TOPICS ON STATIC AND DYNAMIC PROPERTIES OF FRACTALS

by

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Ringraziamenti

Desidero ringraziare:

Giorgio Parisi, per avermi insegnato a non disprezzare le idee più ardite.

Attilio Stella, per la collaborazione attiva e vivace e per avermi insegnato l'importanza di ben esporre i risultati acquisiti.

Roberto Livi e Stefano Ruffo, per l'amicizia dimostratami e avermi insegnato, il primo ad aver fiducia nella bontà della ricerca effettuata e il secondo ad amare la tolleranza e a guardare a "sinistra".

Mario Tonin, per gli incoraggiamenti continui, per avermi iniziato alla ricerca ed insegnato a guardare a "destra".

Desidero infine ringraziare Salomone per avermi insegnato che *"..... tutto è vanità"* (Qo 1,2).

Part 1

Dynamical Properties of Fractal Strucutre

I. Introduction

Only in the recent years much interest has been devoted to structures, called fractals, with a dilatation simmetry (rather than the more usual ones with traslational invariance). Indeed it is a common convinction that many physical systems in nature are fractals over many length scales[1]: the path of a particle in quantum mechanics[2], branched and linear polymers[3], surfaces of materials[4], clusters at the percolation threshold etc[5].

For a fractal one can define at least two dimensions besides to the dimension, d, of the embedding Euclidean space: the first one is the Hausdorff fractal dimension $\bar{d}$[1] and the second one is the spectral or fracton dimension $\tilde{d}$[6].

The latter dimension was introduced only a few year ago in connection with an anomalous temperature dependence experimentally observed for the ESR spin-lattice relaxation time of iron in some proteins[7].

This result was interpreted in terms of an anomalous vibrational density of states arising from the fractal nature of the proteins.

Interesting applications of this new dimension have been found in percolation problem and anomalous diffusion on fractals[8]. In particular it was conjectured that for percolation clusters $\tilde{d} = 4/3$ in all dimension which seems in good agreement with the numerical estimates at disposal, even if some problem on its possible exactness has been raised[6].

Here we review the definitions of the fractal and spectral dimensions, their connection to the anomalous diffusion on fractal structure and the scaling behaviour of the probability of returning to the origin of a random walk. We give also a simple derivation of the Einstein relation for the electrical resistance exponent (sec. II).

Sec. III contains a rather detailed analysis of the scaling behaviour of a random walk on a Sierpinski gasket. Using an exact renormalization group transformation we verify the scaling relation presented in sec. II and determine the behaviour of the mean number of visited sites and the time spent at the origin by a random walk

during a time t. In sec. IV we solve exactly the long time behaviour of a diffusing particle on a 1-D hierarchical structure with an ultrametric topology. That model has been proposed recently for its importance especially in connection with layered computing arrays. Non-universal scaling behaviour has been conjectured for this type of diffusion, termed ultradiffusion, based on an approximate renormalization group analysis. The exact analysis turns out to give such a non-universal scaling--behaviour described in terms of a line of (non-equivalent) fixed points present in our recursion equations.

Sec. V is a paper where we suggest possible general mechanism in order to explain the anomalous behaviour, in the low frequency limit, of the density of the vibrational states in hemoproteins (see the above mentioned problems which drive to the introduction of $\tilde{d}$).

The main result is exemplified in a deterministic fractal where we consider not only short-range forces, among monomers forming the protein, but also long range ones (provided, e.g., by salt bridges). The result, obtained using an exact renormalization group approach, is a non-universal scaling behaviour and, assuming a thermal activation mechanism across energy barriers, a temperature dependent diffusive exponent.

II. Spectral Dimension

A. Definitions

We shall try now to give a rather formal account of the current ideas about the other important "dimension" which characterizes fractals besides to the well known fractal Hausdorff dimension $\bar{d}$ [1].

For statistical or deterministic fractal (self-similar structure) we mean an infinite set of points having a statistical or deterministic dilation symmetry.

This symmetry manifest itself for example in the definition of the fractal dimension, $\bar{d}$: the number of points (mass), M(R), of the fractal structure within a sphere of radius R goes like

$$M(R) \underset{R\to\infty}{\sim} R^{\bar{d}} \qquad (2.1)$$

If we imagine to attach a each site of the fractal pointlike masses, subjected to harmonic forces among them, then the density of states, $\rho(\omega)$, in the low frequency limit ($\omega \to 0$) is assumed to vary like

$$\rho(\omega) \underset{\omega\to 0}{\sim} \omega^{\tilde{d}-1} \qquad (2.2)$$

where $\tilde{d}$ is called spectral dimension or fracton dimension [6,9,10] (phonons on a fractal become fractons!). When the density of states is a singular function of frequency then (2.2) should be considered as an expression which must be used in the statistical averages. For periodic d-dimensional Euclidean space one has simply $d = \bar{d} = \tilde{d}$ but for a not-traslational invariant fractals in general $\bar{d} \neq \tilde{d}$.

B. Connections between vibrational and diffusion problem

Now we want to derive, in a formal way, the link between classical diffusion in a random lattice and vibrational problem on the same lattice [11].

Let's denote with n ($\in \mathbb{Z}^d$) the lattice sites and with $K_{n,m}$ the transition rate per unit time of hopping from the site n to the site m and it will be assumed that

$K_{m,n} = Kn,m \geq 0$ are random variables distributed according to a given probability $\mathcal{P}\{K\}$. Furthermore let $\mathcal{P}\{K\}$ be such that all averages we get at the end, are traslational invariant. Our classical diffusion problem will satisfy the following master equation:

$$\frac{d}{dt} P_n(t) = \sum_m \left(K_{m,n} P_m(t) - K_{n,m} P_n(t) \right)$$
$$= \sum_m K_{n,m} \left(P_m(t) - P_n(t) \right) \quad (2.3)$$

where $P_n(t)$ is the probability to be at site n at time t.

Defining the matrix operator

$$H_{n,m} = - K_{n,m} \qquad n \neq m$$
$$H_{n,n} = \sum_m K_{n,m} \quad (2.4)^+$$

eq. (2.3) can be rewritten simply as

$$\frac{d}{dt} P(t) = - H P(t) \quad (2.5)$$

In the associate vibrational problem one imagines a pointlike particle with unit mass at each lattice site which can vibrate perpendicularly to the lattice itself. Masses at sites n and m are subjected to an elastic force with constant $K_{n,m}$. The eqs. of motion are then

$$\frac{d^2}{dt^2} X(t) = - H X(t) \quad (2.6)$$

The formal solution of eq. (2.5) is given by

$$P(t) = \exp\{- H(t)\} P(0) \quad (2.7)$$

which implies that the Laplace transform of P(t) is

$$\tilde{P}(\varepsilon) = \int_0^\infty dt\, e^{-\varepsilon t} P(t) = (\varepsilon + H)^{-1} P(0) \quad (2.8)$$

+ From this definition it follows that H has a zero eigenvalue ($\sum_m H_{n,m} = 0$) while the remaining eigenvalues are positive.

If the initial condition is $P_n(t=0) = \delta_{n,0}$ (i.e. the diffusing particle is at site 0 at time o) then for a periodic lattice of N ($\to \infty$) sites:

$$-\frac{1}{\pi} Im \langle \tilde{P}_0(-\varepsilon + i0^+) \rangle = -\frac{1}{\pi} Im \langle (H - \varepsilon + i0^+)^{-1}_{0,0} \rangle$$
$$= N^{-1} tr\, \delta(H - \varepsilon) \equiv \mathcal{N}(\varepsilon) \quad , \qquad (2.9)$$

where $\langle \mathcal{O} \rangle = \int \prod_{n,m} dK_{n,m} \mathcal{P}\{K\} \mathcal{O}$ and we have used the translational invariance after the average while $\mathcal{N}(\varepsilon)$ is the eigenvalue density of states for the diffusion problem. Since the Fourier transform of eq. (2.6) is the same as the Laplace transform of eq. (2.5) substituting ε with ω^2 them the density of states $\rho(\omega)$ for the vibrational problem is

$$\rho(\omega) = 2\omega\, \mathcal{N}(\omega^2) \quad . \qquad (2.10)$$

If at low ε we assume the following asymptotic behaviour

$$\tilde{P}_0(\varepsilon) \sim \varepsilon^{\frac{\tilde{d}}{2} - 1} + \text{less singular terms}^{+} \quad , \qquad (2.11)$$

then at low frequency ω

$$\rho(\omega) \sim \omega^{\tilde{d}-1} \quad . \qquad (2.12)$$

The definition of $\mathcal{N}(\varepsilon)$ implies that the spectral dimension $\tilde{d}$ is an intrinsic parameter of the fractal we are considering and it doesn't depend on the embedding space.

Due to the dilation invariance $\langle P_n(t) \rangle$ should be an homogeneous function of n and t i.e.

$$\langle P_n(t) \rangle = t^{-\frac{\tilde{d}}{2}} f(|n| t^{-\nu}) \quad , \qquad (2.13)$$

where for n=o we used eq. (2.11) and ν is the correlation length exponent. Indeed, from the fact we are working on a fractal with Hausdorff dimension $\bar{d}$, we must

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+ As in the case of d-dimensional lattice[12] we expect that $\tilde{P}_0(\varepsilon=0)$ is a finite constant if $\tilde{d} > 2$ and (2.11) holds for the singular part of $\tilde{P}_0(\varepsilon)$.

have

$$\sum_n f(n) = l^{\bar{d}} \sum_n f(ln) \quad , \qquad (2.14)$$

where l is a rescaling factor. Thus from (2.13) and the normalization condition $\Sigma_n P_n(t)=1$ we get

$$d_w \equiv \frac{1}{\nu} = \frac{2\bar{d}}{\tilde{d}} \quad , \qquad (2.15)$$

and

$$\langle n^2(t) \rangle = \sum_n \langle P_n(t) n^2 \rangle \propto t^{2\nu} \quad , \qquad (2.16)$$

where d_w is the Hausdorff dimension of a random walk on the fractal.

Eq. (2.15) was derived in ref. 6 for the first time and will be of great utility in what follows.

C. Einstein relation

Another interesting relation, besides to (2.15), can be derived which gives the scaling law of the average electrical resistance, R(r), between two point at distance r on the fractal, in terms of $\bar{d}$ and d_w.

We shall suppose there is an electrical resistance $r_{n,m} = 1/K_{n,m}$ between sites n and m and that there is an injected current i at the site r and the same one going out at the origin o. If V is the potential at the site n then the Kirchhoff's law states that there is a net current

$$I_n = \sum_m K_{n,m} (V_m - V_n) \quad , \qquad (2.17)$$

at the site n with $I_n = i(\delta_{o,n} - \delta_{r,n})$. Eq. (2.17) can be rewritten in terms of (2.4) as

$$I = -HV \quad . \qquad (2.18)$$

Due to the fact that V is determined apart from an overall additive constant we can invert (2.18) in the space perpendicular to the zero mode of H giving (let $\bar{H}^{-1}$ be the inverse of H in this space)

$$R(r) = \left\langle \frac{V_r - V_0}{i} \right\rangle = 2 \left\langle \bar{H}^{-1}_{00} - \bar{H}^{-1}_{r0} \right\rangle \quad . \qquad (2.19)$$

However from eq. (2.8) with $P_r(0) = \delta_{r,0}$

$$-2 \left\langle \tilde{P}_r(\varepsilon) - \tilde{P}_0(\varepsilon) \right\rangle = -2 \left\langle (\varepsilon + H)^{-1}_{r,0} - (\varepsilon + H)^{-1}_{0,0} \right\rangle , \qquad (2.20)$$

which in the $\varepsilon \searrow 0$ is the same as the r.h.s. of eq. (2.19) because the zero mode of H has space independent component. Thus we get

$$R(r) = 2 \left(\tilde{P}_0(0^+) - \tilde{P}_r(0^+) \right) , \qquad (2.21)$$

which, using the Laplace transform of (2.13)

$$\tilde{P}_r(\varepsilon) = |r|^{d_w (1 - \frac{\tilde{d}}{2})} f\left(|r| \varepsilon^{1/d_w} \right) \qquad (2.22)$$

implies

$$R(r) \sim |r|^{\zeta} \qquad (2.23)$$

with ($\frac{d_w \tilde{d}}{2} = \bar{d}$ from the scaling law (2.15))

$$\zeta = d_w - \bar{d} \qquad (2.24)$$

if $d_w > \bar{d}$ i.e. $\tilde{d} < 2$ otherwise $\zeta = 0$.

The scaling law (2.24) is known as Einstein relation [8,13].

D. Examples

The simplest examples are provided by self-interacting walk with correlation exponent $\nu = 1/\bar{d}$. If contact points are not considered as branching points then a diffusing particle moves at a distance $N \sim t^{1/2}$ along the chain during the time interval t and at a (Euclidean) distance N^{ν}. According to eq. (2.16) this means

$$d_w = 2 \bar{d} \qquad (2.25)$$

i.e., using eq. (2.15), $\tilde{d} = 1$ as one expects since the

chain is topologically equivalent to a straight line.

Also eqs. (2.23) and (2.24) are obvious in this case because the resistance between two points separated by a distance r will be proportional to the (mean) number of steps between them (= $r^{\tilde{d}}$).

In the following sections we are going to present three examples of deterministic fractals for which we shall compute explicitly, besides to other quantities, d_w, $\tilde{d}$ and ζ and verify the scaling relations (2.15) and (2.24). As we have already said in the introduction the last two examples would pretend to exemplify some interesting physical situation.

III. Spectral Dimension for a Sierpinski Gasket and Renewal Theory

A. The R.G. transformation

The figure 1 shows a sequence of the construction of a triangular Sierpinski gasket. At each step its size increase by a factor 2 while its mass increase by a factor 3 meaning that

$$\bar{d} = \frac{\ln 3}{\ln 2} = 1.58\ldots \qquad (3.1)$$

We are going now to present a real space renormalization group, based on a decimation transformation[9], for the (infinite) set of eq. (2.3) with $K_{n,m} = K$ for $|n-m|=1$ and 0 otherwise. With reference to the fig. 2 the Laplace transform of eq. (2.3) is

$$(\lambda+4)\,\tilde{P}_1(\lambda) = \tilde{p}_3(\lambda) + \tilde{p}_2(\lambda) + \tilde{q}_2(\lambda) + \tilde{q}_3(\lambda) + 1, \qquad (3.2a)$$

$$(\lambda+4)\,\tilde{P}_3(\lambda) = \tilde{P}_1(\lambda) + \tilde{P}_2(\lambda) + \tilde{p}_1(\lambda) + \tilde{p}_2(\lambda), \qquad (3.2b)$$

etc.

where in eq. (3.2a) we used the initial condition $P_m(t=0) = \delta_{m,1}$, $\lambda = \varepsilon/K$ and we have rewritten $K\tilde{P}(\varepsilon)$, $K\tilde{p}(\varepsilon)\ldots$ simply as $\tilde{P}(\lambda)$, $\tilde{p}(\lambda)\ldots$. Eliminating the $\tilde{p}$'s and $\tilde{q}$'s from the set of eqs (3.2) and using that

$$\tilde{p}_2 = \frac{2\tilde{P}_2 + (\lambda+4)(\tilde{P}_1+\tilde{P}_3)}{(\lambda+5)(\lambda+2)} \quad \text{etc.}, \qquad (3.3)$$

we get

$$(\lambda'+4)\,\tilde{P}_1'(\lambda') = \tilde{P}_2'(\lambda') + \tilde{P}_3'(\lambda') + \tilde{Q}_2'(\lambda') + \tilde{Q}_3(\lambda') + 1, \qquad (3.4a)$$

$$(\lambda'+4)\,\tilde{Q}_2'(\lambda') = \tilde{Q}_1'(\lambda') + \tilde{Q}_3'(\lambda') + \tilde{P}_3'(\lambda') + N_1'(\lambda'), \qquad (3.4b)$$

etc.

with

$$\lambda' = \lambda(\lambda+5) \quad , \tag{3.5}$$

$$\tilde{P}'_n(\lambda') = \frac{\lambda+6}{(\lambda+5)(\lambda+2)} \tilde{P}_{2n}(\lambda) \quad . \tag{3.6}$$

B. Critical indices

We see that eqs. (3.4) have the same form of eqs. (3.2) with the substitution λ with λ'. This implies that the solution $\tilde{P}'_n(\lambda')$ of (3.4) satisfies to

$$\tilde{P}'_n(\lambda') = \tilde{P}_n(\lambda') \quad , \tag{3.7}$$

which, together eqs (3.5) and (3.6), gives, as $\lambda \to 0$ ($\xi \to 0$),

$$\tilde{P}_n(5\lambda) = \frac{3}{5} \tilde{P}_{2n}(\lambda) \quad . \tag{3.8}$$

Thus for $\lambda \to 0$, using standard arguments[14] eq. (3.8) give rise to the homogeneous function (2.22) with

$$d_w = \frac{\ln 5}{\ln 2} \quad \text{and} \quad \tilde{d} = \frac{2\ln 3}{\ln 5} \quad . \tag{3.9}$$

It is immediate to obtain the resistance exponent ζ using the same renormalization strategy as above for the set of equations (2.17). Now each bond has a resistance K^{-1} and on a rescaled lattice we get a new K' given by

$$K' = \frac{3}{5} K \quad . \tag{3.10}$$

If we call $R(L,K)$ the resistance between sites at distance L we shall have

$$R(2L,K) = R(L,K') \quad , \tag{3.11}$$

but for dimensional reason $R(L,K) = K^{-1} \bar{R}(L)$, which implies together eq. (3.11) (for $L = 2^n$, $n \in \mathbb{N}$)

$$R(L,K) = K^{-1} L^{\zeta} A \tag{3.12}$$

where A is a suitable constant and

$$\zeta = \frac{\ln 5/3}{\ln 2} \qquad (3.13)$$

which satisfies the Einstein relation (2.24).

We strees however that this result is rather obvious in our framework because eq. (3.8) for $\lambda = 0$ could be seen as the recursion equation for potentials at site n and 2n implying directly (3.12) and (3.13).

C. Existence of fixed points and approach to the scaling region

We would like, now, to discuss a little bit more carefully the recursion equation (3.8) for generic λ .

Let's first remark that due to the recursion equation (3.5) the equations of motion (2.3) don't remain invariant in form after one renormalization group iteration since a second derivative with respect to the time is created, corresponding to take into account memory effects. If one wants to have a consistent renormalization transformation one should start from the beginning with a generic function $f(d/dt)$, such that $f(0) = 0$, instead of simply d/dt in the l.h.s. of eq. (2.3).

Thus now the recursion equation (3.5) reads

$$g'(2^{d_w}\lambda) = g(\lambda)\left(g(\lambda) + 5\right) \,, \qquad (3.14)$$

where $g(\lambda) = f(\varepsilon)/K$, $\varepsilon = \lambda K$ and we have imposed to λ to scale as a Wegner variable[15] in such way a non-trivial fixed point exists.

Looking at the existence of a fixed point solution in the form

$$g^*(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} g_n^* \,, \qquad (3.15)$$

we find $2^{d_w} = 5$ (i.e. eq. (3.9)) and

$$g_n^* = \frac{1}{5^n - 5} \sum_{\substack{n_1, n_2 = 1 \\ n_1 + n_2 = n}}^{n} g_{n_1}^* g_{n_2}^* \frac{n!}{n_1! n_2!} \,, \qquad (3.16)$$

with g_1^* an arbitrary parameter (i.e. g_1 is a marginal

variable, see below). From eq. (3.16) one derives that $|g_2| < |g_1|^2$; assuming, by induction, that $|g_m| < |g_1|^m$ for $m < N$ one gets immediately $|g_N| < |g_1|^N$ i.e. the series (3.15) is absolutely convergent for any λ (with $|g^*(\lambda)| < \exp(\lambda|g_1|) - 1$). Furthermore we find a line of fixed points depending on the value of g_1^* . This comes from the recursion equation for g_1 , i.e. $g_1' = g_1$, indicating that g_1 is a marginal variable. The other eigenvalues of the linearized recursion equation near the fixed point for $g(\lambda) = \sum_{n=1}^{\infty} g_n \lambda^n/n!$ are 5^{-n} $n = 1, 2, \ldots$, i.e. the fixed points (3.16) are stable. In other words indipendently on the initial conditions $\{g_n\}_{n>1}$ we reach rather quickly (and remain forever in) the scaling region, characterized by the critical exponents (3.9), appropriates to the fixed point (3.16) which depends on the starting g_1-value; in that region for any λ we can write

$$\tilde{P}_{\ell n}(\lambda) = A\, \ell^{b(\lambda)}\, \tilde{P}_n(\lambda \ell^{d_w}) \quad , \qquad (3.17)$$

with $\ell = 2^j$, A a suitable constant weakly depending on 1,

$$b(\lambda) = \log_2 \frac{(g^*(\lambda)+5)(g^*(\lambda)+2)}{g^*(\lambda)+6} \quad , \qquad (3.18)$$

($b(0) = d_w(1 - \tilde{d}/2)$) and $j > j_0$ where both j_0 and A depend on the initial conditions. Eq. (3.17) implies the asymptotic behaviour (2.13).

D. Mean number of visited sites and returns to the origin of a random walk

Let's consider a random walk during a time t and look at the mean time, $T_{n_0}(t)$, it spends at the starting point, n_0 . It is easy to show that the Laplace transform $\tilde{T}_{n_0}(\varepsilon)$ of $T_{n_0}(t)$ is given by +

$$\tilde{T}_{n_0}(\varepsilon) = \frac{1}{\varepsilon}\, \tilde{P}_{n_0, n_0}(\varepsilon) \quad , \qquad (3.19)$$

+ Indeed the time spent at the site n is given by $\int_0^t P_{n_0,n}(\tau)\, d\tau$ whose Laplace transform is $\varepsilon^{-1}\, \tilde{P}_{n_0,n}(\varepsilon)$.

where now we write explicitly the initial condition i.e.

$P_{n_0 n}(t)$ = Probability that the diffusing particle is found at the site n at the time t if it was initially at the site n_0 .

From eq. (3.17) for $\varepsilon \to 0$ ($\lambda \to 0$) it follows

$$\tilde{T}_{n_0}(\varepsilon) \sim \varepsilon^{\frac{\tilde{d}}{2}-2} \quad \text{or} \quad T_{n_0}(t) \underset{t\to\infty}{\sim} t^{1-\tilde{d}/2} , \qquad (3.20)$$

as fôr as the site n_0 survives decimation (in other words n_0 appears very late in the iterative construction of the Sierpinski gasket). While if $\tilde{d} > 2$ since $\tilde{P}_{n_0 n_0}(0)$ is a finite constant[+] (see the remark in eq. (2.11)), then

$$\tilde{T}_{n_0}(\varepsilon) \sim \varepsilon^{-1} \quad \text{or} \quad T_{n_0}(t) \underset{t\to\infty}{\sim} \text{const.} \qquad (3.21)$$

Even in the case of the mean number of distinct sites, $S_{n_0}(t)$, visited by a random walk, starting at n_0 , in the time internal t, it is not difficult to prove that

$$S_{n_0}(t) = 1 + \sum_{n\neq n_0} \int_0^t P'_{n_0,n}(\tau)\, d\tau , \qquad (3.22)$$

where $P'_{n_0 n}(\tau)d\tau$ is the probability of arriving at n for the first time between τ and $\tau + d\tau$. Since

$$P_{n_0 n}(t) = \int_0^t P'_{n_0 n}(\tau)\, P_{nn}(t-\tau)\, d\tau \quad n \neq n_0 \qquad (3.23)$$

we get immediately that the Laplace transform of $S_{n_0}(t)$ is

$$\tilde{S}_{n_0}(\varepsilon) = \varepsilon^{-1} + \varepsilon^{-1} \sum_{n\neq n_0} \tilde{P}'_{n_0 n}(\varepsilon)$$

$$= \sum_n \varepsilon^{-1}\, \tilde{P}_{n_0 n}(\varepsilon) \,/\, \tilde{P}_{nn}(\varepsilon) \quad . \qquad (3.24)$$

From eq. (3.17) and the property (2.14) it follows for, finite ε ,

$$\tilde{S}_0(\ell^{d_w}\varepsilon) \underset{\ell\gg 1}{\sim} \ell^{-\bar{d}-d_w}\, \tilde{S}_0(\varepsilon) , \qquad (3.25)$$

+ In that case eq. (3.17) holds for the singular part of $\tilde{P}_{n_0 n_0}(\varepsilon)$.

as for as $\tilde{d}<2$, which implies

$$\tilde{S}_0(\varepsilon) \underset{\varepsilon\to 0}{\sim} \varepsilon^{-1-\frac{\tilde{d}}{2}} \quad \text{or} \quad S_0(t) \underset{t\to\infty}{\sim} t^{\tilde{d}/2} \quad . \quad (3.26)$$

While for $\tilde{d}>2$ if $\tilde{P}_{nn}(0)$ has a weak dependence on n, from the normalization condition, $\sum_n \tilde{P}_{no}(\varepsilon) = \varepsilon^{-1}$,

$$\tilde{S}_0(\varepsilon) \underset{\varepsilon\to 0}{\sim} \varepsilon^{-2} \quad \text{or} \quad S_0(t) \underset{t\to\infty}{\sim} t \quad (3.27)$$

The asymptotic behaviours established in eqs (3.20), (3.26) and the one which follows directly from (3.17), $\tilde{P}_{n_o n_o}(t) \underset{t\to\infty}{\sim} t^{-\tilde{d}/2}$ +, were checked by a Monte Carlo simulation on a Sierpinski gasket [16].

E. Conclusions

In conclusion we used an exact renormalization transformation in a Sierpinski gasket which was introduced in ref. 9 for calculating d_w and, on the basis of the conjectured scaling relation (2.15), the spectral dimension $\tilde{d}$. Here we have shown, taking into account "wave function renormalization" (3.6), that, in the same framework, all the exponents defined in sec. II can be determined and are seen to satisfy the predicted scaling relations. Furthermore the existence of the fixed point for the recursion equation (3.14) allowed us to establish the asymptotic behaviours (2.13) for the probability distribution, the eq. (3.20) for the "number" of returns at the origin and eq. (3.26) for the number of distinct visited sites of a diffusing particle in the triangular Sierpinski gasket.

+ They were conjecured in ref. 9. Here they have been proved for the first time.

IV. "Ultra"-diffusion on a One Dimensional Hierarchical Structure[17]

A. Introduction

Recently diffusion on the 1-D hierarchical structure, shown in fig. 3, has been studied both by numerical simulations[18] and by approximate renormalization group techniques[18].

A particle can hop from a cell to another one on a line where energy barriers are distributed in a hierarchical way. The probability rate to cross the i-th barrier will be denoted ε_i (high barrier means low ε)+.

Assuming a thermal activation picture, i.e. $\varepsilon_{i+1}/\varepsilon_i = \exp\{-\text{constant}/T\}$, it was shown (in the same approximation) that the probability for a diffusing particle of returning to the origin after a time t has an anomalous power law decay with a temperature dependent exponent.

This feature together the ultrametric property presented by the hierarchical structure (see below) led the authors of ref 18 to call ultradiffusion the diffusion process.

On hierarchical structures, in general, it can be introduced an ultrametric topology[19] defined by a (ultra) metric d such that whevever $d(p,q) = d(q,r)$ then $d(p,q)$, $d(q,r) \geq d(p,r)$.

For the 1-D structure of fig. 3 an ultrametric can be introduced if the distance between two cell is defined as the highest barrier between them[18].

The interest in ultradiffusion comes mainly from the possibility to observe it in layered computing arrays[18,20]. These structure seem to have the interesting property to be highly free from errors. The ultrametric distance between two inputs is defined as the number of layers that are affected when the inputs are interchanged.

As the authors of ref. 18 noticed the interest in this type of problems should be of very general charac-

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+ Notice that now we have not a fractal set ($\bar{d} = 1$).

ter since physical systems, both natural and artificial, presenting hierarchical organization are rather common: molecular diffusion on complex macromolecules[21], macromolecule motions[22], particular types of 1-D superionic conductors[23], spin glass[24,25] and evolutionary processes[26].

B. Exact renormalization transformation

The master equation for the probability distribution is (see fig. 3)

$$dP_2/dt = \varepsilon_0 (P_3 - P_2) + \varepsilon_1 (P_1 - P_2) ,$$
$$dP_1/dt = \varepsilon_0 (P_0 - P_1) + \varepsilon_1 (P_2 - P_1) ,$$
$$dP_0/dt = \varepsilon_0 (P_1 - P_0) + \varepsilon_3 (P_{-1} - P_0), \ldots , \quad (4.1)$$

or using the Laplace transform, $\tilde{P}_m(\omega, \{\varepsilon\})$ of $P_m(t, \{\varepsilon\})$ with $\lambda = \omega/\varepsilon_0$, $\alpha_i = \varepsilon_i/\varepsilon_0$ and writing $\tilde{P}_m(\lambda)$ instead of $\varepsilon_0 \tilde{P}_m(\omega, \{\varepsilon\})$

$$\lambda \tilde{P}_2 = \tilde{P}_3 - \tilde{P}_2 + \alpha_1 (\tilde{P}_1 - \tilde{P}_2) ,$$
$$\lambda \tilde{P}_1 = \tilde{P}_0 - \tilde{P}_1 + \alpha_1 (\tilde{P}_2 - \tilde{P}_1) ,$$
$$\lambda \tilde{P}_0 = \tilde{P}_1 - \tilde{P}_0 + \alpha_3 (\tilde{P}_{-1} - \tilde{P}_0) + 1 , \quad (4.2)$$

where we used the initial condition $P_m(t=0) = \delta_{m,0}$. The renormalization group transformation, we shall get using a standard decimation, should result in changing the form of the equations (4.2) since the hierarchical structure itself should be modified.

To avoid this untractable problem we perform a new type of decimation transformation[17] where the decimated cells are indicated by a cross in the figure. This means that we must eliminate, for example, $\tilde{P}_1$ and $\tilde{P}_2$ from eqs. (4.2) i.e.

$$\tilde{P}_2 = \frac{(\lambda+1+\alpha_1)\,\tilde{P}_3 + \alpha_1\,\tilde{P}_0}{(\lambda+1+\alpha_1)^2 - \alpha_1^2} \quad ,$$

$$\tilde{P}_1 = \frac{(\lambda+1+\alpha_1)\,\tilde{P}_0 + \alpha_1\,\tilde{P}_3}{(\lambda+1+\alpha_1)^2 - \alpha_1^2} \quad , \qquad (4.3)$$

and rewrite the eqs. for $\tilde{P}_0$, $\tilde{P}_{-1}$, $\tilde{P}_3$, $\tilde{P}_{-4}$ etc. After some simple algebra we get

$$\lambda' \tilde{P}_0'(\lambda') = \tilde{P}_1'(\lambda') - \tilde{P}_0'(\lambda') + \alpha_2' \left(\tilde{P}_{-1}'(\lambda') - \tilde{P}_0'(\lambda') \right) + 1 \quad ,$$

etc. $\qquad (4.4)$

with

$$\lambda' = 2\,\frac{1+2\alpha_1}{\alpha_1}\,\lambda + \frac{3+2\alpha_1}{\alpha_1}\,\lambda^2 + \frac{\lambda^3}{\alpha_1} \quad , \qquad (4.5)$$

$$\alpha_n' = \frac{(\lambda+1)^2 + 2\alpha_1(\lambda+1)}{\alpha_1}\,\alpha_{n+1} \qquad (4.6)$$

and, displaying the dependence on α's,

$$P_n'(\lambda', \{\alpha'\}) = \frac{\alpha_1}{(\lambda+1)^2 + 2\alpha_1(\lambda+1)}\,P_{m(n)}(\lambda, \{\alpha\}) \qquad (4.7)$$

where $m(n)$ ($= 2n+1$ if n is odd and $m(n) = 2n$ if n is even) is a simple relabelling of sites in order to make the new set of equations of the same form as the original one. This implies (compare previous section) that the solution of (4.4) satisfies to

$$P_n'(\lambda', \{\alpha'\}) = P_n(\lambda', \{\alpha'\}) \quad , \qquad (4.8)$$

and from eq. (4.7)

$$P_n(\lambda', \{\alpha'\}) = \frac{\alpha_1}{(\lambda+1)^2 + 2\alpha_1(\lambda+1)}\,P_{m(n)}(\lambda, \{\alpha\}) \qquad (4.9)$$

Near $\lambda = 0$ the recursion equations (4.5-7) becomes

$$\lambda' = 2\,\frac{1+2\alpha_1}{\alpha_1}\,\lambda \quad , \qquad (4.10a)$$

$$\alpha_n' = \frac{1+2\alpha_1}{\alpha_1}\,\alpha_{n+1} \quad , \qquad (4.10b)$$

$$P_{2n}(\lambda', \{\alpha'\}) = \frac{\alpha_1}{1+2\alpha_1} P_{4n}(\lambda, \{\alpha\}) , \qquad (4.10c)$$

$$P_{2n-1}(\lambda', \{\alpha'\}) = \frac{\alpha_1}{1+2\alpha_1} P_{4n-1}(\lambda, \{\alpha\}) , \qquad (4.10d)$$

where we have explicited the site dependence of P'_S. At the fixed points of (4.10b),

$$\alpha_n^* = \alpha_1^* \left(\frac{\alpha_1}{1+2\alpha_1} \right)^{n-1} \qquad (4.11)$$

eqs. (4.10c-d) give rise to the homogeneous function (2.22) with

$$d_w = \ln \frac{2(1+2\alpha_1^*)}{\alpha_1^*} / \ln 2 , \qquad (4.12)$$

$$\tilde{d} = 2 \ln 2 / \ln \frac{2(1+2\alpha_1^*)}{\alpha_1^*} , \qquad (4.13)$$

which satisfy the relation $\tilde{d} = 2\bar{d}/d_w$ (see sec. II), where $\bar{d} = 1$ in this case.

The important feature here is the existence of a whole line of fixed points (4.11) obtained as α_1^* varies in the real positive axis and the implied dependence on α_1^* of the critical exponents (4.12) and (4.13).

At the fixed point (4.11) one has $\alpha_{n+1}^*/\alpha_n^* = \varepsilon_{n+1}^*/\varepsilon_n^* = \alpha_1^*/(1+2\alpha_1^*)$ which is R in the notation of ref. 18 while for large t $P_0(t) \sim t^{-\tilde{d}/2}$ with $\tilde{d}/2 = (1 - \ln R/\ln 2)^{-1}$. Rather surprisingly this exact result has been obtained also in the approximate renormalization group approach of Huberman and Kerszberg[18] and it should explain why their result (they supposed approximate) compares well with numerical simulations.

If a thermal activation picture is assumed, $R \sim \exp\{-C/T\}$ then $\tilde{d}/2 = T/(T + \text{const})$ which goes to zero as the temperature decreases resulting in an anomalous low-frequency noise spectrum.

Let's remark finally that the simple 1-D diffusion is obtained in the $\alpha_i \to \infty$ limit ($d_w = 2$, $\tilde{d} = 1$) i.e. 1-D structure with energy barrier of equal high ε_0.

C. Conclusions

The fixed point analysis of the recursion equations (4.5) and (4.6) is now more complicated than the one of sec. III. Indeed, here, besides to start with a master equation where d/dt is replaced by a more general function of d/dt we should generalize it in order to include terms of the type

$$\int_0^t \alpha(t-\tau)\, P(\tau)\, d\tau \qquad (4.14)$$

due to the λ-dependence of the renormalized α's in eq. (4.6)[27]. However the leading behaviour for large t is dictated by eqs. (4.10a-d) and it is described by the exponents (4.12) and (4.13).

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27. A detailed analysis is in progress.

FIGURE CAPTIONS

Figure 1.
Iterative process involved in the construction of the triangular Sierpinski gasket.

Figure 2.
(a) Part of a triangular Sierpinski gasket.
(b) The same part of the Sierpinski gasket after dynamic decimation.

Figure 3.
(a) 1-dimensional hierarchical structure where "ultra"-diffusion occurs.
(b) The same structure as in (a) after dynamical decimation.

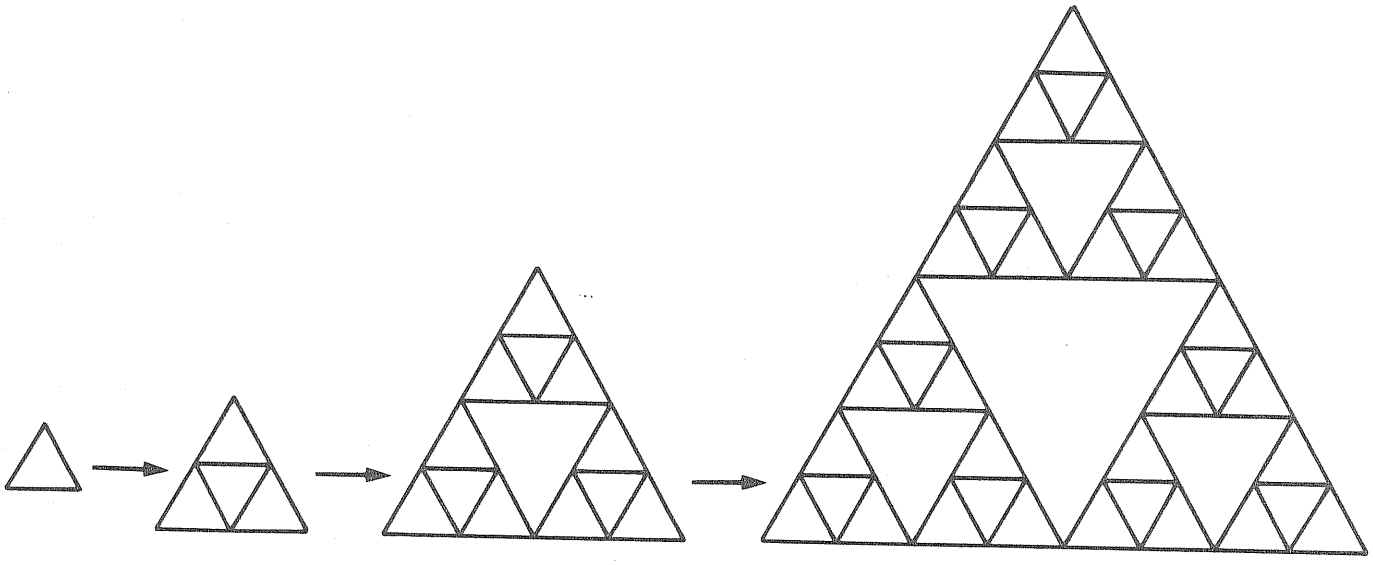


fig. 1

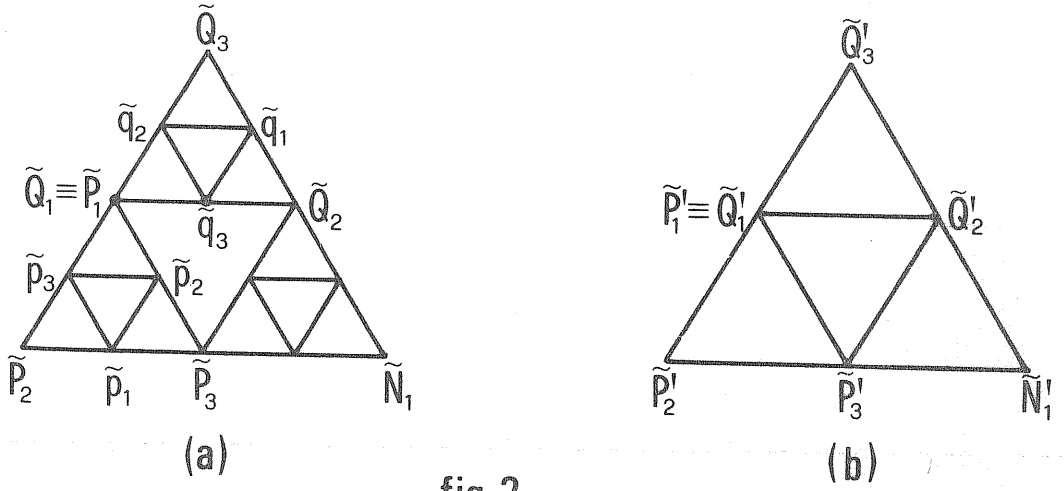
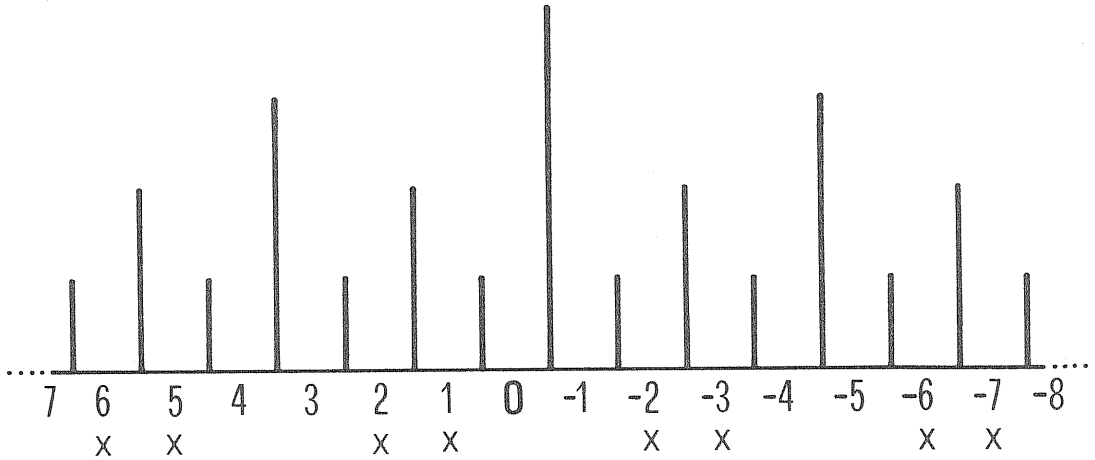
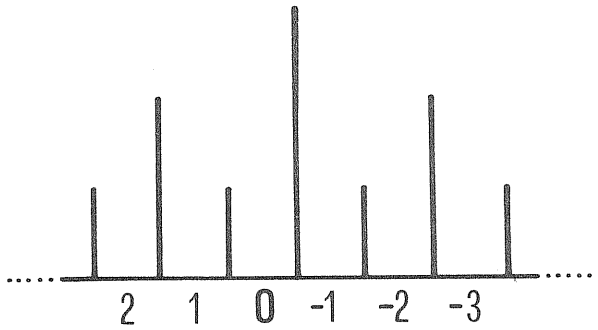

$\tilde{Q}_3$
$\tilde{q}_2$
$\tilde{q}_1$
$\tilde{Q}_1 \equiv \tilde{P}_1$
$\tilde{q}_3$
$\tilde{Q}_2$
$\tilde{p}_3$
$\tilde{p}_2$
$\tilde{P}_2$
$\tilde{p}_1$
$\tilde{P}_3$
$\tilde{N}_1$
(a)
$\tilde{Q}'_3$
$\tilde{P}'_1 \equiv \tilde{Q}'_1$
$\tilde{Q}'_2$
$\tilde{P}'_2$
$\tilde{P}'_3$
$\tilde{N}'_1$
(b)

fig. 2



(a)



(b)

fig. 3

V. # SPECTRAL DIMENSION OF A FRACTAL STRUCTURE WITH LONG-RANGE INTERACTIONS.

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DFPD 17/85 June 1985

PACS numbers 87.15. By, 63.50+X, 76.30.-V, 05.40.+j.

ABSTRACT

An exact renormalization method is applied to a deterministic self--similar structure with long range interactions. Non-universal scaling behaviour is obtained for the spectral properties and an exact result in renewal theory is derived. The model exemplifies possible general mechanisms of explanation of recent experimental findings about the density of vibrational states in hemoproteins.

A similar mathematical framework also allows to analyze exactly recently proposed models for ultradiffusion in hierarchical structures.

3.

The harmonic analysis of structures with a dilation symmetry, or fractals [1], as well as the related diffusion problems, are of much interest in connection with several issues, ranging from electrical conduction in linear polymers [2], to anomalous temperature dependence of ESR spin lattice relaxation times of iron in some proteins [3].

As a consequence of self-similarity, in a fractal the density of vibrational states, $\rho(\omega)$, at small frequency, ω, scales like $\omega^{\tilde{d}-1}$. $\tilde{d}$ is the spectral dimensionality, which differs both from d, the dimension of the embedding Euclidean space, and from $\bar{d}$, the "geometrical" fractal dimension of the structure [4,5].

A simple relation, $\tilde{d}=2\bar{d}/d_w$, connects $\tilde{d}$ with d_w, the dimension associated with a random walk on the fractal. Moreover it has been conjectured that $\tilde{d}$ directly determines the scaling behaviour of various relevant random walk properties, like the range, or the probability, $P_o(t)$, of return to the original site, after time t. The latter is expected to behave as $P_o(t) \sim t^{-\tilde{d}/2}$ for $t \to \infty$ [4,5].

Up to now, spectral properties have been studied both on random fractals, like percolation clusters [6], and on deterministic fractals, like Sierpinski gaskets [5].

In all cases only short-range (harmonic) forces, or hopping probabilities, were considered.

In this article we present and analyze by an exact renormalization method a fractal model allowing for long-range interactions.

Besides filling an obvious gap in the literature on the field, the present investigation was directly motivated by a problematic developed recently in connection with experimental determinations of the fracton density of states in hemo and other proteins in solution [3].

To introduce our model, let us consider a triadic Koch curve in the plane, like the one sketched in fig. 1. Each point on the curve is free

to vibrate perpendicular to the plane and is connected by spring forces (elastic constant K, continuous bonds) to its nearest neighbours. In addition to these, there are other elastic forces of longer range (dotted bonds, constants, K_1, K_2,...), acting according to the self-similar scheme indicated in the figure.

Such a structure has clearly $\bar{d}=\ln 4/\ln 3$, since the number of points increases by a factor 4, whenever the linear size is multiplied by 3.

Putting $\lambda = m\omega^2/K$ and $\alpha_i = K_i/K$, m being the mass of the points, we get the following set of equations

$$\lambda X_i = \sum_j \beta_j(i)\,(X_i - X_j) \qquad (1)$$

for the displacements X_i, at the various sites, appropriate to a vibrational mode $\{X e^{i\omega t}\}$. The sum over j extends to all sites interacting directly with site i, and β_j is equal to 1, or α_m, according to whether the point j is nearest neighbour of i, or is connected to it by a coupling K_m, respectively. The low frequency scaling properties of the above eigenmodes can be studied by a relatively simple renormalization group procedure. With reference to Fig. 1, we eliminate from the system (1) all variables at the vertices of the elementary triangles having the coupling K_1 associated with one of their sides. With this dynamical decimation the system is spatially rescaled by a factor $\ell=3$. The displacements at the "surviving" points (which form a structure of the same type as the original one, after a proper rescaling), can be seen to obey a system of the form (1), with new, effective reduced square frequency, λ', and couplings, $\{\alpha'\}$.

To exemplify this, let us explicitly consider the elimination of x_1, x_2, x_3, x_5, x_6 and x_7 in eqs. (1) (see fig. 1). After some algebra

one gets for $Y=x_4$ an equation of the form:

$$A\,Y = B(W+X) + \alpha_2 (Y-Z) \qquad (2)$$

with A and B suitable functions of λ and α_1. Thus, by rescaling the undecimated displacements by a factor B, eq. (2) becomes of the same form as the one for x_1 or x_5 in (1), if one redefines the frequency and coupling, according to $\lambda' - 2 = A/B$, $\alpha'_1 = \alpha_2/B$.

By inspection one realizes that similar results hold also for the other equations satisfied by the undecimated displacements, the general rule for the effective couplings being $\alpha'_n = \alpha_{n+1}/B$.

The above relations take on simple form in the $\omega \to 0$ $(\lambda \to 0)$ limit

$$\lambda' = 16\,\frac{1+\alpha_1}{1+2\alpha_1}\,\lambda \qquad (3)$$

$$\alpha'_n = 4\,\frac{1+\alpha_1}{1+2\alpha_1}\,\alpha_{n+1} \qquad (4)$$

and allow us to discuss the scaling of the eigenfrequencies according to a basic relation [5], which, for our model, becomes

$$\lambda(\{\alpha'\}) = l^{2a}\,\lambda(\{\alpha\}) \qquad (5)$$

in the limit of an infinite system. Thus the exponent a, which yields $\tilde{d}$ $(\tilde{d} = \bar{d}/a)$, can be determined by discussing the fixed points $\{\alpha^*\}$ of (4). The interesting feature of (4) is that it shows a whole line of physically acceptable fixed points, characterized by the parameter α_1^* ranging from 0 to $+\infty$. Indeed it turns out that

$$\alpha_n^* = \alpha_1^* \left[\frac{1+2\alpha_1^*}{4(1+\alpha_1^*)}\right]^{n-1}, \quad n = 1, 2, \ldots \qquad (6)$$

is a fixed point of eq. (4). The corresponding a exponent follows from (3), and of course depends on α_1^*. Taking into account that $\ell=3$, we get

$$a = \frac{1}{2\ln 3} \ln \frac{1+\alpha_1^*}{1+2\alpha_1^*} + \bar{d} \qquad (7)$$

As α_1^* varies from 0 to $+\infty$, $\tilde{d} = \bar{d}/a$ varies from 1 to 4/3.

The case $\alpha_i^* = 0$, i=1,2,3,... $\tilde{d}=1$, can be seen to attract all initial $\{\alpha\}$'s, for which $\alpha_i = 0$, for i greater than same i_o. This means that all situations in which further neighbour bridges are not extending to all scales, are finally mapped into the dynamics of a nearest neighbour model, with the consequent result $\tilde{d}=1$. Of course, the cross-over to this situation will be the slower, the higher is the (finite) range of the bridges.

One can easily verify that initial interaction patterns $\{\alpha\}$, such that $\lim_{n\to\infty} \alpha_{n+1}/\alpha_n = C$, $(1/4 \le C < 1/2)$ are all attracted by the fixed point (6) with $\alpha_1^* = (1-4C)/(4C-2)$. These $\{\alpha\}$'s, due to the fractal geometry, are long-range interactions with power-law decay at large distance R, i.e. $\alpha(R) \sim R^{-\ln C/\ln 3}$, for $R\to\infty$. Contrary to the short range case, long-range interactions radically change the spectral properties and lead to non-universal dynamical critical behaviour. It is interesting to notice that in the present long-range model the inequality $\tilde{d} \le \bar{d}$, which was expected to hold generally [5,7], is not always satisfied.

With reference to the problem of random walk diffusion on our structure, it is worth remarking, that the above renormalization approach actually allows a direct check of the scaling of $P_o(t)$, the quantity associated with renewal theory on the fractal. When dealing with diffusion, a system like (1) is satisfied by the Laplace transforms $\tilde{P}_i(\omega)$ (replacing x_i) of

the probability that the walker is at site i at time t, after starting at site 0 at t=0. The only difference is that the equation for $\tilde{P}_o(\omega)$ has a 1 added on the right hand side (initial condition). In this case λ stays for ω , and the various α_i's are hopping rates. The same renormalization procedure outlined above applies in this case, and, if site 0 survives decimation, the following relation

$$\tilde{P}_o(l^{2a}\omega, \{\alpha'\}) = \frac{2\alpha_1 + 1}{4(\alpha_1 + 1)} \tilde{P}_o(\omega, \{\alpha\}) \qquad (8)$$

is obtained, for $\omega \to 0$, a being given by eq. (7), with α_1 in place of α_1^*. The fixed point analysis above leads to conclude that indeed $\tilde{P}_o(\omega) \sim \omega^{\tilde{d}/2 - 1}$, with $\tilde{d}$ as obtained above. This is in agreement with a general conjecture by Rammal and Toulouse [5], which was tested by simulation on Sierpinski gaskets [8]. We notice that a formula analogous to (8), and thus a direct analytic test of the scaling behaviour of P_o, can be obtained in the case treated numerically by the authors of ref. 8.

The model presented here, even if rather simple., may bear some light on the possible mechanisms leading to the relatively high values of $\tilde{d}$ ($\tilde{d}$=1.3÷1.7) measured in some hemoproteins and ferrodoxin [3]. It has been suggested that such values of $\tilde{d}$ should be explained on the basis of cross-linking bonds (e.g. H-bridges) among different segments of the folded chain backbone of the protein [9]. While there is general agreement on the importance of such bonds, there is considerable controversy about the specific mechanism by which they could affect $\tilde{d}$ [10,11,12] . In particular recent numerical simulations of diffusion on self-avoiding chains seem to indicate that $\tilde{d}$ stays equal to 1, the value without cross-linking bridges, if these are assumed to be of short range [13,14] . In the model presented here, the Koch triadic could very schematically represent

the backbone of a protein. The elastic couplings K_i could simulate cross-links among different parts of the backbone. The behaviour discussed above suggests that a definite deviation of $\tilde{d}$ from 1, strictly speaking, can be produced only by bridges of infinite range. Forces of long range, namely the salt bridges, are not absent in the proteins . Probably, on the scales actually tested by the experiment, the observed $\tilde{d}$ is just an indication of some crossover of the same type, as the one taking place in the model for finite range $\{\alpha\}$'s.

The non universal scaling behaviours found above can also be seen as a consequence of the infinite hierarchy of time scales present in the model. Considering diffusion, and assuming a thermal activation mechanism across energy barriers, we can think of C as a function of temperature, e.g. $C \sim \exp(-\text{const}/T)$. The above results thus lead to a temperature dependence of the diffusive exponents.

Behaviours of the same type have been recently studied with approximate methods by Huberman and Kerszberg [15] on a particular (non-fractal) model of diffusion with a hierarchy of energy barrier scales, and are expected to be relevant for a variety of physical situations, ranging from molecular diffusion on complex macromolecules [16], to spin glass systems [17,18], or computing structures [19]. It is worth remarking that the methods of the present article can be properly adapted to solve the model of ref. 15 providing an exact confirmation of the approximate and numerical predictions produced there [20].

Concerning the specific model presented above, a problem which will be interesting to undertake in the future is the explicit construction of the spectrum and eigenstates, following e.g. recursive methods of the type applied by Domany et al. [21] to the Schrödinger equation on Sierpinski gaskets.

ACKNOWLEDGEMENT. We thank professor M.W. Cole for a critical reading of the manuscript.

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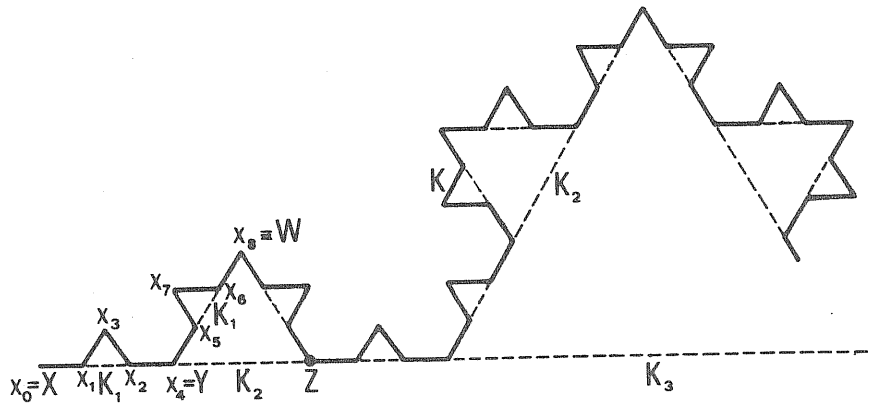
FIGURE CAPTION

Fig. 1 - Sketch of the fractal structure allowing for interactions at all length scales. The continuous curve represents the backbone.


K
K2
X8≡W
X7
X6
K1
X5
X3
X0=X
X1
K1
X2
X4=Y
K2
Z
K3

PART 2

STATISTICAL MECHANICS OF SELF-AVOIDING SURFACES

I. Introduction

Self-avoiding surfaces (SAS) are an interesting problem in lattice statistics both from a theoretical and experimental point of view [1,2].

Self-avoiding walk (SAW) properties are known from long time [3] and the link with spin systems has been extensively studied [4,5] especially in view of the relevance of SAW for polymer physics [6,7,8]. SAS's are an obvious generalization of SAW's and they are expected to play an equally important rôle.

In the strong coupling expansions of lattice gauge theories random surfaces appear in a natural way [9,10] like random walks for standard magnetic systems. However while SAW's or, in general, self interacting walks are obtained as particular limits(number of components of the spin variable going to zero) of a class of magnetic systems [4,5,7], SAS's are not obtained from some limit of the standard lattice gauge theories [11].We shall show that a new class of lattice gauge theories, with n-component fields, in the $n \to 0$ limit give rises to SAS models [12].This should be the first step toward a field theoretical approach to self-interacting surfaces.

In the literature a rather restricted subset of SAS's has been already studied in many context. This is known as solid-on-solid model (SOS) [13]. Besides to be a natural generalization of SOS, SAS's, or self-interacting surfaces in general, should be considered as more realistic models for describing surfaces of materials and interfaces than the SOS does. There is already an experimental evidence that several materials present surfaces with irregularities with a high degree of self-similarity over many scales in the molecular range [14]. This is evidentiated looking at how many molecules, of fixed cross section, are necessary for getting a monolayer coverage of a given surface. Varying the cross section of the adsorbate molecules it was obtained a rather clear experimental signal for non-trivial fractal Hausdorff dimensions, ranging from 2 to 3, for most analyzed materials [14].

Like SAW models linear polymer in a good solvent [8] one could hope that SAS's describe equilibrium properties of a single "sheet" polymer with exclude volume effects [15]. At present, to our knowledge, conformational properties

of any real sheet polymer have not yet been measured; perhaps good candidates for such experimental study could be found among β-sheet, proteins or other organic macromolecules.

Recently much attention has been devoted in the literature to models of planar randon surfaces (PRS) [16,17,18,19]. The interest in those models, even if originally rised from particle physics [16,20], is, at least nowadays, only theoretical (string theories, 2-dimensional quantum gravity, etc.) [21].

However up to now only a few models of PRS seem to display a non-trivial critical behaviour [19]. The relevance of PRS models to the before mentioned physics should become clear as far as the rôle of self-interactions of a PRS is understood.

A short summary of the contents of this work is as follow. In sec. II the basic definitions of SAS models are given together the expected scaling behaviour. It is presented a lattice gauge model with n-component field variables, that in the $n \to 0$ limit gives the statistical mechanics of SAS.

Sec. III contains lattice gauge models which generalize the ones introduced in the previous section in order to weight surface with handles and to describe the diffusion on a SAS.

In Sec. IV SAS models are analyzed in the $d \to \infty$ limit and a Flory type argument is developed which gives the correlation length ν-exponent below the upper critical dimension. A possible pattern of hyperscaling violation is considered which gives the entropic critical index in terms of the correlation length exponent and of the euclidean dimensionality.

A real space renormalization group for SAS is proposed in Sec. V giving approximate recursion equations whose results are compared with the "Flory" ones.

A first order ϵ-expansion around the lower critical dimension is also obtained for the ν -exponent.

A non trivial SAS model on a fractal strucutre is proposed and exactly solved in Sec. VI.

Sec. VII contains conclusions and some speculations on the spectral dimension of SAS.

II. SAS and connection with a system of n-component spins.

A. Definitions and asymptotic behaviours.

We consider a d-dimensional lattice of N sites $X = (X_1, \ldots, X_d)$, which is periodic in each direction $\mu = 1, \ldots, d$. A SAS [1,2], S, is collection of $|S|$ elementary plaquettes, p, which is connected and such that every link of the lattice belongs to, at most, two plaquettes. The boundary ∂S of S is formed by the links belonging to one plaquette of S. The surfaces we consider may be orientable or not and to have handles or not.

Instead of working with all surfaces with fixed area (=number of plaquettes) it is convenient to formulate the problem in a grand-canonical context where a fugacity K is associated to each plaquette.

Typical correlations function of interest are

$$G(\gamma_1, \ldots, \gamma_\ell) = \sum_{S : \partial S = \gamma_1 \cup \ldots \cup \gamma_\ell} K^{|S|} , \qquad (2.1)$$

where $\{\gamma_i\}$ is a set of self-avoiding closed loops.

Of particular interest will be the correlation function associated to two given plaquettes P_1 and P_2

$$G(\partial P_1, \partial P_2) = \sum_L C_L(\partial P_1, \partial P_2) K^{L-2} , \qquad (2.2)$$

where $C_L(\partial P_1, \partial P_2)$ is the number of closed SAS of L plaquettes containing P_1 and P_2.

By summing eq. (2.2) on all $P_2 \neq P_1$ and defining

$$C_L(\partial P) = \sum_{P' \neq P} C_L(\partial P, \partial P') / (L-1) , \qquad (2.3)$$

which is the number of closed SAS of L plaquettes containing P, we get *

$$\frac{d}{dK} G(\partial P) = 1 + \sum_{P' \neq P} G(\partial P, \partial P') . \qquad (2.4)$$

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* If p' and p have a link in common then $G(\partial P, \partial P')$ means $G(\partial(P \cup P'))$.

Let $\vec{r}_p$ be the position of the center of mass of a plaquette p and let's assume the following asymptotic behaviour for $G(\partial P, \partial P')$ as K approach from below some critical value K_c

$$G(\partial P, \partial P') \underset{|\vec{r}_p - \vec{r}_{p'}| \to \infty}{=} f\left(|\vec{r}_p - \vec{r}_{p'}|/\xi\right) |\vec{r}_p - \vec{r}_{p'}|^{-d-(\theta-2)/\nu}, \quad (2.5)$$

where the correlation length ξ diverges like $(K_c - K)^{-\nu}$ and θ and ν are critical exponents. From eq. (2.4) we get

$$G(\partial P) \underset{K \to K_c^-}{\sim} (K_c - K)^{\theta-1} + \text{less singular terms}, \quad (2.6)$$

that, by an inverse Laplace transformation, implies

$$C_L(\partial P) \underset{L \to \infty}{\sim} K_c^{-L} L^{-\theta}, \quad (2.7)$$

from which we shall call θ the entropy exponent.

The average radius of gyration with respect to the center of mass of a closed surface S is defined as

$$R_S^2 = \frac{1}{|S|} \sum_{p \in S} (\vec{r}_p - \vec{r}_c)^2 = \frac{1}{2|S|^2} \sum_{p,q \in S} (\vec{r}_p - \vec{r}_q)^2, \quad (2.8)$$

where in the last equation we have used the definition of the center of mass, $\vec{r}_c = \sum_{p \in S} \vec{r}_p / |S|$.

The grand-canonical average radius of gyration takes the following form

$$\begin{aligned} \langle R_S^2 \rangle &= \sum_{S:\partial S = \phi} R_S^2 K^{|S|} \Big/ \sum_{S:\partial S = \phi} K^{|S|} \\ &= \sum_{p,q} \sum_{S:\partial S = \phi} (\vec{r}_p - \vec{r}_q)^2 \frac{K^{|S|}}{2|S|^2} \Big/ \sum_{S:\partial S = \phi} K^{|S|} \end{aligned} \quad (2.9)$$

Now we observe that

$$\begin{aligned} Z_1 \equiv \sum_{S:\partial S = \phi} K^{|S|} &= \sum_L N_2 \, C_L(\partial P) \frac{K^L}{L} \\ &= N_2 \int_0^K G(\partial P) \, dK \end{aligned} \quad (2.10)$$

where N_2 is the number of plaquettes of the lattice ($= N \cdot d(d-1)/2$) and $N_2 C_L(\partial P)/L$ is the total number of closed surfaces of L plaquettes.

From eqs. (2.9) and (2.10) we derive the following equation for $\langle R_s^2 \rangle$

$$\left(K \frac{d}{dK}\right)^2 \left(Z_1 \langle R_s^2 \rangle\right) = \frac{1}{2K^2} \sum_{P,q} G(\partial P, \partial q)(\vec{x}_p - \vec{x}_q)^2 , \quad (2.11)$$

which, together the assumed asymptotic behaviour (2.5) and (2.7), yields

$$\langle R_s^2 \rangle \underset{K \to K_c^-}{\sim} \xi^2 . \quad (2.12)$$

B. <u>SAS and n→0 limit of gauge models.</u>

It is well known the connection between the self avoiding walk problem and the n→0 limit of an n-component spin model [4,5,7,8].

Here we want to define a new model of n-component spins which in the n→0 limit gives rise to SAS [12].

It was already known that U(n) gauge model with hamiltonian

$$H = K \sum_P R_e \, tr \, U_P , \quad (2.13)$$

where $U_P = \Pi_{l \in \partial P} U_l$, $U_l = U_{-l}^+$ and U_l is an nxn matrix associated to the oriented link l with the constraints $U_l U_l^+ = n \cdot \mathbb{1}$, in the n→0 limit gives rise to oriented SAS with a fugacity K associated to each plaquette and a fugacity n to the Euler characteristic, χ [11]. Thus small value of n should favour SAS with many handles and the n→0 limit does not exist.

The model we are looking for is defined by the hamiltonian [12]

$$H_n = K \sum_P \sum_{\alpha=1}^{n} \prod_{l \in \partial P} S_l^\alpha , \quad (2.14)$$

where the spin variables $\vec{S}_l = (S_l^1, \ldots, S_l^n)$ are defined on the links, l, of the lattice and can be equal only to the 2n states [22].

$$\vec{S}_l = (0, \ldots, 0, \pm\sqrt{n}, 0, \ldots, 0) . \quad (2.15)$$

It is easy to see that the following equalities hold

$$\sum_{\alpha=1}^{n} (S_\ell^\alpha)^2 = n \;, \quad S_\ell^\alpha S_\ell^\beta = \delta^{\alpha\beta} (S_\ell^\alpha)^2 \;,$$

$$(S_\ell^\alpha)^p = n (S_\ell^\alpha)^{p-2} \quad p \geq 3 \;, \quad tr_{S_\ell} (S_\ell^\alpha)^2 \qquad (2.16)$$

$$\equiv \frac{1}{2n} \sum_{\vec{S}_\ell} (S_\ell^\alpha)^2 = 1 \;, \lim_{n\to 0} tr_{S_\ell} (S_\ell^\alpha)^{2p} = \delta_{p,0} + \delta_{p,1} \;.$$

Using eqs. (2.16) it is immediate to verify that in the n→0 limit the partition function becomes

$$Z^{(n)} = Tr_S e^{H_n} = 1 + n\left(Z_1 + \frac{N_2 K^2}{2}\right) + O(n^2), \qquad (2.17)$$

where Z_1, was defined in eq. (2.10) as the sum over all closed orientable and not SAS +. In other words the free energy per site and spin component is

$$\lim_{N\to\infty} \lim_{n\to 0} \frac{\ln Z^{(n)}}{nN} = \frac{d(d-1)}{2}\left[\sum_L c_L(\partial P) \frac{K^L}{L} + \frac{K^2}{2}\right]. \qquad (2.18)$$

In the same way one can show that

$$\lim_{n\to 0} \langle \prod_{\ell\in\gamma} S_\ell^\alpha \rangle = G(\gamma) \;, \qquad (2.19)$$

where γ is a self-avoiding loop and α is an arbitrary chosen index. If now we have two self-avoiding loops γ_1 and γ_2 with no common link it is clear that

$$G(\gamma_1, \gamma_2) = \lim_{n\to 0} \sum_{\beta=1}^{n} \langle \prod_{\ell_1\in\gamma_1} S_{\ell_1}^\alpha \prod_{\ell_2\in\gamma_2} S_{\ell_2}^\beta \rangle \;, \qquad (2.20)$$

where the sum over β is introduced in order to avoid disconnected contributions of SAS S such that $S=S_1 \cup S_2$ with $S_1 \cap S_2 = \phi$ and $\partial S_1 = \gamma_1$, $\partial S_2 = \gamma_2$. Analogously one can get correlation functions for many loops as defined in eq. (2.1)).

The hamiltonian defined in eq. (2.14) besides to be globally invariant under permutation group, whose action on the statistical variables is

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+ The trivial contribution $N_2 K^2/2$ in eq. (2.17) comes from a not self-avoiding surface formed by two plaquettes glued together.

$$S_\ell^\alpha \to S_\ell^{\sigma(\alpha)} \qquad (2.21)$$

where σ is a permutation of n objects, is also invariant under local gauge transformations of the Z_2 group i.e.

$$\vec{S}_{(xy)} \to \varepsilon_x \vec{S}_{(xy)} \varepsilon_y^{-1} \qquad (2.22)$$

where (xy) is a link and $\varepsilon_x, \varepsilon_y = \pm 1$.
This last local symmetry makes the models (2.13) and (2.14) rather similar even if in the former the statistical variables are nxn matrices while in the latter n-component vectors.

The field variables (2.15), we used in the hamiltonian (2.14), was chosen for the rather simple rules (2.16) they satisfy. Another simple choice is to take the n-component vector $\vec{S}_\ell$ satisfying to only the normalization constraint $\vec{S}_\ell^{\,2} = n$.
More general models, which give self-interacting surfaces will be considered elsewhere [12].

III. More on SAS and diffusion on them.

A. Weighting handles.

Now we wish to introduce some variants of the model defined by eq. (2.14) in order to weight also the Euler characteristic, χ of a SAS. For a given surface S, χ is defined as

$$\chi(S) = N_2(S) - N_1(S) + N_0(S) = 2 - 2H - B \,, \qquad (3.1)$$

where N_2, N_1 and N_0 are the number of plaquettes, links and sites respectively belonging to S while H and B are the number of handles and holes of S.
Let us introduce now mxm matrices $U_\ell \in O(m)$ (i.e. $U_\ell \tilde{U}_\ell = \mathbb{1}$) an each oriented link, l, of the lattice with the constraint $U_\ell = \tilde{U}_{-\ell}$. The only properties of the group integration we need is ($tr_U \equiv \int_{O(m)} dU$)

$$tr_U 1 = 1 \,, \quad tr\, U_{ij} U_{k\ell} = \frac{1}{m} \delta_{ik} \delta_{j\ell} \,. \qquad * \qquad (3.2)$$

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* Another way to get the same results is to consider mxm matrices U_ℓ whose element are all zero but only one at a time takes values ±1 and to define $tr_U = \frac{1}{2m} \sum_U$ (cfr. eqs. (2.15,16)).

Defining the new hamiltonian as

$$H_{n,m} = mK \sum_{p} tr_p U_p \sum_{\alpha=1}^{n} \prod_{\ell \in \partial p} S_\ell^\alpha \quad , \qquad (3.3)$$

where $U_p = \prod_{\ell \in \partial p} U_\ell$ and the plaquettes p are considered oriented, one can see readily that in the n→0 limit every SAS,S, in weighted by $K^{|S|} m^{\chi(S)}$. Indeed, just as before, due to the properties (2.16), the n→0 limit selects only SAS, S, while the remaining trace over $\prod_{p \in S} tr\, U_p$, due to eq. (3.2),gives rise to a factor 1/m for each link of S and introduces a free index, i, for each site of S, that means a factor $(mK)^{N_2(S)} m^{-N_1(S)+N_0(S)} = K^{N_2(S)} \cdot m^{\chi(S)}$. In particular

$$\lim_{n\to 0} \frac{\ln Z^{(m,n)}}{n} = \sum_{S:\partial S=\phi} K^{|S|} m^{\chi(S)} + \frac{N_2 K^2}{2} , \qquad (3.4)$$

and for a self-avoiding oriented loop γ

$$G_m(\gamma) = \lim_{n\to 0} \left\langle \prod_{\ell\in\gamma} S_\ell^\alpha \, tr \prod_{\ell\in\gamma} U_\ell \right\rangle = \sum_{S:\partial S=\gamma} K^{|S|} m^{\chi(S)} . \qquad (3.5)$$

If one wish to work with orientable surfaces it is sufficient to introduce complex matrices $U_\ell \in U(m)$ with $U_\ell = U_{-\ell}^{\dagger}$ and use the properties

$$\int_{U(m)} dU \; 1 = 1 \; , \int_{U(m)} dU \; U_{ij} U_{k\ell} = 0 \; , \int_{U(m)} dU \; U_{ij} U^{\dagger}_{k\ell} = \frac{\delta_{i\ell}\delta_{jk}}{m} , \qquad (3.6)$$

together with hamiltonian (3.3) with $R_\ell \, tr\, U_p$ instead of $tr\, U_p$.

The new models we have just defined besides to be invariant under transformation (2.21) and (2.22) are also invariant under local gauge transformations

$$U_{(xy)} \to \gamma_x \, U_{(xy)} \, \gamma_y^{-1} \qquad (3.7)$$

with γ_x, γ_y belonging to the same group of U's.

B. <u>A model for diffusion on SAS</u>

The last model we shall introduce here is the one for the diffusion of a random walk on a SAS [12].

The hamiltoniam is

$$H_w = K \sum_{p} \sum_{\alpha=1}^{n} \prod_{\ell\in\partial p} S_\ell^\alpha \; exp\left\{ \frac{J}{2} \sum_{\substack{x,y\in p \\ |x-y|=1}} \vec{\varphi}_x \vec{\varphi}_y \right\} \qquad (3.8)$$

$$- \frac{1}{2} \sum_x \vec{\varphi}_x^{\,2} \; ,$$

where $\vec{\varphi} = (\varphi^1, \cdots, \varphi^m) \in \mathbb{R}^m$ are vector fields defined on sites of the lattice. It is not difficult to verify that

$$\lim_{m\to 0}\lim_{n\to 0} \frac{\langle \varphi'_x \varphi'_y \rangle}{n} = \lim_{m\to 0}\lim_{n\to 0} \frac{1}{n}\prod_z \int_{\mathbb{R}^m} d^m\varphi_z \, T_{zs}\, e^{Hw}\, \varphi'_x \varphi'_y$$

$$= \lim_{m\to 0} \sum_{\substack{S:\partial S=\phi \\ x,y\in S}} K^{|S|} \prod_{z\in S} \int_{\mathbb{R}^m} d^m\varphi_z \; \varphi'_x \varphi'_y \; \exp\Big\{ -\sum_x \frac{\vec{\varphi}_x^{\,2}}{2} + J \sum_{x,y\in S,\, |x-y|=1} \vec{\varphi}_x \cdot \vec{\varphi}_y \Big\} \qquad (3.9)$$

where the $\vec{\varphi}_z$ integrations for $z \notin S$ have been already performed giving $\prod_{z\notin S} (2\pi)^{m/2} \to 1$ as $m \to 0$. Notice that the partition function is 1 in the $n, m \to 0$ limit. The integral in eq. (3.9) gives

$$(\mathbb{1}_S - \mathbb{J}_S)^{-1}_{xy} \det(\mathbb{1}_S - \mathbb{J}_S)^{-m/2} (2\pi)^{\frac{m}{2} N_0(S)}$$

$$= \sum_{W:\, \partial W = \{xy\},\, W\subset S} J^{|W|} \det(\mathbb{1}_S - \mathbb{J}_S)^{-m/2} (2\pi)^{\frac{m}{2} N_0(S)} \qquad (3.10)$$

where $\mathbb{J}_{Sxy} = J$ ($\mathbb{1}_{Sxy} = 1$) if $x, y \in S$, $|x-y|=1$ ($x=y$) and zero otherwise while W is a random walk of $|W|$ steps on S. Inserting eq. (3.10) in (3.9) we find finally

$$\lim_{m\to 0}\lim_{n\to 0} \frac{\langle \vec{\varphi}_x \vec{\varphi}_y \rangle}{n\cdot m} = \sum_{\substack{S:\partial S=\phi \\ x,y\in S}} K^{|S|} \sum_{\substack{W:\partial W=\{xy\} \\ W\subset S}} J^{|W|} \qquad (3.11)$$

The same modifications as in (3.3) should be done in eq. (3.8) if one wishes to weight also handles of S in eq. (3.11).

Certainly the models we presented in this sections are very difficult to analyze with standard methods of statistical mechanics but besides to be interesting in its own they are the first step toward a possible field theoretical formulation of the SAS models and diffusion on them.

IV. Mean Field, Flory Approximation and Possible Pattern of Hyperscaling Violation.

A. Mean field.[2]

For mean field approximation of SAS we mean the $d\to\infty$ limit of that model. As shown in ref. 23 in the large d limit the dominant graphs contributing to correlation functions in lattice gauge theories are tube-like configurations of plaquettes. Here for SAS is just the same [2]. For example for high d $G(\partial P)$ is a sum of trees of elementary cubes rooted in P making it rather similar to a branched polymer. Since in the considered limit the

self-avoiding contraint becomes irrilevant we can write the following consistency equation for G(∂P)

$$G(\partial P) = K + 2(d-2)[G(\partial P)]^5 \quad , \qquad (4.1)$$

where K is the contribution of the graph with only one plaquette while the remaining contributions comes from tree graphs rooted in the faces of the 2(d-2) cubes which share the plaquette p.

Deriving with respect to K eq. (4.1) we determine K_c as the point where $\partial G(\partial P)/\partial K$ diverges. This gives

$$K_c = \frac{4}{5}[10(d-2)]^{-1/4} \quad , \qquad (4.2)$$

and

$$G(\partial P) \underset{K \to K_c^-}{\sim} (K_c - K)^{1/2} + \text{reg. terms} \quad , \qquad (4.3)$$

which, using eq. (2.6), means $\theta = 3/2$. In order to find how the correlation length diverges at the critical point one has to calculate $G(\partial P_1, \partial P_2)$ in the same approach as above. Now the dominant graphs are trees of cubes joining P_1 to P_2. One can look at these graphs like simple random walk going from P_1 to P_2 whose elementary steps have a fugacity $[G(\partial P)]^4$. Thus the Fourier transform of $G(\partial P_1, \partial P_2)$ is given by

$$\tilde{G}(q) \underset{q \to 0}{=} A / [B(1 - 10(d-2)G^4(\partial P)) + q^2] \qquad (4.4)$$

where A and B are suitable constant which are finite as $K \to K_c$.

From (4.4) it follows that

$$\xi^2 \equiv -\frac{1}{\tilde{G}(q)} \nabla_q^2 \tilde{G}(q)\Big|_{q=0} \propto [1 - 10(d-2)G^4(\partial P)] \qquad (4.5)$$

$$\underset{K \to K_c^-}{\sim} (K_c - K)^{-2\nu} \quad , \quad \nu = 1/4 \quad .$$

One expect that the mean field exponents $\theta = 3/2$ and $\nu = 1/4$ are always the same above some critical dimension, d_c, while SAS have a non trivial critical behaviours below d_c. This hope rises from previous experience with linear and branched polymer and from geometrical critical phenomena in general. Thus the next task is to determine d_c and to give an estimate of θ and ν for $d < d_c$.

B. Flory approximation for sheet polymers.[1,2]

As discussed in the introduction we assume that our surfaces represent sheet polymers in a good solvent. In the same spirit of the Flory approximation for SAW [8,15] one can think that the free energy of a sheet polymer of "area" N and radius of gyration R is the sum of two pieces [1,2]

$$F = F_{rep} + F_{elastic} \quad , \tag{4.6}$$

where

$$F_{rep} \propto \left(\frac{N}{R^d}\right)^2 R^d \tag{4.7}$$

is the interaction energy due to monomer-monomer repulsion which we have evaluated in a "mean-field" view while $F_{elastic}$ is of entropic origin ($F_{elastic}=-TS$) and it tends to favour contract configurations of the sheet polymer. For a linear polymer $F_{elastic} \propto -\ln P(R) \sim R^2/N$ where P(R) is the end-to-end distribution of a Gaussian chain. For random surfaces we expect that what it is Gaussian distributed is not the radius of gyration but a typical area (e.g.the projection of the surface an a given lattice plane)*. Since a typical area should be proportional to R^2 we guess that

$$F_{elastic} \propto R^4/N \tag{4.8}$$

Minimizing eq. (4.6) with respect to R we get that the minimum occurs for $R=\xi \propto N^{\nu_F}$ with

$$\begin{aligned} \nu_F &= \frac{3}{4+d} \qquad 2 \le d \le d_c = 8 \, , \\ \nu_F &= \frac{1}{4} \qquad\quad d \ge d_c \, , \end{aligned} \tag{4.9}$$

where $d_c=8$ is determined requiring that for $d>d_c$ $F_{rep} \ll F_{elastic}$ if $F_{elastic}$ is of order 1 [8]. $d_c=8$ should be not a surprise [10] because it is well known that d_c

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* An indication that this conjecture should be correct comes from an exact solvable, even if rather peculiar, model of random surfaces in d=2 [18].

should be the dimension above which two independent SAS do not intersect each other with probability 1*. If D (= $=1/\nu$) is the fractal dimension of SAS's then the fractal dimension of the intersection of two independent SAS's is $D_I = Max\{2D-d, 0\}$ [24] which implies $d_c=2D(d_c)$.

Because it is expected that above d_c the mean field analysis made at the beginning of this section holds one gets $D(d_c) = \nu^{-1}(d_c) = 4$ and $d_c=8$.

C. Hyperscaling violation.[32]

The next goal is the estimate of the θ exponent. Let's observe first that $G(\partial P)$, in term of the models defined in sec. 2 and 3, is the n→0 limit of an internal energy (apart from a multiplicative constant) and thus one could argue that

$$G(\partial P) \underset{K\to K_c^-}{\sim} (K_c - K)^{1-\alpha} , \qquad (4.10)$$

where α is the specific heat exponent, and that an hyperscaling relation of the usual type holds

$$2-\alpha = d\nu \qquad (4.11)$$

for $d\leq d_c$. However from eqs. (2.6) and (4.10) $2-\alpha=\theta=d\nu$ which does not hold at $d_c=8$. Similar disagreement is found also in lattice animals and percolation, and modified hyperscaling relations have been conjectured [25] [26] which turn out to be in very good agreement with known results and exact near the upper critical dimension.

We shall try now to guess in which way hyperscaling violation for SAS could occurs. The following derivation is rather similar to the one given in ref. 27 for isotropic and directed percolation and lattice animals. In that reference it was used a generalized Ginzburg criteria [28] plus other additional hypothesis; here the only difference is in the minor number of assumptions.

The basic idea is to assume that the rôle of the free energy density for SAS is done by eq. (2.10) i.e.

$$F(K) = \frac{1}{N_2} \sum_{S:\partial S=\phi} K^{|S|} \qquad (4.12)$$

and that for $K\to K_c^-$ and $d\leq d_c$ $F(K) \sim 1/\Omega$ where Ω is a volume over which critical fluctuations are important [28]. The hyperscaling relation (4.11) should follow if $\Omega \sim \xi^d$. However, given a typical self-avoiding surface one expects that fluctuations should occur mainly

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* See e.g. A. Stella, R. DeKeyser and A. Maritan same issue as ref. 32.

perpendicularly to it implying that $\Omega \sim \xi^{d-2}$ and

$$F = \xi^{2-d} + \text{regular term} \quad . \qquad (4.13)$$

From eqs. (2.6), (2.10) and (4.13) the modified hyperscaling relation is obatined

$$\theta = \nu(d-2) \qquad d \leq 8 \quad , \qquad (4.14)$$

which, at least, at $d_c=8$ is exact.

Even if the argument given above to derive (4.14) is rather intuitive we expect that the result is at least a good approximation: this hope is based on the fact that analogous derivations for lattice animals, isotropic and directed percolation brings to very good agreement with the existing numerical estimates or to the exact results if they are at disposal [27].

V. Real Space Renormalization Group for SAS. [1,2]

A. R.G. strategy: isotropic rescaling.

In order to have an alternative estimate of the correlation length exponent for SAS's we are going to consider an approximate mapping, $K \to K'(K)$, for the plaquette fugacity in such a way that near K_c

$$\xi(K') = l^{-1}\, \xi(K) \quad , \qquad (5.1)$$

l being a suitable length rescaling factor [1,2]. If K'(K) is regular from eq. (5.1) and the assumed singularity of $\xi(K) \sim (K_c - K)^{-\nu}$, K_c and ν are given by

$$K'(K_c) = K_c \quad , \qquad dK'/dK|_{K_c} = l^{1/\nu} \quad . \qquad (5.2)$$

It is better to illustrate first the construction of an approximate K'(K) with l=2 in the three-dimensional case. To this purpose we think to the lattice partitioned in cubic cell of side 2 (see fig. 1a). By convention the plaquettes on the boundary between two adjacent cells are assumed to belong to the one with higher values of the coordinates normal to the plaquette.

With reference to the fig. 1a let's consider all connected SAS's made of plaquettes, belonging (with the before mentioned convention) to a given cell, whose

boundary is contained in the 2 and 3-axis and in the upper and right faces. All surfaces we obtain in this way will be considered as realizing an effective full covering of the 2x2 face in the (2,3) plane and they are assumed to contribute to the fugacity K' of the new effective plaquette of a rescaled elementary cell (fig. 1(b)). The reason to keep only surface with part of the border fixed is due to overcounting problem like the one occurring in the "corner rule" for self avoiding walk [29]. There are just fourteen surfaces contributing to the mapping, giving

$$K' = K^4 + K^6 + 4K^7 + 3K^8 + 4K^9 + K^{10} \qquad (5.3)$$

Equations (5.2) together eq. (5.3) give only one non-trivial fixed point $K_c = .651$ (in reasonable agreement with the only existing Monte Carlo estimate $K_c = .588$ [30] and another independent estimate $K_c = .623$ [31]) with $\nu = .37$.

B. Anisotropic rescaling and recursion equations in d-dimension.

If we want to have a rescaling factor l=3 we should have to consider a cell of side 3 and to count surfaces of the same type as above. However it is easy to recognize that the number of such surfaces are of order 2^{18}. At this level it should be more useful to have series expansion for correlation functions rather than approximate recursion equations! At present it is rather difficult (at least for the author) to implement, in an efficient way, a computer program for counting surfaces.

In order to have recursion equations for higher values of d and/or l we had to use a new type of approximation [1,2]. A global rescaling length l is obtained in successive steps: at each step one single lattice direction is rescaled using an anisotropic cell of the type illustrated in fig. 2(a) for generic d.

The operation is repeated for all the d lattice directions, with the same rules used above. In this case we get two recursion equations: one for the fugacity associated to plaquettes perpendicular to the rescaled direction (fig. 2(a) and (b)) and another one for the remaining plaquettes.

With an obvious writing:

$$K'_{\perp} \equiv R_l(K) = K\,\frac{K^{2l}-1}{K^2-1}\,, \qquad (5.4a)$$

$$K'_{\parallel} = K^{\ell} \quad . \tag{5.4b}$$

Successive application of eqs. (5.4 a,b) will give $d(d-1)/2$ new fugacities, one for each coordinate plane: However one can disregard the anisotropy generated by the above procedure and take only one of the derived recursion equations. As confirmed by the results obtained below, and the ones reported in reference 2 for linear and branched polymers, this approximation doesn't seem to cause more inaccuracy than the one already implied by the isotropic rescaling. Furthermore the results are almost independent on which recursion equation is chosen but we found that the best results are obtained using the recursion

$$K' = \left[R_{\ell}^{d-2}(K)\right]^{\ell^2} \quad , \tag{5.5}$$

i.e. applying (5.4a) (d-2)-lines and then (5.4b) two times. For d=2 the model is trivial and our recursion (5.5) gives the exact result $K_c=1$ with $\nu=1/2$. Results for $3\leq d\leq 8$ and $l=2,3$ are reported in Table I together the "Flory" values of the ν-exponent. The agreement of the different ν estimates, even if qualitative, is rather remarkable in the whole range of dimensionalities where the model is expected to present a non-trivial scaling behaviour.

It is appealing to work out eq. (5.5) in the $l\to 1$ limit; the β-function is defined in the usual way as

$$\beta(K) = \frac{dK'}{d\ell}\Big|_{\ell=1} = \frac{2K\ln K}{1-K^2}\left(1-(d-1)K^2\right), \tag{5.6}$$

and K_c is the non-trivial zero of β(k) while $1/\nu = d\beta(K)/dK\,|_{K_c}$ In this way one gets

$$K_c = (d-1)^{-1/2} \quad \text{and} \quad \nu = \frac{d-2}{2(d-1)\ln(d-1)} \tag{5.7}$$

which gives the exact result for d=2 and it is expected to remain exact in $d=2+\epsilon$ like it happens in the same approach for self-avoiding walk and percolation [2,29].

In that limit we get

$$K_c = 1 - \frac{\varepsilon}{2} + \mathcal{O}(\varepsilon^2) \quad , \tag{5.8a}$$

Table I

d		3	4	5	6	7	8
$\ell = 2$	K_c	.636	.491	.412	.360	.323	.295
	ν	.38	.32	.29	.27	.26	.24
$\ell = 3$	K_c	.619	.477	.400	.350	.315	.288
	ν	.39	.34	.31	.29	.28	.27
Flory values		.43	.37	.33	.30	.27	.25

and

$$\nu = \frac{1}{2} - \frac{\varepsilon}{4} + O(\varepsilon^2) \quad . \qquad (5.8b)$$

VI. Exact results for a SAS model on a fractal structure.[32]

A. The model.

It is always interesting and instructive to have at disposal some exatdly solvable model of SAS even if the lattice where it is defined is somewhat unphysical. The main reason for studying such a model is that it should be a good example for testing our renormalization group ideas and Flory approximation previously presented.

The lattice where we are working now is a 3 dimensional structure for which every section parallel, let's say, to the x-y plane is a Sierpinski gasket as fig. 3a shows [32]. We shall call it Toblerone lattice. The shaded triangles of the figure indicate that the corresponding areas are lacking and cannot be used for constructing SAS's.

We are looking for the mean area of SAS's whose boundary contains at least the lines AB and A'B' shown in the figure. We assume furthermore that elementary (not shaded) triangles and squares have the same area.

All surfaces can be constructed recursively starting from an initial configuration $M_1\{\sigma\}$ which is the statistical weight of a surface of the type shown in fig. 3b where we have introduced σ variables for every plaquette which is +1 or -1 according to if the surface tauches or does not the highest edge.

The recursion equations, we get, are

$$M_{n+1}\{\sigma\} = M_n\{\sigma\} \sum_{\{s,s'\}} M_n\{s\} M_n\{s'\} \prod_{i=1}^{2^N} (3 - s_i s'_i - s_i - s'_i)/4 \qquad (6.1a)$$

for $\{\sigma\} \neq \{-1, \ldots, -1\}$ and

$$M_{n+1}\{-1\} = \Big(\sum_{\{s\}} M_n\{s\} \Big)^2 + M_n\{-1\} \sum_{\{s,s'\}} M_n\{s\} \cdot M_n\{s'\} \prod_{i=1}^{2^N} \frac{3 - s_i s'_i - s_i - s'_i}{4} \Big) \qquad (6.1b)$$

where $M_n\{-1\}$ stays for $M_n\{-1, \ldots, -1\}$.

The initial condition is

$$M_1\{\sigma\} = \exp\left\{ \ln K \left[2^{N+1} + \frac{1}{2} \sum_{i=1}^{2^N} (s_i - s_i s_{i+1}) \right] \right\} . \quad (6.1c)$$

The only non-trivial fixed point in the physical region is:

$$M_n\{-1\} = M^* , \quad M_n\{\sigma\} = 0 \qquad \{\sigma\} \neq \{-1, \ldots, -1\} \quad (6.2)$$

with $M^* = (\sqrt{5}-1)/2$.

B. Recursion equation analysis.

It is not difficult to show that the eigenvalues of linearized recursion equations are

$$\lambda_1 = 2 M^* + 3 M^{*2} , \quad (6.3a)$$

$$\lambda_i = M^{*2} \qquad i = 2, 3, \ldots, 2^N . \quad (6.3b)$$

Now let's concentrate on the value of K, K_c, from which we must start in order that $M_n\{\sigma\} \underset{n\to\infty}{\to} M^*$. To this purpose we define the quantities

$$a_n = M_n\{-1\} , \quad b_n = \overline{T_\sigma} M_n\{\sigma\} - a_n \quad (6.4)$$

which satisfy the following inequalities

$$(a_n + b_n)^2 + a_n^2 (a_n + 2 b_n) < a_{n+1} < (a_n + b_n)^2 + a_n^2 (a_n + 2 b_n) + b_n^2 a_n \quad (6.5a,b)$$

$$a_n b_n (a_n + 2 b_n) < b_{n+1} < a_n b_n (a_n + 2 b_n) + b_n^3$$

The important thing of this inequalities is that the recursion equations defined by the left and right hand sides have the same non-trivial fixed point, i.e. $a^* = M^*$ and $b^* = 0$.

For N sufficiently large $a_1 = K^{2^N}$ and $b_1 = \alpha \lambda^{2^N}$, where α is a suitable constant and λ is the highest eigenvalue of the matrix

$$\begin{vmatrix} K^2 & K^{5/2} \\ K^{5/2} & K \end{vmatrix} \quad (6.6)$$

(cf eq. (6.1c)) that is

$$\lambda = \frac{K}{2}\left(1+K+\sqrt{(1-K)^2+4K^3}\right) \quad . \qquad (6.7)$$

Let's call $\bar{K}_c$ and $\bar{\bar{K}}_c$ the values of K for which the recursion equations defined by the l.h.s. and r.h.s. respectively of eqs. (6.5) bring (a_n, b_n) to the fixed point.

For $K<1$ $a_1 \to 0$ as $N\to\infty$ so that the initial critical values $(\bar{a}_{1c}, \bar{b}_{1c})$ and $(\bar{\bar{a}}_{1c}, \bar{\bar{b}}_{1c})$ for the two above mentioned recursion equations must be searched in the a=0 axis. The corresponding $\bar{b}_{1c}$ and $\bar{\bar{b}}_{1c}$ values are obviously independently on N. This meams that $\bar{K}_c = \bar{\bar{K}}_c$ with $\lambda(\bar{K}_c) = 1$, i.e. $\bar{K}_c = .693$. Due to the inequalities (6.5) the initial critical value for b_1 must be finite and independent on N as $N\to\infty$. This implies that $K_c = \bar{K}_c$!

C. Radius of gyration.

In order to determine the ν-exponent associated with the radius of gyration of our surfaces let's consider an infinite Toblerone lattice and all surfaces whose boundaries contain lines of the type AB and A'B' line in fig. 3a with N=1, 2,... and A fixed. For each N the surfaces have a radius of gyration of order 2^N so that the average radius of gyration $\langle R\rangle$ can be defined as

$$\langle R\rangle = \sum_{N=1}^{\infty} 2^N Z_N \Big/ \sum_{N=1}^{\infty} Z_N \qquad (6.8)$$

where $Z_N = Tr_\sigma M_N\{\sigma\}$ $(= a_N + b_N)$ with initial condition (6.1C). If we start with a value of $K = K_c - \delta K$ such that $Z_1(K_c-\delta K) - Z_1(K_c) \overset{N\to\infty}{=} \alpha(\lambda(K_c-\delta K))^{2^N} - \alpha(\lambda(K_c))^{2^N} = \alpha\,\delta K\, 2^N \frac{d\lambda}{dK}\Big|_{K_c} + O(\delta K^2)$

(where we used the fact that $\lambda(K_c) = 1$ for $N\to\infty$) then due to the recursion equations (6.1)

$$Z_N = M^* + \delta K\, 2^N \lambda_1^N C \qquad (6.9)$$

at least for $N_0 < N < N(\delta K)$ with $\delta K(2\lambda_1)^{N(\delta K)} = r << 1$ and N_0 a finite value independent on δK. For $N > N(\delta K)$, Z_N goes to zero rapidly so that the main contribution to eq. (6.8) comes from values of $N \in (N_0, N(\delta K))$ giving

$$\langle R\rangle = C'\left(\frac{\delta K}{2}\right)^{-\nu} \qquad (6.10)$$

with

$$\nu^{-1} = D_{SAS} = \frac{\ln 2\lambda_1}{\ln 2} = 2.252\ldots , \qquad (6.11)$$

and C' is a suitable constant proportional to $r^{-\nu}$ so to make $\langle R \rangle$ independent on the arbitrary choice for the value of r.

It is worthwhile to remark that $\ln\lambda_1/\ln 2$ is the fractal dimension, D_{SAW}, of a self-avoiding walk on a Sierpinski gasket [33] whose fractal dimension is $\bar{d}=\ln 3/\ln 2$ The fractal dimension of the Toblerone lattice is $\bar{d}+1$ and for SAS's we have just obtained

$$D_{SAS}(\bar{d}+1) = 1 + D_{SAW}(\bar{d}) \ . \qquad (6.12)$$

It is curious to observe that the same relation holds in the Flory approximation for $D^{(F)}_{SAS}(d) = (d+4)/3 \ \ (d \le 8)$, established in sec. IV, and for $D^{(F)}_{SAW}(d) = (d+2)/3 \ \ (d \le 4)$ [8].

Furthermore if this last approximation is applied to the Toblerone it gives $D^{(F)}_{SAS}(2.58) = 2.195\ldots$ in good agreement with the result (6.11).

Other interesting quantities to be studied in our Toblerone lattice are θ exponent and string tension behaviour.

In particular for this last it would be interesting to know if at K_c it remains finite like in the $d\to\infty$ limit [23] or goes to zero according to $(K_c - K)^{\mu}$ where μ is a new exponent

VII. Conclusion and Perspectives.

A. Summary

Here we considered conformational properties of SAS on a hypercubic d-dimensional lattice.

Our analysis was based on an approximate real space renormalization group and on a Flory argument for the correlation length exponent. Both methods are in qualitative agreement in the whole range of dimensionality ($2 \le d \le 8$) where the model is expected to exhibits a non-trivial scaling behaviour.

Furthermore our anisotropic rescaling recursion method seems to have an extreme efficiency [2], which compensate our inability to perform exact enumerations when the problem becomes too complicated.

With regard to the entropic exponent, a conjectural argument of how hyperscaling violation could occur is given. Unfortunately we cannot find alternative way to estimate that exponent.

We believe that in the near future both Monte Carlo simulations and exact ennumeration work will be able to show the reliability or not of our results.

The SAS model on the Toblerone lattice, we have solved exactly as far as regard K_c and ν , should be considered already a partial confirmation of the approximate results obtained for more realistic lattices.

A further,certainly important,forward step was to have found a set of n-component spin models which in the $n \to 0$ limit give the statistical mechanics of SAS's. This result, apart from representing a generalization of what was known for SAW's [4,5,7,8], could open the way to a field theoretical approach to the exclude volume problem in surface models.

B. Spectral dimension of SAS.

Besides to the fractal dimension, D, another interesting quantity for surface is the so called spectral dimension, d_s, which gives how the density of states of a fractal scales at low frequencies +. [34,35] (see also part.1). Since SAS have not self-intersections and have the topology of a sphere it is expected that $d_s=2$ [36] in the same way as for SAW $d_s=1$ [34,35]. However while the latter result is obvious, because stretching a SAW we get a straight line, the former is not so trivial. Indeed one should be able to see that the most statistically important surfaces can be stretched, without strain, in a more or less smooth surface.

If this is the case the intersection of a typical SAS in d-dimension with a (d-1)-hyperplane should result in a SAW. Then the Hausdorff fractal dimension of SAS's in d dimension should be

$$D_{SAS}(d) = D_{SAW}(d-1) + 1 \qquad (7.1)$$

+ We are imagining point masses on every site of a surface vibrating perpendicularly to it and connected to their nearest neighbour masses by harmonic forces.

(see eq. (6.12).

Using the Flory approximation $D_{sAS}^{(F)}(d)=(4+d)/3$ and $D_{sAW}^{(F)}(d)=(2+d)/3$ one finds that (7.1) holds for $d\leq 5$! If this result continues to be valid even with the exact values for D_{sAS} and D_{sAW} then it is rather plausible that $d_s=2$ for SAS's at least for $d\leq 5$. However it is not excluded that d_s can become different from two at some $d<8$ (at $d=8$ eq. (7.1) doesn't hold)

If the self-avoiding constraint is released our SAS's become the so called planar random surfaces (PRS)[16,17,18] for which there is some numerical [17] and analitycal [37] evidence that their scaling behaviour is mean field like (see sec. IV) for all $d\geq 2$. Due to the self-intersections one expects that the spectral dimension of PRS's is different from 2, at least in low dimension.

Then the situation for SAS's and PRS's seems rather similar to the one occurring for SAW and RW (random walk) for which we have the following picture

	SAW	RW
D	$\begin{cases} \sim \frac{d+2}{3} & d\leq 4 \\ = 2 & d>4 \end{cases}$ [8]	2 for any d [38,39]
d_s	1 for any d [34,35]	$\begin{cases} \sim \frac{8}{4+d} & d\leq 4 \\ = 1 & d>4 \end{cases}$ [38]

Thus of SAS's it seems rather plausible to have $d_s=2$ for any $d\geq 2$ implying that a random walk diffusing on typical SAS's (see sec.3), after a time t, moves at a mean square distance t^{2/d_w} with $d_w=2D/d_s=D$ (see part 1) (this implies, for example, that the resistence exponent, ζ, given by the Einstein relation $\zeta=d_w-D$, is zero [40]).

Another unclear problem, at least to the author, is the relevance of handles for SAS's. What it is surely known is that if now the self-avoiding constraint is removed the statistical model defined by eq. (3.5) is ill defined [16] +.

+ Due to this problem we think that the (free) random surfaces referred in ref. 36 must be considered just the PRS.

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FIGURE CAPTIONS.

Figure 1 (a). Surface of nine plaquettes contributing to K' in eq. (5.3). The fixed part of the border is contained in the 2 and 3 axis.(b) The corresponding renormalized plaquette in the (2,3) plane.

Figure 2 (a). Anisotropic cell (3x1x...x1) used in d-dimension in order to obtain the recursion eqs. (5.4a,b). It is drawn a surface of five plaquettes contributing to K' . (b) The renormalized plaquette perpendicular to the rescaled direction.

Figure 3 (a). Toblerone lattice at the fourth iteration. (b) A typical configuration contributing to $M_1\{\sigma\}$ (eq. (6.1c)).

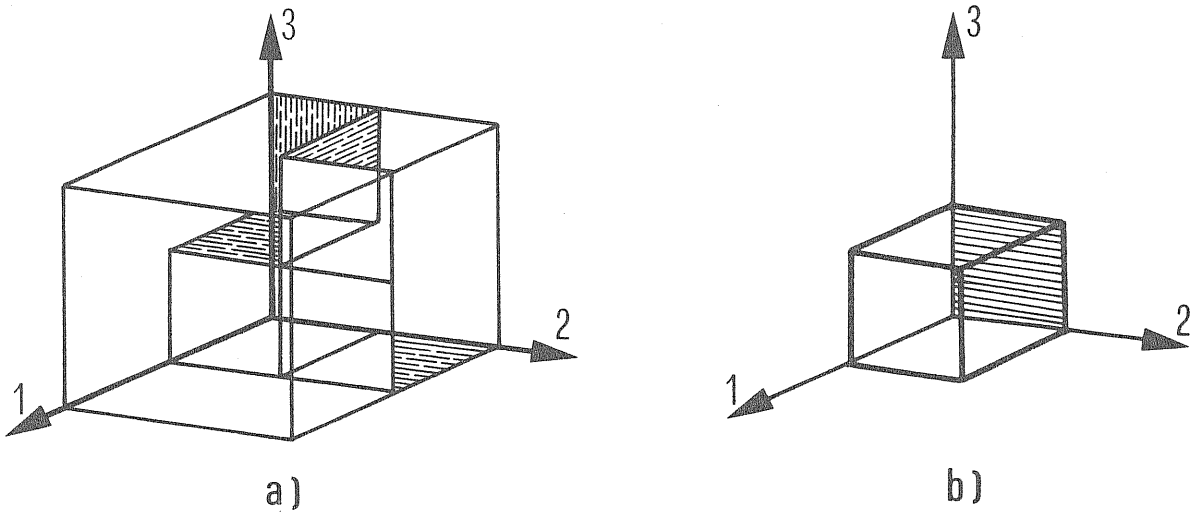

fig. 1

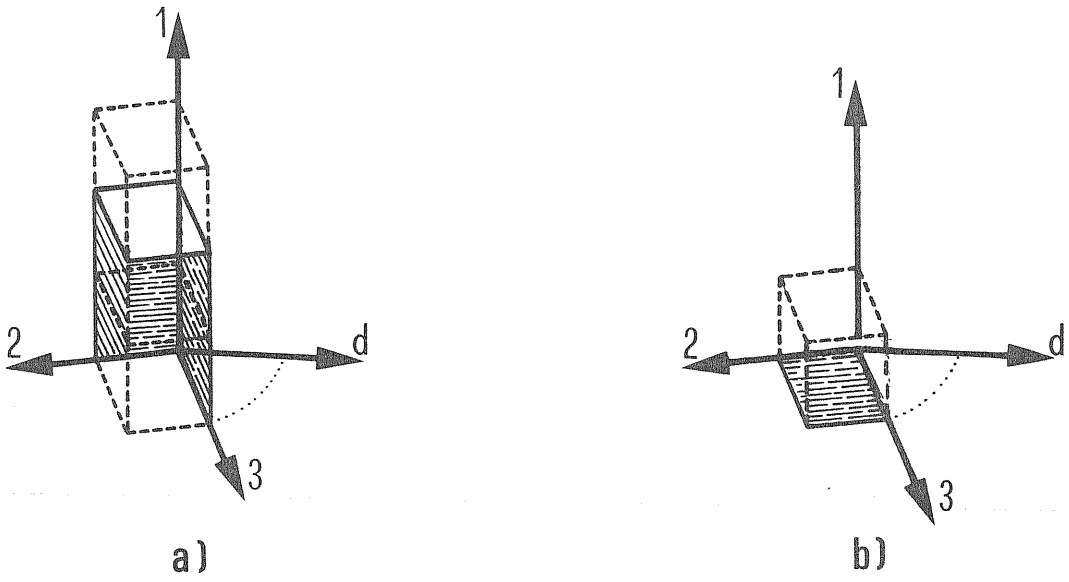

fig. 2

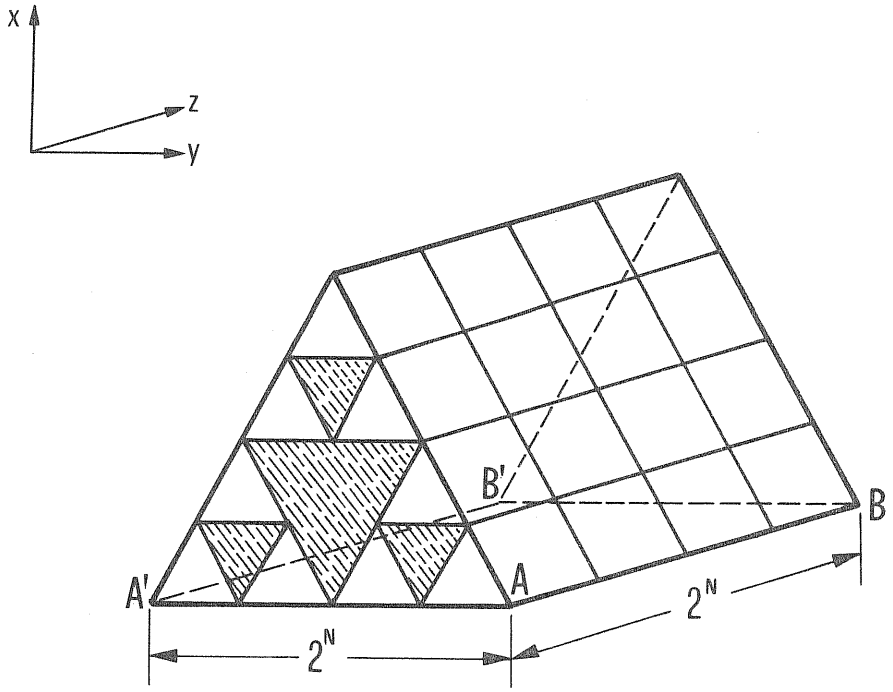

fig. 3 a

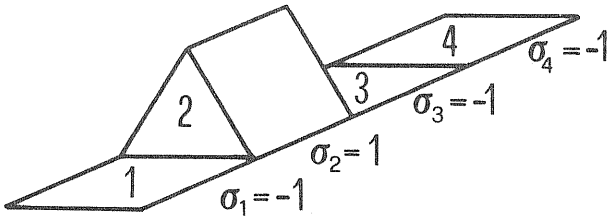

fig. 3 b