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STOCHASTIC PROPERTIES OF SUPERSYMMETRIC FIELD THEORIES: A PERTURBATIVE STUDY

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I. INTRODUCTION

The common belief among physicists at the end of the past century was that Newtonian mechanics could provide an accurate modelling of natural phenomena. If all intial data could be collected for a given system, one would be able to predict completely its future evolution by simply solving the associated deterministic equations of motion.

Later, it became however evident that this approach was not manageable for macroscopic systems, due to the large number of degrees of freedom involved. To study such systems a statistical approach was introduced which makes extensive use of probabilistic concepts [1]. In this way one gives up a detailed, deterministic description of the system in favour of a phenomenological one in which only averaged quantities are considered. These slow-varying variables are however allowed to fluctuate; and in fact their time evolution towards equilibrium is described in general by coupled non-linear differential equations in which the right-hand sides are written as a sum of deterministic and fluctuating forces (Langevin equations). The deterministic Newtonian equations of motion become then stochastic differential equations [*].

On the other hand we now understand that Newtonian mechanics cannot give a detailed description of natural phenomena because of the advent in physics of Special Relativity and Quantum Mechanics.

(*) For a complete introduction to stochastic methods in Physics see for example ref's (2,3,4). Moreover note that in practice the Newton equations are not fully deterministic; in fact, as recently discovered, even quite simple systems of differential equations have the property of giving rise to essentially unpredictable (chaotic) behaviour [5].

As known, Quantum Mechanics has an essential probabilistic content which makes its methods and techniques very similar to those of Statistical Mechanics; and really a Quantum Statistical Mechanics can be consistently constructed [1].

Furthermore, combination of Quantum Mechanics with Special Relativity requires the introduction of Quantum Field Theory. In one of its possible formulations this theory can be elegantly summarized by the Euclidean generating functional [6,7,8], which is written using standard notations in the following way:

$$Z[J] = \int \mathcal{D}\phi \; e^{-\frac{1}{\hbar}\left\{ S[\phi] + \int J\phi \right\}} .$$

Consistency with Special Relativity (or better with SO(4)-invariance) is automatic if the action $S[\phi]$ and the coupling with the external source $\int J\phi$ are scalars. Z is also consistent with Quantum Mechanics since the sum over all configurations, implicit in the definition of the previous functional integral, clearly implements the quantum superposition principle.

What is now remarkable is that Euclidean Quantum Field Theory can be interpreted as a statistical mechanical system. In fact if one makes the correspondences: temperature $\longleftrightarrow \hbar$, Energy $\longleftrightarrow S[\phi]$, then Z becomes the partition function of a statistical field system in 4+1 dimensions [*]. The equilibrium properties of this system can be derived by studying the asymptotic limit of equations explicitly involving the 5th time variable, namely of Langevin-like equations [15]. Then stochastic concepts and methods vigorously enter the domain of quantum field theories contributing, from a new point of view, to the understanding of the

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(*) There is a wide literature about this connection; see for example ref.'s (7-10). For some recent new results see ref.'s (11-14).

physical content of these models [(*)].

Another element which is now playing a more and more important role as an essential ingredient of quantum field models is supersymmetry [(18)]. Supersymmetric field theories exhibit in fact quite remarkable and unique features, and among those their very good ultraviolet behaviour, due to some "miracolous" cancellations of divergences [(19)]. However, another remarkable property of supersymmetric theories has been pointed out recently: if one looks at the corresponding generating functional Z, it is always possible to transform the functional integral into a Gaussian one by mean of a suitable change of integration variables (the Nicolai map) [(20,21)]. In the cases in which this change of variables is local, one can interpret it as a stochastic differential equation (again of the Langevin-type), and hence one can use all standard stochastic techniques to extract further physical information on the theory [(22,23)]. Supersymmetric field theories possess thus a richer stochastic structure than standard field models which makes then the connection between Quantum Field Theory and Statistical Mechanics even more stringent and deep.

This work presents a discussion of some aspects of such interesting connection between supersymmetry and stochastic properties of quantum field theories, with special emphasis on the new perturbative technique that this connection produces. After a short introduction on standard stochastic techniques, with applications to "stochastic quantization", we discuss the definition and the properties of the Nicolai map in supersymmetric theories. Taking a two-dimensional model (N=2 Wess-Zumino model) as an example, the stochastic interpretation of local

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(*) In this context note that, as shown in ref.'s (16,17), there is a strict relation between the different approaches of ref.'s (11-13, 15).

Nicolai mappings is studied in detail, together with the new perturbation expansion which the local transformations automatically produce.

Then a perturbative study of the local Nicolai mapping in the case of four-dimensional N=1 supersymmetric Yang-Mills model is given by computing at one-loop, using the perturbatively inverted map, two- and three-point Green's functions. Only in the light-cone gauge these results coincide with those obtained in ordinary perturbation theory, thus confirming that a local Nicolai mapping for the theory is possible only in this special gauge. Finally analogous computations are performed also for the four-dimensional N=2 supersymmetric Yang-Mills theory in the light-cone gauge; also in this case the results agree with the standard perturbative ones.

II. SOME RESULTS OF THE STOCHASTIC APPROACH TO PHYSICAL PROBLEMS

II.1 The Brownian motion

In the first half of the past century, the famous botanist R. Brown observed that, when suspended in water, small pollen grains are found to be in a very irregular state of motion. A satisfactory explanation of this phenomenon (the so called Brownian motion) came only at the beginning of this century when A. Einstein published his famous work [24]. In it he showed that the motion is caused by the impacts on the pollen grain from the incessantly moving molecules of liquid in which it is suspended. Moreover the motion of these molecules is so complicated that its effect on the grain can only be described in probabilistic way: this recourse to a statistical explanation was really a new thing and can be regarded as the beginning of a stochastic modelling of natural phenomena.

Some time after Einstein's explanation, P. Langevin presented a new simple derivation of it [25]. The starting point is the observation that on a small pollen particle of mass m immersed on a fluid, there should act two forces:

1) a friction force; one can assume that it is given by the same formula of macroscopic hidrodynamics:

$$F = -\alpha v \qquad\qquad \alpha = 6\pi\eta a$$

where v is the velocity of the particle, of transversal dimensions a, and η is the viscosity of the fluid;

2) a fluctuating force h which represents the incessant impacts of the molecules of the fluid on the Brownian particle. This force h is a stochastic or random force, the properties of which are given only in an averaged way.

The Newton's equation for the velocity of the particle is then:

$$\dot{v}(t) + \gamma v(t) = \xi(t) \tag{2.1}$$

with $\gamma = \frac{\alpha}{m}$ and ξ = h/m. Eq.(2.1), the so called Langevin equation, is a first simple example of stochastic differential equation. To proceed further we assume that $\xi(t)$ has a Gaussian distribution and then, by averaging over a large number of equivalent systems one gets,

$$\langle \xi(t) \rangle = 0 \tag{2.2}$$
$$\langle \xi(t)\,\xi(t') \rangle = q\,\delta(t-t') \quad , \quad q = \frac{2\gamma K T}{m}$$

($\langle\ \rangle$ means ensamble average; T is the temperature of the fluid , K the Boltzman's constant). With these specifications, by integrating eq.(2.1) one can calculate for example the diffusion coefficient D, to find the Einstein's results:

$$D = \frac{KT}{6\pi\eta a} \ .$$

Since $\xi(t)$ is a stochastic variable, also v(t) obeying eq.(2.1) is a stochastic variable; then we may ask for the probability density p(v,t)(*). The quantity

$$dP = p(v,t)\ dv$$

is the probability of finding the particle with its velocity in the interval (v,v+dv). The equation of motion for p can be derived from eq.(2.1) and it is given by

$$\frac{\partial p}{\partial t} = \gamma \frac{\partial (v p)}{\partial v} + \gamma \frac{KT}{m} \frac{\partial^2 p}{\partial v^2} \tag{2.3}$$

(*) For a brief introduction to stochastic and probabilistic concepts see Appendix C.

which is a simple example of Fokker-Planck equation [26]. Given the initial distribution $p(v,o)$, one can obtain $p(v,t)$ for any later time by solving eq.(2.3), and thus any average value can be obtained by integration:

$$\langle G(v(t)) \rangle = \int_{-\infty}^{+\infty} dv \; G(v) \; p(v,t)$$

(G is an arbitrary function).

This example, even if extremely simple, shows many of the general features of a stochastic approach to a physical problem. In the following a brief study of 'generalized' Langevin equations will be given and the connection with the corresponding Fokker-Planck equations will be explicitly discussed.

Finally, let us mention that the study and analysis of the Brownian motion have become nowadays much more quantitative and sophisticated; thanks to the use of coherent laser light, one can study the motion of much smaller particles than the traditional pollen by detecting their scattered light. It is with the use of these modern techniques that important results about the sizes of viruses and macromolecules have been obtained [2].

II.2 Langevin equation and Fokker-Planck equation

Given a stochastic variable $x(t)$, a general Langevin equation can be written in the following form [2,3]:

$$\dot{x} = f(x,t) + g(x,t)\, \xi(t) \qquad (2.4)$$

where f is the drift or friction force and g is a given function. The stochastic force $\xi(t)$ is again assumed to be a Gaussian stochastic process

with zero mean and δ-correlation (**):

$$\langle \xi(t) \rangle = 0 \qquad \langle \xi(t)\, \xi(t') \rangle = \delta(t-t') \qquad (2.5)$$

The fundamental property of the Langevin equation (2.4) with δ-correlated noise is that it describes a Markov process: the conditional probability at time t_n depends only on the value of $x(t)$ at the next earlier time t_{n-1}. This fact can be naïvely understood by noting that a) a first-order differential equation like (2.4) is uniquely determined by the initial condition and b) the δ-correlated noise $\xi(t)$ at a time $t < t_{n-1}$ cannot change the conditional probability at a later time $t > t_{n-1}$.

Usually a general solution for eq.(2.4) cannot be given. In order to solve it one has to use some approximation methods to write down a perturbative solution as a series in some small parameter appearing in

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(**) Note that from a rigorous point of view, writing a stochastic differential equation in the form of eq.(2.4) with the specifications of eq.(2.5) is ambiguous and mathematically incorrect (see for example ref.'s (27, 2, 3, 28)). A more precise approach would require the substitution of eq.(2.4) with the following integral equation:

$$x(t) - x(o) = \int_0^t f(x(t'),t')\, dt' + \int_0^t g(x(t'),t')\, dw(t')$$

where $w(t)$ is a Wiener process; the above integrals are stochastic integrals whose precise meaning must be given for example in terms of Ito differential calculus. Only using these precise definitions one can state and prove theorems on the existence and uniqueness of the solutions of stochastic differential equations [29]. However in the framework of perturbation theory the expression (2.4) is sufficiently well defined and consistent once precise boundary conditions are taken into account.

eq. (2.4); we will see later in a specific example involving Quantum Field Theory how this procedure can be explicitly worked out. However there is also another possibility: we can set up a Fokker-Planck equation by which the probability density of the stochastic variable x(t) can be computed [3,30-38]. To do so, one has first to compute the so-called Kramers-Moyal expansion coefficients:

$$D^{(n)}(z,t) = \frac{1}{n!} \lim_{\tau\to 0} \frac{1}{\tau} \left< \left[x(t+\tau) - z\right]^n \right> \Big|_{x(t)=z} \qquad (2.6)$$

where $x(t+\tau)$ ($\tau > 0$) is a solution of eq.(2.4) with initial condition $x(t) = z$. By first writing eq.(2.4) in integral form:

$$x(t+\tau) - z = \int_t^{t+\tau} dt' \left\{ f(x(t'),t') + g(x(t'),t')\,\xi(t') \right\}$$

and then expanding f and g in powers of $(x(t')-z)$, one easily gets the following results:

$$\begin{aligned} D^{(1)}(x,t) &= f(x,t) + g(x,t)\,\partial_x g(x,t) \\ D^{(2)}(x,t) &= \left[g(x,t)\right]^2 \qquad\qquad (2.7) \\ D^{(n)}(x,t) &= 0 \qquad\qquad n \geq 3 . \end{aligned}$$

From the definition of conditional probability $p(x,t+\tau\,|x',t)$, it follows that the probability density $p(x,t+\tau)$ at the time $t+\tau$ ($\tau \geq 0$) and the probability density $p(x,t)$ at the time t, are connected by:

$$p(x,t+\tau) = \int dx'\, p(x,t+\tau\,|x',t)\, p(x',t) \qquad (2.8)$$

Assuming τ small, from this relation it is possible to derive a partial differential equation for p(x,t), the so-called Kramers-Moyal expansion:

$$\frac{\partial p(x,t)}{\partial t} = L_{KM}(x,t) \cdot p(x,t) \qquad (2.9)$$

with

$$L_{KM}(x,t) = \sum_{n=1}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n D^{(n)}(x,t) . \qquad (2.10)$$

Since the probability $p(x,t|x',t')$ is the distribution p(x,t) for the special initial conditions $p(x,t') = \delta(x-x')$, it follows that also $p(x,t|x',t')$ satisfies eq.(2.9), with initial conditions: $p(x,t'|x',t') = \delta(x-x')$.

It is also possible to write down an equation of motion for $p(x,t|x',t')$ in which differential operators with respect to x' and t' appear. This equation is the differential counterpart of the so-called Chapman-Kolmogorov equation for Markov processes, and reads:

$$\frac{\partial}{\partial t'} p(x,t\,|x',t') = - L^{+}_{KM}(x',t') \cdot p(x,t\,|x',t') \qquad (2.11)$$

where

$$L^{+}_{KM}(x',t') = \sum_{n=1}^{\infty} D^{(n)}(x',t') \left(\frac{\partial}{\partial x'}\right)^n \qquad (2.12)$$

is the adjoint operator of (2.10).

A fundamental result concerning the expansions (2.9) and (2.11) is the following [39,3]:

for positive transition probability $p(x,t|x',t')$, the above expansion either stops at first or second term, or it must contain an infinite number of terms. We are mainly interested in the expansions (2.9) and (2.11) which correspond to the Langevin equation (2.4); in that case the expansions stop at the second term and eq.(2.9) reduces to the so-called forward Fokker-Planck equation:

$$\frac{\partial}{\partial t} p(x,t|x',t') = L_{FP}(x,t) \cdot p(x,t|x',t') \tag{2.13}$$

with:

$$L_{FP}(x,t) = -\frac{\partial}{\partial x} D^{(1)}(x,t) + \frac{\partial^2}{\partial x^2} D^{(2)}(x,t) \quad . \tag{2.14}$$

An analogous equation holds for the probability density $p(x,t)$. On the other hand eq.(2.11) reduces to

$$\frac{\partial}{\partial t'} p(x,t|x',t') = -L^{+}_{FP}(x',t') \cdot p(x,t|x',t') \tag{2.15}$$

with

$$L^{+}_{FP}(x',t') = D^{(1)}(x',t')\frac{\partial}{\partial x'} + D^{(2)}(x',t')\frac{\partial^2}{\partial x'^2} \tag{2.16}$$

which is the backward Fokker-Planck equation; it can be obtained from eq.(2.13) by time reversal. The initial conditions in both cases are: $p(x,t|x',t) = p(x,t'|x',t') = \delta(x-x')$.

Many different methods for solving Fokker-Planck equations have been developed [3]; here we would like to mention that solutions can be given in terms of path-integral [3,8]. For example in the case of the probability distribution $p(x,t)$, the solution of eq.(2.13) can be written in the following form:

$$p(x,t) = \int d\Omega(x(t))\, p(x_0,t_0)\, e^{-\int_{t_0}^{t} dt'\, O(x(t'),\dot{x}(t'))} \tag{2.17}$$

$$d\Omega(x(t)) = \prod_{\substack{i=1 \\ N\to\infty}}^{N} \frac{dx(t_i)}{\left[2\pi D^{(2)}(x(t_i),t_i)\tau\right]^{1/2}} \quad , \quad t_i = i\cdot\tau$$

where

$$O(x,\dot{x}) = \frac{1}{2}\frac{(\dot{x}-D^{(1)})^2}{D^{(2)}} + \frac{1}{2}\frac{dD^{(1)}}{dx} \quad . \tag{2.18}$$

The function $O(x,\dot{x})$ is called generalized Onsager-Machlup function; it is a thermodynamic potential related to the rate of entropy production.

II.3 The stochastic quantization

As an application of the previous concepts and definitions to Field Theory, let us briefly consider the new scheme of quantization proposed some time ago by Parisi and Wu [15,40].

Let us take a physical system described by a field $\phi(x)$ in a D-dimensional Euclidean space (extension to Minkowski space is discussed in ref.'s (41,42). For simplicity, we assume that ϕ is a scalar; however suitable generalizations to spinor [43-45], vector [15,46-58] and tensor [59,60] fields can be given). The dynamics of ϕ is described by a Lagrangian $\mathcal{L}$, which, for example, can be taken of the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g}{3}\phi^3 \quad . \tag{2.19}$$

We are interested in computing the Green's functions of the theory, which in a functional approach are given by the following functional integral:

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle_0 = \frac{\int \mathcal{D}\phi \, \phi(x_1) \cdots \phi(x_n) \, e^{-S}}{\int \mathcal{D}\phi \, e^{-S}} \qquad (2.20)$$

where $S = \int d^D x \, \mathcal{L}$ is the action.

When we use eq.(2.20) we choose what one can call a 'static' point of view, in the sense that we assume the system, interpreted as a statistical system, in equilibrium at a given temperature (choosen to be unity for convenience). And in fact eq.(2.20) is nothing else but a suitable Gibbs average in equilibrium statistical mechanics.

However we can also adopt a 'dynamical' point of view and describe the system in terms of non-equilibrium statistical mechanics (*). We assume that the field ϕ is also a function of an additional fictitious time t: $\phi = \phi_t(x) = \phi(x,t)$.

The system is coupled to a heat reservoir at a given temperature, but at the beginning is not in equilibrium with it. It is prepared at the initial time (which can be always taken to be t = 0) in a given configuration $\phi_0(x)$; only for large values of t (really in the limit $t \to \infty$) $\phi(x,t)$ will reach the equilibrium configuration. The crucial point now is to give the 'stochastic law' according to which ϕ_t evolves from the initial configuration ϕ_0 to the equilibrium. There is a large arbitrariness in choosing this law; however one usually assumes that the evolution in the extra time t is governed by a Langevin equation:

(*) A different stochastic approach to field theories can be given along the lines of the so-called stochastic mechanics [61]; for details see ref. (27) and the papers there quoted.

$$\frac{\partial \phi(x,t)}{\partial t} = - \left. \frac{\delta S[\phi]}{\delta \phi} \right|_{\phi = \phi(x,t)} + \xi(x,t) \qquad (2.21)$$

where $\xi(x,t)$ is a multidimensional Gaussian white noise (*), one for each space-time point (and field component):

$$\langle \xi(x,t) \rangle = 0$$
$$\langle \xi(x,t) \, \xi(x',t') \rangle = \delta^{(D)}(x - x') \, \delta(t - t') \qquad (2.22)$$

(as usual $\langle \; \rangle$ means expectation value with respect to the stochastic variable ξ).

Since $\phi(x,t)$ obeys eq.(2.21), it is now a stochastic variable and the relevant correlation functions can be computed in the usual way:

$$\langle \phi(x_1,t_1) \cdots \phi(x_n,t_n) \rangle = \int \mathcal{D}\phi \; p[\phi,t] \; \phi(x_1,t_1) \cdots \phi(x_n,t_n) \qquad (2.23)$$

where $p[\phi,t]$ is the probability density associated with the stochastic process ϕ. From eq.(2.21) it is easy to compute the drift and diffusion coefficients (remember eq.(2.6) and (2.7)):

$$D^{(1)}[\phi,t] = - \left. \frac{\delta S[\phi]}{\delta \phi} \right|_{\phi = \phi(x,t)}$$
$$D^{(2)}[\phi,t] = 1 \qquad ; \qquad (2.24)$$

$p[\phi,t]$ satisfies then the following Fokker-Planck equation (see eq.(2.13)):

(*) Extension to Poisson noises has been proposed in ref.(62). A modification of the Langevin equation (2.21) has been studied in ref. (63).

$$\frac{\partial}{\partial t} p[\phi,t] = \int d^D x \left\{ \frac{\delta}{\delta\phi}\left[\frac{\delta S[\phi]}{\delta \phi}\, p[\phi,t] \right] + \frac{\delta^2 p[\phi,t]}{\delta \phi^2} \right\} \qquad (2.25)$$

(a suitable regularization is understood in the right-hand side of eq.(2.25)).

The stochastic approach to quantization is based on the following property:

$$\lim_{t\to\infty} \langle \phi(x_1,t) \cdots \phi(x_n,t) \rangle = \langle \phi(x_1) \cdots \phi(x_n) \rangle_0 \qquad (2.26)$$

or equivalently,

$$\underset{t\to\infty}{w\text{-}\lim}\; p[\phi,t] = \frac{e^{-S}}{\int \mathcal{D}\phi\, e^{-S}} \; ; \qquad (2.27)$$

in other words at large times the probability distribution reaches the equilibrium one and the 'static' definition of eq.(2.20) is recovered (the limit in eq.(2.27) is a weak limit, and the equilibrium solutions of eq.(2.25) exists only in a weak sense; see ref.(64) for details)(*).

(*) All the advantages of this approach are now transparent. First of all stochastic quantization clearly represents a new independent kind of quantization of field theories, which is particularly relevant for gauge theories [(15,55)]. Moreover the dynamics in the additional time is non trivial and gives rise to the possibility of a new kind of regularization the so-called 'stochastic regularization' [(65,45,66,63,72)]. This produces the possibility of renormalizing the theories at finite extra time (i.e. before the limit $t\to\infty$)[(67,68)], making possible an independent computation of critical exponents [(69)]. Finally, and this is the original motivation, the additional time t can be considered as a computer time; the stochastic quantization approach can be then used as an algorithm for computer simulations in Quantum Field Theory [(70)].

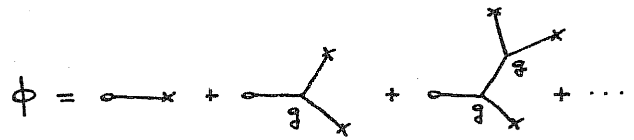
Many different proofs exist of eq.(2.26) and eq.(2.27) [(64,54,71-76)]; here we would like to mention an approach based on a diagrammatic technique which will become useful later [(65,75,76,77)]. The idea is to perturbatively solve the Langevin equation (2.21). First of all this equation can be transformed into an integral equation in momentum space:

$$\phi(k,t) = \int_0^t dt'\, G(k;t-t') \Big\{ \xi(k,t') - g \int \frac{d^D p}{(2\pi)^D}\, \phi(p,t')\, \phi(k-p,t') \Big\} \qquad (2.28)$$

where G is the forward stochastic propagator

$$G(k;t-t') = e^{-(t-t')(k^2+m^2)}\, \theta(t-t') \; ; \qquad (2.29)$$

we choose for simplicity the boundary condition $\phi(x,0) = 0$. Solving eq.(2.28) by iteration one arrives at a power series expansion of ϕ in the coupling constant, which can be diagrammatically written in the following way



(2.30)

where we denote G by a line and ξ by a cross. The n-point Green's function $\langle \phi(x_1,t) \ldots \phi(x_n,t) \rangle$ can be now written in terms of the so-called 'stochastic diagrams', obtained by joining together the various crosses in all possible ways according to the rules (2.22). One can prove that in the limit $t\to\infty$, the sum of all these stochastic graphs coincide with the standard Feynman diagram expansion of the n-point function; and this is another way to express eq.(2.26).

The correlation functions (2.26) of the stochastic variable $\phi(x,t)$ can be formally obtained from a generating functional $Z[J]$ in the following way (*):

$$\langle \phi(x_1,t_1)\cdots\phi(x_n,t_n)\rangle = \left.\frac{\delta^n Z[J]}{\delta J(x_1,t_1)\cdots\delta J(x_n,t_n)}\right|_{J=0} \qquad (2.31)$$

It is easy to check that Z has the following expression (71,16):

$$Z[J] = \int d\Omega(\phi)\, d\Omega(\xi)\, p[\phi,0]\, \delta(\phi-\phi_\xi)\cdot$$

$$\cdot\, e^{-\int_0^t dt'\int d^Dx \left\{\frac{1}{2}\xi^2 + J\phi\right\}} \qquad (2.32)$$

ϕ_ξ is the solution of the Langevin equation (2.21), solved with initial probability $p[\phi,0]$. To define the measure $d\Omega(\phi)$, we slice the interval $[0,t]$ in N infinitesimal parts ε, with $t_n = n\varepsilon$; if ϕ_{t_n} are the field configurations at time t_n, then:

$$d\Omega(\phi) = \lim_{N\to\infty} \prod_0^N{}_n\, \mathcal{D}\phi_{t_n} \quad .$$

In other words this measure is a product of usual four-dimensional functional measures. The δ -function is a formal expression which assures that the integral (2.32) is performed over the solutions of eq.(2.21); it can be more correctly written as:

$$\delta(\phi-\phi_\xi) = \delta\left(\dot\phi + \frac{\delta S}{\delta\phi} - \xi\right)\det\left(\frac{\delta\xi}{\delta\phi}\right)$$

where the determinant is the Jacobian of the transformation $\xi \mapsto \phi$.

(*) Functional approach to stochastic problems has been also studied in ref.'s (78,8,67,79,80).

Choosing propagation forward in time this determinant can be explicitly computed, giving at the end, after integration over $d\Omega(\xi)$, the following expression for Z:

$$Z_{forw.}[J] = \int d\Omega(\phi)\, p[\phi,0]\; e^{-\int_0^t dt'\int d^Dx \{L_+ + J\phi\}} \qquad (2.33)$$

with

$$L_+ = \frac{1}{2}\left(\dot\phi + \frac{\delta S}{\delta\phi}\right)^2 - \frac{1}{2}\frac{\delta^2 S}{\delta\phi^2} \quad . \qquad (2.34)$$

On the other hand, if we choose propagation backward in time , we get for $Z_{back.}[J]$ the same expression of eq.(2.33) but with

$$L_- = \frac{1}{2}\left(\dot\phi + \frac{\delta S}{\delta\phi}\right)^2 + \frac{1}{2}\frac{\delta^2 S}{\delta\phi^2} \qquad (2.35)$$

in place of L_+. The 'Lagrangian' (2.34) is nothing else that the generalized Onsager-Machlup function already encountered in the previous section; and in fact eq.(2.33) is exactly the analogue of eq.(2.17).

However one can also express the Jacobian $\det(\delta\xi/\delta\phi)$ by introducing suitable anticommuting variables ψ , $\bar\psi$:

$$\det\left(\frac{\delta\xi}{\delta\phi}\right) = \int d\Omega(\psi)\, d\Omega(\bar\psi)\; e^{-\int_0^t dt'\int d^Dx\, \bar\psi\left(\frac{\delta\xi}{\delta\phi}\right)\psi} \; .$$

What is remarkable is that if one properly choose $p[\phi,0]$, namely $p[\phi,0] = \delta(\phi(0) - \phi(t))$ (i.e. periodic boundary conditions), then the system reveals an hidden supersymmetry (*) . In fact $Z[J]$ can be now written as:

$$Z[J] = \int d\Omega(\phi)\, d\Omega(\psi)\, d\Omega(\bar{\psi})\, e^{-\int_0^t dt' \int d^Dx \{ \tilde{\mathcal{L}} + J\phi \}} \tag{2.36}$$

with

$$\tilde{\mathcal{L}} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\frac{\delta S}{\delta \phi}\right)^2 + \bar{\psi}\left(\partial_t + \frac{\delta^2 S}{\delta \phi^2}\right)\psi \tag{2.37}$$

This Lagrangian manifests a sort of non-relativistic supersymmetry; in fact it changes only by a total time derivative under the following transformations:

$$\begin{cases} \delta\phi = -(\varepsilon\psi + \bar{\varepsilon}\bar{\psi}) \\ \delta\psi = \varepsilon\left(\dot{\phi} + \frac{\delta S}{\delta\phi}\right) \\ \delta\bar{\psi} = \bar{\varepsilon}\left(\dot{\phi} - \frac{\delta S}{\delta\phi}\right) \end{cases} \tag{2.38}$$

(*) The presence of this underlying supersymmetry was first noted in ref. (22); then it has been studied by many authors, see ref.'s (71,72,67,81,82,83). Finally it is interesting to observe that, by a similar mechanism, any quantum mechanical system reveals an hidden supersymmetry; see ref. (84).

where $\varepsilon, \bar{\varepsilon}$ are infinitesimal anticommuting parameters. And this is clearly a first important connection between stochastic properties of a field theory and supersymmetry.

To finish, let us stress that in this case the full stochastic interpretation of the system is really recovered only in the limit $t \to \infty$ (71, 83,85). In fact having chosen periodic boundary conditions, there is no clear causal interpretation of the Langevin equation (2.21), which on the ohter hand has to be interpreted as a forward-time process if one wants to preserve its stochastic character. If one carefully performs the integration over the anticommuting variables in eq.(2.36), one really discovers that with periodic boundary conditions the generating functional contains both the forward and the backward dynamics in the fictitious time t. Explicitly one gets:

$$Z[J] = \{ Z_{forw.}[J] - Z_{back.}[J] \} .$$

Only in the limit $t \to \infty$, where any trace of specific boundary conditions disappear, one recovers the standard stochastic interpretation for $Z[J]$, since, if the underlying supersymmetry is unbroken:

$$\lim_{t\to\infty} Z[J] \equiv \lim_{t\to\infty} Z_{forw.}[J] .$$

III. STOCHASTIC PROPERTIES OF SUPERSYMMETRIC FIELD THEORIES

III.1 Nicolai map and stochastic identities

Among quantum field theories, those exhibiting supersymmetry invariance have attracted more and more interest due to their peculiar features. It is well known, for example, that supersymmetric field models present very remarkable cancellations of divergences in the perturbative expansion [19]; even more in some special cases of extended supersymmetric theories one obtains vanishing β -functions [86-89]. Furthermore, some relativistic supersymmetric models exhibit remarkable stochastic properties.

This fact was first realized in ref.(22) in which it is shown the possibility of constructing supersymmetric theories from classical stochastic equations. In addition to that another important property of supersymmetric models was first pointed out in ref.'s (20,21). It is suggested there that, after integrating out the fermionic fields in the Euclidean functional integral which defines the theory, the resulting partition function can be rendered Gaussian by performing a suitable change of bosonic variables (Nicolai map).

The existence of the Nicolai mapping is essentially a consequence of the fact that the vacuum energy for supersymmetric models vanishes if the ground state is supersymmetric [90]. In absence of interactions, this just means that the number of fermionic and the number of bosonic degrees of freedom are equal. For interacting theories the vanishing of the vacuum energy is equivalent to the condition that the Euclidean functional integral is a constant [(*)]. It is then quite natural that unbroken supersymmetric models can be also characterized by some properties of the

(*) As it will be apparent in the following, this constant coincides with the winding number of the Nicolai mapping [120].

corresponding functional integral measure [91].

To be specific let us consider an Euclidean supersymmetric theory containing some scalar fields $\phi_i(x)$ and the fermionic partners $\psi_i(x)$, which are usually Majorana spinors. For simplicity we assume that all auxiliary fields have been eliminated. The Lagrangian of the model can be written as the sum of a fermionic and a purely bosonic parts:

$$\mathcal{L}(\phi,\psi;g) = \mathcal{L}_B(\phi;g) + \mathcal{L}_F(\phi,\psi;g) \tag{3.1}$$

where we can write in general:

$$\mathcal{L}_F = \frac{1}{2}\bar{\psi}\, D(\phi;g)\,\psi \tag{3.2}$$

(g is the coupling constant). Since the expression of $\mathcal{L}_F$ is quadratic in the fermions, one can integrate them out in the functional integral [92], obtaining as a result the Matthews-Salam determinant [93]:

$$\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\; e^{-\int dx\,\mathcal{L}_F} = \left[\det D(\phi;g)\right]^{1/2} . \tag{3.3}$$

In this way one can transform the functional integral measure

$$d\mu = \mathcal{D}\phi\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\; e^{-\int dx\,\mathcal{L}}$$

into a purely bosonic one:

$$d\mu = \mathcal{D}\phi\left[\det D(\phi;g)\right]^{1/2} e^{-\int dx\,\mathcal{L}_B} . \tag{3.4}$$

If supersymmetry is an exact symmetry, there exists then a transformation T_g of the bosonic variables:

$$T_g \; : \; \phi_i(x) \longmapsto \xi_i = \xi_i(x;\phi;g) \qquad (3.5)$$

with the following properties:

1) T_g is invertible at least in the sense of formal power series;

2) $\mathcal{L}_B(\phi;g) = \frac{1}{2}\,\xi_i^2$ + total deriv. (3.6)

(this total derivative can be eliminated by choosing suitable boundary conditions);

3) the Jacobian of the transformation (3.5) equals the Matthews-Salam determinant, i.e.

$$\left| \det\left(\frac{\delta \xi_i}{\delta \phi_j}\right)\right| = \left[\det D(\phi;g)\right]^{1/2} \qquad (3.7)$$

From eq.(3.6) it follows that the measure (3.4) becomes, in terms of the Nicolai variables ξ_i, Gaussian with zero mean and covariance one (*).

(*) It is interesting to note that the fundamental property of the supersymmetric functional integral measures to have a Nicolai mapping has a superspace analogue: the Jacobian for the change of variables in an integral over unconstrained superfields is always equal to one [94]; this means that the measure is unique and universal. This property, as shown in ref. (95), has also important consequences for standard non-supersymmetric field theories.

For gauge theories, the presence of a gauge fixing term in the Lagrangian which violates supersymmetry, slighlty modifies the above property 3); in that case [21] the Jacobian of the corresponding Nicolai mapping is equal to the Matthews-Salam determinant times the Faddeev-Popov determinant. This property of admitting a transformation T_g is clearly a fundamental one and it can be taken as characterizing field theories with unbroken supersymmetry.

The simplest example of Nicolai mapping [96,23,98] is obtained in the study of Supersymmetric Quantum Mechanics [97-105,80,85]. The Euclidean Lagrangian is

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 + \frac{1}{2}\left[V'(q)\right]^2 + \bar{\psi}\left(d_t + V''(q)\right)\psi \qquad (3.8)$$

where $q(t)$ is the bosonic variable and $\psi(t), \bar{\psi}(t)$ are the corresponding anticommuting ones; $V(q)$ is the superpotential (a prime means functional derivative with respect to q). The Nicolai mapping is simply:

$$q(t) \longmapsto \xi(t) = \dot{q}(t) + V'(q) \qquad (3.9)$$

In fact when expressed in terms of the Nicolai variable $\xi(t)$, the bosonic part of the Lagrangian becomes:

$$\mathcal{L}_B = \frac{1}{2}\,\xi(t)^2 - d_t V(q)$$

and the Jacobian of the transformation coincides with the Matthews-Salam determinant: $\det\left\{d_t + V''(q)\right\}$.

The proof of the existence of the Nicolai mapping in unbroken supersymmetric models can be based on a recently developed method [106-108,91] which directly constructs the map, step by step in a perturbative way. The central point of the construction is the introduction of an operator R which generates infinitesimal shifts in the coupling constant g of a generic supersymmetric theory such that the Jacobian of the finite shift is equal to the Matthews-Salam determinant originating from the integral of fermionic variables. The Nicolai mapping is then obtained by integrating the coupling constant flow generated by R using the free theory as initial value; it turns out that:

$$T_g^{-1} \sim e^{gR} \quad . \tag{3.10}$$

Note that this procedure has also the advantage of possessing a graphical expansion in terms of suitable tree diagrams (see ref. (109) for an example of application).

From the above method of explicit construction of the Nicolai map, it is also clear that the relation between the Nicolai variables and the old bosonic ones is in general non-local and non-polynomial. Neverthelesss, there are some models for which the mapping becomes local and polynomial; relevant examples are the Supersymmetric Quantum Mechanics (look at eq. (3.9)) and the N=2 supersymmetric Wess-Zumino model in two dimensions (which will be discussed in detail in the next section)[(*)]. In these cases the map can be interpreted as a stochastic differential equation which closely resembles a Langevin equation [(23)]. Thus once a supersymmetric theory possesses a local Nicolai map, we can automatically describe it in terms of stochastic processes; we will return later on this point.

(*) Other lower-dimensional models which admit a local Nicolai mapping are discussed in ref.'s (110,111,127). Furthermore note that at least for non-gauge theories, it is possible to give some general conditions for the existence of a local mapping; see ref.(112) for details.

Another approach to the whole matter is however possible [(113-116)]. It originates from the observation that in some supersymmetric theories certain local combinations of bosonic variables (stochastic variables) satisfy free-field vacuum expectation values, the so-called 'stochastic identities'; relevant examples are certain classes of four-dimensional supersymmetric Yang-Mills theories in the light-cone gauge. The connection with the previous Nicolai approach can be easily found. Since the expectation values of the stochastic variables can be calculated by using the functional measure of the free theory (i.e. by simply calculating white noise avereges), these variables are obviously independent of the coupling constant g, and then they annihilate the infinitesimal generator R of the coupling constant shift. Remembering eq.(3.10), it is thus evident that stochastic variables are local fixed points of the corresponding Nicolai mapping.

The stochastic identities are then elementary consequences of the Nicolai mapping and can be therefore derived from it whenever a closed expression for such a map exists. However it is possible to derive stochastic identities directly from the theory by using the algebraic structure of supersymmetry, independently of the expression of the Nicolai mapping. In particular stochastic identities can be derived even in cases in which the Nicolai map is non-local and not explicitly known (see ref. (114)).

It follows that the detailed study of these stochastic identities clearly represents a fundamental step towards a full understanding of the stochastic properties of supersymmetric models. This observation is even more important if one notices that a complete set of stochastic identities allows a new scheme of perturbative analysis of supersymmetric field theories, along the lines of the classic stochastic perturbation theory [(23)]. The main idea is to compute the Green's functions containing only bosonic fields in terms of the correlation functions involving the stochastic variables, by simply inverting the relation between stochastic and bosonic variables (*); in this way one can fully describe the bosonic sectors of supersymmetric theories by means of the stochastic identities.

In the following sections we will see how powerful this new perturbative technique is in extracting physical information from supersymmetric field theories by applying it to some interesting models.

III.2 A two-dimensional example

To render more transparent all above considerations, we consider now in some detail the study of a specific supersymmetric field theory: the two-dimensional N=2 Wess-Zumino model [23]. It is described in terms of two scalar fields φ_a ($a = 1,2$) and a two-component Dirac spinor ψ together with its conjugate $\bar\psi$. The corresponding Euclidean Lagrangian can be written in the following form (**):

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_a)^2 + F_a (\gamma_1)_{ab} V'_b - \frac{1}{2} F_a^2 + \bar\psi \partial\!\!\!/ \psi + \bar\psi (\gamma_1 \cdot V'') \psi \tag{3.14}$$

(*) Note that, with the conventions used in the definitions of the Nicolai map, this relation practically coincides with the Nicolai transformation.

(**) For a precise definition of Dirac and Majorana spinors in Euclidean space see ref.'s (117,118). We use the following representation for the γ-matrices: $\gamma_1 = \sigma_1$, $\gamma_2 = \sigma_3$, $\gamma_3 = -i\gamma_1\gamma_2 = -\sigma_2$; σ_i ($i=1,2,3$) are the Pauli matrices. Moreover, by convention: $x^\mu \equiv (t,x)$, $\mu = 1,2$.

where F_a are auxiliary fields and V is the superpotential which, to be specific, we take to be:

$$V = m\varphi_1\varphi_2 + \frac{g}{2}\varphi_2\left(\varphi_1^2 - \frac{1}{3}\varphi_2^2\right) \tag{3.15}$$

(m is the mass and g is the coupling constant); finally in eq.(3.14) we have used the following compact notations:

$$V'_a = \frac{\delta V}{\delta \varphi_a}\,, \quad V''_{ab} = \frac{\delta^2 V}{\delta\varphi_a \delta\varphi_b}\,, \quad (\gamma_1 \cdot V'')_{ab} = (\gamma_1)_{ac} V''_{cb}$$

After the elimination of the auxiliary fields the Lagrangian (3.14) takes on the standard form:

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F \tag{3.16}$$

$$\mathcal{L}_B = \frac{1}{2}(\partial_\mu\varphi_a)^2 + \frac{1}{2}m^2(\varphi_a)^2 + \frac{gm}{2}\varphi_1(\varphi_a)^2 + \frac{g^2}{8}(\varphi_a\varphi_a)^2 \tag{3.17a}$$

$$\mathcal{L}_F = \bar\psi\left\{\partial\!\!\!/ + m + g(\varphi_1 + i\gamma_3\varphi_2)\right\}\psi \tag{3.17b}$$

As one can easily check the action $A = \int d^2x\, \mathcal{L}$ is invariant under the following set of supersymmetric transformations (ε, $\bar\varepsilon$ are independent anticommuting parameters):

$$\begin{aligned}
\delta\varphi_1 &= \bar\varepsilon\psi + \bar\psi\varepsilon \\
\delta\varphi_2 &= -i(\bar\varepsilon\gamma_3\psi + \bar\psi\gamma_3\varepsilon) \\
\delta\psi &= \partial\!\!\!/(\varphi_1 - i\gamma_3\varphi_2)\varepsilon - (F_1 - i\gamma_3 F_2)\varepsilon \\
\delta\bar\psi &= -\bar\varepsilon\partial\!\!\!/(\varphi_1 + i\gamma_3\varphi_2) - \bar\varepsilon(F_1 - i\gamma_3 F_2) \\
\delta F_1 &= -(\bar\varepsilon\partial\!\!\!/\psi + \bar\psi\overleftarrow{\partial\!\!\!/}\varepsilon) \\
\delta F_2 &= i(\bar\varepsilon\gamma_3\partial\!\!\!/\psi + \bar\psi\overleftarrow{\partial\!\!\!/}\gamma_3\varepsilon)
\end{aligned} \tag{3.18}$$

The model admits a local Nicolai map which can be explicitly written in the following way [23]:

$$\xi_a(x) = \partial_t \varphi_a(x) + \frac{\delta W[\varphi]}{\delta \varphi_a(x)} \tag{3.19}$$

where

$$W[\varphi] = \int dx \left\{ \frac{1}{2} \varphi_2 \overleftrightarrow{\partial}_x \varphi_1 + V \right\} . \tag{3.20}$$

The bosonic part of the Lagrangian (eq.(3.17a)) becomes in fact quadratic in the new variables ξ_a (we neglect as usual surface terms); moreover the Jacobian of the transformation (3.19), which explicitly is

$$\det \left\{ \gamma_1 \cdot \left[\not{\partial} + m + g\left(\varphi_1 + i\gamma_3 \varphi_2\right)\right]\right\} ,$$

clearly coincides with the Matthews-Salam determinant obtained from the integration of the fermionic variables ψ and $\bar{\psi}$ in the functional integral. As a consequence the variables $\xi_a(x)$ are Gaussian and obey the following stochastic identities (*)

$$\langle \xi_a(x) \rangle_0 = 0$$

$$\langle \xi_a(x)\, \xi_b(x') \rangle_0 = \delta_{ab}\, \delta^{(2)}(x-x') \tag{3.21}$$

(*) The name 'stochastic' given to these relations is now clear: it follows from the observation that these identities can be viewed as the ensamble averages typical of a Langevin system in which $\xi_a(x)$ is the white noise.

and in general:

$$\langle \xi_{a_1}(x_1) \cdots \xi_{a_n}(x_n) \rangle_0 =$$

$$= \sum_{1}^{n}{}_{r,s}\, \delta_{a_r a_s}\, \delta^{(2)}(x_r - x_s) \langle \prod_{1\, p\neq r,s}^{n} \xi_{a_p}(x_p) \rangle_0 . \tag{3.22}$$

In the spirit of ref.'s (113,114) however, these identities can be derived directly from the supersymmetry transformation (3.18), without any reference to the Nicolai mapping. Even better only a subgroup of the full supersymmtry transformation group is really necessary. In fact, for the choice $\varepsilon \equiv 0$, $\bar{\varepsilon} = (\bar{\xi}, 0)$, the transformations (3.18) become:

$$\begin{cases} \delta \eta_a = \bar{\xi}\, \omega_a \\ \delta \bar{\psi}_a = -\bar{\xi}\, \eta_a \\ \delta \omega_a = 0 \end{cases} \tag{3.23}$$

where we have introduced the short notations

$$\eta_a = (\gamma_1)_{ab}\, \xi_a \qquad \omega_a = (\not{\partial} + V'')_{ab} \psi_b . \tag{3.24}$$

The lagrangian (3.16), (3.17), which can be now written in compact form as

$$\mathcal{L} = \frac{1}{2}(\eta_a)^2 + \bar{\psi}_a \omega_a ,$$

is obviously invariant under the transformations (3.23). We now use the property of unbroken supersymmetric models that the vacuum expectation value of the supersymmetry variation of any functional G of the fields is zero:

$$\langle \delta G \rangle_0 = 0 .$$

If we take for G the expression $\overline{\psi}_a \eta_b$, from the above condition, using eq.(3.23), we obtain the following Ward identity:

$$\langle \eta_a(x)\,\eta_b(x')\rangle_0 = -\langle \overline{\psi}_a(x)\,\omega_b(x')\rangle_0 . \qquad (3.25)$$

But $\langle \overline{\psi}_a\,\omega_b\rangle_0 = -\delta_{ab}\,\delta^{(2)}(x-x')$, as simply follows from

$$\frac{\delta}{\delta\overline{\psi}_b(x')}\langle \overline{\psi}_a(x)\rangle_0 = 0 .$$

As a consequence, from eq.(3.25) we reobtain the stochastic identities of eq.(3.21) (*).

It is interesting to observe that really the model admits an entire family of Nicolai maps. Eq.(3.19) can be in fact generalized in the following way (119):

$$\xi_a(x) = \partial_t\varphi_a(x) + \frac{\delta\tilde{W}[\varphi]}{\delta\varphi_a(x)} \qquad (3.26)$$

where now:

$$\tilde{W}[\varphi] = \int dx\,\left\{\frac{\varepsilon}{2}\varphi_2\overleftrightarrow{\partial}_x\varphi_1 + V\cos\alpha + \tilde{V}\sin\alpha\right\} \qquad (3.27)$$

with:

$$\tilde{V} = \frac{m}{2}\left(\varphi_2^2-\varphi_1^2\right) + \frac{g}{2}\varphi_1\left(\varphi_2^2-\frac{1}{3}\varphi_1^2\right)$$

(*) The other identities can be similarly derived by considering for G the expression: $\overline{\psi}_{a_1}(x_1)\,\eta_{a_2}(x_2)\cdots\eta_{a_n}(x_n)$.

and V as in eq. (3.15): moreover $\varepsilon = \pm 1$ and $\alpha \in [0, 2\pi]$. The arbitrariness connected with the parameter α is simply due to the invariance of the fermionic functional measure under the following chiral transformations:

$$\psi \longrightarrow e^{i\alpha P_-}\,\psi \qquad\qquad P_\pm = \frac{1\pm i\gamma_3}{2}$$
$$\overline{\psi} \longrightarrow \overline{\psi}\,e^{-i\alpha P_+} .$$

On the other hand the ' ε -freedom' is probably connected with the presence in the theory of two supersymmetric vacua (at least at the classical level).

Before discussing in detail the limits of the stochastic properties of the model and in particular if eq.(3.15) can be really interpreted as a true Langevin equation, let us spend a few words on the problems connected with the rigorous definition of the Nicolai mapping. Even if these questions are not of immediate relevance for a perturbative study of the Nicolai mapping, they are, on the other hand, crucial for understanding the various aspects of the stochastic interpretation of the supersymmetric theories.

III.3 A word on rigour

It is well known that in order to give meaning to the functional integral involving the Euclidean Lagrangian (3.14)(*), one has to suitably regularize the theory (10). We first suppose then to put our model in an Euclidean box, choosing periodic boundary conditions for both bosonic and fermionic fields (97,120,121,23); note that this choice is crucial if we want to preserve supersymmetry invariance. After that the model has a finite number of degrees of freedom.

(*) To be specific we take as a reference model our two-dimensional theory; all the discussion however has a more general validity. Furthermore, for a rigorous approach to one-dimensional models, see ref.(98).

Furthermore in order to cure ultraviolet divergences we introduce a suitable cut-off Λ by decomposing the fields into Fourier-series components, and then by summing over all the 'frequences' k_μ for which $|k| < \Lambda$ (23). This regularization has the property of respecting all symmetries of the periodic box; moreover it is supersymmetric - invariant if the Fourier decomposition is made on all fields, i.e. before the elimination of the auxiliary fields from the Lagrangian (a different regularization is discussed in ref.(122)).

At this point all objects entering the theory have a mathematical meaning; in particular the bosonic fields are even C^∞ operator valued functions (*). Now a regularized Nicolai mapping clearly exists for the model and can be explicitly constructed; it assumes the form of a map between two well defined functional measures: that of the full interacting theory expressed in terms of the variables φ_a^Λ and the Gaussian one for the variables ξ_a^Λ (the fields depend now on the cut-off Λ).

The problem now is to see what happens if one removes the cut-off Λ (i.e. let $\Lambda \to \infty$) and takes the thermodynamical limit. The question is not yet completely solved, even if, as claimed in ref.(23), one probably cannot expect to obtain a Nicolai map in the infinite volume limit. There are in fact many open problems.

First of all, due to our choice of boundary conditions (periodic both for bosons and fermions), the functional integral has no longer an obvious interpretation as a partition function of a classical statistical system (124).

(*) The fermionic variables are usually integrated out; one then gives meaning to the resulting Matthews-Salam determinant by a suitable renormalization procedure (see ref.(123)).

Moreover for the same reason the model presumably does not satisfy the Osterwalder-Schrader positivity condition (125); thus the analytic continuation of the Euclidean theory into the physical Minkowski sector is not automatically guaranteed. Furthermore the theory possesses two distinct classical supersymmetric ground states (the bosonic potential is double well-like). There is thus the suspect, in analogy with other cases (126), that the quantum theory really has two phases; the corresponding Nicolai map should then show a sector (or phase) structure.

It is clear that all these questions require a deeper study. They are not simply academic problems since they directly involve the relation between supersymmetric models and corresponding statistical systems: their solution would further clarify the stochastic properties of supersymmetric field theories.

III.4 Nicolai vs. Langevin

Let us consider once more the explicit form of the Nicolai mapping for the two-dimensional Wess-Zumino model; it can be written in the following way (eq.(3.19))

$$\dot{\varphi}_a = - \frac{\delta W[\varphi]}{\delta \varphi_a} + \xi_a \quad . \qquad (3.28)$$

As already said, this equation has the general structure of a Langevin equation (compare with eq.(2.21)). However we will show in this section that $\varphi_a(t,x)$ cannot be considered as a full stochastic process in the time t, because eq.(3.28) does not really have the correct causal bahaviour for a transport phenomena (*).

(*) This fact was first pointed out in ref. (23)

To grasp the reason of this phenomena it is sufficient to discuss the free case ($g = 0$). Moreover, since the boundary conditions on eq.(3.28) become now crucial, to be as close as possible to physical interpretation, we will consider the Minkowski-like version of the Nicolai mapping. In this case eq.(3.28) is still valid, but now:

$$W_0[\varphi] = \int dx \left\{ \frac{i}{2} \varphi_2 \overleftrightarrow{\partial_x} \varphi_1 + i m \varphi_1 \varphi_2 \right\} . \tag{3.29}$$

By rearranging a little eq.(3.28) and (3.29) one can write the Nicolai map in the following more compact matrix form (we choose $\gamma_0 = \sigma_1$, $\gamma_1 = i\sigma_3$ for the Minkowski γ-matrices):

$$\eta = (\not{\partial} + i m)\varphi$$

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \qquad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} ; \tag{3.30}$$

η_a is still a white noise.

The solution of eq.(3.30) can now be given in terms of its corresponding Green's function $G_{ab}(x-x')$

$$\varphi_a(x) = \int d^2x' \, G_{ab}(x-x') \, \eta_b(x') \tag{3.31}$$

with

$$(\not{\partial}_x + i m)_{ab} \, G_{bc}(x-x') = \delta_{ac} \, \delta^{(2)}(x-x'). \tag{3.32}$$

As usual eq.(3.32) can be solved through a Fourier transform; setting

$$G_{ab}(x-x') = \int \frac{d^2p}{(2\pi)^2} \, e^{ip\cdot(x-x')} \, \tilde{G}_{ab}(p)$$

we get

$$(\not{p} + m)_{ab} \, \tilde{G}_{bc}(p) = -i \, \delta_{ac} \tag{3.33}$$

The problem now is how to avoid the pole at $p^2 = m^2$; this is clearly connected with the appropriate choice of boundary conditions in eq.(3.31). These are dictated by the physical interpretation of eq.(3.31) as an (inverted) Nicolai mapping. In fact using it one can compute the two-point Green's function $\langle 0 | T\{\varphi_a(x)\varphi_b(x')\} | 0 \rangle$ in terms of the η_a-correlations; the request that this two-point function coincides with the standard propagator, namely with

$$\langle 0 | T\{\varphi_a(x)\varphi_b(x')\} | 0 \rangle = \delta_{ab} \, D(x-x')$$

$$D(x-x') = \int \frac{d^2p}{(2\pi)^2} \, \frac{e^{ip\cdot(x-x')}}{p^2 - m^2 + i\varepsilon} \tag{3.34}$$

directly gives

$$G_{ab}(x-x') = (-\not{\partial}_x + i m)_{ab} \, D(x-x') \tag{3.35}$$

To obtain the physical interpretation of eq.(3.35), we explicitly perform the p_0 integration in eq.(3.34):

$$D(x-x') = -i \int \frac{dp}{(4\pi E_p)} \, e^{-ip(x-x')} \left\{ e^{-iE_p(t-t')} \theta(t-t') + \right.$$

$$\left. + \, e^{iE_p(t-t')} \theta(t'-t) \right\} \tag{3.36}$$

with $E_p = (p^2 + m^2)^{\frac{1}{2}}$. Moreover we imagine our system enclosed in a box of size L; then the momentum integral in eq.(3.36) is replaced by a sum, $\int \frac{dp}{2\pi} \longrightarrow \frac{1}{L} \Sigma_n$ since $p \longrightarrow \frac{2\pi}{L} n \; (n \in \mathbb{Z})$. Introducing the following periodic solutions of the homogeneous Klein-Gordon equation (i.e. the one-particle wave functions):

$$\phi_n^{(\pm)}(x) = \frac{1}{L^{1/2}\sqrt{2E_n}} \, e^{-i\left(\pm E_n t + \frac{2\pi n}{L} x\right)}$$

(the plus or minus sign refers to the sign of the energy), eq.(3.36) can be finally written as

$$D(x-x') \sim -i \sum_n \left\{ \Theta(t-t')\, \phi_n^{(+)}(x) \left[\phi_n^{(+)}(x')\right]^* + \right.$$

$$\left. + \Theta(t'-t)\, \phi_n^{(-)}(x) \left[\phi_n^{(-)}(x')\right]^* \right\} \quad (3.37)$$

From this result and from eq.(3.35), we see that $G_{ab}(x-x')$ propagates positive (negative) frequences, forward (backward) in time. On the other hand a full interpretation of eq.(3.28) as a Langevin equation would require a propagation forward (or backward) in time for both positive and negative frequency solutions. It follows that the Nicolai mapping cannot be interpreted as describing a full stochastic process (*): in agreement with the general causal structure of Quantum Field Theory, it is a forward process for particles with positive energy and a backward process for negative energy particles, i.e. for antiparticles.

This conclusion can also be obtained by comparing our free Nicolai mapping eq.(3.28), (3.29) with the corresponding Langevin equation which correctly describes the field $\varphi_a(t,x)$ as a Gaussian Markov process. In fact generalizing the methods of stochastic mechanics [61], it is possible to give a full stochastic description of the free scalar field $\varphi_a(t,x)$

(*) Note however that a full stochastic interpretation of the Nicolai mapping can be recovered either in the limit $|t| \to \infty$ (see the discussion of section II.3), or by restricting the study of full supersymmetric theories to particular fermion sectors. In this second case one obtains effective purely bosonic theories which remember their supersymmetric origin by the presence of the corresponding Nicolai maps. Many examples of this phenomenon have been discussed [119,127]; of particular interest is the study of a pure Yang-Mills theory in the temporal gauge [128,129].

based on the following stochastic differential equation [27]:

$$\dot{\varphi}_a = i\sqrt{\partial_x^2 + m^2}\, (\gamma_0 \varphi)_a + \xi_a \quad .$$

This equation does not coincide with our free Nicolai mapping.

This conclusion has been obtained in the case of a simple two-dimensional model, but clearly it remains true for any supersymmetric field theory which admits a local Nicolai mapping. Note however that the existence for such theories of mappings with an almost, but not complete, stochastic structure (*) , is still relevant and interesting and helps a lot in understanding their structure. In particular, as we will see in the next section, it gives the possibility of a new perturbative approach to these theories.

III.5 The new perturbative expansion

We have already claimed that one of the most important consequences of having a local Nicolai mapping, or more in general, a set of stochastic identities in a supersymmetric field theory, is the possibility of setting up a new scheme of perturbation analysis of the theory.

When expressed in terms of stochastic variables, a supersymmetric theory looks quite simple, since the correlation functions involving these variables are the Gaussian ones (remember eq.(3.22)). However the physical information on the theory is contained in the old bosonic fields; we are thus interested in computing Green's functions involving these variables. The idea of the new perturbation scheme is then to invert the relation connecting the stochastic variables with the bosonic ones and to use this inverted Nicolai mapping to compute the bosonic Green's

(*) It is called 'parastochastic' in ref. (23).

functions involving these variables. The idea of the new perturbation scheme is then to invert the relation connecting the stochastic variables with the bosonic ones and to use this inverted Nicolai mapping to compute the bosonic Green's functions of the theory in terms of the simple correlation functions of the stochastic variables (*), in strict analogy with what is done in stochastic quantization. That the fermion contributions to these purely bosonic Green's functions are correctly taken into account by this method is a direct consequence of the characteristic properties of the Nicolai mapping.

The cancellations of divergences between bosonic and fermionic loops are then automatic in this formalism and thus the new (stochastic) graphs (**) that this perturbation scheme produces have a better ultraviolet behaviour than the corresponding Feynman diagrams. This fact suggests that this approach can be very useful in understanding how this miracolous cancellation of divergences can produce, in some special cases, completely finite supersymmetric field theories.

For the two-dimensional Wess-Zumino model all the properties of this new perturbation technique become transparent, since the theory is finite.[130] Even if we are really interested in four-dimensional theories, it is interesting to spend a few words to study in more detail the perturbative expansion for this lower dimensional model.

First of all note that the Nicolai map for the model, eq.(3.19), can be rewritten by a suitable redefinition of the stochastic variables (eq.(3.24)) in the following explicit form:

(*) Note that the rest of the physical content of the theory, namely the Green's functions containing explicitly the fermionic fields, can be reconstructed by using for example the Ward identities of supersymmetry.

(**) These diagrams were first considered in ref. (23) where they are called 'infradiagrams'.

$$\eta_a = \{ (\not\partial + m)\varphi \}_a + \frac{g}{2} t_{abc}\, \varphi_b \varphi_c \tag{3.38}$$

where

$$t_{abc} = \frac{1}{2}\left(\delta_{1b}\, \delta_{ac} + \delta_{1c}\, \delta_{ab} - \varepsilon_{ab}\, \delta_{2c} - \varepsilon_{ac}\, \delta_{2b} \right) . \tag{3.39}$$

This map can be easily inverted; going to momentum space one finds:

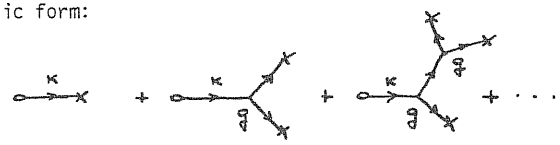
$$\varphi_a(k) = \frac{(m - i\not k)_{ab}}{k^2 + m^2}\, \eta_b(k) - \frac{g}{2} \frac{(m - i\not k)_{ab}}{k^2 + m^2}\, t_{bcd} \int d^2p\, \varphi_c(p)\, \varphi_d(k-p) \tag{3.40}$$

which implicitly defines $\varphi_a(k)$ in terms of the stochastic variables $\eta_a(k)$, obeying now the relations:

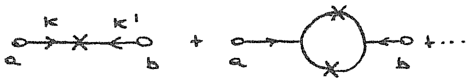
$$\begin{aligned} \langle \eta_a(k) \rangle_0 &= 0 \\ \langle \eta_a(k)\, \eta_b(k') \rangle_0 &= \delta_{ab}\, \delta^{(2)}(k+k') \end{aligned} \tag{3.41}$$

Clearly eq.(3.40) allows to calculate perturbatively Green's functions involving the field φ_a in terms of the correlations (3.41) involving the stochastic variables η_a. For example one can check that at lowest order (i.e. for g = 0) the Green's functions calculated using eq.(3.40) coincide with the tree-level expressions one deduces from the Lagrangian (3.16), (3.17).

To compute higher order contributions it is useful to introduce a diagrammatic expansion. Eq.(3.40) gives φ_a in terms of η_a only in an implicit form. Thus the solution of this equation can be only found by iterations and diagrammatically this solution can be expressed in the following symbolic form:

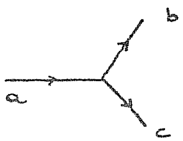
$\varphi_a(k) =$  (3.42)

where the crosses represent the 'stochastic sources' η_a (note the analogy with what was done in sect. II.3). It is not difficult to see that, as in usual stochastic quantization approach, the Green's functions for the field φ_a are now obtained as a diagrammatic expansion in which the crosses of the above tree stochastic graphs are connected in all possible ways, using the rules (3.41), to form closed loops. Then, for instance, one gets:

$\langle \varphi_a(k)\, \varphi_b(k') \rangle_0 =$ 

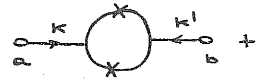
In practice to compute Green's functions one needs only the Feynman rules to build up these 'stochastic graphs' and not the explicit form of the solution of eq.(3.40). Looking at this equation one easily finds the following stochastic Feynman rules:

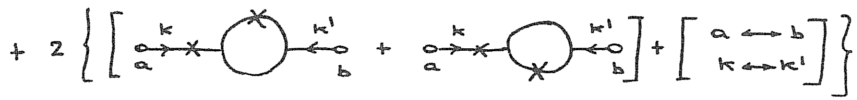
 $\qquad \dfrac{\delta_{ab}}{k^2 + m^2}$

 $\qquad - \dfrac{g}{2} \dfrac{(m - i\not{k})_{ad}}{m^2 + k^2}\, t_{dbc}$ (3.43)

and no explicit propagator is associated to uncrossed lines (a momentum integration for each loop is understood).

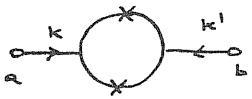
As an example, let us now compute the one-loop corrections to the connected two-point Green's function; taking into account the symmetry factors, one can write:

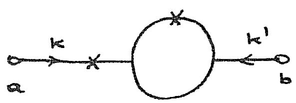
$\langle \varphi_a(k)\, \varphi_b(k') \rangle_0 \Big|_{\text{order } g^2} = 2 \cdot$  (3.44)



the symmetry factor 2 takes into account the two possibilities of contracting the various crosses in the graphs. Moreover it is easy to realize that:

Thus to calculate $\langle \varphi_a\, \varphi_b \rangle_0 \big|_{g^2}$ one has to compute explicitly only two stochastic graphs. Using the previous Feyman rules one immediately obtains:

 $= \dfrac{\delta_{ab}\, \delta^{(2)}(k+k')}{k^2 + m^2} \dfrac{g^2}{2} I$ (3.45)



$$= -\frac{g^2}{4}\,\frac{\delta^{(2)}(k+k')}{(k^2+m^2)^2}\left\{ k^2\delta_{ab} - im(\not{k})_{ab}\right\} I \qquad (3.46)$$

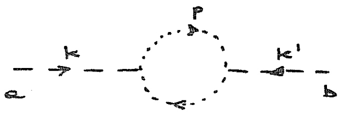
where

$$I = \int d^2p \; \frac{1}{(p^2+m^2)\left[(k-p)^2+m^2\right]} \, .$$

Using eq.(3.44) one finally gets:

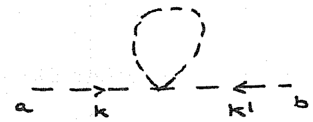
$$\langle \varphi_a(k)\, \varphi_b(k')\rangle_0\Big|_{g^2} = -\, g^2\, \frac{\delta_{ab}\,\delta^{(2)}(k+k')}{(k^2+m^2)^2}\,(k^2-m^2)\, I \, .$$

This is also the result one obtains with usual perturbation techniques, by summing up the following Feynman graphs (fermion and boson propagators are respectively indicated with: ·····➤···· and ——➤——):

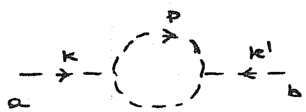


$$= -\,2g^2\,\frac{\delta^{(2)}(k+k')}{(k^2+m^2)^2} \; \cdot$$

$$\cdot \left\{ m^2(\gamma_2)_{ab}\, I - \delta_{ab}\int d^2p\; \frac{p\cdot(k-p)}{(p^2+m^2)\left[(k-p)^2+m^2\right]} \right\}$$



$$= -\,2g^2\,\frac{\delta^{(2)}(k+k')}{(k^2+m^2)^2}\,\delta_{ab}\int d^2p\;\frac{1}{p^2+m^2}$$



$$= m^2 g^2\,\frac{\delta^{(2)}(k+k')}{(k^2+m^2)^2}\left\{2(\gamma_2)_{ab} + 3\delta_{ab}\right\} I$$

Note that while some of the above Feynman diagrams are logarithmically divergent and only their sum is finite, the correpsonding stochastic diagrams (3.45), (3.46) are both finite.

III.6 The classical interpretation of the Nicolai variables

Before discussing in detail the case of more realistic four-dimensional models, it is interesting to describe the classical meaning of the local Nicolai or stochastic variables. Since in practice there are no fermions at the classical level, we can take into account in our present discussion all those models for which the bosonic part of the action can be put in a quadratic form. And in fact, besides the supersymmetric theories which really admits a local Nicolai map, namely the Supersymmetric Quantum Mechanics, the N=2 Wess-Zumino model already considered and the four-dimensional supersymmetric Yang-Mills theories discussed in the following chapter, also for the pure Yang-Mills theory in eight dimensions (131) and for the four-dimensional conformal gravity (132) one is able to find local combinations of fields that quadratize the actions(*).

A part from the gravitational case which will be briefly discussed at the end, it is interesting to note that the above theories are respectively connected with a particular class of abstract algebras, the so-called alternative algebras (133): the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternion numbers $\mathbb{Q}$, the octonion numbers $\mathbb{O}$. These algebras possess respectively one, two, four and eight unities; by means of them one can write in a compact form the transformations between the Nicolai variables (**) and the original bosonic ones for all the four theories. Let us collect these unities as

$$S_\mu = (-e_a^{(D)}, 1) \qquad \bar{S}_\mu = (e_a^{(D)}, 1)$$

$$a = 1, 2, \ldots, D-1$$

$$\mu = 1, 2, \ldots, D$$

(D is the dimension of the space which coincides with the dimension of the corresponding algebra: D = 1,2,4,8) (***). Furthermore one has:

(*) If these combinations of fields are really stochastic variables for the supersymmetric extension of these two theories is still an open problem (see also Appendix B).

(**) By extension, we call here Nicolai variables all local combinations of bosonic fields which put in quadratic form the classical actions.

(***) Obviously there are no $e_a^{(1)}$ in the case of real numbers; for the complex numbers $e^{(2)} \equiv i$, the imaginary unity; finally $e_a^{(4)}$ and $e_a^{(8)}$ coincide respectively with the quaternions ($-i\sigma_a$, σ_a are the Pauli matrices) and the octonions.

$$e_a^{(D)} e_b^{(D)} = -\delta_{ab} + f_{abc}^{(D)} e_c^{(D)} \tag{3.47}$$

where $f_{abc}^{(D)}$ are the structure constants of the various algebras ($f_{abc}^{(1)} = f_{abc}^{(2)} \equiv 0$ since $\mathbb{R}$ and $\mathbb{C}$ are commuting algebras, $f_{abc}^{(4)} = \varepsilon_{abc}$, $f_{abc}^{(8)} = C_{abc}$).

If one indicates with A_μ the standard bosonic field variables of the four theories and with N_μ the corresponding Nicolai variables (in the case of gauge models, A_μ and N_μ belong to the Lie algebra of the gauge group), then the relations between the two can be written in the following compact form

$$(S \cdot N) = (\bar{S} \cdot \partial)(S \cdot A) + g\, \mathcal{V}(S \cdot A, \bar{S} \cdot A). \tag{3.48}$$

$\mathcal{V}$ takes into account the self-interaction between the A_μ (g is the coupling constant), and it is equal to the derivative of the superpotential

$$\mathcal{V} = \frac{\delta V(\bar{S} \cdot A)}{\delta(\bar{S} \cdot A)}$$

for the theories in one and two dimensions, and to

$$\mathcal{V} = -\frac{i}{2} \bar{S}_\mu S_\nu [A_\mu, A_\nu] \tag{3.49}$$

for the gauge theories (in 4 and 8 dimensions). Really in the case of Yang-Mills theories eq.(3.48) holds only in the covariant gauge $\partial_\mu A_\mu = 0$; however it can be easily generalized to a generic gauge F(A) = 0 by substituting in it

$$(\bar{s}\cdot\partial)(s\cdot A) \qquad \text{with} \qquad 2\,\sigma_{\mu\nu}\,\partial_\mu A_\nu + \frac{1}{D}(\bar{s}\cdot s)(F(A))$$

where $\sigma_{\mu\nu} = \frac{1}{4}\left(\bar{s}_\mu s_\nu - \bar{s}_\nu s_\mu\right)$.

Using eq. (3.47), it is easy to see that in all four cases the classical action can be written as

$$A_B = \int d^Dx \left\{ \frac{1}{2} N_\mu N_\mu + \partial_\mu J_\mu \right\}$$

with J_μ a certain local polynomial in the fields. A surface term in the action does not contribute to the equations of motion, which then reduce to

$$\int d^Dx'\; N_\mu(x')\, \frac{\delta N_\mu(x')}{\delta A_\nu(x)} = 0 \quad .$$

From this equation it is apparent that $N_\mu = 0$ corresponds to particular classical solutions of the field equations; these solutions have finite action and energy and moreover possess non-trivial topological properties.

For the four-dimensional gauge theory (D=4) this situation is well known since the equation $N_\mu = 0$ gives the instanton solution [134,135]. In fact in this case the Nicolai variables are simply:

$$N_i^{(\pm)} = F_{0i} \pm \frac{1}{2}\varepsilon_{ijk} F_{jk} \qquad (3.50)$$

$$i,j,k = 1,2,3$$

where $F_{\mu\nu}$ is the field strength (the fourth Nicolai variable gives the gauge fixing condition); then imposing $N_i^{(\pm)} = 0$ amounts to select selfdual

or antiselfdual solutions of the classical equations of motion. The same situation holds in the case of Supersymmetric Quantum Mechanics (D=1)[96] and for the eight-dimensional Yang-Mills theory (D=8) [131], where the instantons corresponding to the condition $N_\mu = 0$ have been found.

In the fourth case (D=2), since the theory is a purely scalar one, one can hope to obtain through the condition $N_\mu = 0$ a soliton-like solution (*). By looking at the Lagrangian of the model (eq. (3.16), (3.17)) one realizes that the theory possesses two distinct classical supersymmetry vacua (obtained by minimizing the bosonic potential):

$$\begin{cases} \varphi_1 = 0 \\ \varphi_2 = 0 \\ \psi = 0 \end{cases} \qquad\qquad \begin{cases} \varphi_1 = -\,2m/g \\ \varphi_2 = 0 \\ \psi = 0 \end{cases} \quad . \qquad (3.51)$$

It is now easy to obtain from the condition $N_\mu = 0$ the soliton solution which interpolates between these two vacua. Looking at the explicit expression of the Nicolai mapping one gets the conditions:

$$\begin{cases} \partial_t \varphi_1 - \varepsilon\, \partial_x \varphi_2 + m\,\varphi_2 + g\,\varphi_1\varphi_2 = 0 \\ \partial_t \varphi_2 + \varepsilon\, \partial_x \varphi_1 + m\,\varphi_1 + \frac{g}{2}\left(\varphi_1^2 - \varphi_2^2\right) = 0 \end{cases}$$

(we have explicitly taken into account the possibility of introducing the parameter ε : $\varepsilon = \pm 1$). Imposing $\varphi_2 = 0$, one obtains

$$\begin{cases} \partial_t \varphi_1 = 0 \\ \varepsilon\, \partial_x \varphi_1 = -\frac{g}{2}\varphi_1^2 - m\varphi_1 \end{cases}$$

(*) Soliton solutions in scalar two-dimensional supersymmetric models were first studied in ref. (136); for more recent results see ref. (137) and the works there quoted.

whose solution is:

$$\varphi_1(x) = \varepsilon \frac{m}{g} \left\{ tgh\left[\frac{m}{2}(x-x_0)\right] - \varepsilon \right\}$$

(x_0 is an arbitrary integration constant). Then equating to zero the Nicolai variables produces:

$$\begin{cases} \varphi_1(x) = \varepsilon \frac{m}{g} \left\{ tgh\left[\frac{m}{2}(x-x_0)\right] - \varepsilon \right\} \\ \varphi_2 = 0 \\ \psi = 0 \end{cases} \tag{3.52}$$

namely the soliton ($\mathcal{E}$ = +1) and antisoliton ($\mathcal{E}$ = -1) solutions of the classical equations of motion. Classical stability of above solution can be checked using standard procedures.

The Nicolai variables for Euclidean conformal gravity can be obtained in strict analogy with the four-dimensional Yang-Mills theory[132]. The Lagrangian is simply given by:

$$\mathcal{L} = \frac{1}{16} \sqrt{g} \; C^{\mu\nu\rho\lambda} C_{\mu\nu\rho\lambda}$$

where $C_{\mu\nu\rho\lambda}$ is the Weyl tensor, the traceless part of the Riemann curvature. One can check that by introducing the following variables (compare with eq.(3.50))(*)

(*) Note that also this model, being in four dimensions, is connected with the quaternionic algebra; in fact in eq.(3.53) the quaternionic structure constant appear.

$$N^{(\pm)}_{ij} = g^{1/4} \left(C_{0i0j} \pm \frac{1}{2} \varepsilon_{ik\ell} C_{0jk\ell} \right) \tag{3.53}$$

the above Lagrangian can be rewritten, up to surface terms, as:

$$\mathcal{L} = \frac{1}{2} N^{(\pm)}_{ij} N^{(\pm)\,ij} \; .$$

The condition $N^{(\pm)}_{ij} = 0$ selects now Weyl selfdual and Weyl antiselfdual solutions of the classical equations of motion, which in analogy with the Yang-Mills case, are called gravitational instantons [138]. A particularly interesting case of gravitational instanton is given by the so-called Fubini-Study metric [139] on the complex projective space $P_2(C)$. The corresponding real four-dimensional metric has the following form:

$$g_{\mu\nu} = \frac{4a^2}{(x^2+a^2)} \left\{ \delta_{\mu\nu} - \frac{x_\mu x_\nu + \tilde{x}_\mu \tilde{x}_\nu}{(x^2+a^2)} \right\}$$

where a is a constant length and

$$x_\mu = \delta_{\mu\nu} x^\nu \qquad x^2 = \delta_{\mu\nu} x^\mu x^\nu$$

$$\tilde{x}_\mu = \gamma_{\mu\nu} x^\nu \qquad \gamma = i\sigma_2 \otimes \mathbb{1}_2 \; .$$

This metric is also a solution of the vacuum Einstein's equations with cosmological constant $\Lambda = 3/2a^2$.

IV. FOUR-DIMENSIONAL GAUGE MODELS

IV.1 Preliminaries

The correspondence between stochastic and supersymmetric properties of field theories discussed in the previous sections in the framework of simple models, can be extended to the case of gauge theories in physical four-dimensional space. In fact a complete set of stochastic identities have been discovered in the case of pure N=1 supersymmetric Yang-Mills theory (113-114) and recently also for theories in which the N=1 gauge multiplet is coupled with a matter scalar multiplet (116) (including thus the N=2 supersymmetric Yang-Mills model).

The presence of gauge invariance clearly makes the study of these identities more complicated than in the case of standard non-gauge theories. In fact it is well known that in order to give meaning to the functional integral involving gauge fields one has to introduce a suitable constraint (gauge fixing) which eliminates the sum over the redundant degrees of freedom in the path integral. This gauge-fixing procedure explicitly breaks supersymmetry and then the derivation of those special supersymmetric Ward identities which are the stochastic identities is clearly more delicate. Really the existence of stochastic identities in above supersymmetric gauge theories has been proved in a particular gauge, the so-called light-cone gauge. However, on the basis of the requirement of gauge invariance of physical results, one might hope that the same set of identities holds also in other gauges, for example in the covariant ones (*).

In the following this problem will be carefully analyzed within a N=1 supersymmetric gauge model by computing, using the stochastic

(*) And indeed this was suggested in ref. (107); see however ref. (108).

identities along the lines described in sect.III.5, two- and three-point bosonic Green's functions at one-loop level. Comparison of these results with the analogous ones obtained using standard perturbative techniques, confirms the full validity of the stochastic identities only in the light-cone gauge (140) (*).

This conclusion can be easily understood if one notices that only in this particular gauge a residual supersymmetry invariance survives in the effective action. Since the stochastic identities can be derived directly from the supersymmetry algebra, the unique role of the light-cone gauge appears evident. However, a direct and independent look at this situation using the new perturbative scheme discussed in sect. III.5 is celarly useful. Furthermore, in view of the substantial simplification that the presence of stochastic identities leads, due to the complete elimination from the theory of fermion fields, the direct computation at one-loop of two- and three-point bosonic Green's functions in the light-cone gauge for the above supersymmetric Yang-Mills models is clearly a non trivial check on the consistency of the new perturbation expansion which the stochastic properties of these theories produce.

Let us first consider the case of the N=1 supersymmetric Yang-Mills theory. In the so-called Wess-Zumino gauge, after the elimination of the auxiliary fields, the model can be described by a gauge field A_μ and a two-component Weyl spinor field λ_α together with its conjugate $\bar{\lambda}^{\dot{\alpha}}$. All these fields belong to the adjoint representation of the internal gauge group, that for simplicity we assume to be SU(2): $A_\mu = A^a_\mu t^a$, $\lambda_\alpha = \lambda^a_\alpha t^a$, $(t^a)_{bc} = i\,\varepsilon_{abc}$, (a,b,c, = 1,2,3). The corresponding Euclidean action for the model is

(*) Note that this conclusion can also be obtained using different techniques; see ref. (141).

$$A = \int d^4x \left(\mathcal{L}_B + \mathcal{L}_F \right) \tag{4.1a}$$

$$\mathcal{L}_B = \frac{1}{4} \left(F^a_{\mu\nu} \right)^2 \qquad \mathcal{L}_F = \lambda^\alpha (\varsigma \cdot D)_{\alpha\dot\alpha} \bar\lambda^{\dot\alpha} \tag{4.1b}$$

where the field strength $F^a_{\mu\nu}$ is defined by

$$\begin{aligned} F_{\mu\nu} = F^a_{\mu\nu} t^a = \partial_{[\mu} A_{\nu]} - ig \left[A_\mu, A_\nu \right] = \\ = \frac{i}{g} \left[D_\mu, D_\nu \right] \quad . \end{aligned} \tag{4.2}$$

$D_\mu = \partial_\mu - ig \left[A_\mu, \ \right]$ is the covariant derivative and $\varsigma^\mu \equiv (i\sigma, \mathbb{1})$ are the quaternionic unities (see sect. III.6). It is easy to check that the action (4.1) is invariant under the following supersymmetry transformations:

$$\begin{aligned} \delta A_\mu &= \varepsilon^\alpha \sigma^\mu_{\alpha\dot\alpha} \bar\lambda^{\dot\alpha} \\ \delta \lambda_\alpha &= \varepsilon^\beta (\sigma_{\mu\nu})_{\beta\alpha} F_{\mu\nu} \quad . \end{aligned} \tag{4.3}$$

As already noticed in sect. III.6, the following variables

$$f^a_i = F^a_{4i} + \frac{1}{2} \varepsilon_{ijk} F^a_{jk} \equiv E^a_i + B^a_i \tag{4.4}$$

$$(i,j,k = 1,2,3)$$

reduce the bosonic Lagrangian $\mathcal{L}_B$ to a quadratic form:

$$\mathcal{L}_B = \frac{1}{2} (f^a_i)^2 + \partial_\mu J_\mu \tag{4.5}$$

(J_μ is the topological current; then the term $\partial_\mu J_\mu$ can be neglected either discarging multi-instanton effects or by choosing suitable boundary conditions). Then the variables (4.4) are good candidates to be Nicolai

variables of the theory. Note however that the correspondence $f^a_i \longleftrightarrow A^a_\mu$ is not one to one, since we have not yet fixed the gauge to eliminate the redundant degrees of freedom.

Let us add then to the action (4.1) the following gauge-fixing term $\frac{1}{2\alpha} \int d^4x \, (G^a[A])^2$; introducing also the corresponding Faddeev-Popov ghosts, the gauge fixing procedure results in the following additional contribution to the action:

$$A' = \int d^4x \left\{ \mathcal{L}'_B + \mathcal{L}'_F \right\} \tag{4.6a}$$

$$\mathcal{L}'_B = \frac{1}{2\alpha} \left(G^a[A] \right)^2 \qquad \mathcal{L}'_F = \bar c^a \frac{\delta G^a}{\delta A^b_\mu} D_\mu c^b \ . \tag{4.6b}$$

At this point the full bosonic action $A_B = \int d^4x \left\{ \mathcal{L}_B + \mathcal{L}'_B \right\}$ can be written in the following form:

$$A_B = \frac{1}{2} \int d^4x \left\{ (f^a_i)^2 + \frac{1}{\alpha} (f^a_4)^2 \right\} \tag{4.7}$$

where:

$$f^a_4 = G^a[A] \quad . \tag{4.8}$$

Moreover the partition function for the theory in the new variable f^a_μ takes the expression:

$$Z = \int d\Omega(f^a_\mu) \det\left(\frac{\delta A_\mu}{\delta f_\nu} \right) \det(\varsigma \cdot D) \cdot \det\left(\frac{\delta G}{\delta A_\mu} D_\mu \right) e^{-A_B[f]} ; \tag{4.9}$$

then eq. (4.4) and eq.(4.8) are the Nicolai map for the theory only if the Jacobian $\det(\frac{\delta f}{\delta A})$ of the transformation $A_\mu \longmapsto f_\mu$ equals the product of the Matthews-Salam determinant det(S·D) times the Faddeev-Popov determinant $\det(\frac{\delta G}{\delta A_\mu} D_\mu)$ (see also sect. III.1). In this case the f^a_μ satisfy the following stochastic identities (*)

$$\left\langle \prod_{1}^{2n}{}_{i} f^{a_i}_{\mu_i}(x_i) \right\rangle_0 = \sum_{r,s} \left\langle f^{a_r}_{\mu_r}(x_r) f^{a_s}_{\mu_s}(x_s) \right\rangle_0 \cdot \tag{4.10}$$

$$\cdot \left\langle \prod_{p \neq r,s} f^{a_p}_{\mu_p}(x_p) \right\rangle_0$$

$$\left\langle f^{a_1}_{\mu_1}(x_1) f^{a_2}_{\mu_2}(x_2) \right\rangle_0 = \delta_{a_1 a_2} \hat{\delta}_{\mu_1 \mu_2} \delta^{(4)}(x_1 - x_2) \tag{4.11}$$

$$\hat{\delta}_{\mu_1 \mu_2} = \begin{cases} 1 & \mu = \nu = 1,2,3 \\ \alpha & \mu = \nu = 4 \\ 0 & \text{otherwise} \end{cases}$$

(the correlations containing an odd number of f_μ's vanish).

By assuming true these identities, one can now compute the Green's functions for the fields A_μ in terms of variables f_μ by simply inverting the relations (4.4) and (4.8). Note that due to the eq.(4.3) this inversion can be explicitly given only after the choice

(*) Note that $G^a[A]$ is surely a stochastic variable as explicitly shown in ref.'s(108, 91).

of a specific gauge fixing function $G^a[A]$.

IV.2 N=1 supersymmetric Yang-Mills theory: the covariant gauges

a) The Lorentz gauge is characterized by the choice:

$$G^a[A] = \partial_\mu A^a_\mu \ . \tag{4.12}$$

We already know from sect.III.6 that the relation between the variables f^a_μ and A^a_μ can be written in the following compact form:

$$(s\cdot f^a) = (\bar{s}\cdot\partial)(s\cdot A^a) + \frac{g}{2}\varepsilon_{abc}(\bar{s}\cdot A^b)(s\cdot A^c) \tag{4.13}$$

where again $s^\mu \equiv (i\underline{\sigma}, 1)$, $\bar{s}^\mu = (s^\mu)^\dagger$. A more convenient formalism can be adopted by introducing four-dimensional Euclidean γ-matrices

$$\gamma_\mu = i\begin{pmatrix} 0 & -s_\mu \\ \bar{s}_\mu & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

After some manipulations eq.(4.13) can be in fact reduced to the following form in momentum space:

$$A^a_\mu(k) = -i\frac{k_\nu}{k^2}\Big\{ \bar{t}_{\mu\nu 4\rho} f^a_\rho(k) - \tag{4.14}$$

$$- \frac{g}{2}\varepsilon_{abc} t_{\mu\nu\lambda\rho}\int\frac{d^4p}{(2\pi)^2} A^b_\lambda(p) A^c_\rho(k-p)\Big\}$$

where

$$\bar{t}_{\mu\nu\lambda\rho} \equiv \frac{1}{4}\mathrm{Tr}\{(1-\gamma_5)\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho\} = \tag{4.15a}$$

$$= \delta_{\mu\nu}\delta_{\lambda\rho} - \delta_{\mu\lambda}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\lambda} - \varepsilon_{\mu\nu\lambda\rho}$$

and

$$t_{\mu\nu\lambda\rho} \equiv \bar{t}_{\mu\nu[\lambda\rho]} \quad . \tag{4.15b}$$

Eq.(4.14) clearly allows to compute perturbatively Green's functions involving the field A^a_μ in terms of the correlation functions of the variables f^a_μ :

$$\langle f^a_\mu(k) \rangle_0 = 0$$

$$\langle f^a_\mu(k)\, f^b_\nu(k') \rangle_0 = \delta_{ab}\, \hat{\delta}_{\mu\nu}\, \delta^{(4)}(k+k') \tag{4.16}$$

$$\langle f^a_\mu(k)\, f^b_\nu(k')\, f^c_\rho(k'') \rangle_0 = 0$$

and so on.

One can then check that at the lowest order (g=0) the Green's functions calculated using eq.(4.14) coincide with the standard tree-level expressions. In particular for the two-point function one has

$$\langle A^{a_1}_{\mu_1}(k)\, A^{a_2}_{\mu_2}(k') \rangle_0 =$$

$$= \delta^{(4)}(k+k')\, \delta_{a_1a_2}\, \hat{\delta}_{\nu_1\nu_2}\, \frac{k_{\lambda_1} k_{\lambda_2}}{(k^2)^2}\, \bar{t}_{\mu_1\lambda_1 4\nu_1}\, \bar{t}_{\mu_2\lambda_2 4\nu_2} \quad ;$$

using eq.(4.15), one easily finds:

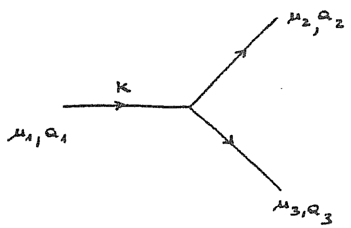
$$\langle A^{a_1}_{\mu_1}(k)\, A^{a_2}_{\mu_2}(k') \rangle_0 = \delta^{(4)}(k+k')\, \frac{\delta_{a_1a_2}}{k^2}\, \Delta_{\mu_1\mu_2}(k) \tag{4.17}$$

$$\Delta_{\mu_1\mu_2}(k) = \left\{ \delta_{\mu_1\mu_2} + (\alpha-1)\, \frac{k_{\mu_1} k_{\mu_2}}{k^2} \right\}$$

which is the standard A_μ - propagator.

As explained in sect. III.5, in order to compute higher order contributions or Green's functions involving more fields A_μ , one can use a diagrammatic expansion in terms of stochastic graphs. Looking at eq.(4.14), one finds that the corresponding Feynman rules to build up these graphs are now (*):

$$\frac{\delta_{ab}}{k^2}\, \Delta_{\mu\nu}(k) \tag{4.18a}$$



$$\frac{ig}{2(2\pi)^2}\, \varepsilon_{a_1a_2a_3}\, t_{\mu_1\nu\mu_2\mu_3}\, \frac{k_\nu}{k^2} \tag{4.18b}$$

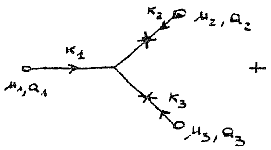
and no explicit propagator is associated to uncrossed lines (a four-dimensional integral for each loop-momentum is understood). It is now

---

(*) Note that eq.(4.14) is not fully covariant. This is a consequence of the choice (4.8) which makes f^a_4 a scalar and not the 4th-component of a vector. However, the following Feynman rules are fully Lorentz-covariant since the non-covariant part of eq.(4.14) is only used to build up the propagator (4.18a). As a consequence any stochastic graph preserves Lorentz-invariance.

easy, for example, to evaluate the three-point vertex using the correspon-ding (amputated) stochastic graphs; one has:

$$T^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3}(k_1, k_2, k_3) = \Big\{$$



$$+ \quad \text{permutations of } 1,2,3 \Big\}$$

$$= \frac{i g}{(2\pi)^2}\, \delta^{(4)}(k_1 + k_2 + k_3)\, \varepsilon_{a_1 a_2 a_3} \cdot \qquad (4.13)$$

$$\cdot \Big\{ \delta_{\mu_2 \mu_1}(k_2 - k_1)_{\mu_3} + \delta_{\mu_1 \mu_3}(k_1 - k_3)_{\mu_2} + \delta_{\mu_3 \mu_2}(k_3 - k_2)_{\mu_1} \Big\}$$

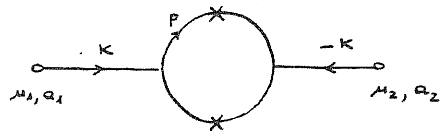
which is the well known standard result.

At this point it is important to observe that, due to its structure, eq.(4.14) allows to compute the full bosonic Green's functions including the self-energy inserctions on the external legs. This observation is by no means trivial: in fact if one tries to compute using eq.(4.14) or the corresponding Feynman rules eq.(4.18), the one-particle irreducible (1PI) part of a given Green's function by considering only the sum of the 1PI stochastic graphs in the corresponding diagrammatic expansion, one finally gets a wrong result (as we will explicitly see later). Clearly this is a general remark, and applies to any gauge, not only to the present Lorentz one; actually this remark is crucial in the light-cone gauge in order to obtain the correct Lorentz structure in the case of the three-point function.

b) Let us now calculate the connected two-point Green's function at one-loop level, using the stochastic graphs. The diagrammatic expansion exactly coincides with that of eq.(3. 44). Thus also here to compute

$$G^{(2)\, a_1 a_2}_{\mu_1 \mu_2}(k) = \langle A^{a_1}_{\mu_1}(k)\, A^{a_2}_{\mu_2}(k') \rangle_0 \Big|_{\text{order } g^2} \qquad (4.20)$$

one has to explicitly evaluate only two graphs, and precisely:

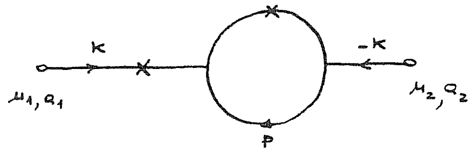


$$\equiv \Gamma^{(I)\, a_1 a_2}_{\mu_1 \mu_2}(k) =$$

$$= \frac{g^2}{2(2\pi)^4}\, \delta_{a_1 a_2}\, k_\alpha k_\beta\, t_{\mu_1 \alpha \nu \rho}\, t_{\mu_2 \beta \nu' \rho'} \cdot$$

$$\cdot \int \frac{d^4 p}{p_+^2\, p_-^2}\, \Delta_{\nu \nu'}(p_+)\, \Delta_{\rho \rho'}(p_-)$$

and



$$\equiv \Gamma^{(II)\, a_1 a_2}_{\mu_1 \mu_2} =$$

$$= \frac{g^2}{2(2\pi)^4}\, \delta_{a_1 a_2}\, k_\alpha\, t_{\mu_1 \nu \rho \sigma}\, t_{\mu_2 \alpha \nu' \rho} \int \frac{d^4 p}{p_+^2\, p_-^2}\, (p_-)_\sigma\, \Delta_{\nu \nu'}(p_+)$$

where $p_\pm = p \pm \frac{k}{2}$. We have multiplied the above graphs by the inverse free propagators of the external legs; in this way at the end we will obtain instead of the two-point Green's function of eq.(4.20), the corresponding 1PI proper part $\Gamma^{(2)\,a_1a_2}_{\mu_1\mu_2}(k)$.

Note that both the contributions are transversal

$$k_{\mu_1}\,\Gamma^{(i)\,a_1a_2}_{\mu_1\mu_2}(k) = k_{\mu_2}\,\Gamma^{(i)\,a_1a_2}_{\mu_1\mu_2} = 0 \qquad i = I,II;$$

then the above graphs can be put in the form

$$\Gamma^{(i)\,a_1a_2}_{\mu_1\mu_2}(k) = \frac{g^2}{(2\pi)^4}\,\delta_{a_1a_2}\,k^2\,P_{\mu_1\mu_2}\,\varphi_i(k)$$

$$P_{\mu_1\mu_2} = \delta_{\mu_1\mu_2} - \frac{k_{\mu_1}k_{\mu_2}}{k^2}$$

with

$$\varphi_I = \frac{1}{6}\,\frac{k_\alpha k_\beta}{k^2}\,t_{\mu\alpha\nu\rho}\,t_{\mu\beta\nu'\rho'}\int\frac{d^4p}{p_+^2\,p_-^2}\,\Delta_{\nu\nu'}(p_+)\Delta_{\rho\rho'}(p_-)$$

$$\varphi_{II} = \frac{1}{6}\,\frac{k_\alpha}{k^2}\,t_{\mu\nu\rho\sigma}\,t_{\mu\alpha\nu'\rho}\int\frac{d^4p}{p_+^2\,p_-^2}\,(p_-)_\sigma\,\Delta_{\nu\nu'}(p_+) \ .$$

The one-loop corrections to the 1PI two-point Green's function can now be written as:

$$\Gamma^{(2)\,a_1a_2}_{\mu_1\mu_2}(k) = \frac{g^2}{(2\pi)^4}\,\delta_{a_1a_2}\,k^2\,P_{\mu_1\mu_2}\,\phi(k) \qquad (4.21)$$

where

$$\phi(k) = 2\varphi_I(k) + 4\left\{\varphi_{II}(k) + \varphi_{II}(-k)\right\} \ .$$

Using eq.(4.15) one can explicitly compute φ_I and φ_{II} ; after a lengthy calculation one gets

$$\phi(k) = \frac{1}{3}\int\frac{d^4p}{p_+^2\,p_-^2}\left\{3(3-\alpha) + (\alpha-1)^2\left[1 - \frac{(p_+\cdot p_-)^2}{p_+^2\,p_-^2}\right] + \right.$$

$$\left. + 5(\alpha-1)\left[1 + 2\,\frac{(p_+\cdot p_-)}{k^2}\left(\frac{(k\cdot p_-)}{p_-^2} - \frac{(k\cdot p_+)}{p_+^2}\right)\right]\right\} \qquad (4.22)$$

This result exactly coincides (both in the finite and divergent part) with the analogous one obtained using standard Feynamn diagrams techniques (*). In particular for the divergent part, one has, using dimensional regularization:

$$\mathrm{Div}\,\Gamma^{(2)\,a_1a_2}_{\mu_1\mu_2}(k) = g^2\,\delta_{a_1a_2}\,k^2\,P_{\mu_1\mu_2}\,(3-\alpha)\,I \qquad (4.23)$$

where

$$I = \mathrm{Div}\int\frac{d^4p}{(2\pi)^4}\,\frac{1}{p_+^2\,p_-^2} = \frac{1}{16\pi^2}\,\frac{2}{\varepsilon} \qquad (4.24)$$

This infinite must be cancelled by the following counterterm

$$= -\left(Z_3 - 1\right)\delta_{a_1a_2}\,k^2\,P_{\mu_1\mu_2} \ .$$

(*) One can also play the inverse game: calculate the correlation function $\langle f^a_\mu\; f^b_\nu\rangle_0$ starting from eq.(4.21) and eq. (4.22), to check the consistency of the procedure up to order g^2. Using directly the 'Nicolai map' eq.(4.4) one obtains: $\langle f^{a_1}_{\mu_1}(k)\; f^{a_2}_{\mu_2}(k')\rangle_0 = \delta_{a_1a_2}\,\hat{\delta}_{\mu_1\mu_2}\,\delta^{(4)}(k+k') + O(g^4)$.

Thus one obtains for the wave function renormalization constant Z_3 the well known standard result

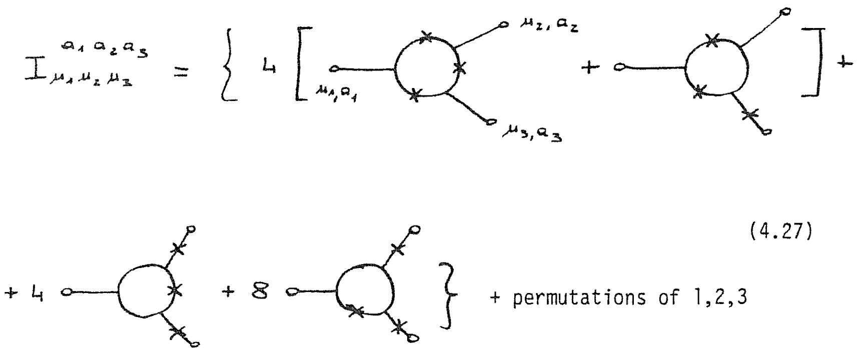
$$Z_3 = 1 + g^2(3-\alpha) I \quad . \tag{4.25}$$

c) Let us consider now the computation of the three-point Green's function

$$G^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3}(k_1, k_2, k_3) = \langle A^{a_1}_{\mu_1}(k_1)\, A^{a_2}_{\mu_2}(k_2)\, A^{a_3}_{\mu_3}(k_3) \rangle_{o} \tag{4.26}$$

at the one-loop level (we will consider for simplicity only its divergent part). As already noticed, the methods of computation based on the Nicolai mapping allows us to calculate the full (connected) Green's function. Then to get the correct final result one has to consider not only the stochastic graphs which are 1PI, but also those graphs which correspond to self-energy insertions on the external legs.

Let us first devote our attention to the 1PI diagrams; the corresponding one-loop contribution to eq.(4.26) is given by:



(4.27)

+ permutations of 1,2,3

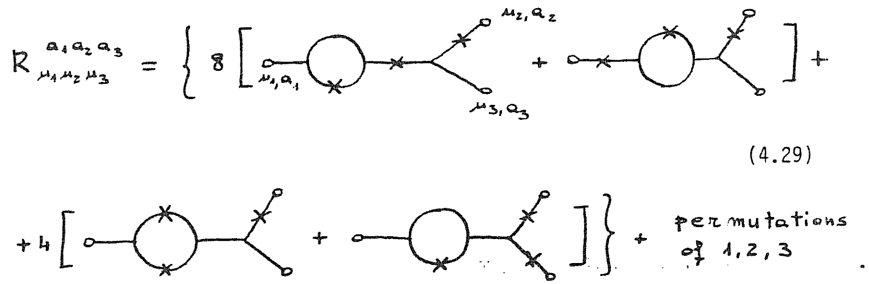
(for simplicity all graphs are supposed without the propagators on the

external legs). However from the explicit expressions of the four graphs one can note that only the last two are divergent. The infinite parts of these two diagrams can be computed using dimensional regularization. After some efforts one finally obtains the following result:

$$\mathrm{Div}\, I^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3}(k_1, k_2, k_3) = T^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3}(k_1, k_2, k_3)\, g^2 \left(\tfrac{1}{2} + 2\alpha\right) I \tag{4.28}$$

where $T^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3}$ is the tree-level contribution to the three-point function of eq.(4.19).

To complete the computation of the one-loop contribution to eq.(4.26), one has to add to the result (4.28) the sum $R^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3}$ of the stochastic graphs corresponding to self-energy insertions on the external legs. Explicitly one gets, with the correct symmetry factors:



(4.29)

Now all the four graphs are divergent; they can be expressed in terms of the functions φ_I and φ_{II} introduced in the evaluation of the one-loop correction to the two-point function.

After some calculations, one finally gets

$$\mathrm{Div}\; R^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3}(k_1,k_2,k_3) = T^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3}(k_1,k_2,k_3)\, g^2 \left(\frac{13-7\alpha}{2}\right) I \; - \qquad (4.30)$$

$$- \; \Sigma^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3}(k_1,k_2,k_3)\, g^2 (3-\alpha)\, I$$

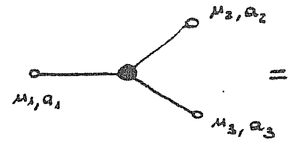
$$\Sigma^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3}(k_1,k_2,k_3) = \left\{ \frac{k^1_{\mu_1}}{(k^1)^2}\, k^1_\lambda\, T^{a_1 a_2 a_3}_{\lambda \mu_2 \mu_3}(k_1,k_2,k_3) + \text{cyclic permutations of } 1,2,3 \right\} .$$

From the sum of eq.(4.28) and eq.(4.30), the complete one-loop correction to eq.(4.26), one can now obtain the corresponding one-loop proper part $\Gamma^{(2)\, a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3}$ (*):

$$\mathrm{Div}\; \Gamma^{(2)\, a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3}(k_1,k_2,k_3) = T^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3}(k_1,k_2,k_3)\, g^2 \left(\tfrac{3}{2}\alpha - 2\right) I \, . \qquad (4.31)$$

(*) Eq.(4.31) clearly shows that the 1PI stochastic graphs do not lead to the 1PI three-point function; there is also a contribution (the first term of eq.(4.30)) coming from the graphs of eq.(4.29), which are apparently reducible. This result is due to the fact that it is really impossible to distinguish between reducible and irreducible stochastic diagrams; in fact in stochastic graphs two kinds of internal lines appear: crossed and uncrossed ones, and irreducibility with respect to the latter cannot be defined.

This divergence contribution is cancelled by the following counterterm



$$= (Z_1 - 1)\, T^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3} \, . \qquad (4.32)$$

Comparison with eq.(4.31) gives for the vertex renormalization constant:

$$Z_1\big|_{\text{stoch.}} = 1 - g^2 \left(\tfrac{3}{2}\alpha - 2\right) I \qquad (4.33)$$

in constrast with the well-known result calculated using Feynman techniques:

$$Z_1 = 1 - g^2\, \tfrac{3}{2}\, (\alpha - 1)\, I \, . \qquad (4.34)$$

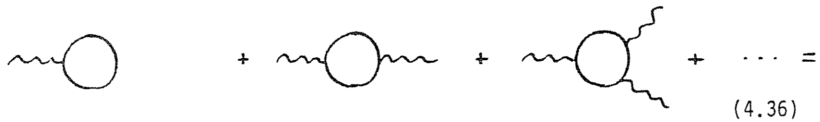
In particular, the result (4.33) would produce the wrong one-loop β-function.

This conclusion clearly suggests that the transformation (4.4) is not the Nicolai mapping for the theory or equivalently that the variables f^a_μ are not stochastic. In particular the fact that $\Gamma^{(2)\, a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3}$ calculated using stochastic diagrams does not coincide with the one calculated using standard Feynman techniques suggests that $\langle fff \rangle_0 \neq 0$. This can be also directly proved by separately computing in a perturbative expansion the Jacobian det $\left(\frac{\delta f_\mu}{\delta A_\nu}\right)$ of the transformation $A_\mu \mapsto f_\mu$ and the product det (S·D) det (∂·D) of the Matthews-Salam determinant times the Faddeev-Popov determinant, considering the field A_μ as external and given. In fact, as already observed, only if the Jacobian coincides, order by order, with the product of the other two determinants the variables f_μ are stochastic.

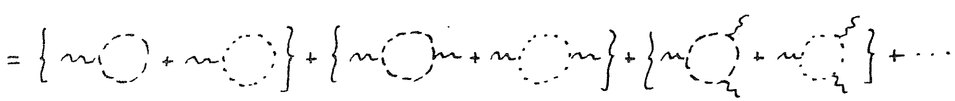
In practice one has to check whether

$$\mathrm{Tr}\ln\left(\frac{\delta f^a_\mu}{\delta A^b_\nu}\right) = \mathrm{Tr}\ln(s\cdot D) + \mathrm{Tr}\ln(\partial\cdot D) \tag{4.35}$$

which diagrammatically means:



(4.36)



The Feynman rules to compute these graphs are derived from the following two actions: the total fermionic action of the theory (see eq.(4.1) and eq.(4.6))

$$A_F = \int d^4x \left\{ \lambda (s\cdot D)\bar\lambda + \bar c(\partial\cdot D)c \right\} \tag{4.37a}$$

and

$$\tilde A = \int d^4x\, \bar\psi^a_\mu \left(\frac{\delta f^a_\mu}{\delta A^b_\nu}\right)\psi^b_\nu \tag{4.37b}$$

where an auxiliary fermionic-type field ψ^a_μ is introduced together with its conjugate $\bar\psi^a_\mu$. In the diagrams (4.36) the $(\bar\psi\psi)$, $(\bar\lambda\lambda)$ and $(\bar c c)$ propagators are respectively indicated with: $\longrightarrow$, $- - \to - -$, $\cdots\to\cdots$.

One can now compute the graphs appearing in the left-hand side of eq.(4.36) to compare them, order by order in g, with the analogous calculations of the diagrams in the right-hand side. It turns out that the two kind of graphs differ at order g^3: the divergent parts of the diagrams with three external A_μ-lines are different.

This fact can be summarized in the following way:

$$\ln\left\{\det\left(\frac{\delta A^b_\nu}{\delta f^a_\mu}\right)\det(s\cdot D)\,\det(\partial\cdot D)\right\} = \tag{4.38}$$

$$= -\int\left[\prod_1^3{}_i\, d^4k_i\, A^{a_i}_{\mu_i}(k_i)\right]\left\{T^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)\,\frac{g^2}{2}I + \text{finite terms}\right\} + O(g^4)\,;$$

$T^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}$ and I are again given by eq.(4.19) and eq.(4.24). Thus eq. (4.35) is violated at order g^3 and consequently the f^a_μ are not stochastic variables. In particular the partition function (4.9) can be now symbolically written (*)

$$Z = \int d\Omega(f)\, e^{-\int\left\{\frac{1}{2}f^2 + \frac{g^3}{2}f^3 + O(g^4)\right\}}\,;$$

and this explicitly shows that $\langle fff\rangle_0 \neq 0$. This result also explains why using eq.(4.27) and eq.(4.29) we obtain the wrong result for the three-point Green's function: in eq.(4.32) we missed the contribution coming from the fact that $\langle fff\rangle_0 \neq 0$. And in fact it is easy to show that the sum of this contribution and of eq.(4.32) gives the correct divergent one-loop correction to the proper three-point function.

(*) Note that while the f^2-term is local the f^3-term is highly non-local!

d) It is possible to repeat all the above considerations by fixing the gauge in a different way. If one chooses the so-called axial gauge(142), then

$$G^a[A] = n_\mu A^a_\mu \quad , \quad n^2 \neq 0 \tag{4.39}$$

(n_μ is a fixed vector), and the definition (4.8) becomes

$$f^a_4 = n \cdot A^a \quad . \tag{4.40}$$

In this gauge it is a little more difficult to express A^a_μ in terms of f^a_μ. However with techniques similar to those used to obtain the result (4.14), one can write in momentum space:

$$A^a_\mu(k) = -i \left\{ \delta_{\mu\sigma} + \frac{i k_\mu k_\sigma - k_\mu n_\sigma}{(k\cdot n)} \right\} \frac{k_\nu}{k^2} t_{\sigma\nu\mu\lambda} f^a_\lambda(k) +$$

$$+ \frac{ig}{2} \varepsilon_{abc} \left\{ \delta_{\mu\sigma} - \frac{k_\mu n_\sigma}{(k\cdot n)} \right\} \cdot \tag{4.41}$$

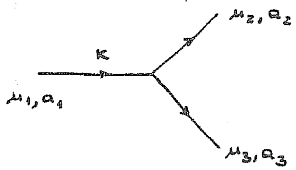
$$\cdot \frac{k_\nu}{k^2} t_{\sigma\nu\lambda\rho} \int \frac{d^4p}{(2\pi)^2} A^b_\lambda(p) A^c_\rho(k-p) \quad .$$

Obviosuly a prespcription should be given here to avoid the pole $1/k\cdot n$; a useful one has been indicated to be the principal value prescription (143). This fact however shows that the rigorous definition of the axial gauge requires some care (see in particular ref.'s (144-146)). In the following we neglect these problems ; our aim is simply to investigate

in a naive way the stochastic properties of the variables f^a_μ .

The Feynman rules to build up the stochastic graphs are now:

$$\frac{\delta_{ab}}{k^2} \tilde{\Delta}_{\mu\nu}(k) \tag{4.42a}$$



$$\frac{ig}{2(2\pi)^2} \varepsilon_{a_1a_2a_3} t_{\lambda\alpha\mu_2\mu_3} \frac{k_\alpha}{k^2} D_{\mu_1\lambda}(k) \tag{4.42b}$$

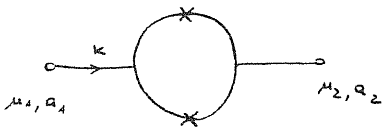
where

$$\tilde{\Delta}_{\mu\nu}(k) = \left\{ \delta_{\mu\nu} + \frac{(\alpha k^2 + n^2)}{(n\cdot k)^2} k_\mu k_\nu - \frac{k_\mu n_\nu + k_\nu n_\mu}{(k\cdot n)} \right\}$$

$$D_{\mu\nu}(k) = \left\{ \delta_{\mu\nu} - \frac{k_\mu n_\nu}{(k\cdot n)} \right\} \quad .$$

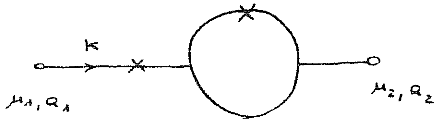
Note that (4.42a) is simply the free propagator of the field A^a_μ .

Let us now evaluate the divergent part of the 1PI two-point function at order g^2: $\Gamma^{(2)a_1a_2}_{\mu_1\mu_2}$. The expansion (3. 44) is still valid; then to obtain $\Gamma^{(2)a_1a_2}_{\mu_1\mu_2}$ one has simply to compute the two (amputated) diagrams:



$$= \frac{g^2}{2(2\pi)^4} \, \delta_{a_1 a_2} \, t_{\mu_1 \alpha \nu \rho} \;\cdot$$

$$\cdot \; t_{\mu_2 \beta \nu' \rho} \, k_\alpha k_\beta \int \frac{d^4p}{p_+^2 \, p_-^2} \; \tilde{\Delta}_{\nu\nu'}(p_+) \; \tilde{\Delta}_{\rho\rho'}(p_-)$$



$$= - \frac{g^2}{(2\pi)^4} \, \delta_{a_1 a_2} \, t_{\lambda \alpha \mu_1 \nu} \;\cdot$$

$$\cdot \; t_{\mu_2 \beta \rho \nu'} \, k_\beta \int \frac{d^4p}{p_+^2 \, p_-^2} \; (p_-)_\alpha \; \tilde{\Delta}_{\nu\nu'}(p_+) \, D_{\rho\lambda}(p_-)$$

$\left(p_\pm = p \pm \frac{k}{2} \right)$. To simplify the calculations we choose $\alpha = 0$ and the external momentum k_μ such that: $k \cdot n = 0$; this is always possible since we are interested only in the divergent parts. Using the principle value prescription to evaluate above integrals, after some calculations one gets:

$$\mathrm{Div}\, \Gamma^{(2)\, a_1 a_2}_{\mu_1 \mu_2}(k) = g^2 \, \delta_{a_1 a_2} \, k^2 \left\{ 6 \, P_{\mu_1 \mu_2} - 4 \, N_{\mu_1 \mu_2} \right\} I \tag{4.43}$$

where

$$P_{\mu_1 \mu_2} = \left\{ \delta_{\mu_1 \mu_2} - \frac{k_{\mu_1} k_{\mu_2}}{k^2} \right\}, \qquad N_{\mu_1 \mu_2} = \frac{n_{\mu_1} n_{\mu_2}}{n^2} \; .$$

This result does not coincide with that obtained using standard Feynman diagrams [147]: the term proportional to $N_{\mu_1 \mu_2}$ should not appear in eq.(4.43). This discrepancy clearly suggests, at least in the framework of this naive treatment, that also in the case of the axial gauge the variables f^a_μ start to loose their 'stochastic properties' at one-loop level(*).

IV.3 N=1 supersymmetric Yang-Mills theory: the light-cone gauge

In dealing with the light-cone gauge it is useful to write the components of the vectors in the so called light-cone basis; the connection between the standard basis and the new basis is given, e. g. for the coordinates, by the following relations:

$$x_\pm = x^\mp = \frac{1}{\sqrt{2}} \left(x_1 \mp i x_2 \right)$$
$$x_{\substack{R\\L}} = x^{\substack{L\\R}} = \frac{1}{\sqrt{2}} \left(x_3 \mp i x_4 \right) \; . \tag{4.44}$$

Then we eliminate the superflous gauge degrees of freedom of the theory described by the action (4.1) by simply putting to zero a component of A^a_μ(**)

$$A^a_R = 0 \qquad \text{or} \qquad A^a_3 = i A^a_4 \; . \tag{4.45}$$

(*) By using different techniques, the same conclusion is also obtained in ref. (141).

(**) This is equivalent to choose $G^a[A] = n \cdot A^a$, with $n_\mu = \frac{1}{\sqrt{2}} (0, 0, 1, -i)$ and let $\alpha \to 0$ in eq.(4.7) and eq.(4.9).

Thus the relation $f_i^a = E_i^a + B_i^a$ between the variables f_i^a and the remaining physical fields A_α^a $(\alpha = +,-,L)$ becomes a one to one mapping; explicitly:

$$f_+^a = -i\,\partial_L A_+^a + i\,\partial_+ A_L^a - ig\,\varepsilon_{abc} A_L^b A_+^c$$

$$f_-^a = i\,\partial_R A_-^a \qquad (4.46)$$

$$f_3^a = i\,\partial_R A_L^a + i\left(\partial_- A_+^a - \partial_+ A_-^a\right) - ig\,\varepsilon_{abc} A_+^b A_-^c$$

(note that for convenience we have introduced the redefinition: $f_\pm \to \sqrt{2}\, f_\pm$).

In this gauge it easy to check directly (113) that the Jacobian $\det\left(\frac{\delta f_i}{\delta A_\alpha}\right)$ exactly coincides with the product of the Matthews-Salam and Faddeev- Popov determinants: $\det(\varsigma \cdot D)\,\det(\partial_R)$. As a consequence the variables $f_\pm$, f_3 surely obey the following stochastic identities

$$\langle f_+^a(x)\, f_-^b(x')\rangle_o = \tfrac{1}{2}\,\delta_{ab}\,\delta^{(4)}(x-x')$$

$$\langle f_3^a(x)\, f_3^b(x')\rangle_o = \delta_{ab}\,\delta^{(4)}(x-x') \qquad (4.47)$$

(the other two-point correlations vanish). These identities can also be obtained by using the residual supersymmetry invariance which is left after the gauge choice of eq.(4.45), in strict analogy with the procedure used in sect. III.2 for the two-dimensional Wess-Zumino model (see ref.'s (113, 114) for details).

Using eq.(4.47) we can now compute in a perturbative expansion Green's functions involving the bosonic field A_α^a; what we need is simply to invert the Nicolai mapping of eq.(4.46). This can be easely

done in momentum space; one explicitly obtains:

$$\begin{aligned}\frac{k^2}{2} A_+^a(k) = {} & k_R f_+^a(k) - k_+ f_3^a(k) + k_+^2 A_-^a(k) + \\ & + ig\,k_R\,\varepsilon_{abc}\int\frac{d^4p}{(2\pi)^2} A_+^c(k-p)\,\cdot \\ & \cdot\left\{-\frac{f_3^b(p)}{p_R} + \left(\frac{k_+}{k_R} + \frac{p_+}{p_R}\right)A_-^b(p) - \frac{p_-}{p_R} A_+^b(p)\right\} + \\ & + g^2 k_R\,\varepsilon_{abc}\,\varepsilon_{bde}\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{p_R} A_+^c(k-p)\,A_+^d(q)\,A_-^e(p-q)\end{aligned} \qquad (4.48a)$$

$$A_L^a(k) = -\frac{f_3^a(k)}{k_R} + \frac{k_+}{k_R} A_-^a(k) - \frac{k_-}{k_R} A_+^a(k) - \frac{ig}{k_R}\varepsilon_{abc}\int\frac{d^4p}{(2\pi)^2} A_+^b(p)\,A_-^c(k-p) \qquad (4.48b)$$

$$A_-^a(k) = -\frac{f_-^a(k)}{k_R}\,. \qquad (4.48c)$$

These relations are however ill-defined unless we give a prescription to avoid the poles in the integrals. Various possibilities have been studied and discussed in the literature (*); we choose the Mandelstam-Leibbrandt prescription (86, 149, 150) essentially for reasons of simplicity: in this way in fact we can use naïve power counting in

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(*) See for example (148) and the references there quoted.

evaluating the divergent part of the integrals which appear in the loop corrections to bosonic Green's functions (*). Note that there are also deeper theoretical reasons for preferring this prescription [151].

We are now ready to compute the one-loop corrections to the connected two- and three-point functions in order to explicitly check the consistency of the new perturbative approach based on eq.(4.48). To do that we simply iterate these relations up to order g^3 and use the stochastic identities of eq.(4.47). This procedure is simpler than the method based on stochastic graphs; in fact the relations (4.48) are non--covariant and thus the corresponding Feynman stochastic rules are now complicated (**).

Let us compute the g^2-corrections to the connected two-point function:

$$G^{ab}_{\alpha\beta}(k,k') = \langle A^a_\alpha(k)\, A^b_\beta(k')\rangle_0 \qquad (4.49)$$

$$\alpha,\beta = +,-,L \, .$$

Note first that at lowest order (i. e. for g=0) eq.(4.48) gives the correct free propagator:

$$\Delta^{ab}_{\alpha\beta}(k,k') = \frac{\delta^{(4)}(k+k')\,\delta_{ab}}{k_R\, k^2} \begin{vmatrix} 0 & k_R & -k_+ \\ k_R & 0 & -k_- \\ -k_+ & -k_- & -2k_L \end{vmatrix} \qquad (4.50)$$

Then substituting in eq.(4.49) the order-g^2 iterations of eq.(4.48), after a lengthy calculation, one reaches the following results [140]:

$$\langle A^a_+(k)\, A^b_-(k')\rangle_0 = \Delta^{ab}_{+-}(k,k') \Big\{ 1 + \qquad (4.51a)$$

$$+ 2g^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2(k-p)^2} \Big[3 - 4\frac{k_R}{p_R} \Big] \Big\}$$

$$\langle A^a_L(k)\, A^b_\pm(k')\rangle_0 = \Delta^{ab}_{L\pm}(k,k') \Big\{ 1 + \qquad (4.51b)$$

$$+ 2g^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2(k-p)^2} \Big(1 + 2\frac{k_R}{k_\pm}\frac{p_\pm}{p_R} - 4\frac{k_R}{p_R} \Big) \Big\}$$

$$\langle A^a_\pm(k) A^b_\pm(k')\rangle_0 \equiv 0 \qquad (4.51c)$$

(*) Analogous computations performed using for example the principal value prescription are considerably more complicated and lead to ambigous results.

(**) Note however that a pseudo-covariant formalism can be developed also in this case following the discussion of the previous section for the axial gauge and the techniques developed in ref.(149, 150) in the framework of standard perturbation scheme.

$$\langle A_L^a(k)\, A_L^b(k')\rangle_o = \Delta_{LL}^{ab}(k,k') \Big\{ 1 -$$

$$- 2g^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2(k-p)^2} \left(1 + 2\, \frac{k_+ p_- + k_- p_+}{k_L\, p_R} - 4\, \frac{k_+ k_-}{k_L\, p_R} \right) \Big\} . \tag{4.51d}$$

These results exactly coincide (both for the finite and infinite parts) with those obtained using standard perturbative techniques [149, 152]. This means that the relations (4.48) connecting the variables A_α^a with f_α^a reproduce the Nicolai mapping for the theory up to order g^2.

By using the Mandelstam-Leibbrandt prescription it is now easy to extract the divergent parts of eq.(4.51). Since the following integrals

$$\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2(k-p)^2\, p_R} \qquad\qquad \int \frac{d^4p}{(2\pi)^4} \frac{p_\alpha}{p^2(k-p)^2\, p_R}$$

are finite, one explicitly obtains, up to order g^2:

$$\mathrm{Div} \langle A_+^a(k)\, A_-^b(k')\rangle_o = \Delta_{+-}^{ab}(k,k') \{ 6\, g^2 I \} \tag{4.52a}$$

$$\mathrm{Div} \langle A_L^a(k)\, A_\pm^b(k')\rangle_o = \Delta_{L\pm}^{ab}(k,k') \{ 2\, g^2 I \} \tag{4.52b}$$

$$\mathrm{Div} \langle A_L^a(k)\, A_L^b(k')\rangle_o = \Delta_{LL}^{ab}(k,k') \{ -2\, g^2 I \} \tag{4.52c}$$

I is again given by eq.(4.24).

With the same techniques one can also compute the divergent part of the one-loop corrections to some three-point functions;using the

relations (4.48) iterated up to order g^3 (*) explicitly one finds:

$$\langle A_+^{a_1}(k_1)\, A_-^{a_2}(k_2)\, A_-^{a_3}(k_3)\rangle_o = T_{+--}^{a_1a_2a_3}(k_1,k_2,k_3) \{ 12\, g^2 I \} \tag{4.53a}$$

$$\langle A_L^{a_1}(k_1)\, A_-^{a_2}(k_2)\, A_-^{a_3}(k_3)\rangle_o = T_{L--}^{a_1a_2a_3}(k_1,k_2,k_3) \{ 8\, g^2 I \} \tag{4.53b}$$

where $T_{\substack{+\\L}--}^{a_1a_2a_3}$ are the tree-level contributions

$$T_{\substack{+\\L}--}^{a_1a_2a_3}(k_1,k_2,k_3) = \pm \frac{2ig}{(2\pi)^2} \frac{\varepsilon_{a_1a_2a_3}}{k_1^2\, k_2^2\, k_3^2} \cdot (k_1)_{\substack{R\\ -}} \left\{ \frac{k_{3-}}{k_{3R}} - \frac{k_{2-}}{k_{2R}} \right\} . \tag{4.54}$$

It is a remarkable fact that the results (4.53) exactly coincide with those obtained using standard Feynman techniques [153, 154], as it has been explicitly checked. This is the proof that,at one-loop,the variables f_α^a are stochastic for the N=1 supresymmetric Yang-Mills theory in the light-cone gauge.

A remark concerning the computation of the three-point function of eq.(4.53) is in order here. As already stressed in discussing the covariant gauge, the inverted Nicolai mapping of eq.(4.48), due to its

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(*) For what concernes the lowest order contribution, one can easely verify that eq.(4.48) reproduces the correct tree-level three-point function.

intrinsic structure, is suitable for computing the full (connected) Green's functions and not their 1PI parts; this is a consequence of the fact that the definition of the 1PI part for stochastic diagrams is ambigous. (Also here if one computes for example only the 1PI stochastic contributions to $\langle A_+ A_- A_- \rangle_c$ a wrong result follows for the corresponding proper three-point function Γ_{+--}).

Moreover in the light-cone gauge there is an additional difficulty: the one-loop corrections to the 1PI Green's functions do not have the same tensorial Lorentz structure of the corresponding tree-level result. And in fact if one computes the one particle reducible contributions, e. g. the self-energy insertions on the external legs of the three-point function, one finds the same result. It is only after having summed reducible and irreducible parts that all the spurious terms which do not have the tree-level Lorentz structure disappear. This unexpected phenomenon is due to the residual symmetry group preserved by the gauge condition (4.45). This little group contains the rotations around the third axis, which distinguish among $+$, $-$ and L components of A^a_α.

The above remark is particularly important in carrying out the renormalization program for supersymmetric Yang-Mills theories in the light-cone gauge (*). Using the results of eq.(4.52) and eq.(4.53) one can easily prove that the theory is multiplicatively renormalizable, at least at one loop. In fact to make finite the two-point functions of eq.(4.52) one can define the following wave function renormalization constants for the gauge fields $A^a_\pm$ and A^a_L (**):

(*) The status of the renormalization program for Yang-Mills theories in the light-cone gauge is summarized in ref.(153).

(**) Note that since the components $A^a_\pm$ and A^a_L of the gauge field have different renormalization constants, it is impossible to define a renormalized Nicolai map; in fact looking at eq.(4.46) it is evident that the stochastic variables f^a_α renormalize in a complicated way.

$$A^a_\alpha = Z^{1/2}_\alpha A^{ren.\, a}_\alpha \qquad \alpha = +, -, L$$

$$Z^{1/2}_\pm = (1 + 3 g^2 I) \tag{4.55}$$

$$Z^{1/2}_L = (1 - g^2 I) \; .$$

As a consequence the three-point Green's functions of eq.(4.53) can be consistently made finite by defining the following connection between bare and renormalized coupling constant:

$$g = g_{ren} \left(1 - 3 g^2_{ren} I\right) \; . \tag{4.56}$$

With this relation one obtains the correct one-loop β-function (155, 89)

$$\beta(g) = -6 \, \frac{g^3}{16\pi^2} \; . \tag{4.57}$$

Additional studies are clearly needed to understand if this renormalization program can be generalized to any loop (153, 156). The above explicit proof that the theory is consistent up to one-loop represents already a non-trivial result: indeed in the literature (*) it is claimed that Yang-Mills theories in the light-cone gauge can not be renormalized in a simple way because, as already observed, the divergent parts of the 1PI Green's functions have a richer tensor structure than the corresponding tree-level contributions.

(*) See however ref.(153).

IV.4 The N=2 supersymmetric Yang-Mills theory

In the previous sections we have studied in detail the properties of stochastic identities of a four-dimensional supersymmetric gauge theory containing only a pure N=1 vector multiplet. Recently it has been established [116] a complete set of stochastic identities also in the case of an analogous theory in six-dimensions (in the light-cone gauge). After naive dimensional reduction this theory describes in four-dimensions a N=2 supersymmetric gauge theory [157,158], i.e. a N=1 vector multiplet coupled with a matter multiplet in the adjoint representation of the gauge group (generalization to an arbitrary representation can be easely obtained).

In this case the relation between the stochastic variables and the old bosonic ones is, strictly speaking, non-local, if we restrict our attention to the physical fields; however one can easily make this relation a local one by introducing suitable auxiliary variables. As before, by inverting this relation, it is possible to compute at one--loop two- and three-point Green's functions for this extended theory and compare these results with those obtained using standard Feynman techniques.

The N=2 supersymmetric Yang-Mills theory is described by a gauge field A_μ^a and a complex scalar field ϕ^a, plus the fermionic partners, two Majorana spinors which for simplicity we combine in a Dirac spinor χ^a (all these fields belong to the adjoint representation of the gauge group, which again we assume to be SU(2)). After the elimination of auxiliary fields, the corresponding Lagrangian reads (in Euclidean space; $P_\pm = \frac{1 \pm \gamma_5}{2}$):

$$\mathcal{L} = \frac{1}{4}\left(F_{\mu\nu}^a\right)^2 + D_\mu \bar{\phi}\, D_\mu \phi - \frac{g^2}{2}\left(\varepsilon_{abc}\, \phi^b \bar{\phi}^c\right)^2 + \tag{4.58}$$

$$+ \bar{\chi}\, \not{\partial} \chi + i\sqrt{2}\, g\, \varepsilon_{abc}\, \bar{\chi}^a \left\{ P_- \phi^c + P_+ \bar{\phi}^c \right\} \chi^b \quad .$$

Using the notations and the conventions of sect. IV.3 we now fix the light--cone gauge by imposing

$$A_R^a = 0 \quad . \tag{4.59}$$

With this condition it is easy to check that by introducing the following transformations

$$\begin{aligned}
f_+ &= i\partial_+ A_L^a - i\partial_L A_+^a + ig\,\varepsilon_{abc}\left(A_+^b A_L^c + \bar{\phi}^b A_s^c\right) \\
f_- &= i\partial_R A_-^a \\
f_3^a &= i\partial_R A_L^a + i\left(\partial_- A_+^a - \partial_+ A_-^a\right) - \\
&\quad - ig\,\varepsilon_{abc}\left(A_+^b A_-^c + \phi^b \bar{\phi}^c\right) \\
F_+^a &= -i\partial_L \phi^a - i\partial_- A_s^a - ig\,\varepsilon_{abc}\left(A_L^b \phi^c + A_-^b A_s^c\right) \\
F_-^a &= i\partial_R \bar{\phi}^a \\
F_3^a &= \partial_R A_s^a - \partial_+ \phi^a - g\,\varepsilon_{abc}\, A_+^b \phi^c \quad ,
\end{aligned} \tag{4.60}$$

the bosonic part of above Lagrangian can be put in a quadratic form, and precisely (*):

$$\mathcal{L} + \frac{1}{2}\left(\partial_R A_S - D_+\phi\right)^2 = \tag{4.61}$$

$$= \frac{1}{2}\left(f_3^a\right)^2 + 2 f_+^a f_-^a + \frac{1}{2}\left(F_3^a\right)^2 + 2F_+^a F_-^a \equiv \tilde{\mathcal{L}} .$$

F_3 and A_S are auxiliary variablesintroduced to write in a simpler way the transformations (4.60); the equation of motion $F_3^a = 0$ defines A_S^a in terms of the physical fields ϕ^a and A_+^a.

To prove that eq.(4.60) is really the Nicolai map for the theory, one has to write the partition function of the model in the following way (with $\mathcal{L}$ evaluated at $A_R = 0$):

$$Z = \int d\Omega\left(A_+, A_-, A_L, \bar\phi, \phi; A_S\right) d\Omega\left(\chi, \bar\chi\right) \cdot \tag{4.62}$$

$$\cdot \left(\det \partial_R\right)^2 e^{-\int d^4x \left\{\mathcal{L} + \frac{1}{2}\left(\partial_R A_S - D_+\phi\right)^2\right\}}$$

(a factor $\left(\det \partial_R\right)$ is the Faddeev -Popov determinant; the remaining factor $\left(\det \partial_R\right)$ is introduced to correctly normalize the functional

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(*) This result can be generalized to the case of the Yang-Mills Lagrangian in any even number of dimensions (G.Veneziano, private comunication).

integral over the auxiliary field A_S). Performing in (4.62) the change of variables (4.60), one gets:

$$Z = \int d\Omega\left(f_3, f_\pm, F_3, F_\pm\right) J^{-1} \Delta_{MS} \cdot \tag{4.63}$$

$$\cdot\, e^{-\int d^4x\, \tilde{\mathcal{L}}}$$

where Δ_{MS} is the Matthews-Salam determinant and J is the following Jacobian: $J = \det \frac{\partial(f_3, f_+, F_3, F_+)}{\partial(\phi, A_S, A_+, A_L)}$ (note that the transformation $(A_-, \bar\phi) \longmapsto (f_-, F_-)$ simply cancells the factor $\left(\det \partial_R\right)^2$ in eq.(4.62)). By choosing now a suitable representation for the Euclidean γ -matrices (e.g. : $\gamma_1 = 1\!\!1^{(2)} \otimes \sigma_2$, $\gamma_2 = \sigma_3 \otimes \sigma_1$, $\gamma_3 = \sigma_2 \otimes \sigma_1$, $\gamma_4 = -\sigma_1 \otimes \sigma_1$) one can show that:

$$J = \Delta_{MS} .$$

As a consequence the variables $\left(f_3, f_\pm, F_\pm\right)$ obey for example the following stochastic identities:

$$\langle f_+^a(x)\, f_-^b(x')\rangle_o = \langle F_+^a(x)\, F_-^b(x')\rangle_o = \frac{1}{2}\delta_{ab}\, \delta^{(4)}(x-x')$$

$$\langle f_3^a(x)\, f_3^b(x')\rangle_o = \delta_{ab}\, \delta^{(4)}(x-x') \tag{4.64}$$

(the remaining two-point correlations vanish).

Going to momentum space eq.(4.60) can be, at least implicitly, inverted; eliminating the auxiliary fields F_3^a and A_S^a , one explicitly gets:

$$\frac{k^2}{2} A_+^a(k) = k_R f_+^a(k) - k_+ f_3^a(k) + k_+^2 A_-^a(k) +$$

$$+ ig\, k_R\, \varepsilon_{abc} \int \frac{d^4p}{(2\pi)^2} \Big\{ A_+^c(k-p) \Big[-\frac{f_3^b(p)}{p_R} + \Big(\frac{k_+}{k_R} + \frac{p_+}{p_R} \Big) A_-^b(p) +$$

$$- \frac{p_-}{p_R} A_+^b(p) \Big] - \Big(\frac{k_+}{k_R} - \frac{p_+}{p_R} \Big) \phi^b(p)\, \bar{\phi}^c(k-p) \Big\} + \tag{4.65a}$$

$$+ g^2 k_R\, \varepsilon_{abc}\, \varepsilon_{bde} \int \frac{d^4q}{(2\pi)^4} \frac{d^4p}{p_R} \Big\{ A_+^c(k-p) \Big[A_+^d(q)\, A_-^e(p-q) +$$

$$+ \phi^d(p)\, \bar{\phi}^e(p-q) \Big] + A_+^d(q)\, \phi^e(p-q)\, \bar{\phi}^c(k-p) \Big\}$$

$$A_L^a(k) = - \frac{f_3^a(k)}{k_R} + \frac{k_+}{k_R} A_-^a(k) - \frac{k_-}{k_R} A_+^a(k) -$$

$$- \frac{ig}{k_R}\, \varepsilon_{abc} \int \frac{d^4p}{(2\pi)^2} \Big\{ A_+^b(p)\, A_-^c(k-p) + \phi^b(p)\, \bar{\phi}^c(k-p) \Big\} \tag{4.65b}$$

$$A_-^a(k) = - \frac{f_-^a(k)}{k_R} \tag{4.65c}$$

$$\frac{k^2}{2k_R} \phi^a(k) = F_+^a(k) + ig\, \varepsilon_{abc} \int \frac{d^4p}{(2\pi)^2} \phi^c(p) \cdot$$

$$\cdot \Big\{ \frac{k_-}{k_R} A_+^b(k-p) + A_L^b(k-p) + \frac{p_+}{p_R} A_-^b(k-p) \Big\} + \tag{4.65d}$$

$$+ g^2 \varepsilon_{abc}\, \varepsilon_{cde} \int \frac{d^4q}{(2\pi)^4} \frac{d^4p}{p_R} A_-^b(k-p)\, A_+^d(q)\, \phi^e(p-q)$$

$$\bar{\phi}^a(k) = - \frac{F_-^a(k)}{k_R} \tag{4.65e}$$

Note that if we put $\phi = \bar{\phi} = 0$, these transformations exactly coincide with those of eq.(4.48) (here again we choose the Mandelstam-Leibbrandt prescription to avoid the poles $1/p_R$ in the integrals).

Using the inverted relations(4.65) one can now compute Green's functions involving the fields A_α^a, ϕ^a, $\bar{\phi}^a$ $(\alpha = +,-,L)$ in terms of the correlation functions of eq.(4.64). For example, at the lowest order one easily finds:

$$\langle A_\alpha^a(k)\, A_\beta^b(k') \rangle_0 \Big|_{g=0} = \Delta_{\alpha\beta}^{ab}(k,k')$$

$$\langle \bar{\phi}^a(k)\, \phi^b(k') \rangle_0 \Big|_{g=0} = \Delta^{ab}(k,k')$$

where $\Delta_{\alpha\beta}^{ab}(k,k')$, explicitly given in eq.(4.50), and $\Delta^{ab}(k,k') = \frac{\delta_{ab}}{k^2}\delta^{(4)}(k+k')$, are the free propagators for the fields A_α^a and ϕ^a respectively.

Iterating eq.(4.65) up to order g^2, one can compute the complete two-point Green's functions at one-loop; they can be written in the following way:

$$\langle A_+^a(k)\, A_-^b(k') \rangle_0 = \Delta_{+-}^{ab}(k,k') \Big\{ 1 + \tag{4.66a}$$

$$+ 4g^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2(k-p)^2} \Big(1 - 2\frac{k_R}{p_R} \Big) \Big\} = \langle \bar{\phi}^a(k)\, \phi^b(k') \rangle_0$$

$$\langle A_L^a(k)\, A_\pm^b(k') \rangle_0 = \Delta_{L\pm}^{ab}(k,k') \Big\{ 1 + \tag{4.66b}$$

$$+ 4g^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2(k-p)^2} \Big(\frac{k_R}{k_\pm} \frac{p_\pm}{p_R} - 2\frac{k_R}{p_R} \Big) \Big\}$$

$$\langle A_L^a(k)\, A_L^b(k')\rangle_0 = \Delta_{LL}^{ab}(k,k') \Big\{ 1 + \tag{4.66c}$$

$$- 4 g^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2(k-p)^2} \left(1 + \frac{k_+ p_- + k_- p_+}{k_L\, p_R} - 2\,\frac{k_+ k_-}{k_L\, p_R} \right) \Big\}$$

$$\langle A_\pm^a(k)\, A_\pm^b(k')\rangle_0 \equiv 0 \quad . \tag{4.66d}$$

These results exactly coincide (both in the finite and infinite parts) with those obtained with standard perturbative techniques (*). Moreover only eq.(4.66a) and eq.(4.66c) have divergent contributions; precisely one has:

$$\mathrm{Div}\, \langle A_+^a(k)\, A_-^b(k')\rangle_0 = \mathrm{Div} \langle \bar\Phi^a(k)\, \phi^b(k')\rangle_0 = \tag{4.67a}$$

$$= \Delta_{+-}^{ab}(k,k') \left\{ 4 g^2 I \right\}$$

$$\mathrm{Div}\, \langle A_L^a(k)\, A_L^b(k')\rangle_0 = \Delta_{LL}^{ab}(k,k') \left\{ -4 g^2 I \right\} \tag{4.67b}$$

where as usual: $I = \frac{1}{16\pi^2}\left(\frac{2}{\varepsilon}\right)$ (see eq.(4.24)).

With a further effort one can also compute the divergent part of some three-point Green's functions:

(*) Note that the Ward identity $\langle A_+ A_- \rangle = \langle \bar\Phi \Phi \rangle$ proved in ref.(114) is exactly verified.

$$\mathrm{Div} \langle A_+^{a_1}(k_1)\, A_-^{a_2}(k_2)\, A_-^{a_3}(k_3)\rangle_0 = T_{+--}^{a_1 a_2 a_3}(k_1,k_2,k_3) \left\{ 8 g^2 I \right\} \tag{4.68a}$$

$$\mathrm{Div} \langle A_L^{a_1}(k_1)\, A_-^{a_2}(k_2)\, A_-^{a_3}(k_3)\rangle_0 = T_{+--}^{a_1 a_2 a_3}(k_1,k_2,k_3) \left\{ 4 g^2 I \right\} \tag{4.68b}$$

where $T_{\pm --}^{a_1 a_2 a_3}$ are the tree-level contributions of eq.(4.54). Also these results exactly coincide with those obtained using standard Feynman techniques. This conclusion can be clearly considered as the perturbative proof, at least at one-loop level, of the validity of the stochastic identities (4.64). Then also the general theory described by the Lagrangian (4.58) in which a Yang-Mills supermultiplet is coupled to a matter multiplet possesses remarkable stochastic properties, if formulated in the light-cone gauge.

Finally note that also in this case using the results of eq.(4.67) and eq.(4.68) one can show that the theory is multiplicatively renormalizable at one-loop (*). In fact introducing in the standard way the wave function renormalization constants:

(*) The same conclusion is obtained in ref.(159), using standard perturbation techniques.

$$A_\alpha = Z_\alpha^{1/2} A_\alpha^{ren.} \qquad \alpha = +,-,L$$

$$\phi = Z_\phi^{1/2} \phi^{ren.} \quad ,$$

the choice:

$$Z_\pm^{1/2} = Z_\phi^{1/2} = (1 + 2g^2 I)$$

$$Z_L^{1/2} = (1 - 2g^2 I)$$

makes finite the two-point Green's functions of eq.(4.66). Moreover the divergent parts of the three-point functions (4.48) can be consistently eliminated if the following relation between the bare and the renormalized coupling constant holds:

$$g = g_{ren} (1 - 2g_{ren}^2 I) \quad .$$

This relation gives the correct one-loop β -function:

$$\beta(g) = -4 \frac{g^3}{16\pi^2} \quad .$$

Note that this is also the exact β-function because for the N=2 supersymmetric Yang-Mills theory there is no divergent coupling constant renormalization above one-loop (89).

IV.5 Discussion

The most striking outcome of the previous computations is that the results of equations (4.51), (4.53), (4.66), (4.68) coincide with those of standard perturbation theory. In this way we have explicitly verified

that the stochastic identities (4.47) and (4.64) persist unchanged up to one-loop order and as a consequence that the considered supersymmetric Yang-Mills theories really possess an underlying stochastic structure.

It is remarkable that those stochastic properties are really present only if the models are formulated in the light-cone guage, as we have explicitly shown. It is well known that this gauge has many other attractive properties. For example the cancellations of the ultraviolet divergences which lead to a completely finite field theory in the case of the N=4 supersymmetric Yang-Mills model is particularly evident in this gauge. Furthermore the infinite-momentum limit (the light-cone) constitutes an optimal frame of reference to obtain simple picture of physical phenomena.The fact that the stochastic identities are completely valid only in the light-cone gauge confirms the special role of this gauge.

Moreover the above results clearly show that the presence of the stochastic identities is a fundamental property of some supersymmetric field theories. These exact constraints on certain bosonic correlation functions can contribute to the full understanding of the intimate structure of supersymmetric models. For example eq.(4.47) and eq.(4.64) should in principle give a detailed information about the characteristic properties of the supersymmetric vacuum (see the discussion of Appendix A). Furthermore stochastic identities can be the basis for new tests on supersymmetric theories, not directly involving the fermionic degrees of freedom, and thus they can be used for new lattice investigations in gauge theories.

However, as repeatedly stressed in the previous sections, one of the most interesting feature of the presence of stochastic identities or equivalently of a local Nicolai mapping in supersymmetric theories is the possibility of a new understanding of these models at the perturbative level. In fact the alternative perturbation expansion based on stochastic diagrams has several advantages, as it has been explicilty shown in various models. In particular stochastic diagrams do not exhibit divergences worse than logarithmic. Moreover a supersymmetric regularization of these graphs can be easely performed since these purely bosonic diagrams automatically take into account the corresponding fermionic contributions.

Clearly all these new techniques based on stochastic identities have some limitations and some problems. For example working in Euclidean space, the stochastic variables ξ_α^a of eq.(4.46) are certainly real; on the other hand the Euclidean light-cone gauge is not completely well defined since the condition (4.45) implicitly assumes that at least the 3- and 4-components of A_μ^a are complex. This problem can not be solved by going into Minkowskispace since in this case the variables ξ_α^a become complex (*).

In spite of these and other difficulties the existence of stochastic identities in a supersymmetric theory clearly constitutes an important and remarkable property, which has a deep and fundamental meaning. A more accurate and complete study of this stochastic structure is obviously needed, since probably other gauge models (e.g. the N=4 supersymmetric Yang-Mills theory) might present similar properties; and from these studies new interesting aspects of the connection between stochastic properties and supersymmetry will surely emerge.

(*) For a discussion on this point see ref.(114).

APPENDIX A

The canonical formalism

In chapters III and IV we have studied the properties of local Nicolai maps and of stochastic variables mainly in the framework of a path integral formulation of supersymmetric quantum field theories. It is however meaningful that a parallel discussion on the stochastic properties of supersymmetric models can also be performed in the framework of a canonical approach[115].

Let us first consider the simple case of Supersymmetric Quantum Mechanics (see sect. III.1; we work now in Minkowski space, as the canonical treatment requires). The Hamiltonian of the system can be written as:

$$\begin{aligned} H &= \tfrac{1}{2}\xi^+\xi - V''(q)\,\bar{\psi}\psi \\ &= \tfrac{1}{2}\xi\,\xi^+ + V''(q)\,\psi\bar{\psi} \end{aligned} \qquad \text{(A.1)}$$

where the Nicolai variables are now:

$$\begin{cases} \xi = p + i\,V'(q) \\ \xi^+ = p - i\,V'(q) \end{cases} \qquad \text{(A.2)}$$

(p is the conjugate momentum to q). p, q, ψ are operators taken at a fixed time (e.g. t=0) in the Schrödinger picture; they obey the following canonical commutation or anticommutation relations:

$$i\,[p, q] = 1 = \{\bar{\psi}, \psi\} \;.$$

From these relations it is easy to check that

$$\{ Q , \bar{Q} \} = 2H$$

where Q , $\bar{Q}$ are the supersymmetric charges.

The possibility of defining the variables ξ , ξ^{+} allows a detailed description of the quantum vacuum of the theory. As shown in ref.(160), the ground states of supersymmetric quantum mechanical models share a simple structure in the empty and filled sectors of the fermion Fock-space, defined by:

$$\psi |+\rangle = \bar{\psi} |-\rangle = 0 \qquad . \qquad \text{(A.3)}$$

$|+\rangle$ is the fermion vacuum and $|-\rangle$ denotes the filled fermion state, with fermion number 1 (*). In these null sectors the supersymmetric invariant zero-energy states $|0, \pm\rangle$ satisfy the condition

$$H |0, \pm\rangle = 0 \quad ;$$

remembering eq.(A.1) one gets

$$\begin{cases} \xi^{+} |0, +\rangle = 0 \\ \xi \, |0, -\rangle = 0 \end{cases} \qquad . \qquad \text{(A.4)}$$

(*) Note that the in the null sectors one can define two different purely bosonic theories described by the following Hamiltonians:

$$H_{\pm} = \langle \pm | H | \pm \rangle = \frac{1}{2} p^2 + (V'(q))^2 \mp V''(q) \ .$$

These effective theories can be obtained in a functional formalism by simply computing respectively with advanced and retarded boundary conditions the Matthews-Salam determinant coming from the integration over the fermions (119, 115).

These conditions are clearly the analogue of the stochastic identity $\langle \xi \rangle_0 = 0$, and can be considered the quantum version of the classical zero-energy equations $\xi = 0 = \xi^{+}$ (see the discussion of sect.III.6). In a Schrödinger picture ($q \longrightarrow x$, $p \longrightarrow -i \frac{\partial}{\partial x}$) eq.(A.4) gives the following ground state wave functionals:

$$\begin{aligned} \psi_{+}(x) &\sim e^{V(x)} \\ \psi_{-}(x) &\sim e^{-V(x)} \end{aligned} \qquad \text{(A.5)}$$

(note that only for some choices of the superpotential $V(x)$ one of these two wave-functions is normalizable and really belongs to the Hilbert space of the physical states of the model; these are just the superpotentials for which the Nicolai variables (A.2) can be consistently defined (96, 120)). Thus in this simple example, the presence of a local Nicolai map, which produces the 'stochastic identities' (A.4), gives us acomplete information about the structure of the vacuum.

For the four-dimensional supersymmetric Yang-Mills theories discussed in chapt.IV the situation is more complicated and obscure and a complete determination of the vacuum wave functional cannot be achieved. Let us consider for example the N=1 supersymmetric gauge model; it is well known that the canonical structure of the theory is more transparent in the so called temporal gauge $A_0=0$ (161, 162). In fact the conjugate momenta to the variables $A_i^a(x)$ (i=1, 2, 3) are simply $\pi_j^a(x) = \partial_0 A_j^a \equiv E_j^a(x)$ and the corresponding non trivial canonical commutation relations read

$$\left[\pi_i^a(\underline{x}, t), A_j^b(\underline{x}', t) \right] = -i \, \delta_{ab} \, \delta_{ij} \, \delta^{(3)}(\underline{x} - \underline{x}') \ .$$

The fermions on the other hand obey canonical anticommutation relations:

$$\left\{ \bar{\lambda}^{\dot{\alpha}}_{a}(\underline{x},t) \, , \, \lambda^{b}_{\alpha}(\underline{x}',t) \right\} = \delta^{\dot{\alpha}}_{\alpha} \, \delta_{ab} \, \delta^{(3)}(\underline{x}-\underline{x}') \, .$$

Introducing now the Nicolai variables (see sect.III.6):

$$\begin{cases} f^{a}_{i}(\underline{x}) = \pi^{a}_{i}(\underline{x}) + i \, B^{a}_{i}(\underline{x}) \\ f^{a\,+}_{i}(\underline{x}) = \pi^{a}_{i}(\underline{x}) - i \, B^{a}_{i}(\underline{x}) \end{cases} \qquad \text{(A.6)}$$

$$B^{a}_{i}(\underline{x}) \equiv \frac{1}{2} \, \varepsilon_{ijk} \, F^{a}_{jk}(\underline{x})$$

the Hamiltonian of the model can be written as

$$H = H_B + H_F \qquad \text{(A.7a)}$$

$$H_B = \int d^3x \, \frac{1}{2} \, f^{a\,+}_{i}(\underline{x}) \, f^{a}_{i}(\underline{x}) = \int d^3x \, \frac{1}{2} \, f^{a}_{i}(\underline{x}) \, f^{a\,+}_{i}(\underline{x}) \qquad \text{(A.7b)}$$

$$H_F = i \int d^3x \, \lambda(\underline{x}) \, \sigma_i \, D_i \, \bar{\lambda}(\underline{x}) = i \int d^3x \, \bar{\lambda}(\underline{x}) \, \bar{\sigma}_i \, D_i \, \lambda(\underline{x}) \qquad \text{(A.7c)}$$

We are here in a Schrödinger picture; thus all our dynamical variables are considered at a fixed time, e.g. t=0. Moreover note that in this gauge

$$B^{a}_{i}(\underline{x}) = \frac{\delta V[\underline{A}]}{\delta A^{a}_{i}(\underline{x})}$$

where (*)

$$V[\underline{A}] = \frac{1}{2} \, \varepsilon_{ijk} \int d^3x \, \mathrm{tr} \left\{ A_i \, \partial_j \, A_k + \frac{2}{3} \, g \, A_i \, A_j \, A_k \right\} .$$

Then eq.(A.6) can be rewritten in the form

$$\begin{cases} f^{a}_{i} = \pi^{a}_{i} + i \, \dfrac{\delta V[\underline{A}]}{\delta A^{a}_{i}} \\ f^{a\,+}_{i} = \pi^{a}_{i} - i \, \dfrac{\delta V[\underline{A}]}{\delta A^{a}_{i}} \end{cases} \qquad \text{(A.8)}$$

and the analogy with eq.(A.2) is now apparent.

Unfortunately the variables f^{a}_{i}, $f^{a\,+}_{i}$ are not the correct Nicolai variables for the theory (see sect. IV.2); then all the discussion previously done in the case of Supersymmetric Quantum Mechanics can not be repeated here. Nevertheless as shown in ref.(128) if we fix our attention only to the null sectors of the theory, defined in analogy with eq.(A.3) by

$$\lambda_{\alpha} \, |+\rangle = \bar{\lambda}^{\dot{\alpha}} \, |-\rangle = 0 \qquad \text{(A.9)}$$

and moreover we consider the theory only in a finite time interval, then the variables (A.8) become stochastic variables. In this case the theory reduces to a purely bosonic effective theory which looking at eq.(A.7) is described by the Hamiltonian

(*) $V[\underline{A}]$ is the integrated Chern-Simons 0-cocycle.

$$H_B = \frac{1}{2}\int d^3x\, f_i^{a+} f_i^a \equiv \int d^3x \left\{ (E_i^a)^2 + (B_i^a)^2 \right\} \quad ; \qquad \text{(A.10)}$$

in other words we have obtained a standard non-supersymmetric Yang-Mills model (*). The ground states in the two null sectors are clearly characterized by

$$\begin{cases} f_i^a\, |0, \pm\rangle = 0 \\ f_i^{a+} |0, \pm\rangle = 0 \end{cases} \qquad \text{(A.11)}$$

In the Schrödinger picture these states are represented by functionals $\Psi_\pm$ of the real functions $A_i^a(x)$ at fixed time; remembering that in this case $\pi_i^a = -i\ \delta/\delta A_i^a$,the solutions of eq.(A.11) are

(*) This phenomenon can be easely understood also in a functional integral formalism; in fact working in the null sectors is equivalent to choose retarded or advanced boundary conditions in computing the determinants involved in the Nicolai procedure. Note that the starting theory is still the supersymmetric one; however the fermion structure is completely masked by that particular choice of boundary conditions. One can now use the 'Nicolai map' (A.8) to set up a perturbative study of the obtained effective bosonic theory; the unnatural choice of boundary conditions (see the discussion of sect. III.4) makes however the whole procedure a little involved (129).

$$\Psi_\pm[\underline{A}] \sim e^{\pm V[\underline{A}]} \quad ; \qquad \text{(A.12)}$$

note again the analogy with the quantum mechanical case. However these solutions cannot be accepted as ground state functionals of the effective theory (i.e. the standard four-dimensional SU(2) gauge model) since $\Psi_\pm$ are not normalizable and do not transform in the correct way under time--independent gauge transformations (161, 163).

This conclusion is relevant also for the full supersymmetric theory, since in practice it means that the null sectors do not contain the supersymmetric vacuum. This can be also understood by noting that the null fermion states $|\pm\rangle$ are orthogonal to the usual field theory vacuum, in which only the Dirac sea is filled (119). Thus the absence of a local Nicolai map for the supersymmetric Yang-Mills theory in the temporal gauge does not allow a detailed understanding of the vacuum functional of the theory. Even more, the previous discussion suggests that the lack of a normalizable ground state functional in ordinary non--supersymmetric gauge theories has something to do with the fact that the variables f_i^a are, in general, not stochastic.

At this point a more careful analysis of all these problems would be clearly necessary, in particular within the framework of a light-cone gauge formulation of supersymmetric gauge models, where a local Nicolai map really exists (*). However the canonical structure of the theory is less

(*) Nevertheless note that, as discussed in sect. III.4, even in this case the local Nicolai map can not be interpreted as a full stochastic process; this is another relevant difference of a gauge field model with respect to the one-dimensional case previously discussed.

transparent in this gauge and then a simple reformulation of above formalism does not seem completely straightforward.

APPENDIX B

The linearized conformal supergravity

In sect. III.6, we have seen that there exists a set of transformations which gives the Lagrangian of conformal gravity a quadratic form. It would be interesting to understand whether these transformations really are the Nicolai mapping for the supersymmetric version of the theory, i.e. for the conformal supergravity (164-168). Here we limit ourselves to a rough study of the question in the linearized version of the N=1 model.

The conformal supergravity multiplet contains besides the gravitational field $h_{\mu\nu}$, connected to the metric by the standard relation $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$, the spin $3/2$ gravitino field ψ_μ, a Majorana spinor, and a chiral U(1) vector field A_μ. The corresponding linearized Lagrangian has the following form:

$$\mathcal{L} = \tilde{C}_{\mu\nu\rho\lambda}\tilde{C}^{\mu\nu\rho\lambda} - \frac{1}{4} F_{\mu\nu}^2 - \qquad \text{(B.1)}$$

$$- \frac{1}{2} \bar{\psi}_\mu \left\{ \not{\partial} \left(\Box \delta_{\mu\nu} - \partial_\mu \partial_\nu\right) - \frac{1}{2} \gamma_5 \gamma_\rho \partial_\sigma \Box \varepsilon^{\mu\nu\rho\sigma} \right\} \psi_\nu$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\tilde{C}_{\mu\nu\rho\lambda}$ is the linearized Weyl tensor(*).

(*) We use here the conventions of ref.(168), with a suitable redefinition of the fields. Hereafter the ten independent components of $h_{\mu\nu}$ will be called h_M, $M=1, 2, \ldots, 10$.

The action $A = \int d^4x\, \mathcal{L}$ is invariant under the following local symmetries:

a) general coordinate transformations $\xi^\mu(x)$

b) Weyl transformations $\omega(x)$

c) chiral transformations $\lambda(x)$

d) Q-supersymmetry $\varepsilon(x)$

e) S-supersymmetry $\vartheta(x)$

(on the right the corresponding infinitesimal parameters are indicated). Due to these local invariances, the partition function

$$Z = \int d\Omega\left(h_M, A_\mu\right) d\Omega\left(\bar{\psi}_\mu, \psi_\nu\right) e^{iA} \qquad \text{(B.2)}$$

is not well defined, unless a suitable gauge-fixing procedure is introduced. To implement the following covariant gauge-fixing conditions:

$$\partial^\mu h_{\mu\nu} = 0 \qquad h^\mu{}_\mu = 0 \qquad \partial \cdot A = 0 \qquad \partial \cdot \psi = 0 \qquad \gamma \cdot \psi = 0 \qquad \text{(B.3)}$$

we use the standard techniques (169); we multiply the previous partition function by the product of the Faddeev-Popov determinant Δ_{FP} times the following functional integrals:

$$\int d\hat{\Omega}(\xi_\mu)\, \mathcal{D}a\; e^{-i\int d^4x\, a_\mu \Box a_\mu} \left(\det \Box\right)^2 \delta\left(a_\mu - \partial_\nu h^\xi_{\mu\nu}\right)$$

$$\int d\hat{\Omega}(\omega)\, \mathcal{D}b\; e^{\frac{i}{6}\int d^4x\, b\, \Box^2 b} \left(\det \Box\right) \delta\left(b - h^\omega_{\mu\mu}\right)$$

$$\int d\hat{\Omega}(\lambda)\, \mathcal{D}c \quad e^{-\frac{i}{2}\int d^4x\, c^2} \quad \delta(c - \partial\cdot A^{\lambda})$$

$$\int d\hat{\Omega}(\varepsilon)\, \mathcal{D}\chi \quad e^{\frac{i}{2}\int d^4x\, \bar{\chi}\not{\partial}\chi} \quad (\det \not{\partial})^{-\frac{1}{2}} \delta(\chi - \partial\cdot\psi^{\varepsilon})$$

$$\int d\hat{\Omega}(\Theta)\, \mathcal{D}\eta \quad e^{-\frac{i}{8}\int d^4x\, \bar{\eta}\,\Box\not{\partial}\eta} \quad (\det \Box\not{\partial})^{-\frac{1}{2}} \delta(\eta - \gamma\cdot\psi^{\Theta})$$

where (a_μ , b , c ; χ , η) are suitable bosonic and fermionic fields (the various determinants are introduced to normalize the corresponding functional measures). $d\hat{\Omega}(\cdot)$ are invariant measures on the corresponding parameter groups.

Since the functional measure in eq.(B.2) is invariant under the symmetries of the action, one easily finds integrating over (a_μ , b , c ; χ , η):

$$Z = \int d\Omega(h_{\mu\nu}, A_\mu)\, d\Omega(\bar{\psi}_\mu, \psi_\nu)\; e^{i\int \mathcal{L}_{eff}\, d^4x} \; .$$
$$\cdot\, \Delta_{FP}\, (\det\Box)^3\, (\det\not{\partial})^{-\frac{1}{2}}\, (\det(\Box\not{\partial}))^{-\frac{1}{2}} \qquad \text{(B.4)}$$

where

$$\mathcal{L}_{eff} = \mathcal{L} + \mathcal{L}_{GF} =$$

$$= \frac{1}{2} h_{\mu\nu}\Box^2 h_{\mu\nu} + \frac{1}{3}\left(h_{\mu\mu}\Box\partial_\lambda\partial_\nu h_{\lambda\nu} + (\partial_\mu\partial_\nu h_{\mu\nu})^2\right) +$$
$$- \frac{1}{2}(\partial_\mu A_\nu)^2 - \frac{1}{2}\bar{\psi}_\mu \Box\left(\not{\partial}\delta_{\mu\nu} + \frac{1}{4}\gamma^\nu\not{\partial}\gamma^\mu\right)\psi_\nu \qquad \text{(B.5)}$$

is the sum of the starting Lagrangian (B.1) and of the gauge fixing one:

$$\mathcal{L}_{GF} = -(\partial_\mu h_{\mu\nu})\Box(\partial_\lambda h_{\lambda\nu}) + \frac{1}{6} h_{\mu\mu}\Box^2 h_{\nu\nu} + \qquad \text{(B.6)}$$
$$-\frac{1}{2}(\partial\cdot A)^2 + \frac{1}{2}\partial\cdot\psi\,\not{\partial}\,\partial\cdot\psi - \frac{1}{8}\gamma\cdot\psi\,\Box\not{\partial}\,\gamma\cdot\psi \; .$$

To compute the Faddeev-Popov determinant Δ_{FP}, we need the explicit expression of the symmetry transformations on the fields. Remember however that we are interested only in the linearized expression of the theory; then introducing suitable ghost fields, we have actually to find the free Faddeev-Popov Lagrangian $\mathcal{L}_{FP}$. For this reason one can show that only the general coordinate and Weyl transformations of $h_{\mu\nu}$, chiral transformations of A_μ and supersymmetry transformations of ψ_μ are explicitly needed

$$\begin{cases} \delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + 2\omega\,\delta_{\mu\nu} \\ \delta A_\mu = \partial_\mu \lambda \\ \delta\psi_\mu = \partial_\mu \varepsilon + \gamma_\mu \Theta \end{cases} \qquad \text{(B.7)}$$

If we call ($\bar{C}^\mu, C_\nu$), ($\bar{D}$, D), ($\bar{H}$, H) the anticommuting scalars ghosts associated to the general coordinate, Weyl and chiral transformations and ($\bar{\xi}$, ξ), ($\bar{\eta}$, η) the spinor commuting ghosts of Q- and S-supersymmetry, one can easily show that

$$\mathcal{L}_{FP} = \bar{C}^\mu\Box C_\mu + \bar{C}^\nu\partial_\nu\partial\cdot C + \bar{C}^\nu\partial_\nu D +$$
$$+ 2\bar{D}\,\partial\cdot C + 4\bar{D}D + \bar{H}\Box H + \bar{\xi}\Box\xi +$$
$$+ \bar{\xi}\not{\partial}\xi + \bar{\eta}\not{\partial}\xi + 4\bar{\eta}\eta \; .$$

Then, up to a numerical multiplicative constant, which we omit for simplicity

$$\Delta_{FP} = \int d\Omega(c, D, H)\, d\Omega(\varrho, \eta)\, e^{i\int d^4x\, \mathcal{L}_{FP}} = (\det \Box) \tag{B.8}$$

Collecting all the results one finally finds for the partition function the following expression:

$$Z = \int d\Omega(h_M, A_\mu)\, d\Omega(\bar{\psi}_\mu, \psi_\nu)\, e^{i\int d^4x\, \mathcal{L}_{eff}} \tag{B.9}$$

By introducing the following variables:

$$f_M \equiv (f_{ij}, f, f_\mu) \qquad i, j=1, 2, 3$$

$$F_\mu \equiv (F_i, F_4) \qquad \mu=1, 2, 3, 4$$

with

$$\begin{cases} f_{ij} = 4\left(\tilde{C}_{oioj} + \frac{i}{2}\varepsilon_{jkl}\tilde{C}_{oikl}\right) \\ f = \frac{1}{\sqrt{3}}\Box\, h_{\mu\mu} \\ f_\mu = \sqrt{2}\, i\, \Box^{\frac{1}{2}}\partial_\nu h_{\mu\nu} \end{cases} \tag{B.10a}$$

$$\begin{cases} F_i = F_{io} + \frac{i}{2}\varepsilon_{ijk}F_{jk} \\ F_4 = i\,\partial\cdot A \end{cases} \tag{B.10b}$$

it is easy to see that the bosonic part of $\mathcal{L}_{eff}$ can be written as:

$$\mathcal{L}_{eff}\Big|_{\text{bosonic}} = \frac{1}{2}(f_M)^2 + \frac{1}{2}(F_\mu)^2 \quad .$$

Then:

$$Z = \int d\Omega(f_M, F_\mu)\, J^{-1}\cdot\Delta_{MS}\, e^{i\int d^4x\left\{\frac{1}{2}f_M^2 + \frac{1}{2}F_\mu^2\right\}} \tag{B.11}$$

where

$$J = \det\left(\frac{\delta f_M}{\delta h_M}\right)\cdot\det\left(\frac{\delta F_\mu}{\delta A_\nu}\right) \tag{B.12}$$

is the Jacobian of the transformations (B.10) and Δ_{MS} is the Matthews-Salam determinant obtained performing the functional integral over the gravitino field ψ_μ.

The direct computation of the determinants of eq.(B.12) (in particular of the first one) is very complicated; however, up to multiplicative numerical constants, we can evaluate them by noting that since our gauge fixing conditions have a formally covariant form, the above determinants must have too a formally covariant form. Furthermore the Nicolai transformations (B.10) are linear in the fields and their (first and second) derivatives; then the Jacobian (B.12) must be proportional to some power of det $\Box$. By looking at the explicit form of eq.(B.10), it is not difficult to find that:

$$\begin{aligned} \det\left(\frac{\delta f_M}{\delta h_M}\right) &= (\det\Box)^{10} \\ \det\left(\frac{\delta F_\mu}{\delta A_\nu}\right) &= (\det\Box)^{2} \end{aligned} \tag{B.13}$$

(the last result can be obtained also by a direct computation thus confirming the argument).

On the other hand the Matthews-Salam determinant can be computed with a suitable redefinition of the fields:

$$\Delta_{MS} = \int d\Omega(\bar{\Psi}_\mu, \psi_\nu)\, e^{i\int d^4x\, \mathcal{L}_{eff}|_{fermionic}} =$$

$$= \left[\det(\Box \not{\partial})\right]^2 \int d\Omega(\tilde{\bar{\Psi}}_\mu, \tilde{\Psi}_\nu)\, e^{i\int d^4x\, \tilde{\mathcal{L}}} \tag{B.14}$$

where

$$\tilde{\mathcal{L}} = \frac{1}{2}\tilde{\bar{\Psi}}_\mu \left\{ \delta_{\mu\nu} - \frac{1}{4}\gamma_\nu\gamma_\mu + \frac{1}{2}\frac{\not{\partial}\gamma^\mu\partial_\nu}{\Box} \right\} \psi_\nu .$$

By the same argument used before, the last functional integral in eq.(B.14) is a numerical constant, and thus:

$$\Delta_{MS} = (\det \Box)^{12} .$$

Then eq.(B.10) is really the Nicolai mapping of the theory. Finally note that the extension of these arguments to the full interacting theory is quite difficult since looking at the explicit form of the Lagrangian(166), one discovers the presence of quartic fermion couplings.

APPENDIX C

Probability concepts

In probability theory (*) one deals with a single experiment $\mathcal{E}$. The results of this experiment (the elements) are some well defined objects ζ which form a space S, called sure event. The events are certain subsets A, B, C, ... of S which form a Borel field $\mathfrak{F}$ (**). Two events A and B are mutually exclusive events if they do not have any element in common, i.e. if $A \cap B = O$. To each event A we assign a number $P(A)$, which we call probability of the event A, such that the following axioms are satisfied:

1) $P(A) \geq 0$

2) $P(S) = 1$

3) if $\{A_i\}_{i\in N}$ is a countable collection of mutually exclusive events, i.e. $A_i \cap A_j = O$ for $i \neq j$, then: $P(\bigcup_i A_i) = \sum_i P(A_i)$.

From these axioms it follows that: $P(\phi) = 0$, $P(\bar{A}) = 1 - P(A)$.

(*) Many introductory reviews on probability theory and stochastic processes are available; for example see ref.'s (2, 3) and references there quoted. Here we follow ref.(170).

(**) A field $\mathfrak{F}$ is a non empty class of sets such that:

1) if $A \in \mathfrak{F}$, then $\bar{A} \in \mathfrak{F}$;

2) if $A, B \in \mathfrak{F}$, then $A \cup B \in \mathfrak{F}$.

It follows that also $A \cap B$, $A - B$, ϕ, $S \in \mathfrak{F}$. $\mathfrak{F}$ is a Borel field if it is closed with respect to union and intersection.

If the event M is such that $P(M) \neq 0$, the probability of the event A conditioned by the event M is:

$$P(A|M) = \frac{P(A \cap M)}{P(M)} \tag{C.1}$$

where $P(A \cap M)$ is the joint probability of the two events A and M ($A \cap M$ is the event containing all the elements common to A and M). Two events A and B are called independent if:

$$P(A \cap B) = P(A) \cdot P(B) .$$

Thus, it turns out that an experiment $\mathcal{E}$ is defined by three objects $S, \mathfrak{F}, P$:

$$\mathcal{E} : (S, \mathfrak{F}, P) . \tag{C.2}$$

Let $\mathcal{E}$ be an experiment as in eq.(C.2). A random variable $\underline{x}$ is a law which assigns a real number $\underline{x}(\underline{s})$ to each result $\underline{s}$ of the experiment $\mathcal{E}$. More rigorously a random variable is a function

$$\underline{x} : S \longmapsto \mathbb{R}$$

such that:

1) the set $\{\underline{x} \leqslant x\}$ (i.e. the set of all the results $\underline{s}$ such that $\underline{x}(\underline{s}) \leqslant x$) is an event for every real x (*);

(*) This is equivalent to say that the function $\underline{x}$ is measurable in the field $\mathfrak{F}$.

2) the probability of the two events $\{\underline{x} = +\infty\}$ and $\{\underline{x} = -\infty\}$ is zero:

$$P(\underline{x} = +\infty) = P(\underline{x} = -\infty) = 0 .$$

Given a random variable $\underline{x}$ we can speak of the probability $P(\underline{x} \leqslant x)$ of the event $\{\underline{x} \leqslant x\}$; this is a function of the number x . We call this function,distribution function of the random variable $\underline{x}$ and we write:

$$F(x) = P(\underline{x} \leqslant x) \qquad x \in \mathbb{R} .$$

In short one can say that $F(x)$ is equal to the probability that $\underline{x} \leqslant x$. It is easy to prove that the distribution function F satisfies the following properties:

1) $F(-\infty) = F(+\infty) = 1$;

2) it is a monotonic function: $F(x_1) \leqslant F(x_2)$ for $x_1 < x_2$;

3) it is continuous on the right: $\lim\limits_{\varepsilon \to 0^+} F(x+\varepsilon) = F(x)$.

The derivative of the distribution function

$$f(x) = \frac{dF(x)}{dx} \tag{C.3}$$

is called density function (or probability density) of the random variable $\underline{x}$ (*). Since the derivative of F is not guaranteed to exist, it is necessary to distinguish various kind of random variables.

Let us suppose first that $F(x)$ is continuous and that the points in which it is not derivable form a denumerable set . In this case the

(*) In the mathematical literature F is called the distribution function, whereas in the physical literature the probability density f is often also called the distribution function. We follow here the mathematical conventions.

random variable $\underline{x}$ is called continuous. Then from the properties of $F(x)$, it follows that

$$f(x) \geqslant 0$$

$$\int_{-\infty}^{+\infty} dx\, f(x) = 1 \quad .$$

Moreover, from the definition (C.3) one gets

$$F(x) = \int_{-\infty}^{x} dy\, f(y)$$

and thus the fundamental relation holds

$$P(x_1 \leqslant \underline{x} \leqslant x_2) = \int_{x_1}^{x_2} dy\, f(y)$$

or in infinitesimal form

$$P(x \leqslant \underline{x} \leqslant x + dx) = f(x)\, dx \quad .$$

Finally:

$$P(\underline{x} = x) = 0 \qquad \forall x \in \mathbb{R} \quad .$$

If the distribution $F(x)$ of the random variable $\underline{x}$ jumps of an amount p_i at the discrete points x_i, then $\underline{x}$ is called discrete. In this case

$$P(\underline{x} = x_i) = p_i \quad , \quad \Sigma_i\, p_i = 1 \quad .$$

Introducing a δ-function, one can write for the density function the following expression

$$f(x) = \Sigma_i\, p_i\, \delta(x - x_i)$$

since

$$\left.\frac{dF(x)}{dx}\right|_{x=x_i} = \{F(x_i) - F(x_i - \varepsilon)\}\, \delta(x - x_i) \qquad (\varepsilon > 0)$$

Then by allowing δ-function singularities in the probability density, we can formally treat the discrete case by the same expressions used in the continuous case. In the following we shall limit to the latter.

Applying eq.(C.1), it is easy to see that the conditional distribution $F(x|M)$ of a random variable $\underline{x}$, conditioned by the event M, is defined as the conditional probability of the event $\{\underline{x} \leqslant x\}$

$$F(x|M) = P(\underline{x} \leqslant x | M) = \frac{P(\underline{x} \leqslant x, M)}{P(M)}$$

where $\{\underline{x} \leqslant x, M\}$ is the event consisting of all the results ζ such that

$$\underline{x}(\zeta) \leqslant x \quad \text{and} \quad \zeta \in M \quad .$$

If the event M is expressed in terms of the random variable $\underline{x}$, then $F(x|M)$ can be expressed in terms of $F(x)$.

In probability theory a very important parameter is the expectation value of a random variable $\underline{x}$. It is defined by the integral

$$\eta = E\{\underline{x}\} = \int_{-\infty}^{+\infty} x\, f(x)\, dx \quad .$$

In the same way one can define the conditional expectation value:

$$E\{\underline{x}|M\} = \int_{-\infty}^{+\infty} x\, f(x|M)\, dx \quad .$$

Another very important parameter is the variance or dispersion σ^2 :

$$\sigma^2 = E\{(x - \eta)^2\} = \int_{-\infty}^{+\infty} (x - \eta)^2 f(x)\, dx \quad .$$

One can easely see that:

$$\sigma^2 = E\{\underline{x}^2\} - E^2\{\underline{x}\} \quad .$$

η and σ^2 give only few informations on the density of $\underline{x}$. A more complete specification of the statistics of $\underline{x}$ is possible if one knows its moments m_k defined by:

$$m_k = E\{ \underline{x}^k \} = \int_{-\infty}^{+\infty} x^k f(x)\, dx \quad ;$$

in particular: $m_0 = 1$, $m_1 = \eta$. The constants

$$\mu_k = E\{ (x-\eta)^k \}$$

are called central moments (or cumulants); e.g. $\mu_0 = 1$, $\mu_1 = 0$, $\mu_2 = \sigma^2$.
One can show that the following relation between μ_k and m_k holds

$$\mu_k = \sum_0^k {}_r \binom{k}{r} (-1)^r \eta^r m_{k-r} \quad .$$

The Fourier transform of the density function $f(x)$ of the random variable $\underline{x}$ is called the characteristic function; it is defined by:

$$\phi(\omega) = E\{ e^{i\omega \underline{x}} \} \quad .$$

From $\phi(\omega)$ one can obtain the moments by differentiation:

$$m_k = \frac{1}{i^k} \left. \frac{d^k \phi(\omega)}{d\omega^k} \right|_{\omega=0}$$

(similarly: $\mu_k = \frac{1}{i^k} \left. \frac{d^k \ln \phi(\omega)}{d\omega^k} \right|_{\omega=0}$). Then if one knows all the moments of a random variable $\underline{x}$, one can build up the characteristic function $\phi(\omega)$ and then the density function $f(x)$ by Fourier transform.

All these cosiderations and definitions can be easily extended to the case of a collection of n real random variables: $\underline{x}_1 , \ldots , \underline{x}_n$.
First of all the marginal or joint distribution is defined in the following way:

$$F(x_1, \ldots, x_n) = P(\underline{x}_1 \leq x_1 , \ldots , \underline{x}_n \leq x_n)$$

where $\{ \underline{x}_1 \leq x_1 , \ldots , \underline{x}_n \leq x_n \} = \{ \underline{x}_1 \leq x_1 \} \cap \cdots \cap \{ \underline{x}_n \leq x_n \}$.
The corresponding density function is obtained from F by derivation with respect to all the variables; this function is positive and moreover

$$\int d^n x \;\; f(x_1, \ldots, x_n) = 1 \quad .$$

Note that from above definition substituting some variables with $+\infty$ in $F(x_1, \ldots, x_n)$ we obtain the joint distribution for the remaining variables.For example: $F(x_1, x_3) = F(x_1, +\infty, x_3, +\infty)$. Analogously integrating $f(x_1, \ldots, x_n)$ with respect to some variables, we obtain the joint density in the remaining variables; e.g. $f(x_1, x_3) = \int dx_2\, dx_4\, f(x_1, x_2, x_3, x_4)$. We indicate with

$$f(x_1, \ldots, x_k \,|\, x_{k+1}, \ldots, x_n)$$

the conditional density of $\underline{x}_1, \ldots , \underline{x}_k$ with respect to $\underline{x}_{k+1}, \ldots , \underline{x}_n$; explicitly one has

$$f(x_1, \ldots, x_k \,|\, x_{k+1}, \ldots, x_n) = \frac{f(x_1, \ldots, x_n)}{f(x_{k+1}, \ldots, x_n)} \quad . \qquad \text{(C.4)}$$

Using this relation one easily concludes that:

$$f(x_1, \ldots, x_n) = f(x_n \,|\, x_{n-1}, \ldots, x_1) \cdots f(x_2 \,|\, x_1)\, f(x_1) \quad .$$

All these properties can be summarized in simple rules. Let us call left and right variables the ones respectively on the left and on the right of the vertical bar in $f(x_1, \ldots, x_k \,|\, x_{k+1}, \ldots, x_n)$. Then to eliminate a certain number of left variables, it is sufficient to integrate with respect to these variables. To eliminate some right variables, it is necessary to multiply f by the density of these variables conditioned by the remaining right ones, and then integrate with respect to the former. For example:

$$f(x_1|x_3) = \int f(x_1,x_2|x_3)\,dx_2$$

$$f(x_1|x_4) = \int f(x_1|x_2,x_3,x_4)\,f(x_2,x_3|x_4)\,dx_2\,dx_3 \; .$$

The random variables $\underline{x}_1, \dots, \underline{x}_n$ are independent if the events

$$\{\underline{x}_1 \leq x_1\}, \; \dots, \; \{\underline{x}_n \leq x_n\}$$

are independent for any $x_1, \dots, x_n$. In this case, distribution and density functions factorize. Moreover we can define the (generalized) moments of order $r = \sum_1^n{}_i\, k_i$, as

$$m_{k_1,\dots,k_n} = E\{\underline{x}_1^{k_1}\cdots\underline{x}_n^{k_n}\} =$$

$$= \int d^n x \; f(x_1,\dots,x_n)\, x_1^{k_1}\cdots x_n^{k_n} \; .$$

As in the case of one random variable:

$$m_{k_1,\dots,k_n} = \frac{1}{i^r}\left.\frac{\partial^r \phi(\omega_1,\dots,\omega_n)}{\partial\omega_1^{k_1}\cdots\partial\omega_n^{k_n}}\right|_{\omega=0}$$

where the characteristic function is now:

$$\phi(\omega_1,\dots,\omega_n) = E\{e^{i\sum_1^n{}_k \omega_k \underline{x}_k}\} \; ;$$

if $\underline{x}_1, \dots, \underline{x}_n$ are independent, then: $\phi(\omega_1,\dots,\omega_n) = \phi_1(\omega_1)\cdots\phi_n(\omega_n)$.

We say that the random variables $\underline{x}_1, \dots, \underline{x}_n$ are uncorrelated if the covariance of any two of them is zero:

$$E\{\underline{x}_i\,\underline{x}_j\} = E\{\underline{x}_i\}\,E\{\underline{x}_j\} \quad , \quad \forall\, i \neq j \; ;$$

in this case: $\sigma^2_{\underline{x}_1+\dots+\underline{x}_n} = \sigma^2_{\underline{x}_1} + \dots + \sigma^2_{\underline{x}_n}$. They are orthogonal if:

$$E\{\underline{x}_i\,\underline{x}_j\} = 0 \quad , \quad \forall\, i \neq j \; ;$$

in this case: $E\{(\underline{x}_1 + \dots + \underline{x}_n)^2\} = E\{\underline{x}_1^2\} + \dots + E\{\underline{x}_n^2\}$.

Stochastic processes

Let us consider an experiment $\mathcal{E}(S, \mathcal{F}, P)$ and imagine to assign the following function of the time $\underline{x}(t,\zeta)$ to each result $\zeta \in \mathcal{E}$. We have then a set of functions, one for each ζ. If for any t, $\underline{x}(t)$ is a random variable, then the family of functions define a stochastic process. Following what we have done for a random variable, we can define for $\underline{x}(t)$ the distribution $F(x,t)$, which in general depends on time:

$$F(x,t) = P(\underline{x}(t) \leq x) \; .$$

The corresponding density function can be obtained deriving $F(x,t)$ with respect to x :

$$f(x,t) = \frac{\partial F(x,t)}{\partial x} \; .$$

But given two different times t_1 and t_2, one can consider the random variables $\underline{x}(t_1)$ and $\underline{x}(t_2)$ and then study the corresponding joint distributions. It follows that a stochastic process is statistically detrmined if one knows the n-order distribution function:

$$F(x_1,t_1;\,x_2,t_2;\,\dots;\,x_n,t_n) = P(\underline{x}(t_1)\leq x_1,\,\dots,\,\underline{x}(t_n)\leq x_n) \tag{C.5}$$

for any n and $t_1, \dots, t_n$ (usually one considers only increasing times: $t_1 \geqslant t_2 \geqslant \dots \geqslant t_n$). These functions are not arbitrary; in particular a distribution of a given order is determined by a distribution of higher order.

The n-order density function

$$f(x_1,t_1;\cdots;x_n,t_n) \qquad \text{(C.6)}$$

of the process $\underline{x}(t)$ is obtained by deriving eq. (C.5) with respect to all the x_i's (*). The expectation value and autocorrelation of a stochastic process are defined in the usual way; one has respectively:

$$\eta(t) = E\{\underline{x}(t)\}$$

$$R(t_1,t_2) = E\{\underline{x}(t_1)\,\underline{x}(t_2)\} \quad .$$

Moreover the quantity

$$E\{\underline{x}(t_1)\,\underline{x}(t_2)\cdots\underline{x}(t_k)\}$$

is a moment of order k.

A stochastic process $\underline{x}(t)$ has uncorrelated (orthogonal, independent) increments if

$$\underline{x}(t_i) - \underline{x}(t_{i+1}) \qquad i \in \mathbb{Z}$$

is a sequence of uncorrelated (orthgonal, independent) random variables; the intervals (t_i, t_{i+1}) are disjoint, but otherwise arbitrary (analogous definitions hold in the case of two or more stochastic variables).

Of particular importance are the Gaussian stochastic processes; $\underline{x}(t)$ is normal if the random variables

$$\underline{x}(t_1), \cdots, \underline{x}(t_n)$$

(*) The generalization to the case of more stochastic processes is straightforward.

are normal for any value of n and $t_1, \dots, t_n$ (*). Then for $\underline{x}(t)$ to be normal it is necessary that the density functions are normal to any order.

A very important concept is that of a stationary stochastic process. We shall say that $\underline{x}(t)$ is stationary (in strict sense) if the two processes $\underline{x}(t)$ and $\underline{x}(t+\varepsilon)$ have the same statistics, for any ε. From this definition it follows that the n-order density function of a stationary process must satisfy the property:

$$f(x_1,t_1;\cdots;x_n,t_n) = f(x_1,t_1+\varepsilon;\cdots;x_n,t_n+\varepsilon) \, .$$

In particular the first-order density function is t-independent; as a consequence

$$E\{\underline{x}(t)\} = \eta = \text{constant}$$

Moreover the second-order density is a function of the variable $\tau = t_1 - t_2$:

$$f(x_1,t_1;x_2,t_2) = f(x_1,x_2;\tau) \quad ;$$

then $f(x_1,x_2;\tau)$ is the joint density of the two random variables $\underline{x}(t+\tau)$

(*) A set of n random variables $\underline{x}_1, \dots, \underline{x}_n$ is Gaussian (or normal) if the joint density function has the following form (using a compact vector notation):

$$f(x_1,\dots,x_n) = \frac{(\det A)^{1/2}}{(2\pi)^{n/2}} \, e^{-(x-\eta)^T\cdot A\cdot(x-\eta)}$$

with

$$E\{\underline{x}_i\} = \eta_i$$

$$E\{(\underline{x}_i-\eta_i)(\underline{x}_j-\eta_j)\} = (A^{-1})_{ij} \quad ;$$

A is a positive-definite, symmetric matrix.

and $\underline{x}(t)$. In the same way, also the autocorrelation $R(t_1, t_2)$ is a function of τ :

$$R(\tau) = E\{\underline{x}(t+\tau)\ \underline{x}(t)\} = R(-\tau) \quad .$$

A stochastic process is weakly stationary if its expectation value is a constant and the autocorrelation is a function only of the variable τ . Obviously if a process is weakly stationary is not in general strictly stationary; however a weakly stationary Gaussian process is also strictly stationary (*). Finally we shall say that $\underline{x}(t)$ is a process with stationary increments if the process $\underline{y}(t) = \underline{x}(t+\varepsilon) - \underline{x}(t)$ is stationary for any value of ε .

We can now classify the various stochastic processes using the conditional probability densities (32).

a) Purely random processes

The process $\underline{x}(t)$ is purely random if the n -order conditional density does not depend on the values of the random variable at earlier times:

$$f(x_1, t_1 \mid x_2, t_2 ; \cdots ; x_n, t_n) = f(x_1, t_1) \quad .$$

Then it follows that:

$$f(x_1, t_1 ; \cdots ; x_n, t_n) = f(x_1, t_1)\ f(x_2, t_2 ; \cdots ; x_n, t_n)$$

and thus the n-order joint density factorizes:

$$f(x_1, t_1 ; \cdots ; x_n, t_n) = f(x_1, t_1) \cdots f(x_n, t_n) \quad .$$

(*) An important example of stationary Gaussian process is the so called white noise; it is characterized by having zero mean value and uniform spectral density (the spectral density of a stationary process is the Fourier transform of its autocorrelation):

$$E\{\underline{x}(t)\} = 0 \qquad E\{\underline{x}(t)\ \underline{x}(t')\} = \delta(t-t') \quad .$$

The complete information on the process is then contained in $f(x, t)$.

b) Markov processes

Whereas for a purely random process there is no memory of values of the stochastic variable at any preceding time, for a Markov process there is only a memory of the value of the stochastic variable for the latest time where we measure $\underline{x}(t)$. In other words one can roughly say that $\underline{x}(t)$ is a Markov process if, known the present, the future behaviour of $\underline{x}(t)$ is not influenzed by the past behaviour. In terms of conditional probabilities this means:

$$P\left(\underline{x}(t_n) \leq x_n \mid \underline{x}(t_{n-1}) \cdots \underline{x}(t_1)\right) =$$

$$= P\left(\underline{x}(t_n) \leq x_n \mid \underline{x}(t_{n-1})\right)$$

$$t_1 > t_2 > \cdots > t_n \quad , \quad \forall n$$

or equivalently:

$$P\left(\underline{x}(t_n) \leq x_n \mid \underline{x}(t), \forall t \leq t_{n-1}\right) =$$

$$= P\left(\underline{x}(t_n) \leq x_n \mid \underline{x}(t_{n-1})\right) \quad .$$

Then one concludes that

$$f(x_n, t_n ; \cdots ; x_1, t_1) = f(x_n, t_n \mid x_{n-1}, t_{n-1}) \cdots$$

$$\cdot f(x_2, t_2 \mid x_1, t_1)\ f(x_1, t_1) \quad .$$

Thus for Markov processes the complete information on their statistics is contained in $f(x_1, t_1 ; x_2, t_2)$ (or equivalently $F(x_1, t_1 ; x_2, t_2)$).
Let us indicate with:

$$p(x, t \mid x_0, t_0) = f_{\underline{x}(t)}\left(x \mid \underline{x}(t_0) = x_0\right)$$

$$t \geq t_0$$

the density of $\underline{x}(t)$ under the condition $\underline{x}(t_0)=x_0$. Obviously one has:

$$p(x,t \mid x_0,t_0) \underset{t \to t_0}{\sim} \delta(x-x_0)$$

and

$$\int_{-\infty}^{+\infty} dx \; p(x,t \mid x_0,t_0) = 1 \quad .$$

If $t > t_1 > t_0$, one can prove the following fundamental relation

$$p(x,t \mid x_0,t_0) = \int_{-\infty}^{+\infty} dx_1 \; p(x,t \mid x_1,t_1) \; p(x_1,t_1 \mid x_0,t_0) \quad ;$$

it is called Chapman-Kolmogorov equation. Finally, if p depends only on $t - t_0$, then the process is called homogeneous.

c) General processes

These are the non-Markovian processes, for which the complete information on them is not contained in $f(x_1,t_1 \,;\, x_2,t_2)$; these processes can be described by a collection of more Markov variables.

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