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ON POSITIVE SOLUTIONS OF TWO-POINT BOUNDARY VALUE PROBLEMS

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TRIESTE

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TESI DI "PHILOSOPHIAE DOCTOR"

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Contents.

CHAPTER 0

INTRODUCTION

The purpose of this thesis is to study various topics linked to the existence of positive solutions for linear and nonlinear two-point boundary value problems.

In chapter 1, we detect some common features among some well known orders, namely
(a) the orders in $\mathbb{R}^N$ obtained by specifying which of the coordinates must be positive and which others must be negative;
(b) the order in the spaces of real symmetric matrices obtained by the positive semi-definite symmetric matrices;
(c) the usual order in the spaces of continuously differentiable functions on a compact topological space;
in order to define the concept of *compactly generated ordered Banach space*.The definition is based on the existence of a family of continuous linear functionals which is compact in a suitable topology. This is used to study the existence of the principal eigenvalue for the two-point boundary value problems located in these spaces. Namely, we consider

$$u'' = \lambda L(t,u); \qquad u(a) = 0 = u(b) \tag{0.1}$$

where, for a compactly generated ordered Banach space X, the operator $L : [a,b] \times X \to X$ verifies the following conditions:
(i) L satisfies the generalised Carathéodory conditions;
(ii) for every bounded $B \subseteq X$, $\overline{L([a,b]\times B)}$ is compact;
(iii) for every t, $L(t,\bullet) : X \to X$ is a continuous positive linear operator.
The main result is applied to the comparison of the principal eigenvalue for different problems as well as to the existence for some nonlinear problems.

In chapter 2, we present some results of existence and multiplicity of positive (nontrivial) solutions for the two-point boundary value problem

$$-u'' = f(x,u) \tag{0.2}$$

$$u(a) = u(b) = 0. \tag{0.3}$$

Problem (0.2)-(0.3) has been widely investigated, both from the point of view of the theory of ordinary differential equations [4,5,7,8] and as a particular case of the second order elliptic problem

$$-\Delta u = f(x,u), \quad x \in \Omega, \tag{0.4}$$

with Dirichlet boundary condition

$$u = 0, \qquad x \in \partial\Omega. \tag{0.5}$$

In particular, for (0.4)-(0.5), a great deal of existence and multiplicity results has been obtained under various assumptions concerning the interaction of $f(\bullet,\bullet)$ with the first eigenvalue λ_1 of the associated linear problem $-\Delta u = \lambda u$ (see for instance [3,6,11,21,24]). Restricting ourselves to problem (0.2)-(0.3), we provide some new conditions of nonresonance with respect to

$$\lambda_1 = (\pi / b\text{-}a)^2, \tag{0.6}$$

in the case of positive solutions. To this end, we combine some classical facts from the theory of elliptic equations [10] with some recent estimates for the "time-map" associated to equation (0.2) (see [17,18,30,33]).

In chapter 3, we provide some new conditions for the solvability of the nonlinear two-point boundary value problem

$$u''(x) + g(u(x)) = p(x) \tag{0.7}$$

$$u(a) = r_1 , u(b) = r_2 , \tag{0.8}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, $p : [a,b] \to \mathbb{R}$ is Lebesgue integrable and $r_1 , r_2 \in \mathbb{R}$.

Starting with Dolph [15], various existence theorems concerning (0.7) - (0.8) have been proved under suitable nonresonance conditions on the nonlinear term g . The usual assumptions require that asymptotically, the range of $\frac{g(u)}{u}$, does not intersect the spectrum of the differential operator $u \to -u''$, subjected to the homogeneous boundary conditions (see for instance [11,12,21,24]). In particular, it is known that (0.7) - (0.8) has at least one solution, for any $p(\bullet)$, r_1, r_2 provided that

$$\lambda_1 < \liminf_{u\to\pm\infty} \frac{g(u)}{u} \le \limsup_{u\to\pm\infty} \frac{g(u)}{u} < \lambda_2 ,$$

where λ_1 and λ_2 are the first two eigenvalues of the associated linear problem.

In some recent papers, [26,27], the growth restrictions on $\frac{g(u)}{u}$ have been relaxed to analogous conditions involving the ratio $\frac{2G(u)}{u^2}$, where

$$G(u) := \int_0^u g(s)\, ds . \tag{0.9}$$

In [26], Mawhin-Ward-Willem found a general necessary and sufficient condition for the solvability of (0.7) - (0.8) between the first two eigenvalues. In particular, a simple corollary of their result in [26], is the following.

Proposition 0.1. *Assume that* $g(u) - \lambda_1 u$ *is nondecreasing and*

$$\lambda_1 < \liminf_{u\to-\infty} \frac{2G(u)}{u^2} \le \limsup_{u\to-\infty} \frac{2G(u)}{u^2} \le \lambda_2 ,$$

$$\lambda_1 < \liminf_{u\to+\infty} \frac{2G(u)}{u^2} \le \limsup_{u\to+\infty} \frac{2G(u)}{u^2} < \lambda_2 .$$

Then (0.7) - (0.8) has at least one solution for all $r_1, r_2 \in \mathbb{R}$ *and* $p \in L^2(a,b)$.

Further extensions were obtained in [27], however, a key assumption in all such results (obtained by variational methods) is that $g(u) - \lambda_1 u$ is nondecreasing. As far as we know, the problem of avoiding such technical condition is still open. Now we propose in this chapter a different approach to this problem. In the same spirit as in [18,30], we prove existence results for (0.7) - (0.8) using topological degree and some estimates for the time map. In particular, as a corollary of our main theorem in chapter 3, we have the following.

Proposition 0.2. *Assume that* $|g(u)| \to \infty$ *for* $|u| \to \infty$, *and*

$$\lambda_1 < \limsup_{u \to -\infty} \frac{2G(u)}{u^2} \leq \limsup_{u \to -\infty} \frac{g(u)}{u} < \lambda_2 ,$$

$$\lambda_1 < \liminf_{u \to +\infty} \frac{g(u)}{u} \leq \liminf_{u \to +\infty} \frac{2G(u)}{u^2} < \lambda_2 ,$$

then (0.7) - (0.8) has at least one solution for all $r_1, r_2 \in \mathbb{R}$ *and* $p \in L^1(a,b)$.

The striking difference between propositions 0.1 and 0.2 is clear. We avoid to require $g(u) - \lambda_1 u$ nondecreasing and we get some improvements allowing better conditions on the ratio $\frac{2G(u)}{u^2}$. In this manner, we can deal with nonlinearities such that on one side (if for instance, $u \geq 0$), $\frac{g(u)}{u}$ may interfer with all the eigenvalues of the linear operator. On the other hand, we pay the price of considering the less general assumptions on $\frac{g(u)}{u}$, and so we cannot generalize completely the previously quoted results. Proposition 0.2 follows from a more general result for the equation

$$u''(x) + h(x,u(x)) = p(x) \tag{0.10}$$

and the case of jumping nonlinearities (c.f. [9,16,19]) is also covered.

Finally, we remark that although we have confined ourselves to the study of the above mentioned two-point boundary value problems; however, similar results can be achieved with respect to other boundary value problems as well. For instance, in [20], we have studied the Neumann problem for nonlinear second order differential equations with singularities at the origin.

CHAPTER ONE

ON THE EXISTENCE OF THE PRINCIPAL EIGENVALUE

1.1 A new type of order — The compactly generated order

In this chapter, we study the following type of order :

Definition 1.1 We say that a Banach space, $X = (X, \|\bullet\|)$ ordered by a cone is *a compactly generated ordered Banach space* if and only if there is a family $(\varphi_i)_{i \in I}$ of continuous, linear functionals on X such that

(i) $\mathbb{I}$ is a compact topological space;
(ii) if $i_n \to i_o$ in $\mathbb{I}$, then $\varphi_{i_n} \to \varphi_{i_o}$ uniformly on bounded sets;
(iii) $x < y$ in $X \Leftrightarrow \varphi_i(x) < \varphi_i(y) \ \forall \ i \in \mathbb{I}$;
(iv) $\|\varphi_i\| \leq 1 \ (i \in \mathbb{I}.)$.

We have at least four examples of well known ordered Banach spaces (OBS) that fit into this general scheme.

Example 1.1 *The Euclidean* N - *space* $\mathbb{R}^N$ *with the usual ordering in* $\mathbb{R}^N$ (that is, $x \leq y \Leftrightarrow x_i \leq y_i \ \forall \ i$.) Now $\mathbb{I}$ corresponds to the set $\{1,\ldots,N\}$ while $\varphi_i(x) := x_i$.

Example 1.2 (*Other order in* $\mathbb{R}^N$). Fix $J \subseteq \{1,\ldots,N\}$ and define

$$x \leq y \Leftrightarrow \begin{cases} x_i \leq y_i & \forall \quad i \in J \\ x_i \geq y_i & \forall \quad i \notin J \end{cases}$$

We easily see that in this case

$$\varphi_i(x) := \begin{cases} x_i & \text{if } i \in J \\ -x_i & \text{otherwise} \end{cases} .$$

Example 1.3 (*Spaces of real symmetric matrices*) In this case, the positive cone is made up of the subspace of positive semi- definite matrices. Now, we let $\mathbf{I}$ be the unit sphere S^{N-1}, then for each $A \in X$ and $y \in S^{N-1}$ we define

$$\varphi_y(A) := (Ay|y)$$

From,

$$(Ay_n|y_n) - (Ay_0|y_0) \pm (Ay_n|y_0)$$

it is easy to deduce condition (ii) of definition 1.1.

Example 1.4 (*Spaces of continuously differentiable functions on a compact space*) Let K be a compact space. We denote by $X = \mathcal{C}^1(K)$ the Banach space of all continuous differentiable real-valued functions on K with the usual maximum norm. This space is endowed with the natural ordering. Here $\mathbf{I} = K$ and

$$\varphi_t(x) := x(t).$$

Condition (ii) in definition (1.1) follows from the fact that the elements of a bounded set in C^1 are uniformly Lipschitz.

1.2 The principal eigenvalue for two-point BVP's in compactly generated ordered Banach spaces

We recall that an eigenvalue λ_1 of the BVP

$$u'' + \lambda L(u) = 0, \qquad B(u) = 0$$

is called the principal eigenvalue when $\lambda_1 > 0$, λ_1 has a positive eigenfunction and λ_1 is the smallest eigenvalue.

In this section, we are concerned with the existence of the principal eigenvalue for the special case

$$-u'' = \lambda L(t,u), \qquad u(a) = 0 = u(b).$$

where $L : [a,b] \times X \to X$ satisfies the assumptions detailed in chapter 0 with X a given compactly generated ordered Banach space. We assume that the order of X is generated by the family $(\varphi_i)_{i \in \mathbf{I}}$, where the φ_i's and $\mathbf{I}$ have the properties listed in definition 1.1. We also assume that X has at least one positive element. We shall denote by

$$\mathbf{B} := \{ u \in C^1([a,b],X) \mid u(a) = u(b) = 0 \}$$

and define a subset P of $\mathbf{B}$ as follows

$$u \in P \Longleftrightarrow 0 \leq u(t) \ \forall \ t \in [a,b].$$

The following result characterizes the interior points of P.

Lemma 1.1 *A point* $u \in P$ *is an interior point of* P *in* $\mathbf{B}$ *if and only if* $u'(a) > 0$, $u'(b) < 0$ *and* $u(t) > 0$ *for* $a < t < b$.

Proof. ($\Rightarrow$) $u \in \overset{\circ}{P}$ implies that $u(a) = 0 = u(b)$ and $u(t) \geq 0 \ \forall \ t$. Assume that $u(t_0) = 0$ for some $t_0 \in]a,b[$. Fix a positive element $x_0 \in X$ and $h \in C^1([a,b])$ with $h(a) = h(b) = 0$, $h'(a) > 0$, $h'(b) < 0$, $h(t) > 0$ for $a < t < b$. The mapping $\overline{x}(t) = h(t)x_0$ is in P. We have

$$u - \frac{1}{n}\overline{x} \to u$$

hence

$$u - \frac{1}{n}\overline{x} \in \overset{\circ}{P} \text{ for sufficiently large } n.$$

Then we have that

$$0 \leq \varphi_i(u(t_0) - \frac{1}{n}\overline{x}(t_0)) \ \forall \ i \in \mathbf{I}$$

$$= \varphi_i(u(t_0)) - \varphi_i(\frac{1}{n}\overline{x}(t_0))$$

$$= -\varphi_i(\frac{1}{n}\overline{x}(t_0))$$

$$< 0$$

which is a contradiction. This shows that $u(t) > 0$ for $a < t < b$.

We now prove that $u'(a) > 0$ (a similar argument works for the case of $u'(b) < 0$). Assume the contrary, that is, $u'(a) \leq 0$. We have that

(I) $$u'(a) - \frac{1}{n}\overline{x}'(a) < 0.$$

The sequence $(u - \frac{1}{n}\overline{x})_n$ converges to u, hence $u - \frac{1}{n}\overline{x} \in \mathring{P}$ for large n. Fix such an n. We have

(II) $$u(t) - \frac{1}{n}\overline{x}(t) \geq 0 \quad \forall \ t.$$

We claim that

(*) there exists $\varepsilon > 0$ such that $\varphi_i(u'(t)) - \varphi_i(\frac{1}{n}\overline{x}'(t)) < 0, \ a \leq t \leq a+\varepsilon, \ i \in \mathbf{I}$.

For, otherwise there exist $t_k \to a$ and $i_k \in \mathbf{I}$ such that

(III) $$\varphi_{i_k}(u'(t_k)) - \varphi_{i_k}(\frac{1}{n}\overline{x}'(t_k)) \geq 0.$$

Since $\mathbf{I}$ is compact, there is a subsequence of $(i_k)_k$ which converges. Assume for simplicity that $i_k \to i_0$. We have $\varphi_{i_k} \to \varphi_{i_0}$ uniformly on bounded sets. So, taking limits in (III) we get

$$\varphi_{i_0}(u'(a)) - \varphi_{i_0}(\frac{1}{n}\overline{x}'(a)) \geq 0.$$

So,

$$u'(a) - \frac{1}{n}\overline{x}'(a) \geq 0$$

and this is contradictary to (I). So (*) holds.

Now fix $t_1 > 0$ such that $a + t_1 \in \,]a,a+\varepsilon[$. For every $i \in \mathbf{I}$, apply the mean value theorem on $[a,a+t_1]$ to the functional $\varphi_i(u(t) - \frac{1}{n}\overline{x})$, we get that

$$\varphi_i(u(a + t_1)) - \varphi_i(\frac{\overline{x}(a + t_1)}{n}) = \varphi_i(u'(\xi) - \frac{\overline{x}'(\xi)}{n})t_1$$

with $\xi \in \,]a,a + t_1[$. We observe that the right hand side of the above equality is negative by virtue of (*) and so we have

$$\varphi_i(u(a + t_1)) < \varphi_i(\frac{\overline{x}(a + t_1)}{n}).$$

Now we take limits with respect to n and we get

$$\varphi_i(u(a + t_1)) \leq 0$$

a contradiction. Hence, we conclude that $u'(a) > 0$.

($\Leftarrow$) Let u be such that $u(a) = 0 = u(b)$, $u > 0$ on $]a,b[$ and $u'(a) > 0$, $u'(b) < 0$. We claim that

(**) there exists $\varepsilon > 0$ and $\delta_1 > 0$ such that $\varphi_i(u'(t)) \geq \delta_1$ (respectively, $\varphi_i(u'(t)) \leq \delta_1$), for all $a \leq t \leq a + \varepsilon$ (respectively, for all $b - \varepsilon \leq t \leq b$) and for all $i \in \mathbf{I}$.

For, on the contrary, there exist sequences $t_n \to a$, $i_n \in \mathrm{I}$ and $\delta_n \downarrow 0$ such that $\varphi_{i_n}(u'(t_n)) \leq \delta_n$. Let i_{n_k} be a subsequence such that $i_{n_k} \to i_0 \in \mathbf{I}$. Taking limits in $\varphi_{i_{n_k}}(u'(t_{n_k})) \leq \delta_{n_k}$ we get $\varphi_{i_0}(u'(a)) \leq 0$. This implies $u'(a) \not> 0$, which is a contradiction. Thus (**) holds

By using a similar argument, it is possible to show that there exists $\delta_2 > 0$ such that

(IV) $$\varphi_i(u(t)) \geq \delta_2 \ \forall\ t \in [a + \varepsilon, b - \varepsilon],\ \forall\ i \in \mathbf{I}.$$

Now we shall prove that the ball $B(u,\delta)$ of centre u and radius δ in the space $C_0^1([a,b],X)$ is contained in P. Consider any $v \in B(u,\delta)$. For every $i \in \mathbb{I}$ and every $t \in [a+\varepsilon, b-\varepsilon]$ we have

$$\begin{aligned}\varphi_i(v(t)) &= \varphi_i(u(t)) + \varphi_i((v-u)(t)) \\ &\geq \varphi_i(u(t)) - \|\varphi_i\| \, \|v-u\|_{C^1} \\ &> \varphi_i(u(t)) - \delta \\ &> 0 \qquad (\text{ by } (IV)).\end{aligned}$$

This means that $v(t) > 0$ for $t \in [a+\varepsilon, b-\varepsilon]$. For every $a < t \leq a+\varepsilon$ and every $i \in \mathbb{I}$, we apply the mean value theorem on $[a,t]$ to the mapping $\varphi_i(v(t))$, and we get

$$\varphi_i(v(t)) = \varphi_i(v'(\xi)).(t-a)$$

for a suitable $\xi \in]a,t[$. It then follows that

$$\begin{aligned}\varphi_i(v(t)) &= \varphi_i(u'(\xi)).(t-a) + \varphi_i(v'(\xi) - u'(\xi)).(t-a) \\ &\geq \varphi_i(u'(\xi)).(t-a) - \|\varphi_i\| \, \|u'(\xi) - v'(\xi)\|_{C^1}(t-a) \\ &\geq [\, \varphi_i(u'(\xi)) - \delta \,](t-a) \\ &> 0 \qquad (\text{ by } (IV)).\end{aligned}$$

This means that $v(t) \geq 0$ in $[a, a+\varepsilon]$. Analogously, we get that $v(t) \geq 0$ in $[b-\varepsilon, b]$, and so we conclude that $v \geq 0$ in $[a,b]$.

Q.E.D.

Lemma 1.2 : *If* $L : [a,b] \times X \to X$ *satisfies the generalized Carathéodory assumptions and if* $L(t,\bullet) : X \to X$ *is a linear operator mapping positive elements into positive elements for each* t, *and if* G *is the Green function of the two-point BVP*

$$-u'' = 0, \qquad u(a) = 0 = u(b),$$

then the mapping $T : \mathbf{B} \to \mathbf{B}$ *defined by*

$$Tu(t) = \int_a^b G(t,s)\, L(s,u(s))ds$$

is strictly positive.

Proof. Let $u \in \mathbf{B}$ be positive. Then $u(t) > 0$ for at least one t. Consequently, $L(t,u(t)) > 0$. It follows that

$$G(t,t)\varphi_i(L(t,u(t))) > 0$$

and hence, by continuity,

$$G(t,\bullet)\varphi_i(L(\bullet,u(\bullet))) > 0$$

in an interval. Therefore

$$\int_a^b G(t,s)\varphi_i(L(s,u(s)))ds > 0 \qquad (i \in \mathbf{I})$$

which implies $Tu(t) > 0$ for $a < t < b$. Writing $\int_a^b . = \int_a^t . + \int_t^b .$ we compute

$$\frac{d}{dt}\varphi_i(T(u)(t)) = -\int_a^t \frac{s-a}{b-a}\varphi_i(L(s,u(s)))ds + \int_t^b \frac{b-s}{b-a}\varphi_i(L(s,u(s)))ds.$$

Taking $t = a$ and $t = b$ we get

$$\varphi_i\left(\frac{d}{dt}\Big|_{t=a} T(u)\right) > 0, \qquad \varphi_i\left(\frac{d}{dt}\Big|_{t=b} T(u)\right) < 0 \qquad (i \in I)$$

that is,

$$\frac{dT(u)}{dt}\Big|_{t=a} > 0, \quad \frac{dT(u)}{dt}\Big|_{t=b} < 0.$$

Therefore we are in position to apply lemma 1.1 and we get $T(u) \in \overset{\circ}{P}$.

Q.E.D.

Theorem 1.1: *If* $L : [a,b] \times X \to X$ *satisfies the generalised Carathéodory conditions, if* $\overline{L([a,b] \times X)}$ *is compact and if* $L(t,\bullet) : X \to X$ *is a linear operator mapping positive elements into positive elements for each* t*, then the two-point BVP*

$$u'' + \lambda L(t,u) = 0, \quad u(a) = 0 = u(b)$$

has a principal eigenvalue.

Proof: It is easily seen from the uniform continuity of G and the compactness of the range of L, that the operator

$$Tu(t) = \int_a^b G(t,s)L(s,u(s))ds$$

is a completely continuous linear operator $\mathbf{B} \to \mathbf{B}$. From lemma 1.2 it follows that T is strictly positive. Then theorem 2 of Ahmad-Lazer [1] implies the conclusion.

Q.E.D.

Corollary 1.1: *Let* L *be as in theorem* 1.1. *If* $[\alpha,\beta] \supsetneq [a,b]$, *then the principal eigenvalue* λ_1^0 *of*

$$u'' + \lambda L(t,u) = 0, \quad u(\alpha) = u(\beta) = 0$$

is strictly greater than the principal eigenvalue λ_1 *of*

$$u'' + \lambda L(t,u) = 0, \quad u(a) = u(b) = 0$$

(i.e., the principal eigenvalue increases as the interval decreases).

Proof: Let G_0 be the Green function associated to

$$-u'' = 0, \quad u(\alpha) = u(\beta) = 0.$$

It is well - known that

$$G_0(t,s) < G(t,s) \qquad (t,s \in [\alpha,\beta] \cap]a,b[ ). \tag{1.1}$$

Let u_0 be an eigenfunction corresponding to λ_1^0. Define

$$u(t) = \begin{cases} 0 & \text{if } a \le t \le \alpha \\ u_0(t) & \text{if } \alpha \le t \le \beta \\ 0 & \text{if } \beta \le t \le b. \end{cases}$$

From (1.1) and from

$$u_0(t) = \lambda_1^0 \int_\alpha^\beta G_0(t,s)L(s,u_0(s))ds$$

it follows that for $a < t < b$ we have

$$\varphi_i(u(t)) < \lambda_1^0 \int_a^b G(t,s)\varphi_i(L(s,u(s)))ds \qquad (i \in \mathbf{I})$$

Let $w = Tu$. By the above, we have $\lambda_1^0 w - u > 0$ and so,by lemma 1.2, $T(\lambda_1^0 w - u) \in \overset{\circ}{P}$. It follows that $\lambda_1^0 T(w) > T(u) = w$. Then corollary 3.1 of Ahmad - Lazer [1] implies $\lambda_1^0 > \lambda_1$.

Q.E.D.

Corollary 1.2: *Let* L_1, L_2 : $[a,b] \times X \to X$ *satisfies the generalized Carathéodory conditions with* $\overline{L_i([a,b] \times X)}$ *compact and* $L_i(t,\bullet) : X \to X$ *a linear operator mapping positive elements into positive elements. If* $L_1(t,u) < L_2(t,u)$ *for all* t *and* u, *then for the principal eigenvalue* $\lambda_1^{\,i}$ *of*

$$u'' + \lambda L_i(t,u) = 0, \quad u(a) = u(b) = 0$$

we have the relationship $\lambda_1^{\,1} > \lambda_1^{\,2}$ (i.e., the principal eigenvalue decreases as the operator L increases).

Proof : Let T_i : $\mathbf{B} \to \mathbf{B}$ be defined by

$$T_i(u)(t) = \int_a^b G(t,s)L_i(s,u(s))ds.$$

Let u be an eigenfunction to $\lambda_1^{\,1}$.By lemma 1.2, $u \in \overset{\circ}{P}$. Moreover, from $L_1 < L_2$ we get for $a < t < b$:

$$u(t) < \lambda_1^{\,1} \int_a^b G(t,s)L_2(s,u(s))ds$$

$$= \lambda_1^{\,1} T(u)(t).$$

Therefore, the conclusion follows from corollary 3.1 of Ahmad - Lazer [1].

Q.E.D.

Theorem 1.2: *Suppose that* $\dim X < \infty$ *and* $f, g : [a,b] \times X \to X$ *satisfy the generalized Carathéodory conditions. Assume that* $\frac{\partial}{\partial u} f(t,u)$ *exists for all* u *and a.e.* t, *satisfies the generalized Carathéodory assumptions and maps positive elements into positive elements. If there exists a positive constant* μ *less than the first eigenvalue* λ_1 *of*

$$u'' + \lambda u = 0, \quad u(a) = u(b) = 0$$

such that $f_u(t,u).v \le \mu v$ *and if* $\lim\limits_{\|u\|\to\infty} \frac{\|g(t,u)\|}{\|u\|} = 0$, *then the two-point boundary value problem*

$$u'' + f(t,u) = g(t,u), \quad u(a) = u(b) = 0$$

has at least one solution.

Proof: The given equation can be rewritten in the form

$$u'' + A(t,u).u = g(t,u) - f(t,0)$$

where

$$A(t,u) = \int_0^1 f_u(t,\xi u)d\xi.$$

Consider the homotopic equations

$$u'' + sA(t,u).u + (1 - s)\mu u = g(t,u) - f(t,0)$$

and let us show the existence of an a priori bound for the corresponding two-point boundary value problem. In the contrary, there are $s_n \in [0,1]$ and solutions u_n such that $\|u_n\|_\infty \to \infty$. Setting $v_n = \frac{u_n}{\|u_n\|_\infty}$, we have

$$(1.2) \qquad v_n'' + s_nA(t,u_n).v_n + (1 - s_n)\mu v_n = \frac{g(t,u_n) - f(t,0)}{\|u_n\|_\infty}.$$

Applying corollary 1.2 to the two eigenvalue problems

$$(1.3) \qquad v'' + \lambda'\{s_nB + (1 - s_n)\mu\}v = 0, \quad v(a) = v(b) = 0$$

$$v'' + \lambda''.\lambda_1 v = 0, \quad v(a) = v(b) = 0$$

we see that for the first eigenvalues the following relation

$$\lambda_1' > \lambda_1''$$

holds for every continuous linear operator B which maps positive elements into positive elements and is such that $B(u) \leq \mu u$. Since $\lambda_1'' = 1$, it follows that $\lambda = 1$ is not an eigenvalue of (1.3), hence $v = 0$ is the only solution to (1.3). From Ascoli's theorem and (1.2) we have that $(v_n)_n$ has a subsequence which converges to v_∞ in C^1.The sequence $(A(\bullet,u_n(\bullet)))_n$ is weakly compact in L^1 and so there is a subsequence which converges pointwise to A_∞. Again, there is a subsequence of $(s_n)_n$ which converges to s_∞. At the end we can take limits in

$$v_{n_k}(t) = \int_a^b G(t,s)\{[s_{n_k}A(s,u_{n_k}(s)) + (1 - s_{n_k})\mu]v_{n_k}(s) - \frac{g(s,u_{n_k}(s)) - f(s,0)}{\|u_{n_k}\|_\infty}\}ds$$

and obtain

$$v_\infty'' + \{s_\infty A_\infty + (1 - s_\infty)\mu\}v_\infty = 0, \quad v_\infty(a) = v_\infty(b) = 0.$$

By (1.3), $v_\infty = 0$, a contradiction. Therefore the a priori bound does exist. Moreover, it follows from (1.3) that the only solution to

$$u'' + \mu u = 0, \quad u(a) = u(b) = 0$$

is $u \equiv 0$. Therefore the conclusion follows from the well - known properties of topological degree.

Q.E.D.

CHAPTER 2

EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR TWO-POINT BOUNDARY VALUE PROBLEMS.

2.1 PREAMBLE

This chapter deals with the existence and multiplicity of positive (non trivial) solutions for the two-point boundary value problem

$$-u'' = f(x,u) \tag{2.1}$$

$$u(a) = 0 = u(b). \tag{2.2}$$

We shall assume throughout this chapter that

$$f : [a,b] \times \mathbb{R}^+ \to \mathbb{R}^+, \quad \mathbb{R}^+ = [0,+\infty),$$

verifies the (L^1-) Carathéodory conditions, i.e., $f(x,\bullet)$ is continuous for a.e. $x \in [a,b]$, $f(\bullet,s)$ is measurable for all $s \in \mathbb{R}^+$ and for each $r > 0$, there is $\gamma_r \in L^1([a,b]; \mathbb{R}^+)$ such that $f(x,s) \leq \gamma_r(x)$ for all $0 \leq s \leq r$ and a.e. $x \in [a,b]$.

Accordingly, solutions to (2.1) are intended in the generalized (Carathéodory) sense, i.e., $u(\bullet)$ verifies (2.1) for a.e. $x \in [a,b]$ with u' absolutely continuous. Further assumptions on f will be explicitly spelt out later.

We note that any solution $u(\bullet)$ of (2.1)-(2.2), with $u \geq 0$ and $u \not\equiv 0$ on $[a,b]$ is such that $u(x) > 0$ for all $x \in \,]a,b[$. This is a consequence of the following

LEMMA 2 1: *Suppose that*

$$-u''(x) = h(x), \quad u(a) = u(b) = 0,$$

with $h \in L^1([a,b]; \mathbb{R}^+)$. *If* $u \not\equiv 0$, *then* $u(x) > 0$ *for all* $x \in]a,b[$ *and* $u'(a) > 0 > u'(b)$.

The proof of lemma 2.1 follows from elementary arguments and so it is omitted.

Let $X = C_0^0([a,b])$, be the Banach space of continuous real valued functions in $[a,b]$ satisfying (2.2), with the sup norm $|\bullet|_\infty$ and let

$$C := \{ u \in X : u(x) \geq 0 \text{ for all } x \in [a,b] \}$$

be the positive cone in X (see [2,10]). For any set $W \subset C$, we denote by $\overline{W}$ and ∂W its closure and boundary relatively to C respectively. We also define, for $R > 0$,

$$B_R := \{ u \in C : |u|_\infty < R \}.$$

Let $k : [a,b]^2 \to \mathbb{R}$ be the Green function for the differential operator $u \to -u''$ with boundary condition (2.2). Then it is well known (see [34]) that the problem (2.1) - (2.2) is equivalent to the (nonlinear) operator equation in X.

$$u = \Phi(u) \tag{2.3}$$

where

$$\Phi(u)(x) := \int_a^b k(x,t)f(t,u(t))dt \tag{2.4}$$

The assumptions on f imply that Φ is completely continuous and $\Phi(C) \subset C$. Then it is clear that $u(\bullet)$ is a solution of (2.1) - (2.2) with $u > 0$ on $]a,b[$ if and only if $u \in C\backslash\{0\}$ is a fixed point of Φ.

In order to find nontrivial solutions of (2.3) we shall use some results from the theory of positive operators in Banach spaces as developed in [2,10,22]. Precisely, we employ the properties of the fixed point index (see [2,31]). In particular, we denote by $i_C(\psi, B_R)$,the fixed point index (relatively to C) of a compact map $\psi : \overline{B_R} \to C$, such that $\psi(u) \not\equiv u$ for $u \in \partial B_R$, with respect to B_R.

2.2 MAIN RESULTS.

Following the previous section, we give some results for the computation of $i_C(\Phi,B_R)$, with Φ defined by (2.4). These results will be employed for obtaining existence and multiplicity theorems to equation (2.1).

Our first result concerns the case in which the map f can be suitably linearized around 0. To this purpose, we recall some basic facts about the linear eigenvalue problem with weight.

Let us consider the linear problem on $[a,b]$

$$-u'' = \mu m(x)u, \qquad u \geq 0, \tag{2.5}$$

$$u(a) = u(b) = 0, \tag{2.6}$$

where $m \in L^\infty([a,b]; \mathbb{R}^+)$ and $m > 0$ on a set of positive measure. Then it is well - known (see [2,10]) that (2.5) – (2.6) has only one eigenvalue $\mu_1(m)$, which is positive and it is equal to the first eigenvalue of the linear problem $-u'' = \mu mu$, $u(a) = u(b) = 0$. Moreover, we have $\mu_1(m) > \mu_1(m^*)$ for $m \leq m^*$, with strict inequality on a set of positive measure, and μ_1 is continuous with respect to m. Then the following result holds true ;

PROPOSITION 2.1: *Assume*

(f_1) $$\limsup_{s \to 0^+} \frac{f(x,s)}{s} \leq m(x),$$

uniformly a.e. in $x \in [a,b]$, *with* $m \in L^\infty([a,b], \mathbb{R}^+)$ *and* $m > 0$ *on a set of positive measure and*

(m_1) $$\mu_1(m) > 1.$$

Then there is $R_0 > 0$ *such that*

$$i_C(\Phi, B_R) = 1$$

holds for all $0 < R \leq R_0$.

Assume

(f_2) $$\liminf_{s \to 0^+} \frac{f(x,s)}{s} \geq m(x)$$

uniformly a.e. in $x \in [a,b]$, *with* $m \in L^\infty([a,b], \mathbb{R}^+)$ *and* $m > 0$ *on a set of positive measure and*

(m_2) $$\mu_1(m) < 1.$$

Then there is $R_0 > 0$ *such that*

$$i_C(\Phi, B_R) = 0$$

holds for all $0 < R \leq R_0$.

The proof is omitted since it can be easily obtained using general results on elliptic equations (see, for instance [10, chapter 3]). Observe that (f_1) implicitly requires that $f(x,0) \equiv 0$.

The next results are based on some estimates for the computation of the time-map used in [18,30]. However,with respect to [18], the proof is considerably simpler.

Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function and define

$$G(s) := \int_0^s g(t)\, dt$$

and

$$\Gamma_g(s) := \frac{1}{2}\lambda_1 s^2 - G(s) \tag{2.7}$$

Then we have

PROPOSITION 2.2: *Assume*

(g_1) $$f(x,s) \leq g(s)$$

for all $s \in \mathbb{R}^+$ *and a.e.* $x \in [a,b]$ *and*

(G_1) $$G(\overline{w}) < \frac{1}{2}\lambda_1 \overline{w}^2$$

for some $\overline{w} > 0$. *Then there is* $0 < w \leq \overline{w}$ *such that* $i_C(\Phi, B_w) = 1$.

Assume

(g_2) $$f(x,s) \geq g(s),$$

for all $s \in \mathbb{R}^+$ *and a.e.* $x \in [a,b]$ *and*

(G_2) $$G(\hat{w}) > (1/2)\lambda_1 \hat{w}^2$$

for some $\hat{w} > 0$. Then there is $0 < w \leq \hat{w}$ such that $i_C(\Phi, B_w) = 0$

Proof. We examine the first part of the statement. From (G_1), we have $\Gamma_g(\overline{w}) > 0$, while $\Gamma_g(0) = 0$ (by definition of Γ_g). Therefore we can choose

$$w := \min \{ s \in [0,\overline{w}] : \Gamma_g(s) = \Gamma_g(\overline{w}) \} \tag{2.8}$$

and $0 < w \leq \overline{w}$. Moreover, by definition of w, $\Gamma_g(s) < \Gamma_g(w)$, for all $s \in [0,w[$, that is,

$$G(w) - G(s) < (1/2)\lambda_1(w^2 - s^2), \quad \text{for } 0 \leq s < w. \tag{2.9}$$

We want to prove that

$$u \neq \theta\Phi(u), \tag{2.10}$$

for all $0 \le \theta \le 1$ and $u \in \partial B_w$. Assume, by contradiction, that for some $0 \le \theta \le 1$ and $u(.) \in \partial B_w$, we have $u = \theta\Phi(u)$. Equivalently, we have

$$-u''(x) = \theta f(x,u(x)), \tag{2.11}$$

for a.e. $x \in [a,b]$, with $u \ge 0$, satisfying (2.2) and such that $\max u(.) = |u|_\infty = w$. Let $x^* \in [a,b]$ be such that

$$u(x^*) = \max u(.) = w.$$

As $w > 0$, we have $a < x^* < b$ and, by lemma 2.1, $u(x) > 0$ for $a < x < b$ and $u'(a) > 0 > u'(b)$.

Now we perform some phase-plane analysis on equation (2.11). Setting $y = u'$, we have

$$u' = y, \qquad y' = -\theta f(x,u),$$

so that $y : [a,b] \to \mathbb{R}$ is a nonincreasing function such that $y(a) > 0 > y(b)$ and $y(x^*) = 0$. Define

$$\alpha := \min\{ x : u(x) = w \}$$
$$\beta := \max\{ x : u(x) = w \}.$$

We have $a < \alpha \le x^* \le \beta < b$ and, moreover,

$$y(x) > 0 \quad \text{for } a \le x < \alpha$$

$$y(x) = 0 \quad \text{for } \alpha \le x \le \beta$$

$$y(x) < 0 \quad \text{for } \beta < x \le b,$$

$0 \le u(x) < w$ for $x \in J := [a,\alpha[\; \cup \;]\beta,b]$. Finally, we define

$$z(x) := (1/2)y^2(x) + \theta G(u(x)).$$

The map $z : [a,b] \to \mathbb{R}^+$ is absolutely continuous and we have, for a.e. $x \in [a,b]$,

$$z'(x) = y(x).y'(x) + \theta g(u(x)).u'(x)$$

$$= \theta\, y(x)\,[\, g(u(x)) - f(x,u(x))\,].$$

Then, by (g_1), $z(.)$ is nondecreasing on $[a,\alpha]$ and nonincreasing on $[\beta,b]$. Hence, for $x \in [a,\alpha]$, we have

$$(1/2)y^2(x) + \theta G(u(x)) \leq (1/2)y^2(\alpha) + \theta G(u(\alpha)) = 0 + \theta G(w)$$

and

$$(1/2)y^2(x) + \theta G(u(x)) \leq (1/2)y^2(\beta) + \theta G(u(\beta)) = 0 + \theta G(w),$$

for $x \in [\beta,b]$. Therefore, (2.9) and $\theta \leq 1$ imply

$$y^2(x) < \lambda_1(w^2 - u(x)^2), \quad \text{for } x \in J$$

and so,

$$\frac{|u'(x)|}{\sqrt{\lambda_1(w^2 - u(x)^2)}} < 1, \quad \text{for } x \in J.$$

Integration on J gives

$$b - a \geq b - \beta + \alpha - a > \int_J \frac{|u'(x)|}{\sqrt{\lambda_1(w^2 - u(x)^2)}}dx$$

$$= 2\int_0^w \frac{d\xi}{\sqrt{\lambda_1(w^2 - \xi^2)}} = \frac{\pi}{\sqrt{\lambda_1}} = b - a,$$

and so, a contradiction is achieved. Thus (2.10) is proved and by the properties of the fixed point index, we get

$$i_C(\Phi,B_w) = i_C(0,B_w) = 1.$$

Now we proceed to the second part of Proposition 2.2. From (G_2) we have $\Gamma_g(\hat{w}) < 0$, while $\Gamma_g(0) = 0$. Therefore we can choose

$$w := \min \{ s \in [0,\hat{w}] : \Gamma_g(s) = \Gamma_g(\hat{w}) \} \tag{2.12}$$

and $0 < w \leq \hat{w}$. Moreover, by definition of w, $\Gamma_g(s) > \Gamma_g(w)$, for all $s \in [0,w[$. This implies $\Gamma'_g(w) = \lambda_1 w - g(w) \leq 0$. Thus we have proved that

$$G(w) - G(s) > (1/2)\lambda_1(w^2 - s^2), \quad \text{for } 0 \leq s < w \tag{2.13}$$

and

$$g(w) > 0. \tag{2.14}$$

Fix $\upsilon > \lambda_1$ and define the operator

$$\Phi^*(u)(x) := \upsilon \int_a^b k(x,t)u(t)dt.$$

It is clear that $\Phi^* : \overline{B_w} \to C$ is compact and $\Phi^*(u) \neq u$ for $u \in \partial B_w$. Moreover, as $\mu_1(\upsilon) < 1$, then $i_C(\Phi^*,B_w) = 0$, according to [10, Prop. 3.7]. We want to prove that

$$u \neq \theta\Phi(u) + (1-\theta)\Phi^*(u) \tag{2.15}$$

for all $0 \leq \theta \leq 1$ and $u \in \partial B_w$. Assume by contradiction, that for some $0 \leq \theta \leq 1$ and $u(\cdot) \in \partial B_w$, we have $u = \theta\Phi(u) + (1-\theta)\Phi^*(u)$, that is

$$-u''(x) = \theta f(x,u(x)) + (1-\theta)\upsilon u(x) \tag{2.16}$$

for a.e. $x \in [a,b]$, with $u \geq 0$, $|u|_\infty = w$ and $u(\cdot)$ satisfying (2.2). Let $x^* \in [a,b]$ be such that $u(x^*) = \max u(\cdot) = w$. Arguing as above, we set $y = u'$ and observe that $y(\cdot)$ is nonincreasing and $y(a) > y(x^*) = 0 > y(b)$. In this case, it is important to observe that

$$0 \leq u(x) < w \quad \text{for } x \neq x^*. \tag{2.17}$$

In fact, $a < x^* < b$ and $g(u(x^*)) > 0$ by (2.14). Then, from (g_2), $f(x,u(x)) \geq \varepsilon > 0$ for some ε and a.e. x in a neighbourhood U of x^*. Hence, by (2.16),

$$u''(x) \leq -\delta < 0 \text{ for a.e. } x \in U,$$

with δ small enough. Then $y(.)$ is decreasing on U and so

$$y(x) > 0 \quad \text{for } a \leq x < x^*$$

$$y(x) < 0 \quad \text{for } x^* < x \leq b$$

and the claim is proved.

Now we define the absolutely continuous function

$$z(x) := (1/2)y^2(x) + \theta G(u(x)) + (1-\theta)(1/2)\upsilon u^2(x).$$

We have for a.e. $x \in [a,b]$,

$$z'(x) = y(x)y'(x) + \theta g(u(x))u'(x) + (1-\theta)\upsilon u(x)u'(x)$$

$$= -\theta y(x)[\, f(x,u(x)) - g(u(x))\,].$$

Then, by (g_2), $z(.)$ is nonincreasing on $[a,x^*]$ and nondecreasing on $[x^*,b]$. Hence x^* is a point of absolute minimum for $z(\bullet)$ and so,

$$(1/2)y^2(x) + \theta G(u(x)) + (1-\theta)(1/2)\upsilon u^2(x) \geq$$

$$\geq \frac{1}{2}y^2(x^*) + \theta G(u(x^*)) + (1 - \theta)\frac{\upsilon u^2(x^*)}{2}$$

$$= 0 + \theta G(w) + (1-\theta)(1/2)\upsilon w^2.$$

Therefore, (2.13), (2.17) and $\upsilon > \lambda_1$, imply

$$y^2(x) \ > \ \lambda_1(w^2 - u(x)^2), \quad \text{for} \ \ x \neq x^*$$

and so,

$$|u'(x)|/(\lambda_1(w^2 - u(x)^2)1/2 \ > \ 1 \quad \text{for } x \neq x^*.$$

Integration on $[a,b]$ gives

$$b - a = b - x^* + x^* - a \ <$$

$$< \int_{x^*}^{b} \frac{-u'(x)}{\sqrt{\lambda_1(w^2 - u(x)^2)}} ds + \int_{a}^{x^*} \frac{u'(x)}{\sqrt{\lambda_1(w^2 - u(x)^2)}} dx$$

$$= 2 \int_0^w \frac{d\xi}{\sqrt{\lambda_1(w^2 - \xi^2)}} = \frac{\pi}{\sqrt{\lambda_1}} = b - a,$$

and so, a contradiction is achieved. Thus (2.15) is proved and by homotopy of the fixed point index, we get

$$i_C(\Phi,B_w) \ = \ i_C(\Phi^*,B_w) \ = \ 0.$$

Therefore, the proof of proposition 2.2 is completed ∎

REMARK 2.1: From the proof it is clear that, under (g_1), we have $i_C(\Phi,B_w) = 1$ for any $w > 0$ such that $\Gamma_g(s) < \Gamma_g(w)$ for all $s \in [0,w[$. Respectively, when (g_2) is assumed, we have $i_C(\Phi,B_w) = 0$ for any $w > 0$ such that $\Gamma_g(s) > \Gamma_g(w)$ for all $s \in [0,w[$.

2.3. EXISTENCE AND MULTIPLICITY THEOREMS.

In this final section of the chapter, we present some results for positive solutions of (2.1) – (2.2) which can be obtained by combining Propositions 2.1 and 2.2.

THEOREM 2.1: *Assume* (f_1), (m_1), (g_2) *and* (G_2). *Then problem* (2.1) – (2.2) *has at least one solution* $u(\bullet)$ *with* $u(x) > 0$ *for* $x \in \,]a,b[$.

THEOREM 2.2: *Assume* (f_2), (m_2), (g_1) *and* (G_1). *Then the same conclusion of Theorem* 2.1 *holds.*

THEOREM 2.3: *Let* $g_1, g_2 : \mathbb{R}^+ \to \mathbb{R}^+$ *be continuous functions such that*

(k_1) $$g_1(s) \leq f(x,s) \leq g_2(s)$$

for all $s \in \mathbb{R}^+$ *and a.e.* $x \in [a,b]$. *For* G_1 *and* G_2 *defined accordingly, suppose that there are* $\overline{w}$, $\hat{w} > 0$ *such that*

$$G_1(\hat{w}) > (1/2)\lambda_1\hat{w}^2, \qquad G_2(\overline{w}) < (1/2)\lambda_1\overline{w}^2.$$

Then the same conclusion of Theorem 2.1 *holds.*

The proof of all the theorems is a straightforward consequence of Propositions 2.1 and 2.2, using the additivity / excision property of the fixed point index.

Our results extend to equation (2.1) some analogous theorems, for the autonomous scalar equation $-u'' = g(u)$, obtained recently in [17]. A particular case in which all the theorems may be applied is the following

COROLLARY 2.1: *Assume that either*

$$\limsup_{s\to 0^+} \frac{f(x,s)}{s} \leq \sigma < \lambda_1 < \zeta \leq \liminf_{s\to +\infty} \frac{f(x,s)}{s}$$

or

$$\liminf_{s\to 0^+} \frac{f(x,s)}{s} \geq \zeta > \lambda_1 > \sigma \geq \limsup_{s\to+\infty} \frac{f(x,s)}{s}$$

holds, uniformly a.e. in $x \in [a,b]$. *Then the same conclusion of Theorem* 2.1 *holds.*

For analogous results for the Dirichlet problem for elliptic equations, see [10,13].

A simple application of Theorem 2.3 can be given to the problem

$$-u'' = g(u) + h(x) \tag{2.18}$$

$$u(a) = u(b) = 0, \tag{2.2}$$

with $g: \mathbb{R}^+ \to \mathbb{R}^+$ a continuous function and $h \in L^\infty([a,b]; \mathbb{R}^+)$. We set

$$G(s) := \int_0^s g(t)dt \quad \text{and} \quad H := |h|_\infty.$$

Then $f(x,s) = g(s) + h(x)$ and (k_1) is trivially satisfied with the choice $g_1 = g$, $g_2 = g + H$ and so we have

COROLLARY 2.2: *Problem* (2.18) – (2.2) *has a (nontrivial) positive solution provided that there are* $\hat{w}, \overline{w} > 0$ *such that*

$$\frac{1}{2}\lambda_1 \hat{w}^2 < G(\hat{w}), \quad G(\overline{w}) < \frac{1}{2}\lambda_1 \overline{w}^2 - H\overline{w}.$$

REMARK 2.2: In [18], the problem

$$(\dagger) \qquad u'' + g(u) = p(x), \quad u(a) = u(b) = 0$$

was considered and an existence theorem was proved under the assumptions $p \in L^{\infty}(a,b)$ and

(#) $$\liminf_{u \to \pm\infty} \frac{2G(u)}{u^2} < \lambda_1.$$

So, the question arises to whether such condition can be relaxed to

(÷) $$\liminf_{u \to \pm\infty} \frac{g(u)}{u} < \lambda_1.$$

By using analogous arguments as developed in [14] (for periodic BVP), we see that in general the answer is in the negation. Indeed, we have

EXAMPLE 2.1: There are $g : \mathbb{R} \to \mathbb{R}$ and $p : [0,\pi] \to \mathbb{R}$ continuous, with

$$\frac{1}{2} = \liminf_{u \to \pm\infty} \frac{g(u)}{u} < \limsup_{u \to \pm\infty} \frac{g(u)}{u} = 1$$

such that there are no solutions to BVP (†).

This example as well as further investigations concerning the nonlinear BVP (†) will appear in a forthcoming paper [28].

In conclusion, we give a multiplicity result for (2.1) – (2.2).

THEOREM 2.4: *Let* f *satisfy* (k_1) *and assume*

$$\Gamma_{g_1}(s) < \limsup_{s \to +\infty} \Gamma_{g_1}(s), \quad \Gamma_{g_2}(s) > \liminf_{s \to +\infty} \Gamma_{g_2}(s) \ \forall \ s \geq 0$$

and

(k_2) $$\limsup_{s \to +\infty} \Gamma_{g_1}(s) > 0 > \liminf_{s \to +\infty} \Gamma_{g_2}(s),$$

(*with* Γ_{g_i} $(i = 1,2)$ *defined as in* (2.7)). *Then problem* (2.1) – (2.2) *has a sequence* $u_n(\cdot)$ *of solutions with* $u_n(x) > 0$ *for* $x \in]a,b[$ *and* $\max u_n(\cdot) \to +\infty$.

Proof. Let us set

$$\rho_1 := \limsup_{s\to+\infty} \Gamma_{g_1}(s), \quad \rho_2 := \liminf_{s\to+\infty} \Gamma_{g_2}(s).$$

By condition (k_2) we can find two increasing sequences $(w_n)_n$ and $(v_n)_n$ with $w_n \to +\infty$, $v_n \to +\infty$, such that $\Gamma_{g_1}(w_n) \to \rho_1$, $\Gamma_{g_2}(v_n) \to \rho_2$ and

$$\Gamma_{g_1}(s) < \Gamma_{g_1}(w_n), \quad \text{for all } s \in [0,w_n[, \tag{2.19}$$

$$\Gamma_{g_2}(s) > \Gamma_{g_2}(v_n), \quad \text{for all } s \in [0,v_n[, \tag{2.20}$$

respectively.

Moreover, we can assume without loss of generality (possibly passing to subsequences), that

$$v_n < w_n < v_{n+1}$$

holds for each $n \in N$. Then, by Proposition 2.2 and Remark 2.1, we have from (2.19) and (2.20), respectively,

$$i_C(\Phi,Bw_n) = 1, \quad i_C(\Phi,Bv_n) = 0$$

for all $n \in N$. Hence the additivity / excision property of the fixed point index guarantees, for each n, the existence of a solution $u_n(\bullet)$ to (2.3) with

$$v_n < |u_n|_\infty = \max u_n(\bullet) < w_n$$

and the theorem is proved.

Q.E.D.

Multiplicity results similar to Theorem 2.4 have been recently obtained in [17,18,32]. However, we point out that our result is independent of those contained in [18] and [32] where the positivity of f is not assumed but further restrictions are considered. With respect to [17], a more general equation is examined. Furthermore, we note that the assumption that f is non-negative may be relaxed to

$$f(x,s) \geq 0 \quad \text{for all } s \geq d > 0 \quad \text{and a.e. } x \in [a,b],$$

provided that supplementary conditions are taken into account.

Finally, we observe that a multiplicity result for equation (2.18) may be easily obtained from Corollary 2.2.

CHAPTER THREE

SOLVABILITY OF THE NONLINEAR BVP BETWEEN THE FIRST TWO EIGENVALUES.

3.1 PREAMBLE

In this chapter, we provide some new conditions for the solvability of the nonlinear two-point boundary value problem

$$u''(x) + h(x,u(x)) = p(x) \tag{3.1}$$

$$u(a) = r_1 , u(b) = r_2 , \tag{3.2}$$

where $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ verifies the Carathéodory conditions, $p : [a,b] \to \mathbb{R}$ is Lebesgue integrable and $r_1 , r_2 \in \mathbb{R}$ .Further assumptions on h will be explicitly required in the next section.

3.2. STATEMENT OF MAIN RESULT.

In this section, we shall state our main result which concerns the boundary value problem (BVP) (3.1) - (3.2). Besides the proof of Proposition 0.2, which is given as a consequence of the main result (Theorem 3.1), we shall also prove some lemmas which will be needed for the proof of the main result in section. 3.3.

Theorem 3.1. *Let* $h : [a,b]\times\mathbb{R} \to \mathbb{R}$ *verify the Carathéodory conditions and denote by* $r := \max\{|r_1|, |r_2|\}$. *Suppose that there are continuous functions* $g_\pm : \mathbb{R} \to \mathbb{R}$ *for which* $|g_\pm(u)| \to +\infty$ *as* $|u| \to +\infty$; *(with* $G_\pm(u) := \int_0^u g_\pm(s)\, ds$*) and constants* $q^\pm > \lambda_1$ *such that the following conditions are satisfied:*

$$h(x,u) \leq g_-(u) \ \forall \ u \leq r \ \textit{and a.e.} \ x \in [a,b] \tag{3.3}$$

$$h(x,u) \leq g_+(u) \ \forall \ u \geq -r \ \textit{and a.e.} \ x \in [a,b] \tag{3.4}$$

$$\limsup_{u \to -\infty} \frac{2G_-(u)}{u^2} \geq \mu^- > \lambda_1 \tag{3.5}$$

$$\liminf_{u \to +\infty} \frac{h(x,u)}{u} \geq \mu^+ > \lambda_1 \ \textit{unif. a.e. in} \ x \in [a,b] \tag{3.6}$$

$$\limsup_{u \to -\infty} \frac{h(x,u)}{u} \leq q^- \ \textit{unif. a.e. in} \ x \in [a,b] \tag{3.7}$$

$$\liminf_{u \to +\infty} \frac{2G_+(u)}{u^2} \leq q^+ \tag{3.8}$$

$$\frac{1}{\sqrt{q^+}} + \frac{1}{\sqrt{q^-}} > \frac{1}{\sqrt{\lambda_1}}. \tag{3.9}$$

Then the BVP

$$\begin{cases} u''(x) + h(x,u(x)) = p(x) & (3.1) \\ u(a) = r_1 , \ u(b) = r_2 & (3.2) \end{cases}$$

has at least one solution for all $r_1 , r_2 \in \mathbb{R}$, *and* $p(\cdot) \in L^1(a,b)$.

We now give the proof of Proposition 0.2 from Theorem 3.1.

Proof of Proposition 0.2. By comparing (0.7) with (3.1) we see that $g(u) = h(x,u)$ $\forall x, u$ and $g_-(u) = g_+(u) = g(u)$. Hence $G_-(u) = G_+(u) = G(u)$. Call

$$q^- := \limsup_{u \to -\infty} \frac{g(u)}{u} < \lambda_2$$

and

$$q^+ := \liminf_{u \to +\infty} \frac{2G(u)}{u^2} < \lambda_2 .$$

Then

$$\frac{1}{\sqrt{q^+}} + \frac{1}{\sqrt{q^-}} > \frac{1}{\sqrt{\lambda_2}} + \frac{1}{\sqrt{\lambda_2}} = \frac{2}{\sqrt{\lambda_2}} = \frac{1}{\sqrt{\lambda_1}} \qquad \text{q.e.d.}$$

Before stating our first lemma whose proof depends mainly on the combination of the Leray-Schauder topological degree and the properties of the Fučik's spectrum, we consider the following parametrized family of problems:

$$\begin{cases} u''(x) + f(x,u(x);\theta) = \theta p(x) & (3.1_\theta) \\ u(a) = \theta r_1 \, , \; u(b) = \theta r_2 \, , \; 0 \le \theta \le 1 & (3.2_\theta) \end{cases}$$

where

$f(x,u(x);\theta) := (1 - \theta)\,[L^+u^+ - L^-u^-] + \theta h(x,u)$;

L^+, L^- are constants : $\dfrac{1}{\sqrt{L^+}} + \dfrac{1}{\sqrt{L^-}} > \dfrac{1}{\sqrt{\lambda_1}}$ with $L^\pm > \lambda_1$;

$u^+ := \max(u,0)$; $u^- := \max(-u,0)$.

Let $K : [a,b]^2 \to \mathbb{R}$ be the Green function for the differential operator $u \to -u''$ with boundary condition $u(a) = u(b) = 0$, that is, for any $w \in L^1([a,b],\mathbb{R})$,

$$u_w(x) := \int_a^b K(x,s)\, w(s)\, ds$$

is the unique solution of the BVP

$$-u''(x) = w(x) \; , \; u(a) = u(b) = 0 \; .$$

Define the C^1-function

$$\psi(x) := \frac{r_2 - r_1}{b - a}(x - a) + r_1 - \int_a^b K(x,s)\, p(s)\, ds$$

with $\psi'(x)$ absolutely continuous. Also, let

$$\phi(u;\theta)(x) := \theta\psi(x) + \int_a^b K(x,s)\, f(s,u(s)\,;\,\theta)\, ds$$

where, $X = \mathcal{C}([a,b],\mathbb{R})$ is the Banach space of continuous real valued functions in $[a,b]$, with the sup norm $|\bullet|_\infty$, $\phi : X\times[0,1] \to X$ is continuous and compact on bounded sets. Then it is clear that $u(x) = \phi(u;\theta)(x)$ if and only if $u(x)$ is a solution of (3.1_θ) - (3.2_θ).

Lemma 3.1. *Assume that there are two constants* $A, B > r$, *such that the following condition holds:*

(i_0) "If $u(\bullet)$ is any solution of (3.1_θ) - (3.2_θ) for some $\theta \in (0,1)$, such that $-A \le u(x) \le B$ $\forall x\in[a,b]$, then $-A < u(x) < B$ $\forall x\in[a,b]$".

Then (3.1) - (3.2) *has at least one solution* $u(\bullet)$ *such that* $-A \le u(x) \le B$ $\forall x\in[a,b]$.

Proof. Define

$$\Omega = \{u\in X : -A < u(x) < B \;\; \forall x\in[a,b]\}$$

we have that

$$\bar{\Omega} = \{u\in X : -A \le u(x) \le B \;\; \forall x\in[a,b]\}.$$

We note that $\Omega \in X$ is open, bounded, $0\in\Omega$ and $\partial\Omega = \bar{\Omega}\setminus\Omega$. Also,$\Phi = \phi(u;\theta)$: $\bar{\Omega}\times[0,1] \to X$ is compact. We also remark that $u = \phi(u;0)$ if and only if u is a solution of (3.1_0) - (3.2_0). This is equivalent to

$$\begin{cases} u'' + L^+u^+ - L^-u^- = 0 \\ u(a) = 0 \;, \quad u(b) = 0 \;. \end{cases}$$

Hence $u \equiv 0$ (see Fučik [19] or Drábek [16]).Then, $u \neq \phi(u;0)$ on $\partial\Omega$ and so

$$\deg_{LS}(I - \phi(\bullet;0), \Omega, 0) \text{ is defined.}$$

Moreover, from Fučik's theorem [19, page 325], $\deg_{LS}(I - \phi(\bullet;0), \Omega, 0) \neq 0$. So, the statement of the lemma also holds for $\theta = 0$. For $\theta = 1$, if $u = \phi(u;1)$ with $u \in \partial\Omega$, then (3.1) - (3.2) has a solution $u(\bullet)$ such that $-A \leq u(x) \leq B \ \forall x \in [a,b]$. Then, we can assume without loss of generality that $u \neq \phi(u;1)$ for $u \in \partial\Omega$. Hence, when $\theta = 0$ and $\theta = 1$, $u \neq \phi(u;\theta)$ for $u \in \partial\Omega$.

Assume as a way of contradiction that $u = \phi(u;\theta)$ for some $u \in \partial\Omega$, $0 \leq \theta \leq 1$. Then u is a solution of (3.1_θ) - (3.2_θ) and $-A \leq u(x) \leq B \ \forall x$. But by the hypotheses of the lemma this implies that $-A < u(x) < B \ \forall x$, and so, $u \in \Omega$, $u \notin \partial\Omega$. This gives a contradiction.

Then we have that

$$u \neq \phi(u;\theta) \ \forall u \in \partial\Omega, \ \forall \theta \in [0,1].$$

Hence

$$\deg_{LS}(I - \phi(\bullet;\theta), \Omega, 0) = \text{constant}$$

with respect to $\theta \in [0,1]$. Therefore

$$\deg_{LS}(I - \phi(\bullet;1), \Omega, 0) = \deg_{LS}(I - \phi(\bullet;0), \Omega, 0) \neq 0.$$

This implies that (3.1) - (3.2) has a solution in Ω. Q.E.D.

Remark 3.1. If $-k \leq u(x) \leq k \ \forall u$ solution of (3.1_θ) - (3.2_θ), $0 < \theta < 1$ then take $A = B = k + 1$ and the hypotheses (i_0) of Lemma 3.1 is fulfilled.

We proceed to find the appropriate A and B. But first, we give the following definitions.

Definition 3 1. For any $u(\bullet)$ solution of (3.1_θ) - (3.2_θ), we define by $u^* \ := \ \max u(\bullet)$ and choose $x^* \in [a,b]$ the point of maximum. So

$$u^* \ := \max u(\bullet) \ = \ u(x^*) \, .$$

Definition 3.2. For any $d > r$, let $J_d^+ \ := \ [\alpha_d,\beta_d]$ be the maximal interval containing x^* and such that $u(x) \geq -d$ .Similarly, J_d^- is defined.

Definition 3. For any $L > 0$, let $J_L \ := \ [\alpha_L,\beta_L]$ be the maximal interval containing x^* and such that $u(x) \geq u^* - L$.

Next we perform some phase-plane analysis on (3.1_θ). Setting $y(x) \ = \ u'(x) \ - \ \theta \ P(x)$ where $P(x) \ := \ \int_a^x p(s)\,ds$ and call $M \ := \ |P|_\infty$ we have

$$u'(x) \ = \ y(x) \ + \ \theta \ P(x) \tag{$3.1_\theta'$}$$

$$y'(x) \ = \ - f(x,u(x);\theta) \, . \tag{$3.1_\theta''$}$$

Lemma 3.2. *Suppose that there is* $c^+ \geq r$ *such that* $f(x,u;\theta) > 0$ *for a.e.* $x \in [a,b]$, $u > c^+$, $0 \leq \theta \leq 1$. *For any* $d \geq r$, *there is* $R_d > c^+$ *such that if* $u(\bullet)$ *is a solution of* (3.1_θ) - (3.2_θ) *with* $u^* > R_d$ *then,*

(i_1) *there are unique* $\alpha_d < \alpha < \gamma_1 \leq x^* \leq \gamma_2 < \beta < \beta_d$ *such that* u *is increasing on* $[\alpha_d,\gamma_1]$; u *is decreasing on* $[\gamma_2,\beta_d]$; $u(\alpha) \ = \ u(\beta) \ = \ c^+$. y *is decreasing on* $[\alpha,\beta]$; $y(\gamma_1) \ = \ M$; $y(\gamma_2) \ = \ -M$. *Also,* $u^* \geq u(x) \geq u^* \ - \ 2M(b \ - \ a)$ *for* $x \in [\gamma_1,\gamma_2]$; $\lim\limits_{u^* \to +\infty} \alpha - \alpha_d = 0$ *and* $\lim\limits_{u^* \to +\infty} \beta_d - \beta = 0$.

Furthermore, if

(i_2) $\lim\limits_{u \to +\infty} f(x,u;\theta) \ = \ +\infty$ *unif. a.e. on* $x \in [a,b]$ *and* $\theta \in [0,1]$, *then, for any* $L > 0$, $\lim\limits_{u^* \to +\infty}$ $(\beta_L \ - \ \alpha_L) \ = \ 0$.

[In particular, if

(i$_3$) $\lim_{u \to +\infty} f(x,u;\theta) = +\infty$ *unif. a.e. in* $x \in [a,b]$ *and* $\theta \in [0,1]$ *then* $\lim_{u^* \to +\infty} \gamma_2 - \gamma_1 = 0$.]

Proof. Let $u(\bullet)$ be a solution of (3.1_θ) - (3.2_θ) for some $0 < \theta < 1$. Assume that $u^* > c^+ + 2M(b - a)$ then $a < x^* < b$ and $u'(x^*) = 0$. So from $(3.1_\theta')$

$$0 = y(x^*) + \theta P(x^*)$$

and hence $|y(x^*)| \leq M$. Define

$$\alpha := \max\{x < x^* : u(x) = c^+\} \text{ and}$$

$$\beta := \min\{x > x^* : u(x) = c^+\}. \text{ Then}$$

$c^+ = u(\alpha) = u(\beta)$; $c^+ < u(x) \leq u^*$ for $\alpha < x < \beta$; and $a < \alpha < x^* < \beta < b$. We further observe that

$$y(\alpha) > M \text{ and } y(\beta) < -M.$$

Indeed, suppose by contradiction that $y(\alpha) \leq M$. Then, by $(3.1_\theta')$

$$\begin{aligned} u^* - u(\alpha) &= \int_\alpha^{x^*} y(s)\,ds + \theta \int_\alpha^{x^*} P(s)\,ds \\ &\leq (x^* - \alpha) M + \theta (x^* - \alpha) M \\ &\leq 2(b - a) M. \end{aligned}$$

So

$$u^* \leq u(\alpha) + 2M(b - a) = c^+ + 2M(b - a)$$

and this gives a contradiction. Similarly, we can show that $y(\beta) < -M$.

Using the fact that if $\alpha < x < \beta$ we have that $u(x) > c^+$, which in turn implies that $f(x,u;\theta) > 0$ (by hypotheses), it follows that $y'(x) < 0$ a.e. $x \in [a,b]$. Hence y is decreasing on $[\alpha,\beta]$. So

$$y(\alpha) > M \geq y(x^*) \geq -M > y(\beta).$$

Therefore, there are unique $\alpha < \gamma_1 \leq x^* \leq \gamma_2 < \beta$ such that

$$y(\gamma_1) = M, \, y(\gamma_2) = -M.$$

Moreover, for $\alpha \leq x < \gamma_1$, we have that

$$y(x) > M, \, |\theta P(x)| \leq M. \tag{3.10}$$

We show that

$$u^* \geq u(x) \geq u^* - 2M(b-a) \text{ for } x \in [\gamma_1,\gamma_2].$$

Let $x_1, x_2 \in [\gamma_1,\gamma_2]$, then

$$\begin{aligned} |u(x_2) - u(x_1)| &\leq \left|\int_{x_1}^{x_2} y(s)\,ds\right| + \theta \left|\int_{x_1}^{x_2} P(s)\,ds\right| \\ &\leq M|x_2 - x_1| + M|x_2 - x_2| \\ &= 2M|x_2 - x_1| \\ &\leq 2M(b-a). \end{aligned}$$

Hence,

$$u^* - 2M(b-a) \leq u(x) \leq u^* \;\forall\; \gamma_1 \leq x \leq \gamma_2.$$

Next, we show that $u(\bullet)$ is increasing on $[\alpha_d,\gamma_1]$. Since $f(x,u;\theta)$ satisfies the Carathèodory condition, there is $\psi_d \in L^1([a,b],\mathbb{R})$ such that $|f(x,u;\theta)| \leq \psi_d(x)$ for a.e. $x \in [a,b]$, $-d \leq u \leq c^+$, $\forall\; \theta \in [0,1]$. Integrating ($3.1_\theta$") for $x \in [\alpha_d, \alpha]$:

$$y(x) = y(\alpha) + \int_x^\alpha f(s,u;\theta)\,ds$$

$$= y(\alpha) + \int_{\substack{x \\ \{s:u(s)>c^+\}}}^{\alpha} f + \int_{\substack{x \\ \{s:u(s)\le c^+\}}}^{\alpha} f$$

$$\ge y(\alpha) + \int_{\substack{x \\ \{s:-d\le u(s)\le c^+\}}}^{\alpha} f$$

$$\ge y(\alpha) - \int_x^\alpha \psi_d(s)\,ds$$

$$\ge y(\alpha) - |\psi_d|_{L^1}\,. \tag{3.11}$$

But on (α,x^*), we have from ($3.1_\theta'$) that

$$u^* - c^+ = u(x^*) - u(\alpha)$$

$$= \int_\alpha^{x^*} y(s)\,ds + \theta \int_\alpha^{x^*} P(s)\,ds$$

$$\le y(\alpha)\,(b - a) + M(b - a)\,.$$

So (3.11) becomes

$$y(x) \ge \frac{u^* - c^+}{b - a} - M - |\psi_d|_{L^1}\,.$$

Hence,

$$\begin{cases} y(x) > M \quad \forall\ \alpha_d \le x \le \alpha \\ \text{provided } u^* > c^+ + (b - a)\,[2M + |\psi_d|_{L^1}] \text{ holds}\,. \end{cases} \tag{3.12}$$

Therefore from (3.10) and (3.12), we have that $u(\bullet)$ is increasing on $[\alpha_d,\gamma_1]$. By similar steps, we can show that $u(\bullet)$ is decreasing on $[\gamma_2,\beta_d]$.

To see that $\lim_{u^* \to +\infty} \alpha - \alpha_d = 0$ and $\lim_{u^* \to +\infty} \beta_d - \beta = 0$, we recall that if $\alpha_d \leq x \leq \alpha$ then

$$y(x) \geq \frac{u^* - c^+}{b - a} - M - |\psi_d|_{L^1}$$

and $-d \leq u(x) \leq c^+$. Integrating $(3.1_\theta')$:

$$c^+ + d \geq u(\alpha) - u(\alpha_d)$$

$$= \int_{\alpha_d}^{\alpha} y(s)ds + \theta \int_{\alpha_d}^{\alpha} P(s)ds$$

$$\geq (\alpha - \alpha_d)\left[\frac{u^* - c^+}{b - a} - 2M - |\psi_d|_{L^1}\right].$$

Hence,

$$0 \leq \alpha - \alpha_d \leq \frac{(c^+ + d)(b - a)}{u^* - [c^+ + (b - a)(2M + |\psi_d|_{L^1})]}.$$

We remark that the above inequality also holds for $\beta_d - \beta$. Hence $\alpha - \alpha_d$ (and $\beta_d - \beta$) $\to 0$ as $u^* \to +\infty$.

For the remaining part of the proof, we note that by (i_2), for any $K > 0$ there is u_K such that $f(x,u;\theta) \geq K$ for $u \geq u_K$. So, let $u(x) \geq u^* - L \geq u_K$ for $x_1 \leq x \leq x_2$ then $f(x,u;\theta) \geq K$ for $x_1 \leq x \leq x_2$. Integrating $(3.1_\theta'')$;

$$y(x) = y(x^*) - \int_{x^*}^{x} f(s,u(s);\theta)ds$$

$$= y(x^*) + \int_{x}^{x^*} f(s,u;\theta)ds$$

$$\geq -M + K(x^* - x)$$

holds for any $x_1 \leq x \leq x^*$ since $|y(x^*)| \leq M$. But,

$$u'(x) = y(x) + \theta P(x)$$

$$\geq -M + K(x^* - x) \quad \text{for } x_1 \leq x \leq x^*.$$

So,

$$u^* - u(x_1) \geq -M(x^* - x_1) + \frac{K}{2}(x^* - x_1)^2$$

and

$$L \geq -M(b - a) + \frac{K}{2}(x^* - x)^2 \quad \text{since } -u(x_1) \leq -u^* + L.$$

Therefore

$$K(x^* - x_1)^2 \leq 2L + 2M(b - a).$$

Hence,

$$x^* - x_1 < \sqrt{\frac{2L + 2M(b - a)}{K}}. \qquad (3.13)$$

Also,

$$y(x) = y(x^*) + \int_{x^*}^{x} (-f(s,u;\theta)ds)$$

$$\leq M - K(x - x^*)$$

since $-f(x,u;\theta) \leq -K$ for $x^* \leq x \leq x_2$. Now, from $(3.1_\theta')$ we also have on integrating that

$$u(x_2) - u^* \leq \int_{x^*}^{x_2} (M - K(s - x^*))ds$$

which implies that

$$- L \leq M(x_2 - x^*) - \frac{K}{2}(x_2 - x^*)^2 .$$

That is,

$$L \geq - M(x_2 - x^*) + \frac{K}{2}(x_2 - x^*)^2$$

$$\geq - M(b - a) + \frac{K}{2}(x_2 - x^*)^2.$$

So,

$$(x_2 - x^*) < \sqrt{\frac{2L + 2M(b - a)}{K}}. \tag{3.14}$$

Summing (3.13) and (3.16) to get

$$0 \leq x_2 - x_1 \leq 2\sqrt{\frac{2L + 2M(b - a)}{K}}.$$

Without loss of generality, we set $x_2 = \beta_L$, $x_1 = \alpha_L$ and the required result follows as $K \to +\infty$.

Q.E.D.

3.3. PROOF OF MAIN RESULT.

In this section, we give the proof of Theorem 3.1. This will be achieved by means of the following steps :

Step I : *Assume that*

(A.1) $\quad h(x,u) \geq L^{-}u$ *for all* $u \leq -c^{+}$ *and a.e.* $x \in [a,b]$..

Then, for any $\varepsilon > 0$, *there is a constant* $R = R(\varepsilon) > c^+$, *such that, if* $\min u(\bullet) < -R$ *and* $u(\alpha^-) = u(\beta^-) = -c^+$, $u(x) < -c^+$ *for* $x \in]\alpha^-, \beta^-[$, $\min_{[a,b]} u(x) = \min_{[\alpha^-,\beta^-]} u(x)$.

Then

(A.2) $$\beta^- - \alpha^- > \frac{\pi}{\sqrt{L^-}} - \varepsilon. .$$

Step II : *Assume that*

(B.1) $$\liminf_{u \to +\infty} \frac{2G_+(u)}{u^2} \leq L^+ .$$

Then, for every $\varepsilon > 0$, *there are an increasing sequence* $B_n(\varepsilon) \to +\infty$ *and a constant* $R = R(\varepsilon) > c^+$ *such that if* $\max u(\bullet) = B_n > R$ *and* $u(\alpha_n^+) = u(\beta_n^+) = c^+$, $u(x) > c^+$ *for* $x \in]\alpha_n^+, \beta_n^+[$, $\max_{[a,b]} u(x) = \max_{[\alpha_n^+,\beta_n^+]} u(x) = B_n$. *Then,*

(B.2) $$\beta_n^+ - \alpha_n^+ \geq \frac{\pi}{\sqrt{L^+}} - \varepsilon \qquad \forall\ n .$$

From steps I and II, we have the following :

Claim 1 ; *There are a constant* $K > c^+$ *and a sequence* $B_n \to +\infty$ *such that there is no solution* $u(\bullet)$ *of* $(3.1_\theta) - (3.2_\theta)$ *with* $\min u(\bullet) < -K$ *and, for some* n, $\max u(\bullet) = B_n > K$.

Proof of claim 1 ; By the assumption (3.9) we know that

$$\frac{\pi}{\sqrt{L^+}} + \frac{\pi}{\sqrt{L^-}} > b - a .$$

Hence, we fix $0 < \varepsilon < \frac{1}{2}\left[\frac{\pi}{\sqrt{L^+}} + \frac{\pi}{\sqrt{L^-}} - (b - a)\right]$. For such ε, from steps I and II we know, respectively, that there are a constant R and a sequence $B_n(\varepsilon) = B_n \to +\infty$. Now, we have that the claim is true for any $K \geq R$. In fact, if by contradiction, $\max u(\bullet) = B_n > K$ and $\min u(\bullet) < -K$ for any $u(\bullet)$ solution of $(3.1_\theta) - (3.2_\theta)$ then,

$$
\begin{aligned}
b - a &> (\beta^+ - \alpha^+) + (\beta^- - \alpha^-) \\
&\geq \left(\frac{\pi}{\sqrt{L^+}} - \varepsilon\right) + \left(\frac{\pi}{\sqrt{L^-}} - \varepsilon\right) \\
&> b - a ,
\end{aligned}
$$

and this is a contradiction. ∎

Step III : *Assume that*

(C.1) $h(x,u) \geq \upsilon^+ u$ *for* $u \geq c^+$ *with* $L^+ \geq \upsilon^+$

and let $J_d^+ = [\alpha_d^+,\beta_d^+]$ *be the maximal interval containing the point of maximum of* $u(\bullet)$ *and such that*

$$u(x) \geq -d \text{ for } x \in J_d^+.$$

Then, for any $\varepsilon > 0$, *there is a constant* $R = R(d,\varepsilon) > c^+$, *such that if,*

$$\max u(\bullet) > R$$

then

$$\beta_d^+ - \alpha_d^+ \leq \frac{\pi}{\sqrt{\upsilon^+}} + \varepsilon .$$

Step IV : *Assume that*

(C.2) $\qquad h(x,u) \geq \upsilon^+ u$ *for* $u \geq c^+$ *with* $L^+ \geq \upsilon^+ > \lambda_1$.

Then, for any $-d \leq -r$ *there is a constant* $R = R(d) > c^+$ *such that*

$$\min u(\bullet) \geq -d \Rightarrow \max u(\bullet) \leq R.$$

Proof of step IV : Take $\varepsilon := \frac{\pi}{2}(\frac{1}{\sqrt{\lambda_1}} - \frac{1}{\sqrt{\upsilon^+}})$ and let $R = R(d,\varepsilon)$. If $\min u(\bullet) \geq -d$ then $J_d^+ = [a,b] = [\alpha_d^+,\beta_d^+]$. Assume, by contradiction that $\max u(\bullet) > R$. Then,

$$b - a \leq \frac{\pi}{\sqrt{\upsilon^+}} + \varepsilon < b - a$$

which is a contradiction. ∎

Combining steps III and IV we have :

Claim 2 : *There is a constant* $D > K$ *such that there is no solution* $u(\bullet)$ *of* $(3.1_\theta)-(3.2_\theta)$ *with* $\min u(\bullet) \geq -K$ *and* $\max u(\bullet) > D$.

Proof of claim 2 : By assumption (3.6) we can find $L^+ \geq \upsilon^+ > \lambda_1$ such that (C.2) holds (for a suitable c^+). Then, it is sufficient to put $d = K$ and then choose $D := R(d)$. ∎

Step V : *Assume that*

(D.1) $$\limsup_{u \to -\infty} \frac{2G_-(u)}{u^2} \geq \upsilon^-$$

and that $J_d^- = [\alpha_d^-,\beta_d^-]$ *be the minimal interval containing the point of minimum of* $u(\bullet)$ *and such that* $u(x) \leq d$ *for* $x \in J_d^-$. *Then, for any* $\varepsilon > 0$ *there are a decreasing sequence* $-A(\varepsilon) \to -\infty$ *and a constant* $R = R(d,\varepsilon) > c^+$ *such that, if*

$$\min u(\bullet) = -A_n < -R,$$

then

$$\beta^-_{(1)n} - \alpha^-_{(1)n} \leq \frac{\pi}{\sqrt{\upsilon^-}} + \varepsilon .$$

Step VI : *Assume that*

(D.2) $$\limsup_{u \to -\infty} \frac{2G_-(u)}{u^2} \geq \upsilon^- \quad \textit{with} \ L^- \geq \upsilon^- > \lambda_1.$$

Then, there are a sequence $-A_n \to -\infty$ *such that, for any* $d \geq r$, *there is a constant* $R = R(d)$ *such that* $\max u(\bullet) \leq d \Rightarrow \min u(\bullet) \neq -A_n < -R$.

Proof of step VI : Take $\varepsilon := \frac{\pi}{2}(\frac{1}{\sqrt{\lambda_1}} - \frac{1}{\sqrt{\upsilon^-}})$, let $R = R(d,\varepsilon)$ and $-A_n \to -\infty$ depending on such ε. If $\max u(\bullet) \leq d$, then $J_d^- = [a,b]$. Assume by contradiction that $\min u(\bullet) = -A_n < -R$, then

$$b - a \leq \frac{\pi}{\sqrt{\upsilon^-}} + \varepsilon < b - a,$$

which is a contradiction. ∎

Combination of steps V and VI gives :

Claim 3 : *For any constant* $d \geq r$, *there is a constant* $-A_d < -K$ *such that there is no solution* $u(\bullet)$ *of* (3.1_θ) - (3.2_θ) *with* $\max u(\bullet) \leq d$ *and* $\min u(\bullet) = -A_d$.

Proof of claim 3 : By assumption (3.5), we can take υ^- as in (D.2). Then, for any $d \geq r$, we can find $-A_d \in \{-A_n : n \in \mathbb{N}\}$ such that $-A_d < -R(d)$ and $-A_d < -K$, hence the claim is proved. ∎

Before proving steps I, II, III and V, we give the proof of theorem 3.1 by using the above claims.

Proof of theorem 3.1 : Let r and c^+ be as defined earlier with $c^+ > r$. We fix the ε's as in claims 1 and 2. Choose $K > c^+$ according to claim 1. Then fix $D > K$ according to claim 2. Since $B_n \to +\infty$, fix one of the B_n's with $B_n > D$. Call B such a B_n and let $-A := -A_B$ according to claim 3.

Now, let $u(\bullet)$ be a solution of $(3.1_\theta) - (3.2_\theta)$ such that $-A \leq u(x) \leq B$. Suppose that $\max u(\bullet) = B > D$, then it follows from claim 2 that $\min u(\bullet) < -K$ and by claim 1, we have the contradiction that $\max u(\bullet) \neq B$. On the other hand, if we suppose that $\min u(\bullet) = -A$, then claim 3 gives that $\max u(\bullet) > B$ which is also false, since $u(x) \leq B \ \forall x$. Hence, our supposition holds.

We conclude the proof by applying lemma 3.1 and obtain the existence of at least one solution $u(\bullet)$ of $(3.1) - (3.2)$ with $-A \leq u(x) \leq B$.

Q.E.D.

We now end this section by proving steps I, II, III and V. Steps I and III will be proved together by using the next lemma; in a like manner also, steps II and V will be proved.

Lemma 3.3 : *Assume that*

$$f(x,u;\theta) \leq \eta u \quad (\geq \eta u) \ \forall\ u \geq c^+.$$

Then, for any $\varepsilon > 0$, *there is* $R = R(\varepsilon) > c^+$ *such that if* $\max u(\bullet) > R$ *and* $u(\alpha) = u(\beta) = c^+$; $u(x) > c^+$ *for* $x \in]\alpha,\beta[$; $\max\limits_{[a,b]} u(\bullet) = \max\limits_{[\alpha,\beta]}$, *we have*

$$\beta - \alpha \geq \frac{\pi}{\sqrt{\eta}} - \varepsilon \quad (\leq \frac{\pi}{\sqrt{\eta}} + \varepsilon).$$

Proof : We shall only prove the first part since the second part follows exactly the same line of argument. Consider the function

$$z(x) := \frac{1}{2}\{\eta u^2(x) + [y(x) - M]^2\}.$$

Then,

$$z'(x) = \eta\, u(x)\, u'(x) + y'(x)[y(x) - M]$$

$$= (\, y(x) - M\,)(\, - f(x,u;\theta) + \eta\, u(x)\,) + \eta\, u(x)(\theta P + M).$$

Since $f(x,u;\theta) \leq \eta\, u(x)$, we have that $z'(x) \geq 0$ and so $z(x)$ is increasing for x such that $y(x) \geq M$. Hence, $z(x) \leq z(\gamma_1)$, which implies that

$$z(x) \leq \frac{1}{2}\eta u^{*2} \quad \forall \quad \alpha \leq x \leq \gamma_1.$$

And so, we have

$$(\, y(x) - M\,)^2 \leq \eta(\, u^{*2} - u^2(x)\,)$$

or

$$u'(x) - \theta P \leq M + \sqrt{\eta(\, u^{*2} - u^2(x)\,)}.$$

So,

$$u'(x) \leq 2M + \sqrt{\eta(\, u^{*2} - u^2(x)\,)}.$$

For any $\eta_1 > \eta$ there is $L_1 = L_1(\eta_1)$ such that

$$u'(x) \leq \sqrt{\eta_1(u^{*2} - u^2(x))} \quad \forall \; x \quad \text{such that} \quad c^+ \leq u(x) \leq u^* - L_1.$$

Define $u(\hat{\alpha}_1) := u^* - L_1$. Then the above inequality holds for $\alpha \leq x \leq \hat{\alpha}_1$ and where $\alpha < \hat{\alpha}_1 < \gamma_1$. Following the proof of lemma 3.2, we see that $\gamma_1 - \hat{\alpha}_1$ tends to zero as $u^* \to +\infty$. Hence

$$\int_{\alpha}^{\hat{\alpha}} \frac{u'(x)}{\sqrt{\eta_1(u^{*2} - u^2(x))}}\, dx \leq \hat{\alpha}_1 - \alpha$$

which shows that

$$\int_{c^+}^{u^* - L_1} \frac{d\xi}{\sqrt{\eta_1(u^{*2} - \xi^2)}} \leq \hat{\alpha}_1 - \alpha.$$

$$= x^* - \alpha + (\hat{\alpha}_1 - x^*).$$

But

$$\int_0^{u^*} \frac{d\xi}{\sqrt{\eta_1(u^{*2} - \xi^2)}} = \frac{\pi}{2\sqrt{\eta_1}},$$

so

$$\int_{c^+}^{u^* - L_1} \frac{d\xi}{\sqrt{\eta_1(u^{*2} - \xi^2)}} = \frac{\pi}{2\sqrt{\eta_1}} - \int_0^{c^+} \frac{d\xi}{\sqrt{\eta_1(u^{*2} - \xi^2)}} - \int_{u^* - L_1}^{u^*} \frac{d\xi}{\sqrt{\eta_1(u^{*2} - \xi^2)}}$$

It is routine to see that

$$\lim_{u^* \to +\infty} \int_{c^+}^{u^* - L_1} \frac{d\xi}{\sqrt{\eta_1(u^{*2} - \xi^2)}} = \frac{\pi}{2\sqrt{\eta_1}}.$$

Repeating the same process for $(\gamma_2, \beta_2]$, we obtain on the whole that

$$\beta(u^*) - \alpha(u^*) \geq \frac{\pi}{\sqrt{\eta_1}}.$$

Hence

$$\liminf_{u^* \to +\infty} [\beta(u^*) - \alpha(u^*)] \geq \frac{\pi}{\sqrt{\eta}}.$$

Also, from the other part,we obtain that

$$\limsup_{u^* \to +\infty} [\beta(u^*) - \alpha(u^*)] \leq \frac{\pi}{\sqrt{\eta}}.$$

Then, using the definitions of lim inf and lim sup we obtain the required conclusion.

∎

Next, we give the lemma for the proof of steps II and V.

Lemma 3.4 *Assume that, either*

$$\begin{cases} h(x,u) \leq g(u) \quad \textit{for} \quad u \geq 0 \\ \liminf\limits_{u \to +\infty} \dfrac{2G(u)}{u^2} \leq \eta \end{cases} \qquad ...(3.15)$$

or

$$\begin{cases} h(x,u) \geq g(u) \quad \textit{for} \quad u \geq 0 \\ \limsup\limits_{u \to +\infty} \dfrac{2G(u)}{u^2} \geq \eta \end{cases} \qquad ...(3.16)$$

with $g(u) \to +\infty$ *for* $u \to +\infty$. *Then, for any* $\varepsilon > 0$, *there are an increasing sequence* $B_n(\varepsilon) \to +\infty$ *and a constant* $R = R(\varepsilon) > c_+$ *such that if* $\max u(\bullet) = B_n > R$ *and* $u(\alpha_n) = u(\beta_n) = c^+$, $u(x) > c^+$ *for* $x \in]\alpha_n,\beta_n[$, $\max\limits_{[a,b]} u(x) = \max\limits_{[\alpha_n,\beta_n]} u(x) = B_n$, *where*

$$u'' + (1 - \theta)\eta u + \theta h(x,u) = \theta p(x),$$

then

$$\beta_n - \alpha_n \geq \frac{\pi}{\sqrt{\eta}} - \varepsilon$$

or (respectively)

$$\beta_n - \alpha_n \leq \frac{\pi}{\sqrt{\eta}} + \varepsilon.$$

Proof . As in the proof of the previous lemma, we shall only give the proof of the first part, that is, the proof of assuming (3.15). Now, let $\eta' > \eta$ (arbitrary), then

$$\limsup_{u \to +\infty} \left(\frac{\eta'}{2}u^2 - G(u)\right) = +\infty.$$

Hence, there is an increasing sequence $B_n \to +\infty$ such that

$$\frac{\eta' u^2}{2} - G(u) < \frac{\eta'}{2}B_n^2 - G(B_n) \quad \forall\ 0 \leq u < B_n.$$

So,

$$G(B_n) - G(u) < \frac{\eta'}{2}(B_n^2 - u^2) \quad \forall\ 0 \leq u < B_n.$$

Consider the function

$$z(x) = (1 - \theta)\frac{\eta}{2}u^2(x) + \theta G(u(x)) + [y(x) - M].$$

$$z'(x) = (y(x) - M)[\theta(g(u) - h(x,u)) + (1 - \theta)(\eta' - \eta)u(x)] +$$

$$+ (M + \theta P)(\theta g(u(x)) + (1 - \theta)\eta' u(x)).$$

We easily see that

$$z'(x) > 0 \qquad \text{for } x \in [\alpha,\gamma_1]$$

and so, $z(x)$ is increasing on $[\alpha,\gamma_1]$. This implies that

$$z(x) \le z(\gamma_1) \qquad \text{for} \quad \alpha \le x \le \gamma_1.$$

That is,

$$(1 - \theta)\eta'\frac{u(x)^2}{2} + \theta G(u(x)) + \frac{1}{2}[y(x) - M]^2 \le$$

$$\le (1 - \theta)\frac{\eta'}{2}u(\gamma_1)^2 + \theta G(u(\gamma_1))$$

$$\le (1 - \theta)\frac{\eta'}{2}u(x^*)^2 + \theta G(u(x^*))$$

$$\le (1 - \theta)\frac{\eta'}{2}u^{*2} + \theta G(u^*).$$

So,

$$\frac{1}{2}[y(x) - M]^2 \le \frac{(1-\theta)}{2}\eta'[u^{*2} - u^2(x)] + \theta[G(u^*) - G(u)]$$

$$\le \frac{(1-\theta)}{2}\eta'[u^{*2} - u^2(x)] + \frac{\theta\eta'}{2}[u^{*2} - u^2(x)].$$

Setting $u^* = B_n$, we obtain,

$$[y(x) - M]^2 \le \eta'[B_n^{\;2} - u^2(x)] \quad \text{for} \quad \alpha \le x \le \gamma_1.$$

And so,

$$u'(x) \le 2M + \sqrt{\eta'(B_n^{\;2} - u(x)^2)} \quad \text{for} \quad \alpha \le x \le \gamma_1.$$

Repeating the same computations as in the previous lemma,we get that

$$\liminf_{n \to +\infty} (\beta_n - \alpha_n) \ge \frac{\pi}{\sqrt{\eta'}}$$

(respectively, for the second part, that is, for (3.16), we get

$$\limsup_{n \to +\infty} (\beta_n - \alpha_n) \leq \frac{\pi}{\sqrt{\eta'}})$$

Finally, it is sufficient to take η' such that

$\dfrac{\pi}{\sqrt{\eta'}} = \dfrac{\pi}{\sqrt{\eta}} - \varepsilon$ where $\eta' > \eta$ (respectively, $\dfrac{\pi}{\sqrt{\eta'}} = \dfrac{\pi}{\sqrt{\eta}} + \varepsilon$ where $\eta' < \eta$).

∎

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