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TRIESTE
Strada Costiera 11

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RENORMALIZATION GROUP TRANSFORMATION AS A MAP IN THE SPACE OF PROBABILITY MEASURES: CONTINUITY AND FIXED POINTS.

Candidate:
Mr. Chai Hok EAB.

Supervisor:
Prof. Gianfausto DELL'ANTONIO.

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TRIESTE

GRUPPO DI RINORMALIZZAZIONE COME INSIEME DI TRASFORMAZIONI SU UNO SPAZIO DI MISURE DI PROBABILITA': CONTINUITA' E PUNTI FISSI

C.H. EAB

RIASSUNTO

Il nostro studio concerne un sistema di spin illimitati su di un reticolo d-dimensionale. Definiamo una proprieta' di compatibilita' asintotica (A.C.) e dimostriamo che ogni famiglia di misure che soddisfa tale proprieta' ha un unico limite proiettivo che e' una misura di probabilita'.

Utilizziamo questa struttura per ridefinire in modo rigoroso il gruppo di rinormalizzazione secondo Kadanoff, e dimostriamo che il suo generatore e' una ben definita applicazione in $\mathfrak{M}(\mathfrak{B})$, lo spazio delle misure di probabilita' su $(X,\mathfrak{B})$. Dimostriamo inoltre che questa applicazione e' continua nella topologia della convergenza debole.

Identifichiamo inoltre, dandone la caratterizzazione, alcuni sottoinsiemi convessi compatti di $\mathfrak{M}(\mathfrak{B})$, invarianti per l'applicazione del gruppo di rinormalizzazione; per questi sottoinsiemi utilizziamo il teorema di punto fisso di Schauder-Tychonoff.

RENORMALIZATION GROUP TRANSFORMATION
AS
A MAP IN THE SPACE OF PROBABILITY MEASURES:
CONTINUITY AND FIXED POINTS.

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ABSTRACT

We study an unbounded spin system on a d-dimensional lattice; it is naturally realized on $X = \mathbb{R}^{\mathbb{Z}^d}$ taken with the product topology. We introduce the concept of Asymptotic Compatibility (AC), and prove that each AC-family of finite-dimensional probability measures has a unique projective limit which is a probability measure on $(X,\mathfrak{B})$

We make use of this structure to redefine rigorously the Kadanoff type Renormalization Group (RG) transformation, and we prove that it is a well defined map on, $\mathfrak{M}(\mathfrak{B})$, the space of all possible probability measures on $(X,\mathfrak{B})$. Moreover, we prove that it is continuous under the weak convergence topology.

We identify and characterize several RG-invariant convex compact sub-set of $\mathfrak{M}(\mathfrak{B})$, for which we can use the Schauder-Tychonoff Fixed Point Theorem. We are able to give some sufficient conditions for the existence of fixed points of any RG-map in $\mathfrak{M}(\mathfrak{B})$.

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0. INTRODUCTION

There is no a priori reason why the critical point properties should be invariant under the Renormalization Group (RG) Transformation; it is however a fact that it does explain the only empirical fact we know namely the Scaling Law [1-14]. It is therefore regarded as a successful technique to obtain the critical point properties, i.e. critical exponenent, critical temperature, etc.. In practice, further approximations and assumptions are made, often without good reason. One may then wonder whether the success is due to the RG-technique or to the approximations.

The number of mathematically rigorous treatments of the RG is somewhat limited [15-38]. Some papers discuss the Hierarchical Models, which had been proposed by Dyson to investigate phase transitions in some Ising-like Modeles in one dimensional lattice [38-42]. The analysis is usually restricted to classical Ising-like Models[21-23], or to scalar fields on Euclidian space with space cut-off [20], e.g. on compact subset of $\mathbb{R}^d$ [28-31], or on a lattice [32-38].

In order to understand the Critical Phenomena, we must investigate the RG-theory rigorously. The out come would be useful as well to the other branches of Physics, to which the RG-technique have been applied [9,12].

There are several ambiguities in the definition of RG-theory; it is differently adapted according to the models, or some approximations are included as part of the theory. Therefore, we take only a special kind to be the subject of our investigation; the so called Real-Space, Kadanoff, Block-Spin, RG-technique[14]. This approach is natural and directly related to the original attemps of Kadanoff [3] to understand the critical point behavior of the Ising model.

It is widely used in the statistical theory of Critical Phenomena on a lattice. We take as the definition the one adapted by Ma in his famous book on Critical Phenomena [14]. We will call it the RG-map induced from Block-Scaling(BS) transformation.

Suggested, and worked out by Wilson, the first RG-theory was applied to investigate critical properties of Ising Model by relating it to the $\lambda\varphi^4$, scalar field on lattice [7,8]. Note that this relation is only approximate, but has fundamentally deep reasons: The details can be found in the work of Langer (1965) [43] on the modified Spherical Model, which was first proposed by Berlin and Kac (1952) [44].

The Ising Model: Brief Historical Review.

The Ising Model is the simplest Statistical Mechanical system on a lattice. It was first proposed by Lenz (1920) as a model of Ferromagnetic systems. The one dimensional problem was dolved by Ising (1925) [45], thus the model has been named after him. Ising found that there is no spontaneous magnetization at non-zero temperature. Then come the celebrated work of Onsagar (1944) [46-48], in which the two dimensional problem was solved:The model does exhibit spontaneous magnetization at non-zero temperature. This convinced the physics community that Statistical Mechanics alone can describe phase transition and critical phenomena; at least in the case of the Ising Model.

Whenever one talks about Ising or Lattice Gas Models, and phase transitions, one has to mention the master piece by Yang and Lee (1952) [49-51]. The general theory was set up to explain the liquid-gas phase transition [50]; it was however motivated [49] and worked out in detail only on the Ising,or the equivalent, Lattice Gas, Models [50].

From then on, the Ising Model became one of the most popular topics of discussion in Statistical Mechanics, and various branches of Physics. It has been generalized in various direction [52-54]: e.g. including more interacting neighbours, assuming a long-range interaction, going beyond the binary form of the Hamiltonian etc.. These are called the Ising-like Models. In some cases a larger state space is associated to each site as e.g. in the Potts Models [54].

Realization of Ising-Like Models.

Here, we will describe some simple aspects of probability theory which are of relevance in all Ising-Like Models. We will clearify certain topological properties, for which the details will be repeated on the continuous spin system. At the end, the theme of this thesis will emerge naturally.

For the details of basic Topology and Probability Theory, we refer to the reference [55-68].

Let (Ω,Σ,μ) be a probability space, and let $\{\xi_m\}_{m\in\mathbb{Z}^d}$ be a collection of random-variables which take value on a set of two elements

$$V = \{u, d\} \quad . \tag{1}$$

Associated with V, a discrete topology

$$\mathfrak{D}^1 = \{\emptyset, u, d, V\} \tag{2}$$

is given.

For each M, a finite sub-set of $\mathbb{Z}^d$, we define a product space

$$V^M = \otimes_{m \in M} V \tag{3}$$

The topology $\mathfrak{D}^M$ is the product topology induced from $\mathfrak{D}^1$. It is trivial that $\mathfrak{D}^M$ is the dicrete topology on V^M.

The σ-field generated from the given collection of all open sets in the discrete topology is equal to the collection itself. Therefore, the $(V^M, \mathfrak{D}^M)$ is a measurable space.

The Ising-Like Models is defined through the joint distribution on each given M: For each given $A \in \mathfrak{D}^M$,

$$\mu\{\omega \in \Omega \mid \{\xi_m(\omega)\}_{m \in M} \in A\} = \frac{\sum_{v \in A} \exp\{-\mathcal{H}^M(v)\}}{\sum_{v \in V^M} \exp\{-\mathcal{H}^M(v)\}} , \tag{4}$$

where $\mathcal{H}^M : V^M \rightarrow \mathbb{R}$, is a pre-assigned function bounded below.

Usually, we give $$\mathcal{H}^M = H^M \circ s , \tag{5}$$

where "s" is a homeomorphism from $V^M \rightarrow I^M$,

$$I = \{+1, -1\} , \tag{6}$$

and $H^M : I^M \rightarrow \mathbb{R}$ is bounded below.

The Ising Model is a particular case: For each $\eta \in I^M$,

$$H^M(\eta) = -\frac{J}{kT} \sum'_{m,n \in M} \eta_m \eta_n , \tag{7}$$

where $\sum'_{m,n\in M}$ denote the nearest neighbours double summation, J is the pair interaction; Ferromagnetism $(J > 0)$, Anti-Ferromagnetism $(J < 0)$, $|J| < \infty$.

There is an algebraic structure in V, but we shall not discuss it here. For the details of the algebraic study of the Ising Model see Dubin (1974) [69] p89-106.

It is not difficult to verify that the probability space in which the Ising Model should be described can be realized as $(I^{\mathbb{Z}^d}, \mathfrak{D})$, where $\mathfrak{D}$ is the product topology induced from $\mathfrak{D}^1$, and $\sigma(\mathfrak{D}) = \mathfrak{D}$. Note that this topology on $I^{\mathbb{Z}^d}$ is not the discrete one. Since any topological space which has a finite number of open set is compact, we have that $(I, \mathfrak{D}^1)$ be compact. By Tychonoff Theorem, we conclude that also $(I^{\mathbb{Z}^d}, \mathfrak{D})$ is compact. Obviously, it is complete, separable metrizable as $(I, \mathfrak{D}^1)$ is.
Let $\mathfrak{M}(\mathfrak{D})$ denote the collection of all possible probability measures defined on $(I^{\mathbb{Z}^d}, \mathfrak{D})$. We have from Pathasarathy (1967) |70| p45, theorem 6.4, that $\mathfrak{M}(\mathfrak{D})$ with the weak convergence topology is also compact and separable
In deed, any transformation from $I^M \rightarrow I^{M'}$ is continuous, thus measurable. Therefore any definition of BS-transform from the finite dimensional space I^M into I^{M_1} induce a transformation of a measure on I^M to a measure on I^{M_1}, which is a finite dimensional RG-map. A non-trivail problem is the uniqueness of the map. If one can resolve this, it is obvious that the RG-map can be defined as a tranformation from $\mathfrak{M}(\mathfrak{D})$ into itself. The RG-map must be proved to be continuous in the weak topology in $\mathfrak{M}(\mathfrak{D})$, which is non-trivail.

Since $\mathfrak{M}(\mathfrak{D})$ is compact convex, and can be embeded into a locally convex linear topological space, then $\mathfrak{M}(\mathfrak{D})$ has the fixed point property by the Schauder-Tychonoff Theorem [57]. Moreover the RG-map is continuous, and therefore there exist at least a measure in $\mathfrak{M}(\mathfrak{D})$ which is a fixed point of the RG-map.

For technically details about the definition and uniqueness of the BS-transformation of Ising-Like Models, one can refer to the review by Niemeijer and van Leeuwen [74].

Unbounded Continuous Spin Systems.

We are not going to investigate the Ising-Like Models directly. We are more interested in studying the unbounded continuous spin, or scalar field on lattice. While this theory is less trivial than the Ising-Like Models, it presents some simplifying structures. The system is purely classical; on each site, the state space is the real line $\mathbb{R}$.

One can proceed as we did before for the Ising-Like Models, and one ens up with the realization of the probability space as $X = \mathbb{R}^{\mathbb{Z}^d}$ where the σ-field $\mathfrak{B}$ is generated by all open sets in the product topology $\mathcal{T}$. Here, $\mathbb{R}$ is given the usual topology.

Before going on deeper, we shall briefly discuss the Langer-Wilson approximation of Ising Model through a $c\eta^4$ scalar field on the lattice [7,8,43].

We shall assume that the Hamiltonian be of the same form as given in (7), the Ising Model, but η_m take value in $\mathbb{R}$ for each $m \in M$. Langer has shown that the system has a different behavior from Ising Model [43]. If we added to H^M (now called H_o^M) by a polynomial $P^M(\eta)$ of the form

$$P^M(\eta) \quad = \quad c\,\Sigma_{m\in M}(\eta_m^2 - 1)^2 - c|M| \qquad (8)$$

then the system behaves when $c \to \infty$ as an approximation of the Ising

Model. Here one could see that the continuous spin or scalar field could give certain feature of the Ising Model. Note that the argument of both Langer and Wilson is not rigorous.

If we rearrange (7) properly, and then added (8), we get

$$H^M(\eta) = J\sum_{m,n\in M}\eta_m(-\Delta|_M)_{mn}\eta_n + 2(c - dJ)\sum_{m\in M}\eta_m^2 + c\sum_{m\in M}\eta_m^4 \quad (9)$$

where Δ is the lattice Laplacian, and $\Delta|_M$ denote its restriction on X^M. It is a special $c\eta^4$ scalar field on the lattice, $c > 0$.

Here on, we will consider only the unbounded continuous spin or scalar field on lattice $\mathbb{Z}^d$. We may occasionally refer to the system modeling by

$$H^M(\eta) = H_o^M(\eta) + P^M(\eta) , \quad (10)$$

$$H_o^M(\eta) = a|M| + h\sum_{m\in M}\eta_m + \sum_{m,n\in M}\{b_o + b'(-\Delta|_M)\}_{mn}\eta_m\eta_n , \quad (11)$$

where $P^M(\eta)$ is general polynomial of degree higher than two; in particular one could have

$$P^M(\eta) = \lambda\sum_{m\ M}\eta_m^4 . \quad (12)$$

No matter how the models are constructed, they should account for the fact that the magnetization can take values from $-\infty$ to $+\infty$. The $\lambda\eta^4$ unbounded continuous spin model is as good as the Ising model to be a model of Ferromagnetism. Moreover, the scalar field theory on a lattice, which is just a synonym for the unbounded continuous spin on lattice, is very interesting by itself. We are not going to discuss this point but refer to a review paper by Guerra et.al. (1975) [75].

The realization on $\mathbb{R}^{\mathbb{Z}^d}$ has several advantages as well as disadvantages in comparing to $I^{\mathbb{Z}^d}$.

On $\mathbb{R}^{\mathbb{Z}^d}$, the BS-transformation can be adapted from the unique formal form in literature, while it has to be defined differently for the Ising-Like Models. This is due to the fact that the block average of spins can take values other than ± 1 [74].

We have the most developed theory of Gaussian measures on $\mathbb{R}^{\mathbb{N}}$, which can be straight forwardly extended to our case.

On the other hand, the root of several difficulty on $\mathbb{R}^{\mathbb{Z}^d}$ lies in the fact that the topology on $\mathbb{R}$ is less trivial. The proof of continuity needs more effort. The fixed point theorem on $\mathfrak{M}(\mathfrak{B})$ is much harder to prove.

Scope of the Problem of Mathematical Rigorous RG.

Here, we would classify the problems related to the RG-technique into the following step:

(i) The definition of the RG-map.
(ii) The topological properties of the map.
(iii) The existence of fixed points,
  (a) Sufficient condition,
  (b) Neccessary condition.
(iv) The stability problem.

We have already indicated how to tackle (i). We will study (ii) with special emphasis on the continuity as it will play an important role in step (iii). Some strong sufficient condition for the existance of fixed points is given. We have discuss (iiib) and (iv) only briefly.

In fact most of the work in the literature, either rigorous or non-rigorous, concentrates on discussing problem (iv) by assume (i) (ii) and (iii). In this way, we provide the background for a rigorous study on the stability problems.

Outline and Main Results.

We aim at a rigorous axiomatic treatment of the Renormalization Group (RG) technique in Critical Phenomena, having particularly in mind the case of Unbounded Continuous Spin, or the Scalar Field on a d-dimensional lattice. The theory is naturally realized on a product space $X = \mathbb{R}^{\mathbb{Z}^d}$ with the product topology $\mathcal{T}$. From the open sets in $\mathcal{T}$, we generate a σ-field $\mathfrak{B}$, thus $(X, \mathfrak{B})$ is a measurable space. The fact that $(X, \mathcal{T})$ is a complete, separable metric space is pointed out and used throughly. This space does not have only nice properties but also some bad ones; e.g. it is not locally compact. Their cure will require some work.

We also point out that the Statistical Mechanics Formalism, in which ussually the measure is given through a potential, does not in general satisfies the compatibility (C) condition of the Komolgorov Consistency Theorem. This forces us to invent and use a weaker condition. We, therefore, introduce an asymptotic compatibility (AC) condition. Indeed, we are able to prove that to any AC-family of finite-dimensional (fd) probability measures one can associate a unique C-family which is asymptotically coincident. In other words each AC-family has a unique projective limit as a measure on $(X, \mathfrak{B})$. This result will allow us to go back and forth between the infinite-dimensional space X, and its fd-sub-spaces $\{X^M\}_{M \in \underline{C}}$

Starting from the conventional Kadanoff type RG-transformation, we rigorously define the Block-Scaling (BS) transformation as a map from each fd-sub-space to a fd-sub-space. Consequently, we prove that it is a measurable map, so we can induce a RG-transformation as a map on fd-measures. The collection of such renormalized measures are proved to be AC-families if the given , initial, family of fd-measures is AC. Eventhough, there are many renormalized families, we have proved that all have the same, and unique, projective limit. We can then define the RG-transform as a map of $\mathfrak{M}(\mathfrak{B})$, the space of all probability measures on $(X, \mathfrak{B})$, into itself. Moreover, we discuss the non-homogeneous scaling case, and point out that it can also well defined if the scaling factors form a Cauchy net.

On $\mathfrak{M}(\mathfrak{B})$, there is a natural topological structure induced from weak convergence. We have clearified that, in our case, weak convergence is well defined and usefull. It turns out that $\mathfrak{M}(\mathfrak{B})$ with the weak convergence topology is a complete, separable metric space. Some bad features of $(X, \mathfrak{T})$ still carry over to $\mathfrak{M}(\mathfrak{B})$, and give us non-compactness of $\mathfrak{M}(\mathfrak{B})$.

The RG-maps, which have been defined on $\mathfrak{M}(\mathfrak{B})$, are then proved to be continuous on $\mathfrak{M}(\mathfrak{B})$ under the weak convergence topology.

The formulae relating characteristics, N-moments, and N-semivariance of the renormalized measures to the ones of the initial, are derived, and repeatedly used through the rest of this thesis.

We classify the space $\mathfrak{M}(\mathfrak{B})$ into classes, related to susceptibility divegence, according to the long distance decay rate of 2-semivariance.

We give a full proof of the fact, formally well known, that the homogeneous scaling RG-map leaves the class (∞,α) invariant for each $\alpha > 0$. We point out that a necessary condition for having a fixed point in the class is to have the scaling factor $\lambda = L^{\alpha/2}$. We also show that a fixed point of a RG-map need not correspond to a critical point state, where the susceptibility diverges, and vice versa. The class (∞,d) is an exceptional case that must be further refined; it is not discussed in this thesis. The class (∞,α), when $0 < \alpha < d$, is RG-invariant, and has a divergent absolute susceptibility. We give an example in which the susceptibility diverges, but the 2-semivariant has a long distance decay faster than any exponential law. So, we conjecture that the critical point, which appears in some Ferromagnetic systems, should not be of this kind.

If the scaling factor of a RG-map is $\lambda = L^{d/2}$, then each Gaussian White Noise (GWN) measure with mean zero is invariant under the RG-map; i.e. a fixed point. There is an infinites of such GWN-measures. For any scaling factor, also some Dirac Delta (DD) measures are fixed points. The existence of these two trivial classes of measures brings up a non-trivial problem, namely how to rule them out. This can be regarded as a stability problem; it is not discussed in this thesis. We expect that its solution will come through a study of properties related to their support.

In order to make use of the Schauder-Tychonoff Fixed Point Theorem, we have shown that the space $\mathcal{M}(\mathcal{B})$ can be embeded in the space of signed measures $\mathcal{M}^{(\pm)}(\mathcal{B})$. We prove that the weak convergence topology on $\mathcal{M}^{(\pm)}(\mathcal{B})$ is a locally convex topology. Hence, $\mathcal{M}^{(\pm)}(\mathcal{B})$ is proved to be a locally compact linear topological space. To apply the theorem we need a convex compact sub-set of $\mathcal{M}^{(\pm)}(\mathcal{B})$ which is RG-invariant.

We are able to give two types of convex compact sub-sets of $\mathcal{M}(\mathcal{B}) \subset \mathcal{M}^{(+)}_{-}(\mathcal{B})$: The first type are the closure of the convex extension of $\Gamma = \{\tau^{k-1}\mu\}_{k \in \mathbb{N}}$, where μ satisfies the condition that the finite volume (site) susceptibility (per unit volume) must converge if an extra fractional power $0 < \beta < 1$ of volume is added, and that the scaling factor of the RG-map τ be $\lambda = L^{(1-\beta)d/2}$. this criteria seem consistent with the previous analysis on power law. We would remark that this condition requires nothing on the convergence of sequence Γ: We allow the sequence to be periodic, etc.. The second type of convex compact sub-sets are related to the concept of Sub-Gaussian measures: $S_{1,\nu}$, $S_{\varkappa,\nu}$, $S_{\varkappa^*,\nu}$. We prove that $S_{\varkappa^*,\nu}$ is a compact convex sub-set of $\mathcal{M}(\mathcal{B})$. It can be invariant under a RG-map if $\tau\nu \in S_{1,\nu}$. The $S_{1,\nu}$ is the central core of the set $S_{\varkappa^*,\nu}$; $S_{1,\nu} \subset S_{\varkappa,\nu} \subset S_{\varkappa^*,\nu}$. Moreover, we give some example showing that in $S_{\varkappa^*,\nu}$ there is some measure which is not a Gaussian, nor DD-measure, nor a convex combination of the two. Especially, a physical interesting of Super-Stable and Lower-Regular Gibbs measures is verified to be in $S_{\varkappa^*,\nu}$ for some $\varkappa > 0$, and some GWN-measure ν.

From the continuity of RG-maps, and the Schauder-Tychonoff Theorem, we ensure that there exists a fixed point $\tau\mu^* = \mu^*$, in each set $\overline{co(\Gamma)}$ and $S_{\varkappa^*,\nu}$ if the conditions described above are satisfied.

Indeed, we have given a solution to problems (i) (ii) and (iiia). The further step on stability of each fixed point can be now be tackled using the results contained in this thesis. We have laid a firm ground for any study about Unbounded Continuous Spin, or Scalar Field on lattice. We have studied so far refers to theories on a lattice, and does not apply therefore directly to the study of continuum case. Any way, we expect that many of our results can be easily extended to the continuum case.

I. THE FUNCTION SPACE $X = \mathbb{R}^{\mathbb{Z}^d}$.

In this chapter, we will study some properties of the space $X = \mathbb{R}^{\mathbb{Z}^d}$. It is a summary since most of the results are well known [55-62]. Our aim is to provide homogeneous notations, while emphasizing only the points we need for our further discussion. The results in standard text books are not in the form we need: e.g. Most of them discussed $\mathbb{R}^{\mathbb{N}}$, but rarely $\mathbb{R}^{\mathbb{Z}}$. What we want is $\mathbb{R}^{\mathbb{Z}^d}$ which is slightly more complicated in practice.

To be definite, we consider only the usual topology on $\mathbb{R}$, denote by $(\mathbb{R}, \mathcal{U})$ or simply $\mathbb{R}$. The space $\mathbb{R}$ has algebraic structure as a field under the usual addition "+" and multiplication ".".

For any given $d \in \mathbb{N}$, we define

$$\mathbb{Z}^d = \{m = (m_1, m_1, \ldots, m_d) \mid m_j \in \mathbb{Z} \text{ for all } j = 1,2,\ldots,d\} \quad (1)$$

and call it the d-dimensional hyper-cubic lattice, or simply d-lattice. It serves us, most of the time, as an index-set, when no topology is given.

On $\mathbb{Z}^d$, there are functions which take value in $\mathbb{R}$. The totality of such functions is denoted by

$$X = \{x : \mathbb{Z}^d \to \mathbb{R}\} \quad (2)$$

Define addition "+", and multiplication "." on X as following: for each $x,y \in X$ and $\alpha, \beta \in \mathbb{R}$, the $\alpha x + \beta y$ means $\alpha[x(m)] + \beta[y(m)]$ for all $m \in \mathbb{Z}^d$. X become then a linear space over the field $\mathbb{R}$.

Obviously, each $x \in X$ is uniquely determined by the collection of real numbers $\{x(m)\}_{m\in\mathbb{Z}^d}$.

For each $n \in \mathbb{Z}^d$, we define an element $e_n \in X$:

$$e_n(m) \quad = \quad \delta_{mn} \quad , \tag{3}$$

where $\delta_{..}$ is the Kronecker delta function. The collection of all these elements $\{e_n\}_{n\in\mathbb{Z}^d}$ is a basis of the linear space X: i.e. For each $x \in X$, we can find a collection of real number $\{x_n\}_{n\in Z^d}$ such that

$$x \quad = \quad \sum_{n\in Z^d} e_n x_n \quad , \tag{4}$$

in fact

$$x_n \quad = \quad x(n) \tag{5}$$

We call $\{e_n\}_{n\in\mathbb{Z}^d}$ the natural basis of X.

Clearly, the function space X is just an infinite cartesian products:

$$X \quad = \quad \bigotimes_{m\in\mathbb{Z}^d} X^m \tag{6}$$

where

$$X^m \quad = \quad \mathbb{R} \quad , \text{ for all } m \in \mathbb{Z}^d. \tag{7}$$

D1 : Let us define, for each $m \in \mathbb{Z}^d$, the projection on the m-coorodinate, a function $p_m : X \rightarrow \mathbb{R}$ as following: For each $x \in X$, $p_m x = x_m$.

D2 : The product topology $\mathcal{T}$ on the X is defined to be the smallest topology such that p_m is continuous for each $m \in \mathbb{Z}^d$.

It can be proved that there is a unique such topology on X. This topology is equivalent to the topology of pointwise convergence: i.e.

For any sequence $\{x^k\}_{k\in\mathbb{N}}$ in X, it converges to an element $x \in X$ iff, for each $m \in Z^d$, $\{x_m^k\}_{k\in\mathbb{N}}$ converges to $x_m \in \mathbb{R}$.

For convenience, we define the set of all finite sub-sets of Z^d:

$$\underline{C} \quad = \quad \{M \in \mathbb{Z}^d | \ |M| < \infty\} \quad , \qquad (8)$$

and the set of all those finite sub-sets which contain a given finite set $\Lambda \in \underline{C}$:

$$\underline{C}(\Lambda) \quad = \quad \{M \in \underline{C} | \ M \supset \Lambda\} \quad . \qquad (9)$$

It can be easily checked that $\underline{C}$, and $\underline{C}(\Lambda)$ are directed sets, where the order is the set inclusion "$\subset$": For any given pair of elements M_1 and M_2 belong to $\underline{C}$ ($\underline{C}(\Lambda)$) there is an element M_3 belong to $\underline{C}$ ($\underline{C}(\Lambda)$) such that $M_1 \subset M_3$ and $M_2 \subset M_3$.

The finite dimensional product space $X^M = \mathbb{R}^M$, $M \in \underline{C}$, can be understood as $\mathbb{R}^{|M|}$, when an ordering is given on M. The product topology as similar to D2 can be given, and denote by $\mathcal{T}^M$.

The space X^M can be embedded into X by fixing an element $x^\circ \in X$:

$$X^M \sim \{x \in X | \ p_m x^\circ = x_m^\circ, \text{ for all } m \in M^c\}, \quad (10)$$

where $M^c = \mathbb{Z}^d \setminus M$; the complement of M. This embedding plays a crucial role in Statistical Mechanics on lattices as a boundary condition. For the sake of simplicity, we fix $x^\circ = 0 \in X$. From now on, it is understood that X^M, $\mathbb{R}^M$, $\mathbb{R}^{|M|}$, or the embedding (10) are synonims.

D3 : For each given $\Lambda \subset M \in \underline{C}$, we define the projection $p_{M\Lambda}: X^M \rightarrow X^\Lambda$ as following: For each $x \in X^M$

$$p_m\{p_{M\Lambda}x\} = \begin{cases} x_m & , \text{ for all } m \in \Lambda \\ 0 & , \text{ for all } m \in M \end{cases} . \qquad (11)$$

A similar definition can be introduced when M is replaced by $\mathbb{Z}^d$, the corresponding projection is denoted by $p_\Lambda : X \rightarrow X^\Lambda$.

P4 : p_m, $p_{M\Lambda}$, p_Λ are continuous and open on their domain of definition.

Whenever Λ is a singleton, we simply denote it by its element if there is no ambiguity: e.g. We write m for $\{m\}$.

From D2, it can be verified that

$$\mathcal{V}_o^M = \left\{ \left\{ p_{Mm}^{-1} A^m \right\}_{A^m \in \mathcal{U}} \right\}_{m \in M} \qquad ; \qquad (12)$$

is a sub-base, and

$$\mathcal{V}^M = \left\{ \left\{ p_{M\Lambda}^{-1} A^\Lambda \right\}_{A^\Lambda \in \mathcal{T}^\Lambda} \right\}_{\Lambda \subset M} \qquad (13)$$

is a base of $\mathcal{T}^M$. A sub-base of $\mathcal{T}$ is similar to (12) by replacing M by $\mathbb{Z}^d$, but the base is not so obvious:

$$\mathcal{V} = \left\{ \left\{ p_\Lambda^{-1} A^\Lambda \right\}_{A^\Lambda \in \mathcal{T}^\Lambda} \right\}_{\Lambda \in \underline{C}} \qquad ; \qquad (14)$$

For each member of the base $\mathcal{V}$, at all but a finite number of $m \in \mathbb{Z}^d$ the projection is either on $\mathbb{R}$ or on the empty set : e.g.

$$p_\Lambda^{-1} A^\Lambda = \ldots\ldots \boxtimes \mathbb{R} \boxtimes \mathbb{R} \boxtimes \ldots\ldots \boxtimes A^{m_1} \boxtimes A^{m_2} \boxtimes \ldots\ldots \boxtimes A^{m_{|\Lambda|}} \boxtimes \mathbb{R} \boxtimes \ldots\ldots \boxtimes \mathbb{R} \boxtimes \ldots\ldots \qquad (15)$$

From $\mathbb{R}$ to X.

Let us recall some well known topological properties of R:

P5 : $\mathbb{R}$ is a separable, complete metric space.

Since every metric space is regular, and separable is equivalent to second countable, we have

P6 : $\mathbb{R}$ is regular and second countable.

Since a countable product space is regular (respectively, second countable) iff each coordinate space is regular (respectively second countable), we have

P7 : $(X, \mathcal{T})$ is regular and second coutable.

We know from the Urysohn metrization theorem that

P8 : Every regular, second countable space is metrizable.

We conclude that $(X, \mathcal{T})$ is metrizable. Again, second countable is equivalent to separable as $(X, \mathcal{T})$ is a metric space.

The completeness of $(X, \mathcal{T})$ is obvious if we use the pointwise convergence topology.

P9 : $(X, \mathcal{T})$ is a separable, complete metric space.

This result plays an important role in this thesis; as we will see later, many results depend on it.

Define $\delta : X \boxtimes X \to \mathbb{R}^+$ as following: For each $x,y \in X$

$$\delta(x,y) = \sum_{m \in \mathbb{Z}^d} \gamma^{-|m|} \frac{d(x_m, y_m)}{1 + d(x_m, y_m)} , \quad \gamma > 2d \qquad (16)$$

where $d(.,.)$ is the distance between two points on $\mathbb{R}$. The $|.|: \mathbb{Z}^d \to \mathbb{R}^+$, defined for each $m \in \mathbb{Z}^d$

$$|m| = \sum_{j=1}^{d} |m_j| \quad . \qquad (17)$$

C10: δ is a metric on X which induce a topology equivalent to $\mathcal{T}$.

Technically, we need to know how many elements are on

$$S_{\ell}^{d,1} = \{ m \in Z^d | \ |m| = \ell \}. \qquad (18)$$

L11: $$|S_{\ell}^{d,1}| = \sum_{j=1}^{\min(d,\ell)} 2^j \frac{d!}{(d-j)!j!} \cdot \frac{(\ell - 1)!}{(\ell - j)!(j-1)!} , \qquad (19)$$

for all $\ell \geq 1$, and $d \geq 1$.

L12: $$|S_{\ell}^{d,1}| \leq (2\ell + 1)^d - (2\ell - 1)^d = S_{\ell}^{d,\infty} \qquad (20)$$
for all $\ell \geq 1$, and $d \geq 1$.

L13: $$\sup_{\ell \in N} |S_{\ell}^{d,1}|^{\frac{1}{\ell}} = 2d, \text{ for all } d \geq 1 \qquad (21)$$

L11-13 will be useful for the discussion in chapter V. The proofs of C10, L11-13 are given in appendix A.

The linear structure of X is compatible with the topology as "+": $X \boxtimes X \to X$, ".": $\mathbb{R} \boxtimes X \to X$ are continuous. Therefore, we conclude that $(X, \mathcal{T})$ is a topological linear space.

The space $(X, \mathcal{T})$ has not only the nice properties mentioned , but also a bad topological property which makes the analysis more difficult.

While $\mathbb{R}$ is locally compact, and so is X^M for each $M \in \underline{C}$, X does not has this property.

It is worth to mention the Tychonoff theorem:

P14: A product space is compact iff each coordinate space is compact.

The attemp to give a similar theory on locally compact leads to a theorem which show a negative answer:

P15: A product is locally compact iff all but a finite number of coordinate spaces are compact, while the remaining ones are locally compact.

If we make the negation of $\Leftarrow$ in P15, we have

P16: $(X, \mathcal{T})$ is not locally compact.

We could now conclude from P9, P16, and linear properties that the space $(X, \mathcal{T})$ is a separable, complete, linear, metric space, but not locally compact.

The Dual of X.

Let X_* be the set of all elements in X which have finite support:

$$X_* = \{x \in X \mid \text{there exist some } M \in \underline{C}, \text{ such that } x_m = 0, \text{ for all } m \in M^c\} = \bigcup_{M \in \underline{C}} X^M \qquad (22)$$

The topology $\mathcal{T}_*$ on X_* is given by relativization.

Let us study the algebraic dual of X, denote by X*. It is a set of all continuous linear function from X into $\mathbb{R}$. It has been proved by Vakhania (1981) [76] p3 that the dual of $\mathbb{R}^{\mathbb{N}}$ is just $R_*^{\mathbb{N}}$. It can be proved by a similar argument that, as a set

$$X^* \quad = \quad X_* \text{ (up to an isometry)} \qquad (23)$$

The topology on the X* , $\mathcal{T}^*$, is defined by giving as the local base the members of the folowing form: For any given g $\in$ X*, $\varepsilon > 0$, and a compact set K in X,

$$U_{\varepsilon,K}(g) = \left\{ f \in X^* \mid \sup_{x \in K} |f(x) - g(x)| < \varepsilon \right\} \qquad (24)$$

is a member of a local base at g. It is the topology of uniformly convergence on a compact set.

The topology $\mathcal{T}^*$ on the dual X* is weaker than the topology $\mathcal{T}_*$ has been given on X_*.

Infinite Dimensional, Banach Sub-Space of X.

Beside the finite dimensional sub-space X^M, and the infinite dimensional with finite support X_*, there are several interesting sub-spaces of X. We mention here a special sub-space that is Banach space.

Let $||.||_p : X — \bar{\mathbb{R}}$ (the extended real numbers) be defined for each x $\in$ X by

$$||x||_p^p = \sum_{m \in \mathbb{Z}^d} |x_m|^p \tag{25}$$

Hence
$$\ell_d^p = \{x \in X|\ ||x||_p < \infty\} \tag{26}$$

It can be easily checked that $||.||_p$ is a norm on ℓ_d^p, and that ℓ_d^p is complete. Therefore ℓ_d^p is a Banach space. An intuitive picture of any $x \in \ell_d^p$ is that its component must decay fast eneough outside some $M \in \underline{C}$, so that the $||x||_p < \infty$.

Another Banach sub-space is ℓ_d^∞, which is defined by a norm

$$||x||_\infty = \sup_{m \in \mathbb{Z}^d} |x_m| \tag{27}$$

An important typical element of ℓ_d^∞ is $\mathbb{1}$ (defined by $\mathbb{1}_n = 1$ for all $n \in \mathbb{Z}^d$); $\mathbb{1}$ does not belong to any ℓ_d^p , $0 < p < \infty$.

Among the Banach space ℓ_d^p, the most important is ℓ_d^2: it is an Hilbert space. Note that our $\ell_{d=1}^2$ is different from the usual ℓ^2, a sub-space of sequences, which is a sub-space of $\mathbb{R}^{\mathbb{N}}$.

II. MEASURES ON $(X, \mathcal{B})$.

Let $\mathcal{B}$ be a σ-field generated by all open sets in the product topology $\mathcal{T}$. We obtain a measurable space $(X, \mathcal{B})$. On $\mathcal{B}$, a measure can be defined as a set function with some special properties: the μ is a measure on $(X, \mathcal{B})$ iff

(i) $\mu(A) \geq 0$, for all A (1)

(ii) $\mu(\emptyset) = 0$, and $\mu(X) < \infty$ (2)

(iii) For any given countable number of disjoint elements of $\mathcal{B}$, $\{B_j\}_{j \in J}$, we have

$$\mu(\bigcup_{j \in J} B_j) = \sum_{j \in J} \mu(B_j) \qquad (3)$$

For any further details of measures theory, we refer to Royden (1968) [56], and Halmos (1978) [77].

There exist a measure such that $\mu(X) = 0$. It is unique with this property, and is denoted by "0"; it is such that $0(A) = 0$ for all $A \in \mathcal{B}$.

Probability measures, and Kolmogorov Consistency Theorem.

Any measure for which $\bar{\mu}(X) > 0$ can be normalized. The normalized measure

$$\mu(.) = \frac{\bar{\mu}(.)}{\bar{\mu}(X)} \qquad (4)$$

is then a probability measure: $\mu(X) = 1$.

When there is no ambiguity, we understand by measure a probability measure.

Similarly, for each $M \in \underline{C}$, we can introduce a measure μ^M on the measurable space $(X^M, \mathcal{B}^M)$. A collection of such measures, $\{\mu^M\}_{M \in \underline{C}}$ is called a family of finite dimensional (fd) measures.

Obviously, for each measure μ on $(X, \mathcal{B})$, we can define a family of fd-measures $\{\mu^M\}_{M \in \underline{C}}$ by defining for each $M \in \underline{C}$,

$$\mu^M = \mu \circ p_M^{-1} \qquad . \qquad (5)$$

One can easily check, for each given $\Lambda \in \underline{C}$, that

$$\mu^M \circ p_{M\Lambda}^{-1} = \mu^{M'} \circ p_{M'}^{-1} \qquad . \qquad (6)$$

for all M, $M' \in \underline{C}(\ )$.

The converse is not obvious, the answer is known as Kolmogorov Consistency Theorem [67,68,72,78]. The theorem is one of the most important theorems in probability theory, related to the existence of an infinite number of independent random variable; it is at the basis of the theory of integration on infinite dimensional spaces [73].

The theorem we are going to state is a little bit different from the version in the texts. The invariance under permutation of the coordinates is assumed here by the definition of product space.

D1 : Let $\{\mu^M\}_{M \in \underline{C}}$ be a family of fd-measures. We say that it is compatible $\overline{(C)}$, if it is coincedent on any given $\Lambda \in \underline{C}$: i.e. for all M_1, $M_2 \in \underline{C}(\Lambda)$, we have

$$\mu^{M_1} \circ p^{-1}_{M_1 \Lambda}(A) \quad = \mu^{M_2} \circ p^{-1}_{M_2 \Lambda}(A)$$

for all $A \in \mathcal{B}^{\Lambda}$.

P2 : (Kolmogorov Consistency Theorem) For any C-family of fd-probability-measures, there exist a unique probability measure on $(X,\mathcal{B})$, such that

$$\mu \circ p_M^{-1}(A) \quad = \mu^M(A) \qquad (8)$$

for each $M \in \underline{C}$, for all $A \in \mathcal{B}^M$.

The proof can be found in Billingsley (1979) [67] p433, theorem 36.1, Shiryayev (1984) [68] p244, theorem 1, Varadhan (1968) [72] p5, theorem 1, or the less formal, know better to physicist, discuss by Reed (1973) [78] p12.

The C-condition of the theorem is too strong. It is not satisfied by most of the physical interesting families of measures in the usual approach to Statistical Mechanics.

Statistical Mechanics v.s. Compatibility.

Let us consider a Canonical Ensemble on a finite sub-set M: It is surrounded by a heat bath at temperature T, and exposed to an homogeneous external field h. We can defined a measure

$$\bar{\mu}^M \quad = \quad e^{-\beta\{U^M(x) \; - \; h\sum_{m \in M} x_m\}} \omega^M(x) \quad , \quad (9)$$

where $\beta = 1/kT$, ω^M is the Lebesgue measure on $(X^M, \mathcal{B}^M)$. Here, U^M: $X^M \rightarrow R$ is a given potential: We requir at this level that U^M be such that the Boltzmann factor is integrable, and gives $0 < \bar{\mu}^M(X^M) < \infty$ for all $\beta > 0$, and $h \geq 0$. The Canonical Partition Function is just the total mass of the measure:

$$Z^M(\beta,h) \quad = \quad \bar{\mu}^M(X^M) \qquad . \qquad (10)$$

As in (4), we define the normalized measure μ^M, for each $M \in \underline{C}$, and get a family of fd-probability-measures $\{\mu^M\}_{M \in \underline{C}}$.

The picture of the problem is clearer if we concern a Gaussian measure [79]. Let S be an infinite matrix,here index by $(m,n) \in \mathbb{Z}^d \otimes \mathbb{Z}^d$, symmetric, positive-definite. It defines a unique Gaussian measure μ with mean zero, and covariance S.

The restriction of μ on X^M, $M \in \underline{C}$ forms a C-family $\{\mu \circ p_M^{-1}\}_{M \in \underline{C}}$ with the covariance $\{S|_M\}_{M \in \underline{C}}$. We can write explicitly the potential

$$U^M(x) \quad = \quad \sum_{m,n \in M} x_m \{(S|_M)^{-1}\}_{m,n} x_n \qquad . \qquad (11)$$

In case S is invertible, we can also define another potential

$$U'^M(x) \quad = \quad \sum_{m,n \in M} x_m \{S^{-1}\}_{mn} x_n \qquad . \qquad (12)$$

This potential defined a family of fd-measures $\{\mu'^M\}_{M \in \underline{C}}$ as given in (9), but it may not be a C-family: We know that $(S|_M)(S|_M)^{-1} = I|_M$, but $(S|_M)(S^{-1}|_M)$ may not be $I|_M$. Still, the $\{\mu'^M\}_{M \in \underline{C}}$ may have the original Gaussian measure μ as the limit: $(S^{-1}|_M)(S|_M)$ may converge to I as M increases to cover $\mathbb{Z}^d$ in certain sense.

Some cases have been explicitly studied by Guerra et.al. (1975) [75]. In this reference, theorem IV.7, p197, confirms of our vague argument.

The massive scalar free field on lattice $\mathbb{Z}^2$ was investigated, and one could see that the C-condition is not satisfied.

We should moreover expect that the RG-map may transform a C-family to a non-C-family.

These two reasons force us to invent a weaker condition than the compatibility, but strong eneough to garantee existence and uniqueness of a measure on $(X, \mathcal{B})$.

Asymptotic Compatibility.

D3 : Let $\{\mu^M\}_{M \in \underline{C}}$ be a family of fd-measures. We say that it is asymptotically compatible (AC), if it is asymptotically coincident on any given $\Lambda \in \underline{C}$: i.e. for any $\varepsilon > 0$, there is an $M_o \in \underline{C}(\Lambda)$, such that for all M_1, $M_2 \in \underline{C}(M_o)$, we have

$$|\mu^{M_1} \circ p_{M_1}^{-1}(A) - \mu^{M_2} \circ p_{M_2}^{-1}(A)| < \varepsilon \qquad (13)$$

for each $A \in \mathcal{B}^\Lambda$.

This definition, intuitively, tells us that, eventhough the family $\{\mu^M\}_{M \in \underline{C}}$ is not a C-family, the deviation of any estimate on any given finite region Λ is small if M is chosen large enough.

P4 : For any AC-family $\{\mu^M\}_{M \in \underline{C}}$, there exist a unique C-family $\{\nu^M\}_{M \in \underline{C}}$ which enjoys the property that, for any given $\Lambda \in \underline{C}$, and $\varepsilon > 0$, there is an $M_o \in \underline{C}(\Lambda)$ such that for all $M \in \underline{C}(M_o)$, we have

$$|\mu^M \circ p_{M\Lambda}^{-1}(A) - \nu^M \circ p_{M\Lambda}^{-1}(A)| < \varepsilon \qquad (14)$$

for each $A \in \mathcal{B}^\Lambda$.

Combining P2 and P4, one dirives the existence of a unique measure ν on $(X,\mathcal{B})$, such that, as in P4,

$$|\mu^M \circ p_{M\Lambda}^{-1}(A) - \nu \circ p_{\Lambda}^{-1}(A)| < \varepsilon \quad . \tag{15}$$

Sometime, we call ν the projective limit of the net $\{\mu^M\}_{M\in \underline{C}}$.

pf : The proof of P4, is immediate from the following lemmas:

L5 : Let $\{\mu^M\}_{M\in \underline{C}}$ be an AC-family. The net $\{\mu^M \circ p_M^{-1}\}_{M\in \underline{C}(\Lambda)}$ converges to a unique measure on $(X^\Lambda, \mathcal{B}^\Lambda)$, for each given $\Lambda \in \underline{C}$.

L6 : The collection of the limiting measures on each $\Lambda \in \underline{C}$, $\{\mu_\Lambda\}_{\Lambda\in \underline{C}}$ of L5 is a C-family.

The proofs of L5 and L6 is given in the appendix B.

Chosing ν^M in P4 to be μ_M in L6 for each $M \in \underline{C}$, we prove P4.||

P7 : Let ν be a given measure on $(X, \mathcal{B})$, and suppose that there is a family of fd-measures $\{\mu^M\}_{M\in \underline{C}}$, such that (15) is satisfied. Then $\{\mu^M\}_{M\in \underline{C}}$ is an AC-family, i.e. (13) is satisfied.

pf : For any given $\varepsilon > 0$, and $\Lambda \in \underline{C}$, there is an $M_o \in \underline{C}(\Lambda)$ such that

$$|\mu^M \circ p_{M\Lambda}^{-1}(A) - \nu \circ p_{\Lambda}^{-1}(A)| < \frac{\varepsilon}{2} \quad , \tag{16.a}$$

and

$$|\mu^{M'} \circ p_{M'\Lambda}^{-1}(A) - \nu \circ p_{\Lambda}^{-1}(A)| < \frac{\varepsilon}{2} \tag{16.b}$$

for each $A \in \mathcal{B}^\Lambda$, for all $M, M' \in \underline{C}(M_o)$. With the use of the Triangle Inequality, and (16a,b), we get (13).||

The proposition P4 and P7 tell us that the AC-condition is both sufficient and necessary for the existence of a unique measure on $(X, \mathcal{B})$. It is clear from D1 and D3 that every C-family are AC. If an AC-family is such that in (13) one can shoose Λ for M_o, then it is a C-family;indeed for each $\Lambda \in \underline{C}$, and any $\varepsilon > 0$, for all M_1, $M_2 \in \underline{C}(\Lambda)$, we have

$$|\mu^{M_1} \circ p^{-1}_{M_1\Lambda}(A) - \mu^{M_2} \circ p^{-1}_{M_2\Lambda}(A)| < \varepsilon$$

for each $A \in \mathcal{B}^{\Lambda}$, which is equivalent to (7).

Concerning the thermodynamic limit, in any sence [80], it deals with a sub-net of the net $\{\mu^M\}_{M \in \underline{C}}$: i.e. $\{M_n\}_{n \in \mathbb{N}}$ is a cofinal sub-set of $\underline{C}$. Usually one assumes M_n to be a restriction of convex set in $\mathbb{R}^d$ onto Z^d, and monotonically increasing. Obviously, a system, defined by a potential, which satisfy the AC-condition, alway has a unique thermodynamic limit, but the converse may not true. Clearly, the existence of a thermodynamic limit implies the existence of a convergent sub-net.

The consistency of the probability measure to the thermodynamics laws is the fundamental aspect of Statistical Mechanics. Any measure with such property is called a Gibbs measure (state). A successful attemps to construct and characterize such measures is the work of Dobrushin, Lanford, and Ruelle (DLR) [81-87]. Any theory that claims to be a theory of Statistical Mechanics must face this problem. We do not take this point as our starting point since we do not believe that at critical points the theory should present the same features as equilibrium thermodynamics. It is very dangerous to require that the limiting probability measure be Gibbs state. The RG may not leave invariant the set of Gibbs states. We will not elaborate on this point in this thesis.

Weak Convergence of Measures.

Let $\mathfrak{M}(\mathcal{S})$ denote the set of all probability measures defined on a measurable space $(S,\mathcal{S})$. We can introduce on $\mathfrak{M}(\mathcal{S})$ a topology through a convergence structure. The notion that has been investigated by various mathematicians is weak convergence. A good review has been given by Billingsley (1968) [71]. The theory concern S as a metric space; most of the time assumes separable, and completeness. This is why we have spent a few pages to verify that our space X is indeed a separable, complete metric space, see chapter I.

Here, the set of all bounded continuous function from S into $\mathbb{R}$ is denoted by C(S).

D8 : A net of measures $\{\mu_j\}_{j\in J}$ in $\mathfrak{M}(\mathcal{S})$ converges weakly to a measure μ in $\mathfrak{M}(\mathcal{S})$ iff, for each $f \in C(S)$, the net $\left\{\int f d\mu_j\right\}_{j\in J}$ converges to $\int f d\mu$. We denote weak convergence by "$\Rightarrow$".

Since $(X,\mathcal{T})$ is a metric space, any measure defined on $(X,\mathfrak{B})$ must be regular. This means the measure is determined by its restriction to closed sub-sets. Consequently, the Urysohn Lemma is applied and we conclude as following:

P9 : For any measures μ and ν in $\mathfrak{M}(\mathfrak{B})$, $\int f d\mu = \int f d\nu$, for each $f \in C(X)$, iff $\mu = \nu$.

This fact is of importance, otherwise, the notion of weak convergence would be very complicated, or useless. Weak convergence can be defined on any measurable space, not necessary a metric space, as far as P9 is valid.

Here, we give a rich alternate version through a theorem.

P10: (Portmanteau Theorem) Let $\{\mu_j\}_{j\in J}$, and μ be measures in $\mathfrak{M}(\mathcal{B})$, then the following are equivalent.

(i) $\{\mu_j\}_{j\in J} \Rightarrow \mu$.

(ii) $\lim_{j\in J}\int f d\mu_j = \int f d\mu$, for all $f \in C(X)$.

(iii) $\limsup_{j\in J}\mu_j(F) \leq \mu(F)$, for all F, closed in $\mathcal{T}$.

(iv) $\liminf_{j\in J}\mu_j(G) \geq \mu(G)$, for all G, open in $\mathcal{T}$.

(v) $\lim_{j\in J}\mu_j(A) = \mu(A)$, for all μ-continuity set $A \in \mathcal{B}$.

Here, the μ-continuity set A, means $\mu(\partial A) = 0$.

A lemma that links weak convergence to $(X, \mathcal{B})$ with weak convergence to $(X^M, \mathcal{B}^M)$ for all $M \in \underline{C}$, is

L11: $\{\mu_j\}_{j\in J} \Rightarrow \mu \in \mathfrak{M}(\mathcal{B})$ iff, for each $M \in \underline{C}$, $\{\mu_j \circ p_M^{-1}\}_{j\in J} \Rightarrow \mu \circ p_M^{-1} \in \mathfrak{M}(\mathcal{B}^M)$.

pf : The prove can be easily obtained from the analogous result proved in Billingsley (1968) [71] p19, for $R = \mathbb{R}^N$: $\{\mu_j\}_{j\in J} \Rightarrow \mu$ iff $\{\mu_j(A)\}_{j\in J} \rightarrow \mu(A)$ for all finite-dimensional μ-continuity set A . Apply P10, (v) $\cong$ (i), we get L11.||

This weak convergence structure is equivalent to the convergence in a topology $\mathcal{W}$ on $\mathfrak{M}(\mathcal{B})$ which has a local base at any point (measure) ν of the form

$$V(\{f_j, \varepsilon_j\}_{j=1}^{k}; \nu) = \{\mu \in \mathfrak{M}(\mathcal{B}) \mid |\int f_j d\mu - \int f_j d\nu| < \varepsilon_j; j=1,2,\ldots,k\} \quad (17)$$

where $f_j \in C(X)$, and $\varepsilon_j > 0$, for all $j = 1,2,\ldots,k$. This topology is called a vague topology, weak topology, or weak convergence topology.

P12: $\mathfrak{M}(\mathfrak{B})$ is a complete, separable metric space.

The proof can be found in Billingsley (1968) [71] appendix III, or Pathasarathy (1967) [70] chapter II, theorem 6.2,and 6.5.

P13: $\mathfrak{M}(\mathfrak{B})$ is not compact.

pf : From ibid. theorem 6.4, I.P16, we conclude as above.||

A very intesting proof, easier to understand, but not fullfil our need, is given by Varadhan (1967) [72] section 4.

Evenif we can give an explicit metric for the weak convergence topology, a direct use of the metric is rarely found. Technically,the convergence structure is used, with the help of Portmenteau theorem P10.

If one looks carefully, weak convergence is very important to physics as we are interested in the convergence of the integral of finite set of observable quantity (the elements of measurable continuous functions from X to R) with respect to some measures in $\mathfrak{M}(\mathfrak{B})$.

III. THE RENORMALIZATION GROUP TRANSFORMATION.

Here, we are going to define a mathematically rigorous RG-transform from the conventional intuitive Kadanoff type [14]. The map will be well define on the set of all probability measures $\mathfrak{M}(\mathfrak{B})$. Finally, we prove that the map is continuous in the weak convergence (Vague) topology [70-73]. This property willplay an important role in the next chapter, where we discuss the existence of fixed points.

Let us define for each odd integer L, and $m_1 \in \mathbb{Z}^d$, a finite sub-set: a Block in $\mathbb{Z}^d$,

$$B_L(m_1) = \left\{ m \in \mathbb{Z}^d \mid \rho_\infty(m, m_1 L) < \frac{L}{2} \right\} . \qquad (1)$$

It is a hyper-cube in $\mathbb{Z}^d$.

Some $M \in \underline{C}$ may have the property that there exists $M_1 \in \underline{C}$ such that

$$M = \bigcup_{m_1 \in M_1} B_L(m_1) \qquad . \qquad (2)$$

The totality of finite sub-sets with this property is denoted by

$$\underline{C}_B = \left\{ M \in \underline{C} \mid \text{there exist } M_1 \in \underline{C} \text{ such that (2) is satisfied} \right\} . \quad (3)$$

Since L is almost all the time fixed, it will dropped when it is clear from the context.

Conventionally, the RG-transformation is considered as a transformation of a potential (Hamiltonian) into another potential, explicitly with a space cut-off. For each $M \in \underline{C}_B$, and any potential $U^M : X^M \rightarrow \mathbb{R}$, for each $\lambda > 0$, the renormalized potential $U^{1 M_1} : X^{M_1} \rightarrow \mathbb{R}$ is defined, formally, as following: For each $x^1 \in X^{M_1}$

$$e^{-U_1^{M_1}(x^1)} = \int_{X^M} e^{-U^M(x)} \otimes_{m_1 \in M_1} \delta(x^1_{m_1} - \lambda L^{-d} \sum_{m \in B_L(m_1)} x_m) \omega^M(dx), \tag{4}$$

where δ is the Dirac delta function, and ω^M is the Lebesgue measure on X^M. Here, we absorb the temperature in the definition of the potentials.

Although the expression makes sense as $|M| < \infty$, after a few iterations it may be ill-defined: After j-iteration, the M_j is no longer belong to $\underline{C}_B$. Even with the best choice of M, it will finally end up with a potential on $X^{\{m\}} = \mathbb{R}$. If we replace M by $\mathbb{Z}^d$, then (4) is just a formal expression.

Let us convert this formal definition into a rigorous one.

BS-Transformation.

D1 : For each $M \in \underline{C}_B$, we define a function $\pi_M : X^M \to X^{M_1}$: For each $x \in X^M$,

$$\pi_M x = \lambda_M L^{-d} \sum_{m_1 \in M_1} \left\{ \sum_{m \in B_L(m_1)} x_m \right\} e_{m_1} \tag{5}$$

where $\lambda_M > 0$, and $\{e_{m_1}\}_{m_1 \in M_1}$ is the natural basis of X^{M_1} call this a finite-dimentional (fd) Block-Scaling (BS) transformation.

The scaling parameter λ_M has a suffix M to denote that at this level it may depend on M.

From II(9) and II(10), assuming h = 0, and $\beta = 1$, a family of fd-measures $\{\mu^M\}_{M \in \underline{C}}$ is defined. From (4), we can easily check that

$$Z^1_{M_1} = Z_M \qquad (6)$$

Using (6), expression (4) can be rewritten as a relation on normalized set function: For each $A \in \mathcal{B}^{M_1}$

$$\mu_1^{M_1}(A) = \frac{1}{Z^1_{M_1}} \int_A e^{-U_1^{M_1}(x^1)}\, \omega^{M_1}(dx^1)$$

$$= \frac{1}{Z_M} \int_{\pi_M^{-1}A} e^{-U^M(x)}\, \omega^M(dx)$$

$$= \mu^M(\{x \in X^M \mid \pi_M x \in A\}) \qquad (7)$$

The set-function $\mu_1^{M_1}$ will be a probability measure only if the BS-transformation π_M is a measurable map from $(X^M, \mathcal{B}^M)$ into $(X^{M_1}, \mathcal{B}^{M_1})$.

P2 : For each $M \in \underline{C}_B$, π_M is a continuous map, thus a measurable map

pf : For each $x \in X^M$, we have $\pi_M x \in X^{M_1}$ by definition. Let the $||.||_M$ denote the Euclidian norm (an ℓ^2-norm) on the space X^M:

$$||\pi_M x||^2_{M_1} = (\lambda L^{-d})^2 \sum_{m_1 \in M_1} \left\{ \sum_{m \in B_L(m_1)} x_m \right\}^2 . \qquad (8)$$

Applying the Cauchy Inequality, see Mitrinovic' (1970) [111] p30, on each term $\{\ldots\}^2_{(m_1)}$, we get

$$||\pi_M x||^2_{M_1} \leq \lambda^2 L^{-d} ||x||^2_M . \qquad (9)$$

Therefore, π_M is continuous at y = 0, which suffices for π_M being continuous at all $y \in X^M$. Any continuous function is measurable.||

C3 : Let us define $||\pi_M|| = \sup_{||x||=1} ||\pi_M x||$, (10)

then $||\pi_M|| = \lambda_M L^{-d/2}$ (11)

pf : From (9), we have clearly $||\pi_M|| \leq \lambda_M L^{-d/2}$. Considering an element $x^* \in X^M$, for all $m \in M$, $x^*_m = 1/\sqrt{|M|}$. It is clear that $||x^*|| = 1$. Since

$$||\pi_M x^*||^2 = (\lambda_M L^{-d})^2 \sum_{m_1 \in M_1} \left(\frac{L^d}{\sqrt{|M|}}\right)^2 = \lambda_M^2 \frac{|M_1|}{|M|} = \lambda_M^2 L^{-d}.$$

This proves (11).||

Note that the map π_M is an isometry $||\pi_M|| = 1$ when $\lambda_M = L^{+d/2}$. The scaling factor of this form will play an importance role later.

The Family of fdRG-transformations.

Now, we shall define the RG-transform as a map on fd-measure as (7). The precise definition is given below.

D4 : Let M, M_1, π_M be given as in D1, we define a map $\tau^M: \mathfrak{M}(\mathcal{B}^M) \to \mathfrak{M}(\mathcal{B}^{M_1})$: For each $\mu^M \in \mathfrak{M}(\mathcal{B}^M)$

$$(\tau^M \mu^M)(A) = \mu_1^{M_1}(A) = (\mu^M \circ \pi_M^{-1})(A) \quad , \quad (12)$$

for all $A \in \mathcal{B}^{M_1}$.

Beware that μ^M is not neccessary defined by a potential as given before. In some sense, we have extended the domain of definition of fdRG-map for each $M \in \underline{C}_B$ to the totality of probability measures on $(X^M, \mathfrak{B}^M)$. The definition is given for $M \in \underline{C}_B$ only. However, it can extended to any $\Omega \in \underline{C}$, by defining its minimum block covering

$$\Omega^* = \inf_{M \in \underline{C}_B \cap \underline{C}(\Omega)} M \qquad (13.a)$$

$$= \{n \in Z^d |\ \text{for each } m \in \Omega, \text{ there exist } m_1 \in Z^d, \text{ such that } m \in B_L(m_1): n \in B_L(m_1) \Rightarrow n \in \Omega^*\} \qquad , \qquad (13.b)$$

then define

$$\mu^{1\Omega_1} = (\tau^\Omega \mu^\Omega) = \mu^\Omega p_{\Omega^*\Omega} \circ \pi_{\Omega^*}^{-1} \qquad . \qquad (14)$$

This will send all collection $\{\mu^\Omega | \Omega$ having the same $\Omega^*\}$ to defferent $\mu^{1\Omega_1}_{(\Omega)}$ depending on Ω .

It is easier to consider the fdRG-maps only on $\underline{C}_B$. This makes no harm to the AC-property as we will prove in P5 that to the possibly non-unique renormalized family of fd-measures is uniquely associated a C-family, and thus a unique renormalized measure on $(X, \mathfrak{B})$.

P5 : Let $\{\tau^M\}_{M \in \underline{C}}$ be a family of fdRG-maps with the scaling factors $\{\lambda_M\}_{M \in \underline{C}}$ independent of M: i.e. $\lambda_M = \lambda$, for all $M \in \underline{C}$. Then, it transform each AC-family $\{\mu^M\}_{M \in \underline{C}}$ to AC-families which have the same projective limit measure on $(X, \mathfrak{B})$

pf : We will organize the proof for the fdRG-transformatios of $\{\mu^M\}_{M \in \underline{C}_B}$, then prove the general case.

For any $\Lambda_1 \in \underline{C}$, and M_1, $M_1' \in \underline{C}(\Lambda_1)$:

$$|\mu^{1^{M_1}} \circ p^{-1}_{M_1\Lambda_1} - \mu^{1^{M'_1}} \circ p^{-1}_{M'_1\Lambda_1}| = |\mu^{M} \circ \pi^{-1}_{M} \circ p^{-1}_{M_1\Lambda_1} - \mu^{M'} \circ \pi^{-1}_{M'}\, p^{-1}_{M'_1\Lambda_1}| \;, \quad (15)$$

where M, M' and M_1, M'_1 are related by (2).

By the assumption of homogeneity of $\lambda_M = \lambda$, we can easily check

that
$$p_{M_1\Lambda_1} \circ \pi_M = \pi_\Lambda \circ p_{M\Lambda} \quad (16)$$

Using this in RHS(15), we get

$$(15) = |\mu^{M} \circ p^{-1}_{M\Lambda} \circ \pi^{-1}_{\Lambda} - \mu^{M'} \circ p^{-1}_{M'\Lambda} \circ \pi^{-1}_{\Lambda}| \quad (17)$$

Since π_Λ is measurable by P2, $\pi^{-1}_\Lambda A \in \mathcal{B}^\Lambda$ for each $A \in \mathcal{B}^{\Lambda_1}$. From the AC-property of $\{\mu^M\}_{M \in \underline{C}}$, we have (17) less than any given $\varepsilon > 0$, if M, $M'_M \in \underline{C}(\Lambda)$ is chosen large enough. This proves the AC-property of $\{\mu^{1^{M_1}}\}_{M_1 \in \underline{C}}$.

If we take the general case, we will have instead of (15)

$$|\mu^{1^{M_1}}_{(M)} \circ p^{-1}_{M_1\Lambda_1} - \mu^{1^{M'_1}}_{(M')} \circ p^{-1}_{M'_1\Lambda_1}| = |\mu^{M} \circ p_{M*M} \circ \pi^{-1}_{M*} \circ p^{-1}_{M_1\Lambda_1} -$$
$$\mu^{M'} \circ p_{M'*M'} \circ \pi^{-1}_{M'*}\, p^{-1}_{M'_1\Lambda_1}| \quad (15^*)$$

Then (17) becomes

$$(15^*) = |\mu^{M} \circ p_{M*M} \circ p^{-1}_{M*\Lambda} \circ \pi^{-1}_{\Lambda} - \mu^{M'} \circ p_{M'*M'} \circ p^{-1}_{M'*\Lambda} \circ \pi^{-1}_{\Lambda}| \quad (17^*)$$
$$= |\mu^{M} \circ p^{-1}_{M\Lambda} \circ \pi^{-1}_{\Lambda} - \mu^{M'} \circ p^{-1}_{M'\Lambda} \circ \pi^{-1}_{\Lambda}| \;.$$

Consequently, one can proceed in the same way as before, and conclude that $\left\{\mu^{1\,M_1}_{(M)}\right\}_{M_1 \in \underline{C}}$ is an AC-family.

There are many M which have M* as minimum covering block, thus there several AC-families as an out come of a given family of fdRG-maps $\{\tau^M\}_{M \in \underline{C}}$ and an AC-family. Nevertheless, all of these AC-families have the same projective limit. This can be verified as following: Let the limit of each net $\left\{\mu^{1\,M_1}_{(M)} \circ p^{-1}_{M_1\Lambda_1}\right\}_{M_1 \in \underline{C}}$ and $\left\{\mu^{1\,M_1}_{(M')} \circ p^{-1}_{M_1\Lambda_1}\right\}_{M_1 \in \underline{C}}$ be $\mu^1_{\Lambda_1}$ and $\mu^{1\,'}_{\Lambda_1}$, we have

$$|\mu^1_{\Lambda_1} - \mu^{1\,'}_{\Lambda_1}| \leq |\mu^1_{\Lambda_1} - \mu^{1\,M_1}_{(M)} \circ p^{-1}_{M_1\Lambda_1}| + |\mu^{1\,'}_{\Lambda_1} - \mu^{1\,N_1}_{(N')} \circ p^{-1}_{N_1\Lambda_1}| + |\mu^{1\,M_1}_{(M)} \circ p^{-1}_{M_1\Lambda_1} - \mu^{1\,N_1}_{(N')} \circ p^{-1}_{N_1\Lambda_1}| \quad (18)$$

Each of the first two terms can be bouned by $\varepsilon/3$ for large M_1 and N_1, respectively, because the $\mu^1_{\Lambda_1}$ and $\mu^{1\,'}_{\Lambda_1}$ are the coresponding limits.

The last term is just $= |\mu^M \circ p^{-1}_{M\Lambda} \circ \pi^{-1}_\Lambda - \mu^{N'} \circ p^{-1}_{N'\Lambda} \circ \pi^{-1}_\Lambda| \leq \varepsilon/3$

for large enough M, N' $\in \underline{C}(\Lambda)$ by the AC-property of $\{\mu^M\}_{M \in \underline{C}}$. Therefore, the LHS(18) is bounded above by an arbitrarily small positive number ε. We can easily say that

$$\mu^1_{\Lambda_1} = \mu^{1\,'}_{\Lambda_1} \quad (19)$$

This is true for any couple of sub-nets of $\{\mu^M\}_{M \in \underline{C}}$, thus all renormalized AC-families must have the same C-family that enjoys the property given in II.P4. There is therefore a unique measure μ^1 on $(X, \mathcal{B})$ as the projective limit of all renormalized AC-families.||

C6 : As in P5, but $\{\mu^M\}_{M\in \underline{C}}$ is a C-family. Then $\{\mu^{1^M}\}_{M\in \underline{C}}$ is a C-family.

pf : Immediate, if we use the fact that, for any $\varepsilon > 0$, and $\Lambda \in \underline{C}$, then the AC-condition is satisfied for all $M_1, M_2 \in \underline{C}(\Lambda)$; i.e. $M_o = \Lambda$, is equivalent to C-condition.||

This corollary rule out, for this case, what we had forecasted in chapter II that an RG-map may send a C-family to a non-C-family.

As had been pointed out in chapter II that usually the initial family of fd-measures satisfies the AC-condition, but not C-condition. Indeed the proposition P5 tells us that $\{\mu^M\}_{M\in \underline{C}_B}$ is enough to define the renormalized measure μ^1 on $(X,\mathfrak{B})$. The order pair (μ,μ^1) define now a well defined function from $\mathfrak{M}(\mathfrak{B})$ into itself.

The RG-map on $\mathfrak{M}(\mathfrak{B})$.

D7 : For each $\mu \in \mathfrak{M}(\mathfrak{B})$, we define the RG-map $\tau: \mathfrak{M}(\mathfrak{B}) \rightarrow \mathfrak{M}(\mathfrak{B})$ by

$$\tau\mu \quad = \quad \mu^1 \tag{20}$$

where μ^1 is the projective limit of $\{\tau^M\mu^M\}_{M\in \underline{C}}$.

We can see that the meaning of RG-map τ, technically dealing with the mapping on sub-net $\{\mu^M\}_{M\in \underline{C}_B}$, abstracly as define in D7.

Here, we are going to prove an important topological property of the map:

P8 : τ is a continuous map in $\mathfrak{M}(\mathfrak{B})$ under the weak convergence topology.

pf : Let $\{\mu_n\}_{n\in N}$ be any sequence of measures on $(X,\mathfrak{B})$ which converges weakly to a measure μ on $(X,\mathfrak{B})$. From II.L11, we have $\{\mu_n \circ p_M^{-1}\}_{n\in N}$ converges weakly to the measures $\mu\circ p_M^{-1}$ on $(X^M, \mathfrak{B}^M)$, for each $M \in \underline{C}$.

Using the fdRG-maps, we are led to consider the family $\{\mu_n\circ p_M^{-1}\circ \pi_M^{-1}\}_{n\in N}$.

For each $M_1 \in \underline{C}$, and $f^{M_1} \in C(X^M)$, we have

$$\int f^{M_1}(x)\mu_n\circ p_M^{-1}\circ \pi_M^{-1}(dx) = \int f^{M_1}\left(\pi_M(y)\right)\mu_n\circ p_M^{-1}(dy) \quad ,(21)$$

which converges to

$$\int f^{M_1}\left(\pi_M(y)\right)\mu\circ p_M^{-1}(dy) = \int f^{M_1}(x)\mu\circ p_M^{-1}\circ \pi_M^{-1}(dx) \quad .(22)$$

This means $\left\{\mu_n^{1^{M_1}}\right\}_{n\in \mathbb{N}} \Rightarrow \mu^{1^{M_1}}$, for each $M_1 \in \underline{C}$.

By P5, for each $n \in \mathbb{N}$, $\left\{\mu_n^{1^{M_1}}\right\}_{M_1\in \underline{C}}$ is an AC-family, thus there exists a projective limit μ_n^1 on $(X,\mathfrak{B})$.

By II.L11, the $\{\mu_n^1\}_{n\in N}$ must converge weakly to μ^1 which is the projective limit of $\left\{\mu^{1^{M_1}}\right\}_{M_1\in \underline{C}}$.

This proves the continuity of the map; indeed convergence of $\{\mu_n\}_{n\in \mathbb{N}}$ to μ implies that $\{\tau\mu_n\}_{n\in N}$ converge to $\tau\mu$. ||

We conclude that the RG-transformation, usually defined on potentials can be extended to be a map on the space of all probability measures $\mathfrak{M}(\mathfrak{B})$ continuous in the weak convergence topology.

Note that we do not know whether the renormalized probability measure is a Gibbs state or not, evenif the initial measure is. We also show that the concept as a transformation of potentials is not well defined since the renormalized measure can be defined by various potentials.

The Non-Homogeneous Scaling RG-transformation.

Suppose we attempt to weaken the condition in P5 from the homogeneous scaling; $\lambda_M = \lambda$ for all $M \in \underline{C}$ to the case that $\{\lambda_M\}_{M \in \underline{C}}$ is a Cauchy net: for each $\varepsilon > 0$, and $\Lambda \in \underline{C}$, there is an $M_o \in \underline{C}(\Lambda)$ such that for all M_1, $M_2 \in \underline{C}(M_o)$, we have

$$|\lambda_{M_1} - \lambda_{M_2}| < \varepsilon \tag{23}$$

It is clear that the net has a unique limit $\lambda > 0$.

Let us denote the fdBS-transformations for the non-homogeneous case by $\bar{\pi}_M$, and for the homogeneous case by π_M. Here, the scaling factor for π_M is the limit λ.

For each $M \in \underline{C}$, and $x \in X^M$, from D1, we have

$$\bar{\pi}_M x = \lambda_M L^{-d} \sum_{m_1 \in M_1} \left\{ \sum_{m \in B_L(m_1)} x_m \right\} e_{m_1} \tag{24}$$

$$p_{M_1 \Lambda_1} \circ \bar{\pi}_M x = \lambda_M L^{-d} \sum_{m_1 \in \Lambda_1} \left\{ \sum_{m \in B_L(m_1)} x_m \right\} e_{m_1} \tag{25}$$

$$= \frac{\lambda_M}{\lambda_\Lambda} \bar{\pi}_\Lambda \, p_{M\Lambda} x \tag{26}$$

Considering, $$\mu^{1^{M_1}} \circ p^{-1}_{M_1\Lambda_1} = \mu^{M} \circ \bar{\pi}^{-1}_{M} \circ p^{-1}_{M_1\Lambda_1} \tag{27}$$

one conclues (26) $\Rightarrow$ $$= \mu^{M} \circ p^{-1}_{M\Lambda}\, \bar{\pi}^{-1}_{\Lambda} \circ \left(\frac{\lambda_M}{\lambda_\Lambda}\right)^{-1} \tag{28}$$

Since $\{\mu^{M}\}_{M \in \underline{C}}$ is an AC-family, it has a measure μ on $(X,\mathcal{B})$ as projective limit, see II.P4. This means, see II (15) that $\{\mu^{M} \circ p^{-1}_{M\Lambda}\}_{M \in \underline{C}(\Lambda)} \to \mu \circ p^{-1}_{\Lambda}$ on each $A \in \mathcal{B}^{\Lambda}$. Obviously, $\{\mu^{M} \circ p^{-1}_{M\Lambda}\}_{M \in \underline{C}(\Lambda)} \Rightarrow \mu \circ p^{-1}_{\Lambda}$.

For each $x \in X^{\Lambda}$, we alway have $\left\{\frac{\lambda_M}{\lambda_\Lambda} \bar{\pi}_{\Lambda} x\right\}_{M \in \underline{C}(\Lambda)}$ converges to $\frac{\lambda}{\lambda_\Lambda} \bar{\pi}_{\Lambda} x = \pi_{\Lambda} x$. Therefore the set for which the sequence $\left\{\frac{\lambda_M}{\lambda_\Lambda} \bar{\pi}_{\Lambda}\right\}_{M \in \underline{C}(\Lambda)}$ fails to converge to π_{Λ} is empty.

From Billingsley (1968) [71] p34, theorem 5.5, we conclude that

$$\left\{\mu^{M} \circ \bar{\pi}^{-1}_{M} \circ p^{-1}_{M_1\Lambda_1}\right\}_{M \in \underline{C}(\Lambda)} \Rightarrow \mu \circ p^{-1}_{\Lambda} \circ \pi_{\Lambda} \quad .$$

We know that $\mu \circ p^{-1}_{\Lambda} \circ \pi_{\Lambda} = \mu^{1} \circ p^{-1}_{\Lambda_1}$ on each $A \in \mathcal{B}^{\Lambda_1}$. It is clear that for each given μ on $(X,\mathcal{B})$, the non-homogeneous-Cauchy-scaling fdRG-map $\{\bar{\tau}^{M}\}_{M \in \underline{C}}$ determines uniquely a measure μ^{1} on $(X,\mathcal{B})$. Therefore the ordered pair (μ,μ^{1}) defines a function $\bar{\tau}$ from $\mathcal{M}(\mathcal{B})$ into itself. Infact, $\bar{\tau} = \tau$ as μ^{1} is determined by the scaling λ.

We can garantee then that the Cauchy case also leads to the same definition D7 and conclution P8.

There is an attempt to use the RG-map implicitly through an equation of state, related to a finite size scaling law [5,88-90]. If it can be written explicitly, it must be a non-homogeneous scaling RG-map. We give the simplest criteria that it must be Cauchy, then it is well defined. Otherwise, one must proceed to a similar prove, case by case.

IV. RG-MAP OF CHARACTERISTIC, MOMENT, AND SEMI-VARIANCE.

The important fact that the dual X* is isomorphic to X_* as discussed in chapter I plays a role here.

The Definition of θ, M, C Functions.

Let us first recall the defintion of Characteristic, Moments, and Semi-variance function [64-73].

D1 : The characteristic function is a function from S* into $\mathbb{C}$, defined for each measure μ on $(S,\mathcal{S})$ as follows: For each $f \in S^*$

$$\theta_\mu(f) = \int e^{if(x)}\mu(dx) \qquad (1)$$

D2 : The N-moment function is an N-linear function from $(S^*)^N$ into $\mathbb{R}$, defined for each measure μ on $(S,\mathcal{S})$ as follows: For any N elements $f_1, f_2, \ldots\ldots, f_N \in S^*$

$$M_\mu^N(f_1,f_2,\ldots,f_N) = E_\mu(\boxtimes_{j=1}^N f_j) \qquad (2)$$

P3 : For any given $N \in \mathbb{N}$, for any N elements $f_1, f_2, \ldots\ldots, f_N \in S^*$, and $\lambda_1, \lambda_2, \ldots\ldots, \lambda_N \in \mathbb{R}$, and each measure μ on $(S,\mathcal{S})$, we have

$$M_\mu^N(f_1,f_2,\ldots,f_N) = (-i)^N \frac{\partial^N}{\partial\lambda_1\partial\lambda_2\cdots\partial\lambda_N}\theta_\mu(\Sigma_{j=1}^N\lambda_j f_j)\Big|\{\lambda_j=0\}_{j=1}^N . \qquad (3)$$

D4 : The N-semi-variant function is an N-linear function from $(S^*)^N$ into $\mathbb{R}$, defined for each measure μ on $(S,\mathcal{S})$ as follows: For any N elements $f_1, f_2, \ldots\ldots, f_N \in S^*$, and $\lambda_1, \lambda_2, \ldots\ldots, \lambda_N \in \mathbb{R}$

$$C_{\mu}^{N}(f_1,f_2,\ldots,f_N) = (-i)^N \frac{\partial^N}{\partial\lambda_1\partial\lambda_2\cdots\partial\lambda_N} \log\theta_{\mu}(\sum_{j=1}^{N} {}_j f_j)\Big|\{\lambda_j=0\}_{j=1}^{N} . \tag{4}$$

The defintion D2, and D4 for some N may be ill-posed for certain spaces $(S,\mathcal{S})$, but this is not our case $(X,\mathcal{B})$. For each N, we know that $f_j \in X^*$ is bouned for all $j = 1,2,\ldots,N$, thus the product is bounded, therefore $E_{\mu}(\boxtimes_{j=1}^{N} f_j)$ always exists. If we consider subspaces of X, such as ℓ_d^p, the validity of D2 and D4 is delicate as the average (2) for some N, and $f_j \in \ell_d^{p*}$, $j = 1,2,\ldots,N$, may not exit for some measure on ℓ_d^p.

P5 : M_{μ}^{N}, and C_{μ}^{N} are symmetric, additive, and homogeneous in each argument.

The proof of P3 and P5 is immediate from definition D1, D2, and D4.

Let $\{e_n^*\}_{n\in\mathbb{Z}^d}$ be a natural basis on X^*: i.e. $e_n^*(e_m) = \delta_{nm}$, where $\{e_m\}_{m\in\mathbb{Z}^d}$ is the natural basis of X. The integral for each measure μ on $(X,\mathcal{B})$

$$\int x_{n_1}x_{n_2}\cdots x_{n_N}\mu(dx) = \int e_{n_1}^*(x)e_{n_2}^*(x)\cdots e_{n_N}^*(x)\mu(dx)$$

$$= M_{\mu}^{N}(e_{n_1}^*,e_{n_2}^*,\ldots,e_{n_N}^*) \tag{5}$$

From time to time when it leads to no ambiguity we use the notation

$$M_{\mu}^{N}(e_{n_1}^*,e_{n_2}^*,\ldots,e_{n_N}^*) = M_{\mu}(n_1,n_2,\ldots,n_N) \tag{6}$$

$$C_{\mu}^{N}(e_{n_1}^*,e_{n_2}^*,\ldots,e_{n_N}^*) = C_{\mu}(n_1,n_2,\ldots,n_N) \tag{7}$$

There are several theorem dealing with characteristic functions, especially the generalized Bochner Theorem; Vakhnia (1981) [76] p7, Skorohod (1974) [73] p15, theorem 1, Shiryayev (1984) [68] II 12, Gel'Fand and Vilenkin (1964) [91] p155, theorem 2, and p157, theorem 3, Pathasarathy (1967) [70] p75, theorem 3.2.

RG-map of Θ, M, C Functions.

Let $\{\mu^M\}_{M \in \underline{C}}$ be an AC-family, and $\{\mu^{1^{M_1}}\}_{M_1 \quad \underline{C}}$ be the renormalized AC-family. Their projective limits are denoted by μ and μ^1, respectively.

P6 : For each $M \in \underline{C}_B$, and $f^1 \in X^{{M_1}^*}$, we have

$$\Theta_{\mu^{1^{M_1}}}(f^1) \quad = \quad \Theta_{\mu^M}(f^1 \circ \eta_M) \qquad (8)$$

pf : Immediate from D1, and the fact that $\mu^{1^{M_1}} = \mu^M \circ \eta_M^{-1}$. ||

P7 : For each $M \in \underline{C}_B$, $N \in \mathbb{N}$, $f_j^1 \in X^{{M_1}^*}$, $j = 1,2,\ldots,N$:

$$\left\{\begin{matrix} M \\ \\ C \end{matrix}\right\}^N_{\mu^{1^{M_1}}} (f_1^1, f_2^1, \ldots, f_N^1) \quad = \quad \left\{\begin{matrix} M \\ \\ C \end{matrix}\right\}^N_{\mu^M} (f_1^1 \circ \eta_M, f_2^1 \circ \eta_M, \ldots, f_N^1 \circ \eta_M) \ . \qquad (9)$$

Especially, when $f_j^1 = e^*_{n_{1j}}$, we have

$$\left\{\begin{matrix} M \\ \\ C \end{matrix}\right\}_{\mu^{1^{M_1}}} (n_{11}, n_{12}, \ldots, n_{1N}) = (\lambda L^{-d})^N \sum_{\{m_j \in B_L(n_{1j})\}_{j=1}^N} \left\{\begin{matrix} M \\ \\ C \end{matrix}\right\}_{\mu^M} (m_1, m_2, \ldots, m_N) \qquad (10)$$

pf : P6 and P3 or D4, we get (9). From the definition of BS-transformation, and the linearity of f_j^1, we have for each $x \in X^M$

$$f_j^1 \circ \pi_M x = \lambda L^{-d} \sum_{n_1 \in M_1} f_{j,n_1}^1 \left\{ \sum_{m \in B_L(n_1)} x_m \right\} . \quad (11)$$

If $f_j^1 = e^*_{n_{1j}}$, we have immediately from (11) that

$$e^*_{n_{1j}} \circ \pi_M = \lambda L^{-d} \sum_{m_j \in B_{L(n_{1j})}} e^*_{m_j} \quad (12)$$

Substituting (12) into (9), then using the fact that both functions are additive, homogeneous, P5, we get (10).||

If $g \in X^*$, then for all $x \in X$

$$g(x) = \sum_{m \in M_1} g_m x_m \quad (13)$$

for some $M_1 \in \underline{C}$. This allows us to replace in P6, and P7 the measures μ^M, $\mu^{1^{M_1}}$ by $\mu \circ p_M^{-1}$, $\mu^1 \circ p_{M_1}^{-1}$, respectively. Therefore, we can write, for example, (8) as

$$\theta_{\mu^1}(g) = \theta_{\mu}(g \circ \pi_M) \quad . \quad (14)$$

V. CLASSIFICATION OF MEASURES IN $\mathfrak{M}(\mathcal{B})$.

As we had mentioned in chapter 0, we are interested in a special kind of phase transition, the second order one, which occurs at the end of a line of first order phase transitions; we call it a critical point [9-14]. Typically, it occurs in Ferromagnetic Systems. At the critical point, the magnetization vanishes while the susceptibility diverges. In Statistical Mechanics, these quantities are related mainly to the second order derivatives of the Thermodynamics Energy Functions (e.g. Gibbs or Helmholtze Free Energy) with respect to independent thermodynamics variables (e.g. external field, temperature, etc..).

For each $M \in \underline{C}$, we know that the (Gibbs) free energy per unit volume (site) is

$$F_M(\beta,h) = -\frac{1}{\beta|M|} \log Z_M(\beta,h) \qquad (1)$$

where $Z_M(\beta,h)$ is the Canonical Partition Function of the lattice system as given in II(10).

Here, we give a list of interesting thermodynamic functions derived from the free energy by differentiating [4,14]. We show explicitly the relation to the probability averages with respect to the measure μ^M, see II(9) and II(4).

Magnetization;

$$\bar{x}_M(\beta,h) = -\frac{\partial}{\partial h} F_M(\beta,h) = \frac{1}{|M|} \sum_{m \in M} E_{\mu^M}(e^*_m) \qquad (2.a)$$

Susceptibility;

$$\chi_M(\beta,h) = \frac{\partial \bar{x}_M}{\partial h}(\beta,h) = \frac{\beta}{|M|} \sum_{m,n \in M} C_{\mu^M}(m,n) \quad . \qquad (2.b)$$

Entropy per unit volume;

$$S_M(\beta,h) = \beta^2 \frac{\partial}{\partial \beta} F_M(\beta,h) = -\beta F_M(\ ,h) + \frac{\beta}{|M|} E_{\mu^M}(U^M - h \sum_{m \in M} e^*_m) . \quad (2.c)$$

Heat Capacity at fixed h;

$$C_M(\beta,h) = -\beta \frac{\partial}{\partial \beta} S_M(\beta,h) = \frac{\beta^2}{|M|} \left\{ E_{\mu^M} \left(\left[U^M - h \sum_{m \in M} e^*_m \right]^2 \right) - \left[E_{\mu^M} \left(U^M - h \sum_{m \in M} e^*_m \right) \right]^2 \right\} \quad . \quad (2.d)$$

At a critical point both susceptibility and heat capacity diverge. The source of this divergence is not in the increase without bound of the correlations, but is rather due to the fact that the fluctuations of the random functions involved increases faster than $\sqrt{|M|}$.

The idealized definition of critical point makes it obviously an unpenetrable region. The closer to the point, the higher the fluctuation, so that any experiment would contain data which are less and less reliable. However, some behaviour can be extrapolated from the results obtained for nearby values of the parameters, for which the fluctuation is not so high.

We will confine ourselves to discuss only the classification of measures with respect to the susceptibility and magnetic correlation [4,14].

Some mathematic notions must be precisely defined:

D1 : For any given collection of real numbers $\{a_n\}_{n\in\mathbb{Z}^d}$, the summation $\sum_{n\in Z^d} a_n$ converges to a real number A iff the net $\{\Sigma_{n\in M} a_n\}_{M\in\underline{C}}$ converges to A.

Ofcourse, one has

$$\sum_{n\in\mathbb{Z}^d} a_n = a_m + \sum_{\ell\in\mathbb{N}} \sum_{n\in S_\ell^{d,k}(m)} a_n \qquad (3)$$

as one can immediatly derive from D1. The usefulness of this expression is in that we know an upper bound of $|S_\ell^{d,k}(0)|$ for $k = 1, 2, \infty$, see appendix A, and I.L12.

D2 : For any given function $f:\mathbb{Z}^d \boxtimes \mathbb{Z}^d \to \mathbb{R}$, the long distance limit exist iff for any given $m \in \mathbb{Z}^d$, the folowing limits exist, and are equal

$$\lim_{\ell\to\infty} \inf_{n\in B_\ell^{d,k}(m)} f(m,n) \quad ; \quad \lim_{\ell\to\infty} \sup_{n\in B_\ell^{d,k}(m)} f(m,n),$$

and are also equal to the corresponding limits obtained from replacing $f(m,n)$ by $f(n,m)$. If this holds, we denote the common value by $\lim_{\rho\to\infty} f(m,n)$

Note that the function f in D2 need not to be symmetric.

Classification of Measures by Susceptibility.

Let us rewrite (2.b) as following:

$$\chi_M = \frac{\beta}{|M|}\sum_{m\in M} {}^C\mu^M(m,m) + \frac{\beta}{|M|}\sum_{\substack{m,n\in M\\ m\neq n}} {}^C\mu^M(m,n) \qquad (4)$$

$$= \quad \chi_M^{\circ} \quad + \quad \chi_M' \qquad (5)$$

We shall call χ_M° and χ_M' the diagonal and off-diagonal average, respectively. We denote formally the limit of $\{\chi_M\}_{M\in \underline{C}}$, $\{\chi_M^{\circ}\}_{M\in \underline{C}}$, and $\{\chi_M'\}_{M\in \underline{C}}$ by χ, χ°, and χ', respectively. Obviously, the convegence of $\bar{\chi}$ is controlled by χ° and χ'.

At the critical point, the susceptibility χ diverges. We conjecture that for the kind of the critical point, we had discussed at the begining, the only diverging part is the off-diagonal one. The diagonal part, χ°, should play an other role related to the stability problem. We will come back to this point when we discuss the Gaussian measures on an Hilbert space.

Instead of studying directly the convergence of χ', we investigate the convergence of an net $\{a(\chi_M')\}_{M\in \underline{C}}$;

$$a(\chi_M') \quad = \quad \frac{\beta}{|M|} \sum_{\substack{m,n\in M\\ m\neq n}} |C_{\mu^M}(m,n)| \qquad (6)$$

It is clear that the convergence of $\{a(\chi_M')\}_{M\in \underline{C}}$ implies the convergence of $\{\chi_M'\}_{M\in \underline{C}}$, but $\{\chi_M'\}_{M\in \underline{C}}$ may converge while $\{a(\chi_M')\}_{M\in \underline{C}}$ diverges. Note that χ_M' is not a partial sum, but an arithematic average, thus the theory of abslutely summability is not applicable [55].

Similarly, we denote the limit of $\{a(\chi_M')\}_{M\in \underline{C}}$ formally by $a(\chi')$. The operation a(.) would be as well define for $\{\bar{\chi}_M\}_{M\in \underline{C}}$ and $\{\chi_M^{\circ}\}_{M\in \underline{C}}$. Note that $a(\chi_M^{\circ}) = \chi_M^{\circ}$, thus $a(\chi_M) = \chi_M^{\circ} + a(\chi_M')$.

At this level, we divide the measures in $\mathfrak{M}(\mathcal{B})$ into two classes.

D3 : Let $\mu \in \mathfrak{M}(\mathcal{B})$. If $a(\chi')$ is defined, we call μ a $a\chi'$-convergent measure, otherwise a $a\chi'$-divergent measure. We denote the set of all $a\chi'$-convergent measures in $\mathfrak{M}(\mathfrak{B})$ by $a\chi'c$, and $a\chi'd$ = $\mathfrak{M}(\mathfrak{B}) \setminus a\chi'c$.

If $a\chi c$ denotes the a χ-convergence class, then we have $a\chi c \subset a\chi' c$.

Classification of Measures by 2-Semivariance, and Exponential-Power Law.

From physical intuition the following condition is usually give on the long distance behaviour of the 2-semivariance

$$\lim_{\rho \to \infty} |C_{\mu}(m,n)| = 0 \quad . \tag{7}$$

Any way, the constant limit case is just a trivial extension of this zero limit case. The information from (7) alone is not enough to say whether $\mu \in a\chi'c$ or $a\chi'd$. We have to compare the rate of decay relative to some functions. As usually done the exponential and power functions are the functions we are going to compare with.

D4 : We say that a measure $\mu \in \mathfrak{M}(\mathfrak{B})$ follows the exponetial law with correlation lenght ξ , if (7) is satisfied, and

$$\lim_{\rho \to \infty} \left\{ - \frac{\log|C_{\mu}(m,n)|}{\rho(m,n)} \right\} = \frac{1}{\xi} \tag{8}$$

exist [4].

Note that, whenever the limit exist, we have $0 \leq \frac{1}{\xi} < \infty$, as the condition (7) forbids ξ to be negative. We make a convention, $\xi = \infty$ denote $1/\xi = 0$.

The case when $\xi = \infty$ is alway refered to as critical behaviour in the literature. We would like to point out that this condition is not sufficient to conclude that $X = \infty$, or even $\mu \in aX'd$. We are then led to refine further the classification by considering inverse power laws which have slower rate of decay than the exponential. It is interesting to test simutaneously the exponential and power law.

D5 : We say that a measure $\mu \in \mathfrak{M}(\mathcal{B})$ follows the exponential-power law (ξ,α), if it follows the exponential law with correlation lengh ξ, and moreover

$$\lim_{\rho \to \infty} \left\{ - \frac{\log|C_{\mu}(m,n)|}{\log\rho(m,n)} - \frac{1}{\xi} \frac{\rho(m,n)}{\log\rho(m,n)} \right\} = \alpha \quad . \qquad (9)$$

exist.

It is clear tha $0 \leq \alpha < \infty$. We can now classify the measures in $\mathfrak{M}(\mathcal{B})$ into classes: The set of all measures in $\mathfrak{M}(\mathcal{B})$ which follow the same exponential-power law (ξ,α) is denote by (ξ,α): i.e. $\mu \in (\xi,\alpha)$ implies μ follows exp-pow law (ξ,α). We call the class $(\infty,0)$ a slow decay correlation. If the theory is so sensitive the class $(\infty,0)$ must be refine further, e.g. by log, loglog, ... In this thesis, we consider only the exponential-power law.

The Relation between the Two Classifications.

Another condition, intuitve by physical argument, is the translational invariance, for which can be stated on the 2-semivariance as

$$C_{\mu}(m,n) = C_{\mu}(0,n-m) \qquad (9)$$

Let us define a summation

$$Q_m = \beta \sum_{n \in Z^d \setminus m} |C_\mu(m,n)| \qquad (10)$$

for each $m \in Z^d$. Obviously, the translational invariance condition (9) ensure the uniformity with respect to $m \in Z^d$ of Q_m; i.e. if Q_m exist then $Q_m = Q_o$, for all $m \in Z^d$. This uniformity simplifies the proof of the convergence of the net $\{a(\chi'_M)\}_{M \in \underline{C}}$.

It is obvious that

L6 : $\sup_{m \in Z^d} Q_m \geq a(\chi'_M)$ for all $M \in \underline{C}$.

L7 : If $\mu \in (\xi, \alpha)$, with

(a) $\xi < \infty$,
(b) $\xi = \infty$, and $\alpha > d$,

then Q_m exists for each $m \in Z^d$.

pf : (a) In this case the μ follows an exponential law with finite correlation lengh ξ: For any given $m \in Z^d$, $\varepsilon > 0$, there exist $M_o \in \underline{C}(m)$ such that for all $n \in M_o^c$,

$$e^{-(1/\xi + \varepsilon)\rho(m,n)} < |C_\mu(m,n)| < e^{-(1/\xi - \varepsilon)\rho(m,n)} \qquad (11)$$

Since $1/\xi > 0$, we can choose ε such that $1/\xi > \varepsilon > 0$, so that the upper bound in (11) decays exponentially. Without loss of any generality, we choose $M_o = B_{l_o}(m)$. The summation (10) can be broken up to be the sum inside and out side $B_{l_o}(m)$: The inside sum is clearly bouned. The outside sum is bounded above by

$$\sum_{\ell=\ell_0+1}\sum_{n\in S_\ell^{d,k}(m)} e^{-(1/\xi-\varepsilon)\ell} \tag{12}$$

$$\leq \sum_{\ell=\ell_0+1}\left\{(2\ell+1)^d - (2\ell-1)^d\right\} e^{-(1/\xi-\varepsilon)\ell} \tag{13}$$

$$- \quad 2d3^{d-1}\sum_{\ell=\ell_0+1}\ell^{d-1}e^{-(1/\xi-\varepsilon)\ell} < \infty \quad . \tag{14}$$

This proves the existence of Q_m for each $m \in \mathbb{Z}^d$.

(b) In this case we have that μ follows a power law with exponent $\alpha > d$; i.e. For any $m \in \mathbb{Z}^d$, $\varepsilon > 0$, there exist $M_o \in \underline{C}(m)$ such that for all $n \in M_o^c$, we have

$$\rho(m,n)^{-(\alpha+\varepsilon)} < |C_\mu(m,n)| < \rho(m,n)^{-(\alpha-\varepsilon)} \tag{15}$$

An argument similar to the one given in (a) can be given up to (14), replacing the exponential upper bound by the power law upper bound from (15): i.e.

$$(14') = 2d3^{d-1}\sum_{\ell=\ell_0+1}\ell^{d-1}\ell^{-(\alpha-\varepsilon)} \tag{16}$$

Since $\alpha-d > 0$, we can choose ε such that $\alpha-d-\varepsilon > 0$. Now, the summation in (16) is a summation of $\ell^{-1-\gamma}$ for some positive $\gamma = \alpha-d-\varepsilon$, which is convergent.

This proves the existence of Q_m for each $m \in \mathbb{Z}^d$.||

P8 : If $\mu \in (\xi,\alpha)$, where ξ and α is the same as in L7, and $\sup_{m \in Z^d} Q_m$ exists, then $\mu \in a\chi'c$.

pf : Immediate from L6, L7, and D3.||

Ofcourse, the replacement of the existence condition of $\sup_{m \in Z^d} Q_m$ by translational invariance gives the same conclusion: $\sup_{m \in Z^d} Q_m = Q_o$.

P9 : If $\mu \in (\xi,\alpha)$ with $\alpha < d$, then $\mu \in a\chi'd$.

pf : Considering the metric ρ^{∞}, see A(1.a), we have

$$a(\chi'_{B_\ell^{d,\infty}(0)}) = \frac{\beta}{(2\ell+1)^d} \sum_{\substack{m,n \in B_\ell^{d,\infty}(0) \\ m \neq n}} |C_\mu(m,n)| \tag{17}$$

$$\geq \frac{\beta}{(2\ell+1)^d} \sum_{\substack{m,n \in B_\ell^{d,\infty}(0) \\ \rho^{\infty}(m,n) \geq \ell_o}} |C_\mu(m,n)|$$

$$\geq \frac{\beta}{(2\ell+1)^d} (2\ell+1)^d \left\{ (2\ell+1)^d - (2\ell_o+1)^d \right\} \inf_{\substack{m,n \in B_\ell^{d,\infty}(0) \\ \rho^{\infty}(m,n) \geq \ell_o}} \cdot |C_\mu(m,n)| \quad .$$

Using the lower bound as given in L7,(15), we get

$$a(\chi'_{B_\ell^{d,\infty}(0)}) > \left\{ (2\ell+1)^d - (2\ell_o+1)^d \right\} (2\ell+1)^{-(\alpha+\varepsilon)}$$

$$> A(2\ell+1)^{d-\alpha-\varepsilon} \tag{18}$$

Since $d-\alpha > 0$, we can choose ε such that $d-\alpha-\varepsilon > 0$. Therefore

$$\lim_{\ell \to \infty} a(\chi'_{B_\ell^{d,\infty}(0)}) = \infty$$

This proves the divergence of the net $\{a(\chi'_M)\}_{M \subset}$ as there is a sub-net $\{a(\chi'_{B_\ell^{d,\infty}(0)})\}_{\ell \in N}$ which diverges.||

Note that the other metric ρ^k, k = 1, 2 would change only the constant A.

From what we have discussed the following general scheme is proved:

	(∞,α)	$\subset$	$a\chi' d$	$\alpha < d$	(19)
$\sup_{m \in \mathbb{Z}^d} Q_m < \infty$	(∞,α)	$\subset$	$a\chi' c$	$\alpha > d$	(20.a)
	(ξ,α)	$\subset$	$a\chi' c$	$\alpha \geq 0, \xi < \infty$	(20.b)
$\{\mu \in \mathfrak{M}(\mathcal{B}) \mid \chi' = \infty\}$		$\subset$	$a\chi' d$		(21)

There are a few questions we would point out here:

Q10: Is the inclusion in (21) infact an equality? The possible situation that $\chi' < \infty$ while $a\chi' = \infty$ could occur when $C_\mu(m,n)$ are negative for some, but not for all $m,n \in \mathbb{Z}^d$. We must rule this case by the help of non-negative defintion of C^2.

Q11: Can one prove P8(a) and (b) without assuming the condition that $\sup_{m \in \mathbb{Z}^d} Q_m < \infty$? The exp-pow decay law may be able not only to give the convergence of Q_m for each $m \in \mathbb{Z}^d$, but also to provide an upper bound for the supremum.

Q12: What is the convergence property of aχ' and χ', when $\mu \in (\infty, d)$? It seem likely that this power law needs further refinement into several classes. Some would be aχ'd. It is hard to immagine what kind of functions to compare with.

The RG-invariant of Power Law.

L13: For any given function $f: \mathbb{Z}^d \boxtimes \mathbb{Z}^d \to \mathbb{R}$, which follows a (ξ, α) law, there exists a function $g: \mathbb{Z}^d \boxtimes \mathbb{Z}^d \to \mathbb{R}$, which follows the $(\infty, 0)$ law, such that

$$f(m,n) = \frac{e^{-\frac{1}{\xi}\rho(m,n)}}{\rho(m,n)} g(m,n) \qquad (22)$$

for all $m \neq n \in Z^d$.

pf : Let us define a function $g = fe^{\frac{1}{\xi}\rho}\rho^{\alpha}$. It satisfies. By a straightforward calculation using D4 and D5, we have $g \in (\infty, 0)$.||

L14: For any given function $g: \mathbb{Z}^d \boxtimes Z^d \to \mathbb{R}$ which follows the $(\infty, 0)$ law, then the function $f: Z^d \boxtimes \mathbb{Z}^d: \to R$ defined by formular (22), and arbitrary when m = n, must follows the (ξ, α) law.

pf : Immediate from D4 and D5.||

P15: If $\mu \in (\xi, \alpha)$, then RG-map τ maps μ to $\tau\mu = \mu^1 \in (\xi/L, \alpha)$. Clearly, (∞, α) is an RG-invariant class.

pf : From D5 and L13, ther exists a slowly decaying function g such that

$$C_{\mu}(m,n) = \frac{e^{-\frac{1}{\xi}\rho(m,n)}}{\rho(m,n)} g(m,n), \qquad (23)$$

for all $m \neq n \in \mathbb{Z}^d$. We know from IV.D7(10) that

$$C_{\mu^1}(m_1,n_1) = (\lambda L^{-d})^2 \sum_{\substack{m \in B_L(m_1) \\ n \in B_L(n_1)}} C_{\mu}(m,n) \qquad (24)$$

From the Triangle Inequality, if $\rho(m_1 n_1) > 1$, then

$$\rho(m,n) \lessgtr \rho(m_1 L, n_1 L) \pm L \qquad (25)$$

Applying (23) into (24), then using (25), after rearranging, we get

$$C_{\mu^1}(m_1,n_1) \lessgtr (\lambda L^{-d})^2 . \frac{e^{-\frac{1}{\xi}\rho(m_1 L, n_1 L)}}{\rho(m_1 L, n_1 L)} . e^{\pm \frac{L}{\xi}} . \left[1 \mp \frac{L}{\rho(m_1 L, n_1 L)} \right]^{-\alpha}$$

$$\cdot \sum_{\substack{m \in B_L(m_1) \\ n \in B_L(n_1)}} g(m,n) \qquad (26)$$

The term $[\ldots]^{-\alpha}$ is bounded, and approaches 1 when $\rho(m_1,n_1) \to \infty$. Since $g \in (\infty,0)$, the sum is bounded by L^{2d}, thus $L^{-2d}\sum g$ is bounded, and certainly belongs to $(\infty,0)$. We can rewrite the (26) as following

$$C_{\mu^1}(m_1,n_1) \lessgtr \lambda^2 L^{-\alpha} e^{\pm \frac{L}{\xi}} . \frac{e^{-\frac{L}{\xi}\rho(m_1,n_1)}}{\rho(m_1,n_1)} G_L(m_1,n_1) . \begin{bmatrix} 2^{\alpha} \\ \left(\frac{2}{3}\right)^{\alpha} \end{bmatrix} , \quad (27)$$

when $\rho(m_1,n_1) \geq 2$, where $G_L = L^{-2d} \sum g \in (\infty,0)$. It is clear that there exists $G_L^{\xi} \in (\infty,0)$ such that

$$C_{\mu^1}(m_1,n_1) = \frac{e^{-\frac{L}{\xi}\rho(m_1,n_1)}}{\rho(m_1,n_1)} (\lambda^2 L^{-\alpha}) G_L^{\xi}(m_1,n_1) \qquad (28)$$

By L14, we have $C_{\mu^1} \in (\xi/L,\alpha)$. Therefore $\mu^1 \in (\xi/L,\alpha)$ by D5.||

If we want the case $\mu \in (\infty,\alpha)$ that

$$\lim_{\rho\to\infty} \frac{C_{\mu^1}(m_1,n_1)}{\rho(m_1,n_1)^{-\alpha}} = \lim_{\rho\to} \left[\frac{C_{\mu^1}(m,n)}{\rho(m_1 L,n_1 L)^{-\alpha}} \right]_{\substack{m \in B_L(m_1) \\ n \in B_L(n_1)}} \qquad (29)$$

we must have $$\lambda^2 L^{-\alpha} = 1 \qquad (30)$$

This is a neccessary condition for $\mu = \tau\mu$, the fixed point. This identification is very important, thus most of the time from now on, we write $\lambda = L^{\alpha/2}$, for some $\alpha > 0$.

One can see that an element of the RG-invariant class (∞, α) may be in a χ' c or in a χ' d, see (19)-(21). The RG-invariant class need not to be the $\chi' = \infty$ class; the fixed point need not be a critical point state.

VI. GAUSSIAN MEASURES ON $(X,\mathcal{B})$.

The simplest type of measures on an infinite dimensional linear space is the Gaussian one. The topic is well studied in Mathematics. It is alway chosen as a testing ground of measure theory. A review by Kuo (1975) [79], a book of 224 pages is entirely devoted to the Gaussian measures on Banach spaces. This would impless us how important and complicated the subject is. Several concrete results are clearly exposed by Vakhania (1981) [76]: He concerns more general measure than Gaussian on some sub-space of $\mathbb{R}^{\mathbb{N}}$. The presentation here is close in spirit to this reference.

Motivated by Quantum Field Theory, Segal (1956,1958) [92,93], and Gross, L.(1960-1970) [94-97] have developed the theory of measure on Hilbert and Banach space. The central core of the development are some Guassian measures , as they are the natural representations of the free field. the topic has become one of the popular topics as it is a blending between two brances, Functional Analysis and Probability Theory; see for example Probability on Banach Space edited by Kuelbs (1978) [98]. The feedback to Physics, the Quantum Field Theory and Statistical Mechanics, can be indicated by the following works: Reed (1973) [78], Nelson (1972-1973) [99-102], Dynkin (1980-1983) [103,104], Dobrushin (1980) [105].

We do not pretended to make a complete review either the mathematic of Gaussian measures on linear space, or of the physics of free fields. The attemp is to present some important results related to our problem.

Before all, we give a definition and a proposition adapted from Vakhania (1981) [76] p25-28.

D1 : A measure μ on $(X,\mathfrak{B})$ is called a Gaussian measure if each $f \in X^*$ induce μ a normal distribution with mean

$$a_f = \int_{\mathbb{R}} s\mu_f(ds) \quad ; \quad |a_f| < \infty \quad , \tag{1}$$

and variance

$$b_f^2 = \int_{\mathbb{R}} (s - a_f)^2 \mu_f(ds) \quad , \tag{2}$$

where $\mu_f(ds) = \mu\{x \in X | \; s - \tfrac{1}{2}ds < f(x) < s + \tfrac{1}{2}ds\}$, (3)

The following position is very important as it tells us that the mean and 2-semivariance determined uniquely a Gaussian measure, and vice versa.

P2 : The characteristic function of an arbitrary Gaussian measure on $(X,\mathfrak{B})$ has the form, for each $f \in X^*$,

$$\Theta(f) = \exp\left\{ i\sum_{m \in Z^d} f_m a_m - \tfrac{1}{2}\sum_{m,n \in Z^d} S_{mn} f_m f_n \right\} , \tag{4}$$

where $a \in X$, and the infinite matrix S is symmetric, and non-negative definite: i.e. The double sum is not negative for each $f \in X^*$. Conversely, the functional Θ express by (4) determines uniquely a Gaussian measure on $(X,\mathfrak{B})$.

We simply denote the space of all Gaussian measures in $\mathfrak{M}(\mathfrak{B})$ by $\mathfrak{M}_G(\mathfrak{B})$.

The Gaussian White Noise Measures in $\mathfrak{M}(\mathcal{B})$.

D3 : Let $\mu \in \mathfrak{M}_G(\mathcal{B})$. If there exist a real number $A > 0$ such that

$$C_\mu(m,n) = A\delta_{mn} \tag{5}$$

for all $m,n \in Z^d$, we call μ a Gaussian White Noise (GWN) measure.

If we were to consitder the long distance behaviour of the 2-semivariance of a GWN-measure μ, we will find that it decays faster than any exponential law, which had been described in chapter V: $\xi = 0 < 1$, the lattice spacing.

Obviously, $\chi' = 0, \chi = \chi^0 = \beta A < \infty$. For that reason, the GWN-measure can not be a critical measure.

Applying IV.P7(10) on the 2-semivariance C^2_μ in (5), we get the renormalized 2-semivariance

$$C_{\mu^1}(m_1,n_1) = A\lambda^2 L^{-d}\delta_{m_1 n_1} \tag{6}$$

for all $m_1,n_1 \in \mathbb{Z}^d$. μ^1 is a GWN-measure. It is clear that $C^2_\mu = C^2_{\mu^1}$, iff $\lambda = L^{d/2}$.

Again, we apply IV.P7(10) on the mean, but fix $\lambda = L^{d/2}$; we find the renormalized mean

$$C^1_{\mu^1}(m_1) = L^{-d/2}\sum_{m \in B_L(m_1)} C^1_\mu(m) \quad . \tag{7}$$

It is clear that the GWN-measure is invariant under the RG-map ($\lambda = L^{d/2}$) if the mean $C^1_\mu = C^1_{\mu^1}$. For the translational invariant case there is only one solution, $C^1_\mu = 0 \in X$.

In conclusion, all GWN-measures with mean zero are invariant under both translation and RG-map with $\lambda = L^{d/2}$. Note that there is an infinite number of such measures.

In literature, this GWN-measures is alway refered to as " a high-temperature trivial fixed point". We would strees that it is not " a fixed point" as one can see from the above discussion. The word "high-temperature" come from the fact that for some sufficiently high T*, depend on the potential, for all T $>$ T*, the RG-transformation would give the fixed point of this form.

Another very important fact is that the GWN-measures are not support by ℓ^2_d, an Hilbert sub-space of X: The C^2_μ is not a trace class operator on ℓ^2_d; i.e.

$$\mathrm{Tr}(C^2_\mu) = \sum_{m \in \mathbb{Z}^d} C(m,m) = \infty , \qquad (8)$$

see Kuo (1975) [79] p29, theorem 2.3 (Prohorov), and p32, corollary 2.1

Consider a class of measures, a bit beyond GWN-measure, defined by a 2-semivariance

$$C_\mu(m,n) = w(m)^2 \delta_{mn} \qquad (9)$$

We would have this μ supported by ℓ^2_d if

$$\sum_{m \in \mathbb{Z}^d} w(m)^2 < \infty \quad ; \qquad (10)$$

i.e. $0 \neq w \in \ell^2_d$. Obviously, C^2_μ defined by (9) is symmetric, positive definite, thus μ is a Gaussian measure in $\mathfrak{M}(\mathcal{B})$ evenif (10) is not satisfied.

P4 : All measure $\mu \in \mathfrak{M}_G(\mathcal{B})$ which are supported by ℓ^2_d have $\chi^\circ = 0$

pf : From the Prohorov theorem, op. cit., μ will concentrated in ℓ^2_d iff C^2_μ is a trace class operator on ℓ^2_d. This fact together with (10), and definition of χ° in IV(4) and IV(5), implies $\chi^\circ = 0.||$

If the support of a Gaussian measure is known in advance that it is in an Hilbert sup-space of X, but not in ℓ^2_d, then χ° may not be zero. If the partial sum LHS(10) diverges faster than $|M|$, it is clear that $\chi^\circ = \infty$. The C^2_μ defined in this way is still positive definite; thus there is a measure μ in $\mathfrak{M}(\mathcal{B})$ with $\chi^\circ = \infty$, $\chi' = 0$, $\chi = \infty$. It is a critical one, but has long distance behaviour decay faster than any exponential law. As we have conjectured at the begining of chapter V, the critical point should not be this kind of measures.

The measure μ define (9) would be an RG-invariant if

$$w(m_1)^2 = (\lambda L^{-d})^2 \left\{ \sum_{m \in B_L(m_1)} w(m)^2 \right\} . \qquad (11)$$

For any given λ, there is a large set of solution of (11), thus is large the number of Gaussian measures of this kind that are invariant under the RG-map.

VII. RG-INVARIANT CONVEX COMPACT SUB-SET IN $\mathfrak{M}(\mathfrak{B})$.

In this chapter, the compactness criteria is studied together with the RG-maps. Our intention is to find out an invariant convex compact sub-set in $\mathfrak{M}(\mathfrak{B})$ for each RG-map.

Obviously, $\mathfrak{M}(\mathfrak{B})$ itself is invariant under all RG-maps which have been defined in chapter III. But $\mathfrak{M}(\mathfrak{B})$ is a non-compact space as has been discussed in II.P13. The problem of finding a RG-invariant compact sub-set of $\mathfrak{M}(\mathfrak{B})$ is very delicate. The product topology on X allows us to relate the compactness to a simpler notion, the uniform tightness. Here, we will briefly review this relation.

D1 : A measure μ on $(X, \mathfrak{B})$ is said to be tight if, for each $\varepsilon > 0$, there exist a compact set $K_\varepsilon \subset X$ such that

$$\mu(X \setminus K_\varepsilon) < \varepsilon \qquad (1)$$

P2 : Every measure μ on $(X, \mathfrak{B})$ is tight.

pf : From Pathasarathy (1967) [70] p16, theorem 3.2, every measure on a complete, separable metric space is tight. We know that $(X, \mathfrak{B})$ does have the property, see I.P9.||

D3 : A closed set $F \subset X$ is called the support of a measure μ on $(X, \mathfrak{B})$ if, for each closed set $D \subset X$ such that $\mu(D) = 1$, we have $F \subset D$.

This definition is very sharp, it defines the support as the smallest closed sub-set that gives the total mass one.

P4 : For each $\mu \in \mathfrak{M}(\mathfrak{B})$, there exist a unique support. Moreover, for each point $x \in F_\mu$, the support of μ, and each open neighbourhood U of x, we have $\mu(U) > 0$.

pf : ibid., p27,28, theorem 2.1, and defintion 2.1.||

P5 : For each $\mu \in \mathfrak{M}(\mathcal{B})$, there exists a σ-compact support and for any given $E \in \mathcal{B}$, and $\varepsilon > 0$, there is a compact set $K_\varepsilon \subset E$

with
$$\mu(E \setminus K_\varepsilon) < \varepsilon \qquad (2)$$

pf : Since any measure $\mu \in \mathfrak{M}(\mathcal{B})$ is tight, see P2, the proof can be given as in ibid., p29, theorem 3.1.||

P4, and P5 tell us conclusively that $\mu \in \mathfrak{M}(\mathcal{B})$ has a unique σ-compact support.

P6 : Let $\Gamma \subset \mathfrak{M}(\mathcal{B})$. Then a neccessary and sufficient condition for $\bar{\Gamma}$ being compact is that for any $\varepsilon > 0$, there should exist a compact set $K_\varepsilon \subset X$ such that

$$\mu(K_\varepsilon) > 1 - \varepsilon \quad , \qquad (3)$$

for all $\mu \in \Gamma$. In otherwords, the sub-set Γ must be uniformly tight.

pf : ibid, p47, theorem 6.7.||

Now, we have the simpler criteria for compactness. Any way, the compact sets in $(X, \mathcal{T})$ require a bit technique.

P7 : In order that a set $F \subset X$ be compact relative to the product topology, it is sufficient and necessary that

(a) F is closed in $\mathcal{T}$.

(b) for each point $m \in Z^d$, the set

$$F[m] = \{p_m x | \; x \in F\} \tag{4}$$

has a compact closure in R.

pf : Using the fact that the product topology and point wise convergence topology are equivalent, and the real number space $\mathbb{R}$ is Hausdorff space, the result is proved as in Kelley (1955) |58| p218, theorem 1.||

Let $\{M_n\}_{n \in \mathbb{N}}$ be an increasing sequence of finite sub-sets of $\mathbb{Z}^d$, and let it be a cofinal of $\underline{C}$. Consider, the collection $\{K^{M_n}\}_{n \in \mathbb{N}}$, where $K^{M_n} \in \mathcal{B}^{M_n}$. For each $n \in \mathbb{N}$, the set $p_{M_n}^{-1} K^{M_n}$ is a cylinder set in X. We construct a measurable sub-set of X as follows:

$$K = \bigcap_{n \in \mathbb{N}} p_{M_n}^{-1} K^{M_n} \tag{5}$$

C8 : If K^{M_n} is a compact sub-set of X^{M_n} for each $n \in \mathbb{N}$, then the set K, given in (5), is a compact sub-set of X.

pf : Since p_{M_n} is continuous, and K^{M_n} is compact, thus closed, $p_{M_n}^{-1} K^{M_n}$ is closed in X. K is a countable intersection of closed set, thus closed in X. This proves the condition (a) of P7.

For each $m \in \mathbb{Z}^d$, the set $K[m]$ as defined in (4) becomes

$$K[m] = p_m \bigcap_{n \in N} p_{M_n}^{-1} K^{M_n} \tag{6}$$

$$\text{RHS}(6) \subset \bigcap_{n=n_o}^{\infty} p_m K^{M_n} \qquad (7)$$

where n_o is the first number that $m \in M_n$. Since p_m is continuous, the RHS(7) is an intersection of compact sets in $\mathbb{R}$, thus compact in $\mathbb{R}$. Therefore, $K[m]$ must be compact in $\mathbb{R}$. This proves the condition (b) of P7.||

Certain classes of measurable sets in the finite dimension sub-spaces of X are particularly interesting.

D9 : For each $M \in \underline{C}$, we denote certain class of measurable set in X^M as following: For each $\bar{x} \in X^M$,

$$A_{\kappa}^{M}(\bar{x}) = \{x \in X^M | \ |\textstyle\sum_{m \in M}(x_m - \bar{x}_m)| \le \kappa(M)\} \qquad (8.a)$$

$$B_{\kappa}^{M,p}(\bar{x}) = \{x \in X^M | \ \textstyle\sum_{m \in M}|x_m - \bar{x}_m|^p \le \kappa(M)^p\} \qquad (8.b)$$

$$C_{a}^{M}(\bar{x}) = \{x \in X^M | \ |x_m - \bar{x}_m| \le a_m(M), \text{ for all } m \in M\} \qquad (8.c)$$

where $\kappa(M) > 0$, and $a(M) \in X^M$, $a_m(M) > 0$ for all $m \in M$. The $\kappa(M)$, and $a_m(M)$, may or may not depend on M.

The set $A_{\kappa}^{M}(\bar{x})$ is a parallel plate centered at $\bar{x}$ with thickness $2\kappa(M)$, where the normal vector to the plate is $r = |M|^{-1}\sum_{m \in M} e_m$.

The set $B_{\kappa}^{M,p}(\bar{x})$ is a ball, with the ℓ_d^p norm, centered at $\bar{x}$ with radius $\kappa(M)$.

The set $C_{a}^{M}(\bar{x})$ is a rectangle in X^M, centered at $\bar{x}$ with the side along the m-coordinate $2a_m(M)$.

A_{κ}^{M} is not compact but closed; B_{κ}^{M} and C_{a}^{M} are compact. Let A_{κ}, B_{κ}^{p} and C_a be the sets defined analogously to K in (5). A_{κ} may or may not be compact; B^p_x and C_a are certainly compact, by C8.

Whenever the center $\bar{x}$ of these sets is understood, we do not mention it explicitly: e.g. $C^M_a(0) = C^M_a$.

For each $M_1 \in \underline{C}$, the set

$$\pi^{-1}_M C^{M_1}_a \quad = \quad \left\{ x \in X^M \mid \pi_M x \in C^{M_1}_a \right\} \tag{9}$$

is a measurable set in X^M, but it is not compact:

$$\pi^{-1}_M C^{M_1}_a \quad = \quad \bigcap_{m_1 \in M_1} \pi^{-1}_M \circ p^{-1}_{M_1 m_1} C^{m_1}_{a_{m_1}} \tag{10}$$

$$= \quad \bigcap_{m_1 \in M_1} p^{-1}_{MB_L(m_1)} \circ \pi^{-1}_{B_L(m_1)} C^{m_1}_{a_{m_1}} \tag{11}$$

Since the set

$$\pi^{-1}_{B_L(m_1)} C^{m_1}_{a_{m_1}} \quad = \left\{ x \in X^{B_L(m_1)} \;\middle|\; \left|\sum_{m \in B_L(m_1)} x_m\right| \le \frac{a_{m_1}}{\lambda L^{-d}} \right\} \tag{12}$$

is a parallel plate centered at $\bar{x} = 0$, it is unbounded; $\pi^{-1}_M C_a$ is clearly unbounded.

P10: For each $M_1 \in \underline{C}$, and a given positive $a \in X^{M_1}$, there exist a positive $b° \in X^M$ such that

$$C^M_b \quad \subset \quad \pi^{-1}_M C^{M_1}_a \tag{13}$$

for all $0 < b \le b°$. Infact, we can choose $b°$ either from

$$\lambda L^{-d} \sum_{m \in B_L(m_1)} b°_m \quad \le \quad a_{m_1} \tag{14}$$

or $$\lambda^2 L^{-d} \sum_{m\in B_L(m_1)} b^{o,2}_m \leq a^2_{m_1} \tag{15}$$

The proof of this proposition is given in appendix C. This proposition tells us that $\pi_M^{-1} C_a^{M_1}$ contains some simple compact sets.

For any measure $\mu \in \mathfrak{M}(\mathcal{B})$, we have

$$\lim_{|a|\to\infty} |\mu \circ p_M^{-1} \circ \pi_M^{-1} C_a^{M_1} - \mu \circ p_M^{-1} C_b^M| = 0 \tag{16}$$

if b is chosen large eneough, and are satisfied (14) or (15): If a is chosen large enough the set $\pi_M^{-1} C_a^{M_1} \setminus C_b^M$ becomes irrelevant. If we have a sufficiently good estimate of the measure $\mu \circ p_M^{-1}$ on the compact set C_b^M, we can estimate the renormalized measure $\mu^1 \circ p_{M_1}^{-1}$ on the compact set $C_a^{M_1}$.

L11: For any given homogeneous scaling RG-map τ, and any measure $\mu \in \mathfrak{M}(\mathcal{B})$, we can give an upper bound of μ and μ^1 on each cylider set of the form $p_{M_1}^{-1}(X^{M_1} \setminus C_a^{M_1})$, for each $M_1 \in \underline{C}$, as following

$$\mu \circ p_{M_1}^{-1}(X^{M_1} \setminus C_a^{M_1}) \leq \sum_{m_1\in M_1} \frac{1}{a^2_{m_1}} E_\mu(e^{*2}_{m_1}) \tag{17}$$

$$\mu^1 \circ p_{M_1}^{-1}(X^{M_1} \setminus C_a^{M_1}) \leq \sum_{m_1\in M_1} \frac{(\lambda L^{-d})^2}{a^2_{m_1}} \sum_{m,n\in B_L(m_1)} E_\mu(e^*_m e^*_n) \tag{18}$$

$$\leq \sum_{m_1\in M_1} \frac{\lambda^2 L^{-d}}{a^2_{m_1}} \sum_{m\in B_L(m_1)} E_\mu(e^{*2}_m) \tag{19}$$

The proof can be found in appendix C. This is a rough estimate. The 2-moments can be replaced by 2-semivariance, when the 1-moments are all zero. The replacement still can be done for the general case by moving the center of all rectangles to the mean: i.e. $\bar{x}_m = E_\mu(e^*_m)$ for all $m \in \mathbb{Z}^d$, see appendix C.

Since the RHS(17)-(19) are upper bounds, we can find the least upper bound of both μ and μ^1 on the cylinder set. However, to obtain a uniform bound with respect to L, one needs the condition that

$$\lim_{L\to\infty} (\lambda L^{-d})^2 \sum_{m,n \in B_L(m_1)} C_\mu(m,n) < \infty \tag{20}$$

This condition is the most delicate. Since we are interested in the critical point measure, see chapter IV, we know that

$$\lim_{L\to\infty} L^{-d} \sum_{m,n \in B_L(m_1)} C_\mu(m,n) = \infty \tag{21}$$

In order to fulfill both (20) and (21), the factor $\lambda^2 L^{-d}$ must be smaller than $L^{-\beta d}$ for some $\beta > 0$ such that

$$\lim_{L\to\infty} L^{-\beta d} L^{-d} \sum_{m,n \in B_L(m_1)} C_\mu(m,n) < \infty \tag{22}$$

The Uniform Tightness of $\{\tau^{k-1}\mu\}_{k \in \mathbb{N}}$.

Let find out a strong sufficient condition which guarantees that

$$\Gamma = \{\tau^{k-1}\mu\}_{k \in \mathbb{N}} \tag{23}$$

is a uniformly tight sub-set of $\mathfrak{M}(\mathcal{B})$.

For each $M_1 \in \underline{C}$, we define

$$\Gamma_{M_1} = \left\{(\tau^{k-1}\mu) \circ p_{M_1}^{-1}\right\}_{k \in \mathbb{N}} \tag{24}$$

L12: It is sufficient for the measure $\mu \circ p_{M_1}^{-1}$ and $\mu^1 \circ p_{M_1}^{-1}$ to be uniformly tight that

$$E_\mu(e_{m_1}^{*2}) \geq (\lambda L^{-d})^2 \sum_{m,n \in B_L(m_1)} E_\mu(e_m^* e_n^*) \qquad (25)$$

for each $m_1 \in M_1$.

pf : As we have discussed in chapter IV, $0 \leq E\,(e_n^{*2}) < \infty$ for all $n \in Z^d$, for each $\mu \in \mathcal{M}(\mathcal{B})$. Therefore,

$$0 \leq \sup_{m_1 \in M_1} E_\mu(e_{m_1}^{*2}) \leq E_{M_1} < \infty . \qquad (26)$$

Chosing a in L10 such that

$$a_{m_1}^2 = \rho^2 E_{M_1} |M_1| \qquad (27)$$

for each $m_1 \in M_1$.

It is very easy to verify that (17) and (18) are bounded above by $1/\rho^2$ if (25) is given: i.e.

$$\mu \circ p_{M_1}^{-1}(X^{M_1} \setminus C_a^{M_1}) \leq \frac{1}{\rho^2} \qquad (28.a)$$

$$\mu^1 \circ p_{M_1}^{-1}(X^{M_1} \setminus C_a^{M_1}) \leq \frac{1}{\rho^2} \qquad (28.b)$$

For any given $\varepsilon > 0$, we can find a compact set $C_{a'}^{M_1}$, where $a' > a$, and chosing $\rho = 1/\sqrt{\varepsilon}$, such that

$$\mu \circ p_{M_1}^{-1}(X^{M_1} \setminus C_{a'}^{M_1}) \leq \varepsilon \qquad (29.a)$$

$$\mu^1 \circ p_{M_1}^{-1}(X^{M_1} \setminus C_{a'}^{M_1}) \leq \varepsilon \qquad (29.b)$$

This proves the uniformly tight of $\mu \circ p_{M_1}^{-1}$ and $\mu^1 \circ p_{M_1}^{-1}$. ||

L13: Γ_{M_1} is uniformly tight if

$$E_{\mu}(e^{*2}_{m_1}) \geq \lim_{L\to\infty}(\lambda L^{-d})^2 \sum_{m,n\in B_L(m_1)} E_{\mu}(e^*_m e^*_n) \qquad (30)$$

for each $m_1 \in M_1$.

pf : Since $\tau_L^{k-1}\mu = \tau^1_{L^{k-1}}\mu$, the uniformly tightness of Γ_{M_1} is immediate from L12.||

P14: $\bar{\Gamma}_{M_1}$ is compact if (30) is satisfied.

pf : Immediate from L13 and P6.||

The condition (30) is very strong. Though the uniformity with respect to $M_1 \in \underline{C}$ is not explicitly given. We expect that $\bar{\Gamma}$ is compact if (30) is satisfy on each $M_1 \in \underline{C}$.

Constructing the set C_a as the same as (5), thus it is compact by C8. We choose $a(M_j) = a^j$, similar to (27), but replace ρ by $\rho(M_j) = \rho^j$, and $E_{M_j} = E_j$:

$$a_m^{j,2} = \rho^{j,2} E_j |M_j| \qquad (31)$$

L15: Γ is uniformly tight if (30) is satisfied for all $M \in \underline{C}$.

pf :
$$\mu(X \setminus C_a) \leq \sum_{j\in N} \mu\circ p_{M_j}^{-1}(X^{M_j} \setminus C^{M_j}_{a^j})$$
$$\leq \sum_{j\in N} \sum_{m\in M_j} \frac{1}{a_m^{j,2}} E_{\mu}(e^{*2}_m) \qquad (32)$$
$$\leq \sum_{j\in N} \frac{1}{\rho^{j,2}} \qquad (33)$$

where the procedure is the same as in the proof of L13.

For any $\varepsilon > 0$, we choose

$$\rho^{j} = \sqrt{\frac{2^{j}}{\varepsilon}} \tag{34}$$

then substitute it in (33), we obtain

$$\mu(X \setminus C_a) < \varepsilon \sum_{j \in \mathbb{N}} 2^{-j} = \varepsilon \tag{35}$$

For each $k \in \mathbb{N}$, we can estimate

$$(\tau^{k-1}\mu)(X \setminus C_a) \leq \sum_{j \in \mathbb{N}} \sum_{m \in M_j} \frac{(\lambda L^{-d})^{2(k-1)}}{a_m^{j,2}} \sum_{n,\ell \in B_{L^{k-1}}(m)} E_{\mu}(e^*_{\ell} e^*_{n}) \tag{36}$$

When $k = 1$, it is the equation (32): $B_{L^0}(m) = m$. It is bounded above by the RHS(32) because of the assumption that (30) must be satisfied for all $M \in \underline{C}$; Therefore it is also on the cofinal $\{M_j\}_{j \in N}$. We can proceed, and obtain a result similar to (35):

$$(\tau^{k-1}\mu)(X \setminus C_a) \leq \varepsilon \tag{37}$$

for all $k \in \mathbb{N}$. In conclusion, for any given $\varepsilon > 0$, there exist a compact set C_a such that (37) is satisfy. This proves the uniform tightness of .||

The RG-Invariant compact set $\bar{\Gamma}$.

Since the set Γ is now uniformly tight, we can immediately state a proposition:

P16: $\bar{\Gamma}$ is compact if (30) is satisfied foe all $M \in \underline{C}$.

pf : Immediate from L15 and P6.||

P17: $\bar{\Gamma}$ is invariant under the RG-map τ : i.e. $\tau\bar{\Gamma} \subset \bar{\Gamma}$.

pf : It is obvious that $\tau\Gamma \subset \Gamma$. Since τ is continuous in the weak convergence topology on $\mathfrak{M}(\mathfrak{B})$, see III.P8, we have $\tau(\bar{\Gamma}) \subset \overline{\tau(\Gamma)}$, see e.g. Eisenberg (1974) [61] p178, corollary 3.6. The consequence is $\tau(\bar{\Gamma}) \subset \bar{\Gamma}$.||

Here, we arive at the conclusion that $\bar{\Gamma}$ is a τ-invariant compact sub-set of $\mathfrak{M}(\mathfrak{B})$, under the strong sufficient condition that (30) must be satisfied for all $M \in \underline{C}$. The question is whether the set of measures which satisfy (30) is not empty.

E18: Let μ be a GWN measure defined on $(X, \mathfrak{B})$ with mean zero, and 2-semivariance as given in VI(5). For any RG-map τ with the scaling factor $\lambda = L^{\alpha/2}$, where $0 < \alpha \leq d$, the conditon (30) is satisfied:

$$L^{\alpha-2d} \sum_{m,n \in B_L(m_1)} E_\mu(e^*_m e^*_n) = L^{\alpha-2d} \sum_{m \in B_L(m_1)} A$$

$$= L^{\alpha-d} A \qquad (38)$$

$$\leq A = E_\mu(e^{*2}_{m_1}) \qquad (39)$$

When $\alpha > d$, the condition is not satisfied.

E19: Considering a DD-measure $\mathfrak{D}_{\bar{x}}$, see in appendix C. The $E_{\mathfrak{D}_{\bar{x}}}(e^*_m e^*_n) = \bar{x}_m \bar{x}_n$, see C(15). The condition (30) for the DD-measure is

$$\bar{x}^2_{m_1} \geq \lim_{L\to\infty} (\lambda L^{-d})^2 \sum_{m,n \in B_L(m_1)} \bar{x}_m \bar{x}_n \qquad (40)$$

$$= \lim_{L\to\infty} (\lambda L^{-d})^2 \left(\sum_{m \in B_L(m_1)} \bar{x}_m \right)^2 \qquad (41)$$

If $\mathfrak{D}_{\bar{x}}$ is conditioned to be translational invariant, we have

$$\bar{x}_o^2 \geq \left(\lim_{L\to\infty}\lambda\right)\bar{x}_o^2 \qquad (42)$$

If $\lambda = L^{\alpha/2}, \alpha > 0$, the RHS(42) is equal to $+\infty$. Therefore, the only translational invariant DD-measure that satisfied the condition (30) must have mean zero: i.e. $\bar{x}_o = 0$.

This two examples gives us confidence that the condition (30) is not so exagerately strong that it can not be satisfied by any measure in $\mathfrak{M}(\mathfrak{B})$ and RG-map. Any way it requires a more detailed study for each given RG-map to give an explicit conclusion that there is a measure which satisfied (30) and is neither GWN nor DD.

The Convex Extension of Γ and Its Closure.

Here, we denote by co(.) the convex extention of any set, see Kelley-Namioka (1963) [106] chapter 1, section 2, and chapter 4, or Dunford-Schwartz Part I (1957) [57] chapter V. We can express any measure $\nu \in co(\Gamma)$ as

$$\nu = \sum_{k\in\mathbb{N}} \gamma_{k-1}(\tau^{k-1}\mu) \qquad (43)$$

where $\gamma_{k-1} \geq 0$, for all $k \in \mathbb{N}$, and

$$\sum_{k\in\mathbb{N}} \gamma_{k-1} = 1 . \qquad (44)$$

L20: $co(\Gamma)$ is uniformly tight if (30) is satisfied for all $M \in \underline{C}$.

pf : Since $$E_\nu(e_m^{*2}) = \sum_{k\in\mathbb{N}} \gamma_{k-1} E_{\tau^{k-1}\mu}(e_m^{*2}) \quad , \qquad (45)$$

where $\{\gamma_{k-1}\}_{k\in N}$ is given in (44), it is obvious that

$$E_{\nu}(e_m^{*2}) \leq E_{\mu}(e_m^{*2}) \qquad (46)$$

then the uniform tightness can be easily proved as in L15.||

P21: $\overline{co(\Gamma)}$ is a compact convex sub-set of $\mathfrak{M}(\mathcal{B})$ if (30) is satisfied, for all $M \in \underline{C}$.

pf : $\overline{co(\Gamma)}$ is compact by L20, and P6. It is convex as the closure of a convex set is convex: Kelley-Namioka (1963) [106] p11o, 13.1 (i).||

P22: $\overline{co(\Gamma)}$ is invariant under the RG-map τ.

pf : Since $\tau\nu = \sum_{k\in N} \gamma_{k-1}\tau^k\mu \in co(\Gamma)$,

the proof can make use of the proof of P17.||

P23: If condition (30) is satisfied for each $M \in \underline{C}$, $\overline{co(\Gamma)}$ is a compact convex sub-set of $\mathfrak{M}(\mathcal{B})$ invariant under the RG-map .

pf : Immediate from P21, and P22.

We have now fulfilled the aim we set up at the begining of this chapter: The set $\overline{co(\Gamma)}$ in P23 has all the properties we want.

Sub-Gaussian Measures.

Motivated by the role of Gaussian measures in Physics and Super-Stability in Statistical Mchanics [80,108-110], we introduce a class of measures on $(X,\mathcal{B})$, the $\varkappa$-Sub-Gaussian class. The concept of Sub-

Gaussian random processes can be found in Jain and Marcus (1978) [112]. We adapt the notion to measures on $(X, \mathfrak{B})$ as we present here.

Let ν be a Gaussian measure in $\mathfrak{M}(\mathfrak{B})$, for each $f \in X^*$, and $\lambda \in \mathbb{R}$, we can evaluate

$$E_{\nu}(e^{\lambda f}) = e^{\lambda C_{\nu}^{1}(f) + \frac{1}{2}\lambda^2 C_{\nu}^{2}(f,f)} . \qquad (47)$$

D24: A measure $\mu \in \mathfrak{M}(\mathfrak{B})$ is $(\varkappa,\nu)$ Sub-Gaussian ($\varkappa\nu$SG) if there exist a $\nu \in \mathfrak{M}_G(\mathfrak{B})$, and $\varkappa > 0$, such that

$$E_{\mu}(e^{\lambda f}) \leq \varkappa e^{\lambda C_{\nu}^{1}(f) + \frac{1}{2}\lambda^2 C_{\nu}^{2}(f,f)} , \qquad (48)$$

for each $f \in X^*$, and $\lambda \in \mathbb{R}$.

Let us define the set of all measures in $\mathfrak{M}(\mathfrak{B})$ which can be dominated by a given Gaussian measure and a positive constant: For any given $\nu \in \mathfrak{M}_G(\mathfrak{B})$, and $\varkappa > 0$, we define

$$S_{\varkappa,\nu} = \{\mu \in \mathfrak{M}(\mathfrak{B}) \mid \mu \text{ is a } \varkappa\nu\text{SG}\} . \qquad (49)$$

Here, we give some simple properties of $S_{\varkappa,\nu}$:

P25: (i) If $\mu \in S_{1,\nu}$, then $E_{\mu}(f) = E_{\nu}(f)$ for each $f \in X^*$.

(ii) If $\varkappa < 1$, then $S_{\varkappa,\nu} = \emptyset$.

(iii) A $\eta \in \mathfrak{M}_G(\mathfrak{B})$ which satisfies the following condition, for each $f \in X^*$,

$$E_{\eta}(f) = E_{\nu}(f) \quad , \quad \text{and} \quad C_{\eta}^{2}(f,f) \leq C_{\nu}^{2}(f,f) \quad , \qquad (50)$$

belong to $S_{\varkappa,\nu}$, for all $\varkappa \geq 1$. In particular when $\varkappa = 1$, the converse is also true

(iv) A DD-measure which is supported at

$$\bar{x} = \sum_{m \in \mathbb{Z}^d} E_\nu(e_m^*) e_m \qquad (51)$$

belong to $S_{\chi,\nu}$, for all $\chi \geq 1$. Inparticular when $\chi = 1$, the converse is also true.

(v) $S_{\chi,\nu} = \bar{S}_{\chi,\nu}$, for all $\chi \geq 1$

pf : (i); The result of Jain and Marcus (1978) [112] p110-111, lemma 5.2, can be immediately extended to this case. Any way, it is worthwise to rproduce the proof here as it is different in presentation:

$$\lim_{\lambda \to 0^{\pm}} E_\mu\left(\frac{e^{\lambda f} - 1}{\lambda}\right) = E_\mu\left(\lim_{\lambda \to 0^{\pm}}\left(\frac{e^{\lambda f} - 1}{\lambda}\right)\right) \qquad (52)$$

by the Lebesgue Convergence Theorem.

$$\text{RHS}(52) = E_\mu(f) \qquad (53)$$

$$\text{LHS}(52) \begin{cases} \leq & E_\nu(f) + \lim_{\lambda \to 0^+}\left(\frac{\chi - 1}{\lambda}\right) \\ \geq & E_\nu(f) + \lim_{\lambda \to 0^-}\left(\frac{\chi - 1}{\lambda}\right) \end{cases} \qquad (54)$$

One can see that the case $\chi = 1$, we must have $E_\mu(f) = E_\nu(f)$. This proves (i)

Note that, the case $\chi > 1$, tell us nothing as (52) (53) (54) would tell us an ovious fact that $E_\mu(f)$ lay between - and + .

(ii); In case $\chi < 1$, (52-54) bring the $E_\mu(f)$ into an impossible region: $\infty \leq E_\mu(f) \leq -\infty$. therefore, the $S_{\chi,\nu}$ can only be empty.

(iii); Applying the conditon (50) into the expression (47) when the Gaussian measure is η, then we have

$$E_{\eta}(e^{\lambda f}) \le E_{\nu}(e^{\lambda f}) \le \varkappa E_{\nu}(e^{\lambda f}) \qquad (55)$$

for all $\varkappa \ge 1$. This clearly means that the Gaussian measure $\eta \in S_{\varkappa,\nu}$.

When $\varkappa = 1$, by (i) , $E_{\eta}(f) = E_{\nu}(f)$. By (48), and positive definite of the 2-semivariance, we have $C^2_{\eta}(f,f) \le C^2_{\nu}(f,f)$, for each $f \in X^*$. This proves that the particular case, $\eta \in S_{1,\nu}$, must satisfy (50).

(iv); Let $\mathcal{D}_{\bar{x}}$ be the DD-measure. From the property of DD-measure, we have

$$E_{\mathcal{D}_{\bar{x}}}(e^{\lambda f}) = e^{\lambda f(\bar{x})} \quad . \qquad (56)$$

Using (51) on the RHS(56), and the fact that 2-semivariance must be positive definite, we have

$$E_{\mathcal{D}_{\bar{x}}}(e^{\lambda f}) = e^{\lambda E_{\nu}(f)} \le e^{\lambda E_{\nu}(f) + \frac{1}{2}\lambda^2 C^2_{\nu}(f,f)} , \qquad (57)$$

for each $f \in X^*$. We can conclude that $\mathcal{D}_{\bar{x}} \in S_{\varkappa,\nu}$ for all $\varkappa \ge 1$.

The particular case, $\varkappa = 1$: Since any DD-measure is completely determinded by its means, and (i), any DD-measure in $S_{1,\nu}$ must be supported by $\bar{x}$.

(v); Let $\{\mu_n\}_{n \in N}$ be a Cauchy sequence in $S_{\varkappa,\nu}$. Obviously it is also a Cauchy sequence in $\mathcal{M}(\mathcal{B})$. Since $\mathcal{M}(\mathcal{B})$ is a complete metric space, it must converge to a unique measure μ in $\mathcal{M}(\mathcal{B})$. Considering, the ratio

$$\frac{E_{\mu}(e^{\lambda f})}{E_{\nu}(e^{\lambda f})} = \left\{\frac{E_{\mu}(e^{\lambda f})}{E_{\mu_n}(e^{\lambda f})}\right\}\left\{\frac{E_{\mu_n}(e^{\lambda f})}{E_{\nu}(e^{\lambda f})}\right\} \tag{58}$$

The first $\{\ldots\}$ converges to unity as the measure μ is the strong limit of the sequence $\{\mu_n\}_{n\in N}$. The second $\{\ldots\}$ is alway $\le \chi$, because $\mu_n \in S_{\chi,\nu}$. Therefore the RHS(55) converge to a number in $(0,\chi]$. After rearranging, we obtain exactly (48). This proves that μ is a $\chi\nu$SG, thus belong to $S_{\chi,\nu}$. As a metric sub-space , this implies that $S_{\chi,\nu}$ is closed in $\mathcal{M}(\mathcal{S})$, i.e. $S_{\chi,\nu} = \overline{S}_{\chi,\nu}$.||

L26: Let $\mu \in S_{\chi,\nu}$. For any given $m \in Z^d$, $\bar{x}_m \in R$, $a_m > 0$, we have

$$\mu\circ p_m^{-1}\left\{X^m \setminus C^m_{a_m}(\bar{x}_m)\right\} \le 2\chi\cosh\left\{\frac{a_m|E_\nu(e^*_m) - \bar{x}_m|}{C_\nu(m,m)}\right\}.\exp\left\{\frac{-a_m^2}{C_\nu(m,m)}\right\}. \tag{59}$$

pf : For any $\alpha > 0$, we can rewrite the set

$$F^m = p_m^{-1}\left\{X^m \setminus C^m_{a_m}(\bar{x}_m)\right\} \tag{60}$$

as a disjoint union of F^m_+ and F^m_- :

$$F^m_\pm = \left\{x \in X|\ e^{\pm\alpha(x_m - \bar{x}_m)} > e^{\alpha a_m}\right\} \tag{61}$$

Using the Chebyshev Inequality, op.cit., we can estimate

$$\mu(F^m_\pm) \le E_\mu(e^{\pm\alpha e^*_m})e^{\mp\alpha\bar{x}_m - \alpha a_m} \tag{62}$$

Applying (48) to the first term in RHS(62), we get

$$\mu(F^m_{\pm}) \leq \varkappa e^{\pm\alpha\left(E_\nu(e^*_m) - \bar{x}_m\right) - \alpha a_m + \frac{1}{2}\alpha^2 C_\nu(m,m)} \quad . \qquad (63)$$

Since we know that the covariance $C_\nu(m,m)$ must be non-zero, positive-definite, and bounded, we can choose the parameter

$$\alpha = \frac{a_m}{C_\nu(m,m)} \quad , \qquad (64)$$

then replace it in (63), we get

$$\mu(F^m_{\pm}) \leq \varkappa \exp\left\{\frac{\pm 2a_m\left[E_\nu(e^*_m) - \bar{x}_m\right] - a_m^2}{2C_\nu(m,m)}\right\} \quad . \qquad (65)$$

After summing up, we get (59).||

C27: As in L26, but $\bar{x}_m = E_\nu(e^*_m)$, then

$$\mu \circ p_m^{-1}\left\{X^m \setminus C^m_{a_m}(\bar{x}_m)\right\} = 2\varkappa \exp\left\{\frac{-a_m^2}{2C_\nu(m,m)}\right\} \qquad (66)$$

pf : Immediate from L26; cosh(0) = 1.||

C28: As in L26, but $a_m > |E_\nu(e^*_m) - \bar{x}_m| \geq 0$, then

$$\mu \circ p_m^{-1}\left\{X^m \setminus C^m_{a_m}(\bar{x}_m)\right\} \leq 2\varkappa \exp\left\{-\frac{\left[a_m - E_\nu(e^*_m) + \bar{x}_m\right]^2}{2C_\nu(m,m)}\right\} \quad , \qquad (67)$$

pf : Instead of chosing α as given in (64), we choose

$$\alpha = \frac{a_m - E_\nu(e^*_m) + \bar{x}_m}{C_\nu(m,m)} > 0 \qquad (68)$$

The proof can proceed forward, and we get (67) as the result.||

P25(i) and C27, together, equivalent to lemma 5.2 in Jain and Marcus, op.cit.

From L26, one can see that the upper bound of RHS(59) is controlled by two factor "a_m" and "$\bar{x}_m$": e.g. one has to choose $|E_\nu(e_m^*) - \bar{x}_m|$, and a_m bounded above. Ofcourse, the choice of C27 is the easiest, and the best, as we will use it in the proof of P29.

P29: $S_{\chi,\nu}$ is a compact sub-set of $\mathfrak{M}(\mathcal{B})$.

pf : Let $\bar{x}$ be given by (51), and

$$a_m^{j,2} = 2\left\{j + \log|M_j| - \log\left[\left(\frac{\varepsilon}{2\chi}\right)\left(\frac{e-1}{e}\right)\right]\sup_{m\in M_j} C_\nu(m,m)\right\} \quad .(69)$$

If proceed in the proof as we did in L15, we will end up with an estimate of any $\mu \in S_{\chi,\nu}$ as following

$$\mu(X \setminus C_a(\bar{x})) < \varepsilon \quad , \qquad (70)$$

where we have used (69) and C27 along the way.

We conclude that $S_{\chi,\nu}$ is a uniformly tight sub-set of $\mathfrak{M}(\mathcal{B})$, because there exists a compact set $C_a(\bar{x})$ such that (70) is true for any given $\varepsilon > 0$, for all $\mu \in S_{\chi,\nu}$. From P25(v) and P6, it follows that the set $S_{\chi,\nu}$ is compact.||

P30: $S_{\chi,\nu}$ is a convex set.

pf : Let μ_1 and μ_0 be any pair of measures in $S_{\varkappa,\nu}$, and $0 \leq \gamma \leq 1$. We define a measure

$$\mu = \gamma\mu_1 + (1-\gamma)\mu_0 \qquad (71)$$

We can evaluate

$$E_{\mu}(e^{\lambda f}) = \gamma E_{\mu_1}(e^{\lambda f}) + (1-\gamma)E_{\mu_0}(e^{\lambda f}) \qquad (72)$$

Since μ_1 and μ_0 are in $S_{\varkappa,\nu}$ we must have

$$\text{RHS}(72) \leq \varkappa E_{\nu}(e^{\lambda f}) \qquad (73)$$

This proves the convexity of the set $S_{\varkappa,\nu}$.||

Here, we arive at the point where we give a sufficient condition for the set $S_{\varkappa,\nu}$ to be invariant under a RG-map .

P31: If $\tau\nu \subset S_{1,\nu}$, then $\tau S_{\varkappa,\nu} \subset S_{\varkappa,\nu}$ for all $\varkappa \geq 1$.

pf : As we have previously discussed that any $f \in X^*$ must be supported by a finite sub-set, e.g. there exist $M \in \underline{C}$ such that $f(x) = \sum_{m \in M} f_m x_m$, for all $x \in X$. This fact together with the definition of RG-map τ, which is induced stepwise from the BS-transformations, then fdRG-maps, we have

$$E_{\tau\mu}(e^{\lambda f}) = E_{\mu \circ p_M^{-1}}(e^{\lambda f \circ \pi_M})$$

$$\leq \varkappa E_{\nu \circ p_M^{-1}}(e^{\lambda f \circ \pi_M})$$

$$= \varkappa E_{\tau\nu}(e^{\lambda f})$$

$$\leq \quad \varkappa E_{\nu}(e^{\lambda f}) \quad . \qquad (74)$$

Therefore $\tau\mu \in S_{\varkappa,\nu}$, for all $\mu \in S_{\varkappa,\nu}$. ||

Note that the condition is very strong as it request $\tau\nu \in S_{1,\nu} \subset S_{\varkappa,\nu}$. If we ask for $\tau\nu \in S_{\varkappa,\nu}$, we will arrive at an upper bound

$$\text{RHS}(74') \quad = \quad \varkappa^2 E_{\nu}(e^{\lambda f}) \quad .$$

This provides us nothing about the invariance.

Ofcourse, if one knows in advanced that the Gaussian measure ν is a fixed point of the RG-map τ, then $\tau\nu = \nu \in S_{1,\nu} \subset S_{\varkappa,\nu}$. Hence, the P31 go through. This point is technically important as we provide an invariant compact convex neighborhood of the fixed point ν, for which the stability of such fixed point can be studied in details; e.g. the contraction of the map in the neighborhood.

E32: Let a function $\Phi: X^* \rightarrow \mathbb{C}$ be given for each $f \in X^*$:

$$\Phi(f) \quad = \quad \exp\left\{ if(\bar{x}) - \tfrac{1}{2} C_{\nu}(f,f) e^{\sum_{m \in \mathbb{Z}^d} f_m^2} \right\} \quad . \qquad (75)$$

One can easily check that $\Phi(0) = 1$, it is positive definite, and it is continuous at zero under the topology of uniform convergence on compact sets. Therefore, there exists a unique measure μ on $(X, \mathcal{B})$, which has this function as the characteristic function: i.e.

$$E_{\mu}(e^{if}) \quad = \quad \Phi(f) \quad . \qquad (76)$$

It is clear that

$$E_{\mu}(e^{\lambda f}) = \exp\left\{\lambda f(\bar{x}) + \tfrac{1}{2}\lambda^2 C_{\nu}(f,f)e^{-\lambda^2\sum_{m\in\mathbb{Z}^d} f_m^2}\right\} \tag{77}$$

$$\leq e^{\left(\lambda f(\bar{x}) + \tfrac{1}{2}\lambda^2 C_{\nu}(f,f)\right)} \tag{78}$$

This means $\mu \in S_{1,\nu}$.

This eaxample shows clearly that there is at least a measure in $S_{1,\nu}$, which is not the mixture of Gaussian measures with DD-measures.

E33: Let μ be a measure on $(X, \mathfrak{B})$, for which a one-dimensional distribution

$$\mu \circ p_m^{-1}(d\eta) = e^{a - b\eta^2 - c\eta^4} \tag{79}$$

and any finite dimentional destribution

$$\mu \circ p_M^{-1}(\otimes_{m\in M} d\eta_m) = \otimes_{m\in M} \mu \circ p_m^{-1}(d\eta_m) \quad :$$

i.e. $\{e_m^*\}_{m\in\mathbb{Z}^d}$ are independent with respect to μ .

Since it is a probability measure we must have

$$\mu \circ p_m^{-1}(R) = 1 = e^a \int e^{-b\eta^2 - c\eta^4} d\eta \tag{80}$$

$$\leq e^a \sqrt{\frac{\pi}{b}} \tag{81}$$

For any $f \in X^*$ which has the support in M, we can estimate

$$E_{\mu}(e^{\lambda f}) = E_{\mu\circ p_M^{-1}}(e^{\lambda f}) \tag{82}$$

$$\leq \left(e^{a} \sqrt{\frac{\pi}{b}} \right)^{|M|} E_{\nu_b \circ p_M^{-1}} (e^{\lambda f}) \quad (83)$$

$$= \left(e^{a} \sqrt{\frac{\pi}{b}} \right)^{|M|} E_{\nu_b} (e^{\lambda f}) \quad (84)$$

The question is whether can one find a "c" as a function of a,b such that (80) and RHS(81) are equal to unity, or not. If this is not possible, then μ is not belong to $S_{\varkappa,\nu}$ for any $\varkappa > 0$: the factor infront of E_{ν_b} in RHS(84) would have no upper bound. This is a very clude estimate. Ofcourse, if one wants the same one-dimentional distribution at each point, one can not succeed: One would have $\gamma^{|M|}$ for some $\gamma > 1$, and ofcourse $\limsup_{M \in \underline{C}} \gamma^{|M|} = +\infty$. For this reason the quartic term must have some "cut-off".

One can see from this example that the definition of $\varkappa\nu$-Sub-Gaussian, we have given, is too strong so that it is not include some interesting case. It is hopefully that the definition of Sub-Gaussian can be weaken such that the compact, convex, and the RG-invariant remain unchange: the proof of P29-31 request weaker than the condition of being $\varkappa\nu$SG.

As a conclusion for $\varkappa\nu$SG, we have also gives a compact, convex, and RG-invariant which has been posed as the aim at the begining of this chapter. Moreover, we can give an example, see E32, which makes clear that the set $S_{\varkappa,\nu}$ contains more than the trivial elements, Gaussian, DD-measures, and their mixture. We point out that $S_{\varkappa,\nu}$ is rich enough for further study on stability of a Gaussian fixed point (measure).

The $(\varkappa^*,\nu)$ Sub-Gaussian

Let us enlarge the set $S_{\varkappa,\nu}$ with the aim to included the measures of the kind given in E33.

D34: A measure $\mu \in \mathfrak{M}(\mathcal{B})$ is a $(\varkappa^*,\nu)$ Sub-Gaussian $(\varkappa^*\nu SG)$ measure if there exist a $\nu \in \mathfrak{M}_G(\mathcal{B})$, and $\varkappa > 0$ such that, for each $M \in \underline{C}$, and each $f \in X^*$ which has support in M, i.e. $f(x) = \sum_{m \in M} f_m x_m$ for all $x \in X$, we have

$$E_\mu(e^{\lambda f}) \leq \varkappa^{|M|} e^{\lambda C^1_\nu(f) + \frac{1}{2}\lambda^2 C^2_\nu(f,f)} . \qquad (85)$$

Similarly , we define

$$S_{\varkappa^*,\nu} = \{\mu \in \mathfrak{M}(\mathcal{B}) \mid \mu \text{ is a } \varkappa^*\nu SG\} . \qquad (86)$$

Let us now reconsider all from begining when the $S_{\varkappa,\nu}$ is replacing by $S_{\varkappa^*,\nu}$:

Since $S_{\varkappa,\nu} \subset S_{\varkappa^*,\nu}$ by the obvious reason, the statements and proofs of P25(i-iv) are the same. The proof of P25(v) have only a slight change; the second $\{\ldots\}$ alway $\leq \varkappa^{|M|}$ instead of $\varkappa$. Then, the conclusion is proves; $\bar{S}_{\varkappa^*,\nu} = S_{\varkappa^*,\nu}$.

Because L26, C27 and C28 are all the estimate related to e^*_m for each $m \in M_j \in \underline{C}$, the conclusions and the proofs have nothing to change: The support of e^*_m is just a point, thus has cardinal number equal to 1.

The convexity of $S_{\varkappa^*,\nu}$ follow the same proof as in P30 without any change because the coefficient $\varkappa$ or $\varkappa^{|M|}$ play no role in the proof.

Obviously, P29 is the same as it depends only on C27, and the choice of a^j, which we keep to be the same.

The sufficient condition in P31 clearly leave the set $S_{\varkappa^*,\nu}$ invariant under the RG-map .

E35: Let us consider a Statistical System defined by Super-Stable and Lower-Regular potential, see Ruelle (1976) [108], and Lebowitz-Presutti (1976) [109]: They concluded, in our language, that

$$\mu^M \circ p_{M\Lambda}^{-1}(dx) = \frac{e^{-U^\Lambda(x)}\omega^\Lambda(dx)}{\bar{\mu}^M(X^M)} \int e^{-U^{M\setminus\Lambda}(y) - W^{\Lambda,M\setminus\Lambda}(x,y)}\omega^{M\setminus\Lambda}(dy) \qquad (87)$$

$$\leq e^{\gamma|\Lambda| - A\sum_{m\in\Lambda} x_m^2}\omega^\Lambda(dx) \qquad (88)$$

for some $\gamma \in R$, and $A > 0$, independent of M and Λ .

Obviously, there is a sub-net of the family of fd-measure that converges to a measure on $(X,\mathcal{B})$, denote by μ. It must has $\mu\circ p_\Lambda^{-1}$ enjoyed (88).

It is clear that such measure is belong to $S_{\varkappa^*,\nu}$ when $\varkappa = e^\gamma$ and ν is a GWN measure with variance A.

Now, we have another convex compact sub-set of $\mathcal{M}(\mathcal{B})$ which is invariance under a given RG-map. Moreover, it contains the very interesting statistical systems, the Super-Stable and Lower-Regular ones.

VIII. THE EXISTENCE OF RG-FIXED POINTS.

In this chapter, we will discuss the existence of fixed points under each given RG-map τ. We will point out clearly , for each case, how one can manage to use the Schauder-Tychonoff Fixed Point Theorem, see Dunford-Schwartz (1957) [57] p456, theorem 5. Indeed, we are able to give some strong sufficient conditions for the existence of fixed points. This is the fruit of the seeds we had sowed in all previous chapters.

As we had shown that the RG-transformation maps a probability measure on $(X,\mathfrak{B})$ to a probability measure on $(X,\mathfrak{B})$, uniquely, one may wonder whether there exist a fixed point in $\mathfrak{M}(\mathfrak{B})$, or not. This question can be translated into a precise language as following: For a given RG-map τ, we ask whether there exist an invariant sub-set Ω, i.e. $\tau\Omega \subset \Omega$, in $\mathfrak{M}(\mathfrak{B})$, and it is a singleton, or equivalently saying that there exist a $\mu \in \mathfrak{M}(\mathfrak{B})$ such that $\tau\mu = \mu$.

Let us recall a definition and a theorem from Dunford-Schawrtz, ibid.:

D1 : A topological space Y is said to have the fixed point property, if for every continuous map $T:Y \rightarrow Y$, there is a $y \in Y$ with $y = T(y)$.

P2 : (Schauder-Tychonoff) A compact convex sub-set of a locally convex linear topological space has the fixed point property.

This proposition draws our attention to a compact convex sub-set of $\mathfrak{M}(\mathfrak{B})$ which must be invariant under the given RG-map τ. Any way, we have to clearify that the set $\mathfrak{M}(\mathfrak{B})$ with the weak convergence topology can be embedded into a locally convex linear topological space.

Let us first embed $\mathfrak{M}(\mathcal{B})$ into the space of finite measures $\mathfrak{M}^{+}(\mathcal{B})$. It is a positve cone with the zero-measure as the tip:

$$\mathfrak{M}^{+}(\mathcal{B}) = \{\lambda\mu \mid 0 \leq \lambda < \infty, \mu \in \mathfrak{M}(\mathcal{B})\} \tag{1}$$

The linear span of $\mathfrak{M}^{+}(\mathcal{B})$, over the field of real number $\mathbb{R}$, is obviously a linear space. Infact, it is the space of all signed measures on $(X, \mathcal{B})$, see Halmos (1974) [77], we denote it by $\mathfrak{M}^{(\pm)}(\mathcal{B})$.

The weak convergence topology, see chapter II, can be similarly introduce on $\mathfrak{M}^{+}(\mathcal{B})$ and $\mathfrak{M}^{(\pm)}(\mathcal{B})$: i.e. the local base at $\nu \in \mathfrak{M}^{(\pm)}(\mathcal{B})$ has the following form for each $n \in \mathbb{N}$, $f_j \in C(X)$ and $\varepsilon_j > 0$, for each $j = 1,2,\ldots,n$,

$$V(\{f_j, \varepsilon_j\}_{j=1}^{n}; \nu) = \{\mu \in \mathfrak{M}^{(\pm)}(\mathcal{B}) \mid \left| \int f_j d\mu - \int f_j d\nu \right| < \varepsilon_j \text{ for each } j = 1,2,\ldots,n\}, \tag{2}$$

compare to II(17).

P3 : The local base at any $\nu \in \mathfrak{M}^{(\pm)}(\mathcal{B})$ has elements which are all convex, therefore $\mathfrak{M}^{(\pm)}(\mathcal{B})$ is a locally convex linear topological space.

pf : Let μ_0 and μ_1 belong to a member fo the local base at a $\nu \in \mathfrak{M}^{(\pm)}(\mathcal{B})$, e.g. $\mu_0, \mu_1 \in V(\{f_j, \varepsilon_j\}_{j=1}^{n}; \nu)$, which has been given in (2). All element in the line segment joint the points μ_0 and μ_1, e.g. for $0 \leq \gamma \leq 1$, an element

$$\mu_\gamma = \gamma\mu_1 + (1-\gamma)\mu_0 \tag{3}$$

are obviously elements of $\mathfrak{M}^{(\pm)}(\mathcal{B})$. Considering for each $j = 1,2,\ldots,n$, we have

$$\left|\int f_j d\mu_\gamma - \int f_j d\nu\right| \leq \gamma \left|\int f_j d\mu_1 - \int f_j d\nu\right| + (1-\gamma)\left|\int f_j d\mu_0 - \int f_j d\nu\right| \quad (4)$$

where the Triangle Inequality has been used. Since μ_0 and μ_1 $V(\{f_j,\varepsilon_j\}_{j=1}^n;\nu)$, we must have

$$\text{RHS}(4) \leq \varepsilon_j \quad (5)$$

This means the signed measure $\mu_\gamma \in V(\{f_j,\varepsilon_j\}_{j=1}^n;\nu)$, thus the set $V(\{f_j,\varepsilon_j\}_{j=1}^n;\nu)$ is convex.

Since this is true for every member of the local base at ν, the topology generated out of this local base must be a locally convex topology on $\mathcal{M}^{(+)}_{-}(\mathcal{B})$, see Kelley-Namioka (1963) [106]. Since $\mathcal{M}^{(+)}_{-}(\mathcal{B})$ is a linear space by construction, and the continuity of the vector addition and multiplication (with scalar) under the weak convergence topology is obvious, we can conclude that the space of signed measure on $(X,\mathcal{B})$ is a locally convex linear topological space.||

Any element in the line segment joining any pair of probability measures is a probability measure, therefore the set $\mathcal{M}(\mathcal{B})$ is convex. On the other hand, it is not compact, see II.P3. The only way to make use of the fixed point theorem is to find out a compact convex sub-set of $\mathcal{M}^{(+)}_{-}(\mathcal{B})$ which is invariant under the extension of the RG-map on it. We are interested in the case where the sub-set is inside the $\mathcal{M}(\mathcal{B})$. In chapter VII, we had introduced two types of sub-sets of the kind: $\overline{\text{co}(\Gamma)}$ and $S_{\chi^*,\nu}$.

P4 : For each RG-map τ, and $\mu \in \mathcal{M}(\mathcal{B})$, the closure of the convex extension of the set $\Gamma = \{\tau^{k-1}\mu\}_{k\in N}$ contains at least a fixed point, if for each $M \in \underline{C}$, and $m_1 \in M$,

$$\lim_{L\to\infty}(\lambda L^{-d})^2\sum_{m,n\in B_L(m_1)} C_\mu(m,n) < E_\mu(\epsilon^{*2}_{m_1}), \quad (6)$$

where λ is the homogeneous scaling factor of τ.

pf : From VII.P23, if μ satisfies the condition (6), we know that $\overline{co(\Gamma)}$ is a convex compact sub-set of $\mathcal{M}(\mathcal{B}) \subset \mathcal{M}^{(\pm)}(\mathcal{B})$ which is a locally convex linear topological space, P3, thus it has the fixed point property, by the Schauder-Tychonoff Theorem, P2. Since the RG-map τ leaves the set invariant, and is continuous, it is clear that there exist an element $\mu^* \in \overline{co(\Gamma)}$ such that $\tau\mu^* = \mu^*$. ||

P5 : For any given RG-map τ, a Gaussian measure ν, and $K > 0$, if $\tau\nu \in S_{1,\nu}$, then there exist an element in $S_{K^*,\nu}$ as a fixed point of the RG-map.

pf : Since $S_{K^*,\nu}$ is a τ-invariant, convex compact sub-set of $\mathcal{M}(\mathcal{B}) \subset \mathcal{M}^{(\pm)}(\mathcal{B})$, by VII.P29-31*, we can use an argument as in P4 to conclude that there is an element $\mu^* \in S_{K^*,\nu}$ such that $\tau\mu^* = \mu^*$. ||

We know from Kelley-Namioka (1963) [106] p100, 13.1(iii), that the closure of a convex extension of a union of two convex compact sets is compact. Therefore $\overline{co(S_{K^*,\nu_1} \cup S_{\gamma^*,\nu_2})}$ is compact and convex. The problem is how to give the τ- invariant condition as the property VII(48) which work well for each component is no longer true in general for the mixture: It may occur that $\overline{co(S_{K^*,\nu_1} \cup S_{\gamma^*,\nu_2})}$ is not mapped into itself by τ. In this case the fixed point theorem is not applicable (through P5)

If we carefully look at the fixed point theorem, it gives us more than need; It tell us that there is a fixed point for some other than the given RG-map τ. On the other hand, it does not tell us how many fixed points are contained in the set, for each given continuous map. Ofcourse, if one is able to prove that the map τ is contractive in the sub-set, the uniqueness is guaranteed by the Banach Contractive Principle.

The sub-set $S_{\kappa^*,\nu}$ is a neighborhood of the Gaussian measure ν. If $\tau\nu = \nu$, the stability of the Gaussian fixed point can be studied: In general, it is hard to find an invariant neighborhood with such a nice property which is necessary for the study of stability of the fixed point.

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APPENDIX A.

In this appendix we prove C10, L11-13 of chapter I, and discuss certain feature of $\mathbb{Z}^d$, and $\underline{C}$.

$\mathbb{Z}^d$ is a commutative group under addition "+" with "0" as identity. It is not a linear space because there are some $\alpha \in R$, and $m \in \mathbb{Z}^d$ such that α m does not belong to $\mathbb{Z}^d$.

It is clear that the relative topology on $\mathbb{Z}^d$ from $\mathbb{R}^d$ is equivalent to the discrete topology. This should remind us that the topology is less important on $\mathbb{Z}^d$. The geometry comes and plays an important role

On $\mathbb{R}^d$, one can give several metric which give equivalent topologies: For each a,b $\in \mathbb{R}^d$, we define $\rho_\chi : \mathbb{R}^d \otimes \mathbb{R}^d \rightarrow \mathbb{R}^+$ as follows

$$\rho_\infty(a,b) = \max_{j=1,2,\ldots,d} |a_j - b_j| \qquad (1.a)$$

$$\rho_1(a,b) = \sum_{j=1}^{d} |a_j - b_j| \qquad (1.b)$$

$$\rho_2(a,b) = \sqrt{\sum_{j=1}^{d} |a_j - b_j|^2} \qquad (1.c)$$

It can be easily checked that they are metrics, and give the usual topology on $\mathbb{R}^d$.

Let us define, for each κ, a ball centered at a_o with radius $\gamma > 0$.

$$B_\gamma^{d,\chi}(a_o) = \{ a \in \mathbb{R}^d | \rho_\chi(a,a_o) \leq \gamma \} \quad . \qquad (2)$$

The ball is a hyper-cubic, a hyper-octahedral, and a hyper-sphere for the correspondance $\chi = \infty$, 1, and 2.

What we concern as important is the degeneracy $g_{\ell}^{d,\chi}$, the number of solutions of

$$\rho_{\chi}(m,0) = \ell \quad , \quad m \in \mathbb{Z}^d \qquad (3)$$

If we retricted the ball to $\mathbb{Z}^d$, then the surface of each ball meets elements of $\mathbb{Z}^d$; then $g_{\ell}^{d,\chi}$ is exactly their number.

I.L12: $g_{\ell}^{d,\chi} \leq (2\ell + 1)^d - (2\ell - 1)^d$ (4)

pf : It is obvious that $g_{\ell}^{d,\infty} \geq g_{\ell}^{d,2}$ and $g_{\ell}^{d,1}$.

Here we are going to calculate $g_{\ell}^{d,\infty}$. Since the ball is now a hyper-cube; surrounding each $m \in Z^d$ by ball $B_{1/2}^{d,\infty}(m)$, we have immediately

$$g_{\ell}^{d,\infty} = \frac{\text{volume}\left(B_{2\ell+1/2}^{d,\infty}(0)\right) - \text{volume}\left(B_{2\ell-1/2}^{d,\infty}(0)\right)}{\text{volume}\left(B_{1/2}^{d,\infty}(0)\right)} \qquad (5)$$

$$= (2\ell + 1)^d - (2\ell - 1)^d \qquad (6)||$$

I.L11: $g_{\ell}^{d,1} = \sum_{j=1}^{\min(\ell,d)} 2^j \cdot \frac{d!}{(d-j)!j!} \cdot \frac{(\ell-1)!}{(\ell-j)!(j-1)!}$, $\ell \geq 1$, $d \geq 1$ (7)

pf : The problem is to find the number of solutions of

$$\sum_{j=1}^{d} |m_j| = \ell, \quad m_j \in \mathbb{Z}, \; j=1,2,\ldots,d \qquad (8)$$

Consider first positive solutions of

$$\sum_{j=1}^{k} n_j = \ell, \quad n_j \in N, \; j=1,2,\ldots,k \leq d \qquad (9)$$

The number of solusions of (9) for each k is

$$c(\ell,k) = \frac{(\ell-1)!}{(\ell-k)!(k-1)!} \qquad (10)$$

by the theorem 41 in Andrews (1976) |110|.

Since the k number can take any value 1,2,...,d, the number of ways is

$$w(d,k) = \frac{d!}{(d-k)!k!} \qquad (11)$$

The degeneracy of k number is 2^k as there are two posibilities for each, + or −. Therefore

$$g_\ell^{d,1} = \sum_{k=1}^{\min(\ell,d)} 2^k w(d,k)c(\ell,k) \qquad (12)$$

because ℓ can distributed non-zero at most all solution (9) have all $n_j = 1$, $k = \ell$, thus $k \leq \min(\ell,d)$.

Writing (12) explicitly, we get (7).||

I.L13: $\sup_{\ell\in N}(g_\ell^{d,1})^{\frac{1}{\ell}} = 2d \qquad (13)$

pf : From I.L12, we know that

$$g_\ell^{d,1} \leq (2\ell+1)^d - (2\ell-1)^d \leq 4d3^{d-1}\ell^{d-1} \qquad (14)$$

It is an increasing sequence for each d, at most of power d-1. Immediately, there exists $\ell_0 \geq 1$ such that , for all $\ell \geq \ell_0$, $(g_\ell^{d,1})^{\frac{1}{\ell}}$ is a non-increasing sequence: i.e. $\sup_{\ell\geq\ell_0}(g_\ell^{d,1})^{\frac{1}{\ell}} = (g_{\ell_0}^{d,1})^{\frac{1}{\ell_0}}$. We claim $\ell_0 = 1$:

Let us consider the formulae (12) in I.L11: when $\ell+1 \leq d$, we have

$$g^{d,1}_{\ell+1} = \sum_{j=1}^{\ell} 2^j w(d,j)c(\ell+1,j) + 2^{\ell+1}w(d,\ell+1)c(\ell+1,\ell+1) \quad (15)$$

$$= \sum_{j+1}^{\ell} 2^j w(d,j)c(\ell,j)\left\{\frac{\ell}{\ell-j+1}\right\} + 2^{\ell+1}d(d-1)\ldots(d-\ell)$$

$$\leq \ell g^{d,1}_{\ell} + 2^{\ell+1}d(d-1)\ldots.(d-\ell) \quad (16)$$

$$\leq \ell(\ell-1)g^{d,1}_{\ell-1} + \ell 2^{\ell}d(d-1)\ldots(d-\ell+1) + 2^{\ell+1}d(d-1)\ldots(d-\ell)$$

$$\leq d(\ell-1)g^{d,1}_{\ell-1} + 2^{\ell+1}d^2(d-1)\ldots(d-\ell+1)$$

$$\leq (2d)\left\{(\ell-1)g^{d,1}_{\ell-1} + 2^{\ell}d(d-1)\ldots(d-\ell+1)\right\} \quad . \quad (17)$$

The $\{\ldots\}$ is just the same as the RHS(16), thus repeating the procedure, we get

$$g^{d,1}_{\ell+1} \leq (2d)^{\ell-1}\left\{(1)g^{d,1}_{1} + 2^2 d(d-1)\right\} = (2d)^{\ell}(2d-1)$$
$$\leq (2d)^{\ell+1}$$

Therfore $\sup_{\ell \in \mathbb{N}} (g^{d,1}_{\ell})^{\frac{1}{\ell}} \leq 2d$ (18)

Since $g^{d,1}_1 = 2d$, we must have $\sup_{\ell \in \mathbb{N}} (g^{d,1}_{\ell})^{\frac{1}{\ell}} = 2d$.||

I.C10: For each $x,y \in X$, we define

$$\delta(x,y) = \sum_{m \in Z^d} \gamma^{-|m|} \frac{d(x_m,y_m)}{1+d(x_m,y_m)} \quad ,\gamma \geq 2d \quad (19)$$

where $d(.,.)$ is a metric on R, and $|.| = \rho_1(.,0)$. is a metric on X which induces a topology equivalent to $\mathcal{T}$.

pf : First of all, we have to clarify that the summation given as in

(19) has meaning; i.e. converges.

The expression (19) can be rewritten as

$$\delta(x,y) = \frac{d(x_o,y_o)}{1 + d(x_o,y_o)} + \sum_{\ell \in N} \gamma^{-\ell} \sum_{m \in S_\ell^{d,1}} \frac{d(x_m,y_m)}{1 + d(x_m,y_m)} \quad . \quad (20)$$

$$< 1 + \sum_{\ell \in N} \gamma^{-\ell} |S_\ell^{d,1}|$$

$$< 1 + \sum_{\ell \in N} \left(\frac{2d}{\gamma}\right)^\ell < \infty \quad .$$

where we have use the fact that

$$0 \leq \frac{d(x_m,y_m)}{1 + d(x_m,y_m)} < 1 \text{ , for all } m \in Z^d, \; x,y \in X \quad (21)$$

then L13, and the radius of convergence of a geometric serie.

It is obvious that δ is positive, symmetric, and $\delta(x,y) = 0$, if $x = y$. Now, we are going to prove the triangle inequality:

From (19), we have for each $x,y,z \in X$

$$\delta(x,z) + \delta(y,z) = \sum_{m \in \mathbb{Z}^d} \gamma^{-|m|} \left\{ \frac{d(x_m,z_m)}{1+d(x_m,z_m)} + \frac{d(y_m,z_m)}{1+d(y_m,z_m)} \right\} \quad (22)$$

Because of the fact that for any positive real number a,b, we have

$$\frac{a}{1+a} + \frac{b}{1+b} \geq \frac{a+b}{1+a+b} \quad (23)$$

$$\frac{a}{1+a} \geq \frac{b}{1+b} \text{ , if } a \geq b \quad (24)$$

we can conclude each term in $\{\ldots\}_m \geq \frac{d(x_m,y_m)}{1+d(x_m,y_m)}$, thus

$$\sigma(x,z) \; + \; \sigma(z,y) \; \geq \; \sigma(x,y) \qquad (25)$$

Here, we can conclude that σ is a metric on X, thus (X,σ) is a metric space.

Now we are ready to prove that (X,σ) is equivalent to $(X,\mathcal{T})$. Equivalence means that σ is finer than $\mathcal{T}$ and $\mathcal{T}$ is finer than σ.

From the uniqueness of topology generated from a given subbase, we can just prove that each element in a subbase of the topology is open in the other topology.

Let us consider an open neighbourhood of any point x in σ and $\mathcal{T}$ toplogy:

$$B_\varepsilon(x) \; = \left\{ y \in X \mid \sigma(y,x) < \varepsilon \right\} \quad , \quad \varepsilon > 0 \qquad (26)$$

$$V_a^M(x) \; = \left\{ y \in X \mid d(y_m,x_m) < a_m \right\} \quad , \; M \in \underline{C} \; , \; a \in X^M \qquad (27)$$

It suffices to prove that for any given $\varepsilon > 0$, $x \in X$, we can find $M \in \underline{C}$, and $a \in X^M$ such that $x \in V_a^M(x) \subset B_\varepsilon(x)$ for the proof that $\mathcal{T}$ is finer than σ , see Boubaki (1966) |60| p29, proposition 3, and vice versa.

$\mathcal{T}$ is finer than σ : For any gven $\varepsilon > 0$, $x \in X$, we are able to find ℓ_0 such that

$$\sum_{\ell > \ell_0} \gamma^{-\ell} \sum_{m \in S_\ell^{d,1}} \frac{d(y_m,x_m)}{1+d(y_m,x_m)} < \tfrac{1}{2}\varepsilon \qquad (28)$$

for all $y \in X$, because the summation (20) is uniformly bounded. If we choose $a_m > 0$, for all $m \in M$ such that

$$\sum_{\ell=0}^{\ell_0} \left(\frac{2d}{\gamma}\right)^{\ell} a_m \; < \; \tfrac{1}{2}\varepsilon \qquad (29)$$

we have $V_a^{B_{\ell_o}^{d,1}(0)}(x) \subset B_\varepsilon(x)$: It is clear from (28) and (29) that $\sigma(x,y) < \varepsilon$ for all $y \in V_a^{B_{\ell_o}^{d,1}(0)}(x)$.

σ is finer than $\mathcal{T}$: It suffices to consider $M = \{m\}$. For any $m \in \mathbb{Z}^d$, $a_m > 0$, $x \in X$, we can simply choose $\varepsilon = a_m$, we have $B_{a_m}(x) \subset V^m_{a_m}(x)$.

Here, we can conclude that the metric topology (X,σ) is equivalent to the product topology $(X,\mathcal{T})$.||

APPENDIX B.

This appendix contains the proofs of L5 and L6 of chapter II.

The concept of net and sub-net plays an important role in the proofs. We refer to the book by Kelley (1955) [58] for definition and details. We try to make each step explicitly related to the concept of sequence which is more familiar to physicists. The special sub-net, that totally directed is intuitively use by physicists in Statistical Mechanics related to the thermodynamic limit.

The aim of L5 and L6 is to give a unique C-family which satisfies II(15), by construction.

II.L5: Let $\{\mu^M\}_{M\in \underline{C}}$ be an AC-family. The net $\{\mu^M \circ p^{-1}_{M\Lambda}\}_{M\in \underline{C}(\Lambda)}$ converges to a unique measure on $(X^\Lambda, \mathcal{B}^\Lambda)$ for each $\Lambda \in \underline{C}$.

pf : It is clear that $\mu^M \circ p^{-1}_{M\Lambda}$ is a probability measure on $(X^\Lambda, \mathcal{B}^\Lambda)$ for each $M \in \underline{C}(\Lambda)$.

We consider a sub-net, totally directed, of measures on $(X^\Lambda, \mathcal{B}^\Lambda)$, $\{\mu^{M_n} \circ p^{-1}_{M_n\Lambda}\}_{n\in N}$: In fact it is a sequence. The $\{M_n\}_{n\in N}$ can be explicitly given by the condition that $M_{n+1} \supset M_n$, and for any M $\in \underline{C}(\Lambda)$, there is at least an $m \in N$ such that $M \subset M_m$.

Obviously from the AC-property, for each $A \in \mathcal{B}^\Lambda$, the sub-net of real number $\{\mu^{M_n} \circ p^{-1}_{M_n\Lambda}(A)\}$ is a Cauchy sequence in $[0,1]$. Therefore it converges to a unique limit in $[0,1]$.

Now, we can define a set-function on $\mathcal{B}^\Lambda$ by

$$\mu_{\Lambda} \doteq \lim_{n\to} \mu^{M_n} \circ p^{-1}_{M_n \Lambda} \qquad (1)$$

Here, we are going to prove that μ_{Λ} is a probability measure on $(X^{\Lambda}, \mathcal{B}^{\Lambda})$.

It is obvious that $\mu_{\Lambda}(\phi) = 0$, and $\mu_{\Lambda}(X^{\Lambda}) = 1$.

The finitely additive property of μ_{Λ} can be seen from the following: Let A and B be two disjoint measurable sub-sets of X . By the Triangle inequality, we have

$$|\mu_{\Lambda}(A\cup B) - \mu_{\Lambda}(A) - \mu_{\Lambda}(B)| \leq |\mu_{\Lambda}(A\cup B) - \mu^{M_n} \circ p^{-1}_{M_n \Lambda}(A\cup B)|$$
$$+ |\mu_{\Lambda}(A) - \mu^{M_n} \circ p^{-1}_{M_n \Lambda}(A)| + |\mu_{\Lambda}(B) - \mu^{M_n} \circ p^{-1}_{M_n \Lambda}(B)|$$
$$+ |\mu^{M_n} \circ p^{-1}_{M_n \Lambda}(A\cup B) - \mu^{M_n} \circ p^{-1}_{M_n \Lambda}(A) - \mu^{M_n} \circ p^{-1}_{M_n \Lambda}(B)| \qquad (2)$$

Since $\mu^{M_n} \circ p^{-1}_{M_n \Lambda}$ is a measure on $(X^{\Lambda}, \mathcal{B}^{\Lambda})$, the last term of (2) is zero. The first three terms can be bounded above by arbitrary small number ε, each by $\varepsilon/3$, if we choose n large enough. This implies that the LHS(2) must be zero; i.e.

$$\mu_{\Lambda}(A\cup B) = \mu_{\Lambda}(A) + \mu_{\Lambda}(B) \qquad (3)$$

To prove σ-additvity, it is sufficient to prove that the set-function μ_{Λ} is continuous at any measurable set $A \in \mathcal{B}^{\Lambda}$. Let $\{A_n\}_{n \in N}$ be any monotonic sequence of measurable sets in $\mathcal{B}^{\Lambda}$, either increasing or decreasing to the set A. Considering

$$|\mu_\Lambda(A) - \mu_\Lambda(A_n)| \leq |\mu_\Lambda(A) - \mu^{M_k} \circ p^{-1}_{M_k\Lambda}(A)|$$

$$+ \quad |\mu_\Lambda(A_n) - \mu^{M_k} \circ p^{-1}_{M_k\Lambda}(A_n)| \quad + \quad |\mu^{M_k} \circ p^{-1}_{M_k\Lambda}(A) - \mu^{M_k} \circ p^{-1}_{M_k\Lambda}(A_n)| \quad (4)$$

For any given $\varepsilon > 0$, we can bound the first two term, each by $\varepsilon/3$, if we choose k large enough, and the last term by $\varepsilon/3$, if we choose n large eneough by the continuity of the measure $\mu^{M_k} \circ p^{-1}_{M_k\Lambda}$ itself. Therefore, (4) can be bounded by any given $\varepsilon > 0$ if we choose n large enough.

This proves the continuity of μ_Λ on each $A \in \mathcal{B}^\Lambda$, thus μ_Λ is a σ-additive set-function.

Conclusively, we have shown that μ_Λ is a probability measure on $(X^\Lambda, \mathcal{B}^\Lambda)$.

Let $\{\mu^{M'_k} \circ p^{-1}_{M'_k\Lambda}\}_{k \in \mathbb{N}}$ be another sub-net. Similarly, we get a measure μ'_Λ on $(X^\Lambda, \mathcal{B}^\Lambda)$.

Considering, for each $A \in \mathcal{B}^\Lambda$.

$$|\mu_\Lambda(A) - \mu'_\Lambda(A)| \leq |\mu_\Lambda(A) - \mu^{M_n} \circ p^{-1}_{M_n\Lambda}(A)|$$

$$+ \quad |\mu'(A) - \mu^{M'_k} \circ p^{-1}_{M'_k\Lambda}(A)| \quad + \quad |\mu^{M_n} \circ p^{-1}_{M_n\Lambda}(A) - \mu^{M'_k} \circ p^{-1}_{M'_k\Lambda}(A)| \quad (5)$$

Each of the first two terms can be bounded by $\varepsilon/3$ if we choose n and k large enough, by the convergence of the sub-net to the limit. The last term is bounded by $\varepsilon/3$ if n and k are large enough; i.e. such that the M_0 in the AC-condition contain in both M_n and M_k. Summing up, we get that the LHS(5) is less than arbitrary small positive number ε, thus

$$\mu_\Lambda(A) \quad = \quad \mu'_\Lambda(A) \quad , \quad \text{for each } A \tag{6}$$

This means that all sub-nets of the net $\{\mu^M \circ p_{M\Lambda}^{-1}\}_{M \in \underline{C}(\Lambda)}$ converge to the same measure, μ_Λ. This proves the uniqueness of μ_Λ.||

II.L6: The collection of the limiting measure on each $\Lambda \in \underline{C}$; $\{\mu_\Lambda\}_{\Lambda \in \underline{C}}$ of L5 is a C-family.

pf : Considering any measurable set in $\mathcal{B}^\Omega$, for each $\Omega \in \underline{C}$, we withdraw from every where in the following as it has been fixed. For any Λ , $\Lambda' \in \underline{C}(\Omega)$, we have

$$|\mu_\Lambda \circ p_{\Lambda\Omega}^{-1} - \mu_{\Lambda'} \circ p_{\Lambda'\Omega}^{-1}| \quad \leq \quad |\mu_\Lambda \circ p_{\Lambda\Omega}^{-1} - \mu^M \circ p_{M\Lambda}^{-1} \circ p_{\Lambda\Omega}^{-1}|$$
$$+ \quad |\mu_{\Lambda'} \circ p_{\Lambda'\Omega}^{-1} - \mu^{M'} \circ p_{M'\Lambda'}^{-1} \circ p_{\Lambda'\Omega}^{-1}| \quad + \quad |\mu^M \circ p_{M\Lambda}^{-1} \, p_{\Lambda\Omega}^{-1} - \mu^{M'} \circ p_{M'\Lambda'}^{-1} \, p_{\Lambda'\Omega}^{-1}| \tag{7}$$

where M, $M' \in \underline{C}(\Lambda) \cap \underline{C}(\Lambda')$. It is very easy to check that $p_{M\Lambda}^{-1} \circ p_{\Lambda\Omega}^{-1} = p_{M\Omega}^{-1}$, so do for the similar terms, we have the last term in (7) become $|\mu^M \circ p_{M\Omega}^{-1} - \mu^{M'} \circ p_{M'\Omega}^{-1}|$. By the AC-property, it can be bounded upper by $\varepsilon/3$ for large enough M and M'. Since $p_{\Lambda\Omega}^{-1}$ and $p_{\Lambda'\Omega}^{-1}$ are measurable in $\mathcal{B}^\Lambda$ and $\mathcal{B}^{\Lambda'}$, we can bounded each of the first two terms by $\varepsilon/3$ for large enough M and M': This comes from the convergence of the net to a limit as has been proved in L5.

Therefore the LHS(7) is bounded by arbitrary small number ε. This means

$$\mu_\Lambda \circ p_{\Lambda\Omega}^{-1} \quad = \quad \mu_{\Lambda'} \, p_{\Lambda'\Omega}^{-1} \tag{8}$$

which is just the C-conditon in II.D1.||

APPENDIX C.

This appendix contains some details of chapter VII; the proofs of P10, and L11; a brief discusion on Dirac Delta (DD) measures on $(X,\mathcal{B})$.

VII.P10: For each $M_1 \in \underline{C}$, and a given positive $a \in X^{M_1}$, there exists a positive $b° \in X^M$ such that

$$C_b^M \subset \pi_M^{-1} C_a^{M_1} \qquad \text{VII(13)}$$

for all $0 < b \leq b°$. In fact, we can choose b° either from

$$\lambda L^{-d} \sum_{m \in B_L(m_1)} b^{\circ}_m \leq a_{m_1} \qquad \text{VII(14)}$$

or from

$$\lambda^2 L^{-d} \sum_{m \in B_L(m_1)} b^{\circ 2}_m \leq a^2_{m_1} \qquad \text{VII(15)}$$

pf : Let $x \in C_b^M$, we calculate $|(\pi_M x)_{m_1}|$ by III.D1(5). The triangle inequality provides

$$|(\pi_M x)_{m_1}| \leq \lambda L^{-d} \sum_{m \in B_L(m_1)} |x_m| \qquad (1)$$

Since $x \in C_b^M$, we have $|x_m| \leq b_m$ for each $m \in M$, thus

$$\text{RHS}(1) \leq \lambda L^{-d} \sum_{m \in B_L(m_1)} b_m \qquad (2)$$

If we choose b° such that the RHS(2) is equal to a_{m_1}, for each $m_1 \in M_1$, then, for any $b \leq b°$ we have $\pi_M x \in C_a^{M_1}$. This prove VII(13), and the choice VII(14).

Considering $|(\pi_M x)_{m_1}|^2$, then applying the Cauchy's Inequality, see Mitrinovic (1970) [111] p30, we get

$$|(\pi_M x)_{m_1}|^2 \leq \lambda^2 L^{-d} \sum_{m \in B_L(m_1)} |x_m|^2 \qquad (3)$$

$$\leq \lambda^2 L^{-d} \sum_{m \in B_L(m_1)} b_m^2 \qquad (4)$$

If we choose b° such that the RHS(4) is equal to $a_{m_1}^2$, for each $m_1 \in M_1$, then, for any $b \leq b°$, we have $\pi_M x \in C_a^{M_1}$. This proves VII(13), and the choice VII(15).||

VII.L11: For any given homogeneous scaling RG-map τ, and any measure $\mu \in \mathcal{M}(\mathcal{D})$, we can estimate the upper bound of μ and μ^1 on each cylinder set of the form $p_{M_1}^{-1}(X^{M_1} \setminus C_a^{M_1})$, for each $M_1 \in \underline{C}$, as following

$$\mu \circ p_{M_1}^{-1}(X^{M_1} \setminus C_a^{M_1}) \leq \sum_{m_1 \in M_1} \frac{1}{a_{m_1}^2} E_\mu(e^{*2}_{m_1}) \qquad \text{VII(17)}$$

$$\mu^1 \circ p_{M_1}^{-1}(X^{M_1} \setminus C_a^{M_1}) \leq \sum_{m_1 \in M_1} \frac{(\lambda L^{-d})^2}{a_{m1}^2} \sum_{m,n \in B_L(m_1)} E_\mu(e^*_m e^*_n) \qquad \text{VII(18)}$$

$$\leq \sum_{m_1 \in M_1} \frac{\lambda^2 L^{-d}}{a_{m_1}^2} \sum_{m \in B_L(m_1)} E_\mu(e^{*2}_m) \qquad \text{VII(19)}$$

From VII.D9, and standard set theory, we have

$$p_{M_1}^{-1}(X^{M_1} \setminus C_a^{M_1}) = p_{M_1}^{-1}\left[\bigcup_{m_1 \in M_1} p_{M_1 m_1}^{-1}(X^{m_1} \setminus C_{a_{m_1}}^{m_1})\right] \qquad (1)$$

$$= \bigcup_{m_1 \in M_1} p_{m_1}^{-1}(X^{m_1} \setminus C_{a_{m_1}}^{m_1}) \quad . \qquad (2)$$

$$p^{-1}_{m_1}(X^{m_1} \setminus C^{m_1}_{a_{m_1}}) = \{x \in X \mid |x_{m_1}| > a_{m_1}\} , \qquad (3)$$

$$= \{x \in X \mid x^2_{m_1} > a_{m_1}\} \qquad (4)$$

By the Chebyshev Inequality, see Shiryayev (1984) [68] p190, we have

$$\mu \circ p^{-1}_{m_1}(X^{m_1} \setminus C^{m_1}_{a_{m_1}}) \leq \frac{1}{a^2_{m_1}} E_{\mu}(e^{*2}_{m_1}) \qquad (5)$$

From the property of a measure on unions of measurable sets, and (4), we get VII(17).

From IV.P7(10), we have that

$$E_{\mu^1}(e^{*2}_{m_1}) = (\lambda L^{-d})^2 \sum_{m,n \in B_L(m_1)} E_{\mu}(e^*_m e^*_n) \qquad (6)$$

Applying (6) into the RHS(VII(17)), when μ has been replaced by μ^1, we get VII(18).

By the Cauchy's Inequality, op.cit., we know that

$$(\sum_{m \in B_L(m_1)} x_m)^2 \leq L^d \sum_{m \in B_L(m_1)} x^2_m \qquad (7)$$

Integraling both sides with respect to μ, then applying the result VII(18); we obtain VII(19).||

When the center is not zero; i.e. $\bar{x}|_{M_1} \in X^{M_1}$, (5) becomes

$$\mu \circ p^{-1}_{m_1}\left[X^{m_1} \setminus C^{m_1}_{a_{m_1}}(\bar{x}_{m_1})\right] \leq \frac{1}{a^2_{m_1}} \int |x_{m_1} - \bar{x}_{m_1}|^2 \mu \circ p^{-1}_{m_1}(dx_{m_1}) \qquad (8)$$

$$= \frac{1}{a^2_{m_1}} \left\{ E_\mu(e^{*2}_{m_1}) - 2\bar{x}_{m_1} E_\mu(e^*_{m_1}) + \bar{x}^2_{m_1} \right\} \quad (9)$$

If the $\bar{x} \in X$ has been chosen to be the mean: i.e.

$$\bar{x}_{m_1} = E_\mu(e^*_{m_1}) \quad (10)$$

we have

$$(9) = a^{-2}_{m_1} C_\mu(m_1, m_1) \quad (11)$$

The L11, can still be work, but replace the 2-moments by 2-semi-variance, with a condition that the mean is invariant under the RG-map ,i.e.

$$\bar{x}_{m_1} = \lambda L^{-d} \sum_{m \in B_L(m_1)} \bar{x}_m \quad (12)$$

The Dirac Delta Measures on $(X, \mathcal{B})$.

Let us define a set-function $\mathcal{D}_{\bar{x}} : \mathcal{B} \to [0,1]$, where $\bar{x} \in X$;

$$\mathcal{D}_{\bar{x}}(A) = \begin{cases} 1 & ; \quad \bar{x} \in A \\ 0 & ; \quad \bar{x} \notin A \end{cases} \quad \text{for all } A \in \mathcal{B}. \quad (13)$$

It can be easily checked that it is a measure in $\mathcal{M}(\mathcal{B})$.

It has the characteristic function define on each $f \in X^*$,

$$\Theta_{\mathcal{D}_{\bar{x}}}(f) = e^{if(\bar{x})} \quad (14)$$

The N-moment, and N-semivariant function define on each $(f_1, f_2, \ldots, f_N)$

$(X^*)^N$ are

$$M^N_{\mu}(f_1,f_2,\ldots,f_N) \quad = \quad f_1(\bar{x})f_2(\bar{x})\ldots.f_N(\bar{x}) \tag{15}$$

$$C^N_{\mu}(f_1,f_2,\ldots,f_N) \quad = \begin{cases} f_1(\bar{x}) & ; \quad N = 1 \\ 0 & ; \quad N \geq 2 \end{cases} \tag{16}$$

In fact $\bar{x}$ is the mean of $\mathcal{D}_{\bar{x}}$: $\bar{x}_m = E_{\mathcal{D}_{\bar{x}}}(e^*_m)$ for all $m \in Z^d$. Since X is a metric space it is a Hausdorff, we know that all finite set in X must be closed, Munkress (1975) [62] p98, theorem 6.8. by (13), we have $\mathcal{D}_{\bar{x}}(\ \bar{x}\ ) = 1$, and it is the smallest closed set, thus the support of $\mathcal{D}_{\bar{x}}$ is $\{\ \bar{x}\ \}$, see VII.D3.

Let us apply the RG-map τ on $\mathcal{D}_{\bar{x}}$, we get

$$\tau\,\mathcal{D}_{\bar{x}} \quad = \quad \mathcal{D}_{\bar{x}^1} \tag{17}$$

where

$$\bar{x}^1 \quad = \quad \lambda\, L^{-d} \sum_{m \in B_L(m_1)} \bar{x}_m \tag{18}$$

for all $m_1 \in Z^d$.

If we ask for a translational invariant DD-measure, the renormalized DD-measure must also be translational invariant; i.e. $\bar{x}_m = \bar{x}_o$ imply

$$\bar{x}^1_{m_1} = \lambda\,\bar{x}_o \tag{19}$$

by (18). If we ask the $\mathcal{D}_{\bar{x}}$ to be invariant under a RG-map, we must have either $\bar{x}_o = 0$, or $\lambda = 1$. In case $\lambda < 1$, any translational invariant $\mathcal{D}_{\bar{x}}$ would generate a sequence $\mathcal{M} = \{\tau^{k-1}\mathcal{D}_{\bar{x}}\}_{k \in N}$ which converges to $\mathcal{D}_0$.

If we do not request the $\Phi_{\bar{x}}$ to be translational invariant, the condition for $\Phi_{\bar{x}}$ to be RG-invariant is

$$\bar{x}_{m_1} = \lambda L^{-d} \sum_{m \in B_L(m_1)} \bar{x}_m \qquad (20)$$

We can easily find out the solution by construction; First fixed $\bar{x}_o$, then solved for the other from (20), we get infinite number of solutions of (20). This means there is an enormous number of non-equivalent of fixed point of DD-measures for a given RG-map .