^2, also we are working in $L^1(0,T; \mathbf{R}^N)$, which is not a reflexive space).

9.3. Remark. In the paper [4], also results on existence of T-periodic solutions for a complete Hamiltonian system

$$- Jz' = \nabla_z H(t,z) \tag{H}$$

are given, where, roughly speaking, $H: \Omega\times\mathbf{R}^N \to \mathbf{R}$, Ω subset of $\mathbf{R}^N$, and $H \to +\infty$ as $(p,q) \to \partial(\Omega\times\mathbf{R}^N)$.

Chapter IV: Perturbations of Hamiltonian systems.

In this last chapter, we are going to discuss some results on perturbation of Hamiltonian systems contained in the papers [7, 31]. We will use the abstract tools developed in §3, and we will describe the results contained in the papers [7, 31]. In particular we will be interested in existence of T-periodic solutions for time-independent Hamiltonian systems perturbed by a small forcing term.

§10. Morse theory and perturbation of Hamiltonian systems.

We start by describing some results on second order Hamiltonian systems which are contained in [31]. Let $V\colon \mathbf{R}\times\mathbf{R}^N \to \mathbf{R}$ be such that

(10.1) $\qquad V \in C^2(S^1\times\mathbf{R}^N;\mathbf{R}),\quad V(t,0) = 0 \ \ \forall\, t \in \mathbf{R};$

(10.2) $\qquad V(t,y) = (1/2)k|y|^2 + U(t,y),\quad$ where $k \in\,]0,1[$ and $|\nabla_y U(t,y)| \le \phi(|y|)\ \ \forall\, (t,y) \in \mathbf{R}\times\mathbf{R}^N$ and where $\phi(s)/s \to 0$ as $s \to +\infty$.

(10.2) implies that

(10.3) $\qquad \forall\, \varepsilon > 0\ \exists\, A = A(\varepsilon) > 0$ such that $|\nabla_y U(t,y)| \le \varepsilon|y| + A(\varepsilon)\ \ \forall\, (t,y) \in \mathbf{R}\times\mathbf{R}^N$ and $|U(t,y)| \le (1/2)\,\varepsilon|y|^2 + A(\varepsilon)\,|y|$ perongi $y \in \mathbf{R}^N$.

Let, as in §4,5, $E = H^1(S^1;\mathbf{R}^N)$ and $f\colon E \to \mathbf{R}$ be defined by (4.5). We will write here, always as in §4,5,6,7, $\forall\, u \in E\ \ u = w_u + \xi_u$.

We start by proving

10.1. Lemma. Let V satisfy (10.1) and (10.2). Then f is such that
(i) $Z(f)$ is bounded;
(ii) $\exists\, c_0 > 0$ such that $\forall\, u \in Z(f)\ \ |f(u)| \le c_0$;
(iii) f satisfies PS.

Proof. See [31, lemma 3.2].

10.1. Lemma. Under the assumptions of Lemma 3.2, and for c_0 given by Lemma 10.1 (ii), the localization assumption (seeRemark 3.5) holds with $c_1 = -\,c_0$, $c_2 = c_0$.

Proof. See [31, Lemma 3.4]. Here we will just sketch the proof.
First of all, one shows that $\exists\ R_1, k_1, k_2, k_3, k_4 > 0$ such that $\forall\ u \in \{-c_0 \leq f \leq c_0\} \cap \{u \in E: \|u\| > R_1\}$ one has $1 < \|u\|\ \|w_u\|^{-1} < k_1$, $1 < \|u\|\ \|\xi_u\|^{-1} < k_2$, $k_3 < \|w_u\|\ \|\xi_u\|^{-1} < k_4$. Then one defines a deformation $\Psi: [0,1] \times (f^b \setminus B_R) \to E$ as follows

$$\Psi(s,u) = \begin{cases} (\|u\|/\|w_u\|) \sin(\theta s - \theta)\, w_u + (\|u\|/\|\xi_u\|) \cos(\theta s - \theta)\, \xi_u & \text{if } w \neq 0, \\ \xi_u & \text{if } w = 0. \end{cases} \tag{10.4}$$

where $\theta \in]-\pi/2, 0[$ is defined by

$$\sin(-\theta) = \|w_u\|/\|u\|, \qquad \cos(-\theta) = \|\xi_u\|/\|u\|. \tag{10.5}$$

Such a deformation is such that

$$\Psi(0,u) = u \quad \forall\ u \in E, \tag{10.6}$$

$$\Psi(1,u) = \|u\|/\|\xi_u\|\ \xi_u \quad \forall\ u \in E, \tag{10.7}$$

$$\|\Psi(s,u)\| = \|u\| \quad \forall\ s \in [0,1],\ \forall\ u \in E. \tag{10.8}$$

One can also prove that, for R large,

$$\frac{df}{ds}(\Psi(s,u))_{|s=0} < 0 \qquad \forall\ u \in \{-c_0 \leq f(u) \leq b \leq c_0\} \setminus B_R$$

Using this deformation, togheter with the fact that $\Psi(0,u) \geq -c_0$ implies that $\Psi(1,u) \leq -c_0$ for $\|u\|$ large, one can show that $\exists\ R > 0$ such that $\forall\ a,b \in [-c_0, c_0]$ $a \leq b$, $f^a \cap C(R,R+1)$ is a deformation retract of $f^b \cap C(R,R+1)$ and $f^a \setminus \mathrm{cl}\ B_R$ is a deformation retract of $f^b \setminus \mathrm{cl}\ B_R$, so that Proposition 3.6 apllies to prove the lemma.

Finally, using again the deformation (10.4), one finds that [31, Lemma 3.5]

10.2. Lemma. S^{n-1} is a deformation retract of f^{-c_0}.

We can now state the following

10.3. Theorem. Suppose V satisfies (10.1), (10.2) and

(10.9) $\qquad V''_{yy}(t,0) = K\,I, \quad K \notin \mathbf{Z}, \ \nabla_y V(t,0) = 0 \quad \forall\ t \in \mathbf{R}$. Then

(i) If $K < 0$ or $K > 1$ (V) has at least one non-trivial 2π-periodic solution;
(ii) If $K > 4$ (V) has at least two non-trivial 2π-periodic solutions;
(iii) If $V(\varepsilon,t,y) = V(t,y) + \varepsilon U_1(t,y)$ satisfies (10.1) and (10.2), then $\exists\ \varepsilon^* > 0$ such that $\forall\ \varepsilon \in [0,\varepsilon^*[$,

(10.10) $\qquad -\ddot{y} = \nabla_y V(\varepsilon,t,y)$

has, if $K < 0$ or $K > 1$, at least two 2π-periodic solutions and, if $K > 4$, at least threee 2π-periodic solutions.

Proof. The proof of part (i) and (ii) is essentially the same of that of Proposition 5.4 and Theorem 5.6, once we observe that from Lemma 10.2 follows that $H_q(f^{-c_0})$ isomorfo $H_q(S^{n-1})$ and that PS holds. Part (iii) follows noticing that the functional

$$f_\varepsilon(u) = f(u) + \varepsilon\int U_1(t,u)$$

still has a nondegenerate critical point near zero of index equal to the index of 0 with respect to the unperturbed functional f, and then apply the same reasoning ad in (i), (ii).

Also the case V time-independent can be dealt with using the same tecniques; in this case, though, we will find nontrivial homology groups for f imposing a non-degeneracy condition on the critical points of f. We find

10.4. Theorem. Suppose V is time-independent and satisfy (10.1), (10.2), (10.9) and

(10.11) $\qquad \nabla_y V(y) \neq 0 \quad \forall\ y \neq 0;$

(10.12) $\qquad$ if $y(t)$ is a 2π-periodic solution of (V), then $y'(t)$ is the only 2π-periodic solution of

$$-\ddot{u} = \nabla_y V(y(t))u.$$

Suppose that $V(\varepsilon,t,y) = V(y) + U_1(\varepsilon,t,y)$ satisfies (10.1) and (10.2) $\forall\ \varepsilon > 0$, that $U_1(\varepsilon,t,y) \to 0$ as $\varepsilon \to 0$ uniformly in (y,t), and that U_1 is continuous in ε. Then $\exists\ \varepsilon^* > 0$ such that $\forall\ \varepsilon \in [0,\varepsilon^*[$, (10.10) has at least $m^* = \min\{m \in N\colon\ m \geq (2N[k^{1/2}]/(2N+1)\}$, ($[\alpha]$ denoting the integer part of α) 2π-periodic solutions.

Proof. The Proof will be carried on in two steps.

Step 1. Use of Morse theory to find the nondegenerate (in the sense that satisfy (iii)) solutions of the unperturbed, time-independent equation.

First of all, from Lemma 10.1, (10.9) and (ii) follows that the functional f, defined in (4.5), is such that

$$Z(f) = \{0\} \cup N_1 \cup \ldots \cup N_j,$$

where

(a) 0 is a nondegenerate critical point of index $N + 2[K^{1/2}]$;

(b) $\forall\, i = 1,\ldots,j$ N_i is diffeomorphic to S^1 and is a nondegenerate critical manifold for f ((iii) is the nondegeneracy condition: see [42]).

Now one can apply the Morse inequalities (2.9), taking a coefficent G such that every negative bundle is orientable (this is possible, choosing, for example, $G = \mathbf{Z}_2$), so that $\mathcal{P}_t(N_i;\theta^-\otimes G) = \Sigma_i t^{\lambda(i)} H_{\lambda(i)}(N_i;\theta^-\otimes G) = (1 + t)\, t^{\lambda(i)}$, $\lambda(i)$ being the index of N_i. It is easy to deduce that f must have at least $N + 2[K^{1/2}]$ critical orbits, nondegenerate by (iii), at least one for every index $= N + 2i$, $i = 0, 1, \ldots, N + 2[K^{1/2}] - 1$.

Step 2. Critical points of the perturbed functional.

From step 1, using again, at every critical level (which is isolated) the Morse inequalities 2.12, (which implies that $H_q(f^{c+\varepsilon}, f^{c-\varepsilon}) \neq 0$ for some q) and Theorem 3.7, one finds the critical points of the perturbed functional. We only remark that, whenever at the level c one has more then one critical orbit, one must, in order to prove the desired multiplicity for the critical points of the perturbed functional, use Theorem 2.22.

10.5. Remark. Using Theorem 3.9, one could find two solutions of the perturbed system for each solution of the unperturbed one, but one should require more regularity in ε.

Finally, we come to general Hamiltonian systems. More precisely, we will consider time-dependent perturbations of conves, autonomous Hamiltonian systems. More precisely, consider

$$(H_\varepsilon) \qquad -Jz' = H'(z) + \varepsilon h(t),$$

where $z = (p,q) \in \mathbf{R}^{2N}$, H' is the gradient of H and J is the symplectic matrix defined by $J(p,q) = (-q,p)$. We assume that h is T-periodic, and we look for T-periodic solutions of H_ε when ε is small.

On H we will assume

(10.13) $H \in C^1(\mathbf{R}^{2N},\mathbf{R})$, is strictly convex, $H(y) \geq H(0) = 0$, and $\exists\, \gamma > 0$ and c_3 such that

$$<H'(z),z> \geq \gamma |z| - c_3;$$

(10.14) $\exists\, k, c_4$ such that

$$H(z) = \leq (k/2)|z|^2 + c_4;$$

(10.15) $\liminf_{x\to 0} [H(z)/|z|^2] \geq K/2$ for some $K > k$;

(10.16) esite $g \in L^\infty$ such that $dg/dt = h - \xi_h$ in the sense of distributions.

We remark that (10.16) allows, for instance, h to be a Dirac mass.

To be able to apply Theorem 3.11, we have to use a slight modification of the Dual Action Principle. Let G be the Legendre transform of H, i.e. $G(y) = \sup \{<z,y> - H(z): z \in \mathbf{R}^{2N}\}$. We have that such a G is a well defined convex function, strictly convex ond of class C^1, and that $G(y) \geq G(0) = 0$. We consider the Hilbert space

$$E = \{u \in L^2(0,T;\mathbf{R}^{2N}) : \int u = 0\},$$

and we define a linear, self-adjoint operator $\mathcal{L} \in L(E)$ by

$$\mathcal{L}u = v \quad \text{iff} \quad -Jv' = u,$$

and a functional $f_\varepsilon: E \to \mathbf{R}$ by

$$f_\varepsilon(u) = \int [G(u - \varepsilon\xi_h) - \varepsilon<u,Jg(t)> - (1/2)<u,\mathcal{L}u>].$$

This functional is the sum of a convex term and a quadratic part. While the quadratic part is of class C^∞, the convex term is not C^1, so that we will have to use the subdifferentials.
Suppose $0 \in \partial f_\varepsilon(u)$. Then

(10.17) $G'(u(t) - \varepsilon\xi_h) - \varepsilon Jg(t) - \mathcal{L}u = \xi_\varepsilon$ a.e.,

where $\xi_\varepsilon \in \mathbf{R}^{2N}$ is some constant vector. Inverting G' one gets

(10.18) $\quad u(t) - \varepsilon\xi_h = H'(\mathcal{L}u + \varepsilon Jg(t) + \xi_\varepsilon) \quad$ a.e..

Set now $z = \mathcal{L}u + \varepsilon Jg(t) + \xi_\varepsilon$. Differentiating, we get

(10.19) $\quad -Jz' = u + \varepsilon(h - \xi_h) \quad$ a.e.,

so that equation (10.18) becames

$$-Jz' = H'(z) + \varepsilon h(t).$$

It is well known (see [28]) that, for $2\pi K^{-1} < T < 2\pi k^{-1}$, f identico f_0 has a global minimum $u^* \neq 0$, with

(10.20) $\quad c^* = f(u^*) = \min f < f(0) = 0.$

Using now some a priori estimates ([7, Lemma 3.2 and 3.3]), one can prove that Theorem 3.11 holds, so that

10.6. Theorem. Assume (10.13), (10.14), (10.15), (10.16) and let $2\pi K^{-1} < T < 2\pi k^{-1}$. Moreover suppose c^* (given by (10.20)) is an isolated critical level for f and that there exists $u^* \in Z_{c^*}(f)$ such that the orbit $\{u^*(\bullet+q),\ \theta \in S^1\}$ is isolated in $Z_{c^*}(f)$. Then there exist δ and $\varepsilon^* > 0$ such that, $\forall\ \varepsilon \in\ ]-\varepsilon^*, \varepsilon^*[$, $\varepsilon \neq 0$, the system (H_ε) has at least two T-periodic solutions, whose corresponding critical point lies in $f^{c^*+\delta}$.

Proof. see [7, Theorem 3.4].

We finally describe an application of Theorem 3.9 contained in the paper [7].

10.7. Theorem. Suppose

(10.21) $\quad H \in C^\infty(\mathbf{R}^{2N},\mathbf{R})$ and $\exists\ c_5 > 0$: $\langle H''(z)y,y\rangle \geq c_5|y|^2\ \ \forall\ z,y \in \mathbf{R}^{2N}$,

and let z_0 be a nondegenerate T-periodic solution of (H0). Then there exists $\varepsilon^* > 0$ such that $\forall$ $\varepsilon \in\ ]-\varepsilon^*,\varepsilon^*[$, $\varepsilon \neq 0$, the forced system (H_ε) has at least two T-periodic solutions near $\{z_0(t+\theta),\ \theta \in S^1\}$.

Proof. ([7, Theorem 3.6]). We only remark that, now the perturbation is local (it can be seen as a bifurcation problem from the unperturbed solution - see [7, §4]). This permits to modify the Hamiltonian in such a way to have a strictly convex Hamiltonian which coincide with H in a neighborhood of z_0. To this modified Hamiltonian one applies the Dual Action Principle, finding a functional g_ε, very similar to the functional f_ε used above, to which Theorem 3.9 can be applied.

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