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# SPIN STRUCTURES AND GLOBAL CONFORMAL TRANSFORMATIONS


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## 1. INTRODUCTION.

P.A.M. Dirac was among the founders of two basic ingredients of the mathematical methods needed in today's physics. His relativistic equation [ 2 ] described the electron in terms of spinor; the object discovered in 1913 by Cartan [ 1 ] (The spinor was given its name when Uhlenbeck and Goudsmith identified the electron extra degrees of freedom, necessary to explain the anomalous Zeeman effect, with the part of angular momentum of the electron which is independent on its distance from the centre of rotation). His magnetic monopole paper [ 3 ] and the Hopf fibering [ 4 ], both published in 1931, opened the era of fibre bundle description of gauge theories.

We shall be concerned with the combination of both above tools needed for a consistent introduction of spinors in curved or topologically nontrivial space-times. Our original results on global description of conformal transformations of spinor fields, and the detailed discussion of spin structures on the minimal conformal compactification $\bar{M}$ of the Minkowski space-time $R^{1,3}$ will be presented in ch.(5). In the preceding ch.(3) and ch.(4) we shall review criteria for the existence, and physical implications of inequivalent spin structures. The attempt will be to give the subject a coherence, which could be difficult to recover from individual papers; however, the preliminary comparison of global properties of this formalism with the Dirac-Kähler approach to fermions, and comments on the relevance of inequivalent spinors in supergravity and Kaluza-Klein theories will be added in (3.6) and (4.5). When recalling in ch.(2) the basic algebraic material, we shall clarify in (2.3) the relation between signatures (s,t) of space-times admitting Majorana spinors and types of the Wedderbrun skew-field of relative Clifford algebras, and also propose in (2.5) to generalize the concept of pure spinor to arbitrary $R^{s,t}$.

Spinors are often thought of to be basically simpler and perhaps more deep-rooted than tensors [44,39,5,6]. However, since the direct geometrical interpretation is difficult, they remain a little mysterious, even if the formal rules to deal with them are highly developed. Because of the most intriguing property: the sign reversion under the $2\pi$-rotation, spinors were sometimes considered to be unphysical. The way out of apparent troubles is to pass over to the double cover of the rotations group, and admit the object, which changes sign after a transformation covering the continuous $2\pi$- rotation, while the observed vectors and tensors remain identical. It can be also argued that the relative $2\pi$ rotations of two parts of a system are not necessarily unobservable. The macroscopic construction (atributed also to Dirac) is a solid cube, with extendible strings attached to its faces, and to the walls of "laboratory". After a $2\pi$ rotation of the cube, strings cannot be disentangled, whereas after a $4\pi$ rotation they can. For illuminating drawings see [ 7 ]; c.f. also [ 8 ] from which we quote:

"It is a kinematic property of the real world that (a) a coordinate frame under rotation by $2\pi$ about any axis is in principle distinguishable from an unrotated coordinate frame; and (b) a coordinate frame under rotation by $4\pi$ about any axis is indistinguishable from an unrotated frame". One has to be careful with analogies between a (pointlike) spinor and a finite size object in the Dirac construction, or other similar "plate of soup" and "belt twisting" experiments. Nevertheless, the sign ambiguity should be considered seriously. The opinion presented from time to time (c.f. for instance [9]) that spinor is defined "up to a sign", i.e. it is an equivalence class under the $Z_2$ identification can not be accepted, unless one gives up the linear structure of the theory, or the Pauli principle. Finally, there is an experiment [10] with a split neutron beam, part of which passes through a region of magnetic field causing the rotation due to the spin precession effect. When the relative rotation is an odd multiple of $2\pi$, the interference pattern changes with respect to a rotation of even multiple of $2\pi$. For a proposal of another experiment, which can "measure" the spin structure of space-time, see [11].

In three-dimensional Euclidean and four-dimensional pseudo-Euclidean spaces, the double coverings of orthogonal groups are universal. To spinors in spaces of other signatures or dimensions we shall always attribute the double (not necessarily) universal covering, which can be obtained by the Clifford scheme (see chapter 2.) This can be motivated either by considering spinor as a "square root" of vector, or similarly the Dirac equation as a square root of Klein-Gordon equation. Being interested in finite component spinors, we do not discuss "bandors", carrying the double-valued representation of SL(n,R) or GL(n,R), c.f. [12]. In special relativity the structure group is a (double cover of) pseudo-orthogonal group, and the metric character of spinor requires some care in presence of the gravitational field. A consistent way to implement spinors in curved spaces is based on the vierbein (n-bein, orthonormal frame ) formalism. Its essence is that spinors behave like spinors only with respect to independent Lorentz rotations of the orthonormal (o.n.) frame at each point of the manifold. In fact, in the developed in this spirit variational principles for field equations, the o.n. frame has been given its own life and became a dynamical field. Furthermore gravitation has been interpreted as a kind of gauge theory. In the present thesis we completely omit the back reaction problem, and consider only spinor field in a fixed background metric. In local coordinates the problem of propagation and covariant derivative of spinors has been solved by Fock-Ivanenko coefficients, i.e. the metric connection composed with the Lie algebras isomorphism $so(s,t) \approx spin(s,t)$. If the space-time M has a nontrivial topology, the situation is more complicated. The various charts on M have to be consistently patched together. At the rigorous level, this is equivalent to introducing the spin structure over M, which will be discussed in chapter 3. The known feature is that not all manifolds admit spin structure. (The physical arguments, based on homotopical considerations will be presented in (3.1), while the rigorous definitions and necessary and sufficient cohomological conditions for the spin structure to exist, in (3.2) and (3.3)). In principle, by considering the theory of geometrical objects as a sort of Spin(s,t) or SO(s,t)-gauge theory, topological obstructions against spinors could be expected. Similarly like charges of particles propagating in a field of magnetic monopole must be quantized, also the global definition of half-integer spin objects is not always possible. The analogy is even more pronounced by observing that the prolongation of the structure group SO(s,t) to Spin(s,t) is of the same character as introducing the SU(3)-isospinors transforming nontrivially under the centre $Z_3$, while the observed particles in Q.C.D. transform only under $SU(3)/Z_3$. Moreover both gauges can be combined in the generalized $spin_G$ structures, which will be presented in (3.5). Then obstructions are weaker, or completely vanish.

Recently, there has been a growing interest in a nonstandard approach to fermions, by means of the Kähler-Dirac equation [13,14, 15,16]. It is written in the language of differential forms, and seems to be conceptually different from the geometric approach, even if it is directly related to Clifford algebras and algebraic spinors. By geometric description of spinors we mean the approach in which the faithful transformations under Spin, covering the orthogonal group SO, are most important. Hence, spinor is the object which 'transforms as spinor', i.e. the definite assignment of spinor coordinates to each spinor frame. This approach culminates in the definition of spin structure over manifolds. The algebraic spinors instead, are strongly based on Clifford algebras. Differently to geometric spinors, for which the Clifford algebra is helpful to define and represent spinorial groups, the algebraic spinors are directly incorporated into the Clifford algebra itself. This results in fields as sections of the subbundle of minimal left ideals in the Clifford bundle. Obstructions in these approaches are in general different.

The existence conditions are by now widely accepted when considering spinors on various spaces in general relativity, and in modern Kaluza-Klein and supersymmetry theories. Another straightforward consequence of the definition, the possibility of inequivalent spin structures in multiply connected spaces, is less known.
The physical implications of inequivalent spinors have been discussed mostly by Petry [ 17 ], by Isham and his colaborators [18,19,20,21 ]. Apart from possibility of inequivalent Spin(s,t)-bundles, the difference occurs in different covariant derivatives of spinors. This can be also translated into different periodicity or antiperiodicity conditions for spinors, which are easy to handle in the functional formalism. The possibility of antiperiodic boundary conditions has nothing to do with the folklore statement that "spinor is defined up to sign". It holds only along exactly specified class of paths in M, and by no means along all paths. Until now, the absolutely convincing experimental evidence in favour of inequivalent spinors is not known. Also Witten [ 22 ] has pointed out, that in general relativity, unlike gauge theories, the cluster expansion cannot be used to justify, that inequivalent topological configurations have to be included in the path integral formalism. Nevertheless, Isham has advocated the opinion that all possible consistent configurations should be taken into considerations. Also the "democracy" of Nature

can be invoked, since in general no particular spin structure is distinguished. In any case, modes of the Dirac operator, and the corresponding Green functions appearing in perturbative corrections of quantum field theory, depend on the choice of the bundle. Rigorously, the test functions needed to smear the field operators should be sections of the dual bundle, associated to the spin structure bundle. Chapter 4 will be devoted for presentation of various examples of spin, non-spin and $Spin_c$ manifolds. The known physical effects of inequivalent spin structures will be collected, and few comments added on their relevance in higher dimensional Kaluza-Klein and supergravity theories.

Chapter 5 will be devoted to study the behaviour of spinors under conformal mappings. In 5.1 we investigate the problem of lifting the conformal map between M and M' to a map between spin structures over M and M'. We prove the general result

, that any conformal map can be lifted , provided we admit all possible inequivalent structures over M. This assignment preserves (up to a sign) the composition in the connected group $Conf_0(M)$ of conformal automorphisms of M. The sign ambiguity is removed by passing over to the double covering $\overline{Conf_0(M)}$ of $Conf_0(M)$. This induces the representation in the space of spinor fields on M. Assuming that in M (of signature (+,-,...,-) the maximal space-like hypersurface exists, the unitary representation in the Hilbert space of solutions of the massless Dirac equation can be obtained in a standard way . With respect to the Dirac operators (related to a given spin structure) we prove that in general they are intertwined (up to a conformal factor) by a conformal map.

Next we shall apply these general results to the conformal compactification $\bar{M}$ of the Minkowski space-time $R^{1,3}$. In order to have a well defined action of $Conf(R^{1,3})$ one has to enlarge $R^{1,3}$. We are interested in the minimal enlarging $\bar{M}$, by adding only some boundary to $R^{1,3}$. There are two convenient realizations of $\bar{M}$ as a projective null cone in $R^{2,4}$ or as the group manifold of U(2). Usually the conformal spinors are introduced by starting from 8-component spinors of $R^{2,4}$ by the Dirac-Hepner-Mack-Salam method [23,24,25], and we partially settle in section 5.3 the interesting hypothesis [27] that one of two reduced 4-component spinors can be interpreted as the 'exotic' (inequivalent) one. In sect. 5.5 we discuss in detail the (two) inequivalent spin structures over $\bar{M}$, and relate them to left (or right respectively), invariant global o.n. frames on $\bar{M} \approx U(2)$. The spectrum of the Dirac operator is computed in sect. 5.6, and the space of solutions of massless Dirac equation shown to be $\{0\}$. The possibility of overcoming the triviality of the resulting unitary representation of $Conf_0(\bar{M})$ [28] by coupling to electromagnetic field, is strongly suggested.

The labelling of formulas is as follows. The number before the dot in ( . , ) denotes the chapter, after the dot and before the comma - section, and after the comma - definite formula.

## 2. ALGEBRAIC PRELIMINARIES.

### 1. Clifford algebras, orthogonal and spinorial groups.

The standard references for Clifford algebras are [29,30,31]. Let $V = R^{s,t}$ or $C^n$ be an n-dimensional linear space over $K=R$ or $C$, equipped with the nondegenerate bilinear form $\langle\ ,\ \rangle$ (of signature $+,\ldots,+$ (s factors) and $-,\ldots,-$ (t factors) for $K=R$ ). The linear map $f\colon V\longrightarrow A$, from $V$ into an associative algebra $A$ with unity $1_A$, has the <u>Clifford</u> property iff $(f(v))^2 = \langle v,v\rangle 1_A$ for all $v \in V$. The <u>Clifford algebra</u> $C(V, \langle\ ,\ \rangle)$ (shortly: $C(V)$) is an associative algebra with $1_{C(V)}$ together with a Clifford map $i_V\colon V \longrightarrow C(V)$, such that for every Clifford map $f\colon V\longrightarrow A$ there exists a unique homomorphism $f'\colon C(V)\longrightarrow A$ and $f = f' \circ i_V$ (the uniqueness of $f'$ can be replaced by the condition that the range of $i_V$ generates $C(V)$ as an algebra). By the universality property of the definition, $C(V)$ is unique up to an isomorphism. Its existence follows from the isomorphism $C(V) = T(V)/J$ , where $T(V)= \bigoplus_{i=1}^{\infty} V\otimes\ldots\ldots\otimes V$ and $J$ is a two-sided ideal in $T(V)$ generated by $\{v\otimes v - \langle v,v\rangle | v \in V\}$ . These definitions imply the well known anticommutation rules

$\{v,w\}_+ = 2\langle v,w\rangle$, where henceforth we denote the Clifford product by a juxtaposition, and identify $V$ with $i_V(V)$ and $K$ with $K\,1_{C(V)}$. Denote by $R_{s,t}$ and $C_n$ the Clifford algebras $C(V)$ for $V=R^{s,t}$, $K=R$ and $V=C^n$ , $K=C$ respectively; then for any $s+t=n$ $C_n = C(C^n) = C(R^{s,t}\otimes C) = C(R^{s,t})\otimes C = R_{s,t}\otimes C$ , where the real bilinear form is complexified to a C-bilinear form.

There are following isomorphisms

(1) $$R_{p+1,q+1} = R_{p,q} \otimes R_{1,1} \quad ,$$

(2) $$R_{p+1,q} = R_{q+1,p} \quad ,$$

(3) $$R_{p,q+3} = R_{q,p+3} \quad ,$$

the proof of which is based on the fact that all properties of C(V) are determined by the <u>orthonormal</u> <u>subset</u> defined as a set of mutually anticommuting elements with squares $\pm 1$, which generates C(V). If $\{e_1, \ldots, e_p; e_{p+1}, \ldots, e_{p+q}\}$ , $\{e'_1; e'_2\}$ , $\{e_1, \ldots, e_q, e_{q+1}; e_{q+2}, \ldots, e_{p+q+1}\}$ , $\{e_1, \ldots, e_q; e_{q+1}, \ldots, e_{p+q+3}\}$ are orthonormal subsets of $R_{p,q}$, $R_{1,1}$, $R_{q+1,p}$, $R_{q,p+3}$ , then

$\{e_1 \otimes e'_1 e'_2, \ldots, e_p \otimes e'_1 e'_2 , 1 \otimes e'_1; e_{p+1} \otimes e'_1 e'_2, \ldots, e_{p+q} \otimes e'_1 e'_2, 1 \otimes e'_2\}$ ,

$\{e_{q+2} e_{q+1}, \ldots, e_{q+p+1} e_{q+1}, e_{q+1}; e_1 e_{q+1}, \ldots, e_q e_{q+1}\}$ ,

$\{e_{q+1} e_{123}, \ldots, e_{q+p} e_{123}; e_1 e_{123}, \ldots, e_1 e_{123}, e_{p+q+1}, e_{q+p+2}, e_{q+p+3}\}$ ,

where $a_{123} = e_{q+p+1} e_{q+p+2} e_{q+p+3}$ , are orthonormal subsets of $R_{p+1,q+1}$ , $R_{p+1,q}$ , $R_{p,q+3}$ respectively.

The lowest dimensional real Clifford algebras are isomorphic to $R_{s,t} = {}^2R$, R, C, H, ${}^2H$, $L(R^2)$ for (s,t) = (1,0),(0,0),(1,0),(2,0),(3,0),(1,1).

The tensoring rules (over R)

$R \otimes R = R$ , $R \otimes C = C$ , $R \otimes H = H$

$C \otimes C = {}^2C = C \oplus C$ , $C \otimes H = L(C^2)$ , $H \otimes H = L(R^4)$

$L(R^n) \otimes L(R^m) = L(R^{nm})$ , $L(R^n) \otimes D = L(D^n)$ for D = R, C, H ;

permit to identify all $R_{s,t}$ by aplying ( 1 ),( 2 ) and ( 3 ) .

In the table ( 1 ) it amounts to go down one step and/or perform inversions w.r.t. the column s-t=1 and s-t=-3 . Similarly $C_{2k} = L(C^{2^k})$, $C_{2k+1} = {}^2L(C^{2^k}) = L(C^{2^k}) \oplus L(C^{2^k})$ .

Denote by: $\alpha$ the degree involution in C(V) induced by $-id_V$ ; $\beta$ the reversion anti-involution of C(V) induced by $id_V$; and " $-$ " the conjugation anti-involution $\bar{a} = \alpha \circ \beta(a)$. Let $C^{(\pm)}(V) = \ker(\alpha \mp id)$ be the even (odd) part of C(V); $C^{(+)}V$ is isomorphic to C(V') where V' is of one (timelike) dimension less than V: $C_n^{(+)} = C_{n-1}$, $R_{s,t}^{(+)} = R_{s,t-1} = R_{t,s}^{(+)}$ (to prove the isomorphism take the orthonormal subset

$$\{e_1 e_{s+t}, \ldots, e_s e_{s+t}; e_{s+1} e_{s+t}, \ldots, e_{s+t-1} e_{s+t}\} ).$$

The canonical element in C(V) defined by $e_n^{\pi} = \prod_{i=1}^{n} e_i$ , where $e_1, \ldots, e_n$ is (possibly oriented) orthonormal basis of V, plays an important rôle. For odd n the centre Z(V) of C(V) is $K \oplus K e_n^{\pi}$ and the anticentre $AZ(V) \equiv \{a \in C(V) \mid ab = \alpha(b)a, \text{ for all } b \in C(V)\}$ is $\{0\}$ , while for even n $Z(V) = K$ and $AZ(V) = K e_n^{\pi}$. It follows that for even n $R_{s,t}$ is simple, and for odd n either simple when $(e_n)^2 = -1$ i.e. s-t=3 (mod 4), or a direct sum of simple eigenspaces of $1/2(1 \pm e_n^{\pi})$ when $(e_n^{\pi})^2 = 1$ i.e. s-t=1 (mod 4). Similarly for even n $R_{s,t}^{(+)}$ is either simple when s-t= 2 (mod 4), or a direct sum when s-t= 0 (mod 4). For even n $C_n$ is always simple, and for odd n $C_n$ is a direct sum of eigenspaces of $1/2(1 \pm \lambda e_n^{\pi})$, where $\lambda$ is fixed by $(\lambda e_n^{\pi})^2 = 1$. Similarly $C_n^{(+)}$ is a direct sum for even n, and is simple for odd n. All that can be immediately seen in the Table( 1 ).

The descending sequence of groups can be defined in $C(V)$

(4) $\underline{C^*(V)} = \{a \in C(V) \mid \exists\, a^{-1}\}$ ,

(5) $\underline{\Gamma(V)} = \{a \in C^*(V) \mid \rho(a)v \in V, \text{ for all } v \in V\}$, (Clifford group)

(6) $\underline{Pin(V)} = \{a \in \Gamma(V) \mid N(a) = \pm 1\}$ ,

(7) $\underline{Spin(V)} = \{a \in Pin(V) \mid \alpha(a) = a\}$ , (even part of Pin)

(8) $\underline{Spin_0(V)} = \{a \in Spin(V) \mid N(a) = +1\}$ , (connected component of Spin)

where $\rho : C^*(V) \longrightarrow L(C(V))$ defined by $\rho(a)b = \alpha(a)ba^{-1}$ is the <u>twisted adjoint representation</u> of $C^*(V)$ in $C(V)$ with kernel $K^* = K \setminus \{0\}$, and the homomorphism $N : \Gamma(V) \longrightarrow K^*$ is called spinor norm $N(a) = \bar{a}a$ (#). The group $\Gamma(V)$ is spanned by $v \in V$ of norm $N(v) \neq 0$, and $Pin(V)$ by $v \in V$ of norm $N(v) = \pm 1$. Groups $Spin(V)$ and $Spin_0(V)$ are spanned by products of even number of $v \in V$, such that their norms are $N(v) = \pm 1$ and $N(v) = 1$ respectively. The Lie algebra of $C^*(V)$ is just $C(V)$ with a Lie bracket beeing the Clifford commutator. The algebra $spin(V)$ is linearly spanned by $e_j e_k$, $j \neq k$.

For a fixed $a$, the restriction of $\rho(a)$ to $V$ is the isometry of $(V, \langle\ \rangle)$, inducing the exact sequences of isomorphisms:

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(#) Sometimes the spinor norm is defined as $N'(a) = \beta(a)a$, however the resulting $Pin'(V)$ covers only $SO(V)$ for odd-dimensional $V$.

(9) $$0 \longrightarrow Z_2 \longrightarrow Pin(V) \longrightarrow O(V) \longrightarrow 0$$

(10) $$0 \longrightarrow Z_2 \longrightarrow Spin(V) \longrightarrow SO(V) \longrightarrow 0$$

(11) $$0 \longrightarrow Z_2 \longrightarrow Spin_o(V) \longrightarrow SO_o(V) \longrightarrow 0 \quad .$$

In the case of $C(R^{s,t})$ , these nontrivial (for $\max(s,t) \geqslant 2$) double coverings of $O(s,t)$ , $SO(s,t)$ and $SO_o(s,t)$ are denoted by $Pin(s,t)$, $Spin(s,t)$ and $Spin_o(s,t)$ respectively.

Observe that they are in general <u>not</u> universal, since $O(s,t) = O(t,s)$ and the first homotopy groups for $t \geqslant s$ are

(12) $$\pi_1(SO_o(s,t)) = \begin{cases} 0 & \text{if} \quad s = 0,1 ; \quad t = 1 \\ Z & \text{if} \quad s = 0,1 ; \quad t = 2 \\ Z_2 & \text{if} \quad s = 0,1 ; \quad t \geq 3 \\ Z \times Z & \text{if} \quad s = 2; \quad t = 2 \\ Z \times Z_2 & \text{if} \quad s = 2; \quad t \geq 3 \\ Z_2 x Z_2 & \text{if} \quad s = 3; \quad t \geq 3 \ . \end{cases}$$

This follows from the fact that $SO(s) \times SO(t)$ is a maximal compact subgroup of $SO_o(s,t)$; $\pi_1(SO_o(s,t)) = \pi_1(SO(s)) \times \pi_1(SO(t))$ and

(13) $$\pi_1(SO(n)) = \begin{cases} 0 & & n = 1 \\ Z & \text{if} & n = 2 \\ Z_2 & & n \geq 3 \end{cases} \quad .$$

For $\min(s,t) \geq 1$ $SO(s,t)$ consists of two disconnected components and is a semidirect product of $SO_o(s,t)$ with $Z_2$. Then $Spin(s,t)$ consists also of two disconnected components (except of $s=t=1$) and is a semidirect product of $Spin_o(s,t)$ with $Z_2$.

Similarly O(s,t) (Pin(s,t)) are semidirect products of SO(s,t) (Spin(s,t)) with $Z_2$ and have twice as many components as SO(s,t) (Spin(s,t)) respectively.

Also the double coverings of the subgroups of O(s,t) preserving separately the time or space orientation can be defined.

Let us mention, that the spinorial groups arising from the Clifford scheme are not only the particular double coverings of connected orthogonal groups, but also specific coverings of the discrete part ($Z_2$ or $Z_4$) – out of many (eight for $s,t \geq 1$) nonisomorphic possibilities corresponding to various sign combinations of squares of elements which cover P,T and PT inversions.

For $C(C^n)$ the groups defined by ( 6 ) and ( 7 ) are denoted by Pin(n,C) and Spin(n,C). However, other useful groups can be defined in $C(C^n) \approx C(R^{s,t}) \otimes C$, which depend on the signature (s,t) [65] :

(14) $$\Gamma_C(s,t) = \{a \in C(R^{s,t}) \otimes C \mid \exists\, a^{-1},\ \alpha(a)R^{s,t}a^{-1} \subset R^{s,t}\}$$

(15) $$Pin_C(s,t) = \Gamma_C(s,t)/R_+ \approx \{a \in \Gamma \mid \hat{a}\, a = 1\} ,$$

where $R_+ = \{r \in R \mid r > 0\}$ and '^' is the conjugation $(-)$ in $R_{s,t}$ composed with complex conjugation in C. Then $Spin_C(s,t)$ and $Spin_o^e(s,t)$ are defined as inverse images $(\rho_C)^{-1}$ of SO(s,t) and $SO_o(s,t)$ respectively, where $\rho_C(a)v \overset{df}{=} \alpha(a)va^{-1}$ for $v \in R^{s,t}$ yields the exact sequence

(16) $$0 \longrightarrow GL(1,C) \longrightarrow \Gamma_C(s,t) \longrightarrow O(s,t) \longrightarrow 0$$

Since $GL(1,C) \approx R_+ \times U(1)$, from (16 ) follows

(17) $$0 \longrightarrow U(1) \longrightarrow Pin_c(s,t) \longrightarrow O(s,t) \longrightarrow 0$$

and similarly

(18) $$0 \longrightarrow U(1) \longrightarrow Spin_c(s,t) \longrightarrow SO(s,t) \longrightarrow 0$$

(19) $$0 \longrightarrow U(1) \longrightarrow Spin_o^c(s,t) \longrightarrow SO_o(s,t) \longrightarrow 0$$

In fact just defined spinorial groups are 'twisted' products of real spinorial groups with U(1)

(20) $$Pin_c(s,t) = Pin(s,t) \times U(1) / Z_2$$

(21) $$Spin_c(s,t) = Spin(s,t) \times U(1) / Z_2$$

(22) $$Spin_o^c(s,t) = Spin_o(s,t) \times U(1) / Z_2 \; ,$$

where $Z_2 = \{(1,1),(-1,-1)\} \subset Spin_o(s,t) \times U(1)$ .

## 2. The matrix forms of $Spin_o(s,t)$ for $1 \leq s+t \leq 6$.

In lower dimensions $s + t \leq 6$ the $Spin_o(s,t)$ groups can be identified with some well known groups. Let us recall the construction of suitable isomorphisms. This is equivalent to determining $Spin_o(s;t)$ explicitly in the matrix representation of Clif(s,t), but gives interesting insight in related problems [32,33].

Let us start with $s + t = 6$. Consider the sixdimensional complex vector space $C^6 \approx \Lambda^2 C^4$ equipped with the nondegenerate quadratic form $Q : C^6 \times C^6 \ni (T,T) \rightarrow Q(T,T) \in C$ given by

(1) $$Q(T,T) \times vol = T \wedge T ,$$

where $vol \in \Lambda^4 C^4$ and $\wedge$ denotes the standard wedge product in $\Lambda C^4$. The GL(4,C) tranformation U induces the transformation $T \rightarrow UTU^T$ with the property

(2) $$UTU^T \wedge UTU^T = \det(U) T \wedge T .$$

Therefore, there is a homomorphism of SL(4,C) into SO(6,C), which can be shown to be surjective with kernel $Z_2$

(3) $$0 \rightarrow Z_2 \rightarrow SL(4,C) \rightarrow SO(6,C) \rightarrow 0$$

Restricting both T and U to be real

(4) $$0 \rightarrow Z_2 \rightarrow SL(4,R) \rightarrow SO_o(3,3) \rightarrow 0$$

shows that $Spin_o(3,3) \approx SL(4,R)$ .

Assume now that a hermitian form $H$ is given on $C^4$; $H = H^+$ in matrix notation. Denote the group $\{U \in C(4) \mid UHU^+ = H,\ \det U = 1\}$ by SU(4) or SU(2,2) if the signature of $H$ is (+,+,+,+) or (+,+,-,-) respectively. Introduce the duality involution on $\Lambda^2 C$, $* \circ * T = T$ given by

(5) $$(*T)_{ij} = 1/2\ \epsilon_{ijkl} T_{kl} ,$$

where $\epsilon_{ijkl}$ is the completely skewsymmetric Levi-Civita symbol, and observe that

(6) $$*(UTU^T) = U^{-1T} * TU^{-1} .$$

Then, the C-antilinear involution $J$, $J^2 = 1$ given by

(7) $$JT = H * \overline{T} H^T ,$$

where $\overline{T}$ is complex conjugation of $T$, commutes with the action of $U$ because of ( 6 ), and defines the sixdimensional real, invariant subspace $R^6 \approx \{T \mid JT = T\}$ in $C^6$ with metric of signature (+,+,+,+,+,+) or (+,+,-,-,-,-) respectively.

This construction yields

(8) $$0 \longrightarrow Z_2 \longrightarrow SU(4) \longrightarrow SO(6) \longrightarrow 0$$

and

(9) $$0 \longrightarrow Z_2 \longrightarrow SU(2,2) \longrightarrow SO_o(2,4) \longrightarrow 0$$

which shows that

(10) $$Spin(6) \approx SU(4) \quad \text{and} \quad Spin_o(2,4) \approx SU(2,2) .$$

The embedding of $R^5$ in $H(2)$ given by

(11) $$(x_1, x_2, x_3, x_4, x_5) \longrightarrow \begin{bmatrix} x_5, & q \\ \bar{q}, & -x_5 \end{bmatrix} \equiv A ,$$

where $q = x_1 + ix_2 + jx_3 + kx_4$; together with the observation that

(12) $$UA^2U^+ = A^2 = \left(\sum_{i=1}^{5} x_i^2\right) \times \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix}_4$$

for $U \in Sp(2)$ yield

(13) $$0 \longrightarrow Z_2 \longrightarrow Sp(2) \longrightarrow SO(5) \longrightarrow 0$$

hence

$$Spin(0,5) = Sp(2)$$

Embedding of $C^4$ in $C(2)$ given by

(14) $$(z_1, z_2, z_3, z_4) \longrightarrow \begin{bmatrix} z_4 + iz_3, & iz_1 + z_2 \\ iz_1 - z_2, & z_4 - iz_3 \end{bmatrix} \equiv A$$

and observation that $\det(UAV^T) = \det(A) = \sum_{i=1}^{4} z_i^2$ for $U,V \in SL(2,C)$

leads to

(15) $$0 \longrightarrow Z_2 \longrightarrow SL(2,C) x SL(2,C) \longrightarrow SO(4,C) \longrightarrow 0 .$$

Taking various slices of $C^4$ yields all fourdimensional spin groups.

When $z_i$ are real then $\bar{A}\varepsilon = \varepsilon A$ ( $\varepsilon = \begin{bmatrix} 0, & 1 \\ -1, & 0 \end{bmatrix}$ ) and

$U,V \in SU(2) = Sp(1)$, hence

(16) $$0 \longrightarrow Z_2 \longrightarrow SU(2) x SU(2) \longrightarrow SO(4,0) \longrightarrow 0 ,$$

When $z_1, z_2, z_3$ are imaginary and $z_4$ real, then $A = A^+$, and $V^T = U^+$,

hence

(17) $$0 \longrightarrow Z_2 \longrightarrow SL(2,C) \longrightarrow SO_o(1,3) \longrightarrow 0 .$$

When $z_1, z_3$ are imaginary and $z_2, z_4$ real, then $A = \bar{A}$, and $U,V \in SL(2,R)$, hence

(18) $$0 \longrightarrow Z_2 \longrightarrow SL(2,R) \times SL(2,R) \longrightarrow SO_o(2,2) \longrightarrow 0$$

Therefore

(19) $Spin(0,4) = SU(2)xSU(2)$ , $Spin_o(1,3) = SL(2,C)$, $Spin_o(2,2) = SL(2,R)xSL(2,R)$ .

Setting $z_4 = 0$ in (14 ) yields: $Tr\ A = 0$ , $V^T = U^{-1}$, hence

(20) $$0 \longrightarrow Z_2 \longrightarrow SL(2,C) \longrightarrow SO(3,C) \longrightarrow 0 .$$

Similar considerations as in (16 ) and (18 ) yield

(21) $$0 \longrightarrow Z_2 \longrightarrow SU(2) \longrightarrow SO(0,3) \longrightarrow 0$$

and

(22) $$0 \longrightarrow Z_2 \longrightarrow SL(2,R) \longrightarrow SO(1,2) \longrightarrow 0$$

Therefore

(23) $$Spin(0,3) = SU(2) \quad \text{and} \quad Spin(1,2) = SL(2,R)$$

C.f. the Table ( 2 ) for a list of matrix forms of $Spin_o(s,t)$ for $1 \leqslant s+t \leqslant 6$ , including also the isomorphisms

(24)
$$Spin_o(1,5) \approx SL(2,H)$$
$$Spin_o(1,4) \approx Sp(1,1,H) , \quad Spin_o(2,3) \approx Sp(4,R)$$
$$Spin(0,2) \approx U(1) \qquad Spin(1,1) \approx GL(1,R), \quad Spin(0,1) \approx Z_2$$

Recall, that $GL(n,D)$ is a group of invertible $n \times n$ matrices with entries in D. The $SL(n,D)$, is a commutator subgroup of $GL(n,D)$ generated by all $a^{-1}b^{-1}ab$, where $a,b \in GL(n,D)$. For abelian D=R,C , $A \in SL(n,D)$ if $\det A = 1$ . For D = H , $A \in SL(n,H)$ iff $|\det A| = 1$ , where $\det A \overset{df}{=} \det\begin{bmatrix} B, -C \\ -C, B \end{bmatrix}$, and $B,C \in GL(n,C)$ , $B + jC = A$ . (Sometimes $DET(A) \overset{df}{=} \exp(Re\ trA) = 1$ is here used [ 34 ]).

## 3. Spinors in n-dimensions.

Elements of the D-linear (or ${}^2D$-linear, ${}^2D = D \oplus D$ , D = R,C or H) space B(V), on which the Clifford algebra C(V) is isomorphically represented as D or ${}^2D$-linear endomorphisms (i.e. matrices with respect to fixed basis), are called generally spinors of V, since their transformations under spinorial groups included in C(V) are determined. More specifically, we shall use the following names, which are however nonstandard in the literature c.f.[30,35,36]. Elements of the D-irreducible part S(V) of B(V) are called pinors; and of B(V) (when $D = {}^2R, {}^2C$ or ${}^2H$) — binors. Elements of the representation space of $C^{(+)}(V)$ are called spinors and of its irreducible part (when nontrivial) <u>Weyl spinors</u> or, since they have half as many components as spinors, halfspinors. It can be seen either by a computation of $(e_{\pi}^{\pi})^2$, or by an inspection of the Table ( 1 ), that Weyl spinors exist if K = C for any even dimension, and if K = R for s-t=0 (mod 4). Then, consistency with the Dirac equation

(1) $$[ i \gamma_k \partial^k - e \gamma_k A^k - m ] \psi = 0$$

requires that m = 0, where we denoted by $\gamma_k$ the matrices representing the o.n. basis $e_k$ of V.

When V is over K = C, spinors of V are called complex, and the standard name for $2^{[n/2]}$-component complex pinors is <u>Dirac spinors</u>. When V is over K = R, spinors are called 'real' (despite of that, D= R, C, H ). They should be distinguished from <u>Majorana spinors</u>, which are eigenstates of an appriopriate charge conjugation operator, and in some basis have real components, hence half as many components as Dirac spinors.

The Majorana conjugated $\psi_M$ spinor has to obey the Dirac equation that $\psi$ obeys, but with the opposite charge, hence $\psi_M = B^{-1}\psi^*$, where $B\gamma_i = -\gamma_i^* B$ and "*" denotes complex conjugation. For massless spinors $B\gamma_i = \pm\gamma_i^* B$ is suffcient. Only for dimensions and signatures for which $(\psi_M)_M = \psi$ i.e. $B^*B = 1$ the Majorana spinors (Majorana selfconjugated spinors) can be defined. A rather lenghty explicit computation shows that this is possible iff s–t = 0,6,7 (mod 8) for arbitrary m, and in addition if s–t = 0,1,2 (mod 8) for m = 0 [37, 38, 34 , 40]. Coquereaux [ 41 ] proposed to relate these Majorana 'reality' conditions with the D = R type of the Clifford algebra (c.f. Table ( 1 )), which is the case iff s–t = 0,1,2 (mod 8). Since this relation is not immediately clear, and yields only part of signatures admitting (only massless ) Majorana spinors, we shall show shortly how the complete list can be obtained.

It is known, that Majorana spinors have in the particular (Majorana) representation real components. From the universality and simplicity properties of Clifford algebras follow that the element B, fixed up to a phase, can be determined in any convenient representation. By inspection of Table ( 1 ) we see that a family of real matrices $\gamma_j$ representing the basis $e_i$ of $R^{s,t}$ can be picked up in $C_n$ iff s-t = 0,1,2 (mod 8). Moreover, in $C_n$ a similar family $\gamma_j$ with purely imaginary entries exists (the bilinears $\gamma_j\gamma_k$ spanning spin(s,t) and spinors are purely real) iff s-t = 0,6,7 (mod 8). The 'if' follows because $i\gamma_j$ represent the basis $e_j$ of $R^{t,s}$ with opposite signature; the 'only if', because for every family of purely imaginary $\gamma_j$, $i\gamma_j$ would be a real family spanning the real representation of C(V), what is possible only for s-t = 0,1,2 (mod 8).

Consider s-t = 0,1,2 (mod 8) and real $\gamma_j$. For $m \neq 0$ the only candidate for B is $e_n^{\pi}$, however since

(2) $$(e_n^{\pi})^* e_n^{\pi} = (e_n^{\pi})^2 = \begin{cases} 1 & \text{if } s-t = 0 \pmod 4 \\ -1 & \text{if } s-t = 2 \pmod 4 \end{cases},$$

Majorana spinors exist if s-t = 0 (mod 8). For m = 0 B can be taken as $1 \in R$, hence massless Majorana spinors exist also for s-t = 0,1,2 (mod 8). Consider now s-t = 0,6,7 (mod 8) and purely imaginary $\gamma_j = -\gamma_j^*$. For $m \neq 0$ $B = 1 = B^*B$ is suitable and Majorana spinors exist. For m = 0 also $B = e_n^{\pi}$ can be taken, but this yields (massless) Majorana spinors only for s-t = 0 (mod 8).

It easily follows that Weyl- Majorana spinors, with $1/4\ 2^{[n/2]+1}$ independent components, exist only if s-t = 0 (mod 8), c.f. Table ( 1 ).

The important property of higher-dimensional spinors is that they form multiplets, when reduced to lower-dimensions (see(4.5)). This can be seen in the representation of $e_j \in C^{2k}$ as

(3) $$\gamma_i = 1 \times \ldots \times 1 \times \sigma_1 \times \sigma_3 \times \ldots \times \sigma_3\ , \quad \gamma_{k+i} = 1 \times \ldots \times 1 \times \sigma_2 \times \sigma_3 \times \ldots \times \sigma_3$$

where there are i-1 factors 1 , k-i factors $\sigma_3$ and $1 \leq i \leq k$ .

## 4. Algebraic spinors

In order to discuss the algebraic spinors it is convenient to consider the isomorphic matrix forms of Clifford algebras as the particular case of more general Wedderburn decomposition theorem, c.f [74]. Obviously, the finite dimensional Clifford algebras $C(V)$ belong to the class of rings with a minimum condition, and their natural action as left multiplications on a fixed minimal left ideal (m.l.i.) $S(V) \subset C(V)$ provides the irreducible representation of $C(V)$. Since $C(V)$ does not contain any nilpotent ideal, the three following conditions are equivalent:

$1^{\circ}$ $C(V)p$ is a m.l.i. for $p \in C(V)$

$2^{\circ}$ $p$ is a primitive idempotent, i.e. $p = p^2 \neq 0$ can not be written as a sum of $p' = (p')^2 \neq 0$ and $p'' = (p'')^2 \neq 0$ such that $p'p''=0$

$3^{\circ}$ $pC(V)p = \{pap \mid a \in C(V)\}$ is a sfield i.e. a (possibly) noncommutative field.

In addition, there is also the right action of $D = pC(V)p$ on $S(V)$, which commutes with the left action of $C(V)$ due to the associativity of $C(V)$. Therefore $C(V)$ can be represented as D-linear automorphisms of the right D-linear space $S(V) = C(V)p$

As it is already suggested by the notation, and will be clear in a moment, the sfield $D$ is isomorphic to $D$ of previous section (c.f. Tab( 1 )), and $S(V)$ to the D-linear space of $2^{\chi} x1$

matrices (columns) with entries from D (all these isomorphisms depend on the choise of basis). When C(V) is simple, the group C*(V) acts transitively on the set of primitive idempotents i.e. any p.i. can be obtained from the fixed one by a similarity transformation. When C(V) is a direct sum of simple (nonuniversal) Clifford algebras, every p.i. necessarily belongs to one of two direct summands $1/2(1 \pm e_n^{\pi})C(V)$, and in these subalgebras the transitivity property holds.

To find at least one p in C(V) observe that the algebras $R_{0,0}$, $R_{0,1}$, $R_{0,2}$ and $C_0$ contain only one trivial p.i. p=1 . Also the only p.i. in $R_{1,0}$, $R_{0,3}$ and $C_1$ are projectors on one of two simple direct summands: $p = 1/2(1 \pm e_n^{\pi})$. Next, the p.i. in every $R_{s,t}$ and $C_n$ can be constructed using (1,1) (1,2) and (1,3). If $p = 2^{-\chi}\prod_{i=1}^{\chi}(1+\omega_i)$ is a p.i. in $R_{s,t}$, and p' in $R_{1,1}$ (for instance $p' = 1/2(1 + e_1')$, then

(1) $$p'' = (p \otimes 1)(1 \otimes p')$$

is a p.i. in $R_{s+1,t+1}$ . Now the p.i. in every $R_{s,t}$ can be obtained substituting the o.n. subset of $R_{s,t+s}$ (or $R_{s+1,t}$) by its expression in terms of the o.n. subset of $R_{t,s+3}$ (or $R_{s,t+1}$) according to (1,3 ) and (1,2). The p.i. in $C_n \approx R_{s,t} \otimes C$ , s+t=n can be taken from $R_{s,t}$ , for instance.

From the above considerations follows that every p.i. in $R_{s,t}$ can be written as

(2) $$p = 2^{-\chi}\prod_{i=1}^{\chi}(1 + \omega_i) \quad ,$$

where $\omega_i^2 = 1$, $[\omega_i, \omega_j] = 0$ for $i,j = 1,\ldots,\chi$ ; and $\chi$ given by

(3) $$\chi = \begin{cases} s+t - \left[\frac{s+t}{2}\right] \\ s+t - \left[\frac{s+t}{2}\right] - 1 \end{cases} \qquad \text{for } s - t = \begin{cases} 0,1,2 \\ 3,4,5,6,7 \end{cases}$$

is related to the Radon-Hurwitz number [ 31 ]. The sfield $D = pR_{s,t}p$ is isomorphic to

(4) $$D = \begin{cases} R \\ C \\ H \end{cases} \qquad \text{if} \quad s-t \overset{8}{=} \begin{cases} 0,1,2 \\ 3,7 \\ 4,5,6 \end{cases} \qquad (\#).$$

By varying the signs of $\omega_i$ in (2) one obtains the complete system of $2^\chi$ mutually commuting primitive idempotents [35, 36]

(5) $$p_\epsilon \overset{df}{=} 2^{-\chi} \prod_{i=1}^{\chi} (1 + \epsilon_i \omega_i)$$

satisfying

(6) $$p_\epsilon p_{\epsilon'} = \delta_{\epsilon\epsilon'} \equiv \prod_{i=1}^{\chi} \delta_{\epsilon_i, \epsilon'_i}$$

(7) $$\sum_\epsilon p_\epsilon = 1 ,$$

where $\epsilon \equiv (\epsilon_1, \ldots \epsilon_\chi)$ , $\epsilon_i = \pm 1$ , $i = 1, \ldots \chi$ ; and $p_{(1,1,\ldots,1)} \equiv p$. Hence, $C(V)$ splits into the direct sum of $2^\chi$ minimal left ideals $S_\epsilon(V) \overset{df}{=} C(V)P_\epsilon$ . Each $S_\epsilon(V)$ is an irreducible representation space of $C(V)$ acting from left, and also a right D-linear space of dimension $2^\chi$, since $p_{\epsilon'}C(V)p_\epsilon$ are 1-dimensional right spaces for

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(#) For $R_{s,s}$ (and $C_{2s}$) the particular primitive idempotent

(8) $$p = 2^{-s} \prod_{i=1}^{s} (1 + e_i e_{i+s}) = \pm \prod_{i=1}^{s} e_i \, 2^{-s} \prod_{j=1}^{s} (e_j + e_{j+s})$$

yields $D = R$ (or $D = C$), and the known Cartan and Chevalley minimal left ideal, related to the Witt decomposition of $R^{s,s}$ (or $C^{2s}$ respectively) into the direct sum of maximal isotropic subspaces.

each $\epsilon'$, and $C(V)p_\epsilon = \bigoplus_{\epsilon'} p_{\epsilon'} C(V)p_{\epsilon}$ .

The isomorhpisms of sfields $p_{\epsilon'}C(V)p_{\epsilon'} \approx p_\epsilon C(V)p_\epsilon$ , and the isomorphisms of 1-dimensional right spaces $p_{\epsilon'}C(V)p_\epsilon \approx p_{\epsilon''}C(V)p_{\epsilon''}$ over $p_\epsilon C(V)p_\epsilon$ and $p_{\epsilon''}C(V)p_{\epsilon''}$ respectively, easily follow from the fact that all p.i. in the simple (sub)algebra of C(V) are similar, and the direct summands $(1 \pm e^{\pi}_{n})C(V)$ are isomorphic for nonsimple C(V).

For a future reference let us pick up the family of $t_\epsilon \in C^*(V)$, satisfying for p and $p_\epsilon$ in the same simple subalgebra

(9) $$t_\epsilon p_\epsilon = p\, t_\epsilon \; .$$

The elements $t_\epsilon$ can be constructed by induction, in agreement with our convention for p (c.f. ( 1 )): if $t_\epsilon$ is a relevant family for $R_{s,t}$ and $t'_\epsilon$ for $R_{1,1}$, then the suitable family for $R_{s+1,t+1}$ is

(10) $$t''(\epsilon,\epsilon_{x+1}) \overset{df}{=} t_\epsilon \otimes t'_{\epsilon_{x+1}}$$

It is clear that chosen in this manner $t''_\epsilon$ belong to cannonical basis constructed from the o.n. subset of C(V), and moreover for a fixed $\epsilon$ both signs in $\beta(t''_\epsilon) = \pm t''_\epsilon$, and similarly in $\alpha\beta(t''_\epsilon) = \pm t''_\epsilon$, can be achieved by taking $t'_1 = 1$, and $t'_{-1} = e'_2$ or $t'_{-1} = e'_1 e'_2$. Using the isomorphisms (1,2) and (1,3) $t_\epsilon$, out of the basis, can be obtained for arbitrary $R_{s,t}$; however no longer both ( $\pm$ ) signs of transformations under $\alpha$ and $\alpha\beta$ are realized.

For fixed $\epsilon$ , $b_{\epsilon'} \overset{df}{=} t^{-1}_{\epsilon'} p\, t_\epsilon$ form basis of $S_\epsilon(V)$, i.e. every $a \in S_\epsilon(V)$ can be uniquely decomposed as $a = b_{\epsilon'} a_{\epsilon'}$ , where $a_{\epsilon'} \in p_\epsilon C(V)p_\epsilon \approx D$. Then the assignment $a \to a_{\epsilon'',\epsilon'} \in D$ , given by

(11) $$a b_{\epsilon'} = b_{\epsilon''}\, a_{\epsilon'',\epsilon'} \quad ,$$

yields the matrix representation of C(V) (see. Appendix A).

Observe, that starting with real vector spaces and Clifford algebras, the complex or quaternionic character of geometric objects appears naturally. From the other side, the abstract C or H numbers gain the geometrical interpretation (c.f. [ 50 ]).

All above considerations hold for every $p' = apa^{-1}$, $a \in C^*(V)$. However, it is important to realize, that the physical properties of different p's will be in general different, for instance with respect to the spinorial transformations and involutions $\alpha$ and $\beta$. This follows because $S(V) = C(V)p$ can be variously placed in $C(V)$ with respect to $V$, and is important for defining a scalar products of spinors [29, 35 ]

(12) $$\langle ap, bp \rangle = t\,\beta(ap)bp = pt\,\beta(a)bp \in D ,$$

(13) $$\langle ap, bp \rangle_\alpha = t_\alpha \alpha\beta(ap)bp = pt_\alpha \alpha\beta(a)bp \in D ,$$

where $t$ or $t_\alpha$ which obey

(14) $$t\,\beta(p) = pt ,$$

(15) $$t_\alpha \alpha\beta(p) = pt_\alpha ,$$

are introduced in order to "improve" the noninvariance of p under antiinvolutions $t$ or $t_\alpha$, respectively. The existence of $t$ and $t_\alpha$ is verified by observing that $\beta$ (or $\alpha$) applied to p, chosen according to our convention, varies only the signs of $\omega_i$ in a prescribed way, i.e. $p \to p_\epsilon$ for some $\epsilon$. Then the rôle of $t$ (or $t_\alpha$) can be played by $t_\epsilon$. For a general $p' = apa^{-1}$, $a \in C^*(V)$, $at_\epsilon\beta(a)$ (or $at_\epsilon\alpha\beta(a)$) is suitable. We know that for our conventional p, $t_\epsilon$ and then $t$ or $t_\alpha$, can be chosen to get a definite sign (and often both signs) in formulas

(16) $$\beta(t) = \pm t \qquad \text{or} \qquad \alpha\beta(t_\alpha) = \pm t_\alpha .$$

In fact, for $t_\alpha$ and $t$ this holds also for arbitrary $p' = apa^{-1}$, since taking $t$ and $t_\alpha$ suitable for $p'$ : $\beta(at\beta(a)) = \pm at\beta(a)$ and $\alpha\beta(at_\alpha\alpha\beta(a)) = \pm at_\alpha\alpha\beta(a)$ . Therefore, the scalar products ( 12 ) or ( 13 ), in general without definite symmetry, can be chosen to be symmetric or antisymmetric over the antiinvolutions of $D = pC(V)p$,

$$a \longrightarrow t\beta(a)t^{-1} \qquad \text{or} \qquad a \longrightarrow t_\alpha\alpha\beta(a)t_\alpha^{-1} \qquad \text{respectively:}$$

(17) $$t\beta(\langle ap,bp\rangle)t^{-1} = \pm\langle bp,ap\rangle$$

(18) $$t_\alpha\alpha\beta(\langle ap,bp\rangle_\alpha)t_\alpha^{-1} = \pm\langle bp,ap\rangle_\alpha$$

Other properties of (12 ) and (13 ) can be uniformly described when in the case of non-simple $C(V)$, one replaces $S(V)$ by $P(V) \equiv S(V) + \alpha(S(V))$ over the ring $rC(V)r$, and $p$ by $r \equiv p + \alpha(p)$, where $\alpha(S(V)) \equiv \{\alpha(a) \mid a \in S(V)\}$ ($P(V) = S(V)$ for simple $C(V)$). Then the resulting scalar products (12 ) and (13 ) on $P(V)$ are nondegenerate, (12 ) can be made positive definite only for $(s,t)=(n,0)$ and otherwise is neutral except for $(s,t) = (0,1),(0,2)$ and $(0,3)$; and (13 ) can be made positive only for $(s,t)=(0,n)$ and otherwise is neutral except for $(s,t)=(1,0)$. Moreover the automorphism groups, defined as $rC(V)r$-linear homomorphisms of $P(V)$ preserving these scalar products contain $Spin_o(V)$, and can be realized as

(19) $$\{a \in C(V) \mid \beta(a)a = 1\},$$

and similarly for $\alpha\beta$ . The Lie algebra

(20) $$\{a \in C(V) \mid \beta(a) + a = 0\}$$

of (19) is generated linearly by elements of the form $a\beta(b) - b\beta(a)$, where $a,b \in P(V)$, and similarly for $\alpha\beta$ , c.f. also the Appendix A.

5. Correspondence between spinors and tensors. Pure spinors.

Consider $S = C^2$, which can be interpreted as the space of Weyl spinors of $R_{1,3} \otimes C \approx C^4$, or the space of pinors of $R^{3,0}$. Then (4,12) and (4,13), with the choice $t = 1$ and $t = e_1$ are scalar products which are hermitian positive definite, and bilinear skew (symplectic), respectively. In the latter case let $\{b_1, b_2\}$ be the suitably normalized basis $\langle b_A, b_B \rangle_\alpha = \epsilon_{A,B}$, where $A,B = 1,2$ and $\epsilon_{AB} = \begin{bmatrix} 0,1 \\ -1,0 \end{bmatrix}$. Introduce the space $\hat{S}$ of C-valued C-antilinear functionals on S, and $\bar{S}$ the space of C-valued C-linear functionals on $\hat{S}$. Let $S \ni d \rightarrow \bar{d} \in \bar{S}$ be defined by $\bar{d}(\xi) = \xi(d)$, for any $\xi \in \hat{S}$. Then $\bar{b}_{A'}$ is a symplectic basis in $\bar{S}$, and there is an isomorphism of the hermitian part of the tensor product $S \otimes_H \bar{S} = \{ Z = Z^{AB'} b_A \otimes b_{B'} \mid \overline{Z^{B'A}} = Z^{AB'} \}$ with $R^{1,3}$, given by $\sigma_\mu^{AB} h_A \otimes h_{B'} \rightarrow e_\mu$, where $\sigma_\mu^{AB}$ are (hermitian) Pauli matrices and $e_\mu$ is the orthonormal (o.n.) basis of $R^{1,3}$. The primed indices conventionally indicate the transformations under the complex conjugate representation of $SL(2,C) \approx Spin_0(1,3)$. Consequently, components of arbitrary tensor can be expressed in a spinor equivalent form as

(1) $$T^{A_1 B_1', \dots, A_p B_p'}{}_{C_1 D_1', \dots, C_q D_q'} = \sigma^{A_1 B_1'}_{\mu_1} \dots \sigma^{A_p B_p'}_{\mu_p} \sigma^{\nu_1}_{C_1 D_1'} \dots \sigma^{\nu_q}_{C_q D_q'} T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q},$$

where

(2) $$T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} = \sigma^{\mu_1}_{A_1 B_1'} \dots \sigma^{\mu_p}_{A_p B_p'} \sigma^{C_1 D_1'}_{\nu_1} \dots \sigma^{C_q D_q'}_{\nu_q} T^{A_1 B_1' \dots A_p B_p'}{}_{C_1 D_1' \dots C_q D_q'},$$

The unprimed, primed and vector indices are lowered by symplectic tensors $\epsilon_{AB}$, $\epsilon_{A'B'}$ and metric tensor $g_{\mu\nu}$ respectively, and raised by their inverses. The expression (1) can be simplified for tensors with definite symmetries, like $g_{\mu\nu}, \epsilon_{\mu\nu\varrho\lambda}$ , $R^{\varrho}{}_{\lambda\mu\nu}$, $F_{\mu\nu}$ . The spinor equivalent form is particularly convenient to discuss the duality properties or to classify the Petrov type of Weyl or Maxwell tensors [ 5, 43 ].

The interesting question is up to what extent the spinor can be described in terms of tensor-like objects. A prototype is the Cartan [ 44 ] way of introducing spinor as "polarized" isotropic vector, or the well known Penrose flag [ 45 ]. It turns out, that most what can be achieved, is a 1:2 correspondence, with a spinor defined up to a sign.

Let us consider in more detail the Cartan concept of spinor. The components of an isotropic vector in $C^3$ can be expressed as $x_1 = \xi_0^2 - \xi_1^2$ , $x_2 = i(\xi_0^2 + \xi_1^2)$ $x_3 = -2\,\xi_0\xi_1$ , where $\xi_0$ and $\xi_1$ are components of a spinor $\xi$ . Also $\xi_0 = \pm\sqrt{\frac{x_1 + i x_2}{2}}$ , $\xi_1 = \sqrt{\frac{-x_1 - i x_2}{2}}$ , where it is impossible to fix signs in a continuous way. It follows that isotropic line determines a spinor (up to nonzero factor) by $x_i\,\sigma^i\xi = 0$ . This construction can be generalized to $C^{2k+1}$. The maximal isotropic plane $\Sigma_k$ in $C^{2k+1}$, spanned by vectors $x^{(1)},\ldots,x^{(k)}$ determines a spinor $\xi = \{\xi_0, \xi_i, \ldots \xi_{i_1,\ldots,i_k}\}$ , $i_j = 1,\ldots,k$ (up to nonzero factor), and vice versa. Because, the number of components of $\xi$ grows up faster than of $\Sigma_k$ with $k \to \infty$ , it is clear that they are not independent in general, but fulfil some recursive relations. It has been proposed, that these relations may have interesting physical interpretation as vanishing of particular transition amplitudes [ 6 ]. Restricted in this way spinors are called <u>pure</u>. Pure spinors form an orbit of the connected Clifford group, and any spinor is a sum of pure spinors. Chevalley [ 29 ] generalized this concept also to real spaces equipped with a quadratic form of maximal index i.e. $V = C^{2s}$, $C^{2s+1}$, $R^{s,s}$ or $R^{s,s+1}$. Pure spinors are defined as elements of a fixed minimal left ideal of the particular form $C(V)\prod_{i=1}^{s} x^{(i)}$ (c.f. footnote on p.25 ) which belong also to some minimal right ideal $\prod_{i=1}^{s} y^{(i)}C(V)$, where $y^{(i)}$ span a second maximal isotropic plane in V. In fact this concept can be generalized to arbitrary V. Let us observe that the intersection of any minimal left ideal with any minimal right ideal in C(V) is either $\{0\}$, or of dimension (over K) equal to $\dim_K D$. The $\{0\}$ intersection is possible only for s-t=1 (mod 4) if K = R, or odd n if K = C; when these ideals belong to different simple summands of C(V). This can be proved by noting the "factorization" property of any element of minimal left (or right) ideal [46, 127] (In any matrix representation it factorizes as U x V, where $U,V \in D^{2^x}$ are regarded as row and column, respectively). Now, for any given p, $d \in C(V)p$ will be called pure iff $d \in p'C(V)$, where $p' = apa^{-1}$, $a \in \Gamma(V)$. Then components of a pure spinor d are constrained in a similar way as in the case discussed by Cartan and Chevalley.

# 3. SPIN STRUCTURES.

## 1. Motivation.

There is essentially one way to introduce consistently spinors in the curved n-dimensional space M, with the pseudoriemannian metric g, of signature (s,t). It is based on the notion of a spin structure over M. Let us present some physical motivation for this construction before we shall recall the specific definitions in next sections. Assume that M is space and time oriented (however later the orientability conditions will be released), and let F be the $SO_o(s,t)$-bundle of oriented orthonormal (o.n.) frames over M. One is interested in the isomorphism of the tangent to M bundle TM with the bundle $\tilde{F} \times_{Spin_o(s,t)} R^{s,t}$, which is associated by the vector (i.e. spin-1) representation to some principal $Spin_o(s,t)$-bundle $\tilde{F}$ over M. It turns out however, that then the bundle $\tilde{F}$, called the bundle of spinor frames, has to be 'properly' placed over the bundle F; namely $\tilde{F}$ regarded as a $Z_2$ bundle over F has to restrict on each fibre to the nontrivial double covering, identical to the covering $\rho$ of relative typical fibers

(1) $$0 \longrightarrow Z_2 \longrightarrow Spin_o \longrightarrow SO_o \longrightarrow 0$$

This is always possible to be achieved locally over any sufficiently small (contractible) $U_\alpha$ out of the open covering $\{U_\alpha\}$ of M, but patching together can be inconsistent.

To visualize the sort of pathologies, which can be encountered, we recall two necessary conditions for a spin structure to exist over manifolds with signature (s,t) ; s= 0,1 ; $t \geq 3$ of the metric ($\pi_1(SO(s,t) = Z_2$, c.f. (2.1,12)). The first condition is that noncontractible loops in every fibre of F should not be contractible as loops in the total space F, for instance by going out of the fibre [47] (*). In other case some discontinuity must occur; lifts of noncontractible loops l are open paths l in fibres of F, and by no continuous deformation in F l can be made closed (equal to the lift of some contractible loop). The second condition states that the following situation can not happen [48]. Consider the parallelly transported vector along the loop l in M. Its final position differs from the starting one by $g \in O(s,t)$ . Now let $l_s$ , $s \in [0,1]$ be a family of loops, i.e. a closed 2-surface $\Sigma$ in M, such that $l_o$ and $l_1$ are trivial i.e. consist of a point. Then the relative $g_s$ form a loop $l_\Sigma$ in O(s,t) such that $g_o = g_1 =$ id . However, if this is the odd loop then the parallel transport of a spinor $\psi$ can not be defined: its final position after the transport along the (trivial) loop $l_1$ has to be both $\psi$ (since nothing happens) and $-\psi$ , by continuity with respect to other $l_s$ transports.

(*) In the case of other signatures it has been proposed [75] to demand, instead of the condition I, that odd loops $l \notin 2\pi_1(p^{-1}(x))$ should not be deformable in F to even loops $l \in 2\pi_1(p^{-1}(x))$.

This suggests that such inconsistencies on some manifolds, precluding the introduction of spinors, have something to do with holes or obstructions against the smooth contraction of two-dimensional surfaces in M. The condition I is equivalent to the condition II . This follows from the exact homotopy sequence

(2) $$\cdots \longrightarrow \pi_m(M,x) \xrightarrow{j^*} \pi_{m-1}(p^{-1}(x),E_x) \xrightarrow{i^*} \pi_{m-1}(F,E_x) \xrightarrow{p^*} \pi_{m-1}(M,x) \longrightarrow \cdots,$$

where $E_x$ is arbitrary o.n. frame over x, $E_x \in p^{-1}(x) \subset F$; and the maps $p^*$ , $i^*$ and $j^*$ are induced respectively by the bundle projection $p: F \longrightarrow M$ , fiber inclusion i into F, and maps similar to the map $j: \Sigma \longrightarrow l_\Sigma$ ( j described in the second condition, is clearly homotopy independent i.e. $[l_\Sigma]=[l_{\Sigma'}] \in \pi_1(p^{-1}(x))$ if $\Sigma$ and $\Sigma'$ are deformable one into another). Specify m=2 in(2). Then $i^*$ is injective (noncontractible loop in $p^{-1}(x)$ is not contractible in F) if and only if $j^*$ is zero (no family $\Sigma$ of parallel transports $l_s$ in M generates the odd loop l in $SO_o$). Also from (2) follows that for simply connected M ( $\pi_1(M) = 0$) and metrics of signatures for which $\pi_1(SO_o(s,t)) = Z_2$ (c.f. 2.1,12). the first and/or second conditions are sufficient. In fact, since $\pi_2(SO_o(s,t)) = 0$ (as for every Lie group) and $\pi_1(SO_o(s,t)) = Z_2$, the relevant part of (2) is

(3) $$0 \longrightarrow \pi_2(F) \xrightarrow{p^*} \pi_2(M) \xrightarrow{j^*} Z_2 \xrightarrow{i^*} \pi_1(F) \longrightarrow 0 \quad .$$

Then by exactness either $\pi_1(F) = 0$ , which is excluded by condition (1), or $\pi_1(F) = Z_2$, which allows $\tilde{F}$ to be defined as the (unique) universal covering of F [ 47 ].

Geroch [ 47 ] discussed various criteria for a fourdimensional noncompact , oriented and time oriented Lorentz space-time to admit a spin structure. The equivalent condition is parallelizability of M (then moreover the global o.n. frame lifts to a global section of $\tilde{F}$ ). The short cohomological proof we shall present in the next section. This result is trivial for dim $M \leq 3$ since then orientability implies parallelizability [ 49 ] and false for dim $M \geq 5$ . Other sufficient conditions are the Weyl tensor to be of particular Petrov type and the integral over M of curvature not to exceed some critical amount. The last result is rather surprising, since we shall show in a moment that the existence condition is a purely topological property.

Therefore it seems that non-spin manifolds are sufficiently twisted in order any metric on M to be of a high curvature.

In the riemannian case there is a stronger result [ 48 ] : if M is not a spin manifold then there exist a 2-sphere $S^2$ such that $1/2 \int_{S^2} ({}^{\pm}R_{\mu\nu\varsigma\sigma}\, {}^{\pm}R^{\mu\nu}{}_{\alpha\beta}\, d\Sigma^{\alpha\beta}\, d\Sigma_{\varsigma\sigma})^{1/2} \geq 2\pi$ , where ${}^{\pm}R = 1/2(R \pm {}^{*}R)$ are selfdual (anti-s.d respectively) parts of the curvature tensor R . Therefore, spaces of anti- or selfdual curvature always admit spinors.

Still another characterization is based on conditions I/II and Geroch construction of generic example, which we shall present in ch.4 . The idea is to 'thicken' a 2-sphere $S^2$ in M and consider the resulting neghbourhood as a plane bundle over $S^2$. This bundle is associated to some principal $SO(2)(\approx U(1))$ bundle over $S^2$, which is classified by the winding number m . If for at least one $S^2$ in M the relative m is odd, then M is not a spin manifold.

For charged spinors, coupled to some gauge field in M, the existence conditions are weaker in general; we shall discuss this case at the and of this chapter.

Rather different way to avoid problems when introducing spinors globally in M [ 51 ] , (see also [ 52 ] and [ 53 ] in case of isospinors) is to require vanishing of spinors on some lower dimensional subset of M . We shall not pursue this approach, since it is not clear whether it means that the space is cut out, or the external boundary conditions are imposed. Let us mention only that usually in such situations the formally defined quantum mechanical Hamilton operator may have many-parameter family of different selfadjont extensions [ 54 ] , which can be even of a nonlocal type [ 55 ].

Summarizing, to introduce spinors on a (pseudo)-riemannian space in a global way and to ensure the consistent distribution of the well known sign ambiguity, the spin structure is necessary.

## 2. Definition of spin structure.

Let M be a connected oriented pseudoriemannian manifold with metric tensor of signature $s + t = n$, and F the $SO_o(s,t)$-bundle of orthonormal (oriented) frames in M.
The spin structure $(\tilde{F}, \eta)$ on M consists of the principal $Spin_o(s,t)$-bundle $\tilde{F}$ over M together with the $\rho$-equivalent bundle morphism $\eta$ onto F such that the following diagram commutes

(1)

$$\begin{array}{ccc} \tilde{F} \times Spin_o(s,t) & \longrightarrow & \tilde{F} \\ \eta \times \rho \downarrow & & \downarrow \eta \\ F \times SO_o(s,t) & \longrightarrow & F \end{array}$$

where $\rho$ is the covering homomorphism (1.1,11), $\rho : Spin_o(s,t) \longrightarrow SO_o(s,t)$ and horizontal arrows denote right actions of groups on relative bundles[c.f. 56,57].

Then, M is called a spin manifold if it admits spin structure.

(2) Theorem

An oriented riemannian manifold M admits spin structure iff the second Stiefel-Whitney class of M ( $w_2(TM)$) vanishes.

The proof of the Theorem(2) will be given in a more general context in the next section.

Now let us present the short and abstract sheaf theoretic justification, essentially due to Haefliger [ 58 ] , c.f. also [59] for a Lorentzian case. The exact sequence

(3) $$0 \longrightarrow Z_2 \longrightarrow Spin(n) \longrightarrow SO(n) \longrightarrow 0$$

yields a long exact cohomology sequence (Appendix B)

(4) $$\ldots \longrightarrow H^1(M,Z_2) \longrightarrow H^1(M,Spin(n)) \longrightarrow H^1(M,SO(n)) \longrightarrow H^2(M,Z_2) \longrightarrow \ldots$$

The collection of SO(n)-valued transition functions of bundle of o.n. frames over M forms the Čech 1-cocycle k on M which fixes some cohomology class $[k] \in \check{H}^1(M,SO(n))$.
It can be shown that image of k in $\check{H}^2(M,Z_2)$ is $w_2(M)$.
Now because of the exactness: $w_2(M) = 0$ iff k is image of some $\tilde{k}$ in $\check{H}^1(M,Spin(n))$. Any Čech 1-cocycle representing $\tilde{k}$ is a collection of Spin(n)-valued transition functions and determines a spin structure.

The equivalent definition of a spin structure, as the particular $Z_2$ covering $\eta : \tilde{F} \longrightarrow F$ (such that $\eta \in \check{H}^1(F,Z_2)$ is nontrivial) has been already mentioned in Sect.(3.1). It yields another justification of (2) [ 60 ]; the exact sequence

(5) $$0 \longrightarrow SO(n) \xrightarrow{i} F \xrightarrow{p} M \longrightarrow 0 \ ,$$

where i is the inclusion map into some fibre, induces the short Serre exact sequence

(6) $$0 \longrightarrow \check{H}^1(M,Z_2) \xrightarrow{p^*} \check{H}^1(F,Z_2) \xrightarrow{i^*} \check{H}^1(SO(n),Z_2) \xrightarrow{\tau} \check{H}^2(M,Z_2)$$

Now $\eta$ is nontrivial iff $i^*([\eta]) = [\delta] \neq 0 \in \check{H}^1(SO(n), Z_2)$, but then $\tau([\delta]) \equiv w_2(M)$ has to be zero.

Let us show now how the Geroch result, that parallelizability is equivalent to the existence of spin structure, follows for oriented and time oriented 4-dim. noncompact lorentz manifolds. If M is parallelizable then there exist a global o.n. frame in M and F is trivial $F = M \times SO_o(1,3)$. Then $(\tilde{F}, \eta)$ can be obviously defined as $\tilde{F} = M \times SL_2C$ and $\eta(p,h) = (p, \rho(h))$.
Now let M admit the spin structure $(\tilde{F}, \eta)$. The structure group $SL_2C$ of $\tilde{F}$ can be restricted to SU(2) since SU(2) is a maximal compact sugroup of $SL(2,C) \approx SU(2) \times R^3$ (topologically).
Principal SU(2)-bundles over M are classified by homotopy classes of maps from M into $S^4$ (since the Hopf fibering $S^7 \xrightarrow{SU(2)} S^4$ is 4-universal); that is by $H^4(M,Z)$, which is trivial for noncompact M. Therefore F must be trivial and admit a global section $\tilde{\sigma}$. Then $\eta \circ \tilde{\sigma}$ is a global section of F and F is also trivial, hence parallelizable.

Let us observe that M is a spin manifold if and only if any of its coverings $\tilde{M}$ is $(w_2(M) = w_2(\tilde{M}))$, whereas if M is not a spin manifold then for any N also $M \times N$ is not, what is an "opposite" behaviour to the vector field problem on spheres:

parallelizable are group manifolds ($S^1$ and $S^3$) and $S^7$ but also any $S^{odd} x\, S^p x \ldots x S^q$. However, boundaries and direct products of spin manifolds are also spin manifolds [ 61 ].

(7) Definition

Two spin structures on $M$ $(\tilde{F}, \eta)$ and $(\tilde{F}', \eta')$ are equivalent iff there exist a (strong) bundle morphism $\beta : \tilde{F}' \longrightarrow \tilde{F}$ which intertwines $\eta$ and $\eta'$ : $\eta \circ \beta = \eta'$ .

(8)
$$\begin{array}{ccc} \tilde{F}' & \xrightarrow{\ \beta\ } & \tilde{F} \\ & \searrow^{\eta'} \quad {}^{\eta}\swarrow & \\ & F & \\ & \downarrow p & \\ & M & \end{array}$$

(9) Lemma

Inequivalent spin structures over riemannian $M$ are in 1-1 correspondence with elements of $\check{H}^1(M,Z_2)$.

The cohomological justification is as follows. Let $\tilde{k}$, $\tilde{k}' \in \check{H}^1(M,Spin)$ yield two spin structures over $M$. Then image of $\tilde{k} - \tilde{k}'$ is $0 \in \check{H}^1(M,SO\ )$ and by exactness of (4) $\tilde{k} - \tilde{k}'$ is the image of some class in $\check{H}^1(M,Z_2)$. Similarly, by the equivalent definition, if $\eta$ and $\eta'$ are two spin structures then $i^*(\eta' - \eta) = 0$ and by exactness of (6) $\eta' - \eta = p^*(\alpha)$ for some $\alpha \in \check{H}^1(M,Z_2)$. Hence inequivalent spin structures are labeled by classes of $\check{H}^1(M,Z_2)$.

Detailed proofs (2) and (9) which are more suited for a physicist's point of view at fibre bundles, we shall present in the next section. They will be valid for a wide variety of similar situations, requiring the prolongation of the structure group to some covering of it. In fact the pseudo- or riemannian character and the signature of the metric are irrelevant in the following sense. There can be topological obstructions to equip M with the pseudoriemannian metric, the orientation or time orientation, but when it is done, the prolongation problem will be precisely governed by the theorems. Furthermore it will apply to the full O(s,t) and its three subgroups containing one of P,T or PT transformations, c.f. also [62].
In these cases M no longer need to be space or time orientable.
Similarly other interesting homomorphisms will be covered

(10) $$0 \longrightarrow Z_n \longrightarrow SU(n) \longrightarrow SU(n)/Z_n \longrightarrow 0$$

(11) $$0 \longrightarrow Z \longrightarrow R \longrightarrow U(1) \longrightarrow 0 \quad .$$

There is one common property in all above cases; the kernel of the homomorphism $\rho : \bar{G} \rightarrow G$ is discrete and contained in the centre of $\bar{G}$. Let us present now following [ 63 ] the existence and inequivalence theorems for the prolongation of G to $\bar{G}$ . This will fix also our notation in subsequent sections.

3. $\tilde{G}$ structure, existence and uniqueness.

Let $\rho : \tilde{G} \to G$ be a homomorphism of Lie groups $\tilde{G}$ and G, such that its kernel K is discrete and contained in the centre of $\tilde{G}$. Let $F = [F,p,M,G]$ be a principal G-bundle over M. Then $\tilde{G}$ structure is a pair $(\tilde{F},\eta)$, where $\tilde{F} = [\tilde{F},\tilde{p},M,\tilde{G}]$ is a principal $\tilde{G}$-bundle over M and $\eta$ is the (strong) bundle morphism $\eta : \tilde{F} \to F$ $(p \circ \eta = \tilde{p})$, which is equivariant under the right actions of relative groups.

Two $\tilde{G}$ structures $(\tilde{F},\eta)$ and $(\tilde{F}',\eta')$ are equivalent iff there is a strong bundle morphism $\beta : \tilde{F} \to \tilde{F}'$, $(\tilde{p}' \beta = \tilde{p})$ such that $\eta' \circ \beta = \eta$.

Let $\{U_\alpha\}$ be a simple open covering of M (i.e. all nonempty intersections $U_\alpha \cap \ldots \cap U_\beta = U_{\alpha\ldots\beta}$ are contractible), and let $\sigma_\alpha : U_\alpha \to F$ be local sections with transition functions $\varphi_{\alpha\beta} : U_{\alpha\beta} \to G$.

The existence of $\tilde{G}$ structure is equivalent to the possibility of lifting $\varphi_{\alpha\beta}$ to $\tilde{\varphi}_{\alpha\beta} : U_{\alpha\beta} \to \tilde{G}$, $\rho \circ \tilde{\varphi}_{\alpha\beta} = \varphi_{\alpha\beta}$ in such a way that $\tilde{\varphi}_{\alpha\beta}$ are transition functions in $\tilde{F}$. Define $k_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to \tilde{G}$ by

(1) $$k_{\alpha\beta\gamma} = \tilde{\varphi}_{\beta\gamma} \tilde{\varphi}_{\alpha\gamma}^{-1} \tilde{\varphi}_{\alpha\beta} .$$

Since $\rho \circ k_{\alpha\beta\gamma} = \mathrm{id} \in G$, then $k_{\alpha\beta\gamma} \in K$. by a straightforward computation it can be shown that $k_{\alpha\beta\gamma}$ is closed $(dk)_{\alpha\beta\gamma\delta} = \mathrm{id}$. Its cohomology class is independent on the choice of local sections and liftings $\tilde{\varphi}_{\alpha\beta}$ and $k_{\alpha\beta\gamma}$ determines $k(F) \in \check{H}^2(M,K)$, called $\tilde{G}$ obstruction (see App.B). These observations leed to

(2) Theorem: F admits $\tilde{G}$ structure if and only if $k(F) = 0$.

It can be seen that for $G = O(n)$ and bundle $F$ of o.n. frames in $M$ $k(F) = w_2(M)$, i.e. the class $k(F) \in \check{H}^2(M,Z_2)$ is equal to the second Stiefel-Whitney class of the associated to $F$ tangent bundle $TM$. This can be proved starting from $O(1)$-bundles for which $k(F) = w_2(F) = 0$, and then by induction [ 63 ].

When $M$ is orientable ( iff the first Stiefel-Whitney class of $F$ is zero $w_1(M) = 0$ ), then the structure group $O(n)$ of $F$ can be reduced to $SO(n)$ and the obstruction class $k(F)$ for $SO(n)$-bundle is equal to that of $O(n)$-bundle. Hence the $Spin(n)$ structure on $M$ exists if and only if $w_1(M) = 0 = w_2(M)$. Similarly, provided that the reduction of $F$ to $SO_o(s,1)$ ( $SO(s,1)$, $O(s,1)$ ) does exist, the $Spin_o(s,1)$ ( $Spin(s,1)$, $Pin(s,1)$ respectively ) structure over $M$ exists iff $w_2(M) = 0$.

Now assume that there are two $\tilde{G}$ structures $(\tilde{F},\eta)$ and $(\tilde{F}',\eta')$ over $M$. Let $\tilde{\sigma}_\alpha$ and $\tilde{\sigma}'_\alpha$ be relative local sections with transition functions $\tilde{\varphi}_{\alpha\beta}$ and $\tilde{\varphi}'_{\alpha\beta}$ respectively, such that $\eta \circ \tilde{\sigma}_\alpha = \sigma_\alpha = \eta' \circ \sigma'_\alpha$. Define $\delta_{\alpha\beta} : U_{\alpha\beta} \to \tilde{G}$ by

(3) $$\delta_{\alpha\beta} = \tilde{\varphi}_{\alpha\beta} \tilde{\varphi}'^{-1}_{\alpha\beta} .$$

Then $\rho \circ \delta_{\alpha\beta} = id$, and in fact $\delta_{\alpha\beta} : U_{\alpha\beta} \to K$. The $\delta_{\alpha\beta}$ determines the element $\delta(\tilde{F}',\tilde{F}) \in \check{H}^1(M,K)$, called the difference class.

(4) <u>Lemma</u>: $(\tilde{F},\eta)$ and $(\tilde{F}',\eta')$ are equivalent iff $\delta(\tilde{F},\tilde{F}') = 0$.

<u>Proof.</u>

( $\Rightarrow$ ) Let $\beta$ be the isomorphism $\beta : \tilde{F} \to \tilde{F}'$. With respect to local sections $\tilde{\sigma}_\alpha$ in $\tilde{F}$ and $\tilde{\sigma}'_\alpha$ defined by $\tilde{\sigma}'_\alpha = \beta \circ \tilde{\sigma}_\alpha$ , $\delta_{\alpha\beta} = 0$.

($\Leftarrow$) Let $\delta(\tilde{F},\tilde{F}') = 0$ , then there exist $\lambda_\alpha : U_\alpha \longrightarrow \tilde{G}$ such that $\delta_{\alpha\beta} = \lambda_\beta \lambda_\alpha^{-1}$ . Then the map $\beta_\alpha : \tilde{p}^{-1}(U_\alpha) \longrightarrow \tilde{F}'$ given by

(5) $$\beta_\alpha(\tilde{E}) = \tilde{\sigma}'_\alpha(\tilde{p}(E))\ \lambda_\alpha(\tilde{p}(\tilde{E}))\ h,$$

where the unique h satisfies $\tilde{E} = \tilde{\sigma}'_\alpha(\tilde{p}(\tilde{E}))\ h$ is $\alpha$ independent. It defines the global map $\beta : \tilde{F} \longrightarrow \tilde{F}'$ such that $\beta(\tilde{E}h) = \beta(\tilde{E})h$ and $\eta' \circ \beta = \eta$ , what proves the equivalence of $(\tilde{F},\eta)$ and $(\tilde{F}',\eta')$.

(6) Theorem:

The inequivalent $\tilde{G}$ structures are in one to one correspondence with elements of $\check{H}^1(M,K)$.

Proof:

Observe that $\delta(\tilde{F},\tilde{F}')\ \delta(\tilde{F}',\tilde{F}'') = \delta(\tilde{F},\tilde{F}'')$. Then given $(\tilde{F},\eta)$ and $\delta \in \check{H}^1(M,K)$ the proof amounts to the explicit construction of $(\tilde{F}',\eta')$ such that $\delta(\tilde{F},\tilde{F}') = \delta$ . Set $\tilde{\varphi}'_{\alpha\beta} = \delta^{-1}_{\alpha\beta}\tilde{\varphi}_{\alpha\beta}$ . Then because of $\delta_{\alpha\beta} \in K \subset$ centre of $\tilde{G}$, $(\delta\tilde{\varphi}')_{\alpha\beta\gamma} = id$ and $\tilde{\varphi}_{\alpha\beta}$ are transition functions in some bundle $\tilde{F}'$. The bundle $\tilde{F}'$ can be constructed by Steenrod method (see the section (5.1)). Assume now that $\tilde{\sigma}'_\alpha$ are local sections in $\tilde{F}'$ relative to $\tilde{\varphi}'_{\alpha\beta}$ .

Then because of $\rho \circ \tilde{\varphi}'_{\alpha\beta} = \rho \circ \tilde{\varphi}_{\alpha\beta} = \varphi_{\alpha\beta}$ the map $\eta' : \tilde{F}' \longrightarrow F$ defined locally by $\eta'_\alpha[\tilde{\sigma}'_\alpha h] = \sigma_\alpha \rho(h)$ is well defined and equivariant. Moreover $\delta(\tilde{F},\tilde{F}') = \delta$ .

## 4. Spinors, Dirac equation and probability current.

Given a spin structure, or a pin structure in general, $(\tilde{F},\eta)$ over M the spinor fields on M are defined as sections of some associated bundle to $\tilde{F}$. In particular, spinor field of type $\gamma$ is a section of the bundle $\tilde{F}\times_{\gamma}V$, where $\gamma$ is a representation of Spin(s,t) (or Pin(s,t) ) in the spinor space V.
Equivalently, the spinor field $\psi : M \longrightarrow \tilde{F}\times_{\gamma}V$ can be described by the $\gamma$-equivariant V-valued map $\Phi : \tilde{F} \longrightarrow V$

(1) $$\Phi(\tilde{E}\, h) = \gamma(h^{-1})\, \Phi(\tilde{E}),$$

where $\tilde{E} \in \tilde{F}$, $h \in Pin(s,t)$.

Given a connection $\Gamma$ on F, there is a natural connection $\tilde{\Gamma}$ on $\tilde{F}$ obtained as a pull back of $\Gamma$ by $\eta^*$

(2) $$\tilde{\Gamma} = \rho_*^{-1}\eta^*\Gamma \quad ,$$

where $\rho_*$ is the isomorphism of Lie algebras spin(s,t) and so(s,t), induced by the homomorphism $\rho$ . Henceforth we assume that $\Gamma$ is the unique Levi-Civita connection, preserving the metric and torsionfree.

Let us check that $\tilde{\Gamma}$ is indeed a connection on the principal Pin-bundle $\tilde{F}$. The continuity properties are rather obvious.
Now let $\tilde{E} \in \tilde{F}$ , $A \in spin$ , then from the relation between the right actions of structure groups

(3) $$\eta(\tilde{E}\exp sA) = \eta(\tilde{E})\,(\exp sA) = \eta(\tilde{E})\,\exp(s\,\rho_*A)$$

follows

(4) $$\eta_*(A_{\tilde{E}}) = (\rho_*A)_E \quad ,$$

where $E = \eta(\tilde{E})$ and $A_{\tilde{E}}$ on the l.h.s. denotes the relative vertical vector at $\tilde{E}$, and similarly on the r.h.s.. Hence

(5) $$\tilde{\Gamma}(A_{\tilde{E}}) = \rho_*^{-1}\left[\Gamma(\eta_*(A_E))\right] = \rho_*^{-1}\left[\Gamma(\rho_* A_E)\right] = \rho_*^{-1}\rho_* A = A$$

and

(6) $$R^*_h \tilde{\Gamma} = \tilde{\Gamma} \circ (R_h)_* = \rho_*^{-1} \circ \Gamma \circ \eta_* \circ (R_h)_* = \rho_*^{-1} \circ \Gamma \circ (R_{\rho(h)})_* \circ \eta_* =$$

$$= \rho_*^{-1} \, ad_{\rho(h^{-1})} \Gamma \circ \eta_* = ad_{h^{-1}} \tilde{\Gamma} \quad .$$

Therefore $\tilde{\Gamma}$ is a well defined connection cf. [ 64 ], which in a standard way determines the covariant derivative $d^H \Psi$ of $\Psi: \tilde{F} \to V$, where $d^H$ is the horizontal part of d. Given a local section $\tilde{\sigma}_\alpha : U_\alpha \to \tilde{F}$ , we shall denote the components of a covariant derivative of $\Psi$ in the direction $X \in TM$ by

(7) $$\nabla_X \psi_\alpha = X\psi_\alpha + \Gamma_\alpha(X)\psi_\alpha$$

where $\Gamma_\alpha(X) = \gamma \circ \rho_*^{-1}\left[\sigma_\alpha^* \Gamma(X)\right]$ and $\sigma_\alpha = \eta \circ \tilde{\sigma}_\alpha : U_\alpha \to F$ .

Next , the Dirac '$\gamma$' matrices can be generalized as follows. We are interested in Dirac spinors, for which the representation $\gamma$ in ( 1 ) comes from the (irreducible) representation of the complex Clifford algebra $C_n$ in the space $L(C^{2^{[n/2]}})$, n=s+t. Given a local section $\sigma_\alpha$ , define the $L(C^{2^{[n/2]}})$ valued one form

$$\gamma_\alpha : TM \to L(C^{2^{[n/2]}}) \quad \text{by}$$

(8) $$\gamma_\alpha(X) = \gamma\left[\sigma_\alpha^* \,\theta(X)\right] ,$$

where $\theta : TF \to R^{s,t}$ is the canonical form defined on arbitrary vector $\mathcal{V}$ tangent to F at $E_x \in F$, $(E_x : R^{s,t} \to T_x M)$ by

(9) $$\theta(\mathcal{V}) = E_x^{-1}(p_*(\mathcal{V})) .$$

Then $\gamma_\alpha(X) = \gamma(\sigma_\alpha^{-1}(X))$, where on the r.h.s. $\sigma_\alpha^{-1} : TM \to R^{s,t}$.

The standard anticommutation rules hold

(10) $$\{\gamma_\alpha(X), \gamma_\alpha(Y)\}_+ = 2g(X,Y).$$

Finally, the Dirac operator is defined by

(11) $$\not{D}\psi_\alpha = i\sum_{a,b=1}^{n}\eta^{ab}\,\gamma_\alpha(E_a)\nabla(E_b)\psi_\alpha$$

where $E_a$, $a,b=1,\ldots,n$ are components of the arbitrarily chosen o.n. frame $E$. The Dirac equation is:

(12) $$\not{D}\psi_\alpha = m\,\psi_\alpha \quad .$$

In practice, given a local frame $E = \{E_a\}$ in $U_\alpha$ we can take $\gamma(E_a)$ to be constant matrices $\gamma_a$ in $L(C^{2^{[n/2]}})$ representing the pseudo-Euclidean basis in $R^{s,t} \in R_{s,t}$ .

Then the spin connection matrix $\tilde{\Gamma}(X)$ is uniquely determined by two properties. It has to be a linear combination of comutators of $\gamma_a$ matrices, that is generators of $Spin(s,t) \subset R_{s,t}$ .
This guaranties that the anticommutation rules (10) are preserved, and the metric $g$ is covariantly constant. Also the torsion of $\tilde{\Gamma}$ (or rather of $\Gamma$ which comes from $\tilde{\Gamma}$ by pushing forward by $\eta_*$ the horizontal subspaces in $T\tilde{F}$) has to vanish

(13) $$d\,\Theta + \Gamma\wedge\Theta = 0 \ .$$

Composing (13) with $\sigma_\alpha^*$

(14) $$d\sigma_\alpha^*\Theta + \sigma_\alpha^*\Gamma\wedge\sigma_\alpha^*\Theta = 0 \ ,$$

and then with the Clifford representation $\gamma$ , one gets

(15) $$d\gamma_\alpha + \tilde{\Gamma}_\alpha\wedge\gamma_\alpha = 0 \ .$$

Applying (15) to a pair of arbitrary basis vectors $X,Y \in \{E_a\}$ yields

(16) $$d\,\gamma_\alpha(X,Y) + [\tilde{\Gamma}_\alpha(X), \gamma_\alpha(Y)] - [\tilde{\Gamma}_\alpha(Y), \gamma_\alpha(X)] = 0 \quad ,$$

which, because $\gamma(E_a)$ are constant, becomes

(17) $$[\tilde{\Gamma}(X), \gamma(Y)] - [\tilde{\Gamma}(Y), \gamma(X)] - \gamma([X,Y]) = 0 ,$$

where we have ommited the reference to a particular local section From (17 ) $\tilde{\Gamma}$ can be solved, or easily guessed.

Let us restrict our attention to the generalized Lorentzian case of signature (1,n-1), and define the Dirac conjugated spinor field $\overline{\Psi}$ by local components as $\overline{\psi}_\alpha = \psi_\alpha^+ \gamma_0$ , where $\gamma_0$ is the constant matrix in $R_{1,n-1}$ , and $\overline{\Psi}(Eh) = \overline{\Psi}(E)\gamma(h)$ . The Dirac equation (12) can be obtained by the variational principle from action integral $S = \int \mathcal{L}\, d\mu$ , where the Lagrangian density $\mathcal{L}$ is

(18) $$\mathcal{L} = i/2\, \eta^{ab}\left[\overline{\psi}\gamma_a \nabla_b \psi - (\overline{\nabla_b \psi})\gamma_a \psi\right] - m\overline{\psi}\psi .$$

The current 1-form $j(\psi, \psi')$ locally given by

(19) $$j_\alpha(\psi, \psi')(X) = \overline{\psi}_\alpha \gamma(X)\psi'_\alpha$$

is $\alpha$-independent and divergencefree if $\psi$ and $\psi'$ are solutions of (12)

(20) $$\delta j(\psi, \psi') = \eta^{ab}\left[E_a(\overline{\psi}\gamma(E_b)\psi') - \overline{\psi}\gamma(\nabla_{E_a}E_b)\psi'\right] =$$

$$= \eta^{ab}\left\{E_a \overline{\psi}\gamma(E_b)\psi' + \overline{\psi}\gamma(E_a)E_b\psi' - \overline{\psi}\left[\Gamma(E_a), \gamma(E_b)\right]\psi'\right\} =$$

$$= \eta^{ab}\left\{\overline{\gamma(E_a)\left[E_b + \Gamma(E_b)\right]\psi}\, \psi' + \overline{\psi}\gamma(E_a)\left[E_b + \Gamma(E_b)\right]\psi'\right\} =$$

$$= \overline{\psi}\not{D}\psi' - \overline{\not{D}\psi}\psi' = 0 .$$

Let $*j$ denotes the Hodge dual of $j$, defined in a coordinate frame by

(21) $$(*j)_{\nu_1 \cdots \nu_{n-1}} = \sqrt{|g|}\, \epsilon_{\nu_1 \cdots \nu_n} j_\mu g^{\mu\nu_n} .$$

Then given any maximal space-like n-hypersurface S in M, the expression

(22) $$\langle \psi, \psi' \rangle_S = - \int_S *j(\psi, \psi')$$

is because of (20) S-independent (in a homotopy class), and defines the nonnegative scalar product $\langle \psi, \psi' \rangle$ of $\psi$ and $\psi'$.

## 5. $Spin_C$ and generalized $spin_G$ structures.

(1) Definition

Let $G$ be a Lie group with the centre containing $Z_2$. Let $M$ be a connected, pseudoriemannian, oriented and time oriented manifold, and $F$ the bundle of oriented o.n. frames over M. The $spin_G$ structure $(F_G, \eta_G)$ consists of

1° the principal $G \times Spin/ Z_2$ -bundle $F_G$ over M, where the equivalence group $Z_2$ is the diagonal of $Z_2 \times Z_2 \subset Spin \times G$,

2° the equvariant with respect to group actions (strong) bundle map $\eta_G : F_G \to F$ .

(2) Definition

Two $spin_G$ structures $(F'_G, \eta_G')$ and $(F_G, \eta_G)$ are eqiuvalent iff there is a (strong) bundle morphism $\beta_G : F'_G \to F_G$ such that $\eta_G \circ \beta_G = \eta'_G$ .

In anology to $\tilde{F}$, the alternative definition of $F_G$ would be the covering $\varkappa_G$ of the (direct) Whitney sum of $F$ and the principal $G/Z_2$-bundle $\mathcal{E}$ , which on each fiber restricts to the covering $Spin_G = (Spin \times G)/Z_2 \to SO \times G/Z_2$ . However, the associated notion of 'equivalence' as the bundle map between $F_G$ and $F'_G$ which intrwines $\varkappa_G$ and $\varkappa'_G$ , is slightly different from the definition (2) [ 60 ].

From the latter definition of $\mathcal{H}_G$ follows that in order to specify the covariant derivative of G-spinors, the (Levi-Civita) connection on F has to be combined with some connection on the principal $G/Z_2$-bundle $\mathcal{E}$. Then the product connection $SO \times G/Z_2$ pulls back by $\mathcal{H}^*$ to a connection on $F_G$. This means, that some $G/Z_2$-gauge field lives on M and spinors have to be charged. Since representations of $Spin_G$ are constucteed from (trivial on $Z_2$) representations of Spin x G, the relation between the internal spin and statistics. arises[106].

The motivation to introduce the $spin_G$ structure is to weaken or evade the obstructions against the usual spin structure, by compensation of the nonvanishing $w_2(M)$ with the $H^2(M,Z_2)$ characteristic class of $\mathcal{E}$.

Forger et. al.[42] have shown that for the properly chosen G, such that $Z_2 \subset \mathrm{Ker}G$ and there is a homomorphism $\chi : Spin(2n) \to G$ with the bijective restriction $\chi|_{\mathrm{Ker}\ Spin(2n)} \to Z_2 \subset \mathrm{Ker}G$, $spin_G(2n)$ structure always exists. In fact, $Spin_G(4)$ structure can be always defined for simply connected G.

In a rather complicated classification of inequivalent $Spin_G(4)$ structures the group $H^4(M,\pi_3(G))$ plays an important rôle [60].

In general, $Spin_G$ forms multiplets of spinors, but the particular case G = U(1) yields $\underline{Spin}_{U(1)}$ denoted by $\underline{Spin}_c$, which is a relatively mild deviation from standard Spin.

The group $Spin_c$ arises naturally in the (complexified) Clifford scheme (see ch.(2.1)), and the definition of Dirac spinors ( 2.3 ) is still valid.

The $spin_c$ structure $F_c$ exists iff $w_2(M)$ is reduction mod 2 of some integral cohomology class, i.e. there exists $c_1 \in H^2(M,Z)$ such that $w_2(M)$ is the image of $c_1$ under the homomorphism $H^2(M,Z) \longrightarrow H^2(M,Z_2)$, induced by the homomorphism of the coefficient groups $Z \rightarrow Z_2$ [ 66 ]. Then the map $Spin \times U(1) \ni (h',h) \rightarrow h^2 \in U(1)$ passes to the homomorphism $Spin_c \rightarrow U(1)$, and yields the complex line bundle L associated to $F_c$ with $c_1$ being the first Chern class of L. The U(1) - gauge is usually interpreted as the electromagnetic field on M. The obstructions against the $spin_c$ structure still do not vanish, but are weaker than obstructions against the spin structure (any spin manifold has also the canonical $spin_c$ structure, which need not to be unique).

The $spin_c$ structure can be equivalently defined as the U(1)-bundle $F_c$ over F

(3) $$0 \rightarrow U(1) \longrightarrow F_c \xrightarrow{\eta_c} F \longrightarrow 0 \quad ,$$

which restricts over each fibre to the homomorphism

(4) $$0 \rightarrow U(1) \longrightarrow Spin_c \longrightarrow SO \longrightarrow 0$$

induced by

(5) $$Spin \times U(1) \ni (h,h') \longrightarrow \rho(h) \in SO \quad .$$

For s=0,1 and $t \geq 3$ the U(1) fibration (4) over SO is nontrivial, because if it was trivial then $\pi_1(\mathrm{Spin}_c)$ would be $Z_2 \times Z$ which is not the case; $\pi_1(\mathrm{Spin}_c) = Z$, since topologically (not as group) $\mathrm{Spin}_c \approx \mathrm{Spin} \times U(1)$.

Now, U(1) bundles over F are in 1-1 correspondece with elements of $H^2(F,Z)$. The short exact sequence

(6) $$0 \longrightarrow SO \xrightarrow{i} F \xrightarrow{p} M \longrightarrow 0$$

induces

(7) $$\ldots \longrightarrow H^1(SO\ ,Z) \longrightarrow H^2(M,Z) \xrightarrow{p^*} H^2(F,Z) \xrightarrow{i^*} H^2(SO,Z) \xrightarrow{\tau} H^3(M,Z) \longrightarrow \ldots.,$$

and $[\eta_c] \neq 0 \in H^2(F,Z)$ is a $\mathrm{spin}_c$ structure if and only if $i^*[\eta_c] \neq 0 \in H^2(SO,Z)$. But $w_3(M) = \tau \circ i^*[\eta_c]$, and the $\mathrm{spin}_c$ structure exists if and only if the third integral Stiefel-Whitney class of M vanishes [ 67 ].

The classifiacation of inequivalent $\mathrm{Spin}_c$ structures still is complicated. It has been shown that in the case of $\mathrm{spin}_c(4)$ structure the inequivalence of $F_c$ bundles is responsible for the inequivalence of $\mathrm{spin}_c$ structures (up to the 2-torsion in $H^2(M,Z)$) [60 ].

## 6. Fermions without spinors?

Another, conceptually different, attempt to describe fermions c.f.[13, 68, 14, 15, 16, 83] is based on a globalization of the concept of algebraic spinor. Let M be a manifold with a (pseudo) riemannian metric g. Then the Clifford bundle Clif(M), with fibers $p^{-1}(x)$, $x \in M$ isomorphic to $C(T_x M)$, can be defined as $\mathrm{Clif}(M)=\frac{\mathrm{Tens}(M)}{J(M)}$, where Tens(M) is the full bundle of tensors and J(M) is a 2-sided ideal generated by $X \otimes X - g(X,X)$ for arbitrary vectors X. One can work also in linearly isomorphic bundle $\Lambda(M)$ of differential forms, equipped with a metric $\tilde{g}$ induced by g, exterior product, and a Clifford product (defined by $a \vee b = a \wedge b + \tilde{g}(a,b)$ for $a,b \in \Lambda(M)$, and extended to $\Lambda(M)$ by associativity and linearity). Then the square of $d-\delta$ (where d is exterior derivative, $\delta = (*)^{-1} d *$, and $*$ is a Hodge dual) is a Laplace-Beltrami operator. In a flat space this is a common property with a Dirac operator. Kähler [ 13 ] showed that in a flat space, in presence of electromagnetic potential A, the equation $(d - \delta - m)a - ieA \vee a = 0$ is equivalent to Dirac equation, provided that a is restricted to belong to minimal left ideal of $\Lambda(M) \otimes C$. In general, this equation describes a flavour multiplet of fermions (number of fermions equals number of minimal left ideals), and it is not always possible to globally split $\Lambda(V) \otimes C$ into subbundles of minimal left ideals. If this can be done, another pecularity is that the Dirac-Kähler equation in presence of gravitation mixes spinors belonging to different minimal left ideals (see however [69] for modified equations). The conceptually most intriguing aspect of the above approach, is its aim to describe the halfinteger-spin objects in terms of integer-spin tensors. This exploites the fact that the fixed Clifford number $a \in \Lambda(M)$ can be viewed either as a spinor i.e. an element of a minimal left ideal $I(M) \subset \Lambda(M)$, or as an inhomogeneous tensor, i.e. element of $\Lambda(M)$, considered as the bundle of automorphisms of I(M). Also the rôle played by the primitive idempotent, necessary to split $\Lambda(M)$ at least locally, is interesting. To illustrate this point consider a 4-dimensional Lorentzian time-oriented M, of signature (+---). Then, given local coordinates, p can be taken as $p=1/2(1+ g_{oo}dx^{o})$. Given a local (dual) vierbein $E^{a}$, p can be also 'aholonomic'; $p=1/2(1+E^{o})$. Finally, any $p=1/2(1+a)$ with (not necessarily vector) $a \in \Lambda(M)$, $a^{2}=1$, is possible. This last possibility can be called "second aholonomization", and the primitive idempotent p considered as a dynamical entity, with the local gauge group $C^{*}(R_{1,3})$. This corresponds to gauging the freedom of choice of the representation of $\gamma$-matrices.

Whether the Dirac or Dirac-Kähler equation is better suited for a description of physical particles in gravitational field, the lattice theories [ 14 ], or in higher-dimensional Kaluza-Klein theories [ 70 ], should be a matter of detailed examination. Let us point out few differences with the standard approach to spinors presented in previous sections. First of all, the spin structure may be helpful to interpret the sections of the minimal left ideal subbundle of $\Lambda(M)$ as spinors, but is not necessary. In fact, the sufficient condition for a global splitting of $\Lambda(M)$ is the existence of a global primitive idempotent in $\Lambda(M)$. However, in the case of real Clifford bundles $Clif(M) \approx \Lambda(M)$ , and riemannian oriented fourdimensional M, p can be always defined as $p = 1/2(1+\omega)$, where $\omega$ is the volume form. In the Lorentzian case of signature (+ – – –), time orientability or orientability are sufficient: $p = 1/2(1+e_o)$ or $p = 1/2(1+\omega)$. For (complex) $Clif(M) \otimes C$, and Lorentzian, time oriented and oriented M, p can be defined as $p = 1/4(1+e_o)(1+\omega)$ for signature (+ – – –) and $p = 1/4(1+ie_o)(1+\omega)$ for signature (– + + +). As follows, these conditions are rather different from the condition for a spin structure to exist. In particular cases, there can be some coincidence, like the Geroch results (c.f.( 3, 1)). Also, according to [71,72], the existence of a global field of maximally isotropic hyperplanes in the complexified tangent bundle $TM \otimes C$ is equivalent to the existence of spin structure (#). In the case of 4-dimensional Lorentzian oriented and time oriented M, this implies that the structure Lorentz group reduces to the spinoriality group of topology $R^2$ [73], therefore: parallelizability = spin structure.

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(#) However, the definition of a spinor frame $\tilde{E}$ in [71] is in variance with the usual one, and the topology of $\{\tilde{E}\}$ is $SO_o(1,3)$, instead of $Spin_o(1,3)$.

# 4 EXAMPLES AND APPLICATIONS.

## 1. Spin structure on nonorientable manifolds and on spheres.

In this section we first illustrate briefly how spinors on nonorientable manifolds can be introduced, and then we recall following Trautman [ 75 ], some interesting relations between the canonical spin connections and nontrivial gauge configurations over spheres.

Let us start with the nonorientable $M = R \times \Sigma \times R$, where $\Sigma$ is the Klein bottle, i.e. $R \times R$ with the points $(x^1, x^2)$ and $(x^1 + mB^1, (-1)^m x^2 + nB^2)$ identified for all $m,n \in Z$. By enlarging the structure group from Spin (1,3) to Pin(1,3), the spin structure can be defined: an element of Pin(1,3) corresponding to the $x^3 \to -x^3$ inversion in O(1,3) can be used for defining the transition functions in $\tilde{F}$. The Dirac spinors have to obey then [c.f. 20 ]

$$\psi(x^0, x^1, x^2, x^3) = (\gamma^0 \gamma^1 \gamma^2)^m \psi(x^0, x^1 + mB^1, (-1)^m x^2 + nB^2, x^3).$$

(In fact, in this case it would be enough to consider the orthochronous Lorentz group $O^{\uparrow}(1,3)$ and its covering $Pin^{\uparrow}(1,3)$). Another example of a nonorientable space is provided by the 2-dimensional real projective space RP(2) ($R^2$ with the identification

$$(x^1, x^2) \sim ((-1)^n x^1 + mB^1, (-1)^m x^2 + nB^2) \quad \text{for all } m,n \in Z\ )$$

The O(2) bundle of o.n. frames is $F = SO(3)$, since $SO(3)/O(2) \approx RP(2)$, and $\tilde{F} = SU(2)$. The spin structure map $\eta$ agrees with the standard homomorphism $\rho : SU(2) \to SO(3)$.

There are known examples of (unique for $n \geq 2$) spin structures on spheres $S^n$. The total space F of the bundle of o.n. frames $[F,p,S^n,SO(n)]$ has the topology of SO(n+1), since $SO(n+1)/SO(n) \approx S^n$.

Then the total space $\tilde{F}$ of the bundle of spinor frames $[\tilde{F},p,S^n,Spin(n)]$ is Spin(n+1), since this is the only nontrivial covering of SO(n+1). Let $\Gamma$ be the Levi-Civita connection on F and $\tilde{\Gamma}$ the induced one on $\tilde{F}$.

<u>n = 1</u> $SO(1) \approx \{1\}$, $Spin(1) = Z_2 \approx S^0$

$$F = SO(2) \approx U(1) \approx S^1 , \qquad \tilde{F} = Spin(2) \approx U(1) \approx S^1 ,$$

the map $U(1) \ni z \xrightarrow{\eta} z^2 \in U(1)$

has the winding number 2 and corresponds to the Mobius strip. There is also the second inequivalent spin structure over $S^1$ given by the trivial $F' = S^1 x Z^2$ and $\tilde{F} \ni (z,\pm 1) \xrightarrow{\eta'} (z,1) \in S^1 x \{1\}$ . This is the only case known to the author when the inequivalence of spin structures comes from nonisomorphic bundles $\tilde{F}$, and not from inequivalent maps $\eta$ , c.f. sect.4 .

<u>n = 2</u>

$$F = SO(3) = RP(3) = S^3/Z_2 , \qquad \tilde{F} = Spin(3) = SU(2) \approx S^3 ,$$

$\tilde{p} = p \circ \eta$ is the well known Hopf fibering and $\tilde{\Gamma}$ describes the magnetic monopole on $S^2$ of unit strenght.

n = 3

$F = SO(4) \approx SU(2) \times SU(2)/Z_2 = SU(2) \times SO(3)$ , $\tilde{F} = Spin(4) \approx SU(2) \times SU(2)$ are trivial principal bundles over $S^3$, since $S^3$ is parallelizable; the su(2)-valued connection one-form $\tilde{\Gamma}$ describes the meron solution [ 76 ].

n = 4

$F = SO(5)$ , $\tilde{F} = Spin(5)$

$\tilde{\Gamma}$ , a connection 1-form on F with values in spin(4)=su(2)+su(2), can be split into two components, and projected down to a connection on $S^7 \approx Sp(2)/Sp(1)$ with values in sp(2)= su(2) (the t'Hooft symbols $\eta^i_{[\mu,\nu]}$ are involved). This corresponds to the BPST instanton [ 77 ].

n = 5

$F = SO(6)$ $\tilde{F} = Spin(6) = SU(4)$

shows that $SU(4)/Sp(2) \approx S^5$. Moreover F is the only nontrivial Sp(2) bundle over $S^5$ since $\pi_4(Sp(2)) = Z_2$ .

n = 6

$F = SO(7)$ $\tilde{F} = Spin(7)$

shows that $Spin(7)/SU(4) \approx S^6$

n = 7

$F = SO(8)$ $\tilde{F} = Spin(8)$

Since $S^7$ is paralelizable and simply connected, the unique spin(7) structure over $S^7$ is trivial, which shows that $Spin(8) \approx Spin(7) \times S^7$ and $SO(8) \approx SO(7) \times S^7$ .

## 2. Non-spin manifolds.

The 'generic' examples of four-dimensional manifolds $M_m$ without spin structure have been described by Geroch [ 47 ]. Consider plane bundles over $S^2$ associated with some principal SO(2) bundle over $S^2$. These are classified by the winding number, i.e. homotopy class of a map from the equator $S^1$ of $S^2$ into $SO(2) \approx S^1$. The explicit construction of $M_m$ is as follows. Let $D_i x C_i = \{(z_i, z'_i) \in CxC \mid |z_i| \leq 1\}$ be two copies (i=1,2) of a direct product of a 2-disk with a 2-plane. Identify boundaries of disks $\{z_i = 1\}$ (obtaining $S^2$) and relative planes according to $z_1 = z_2$ ; $z'_1 = z'_2 z_1^m$ for fixed integer $m$ . It can be seen that wrapping over the $S^2$ adds the twist $2\pi m$ to any loop in the fiber of the frame bundle over $M_m$, and therefore for $m$ odd interchanges even and odd loops.

The important observation is that if some noncontractible loop l in $p^{-1}(x) \approx O(s,t) \subset F$, where $s = 0,1$ and $t \geqslant 3$ can be contracted to a point in the total space F of the o.n. frame bundle over M (by going outside the fiber for instance), then the spin structure cannot be defined over M (c.f. (3.1)). This follows because noncontractible loops lift to open paths in $\tilde{F}$ which connect two distinct points (counterimages of the starting point in F ). If these points are kept fixed, the lifted path cannot be deformed into the closed lift of some contractible loop.

The $M_m$ for m odd are generic non-spin manifolds in four dimensions in the sense that given a 2-sphere in any manifold one can thicken it to a neighbourhood which is topologically a $R^2$-bundle over $S^2$. If for at least one $S^2$ this bundle is classified by odd m, then M is not a spin manifold.

The twisted $S^2$-bundle over $S^2$ is a manifold of that type. It can be given a nonsingular positive definite metric [ 78 ]

(1)
$$ds^2 = 3\Lambda^{-1}(1+v^2)\left\{\frac{1-v^2\cos^2\theta}{3+6v^2-v^4}(d\chi^2 + \sin^2\chi\, d\eta^2) + \frac{(1-v^2\cos^2\theta)\, d\theta^2}{(3-v^2)^2 - v^2(1+v^2)\cos^2\theta} + \right.$$
$$\left. + \frac{(3-v^2) - v^2(1+v^2)\cos^2\theta}{(3+v^2)^2(1-v^2\cos^2\theta)}\sin^2\theta\,(d\Phi - n\sin^2\frac{\chi}{2}\, d\eta)^2\right\} ,$$

where $0 \leq \chi \leq \pi$ , $0 \leq \theta \leq \pi$ , $(\chi, \eta + 2\pi j, \Phi + 2\pi k)$ are identified for $j,k \in Z$; $n = 4v(3+v^2)(3+6v^2-v^4)^{-1}$ , $v = -1-(2+a-b)^{-1/2} + [4-a+b+8(a-b)^{-1/4}(a+b)^{-1}]^{1/2}$ and $a = (\sqrt{2}+1)^{1/3}$ , $b = a^{-1} = (\sqrt{2}-1)^{1/3}$. This metric solves the Einstein equations

(2)
$$R_{\mu\nu} - 1/2\, Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0 .$$

The topological signature is $\tau = \int RR^* = 0$, the Euler number $\chi = \int RR^{**} = 4$, and spin structure does not exist. The total volume is

(3)
$$V = \int g^{1/2} d^4x = 48\pi^2(3-v^2)(1+v^2)^{-1}(3+v^2)^{-1}(3+6v^2-v^4)^{-1}\Lambda^{-2} ,$$

and the action

(4)
$$S = -(16\ )^{-1}\int (R-2\Lambda)g^{1/2} d^4x = (8\pi)^{-1}\Lambda V$$

is a little less negative than the action $2\pi\Lambda^{-1}$ of $S^2 x S^2$. This latter space with $\tau = 0$, $\chi = 4$ and the metric (1), where v=0, also solves (2), and admits the unique spin structure.

Apart from the two spaces above, only three other gravitational instantons are known explicitly: $S^4$, CP(2), and $T^4$ . The 4-spheres $S^4$, with $\tau = 0$ and the metric ( 1 ) where v=1 and $\varphi$ is $4\pi$ identified rather then $2\pi$ [ 78], admits a unique spin structure.

The flat $T^4$, which admits 16 inequivalent spinors, and CP(2), which admits a $spin_c$ structure, but no spin structure, will be discussed separately in next sections. Here let us mention only that to obtain a pseudoriemannian model without spin-structure, one can consider [79] a real $C^\infty$ function on CP(2) with only isolated critical points. With these points removed, its gradient is a nonvanishing vector field U which can be used to define a metric

$$g'(W,V) = g(W,V) - 2g(U,W)g(U,V)g(U,U)^{-1}$$

of signature $(-,+, \ldots, +)$. The resulting space is orientable and time orientable without spin structure.

The examples [80] of <u>compact</u> oriented and time oriented Lorentz manifolds without spin structure are 2k copies of $CP^2$ of topological signature 2k and Euler number 6k. By adding 2k-1 handles of topology $S^3 xR$ to connect the manifold, and then still k+1 handles, the Euler number is reduced to zero and the Lorentz metric exists. Since the signature is preserved and not divisibile by 16 (when $k<8$), the resulting manifold has no spin structure.

In order to complete the list of gravitational instantons let us mention the infinite family $V_{2m}$ of compact fourdimensional algebraic submanifolds of CP(3), satisfying the homogeneous polynomial equation of degree 2m. They admit spinors and $\tau(V_{2m}) = -16\binom{n+1}{3}$ [81].

The $V_4$ is the famous K(3) surface, known to admit a Ricci flat metric, whose explicit form (without Killing vectors and involving 58 parameters) has not been obtained yet.

3. CP(2) and $\text{spin}_C$ structure.

The complex two-dimensional (real fourdimensional) manifold CP(2) consists of complex one-dimensional linear subspaces in $C^3$:

$CP(2) \approx C^3/\sim$ , where for any $\lambda \in C \setminus \{0\}$ $(z_1, z_2, z_3) \sim (\lambda z_1, \lambda z_2, \lambda z_3$

Since U(3) acts in this set transitively, and the stability subgroup is U(2)xU(1) there is a diffeomorphism

(1) $$CP(2) \approx U(3)/U(2)xU(1)$$

Different partitions of (1)

(2) $$CP(2) \approx S^5/U(1) \qquad \text{(Hopf fibering)}$$

and

(3) $$CP(2) \approx {}^{SU(3)/Z_3}\big/_{U(2)}$$

are useful to discuss tensorial (i.e. metric, curvature, connection), and spinorial properties of CP(2) respectively.

The 5-sphere $S^5 = \left\{ \sum_{i=0}^{2} Z_i \bar{Z}_i = 1 \right\}$ in $C^3 \approx R^6$ is invariant under transformations $Z_i \to e^{i\beta} Z_i$. The set of orbits is a quotient manifold covered by coordinates $z_j = Z_j/Z_0$ , $j=1,2$ , with exception of $\{ Z \mid Z_0 = 0 \} \approx CP(1) \approx S^2$; hence CP(2) can be thought of as a compactification of $R^4$ by a 2-sphere at infinity. The (Fubini-Study) metric on CP(2)

(4) $$ds^2 = (1 + z_j \bar{z}_j)^{-1} dz_k d\bar{z}_k - (1 + z_j \bar{z}_j)^{-2} d\bar{z}_j dz_k z_j \bar{z}_k \qquad \text{(sum over } j,k)$$

can be obtained projecting along orbits of U(1) the standard U(1)-invariant metric on $S^5$ (coming from $dz_i d\bar{z}_i$ in $C^3$) :

$$ds^2 + (d\tau - A)^2$$

where $Z_o |Z_o|^{-1} = e^{i\tau}$, and

(5) $$A = 1/2\ i(1 + z_j \bar{z}_j)(\bar{z}_k dz_k - z_k d\bar{z}_k)$$

In coordinates $x_1 + ix_2 = 1/2\ z_1$ ; $x_3 + ix_4 = 1/2\ z_2$ (4) becomes the Eguchi-Freund [ 82 ] gravitational instanton

(6) $$g_{\mu\nu} = (x^2 + 1/4)^{-1} \delta_{\mu\nu} - (x_\mu x_\nu + \tilde{x}_\mu \tilde{x}_\nu)(x^2 + 1/4)^{-1}$$

where $x^2 = x_\nu x_\nu$, $\tilde{x}_\mu = C_{\mu\nu} x_\nu$ and

$$C_{\mu\nu} = \begin{bmatrix} . & 1 & . & . \\ -1 & . & . & . \\ . & . & . & 1 \\ . & . & -1 & . \end{bmatrix}$$

The metric (4) (or (6)) is Einstein $g_{\mu\nu} = 1/6\ R_{\mu\nu}$, and solves the Einstein equations with a cosmological constant $\Lambda = 6$

(7) $$R_{\mu\nu} - 1/2\ Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

(had one started with $S^5$ of radius $6\Lambda^{-1}$, one could solve eq.(7) for any $\Lambda$ ). The Weyl tensor is selfdual (since $\Lambda \neq 0$ the Riemann tensor cannot) and its Weyl spinor is of Petrov type D i.e.

(8) $$\Psi_{A'B'C'D'} = \xi_{(A'}\xi_{B'}\varsigma_{C'}\varsigma_{D')}$$

The total volume is $1/2\pi^2$, the index $\tau = \int RR^* = 1$ and the Euler characteristic $\chi = \int RR^{**} = 3$ . Since the maximal isometry group is $SU(3)/Z_3$, CP(2) is an attractive candidate for an internal Kaluza-Klein space; however it is not a spin manifold [ 48 ].

To see this let us follow the reasoning of previous section and find the odd loop in the fibre , which can be contracted to a point in the total space of the bundle of o.n. frames over CP(2), (CP(2n) in general). From a formula similar to (3) follows that $SU(2n+1)/Z_{2n+1}$ can be considered as the total space of the bundle of unitary frames in CP(2n). Because of the inclusion $U(2n) \subset O(4n)$, this is a subbundle of the bundle of o.n. frames in CP(2n) considered as a 4n-dimensional real manifold with a riemannian metric. Since $\pi_1(SU(2n+1)/Z_{2n+1}) = Z_{2n+1}$, the noncontractible loop 1, $[1] \in (2n+1)\,\pi_1(U_{2n})$, being (2n+1) power of some loop, must be contractible in $SU(2n+1)/Z_{2n+1}$ . It remains only to find at least one loop as above, such that it is also noncontractible as a loop in the group O(4n).

The example is the loop

(9) $$\langle 0,2\pi\rangle \ni s \xrightarrow{1} \begin{bmatrix} z & , 0, & \dots, & 0 \\ 0 & , 1, & \dots, & 0 \\ \vdots & & & \\ 0 & , 0, & \dots, & 1 \end{bmatrix} \in U(2n), \quad z = \exp[(2n+1)s]$$

which is noncontractible in U(2n) and $[1] \in (2n+1)\,\pi_1(U(2n))$. This can be easily seen from the set diffeomorphism (which is not a group isomorphism)

$$U(1) \times U(2n) \ni (\lambda, A_{ij}) \longleftrightarrow A(\lambda)_{ij} \in U(2n)$$

where

(10) $$A(\lambda)_{ij} = \begin{bmatrix} \lambda A_{1,1}, & A_{1,2}, & \dots, & A_{1,2n} \\ \lambda A_{2,1}, & A_{2,2}, & \dots, & A_{2,2n} \\ \vdots & \vdots & & \vdots \\ \lambda A_{2n,1}, & A_{2n,2}, & \dots, & A_{2n,2n} \end{bmatrix}$$

$\lambda$, $A_{ij} \in C$ ; $i,j = 1, \ldots 2n$ ; $|\lambda| = 1$ , $\sum_k A_{ik}\overline{A}_{jk} = \delta_{ik}$, $\det(A_{ij}) = 1$ .

As an O(4n) transformation 1 is still noncontractible, and generates the $(2n+1)2\pi$ rotation in the (1-2) plane of $R^{4n}$ (i.e. in the first component of $C^{2n}$). This completes the proof.

Obviously, the lack of spin structure on CP(2) prevents us from aplying powerful topological identities in which spinors are involved. It is known that by ignoring this point one is led to selfcontradictions; these can serve also as a "physical" proof that $w_2(CP(2)) \neq 0$. For instance, for the metric (6)

(11)
$$\int R_{\mu\nu\rho\sigma} {}^{*}R^{\mu\nu\rho\sigma} \sqrt{g}\, d^4x = 48\pi^2$$

Then by the index theorem, the difference between the number of righthanded and lefthanded helicity zero-eigenmodes of the Dirac operator seems to be noninteger [ 48 ]

(12)
$$n_R - n_L = -\frac{1}{384\pi^2} \times 48\pi^2 = -\frac{1}{8}$$

A similar inconsistency appears [ 126 ] when "introducing" the action functional with chiral fermions and topological terms on $R^{3,1} \times CP(2)$, and discussing the resulting reduced fourdimensional theory.

However, the charged spinors can be defined on CP(2); i.e. the $spin_c$-structure exists[ 84,85 ]. The idea is to introduce a U(1) gauge potential, which cancels the unwanted sign factor of the parallel transport of spinors. Therefore instead of the "integer" Dirac condition, $F_{\mu\nu}$ should satisfy

(13)
$$\int_{\partial S} A_\mu \, dx^\mu = \int_S F_{\mu\nu} \, d\Sigma^{\mu\nu} = \frac{2\pi}{e}(n+1/2) \quad .$$

Given by (5) A is a perfect candidate for a U(1) gauge field on CP(2). It comes from the standard metric connection on $S^5$ which is a total space of nontrivial Hopf fibering [ 86 ]. The CP(2), and CP(n) in general, are most important examples of Kähler manifolds. To appresciate their properties, let us recall the appropriate definitions. A Kähler manifold is a complex manifold equiped with:

1° almost complex structure J i.e. type (1,1) tensor field J such that $J^2 = -1$, considered as a linear operator on vectors;

2° Kähler metric i.e. the riemannian, hermitian (g(JU,JV)=g(U,V)) metric g, such that the corresponding 2-form K(U,V)=g(JU,V) is closed dK=0 .

In local coordinates $g = g_{jk} dz_j d\bar{z}_k$, where $g_{jk}$ is a positive definite hermitian matrix, and its Kähler form is

(14) $$K = 1/2\, i g_{jk} dz_j d\bar{z}_k \ .$$

On CP(n) the standard Kähler metric is

(15) $$g_{ij} = \frac{\partial^2 \ln(1 + z_i \bar{z}_i)}{\partial z_j \, \partial \bar{z}_k} ,$$

where $z_j = Z_j/Z_o$ and $Z_j$, j=0, ..., n are coordinates in $C^{n+1}$. It can be seen that (4) is in fact (proportional to) the Kähler metric on CP(2), and the curvature F=dA of A (5) is its Kähler form [ 86 ].

Being closed, $F$ solves the Maxwell equations. Because the volume form on $CP(n)$ is proportional to $F\wedge \ldots \wedge F$ (2n factors) the dual of $F$ is proportional to $F\wedge .. \wedge F$ (2n-1 factors), and $F$ is source-free $d{*F} = 0$ .

In the $n = 2$ case $F$ is selfdual and its energy-momentum tensor vanishes. Hence the Einstein equations still hold in presence of $F$ and $CP(2)$ is an example of a 'gravito-electromagnetic' instanton with the second Chern number $\int F \wedge F = 4\pi^2$. To avoid inconsistencies in presence of charged scalars, one puts the Dirac condition $\frac{e}{2\pi}\int F d\Sigma = n \in Z$ . Now the fermions with halfinteger charge are well defined on $CP(2)$ . The additional contribution to (12 )

(16) $$\frac{e^2}{16\pi^2}\int F\, F^* \sqrt{g}\, d^4x = 1/2(n+1/2)^2$$

makes $n_R - n_L$ integer, as it should be.

## 4. Twisted spinors.

In this section various physical consequences of the mathematically inequivalent spin structures, known in the literature, will be collected.

As a model space, we shall consider the generalized torus $M = T^n \times R^m$, where

(1) $$T^n = S^1 \times \ldots \times S^1 = \{(z_1, \ldots, z_n) \in C^n \mid |z_i| = B^i > 0\}.$$

M is a flat parallelizable multiply connected manifold with $\pi_1(T^n) = Z \times \ldots \times Z$ (n factors), admitting a pseudoriemannian metric of arbitrary signature (t,s), $t + s = n + m$; and time and space orientations.

Since $H^1(T^n, Z_2) = Z_2 \times \ldots \times Z_2$ (n factors) there are $2^n$ inequivalent spin structures $(\tilde{F}, \eta_{\underline{\epsilon}})$ on M, all with the trivial bundle $\tilde{F} = M \times Spin_0(t,s)$ and inequivalent maps $\eta_{\underline{\epsilon}}$ only, where $\underline{\epsilon}$ is a multiindex $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)$, $\epsilon_i = 0$ or 1, for $i = 1, \ldots, n$. To describe $\eta_{\underline{\epsilon}}$ let us denote by $\iota$ one of the obvious inclusions $U(1) \hookrightarrow SO_0(s,t)$ coming from identification of $U(1) \approx SO(2)$ with a subgroup of $SO_0(s,t)$, where to avoid inessential complications, we assume $\max(t,s) \geqslant 2$. Then $\eta_{\underline{\epsilon}}$ are given by

(2) $$\eta_{\underline{\epsilon}}((z,x), h) = ((z,x), \tilde{\eta}_{\underline{\epsilon}}(z)\, \varrho(h))$$

where

(3) $$\tilde{\eta}_{\underline{\epsilon}}(z) = \iota\left(\prod_{i=1}^{n} z_i^{\epsilon_i}\right) \in SO_0(s,t) .$$

These inequivalent spin structures can be most easily visualised by considering M as a box in $R^{n+m}$ with periodically identified points in the first n directions.

The map $\tilde{\eta}_{\underline{\epsilon}}$ determines a (n + m)-bein in M, which is twisted with respect to the standard one, in such a way that it performs the full $SO_0(s,t)$-rotation by 2 along every direction for which $\epsilon_i = 1$ .

Alternatively, by performing a local $Spin_0(s,t)$ rotation, which covers $\tilde{\eta}_{\underline{\epsilon}}$ , the inequivalent spinors can be viewed as obeying antiperiodic boundary conditions along the $\epsilon_i = 1$ directions.

In the simplest four-dimensional cases: the trivial compact riemannian gravitational instanton (c.f. (4,2)) $T^4$, the space-time $T^3 \times R$ with the compactified space part, and the 'toy' space $S^1 \times R^2 \times R$ there are 16, 8, and 2 inequivalent spin structures respectively.

The space $T^4$ has been employed in the t'Hooft [ 53 ] treatment of confinement, which is similar to the inequivalent spinors problem. Twisted versions of tori, including the Klein bottle and RP(2), have been also considered [ 20 ]. Exotic spinors on the conformally compactified Minkowski space will be discussed in detail in the next chapter. However, the physical implications of mathematically inaquivalent spinors have been most often discussed in the space $S^1 x R^2 x R$ .

Petry [17] has assumed that the electrons of the superconducting ring are confined to live in the space-time of topology $S^1 \times R^2 \times R$. In his model the relative nontrivial Lorentz twist is replaced by the nontrivial rotation of the phase. This can be thought of as introducing the $\text{spin}_c$ structure. Then the only difference between normal and exotic spinors appears in the covariant derivatives

(4) $\nabla_\mu = \partial_\mu$ or $\nabla_\mu' = \partial_\mu - (\lambda^{-1}\partial_\mu \lambda)$, where $\lambda : M \longrightarrow C$, $|\lambda| = 1$ cannot be gauged away. It turns out that under some technical assumption, neglecting the quantum fluctuations of the electromagnetic field, and assuming the Meissner effect inside the superconductor, the two inequivalent spin structures leads to a novel explanation of the observed quantization (the famous factor 2 included) of the magnetic flux trapped in the ring. Usually it is described by the pairing hypothesis based on the B.C.S. dynamical theory [87]. Also the correct periodicity of the Josephson junction current can be explained.

It should be noted, that in the Petry model the description of the mechanism of superconductivity is not intended, but rather the periodicity effect and the extremal precision of the measurement of the factor 2e in superconductors and Josephson junction is attributed to the nontrivial topology of the system. A quite similar point of view has been also proposed in recent discussions of the quantized Hall effect, where the stability and precision of the measurement relates to the topological properties [88].

Whereas the laboratory systems are always embedded in flat surroundings and the nontrivial topology of rings or circuits has to be only approximate, it is difficult to answer on which grounds in the whole Universe some particular type could be distinguished and others inequivalent rejected. This yields the possibility of more advanced speculations, which are rather difficult to be confirmed experimentally but may offer a new insight and influence other domains like the quantum field theory. In the gravitation theory and cosmology the topological considerations are essential, and several solutions of the Einstein's equations admit multiply connected space-times [ 39 ]. Before discussing the most challenging quantum field theory in curved space-time, some effects of the global nontrivial topology can be seen even on metrically flat manifolds(⋕).

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(⋕) The true mulitple connectivity of the space has to be distinguished from the completely artificial string-like singularity caused by adopting a particular coordinate system, which does not cover the whole space. For instance, working in the cylindrical coordinates, the antiperiodic angle conditions for spinors are excluded if the space remains simply connected. Therefore, the different spectra of hydrogen atoms due to twisted spinors (c.f.[ 90 ]) are not possible in this case.

Several consequences are known for boson fields : the Casimir effect, the generation of topological mass, finite temperature effects, the suppression of spontaneous symmetry breaking, and relations to the confinement problem [18, 91, 93, 94, 95,53]. For spinors interesting effects have been also predicted.

Assuming that spinors exist, Isham [96] discussed various possibilities to accomodate inequivalent spinors in the functional integral formalism. The Dirac lagrangian in a gravitational background (3.4,8), written in the natural basis, takes the form

(5) $$\mathcal{L}^E(\psi) = \left\{\frac{i}{2}(\bar{\psi}\gamma_a\nabla_\mu\psi - \nabla_\mu\bar{\psi}\gamma_a\psi)E^{a\mu} - m\bar{\psi}\psi\right\} \det(E),$$

where $E = E_a^{\mu}$ is the n-bein field, $\nabla_\mu\psi = \partial_\mu\psi + i\Gamma_\mu\psi$, $\nabla_\mu\bar{\psi} = \partial_\mu\bar{\psi} - i\bar{\psi}\Gamma_\mu$, $\Gamma_\mu = \frac{i}{8}(E_{b\alpha}E^{\alpha}_{a,\mu} + E_{b\alpha}E_a^{\nu}\{^{\alpha}_{\nu,\mu}\})[\gamma^a,\gamma^b]$, and $\{^{\alpha}_{\mu,\nu}\}$ are Christoffel symbols. Next assume that the generating functional

(6) $$Z_E = \int D\bar{\psi}\, D\psi \quad \exp i\int\mathcal{L}_E(\psi)$$

is given a meaning by some method. On simply connected $M$, $Z_E$ is invariant under the Lorentz gauge rotation $L$ of a vierbein $E \longrightarrow E' = EL$. In fact, $L$ can be lifted to a Spin transformation and then shifted on spinor fields, with no effect on $Z_{E'} = Z_E$, since the integration measure is invariant. This is not the case for multiply connected $M$ and $L$, which can not be lifted to a continuous spin transformation. Therefore $Z_{EL} \neq Z_E$, and in general $Z_E^k \neq Z_E$, where $k$ labels the inequivalent spin structures which are in 1-1 correspondence with $H^1(M,Z_2)$.

Before presenting different physical effects of $Z_E^k$, let us note that in principle the gauge invariance can be restored in several ways [ 96 ]. One can write the weighted sum

(7) $$Z_E^{\chi} = \sum_k \chi(k) Z_E^k \quad ,$$

where the character $\chi : H^1(M,Z_2) \longrightarrow \{-1,1\}$ plays the rôle analogous to $\theta$ labelling different vacua of the instanton field [ 97 ]. For inequivalent spinors the supression of higher k 's does not seem to hold, while the tunnelling phenomenon could be found some interpretation in the Petry model.

Another possibility is to work with the product

(8) $$Z_E = \prod_k Z_E^k$$

which is equivalent to summing up Lagrangians of inequivalent spinors. They are therefore treated as a multiplet of distinct fields in the theory. There are also intermediate possibilities, which combine ( 7 ) and ( 8 ).

The vacuum expectation values of the energy-momentum tensor $T_{\mu\nu}$ have been computed both by the image method and $\zeta$-function regularization, for two inequivalent spinors on $M = S^1 \times R^2 \times R$ [ 20 ]. The exotic spinor is assumed to be antiperiodic on $S^1$ of a circumference B. The first regularization method amounts to the rejection of n=0 term in the Green function expansions

(9) $$G_{ren}(x^{\mu}, x'^{\mu}) = \sum_{0 \neq n \in Z} G_o\left(x^{\mu}, x'^{\mu} + \delta_1^{\mu} nB\right) \times \begin{cases} 1 & \text{(normal)} \\ (-1)^n & \text{(twisted)} \end{cases}$$

where

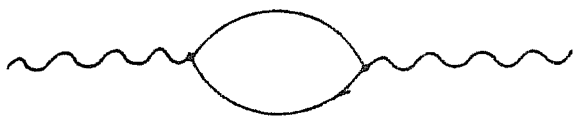
(10) $$G_o(x^{\mu}, x'^{\mu}) = \frac{i}{(2\pi)^2}\left[(x-x')^{\mu}(x-x')_{\mu} + i0\right]^{-1} .$$

Then

(11) $$T_{\mu\nu} = \left[\frac{-2i\,\partial^2}{\partial x^{\mu}\,\partial x^{\nu}} G_{ren}(x,x')\right]_{x=x'} = \frac{\pi^2}{45B}\begin{bmatrix}1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 3\end{bmatrix} \times \begin{cases} 1 & \text{(normal)} \\ -7/8 & \text{(twisted)}\end{cases}$$

It turns out that the twist lowers the energy of spinors. Moreover the normal (and twisted) Majorana spinors have the vacuum expectation value of T minus twice as large as the normal (and twisted) scalar fields on $S^1 \times R^3$, which is a typical supersymmetric outcome (c.f. sect. 5).

It is not yet known whether the field theories remain renormalizable when put on topologically nontrivial spaces. Ford [ 19 ] has performed a detailed study of spinor electrodynamics on $S^1 \times R^2 \times R$, where the one loop corrections do not spoil renormalizability. Again, the method relies upon different periodicity conditions for inequivalent spinors. In all momentum-space computations one integral is replaced by a sum, which runs over integers or halfintegers respectively. All quantities are computed by dimensional regularization in the same way for both kinds of spinors .
The one loop correction to the vacuum polarization tensor



(12) $$\Pi_{\mu\nu}(q) = \frac{e_B}{2\pi^3 B}\sum_{n\in Z}\int d^3p\ Tr\left[\frac{\gamma_{\mu}(\not p + m)\gamma_{\nu}(\not p + m - \not q)}{(p^2-m^2+i0)[(p-q)^2-m^2+i0]}\right]$$

gives the finite part in the form $\Pi^{o}_{\mu\nu}(q) + \Pi^{1}_{\mu\nu}(q)$ ,

where the first term is identical to the correction in the Minkowski space-time (B independent), and the second $\Pi^1_{\mu\nu} \xrightarrow{B\to\infty} 0$ .
For twisted spinors $^t\Pi_{\mu\nu}(B) = 2\Pi_{\mu\nu}(2B) - \Pi_{\mu\nu}(B)$ , $^t\Pi^\circ_{\mu\nu} = \Pi^\circ_{\mu\nu}$ and the B-dependent part is different: $^t\Pi^1_{\mu\nu} \xrightarrow{B\to\infty} -\Pi^1_{\mu\nu}$ .
It turns out that for normal spinors the one loop corrections generate the imaginary mass term of the photon and the tachyonic character can spoil the causality. This is not the case for twisted spinors; however both types mixed in any way again lead to tachyonic modes. Ford has computed also the vacuum energy for massive Dirac spinors on $S^1 \times R^3$ and confirmed the results of De Witt et al. In the $B \to \infty$ limit the vacuum energy density goes like $1/45B^2$ for normal and $-7/360B^2$ for twisted spinors.

It has been proposed that inequivalent snimors can be related to the different flvours of fermions [ 98 ]. In order to be nearest to the observed number of flavours, the quantum chromodynamics was considered in the space $R \times T^3$. The effects of eight inequivalent spinors were summed in the one-loop corrections to the gluon propagator, which again exhibits $m^2 < 0$ modes.

Let us observe, that the possibility of inequivalent spin structures appears also when momentum cut-off (box in Fourier space) is introduced to regularize divergent quantities. One can speculate also about the possibility of adding toroidal flat, but compact, additional dimensions in the dimensional regularization scheme.

## 5. Spin structures in Kaluza-Klein and supergravity theories.

In modern higher dimensional supergravity and Kaluza-Klein theories, or combinations of both, there is enough room to consider various nontrivial space-times (c.f.[99,100,101,102,103,104 ]). Problem of spinors seems to be one of the most acute in "realistic" higher-dimensional theories, and we shall discuss only two basic aspects of it : the existence and inequivalent spin structures. In supergravity one works from the begining with spinors. Obviously, this means that the underlying space-time carries some sort of spin structure. For a simple $N = 1$ supergravity this cannot be neither the generalized, nor $spin_c$ structure, since most often spinors are Majorana in order to balance the number of bosonic and fermionic degrees of freedom. In the $N = 2$ case, spinors can in general build pseudo-Majorana multiplets and the generalized spin structure can be admited. The global existence of spinors in the Kaluza-Klein theories is a reasonable physical postulate. The space-time is assumed there to be a direct product $M \times Q$, where $M$ is a four-dimensional maximally symmetric Lorentzian space-time which we observe (Minkowski, anti-, or de Sitter) and $Q$ is an internal riemannian space, "spontaneouosly" compactified, and of very small volume ( the inverse Planck mass being the typical order of length). As we already discussed, the spin structure exists on $M \times Q$ (and $M$) iff it exists on $Q$ (this is not immediate for models with twisted M x Q products). Because of (2.3, 3) and the form $\not{D}^{M\times Q} = \not{D}^{M} + \not{D}^{Q}$ of Dirac operator, it follows that there will be an infinite tower of multiplets of $2^{d/2}$ spinors on M (d = dim Q), with masses equal to eigenvalues of $\not{D}^{Q}$. (All classical parameters are subject to quantum corrections, rather difficult to predict). Since starting from pure gravity and fermions in d+4 it is difficult [ 102 ] to obtain the observed chiral asymmetry in 4 dimensions, often the extra gauge fields are also intoduced in d+4 dimensions [105]. Then one needs only a generalized spin structure to exist, but not always physical relations between charges of bosons and fermions, or their multiplicities, arise [106 ]. In supersymmetric Kaluza-Klein theories the upper bound $d \leq 7$ comes from the inconsistency of coupling the spin higher than two. Also d=7 is minimal for obtaining the phenomenological SU(3) x SU(2) x U(1) gauge symmetry purely from isometries of Q.

In [ 107 ] the existence of spin structures on the Witten [101] spaces $M^{pqr} = G/H$, where G = SU(3) x SU(2) x U(1), H = SU(2) xU(1)xU(1), and p,q,r determine the details of the embedding, is discussed. The method is based on the condition II (c.f. (3.1)), which is necessary for a spin structure to exist, and the conjecture is made, that it is in fact sufficient for homogeneous spaces. The $S^2$ representing the generator of $\pi_2(M^{p,q,r})$ is explicitly constructed, and the transitive action of the isometry group of $M^{p,q,r}$ (at least SU(3) x SU(2) x U(1)) is used to spread out the orthonormal frame over $S^2$ but a single point x . This determines the local parallelization. The loops wrapping $S^2 \setminus \{x\}$ induce loops in the orthonormal frame bundle F, which by continuity arguments can be extended to whole $S^2$, by adding some loop in $p^{-1}(x) \subset F$. The noncontractability of this loop implies the nonexistence of spin structure. In conventions of [107]; this holds when p/s is even, where s is the highest common factor of p and q. For instance $M^{0,1,0} \approx CP(2) \times S^3$ is without spin structure (c.f.(4.3)). As follows, on homogeneous spaces the rigorous cohomological arguments can be simplified. For CP(2) also the group theoretical justification has been presented [99] .

The original Kaluza-Klein internal space $S^1$ , as well as many other candidates of the form $Q = Q' \times S^1$ are multiply connected and admit nonequivalent spin structures. These spaces are unfavoured by Einstein equations, without presence of matter. However, many other possibilities exist, for instance in conventions of [107], $\pi_1(M^{p,q,r}) = Z_l$, where $l = (3p^2 + q^2)/k$, and k is a highest common divisor of 2pr, rq and $(3p^2 + q^2)$. Provided that the spin structure exists, there will be only one for odd l, and exactly two for even l. The number of local Killing spinors determines the supersymmetries. In general, they are suppressed by the global ("boundary") periodicity conditions. At this point, the twisted spinors could in principle increase the number of globally allowed Killing spinors, by the possibility of antiperiodic conditions (unfortunately this is not the case for $M^{p,p,r}$ ).

The important observation [ 18 ], concerning inquivalent spinors, is that the numbers of inequivalent spin structures and different 'twisted' scalar field configuration spaces are both labelled by $H^1(M,Z_2)$. Chockalingham and Isham have shown [108] that twisted scalars and spinors can be meaningfully combined to built the twisted supersymmetric multiplet. This requires that the sum of twists of individual fields in the multiplet vanishes modulo $Z_2$ in order the Lagrangian density be a normal function. Also the fermionic infinitesimal group parameter $\epsilon$ has to be twisted. For instance, The Lagrangian of N = 1 supergravity multiplet [109 ] containing vierbein $E_a$, 3/2-spin Majorana spinor $\psi_\mu$, auxiliary scalars M, N and vector $b_\mu$ fields, does not impose any condition on the twists. The supersymmetry transformations with the parameter $\epsilon$ restrict the twists of different fields to be $t(\epsilon) = t(\psi_\mu)$ and $t(M) = t(N) = t(b_\mu) = 0$. In the quantized theory the ghosts $C^\rho$, $C^{ab}$, and (complex spinor) C must be included [ 110 ]. Because of the B.R.S. symmetry $t(C) = t(\psi_\mu)$ and $t(C^\rho) = t(C^{ab}) = 0$ . Coupling the N = 1 supergravity [111,112 ] to the Wess-Zumino scalar multiplet $(A,B,\chi,F,G)$, where A,B,F,G are scalars and $\chi$ is a Majorana spinor, yields $t(\bar{\psi}_\mu) + t(G) + t(\chi) = 0$ from the consistency of the trilinear term in the Lagrangian. It turns out that, including the supersymmetry transformations, there are left many possibilities, which have to be consistent only with $t(\epsilon)=t(\psi_\mu)$ and $t(A) = t(B) = t(G) = t(\chi) + t(\epsilon)$ . Also the twisted vector multiplets coupled to the N=1 supergravity can be constructed [108 ].

# 5 SPINORS ON CONFORMALLY COMPACTIFIED MINKOWSKI SPACE-TIME.

## 1. Conformal transformations of spin structures

In this section we shall investigate the way, in which spin structures transform under the conformal mappings of manifolds [129]. Since the metric is explicitly involved in the definition of spinors, the isometries are a natural class of transformations to be considered. Also the conformal transformations can be easily included, which is of some relevance for various massless systems in physics.

Let $M$ and $M'$ be n-dimensional, connected, pseudoriemannian manifolds with metric tensors $g$ and $g'$ respectively, both of signature $(s,t)$. The map $f: M \to M'$ is conformal if the pulled back metric $g'$ is conformally related to $g$,

(1) $$f^*g' = \Omega^2 g$$

where $\Omega: M \to R$ is a conformal factor. The group of all conformal (isometric if $\Omega=1$) transformations of $(M,g)$ onto itself is called conformal (isometry) group of $M$ and denoted by $Conf(M)$ $(Isom(M))$ respectively. Let $Conf_o(M)$ and $Isom_o(M)$ be connected components of the identity.

There is a natural lift of $f$ to a bundle map $\hat{f}$ between principal o.n. frame bundles over $M$ and $M'$ respectively, given by

(2) $$\hat{f}(E_x) = \Omega_x^{-1} \circ f_* \circ E_x$$

where the o.n. frame $E_x$ at $x$ is identified with the isometry

$E_x : R^{s,t} \longrightarrow T_x M$ of the standard pseudoeuclidean space $R^{s,t}$ onto the tangent to M space at $x \in M$. The $\hat{f}(E_x)$ given by ( 2 ) is a well defined frame at $f(x) \in M'$ (i.e. isometry $R^{s,t} \longrightarrow T_{f(x)}M'$). The map $\hat{f}$ is naturally equivariant with respect to the actions of O(s,t) on F and F' given by

(3) $$F \times O(s,t) \ni (E,h) \longrightarrow E \circ h \in F ,$$

and similarly for F'. This uniquely determines transformations of tensorlike objects. However, in order to determine transformations of spinors, also the analogous map between spin structures, compactible with $\hat{f}$ has to be given.

In order to include the orientation-reversing conformal maps, or to work from the begining with not necessarily oriented manifolds, we consider the full O(s,t)-bundles F and assume that M and M' are pin manifolds. We are interested whether the conformal map f and the bundle map $\hat{f}$ can be lifted to a bundle map $\tilde{f}$ between spin structures over M and M' respectively.

(4) $$\begin{array}{ccc} \tilde{F} & \overset{\tilde{f}}{\dashrightarrow} & \tilde{F}' \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta'} \\ F & \overset{\hat{f}}{\longrightarrow} & F' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ M & \underset{f}{\longrightarrow} & M' \end{array}$$

The diagram ( 4 ) has to commute.

To answer this question we shall introduce some 1-cocycle, which represents a class in $H^1(M,Z_2)$. Let $U_\alpha$ and $U'_\alpha$ be simple coverings of M and M', and let for any $\alpha$ there exist $\hat{\alpha}$ such that $f(U_\alpha) \subset U'_{\hat{\alpha}}$.

Let $\sigma_\alpha' : U'_\alpha \to \tilde{F}'$ be local sections with transition functions $\tilde{\varphi}' : U_\alpha \cap U_\beta \to Pin(s,t)$. Then $\sigma'_\alpha \equiv \eta' \circ \tilde{\sigma}'_\alpha$ are corresponding sections in F' with transition functions $\varphi'_{\alpha\beta} = \rho(\tilde{\varphi}'_{\alpha\beta})$, where $\rho$ is the standard 2:1 covering map $\rho : Pin(s,t) \longrightarrow O(s,t)$.

Now set

(5) $$\sigma_\alpha(x) = \hat{f}^{-1} \circ \sigma'_{\hat{\alpha}}(f(x)) ,$$

which makes sense because $\hat{f}$ can be inverted on $p^{-1}(f(U_\alpha))$ provided sufficiently fine covering is chosen. Then $\sigma_\alpha = \varphi'_{\hat{\alpha}\hat{\beta}}\,\sigma_\beta$ on $U_{\alpha\beta}$. Choose now liftings $\tilde{\sigma}_\alpha$ such that $\eta \circ \tilde{\sigma}_\alpha = \sigma_\alpha$ and

$\tilde{\sigma}_\alpha = \tilde{\varphi}_{\alpha\beta}\,\tilde{\sigma}_\beta$ , $\rho(\tilde{\varphi}_{\alpha\beta}) = \varphi_{\alpha\beta}$.

Define

(6) $$k_{\alpha\beta}(x) = \tilde{\varphi}^{-1}_{\alpha\beta}(x)\,\tilde{\varphi}'_{\hat{\alpha}\hat{\beta}}(f(x))$$

From $\rho(k_{\alpha\beta})=1$ follows $k_{\alpha\beta} = \pm 1$ (constant on $U_{\alpha\beta} \equiv U_\alpha \cap U_\beta$).

Straightforward computation shows that

(7) $$(\delta k)_{\alpha\beta\gamma} \overset{df}{=} k_{\beta\gamma} k^{-1}_{\alpha\gamma} k_{\alpha\beta} = 1 ,$$

hence $k_{\alpha\beta}$ define a 1-cocycle with values in $Z_2$.

(8) LEMMA

The bundle map $\tilde{f} : \tilde{F} \to \tilde{F}'$ of the commutative diagramm ( 4 ) exists if and only if $k_{\alpha\beta}$ is coexact, $k_{\alpha\beta} = (\delta k)_{\alpha\beta} = k_\beta k_\alpha^{-1}$ for some $k_\alpha : U_\alpha \to Z_2$.

Proof.

$(\Leftarrow)$ . We shall construct $\tilde{f}$ assuming $k_{\alpha\beta} = k_\beta k_\alpha^{-1}$ .

Let $x = p(\tilde{E}_x) \in U_\alpha$ , $\tilde{E}_x = \tilde{\sigma}_\alpha(x)h_\alpha$ . Define

(9) $$\tilde{f}\ (\tilde{E}_x) = \tilde{\sigma}'_{\hat{\alpha}}(f(x))h_\alpha k_\alpha$$

and observe that this definition is " $\alpha$ " independent. Indeed, let $x = p(\tilde{E}_x) \in U_\alpha \cap U_\beta \neq \emptyset$, $\tilde{E}_x = \tilde{\sigma}_\beta(x)h_\beta$ and $h_\beta = \tilde{\varphi}_{\alpha\beta}^{-1}(x)h_\alpha$ .

Then

(10) $$\tilde{f}_\beta(\tilde{E}_x) = \tilde{\sigma}'_{\hat{\beta}}(f(x))h_\beta k_\beta = \tilde{\sigma}'_{\hat{\alpha}}\ \tilde{\varphi}'_{\hat{\alpha}\hat{\beta}}(f(x))\ \varphi^{-1}_{\alpha\beta}(x)k_\beta\ k_\alpha\ k_\alpha^{-1}h_\alpha = \\ = \tilde{\sigma}'_{\hat{\alpha}}(f(x))h_\alpha\ k_\alpha = \tilde{f}_\alpha(E_x).$$

The equivariance property $\tilde{f}(\tilde{E}_x h) = \tilde{f}(\tilde{E}_x)h$ is obvious, and $\tilde{f}$ is a well defined bundle map, which makes the diagram ( 4 ) commute

(11) $$\eta' \circ \tilde{f}(\tilde{E}_x) = \eta'(\ \tilde{\sigma}'_{\hat{\alpha}}(f(x))h_\alpha k_\alpha\ ) = \sigma'_{\hat{\alpha}}(f(x))\,\rho(h_\alpha k_\alpha\ ) = \\ = \hat{f}\ \hat{f}^{-1}\circ\ \sigma'_{\hat{\alpha}}(f(x))\,\rho(h_\alpha) = \hat{f}(\ \sigma_\alpha(x)\,\rho(h_\alpha\ )) = \\ = \hat{f}\circ\eta(\tilde{E}_x)\ .$$

$(\Rightarrow)$ Assume that f exists and ( 4 ) commutes. Define (unique) $k_\alpha : U_\alpha$ Pin(s,t) by $\tilde{f}(\tilde{E}) = \tilde{\sigma}'_{\hat{\alpha}}(f(x))h_\alpha k_\alpha(x)$ where $\tilde{E}_x = \tilde{\sigma}_\alpha(x)h_\alpha$ From (11) (commutativity of (4)) follows $\rho(k_\alpha(x)=1$, hence $k_\alpha = \pm 1$ is a 0-cochain, and from (10) $k_{\alpha\beta} = k_\beta k_\alpha^{-1}$ . Therfore $k_{\alpha\beta}$ is coexact, which finishes the proof of the lemma. □

(12) LEMMA

$\tilde{f}$ is unique up to a sign.

Proof.

Assume that there are $\tilde{f}$ and $\tilde{f}'$ such that (4) commutes. Again we can invert $\tilde{f}$ on $p^{-1}(f(U_\alpha))$ and define smooth, equivariant, fibre-preserving map of $p^{-1}(U_\alpha)$ onto itself. Because of properties of $\tilde{f}$ and $\tilde{f}'$ new sections defined as $\tilde{\sigma}'_\alpha = \tilde{f}^{-1} \circ \tilde{f}' \circ \tilde{\sigma}_\alpha$ satisfy

$$\eta \tilde{\sigma}'_\alpha = \hat{f}^{-1} \eta' \tilde{f}' \tilde{\sigma}_\alpha = \hat{f}^{-1} \hat{f} \eta \tilde{\sigma}_\alpha = \sigma_\alpha \,, \quad \text{hence}$$

$\tilde{\sigma}'_\alpha = \tilde{\sigma}_\alpha k_\alpha$ for $k_\alpha = \pm 1$. Moreover, the choise of the sign of $k_\alpha$ for some $U_\alpha$ determines that for others $U_\beta$ . Indeed on $U_\alpha \cap U_\beta \neq \emptyset$ $k_\alpha = k_\beta$ because of

$$\tilde{\sigma}'_\beta = \tilde{\sigma}_\beta \, k_\beta = \tilde{\sigma}_\alpha \, \tilde{\varphi}_{\alpha\beta} k_\beta \qquad \text{and} \qquad \tilde{\sigma}'_\beta = \tilde{f}^{-1} \tilde{f}' (\tilde{\sigma}_\alpha \tilde{\varphi}_{\alpha\beta}) = \tilde{\sigma}_\alpha k_\alpha \tilde{\varphi}_{\alpha\beta}$$

Hence there are exactly two liftings of $\hat{f}$ : $\tilde{f}$ and $-\tilde{f}$. □

Obstructions of the Lemma ( 8 ) can be evaded when one replaces the pin-structure $(\tilde{F},\eta)$ by some nonequivalent one. Given a nonzero element $[k_{\alpha\beta}] \in \check{H}^1(M,Z_2)$ the construction of the nonequivalent pin-structure $\tilde{F}(k)$ was described by Greub and Petry [ 63 ] . Moreover, all nonequivalent classes can be obtained in that way. We shall fill up some points of their construction, omitting again the continuity questions.

Functions

(13) $$\tilde{\varphi}^k_{\alpha\beta} : \; U_\alpha \cap U_\beta \longrightarrow Z_2 \quad \text{defined by} \quad \tilde{\varphi}^k_{\alpha\beta} = \tilde{\varphi}_{\alpha\beta} k^{-1}_{\alpha\beta}$$

fulfil the cyclicity property $\tilde{\varphi}^k_{\beta\gamma} (\tilde{\varphi}^k_{\alpha\gamma}) \tilde{\varphi}^k_{\alpha\beta} = 1$ and

(14) $$\rho (\tilde{\varphi}^k_{\alpha\beta}) = \rho (\tilde{\varphi}_{\alpha\beta}) = \varphi_{\alpha\beta} \,.$$

The total space $\widetilde{F}(k)$ with transition functions $\widetilde{\varphi}^k_{\alpha\beta}$ is constructed following Steenrod [ 113 ] as

(15) $$\widetilde{F}(k) = \bigcup_{\alpha} (U_\alpha \times \text{Pin})/\sim ,$$

where $(x,h)_\alpha \in U_\alpha \times \text{Pin}$ and $(x',h')_\beta \in U_\beta \times \text{Pin}$ are $\sim$ related iff

(16) $$x = x' \in U_\alpha \cap U_\beta \neq \emptyset \quad \text{and} \quad h = \widetilde{\varphi}^k_{\alpha\beta} h' \quad .$$

Denote by $[\quad]$ the equivalence class with respect to $\sim$ .

The action of the Pin group given by

(17) $$[(x,h)_\alpha]h' = [(x,hh')_\alpha] \quad , \quad h' \in \text{Pin}$$

does not depend on the representative element

(18) $$[(x,(\widetilde{\varphi}^k_{\alpha\beta})^{-1}h)_\beta]\, h' = [(x,hh')_\alpha]$$

Similarly the projection $p : \widetilde{F}(k) \longrightarrow M$

(19) $$p\,[(x,h)_\alpha] = x \in M$$

is well defined. The local sections $\widetilde{\sigma}_\alpha(x) = [(x,1)_\alpha]$ have precisely $\widetilde{\varphi}^k_{\alpha\beta}$ as transition functions.

(20) $$\widetilde{\sigma}_\alpha(x) = [(x,1)_\alpha] = [(x, \widetilde{\varphi}^k_{\alpha\beta})_\beta] = [(x,1)_\beta]\widetilde{\varphi}^k_{\alpha\beta} = \widetilde{\sigma}_\beta(x)\,\widetilde{\varphi}^k_{\alpha\beta} \quad .$$

Finally, the pin-structure map $\eta(k) : \widetilde{F}(k) \longrightarrow \widetilde{F}$

(21) $\eta_\alpha(k)\,[(x,h)_\alpha] = \sigma_\alpha(x)\,\rho(h)$ is also $\alpha$ -independent and $\rho$ -equivariant $\eta(k)\,\big([(x,h)_\alpha]h'\big) = \sigma_\alpha(x)\,\rho(hh') = \eta(k)\,\big([(x,h)_\alpha]\big)\,\rho(h')$ .

Moreover,

(22) $$\eta(k)\circ\widetilde{\sigma}^k_\alpha = \eta\circ\widetilde{\sigma}_\alpha = \sigma_\alpha \,.$$

From Lemmas ( 8 ), (12 ) and the observation that "$k_{\alpha\beta}$" computed for $\widetilde{F}(k)$ and $\widetilde{F}'$ is exact, follows our main result:

(23) PROPOSITION

Let the situation be that of a commutative diagram ( 4 ). A conformal map $f: M \to M'$ and a bundle map $\hat{f}: F \to F'$ can be lifted to a pin-structure bundles map $\tilde{f}: \tilde{F}(k) \to \tilde{F}'$ for exactly one (up to equivalence) pin-structure $\tilde{F}(k)$ where $k_{\alpha\beta}$ is given by (6). The lifting $f$ is unique up to a sign.

Consider now the case $M = M'$, $F = F'$ and the group $Conf(M)$ of conformal maps $f: M \to M$.

According to proposition 1 the possibility appears that there exist spin structure changing transformations. By continuity arguments this is not the case for the connected component of $Conf(M)$, but can happen, as we shall show in ch.(5), for some orientation reversing transformations.

Let us check now, that the assignment $f \to \tilde{\hat{f}}$ yields the representation of the double cover of $Conf_o(M)$ in the case of spin manifolds.

Consider fixed $x \in M$, then the assignment $f \to \hat{f}$ and restriction of $\hat{f}$ to the fiber $p^{-1}(x) \subset F$ determines some element $\Lambda(f) \in SO_o$ by

(24) $$\hat{f}(\sigma_{\alpha}(x)) = \sigma_{\hat{\alpha}}(f(x))\ \Lambda(f)\ .$$

We want to lift this assignment to $\tilde{\Lambda}(f) \in Spin_o$

(25) $$\begin{array}{ccc} & & \tilde{\Lambda}(f) \in Spin_o \\ & \nearrow & \downarrow \\ Conf_o \ni f & \longrightarrow & \Lambda(f) \in SO_o \end{array} \quad .$$

For any path connecting id with f in $Conf_o$ there is a unique lifted path in $Spin_o$ starting at id, the end point of which determines $\tilde{\Lambda}(f)$. Now for a closed path c at $id \in Conf_o$ the above construction yields $\tilde{\Lambda}(id) = \pm id|_{Spin_o}$.

In fact, we have a map $c \longrightarrow k_c \in Z_2 \subset Spin_o$ such that

i) $c + c' \longrightarrow k_c \; k_{c'}$

ii) c and c' are homotopic $\Rightarrow k_c = k_{c'}$,

i.e. the group homomorphism $\alpha : \pi_1(Conf_o) \longrightarrow Z_2$.

If $\alpha$ is trivial, there is obviously the unique lifting $f \to \tilde{\Lambda}(f)$, and then $f \to \tilde{f}$ preserves the group structure. If $\alpha$ is not trivial, take the universal covering $Conf_o^u$ of $Conf_o$ given by $\{(f,c)\}/\sim$, where $f \in Conf_o$, c is a path in $Conf_o$ connecting f with the identity and $(f,c) \sim (f',c')$ iff $f = f'$ and c and c' are homotopic. There is a natural action of $\pi_1(Conf_o)$ on $Conf_o^u$ coming from $(f,c)c' = (f, c \circ c')$, where c' is closed loop at $id|_{Conf_o}$. It can be seen that $\overline{Conf}_o = Conf_o^u \times_\alpha K$ is a double covering of $Conf_o$.

Repeating the previous construction of $\tilde{\Lambda}(f)$ for $f \in \overline{Conf}_o$ leads to a trivial group homomorphism $\alpha : \pi_1(\overline{Conf}_o) \longrightarrow Z_2$ and unambiguous homomorphic lifting $f \longrightarrow \tilde{f}$.

Results of this section generalize [114] for inequivalent spin structures.

## 2. Conformal properties of Dirac operator.

The map $\tilde{f}: \tilde{F} \to \tilde{F}'$ between spin structures over M and M' can be used to define the transformation (pull back) of spinor fields

(1) $$\Psi^{f}(\tilde{E}_x) = \Omega^{\omega}(x)\,(\tilde{f}^{*}\Psi)(\tilde{E}_x),$$

where the scaling degree $\omega$ will be fixed by the invariance of the massless Dirac equation. In local components

(2) $$\Psi^{f}_{\alpha} \equiv \tilde{\sigma}_{\alpha}^{*}\Psi^{f} = \Omega^{\omega}\tilde{\sigma}_{\alpha}^{*}\,\tilde{f}^{*}\,\Psi = \Omega^{\omega}(\tilde{f}\circ\tilde{\sigma}_{\alpha})^{*}\,\Psi = \Omega^{\omega}(\tilde{\sigma}_{\hat{\alpha}}S)^{*}\Psi = \Omega^{\omega}\gamma(S^{-1})f^{*}\psi_{\hat{\alpha}}$$

where $S: U_{\alpha} \to Pin(s,t)$ is determined by

(3) $$\tilde{f}\circ(\tilde{\sigma}_{\alpha}S^{-1}) = \tilde{\sigma}_{\hat{\alpha}}\circ f$$

(for special choice (1,5) of sections $S = id\,|_{Pin}$)

It is clear, that the assignment $f \to \tilde{f}$ yields for M=M' the representation of the double cover of $Conf_0(M)$ in the space of spinor fields on M. Now, we shall proceed in the well known way in order to obtain the unitary representation in the space of solutions of the massless Dirac equation. First we shall find the transformations of $\gamma_{\alpha}$ and $\Gamma_{\alpha}$ of (3.4,17) and (3.4,8)

(4) $$(f^{*}\gamma_{\hat{\alpha}})(X) = \gamma_{\hat{\alpha}}(f_{*}X) = \gamma(\sigma_{\hat{\alpha}}^{-1}(f_{*}X)) = \gamma\left[(\hat{f}\sigma_{\alpha}\rho(S^{-1}))^{-1}(f_{*}X)\right] =$$
$$= \gamma\left[\Omega\,\rho(S)\,\sigma_{\alpha}^{-1}(X)\right] = \Omega\,\gamma(S)\,\gamma_{\alpha}(X)\,\gamma(S^{-1}),$$

where S is given by ( 3 ). Then the unique $f^{*}\Gamma_{\hat{\alpha}}$ has to be

(5) $$f^{*}\Gamma_{\hat{\alpha}} = \gamma(S)\Gamma_{\alpha}\gamma(S^{-1}) + \gamma(S\,dS^{-1}) + \gamma(S)\,B\,\gamma(S^{-1})$$

where $B(x) \overset{df}{===} -1/4 \ \Omega^{-1}[\gamma_\alpha(\nabla\Omega), \gamma_\alpha(x)]$ and the gradient of $\Omega$ satisfies $g(\nabla\Omega, X) = d\Omega(X) = X\Omega$ for any $X \in TM$.

Indeed, a detailed computation shows that $f^*\Gamma_{\hat\alpha}$ is torsion-free

(6) $$d[\Omega\gamma(S)\gamma_\alpha \ \gamma(S^{-1})] + [\gamma(S)\Gamma\gamma(S^{-1}) + \gamma(SdS^{-1}) + \gamma(S)B\gamma(S^{-1})] \overset{\wedge}{\wedge} \Omega\gamma(S)\gamma_\alpha\gamma(S^{-1}) = 0$$

where the "double" wedge $\overset{\wedge}{\wedge}$ indicates the (adjoint) action by a commutator. This follows, because $\Gamma_\alpha$ is torsion free and

(7) $$(\Omega B \overset{\wedge}{\wedge} \gamma)(X,Y) = -1/4[[\gamma(\nabla\Omega), \gamma(X)], \gamma(Y)] - (X \leftrightarrow Y) =$$

$$= 1/2 \ \{\gamma(\nabla\Omega), \gamma(Y)\}_+ \gamma(X) - (X \leftrightarrow Y) =$$

$$= d\Omega(Y)\gamma(X) - (X \leftrightarrow Y) = d\Omega(Y)\gamma(X) - d\Omega(X)\gamma(Y) =$$

$$= -(d\Omega \wedge \gamma)(X,Y) .$$

Now the Dirac operator is transformed as

(8) $$(\not{D}\psi_\alpha)^f(x) = \Omega^\omega\gamma(S^{-1})\not{D}\psi_{\hat\alpha}(f(x)) = \Omega^\omega\gamma(S^{-1})\eta^{ab}\gamma_{\hat\alpha}(E_a)[E_b + \Gamma_{\hat\alpha}(E_b)]\psi_{\hat\alpha}(f(x)) =$$

$$= \Omega^{\omega-2}\gamma(S^{-1})\eta^{ab}\gamma_{\hat\alpha}(df_xE_a)[df_xE_b + \Gamma_{\hat\alpha}(df_xE_b)]\psi_{\hat\alpha}(f(x)) =$$

$$= \Omega^{\omega-2}\gamma(S^{-1})\eta^{ab}f^*\gamma_{\hat\alpha}(E_a)[E_b + f^*\Gamma_{\hat\alpha}(E_b)] f^*\psi_{\hat\alpha}(x) =$$

$$= \Omega^{\omega-1}\eta^{ab}\gamma(E_a)\{\gamma(S^{-1})E_b + \Gamma_\alpha(E_b)\gamma(S^{-1}) + E_b\gamma(S^{-1}) +$$

$$- 1/4 \ \Omega^{-1}[\gamma_\alpha(\nabla\Omega), \gamma_\alpha(E_b)] \ \gamma(S^{-1})\} \ f^*\psi_{\hat\alpha}(x) =$$

$$= \Omega^{-1}\not{D}\psi_\alpha^f(x) + \Omega^{-2}\left(\frac{n-1}{2} - \omega\right) \gamma_\alpha(\nabla\Omega) \ \psi_\alpha^f \quad ,$$

Where the last equality follows from the identity

$$\eta^{ab}\gamma_a[\gamma_c, \gamma_b] = -2(n-1)\gamma_c \quad , \quad \eta_{ab} = \text{diag(s factors +1; t factors -1)}.$$

It follows, that under conformal transformations f, which do not change the spin structure, the massless Dirac operator is invariant (up to a factor $\Omega$ ) for exactly one scaling dimension $\omega = \frac{n-1}{2}$ . If the spin structure is changed, then for $\omega = \frac{n-1}{2}$ $\tilde{f}$ <u>intertwines</u> the relative Dirac operators (again up to a factor $\Omega$ ). In particular their kernels are mapped 1-1 one onto another. For $m \neq 0$ the conformal factor $\Omega$ forbids $\not{D} + m$ to be invariant.

Now because of the conformal transformation rules

$$\sqrt{f^* |g|} = \Omega^n \sqrt{|g|}\,, \qquad (f^*g)^{\mu\nu} = \Omega^{-2} g^{\mu\nu} \qquad \text{and}$$

$$j^f(\psi,\psi')(X) = j(\psi^f, \psi'^f)(X) = \Omega^{n-1}\overline{f^*\psi}\;\gamma(X) f^*\psi' = \Omega^{n-2} f^* j\;(\psi,\psi')(X)$$

the scalar product (3.4, 22) is conformally invariant in every dimension

$$\langle \psi^f, \psi'^f \rangle = \int_S * j^f(\psi,\psi') = \int_S \Omega^{n-2} * f^* j\,(\psi,\psi') = \int_S f^* * j(\psi,\psi') =$$

$$= \int_{f(S)} * j\,(\psi,\psi') = \langle \psi, \psi' \rangle\,.$$

After passing to equivalence classes with respect to the norm induceed by $\langle\;,\;\rangle$ and completion, one can construct the Hilbert space of spinor fields fulfilling the massless Dirac equation. Then the double covering of the connected conformal group can be unitarily implemented in the space of massless, "free" Dirac spinors, i.e. obeying (3.4,12) with m=0.

## 3. Conformal compactification of $R^{1,3}$.

The idea of conformally compactified Minkowski space-time $\bar{M}$ has been considered in physics with various motivations; for instance just to avoid the singularities in the action of the conformal group in $R^{1,3}$, to study cosmological models, asymptotic flatness or gravitational radiation, and to gain the compact space-time with the Lorentzian analogs of the Euclidean space instantons [115, 116, 45, 33, 117], c.f. also [118] . Attempts to attribute the direct physical meaning to $\bar{M}$ are limited by the well known existence of time-like loops in this space and lack of the global causality. However $\bar{M}$ can be at least interpreted as imposition of special asymptotic conditions in $R^{1,3}$. Conformally related to the flat Minkowski space-time, $\bar{M}$ can be used to study the conformally invariant Yang-Mills systems, also coupled to massless fermions [33,119]. Tensorlike objects has been in detail discussed on $\bar{M}$ [ 120 ], and $\bar{M}$ plays a prominent rôle in the Penrose twistor theory, however the <u>global</u> properties of spinor fields on $\bar{M}$ have not been discussed at least in the literature available to the author.

Apart from cosmological models the relevance of $\bar{M}$ is sometimes questioned [ 121 ], and the conformal symmetry postulated only infinitesimally at the Lie algebra level. To deal with the finite transformations the notion of a 'local group action' has been introduced

[122]. As is well known, every special conformal transformation in $R^{1,3} \equiv \{x = x^{\mu}\sigma_{\mu}\}$ parametrized by $k = k^{\mu}\sigma_{\mu} \neq 0$

(1) $$x \longrightarrow \frac{x - k\langle x,x\rangle}{1 - 2\langle k,x\rangle + \langle k,k\rangle\langle x,x\rangle}$$

where $\langle x,x\rangle = \det(x) = x^{\mu}\eta_{\mu\nu}x^{\nu}$, $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$, $|k| = \langle k,k\rangle$, is well defined only outside the light cone $\{x \mid (x - \frac{k}{|k|})^2 = 0\}$ if $|k|^2 \neq 0$, and outside the hyperplane $\{x \mid k^{\mu}x_{\mu} = 1/2\}$ if $|k|^2 = 0$.

We are of the opinion, that generally singularities may indicate some nontrivial situation to occur, and therefore should be treated carefully. Quite often in mathematical physics one encounters the cases when the infinite curvature is concentrated only in a discrete subset, gauge fields are string-like singular, or the wave function is discontinuous on zero-measure surfaces. This can happen to be the key point, which can be consistently worked out by passing to the topologically nontrivial point of view.

We are interested in the <u>minimal</u> compactification $\bar{M} = S^1 \times S^3/Z_2$, in which the standard Minkowski space $R^{1,3}$ is densely embedded. The double covering of $\bar{M}$, $\bar{\bar{M}} = S^1 \times S^3$ is in fact diffeomorphic to $\bar{M}$, but the loops generating the first homotopy group $\pi_1(\bar{M}) = Z$ are of different metric character, as we shall see. Since inequivalent spinors arise only when $\pi_1(M) \neq 0$, we shall not be working with the universal (infinite) open covering $M^u$, which is often discussed because of its global causal order (c.f. [116]). Different compactifications are locally equal, but their global properties are distinct. This results in different boundary conditions, which are well known to form the part of dynamics of fields or states of interest.

The conformally compactified Minkowski space time can be most conveniently realized either as the projective null cone in $R^{2,4}$, or as the group manifold of U(2). In this section we recall the first picture, which based on the Lie algebra isomorphism $o(2,4) \approx conf(1,3)$. Let $J_{ab}$, a,b = 0,1,2,3,5,6 be generators of so(2,4)

(2) $$[J_{ab}, J_{cd}] = (\eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac})$$

where $\eta_{ab} = \mathrm{diag}(+1,-1,-1,-1,-1,+1)$. Then the linear combinations of $J_{ab}$

(3) $$M_{\mu\nu} = J_{\mu\nu} \ , \quad P_\mu = J_{4\mu} + J_{5\mu} \ , \quad D = J_{45} \ , \quad K_\mu = J_{5\mu} - J_{4\mu} \ ,$$

obey the commutation relations of conf(1,3)

(4) $$\begin{aligned}
&[M_{\mu\nu}, M_{\alpha\beta}] = \eta_{\mu\beta} M_{\nu\alpha} + \eta_{\nu\alpha} M_{\mu\beta} - \eta_{\mu\alpha} M_{\nu\beta} - \eta_{\nu\beta} M_{\mu\alpha}, \\
&[M_{\mu\nu}, P_\alpha] = \eta_{\mu\alpha} P_\nu - \eta_{\nu\alpha} P_\mu, \\
&[D, P_\mu] = -P_\mu, \\
&[D, K_\mu] = K_\mu, \\
&[M_{\mu\nu}, K_\alpha] = \eta_{\mu\alpha} K_\nu - \eta_{\nu\alpha} K_\mu, \\
&[P_\mu, K_\nu] = 2\eta_{\mu\nu} D + 2M_{\mu\nu} \ ,
\end{aligned}$$

where $\mu,\nu,\alpha,\beta = 0,1,2,3$ and $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$. Next, $\bar{M}$ is defined as the set of one dimensional isotropic linear subspaces of $R^{2,4}$

(5) $$\bar{M} = \{\omega \in R^{2,4} \mid \omega \neq 0 \ , \langle\omega,\omega\rangle = 0\} / \sim \ ,$$

where $[\omega] = [\omega']$ i.e. $\omega \sim \omega'$ iff $\exists \ 0 \neq r \in R$, $\omega' = r\omega$ .

By picking up a pair $\pm\omega$, $\omega_0^2 + \omega_6^2 = 1 = \omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_5^2$

from each ray in the null cone $\omega_0^2 + \omega_6^2 - \omega_1^2 - \omega_2^2 - \omega_3^2 - \omega_5^2 = 0$, it can be seen that $\bar{M}$ is the direct product of a circle $S^1$ in 0,6 plane and a 3-sphere $S^3$ in 1,2,3,5 coordinates with identified opposite points i.e. $\bar{M} = S^1 \times S^3/Z_2$. While the topological properties of $\bar{M}$ are most easily discussed in the second realization (U(2)), the action of the conformal group can be seen now. The null cone in $R^{2,4} \setminus \{0\}$ forms an orbit of the natural (and linear) action of O(2,4) in $R^{2,4}$, which induces well defined transformations of $\bar{M}$ $\omega \to [h\omega]$.

Since the centre of O(2,4), $Z_2 = \{1,-1\}$ acts trivially, the effective group is $O(2,4)/Z_2$, identified with the full conformal group Conf(1,3). The centre $Z_2$ belongs also to $SO_0(2,4)$ (both space and time dimensions in $R^{2,4}$ are even), and there is a similar isomorphism of the connected components

(6) $$\mathrm{Conf}_0(1,3) \approx SO_0(2,4)/Z_2$$

The four disconnected components of Conf(1,3) are generated by $Z_4 = O(2,4)/SO_0(2,4) = \{I,P,T,PT\} \subset O(2,4)$, where the 6 x 6 matrices I,P,T, and PT have the fourdimensional reversions embedded in the left upper corner and $1_2$ in the right lower corner.

The standard Minkowski space $R^{1,3}$ is injected in $\bar{M}$ by the map

(7) $$j : x^\mu \to [(x^\mu,\ 1/2(1+x^2),\ 1/2(1-x^2))] ,$$

which can be inverted on its range (the dense in $\bar{M}$ set of rays $[\omega]$ for which $\omega^5 + \omega^6 \neq 0$ ) and defines the local coordinates [ 23 ]

(8) $$x^\mu = j^{-1}([\omega])^\mu = \omega^\mu(\omega^5 + \omega^6)^{-1}$$

The rest $\bar{M} \setminus j(R^{1,3})$ of $\bar{M}$ forms the light cone at 'infinity', of topology

(9) $$(S^1 \times S^3/Z_2) \smallsetminus R^{1,3} = \{[\omega] \mid \omega_0^2 - \omega_1^2 - \omega_2^2 - \omega_3^2 = 0,\ \omega_6 - \omega_5 = 0\}$$

In terms of objects in $R^{1,3}$ $\bar{M} \smallsetminus j(R^{1,3})$ can be thought of as follows. Consider straight lines $u + rv$ in $R^{1,3}$ passing through $u \in R^{1,3}$ in the direction $0 \neq v \in R^{1,3}$ parametrized by $r \in R$. Then, since

(10) $$\lim_{r \to \pm\infty} j(u + rv) = \begin{cases} [\omega^\mu = 0\ ,\ 1\ ,\ -1] & v^2 > 0 \\ [\omega^\mu = 0\ ,\ -1\ ,\ 1] \quad \text{if} & v^2 < 0 \\ [\pm p v^\mu\ ,\ \pm p u^\nu v_\nu\ ,\ \mp p u^\nu v_\nu] & v^2 = 0\ , \end{cases}$$

where $p = [v_0^2 + (u^\nu v_\nu)^2]^{-1/2}$, any point $[\omega] \in S^1 \times S^3/Z_2 \smallsetminus R^{1,3}$ can be obtained as a limit (10) taking

(11) $$\begin{cases} \text{arbitrary} \quad u \in R^{1,3} \quad \text{and} \quad \text{arbitrary}\ 0 \neq v \in R^{1,3} & \text{if}\ \omega^0 = 0 \\ u^\mu = (\omega^5/\omega^0, 0, 0, 0) \quad \text{and} \quad v^\mu = \omega^\mu & \text{if}\ \omega^0 \neq 0 \end{cases}$$

Since the limits $r \to +\infty$ and $r \to -\infty$ coincide for $v^2 \neq 0$, all nonisotropic lines in $R^{1,3}$ form closed loops. Similarly all isotropic lines approach asymptotically points differing only by sign and therefore $Z_2$ identified. It turns out that every isotropic line represents the generator of $\pi_1(\bar{M})$ [123], what will be seen easily in the next section.

Now, we shall show that conformally covariant 1/2-spin fields on the Minkowski space $R^{1,3}$ constructed following [25] from $SO_0(2,4)$ covariant spinor fields in $R^{2,4}$ cannot be extended to the conformally compactified Minkowski space $\bar{M}$ [123]. Spinor field $\chi(\omega)$ in $R^{2,4}$ has eight complex components as an element of the representation space of a relative Clifford algebra. In order to be well defined on rays in the null cone of $R^{2,4}$ $\chi(\omega)$ is required to be homogeneous of degree n

(12) $$\chi(r\omega) = r^n \chi(\omega) , \quad r \in R .$$

The SO(2,4) algebra is realized in spinor space as $J_{ab} = M_{ab} + s_{ab}$ , where $M_{ab} = i(\omega_a \frac{\partial}{\partial \omega^b} - \omega_b \frac{\partial}{\partial \omega^a})$ is the orbital part taking in coordinates (8) the form

(13) $$M_{\mu\nu} = i(x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}), \; M_{6\mu} + M_{5\mu} = i\frac{\partial}{\partial x^\mu} , \; M_{65} = i(x_\nu \frac{\partial}{\partial x_\nu} - n)$$
$$M_{6\mu} - M_{5\mu} = i(2x_\mu x_\nu \frac{\partial}{\partial x_\nu} - x^2 \frac{\partial}{\partial x^\mu} - 2nx_\mu)$$

and the spin part is chosen in the 8x8 matrix representation

(14) $$S_{\mu\nu} = [\gamma_\mu, \gamma_\nu] \otimes i/4 , \; S_{\mu 5} = -i/2 \gamma_\mu \otimes \sigma_2 , \; S_{\mu 6} = 1/2 \gamma_\mu \otimes \sigma_1 , \; S_{65} = i/2 \otimes \sigma_3 .$$

Usually the spin part $S_{5\mu} + S_{6\mu}$ of translation generators is removed by the x-dependent similarity transformation

(15) $$T(x) = \exp\left[ix^\mu (S_{\mu 5} + S_{\mu 6})\right] = \begin{bmatrix} 1_4 , & 0 \\ -x^\nu \gamma_\nu , & 1_4 \end{bmatrix}$$

which obeys

(16) $$T^{-1}(x) \left[ i\frac{\partial}{\partial x^\mu} + S_{\mu 5} + S_{\mu 6} \right] T(x) = i\frac{\partial}{\partial x^\mu} .$$

Generators of Lorentz transformations and a dilatation remain unchanged. Now a pair of spinors depending only on $x_\mu$ variables can be defined as eigenvectors of $S_{65}$

(17) $$\psi_\pm(x) = 1/2(1 \pm S_{65})(\omega_5 + \omega_6)^{-n} \, T(x) \chi(\omega)$$

where $\omega = \omega(x)$ is given by (7).

The eigenspace $S_{65} = +1/2$ of spinors $\psi_+$ , with only lowest four components nonzero, is preserved under the action of all conformal generators and for n = -2 has the physical scaling dimension $n + 1/2 = -3/2$ .

This conformally covariant spin-1/2 fields on $R^{1,3}$, because of a construction, can be obviously extended to $\bar{M}$ provided T(x) transformation is legitimate. From the homogenity condition (12) follows the asymptotic behaviour of $(\omega_5+\omega_6)^{-n}\chi(\omega)$ on any straight line $u + rv$ ; $u \in R^{1,3}$, $0 \neq v \in R^{1,3}$, $r \in R$

(18)
$$\lim_{r\to+\infty} (\omega_5+\omega_6)^{-n}\chi(\omega) = (-1)^n \lim_{r\to-\infty}(\omega_5+\omega_6)^{-n}\chi(\omega) = \begin{cases} r^{2n} \; \chi(0,0,0,0,-\frac{v^2}{2},\frac{v^2}{2}) & \text{if} \quad v^2 \neq 0 \\ r^n \; \chi(v_0,v_1,v_2,v_3,-uv,uv) & \text{if} \quad v^2 = 0 \end{cases}$$

where $\omega = \omega(u + rv)$.

Hence for $n = -2$ both $\chi_\pm = 1/2(1 \pm iS_{65})$ are periodic (because of asymptotic vanishing derivatives play a rôle) and can be called normal spinors on $\bar{M}$. But the $\psi_+$ field due to the appereance of x in T(x) (15) is a linear combination of periodic and completely antiperiodic parts. Thus, it cannot be interpreted neither as a normal, nor as an exotic spinor field on $\bar{M}$, since the periodicity condition on any timelike as well as space like lines is not fulfilled. It can be seen that replacement of (12) by $\chi(|r|\omega) = |r|^n \chi(\omega)$, $\chi(-\omega) = -\chi(\omega)$ or taking the field $\psi_-$ for $n = -1$ do not help.

## 4. $\bar{M}$ as U(2).

In the second realization as the group manifold U(2), it is evident that $\bar{M}$ is connected, compact and parallelizable. It can be given the Lorentz metric, and time and space orientations. There are two natural global sections of the frame bundle over $\bar{M}$. Consider a bijective isomorphism $X \leftrightarrow {}_R X$ of hermitian 2x2 matrices and the tangent space $T_u$ at each $u \in U(2)$ given by

(1) $$_R X\ \varphi(u) = \frac{\partial}{\partial\tau}\Big|_{\tau=0} \varphi\left(e^{i\tau X} u\right) ,$$

and another one $X \leftrightarrow {}_L X$ given by

(2) $$_L X\ \varphi(u) = \frac{\partial}{\partial\tau}\Big|_{\tau=0} \varphi\left(u\, e^{i\tau X}\right) ,$$

where we identify vectors with differential operators on functions $\varphi : U(2) \to C$. Then define the global four independent vector fields (frame) by $\{_R\sigma_a\} \equiv {}_R\sigma$, and the other frame by $\{_L\sigma_a\} \equiv {}_L\sigma$, where

(3) $$\sigma_0 = 1_2 , \quad \sigma_1 = \begin{bmatrix} 0, & 1 \\ 1, & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0, & -i \\ i, & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1, & 0 \\ 0, & -1 \end{bmatrix}$$

are the Pauli matrices. The time orientation, orientation and metric on U(2) are fixed by setting ${}_R\sigma_0$ to point into the future, and ${}_R\sigma_a$ to be oriented and orthonormal

(4) $$g({}_R\sigma_a, {}_R\sigma_b) = \eta_{ab} = \mathrm{diag}(+1,-1,-1,-1) .$$

Then also ${}_L\sigma_a$ are orthonormal of the same orientation and time orientation, and $g(X,Y)$ can be obtained by polarization of

(5) $$g({}_RX,{}_RX) = g({}_LX,{}_LX) = \det(X) \ .$$

The vectors ${}_R\sigma_a$ fulfil

(6) $$[{}_R\sigma_\circ, {}_R\sigma_j] = 0 \ , \quad [{}_R\sigma_j, {}_R\sigma_k] = 2\,\epsilon_{jkl}\ {}_R\sigma_l \quad ; \quad j,k,l \in \{1,2,3\} \ .$$

Denote by ${}_R\tau^a$ the dual to ${}_R\sigma_a$ basis of right invarient 1-forms in $T^*U(2)$

(7) $${}_R\tau^a({}_R\sigma_b) = \delta^a_b \ .$$

They fulfil

(8) $$d\,{}_R\tau^\circ = 0 \ , \qquad d\,{}_R\tau^j = -\epsilon^{jkl}\,{}_R\tau^k \wedge {}_R\tau^l$$

and the Maurer-Cartan right invariant 1-form on $U(2)$ is

(9) $$\omega(u) = \sum_a {}_R\tau^a\,\sigma_a \ .$$

The connection 1-form $\Gamma^a_{\ b}$ defined by

(10) $$d\,{}_R\tau^a + \Gamma^a_{\ b} \wedge {}_R\tau^b = 0$$

is

(11) $$\Gamma^j_{\ l} = \epsilon^{jkl}\,{}_R\tau^k \ , \qquad \Gamma^\circ_{\ j} = \Gamma^j_{\ \circ} = \Gamma^\circ_{\ \circ} = 0 \ .$$

the curvature 2-form

(12) $$R^a_{\ b} \equiv R^a_{\ bcd}\ {}_R\tau^c \wedge {}_R\tau^d = d\,\Gamma^a_{\ b} + \Gamma^a_{\ c} \wedge \Gamma^c_{\ b}$$

yields the curvature tensor zero when any $a,b,c,d = 0$, and

(13) $$R_{jklm} = -\,\delta_{jk}\,\delta_{lm} + \delta_{jl}\,\delta_{km} \ ,$$

Ricci tensor zero when $a$ or $b = 0$, and

(14) $$R_{km} = R^j_{\ kjm} = -2\,\delta_{km},$$

and curvature scalar

(15) $$R = R^a{}_a = 6 \ .$$

Now as a group $U(2) = (U(1) \times SU(2))/Z_2$, but topologically $\overline{M} = S^1 \times S^3$, which follows from the set diffeomorphism (c.f. 4.3,10)

(16) $$U(1) \times SU(2) \ni \left(\lambda, \begin{bmatrix} a, & b \\ c, & d \end{bmatrix}\right) \longleftrightarrow \begin{bmatrix} \lambda a, & b \\ \lambda c, & d \end{bmatrix} \in U(2)$$

where $a,b,c,d \in C$, $|\lambda| = |a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$

$a\bar{c} + b\bar{d} = 0$, $ad - bc = 1$. It follows that the first homotopy group of $M$ is $\pi_1(M) = \pi_1(S^1) \times \pi_1(S^3) = Z$ and its generator can be represented by the loop

(17) $$\langle 0, 2\pi \rangle \ni r \longrightarrow \exp\left[ir/2(\sigma_0 + \sigma_3)\right] = \begin{bmatrix} \exp ir, & 0 \\ 0 & , 1 \end{bmatrix} \in U(2)$$

Let us introduce the chart $\{U_\alpha, z_\alpha\}$ on $U(2) = \bigcup_\alpha U_\alpha$, where $U_\alpha = \{u \in U(2) \mid \det(u + \alpha) \neq 0\}$, $\alpha \in \{1, -1, i\}$ and $z_\alpha : U_\alpha \rightarrow R^{1,3} \equiv \{z = z^\nu \sigma_\nu\}$ is given by

(18) $$z_\alpha(u) = i\, \frac{\alpha - u}{\alpha + u}$$

The inverse $z_\alpha^{-1}$ of $z$ is

(19) $$u(z_\alpha) = \alpha\, \frac{i - z_\alpha}{i + z_\alpha}$$

(in the notation we notoriously mix the maps and their values, the meaning should be clear from the context).

The transition functions $\varphi_{\alpha\beta}$ on

(20) $$z_\alpha(U_\alpha \cap U_\beta) = \left\{ z_\alpha = z_\alpha^\nu \sigma_\nu \,\middle|\, \det\left[z_\alpha(\beta - \alpha) + i(\beta + \alpha)\right] \neq 0 \right\}$$

are

(21) $$\varphi_{\alpha\beta} = z_\alpha \circ z_\beta^{-1} = \frac{i(\alpha-\beta)+(\alpha+\beta)z_\beta}{\alpha+\beta-i(\alpha-\beta)z_\beta} .$$

Denote $z^{-1} \equiv z_1^{-1}$, and observe that $z^{-1}$ is the well known Cayley map, thought of as the injection of the Minkowski space $R^{1,3}$ into $U(2)$. Then $z$ are coordinates on $U_1$ , the dense in $U(2)$ image of $z^{-1}$. The light cone at infinity is $\{u \in U(2) \mid \det(1+u) = 0\}$. It turns out that the flat metric $\eta$ on $R^{1,3}$ , $\eta(z,z) = \det(z)$ (we identify $R^{1,3}$ with $TR^{1,3}$) and the metric $g$ on $U(2)$ defined by ( 4 ) are conformally related by the Cayley map $z^{-1}$

(22) $$(z^*\eta) = 4\det(u)\,\det^{-2}(1+u)g ,$$

and

(23) $$(z^{-1})^*g = 4\det(1+z^2)^{-1}\eta ,$$

which follows either from

(24) $$z_*({}_RX) = 2(1+u)^{-1}Xu(1+u)^{-1} = 1/2(z+i)X(z-i)$$

or

(25) $$z_*({}_LX) = 2(1+u)^{-1}uX(1+u)^{-1} = 1/2(z-i)X(z+i) .$$

where $z^*$ and $z_*$ denote the induced by $z$ pulled back, or push forward maps, respectively.

Let us note, that the standard o.n. frame (coming from coordinates) in $R^{1,3}$ does not behave asymptotically in a proper way, in the sense that for any $\alpha$ the conformally rescaled frame $\sigma_\alpha$ in $R^{1,3}$

(26) $$E_a = \det(z_\alpha + i)(z_\alpha^{-1})_* \sigma_a$$

can not be extended to the global frame in U(2). Indeed, on $z_\beta(U_\alpha \cap U_\beta)$ in coordinates $z_\beta$ it becomes

(27) $$(\varphi_{\alpha\beta})_*(E_a) = \det(i + z_\beta) \det^{-1}\left[ z_\beta(\beta - \alpha) + i(\beta + \alpha)\right] \times$$
$$\left[z_\beta(\beta - \alpha) + i(\beta + \alpha)\right] \sigma_a \left[z_\beta(\beta - \alpha) + i(\beta + \alpha)\right].$$

On $z_\beta(U_\alpha \cap U_\beta)$ this is the rescaled and Lorentz rotated $\sigma_a$, which in general becomes singular on $z_\beta(U_\beta \setminus U_\alpha \cap U_\beta)$.

Examples of frames on $R^{1,3}$, adapted to the global topology of U(2), are provided by

(28) $$z_*({}_{\substack{R\\L}}\sigma_a) = 1/2(z \pm i)\, \sigma_a (z \mp i),$$

which are suitable rescaled and Lorentz rotated standard frames in $R^{1,3}$.

The group $Spin_o(2,4) \approx SU(2,2)$ and the homomorphism

(29) $0 \to Z_2 \to SU(2,2) \to SO_o(2,4) \to 0$ have been described in (2.2).

The composition of (29) with the Klein homomorphism (3,6)

(30) $$0 \to Z_2 \to SO_o(2,4) \to Conf_o(1,3) \to 0$$

leads to

(31) $$0 \to Z_4 \to SU(2,2) \to Conf_o(1,3) \to 0$$

where $Z_4 = \{1,-1,i,-i\}$ is the Centre of SU(2,2).

In the basis in which $H = i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ one has the following block diagonal SU(2,2) realization of conformal transformations:

$$\text{translation}, \qquad z \longrightarrow z+t, \qquad \begin{bmatrix} 1, & t \\ 0, & 1 \end{bmatrix}, \qquad t = t^\nu \sigma_\nu ,$$

$$\text{Lorentz transformation}, \qquad z \longrightarrow \rho(A) z, \qquad \begin{bmatrix} A, & 0 \\ 0, & (A^+)^{-1} \end{bmatrix}; \qquad A \in SL(2,C),$$

(32)

$$\text{dilatation}, \qquad z \longrightarrow e^\lambda z, \qquad \begin{bmatrix} e^{\lambda/2}, & 0 \\ 0, & e^{-\lambda/2} \end{bmatrix}, \qquad \lambda \in R ,$$

special conformal transformation ,

$$z \longrightarrow \frac{z - k z^\nu z_\nu}{1 - k^\nu z_\nu + k^\nu k_\nu z^\mu z_\mu} \qquad \begin{bmatrix} 1, & 0 \\ k, & 1 \end{bmatrix} \qquad k = k^\nu \sigma_\nu ,$$

where the action of $\begin{bmatrix} a, b \\ c, d \end{bmatrix} \in U(2,2)$ on $z$ is

$$z \longrightarrow z' = (az + b)(cd + d)^{-1}.$$

The globally defined action on $U(2)$ is given by

(33)

$$u \longrightarrow (Au + B)(Cu + D)^{-1}$$

where

$$A = d + a + i(b - c) , \qquad B = d - a + i(b + c) ,$$

(34)

$$C = d - a - i(b + c) , \qquad D = d + a - i(b - c)$$

Disconnected part of Conf(1,3) is generated by the reversions P, T and PT. In matrix notation their action on $R^{1,3}$ and $U(2)$ is respectively

$$T(z) = -\epsilon z^* \epsilon^{-1} , \qquad T(u) = \epsilon u^* \epsilon^{-1}$$

$$P(z) = \epsilon z^* \epsilon^{-1} , \qquad P(u) = \epsilon u^T \epsilon^{-1}$$

(35)

$$TP(z) = -z , \qquad TP(u) = u^{-1}$$

$$R(z) = \frac{z}{\det z} = \frac{-1}{P(z)} , \qquad R(u) = -\epsilon u^T \epsilon^{-1} ,$$

where $\epsilon = i\sigma_2 = \begin{bmatrix} 0, & -1 \\ 1, & 0 \end{bmatrix}$, $z^*$ and $z^T$ denote complex conjugation and transposition respectively, and we have included also the conformal inversion $R(z) = \frac{-z}{z^\nu z_\nu}$ which sits in the P sector of Conf(M)

## 5. Spin structures on $\overline{M}$.

Since $\overline{M} = U(2)$ is parallelizable, the $G$-bundle $F$ of orthonormal frames $(G = O(1,3), SO(1,3), SO_0(1,3))$ can be trivialized as a direct product $F = U(2) \times G$ with respect to some global frame $E(u)$. Then the $\widetilde{G}$ structure $(\widetilde{F}, \eta)$ exists $(\widetilde{G} = Pin(1,3), Spin(1,3), Spin_0(1,3))$, where

$\widetilde{F} = U(2) \times \widetilde{G}$ , and $\eta : \widetilde{F} \to F$ is given by

(1) $$\eta(u,h) = (u, \rho(h)) .$$

To determine the number of inequivalent $\widetilde{G}$ structures observe that there is a 1 - 1 correspondence between $H^1(M,Z_2)$ and $Hom(\pi_1(M), Z_2)$ [ 80 ]. Since $\pi_1(\overline{M}) = Z$ therefore there are exactly two inequivalent spin structures on $\overline{M}$. To find the second one note that the generator of $\pi_1(\overline{M})$ is explicitly known and we can follow the Isham [ 18] argumentation. Suppose we succeeded in finding another global o.n. frame (vierbein) $E'(u)$ in $\overline{M}$ such that $E'(u) = L(u)E(u)$, where $L : \overline{M} \to G$ is not liftable to a continuous $\widetilde{L} : \overline{M} \to \widetilde{G}$, i.e. makes odd number of $2\pi$ rotations along some "generic" loop $l$ in $\overline{M}$. Then another $\widetilde{G}$ structure $(\widetilde{F}', \eta')$, defined by $\widetilde{F}' = \widetilde{F} = M \times \widetilde{G}$ and $\eta'(u,h) = (u, L^{-1}(u)\rho(h))$, is inequivalent to $(\widetilde{F}, \eta)$. Indeed, assuming the contrary that some equivariant $\beta : \widetilde{F} \to \widetilde{F}'$ exists, which in case of trivial bundles is necessarily of the form $\beta(u,h) = (u, \overline{\beta}(u)h)$ for some $\overline{\beta} : M \to \widetilde{G}$, we are led to contradiction with the property $\eta' \circ \beta = \beta \circ \eta$ .

Namely $\rho(h) = L^{-1}(u)\rho(\bar{\beta}(u))\rho(h)$ , and then $L(u) = \rho(\bar{\beta}(u))$ cannot hold along the loop 1.

It turns out that in our case such two global frames are already given: the left and right invariant ${}_L\sigma$ and ${}_R\sigma$, defined by (4,1) and (4,2). They are related one with another by a point dependent rotation

(2) $${}_R\sigma_a(u) = {}_L(u^{+}\sigma_a u)(u)$$

(3) $${}_L\sigma(u) = {}_R(u\,\sigma_a u^{+})(u)$$

(The phase of u annihilates, and only the SU(2) part acts effectively). This is a nontrivial rotation which can not be lifted to a continuous $SL_2C$ transformation. It performs the full $2\pi$ angle along the loop

(4) $$\langle 0,2\pi\rangle \ni r \longrightarrow u(r) = e^{i r/2\,(\sigma_0+\sigma_3)} \in U(2),$$

which, as we have seen in last section, generates the homotopy group $\pi_1(\bar{M})$.

Summarizing , the two inequivalent spin structures $(\tilde{F},\eta)$, $(\tilde{F}',\eta')$ on $\bar{M} = U(2)$ are given by

$$\tilde{F} = U(2) \times \tilde{G} = \tilde{F}'$$

(5) $$\eta(u,h) = (u,{}_L\sigma(u)\rho(h)) = (u,{}_R\sigma(u)\rho(u^{+})\rho(h)) \equiv (u,\rho(u^{+}h))$$

and

(6) $$\eta'(u,h) = (u,{}_R\sigma(u)\rho(h)) \equiv (u,\rho(h))$$ ,

where on the right hand sides of (5) and (6) we have assumed F to be trivialized by ${}_R\sigma_a$ , and ${}_R\sigma_a\rho(h) \equiv {}_R(h^+\sigma_a h)$.

The inequivalence of $(\tilde{F},\eta)$ and $(\tilde{F}',\eta')$ is clearly ~~the~~ a global property. Locally, over simple connected regions in $\bar{M}$ , for instace over the dense in $\bar{M}$ image of $R^{1,3}$, one can undo the relative twist of (5) with respect to (6). By performing a gauge rotation of spin frames which covers (3) one can achieve the same spin structure over $R^{1,3}$, but then the spinors have to obey antiperiodic boundary conditions along any loop which can be continuously deformed into the loop (4,17) representing the generator of $\pi_1(\bar{M})$. In terms of objects in $R^{1,3}$, this can be any isotropic line, for instance $\langle 0,2\ \rangle \ni r \to z\,(\exp ir/2(\sigma_0+\sigma_3)) = 4\,tg(r/2)(\sigma_0+\sigma_3) \in R^{1,3}$, and one can speculate that the difference between the inequivalent spinors should be manifested in the massless sector of the theory.

Now in order to include the full group of conformal transformations we consider the case $\tilde{G} = Pin(1,3)$ and $G = O(1,3)$ . It turns out that $Conf_0(\bar{M})$ and the time reversion T (4,35) lift to the automorphisms of the pin structure $(\tilde{F},\eta)$ onto itself, and similarly for $(\tilde{F}',\eta')$. This is not the case for P, PT and R reversions. For instance the parity inversion (4,35)

(7) $$P : u \to P(u) = \epsilon u^T \epsilon^{-1}$$

interchanges frames ${}_R\sigma$ and ${}_L\sigma$

(8) $$P_*({}_R\sigma_a)(u) = {}_L(\epsilon\sigma_a^T\epsilon^{-1})(P(u)) = ({}_L\sigma P)_a(P(u)),$$

where in order to avoid confusion with different P's we have to recall that:

$P_* : T\bar{M} \to T\bar{M}$ is the derivative of parity inversion $P : \bar{M} \to \bar{M}$ and ${}_L\sigma P$ denotes the result of the right multiplication of the frame ${}_L\sigma_a \in F$ by the element $P \in Pin(1,3)$.

In the ${}_R\sigma$ trivialization of $F$ (8) becomes $\hat{P}(u,h) = (u, \rho(u)Ph)$.

Because of (2) the lifting of $\hat{P}$ to $\tilde{P} : (\tilde{F},\eta) \to (\tilde{F},\eta)$ can not exist, but the lifted maps between inequivalent pin structures $\tilde{P} : (\tilde{F},\eta) \to (\tilde{F},\eta')$ and $\tilde{P}' : (\tilde{F},\eta') \to (\tilde{F},\eta)$ do exist.

Let $(u,h) \in \tilde{F} \times Pin(1,3)$, define

(9) $$\tilde{P}(u,h) = (u,\bar{P}h)$$

then (1,4) is satisfied

(10) $$\eta' \circ \tilde{P}(u,h) = (u, P\rho(h)) = (u, \rho(u^+)\rho(u)P\rho(h)) = \hat{P} \circ \eta(u,h)$$

where $\bar{P} \in Pin(1,3)$, $\rho(\bar{P}) = P \in O(1,3)$.

Therefore we have found the example of pin structure changing tranformation of the space-time, which proves the content of the theorem (1,23) to be nontrivial.

Let us observe that in our case there is no preferred spin structure over $\bar{M}$, and moreover, if one wants to implement some basic geometrical tranformations of the space-time, then <u>both</u> of them have to be included into considerations.

## 6. Spectrum of $\not{D}$ on $\bar{M}$.

On a parallelizable manifold $\bar{M}$, the spinor field is described by its components with respect to the global spinor frame $\tilde{\sigma} \equiv \tilde{E}$ in $\tilde{F}$, which we choose to be such that $\eta \circ \tilde{\sigma} = {}_R\sigma$. Hence spinor field is a function $\psi : \bar{M} \to V$. Under the change $\tilde{\sigma}' = \tilde{\sigma} h$ for some h: M→Pin(1,3), the o.n. frame transforms as ${}_R\sigma \to \sigma' = {}_R\sigma\, \rho(h)$ and the spinor components as

(1) $$\psi \to \psi' = \gamma(h^{-1})\,\psi$$

For Dirac spinors the representation $\gamma : Pin(1,3) \to L(C^4)$ is chosen according to ch.(2), with the only exception for the time reversion, which is more appropriate [ 124 ] to be defined as the C-antilinear operator

(2) $$\psi'(z) = \gamma_o \psi^*(T(z))$$

To write the Dirac equation we follow the prescription presented in ch.( 3 ). Let $E_a$ be some o.n. frame. Set $\gamma(E_a) = \gamma_a = \begin{bmatrix} 0, & \sigma_a \\ \underline{\sigma}_a, & 0 \end{bmatrix}$ where $\underline{\sigma}_o = \sigma_o = 1_2$ and $-\underline{\sigma}_i = \sigma_i$ are Pauli matrices. A straightforward computation shows that

(3) $$\Gamma_o \equiv \Gamma(E_o) = 0\ , \quad \Gamma_j \equiv \Gamma(E_j) = \Sigma_j\ ,$$

where $\Sigma_j = \begin{bmatrix} \sigma_j \ ; & 0 \\ 0 \ ; & \sigma_j \end{bmatrix}$ , fulfils the condition (3.5,20) and is the unique spin connection matrix.

The Dirac operator is

(4) $$\not{D} = i\,\eta^{ab}\,\gamma_a(E_b + \Gamma_b)$$

After some rearrangement and similarity transformation ( 4 ) becomes

(5) $$E_o + \gamma_5 ( \Sigma_j E_j - 3/2\, i),$$

where $\gamma_5 = i\, \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

The spectrum of ( 4 ) can be obtained analogously to the standard quantum machanical treatment of the angular momentum. Let $L_j = +i/2\, E_j$, $s_j = \frac{1}{2} \Sigma_j$ . Then

(6) $$[L_j, L_k] = i\, \epsilon_{jkl} L_l \quad , \quad [s_j, s_k] = i\, \epsilon_{jkl} s_l \ ,$$
$$[L_j, E_o] = 0 = [s_j, E_o] \ , \ [L_j, s_k] = 0$$

Set $J_k = L_k + s_k$ and observe that $\Sigma_k E_k = -2i(J^2 - L^2 - s^2)$.

Now, from the Casimir eigenvalues; $E_o = im$ , $s^2 = 3/4$, $L^2 = l(l+1)$, and $J^2 = (l + 1 \pm \frac{1}{2})(l \pm \frac{1}{2})$, where $m \in Z$ , $l$ is nonnegative halfinteger, and for $l = 0$ only (+) is possible, follows that the possible spectrum of $\not{D}$ is

(7) $$\begin{cases} m \mp (2l + 1) - \frac{1}{2} & \text{for} \quad \gamma_5 \psi = \psi \\ m \pm (2l + 1) + \frac{1}{2} & \text{for} \quad \gamma_5 \psi = -\psi \end{cases}$$

However, in order to be well defined on $\bar{M} \approx U(1) \times SU(2)/Z_2$ the eigenfunctions have to assume the same values at the points in $Z_2 = \{(1,1), (-1,-1)\}$. Therefore only the following two combinations out of ( 7 ) are appropriate

(8) $$\begin{cases} m \ \text{even}, & l \ \text{integer} \\ m \ \text{odd}, & l \ \text{half integer} \end{cases}$$

The inequivalent spin structure on $\bar{M}$ yields the covariant derivative $\nabla'$ and the Dirac operator D', related to $\nabla$ and D by

(9) $$\nabla' = \tilde{L} \nabla \tilde{L}^{-1} \quad \text{and} \quad \not{D}' = \tilde{L} \not{D} \tilde{L}^{-1} ,$$

where $\tilde{L}(u) = u \det(u)^{-1/2}$ covers the Lorentz transformation (5,3), by which $_L\sigma$ and $_R\sigma$ are related. This yields the spectrum

(10) $$\begin{cases} m \mp (2l+1) + \frac{1}{2} & \text{for} \quad \gamma^5 \psi = \psi \\ m \pm (2l+1) - \frac{1}{2} & \text{for} \quad \gamma^5 \psi = -\psi \end{cases} ,$$

out of which again only the combination ( 8 ) is possible. However, this exactly completes the table ( 7 ) . This can be also seen by gauging away the relative twist of inequivalent spin structure maps $\eta$ and $\eta'$, which results in antiperiodicity of spinors (note that L is discontinuous due to the presence of a square root). Therefore in ( 7 )

(11) $$\begin{cases} m \quad \text{even,} \quad l \ \text{half integer} \\ m \quad \text{odd,} \quad l \ \text{integer} \end{cases}$$

are also possible for 'exotic' spinors.

None of these possibilities gives zero eigenmodes of the massless Dirac operator on $\bar{M}$. The reason is the presence of 1/2 in the spectrum. If follows, that the kernel of the Dirac operator is trivial, and the most straightforward way to get unitary representations of Conf(M) on spinor fields gives the trivial representation.

There may be some ways out. One can admit the torsion, work on the open covering $M^{u}$ [125], or on $\bar{\bar{M}}$ = U(1) x U(2) where "exotic", i.e. antiperiodic on U(1) spinors admit zero eigenmodes. One can minimally couple the spinors to some extra gauge field on $\bar{M}$

in order to kill the 1/2 summand. The simplest possibility is the U(1)-gauge potential

(12) $$A = 1/4 \det^{-1}(u) d \det(u),$$

i.e. $A^{o} = i/2\, E^{o}$, $A^{i} = 0$. Then the zero eigenmodes of the Dirac operator $i\not{D} + \not{A}$ on M have +1 helicity for "normal" spinors and (-1) for "exotic" spinors, or vice versa for A'=-A. For a fixed gauge field A (12) the conformal invariance is explicitly broken and to restore it one should go over to the corresponding field theory with additional conformally invariant equations for A itself. This may be interesting for a Yang-Mills system with massless fermions [33, 119]. It seems that the conformal compactification may be relevant only for spinors in Yang-Mills theories.

At this point some remarks are in order. There exists a well established theory of induced representations of the conformal group, and all unitary representations should be included at this list, c.f. [128]. On the other hand, the unitary representations of the conformal group induced from the representation of the little (Weyl) subgroup, or rather from the $(1/2,0) \oplus (0,1/2)$ representation of the Lorentz group, are related to Dirac spinors. According to our result, the obtained in that way equations of motion either are nonfree (in the sense described in a moment), or are not written on $\bar{M}$.

Let us stress again, that we followed the standard differential-geometric way to introduce Dirac spinors in pseudo-riemannian spaces, and unitarily implement the conformal group. The negative result, i.e. the triviality of such representation, concerns only the case of massless, free, Dirac spinors on $\bar{M}$, by which we mean solutions of $\not{\nabla}\psi = 0$, where $\not{\nabla}$ is defined by (4). These spinors are subject only to interact with a fixed conformally flat metric, and topology of $\bar{M}$. This result does not concern the covering spaces of $\bar{M}$, and the interacting spinors on $\bar{M}$, by which we mean either presence of torsion, or additional gauge fields on $\bar{M}$. In fact this last possibility seems to be the most interesting one, and can make a virtue of our negative result. We intend to investigate these additional forces, and the conformal invariance of the coupled system, in future.

## 6. CONCLUSIONS.

We have considered spin structures over topologically nontrivial spaces M, and collected examples of non-spin manifolds. The generalized spin structure, when charged (single or multiplet) spinors propagate in a background gauge field A on M, can be easier defined on M. However, then some relations between charges of particles with different parity of spin arise, and it is not known what happens if A becomes a dynamical field. Another approach to fermions, by the Dirac-Kähler equation, seems to be conceptually different, also from the global point of view.

As a consequence of the definition, the possibility of inequivalent spin structures on multiply connected M arises. They yield in general different physical results. We have investigated the way in which spin structures, and spinor fields transform under conformal mappings. The main result is that the double covering of the connected conformal group of M can be represented in the chosen class of spinors, while disconnected transformations lift, in general, to maps which change the spin structure. In particular, the known conformal invariance (up to a conformal factor) of the Dirac operator has to be generalized: conformal maps intertwine the Dirac operators of (possibly) nonequivalent spin structures.

The spinor fields on the minimal conformal compactification $\bar{M}$ of $R^{1,3}$ were discussed in detail. The two inequivalent spin structures correspond to left or right invariant global frames on $\bar{M}$, realized as U(2). In the Minkowski space-time $R^{1,3}$, they correspond to two frames, which are Lorentz rotated one with respect to another. The total twist along every isotropic line is an odd multiple of $2\pi$, while along every space- or timelike line is an even multiple of $2\pi$. This can be translated into relative periodicity conditions for spinors. The "parity" inversion on U(2) is shown to interchange these inequivalent spin structures, hence both of them should be included in a theory, which implements this fundamental symmetry.

Applications of $\bar{M}$ are limited by the existence of time like loops, i.e. the lack of global causality. However $\bar{M}$ can be modelled as particular asymptotic conditions in $R^{1,3}$. The kernel of $\not{\partial}$ on $\bar{M}$ is shown to be trivial. Therefore in order to unitarily implement the conformal group in the space of spinors on $\bar{M}$, the additional gauge fields should be introduced. In the simplest considered case of U(1)-field, the solutions of $\not{D}\psi = 0$ are of definite chirality (different for different spin structures), and obviously are interchanged by the parity inversion.

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# 7. APPENDIX A

TABLE 1. Matrix forms of $R_{s,t}$ for $s+t \leq 7$ ( $D(n) = L(D^n)$ ).

| s+t \ s-t | -7 | -6 | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | R | | | | | | | |
| 1 | | | | | | | C | | $^2$R | | | | | | |
| 2 | | | | | | H | | R(2) | | R(2) | | | | | |
| 3 | | | | | $^2$H | | C(2) | | $^2$R(2) | | C(2) | | | | |
| 4 | | | | H(2) | | H(2) | | R(4) | | R(4) | | H(2) | | | |
| 5 | | | C(4) | | $^2$H(2) | | C(4) | | $^2$R(4) | | C(4) | | $^2$H(2) | | |
| 6 | | R(8) | | H(4) | | H(4) | | R(8) | | R(8) | | H(4) | | H(4) | |
| 7 | $^2$R(8) | | C(8) | | $^2$H(4) | | C(8) | | $^2$R(8) | | C(8) | | $^2$H(4) | | C(8) |

TABLE 2. Matrix forms of $Spin_0(s,t) = Spin_0(t,s)$ for $s+t \leq 6$

| s+t \ s-t | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | | $Z_2$ | | | | | |
| 2 | GL(1,R) | | U(1) | | | | |
| 3 | | SL(2,R) | | SU(2) | | | |
| 4 | SL(2,R)xSL(2,R) | | SL(2,C) | | SU(2)xSU(2) | | |
| 5 | | Sp(4,R) | | Sp(1,1;H) | | Sp(2,H) | |
| 6 | SL(4,R) | | SU(2,2) | | SL(2,H) | | SU(4) |

TABLE 3. Automorphism groups (2.4,19) of the scalar product < , >, c.f. [35].

| s+t \ s-t | -7 | -6 | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | O(1) | | | | | | | |
| 1 | | | | | | | O(1,C) | | $^2$O(1) | | | | | | |
| 2 | | | | | | SO*(2) | | O(1,1) | | O(2) | | | | | |
| 3 | | | | | GL(1,H) | | U(1,1) | | GL(2,R) | | U(2) | | | | |
| 4 | | | | Sp(2,2) | | Sp(2,2) | | Sp(4,R) | | Sp(4,R) | | Sp(4) | | | |
| 5 | | | Sp(4,C) | | $^2$Sp(2,2) | | Sp(4,C) | | $^2$Sp(4,R) | | Sp(4,C) | | $^2$Sp(4) | | |
| 6 | | Sp(8,R) | | Sp(4,4) | | Sp(4,4) | | Sp(8,R) | | Sp(8,R) | | Sp(4,4) | | Sp(8) | |
| 7 | GL(8,R) | | U(4,4) | | GL(4,H) | | U(4,4) | | GL(8,R) | | U(4,4) | | GL(4,H) | | U(8) |

TABLE 4. Automorphism groups (2.4,19) of the scalar product $< , >_\alpha$, c.f. [35].

| s+t \ s-t | -7 | -6 | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | O(1) | | | | | | | |
| 1 | | | | | | | U(1) | | GL(1,R) | | | | | | |
| 2 | | | | | | Sp(2) | | Sp(2,R) | | Sp(2,R) | | | | | |
| 3 | | | | | $^2$Sp(2) | | Sp(2,C) | | $^2$Sp(2,R) | | Sp(2,C) | | | | |
| 4 | | | | Sp(4) | | Sp(2,2) | | Sp(4,R) | | Sp(4,R) | | Sp(2,2) | | | |
| 5 | | | U(4) | | GL(2,H) | | U(2,2) | | GL(4,R) | | U(2,2) | | GL(2,H) | | |
| 6 | | O(8) | | SO*(8) | | SO*(8) | | O(4,4) | | O(4,4) | | SO*(8) | | SO*(8) | |
| 7 | $^2$O(8) | | O(8,C) | | SO*(8) | | O(8,C) | | $^2$O(4,4) | | O(8,C) | | $^2$SO*(8) | | O(8,C) |

5. Examples of the construction of matrix representation of $R_{s,t}$ by (1.4,11).

$\underline{R_{3,0}} \approx R^{(+)}_{3,1}$ , $R^{3,0} = \mathrm{span}_R(e_1,e_2,e_3)$ , $p = 1/2(1 + e_3)$

$pR_{3,0}p \approx \mathrm{span}_R \{1, e_{123}\} p \approx C$ , where $1 \leftrightarrow 1$ , $e_{123} \longleftrightarrow i$ . Here,

$i \rightarrow -i$ under every reflection in $R^{3,0}$ and $\beta(p) = p, \alpha\beta(p) \neq p$.

Then $t_{+1} = 1$ , $t_{-1} = e_2e_3$ , and a basis of $S_{+1}(V)$ is $b_{+1} = p$ ,

$b_{-1} = (e_2e_3)^{-1}p = e_3e_2p$ .

$e_1b_{+1} = b_{-1}e_{123}p$ , $e_1b_{-1} = b_{+1}(-e_{123}p)$, $e_1 \leftrightarrow \begin{bmatrix} 0,-i \\ i,\ 0 \end{bmatrix}$ ,

$e_2b_{+1} = -b_{-1}$ , $e_2b_{-1} = -b_{+1}$ $e_2 \leftrightarrow \begin{bmatrix} 0,-1 \\ -1,\ 0 \end{bmatrix}$ ,

$e_3b_{+1} = b_{+1}$ , $e_3b_{-1} = -b_{-1}$ $e_3 \leftrightarrow \begin{bmatrix} 1,\ 0 \\ 0,-1 \end{bmatrix}$ .

$\underline{R_{1,3}}$ , $R^{1,3} = \mathrm{span}(e_o,e_1,e_2,e_3)$ , $p = 1/2(1 + e_o)$ ,

$pR_{1,3}p = \mathrm{span}(1,e_{12},e_{23},e_{31}) \approx H$ , where $1 \leftrightarrow 1$ , $e_{12} \leftrightarrow i$ , $e_{23} \leftrightarrow j$, $e_{31} \leftrightarrow k$.

Here $\beta(p) = p$, $\alpha\beta(p) \neq p_o$ . Then $t_{+1} = 1$ , $t_{-1} = e_3$, $b_{+1} = p$, $b_{-1} = e_3p$ .

$e_ob_{+1} = b_{+1}$, $e_ob_{-1} = -b_{-1}$ , $e_o \leftrightarrow \begin{bmatrix} 1,\ 0 \\ 0,-1 \end{bmatrix}$

$e_1b_{+1} = b_{+1}(-k)$ , $e_1b_{-1} = b_{+1}(-k)$ , $e_1 \leftrightarrow \begin{bmatrix} 0,-k \\ -k,\ 0 \end{bmatrix}$

$e_2b_{+1} = b_{-1}j$ , $e_2b_{+1} = b_{+1}j$ , $e_2 \leftrightarrow \begin{bmatrix} 0,\ j \\ j,\ 0 \end{bmatrix}$

$e_3b_{+1} = b_{-1}$ , $e_3b_{-1} = -b_{+1}$ , $e_3 \leftrightarrow \begin{bmatrix} 0,-1 \\ 1,\ 0 \end{bmatrix}$

$\underline{R_{3,1}}$ , $R^{3,1} = \mathrm{span}(e_1,e_2,e_3,e_4)$ , $p = 1/4(1 + e_1)(1 + e_3e_4)$ ,

$pR_{3,1}p \approx \mathrm{span}\{1\} \approx R$ .

$t_{(1,1)} = 1$ , $t_{(1,-1)} = e_2e_3$ , $t_{(-1,1)} = e_2$ , $t_{(-1,-1)} = e_3$ ,

$b_{(1,1)} = p$ , $b_{(1,-1)} = e_2e_3p$ , $b_{(-1,1)} = e_2p$ , $b_{(-1,-1)} = e_3p$ ,

$e_1b_{(1,1)} = b_{(1,1)}$, $e_1b_{(1,-1)} = b_{(1,-1)}$, $e_1(b_{-1,1}) = -b_{(-1,1)}$, $e_1(b_{-1,-1}) = -b_{(-1,-1)}$

$e_2b_{(1,1)} = b_{(-1,1)}$, $e_2b_{(1,-1)} = b_{(-1,-1)}$, $e_2b_{(-1,1)} = b_{(1,1)}$ , $e_2b_{(-1,-1)} = b_{(1,-1}$

$e_3b_{(1,1)} = b_{(-1,-1)}$, $e_3b_{(1,-1)} = -b_{(-1,1)}$, $e_3b_{(-1,1)} = -b_{(1,-1)}$, $e_3b_{(-1,-1)} = b_{(1,1)}$

$e_4b_{(1,1)} = b_{(-1,-1)}$, $e_4b_{(1,-1)} = b_{(-1,1)}$, $e_4b_{(-1,1)} = -b_{(1,-1)}$, $e_4b_{(-1,-1)} = -b_{(1,1)}$

$e_1 \leftrightarrow \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{bmatrix}$, $e_2 \leftrightarrow \begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}$, $e_3 \leftrightarrow \begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot \\ \cdot & -1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}$, $e_4 \leftrightarrow \begin{bmatrix} \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}$ .

## APPENDIX B

## ČECH COHOMOLOGY

Any two sheaf cohomology theories on M are uniquely isomorphic. The Čech cohomology for differetiable manifolds seems to be best suited for a physical point of view at fibre bundles. Let us sketch now basic facts of its description; for details see ch. 5 of a beautiful Warner's book [26].

The q-simplex $\sigma$ is a collection $(U_o, \dots, U_q)$ of open sets (out of the given cover $\{U_\alpha\}$ of M) with nonempty intersection $|\sigma| = U_o \cap U_1 \dots U_q \neq \emptyset$. The i-th face of $\sigma$ is the q-1 simplex $\sigma^i = (U_o, \dots, U_{i-1}, U_{i+1}, \dots, U_q)$.

Let for $q < 0$ $C^q(\{U_\alpha\}, S)$ be zero, and for $q \geqslant 0$ be the K-module of cochains, that is functions which assign to each q-simplex $\sigma$ an element of $\Gamma(|\sigma|, S)$. The K-module $\Gamma(|\sigma|, S)$ consists of sections $f: |\sigma| \rightarrow S$, $\pi \circ f = id$ from $|\sigma| \subset M$ to a sheaf $S = \{S, \pi, |\sigma|\}$ of K-modules over $|\sigma|$, where:

(i) $\pi: S \rightarrow |\sigma|$ is a local homeomorphism onto $|\sigma|$,

(ii) $\pi^{-1}(p)$ is a K-module for each $p \in |\sigma|$,

(iii) the composition laws are continuous in topology of S.

The coboundary homomorphism $d: C^q(\{U_\alpha\}, S) \rightarrow C^{q+1}(\{U_\alpha\}, S)$ defined (with a little abuse of notation) by

(B.1) $$df(\sigma) = \sum_{i=0}^{q} (-1)^i f(\sigma^i)$$

with a property $d \circ d = 0$, determines the cochain complex $C^*(\{U_\alpha\}, S)$ whose q-th cohomology module is $\check{H}^q(\{U_\alpha\}, S) = Z^q/B^q$ , a quotient of q-cocycles (ker d) by q-boundaries (im d). The whole construction behaves in a proper way under the refinement of covers $\{U_\alpha''\} < \{U_\alpha'\} < \{U_\alpha\}$, and there exist canonical homomorphisms such that the diagram (B.2) commutes.

$$\text{(B.2)} \qquad \begin{array}{ccc} \check{H}^q(\{U_\alpha\},S) & \longrightarrow & \check{H}^q(\{U_\alpha'\},S) \\ & \searrow \quad \swarrow & \\ & \check{H}^q(\{U_\alpha''\},S) & \end{array}$$

Then one defines the q-th Čech cohomology module for M with coefficients in the sheaf of K-modules S as a direct limit

$$\check{H}^q(M,S) = \operatorname{dir} \lim_{\{U_\alpha\}} \check{H}^q(\{U_\alpha\},S).$$

It can be shown that the Čech cohomology satisfies the axioms of sheaf cohomology theory. Among them we make use of the following property: given a short exact sequence of sheaf homomorphisms (in each K-module $\pi^{-1}(p)$ the image of a given homomorphism is the kernel of the next)

$$\text{(B.3)} \qquad 0 \rightarrow S' \rightarrow S \rightarrow S'' \rightarrow 0$$

there is an exact sequence of homomorphisms

$$\text{(B.4)} \qquad \ldots \rightarrow \check{H}^q(M,S') \rightarrow \check{H}^q(M,S) \rightarrow \check{H}^q(M,S'') \rightarrow \check{H}^{q+1}(M,S') \rightarrow \ldots$$

For applications the ring K is usually taken to be $Z_2$, Z, or R; hence the K-modules are abelian groups or a real vector space respectively.

From considering local sections $\sigma_\alpha : U_\alpha \rightarrow F$ and their transition functions $\varphi_{\alpha\beta}$ with values in the Lie group G, follows that there is a 1-1 correspondence between isomorphism classes of principal G-bundles and the Čech cohomology set $\check{H}^1(M,G)$ of M with coefficients in the constant sheaf G.

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