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USING INDEX THEOREMS IN SUPERSYMMETRIC FIELD THEORIES AND STRINGS

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FIELD THEORIES AND STRINGS

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CONTENTS

I. INDEX THEOREMS

Index theorems demonstrate the existence of a relationship between analytical properties of differential operators on fiber bundles and the topological properties of the fiber bundles themselves. The analytical properties in question concern the number of solutions of differential equations related to the differential operators. The topological properties are expressed by characteristic classes of the fiber bundles involved.

In this chapter we will first review some general mathematical concepts related to the index of differential operators. We will then discuss some of the index theorems for the classical elliptic complexes over compact manifolds without boundary. We will proceed studying the less popular index theorems for operators over non-compact spaces. We will finally comment on the connection between the Witten index for supersymmetric theories and the analytical index of linear operators.

I.1 Mathematical preliminaries

In this section we will briefly review the mathematical concepts involved into the formulation of the index theorems. Our exposition will follow ref. [1] to which we remand for a more detailed discussion of the subject and for further references.

Let M be a smooth manifold without boudary of dimension n. Let E and F be vector bundles over M and

$$D: C^{\infty}(E) \longrightarrow C^{\infty}(F)$$

be a first order differential operator. Using local coordinates $\{x_i\}$ on M,we can write D in the form:

$$D = a_j(x_i)\frac{\partial}{\partial x^j} + b(x_i)$$

where a_j and b are matrix valued.

Definition 1. The symbol of D is defined as its Fourier trasform.

Let (x,k) be local coordinates for the cotangent bundle T*(M). We will refer to k as the Fourier trasform variable.

Definition 2. The leading symbol $\tilde{D}$ of D is the highest order part of its Fourier trasform:

$$\tilde{D}(x,K) \equiv i\, a_j(x)\, K_j$$

Clearly, $\tilde{D}$ defines a linear map from E to F.

Definition 3. D is said to be an elliptic operator if for E=F and $k \neq 0$, $\tilde{D}(x,k)$ is invertible.

Definition 4. Let $\{E_p\}$ be a finite sequence of vector bundles over M. Let D_p: $C^\infty(E_p) \to C^\infty(E_{p+1})$ a sequence of differential operators. The sequences $\{E_p\}$, $\{D_p\}$ define a complex, denoted by (E,D), iff $D_{p+1} D_p = 0$.

Definition 5. Let D_p^* : $C^\infty(E_{p+1}) \to C^\infty(E_p)$ the dual map of D_p. Define the associated

laplacian as:

$$\Delta_p \equiv D_p^* D_p + D_{p-1} D_{p-1}^* \qquad \text{(I.1)}$$

The complex (E,D) is elliptic iff Δ_p is an elliptic operator on $C^\infty(E_p)$.

Equivalently, (E,D) is an elliptic complex if:

$$\mathrm{Ker}\ \tilde{D}_p(x,K) = \mathrm{Img}\ \tilde{D}_{p-1}(x,K)\,,\quad K \neq 0 \tag{I.2}$$

The De Rham complex is the most familiar example of elliptic complex. In this case $E_p = \Lambda^p(M)$, the bundle of the p-forms on M. $D_p = d$, the exterior derivative acting on p-forms. The laplacian $\Delta_p = dd^* + d^*d$ is clearly elliptic since its symbol is $\tilde{D}(x,k) = |k|^2$.

The following theorem is the generalization to elliptic complexes of the Hodge decomposition theorem:

Theorem 1. If $f_p \in C^\infty(E_p)$, f_p can be uniquely decomposed as:

$$f_p = D_p\ f_{p-1} + D_p^*\ f_{p-1} + h_p \tag{I.3}$$

where h_p is harmonic, i.e. $\Delta_p h_p = 0$.

Note that $\mathrm{Ker}\ D_p \supset \mathrm{Img}\ D_{p-1}$ since $D_p\ D_{p-1} = 0$. We may thus define the cohomology groups for the elliptic complex (E,D) as:

$$H^p(E,D) \equiv \mathrm{Ker}(D_p)/\mathrm{Img}(D_{p-1}) \tag{I.4}$$

From the previous theorem it follows that each cohomology class contains a unique harmonic representative. Therefore:

$$H^p(E,D) \simeq \mathrm{Ker}\ \Delta_p \tag{I.5}$$

Cohomology groups are finite dimensional.

We are now in condition to formulate the following

Definition 6. The index of an elliptic complex (E,D) is defined as:

$$\begin{aligned} \operatorname{index}(E,D) &= \sum_p (-1)^p \dim H^p(E,D) \\ &= \sum_p (-1)^p \dim \operatorname{Ker} \Delta_p \end{aligned} \tag{I.6}$$

In the case of the De Rham complex one has

$$H^p(E,D) = H^p_{DR}(M) \tag{I.7}$$

where H^p_{DR} are the De Rham cohomology classes (the p-forms on M which are closed but not exact). The De Rham theorem states that the De Rham cohomology is actually isomorph to the symplicial cohomology $H^p(M,R)$. The number

$$b_p = \dim H^p(M,\mathbb{R}) = \dim H_p(M,\mathbb{R}) \tag{I.8}$$

is defined to be the p-th Betti number of M. The index of the De Rham complex can therefore be written as:

$$\begin{aligned} \operatorname{index}(\Lambda^*,d) &= \sum_p (-1)^p \dim H^p_{DR}(M,\mathbb{R}) \\ &= \sum_p (-1)^p b_p \equiv \chi(M) \end{aligned} \tag{I.9}$$

$\chi(M)$ is the Euler charcteristic of the manifold M. We see how the De Rham theorem allows one to relate topological quantities - the Betti numbers defined in terms of the symplicial cohomologies $H^p(M,R)$ - to solutions of differential equations -the harmonic forms on M.

It is sometimes convenient, given an elliptic complex (E,D), to build a two-terms elliptic complex with the same index. Define

$$F_0 \equiv \bigoplus_p E_{2p} \qquad , \quad F_1 \equiv \bigoplus_p E_{2p+1} \tag{I.10}$$

to be the even and odd bundles respectively. Consider the operators:

$$L = \bigoplus_p \left(D_{2p} + D^*_{2p-1} \right) \quad , \quad L : C^\infty(F_0) \longrightarrow C^\infty(F_1)$$

(I.11)

$$L^* = \bigoplus_p \left(D^*_{2p} + D_{2p-1} \right) \quad , \quad L^* : C^\infty(F_1) \longrightarrow C^\infty(F_0)$$

The laplacians corresponding to F_0 and F_1 are:

$$\Box_0 \equiv L^* L = \bigoplus_p \Delta_{2p}$$

$$\Box_1 \equiv L L^* = \bigoplus_p \Delta_{2p+1}$$

The index of the complex (F,L) is therefore:

$$\begin{aligned} \text{Index}\,(F,L) &= \dim \ker \Box_0 - \dim \ker \Box_1 = \\ &= \sum_p (-1)^p \dim \ker \Delta_p = \\ &= \text{index}\,(E,D) \end{aligned} \tag{I.12}$$

We will often refer to the index of the two-components complex (F,L) more simply as the index of L. Eq.(I.12) shows how every elliptic complex can be "rolled up" to give an operator L as in eq.(I.11) whose index equals the index of the original complex.

In the example of the De Rham complex, F_0 and F_1 are the bundles of the even and odd forms respectively. The operator L is given by $d + d^*$ and the Euler characteristic is the sum of the even Betti numbers minus the sum of the odd Betti numbers.

I.2 Index theorems on compact manifolds

Atiyah and Singer gave a general formula for the index of any elliptic complex on compact manifolds without boundary, in terms of characteristic classes of fiber bundles related to the complex [2].

Despite its mathematical beauty and compactness, the Atiyah-Singer index theorem, in its general formulation, is probably much too abstract and unmanageable for most physicists. Therefore we will prefer to consider separately the particular complexes which will be of interest to us in the next chapters and to give explicit formulas for the indices of each of them.

The index of any elliptic complex over <u>odd</u> <u>dimensional</u> <u>compact</u> manifold without boundary is zero. Therefore we will be always considering, in this section, manifold of <u>even</u> dimension.

I.2.i The De Rham complex

We already met the De Rham complex in the previous paragraph. One can roll it up introducing the bundles of the even and odd forms:

$$F_0 \equiv \bigoplus_p \Lambda^{2p}(M) \quad , \quad F_1 \equiv \bigoplus_p \Lambda^{2p+1}(M)$$

The differential operator defined in (I.11) will be

$$L = d + d^*$$

As we pointed out, the index of the De Rham complex, by virtue of the De Rham theorem, equals the Euler characteristic of the manifold:

$$\text{index}(\Lambda^*, d+d^*) = \chi(M)$$

The <u>Gauss-Bonnet index theorem</u> expresses the index of the De Rham complex in terms of the integral of the <u>Euler characteristic class</u>: :

$$\text{index}(\Lambda^*, d+d^*) = \chi(M) = \int_M e(M) \tag{I.13}$$

where e(M) is the Euler form defining the Euler characteristic classes in the following way. Consider a Riemannian connection on a n=2r dimensional (oriented) real vector bundle. The corresponding curvature two-form

$$R^{ab} = \frac{1}{2} R^{ab}{}_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu} \tag{I.14}$$

($R^{ab}{}_{\mu\nu}$ is the Riemann tensor, $a,b,\mu,\nu = 1,\ldots,n$) will take values into the SO(n) Lie algebra. Being an anti-symmetric real matrix (whose elements are two-forms)it can formally be put into the "block-diagonal" form:

$$\frac{R^{ab}}{2\pi} = \begin{pmatrix} \begin{matrix} 0 & X_1 \\ -X_1 & 0 \end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix} 0 & X_r \\ -X_r & 0 \end{matrix} \end{pmatrix} \tag{I.15}$$

The <u>Euler form</u> is defined as the following n-form:

$$e(M) \equiv X_1 \cdots X_r \tag{I.16}$$

One defines e(M) = 0 for odd dimensional manifolds.

A polynomial $P(\alpha)$ of a complex nxn matrix α is said to be an <u>invariant</u> or <u>characteristic</u> polynomial if $P(\alpha) = P(g^{-1}\alpha\, g)$, $g \in GL(n,C)$. From its definition it is evident that the Euler form e(M) is a SO(n)-invariant, real, polynomial in the curvature two-form R^{ab}.

If $P(\alpha)$ is an invariant polynomial and Ω indicates a matrix valued curvature two-form (e.g. a Riemannian curvature or a gauge field strength), then one can show the following important properties:

(a) $dP(\Omega)=0$, that is $P(\Omega)$ is closed.

(b) $P(\Omega)$ has topologically invariant integrals. This means that if Ω and Ω' are two different curvatures corresponding to the connections ω and ω', then

$$P(\Omega') - P(\Omega) = dQ(\omega', \omega)$$

Let us indicate by $p_j(\Omega)$ the homogeneous polynomial of degree j in in the expansion of a characteristic polynomial $P(\Omega)$. Since $P(\Omega)$ is closed, $p_j(\Omega)$ are closed too. Thus the $p_j(\Omega)$ define 2j-cohomology classes: $p_j(\Omega) \in H^{2j}(M)$. The cohomology classes defined by $p_j(\Omega)$ are called <u>characteristic</u> classes. Because of (b) they do not depend on the particular connection chosen on the bundle.

Let us concluding this sub-section giving an explicit expression for the Euler form in terms of components:

$$e(M) = \frac{1}{2^r(2\pi)^r} \frac{1}{r!} \epsilon^{i_1 \cdots i_n} R^{i_1 i_2} \wedge \cdots \wedge R^{i_{n-1} i_n} \tag{I.17}$$

I.2.ii The signature complex

The De Rham complex is associated to the decomposition of the exterior bundle $\Lambda^* M$ into the bundles of the even and odd forms. $\Lambda^*(M)$ can be splitted in a second way which gives rise to the signature complex. One has to restrict oneself to oriented, n=2r dimensional manifolds. Consider the operator ω acting on p-forms

$$\omega \equiv i^{p(p-1)+\frac{n}{2}} \quad * \tag{I.18}$$

where $*$ is the Hodge duality trasformation. One has:

$$\begin{aligned} \omega\,(d+d^*) &= -\,(d+d^*)\,\omega \\ \omega^2 &= 1 \end{aligned} \tag{I.19}$$

Let $\Lambda^\pm$ be the ± 1 eigenspaces of ω. Because of (I.19)

$$d+d^* \;:\; C^\infty(\Lambda^+) \longrightarrow C^\infty(\Lambda^-)$$

The complex $(\Lambda^\pm, d+d^*)$ is called the signature complex. Because of the Poincaré duality between $\Lambda^p(M)$ and $\Lambda^{n-p}(M)$ the contributions to the index of the signature complex of the harmonic forms with eigenvalues ± 1 for cancel except for $\Lambda^r(M)$. Also it is easy to see that the index is zero unless r is even, that is unless n=4k. Thus one gets for the index of the signature complex

$$
\begin{aligned}
\operatorname{index}(\Lambda^{\pm}, d+d^{*}) &= \dim H_{+}^{2k}(M,\mathbb{R}) \\
&\quad - \dim H_{-}^{2k}(M,\mathbb{R}) \\
&\equiv \tau(M) \\
&\equiv \text{Hirzebruch signature of } M
\end{aligned}
\tag{I.20}
$$

where $H_{\pm}^{2k}(M,R)$ are the harmonic 2k-forms with eigenvalues ± 1 under ω.

The index theorem for the signature complex is known as the Hirzebruch index theorem and reads like:

$$\tau(M) = \int_M L(M) \tag{I.21}$$

where L(M) is an invariant polynomial called the Hirzebruch characteristic L-polynomial. L(M) is defined in terms of the formal eigenvaules of the matrix of the curvature two-form R^{ab}:

$$L(M) \equiv \prod_j \frac{x_j}{\tanh x_j} \tag{I.22}$$

the x_j are the same as in eq.(I.15).

I.2.iii The spin complex

The Dirac operator on an euclidean-signature Riemannian spin manifold is defined through the covariant derivative with respect to the basis of orthonormal frames of the cotangent bundle T*(M):

$$
\begin{aligned}
D &= \gamma^a e^{\mu}{}_a(x) \; D_\mu(x) \\
&= \gamma^a e^{\mu}{}_a(x) \left(\partial_\mu + \tfrac{1}{4} [\gamma^b, \gamma^c] \, \omega_\mu^{bc} \right)
\end{aligned}
\tag{I.23}
$$

where $e^{\mu}{}_a$ are the inverse of the "vielbein" $e^a{}_\mu$, $e^a{}_\mu e_{\nu a} = g_{\mu\nu}$, ω_μ^{bc} is the spin-connection, γ^a are the Dirac (euclidean) matrices in n dimension. D is an elliptic operator for metrics with euclidean signature. n being even, we can split the Dirac spinors ψ (x) into the eigenspaces with chirality ±1:

$$
\gamma_{n+1} \, \psi_\pm(x) = \pm \, \psi_\pm(x) \tag{I.24}
$$

γ_{n+1} is the generalization to n dimensions of the usual four-dimensional γ_5 matrix. The spinors $\psi_\pm$ (x) are sections of the spin bundles $\Delta_\pm$. The spin complex is defined by:

$$
\begin{aligned}
D &: C^\infty(\Delta_+) \longrightarrow C^\infty(\Delta_-) \\
D^* &: C^\infty(\Delta_-) \longrightarrow C^\infty(\Delta_+)
\end{aligned}
\tag{I.25}
$$

The index of the spin complex is:

$$\text{index}\,(\Delta_\pm, D) = \dim \operatorname{Ker} D - \dim \operatorname{Ker} D^* = n_+ - n_-$$

where $n_\pm$ are the numbers of normalizable zero modes of the Dirac operator with chirality ± 1.

The index theorem for the spin complex is:

$$n_+ - n_- = \int_M \hat{A}(M) \tag{I.26}$$

where $\hat{A}$ (M) is the <u>A-roof genus</u>, a characteristic polynomial defined as follows:

$$\hat{A}(M) = \prod_{i=1}^{n/2} \frac{x_i/2}{\sinh x_i/2} \tag{I.27}$$

where the x_i are defined as in eq.(I.15).

Physicists often consider spinors $\psi^A(x)$ with an additional "gauge" index A, A=1,...,m, labeling a m-dimensional representation $\underline{r}$ of some Lie group G. This corresponds to take the tensor product $\Delta_\pm \otimes V$ of the spin bundles $\Delta_\pm$ with the vector bundle V, whose fibers are isomorphic to the linear spaces of the representation $\underline{r}$. On $\Delta_\pm \times V$ one considers the

Dirac operator D_V which includes the gauge connection on V:

$$D_V \equiv \gamma^a e_a{}^\mu \left(\partial_\mu + \frac{1}{4}[\gamma^b,\gamma^c]\,\omega_\mu^{bc} + i\,(T^i)_{AB} A^i_\mu \right) \qquad (I.23)$$

where $(T^i)_{AB}$, i=1,...,dimension of G, are the mxm matrices relative to the representation r of the Lie algebra of G. A^i_μ are the usual gauge fields.

The index theorem for the twisted spin complex ($\Delta_\pm$ x V, D_V) is:

$$\mathrm{index}\,(\Delta_\pm \otimes V,\, D_V) = \int_M \hat{A}(M) \wedge ch(V) \qquad (I.29)$$

ch(V) is the Chern characteristic polynomial of V. It is defined as:

$$ch(V) = \mathrm{tr}\,\exp\,\frac{F}{2\pi} \qquad (I.30)$$

where $F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$ are the gauge curvature two-forms with value in the Lie algebra of the gauge group G.

I.3 Index theorems on non-compact manifolds

In the context of quantum field theory the Atiyah-Singer index theorem has found applications to problems admitting a compactified formulation. For example, in the case of istantons, the euclidean space-time compactification is necessary in order to make the problem well defined and the Atiyah-Singer index theorem applicable.

On the other hand, there exist classes of operators (which include the Dirac operator in Minkowski space-time) which have well defined and non vanishing index when defined on non-compact, odd dimensional manifolds. The corresponding compactified index problems are obviously non equivalent since, as we mentioned in section I.2, the index of any elliptic differential operator on odd dimensional manifolds is zero.

We will see in the next chapter how differential operators on non-compact manifolds naturally arise in the topological non-trivial sectors of quantum field theories.

When dealing with operators on open spaces one has to resort to index formulas others than those discussed in section I.2. In the following of this section we will consider index theorems due to Callias, Bott and Seeley [3] and to E.Weinberg [4].

A general method to derive an index theorem for a linear operator L is to evaluate traces of the type:

$$Z_s(\beta) = tr\left(e^{-\beta L^* L} - e^{-\beta L L^*} \right) \tag{I.31}$$

where β is a positive real parameter, or else of the type:

$$J(z) = tr\left(\frac{z}{z + L^*L} - \frac{z}{z + LL^*} \right) \tag{I.32}$$

where z is a complex variable. (Notice that J(z)/z is just the Laplace trasform of $Z_s(\beta)$).

If L is defined on a compact manifold the traces above are actually independent on β and z and equal to the index of L. This is easy to show: on compact manifolds the spectra of the self-adjoint operators LL* andL*L are discrete. Thus the traces (I.31) and (I.32) can be written as the sum over the eigenstates of LL* and L*L. For any given eigenstate $|\lambda\rangle$ of L*L, with non zero eigenvalue λ , there exists a corresponding eigenstate of LL* with the same eiegenvalue, and vice-versa. In fact

$$LL^*\left(\frac{L|\lambda\rangle}{\sqrt{\lambda}}\right) = L\,\frac{\lambda\,|\lambda\rangle}{\sqrt{\lambda}} = \lambda\left(\frac{L|\lambda\rangle}{\sqrt{\lambda}}\right) \quad ;$$

$$\left\| \frac{L|\lambda\rangle}{\sqrt{\lambda}} \right\| = 1$$

Hence, the contribution of the LL* and L*L eigenstates with non zero eigenvalues cancel in the traces (I.31) and (I.32). One is left with the eigenstates with zero eigenvalue only. So, for operator over compact manifolds, one has

$$\begin{aligned} J(z) = Z_s(\beta) &= \dim \operatorname{Ker} L^*L - \dim \operatorname{Ker} LL^* \\ &= \operatorname{index}(L) \end{aligned} \tag{I.33}$$

One interesting feature of the index theorems for operator defined on open spaces is that the contributions of the L*L and LL* eigenstates with non zero eiegenvalues to traces like $Z_s(\beta)$ and $J(z)$ do <u>not</u>, in general, cancel. Thus the traces (I.31) and (I.32) do depend, generally, non-trivially on the parameters β and z. That is due to the fact that, in this case, the spectra of L*L and LL* has <u>continuous</u> parts. The <u>densities</u> of the eiegenstates of L*L and LL* with eigenvalues belonging in the continuous spectrum do not necessarely match. One can rewrite the trace in (I.32), for example, as follows:

$$J(z) = \mathrm{tr}\left(\frac{z}{z+L^*L} - \frac{z}{z+LL^*}\right) =$$

$$= \sum_{\substack{\text{discrete}\\ \text{spectrum}}} \left(\frac{z}{z+\lambda_n} - \frac{z}{z+\lambda_n} \right) +$$

$$+ \int_0^\infty \left(\frac{dn_+(\lambda)}{d\lambda} - \frac{dn_-(\lambda)}{d\lambda}\right) \frac{z}{z+\lambda}\, d\lambda \tag{I.34}$$

$$= \text{index } L + \int_0^\infty \left(\frac{dn_+(\lambda)}{d\lambda} - \frac{dn_-(\lambda)}{d\lambda}\right) \frac{z}{z+\lambda}\, d\lambda$$

where $\frac{dn_+}{d\lambda}$, $\frac{dn_-}{d\lambda}$ are respectively the densities of eigenstates of L*L and LL* with eigenvalue λ .(The index of L counts the difference of the normalizable zero eigenvalue eigenstates of L*L and LL*, which thus belong in the discrete part of the spectrum.)

If the difference $\left(\frac{dn_+}{d\lambda} - \frac{dn_-}{d\lambda}\right)$ is not too singular as $\lambda \to 0$, it is easy to show that the limit $z \to 0$ in the trace $J(z)$ will give the index of L

$$\text{index } L = \lim_{z \to 0} J(z) \tag{I.35}$$

Analogously, if one considers the trace (I.31):

$$\text{index } L = \lim_{\beta \to \infty} Z_s(\beta) \tag{I.36}$$

The index theorems that we are going to discuss in the next paragraphs provide closed expressions not only for the indices of the operators L but also for the traces $J(z)$ and $Z_s(\beta)$. Therefore they give informations about the entire spectra of L^*L and LL^*, in contrast to the Atiyah-Singer theorem. This fact will be exploited in chapter II to evaluate the quantum corrections to the masses of solitons and monopoles in supersymmetric field theories.

I.3.i The Callias-Bott-Seeley theorem [3]

Consider the following linear differential operator on R^n (n odd):

$$L \equiv i\gamma^i \partial_i \otimes 1\!\!1_m + i\, 1\!\!1_q \otimes \Phi(x) \tag{I.37}$$

where γ^i are the qxq ($q = 2^{\frac{n-1}{2}}$) dimensional γ-matrices in n dimensions; $\Phi(x)$ is a unitary mxm matrix of C^∞ functions, asymptotically homogeneous of order zero as $|x| \to \infty$ .Such a condition on Φ assures that L be a Fredholm operator, that is that zero is an isolated point in the spectra of L*L and LL*. This makes the index problem for the operator in (I.37) well defined and guarantees that eqs. (I.35) and (I.36) hold.

The Callias-Bott-Seeley index theorem states that:

$$\mathrm{tr}\left(\frac{z}{z+L^*L} - \frac{z}{z+LL^*}\right) = \mathrm{index}(L) \cdot \frac{1}{(1+z)^{n/2}} \tag{I.38.a}$$

and

$$\mathrm{index}\, L = \frac{1}{2\left(\frac{n-1}{2}\right)!}\left(\frac{i}{8\pi}\right)^{\frac{n-1}{2}} \int_{S_\infty^{n-1}} \mathrm{tr}\left(\phi(x)\, d\phi\right)^{n-1} \tag{I.38.b}$$

$(d\phi)^{n-1}$ is the n-1 power of the matrix $d\phi$ with the differentials being multiplied by exterior derivation. S_∞^{n-1} is the n-1 dimensional sphere at infinity $|x| \to \infty$.

Some comments on the trace formula (I.38) are in order.

It is easily seen that the index of L is indeed recovered taking the limit lim J(z), as expected. Moreover the index formula (I.38.a) gives zero whenever n is even.

The index of L, being a surface integral, is a topological invariant, i.e. it depends only on the behaviour of the "potential" Φ (x) at infinity $|x| \to \infty$ and it remains invariant under "compact" perturbations of Φ (x).

The trace J(z) depends non trivially on z when the index of L is non vanishing. This means, according to eq.(I.34), that the densities of the eiegenstates of L*L and LL* with eigenvalues in the continuous part of the spectrum are not equal when Φ (x) is topologically non trivial at infinity. It is remarkable that for each fixed value of z the trace J(z) is a topological invariant. This is not obvious but it is has been proven to be a general phenomenon.

The formula (I.38) allows to evaluate directly the difference $\frac{dn_+}{d\lambda} - \frac{dn_-}{d\lambda}$ defined in eq.(I.34):

$$J(z) = \text{index}(L) + \int_{0^+}^{\infty} \left(\frac{dn_+}{d\lambda} - \frac{dn_-}{d\lambda} \right) \frac{z}{z+\lambda}\, d\lambda \qquad \text{(I.34)}$$

(Recall that the asymptotic conditions on Φ (x) imply that $\frac{dn_\pm}{d\lambda} \neq 0$ only for $\lambda > 1$.) Eq.(I.38) becomes:

$$\int_{0^+}^{\infty} \left(\frac{dn_+}{d\lambda} - \frac{dn_-}{d\lambda} \right) \frac{z}{z+\lambda}\, d\lambda = \qquad \text{(I.39)}$$

$$= \frac{1}{z} \left(\frac{1}{(1+z)^{n/2}} - 1 \right) \text{index}(L)$$

$$\equiv f(z)$$

which looks like a dispersion relation for $\frac{dn_+}{d\lambda} - \frac{dn_-}{d\lambda}$. The function f(z) of the complex variable z defined in (I.39) has a cut on the real axis from -1 to $-\infty$, and it is analytic on the rest of the complex plane. Consider the Cauchy formula for f(z):

$$\frac{1}{2\pi i} \oint_C \frac{f(\omega)}{\omega - z} \, d\omega = f(z) \qquad \text{(I.40)}$$

where the integration is taken along the contour C depicted in fig.1.

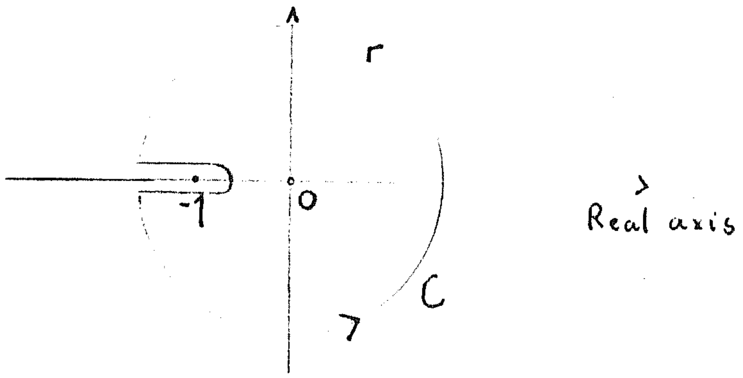

fig.1

Putting

$$\Delta f(\lambda) \equiv \lim_{\epsilon \to 0^+} \left[f(\lambda + i\epsilon) - f(\lambda - i\epsilon) \right] \qquad \text{(I.41)}$$

for the discontinuity of f(z) at the cut, one obtains:

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{-1} \frac{\Delta f(\lambda)}{\lambda - z} \, d\lambda$$

Comparing with eq.(I.39), one concludes that

$$\frac{dn_+}{d\lambda}(\lambda) - \frac{dn_-}{d\lambda}(\lambda) = - \frac{1}{2\pi i} \Delta F(-\lambda) \tag{I.42}$$

As expected the difference $\frac{dn_+}{d\lambda} - \frac{dn_-}{d\lambda}$ receives a non vanishing contribution only from the eigenstates of L*L and LL* with eiegenvalues $\lambda > 1$.(The normalization of Φ (x) has been chosen such that the continuous spectrum starts from $\lambda=1$.)

I.3.ii An index theorem for the Dirac operator in the monopole background

Operators of the typer considered in section I.3.i arise when one works with the static Dirac equation in Minkowski space, in presence of a matrix valued "scalar" field background ϕ (x). ϕ(x) has to be such that L be Fredholm, that is that the continuous spectra of L*L and LL* are separated from the zero by a finite gap. E. Weinberg worked out a generalization of the Callias trace formula which holds for static Dirac operators with both scalar and gauge monopole-like backgrounds. Notice that , in this case, the spectra of L*L and LL* will be continuous starting from zero.

The operator L under consideration is:

$$L \equiv -i\sigma^i D_i + S^a T^a$$

$$\left(D_i \equiv \partial_i + A_i^a T^a \right) \tag{I.43}$$

σ^i are the Pauli matrices, $A_i = A_i^a T^a$(i=1,2,3) are the spatial components of the gauge fields. The gauge choice is $A_0 = 0$. $(T^a)_{bc}$ are the generators of the gauge group SU(2) in the adjoint representation. S^a are scalar field backgrounds trasforming according the adjoint representation of SU(2). The backgrounds A_i^a and S^a are taken to assume a (multi)monopole configuration and to satisfy the Bogomolny equations:

$$(D_i S)^a = B_i^a \equiv \frac{1}{2} \epsilon_{ijk} F_{jk}^a \tag{I.44}$$

F_{ij}^a are the gauge field strengths. The behaviour of the backgrounds at spatial infinity is:

$$B_i^a \xrightarrow{r \to \infty} C \frac{\hat{x}_i \hat{S}^a}{r^2} + O\left(\frac{1}{r^2}\right)$$

$$S^a \xrightarrow{r \to \infty} v \hat{S}^a + O\left(\frac{1}{r}\right) \tag{I.45}$$

$$\hat{S}^a \equiv \frac{S^a(x)}{\left[(S^a)^2\right]^{1/2}}$$

The magnetic charge of the fields configuration is defined as the following surface integral:

$$Q_M \equiv \oint d^2\sigma^i \; S^a B_i^a \tag{I.46}$$

Q_M is a topologically invariant and it is quantized

$$Q_M = 4\pi n \quad , \quad n \in \mathbb{Z}$$

Comparing with eq.(I.45), one sees that C=n.

Weinberg's trace formula is:

$$J(z) = \frac{2ngv}{\left((gv)^2 + z\right)^{1/2}} \tag{I.47a}$$

Once again the index of L is recovered taking the limit for $z \to 0$:

$$\text{index}(L) = \lim_{z \to 0} J(z) = 2n \tag{I.47b}$$

Notice that, in this case too, the trace J(z) is a topological invariant for every fixed value of z.

Let us briefly describe the derivation of eq.(I.47) - as by-products we will get formulas that will be useful in the next chapter. It is convenient to introduce a "four-dimensional" notation:

$$A_\mu \equiv \begin{cases} A_i^a & , \quad \mu = i \\ S^a & , \quad \mu = 4 \end{cases}$$

and (I.48)

$$D_\mu \equiv \begin{cases} D_i & , \quad \mu = i \\ g\, S^a T^a & , \quad \mu = 4 \end{cases}$$

The trace in (I.47) ca be rewritten as:

$$J(z) = - \mathrm{Tr}\ \gamma_5 \frac{z}{-(\gamma\cdot D)^2 + z} \tag{I.49}$$

as it can be seen using the basis

$$\gamma_0 = \begin{pmatrix} 0 & 1\!\!1_2 \\ 1\!\!1_2 & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1\!\!1_2 & 0 \\ 0 & -1\!\!1_2 \end{pmatrix}$$

for which

$$\gamma\cdot D = \begin{pmatrix} 0 & L \\ L^* & 0 \end{pmatrix}$$

Define the non-local "current":

$$I_i(x,y) \equiv \mathrm{tr} \langle x | \gamma_5 \gamma_i \frac{1}{\gamma\cdot D + z^{1/2}} | y \rangle = \mathrm{tr}\ \gamma_5 \gamma_i S(x-y)$$

where the symbol tr indicates the trace over the spinor and color indices only. S (x-y) is the propagator in the monopole background:

$$\begin{aligned}(\gamma^i \partial_{i,x} + \gamma_\mu A_\mu(x) + z^{1/2})\, S(x-y) &= S(x-y)\,(-\overleftarrow{\partial}_{i,y}\gamma^i + \gamma_\mu A_\mu(y) + z^{1/2}) \\ &= \delta(x-y)\end{aligned}$$

Taking the derivatives respect to x_i and y_i ,one gets:

$$\left(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_i}\right) I_i(x,y) = \mathrm{tr}\ \gamma_5\gamma_i\left(\partial_{i,x_i}+\partial_{i,y_i}\right) S(x-y) =$$

$$= -\mathrm{tr}\ \gamma_5\ 2z^{1/2} S(x-y) - \mathrm{tr}\ \gamma_5\gamma_\mu \left[A_\mu(x)-A_\mu(y)\right]$$

(I.50)

Consider now the limit of the expression above for $x \to y$. In three dimensions the second term in the last line of (I.50) vanishes instead of giving an anomaly as it would in four dimensions. Hence:

$$\partial_i I_i(x,x) = -2z^{1/2}\,\mathrm{tr}\ \gamma_5\ S(x,x) =$$

$$= -2z^{1/2}\,\mathrm{tr}\ \langle x|\ \gamma_5 \frac{1}{\gamma\cdot D+z^{1/2}} |x\rangle$$

$$= -2z^{1/2}\,\mathrm{tr}\ \langle x|\ \gamma_5 \frac{-\gamma\cdot D+z^{1/2}}{-(\gamma\cdot D)^2+z} |x\rangle$$

$$= -\ 2\mathrm{tr}\ \langle x| \frac{z}{-(\gamma\cdot D)^2+z} |x\rangle$$

Comparing with eq.(I.49), one concludes:

$$J(z) = \frac{1}{2}\int d^3x\ \partial_i I_i(x,x) = \frac{1}{2}\oint d^2\sigma^i\ I_i(x,x)$$

To evaluate the surface integral on the right-hand side it will be enough to expand the current $I_i(x,x)$ in powers of $1/|x|^2$. Only the terms of order $1/|x|^2$ will give a non-vanishing contribution to the surface integral. Since

$$-(\gamma\cdot D)^2 + z = -D_i^2 - S^2 - \sigma^{\mu\nu}F_{\mu\nu} + z$$

it follows that

$$\frac{1}{-(\gamma\cdot D)^2 + z} = + \frac{1}{-D^2 - S^2 + z} + \frac{1}{-D^2 - S^2 + z}\,\sigma\cdot F\,\frac{1}{-D^2 - S^2 + z} + \cdots \qquad \text{(I.52)}$$

where the subsequent terms fall off more rapidly than $1/|x|^2$ and can be neglected. Asymptotically the propagators are:

$$\begin{aligned}\frac{1}{-D^2 - S^2 + z} = {} & i(\hat{S}\cdot T)\,\frac{1}{-\partial^2 + g^2v^2 + z}\,i(\hat{S}\cdot T) + \\ & + (\mathbb{1} - i\,\hat{S}\cdot T)\,\frac{1}{-\partial^2 + z}\,(\mathbb{1} - i(\hat{S}\cdot T)) \\ & + \cdots\end{aligned} \qquad \text{(I.53)}$$

Plugging eqs.(I.52) and (I.53) into eq.(I.51) and performing the trace over the spinor indices, one gets:

$$J(z) = \frac{1}{2}\oint \hat{x}_i I_i = -\oint \hat{x}_i \epsilon_{i\lambda\mu\nu} \cdot$$

$$\cdot \, tr \langle x | D_\lambda \frac{1}{-D^2-S^2+z} \cdot F_{\mu\nu} \frac{1}{-D^2-S^2+z} |x\rangle$$

Recalling the asymptotic expressions (I.45) for the backgrounds, one finally obtains:

$$J(z) = -\oint \hat{x}_i \frac{4n v}{|x|^2} \, tr(\hat{S}\cdot T)^6 \langle x | \left(\frac{1}{(-\partial^2+g^2v^2+z)}\right)^2 | x\rangle$$

$$= \oint \hat{x}_i \frac{8nv}{|x|^2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2+g^2v^2+z)^2} = \frac{2nvg}{((gv)^2+z)^{1/2}}$$

I.4 Index theorems and supersymmetry

Witten pointed out the existence of a deep connection between index theorems and supersymmetric theories [5].

A supersymmetric quantum mechanical system is characterized by the existence of one or more (super)charges Q mapping bosons into fermions and vice-versa:

$$\mathcal{H} = F_0 \oplus F_1 \qquad ; \qquad F_0 \overset{Q}{\longleftrightarrow} F_1$$

F_0 and F_1 are, respectively, the bosonic and the fermionic subspaces of the whole Hilbert space $\mathcal{H}$ of the quantum theory. F_0 and F_1 are the eigenspaces with eigenvalues ± 1 of the operator $(-1)^F$,with F being the fermionic number operator.

The (hermitian) supersymmetry charge Q is connected to the Hamiltonian of the system via an anti-commutation relation:

$$\{Q, Q\} = 2H$$

In a basis for which $(-1)^F$ is diagonal

$$(-1)^F = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix}$$

the hermitian supercharge takes the form

$$Q = \begin{pmatrix} 0 & L^* \\ L & 0 \end{pmatrix}$$

L is an operator mapping the bosonic subspace into the fermionic subspace and L* is its adjoint:

$$L : F_0 \longrightarrow F_1 \qquad ; \quad L^* : F_1 \longrightarrow F_0 \tag{I.54}$$

The Hamiltonian written in terms of L and L* is

$$H = \begin{pmatrix} L^*L & 0 \\ 0 & LL^* \end{pmatrix} = L^*L \oplus LL^* \tag{I.55}$$

(the direct sum is relative to the decomposition of the total Hilbert space $\mathcal{H} = F_0 \oplus F_1$). It is evident from eqs.(I.54) and (I.55) that $\mathcal{H} = F_0 \oplus F_1$ together with the operator L has the structure of a two-component complex, the Hamiltonian playing the role of the generalized laplacian on F_0 and F_1 . If the quantum system has a finite numbers of degrees of freedom, the quantum states are represented by sections of some finite dimensional vector bundle and the complex just defined is of the type studied in the previous sections. Field theories (or any theory with infinite number of degrees of freedom), on the other hand, give rise to complexes over infinite dimensional vector bundles: nevertheless many of the concepts we learnt about elliptic complexes still apply.

The index of the supersymmetrical complex ($\mathcal{H} = F_0 \oplus F_1$, L)

$$\text{index}\,(F_0 \oplus F_1, L) = \dim \ker L^* L - \dim \ker L L^* = \tag{I.56}$$

$= \;$ n°(bosonic zero energy states) - n°(fermionic zero energy states)

is known as the Witten index [5] of the corresponding supersymmetric theory. The knowledge of the Witten index gives informations about the underlying quantum theory: supersymmetry is unbroken whenever the Witten index is different from zero, since, then, states with zero energy necessarely exist. For theories with a discrete energy spectrum the Witten index can be represented as a trace over the whole Hilbert space:

$$\text{Witten index} = \mathrm{tr}(-1)^F = \underset{\beta \in \mathbb{R}^+}{\mathrm{tr}(-1)^F e^{-\beta H}} \tag{I.57}$$

as explained in section I.2. This formula allows to derive a useful representation of the Witten index in terms of functional integrals [6].

The fact that supersymmetric theories have, in some sense, built in their own structure the concept of complex has interesting consequences both from the physical and the mathematical point of view. On one side, the known mathematical results about indices of linear operators - which have been partially summarized in this chapter - become directly applicable to physical problems arising in the context of supersymmetric quantum theories [10], [11], [12]. The rest of the present thesis is devoted to illustrate some of such applications. On the other hand, methods independently developed by physicists to evaluate the Witten index [6], [7], [8], [9], [13] turned out to suggested novel derivations of the classical index theorems [7], [14], as well as other results in different mathematical areas [15].

Let us now see on a particular example how the general relationship between supersymmetric theories and analytical complexes works. We will get results which will turn out useful in chapter III. Consider the supersymmetric generalization of the non linear -model in 0+1 dimensions [34]:

$$\mathcal{L} = \frac{1}{2} g_{ij}\, \partial_t \varphi^i \partial_t \varphi^j + \frac{i}{2} \bar{\psi}^i \gamma^0 \frac{D^{ij}}{t} \psi^j + \frac{1}{6} R_{ijkl}(\varphi)\, \bar{\psi}^i \psi^k \bar{\psi}^j \psi^l \tag{I.58}$$

where φ^i $(i=1,\ldots,n)$ are real scalar fields parametrizing the n-dimensional compact manifold M. $g_{ij}(\varphi)$ is the metric on M and $R_{ijkl}(\varphi)$ is the Riemann tensor. ψ^i are two-components Majorana spinors and $\frac{D}{Dt}$ is the covariant derivative acting on them:

$$\left(\frac{D}{Dt}\psi\right)^i \equiv \partial_t \psi^i + \Gamma^i_{k\ell}\left(\partial_t X^k\right)\psi^\ell$$

The model in (I.58) can be obtained via dimensional reduction from the supersymmetric N=1 non linear σ-model in 2 dimensions. Because of their topological invariance, the indices for the one-dimensional and the two-dimensional models are the same [5], [8].

In a basis in which $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ the Majorana spinors ψ^i will be of the form

$$\psi^i = \begin{pmatrix} \chi^i \\ \chi^{i*} \end{pmatrix} \qquad i = 1, \ldots, n$$

where the lower component is the hermitian conjugate of the upper one. In the same basis the supersymmetry algebra is

$$H = Q Q^* + Q^* Q$$

$$Q^2 = Q^{*2} = 0 \tag{I.59}$$

The χ^i and χ^{i*} satisfy upon quantization the anti-commutation relations

$$\{\chi^i, \chi^j\} = 0 = \{\chi^{i*}, \chi^{j*}\}$$

$$\{\chi^i, \chi^j\} = g^{ij}(\varphi) \qquad \text{(I.60)}$$

We will first work in a Fock-type representation in which the quantum operators χ^i, χ^{i*} are regarded as annihilation and creation operators. The supersymmetry charges are

$$Q = i \sum_i \chi_i^* p_i \quad ; \quad Q^* = -i \sum \chi_i^* p_i \qquad \text{(I.61)}$$

where p_i is the momentum conjugate to and will be represented by the appropriate covariant derivative:

$$p_i = -i \frac{D}{D\varphi^i}$$

In the chosen representation, the Hilbert space of the quantum states is the tensor product of the Fock space relative to the χ^i and χ^{i*} with the space of the smooth functions on M. States with p fermionic excitations will correspond to functions $f_{i_1 \dots i_p}(\varphi)$ ($i_k = 1,\dots,n$) on M, with p indices specifying the type of the fermionic creation operators. Because of the Fermi statistic, $f_{i_1 \dots i_p}(\varphi)$ will be antisymmetric in the indices $i_1, \dots, i_p$, thus it will be an element of $\Lambda^p(M)$, the bundle of the p-forms on M.

The total Hilbert space will be represented by the exterior bundle on M

$$\mathcal{H} = \Lambda^*(M) = \bigoplus_{p=0}^{n} \Lambda^p(M) \tag{I.62}$$

It is easy to convince oneself that the supercharges Q and Q* in (I.61) will be represented by the exterior derivative and its adjoint, respectively:

$$Q = d \qquad \text{and} \qquad Q^* = d^* \tag{I.63}$$

so that the Hamiltonian is given by the laplacian on M

$$H = d d^* + d^* d$$

Hence, the complex ($\mathcal{H}$, Q) relative to the N=1 supersymmetric σ-model is just the De Rham complex. Accordingly, its Witten index is the Euler characteristic of the manifold M (see eq.(I.13)):

$$\mathrm{tr}\,(-1)^F = \chi(M) \tag{I.64}$$

In section I.2 we learnt of a second way to split up the exterior bundle $\Lambda^*(M)$ which gives rise to the signature complex. In the context of the supersymmetric σ-model this corresponds to consider the quantum operator Q_5 implementing the following discrete symmetry of the Lagrangian (I.58):

$$Q_5 \;:\; \psi^i \longrightarrow \gamma_5 \psi^i \tag{I.65}$$

In the representation defined in eqs.(I.62) and (I.63), Q_5 is given by the hodge star duality operator exchanging p forms with n-p forms. Decomposing the Hilbert space of the quantum states into eigenspaces of Q_5 with eigenvalues ± 1 produces the signature complex (see section 1.2.ii). Thus

$$\begin{aligned} \mathrm{tr}\, Q_5 &= n^{\circ}(\text{zero energy states with } Q_5 = +1) \\ &\quad - n^{\circ}(\text{zero energy states with } Q_5 = -1) \\ &= \dim H_+(M,\mathbb{R}) - \dim H_-(M,\mathbb{R}) \\ &= \tau(M) \end{aligned} \tag{I.66}$$

with $\tau(M)$ being the Hirzebruch signature of M and $H_\pm(M,R)$ being the spaces of the harmonic forms on M with eigenvalues ± 1 under $\omega \equiv Q_5$, defined in eq.(I.18).

Let us describe another representation for the quantum system (I.58) which also will be useful in the next chapters. Instead of working with the complex fermionic operators λ^i and their conjugate λ^{i*}, we will consider the equivalent set of hermitian fermionic variables λ^i_1 and λ^i_2. n being even, one can represent λ^i_1 and λ^i_2 by the following matrices:

$$\begin{aligned} \lambda^i_1 &= \gamma^i \otimes \mathbb{1} \\ \lambda^i_2 &= \gamma^{n+1} \otimes \gamma^i \end{aligned} \qquad \{\gamma^i, \gamma^j\} = g^{ij} \tag{I.69}$$

γ^i are the γ-matrices with "curved" indices which are related to the "flat" γ-matrices γ^a (a=1,...,n) through the "vielbein":

$$\gamma^i = e^i{}_a \gamma^a \quad , \quad e^i{}_a e^{ja} = g^{ij}$$

γ_{n+1} is the chirality matrix in n dimensions. The supercharge Q is represented by the Dirac operator acting on "double" spinors in n dimensions

$$Q = \gamma^i \otimes 1\!\!1 \left(i\partial_i + \omega_i^{ab} (\gamma^{ab} \otimes 1\!\!1 + 1\!\!1 \otimes \gamma^{ab}) \right) \tag{I.70}$$

$\gamma^{ab} \equiv \frac{1}{4}[\gamma^a, \gamma^b]$ and ω_μ^{ab} is the spin connection on M. The operators $(-1)^F$ and Q_5 correspond to the matrices

$$\begin{aligned} (-1)^F &= \gamma_{n+1} \otimes \gamma_{n+1} \\ Q_5 &= \gamma_{n+1} \otimes 1\!\!1 \end{aligned} \tag{I.71}$$

Comparison with formulas (I.64) and (I.66) gives

$$\mathrm{tr}\, \gamma_{n+1} \otimes \gamma_{n+1} = \chi(M) \tag{I.72}$$

$$\mathrm{tr}\, \gamma_{n+1} \otimes 1\!\!1 = \tau(M) \tag{I.73}$$

where the traces are taken over the space of the eigenfunctions of the Dirac operator (I.70).

II. INDEX THEOREMS IN SUPERSYMMETRIC QUANTUM FIELD THEORIES

One of the most remarkable properties of supersymmetric quantum field theories is their improved ultra-violet behaviour. Cancellations between bosons and fermions make supersymmetric Feynman diagrams less divergent. The simplest example where boson-fermion cancellations occur is the vacuum energy: since 1975 [16] it has been known that the vacuum energy of a supersymmetric field theory is zero at all orders in perturbation theory when supersymmetry is unbroken. At tree level the energy of the vacuum is zero because the zero point energies of the bosonic and of the fermionic oscillators exactly balance. The vacuum energy remains zero even when interactions are taken into account because of the vanishing of the sum of the "bubble" diagrams at any given order in the loop expansion.

The issue of this chapter is the problem of the quantum corrections to the ground state energy in the topologically non trivial sectors of of supersymmetric field theories.

This question has been analysed for the first time in ref. [17] in two dimensions and in ref. [18] in four. The authors concluded that the $O(\hbar)$ correction to the mass of solitons and monopoles in supersymmetric theories vanishes. This seemed to be the generalization to topologically non trivial sectors of the result known for the vacuum sector.

The starting point of the analysis of refs. [17] and [18] is the following expression for the $O(\hbar)$ correction to the mass of a soliton:

$$\Delta^{(1)} M = \frac{\hbar}{2} \sum_{n} \left(\omega_B^{(n)} - \omega_F^{(n)} \right) \tag{II.1}$$

where $\omega_B^{(n)}$ and $\omega_F^{(n)}$ are the eiegenvalues of the differential operators coming from the quadratic expansion of the bosonic and fermionic lagrangians about the classical solution. The classical soliton configuration can be shown to be invariant under half of the supersymmetry trasformations of the theory. Thus, in ref. [17] and [18] it is argued that every term $\omega_B^{(n)} - \omega_F^{(n)}$ in the sum (II.1) vanishes, since for each eigenstate of the differential operator governing the bosonic fluctuations around the background, with eigenvalue $\omega_B^{(n)} \neq 0$, there is a corresponding eiegenstate of the differential operator governing the fermionic fluctuations with the same eigenvalue $\omega_F^{(n)} = \omega_B^{(n)}$.

The previous argument is incorrect because the relevant differential operators, being defined on non-compact manifolds, have continuous spectra. The O(h) quantum correction to the mass of solitons should be more properly written as an integral:

$$\Delta^{(1)} M = \frac{\hbar}{2} \int \left(\frac{dn_B}{d\omega} - \frac{dn_F}{d\omega} \right) \omega \, d\omega \qquad \text{(II.2)}$$

where $\frac{dn_B}{d\omega}$ and $\frac{dn_F}{d\omega}$ are the densities of the eigenfunctions of the differential operators governing the bosonic and the fermionic quantum fluctuations. We will show in the next paragraphs that the difference $\frac{dn_B}{d\omega} - \frac{dn_F}{d\omega}$ is non vanishing on topologically non trivial backgrounds: it is calculable by means of the index theorems for operators on non compact spaces discussed in section I.3. This method has the advantage respect to others appeared in the literature [19] that it avoids any explicit reference to boundary conditions and it clarifies the topological nature of the mass correction.

The fact that the masses of supersymmetric solitons receive, in general, non zero quantum correction does not imply that supersymmetry is broken in the corresponding topological sectors. It has been shown [20] that, in presence of solitons, the supersymmetry algebra includes central charges which, at the classical level, are proportional to the topological charges. From the algebra one can derive bounds for the Hamiltonian which are the quantum mechanical version of the Bogomolny bound for solitons in classical field theories. It can be shown that unbroken supersymmetry is equivalent to the saturation of such bounds on the ground state: in the soliton sector this means that supersymmetry is preserved if and only if the mass of the soliton equals the value of the central charge, at quantum level. A non vanishing quantum correction to the soliton mass does not trigger spontaneous supersymmetry breaking as long as the central charge receives a corresponding renormalization. We will show that this is what happens in some supersymmetric field theories in two and four dimensions.

II.1 An application of the Callias index theorem: quantum corrections to the mass of solitons in 1+1 dimensional supersymmetric field theories [10]

We consider the supersymmetric theory in two dimensions

$$S = \int d^2x \left[\frac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi + \frac{i}{2} \bar{\Psi} \not{\partial} \Psi - \frac{1}{2} W'(\varphi)^2 - \frac{1}{2} W''(\varphi) \bar{\Psi} \Psi \right] \tag{II.3}$$

where φ is a real scalar field and ψ is a Majorana spinor. $W(\varphi)$ is a superpotential chosen such that the above theory admits topological solitons, and the prime denotes a derivative with respect to the argument.

An example is

$$W'(\varphi) = \sqrt{\tfrac{1}{2}\lambda}\,\left(\varphi^2 - \mu^2/\lambda\right) \tag{II.4}$$

The classical soliton (antisoliton) $\varphi_S(x)$ satisfies the Bogomolny equation:

$$\frac{d\varphi_s(x)}{dx} = \underset{(-)}{+}\, W'(\varphi_s(x)) \tag{II.5}$$

The classical soliton mass is:

$$\begin{aligned} M_0 &= \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}\left(\frac{d\varphi_s}{dx}\right)^2 + \frac{1}{2} W'(\varphi_s(x))^2 \right] \\ &= -\int_{-\infty}^{+\infty} dx\ \frac{\partial \varphi_s}{\partial x}\, W'(\varphi_s(x)) \\ &= -\int_{-\infty}^{+\infty} dx\ \frac{\partial W}{\partial x}(\varphi_s(x)) \\ &= -\Big[W(\varphi_s(x)) \Big]_{-\infty}^{+\infty} \end{aligned} \tag{II.6}$$

As explained in the introduction, the O(h) correction to the soliton mass is given by

$$\Delta^{(1)} M = \frac{1}{2} \hbar \int \left(\frac{dn_B}{d\omega} - \frac{dn_F}{d\omega} \right) \omega \, d\omega \qquad \text{(II.7)}$$

apart from renormalization counterterms. The bosonic fluctuations $\xi(x)$ satisfy

$$\left[-\frac{d^2}{dx^2} + \frac{1}{2} \left(W'^2(\varphi_s(x)) \right)'' \right] \xi(x) = \omega_B^2 \, \xi(x) \qquad \text{(II.8)}$$

while the fermionic eigenfunctions $\psi(x) = \begin{pmatrix} u_+(x) \\ u_-(x) \end{pmatrix}$ satisfy

$$L u_+ \equiv \left[i \frac{d}{dx} + i W''(\varphi_s(x)) \right] u_+ = \omega_F u_-$$

$$L^* u_- = \left[+ i \frac{d}{dx} - i W''(\varphi_s(x)) \right] u_- = \omega_F u_+ \qquad \text{(II.9)}$$

We have used the representation $\gamma^0 = \sigma_2$, $\gamma^1 = i\sigma_3$.

From eq.(II.9) one obtains the decoupled equations

$$L^* L\, u_+(x) = \omega_F^2\, u_+(x)$$

$$L\, L^* u_-(x) = \omega_F^2\, u_-(x) \tag{II.10}$$

Observe that

$$L^* L = -\frac{d^2}{dx^2} - W'''(\varphi_s)\frac{d}{dx} + \left(W''(\varphi_s)\right)^2 \tag{II.11}$$

which, using eq.(II.5) for the soliton, becomes

$$L^* L = -\frac{d^2}{dx^2} + \frac{1}{2}\left[\left(W'(\varphi_s(x))\right)^2\right]'' \tag{II.12}$$

so that the bosonic fluctuations equations can be rewritten as

$$L^* L\, \xi(x) = \omega_B^2\, \xi(x) \tag{II.13}$$

The differential operator L*L and LL*, appearing in eqs.(II.10) and (II.13) are defined on the non compact space $\mathbb{R}$. They have a continuous portion in the spectrum which is separated from zero by a finite gap (and hence are Fredholm) - because the "potential" $W''(\varphi_s)$ goes to finites constants at spatial infinity.

The quantities in which we are interested are the bosonic (fermionic) densities of eigenstates $\frac{dn_B}{d\omega}$ ($\frac{dn_F}{d\omega}$)of the continuous part.

Let us define $\frac{dn_+}{d\omega}$ ($\frac{dn_-}{d\omega}$)as the densities of the eiegenfunctions of the operators L*L (LL*). From eq.(II.13) it is clear that

$$\frac{dn_B}{d\omega} = \frac{dn_+}{d\omega} \tag{II.14}$$

The relation between $\frac{dn_F}{d\omega}$ and $\frac{dn_\pm}{d\omega}$ is more subtle. $\frac{dn_F}{d\omega}$ is the density of eigenstates corresponding to the operator

$$Q = \begin{pmatrix} 0 & L^* \\ L & 0 \end{pmatrix}$$

This is exactly half the densities of the eiegenstates of

$$Q^2 = \begin{pmatrix} L^*L & 0 \\ 0 & LL^* \end{pmatrix}$$

because of the existence of the operator

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which anticommutes with Q and hence commutes with Q^2. Thus to each eigenvalue ω^2 of Q^2 there correspond two eigenstates $|\omega\rangle$ and $P|\omega\rangle$, only one of which is a positive-frequency eigenstate of Q with eigenvalue ω. The density of eigenstates of Q^2 is obviously $\frac{dn_+}{d\omega} + \frac{dn_-}{d\omega}$, so that

$$\frac{dn_F}{d\omega} = \frac{1}{2}\left(\frac{dn_+}{d\omega} + \frac{dn_-}{d\omega}\right) \tag{II.15}$$

The conclusion is that

$$\frac{dn_B}{d\omega} - \frac{dn_F}{d\omega} = \frac{1}{2}\left(\frac{dn_+}{d\omega} - \frac{dn_-}{d\omega}\right) \tag{II.16}$$

We can easily evaluate the right-hand side of this equation using the Callias-Bott-Seeley trace formula (I.38) applied to this special case (n=1 and L defined in (II.9)):

$$tr\left(\frac{z}{z + L^*L} - \frac{z}{z + LL^*}\right) = \frac{1}{2}\left(\frac{a_+}{\sqrt{z + a_+^2}} - \frac{a_-}{\sqrt{z + a_-^2}}\right) \tag{II.17}$$

here $a_{\pm} = W''(\varphi_s(x = \pm\infty))$ and $z \in \mathbb{C}$. We can define a function f(z) as in (I.39):

$$f(z) = \frac{1}{2z}\left[\frac{a_+}{\sqrt{a_+^2+z}} - \frac{a_-}{\sqrt{a_-^2+z}}\right] - \frac{1}{z}$$

The discontinuity at the cut is

$$\Delta f(-\omega^2) = \lim_{\epsilon\to 0^+}\left[f(-\omega^2+i\epsilon) - f(-\omega^2-i\epsilon)\right]$$

$$= -\frac{i}{\omega^2}\left[\frac{a_+}{\sqrt{\omega^2-a_+^2}}\,\theta(\omega^2-a_+^2) - \frac{a_-}{\sqrt{\omega^2-a_-^2}}\,\theta(\omega^2-a_-^2)\right]$$

from which one gets, recalling formula (I.42):

$$\frac{dn_+}{d\omega^2} - \frac{dn_-}{d\omega^2} =$$

$$= -\frac{1}{2\pi\omega^2}\left[\frac{a_+}{\sqrt{\omega^2-a_+^2}}\,\theta(\omega^2-a_+^2) - \frac{a_-}{\sqrt{\omega^2-a_-^2}}\,\theta(\omega^2-a_-^2)\right] \quad \text{(II.18)}$$

Thus the soliton mass correction is

$$\Delta^{(1)}M = \frac{\hbar}{2}\int\left(\frac{dn_B}{d\omega} - \frac{dn_F}{d\omega}\right)\omega\, d\omega$$

$$= \frac{\hbar}{4}\int_0^{\infty}\left(\frac{dn_+}{d\omega^2} - \frac{dn_-}{d\omega^2}\right)\omega\, d\omega^2$$

which after a change of variable gives the result

$$\Delta^{(1)} M = -\frac{\hbar}{4\pi}\int_0^\infty dk\left[\frac{a_+}{\sqrt{k^2+a_+^2}} - \frac{a_-}{\sqrt{k^2+a_-^2}}\right] \qquad \text{(II.19)}$$

It is readly seen that,for the super $\lambda\varphi^4$-model,

$$W(\varphi) = \sqrt{\tfrac{1}{2}\lambda}\left(\tfrac{1}{3}\varphi^2 - \mu^2\varphi/\lambda\right) \qquad \text{(II.20)}$$

$a_+ = - a_- = \sqrt{2}\mu$,our result agrees with that of ref. [19] which has been obtained with finite volume techniques.

From the expression (II.19) it is evident that $\Delta^{(1)}$ M is a topological invariant - it depends only on the asymptotic values a_+ and a_- of the "potential" $W''(\varphi_s(x))$ and not on its detailed form.

One also sees that the non-vanishing of the mass correction is intimately linked to the topologically non-triviality of the classical solution - the same formula evidently gives zero for the correction of the vacuum energy at $O(\hbar)$.

Putting together eqs.(II.6) and (II.19) on obtains for the unrenormalized mass of the soliton at $O(\hbar)$:

$$M^{(1)} = -\Big[W(\varphi_s(x))\Big]_{-\infty}^{+\infty} - \frac{\hbar}{4\pi}\int_0^\infty\left[\frac{a_+}{\sqrt{a_+^2+k^2}} - \frac{a_-}{\sqrt{a_-^2+k^2}}\right]dk \qquad \text{(II.21)}$$

To get the renormalized mass, which is expected to be a finite function of the renormalized parameters of the theory, one has to substitute into the expression (II.21) the unrenormalized parameters in terms of the renormalized ones. For simplicity, we will specify the superpotential to be as in (II.20). Eq.(II.21) becomes

$$M^{(1)} = \frac{2\sqrt{2}}{3} \frac{\mu_0^3}{\lambda} \left(1 - \frac{3}{2} \hbar \, I \, \frac{\lambda}{\mu^2} \right) \tag{II.22}$$

where I is the divergent integral

$$I = \frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{dk}{(k^2 + 2\mu^2)^{1/2}} \tag{II.23}$$

is the bare mass of the meson of the theory. The renormalization of μ_0^2 is easily computed to be [19] :

$$\mu_0^2 = \mu_R^2 \left(1 + \frac{\lambda\hbar}{\mu^2} \left(I + \frac{1}{\sqrt{12}\pi} \right) \right)$$

where μ_R is the physical mass of the meson renormalized on-shell. Plugging this expression for into (II.22) one obtains for the renormalized soliton mass:

$$M^{(1)} = \frac{2\sqrt{2}\,\mu_R^3}{3\lambda} + \frac{\sqrt{2}\,\mu_R \hbar}{\sqrt{12}\,\pi} \tag{II.24}$$

II.2 Monopole mass correction in 3+1 dimensional supersymmetric field theories [11]

II.2.i N=2 super Yang-Mills

In four dimensions the simplest supersymmetric field theory admitting monopoles is the N=2 super Yang-Mills theory. Such a theory can be obtained by dimensional reduction from the (5+1)-dimensional N=1 theory [32] :

$$\mathcal{L} = -\frac{1}{4} F^a_{AB} F^{AB,a} + i \bar{\lambda} \Gamma^A D_A \lambda$$

$$A, B = 0, 1, \ldots, 5 \qquad \text{(II.25)}$$

$$(1 + \Gamma_7) \lambda = 0$$

To perform the reduction we use the following representation for the γ-matrices:

$$\Gamma_0 = \gamma_5 \otimes \sigma_1$$

$$\Gamma_i = i \gamma_i \otimes \mathbb{1} \qquad i = 1, \ldots, 4$$

$$\Gamma_5 = -\gamma_5 \otimes i\sigma_2 \qquad \text{(II.26)}$$

$$\Gamma_7 = \Gamma_0 \ldots \Gamma_5 = -\gamma_5 \otimes \sigma_3$$

where γ^i are the four dimensional euclidean Dirac matrices for which we chose

$$\gamma^{i} = \begin{pmatrix} 0 & \tilde{\sigma}_{i}^{+} \\ \tilde{\sigma}_{i} & 0 \end{pmatrix} \qquad \gamma_5 = -\gamma_1 \cdots \gamma_n = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & 1\!\!1 \end{pmatrix}$$

(II.27)

$$\tilde{\sigma}_i \equiv \left(-i\sigma_1 , -i\sigma_2 , -i\sigma_3 , 1\!\!1 \right)$$

σ_i are the Pauli matrices. Armed with these conventions, it is easy to write down the eigenvalue equations for the bosonic and fermionic fluctuations in the field of a Bogomolny-Prasad-Sommerfield monopole solution for our theory.

We will use in the following the 6-dimensional notation of ref. [18]. The 6-vectors parametrizing the monopole solution are chosen to be

$$m_A = (0,0,0,0,0,1)$$

$$n_A = (1,0,0,0,0,0)$$

$$r_A = (0,0,0,0,1,0)$$

so that the monopole has no electric charge. The symmetry-breaking Higgs field is $P^a \equiv A_4^a$ and the monopole field strength tensor $\bar{F}_{AB}$ is nonvanishing only when its components are i,j (i,j = 1,...,4). Further, it is self-dual:

$$\bar{F}_{ij} = \frac{1}{2} \epsilon_{ijkl} \bar{F}_{kl}$$

Let us consider the fermionic fluctuations. They satisfy

$$\Gamma^A D_A \lambda = 0 \qquad \text{with} \qquad (1 + \Gamma_7) \lambda = 0 \tag{II.28}$$

It follows from the Weyl condition that, in our representation, the 8-component spinor λ may be written

$$\lambda = \begin{bmatrix} \psi_+ \\ 0 \\ 0 \\ \psi_- \end{bmatrix}$$

where $\psi_\pm$ are two independent <u>complex</u> two component spinors. Writing $\psi_\pm(\vec{x},t) = e^{-i\omega t}\chi_\pm(\vec{x})$, The Dirac equation (II.28) becomes

$$\begin{aligned} \tilde{\sigma}_i \bar{D}_i \chi_+ &= \omega \chi_- \\ -\tilde{\sigma}_i^\dagger \bar{D}_i \chi_- &= \omega \chi_+ \end{aligned} \tag{II.29}$$

where

$$\bar{D}_i^{ab} \equiv \partial_i \delta^{ab} + g f^{abc} \bar{A}_i^c$$

$$\bar{D}_4^{ab} \equiv g f^{abc} \bar{p}^c$$

Define

$$L_{\alpha\beta}^{ab} \equiv (\tilde{\sigma}_i)_{\alpha\beta} \bar{D}_i^{ab}$$

Then

$$\begin{aligned} L \chi_+ &= \omega \chi_- \\ L^* \chi_- &= \omega \chi_+ \end{aligned} \qquad \text{(II.31)}$$

This is identical in form to eq.(II.9) of the previous section, and we may repeat the steps performed there.

The equations may be decoupled by iteration to give

$$L^* L \chi_+ = \omega^2 \chi_+$$
$$L L^* \chi_- = \omega^2 \chi_- \qquad \text{(II.32)}$$

Associating the densities of eigenstates $\frac{dn_+}{d\omega}$, $\frac{dn_-}{d\omega}$ with the operators L*L,LL* respectively, we find that the fermionic density of eigenstates $\frac{dn_F}{d\omega}$ may be expressed in terms of these quantities. The solutions of eq.(II.32) above have the associated density $2\left(\frac{dn_+}{d\omega} + \frac{dn_-}{d\omega}\right)$ since each equation for the complex is equivalent to two real equations. Just as in paragraph II.1 the density for the solutions of eq.(II.31) is <u>half</u> that for eq.(II.32), thus

$$\frac{dn_F}{d\omega} = \frac{dn_+}{d\omega} + \frac{dn_-}{d\omega} \qquad \text{(II.33)}$$

We now turn to the bosonic fluctuations. In the monopole back ground field gauge, the gauge field quadratic fluctuation equation is, in the 6-dimensional notation,

$$(\bar{D}_i \bar{D}_i \xi_A)^a + 2g f^{abc} \bar{F}^c_{AB} \xi^{B,b} = -\omega^2 \xi^a_A \qquad \text{(II.34)}$$

For A = 0 and 5, $\bar{F}^a_{AB}$ vanishes and we get

$$(\bar{D}_i \bar{D}_i \eta_\alpha)^a = -\omega^2 \eta^a_\alpha \qquad \text{(II.35)}$$

where we have written η_α (α =1,2) for $\xi_{0,5}$. With L as in eq.(II.30) it is easy to check that

$$(L L^*)^{ab}_{\alpha\beta} = -(\bar{D}_i \bar{D}_i)^{ab} \delta_{\alpha\beta} \tag{II.36}$$

(One uses the fact that $\sigma_{ij} \equiv \frac{1}{4i}(\tilde{\sigma}_i^\dagger \tilde{\sigma}_j - \tilde{\sigma}_j^\dagger \tilde{\sigma}_i)$, $\bar{\sigma}_{ij} \equiv \frac{1}{4i}(\tilde{\sigma}_i \tilde{\sigma}_j^\dagger - \tilde{\sigma}_j \tilde{\sigma}_i^\dagger)$ are self dual and anti self-dual respectively).

Thus η satisfies

$$L L^* \eta = \omega^2 \eta \tag{II.37}$$

and its associated density is $\frac{dn_-}{d\omega}$.

The components ξ_i , i=1,...,4 require more work. Let us define a new set of matrices $\tilde{\Sigma}_i$ which forms a real 4x4 representation of the $\tilde{\sigma}_i$ of eq.(II.27)

$$\tilde{\Sigma}_i \equiv (-i\Sigma_1, -i\Sigma_2, -i\Sigma_3, \mathbb{1})$$

with, for example,

$$\Sigma_1 = \sigma_1 \otimes \sigma_2 , \quad \Sigma_2 = -\sigma_3 \otimes \sigma_2 , \quad \Sigma_3 = \sigma_2 \otimes \mathbb{1}$$

Having put

$$\Sigma_{ij} \equiv \frac{1}{4i} \left(\tilde{\Sigma}_i^{+} \tilde{\Sigma}_j - \tilde{\Sigma}_j^{+} \tilde{\Sigma}_i \right)$$

it is straightforward to prove that

$$\frac{1}{2} i \left(\Sigma_{\ell m} \right)_{jk} \bar{F}_{\ell m} = \bar{F}_{ij}$$

for self-dual $\bar{F}$. Eq.(II.34) for A,B = i,j can be written

$$\left(\bar{D}_i \bar{D}_i \xi_j \right)^a + i \left(\Sigma_{\ell m} \right)_{jk} \bar{F}^c_{\ell m} f^{abc} \xi^b_k = -\omega^2 \xi^a_j$$

Defining the operator $M^{ab}_{jk} = (\tilde{\Sigma}_i)_{jk} \bar{D}^{ab}_i$, it follows that

$$\left(M^{+} M \xi \right)^a_i = \omega^2 \xi^a_i$$

But since Σ_i is just a 4x4 representation of the Pauli matrices, there exists a unitary trasformation

$$\Sigma_i \longrightarrow \Sigma_i' = \sigma_i \otimes 1\!\!1$$

so that

$$\tilde{\Sigma}_i \longrightarrow \tilde{\Sigma}_i{}' = \tilde{\sigma}_i \otimes \mathbb{1}$$

and M becomes

$$M = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} , \quad L = \tilde{\sigma}_i \bar{D}_i$$

Writing

$$\xi = \begin{pmatrix} a_\alpha \\ b_\alpha \end{pmatrix} \qquad \alpha = 1, 2$$

we finally get

$$\begin{aligned} L^* L a &= \omega^2 a \\ L^* L b &= \omega^2 b \end{aligned} \tag{II.38}$$

So the densities of the eigenstates for $A_1, \ldots, A_4$ is $2\frac{dn_+}{d\omega}$. Finally the ghost fluctuations satisfy

$$\bar{D}_i \bar{D}_i C = -\omega^2 C$$

where C^a is a complex ghost field. Writing

$$d_\alpha = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

where $C = C_1 + iC_2$,it follows that

$$L L^* d = \omega^2 d \tag{II.39}$$

and hence the density of the ghost fluctuations is $\frac{dn_-}{d\omega}$. Collecting the results from eq.(II.33),(II.37),(II.38) and (II.39), and counting the ghosts as fermions, we get

$$\begin{aligned} \frac{dn_B}{d\omega} - \frac{dn_F}{d\omega} &= \frac{dn_-}{d\omega} + 2\frac{dn_+}{d\omega} - \\ &\quad - \left(\frac{dn_+}{d\omega} + \frac{dn_-}{d\omega} \right) - \frac{dn_-}{d\omega} \\ &= \frac{dn_+}{d\omega} - \frac{dn_-}{d\omega} \end{aligned} \tag{II.40}$$

We may now calculate

$$\Delta^{(1)} M = \frac{\hbar}{2} \int \left(\frac{dn_B}{d\omega} - \frac{dn_F}{d\omega} \right) \omega \, d\omega$$

$$= \frac{\hbar}{2} \int \left(\frac{dn_+}{d\omega} - \frac{dn_-}{d\omega} \right) \omega \, d\omega \qquad \text{(II.41)}$$

The quantity

$$\frac{dn_+}{d\omega} - \frac{dn_-}{d\omega}$$

is obtained from the trace theorem (I.47) of E.Weinberg, discussed in section I.3.ii:

$$\mathrm{tr} \left(\frac{z}{L^* L + z} - \frac{z}{L L^* + z} \right) = \frac{2 g v n}{((gv)^2 + z)^{1/2}} \qquad \text{(I.47)}$$

with L $= \tilde{\sigma}_i \bar{D}_i$. g is the gauge coupling, v the value taken by the modulus of the Higgs field P at spatial infinity and n is the topological winding number of the monopole classical configuration, which for our case equals 1. Rewriting formula (I.47) as

$$\int_0^\infty \frac{z}{z + \omega^2} \left(\frac{dn_+}{d\omega^2} - \frac{dn_-}{d\omega^2} \right) d\omega^2 = \frac{2 g v n}{((gv)^2 + z)^{1/2}} - \frac{2}{z}$$

and following the steps performed in section I.3.i (eqs.(I.41) and (I.42)), one gets

$$\frac{dn_+}{d\omega^2} - \frac{dn_-}{d\omega^2} = -\frac{2gv}{\pi\omega^2(\omega^2-(gv)^2)^{1/2}}\,\theta\left(\omega^2-(gv)^2\right)$$

Inserting this in eq.(II.41) and renaming gv=m (the meson mass), we have finally

$$\Delta^{(1)}M = -\frac{2\hbar m}{\pi}\int_0^\infty \frac{dk}{\sqrt{k^2+m^2}} \tag{II.42}$$

This is divergent and non-vanishing contrary to the claim of ref. [18] . A regularization $e^{i\alpha\cdot k}$ is intended for the divergent integral, which we can re express, for further purposes, as an integral over four dimensional momentum, as follows:

$$\Delta^{(1)}M = 16\pi i\hbar m\, I \tag{II.43}$$

with

$$I \equiv \lim_{\alpha_\mu \to 0}\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{(k^2-m^2)^2}\,e^{i\alpha\cdot k} \tag{II.44}$$

To obtain the renormalized monopole mass one has to express the bare coupling constant g_B and the bare mass m_B in terms of the renormalized g_R and m_R :

$$g_B = Z_g \, g_R \quad ; \quad m_B = Z_m \, m_R \tag{II.45}$$

It is a general belief that, once the fundamental parameters of a field theory are renormalized, derived quantities, such as soliton masses, are finite functions of these renormalized parameters. We will see that in our example this holds true.

At classical level the mass of the monopole is

$$M_{classical} = \frac{4 m \pi}{g^2} \tag{II.46}$$

At $O(\hbar)$ one has to take into account the correction (II.43) we just computed:

$$M^{(1)} = \frac{4\pi m_B}{g_B^2} \left(1 + 4 \pi i \hbar g^2 I \right) \tag{II.47}$$

Recalling eq.(II.45), one obtains for the renormalize monopole mass at $O(\hbar)$

$$M^{(1)} = \frac{4\pi m_R}{g_R^2} \left(Z_m Z_g^{-2} + 4 i g^2 \hbar I \right) \tag{II.48}$$

An explicit computation gives

$$Z_m = 1 \tag{II.49}$$

which is one of the non-renormalization theorems due to supersymmetry, and

$$Z_g = 1 + 2ig^2\hbar\,(I + iC) \tag{II.50}$$

where C is a finite constant and the divergent integral I is defined in (II.44) Putting eqs.(II.49) and (II.50) into (II.48) one verfies that the mass of the monopole is a finite function of the renormalized parameters:

$$M^{(1)} = \frac{4\pi m_R}{g_R^2}\left(1 + 2g^2\hbar C \right) \tag{II.51}$$

Unfortunately determining the finite constant C is technically arduous. C depends not only on the renormalization prescription - since the theory is massive it is natural to renormalize on-shell - but also on the gauge fixing term, contrary to the infinite part of Z_g, responsible for the gauge invariant $\beta(g)$ function. The $O(\hbar)$ correction has been computed in the monopole-background-field gauge, which was necessary to make the index theorem applicable. Therefore to determine uniquely C one should perform the 1-loop renormalization of g in the same gauge, that is one should evaluate 1-loop diagrams with explicitly space-dependent propagators: in practice this would be very hard. There are arguments, based on the saturation of the Bogomolny bound (see section II.3) and on the assumption of the Dirac quantization condition, which imply that the constant C is actually zero [11].

II.2.ii N=4 super Yang-Mills

The N=4 super Yang-Mills theory in 3+1 dimensions is obtained by dimensional reduction of the 9+1-dimensional N=1 theory [32]

$$\mathcal{L} = -\frac{1}{4} F^a_{AB} F^{AB,a} + \frac{i}{2} \bar{\lambda} \Gamma^A D_A \lambda$$

$$C\bar{\lambda}^T = \lambda \qquad (1 + \Gamma_{11})\lambda = 0 \tag{II.52}$$

The dimensionally reduced theory has the BPS monopole just as in the N=2 case - in fact the very same solution is simply embedded in the larger theory. The only change is the number of fluctuation equations. Instead of deriving them in detail we will use the knowledge gained in the previous section to obtain the answer.

In section II.2.i we found that:

(a) The densities of the eigenstates for the fermions is $C^{(1)}\frac{1}{2}\left(\frac{dn_+}{d\omega} + \frac{dn_-}{d\omega}\right)$ where $C^{(1)}$ is the number of independent complex two component spinors in the theory.

(b) The densities for A_0 and the scalars others than the Higgs field taking part of the monopole configuration, is $C^{(2)}\frac{dn_-}{d\omega}$, where $C^{(2)}$ is the number of real fields of this type.

(c) The density for the gauge fields $A_{1,2,3}$ and for the Higgs field A_4 is $2\,\frac{dn_+}{d\omega}$.

(d) The density for the ghosts is $\frac{dn_-}{d\omega}$.

For the N=2 theory we had $C^{(1)}=2$ and $C^{(2)}=2$ and hence

$$\frac{dn_B}{d\omega} - \frac{dn_F}{d\omega} = \frac{dn_-}{d\omega} + 2\,\frac{dn_+}{d\omega} - \left(\frac{dn_+}{d\omega} + \frac{dn_-}{d\omega}\right) - \frac{dn_-}{d\omega}$$

(this is eq.(II.40)). The N=4 theory, on the other hand, has twice as many fermions, hence $C^{(1)} = 4$, while the number of fields contributing to $C^{(2)}$ is 6. Thus

$$\frac{dn_B}{d\omega} - \frac{dn_F}{d\omega} = 3\,\frac{dn_-}{d\omega} + 2\,\frac{dn_+}{d\omega} - 2\left(\frac{dn_+}{d\omega} + \frac{dn_-}{d\omega}\right) - \frac{dn_-}{d\omega} \tag{II.52}$$

$$= 0 \quad !$$

We conclude that for the N=4 super Yang-Mills theory, the unrenormalized mass correction $\Delta^{(1)}M = \frac{\hbar}{2}\sum(\omega_B-\omega_F)$ is actually vanishing. It appears to be the unique theory for which this is true.

The result (II.52) is consistent with the expected fact the mass of the monopole be a <u>finite</u> function of the renormalized gauge coupling constant and meson mass. In fact formula (II.53) reads in this case:

$$M^{(1)} = \frac{4\pi m_R}{g_R^2}\left(Z_g^{-2} + O \right)$$

and

$$Z_g \Big|_{\text{divergent}} = 1$$

for the N=4 theory.

II.3 The quantum Bogomolny bound

It has been shown [20] that the supersymmetry algebra corresponding to the two-dimensional action (II.3) is

$$\left\{ \overline{Q}_\alpha , Q_\beta \right\} = -2\, \gamma^\mu_{\alpha\beta} P_\mu + i\, \gamma_{3\,\alpha\beta} T \qquad \text{(II.53)}$$

where $\gamma_3 = \gamma^0\gamma^1$, $\alpha, \beta = 1,2$. Q_α are the Majorana supercharges and T is a central charge which turns out to be a surface integral:

$$T \equiv \int_{-\infty}^{+\infty} dx \, \frac{d}{dx}\left(2\, W(\varphi(x)) \right) \qquad \text{(II.54)}$$

T is different from zero only in topologically non trivial sectors of the theory.

Central charges analogously appear in the supersymmetry algebra for the four-dimensional model (II.25):

$$\{\bar{Q}^i_\alpha, Q^j_\beta\} = \delta^{ij}\gamma^\mu_{\alpha\beta}P_\mu + i\,\epsilon^{ij}\left(\delta_{\alpha\beta}Z_1 + i(\gamma_5)_{\alpha\beta}Z_2\right) \tag{II.55}$$

The indices i,j = 1,2 label the number of supercharges, ϵ_{ij} is the antisymmetric two-index symbol (ϵ_{12} = +1), Q^i_α are the Majorana spinorial generators. $Z_{1,2}$ are the central charges:

$$Z_1 = \int d^3x\; \partial^i\left(S^a F^a_{0i} + P^a \frac{1}{2}\epsilon^{ijk}F^a_{jk}\right)$$

$$Z_2 = \int d^3x\; \partial^i\left(\frac{1}{2}S^a\epsilon_{ijk}F_{jk} - P^aF^a_{0i}\right) \tag{II.56}$$

(S^a and P^a are the scalar and pseudoscalar fields belonging in the N=2 super Yang-Mills multiplet).

From the superalgebras (II.53) and (II.55) one can derive bounds for the Hamiltonians which are the quantum generalizations of the Bogomolny bounds satisfied from the classical soliton configurations. In the rest frame $P_\mu = (H, \vec{0})$, the superalgebra (II.53) becomes:

$$\{ Q^\dagger_\alpha , Q_\beta \} = -2H\delta_{\alpha\beta} + i(\gamma_0\gamma_3)_{\alpha\beta} T \qquad \text{(II.57)}$$

($\bar{Q} \equiv Q^* \gamma^o$). Since the left-hand side is positive definite, it follows that

$$2H \geq |T| \qquad \text{(II.58)}$$

which is intended in the sense of matrix elements.

If the bound is saturated on a given soliton state $|s\rangle$, eq.(II.57) implies:

$$\langle s| \{ Q^\dagger_\alpha , Q_\beta \} |s\rangle = -2 \langle s| T |s\rangle P_{\alpha\beta} \qquad \text{(II.59)}$$

where the matrix $P_{\alpha\beta} = \left(\frac{1 - i\gamma_0\gamma_3}{2} \right)_{\alpha\beta}$ is a <u>projector</u> :

$$P^\dagger = P \qquad ; \qquad P^2 = P$$

Let v be a two-vector living in the subspace "orthogonal" to P, i.e. such that P v = 0. Then, the linear combination of supercharges

$$\tilde{Q} \equiv v_\alpha Q_\alpha$$

annihilates $|s\rangle$. In fact, from eq.(II.59) one has

$$\frac{1}{2}\, v^{\alpha} \langle s | \{ Q^{+}_{\alpha}, Q_{\beta} \} | s \rangle v^{\beta} = 0 =$$

$$= \frac{1}{2} \langle s | \{ \tilde{Q}^{+}, \tilde{Q} \} | s \rangle = \| \tilde{Q} | s \rangle \|^{2}$$

It is easy to show that the reverse is also true: if a state $|s\rangle$ is annihilated by the supercharge $\tilde{Q} = v^{\alpha} Q^{\alpha}$ then the quantum Bogomolny bound (II.58) is saturated on $|s\rangle$. Thus , saturation of the bound (II.58) on a given state is equivalent to the invariance of that state under supersymmetry. Only half of the supersymmetries of the theory leave the state invariant if the central charge is different from zero (this corresponds to the fact that half of the eiegenvectors of P have zero eiegenvalue).

Quantum bounds analogous to (II.58) can be derived also in the four-dimensional theory (II.25). The superalgebra in the rest frame is

$$\{ Q^{+}_{\alpha}, Q_{\beta} \} = \delta_{\alpha\beta}\, \delta_{ij}\, H + i\, \epsilon_{ij} \left((\gamma_0)_{\alpha\beta} Z_1 + (i \gamma_0 \gamma_5)_{\alpha\beta} Z_2 \right) \tag{II.60}$$

The positivity of the left-hand side implies that on any given state

$$H^2 \geqslant Z_1^2 + Z_2^2 \tag{II.61}$$

As before, the equality sign holds in (II.61) if and only if the considered quantum state is left invariant by some supersymmetry generator. If the central charges are different from zero, a state for which the bound (II.61) is saturated is invariant under half of the supersymmetry charges of the theory. In fact, let $|s\rangle$ be annihilated by the supercharge $Q^{(v)} = \sum_{i,\alpha} v^i_\alpha \; Q^i_\alpha$ .Evaluating (II.60) on $|s\rangle$ and multiplying by v^j_β ,one gets:

$$\langle s| H |s\rangle v = \Big[-i\, \epsilon \otimes \gamma_0 \; \langle s| Z_1 |s\rangle + \\ + \epsilon \otimes \gamma_0 \gamma_5 \langle s| Z_2 |s\rangle \Big]$$

where v is a vector with components v^i_α , ϵ is the matrix with elements ϵ_{ij} and the Dirac matrices act on the α , β indices. The matrix on the right-hand side has eigenvalues

$$\pm \sqrt{\langle Z_1\rangle^2 + \langle Z_2\rangle^2}$$

since its square is

$$\mathbb{1} \left(\langle Z_1\rangle^2 + \langle Z_2\rangle^2 \right) + \mathbb{1} \otimes \{ \gamma_0 , \gamma_0 \gamma_5 \} \langle Z_1\rangle \langle Z_2\rangle$$

$$= \left(\langle Z_1\rangle^2 + \langle Z_2\rangle^2 \right) \mathbb{1}$$

Thus the bound is saturated on $|s\rangle$. The reverse statement is proved in a similar fashion.

At classical level the bounds (II.58) and (II.61) are saturated on the soliton and monopoles configurations (see eqs.(II.6), (II,54) and eqs.(II.46), (II.56)):

$$\langle T \rangle_{classical} = \int_{-\infty}^{+\infty} dx \frac{d}{dx} 2 W(\varphi_s(x)) = M_{classical} \tag{II.62}$$

and

$$\langle Z_2 \rangle_{classical} = 0$$

$$\langle Z_1 \rangle_{classical} = \oint \bar{S}^a \bar{B}_i^a d^2\sigma^i = \frac{4\pi m}{g^2} = M_{classical} \tag{II.63}$$

One can also verify that the classical solutions are invariant under half of the supersymmetries of the theories. In d=2 the supersymmetry trasformation laws are

$$\begin{aligned} \delta\varphi &= \bar{\epsilon}\psi \\ \delta\psi &= (i \not{\partial}\varphi - W'(\varphi))\epsilon \end{aligned} \tag{II.64}$$

where $\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$ is a two-component Majorana spinor. It is easy to check that the configuration

$$\psi = 0 \qquad \varphi = \varphi_s(x) \tag{II.65}$$

is left invariant under the trasformations (II.64) with $\epsilon = \begin{pmatrix} 0 \\ \epsilon_2 \end{pmatrix}$ by virtue of the Bogomolny eq.(II.25). The supersymmetry trasformations of the N=2 super Yang-Mills theory are

$$\begin{aligned}
\delta A_\mu^a &= i\left[\bar{\epsilon}\gamma_\mu\chi - \bar{\chi}\gamma_\mu\epsilon\right] \\
\delta P^a &= \bar{\epsilon}\gamma_5\chi - \bar{\chi}\gamma_5\epsilon \; ; \quad \delta S^a = i(\bar{\epsilon}\chi - \bar{\chi}\epsilon) \\
\delta \chi^a &= \Big[\sigma^{\mu\nu}F_{\mu\nu}^a - \gamma^\mu(D_\mu S)^a + i\gamma^\mu(D_\mu P)^a\gamma_5 \\
&\qquad - ig\, f^{abc}\, P^b S^c \gamma_5\Big]\epsilon
\end{aligned} \tag{II.66}$$

where ϵ is a Dirac spinor. On the monopole background the variations of the bosonic fields are trivially zero, while for the fermions one has

$$\begin{aligned}
\langle \delta\chi^a \rangle_{classical} &= (\sigma^{ij}\bar{F}_{ij} - \gamma^i D_i \bar{S})\epsilon \\
&= \gamma^i \cdot \bar{B}_i^a \; P\epsilon
\end{aligned}$$

where $\bar{B}_i^a$ is the magnetic field of the monopole, the matrix P is a projector, and we made use of the Bogomolny equation (I.44). Thus, $<\delta_\epsilon \chi^a> = 0$ for a supersymmetry trasformation with parameter ϵ orthogonal to the projector P, i.e. such that $P\epsilon = 0$.

In sections II.1 and II.2 we have shown that the Hamiltonian on the soliton and on the monopole states does receive, in general, a non vanishing quantum correction. Nevertheless the bounds (II.58) and (II.61) could still hold at quantum level if also the central charges renormalize in the corresponding way. In what follows we will discuss in detail this question for the two-dimensional theory. The analysis in four dimensions is conceptually similar but technically more complicated and will be only briefly sketched.

II.3.i d=2

Let us start from the lagrangian obtained expanding the lagragian in (II.3) around the soliton background:

$$\mathcal{L} = -W'(\varphi_s) + \frac{1}{2}(\partial_\mu \eta)^2 + \frac{i}{2}\bar{\psi}\not{\partial}\psi + \\ -\frac{1}{2}\left[W'(\varphi_s+\eta)^2 - W'(\varphi_s)^2 - 2W'(\varphi_s)W''(\varphi_s)\eta\right] \\ -\frac{1}{2}W''(\varphi_s+\eta)\bar{\psi}\psi \tag{II.67}$$

where η is the quantum bosonic field and $\varphi_s(x)$ is the classical background. The central charge density Υ ($T = \int \Upsilon \, dx$) is , in the same expansion

$$\frac{1}{2}\Upsilon = \frac{\partial}{\partial x}\left[W(\varphi_s) + \left[W(\varphi_s + \eta) - W(\varphi_s)\right]\right]$$

In $O(\hbar)$ the correction to $<\frac{1}{2}T>$ is

$$\Delta^{(1)}\left\langle \frac{1}{2}T \right\rangle = \int_{-\infty}^{+\infty} dx \frac{d}{dx}\left(\frac{1}{2} W''(\varphi_s)\langle \eta^2(x)\rangle \right)$$

(the term in T linear in the quantum field $\left[W'(\varphi_s)\,\eta(x)\right]_{-\infty}^{+\infty}$ vanishes because $W'(\varphi_s) = 0$ at $x = \pm\infty$).

To this order the propagator for η is

$$\langle \eta(x)\,\eta(y)\rangle = \frac{i\hbar}{-\Box - \frac{1}{2}\left(W'(\varphi_s(x))^2\right)''}\,\delta^{(2)}(x-y)$$

and hence

$$\Delta^{(1)}\left\langle \frac{1}{2}T\right\rangle = \left\{ \frac{1}{2} W''(\varphi_s(+\infty))\cdot\left[\frac{i\hbar}{-\Box - \frac{1}{2}\left[W'(\varphi_s(+\infty))^2\right]} \times \delta^2(x-y)\right] - (+\infty \to -\infty)\right\}_{x=y}$$

$$= \frac{1}{2} i\hbar \left[\frac{a_+}{-\Box - a_+^2} \delta^2(x-y) - \frac{a_-}{-\Box - a_-^2} \delta^2(x-y) \right]_{x=y}$$

$$= \left\{ \frac{1}{2} i\hbar \int \frac{d^2k}{(2\pi)^2} \left[\frac{a_+}{k^2 - a_+^2} - \frac{a_-}{k^2 - a_-^2} \right] e^{ik(x-y)} \right\}_{x=y}$$

Performining the integration over k_0 we obtain finally

$$\Delta\langle \tfrac{1}{2} T \rangle = - \frac{\hbar}{4\pi} \int_0^\infty dk \left(\frac{a_+}{\sqrt{k^2+a_+^2}} - \frac{a_-}{\sqrt{k^2+a_-^2}} \right) \qquad \text{(II.69)}$$

Comparing with eq.(II.19) one gets

$$\langle H \rangle - \langle \tfrac{1}{2} T \rangle = 0 \qquad \text{(II.69)}$$

to $O(\hbar)$. Observe that the equality of $\langle H \rangle$ and $\langle \frac{1}{2} T \rangle$ up to $O(\hbar)$ is an equality between unrenormalized quantities, but renormalization amounts to express the bare mass in terms of the renormalized mass so that the equality (II.69) is also true after renormalization(f1).

One can ask if the result (II.69) stays true at higher orders in the semiclassical expansion or even non-perturbatively.

In the vacuum sector the analogous question is if the vacuum energy remains zero after the interactions are switched on. It has indeed been proven [16] that, thanks to supersymmetry Ward identities, perturbative radiative corrections do not break supersymmetry if it is unbroken at the tree level. At non-perturbative level, general theorems are not available. In certain classes of theories the non-vanishing of the Witten index excludes the possibility of dynamical, non-perturbative breaking of supersymmetry [5].

In the following we will show that, the soliton sector action (II.67) is not invariant under supersymmetry trasformations because of the presence of surface terms which are non-zero in a topologically non-trivial background. Therefore, it is not even possible to write down for the lagrangian (II.67) supersymmetric Ward identities which would prove (at least formally) the cancellation of the "vacuum" diagrams. This is not much of a surprise since we just proved that the Hamiltonian H does get renormalized. At the same time we will see that the combination $\tilde{\mathcal{L}} = \mathcal{L} - \frac{1}{2}\Upsilon$ is truly supersymmetric, due to a cancellation of the relevant surface terms. The question of the saturation of the Bogomolny bound on the soliton state may be studied in terms of the spontaneous supersymmetry breaking in the lagrangian $\tilde{\mathcal{L}}$. By analogy with what happens in the vacuum sector one would conclude that radiative corrections do not break the supersymmetry of $\tilde{\mathcal{L}}$ (as we have already proved at $O(\hbar)$. See also ref. [33]). On the other hand non-perturbative breaking cannot be excluded. Topologically arguments à la Witten do not give any insight into the problem as discussed if ref. [20]. In general representations of the supersymmetry algebra which saturate the Bogomolny bound are smaller than those which do not. Witten and Olive used this to argue the non-perturbative saturation of the Bogomolny bound in four dimensions. Unfortunately the same arguments is not available in two dimensions, since in this case all representations are two dimensional (for non-vanishing central charge).

The previous considerations are better discussed in terms of superfields. The lagrangian in (II.3) may be written as

$$\mathcal{L} = \frac{1}{2i} \int d^2\theta \left[\frac{1}{2} \bar{D} \Phi D \Phi + 2W(\Phi) \right] \qquad \text{(II.70)}$$

where

$$\Phi(x, \theta) \equiv \varphi(x) + \bar{\theta} \psi(x) + \frac{1}{2} \bar{\theta} \theta F(x) \qquad \text{(II.71)}$$

is a real superfield whose components are a real scalar $\varphi(x)$, a Majorana spinor $\psi_\alpha(x)$ and an auxiliary field $F(x)$.

The supercovariant derivative is

$$D_\alpha = \frac{\partial}{\partial \bar{\theta}_\alpha} - i (\gamma^\mu \theta)_\alpha \partial_\mu$$

and the supersymmetry generator is

$$Q_\alpha = \frac{\partial}{\partial \bar{\theta}_\alpha} + i(\gamma^\mu \theta)_\alpha \partial_\mu$$

The soliton in superspace is

$$\Phi_s(x, \theta) = \varphi_s(x) - \frac{1}{2}\bar{\theta}\theta W'(\varphi_s(x))$$

and it satisfies the super-Bogomolny equation

$$D_2 \Phi_s(x, \theta) = 0 = Q_2 \Phi_s(x, \theta)$$

Decompose the superfield

$$\Phi(x, \theta) = \Phi_s(x, \theta) + \Phi_q(x, \theta)$$

where Φ_q is the quantum fluctuation. The Lagrangian becomes

$$\mathcal{L} = \frac{1}{2i} \int d^2\theta \left[\frac{1}{2} \bar{D} \Phi_q D \Phi_q + 2V(\Phi_s, \Phi_q) \right]$$

where

$$V(\phi_s, \phi_q) \equiv W(\phi_s + \phi_q) - W(\phi_s) - W'(\phi_s)\phi_q$$

(we have dropped the zero order term.)

The supersymmetry variation of the quantum superfield is defined by

$$\delta \phi_q = \bar{\epsilon} Q \phi_q = (-i\epsilon_i Q_2 + i\epsilon_2 Q_1)\phi_q$$

Performing the supersymmetry trasformation on the interaction term

$$\delta V(\phi_s, \phi_q) = \frac{\delta V}{\delta \phi_q} \bar{\epsilon} Q \phi_q$$

it is seen that Q_2 is no longer a symmetry of the theory since

$$Q_2 \phi_s \neq 0$$

while the Q_1-variation produce the total derivative

$$\delta V(\phi_s,\phi_q) = i\,\epsilon_2\, Q_1\left(V(\phi_s,\phi_q)\right)$$

Of the derivatives contained in Q_1 , the θ-derivative vanishes on integration with respect to $d^2\theta$,while the time derivative also vanishes on integration because the soliton is static. We are left with the spatial derivative

$$\delta\mathcal{L} = -\,\epsilon_2\int d^2\theta\;\theta_1\,\frac{d}{dx}V(\phi_s,\phi_q) \tag{II.72}$$

This in fact gives rise to a non-vanishing surface term in the variation of the action, since the soliton is not periodic in x.

The central charge density expanded about the soliton can be written in superfield notation as

$$\frac{1}{2}\,\tau = \frac{1}{2i}\int d^2\theta\;\bar\theta\theta\,\frac{d}{dx}\Big[W(\phi_s+\phi_q) - $$

$$- W(\phi_s) - W'(\phi_s)\,\phi_q\Big] = \frac{1}{2i}\int d^2\theta\;\bar\theta\theta\,\frac{d}{dx}V(\phi_s,\phi_q)$$

(We have dropped the zero order term which is identical to that in $\mathcal{L}$, and also the linear term as it vanishes by itself.)

Under Q_1 -trasformation

$$\frac{1}{2}\,\delta T = \frac{1}{2i}\int d^2\theta\,\bar{\theta}\theta\,\frac{d}{dx}\left(i\,\epsilon_2\,Q_1 V\right)$$

The only term which contributes in

$$Q_1 \equiv -i\,\frac{d}{d\theta_2} + \theta_2\,\frac{d}{dt} - \theta_1\,\frac{d}{dx}$$

is the θ-derivative, as the other two are annihilated by the factor $\bar{\theta}\theta$. Thus

$$\begin{aligned}\frac{1}{2}\,\delta T &= -\,\epsilon_2\int d^2\theta\;\theta_1\theta_2\,\frac{d}{dx}\left(\frac{d}{d\theta_2}V\right)\\ &= -\,\epsilon_2\int d^2\theta\;\theta_1\,\frac{d}{dx}\,V\left(\Phi_5,\Phi_9\right)\end{aligned} \qquad \text{(II.73)}$$

exactly equal to $\delta\mathcal{L}$ eq.(II.72).

We conclude that although $\mathcal{L}$ and T are not individually invariant under Q_1-supersymmetry, the combination

$$\tilde{\mathcal{L}} \equiv \mathcal{L} - \frac{1}{2}\,T$$

is exactly invariant, as stated above.

II.3.ii d=4

Let us recall the expression for the central charges in the N=2 super Yang-Mills theory:

$$Z_1 = \int d^3x\, \partial_i \left(S^a F^a_{oi} + P^a \tfrac{1}{2} \epsilon_{ijk} F^a_{jk} \right)$$

(II.74)

$$Z_2 = \int d^3x\, \partial_i \left(S^a \tfrac{1}{2} \epsilon_{ijk} F^a_{jk} - P^a F^a_{oi} \right)$$

At the classical level, on the monopole

$$\langle Z_1 \rangle = \frac{4\pi m}{g^2} = M_{\text{classical}}$$

$$\langle Z_2 \rangle = 0$$

Expanding around the classical configuration, one finds that the $O(\hbar)$ correction to the right-hand sides of eq.(II.74) comes from

$$\Delta^{(1)} \langle Z_1 \rangle = \int d^3x\, \partial_i \left(\bar{P}^a \tfrac{1}{2} \epsilon_{ijk} \langle f^a_{jk} \rangle \right)$$

(II.75)

$$f^a_{jk} \equiv \partial_j \xi^a_k - \partial_k \xi^a_j + g f^{abc} \xi^b_j \xi^c_k$$

(ξ_i, i=1,2,3 are the fluctuations of the vector field space components.)

The only non-vanishing term in $< F^a_{jk} >$ is $g\, f^{abc} < \xi^b_j \xi^c_k >$. Moreover, since the integral of a divergence can be trasformed into a surface integral over the sphere at infinity, only the asymptotic propagator is needed:

$$\Delta^{(1)} \langle Z_1 \rangle = \int r^2 \sin\Theta \, d\Theta \, d\hat{\phi} \; \hat{x}^i \cdot$$

$$\cdot \left(\frac{1}{2} g \, \epsilon_{ijk} \, f^{abc} \; \bar{P}^a \langle \xi^b_j \; \xi^c_k \rangle \right) \tag{II.76}$$

From the lagrangian, one gets

$$\left\langle \xi^b_j(x) \, \xi^c_k(y) \right\rangle = i\hbar \left(\bar{D}_\mu \bar{D}^\mu + 2g \, \bar{\bar{F}} \right)^{-1\, bc}_{jk} \delta^4(x-y)$$

$$= i\hbar \Big[(\bar{D}^2)^{-1\, bc} g_{jk} - (\bar{D}^2)^{-1\, bd} \, 2g \, \bar{\bar{F}}^{de} (\bar{D}^2)^{-1\, ec} + \tag{II.77}$$

$$+ \; \cdots \Big] \, \delta^{(4)}(x-y)$$

Asymptotically (see eq.(I.53))

$$\left(\frac{1}{\bar{D}^2} \right)^{ab} \xrightarrow{|x| \to \infty} \frac{1}{\Box} \hat{x}^a \hat{x}^b + \frac{1}{\Box + m^2} \left(\delta^{ab} - \hat{x}^a \hat{x}^b \right) \tag{II.78}$$

The only term in the expansion equation (II.77) which contributes to $\Delta^{(1)}\langle Z_1 \rangle$ is the second one. The first term is symmetric in jk while the higher order terms fall off too fast at infinity to contribute. The relevant term may be written asymptotically

$$\epsilon_{ijk} \langle \xi_j^b(x)\, \xi_k^c(y) \rangle = i\hbar \left[(\bar{D}^2)^{-1\,bd}\, 2g\, \epsilon_{ijk}\, \bar{F}_{jk}^e\, f^{dce}\, \delta^{(4)}(x-y) \right]$$

$$\xrightarrow{|x| \to \infty} -4i\hbar\, (\bar{D}^2)^{-1\,bd}\, \frac{x_i x^e}{r^4}\, f^{dce}$$

Inserting $\left(\frac{1}{\bar{D}^2}\right)^{ab}$ from (II.78) and $\bar{P}^a \to -\frac{m}{g}\hat{x}^a$ in eq.(II.76) one finally gets

$$\Delta^{(1)}\langle Z_1 \rangle = 16\pi i\hbar m \lim_{x \to y} \frac{1}{(\Box + m^2)^2}\, \delta^{(4)}(x-y) \qquad \text{(II.79)}$$

$$= 16\pi i\hbar m\, I$$

where I is the logarithmic divergent integral defined in (II.44). Comparing with eq.(II.43) we find that, to O(h)

$$\Delta^{(1)} M = \Delta^{(1)}\langle Z_1 \rangle \qquad \text{(II.80)}$$

and the quantum Bogomolny bound is saturated at this level.

We mentioned before that there exist topological arguments due to Witten and Olive [20] according to which the bound should be saturated by the monopole for the full quantum theory. Now it is believed [5], [6]

,that such reasonings may run into problems if the associated Hamiltonian has a continuous spectrum going down to zero, which is just the case for the N=2 theory. The calculation above reveals that,nevertheless, the argument holds at O(h).

III. FERMION CHIRALITY IN SUPERSTRING THEORIES

Superstring theories [21] , [22] ,are naturally formulated in ten-dimensional space-time. The attempts to make them realistic are based on the assumption that six of the space-like dimensions "curl up" to describe a compact internal space small enough to be not directly observable.

The compactification mechanism being, at present, largely unknown, it is reasonable to address problems whose resolution does not depend critically on the details of the dynamics. A problem of this kind is to determine the quantum numbers of the light fermions coming out of the compactification. The spectrum of the observed "light" fermions (that is the fermions which are massless as far as one forgets about the SU(2)xU(1) breaking) is known to be chiral: quarks and leptons live into complex representations of the standard gauge group. One of the major difficulties of most of the Kaluza-Klein theories investigated in the past is their inhability to produce chiral fermions in four dimensions [23] . In this respect, the recently formulated heterotic string theory [24] seems to be in a better shape since it can naturally give rise to chiral fermion families.

The question of the chirality content of the compactified heterotic superstring has been first discussed by Candelas et.al. [29] . The analysis of ref. [29] will be reviewed in the next section. Its starting point is the so-called "zero slope" limit of the heterotic string theory. In this limit, one first computes the spectrum of the free string in flat ten-dimensional space. The massive excitations -which have masses of the order of the inverse of the string size which can be argued to be comparable to the Planck length -are disregarded. To reproduce the low-energy string physics one introduces an effective ten-dimensional field theoretical lagrangian whose field content is just given by the supergravity multiplet of the zero mass string states. This supergravity lagrangian is given as an infinite expansion in powers of α' (the dimensional parameter of the string), terms of higher order in α' containing higher number of derivatives of the fields. For energies much lower than $(\alpha')^{-1/2}$ one can forget about the higher derivative terms and one is left with the usual ten-dimensional supergravity lagrangian. At this point one makes the Kaluza-Klein ansatz [26] :the ground state configuration of the gravitational field. is assumed to describe a space time of the type M^4xC^6 where M^4 is the flat four dimensional Minkowski space while C^6 is an internal compact six dimensional space. In this approach, the spectrum of the chiral four dimensional fermions is obtained looking at the zero modes of the Dirac operator on the six dimensional internal manifold, as explained in the next section.

A more "stringy" way to attack the same problem will be illustrated in section III.2 [12] . There we will look directly at the spectrum of the first quantized string action in given gravitational and gauge backgrounds. Since, as we will discuss in detail, world-sheet fermion number and spacetime chirality are closely related, the number of chiral fermionic states is determined by the Witten index of a related two-dimensional N=1 supersymmetric non-linear σ-model. In general, it is not obviuous that this method is equivalent to the Kaluza-Klein field theoretic one:nevertheless, for the chirality problem, we will show that one indeed obtains the same result because of the topological invariance of the Witten index of the two-dimensional supersymmetric σ-model.

III.1 The field theoretical approach

The low energy effective ten-dimensional supergravity obtained from the zero-slope limit of the heterotic string contains as only chiral field a left-handed Majorana-Weyl spinor λ^a in the adjoint representation of the gauge group G. Consistency of the string theory restricts G to be either O(32) or $E_8 x E_8$. As mentioned, one assumes the vacuum geometry to be of the type $M^4 x C^6$ where C^6 is some small six dimensional space and M^4 is the Minkowski space-time. One also allows the gauge fields to take non-trivial vacuum expectation values on C^6. Let F be the subgroup of G in which the background gauge fields live, and H the maximal subgroup of G such that

$$G \supset H \times F \tag{III.1}$$

One can perform an harmonic expansion on C^6 of the ten-dimensional spinor field $\lambda(x,y)$

$$\lambda(x,y) = \sum_n \Psi^{(n)}_{(\alpha)}(x)\,\chi^{(n)}_{(\beta)}(y) \tag{III.2}$$

where the coordinates x parametrize the Minkowski space M^4 and y the internal space C^6. $\Psi^{(n)}_{(\alpha)}(x)$ are four-dimensional spinor fields in the representation $\underline{\alpha}$ of H while $\chi^{(n)}_{(\beta)}(y)$ are a complete set of spinor fields on C^6 in the representation $\underline{\beta}$ of F. $\underline{\alpha}$ and $\underline{\beta}$ are such that the adjoint of G decomposes under the embedding (III.1) into the ($\underline{\alpha}$, $\underline{\beta}$) representation of HxF. As complete set of functions $\chi^{(n)}_{(\beta)}(y)$ on C^6 it is convenient to take the (orthonormal) eigenfunctions of the Dirac operator on C^6

$$i \not{D}_6 \, \chi^{(n)}_{(\beta)}(y) = m_n \chi^{(n)}_{(\beta)}$$

$$(\chi^{(n)}, \chi^{(m)}) \equiv \int_{C^6} d^6y \; \bar{\chi}^{(n)} \chi^{(m)} = \delta^{nm} \tag{III.3}$$

where the Dirac operator D_6 includes the spin connection on C^6 and the background gauge connection acting on spinors in the representation β of F. Plugging the decomposition (III.2) into the ten-dimensional Dirac equation for $\lambda(x,y)$

$$i \not{D}_{10} \, \lambda(x,y) = 0 \tag{III.4}$$

one obtains the following set of equations:

$$\sum_n \left[\left(\not{D}_4 \psi^{(n)}_{(\alpha)}(x) \right) \chi^{(n)}_{(\beta)}(y) + \gamma_5 \psi^{(n)}_{(\alpha)}(x) \not{D}_6 \chi^{(n)}_{(\beta)} \right] = 0 \tag{III.5}$$

To derive (III.5) we have chosen the following representation for the γ-matrices in ten dimensions:

$$\Gamma^{\mu} = \gamma^{\mu} \otimes 1\!\!1 \qquad\qquad \mu = 0, 1, \ldots, 3$$

$$\Gamma^{m} = \gamma_5 \otimes \gamma^{m} \qquad\qquad m = 1, \ldots, 6$$

γ^{μ} and γ^{m} are respectively the γ-matrices in 4 and 6 dimensions and γ_5 is the chirality matrix in four dimensions. Multiplying eq.(III.5) by ${}^{(m)}(y)$, integrating over C^6 and taking into account the orthonormality of the ${}^{(n)}(y)$, one obtains an infinite set of decoupled Dirac equations for the four-dimensional spinors:

$$i \not{D}_4 \, \psi^{(n)}_{(\alpha)}(x) + \gamma_5 \, m_n \, \psi^{(n)}_{(\alpha)}(x) = 0 \tag{III.6}$$

It follows that the massless fermions in four dimensions are just given by the zero modes of the Dirac operator D_6 on the internal space. The number of chiral fermion in the representation α ,i.e. the number of left-handed spinors in the representation α minus the number of right-handed spinors in the same representation, is the index of the operator D_6:

$$\begin{aligned} N_{(\alpha)} &\equiv n^{o}_{L}(\alpha) - n^{o}_{R}(\alpha) = \\ &= n^{o}_{L}(\beta) - n^{o}_{L}(\beta) = \text{index } D_6 \end{aligned} \tag{III.7}$$

(Note that $N_{(\alpha)}$ is what is usually called the number of families only in the case when $\underline{\alpha}$ is irreducible. If $\underline{\alpha}$ is reducible there will be a number $N_{(\alpha)}$ of chiral generations in each of the complex representations of the unbroken H which appear in the decomposition of $\underline{\alpha}$ into irreducible reprpesentations.)

The index of the Dirac operator D_6 can be evaluated by means of the general Atiyah-Singer theorem discussed in section I.1.iii:

$$\text{index } D_6 = \int_{C^6} \hat{A}(R) \wedge ch(F) \tag{III.7}$$

The relevant terms in the expansion of the $\hat{A}$ - genus and of the Chern character are:

$$\hat{A}(R) = 1 + \frac{1}{12} c_2(R) - \frac{1}{24} c_1^2(R) + \cdots$$

$$\begin{aligned} ch(F) &= tr_\beta \left(\exp \frac{F}{2\pi} \right) = \\ &= \dim \underline{\beta} + tr_\beta \frac{F}{2\pi} + \\ &\quad + tr_\beta \frac{F^2}{(2\pi)^2} + tr_\beta \frac{F^3}{(2\pi)^3} + \cdots \end{aligned} \tag{III.9}$$

where $c_{1,2}(R)$ are the Chern characteristic classes defined by the expansion of the total Chern form

$$c(R) \equiv \det\left(1\!\!1 + \frac{R}{2\pi}\right) = 1 + c_1(R) + c_2(R) + \cdots$$

Substituting these expressions into (III.8) one gets:

$$N_{(\alpha)} = \text{index } D_6 = \frac{1}{3!}\int_{C^6} \frac{\mathrm{tr}\, F^3}{(2\pi)^3} + \frac{1}{12}\int_{C^6} \frac{\mathrm{tr}\, F}{2\pi} c_2(R) \qquad \text{(III.10)}$$

The formula can be reduced if one takes into account that consistency of the string theory requires that [25] :

$$c_2(R) = \frac{1}{30} c_2(F) \qquad \text{(III.11)}$$

The gauge second Chern class is

$$c_2(F) = -\frac{1}{2}\frac{1}{(2\pi)^2}\,\mathrm{tr}_{\substack{\text{adjoint}\\ \text{of } G}}(F^2)$$

so that [27] :

$$N_\alpha = \frac{1}{(2\pi)^3}\left\{\frac{1}{6}\int tr_\beta\, F^3 \;-\; \frac{1}{24}\cdot\frac{1}{30}\int tr_\beta(F)\, tr_{adj}\, F^2\right\} \qquad \text{(III.12)}$$

So far we did allow for non vanishing torsion on the internal six-dimensional manifold. In other words the connection in the Riemann tensor is not necessarely the Christoffel connection. Actually , which connection one uses in the index formula (III.8) is irrelevant since, as pointed out in section I.1, characteristic classes are independent on the particular connection chosen. The vanishing of the background torsion can be shown to be equivalent to the vanishing of the vacuum expectation value of the antisymmetric field appearing in the ten-dimensional supergravity. There are phenomenological requirements suggesting that the background antisymmetric field is zero [25] . In that case, the absence of chiral anomalies in the two-dimensional field theory describing the compactified string requires the equality of the spin connection with the gauge field [25] (see also next section)

$$\omega_\mu = A_\mu \qquad \text{(III.13)}$$

This ansätz satisfies the consistency constraint (III.11) for the gauge groups O(32) and $E_8 x E_8$ and O(32). When $\omega_\mu = A_\mu$, H will be a subgroup of O(6), the holonomy group of a six-dimensional Riemannian manifold. It is easily seen that if G=O(32), one never gets chiral fermions. For $G=E_8 \times E_8$ (and H=O(6) or SU(3)) one has

$$N_{(\alpha)} = \text{index } D_6(\omega = A) = \chi/2 \qquad \text{(III.14)}$$

where χ is the Euler characteristic of C^6. Eq.(III.14) can be proven either by direct evaluation of the formula (III.8) or noticing that the six-dimensional Dirac operator D_6 becomes, after the identification (III.13):

$$D_6 = \gamma^i \otimes P_+ \cdot \left(\partial_i + \omega_i^{ab} (\gamma^{ab} \otimes 1\!\!1 + 1\!\!1 \otimes \gamma^{ab}) \right) \qquad \text{(III.15)}$$

where

$$P_+ \equiv \frac{1\!\!1 + \gamma_7}{2}$$

is the projector into the right-handed spinor space. In fact, for F=O(6) the representation β in eq.(III.2) is the spinorial $\underline{4}$ of O(6). The index of the operator in eq.(III.15) can be formally written as

$$\text{index } D_6 = \mathrm{tr} (\gamma_7 \otimes P_+)$$

where the trace is taken over all the eigenstates of the operator

$$Q = \gamma^i \otimes 1\!\!1 \cdot \left(\partial_i + \omega_i^{ab} (\gamma^{ab} \otimes 1\!\!1 + 1\!\!1 \otimes \gamma^{ab}) \right) \qquad \text{(III.16)}$$

In section I.3 we saw that the operator Q in (III.16) gives a representation of the supercharge of the supersymmetric N=1 σ-model in 0+1 dimensions. We learnt that (see eqs.(I.72),(I.73))

$$\begin{aligned} \mathrm{tr}\,(-1)^F &= \mathrm{tr}\ \gamma_7 \otimes \gamma_7 = \chi(C^6) \\ \mathrm{tr}\ Q_5 &= \mathrm{tr}\ \gamma_7 \otimes \mathbb{1} = \tau(C^6) \end{aligned} \tag{III.17}$$

(Remember that the Hirzebruch signature of a 4k+2 dimensional manifold is zero.) From (III.17) it follows that

$$\begin{aligned} \mathrm{index}\ D_6 &= \mathrm{tr}\ \gamma_7 \otimes P_+ = \\ &= \frac{1}{2}\,\mathrm{tr}\ \gamma_7 \otimes \mathbb{1} + \frac{1}{2}\,\mathrm{tr}\ \gamma_7 \otimes \gamma_7 = \\ &= \frac{1}{2}\,\chi(C^6) \end{aligned}$$

in agreement with eq.(III.13).

III.2 World-sheet Witten indices and fermion chirality for the compactified heterotic string [12]

In this section we show how to derive the chiral content of the compactified heterotic string without making any reference to the low-energy effective field theory coming from the zero-slope limit.

A first-quantized superstring propagating in a given gravitational and gauge background is described by a two-dimensional supersymmetric σ-model on a curved manifold, coupled to external gauge fields. The background fields themselves originate as vacuum expectation values of the operators which, in the second-quantized string theory, create string states corresponding to gravitons and gauge bosons [28] , [29] , [30] , [31] .

Considering open string in this formalism is technically problematic since the coupling to the background gauge fields should be introduced by adding to the σ-model action line integrals along the string boundary. Therefore we are going to stick to closed strings with the gauge fields introduced via the heterotic string mechanism - in any case open string theory are believed not to be too interesting for phenomenology.

Of the various formulations of the heterotic string theory the one which will best suit us is the fermionized version. All fermionic variables (gauge as well Lorentz) will be treated à la Neveu-Schwarz-Ramond.

The σ-model action corresponding to the heterotic string, in the light-cone gauge and in the fermionized version, is [30]

$$S = \frac{1}{4\pi\alpha'}\int d\sigma\, d\tau \Big[g_{\mu\nu}\partial_\alpha X^\mu \partial^\alpha X^\nu + i\,\psi^a (D_+)^{ab}\psi^b + \tag{III.18}$$

$$+ i\, c^I (D_-)^{IJ} c^J + \frac{1}{2} F^m_{\mu\nu} (T^m)_{IJ} \psi^\mu \psi^\nu c^I c^J \Big]$$

σ and τ are the coordinates parametrizing the world-sheet of the string. $X^\mu(\sigma,\tau)$, $\mu = 1,\ldots,8$ are the commuting string variables, while $\psi^a(\sigma,\tau) \equiv e^a_\mu \psi^\mu$ (e^a_μ are the "vielbein": $e^a_\mu e_{a\nu} = g_{\mu\nu}$) are the anticommuting variables. Both X^μ and ψ^μ carry a vector index of the transverse Lorentz group O(8). The c^I , I = 1,...,32 are 32 anticommuting variables whose group properties depend on the specific choice, Spin $(32)/Z_2$ or $E_8 x E_8$, for the gauge group G.

$g_{\mu\nu}(X)$ and $A^m_\mu(x)$ are, respectively, the gravitational and gauge backgrounds, $F^m_{\mu\nu}$ is the field strength corresponding to A^m_μ and T^m_{IJ} are the generators of the gauge group G in the representation according to which the c^I trasform.

The covariant derivatives are defined as follows:

$$D^{ab}_+ \psi^b \equiv \partial_+ \psi^a + \omega^{ab}_\mu \partial_+ X^\mu \psi^b$$

$$(D_-)^{IJ} c^J \equiv \partial_- c^I + A^m_\mu (T^m)_{IJ} \partial_- X^\mu c^J \qquad \text{(III.19)}$$

where

$$\partial_\pm \equiv \partial_\tau \pm \partial_\sigma$$

The trasformation laws of the two-dimensional N=½ supersymmetry are:

$$\delta X^\mu = i \epsilon \psi^\mu \quad , \quad \delta \psi^\mu = - \partial_- X^\mu \epsilon$$

$$\delta c^I = i \epsilon A^m_\mu (T^m)_{IJ} c^J \psi^\mu \qquad \text{(III.20)}$$

Under these trasformations, the action varies in general by a surface term, which vanishes only for periodic boundary conditions on ψ^{μ}.

When G = Spin $(32)/Z_2$, the C^I are taken to transform in the vector representation of G. For G = E_8xE_8, only the subgroup O(16)xO(16) is linearly realized on the fields. The C^I split into two sets of variables, C^A and $\tilde{C}^A$, A = 1,...,16, trasforming according to the (16,1) and (1,16) of O(16)xO(16).

For the sake of clarity, we briefly review the less familiar Neveu-Schwarz-Ramond formulation of the heterotic string [24] . To define a theory which is supersymmetric in ten dimensions, one must consider both periodic and anti-periodic boundary conditions (PBC and APBC) on the Grassman variables ψ^{μ} and C^I. These define different sectors of the theory, within each one must perform a suitable projection to obtain the correct physical spectrum.

For vanishing background, the model splits into a left-handed $(X^{\mu}(\tau+\sigma), C^I(\tau+\sigma))$ and right-handed $(X^{\mu}(\tau-\sigma), C^I(\tau-\sigma))$ sector. In the right-handed sector, PBC gives rise to fermionic (Ramond) states with respect to the ten-dimensional Lorentz group. They transform according to both spinorial (left-chiral) and (antispinorial (right-chiral) representations of the trnsverse Lorentz group O(8). To obtain the desidered spectrum, the anti-spinors are eliminated by a projection. APBC lead to bosonic (Neveu-Schwarz) states belonging to representations in the vector and singlet conjugacy classes of O(8). In this case the singlet-type representation are removed.

In the left-handed sector, when G= Spin $(32)/Z_2$, one imposes either PBC or APBC on all the C^I . The choice of the factor group Z_2 is such that the Ramond (PBC) sector has spinor-like representations (and no antispinors), while the Neveu-Schwarz (APBC) sector has representations in the singlet conjugacy class and none in the vector class. For G = $E_8 x E_8$, we have four sectors corresponding to each of the two possible boundary conditions on each set of 16 c^A, $\tilde{c}^A$. One now picks out spinorial and singlettype representations separately for each factor group.

Concentrating now on the Ramond-type sectors, we show that, in terms of the two-dimensional supersymmetry of the Lagrangian, the spinorial and antispinorial representations correspond to the eigenvalues ± 1 of the operator $(-1)^F$, where F is the two-dimensional fermion number.

In terms of operators:

$$(-1)^F = \gamma_{11} \otimes (-1)^{\sum_n d^\mu_{-n} d^\mu_n} \qquad \text{(III.21)}$$

The operator on the right was introduced in the pioneering article of Gliozzi, Olive and Scherk [21] to distinguish the two classes within the Ramond model. The supersymmetry generator of this two-dimensional model, model, in terms of the usual transverse oscillators, is

$$Q = \frac{1}{2} p^\mu \gamma^\mu + \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} d^\mu_n \alpha^\mu_{-n} \qquad \text{(III.22)}$$

The fermion number operator F may be defined through:

$$\{(-1)^F, Q\} = 0 \quad , \quad ((-1)^F)^2 = 1 \tag{III.23}$$

It is evident that the definition in eq.(III.21) satisfies these properties.

Thus removing states of the Ramond sectors which are in anti-spinorial representations is equivalent to projecting the theory onto the even world sheet fermion number subspace. A similar result holds for the Neveu-Schwarz sectors, altough there is no supersymmetry which transforms these subspaces into each other.

Such a characterization of the projections required to make the spectrum space time supersymmetric is essential when the backgrounds in the action (III.18) are non-trivial. In that case, the fields are not decomposable in terms of free oscillators; nevertheless, the operator $(-1)^F$ is well-defined and the physical spectrum is <u>defined</u> as the projection onto the subspace $(-1)^F=1$. This procedure must be carried out <u>separately</u> for all the sectors: those pertaining to the ψ^μ, which carry a Lorentz index, as well as those pertaining to the c^I, which have a gauge index.

For a string theory which compactifies down from ten to four dimensions, the spin connection ω_μ in the internal six dimensions in general takes values in the Lie algebra of O(6). It has been shown [25] that, in order to avoid chiral anomalies, the gauge field and the spin connection must be equated:

$$A_\mu^{ab} T^{ab} = \omega_\mu^{ab} \sigma^{ab} \tag{III.24}$$

where σ^{ab} are the generators of the O(6) internal Lorentz group, while T^{ab} are the generators which span an O(6) subgroup of the gauge group G. G is broken to the maximal subgroup H such that:

$$G \supset O(6) \times H \tag{III.25}$$

Under this decomposition, a subset c^P (p = 1,...,6) of the c^A transforms as the (6,1), while the remaining set, $c^{\bar{A}}$ (A = 1,...,10) and $\tilde{c}^A$ (A = 1,...,16) are inert under O(6). Since the metric is flat in two directions we can also split $X^\mu = (X^{\bar{\mu}}, X^i)$ where $\bar{\mu}$ = 1,2 and i=1,...,6 and $\psi^\alpha = (\psi^{\bar{\alpha}}, \psi^p)$ with $\bar{\alpha}$ =1,2 and p = 1,...,6. Finally we define the real two-component spinor $\lambda^P = \binom{\psi^P}{c^P}$ and the action (III.18) becomes:

$$S = S_- + S_+ + S_3$$

$$S_- = \frac{1}{4\pi\alpha'} \int d\sigma\, d\tau \left((\partial_\alpha X^{\bar{\mu}} \partial^\alpha X^{\bar{\mu}})_- + i\, \psi^{\bar{\mu}} \partial_+ \psi^{\bar{\mu}} \right)$$

$$S_+ = \frac{1}{4\pi\alpha'} \int d\sigma\, d\tau \left((\partial_\alpha X^{\bar{\mu}} \partial^\alpha X^{\bar{\mu}})_+ + i c^{\bar{A}} \partial_+ c^{\bar{A}} + i\, \tilde{c}^A \partial_+ \tilde{c}^A \right) \tag{III.26}$$

$$S_3 = \frac{1}{4\pi\alpha'} \int d\sigma\, d\tau \Big(g_{ij}\, \partial_\alpha X^i \partial^\alpha X^j + i\, \bar{\lambda}^p \gamma^\alpha (D_\alpha \lambda)^p$$

$$+ \frac{1}{6} R_{pqrs}\, \bar{\lambda}^p \lambda^r \bar{\lambda}^q \lambda^s \Big)$$

S_- and S_+ are free theories which describe, respectively, the decoupled roght-handed and left-handed variables. S_3 is an N=1 supersymmetric σ-model on the compact six-dimensional internal manifold. The Hilbert space of the theory is the tensor product of the Hilbert spaces corresponding to the three decoupled actions.

To study the spacetime fermionic states of the string, one restricts oneself to periodic boundary conditions on the $\psi^{\bar{\mu}}$, ψ^{i} . For the C^{I} variables, the relevant boundary conditions depend on the choice of the gauge group, as mentioned earlier.

Consider first the more interesting case G = $E_8 x E_8$. The linearly realized O(16)xO(16) breaks to O(16)xO(16), with respect to which the $C^{\bar{A}}$ and $\tilde{C}^{A}$ transform in the (10,1) and (1,16) representations, respectively. Chiral fermion generations come from zero mass states in complex representations of the unbroken group. The only possibility is the spinorial of O(10), the (16,1), which belongs in the sector:

$$C^{A} = C^{1}, \dots C^{16}: \text{Ramond (PBC)}$$

$$\tilde{C}^{A} = C^{17} \dots C^{32}: \text{Neveu-Schwarz (APBC)}$$

It is easy to see that the other three sectors give either real representations or massive states.

The result of all this is that we need to consider the three actions S_-, S_+, S_3 with the following boundary conditions:

S_- ,S_3 : PBC on all fields

S_+ : PBC on 10 $c^{\bar{A}}$'s

APBC on 16 $\tilde{c}^{A}$'s

It is important to note that, with these boundary conditions, S_3 is truly invariant under N=1 supersymmetry, without producing surface terms. Thus its spectrum is positive definite. For the remaining (free) modes, one can verify that the ground state energy is zero. So the ground states of S are massless if the N=1 supersymmetry is unbroken.

Now it is easy to derive the chiral fermion content. The ground states of S_- are a left-handed and a right-handed spinor, |L> and |R>, with respect to the four-dimensional Lorentz group, with eigenvalues ± 1 of $(-1)^{F_\psi}$: F_ψ being the fermion number relative to the "Lorentz" Grassman variables ψ^{μ} and ψ^{i}. The ground states S_+ form a (16,1) and $(\overline{16},1)$ representation of O(10)xO(16), with $(-1)^{F_c} = \pm 1$ respectively: F_c being the fermion number for the first 16 "gauge" variables $c^1,\ldots,c^{16}$. The action S_3 is a non-trivial field theory, and its spectrum cannot be computed exactly. Let $n_{\alpha\beta}$ ($\alpha,\beta = \pm 1$) be the total number of ground states of this action with eigenvalues $(-1)^{F_\psi} = \alpha$, $(-1)^{F_c} = \beta$. The Lorentz and gauge properties of the ground states are summarized in Table 1.

The number of chiral fermion generations is

$$n(16_L) - n(\overline{16}_L) = n_{++} - n_{+-} \tag{III.27}$$

We now exploit the topological properties of the N=1 supersymmetric σ-model S_3 [5] we learnt in section I.3. Let us recall eqs.(I.72),(I.73):

$$\text{Witten index} = \mathrm{tr}\,(-1)^{F_4} \otimes (-1)^{F_c}$$

$$= n_{++} + n_{--} - n_{-+} - n_{+-}$$

$$= \chi\,,\ \text{the Euler characteristic of the six-dimensional manifold} \qquad \text{(III.28)}$$

$$\mathrm{tr}\,(-1)^{F_c} = n_{++} - n_{+-} + n_{-+} - n_{--}$$

$$= \text{Hirzebruch signature of the manifold}$$

$$= 0 \quad \text{for } 4K+2 \text{ dimensional manifolds} \qquad \text{(III.29)}$$

the last equation implies

$$n(16_L) + n(\overline{16}_L) = n(16_R) + n(\overline{16}_R) \qquad \text{(III.30)}$$

which states the CPT invariance of the spectrum. Combining eqs.(III.28) and (III.29) one finally obtains:

$$n(16_L) - n(\overline{16}_L) = \chi/2 \qquad \text{(III.31)}$$

If the holonomy group of the internal six-dimensional manifold is a proper subgroup of O(6), such as SU(3), the unbroken gauge group may be bigger. For example, one may have $E_8 \supset SU(3) \times E_6$, for which the previous formula holds, with the 16 of the O(10) replaced by the 27 of E_6. Our result (III.31) is equally applicable to holonomy groups others than O(6) and SU(3), provided that the complex representations of the unbroken gauge group H do not arise from the adjoint representation of the Spin $(32)/Z_2$ or O(16)xO(16).

Thus we have obtained, in the quantum string theory, the result which in ref. [25] ,was derived from field theory considerations, disregarding the excited states of the string.

The reason why these two results coincide is precisely the topologically invariance of the Witten indices (III.28) and (III.29) in two dimensional supersymmetric field theories, which permits one to take the limit in which the length of the string vanishes.

For G = Spin $(32)/Z_2$, fermions in complex representations could appear only in the sector with periodic boundary conditions on all 32 c^I 's. Unfortunately, in this sector, all states are massive, and there are no chiral fermions.

At first sight, it may seem that the same mechanism just described would also produce massive chiral fermions in four dimensions. Fortunately this is not so. The crucial observation is that the constraint of vanishing momentum in the σ-direction 24 implies that the massive states correspond to excited states of either S_3 or S_- (or both). Since $(-1)^{F_\psi} = 1$ and $(-1)^{F_\psi}$ $= (-1)^{F_\psi}\Big|_{S_3} \otimes (-1)^{F_\psi}\Big|_{S_-}$, it follows that

$$\gamma_5 = (-1)^{F_\psi}\Big|_{S_3} \otimes (-1)^{\sum_n d^{\bar{\mu}}_{-n} d^{\bar{\mu}}_n} \qquad \text{(III.32)}$$

But the supersymmetries of S_3 and $S\_$ guarantee that $\mathrm{tr}(-1)^{F_\psi}\big|_{S_3} = 0$ and $\mathrm{tr}\ (-1)^{\sum_n d^{\bar{\mu}}_{-n} d^{\bar{\mu}}_n} = 0$ on all excited leves, there are no "massive chiral fermions".

Let us end with the following remark: sectors of the model in which Neveu-Schwarz (antiperiodic) boundary conditions are imposed on any of the variables appearing in S_3 are not N=1 supersymmetric, and our topological considerations do not seem to apply directly. If indeed there is no appropriate topological invariance, one would suspect that field theory considerations for these sectors are not in general valid.

Footnotes

f1 The saturation of (II.58) at the quantum level has been studied with finite-volume techniques in a number of papers 33 . Unfortunately, because of the ambiguities related to the choice of the boundary conditions, contradictory results have been obtained: in finite volume the question does not seem to be definetly settled. In this respect our infinite-volume calculation has the advantage of providing a unique answer.

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S_-	S_+	S_3	S
L	16	n_{++}	$n_{++}16_L$
R	16	n_{+-}	$n_{+-}16_R$
L	$\overline{16}$	n_{-+}	$n_{-+}16_L$
R	$\overline{16}$	n_{--}	$n_{--}16_R$

Table I: Quantum numbers of the ground states