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To Scimmia, Citto, Jesus & Chetty

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Goffredo Chirco

Abstract

In this thesis we consider a scenario where gravitational dynamics emerges from the holographic hydrodynamics of some microscopic, quantum system living in a local Rindler wedge. We start by considering the area scaling properties of the entanglement entropy of a local Rindler horizon as a conceptually basic realization of the holographic principle. From the generalized second law and the Bekenstein bound we derive the gravitational dynamics via the entropy balance approach developed in [Jacobson 1995]. We show how this setting can account for the equilibrium and the non-equilibrium features associated with the gravitational dynamics and extend the thermodynamical derivation from General Relativity to generalized Brans-Dicke theories. We then concentrate on the possibility to define a version of fluid/gravity correspondence within the local Rindler wedge setting. We show how the hydrodynamical description of the horizon can be directly associated to a hydrodynamical description of the thermal fields. Because of the holographic behavior, the properties of the Rindler wedge thermal gauge theory are effectively encoded in a codimension one system living close to the Rindler horizon. In a large scale analysis, this system can be thought of as a fluid living on a codimension one stretched horizon membrane. This sets an apparent duality between the horizon local geometry and the fluid limit of the thermal gauge theory. Beyond the connection between the classical Navier-Stokes equations and a classical geometry, we discuss the possibility to realize such a duality at any point in spacetime by means of the equivalence principle. Given the shared local Rindler geometric setting, we eventually deal with the intriguing possibility to link the fluid/Rindler correspondence to the derivation of the gravitational field equations from a local non-equilibrium spacetime thermodynamics.

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Chapter 1

Introduction

1.1 State of Art

Einstein's general theory of relativity describes how spacetime makes matter move, at a kinematic level, as well as how matter tells spacetime to curve, at a dynamic level. The former case - kinematics - essentially derives from the basic assumption of the *equivalence principle* [Einstein 1907]. As far as the dynamics, on the other hand, a funding principle is effectively still missing.

In the last two decades, several results, from different theoretical frameworks, have indicated that this principle may be understood within the apparently deep connection between gravity and thermodynamics.

A first hint in this direction is provided by kinematics itself. By stating that gravity is nothing but acceleration in disguise, the equivalence principle suggests that *horizons*, as direct byproduct of acceleration, necessarily play a fundamental role. Interestingly, this role becomes clear only when quantum physics is considered.

The work of Hawking and Unruh in the early 1970s shows that uniform acceleration in flat spacetime, corresponding, via equivalence principle, to gravitational forces, acts as to thermalize quantum fields (Hawking-Unruh effect)[Hawking 1975a; Unruh 1976; Sewell 1982]. In the local accelerating frame, described by a *local Rindler* metric, the Minkowski vacuum appears as a thermal state. Therefore, one can study the entropy associated with this system. This idea originally goes back to two different calculations by 't Hooft ['t Hooft 1985] and Bombelli, Koul, Lee, and Sorkin (BKLS)[Bombelli 1986; Srednicki 1993]. In particular, 't Hooft calculates the thermal partition function at the Hawking temperature for a scalar field outside a very massive Schwarzschild black hole, which the Rindler spacetime closely approximates. Since the local Unruh temperature $T = \hbar a/2\pi$ diverges at the horizon, the entropy diverges as well and must be regularized by replacing the horizon with a *brick wall* boundary condition. The resulting entropy scales like the cross-sectional area of the horizon.

Separately, BKLS point out that the entropy of the fields localized by the horizon can be given also in terms of a generic statistical von Neumann entropy, $S_{ent} = -Tr\hat{\rho}\ln\hat{\rho}$, where $\hat{\rho}$ is an entanglement density matrix that results from tracing over the unobservable regions of the Hilbert space. Again, this entropy must be regularized with a ultraviolet (UV) cutoff, which yields in four spacetime dimensions $S_{ent} \sim A/\ell_c^2$. Interestingly, the thermal and quantum pictures of the entropy turn out to be equivalent, because the density matrix $\hat{\rho}$ is precisely a thermal Gibbs state.

Quantum fields thermal character and entropy area scaling behavior are at the root of the thermodynamical description of black holes dynamics developed in 1970s [Israel 1967; Bardeen 1973; Carter 1973; Hawking 1971; Hawking 1975a]. At classical level, Hawking's *area law*, associated with the general presence of *irreversible* processes in black hole dynamics [Penrose 1971; Christodoulou 1971; Hawking 1971], suggests the idea that a black hole should carry an entropy proportional to its horizon area. The intuition of Bekenstein then produces the expression $S_{BH} = \alpha A/\hbar G$, measuring the area in units of the squared Planck length $\ell_p^2 = \hbar G/c^3$, with an interpretation of the horizon entropy[1] as the logarithm of the number of ways in which black hole might have been formed [Bekenstein 1973; Hawking 1976]. Later on, the discovery of the Hawking-Unruh effect clarifies the thermal nature of black holes and fix the value of the horizon entropy to the universal Bekenstein-Hawking value, $S_{BH} = A/4G$. The ultimate evidence of a consistent characterization of the black hole entropy with the laws of classical thermodynamics is then provided by Bekenstein's conjecture of the *generalized second law*, stating that the sum of the entropy outside the black hole and the entropy of the black hole itself will never decrease, $d(S_{out} + \alpha A/\hbar G) \geq 0$ [Bekenstein 1973].

Horizons appear then to be deeply related to the notion of entropy. In particular, the fact that entropy in general scales like the area of the horizon, provides a fundamental relation between thermodynamics and spacetime geometry.

[1] A statistical mechanics derivation of such horizon entropy is originally attempted in [Gerlach 1976]. Also, in [Bekenstein 1975] the entropy is interpreted as the number of internal black hole states consistent with a single black hole exterior, while as the number of horizon quantum states in ['t Hooft 1990; Susskind 1994]. More formal or geometrical interpretations are given in [Jacobson 1994c; Visser 1993; Bañados 1994] and in [Frolov 1997a; Frolov 1997b], based on thermo-field theory. Also, a considerable amount of work has been done in calculating black hole entropy based on different candidate models for quantum gravity [Rovelli 2004; Das 2000].

A fundamental step forward consists in realizing that the physical meaning of this relation should go beyond the framework of black hole dynamics. In fact, the Hawking-Unruh effect itself, for its local nature, contributes to shift the focus from the thermal properties of black holes toward the thermodynamics of horizons. As a general consequence of quantum physics involving horizons, the framework of black hole thermodynamics is then extended to general spacetimes, whenever these are endowed with a stationary horizon, associated with a global or local approximate notion of Killing symmetry [Jacobson 2003; Wald 1992].

Now, being rooted on gravitational kinematics, *horizon thermodynamics* has no apparent relation with gravitational dynamics, since no gravitational field equations are involved in the picture. However, in the last decade, several investigations have shown that this relation indeed exists and is apparently very deep, as it seems to relate different newly recognized patterns concerning gravity.

At a formal level, this relation appears in the equivalence between the gravitational field equations evaluated on a horizon and the thermodynamic identity $TdS = dE + PdV$ [Padmanabhan 2002]. Such equivalence is demonstrated for a wide class of models including stationary axisymmetric and spherically symmetric horizons in Einstein gravity [Kothawala ; Paranjape 2006], generic static horizons and dynamical apparent horizons in Lanczos-Lovelock gravity [Cai 2008; Kothawala 2009], three dimensional BTZ black hole horizons [Jamil 2009], FRW cosmological models in various gravity theories [Cai 2005], up to the case of Hořava-Lifshitz gravity [Cai 2010].

Further, second order gravitational action functionals have been shown to be generally characterized by a holographic relationship between bulk and surface terms. Interestingly, when surface terms are evaluated at the horizon, in any given solution, they give the horizon entropy [Padmanabhan 2002; Mukhopadhyay 2006]. Again, this result extends far beyond Einstein's theory to situations in which the entropy is not proportional to horizon area. Because the surface term has the thermodynamic interpretation, as horizon entropy, and is related holographically to the bulk term, these results again led to suspect an indirect connection between spacetime dynamics and horizon thermodynamics [Padmanabhan 2010].

In this line, it is interestingly shown the gravitational field equations of any diffeomorphism invariant theory can be effectively derived from a principle of maximization of the horizon entropy [Padmanabhan 2002]. Note that, within this context, a thermodynamical derivation of gravitational dynamics is always derived from an "on shell" argument, the gravitational dynamics being already encoded in the Noether charge entropy defined in terms of the gravitational action [Wald 1992; Padmanabhan 2010].

In this sense, the thermodynamical derivation of the Einstein equations appears close to a formally equivalent rewriting of the hamiltonian formulation.

Beyond the formal equivalence, gravitational dynamics can be deduced from a thermodynamic principle even without any direct knowledge about the gravitational action. In any neighborhood of spacetime, the proportionality between the area of the local acceleration horizon and entropy of the locally Minkowski vacuum fields provides a constitutive relation between spacetime geometry and matter. Perturbations of the quantum fields thermal state, described in terms of entropy variations at thermodynamical level, intuitively reflect in a perturbation of the horizon causal structure, measured by an area change. In this sense, one can expect spacetime dynamics to be induced by matter via a thermodynamical principle associated with the entropy variation. It turns out that this principle consist in the general equilibrium *entropy balance* law, $TdS = \delta Q$, with S, δQ and T interpreted as the entanglement entropy, the energy flux and temperature associated with an accelerated observer just inside the horizon. For the equivalence principle, then, this relation holds for all the local Rindler causal horizons through each spacetime point. Viewed in this way, the Einstein equation is an *equation of state*[1] [Jacobson 1995].

The variety and coherence of thermodynamical aspects characterizing the gravitational dynamics then starts being considered as an effective viable road to understand gravity, in parallel to quantum gravity. In particular, the setting provides an intriguing and intuitive way to re-interpret the nature of some possible microscopic quantum degrees of freedom of spacetime, coming from various corners of quantum gravity research, in light of the gravity/thermodynamics correspondence.

A fundamental step towards the idea of geometrodynamics as a form of thermo/hydro dynamics was accomplished in the 70s, thanks to the introduction of the so called *membrane paradigm* approach [Damour 1979; Thorne 1986]. The membrane paradigm regards the event horizon as a two-dimensional membrane that resides in three dimensional space. This membrane is formally described as a viscous fluid with charge, finite entropy and temperature, entailing the fundamental properties of the quantum fields leaving close to the effective horizon surface. The interaction of the horizon with the external universe is therefore described in terms of familiar laws of the horizon's fluid, i.e., Navier-Stokes equation, Ohm's law, tidal force equation, and first and second laws of thermodynamics [Thorne 1986].

[1] The same approach holds when a local generalization of the Noether charge entropy is used, in place of entanglement entropy. This generalizes the"on-shell" thermodynamic derivation [Padmanabhan 2002] to a more general local approach [Parikh 2009].

Now, for black holes, the relationship between the dynamics of a fluid and the dynamics of event horizon is just an analogy. The reason is that hydrodynamics is only a valid effective theory of many-body systems on long spatial and temporal scales [Forster 1995; Lifshitz 2000]. In order for hydrodynamics to be a valid description, the characteristic wavelength and time scale of perturbations to the system must be much larger than the microscopic scale set by a correlation length (or mean free path). This basic criterion cannot be fulfilled even in the familiar example of a spherically symmetric Schwarzschild horizon.

However, the more general Rindler horizons do not have an intrinsic curvature scale. Therefore, a large scale hydrodynamic limit exists in this case. Hence, in particular, the horizon hydrodynamics can be used to extend the framework of horizon thermodynamics to the non-equilibrium regime, where dissipative effects can be taken into account [Eling 2008]. In this case, the thermodynamical derivation of the gravitational equations of motion via the entropy balance principle is generalized to include dissipative effects associated with the propagation of gravitational degrees of freedom which are turned off in the equilibrium regime. Within this setting, the thermodynamical derivation of gravitational dynamics can be extended to generalized theories of gravity, whenever the entanglement entropy can be casted in a form proportional to the horizon area. [Eling 2006; Chirco 2010b; Chirco 2011b].

In this sense, the interplay between gravitational dynamics and thermodynamics is strengthened not only beyond Einstein's general theory of relativity, but, interestingly, also beyond the equilibrium setting. However, to the striking breakthrough given by the thermodynamical interpretation for gravity corresponds the conceptually mind boggling puzzle, regarding the presence of an underlying microscopic level of description for spacetime.

In fact, this puzzle introduces a new dimension to the problem: a funding principle for gravitational dynamics will necessarily have to do with the way gravity *emerges* from a consistent characterization of the microscopic degrees of freedom associated with the thermodynamical behavior at the macroscopic scale, though, one should expect such a general principle to be independent of the specific details of the underlying microscopic theory.

In quantum gravity, the problem of a statistical mechanical level of description for gravity deals with the idea that a microscopic, or quantum, description of gravity should derive from a quantization of space and time, that is, from a quantization of geometry. In this sense, *string theory*, *loop quantum gravity*, *group field theory* and *lattice* models, attempt to develop a quantum pre-geometric replacement for classical differential geometry. Such pre-geometric level of description is supposed to be relevant at extremely small distances, where the quantum aspects of the theory would be expected to dominate, while at larger distances (where the classical aspects dominate) one would hope to recover both ordinary differential geometry and specifically Einstein gravity or some generalization of it [Carlip 2001; Smolin 2003; Friedan 2003; Rovelli 2004; Bousso 2004; Oriti 2005; Freidel 2005].

An alternative point of view consists in looking at the emergence of gravitational dynamics as a direct manifestation on a macroscopic scale of the thermodynamics of the vacuum. The idea follows the suggestion, typically attributed to Sakharov [Sakharov 1968; Visser 2002], that gravity itself may not be fundamental physics, in the same way fluid mechanics, as an example, is only the low-energy low-momentum limit of a known underlying microphysics, given by molecular dynamics. In this case, the concepts of density and velocity field make no sense at the microphysical level and emerge only as one averages over time-scales and distance-scales larger than the mean free time and mean free path. Analogously, in such *induced gravity* picture, the description of spacetime in terms of a metric, $g_{ab}(t, \mathbf{x})$, is considered as an *emergent* phenomenon, valid at scales large compared with some critical length, which possibly could be the Planck length [Adler 1982; Novozhilov 1991; Fursaev 2004]. In particular, in the emergent perspective, interesting insights are given by the study of non gravitational *analogue models*, which set several parallels between induced gravity and condensed matter systems [Barceló 2005].

In all these cases, a fundamental element for the understanding of a possible emergent scenario is given by the area scaling behavior of the horizon entropy. From a statistical thermodynamics point of view, the entropy measures the number of independent quantum states compatible with the macroscopic parameters characterizing a thermal state, that is $\mathcal{N} = e^S$. This number corresponds to the number of microscopic degrees of freedom, say $N = \ln \mathcal{N}$, which are necessary to fully describe the physics of the thermal state. Now, for the thermalized quantum fields responsible for the Hawking-Unruh effect the area/entropy proportionality implies that $N \propto A$, whereas, for a general local quantum field theory this number generally is proportional to the volume of the system. The *holographic* behavior of the localized fields then suggests that gravity is associated with a reduction of the degrees of freedom necessary to describe the ambient quantum fields, whatever is their detailed nature.

This phenomenon originally led t'Hooft and Susskind to conjecture a description of the microscopic degrees of freedom associated with the back hole entropy given by means of a quantum field theory in one dimension less ['t Hooft 1993; Susskind 1995].

More radically, this conjecture is then formalized in a *holographic principle* stating that the physics of a region delimited by a boundary of area A should be fully describd by no more than $A/4$ degrees of freedom. For the black hole solution, for example, this implies that the lower dimensional quantum field theory should contain all the physical information of the gravitational solution. In this sense, the holographic principle formally interpret the effect of gravitational dynamics on the quantum field Hilbert space in terms of an effective *duality* between gravity in $(d+1)$ dimensions and quantum field theories in (d) dimensions.

The most prominent example of holographic duality is realized by the AdS/conformal field theory (CFT) correspondence [Maldacena 1998; Gubser 1998; Aharony 2000] in which the gravity theory, given by Type IIB string theory in asymptotically Anti-de Sitter spacetimes, $AdS_5 \times S^5$, is dual to $SU(N)$ $\mathcal{N} = 4$ Super Yang-Mills (SYM) conformal field theory on the boundary.

In such a duality, a classical black hole in AdS spacetime corresponds to a strongly coupled thermal CFT at the Hawking temperature leaving on the spacetime boundary. In particular, in reminiscence of the membrane paradigm, the large scale dynamics of the black hole therefore is dual to the hydrodynamics of the thermal gauge theory [Bhattacharyya 2008a; Rangamani 2009]. The resulting map between gravity and fluid dynamics has come to be known as the *fluid/gravity correspondence*. Hydrodynamic transport coefficients such as viscosities are calculated from a microscopic theory using "Kubo formulas", which involve finite temperature Green's functions of conserved currents. The duality picture allows one to determine the transport coefficients of these strongly coupled theories by mapping the calculation of Green's functions into a classical boundary value problem in the bulk spacetime [Son 2007]. As a consequence, the transport coefficients of the dual gauge theory can be calculated directly at the black hole horizon from the membrane paradigm [Kovtun 2003a; Saremi 2007; Starinets 2009; Iqbal 2009]. In particular, a key early result that emerged from this work is that, in the limit of infinite coupling, any (not necessarily conformal) gauge theory with an Einstein gravity dual has a shear viscosity to entropy density ratio of $\eta/s = \hbar/4\pi k_B$. This value was conjectured by Kovtun, Son, and Starinets (KSS) to be a universal lower bound [Kovtun 2005].

Again, the fluid/gravity map suggests a strong connection between gravity and statistical physics [Hubeny 2011].

Indeed, the picture described above conceptually leads to think of Einstein's equations in the presence of a regular event horizon as the strong coupling analogue of the Boltzmann transport equations. In particular, the analogy with the Boltzmann transport equations implies that these equations are irreversible, somehow in line with the non-equilibrium thermodynamical description. In analogy with the Boltzmann H-theorem, which asserts that a certain functional of kinetic variables called H always increases in time and is maximum in equilibrium, Einstein's equations, together with the assumption of regularity of future event horizons (and physical energy conditions), always obey the classic area increase theorem of general relativity. The condition of regularity of the future event horizon then seems to break the time reversal invariance of Einstein's equations. The Boltzmann theorem has a local analogue in fluid dynamics, which maps to the statement that the equations of fluid dynamics are accompanied by a local entropy current with everywhere non-negative divergence. The area increase theorem of general relativity can be used to construct such an entropy current for the fluid dynamics generated from the fluid/gravity map.

Overall then, the holographic duality scenario provides further support to the idea of gravity as an emergent phenomenon, arising from some microscopic description that does not know about its existence. The effective thermodynamical or more generally hydrodynamical character of gravitational dynamics seems to originate from some interplay between irreversibility and causality, which is encoded in the fundamental notion of entropy and its generalized second law and somehow reflected in a statistical mechanical description of the fundamental microscopic level. When a geometric structure of spacetime is introduced in the game the notion of entropy is associated with the notion of horizon while somehow irreversibility is encoded in the holographic principle.

In this sense, as far as the query for a funding principle for gravitational dynamics, the holographic principle seems to be more fundamental than thermodynamics itself. Following this line of thought, a further different emergent gravity argument has been recently proposed by Verlinde [Verlinde 2011]. In this approach, inspired by the AdS/CFT and open closed string correspondence, the nature of the degrees of freedom of the fundamental microscopic theory is generally treated as information, to be associated with matter and its location on the emergent level of description. This information is measured in terms of entropy associated with the general boundary/screen constraining the matter volume, by means of the holographic principle. Changes in the entropy, associated with matter displacement at macroscopic level, are consequence of the statistical averaged random dynamics at the microscopic level. Therefore, gravitational dynamics is interpreted as the ultimate effect of an entropic force. In this sense, the thermodynamical description of gravity *effectively* derives from the holographic principle together with the equivalence principle.

1.2 A Very Local Point of View

Out of this wide picture, gravity seems to loose its status of fundamental force. Despite the number of different theoretical approaches involved, the set of fundamental principles which lead to such a suggestion, is small and very general. Given that the interplay between gravity, quantum mechanics and thermodynamics arises at a *local level*, with the Hawking-Unruh effect at first, we wonder whether it is possible to reproduce the conceptual series of arguments which led us from the equivalence principle to the holographic principle at a local level, starting from the point of view of an accelerated observer, that is from the physics of a Rindler acceleration horizon in flat Minkowski spacetime. Despite the apparently oversimplified emergent scenario, an affirmative answer to this question would be quite interesting, given that, being locally Minkowski, gravity would be actually absent.

In this thesis we shall consider this point of view by starting from the *local* horizon thermodynamics framework. We will take the area scaling properties of the entanglement entropy of the Rindler horizon as a conceptually basic realization of the holographic principle. From the generalized second law and the Bekenstein bound we derive the gravitational dynamics via the entropy balance approach developed in [Jacobson 1995].

Here, we show how this setting can account for the equilibrium and the non-equilibrium features associated with the gravitational dynamics, in relation with the activation and the propagation of the gravitational degrees of freedom entailed in the definition of the entropy functional, via the area-entropy proportionality relation. We suggest that the allowed gravitational degrees of freedom can be fixed by the kinematics of the local spacetime causal structure, through the specific equivalence principle formulation used, providing an interpretative advance in support of the thermodynamical derivation of gravity. These considerations lead us to a first extension of the thermodynamical derivation of the gravity field equations from General Relativity to generalized Brans-Dicke theories.

We concentrate then on the possibility to define a version of fluid/gravity correspondence within the local Rindler wedge setting. We show how the hydrodynamical description of the horizon can be directly associated with a hydrodynamical description of the thermal fields. Because of the holographic behavior, the properties of the Rindler wedge thermal gauge theory are effectively encoded in a codimension one system living close to the Rindler horizon. In a large scale analysis, this system can be thought of as a fluid living on a codimension one stretched horizon membrane. This sets an apparent duality between the horizon local geometry and the fluid limit of the thermal gauge theory.

From this field theory, with a classical linear response approach, we derive the horizon viscosity via a holographic Kubo formula in terms of a two-point function of the stress tensor of matter fields in the bulk. The entanglement viscosity over entropy density ratio turns out to satisfy the universal Kovtun-Son-Starinets value in four dimensions, $1/4\pi$, suggesting the universal ratio may be a fundamental property of quantum entanglement and its associated holography more than being rooted in quantum gravitational physics [Chirco 2010a].

This approach naturally identifies the underlying fundamental system with the Minkowski vacuum, in an induced gravity flavor, in line with the work done by Candelas and Sciama [Candelas 1977], where black holes are shown to be effectively conform to the principles of *nonequilibrium* and *irreversible thermodynamics*, in the form of a fluctuation-dissipation theorem and the dissipation of a gravitational perturbation [Hawking 1972] can be measured in terms of the quantum fluctuations of the black hole gravitational field. Note, however, that the actual nature of the microscopic degrees of freedom is not central in our analysis.

Beyond the connection between the classical Navier-Stokes equations and a classical geometry, based on some recent works [Bredberg 2011b; Bredberg 2011a; Compere 2011] suggesting the possibility of an underlying holographic duality relating a theory on fixed r_c to the interior bulk of the Rindler spacetime, we discuss the possibility to realize such a duality at any point in spacetime by means of the equivalence principle. Given the shared local Rindler geometric setting, the fluid/Rindler correspondence seems to be naturally linked to derivations of the gravitational field equations from a local non-equilibrium spacetime thermodynamics. Interestingly, given the possibility to derive the gravitational dynamics from the horizon hydrodynamics and contemporarily define a duality between the horizon fluid and a dimensionally reduced quantum field theory, we consider the opportunity to make gravity emerge in an holographic way from the dual theory fluid dynamics. Therefore, we eventually deal with the intriguing possibility to produce a gravitational dynamics as a dual of a non relativistic emergent field theory and discuss the possible consequence of these results within the context of the emergent gravity scenario.

1.3 Plan of the Thesis

The thesis content is organized as follows. Chapter 2 will provide a toolbox for the basic physics involved in the general framework of emergent gravity thermodynamics. Here we set a basic conceptual chain of arguments connecting the equivalence principle to the holographic principle, which will be successively reproduced in our local approach. Chapter 3 will introduce and characterize the notion of local Rindler frame associated with a local acceleration horizon. This frame will constitute a sort of local gedanken laboratory, where gravitational dynamics can be studied as an effectively apparent phenomenon. In this chapter we introduce the thermodynamical derivation of Einstein's equation based on the thermal properties of the local Minkowski vacuum. This approach will be repeatedly adopted throughout the thesis work. The next three chapters contain the work developed during the PhD studies. In Chapter 4 we argue on the necessity of a non-equilibrium treatment of the thermodynamical derivation and discuss its extension to generalized gravity theories. At this stage we deal with a scenario where gravity emerges from the holographic hydrodynamics of some microscopic, quantum system. In Chapter 5 we characterize the link between horizon hydrodynamics and vacuum fluctuations by proposing an entanglement derivation of the horizon viscosity coefficient. Chapter 7 proposes an extension of the fluid/gravity duality to higher order gravity theories and discuss about the possibility to realize a dual description at a local level, with a consequential connection to the local thermodynamical derivation of gravity form the setting previously discussed. Eventually, in Chapter 8 we will conclude with a comment on the reliability of the proposed scenario, its possible connections with the different approaches reviewed and its future perspectives.

The content of the thesis is based on the following selected research papers:

- G. Chirco, S. Liberati,
  "Non-equilibrium Thermodynamics of Spacetime: The Role of Gravitational Dissipation."
  Phys. Rev. D, 81, 024016, (2010).
  e-Print: arXiv:0909.4194 [gr-qc]

- G.Chirco, C. Eling, S.Liberati,
  "Reversible and Irreversible Spacetime Thermodynamics for General Brans-Dicke Theories."
  Phys. Rev. D 83, 024032 (2011).
  e-Print: arXiv:1011.1405 [gr-qc]

- G. Chirco, C. Eling, S. Liberati,
  "The universal viscosity to entropy density ratio from entanglement."
  Phys. Rev. D, 82, 024010 (2010).
  e-Print: arXiv:1005.0475 [hep-th]

- G.Chirco, C. Eling, S.Liberati,
  "Higher Curvature Gravity and the Holographic fluid dual to flat spacetime."
  JHEP Number 8, 9, (2011).
  e-Print: arXiv:1105.4482v2 [hep-th]

Chapter 2

Gravity as Thermodynamics: a Toolbox

In this chapter we collect the minimal set of arguments necessary to conceptually relate the equivalence principle to the holographic principle in a spacetime neighborhood. From this set, by mean of a *local* horizon thermodynamics we will start, in the following chapters, to investigate the problem of gravitational dynamics in a way similar, in spirit, to using freely falling observers to determine the kinematics of gravity.

2.1 Equivalence Principle ⇔ Acceleration

The Equivalence Principle is already incorporated in Newtonian gravity. Newton points out in the *Principia* that the "mass" of any body — meaning the quantity that regulates its response to an applied force — and the "weight" of the body — the property regulating its response to gravitation — should be equal. This equivalence is later reformulated [Bondi 1957] in terms of a distinction in the definition of mass: the *inertial mass*, the ratio between force and acceleration in Newton's second law, which measures a particle's resistance to acceleration, and the *gravitational mass*, gravitational analog of electric charge, which appears in the equation for the attractive force

$$f = -\frac{Gmm'}{r^2}, \tag{2.1}$$

G being the gravitational constant. Because of the symmetry of Eq. (2.1) (Newton's third law), in Newton's theory the two masses are essentially equivalent. In this sense, the Newton's equivalence principle states that for *all* particles, inertial and gravitational masses are in the same proportion. In particular, given that *all* particles experience the same gravitational field, the path followed by a particle in space and time is entirely independent of its internal structure and composition.

Newton's equivalence principle, or *Weak Equivalence Principle* (WEP), thus predicts that mechanics will take *precisely* the same course in a freely falling laboratory (e.g. the famous elevator cabin) as in a laboratory that is unaccelerated and far away from all attractive masses, i.e., as in a strict *inertial frame*.

In the general theory of relativity (GR), Einstein essentially assumes that the rest of physics goes along with mechanics and thereby postulates that [Einstein 1907] " *all* local freely falling, non rotating laboratories are fully equivalent for the performance of *all* physical experiments".

More technically, Einstein's reformulation of the equivalence principle, known as *Strong Equivalence Principle* (SEP), states that [Will 1981]: (i) the WEP is valid for self-gravitating bodies as well as for test bodies, (ii) the outcome of *any* local test experiment is independent of the velocity of the freely-falling apparatus and (iii) the outcome of any local test experiment is independent of where and when in the universe it is performed.

Therefore, the SEP essentially extends the validity of the WEP to self-gravitating bodies and adds two more important statements: Local Lorentz Invariance (LLI) and Local Position Invariance (LPI).

A freely-falling observer carries a local frame in which test bodies have unaccelerated motions, i.e. a *local inertial frame* (LIF). According to the requirements of the LLI, the outcomes of *all* experiments are independent of the velocity of the LIF and therefore if two such frames located at the same event $\mathcal{P}$ have different velocities, this should not affect the predictions for identical experiments. LPI requires that the above should hold for all spacetime points. Therefore, roughly speaking, in local freely falling frames the theory should reduce to Special Relativity.

This implies that there should be at least one second rank tensor field which in LIF reduces to the Minkowski metric η_{ab}. In particular, since at each event $\mathcal{P}$ there exist local frames called *local Lorentz frames*, one can find suitable coordinates at $\mathcal{P}$, by which

$$g_{ab} = \eta_{ab} + \mathcal{O}\left(\sum_{a} |x^a - x^a(\mathcal{P})|^2\right) \tag{2.2}$$

and $\partial g_{ab}/\partial x^a = 0$. Therefore, in local Lorentz frames, the geodesics of the metric g_{ab} are straight lines. In particular, free-fall trajectories are straight lines in a local freely-falling frame[1]. This is extremely important as it ensures the possibility to define light cones and hence causality at local level.

The extension of LLI and LPI to all the local experiments, including local gravitational experiments, is a quite strong requirement. For the time being there is no theory other than GR that satisfies the SEP [2].

However, one can consider a weaker formulation of the equivalence principle, where only local *non* gravitational experiments are taken into account. Such a formulation is provided by the *Einstein Equivalence Principle (EEP)*: (i) the WEP is valid, (ii) the outcome of any local *non-gravitational* test experiment is independent of the velocity of the freely-falling apparatus (LLI) and (iii) the outcome of any local non-gravitational test experiment is independent of where and when in the universe it is performed (LPI).

Again, EEP predicts that in local freely falling frames the theory should reduce to Special Relativity. However, in this case, the second rank tensor field in the local freely falling frame can reduce to a metric which is *conformal* with the Minkowski one. The freedom of having an arbitrary conformal factor is due to the fact that the EEP does not forbid a conformal rescaling in order to arrive to special-relativistic expressions for the physical laws in the local freely-falling frame[3].

One can then conclude that rescaling coupling constants and performing a conformal transformation leads to a metric g_{ab} which, in every freely falling local frame reduces (locally) to the Minkowski metric η_{ab}. It should be stressed that all conformal metrics φg_{ab} (φ being the conformal factor) can be used to write down the equations or the action of the theory. However, since at each event $\mathcal{P}$ there exist local frames called *local Lorentz frames*, one can find suitable coordinates in which the expression (2.2) holds [Sotiriou 2007a].

The possibility of considering different formulations of the equivalence principle will be extremely important for our approach to gravitational dynamics (see Chapter 3). However, at this stage, we want to essentially focus on the basic implication of the equivalence between gravitational and inertial mass, that is the fact that no external

[1]Identifying the two frames we realize that the geodesics of g_{ab} coincide with free falling trajectories.

[2]It should be stressed that in principle it is possible to build up other theories that satisfy the SEP but up to this point only one is actually known: Nordström's conformally-flat scalar theory, which dates back to 1913 [Nordström 1913]. However, this theory is not observationally viable since it predicts no deflection of light.

[3]Note however, that while one could think of allowing each specific matter field to be coupled to a different one of these conformally related second rank tensors, the conformal factors relating these tensors can at most differ by a multiplicative constant if the couplings to different matter fields are to be turned into constants under a conformal rescaling as the LPI requires (this highlights the relation between the LPI and varying coupling constants).

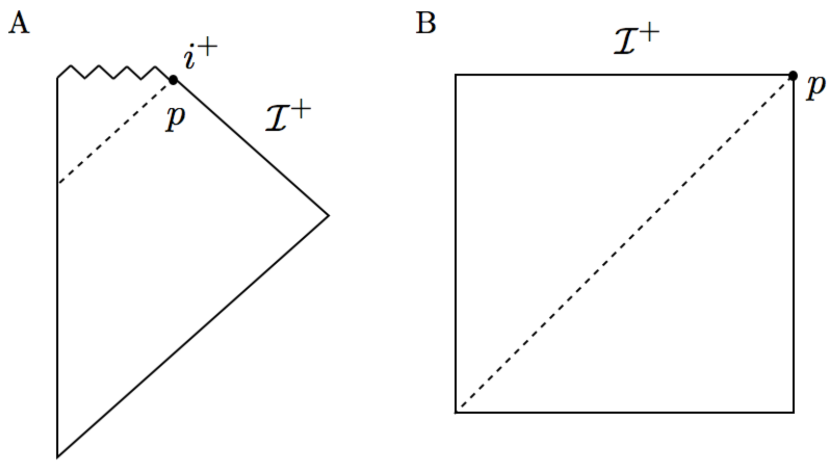

Figure 2.1: Carter Penrose diagrams of causal horizon in conformal compactification of A) Asymptotically flat black hole spacetime and B) de–Sitter spacetime. Future null infinity is denoted by $\mathcal{I}^+$, while future time-like infinity by i^+.

static homogeneous gravitational field can be detected in a *local inertial frame*. This effectively implies that one can not only *locally* eliminate gravity by free fall, but equivalently *locally* create it by *acceleration*.

2.2 Acceleration ⇔ Horizons

In a local inertial frame, an accelerating observer will necessarily outrun photons. Clearly, this observer will never reach the speed of light; however, if light starts out from a point sufficiently far behind her, it will never catch her up. As a consequence, there will be a causally hidden region of events whose forward light cones never intersect the observer's world line. In the observer's accelerating frame, the boundary of the hidden region defines a *causal horizon*.

2.2.1 Causal Horizons

A causal horizon is generally defined as the boundary of the past of any time-like curve λ of infinite proper length in the future direction [Gibbons 1977].

For a black hole in asymptotically flat spacetime, e.g., the event horizon is the boundary of the past of all of $\mathcal{I}^+$. However, an equivalent definition is given by the boundary of the past of *any* time-like curve that goes to future time-like infinity i^+. Thus, the event horizon can be equivalently defined as the boundary of the past of i^+ itself (see Fig. 2.2). In this sense, the black hole horizon is defined with reference to the intrinsic asymptotic structure of the spacetime, without referring to a particular class of observers.

Differently, in a spacetime that is asymptotically de Sitter in the future, an asymptotic de Sitter horizon is defined by the boundary of the past of a time-like worldline reaching a point p at space-like $\mathcal{I}^+$, or equivalently the boundary of the past of p. Since there are many different points at $\mathcal{I}^+$, in this case there will be many inequivalent asymptotic de Sitter horizons. Therefore, an asymptotic de Sitter horizon is said to be *observer-dependent*. The observer dependent character of this notion of horizon will be important for a universal characterization of the thermodynamical description of gravity.

In particular, in asymptotically flat spacetime, one can generally define an observer dependent asymptotic Rindler horizon (ARH) by the boundary of the past of an *accelerated* worldline that goes to a point p on $\mathcal{I}^+$ (which is here null), or equivalently the boundary of the past of p.

In Minkowski spacetime an ARH is just what is usually meant by a *Rindler horizon*. This is the simplest context where horizons arise for a class of observers. In particular, having in mind the conceptual expedient by which in a local inertial frame gravity can be "created" by means of acceleration, the properties of the Rindler horizon associated with an accelerated observer in Minkowski spacetime will constitute the basic setting for our discussion.

2.2.2 Rindler Wedge

The Minkowski line element in general $(d+1)$-dimensions can be written in both *Cartesian* (Minkowski), $x^a = (t, z, x^i)$, and *polar* (Rindler) coordinates, $y^a = (\tau, \xi, x^i)$, where $i = 1..d$ (d is the number of transverse spatial dimensions),

$$ds^2 = -dt^2 + dz^2 + dx^i dx_i = -\kappa^2 \xi^2 d\tau^2 + d\xi^2 + dx^i dx_i, \tag{2.3}$$

the Rindler metric being obtained by the coordinate transformation,

$$\begin{aligned} t &= \xi \sinh(\kappa\tau) \\ z &= \xi \cosh(\kappa\tau) \\ x^i &= x^i \end{aligned} \tag{2.4}$$

Here κ is an arbitrary constant with dimensions $[L]^{-1}$ associated with the normalization of the time-like Killing vector ∂_τ. The opportunity to introduce the constant κ will be discussed in Chapter 3.

Unlike the global inertial coordinates x^a, the Rindler coordinates only cover a "wedge" subregion of Minkowski space where $z > |t|$, the so called *Rindler wedge*.

In the first form of the line element the translation symmetries generated by the Killing vectors ∂_t and ∂_z are manifest, while, in the second form, the boost (hyperbolic rotation) symmetry generated by the Killing vector ∂_τ is manifest. The latter is clearly analogous to rotational symmetry in Euclidean space. In particular, the time-like Killing flow ∂_τ is equivalent to a continuous boost in the z direction. The respective boost time parameter τ is proportional to the proper time along the worldlines of the uniformly accelerated observer defined by the $\xi = const$ hyperbolas.

Therefore, the Rindler acceleration horizon associated to a uniformly accelerated observer in flat spacetime is a bifurcate Killing horizon[1]. The pair of planes at $z = t$ (in Cartesian coordinates) defines a bifurcate Killing horizon associated to the boost Killing flow ∂_τ, the bifurcation surface coinciding with the x^i plane at $z = 0$ and $t = 0$.

This symmetry has a fundamental role when quantum physics comes into play.

2.3 Killing Horizon Symmetry ⇔ Thermal Quantum Fields

The Rindler wedge of Minkowski, where $\xi^a = (\partial_\tau)^a$ is time-like, can be seen locally as a spacetime in its own right. In particular, one can construct a quantum field theory for this region, using ξ^a as the time-like Killing vector field. However, in the 1970's, it was realized that quantization of fields on the Rindler spacetime (2.3) is inequivalent to the usual field quantization in full Minkowski spacetime. The reason is that the Rindler

[1] A causal horizon whose generators coincide with some Killing flow is a Killing horizon, i.e. a null hypersurface generated by a Killing flow. A bifurcate Killing horizon (in four dimensions) is a pair of Killing horizons which intersect in a particular two dimensional space-like cross section - called the bifurcation surface - on which the Killing vector vanishes. Examples of these occur in Minkowski, de Sitter, Anti de Sitter, and Schwarzschild spacetimes. Given that the various examples of Killing horizons are all indistinguishable in a neighborhood of the bifurcation surface smaller than the curvature scale, the acceleration horizon provides a universal template for all of them.

Hamiltonian generates a flow in boost time. It follows that the notion of the vacuum for the Rindler quantization must be different than the usual Minkowski vacuum $|0\rangle$. In particular, a non-inertial observer will determine a different zero energy state, called the Fulling-Rindler vacuum $|F\rangle$ [Fulling 1973].

2.3.1 The Unruh Effect

On the other hand, Unruh's discovery [Unruh 1976] was that when expressed in the Rindler wedge Fock space the restriction of the Minkowski vacuum $|0\rangle$ appears to be a *thermal state*, characterized by a density matrix $\rho = Z^{-1} \exp(-H_B/T)$, where H_B is the operator generating Lorentz boost on the quantum fields and the "temperature" is $T = \hbar\kappa/2\pi c$. This temperature does not have dimensions of energy. However, if one rescales H_B to generate proper time translations defined by ξ^a on the world line of the uniformly accelerated observer, this temperature becomes the Unruh temperature,

$$T_U = T\,(\kappa\xi)^{-1} = \hbar a/2\pi c, \tag{2.5}$$

associated to an observer with acceleration $a = \xi^{-1}$. Thus there is already something "thermal" about the vacuum fluctuations even in flat spacetime and effect is rooted in the existence of the causal horizon and its stationarity properties.

Owing to the symmetry of the Minkowski vacuum under translations and Lorentz transformations, the vacuum will appear stationary in a uniformly accelerated frame. Moreover, since it is the ground state, it is stable to dynamical perturbations. Stationarity and stability of the state alone are sufficient to indicate that the state is a thermal one, as shown by Haag [Haag 1992] in axiomatic quantum field theory.

We provide two derivations of the Unruh effect, both of which are valid for arbitrary interacting scalar fields in spacetime of any dimension. The arguments will closely follow a simple and clear exposition provided by Jacobson in his notes on Black Hole Thermodynamics [Jacobson 2005].

2.3.2 Two-point Function and KMS Condition

A thermal density matrix $\rho = Z^{-1}\exp(-\beta H)$ has two identifying properties. First, it is stationary, since it commutes with the Hamiltonian H. Second, because $\exp(-\beta H)$ coincides with the evolution operator $\exp(-itH)$ for $t = -i\beta$, expectation values in the state ρ possess a certain symmetry under translation by $-i\beta$, known as the KMS condition [Sewell 1986; Haag 1992]: Let $\langle A\rangle_\beta$ denote the expectation value $tr(\rho A)$, and let A_t denote the time translation by t of the operator A. Using cyclicity of the trace we have

$$\begin{aligned}\langle A_{-i\beta}B\rangle_\beta &= Z^{-1}tr\left(e^{-\beta H}(e^{\beta H}Ae^{-\beta H})B\right) \\ &= Z^{-1}tr\left(e^{-\beta H}AB\right) \\ &= \langle AB\rangle_\beta\end{aligned} \tag{2.6}$$

Note that for nice enough operators A and B, $\langle A_{-i\tau}B\rangle_\beta$ will be analytic in the strip $0 < \tau < \beta$. Now let us compare this behavior with that of the two-point function along a uniformly accelerated worldline in the Minkowski vacuum.

If, as usual, the vacuum state shares the symmetry of Minkowski spacetime, then, in particular, the 2-point function $G(x, x') = \langle\varphi(x)\varphi(x')\rangle$ must be a Poincaré invariant function of x and x'. Thus it must depend on them only through the invariant interval, so one has $G(x, x') = f((x - x')^2)$ for some function f. Now consider an observer traveling along the hyperbolic trajectory $\xi = a^{-1}$. This worldline has constant proper acceleration a, and $a\tau$ is the proper time along the world line. Let us examine the 2-point function along this hyperbola

$$\begin{aligned}G(\tau, \tau') &\equiv G(x(\tau), x(\tau')) \\ &= f\left([x(\tau) - x(\tau')]^2\right) \\ &= f\left(4a^{-2}\sinh^2[(\tau - \tau')/2]\right),\end{aligned} \tag{2.7}$$

where the third equality follows from (2.4)[1]. Now, since $\sinh^2(\tau/2)$ is periodic under translations of τ by $2\pi i$, it appears that $G(\tau, \tau')$ is periodic under such translations in each argument. In terms of the 2-point function the KMS condition implies $G(\tau - i\beta, \tau') = G(\tau', \tau)$, which is not the same as translation invariance by $-i\beta$ in each argument. This does not mean that in fact the 2-point function in the Minkowski vacuum along the accelerated worldline is not thermal. First of all, a Poincaré invariant function of x and x' need not depend only on the invariant interval. It can also depend on the invariant step-function $\theta(x^0 - x'^0)\theta((x - x')^2)$. More generally, the analytic properties of the function f have not been specified, so one cannot conclude from the periodicity of $\sinh^2(\tau/2)$ that f itself is periodic[2].

To reveal the analytic behavior of $G(x, x')$, it is necessary to incorporate the con-

[1] Note that we will put $\kappa, \hbar, c = 1$ all through the following argument.

[2] For example, f might involve the square root, $\sinh(\tau/2)$, which is anti-periodic. In fact, this is just what happens.

ditions that the spacetime momenta of states in the Hilbert space lie inside or on the future light cone and that the vacuum has no four-momentum. One can show (by inserting a complete set of states between the operators) that these imply the existence of an integral representation for the 2-point function of the form

$$G(x,x') = \int d^n k \theta(k^0) J(k^2) e^{-ik(x-x')} \tag{2.8}$$

where $J(k^2)$ is a function of the invariant k^2 that vanishes when k is space-like. Now let us evaluate $G(\tau,\tau')$ along the hyperbolic trajectory. Lorentz invariance allows us to transform to the frame in which $x - x'$ has only a time component which is given by $2a^{-1}\sinh[(\tau-\tau')/2])$. Thus we have

$$G(\tau,\tau') = \int d^n k \theta(k^0) J(k^2) e^{-i2a^{-1}k^0 sinh[(\tau-\tau')/2]} \tag{2.9}$$

Now consider analytic continuation $\tau \to \tau - i\theta$. Since only $k^0 > 0$ contributes, the integral is convergent as long as the imaginary part of the sinh is negative. One has $\sinh(x+iy) = \sinh x\, \cos y + i\, \cosh x\, \sin y$, so the integral converges as long as $0 < \theta < 2\pi$. Since $\sinh(x - i\pi) = \sinh(-x)$, we can finally conclude that $G(\tau - i2\pi, \tau') = G(\tau', \tau)$, which is the KMS condition (2.7).

2.3.3 The Vacuum State as a Thermal Density Matrix

The essence of the Unruh effect is the fact that the density matrix describing the Minkowski vacuum, traced over the states in the region $z < 0$, is precisely a Gibbs state for the boost Hamiltonian H_B at a temperature $T = 1/2\pi$,

$$tr_{z<0}|0\rangle\langle 0| = Z^{-1}\exp(-2\pi H_B), \tag{2.10}$$

with

$$H_B = \int T_{ab}(\partial_\tau)^a d\Sigma^b. \tag{2.11}$$

This rather amazing fact has been proved with varying degrees of rigor by many different authors. A proof of such a result is provided here by a path integral argument.

As anticipated at the top of the section, since the boost Hamiltonian has dimensions of an action rather than energy, so does the temperature. Note from (2.4) that the norm of the Killing field ∂_τ on the orbit $\xi = a^{-1}$ is a^{-1}, whereas the observer has unit 4-velocity. If the Killing field is scaled by a so as to agree with the unit 4-velocity at

$\xi = a^{-1}$, then the boost Hamiltonian (2.11) and temperature are scaled in the same way. Thus the temperature appropriate for the observer at $\xi = a^{-1}$ is $T = a/2\pi$. Since a is the proper acceleration of this observer, we recover the Unruh temperature defined above. Alternatively, the two-point function defined by (2.10) along the hyperbola obviously satisfies the KMS condition relative to boost time τ at temperature $1/2\pi$. When expressed in terms of proper time $a\tau$, this corresponds to the temperature $a/2\pi$.

In particular, one can view the relative coolness of the state at larger values of ξ as being due to a redshift effect - in this case a Doppler shift - as follows. Suppose a uniformly accelerated observer at ξ_0 sends some of the thermal radiation he sees to another uniformly accelerated observer at $\xi_1 > \xi_0$. This radiation will suffer a redshift given by the ratio of the norms of the Killing field: say p is the spacetime momentum of the radiation, then $p \cdot \partial_\tau$ is conserved [Wald 1984], but the energy locally measured by the uniformly accelerated observer is $p \cdot \partial_\tau / |\partial_\tau|$, so that $E_1/E_0 = |\partial_\tau|_0 / |\partial_\tau|_1$. This is precisely the same as the ratio T_1/T_0 of the locally measured temperatures. At infinity $|\partial_\tau| = \xi$ diverges, so the temperature drops to zero, which is consistent with the vanishing acceleration of the boost orbits at infinity.

The path integral argument to establish (2.10) goes like this: Let H be the Hamiltonian generating ordinary time translation in Minkowski space. The vacuum $|0\rangle$ is the lowest energy state, and we suppose it has vanishing energy: $H\,|0\rangle = 0$. If $|\psi\rangle$ is any state with nonzero overlap with the vacuum, then $\exp(-\tilde{t}H)|\psi\rangle$ becomes proportional to $|0\rangle$ as the imaginary time $\tilde{t}$ goes to infinity. That is, the vacuum wavefunctional $\Psi_0[\varphi]$ for a field φ is proportional to $\langle\varphi|\exp(-\tilde{t}H)|\psi\rangle$ as $\tilde{t} \to \infty$. Now this is just a matrix element of the evolution operator between imaginary times $\tilde{t} = -\infty$ and $\tilde{t} = 0$, and such matrix elements can be expressed as a path integral in the "lower half" of Euclidean space,

$$\Psi_0[\varphi] = \int_{\varphi(-\infty)}^{\varphi(0)} \mathcal{D}\varphi \exp(-I), \tag{2.12}$$

where I is the Euclidean action. The key idea in recovering (2.10) is to look at (2.12) in terms of the angular "time"-slicing of Euclidean space instead of the constant $\tilde{t}$ slicing. The relevant Euclidean metric (restricted to two dimensions for notational convenience) is given by

$$ds^2 = d\tilde{t}^{\,2} + d\sigma^2 = \rho^2 d\theta^2 + d\rho^2. \tag{2.13}$$

Adopting the angular slicing, the path integral (2.12) is seen to yield an expression for the vacuum wavefunctional as a matrix element of the boost Hamiltonian (2.11) which coincides with the generator of rotations in Euclidean space,

$$\langle\varphi_L\varphi_R|0\rangle = \mathcal{N}\,\langle\varphi_R|\exp(-\pi H_B)|\varphi_L\rangle \tag{2.14}$$

where φ_L and φ_R are the restrictions of the boundary value $\varphi(0)$ to the left and right half spaces respectively, and a normalization factor $\mathcal{N}$ is included. The Hilbert space $\mathcal{H}_R$ on which the boost Hamiltonian acts consists of the field configurations on the right half space $z > 0$, and is being identified via reflection (actually composed with CPT [Bisognano 1975; Bisognano 1976; Sewell 1982]) with the Hilbert space $\mathcal{H}_L$ of field configurations on the left half space $z < 0$. The entire Hilbert space is $\mathcal{H} = \mathcal{H}_L \bigotimes \mathcal{H}_R$, modulo the degrees of freedom at $z = 0$.

Using the expression (2.14) for the vacuum wave functional we can now compute the reduced density matrix for the Hilbert space $\mathcal{H}_R$: Now consider the vacuum expectation value of an operator $\mathcal{O}_R$ that is localized on the right half space,

$$\begin{aligned}
\langle\varphi'|(tr_L|0\rangle\langle 0|)|\varphi\rangle &= \sum_{\varphi_L}\langle\varphi_L\varphi'|0\rangle\langle 0|\varphi_L\varphi\rangle \\
&= \mathcal{N}^2\sum_{\varphi_L}\langle\varphi'|\exp(-\pi H_B)|\varphi_L\rangle\langle\varphi_L|\exp(-\pi H_B)|\varphi\rangle \\
&= \mathcal{N}^2\langle\varphi'|\exp(-2\pi H_B)|\varphi\rangle
\end{aligned} \tag{2.15}$$

where (2.14) was used in the second equality.

This shows that, as far as observables located on the right half space are concerned, the vacuum state is given by the thermal density matrix (2.10). More generally, this holds for observables localized anywhere in the Rindler wedge, as follows from boost invariance of (2.10). This path integral argument directly generalizes to all static spacetimes with a bifurcate Killing horizon, such as the Schwarzschild and deSitter spacetimes [Laflamme 1989; Jacobson 1994a]. In the general setting, the state defined by the path integral cannot be called *the* vacuum, but it is a natural state that is invariant under the static Killing symmetry of the background and is nonsingular on the time slice where the boundary values of the field are specified, including the bifurcation surface.

2.4 Thermal Field Theory ⇔ Black Hole Thermodynamics

One can generally consider a black hole as being analogous to an ordinary dynamical system with many degrees of freedom [Wald 1992].

In a general non-equilibrium setting, the state of an ordinary dynamical system can be defined only given a complete knowledge of its microscopic structure and dynamics. However, in thermal equilibrium, the same state can be characterized by a small number of macroscopic extensive parameters such as total energy, E, entropy, S, volume, V, and particle number, N.

Analogously, the description of a general non stationary black hole state requires the detailed initial data of general relativity, as well as the full Einstein equation to determine its time evolution. On the other hand, the black hole *uniqueness* theorem states that a stationary black hole can be characterized by only three parameters: mass, M, angular momentum, J, and charge, Q. [Israel 1967; Hawking 1971; Carter 1973; Wald 1984].

For a slight deviation from stationarity, the black hole state parameters are related one to each other by,

$$dM = \Omega dJ + \Phi dQ + \kappa dA/8\pi G, \tag{2.16}$$

where Ω and Φ are the horizon angular velocity and electric potential, while κ is the *surface gravity* of the horizon[1]. Stationarity, from Killing symmetry, assures that κ, Ω and Φ, though defined locally, will stay constant over the black hole horizon.

In fact, the horizon quantities characterize the black hole state in the same way a thermal equilibrium state is characterized by the intensive parameters such as temperature, T, chemical potential, μ, and pressure, P.

In this sense, equation (2.16) is mathematically equivalent to the ordinary first law of thermodynamics, relating two states in thermal equilibrium connected by a reversible transformation,

$$TdS = dE + PdV - \mu dN. \tag{2.17}$$

The formal analogy effectively extends to the ordinary second law of thermodynamics — non decreasing total entropy — in relation to the black hole *area theorem* [Hawking 1971], which states that the area of a black hole event horizon never decreases

[1]The surface gravity is defined as the magnitude of the gradient of the norm of the horizon generating Killing field $\chi^a = \xi^a + \Omega\psi^a$ at the horizon, $\kappa^2 \equiv -\chi_{;a}\chi^{;a}$, or more physically as the magnitude of the acceleration, with respect to the Killing time, of a stationary zero angular momentum particle just outside the horizon.

with time, assuming Cosmic Censorship and a positive energy condition. The presence of *irreversibility* in the two contexts is what suggests the connection between black hole physics and thermodynamics has a deep physical content.

This connection is strengthened by the further consistence with the *zeroth law* of thermodynamics, which states that for a body in thermal equilibrium the temperature must be uniform over the body [Carter 1973; Racz 1992], in analogy with the constant value of the surface gravity along the stationary horizon. While, the *third law* of thermodynamics states that a thermal system can not be lowered at absolute zero temperature by an equilibrium process, with an analog, in black hole physics, given by the *Israel theorem*, stating that the surface gravity of the horizon cannot be reduced to zero in a finite nuber of steps [Israel 1986]. Therefore, remarkably, classical general relativity predicts a relation between κ and T, as well as for A and S, whose physical meaning is set by quantum physics [Bekenstein 1973; Bardeen 1973].

2.4.1 Hawking Effect

One can onsider an accelerated nonrotating observer sitting at fixed radius r outside a Schwarzschild black hole. For r very near the horizon R_s, the acceleration a is very large, and the associated timescale a^{-1} is very small compared to R_s. The curvature of the spacetime is negligible on this timescale, so one expects the vacuum fluctuations on this scale to have the usual flat space form, provided the quantum field is in a state which is regular near the horizon. A freely falling observer will describe the state at these scales as the vacuum. However, under these assumptions, the accelerated observer will experience the Unruh effect: the vacuum fluctuations will appear to this observer as a thermal bath at a temperature $T = (\hbar/2\pi)a$. The outgoing modes of this thermal bath will be redshifted as they climb away from the black hole. The ratio of the temperatures measured by static observers at two different radii is $T_2/T_1 = \chi_1/\chi_2$, where χ is the norm of the time-translation Killing field. At infnity $\chi_\infty = 1$, so one has an outgoing thermal flux in the rest frame of the black hole at the temperature [Hawking 1975a]

$$T_\infty = \chi_1 \, \hbar a/2\pi = \hbar\kappa/2\pi \tag{2.18}$$

where κ is the surface gravity. Therefore, a black hole radiates exactly like a black body at temperature $\hbar\kappa/2\pi$. In this sense, κ is not merely a mathematical analogue of temperature, but literary the physical temperature of the black hole.

2.4.2 Black Hole Entropy

From the point of view of an outside observer, the formation of a black hole appears to violate the second law of thermodynamics. The collapsing system may have arbitrary entropy, while the final state has none. So the phase space of the collapsing system appear to be drastically reduced. However, during the collapse, the area of the event horizon will grow. The area theorem then suggests to use the area increase of the event horizon as a compensation for the loss of matter entropy.

With this reasoning, Bekenstein proposes that a black hole should carry an entropy proportional to its horizon area, $S_{BH} = \alpha A/\hbar G$, measured in units of the squared Planck lenght $\ell_p^2 = \hbar G/c^3$ [Bekenstein 1973]. Later on, via the first law of horizon thermodynamics, Eq. (2.16), the Hawking effect described above then fixes the coefficient α in the Bekenstein entropy formula to be $1/4$.

Bekenstein's original idea was that the entropy of a black hole is the logarithm of the number of ways it could have formed. This is closely related to the Boltzman definition of entropy as the number of microstates compatible with the macrostate. Hawking noted that a potential problem arises if one contemplates increasing the number of species of fundamental fields. There would seem to be more ways of forming the black hole, however the entropy is fixed at $A/4$. Hawking's resolution of this was that the black hole will also radiate faster because of the extra species, so that there would be less phase space per species available for forming the hole. Presuming these two effects balance each other, the puzzle would be resolved. This argument was further developed by Zurek and Thorne [Zurek 1985], whose analysis makes it uneccessary to presume that the two effects cancel.

2.4.2.1 Thermal Entropy of Unruh Radiation

Another proposed interpretation is that black hole entropy actually should be identified with the entropy of the thermal bath of quantum fields outside the horizon ['t Hooft 1985]. Let us assume the black hole is nonrotating for simplicity. Recall that the quantum field outside the horizon is in a thermal state with respect to the static vacuum. More precisely, in the Unruh state which results from collapse this is true only for the outgoing modes, while it is strictly true for the Hartle–Hawking state which has incoming thermal radiation as well. Since the outgoing radiation dominates the calculation, we use the Hartle–Hawking state for convenience.

The density matrix ρ for the field outside in the Hartle–Hawking state $|HH\rangle$ can be obtained by a calculation similar to the one which yields the Minkowski vacuum as a thermal state, with the result

$$\rho_{ext} \equiv Tr_{int}|HH\rangle\langle HH| = exp(-\beta H). \tag{2.19}$$

Here $\beta = 1/T_H$, and H is the static Hamiltonian

$$H = \int T_{ab}\chi^a d\Sigma^b, \tag{2.20}$$

where χ^a is the static Killing field, and the integral is over a spatial slice extending from the horizon to infinity.

The entropy associated with this thermal state can be evaluated as for any thermal state. However, since it is infinite, some regulator is required. One can then give a simple argument displaying the nature of the divergence, in the same line of the calculation performed by 't Hooft, where the entropy is evaluated using a mode sum and the cutoff at height ℓ_c above the horizon is referred to as a "brick wall".

The total entropy of the bath is the integral of the local entropy density s over the volume outside the black hole,

$$S = \int s\, 4\pi r^2 dl, \tag{2.21}$$

where $d\ell$ is the proper length increment in the radial direction and we have assumed spherical symmetry. The local temperature T is given by $T = T_H/\chi \simeq (\kappa/2\pi)/(\kappa\ell) = 1/2\pi\ell$, which diverges as the horizon is approached. Therefore it suffices to consider massless radiation, for which $S \propto T^3$, and the dominant contribution (in a finite box) will come from the region near the horizon. Cutting off the integral at a proper height ℓ_c, we thus have

$$S \sim A\int_H \ell^{-3} d\ell \sim A/\ell_c^2. \tag{2.22}$$

Because of the local divergence at the horizon, the result comes out proportional to the area. It is remarkable that this simple estimate gives an area law for the entropy. If the cutoff height is identified with the Planck length, then the entropy even has the correct order of magnitude.

2.4.2.2 Entanglement Entropy

Another proposal is that the black hole entropy is a measure of the information hidden in correlations between degrees of freedom on either side of the horizon [Bombelli 1986; Srednicki 1993]. For instance, although the full state of a quantum field may be pure, the reduced density matrix ρ_{ext} (defined above for the Hartle–Hawking state) will be mixed. The associated information-theoretic entropy,

$$S_{ent} = -Tr \rho_{ext} \ln \rho_{ext}, \tag{2.23}$$

should perhaps thus be part of the black hole entropy. This entropy is sometimes called entanglement entropy. (It has also been called geometric entropy.) If the formal calculation establishing (2.19) can be trusted, we know that entanglement entropy is identical to the thermal entropy of the quantum field outside the horizon as defined above. In particular, it will diverge in the same way. Instead of thinking of this as an infinite temperature divergence, one can think of it as due to the correlations between the infinite number of short wavelength degrees of freedom on either side of the horizon. These correlations are evident from the form of the state near the horizon when expressed in terms of excitations above the inside and outside static vacua. This notion is quite appealing since it traces back the entropy directly to the defining character of the horizon as a causal barrier that hides information, and it also naturally accounts for the scaling with area. Moreover, it allows the generalized second law to be understood as a consequence of causality [Sorkin 1986; Bombelli 1986]. However, it is problematic in the quantitative measure of missing information (which appears to be infinite in ordinary field theory and to depend on the number of species) and in the neglect of the quantum fluctuations of the horizon itself. These issues are tied up with the renormalization of Newton's constant.

2.4.2.3 Species Problem

Besides the divergence, which might be cut off in some way, there is another problem with the idea that the thermal or entanglement entropies of quantum fields be identified with black hole entropy. Namely, this entropy depends on the number of different fields in nature, whereas the black hole entropy is universal, always equal to $A/4\hbar G$.

Various resolutions to the species problem have been suggested. The most natural one is that the renormalized Newton constant, which appears in the Bekenstein-Hawking entropy $A/4\hbar G$, depends on the number of species in just the right way to absorb all species dependence of the black hole entropy [Jacobson 1994b; Susskind 1994]. To understand this point, one must include the gravitational degrees of freedom in the description.

2.4.3 Entropy from Horizon Fluctuations

Another clue to the nature of the horizon states counted by $A/4$ comes from an old analysis by Candelas and Sciama [Candelas 1977; Sciama 1981]. They showed how the relationship between near equilibrium transition rates for a system in contact with a horizon and horizon area is extended to non-equilibrium processes. They interpreted the viscous dissipation rate of a shearing horizon, via the fluctuation-dissipation theorem, in terms of the quantum gravitational spectrum of shear fluctuations.

This explains "why" a horizon has a coefficient of viscosity, and suggests that it is, at least in part, the quantum shear states of the fluctuating horizon that the entropy counts. We will discuss about a characterization of the horizon viscosity in chapters 4 and 5. In particular, we will consider the physical meaning of a horizon bulk viscosity, in addition to the shear viscosity, within the thermodynamical derivation of the Einstein's equations. Note that even when both shear and bulk viscosity are absent the horizon acts as a "perfect dissipator" [Candelas 1977; Sciama 1981] via just the area expansion.

2.4.4 Quantum Gravitational Statistical Mechanics

After the discovery of the Hawking effect, Gibbons and Hawking proposed a formulation of quantum gravitational statistical mechanics that enabled them to compute the black hole entropy [Gibbons 1977].

The basic idea is to imitate standard methods of handling thermodynamic ensembles in other branches of physics. Thus, the goal is to compute the partition function $Z = Tr\exp(-\beta H)$ for the system of gravitational and matter fields in thermal equilibrium at temperature T, from which the entropy and other thermodynamic functions can be evaluated. In fact, it is better in principle to consider the microcanonical ensemble rather than the canonical one. This is because the canonical ensemble is unstable for a gravitating system. If a black hole is in a large heat bath at the Hawking temperature, a small fluctuation to larger mass will cause its temperature to drop, which leads to a runaway growth of the hole. Conversely, a small fluctuation to smaller mass will lead to a runaway evaporation of the hole.

This instability can be controlled by putting the black hole in a very small container, with radius less than $3/2$ times the Schwarzschild radius (for a Schwarzschild black hole), and somehow holding the temperature at the box fixed[1]. Alternatively one can

[1] The reason this eliminates the instability is interesting: although a fluctuation to (say) larger mass causes the Hawking temperature to drop, this is more than compensated by the fact that the horizon has moved out, so the local temperature at the box is less redshifted than before, so the hole is in fact locally hotter than the box [Jacobson 2005].

work with the more physical microcanonical ensemble, in which the total energy is fixed.

An actual computation of Z would seem to require an understanding of the Hilbert space of quantum gravity, something still missing. Gibbons and Hawking sidestepped this difficulty by passing to a path integral representation for Z whose semiclassical approximation could be plausibly evaluated. Thus, one writes

$$Z = Tr\exp(-\beta H) = \int DgD\varphi e^{-I[g,\varphi]}, \tag{2.24}$$

where g and φ stand for the metric and matter fields respectively and I is the Euclidean action. The stationary point of the action is the Euclidean black hole, with mass determined by the condition that there be no conical singularity in the $r-t$ plane at the Euclidean horizon. The Euclidean Rindler coordinates are just polar coordinates, $ds^2 = \xi^2 d\tau^2 + d\xi^2$, so this means the period of the "angular" coordinateτ must be 2π. Since $\tau = \kappa t$ (cf. section 3.2.3), it follows that $\kappa = 2\pi/\beta$. The zeroth order contribution to the entropy is then obtained as

$$S_0 = \left(\beta\frac{\partial}{\partial\beta} - 1\right) I[g_0, \varphi_0] \tag{2.25}$$

where (g_0, φ_0) is the classical stationary point.

To include quantum fluctuations one could write $g = g_0 + \tilde{g}$ and $\varphi = \varphi_0 + \tilde{\varphi}$, and integrate over $\tilde{g}$ to obtain an effective action $I_{eff}[g_0, \varphi_0] = -\ln Z$.

This effective action will contain a Ricci scalar term with a coefficient $1/16\pi G_{ren}$, where G_{ren} is the renormalized Newton constant, as well as higher curvature terms, non-local terms etc. The contribution of the fluctuations to the entropy is primarily through their effect on the renormalization of G.

Viewed in a different way, the fluctuation contribution can be related to the thermal entropy of acceleration radiation or the (formally equivalent) entanglement entropy discussed earlier. The path integral over $\tilde{g}$ and $\tilde{\varphi}$ formally gives $Tr\exp(-\beta H_0[\tilde{g}, \tilde{\varphi}])$, where H_0 is the evolution operator for the fluctuations in the background $(g_0(\beta), \varphi_0(\beta))$. Thus, the contribution S' of the fluctuations to the entropy

$$S = S_0 + S' = (\beta\frac{\partial}{\partial\beta} - 1)I_{eff}[g_0, \varphi_0] \tag{2.26}$$

looks at first just like the entanglement entropy.

However, in computing the entanglement entropy only the period β of the background is varied, while otherwise the background is fixed. By contrast, in computing S as above, one also must differentiate with respect to the β-dependence of the background $(g_0(\beta), \varphi(\beta))$ [Frolov 1996]. Formally, this extra variation makes no contribution, since (g_0, φ_0) is chosen to be a stationary point of the effective action. Thus the two computations might yield the same result. However, the calculation in which only the period is varied introduces a conical singularity at the horizon, and this can lead to some difference.

2.4.5 Horizon Entropy and Noether Charge

In alternative to the statistical interpretation, Wald showed that in general theories of gravity the entropy of stationary black holes with bifurcate Killing horizons is a Noether charge [Wald 1992]. We shall now briefly describe how these results arise in a class of theories which are natural generalizations of Einstein gravity. The argument follows the exposition given in [Wald 1993; Padmanabhan 2010].

Consider a theory for gravity described by the metric g_{ab} coupled to matter. We will take the action describing such a theory in D dimensions to be

$$A = \int d^D x \sqrt{-g}[L(R^{ab}_{cd}, g_{ab})] + L_{matt}(g_{ab}, q_A), \tag{2.27}$$

where L is any scalar built from metric and curvature and L_{matt} is the matter Lagrangian depending on the metric and some matter variables q_A [1]. Varying g_{ab} in Eq. (2.27) we get $\delta(L_{matt}\sqrt{-g}) = -(1/2)\sqrt{-g}T^{ab}\delta g_{ab}$ and

$$\delta(L\sqrt{-g}) = \sqrt{-g}(\mathcal{G}^{ab}\delta g_{ab} + \nabla_a \delta v^a). \tag{2.28}$$

The variation of the gravitational Lagrangian density generically leads to a surface term which is expressed by the $\nabla_a \delta v^a$ term. Ignoring this term for the moment (we will comment on this later) we get equations of motion to be $2\mathcal{G}_{ab} = T_{ab}$ where the explicit form of $\mathcal{G}_{ab}$ is

$$\mathcal{G}_{ab} = P_a{}^{cde} R_{bcde} - \frac{1}{2} L g_{ab} - 2\nabla^c \nabla^d P_{acdb} \equiv \mathcal{R}_{ab} - 2\nabla^c \nabla^d P_{acdb}, \tag{2.29}$$

where

$$P_{abcd} \equiv \frac{\partial L}{\partial R_{abcd}}. \tag{2.30}$$

(Our notation is based on the fact that $\mathcal{G}_{ab} = G_{ab}$ and $\mathcal{R}_{ab} = R_{ab}$ in Einstein's gravity.)

[1] We have assumed that L does not involve derivatives of curvature tensor, to simplify the discussion

For any Lagrangian L, the functional derivative of $\mathcal{G}_{ab}$ satisfies the generalized Bianchi identity: $\nabla_a \mathcal{G}^{ab} = 0$.

Many such theories have been investigated in the literature and most of them have black hole solutions. Whenever the black hole metric can be approximated by a Rindler metric near the horizon, it is possible to associate a temperature with the horizon, using the procedures described earlier. On the other hand, the entropy can be introduced by the following argument.

In any generally covariant theory, the infinitesimal coordinate transformations $x^a \to x^a + \xi^a$ leads to conservation of a Noether current which depends on ξ^a. To derive the expression for the Noether current, let us consider the variations in δg_{ab} which arise through the diffeomorphism $x^a \to x^a + \xi^a$. In this case, $\delta(L\sqrt{-g}) = -\sqrt{-g}\nabla_a(L\xi^a)$, with $\delta g^{ab} = (\nabla^a \xi^b + \nabla^b \xi^a)$. Substituting these in equation (2.28) and using $\nabla_a \mathcal{G}^{ab} = 0$, we obtain the conservation law $\nabla_a J^a = 0$, for the current,

$$J^a \equiv 2\mathcal{G}^{ab}\xi_b + L\xi^a + \delta\delta_\xi v^a = 2\mathcal{R}_{ab}\xi^a + \delta_\xi v^a, \tag{2.31}$$

where $\delta_\xi v^a$ represents the boundary term which arises for the specific variation of the metric in the form $\delta g^{ab} = (\nabla^a \xi^b + \nabla^b \xi^a)$. Quite generally, the boundary term can be expressed as [Deruelle 2004; Lopes Cardoso 2000],

$$\delta v^a = \frac{1}{2}\alpha^{a(bc)}\delta g_{bc} + \frac{1}{2}\beta_d^{a(bc)}\delta\Gamma^d_{bc}. \tag{2.32}$$

The coefficient β^{abcd} arises from the derivative of L_{grav} with respect to R^{abcd} and hence possesses all the algebraic symmetries of the curvature tensor. In the special case of diffeomorphisms, $x^a \to x^a + \xi^a$, the variation $\delta_\xi v^a$ is given by Eq. (2.32) with

$$\delta g_{ab} = -\nabla_{(a}\xi_{b)}; \quad \delta\Gamma^d_{bc} = -\frac{1}{2}\nabla_{(b}\nabla_{c)}\xi^d + \frac{1}{2}R^d_{(bc)m}\xi^m. \tag{2.33}$$

Using the above expressions in Eq. (2.31), it is possible to write an explicit expression for the current J^a for any diffeomorphism invariant theory. It is also convenient to introduce an antisymmetric tensor J^{ab} by $J^a = \nabla_b J^{ab}$. For the general class of theories we are considering the J^{ab} and J^a can be expressed in the form

$$J^{ab} = 2P^{abcd}\nabla_c\xi_d - 4\xi_d(\nabla_c P^{abcd}), \tag{2.34}$$

$$J^a = -2\nabla_b(P^{adbc} + P^{acbd})\nabla_c\xi_d + 2P^{abcd}\nabla_b\nabla_c\xi_d - 4\xi_d\nabla_b\nabla_c P^{abcd}, \tag{2.35}$$

where $P^{abcd} \equiv (\partial L/\partial R^{abcd})$. (The expressions for J^a, J^{ab} are not unique. This ambiguity has been extensively discussed in the literature but for our purpose we will use the J^a defined as above.) We shall see that, for most of our discussion, we will not require the explicit form of $\delta_\xi v^a$ except for one easy to prove result: $\delta_\xi v^a = 0$ when ξ^a is a Killing vector and satisfies the conditions

$$\nabla_{(a}\xi_{b)} = 0; \quad \nabla_a \nabla_b \xi_c = R_{cbad}\xi^d. \tag{2.36}$$

The expression for Noether current simplifies considerably when ξ^a a satisfies Eq. (2.36) and is given by

$$J^a = 2\mathcal{G}^{ab}\xi_b + L\xi^a = 2\mathcal{R}^{ab}\xi_b. \tag{2.37}$$

The integral of J^a over a space-like surface defines the conserved Noether charge, $\mathcal{N}$.

To obtain a relation between the horizon entropy and Noether charge, we first note that on-shell, i.e. when field equations hold ($2\mathcal{G}_{ab} = T_{ab}$), we can write

$$J^a = (T^{ab} + g^{ab}L)\xi_b. \tag{2.38}$$

Therefore, for any vector k_a which satisfies $k_a\xi^a = 0$, we get the result

$$(k_a J^a) = T^{ab}k_a\xi_b. \tag{2.39}$$

The change in this quantity, when T^{ab} changes by a small amount δT^{ab}, will be $\delta(k_a J^a) = k_a\xi_b\delta T^{ab}$. It is this relation which can be used to obtain an expression for the horizon entropy in terms of the Noether charge. When some amount of matter with energy-momentum tensor δT^{ab} crosses the horizon, the corresponding energy flux can be thought of as given by integral of $k_a\xi_b\delta T^{ab}$ over the horizon where ξ^a is the Killing vector field corresponding to the bifurcation horizon and k_a is a vector orthogonal to ξ^a which can be taken as the normal to a time-like surface, infinitesimally away from the horizon (In Chapter 3 we will treat this surface as a "stretched horizon", defined by the condition $N = \epsilon$, where N is the lapse function with $N = 0$ representing the horizon.) In the $(D-1)$- dimensional integral over this surface, one coordinate is just time; since we are dealing with an approximately stationary situation, the time integral reduces to multiplication by the range of integration. Based on our discussion earlier we will assume that the time integration can be restricted to the range $(0, \beta)$ where $\beta = 2\pi/\kappa$ and κ is the surface gravity of the horizon. (The justification for this requires a much more detailed mathematical analysis which we shall not get into here.)

Thus, on integrating $\delta(k_a J^a)$ over the horizon we get

$$\delta \int_H d^{D-1}x\sqrt{h}(k_a J^a) = \int_H D^{D-1}x\sqrt{h}k_a\xi_j\delta T^{aj} = \beta \int_H d^{D-2}x\sqrt{h}k_a\xi_j\delta T^{aj}, \qquad (2.40)$$

where the integration over time has been replaced by a multiplication by $\beta = (2\pi/\kappa)$ assuming approximate stationarity of the expression. The integral over δT^{aj} is the flux of energy δE through the horizon so that $\beta\delta E$ can be interpreted as the rate of change in the entropy associated with this energy flux. One can obtain, using these facts, an expression for entropy, given by

$$S_{Noether} \equiv \beta\mathcal{N} = \beta \int d^{D-1}\Sigma_a J^a = \frac{\beta}{2}\int d^{D-2}\Sigma_{ab}J^{ab}, \qquad (2.41)$$

where $d^{D-1}\Sigma_a = d^{D-1}x\sqrt{h}k_a$, the Noether charge is $\mathcal{N}$ and we have again introduced the antisymmetric tensor J^{ab} by $J^a = \nabla_b J^{ab}$. In the final expression, the integral is over any surface with $(D-2)$ dimension which is a space-like cross-section of the Killing horizon on which the norm of ξ^a vanishes.

As an example, consider the special case of Einstein gravity in which Eq. (2.35) reduces to

$$J^{ab} = \frac{1}{16\pi}\left(\nabla^a\xi^b - \nabla^b\xi^a\right). \qquad (2.42)$$

If ξ^a is the time-like Killing vector in the spacetime describing a Schwarzschild black hole, we can compute the Noether charge $\mathcal{N}$ as an integral of J^{ab} over any two surface which is a space-like cross-section of the Killing horizon on which the norm of ξ^a vanishes. The area element on the horizon can be taken to be $\delta\Sigma_{ab} = (l_a\xi_b - l_b\xi_a)\sqrt{\sigma}d^{D-2}x$ in Eq. (2.41) with l_a being an auxiliary vector field satisfying the condition $l_a\xi^a = -1$. Then the integral in Eq. (2.41) reduces to

$$S_{Noether} = -\frac{\beta}{8\pi}\int \sqrt{\sigma}d^{D-2}x(l_a\xi_b)\nabla^b\xi^a = -\frac{\beta\kappa}{8\pi}\int \sqrt{\sigma}d^{D-2}x = \frac{1}{4}A_H, \qquad (2.43)$$

where we have used equation $\xi^b\nabla_b\xi^a = \kappa\xi^a$, the relation $l_a\xi^a = -1$, and the fact that ξ^a is a Killing vector. The result, of course, agrees with the standard one.

It is also possible to show, using the expression for J^{ab}, that the entropy in Eq.(2.41) is also equal to

$$S_{Noether} = \frac{2\pi}{\kappa}\oint_S \left(\frac{\partial L}{\partial R_{abcd}}\right)\epsilon_{ab}\epsilon_{cd}d\Sigma, \qquad (2.44)$$

where κ is the surface gravity of the horizon and the $(D-2)$-dimensional integral is on a space-like bifurcation surface with ϵ_{ab} denoting the bivector normal to the bifurcation surface.

The variation is performed as if R_{abcd} and the metric are independent and the whole expression is evaluated on a solution of the equation of motion. A wide class of theories have been investigated using such a generalization in order to identify the thermodynamic variables relevant to the horizon.

The validity of Walds proposal has been checked in many examples in a string theory context where the direct counting of microstates can be compared explicitly to the Noether charge entropy [Lopes Cardoso 2000].

The notion of black hole Boltzmann entropy, at the quantum statistical level, extends of to stationary states of all causal horizons, i.e. generally to Killing horizons. Therefore, any Killing horizon is endowed with a surface entropy density of 1/4 [Jacobson 2003]. In particular, this will be true for a a fundamental step toward the generalization of the black hole thermodynamic framework to a general spacetime neighborhood.

2.5 Causal Horizon Entropy ⇔ Holographic Principle

The realization that horizon entropy is an intrinsically observer dependent notion raises a natural interpretative question on the states that the horizon entropy counts. In particular, the nonextensive nature of the Bekenstein horizon entropy, together with its quantum statistical interpretation, provide an intriguing puzzle on the effective number of degrees of freedom characterizing the fundamental level of description underlying gravity and quantum fields ['t Hooft 1993; Susskind 1995].

In the previous sections an effective framework for such a "fundamental system" was given by local quantum field theory on a classical background spacetime satisfying Einstein's equations [Birrell 1982; Wald 1992; Jacobson 1994b]. One can define the *number of degrees of freedom* of a quantum-mechanical system[1], N, to be the logarithm of the dimension $\mathcal{N}$ of its Hilbert space $\mathcal{H}$:

$$N = \ln \mathcal{N} = \ln \dim(\mathcal{H}). \tag{2.45}$$

[1]The number of degrees of freedom is equal (up to a factor of $\ln 2$) to the number of bits of information needed to characterize a state. For example, a system with 100 spins has $\mathcal{N} = 2^{100}$ states, $N = 100 \ln 2$ degrees of freedom, and can store 100 bits of information.

Now, a general quantum-mechanical system has an infinite-dimensional Hilbert space. Thus, the answer to our question appears to be $N = \infty$. However, these considerations do not take into account the effects of gravity altogether.

A finite estimate is obtained by including gravity at least in a crude, minimal way [Bousso 2002]. One might expect that distances smaller than the Planck length, $\ell_p = 1.6 \times 10^{-33}$cm, cannot be resolved in quantum gravity. So let us discretize space into a Planck grid and assume that there is one oscillator per Planck volume. Moreover, the oscillator spectrum is discrete and bounded from below by finite volume effects. It is also bounded from above because it must be cut off at the Planck energy, $M_p = 1.3 \times 10^{19}$GeV. This is the largest amount of energy that can be localized to a Planck volume without producing a black hole. Thus, the total number of oscillators is V (in Planck units), and each has a finite number of states, n. (A minimal model one might think of is a Planckian lattice of spins, with $n = 2$.) Hence, the total number of independent quantum states in the specified region is

$$\mathcal{N} \sim n^V. \tag{2.46}$$

The number of degrees of freedom is given by

$$N \sim V \ln n \gtrsim V. \tag{2.47}$$

This result successfully captures our prejudice that the degrees of freedom in the world are local in space, and that, therefore, complexity grows with volume. It turns out, however, that this view conflicts with the laws of gravity.

2.5.1 Bekenstein Entropy and Generalized Second Law

When a matter system is dropped into a black hole, its entropy is lost to an outside observer. That is, the entropy S_{matter} starts at some finite value and ends up at zero. But the entropy of the black hole increases, because the black hole gains mass, and so its area A will grow. Thus it is at least conceivable that the total entropy, $S_{\text{matter}} + \frac{A}{4}$, does not decrease in the process. With this reasoning Bekenstein conjectures that in a consistent generalization of the second law of thermodynamics, accounting for black hole physics, the sum of the entropy outside the black hole and the entropy of the black hole itself will never decrease,

$$d(S_{out} + \alpha A/\hbar G) \geq 0. \tag{2.48}$$

This defines the *generalized second law* (GSL).

However, it is by no means obvious that the generalized second law will hold. The growth of the horizon area depends essentially on the mass that is added to the black hole; it does not seem to care about the entropy of the matter system. If it were possible to have matter systems with arbitrarily large entropy at a given mass and size, the generalized second law could still be violated[1]

In this sense, one would expect that requiring the validity of GSL would lead to a universal bound on the entropy of matter systems in terms of their extensive parameters.

2.5.2 Bekenstein Bound

If the thermodynamic properties of black holes developed above, including the assignment of entropy to the horizon, are sufficiently compelling to be considered laws of nature, then one may demand that the generalized second law hold in all processes. In this line of thought, for any weakly gravitating matter system in asymptotically flat space, Bekenstein [Bekenstein 1981] argues that the GSL implies the following bound:

$$S_{\text{matter}} \leq 2\pi E R. \tag{2.49}$$

In full, $S \leq 2\pi k E R/(\hbar c)$ (note that Newton's constant does not enter). Here, E is the total mass-energy of the matter system. The circumferential radius R is the radius of the smallest sphere that fits around the matter system. Therefore, the *Bekenstein bound* define an upper limit on the matter entropy that can be contained within a given finite region of space associated with a finite amount of energy. On the other hand, this bound remarkably implies that the information necessary to fully describe the system must be *finite*, if the region of space and the energy is finite.

2.5.3 Complexity According to the Area Scaling Entropy

Thermodynamic entropy has a statistical interpretation. Let S be the thermodynamic entropy of an isolated system at some specified value of macroscopic parameters such as energy and volume. Then e^S is the number of independent quantum states compatible with these macroscopic parameters. Thus, entropy is a measure of our ignorance about the detailed microscopic state of a system. In particular, the number of states will

[1]The GSL is automatically protected by Unruh radiation as long as the entropy of the matter system does not exceed the entropy of unconstrained thermal radiation of the same energy and volume. This is plausible if the system is weakly gravitating and if its dimensions are not extremely unequal [Unruh 1982].

simply be $\mathcal{N} = e^S$.

Considering an interpretation of horizon entropy as the entropy of the quantum fields localized by the causal horizon, the saturated bound,

$$S_{\text{BH}} = \frac{A}{4}, \tag{2.50}$$

suggest that the number of degrees of freedom in a region bounded by the horizon of area A is given by

$$N = \frac{A}{4}; \tag{2.51}$$

the number of states is

$$\mathcal{N} = e^{A/4}. \tag{2.52}$$

Therefore this result would suggest that, because of gravity, not all degrees of freedom that field theory apparently supplies can be used for generating entropy, or storing information. In particular, one can generally argue that $A/4$ degrees of freedom are sufficient to fully describe any stable region in asymptotically flat space enclosed by a sphere of area A [Bousso 2002]. This invalidates the field theory estimate, Eq. (2.47), and suggest an holographic interpretation.

2.5.4 Unitarity and a Holographic Interpretation

In a field theory description, there are far more than $A/4$ degrees of freedom. The restriction to a finite spatial region provides an infrared cut-off, precluding the generation of entropy by long wavelength modes. Hence, most of the entropy in the field theory estimate comes from states of very high energy. But a spherical surface cannot contain more mass than a black hole of the same area. According to the Schwarzschild solution, the mass of a black hole is given by its radius. Hence, the mass M contained within a sphere of radius R obeys

$$M \lesssim R. \tag{2.53}$$

For example, consider a sphere of radius $R = 1\,\text{cm}$, or 10^{33} in Planck units. Suppose that the field energy in the enclosed region saturated the naive cut-off in each of the $\sim 10^{99}$ Planck cells. Then the mass within the sphere would be $\sim 10^{99}$. But the most massive object that can be localized to the sphere is a black hole, of radius and mass 10^{33}.

Thus, most of the states included by the field theory estimate are too massive to be gravitationally stable. Long before the quantum fields can be excited to such a level, a black hole would form.[1] If this black hole is still to be contained within a specified sphere of area A, its entropy may saturate but not exceed the entropy bound.

Therefore any attempt to excite more than $A/4$ of these degrees of freedom is thwarted by gravitational collapse. From the outside point of view, the most entropic object that fits in the specified region is a black hole of area A, with $A/4$ degrees of freedom [Bousso 2002].

A conservative interpretation of this result is that the demand for gravitational stability merely imposes a practical limitation for the information content of a spatial region. If we are willing to pay the price of gravitational collapse, we can excite more than $A/4$ degrees of freedom—though we will have to jump into a black hole to verify that we have succeeded. Still, with this interpretation, all the degrees of freedom of field theory should be effectively retained. The region will be described by a quantum Hilbert space of dimension e^V.

However, unitarity provides a compelling consideration in this sense. Quantum mechanical evolution preserves information; it takes a pure state to a pure state. Let's suppose a region was described by a Hilbert space of dimension e^V, and suppose that region was converted to a black hole. According to the Bekenstein entropy of a black hole, the region is now described by a Hilbert space of dimension $e^{A/4}$. The number of states would have decreased, and it would be impossible to recover the initial state from the final state. Thus, unitarity would be violated. Hence, the Hilbert space must have had dimension $e^{A/4}$ to start with.

The insistence on unitarity in the presence of black holes led 't Hooft ['t Hooft 1993] and Susskind [Susskind 1995] to embrace a more radical, "holographic" interpretation of Eq. (2.51).

Holographic principle [Bousso 2002]. *A region with boundary of area A is fully described by no more than $A/4$ degrees of freedom, or about 1 bit of information per Planck area. A fundamental theory, unlike local field theory, should incorporate this counterintuitive result.*

2.5.5 Implications of the Holographic Principle

The holographic principle implies a radical reduction in the number of degrees of freedom we use to describe nature. It exposes quantum field theory, which has degrees of

[1] Thus, black holes provide a natural covariant cut-off which becomes stronger at larger distances. It differs greatly from the fixed distance or fixed energy cutoffs usually considered in quantum field theory.

freedom at every point in space, as a highly redundant effective description, in which the true number of degrees of freedom is obscured. In particular, the holographic principle challenges to formulate a theory in which a covariant formulation of the entropy bound is manifest [Bousso 2002].

Clearly, since physics appears to be local to a very good approximation to do so, such a formulation will have to deal with the problem of locality. A possibility would be to retain locality and consider that a local theory could be rendered holographic if an explicit gauge invariance is identified, leaving only as many physical degrees of freedom as dictated by the covariant entropy bound. The challenge, in this case, is to implement such an enormous and rather peculiar gauge invariance ['t Hooft 1999; 't Hooft 2001; 't Hooft 2003]. Otherwise, one can regard locality as an emergent phenomenon without fundamental significance. In this case, the holographic data are primary. In particular, since light-sheets are central to the formulation of the holographic principle, one would expect null hypersurfaces to play a primary role in the classical limit of an underlying holographic theory.

An argument in support of the latter type of approach is given by the AdS/CFT correspondence. The AdS/CFT correspondence defines quantum gravity - albeit in a limited set of spacetimes. Anti-de Sitter space contains a kind of holographic screen, a distant hypersurface on which holographic data can be stored and evolved forward using a conformal field theory (Sec. 2.5.6).

2.5.6 The AdS/CFT Correspondence

The most prominent example of the AdS/CFT correspondence concerns type IIB string theory in an asymptotically $AdS_5 \times \mathbf{S}^5$ spacetime (the *bulk*), with n units of five-form flux on the five-sphere[1] [Maldacena 1998; Gubser 1998; Witten 1998a]. This theory, which includes gravity, is claimed to be non-perturbatively defined by a particular conformal field theory without gravity, namely 3+1 dimensional supersymmetric Yang-Mills theory with gauge group $U(n)$ and 16 real supercharges. One generally refers to this theory as the *dual CFT*.

The metric of $AdS_5 \times \mathbf{S}^5$ is

$$ds^2 = R^2 \left[-\frac{1+r^2}{1-r^2} dt^2 + \frac{4}{(1-r^2)^2} \left(dr^2 + r^2 d\Omega_3^2 \right) + d\Omega_5^2 \right], \tag{2.54}$$

where $d\Omega_d$ denotes the metric of a d-dimensional unit sphere. The radius of curvature

[1]There is a notational conflict with most of the literature, where N denotes the size of the gauge group. In this review, N is reserved for the number of degrees of freedom.

is related to the flux by the formula

$$R = n^{1/4}, \tag{2.55}$$

in units of the ten-dimensional Planck length.

The proper area of the three-spheres diverges as $r \to 1$. After conformal rescaling [Hawking 1975b], the space-like hypersurface, $t = \text{const}, 0 \leq r < 1$ is an open ball, times a five-sphere. (The conformal picture for AdS space thus resembles the world volume occupied by a spherical system) Because the five-sphere factor has constant physical radius, and the scale factor vanishes as $r \to 1$, the five-sphere is scaled to a point in this limit. Thus, the conformal boundary of space is a three-sphere residing at $r = 1$.

It follows that the conformal boundary of the spacetime is $\mathbb{R} \times \mathbf{S}^3$. This agrees with the dimension of the CFT. Hence, it is often said that the dual CFT "lives" on the boundary of AdS space.

The idea that data given on the boundary of space completely describe all physics in the interior provide an effective realization of the holographic principle. The dual CFT seems to achieve what local field theory in the interior could not do. It contains an area's worth of degrees of freedom, avoiding the redundancy of a local description.

However, to check quantitatively whether the holographic bound really manifests itself in the dual CFT, one must compute the CFT's number of degrees of freedom, N. This must not exceed the boundary area, A, in ten-dimensional Planck units. Also, one must actually verify that there is a light-sheet that contains all of the entropy in the spacetime.

It is worth to provide here a qualitative description of the problem in some details. We will follow the synthetic exposition given in [Bousso 2002].

The proper area of the AdS space boundary is divergent. The number of degrees of freedom of a conformal field theory on a sphere is also divergent, since there are modes at arbitrarily small scales. In order to make a sensible comparison, Susskind and Witten [Susskind 1998] regularized the bulk spacetime by removing the region $1 - \delta < r < 1$, where $\delta \ll 1$. This corresponds to an infrared cutoff. The idea is that a modified version of the AdS/CFT correspondence still holds for this truncated spacetime.

The area of the $\mathbf{S}^3 \times \mathbf{S}^5$ boundary surface[1] is approximately given by

$$A \approx \frac{R^8}{\delta^3} \tag{2.56}$$

In order to find the number of degrees of freedom of the dual CFT, one has to understand how the truncation of the bulk modifies the CFT. For this purpose, Susskind and Witten [Susskind 1998] identified and exploited a peculiar property of the AdS/CFT correspondence: infrared effects in the bulk correspond to ultraviolet effects on the boundary.

There are many detailed arguments supporting this so-called *UV/IR relation* (see also, e.g., [Balasubramanian 1999b; Peet 1999]. Here we give just one example. A string stretched across the bulk is represented by a point charge in the dual CFT. The energy of the string is linearly divergent near the boundary. In the dual CFT this is reflected in the divergent self-energy of a point charge. The bulk divergence is regularized by an infrared cut-off, which renders the string length finite, with energy proportional to δ^{-1}. In the dual CFT, the same finite result for the self-energy is achieved by an ultraviolet cutoff at the short distance δ.

We have scaled the radius of the three-dimensional conformal sphere to unity. A short distance cut-off δ thus partitions the sphere into δ^{-3} cells. For each quantum field, one may expect to store a single bit of information per cell. A $U(n)$ gauge theory comprises roughly n^2 independent quantum fields, so the total number of degrees of freedom is given by

$$N \approx \frac{n^2}{\delta^3}. \tag{2.57}$$

Using Eq. (2.55) we find that the CFT number of degrees of freedom saturates the holographic bound,

$$N \approx A, \tag{2.58}$$

where we must keep in mind that this estimate is only valid to within factors of order unity. Thus, the number of CFT degrees of freedom agrees with the number of physical degrees of freedom contained on any light-sheet of the boundary surface $\mathbf{S}^3 \times \mathbf{S}^5$.

One must also verify that there is a light-sheet that contains all of the entropy in the spacetime. If all light-sheets terminated before reaching $r = 0$, this would leave the possibility that there is additional information in the center of the universe which is not encoded by the CFT. In that case, the CFT would not provide a complete description of

[1]Unlike Susskind and Witten [Susskind 1998], we do not compactify the bulk to five dimensions in this discussion; all quantities refer to a ten-dimensional bulk. Hence the area is eight-dimensional.

the full bulk geometry—which is, after all, the claim of the AdS/CFT correspondence.

Thus, the CFT state on the boundary (at one instant of time) contains holographic data for a complete slice of the spacetime. The full boundary of the spacetime includes a time dimension and is given by $\mathbb{R} \times \mathbf{S}^3 \times \mathbf{S}^5$. Each instant of time defines an $\mathbf{S}^3 \times \mathbf{S}^5$ boundary area, and each such area admits a complete future directed light-sheet. The resulting sequence of light-sheets foliate the spacetime into a stack of light cones. There is a slice-by-slice holographic correspondence between bulk physics and dual CFT data.

To summarize, the AdS/CFT correspondence exhibits the following features:

- There exists a slicing of the spacetime such that the state of the bulk on each slice is fully described by data not exceeding A bits, where A is the area of the boundary of the slice.
- There exists a theory without redundant degrees of freedom, the CFT, which generates the unitary evolution of boundary data from slice to slice.

Assuming the validity of the covariant entropy bound in arbitrary spacetimes, Bousso [Bousso 1999b] showed that a close analogue of the first property always holds. The second, however, is not straightforwardly generalized. It should not be regarded as a universal consequence of the holographic principle, but as a peculiarity of Anti-de Sitter space.

An important consequence of the AdS/CFT correspondence is that the dynamics of the stress-energy-momentum tensor in a large class of d-dimensional strongly coupled quantum field theories is governed by the dynamics of Einstein's equations with negative cosmological constant in $d + 1$ dimensions. As a consequence, the relativistic hydrodynamics of the gauge theory can be effectively described by the long time, long wavelength dynamics of a black hole in AdS [Bhattacharyya 2008a]. In this framework, the relativistic Navier–Stokes equations turn out to be equivalent to the subset of the General Relativity (GR) field equations called the momentum constraints, which constrain "initial" data on the time-like AdS boundary. Moreover, the incompressible Navier–Stokes equations describing ordinary, everyday fluids can be obtained by taking a particular non-relativistic limit of these results [Bhattacharyya 2009].

Again, it's worth to consider this scenario in some details, as many of the concepts involved will be used throughout the thesis work. In the exposition we follow the arguments given in the recent rewiev [Hubeny 2011].

2.6 Holographic Principle ⇔ Fluid/Gravity Duality

In the realization of the AdS/CFT correspondence discussed above, the gauge theory reduces to a classical Type IIB string theory in the 't Hooft limit ($N \to \infty$, fixed λ). Even classically, string theory has complicated dynamics; however in the strong gauge coupling ($\lambda \to \infty$) regime, it reduces to the dynamics of Type IIB supergravity (by decoupling the massive string states). Now, Type IIB supergravity on $\mathrm{AdS}_5 \times \mathbf{S}^5$ admits several consistent truncations. Among these, the simplest and most universal one is the truncation to Einstein's equations with negative cosmological constant,

$$E_{ab} \equiv R_{ab} - \frac{1}{2} R\, g_{ab} + \mathcal{L}\, g_{ab} = 0, \quad \mathcal{L} \equiv -\frac{d\,(d-1)}{2\, R_{AdS}^2}. \tag{2.59}$$

In particular, the AdS/CFT dictionary provides a one-to-one map between single particle states in the classical Hilbert space of string theory and single-trace operators in the gauge theory. For instance the bulk graviton maps to the stress tensor of the boundary theory. Therefore, by taking the collection of such single trace operators as a whole, one can try to formulate dynamical equations for their quantum expectation values in the field theory. While this can be done in principle, the resulting system is non-local in terms of the intrinsic field theory variables themselves.

However, because we can associate the quantum operators (and their expectation values) of the gauge theory at strong coupling to the classical fields of string theory/supergravity, the set of classical equations one actually need to look for are just the local equations of Type IIB supergravity on $\mathrm{AdS}_5 \times \mathbf{S}^5$. This reduction, whilst retaining lots of interesting physics, still turns out to be too complicated from the field theory perspective. For one, the space of single trace operators is still infinite dimensional (at infinite N), and relatedly attempting to classify the solution space of Type IIB supergravity is a challenging problem.

However, the fact that on the string side one can reduce the system to (2.59), implies that there is a decoupled sector of stress tensor dynamics in $\mathcal{N} = 4$ SYM at large λ.

Actually, there is an infinite number of conformal gauge theories which have a gravitational dual that truncates consistently at the two-derivative level to Einstein's equations with a negative cosmological constant; $\mathcal{N} = 4$ SYM theory is just a particularly simple member of this class. Thus (2.59) describes the *universal decoupled* dynamics of the stress tensor for an infinite number of different gauge theories.

In particular, one can try to characterize the behavior of quantum field theory stress tensors starting from the basic organizing principle of physics: separation of scales. Indeed, analogously to many situations in physics, complicated UV dynamics results in relatively simple IR dynamics. As a general principle of finite temperature physics, the dynamics of nearly equilibrated systems at high enough temperature may be described by an effective theory called hydrodynamics. The key dynamical equation of hydrodynamics is the conservation of the stress tensor

$$\nabla_a T^{ab} = 0, \tag{2.60}$$

where ∇_a is the covariant derivative compatible with the background metric γ_{ab} on which this fluid lives. As this equation is an autonomous dynamical system involving just the stress tensor, it should lie within the sector of universal decoupled stress tensor dynamics.

Given that the AdS/CFT correspondence asserts that this universal sector is governed by (2.59), we are led to conclude that (2.59) must, in an appropriate high temperature and long distance limit which we refer to as the *long wavelength regime*, reduce to the equations of d-dimensional hydrodynamics. Indeed, this expectation has been independently verified in [Bhattacharyya 2008a] and the resulting map between gravity and fluid dynamics has come to be known as the *fluid/gravity* correspondence.

Given any solution to the these fluid dynamical equations, the fluid/gravity map *explicitly* determines a solution to Einstein's equations (2.59) to the appropriate order in the derivative expansion. The solutions in gravity are simply inhomogeneous, time-dependent black holes, with slowly varying but otherwise generic horizon profiles.

2.6.1 The Fluid/Gravity Correspondence

The connection between the fluid/gravity map at the full non-linear level is established and extensively studied much earlier at the linearized level in the AdS/CFT context [Policastro 2001]. The first hints of the connection between fluid dynamics and gravity at the non-linear level are obtained in attempts to construct non-linear solutions dual to a particular boost invariant flow [Janik 2006], which provided inspiration for the fluid/gravity map. Such a map was also suggested by the observation that the properties of large rotating black holes in global AdS space are reproduced by the equations of non-linear fluid dynamics [Bhattacharyya 2008b; Rangamani 2009].

According to the gauge/gravity dictionary, distinct asymptotically AdS bulk geometries correspond to distinct states in the boundary gauge theory. The pure AdS geometry, i.e., the maximally symmetric negatively curved spacetime, corresponds to the vacuum state of the gauge theory. A large[1] Schwarzschild-AdS black hole corresponds to a thermal density matrix in the gauge theory. This can be easily conceptualized in terms of the late-time configuration a generic state evolves to: in the bulk, the combined effect of gravity and negative curvature tends to make a generic configuration collapse to form a black hole which settles down to the Schwarzschild-AdS geometry, while in the field theory, a generic excitation will eventually thermalize.

On the boundary, the essential physical properties of the gauge theory state (such as local energy density, pressure, temperature, entropy current, etc.) are captured by the expectation value of the *boundary stress tensor*, which in the bulk is related to normalizable metric perturbations about a given state. It can be extracted via a well-defined Brown-York type procedure [Balasubramanian 1999a].

To describe gravity duals of fluid flows, a useful starting point is the map between the boundary and bulk dynamics in global thermal equilibrium. In the field theory, one characterizes thermal equilibrium by a choice of static frame and a temperature field. On the gravity side, the natural candidates to characterize the equilibrium solution are static (or more generally stationary) black hole spacetimes, as can be seen by demanding regular solutions with periodic Euclidean time. The temperature of the fluid is given by the Hawking temperature of the black hole, while the fluid dynamical velocity is captured by the horizon boost velocity of the black hole. For planar Schwarzschild-AdS black holes the temperature grows linearly with horizon size; the AdS asymptotics thus ensures thermodynamic stability as well as providing a natural long wavelength regime.

Now one can consider to move away from the equilibrium configuration. Starting with the stationary black hole (namely the boosted planar Schwarzschild-AdS$_{d+1}$) solution, one wish to use it to build solutions where the fluid dynamical temperature and velocity are slowly-varying functions of the boundary directions. Intuitively, this mimics patching together pieces of black holes with slightly different temperatures and boosts in a smooth way so as to get a regular solution of (2.59). In order to obtain a true solution of Einstein's equations, the patching up procedure cannot be done arbitrarily; one is required at the leading order to constrain the velocity and temperature fields to obey the equations of ideal fluid dynamics.[2] Further, the solution itself is corrected order by order in a derivative expansion, a process that likewise corrects the fluid

[1] Note that AdS is a space of constant negative curvature, which introduces a length scale, called the AdS scale R_{AdS}, corresponding to the radius of curvature. The black hole size is then measured in terms of this AdS scale; large black holes have horizon radius $r_+ > R_{\text{AdS}}$.

[2] These constraints are actually the radial momentum constraints for gravity in AdS and imply (2.60). In contrast to the conventional ADM decomposition, we imagine foliating the spacetime with time-like leaves and "evolve" into the AdS bulk radially.

equations. All these steps may be implemented[1] in detail in a systematic boundary gradient expansion. The final output is a map between solutions of negative cosmological constant gravity and the equations of fluid dynamics in one lower dimension, i.e. the fluid/gravity map.

2.7 Within a Spacetime Neighborhood

In the Rindler wedge setting the notion of entanglement entropy naturally accounts for the scaling with the area. This notion allows the generalized second law, and consequently the entropy bound, to be understood as a consequence of causality and ultimately associated with the presence of a horizon. Also, we saw that when the Minkowski vacuum is restricted to the Rindler wedge, the quantum fluctuations of this step have a dual, thermal description associated with the horizon. Moreover, given that the vacuum quantum fields degrees of freedom in this thermal atmosphere are essentially piled up near the horizon boundary, one may consider the possibility to construct an equivalent holographic description in terms of a dimensionally reduced theory, living on a codimension one stretched horizon membrane.

In particular, given that an effective description of the large scale dynamics of the vacuum thermal state is always provided by hydrodynamics, one could even imagine a duality between the dimensionally reduced thermal gauge theory and the long time, long wavelength dynamics of the causal horizon. In this sense, the series of conceptual steps presented above seems to be a priori reproducible starting from the physics of quantum field theory in a Rindler wedge.

However, the Rindler wedge physics provide a fundamental extra ingredient. Gravitational dynamics can be directly derived from the thermodynamical properties of the Minkowski vacuum [Jacobson 1995; Eling 2006; Eling 2008; Chirco 2010b]. For the intrinsic observer dependent nature of the causal horizon, this framework gives the possibility to characterize the emergence of gravitational dynamics at a local level, starting from an accelerated observer in the locally flat surroundings of any point in spacetime.

In this sense, in the following chapters, we will consider a scenario where gravitational dynamics emerges from the holographic hydrodynamics of some microscopic, quantum system leaving in a local Rindler system. We will then consider the possibility

[1]In the technical implementation of this program, it is important that one respects boundary conditions. We require that the bulk metric asymptote to γ_{ab} (up to a conformal factor) and further be manifestly regular in the part of the spacetime outside of any event horizon.

to develop a local Rindler wedge fluid/gravity duality on the basis of the thermodynamical properties of the local causal horizons and the universality of the holographic principle. Eventually, we will combine the two pictures and discuss on the possibility to use them to provide an idea of how emergent gravity might arise from a local holographic behavior and be extended to any spacetime, via the equivalence principle.

Chapter 3

Local Rindler Setting

The notion of causal horizon can be generalized at local level, by considering the boundary of the past of any set of events, with no reference to the infinite future. Therefore, in principle, any locally accelerating frame in a local Lorentz frame will be associated with a *local Rindler horizon*

In a local Lorentz frame, the spacetime is locally Minkowski and the Minkowski isometries can be reproduced as a local approximation. An observer who accelerates *uniformly* in this frame will follow an orbit associated with a one-parameter group of *Lorentz boost* isometries generated by an approximate Killing field ξ^a.

Similarly, the global notion of bifurcate Killing horizon can be localized by focusing on a neighborhood of the bifurcation surface or by extending a neighborhood of a piece of a single Killing horizon to a neighborhood of a bifurcate Killing horizon including the bifurcation surface, provided the surface gravity is constant and nonvanishing within the neighborhood [Racz 1992]. The constancy of the surface gravity can be derived either using Einstein equations and the dominant energy condition [Bardeen 1973] or from the assumption that a neighborhood of the horizon is static or stationary-axisymmetric with a $t-\varphi$ reflection isometry [Carter 1973; Racz 1996].

In this sense, the whole phenomenology characterizing the interplay between gravity, quantum field theory and thermodynamics should be somehow reproducible at a very local level, as a consequence of the quantum vacuum physics in presence of a local acceleration horizon with its symmetries.

3.1 Local Rindler Frame

At every event in spacetime, one can introduce uniformly accelerating observers and use the horizon thermodynamics perceived by these local Rindler observers to constrain the geometry of the background dynamics.

Given a point p in a generic spacetime $(\mathcal{M}, g_{ab})$, one can generally consider a local causal horizon at p, as one side of the boundary of the past of a space-like 2-surface patch $\mathcal{B}$ including p. Thereby, in a neighborhood of p, the local horizon will be constituted by the congruence of null geodesics orthogonal to $\mathcal{B}$, characterized by the past pointing tangent null vector k^a.

With respect to the point p, one can then invoke Local Lorentz Invariance of spacetime[1] to introduce a local inertial frame (LIF) with normal Riemann coordinates $\{x^a\}$, such that p stays at $x^a = 0$. This is allowed as long as one restricts to a region of size $L \ll R(p)^{-1/2}$, where $R(p)$ gives the value of the smaller scale associated with the radius of curvature at p (which will be generically non zero).

Within this region the metric will be approximately Minkowski, that is

$$g_{ab} = \eta_{ab} + \mathcal{O}(\epsilon^2), \tag{3.1}$$

the order of approximation ϵ being fixed by the local curvature.

On the introduced LIF, one can construct a local Rindler frame (LRF) by the coordinate transformations introduced in the previous chapter (see 2.3). Therefore, with $x = \xi \cosh(\tau\kappa)$ and $t = \xi \sinh(\tau\kappa)$, the LRF metric[2] takes the form

$$ds^2 = -\kappa^2\xi^2 d\tau^2 + d\xi^2 + dy^2 + dz^2. \tag{3.2}$$

The metric above corresponds to the action of a Lorentz boost, with acceleration $a = 1/\xi$, associated with the approximate Killing vector $\xi^a = \partial_\tau$. The lapse function is $N = \kappa\xi$, where κ is an arbitrary constant associated with the normalization of boost time τ[3].

Therefore, within a small neighborhood of p, one can associate the boundary of the

[1] Note that the construction of the local Rindler frame requires to fully use the equivalence principle, at least in its Einstein formulation, in order to identify geodesics motion.

[2] Essentially, the introduction of the LRF uses the fact that we have two length scales in the problem at any event. The first one is the length scale $R^{-1/2}$, associated with the curvature components of the background metric, over which we have no control while the second is the length scale κ^{-1} associated with the accelerated trajectory which we can choose. Hence we can always ensure that $\kappa^{-1} \ll R^{-1/2}$.

[3] It is convenient here to introduce an arbitrary rescaling factor κ for the proper time, in order to have a clear label for the Rindler wedge temperature in the following derivation.

past of the patch $\mathcal{B}$ to a section of the approximate Killing horizon. The future pointing approximate boost Killing vector ξ^a, is tangent to the null congruence comprising the causal horizon and, by definition, leaves invariant the tangent plane to $\mathcal{B}$ at p.

In terms of the approximate Killing vector ξ^a, one can introduce a time[1] label v along the horizon null hypersurface, defined by $\xi^a \nabla_a v = 1$.

Given the approximate Killing symmetry, the parameter v can be expressed in terms of the null congruence affine parameter λ, via the standard relation

$$\lambda = -e^{-\kappa v}. \tag{3.3}$$

Thereby, the point p will be located at $v = \infty$ with respect to the Killing flow, while at $\lambda = 0$ for the affinely parametrized flow.

As a consequence, one gets $\xi^a = (d\lambda/dv)\, k^a$, with $(d\lambda/dv) = -\kappa\lambda$ and, in the same way,

$$\hat{\theta} = \left(\frac{d\lambda}{dv}\right)\theta = -\kappa\lambda\,\theta \quad \text{and} \quad \hat{\sigma} = \left(\frac{d\lambda}{dv}\right)\sigma = -\kappa\lambda\,\sigma, \tag{3.4}$$

which gives some helpful relations between the Killing expansion $\hat{\theta}$ and shear $\hat{\sigma}$ and the respective affine geodesics quantities.

3.1.1 Local Horizon Temperature

By assuming that the ground state of the fields in the LIF is locally approximated by the Minkowski vacuum, stability of the vacuum and local Lorentz summetry, guarantee that, the system of quantum field fluctuations at short distances outside the local Rindler horizon is well described by a canonical ensemble, with an approximate temperature[2]

$$T \approx T_U\, N = \frac{\hbar\kappa}{2\pi}, \tag{3.5}$$

where T_U is the Unruh temperature defined in Chapter 2. The expression in (3.5) shows that T stays constant throughout the Rindler wedge, because of the gravitational Doppler factor (lapse function) $N = \kappa\,\xi$ associated with the Unruh temperature, and it is well defined on the horizon. Therefore, the thermal character of the Rindler state is effectively extended from the single Rindler observer to the whole wedge.

[1]Notice that there is no relation between the proper time τ defined in the wedge and the Killing parameter v along the horizon. Nevertheless we need to keep the same scaling κ for dimensional consistence.

[2]It can be proved formally that the stability of the Rindler metric (Lorentz invariance), hence the Rindler horizon stationarity, actually implies the KMS conditions [Haag 1992; Haag 1977], responsible for the thermal character of the vacuum energy fluctuations as measured by a Rindler observer.

Alternatively, one can introduce a general temperature for the local horizon by moving to the Euclidean analogue of the Rindler space. The region around p is represented in the Euclidean sector obtained by analytically continuing to imaginary values of t by $t_E = it$. The horizons $t = \pm x$ reduce to the origin of the Euclidean section and the hyperbolic trajectory of a Rindler observer becomes a circle of radius κ^{-1} around and arbitrarily close to the origin. The Rindler coordinates (t, x) become — on analytic continuation to $t_E = it$ — the polar coordinates $(r = x, \theta = \kappa t_E)$ near the origin. The local temperature on the stretched horizon will be $\beta_{loc}^{-1} = \beta^{-1}N = \kappa/2\pi N$ so that $\beta^{-1} \equiv \kappa/2\pi$.

3.1.2 Local Horizon Entropy

As a further step, one needs to introduce a notion of entropy for the local horizon system. Such a notion can be generally introduced via an entanglement argument. In the Rindler wedge, an accelerated observer can only access information on space-like slices bounded by the bifurcation plane. Thereby, since vacuum fluctuations between the inside and the outside of the wedge are correlated, she will perceive an entanglement entropy, which scales with the area of the local boundary and diverges with the density of field states in the UV limit. However, with the introduction of an UV cut-off (generically justified via the quantum fluctuations of the horizon at the Planck scale, the so called *zitterbewegung*) one can make this entropy become actually proportional to the area, that is

$$S = \alpha A, \tag{3.6}$$

where the proportionality factor α can *a priori* depend on the nature of the quantum fields as well as be some complicate function of the position in spacetime ['t Hooft 1985; Bombelli 1986]. As we discussed in the previous chapter, the entanglement entropy is equivalent to the thermal entropy of the fields outside the horizon. Then, together with the temperature T, this notion of entropy is enough to consider the local Rindler wedge with its Killing horizon as an analogue of a canonical ensemble (Gibbs state) bounded by a diathermic wall.

3.2 Local Horizon Thermodynamics

In Chapter 2 we showed how the thermal properties of the Rindler wedge can be effectively encoded in the thermodynamics of its horizon boundary [Eling 2008]. At local level, the notion of equilibrium for the Rindler thermal fields will be associated with the stationarity given by the approximate Killing symmetry of the local causal horizon.

In particular, the equilibrium state for the system is identified with the horizon cross-section at p, the bifurcation surface, where the Killing expansion $\hat{\theta}$ and shear $\hat{\sigma}$ vanish and the horizon is instantaneously stationary with respect to λ, that is

$$\frac{dS}{d\lambda} = \frac{d(\alpha A)}{d\lambda} = 0. \tag{3.7}$$

Moving away from the bifurcation surface at p, the spacetime will become dynamical and the presence of matter will eventually distort the local Rindler causal structure, perturbing the Minkowski vacuum thermal state at the same time. In this sense, the affine parameter λ measures the distance from the equilibrium configuration along the null geodesic congruence.

From a geometrical point of view, the realization of equilibrium just requires that the affine quantities (θ, σ) are not diverging in p. This condition can be read as a restriction on the local curvature of $\mathcal{B}$ at p, which is responsible for the geometrical properties of the horizon null congruence and consequently the value of affine expansion and shear at p. In this sense, in order to define an equilibrium surface one just need to require a suitably smooth curvature for $\mathcal{B}$ at p, without fixing *a priori* the values of affine expansion and shear.

This is a very delicate point, as we will see that the properties of $\mathcal{B}$ at p, and the corresponding values of the optical scalars of the associated null congruence, actually select the theories of gravity which may arise from the thermodynamical approach by fixing the gravitational degrees of freedom of the theory. In this sense, we will show that the choice of $\mathcal{B}$ can be directly related to the Equivalence Principle formulation, which plays a fundamental role in the argument.

3.2.1 Local Rindler Wedge Perturbation

As long as the departure from equilibrium is small and slow, it can be described in analogy with a quasi-static process where a suitably small amount of energy is thrown through the horizon, in analogy with the "physical process version" for the first law of black hole horizon. In this way, spacetime geometry deformations will be related to variations of the fields energy content.

Suppose some energy is added to the system, in such a way that $\delta\rho \ll \rho$. Then, the respective variations in entropy and energy will be related by

$$\delta S = \delta E/T. \tag{3.8}$$

In the Rindler frame the appropriate energy-momentum density is $T^a_b\,\xi^b$. The integral of the energy momentum density gives the Rindler Hamiltonian

$$H_R = \int T^a_b\,\xi^b d\Sigma_a, \tag{3.9}$$

which leads to evolution in Rindler time τ and appears in the thermal density matrix $\rho = \exp(-\beta H_R)$.

A local Rindler observer, moving along the orbits of the Killing vector field ξ^a with four velocity $u^a = \xi^a/N$, will associate an energy density $u^a(T_{ab}\xi^b)$ and an energy

$$\delta E = \int (T_{ab}\,\xi^b)\,d\Sigma^a \tag{3.10}$$

where $d\Sigma^a = u^a\,dv\,\widetilde{\boldsymbol{\epsilon}} = u^a dV_{prop}$, with v being the time along the observer trajectory and $\widetilde{\boldsymbol{\epsilon}}$ indicating the area element of the associated 2-dim cross-section .

Thus, if this energy gets transferred across the horizon, the corresponding entropy transfer will be given by Eq. (3.8), with $T = \beta_{loc}^{-1}$ as the local (redshifted) temperature of the horizon. Since $\beta_{loc}\,u^a = (\beta N)\,(\xi^a/N) = \beta\,\xi^a$, one finds that

$$\delta S = \beta\,\xi^a\xi^b\,T_{ab}\;dV_{prop}. \tag{3.11}$$

Therefore, given that the area-scaling entanglement entropy is equivalent to the thermal entropy of the fields outside the horizon, one can interpret the gravitational entropy as giving the response of the spacetime deformations due to the presence of matter.

From (3.6), the variation on the horizon entropy can be written as

$$\delta S = \alpha \delta A, \tag{3.12}$$

where a UV cut-off is implicitly introduced via the proportionality constant α. With this choice, changes in the entanglement entropy of the fields in the wedge, can be effectively described in terms of geometrical variations of the horizon cross-section.

Note that, in general one would have $dS = \delta(\alpha A)$ (i.e. α can be some spacetime function). The $\alpha =$ constant assumption, made in the GR derivation of [Jacobson 1995; Eling 2006], can be indeed recast as an explicit choice of a specific formulation of the

Equivalence principle. As said, generally the UV cut-off α is fixed at the quantum gravity scale. This can be identified as the scale at which the gravitational action is of the order of the quantum of action $\hbar$. For GR this is the standard Planck length $l_p = \sqrt{G\hbar/c^3}$ and hence it is directly related to the Newton constant. However, for a general scalar-tensor theory, i.e. a theory compatible just with the Einstein Equivalence principle (EEP) [Will 2005], G is promoted to a spacetime field. As a consequence of this, one should expect that the cut-off will be generically position dependent. In this sense, assuming $dS = \alpha\delta A$ is equivalent to assume the strong formulation of the Equivalence Principle (SEP) [Will 2005], hence this implies that one will be able to recover the dynamics of at most one of the two SEP-compatible gravity theories: GR and Nordström gravity [Gerard 2007].

Given the assumption in (3.12), a quantitative expression for the system entropy variation is obtained just by applying the definition for the change of the horizon area in terms of the expansion rate of the null geodesics comprising it, that is

$$\delta A = \int_H \widetilde{\boldsymbol{\epsilon}}\, \theta \, d\lambda. \tag{3.13}$$

Moving away along the null congruence, from the equilibrium surface at $\lambda = 0$, the infinitesimal evolution of θ is given by a linear expansion around its equilibrium value at p, up to the first order in λ,

$$\theta \approx \theta_p + \lambda \left. \frac{d\theta}{d\lambda} \right|_p + \mathcal{O}(\lambda^2)\,. \tag{3.14}$$

This first order coefficient will be determined as usual by the Raychaudhuri equation,

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \|\sigma\|^2 - R_{ab}k^a k^b\,, \tag{3.15}$$

where $\|\sigma\|^2$ stands for the squared congruence shear $\sigma^{ab}\sigma_{ab}$.[1]

In this way, the entropy variation, up to $\mathcal{O}(\lambda^2)$, is given by

$$\delta S = \alpha \int_H \widetilde{\boldsymbol{\epsilon}}\, d\lambda \left[\theta - \lambda \left(\frac{1}{2}\theta^2 + ||\sigma||^2 + R_{ab}k^a k^b\right)\right]_p. \tag{3.16}$$

Now, with respect to the null congruence parameters, or equivalently in the limit

[1] Here we consider a vanishing twist, as the null congruence is taken hypersurface orthogonal.

for $\xi^a \xi_a \to 0$, Eq. (3.10) reads

$$\delta Q = \int_H \widetilde{\boldsymbol{\epsilon}}\, d\lambda \, (-\lambda\kappa)\, T_{ab} k^a k^b \,. \tag{3.17}$$

Therefore by asking that (3.8) holds, one gets an effective relation between matter and geometry, without making any use of the gravitational equations of motion.

3.2.1.1 Entropy Balance with Noether Charge

A similar relation can be obtained, by considering a local restriction of the Noether charge entropy. Since ξ^a is locally a Killing vector, $\beta_{loc}\, J^a$ can be thought of as local entropy current, with $J^a = \left(L\xi^a + 2G^{ab}\xi_b\right)$ (see (2.37) and (2.38) in Chapter 2). On-shell, i.e. when field equations hold ($2G_{ab} = T_{ab}$), one can write:

$$J^a = \left(T^{aj} + g^{aj} L\right) \xi_j \tag{3.18}$$

Therefore, for any vector k_a which satisfies $k_a \xi^a = 0$, it follows,

$$(k_a J^a) = T^{aj} k_a \xi_j . \tag{3.19}$$

The change in this quantity, when T^{aj} changes by a small amount δT^{aj}, will be $\delta(k_a J^a) = k_a \xi_j \delta T^{aj}$. It is this relation which can be used to obtain an expression for horizon entropy in terms of the Noether charge. In fact, on integrating $\delta(k_a J^a)$ over the horizon we get

$$\begin{aligned} \delta \int_{\mathcal{H}} d^{D-1}x \sqrt{h}(k_a J^a) &= \int_{\mathcal{H}} d^{D-1}x \sqrt{h} k_a \xi_j \delta T^{aj} \\ &= \beta \int_{\mathcal{H}} d^{D-2}x \sqrt{h} k_a \xi_j \delta T^{aj} \end{aligned} \tag{3.20}$$

where the integration over time has been replaced by a multiplication by $\beta = (2\pi/\kappa)$ assuming approximate stationarity of the expression. The integral over δT^{aj} is the flux of energy δE through the horizon so that $\beta \delta E$ can be interpreted as the rate of change of the entropy associated with this energy flux.

Therefore, one can take $\delta S_{grav} = \beta_{loc}\, u_a\, J^a\, dV_{prop}$ as the gravitational entropy associated with a volume dV_{prop} as measured by a local Rindler observer, moving along the orbits of the Killing vector field ξ^a with four velocity $u^a = \xi^a / N$ [1].

[1]Note that the conservation of J^a ensures that there is no irreversible entropy production in the spacetime.

In particular, one has [Padmanabhan 2010]

$$\delta S_{grav} = \beta_{loc} N\, u_a\, J^a\, dV_{prop} = \beta\, \xi_a\, J^a\, dV_{prop} = \beta[\xi_j\, \xi_a\, (2G^{aj}) + L(\xi_j \xi^j)]\, dV_{prop} \quad (3.21)$$

As one approaches the horizon, $\xi^a \xi_a \to 0$ making the second term vanish and we find that

$$\delta S_{grav} = \beta\, [\xi^j \xi^a (2G_{aj})]\, dV_{prop} \quad (3.22)$$

In the same limit ξ^j will become proportional to the original null vector k^j.

Therefore, in this case, assuming the equations of motion $(2G_{ab} = T_{ab})$ turns out to be equivalent to the condition $\delta S_{grav} = \delta S_{matter}$.

In this sense, in the same way the local inertial frame was originally introduced to study how gravity couples to matter, one can use local Rindler frames to interpret the physical content of the field equations.

The thermodynamical characterization of the local Rindler wedge system is not based just on the properties of the local causal horizon hypersurface neither on the physics of the local thermal gauge theory describes interaction of the local horizon and wedge fields, but effectively on the interaction of the two.

In particular, from a dynamical point of view, the interaction between the wedge fields and the local horizon is well captured by a membrane paradigm approach, where the acceleration horizon, at $N = 0$, is approximated by the time-like surface associated with the orbits of the Killing vector field ξ^a, with $N = constant$, in the limit $N = \xi^a \xi_a \to 0$. Such a time-like stretched horizon can be formally associated with a fictitious *fluid membrane* living on the horizon which can be used to study the dynamics the boundary of the past of $\mathcal{B}$ in a more general non-equilibrium setting. In particular, the mechanical interaction of the horizon geometry with the outside fields will be captured by the transport coefficients of the fluid.

3.2.2 Rindler Fluid Membrane

Imagine to perform a 2+1+1 split of the local wedge where the foliating space-like surfaces are surfaces of constant time according to a family of accelerated observers with 4-velocity u^a. The time-like surface associated with the orbits of the Killing vector field ξ^a, at $N = constant$, has unit space-like normal n_a. Since this vector field can be extended throughout the local spacetime as the normal to all surfaces of constant N, we have a $2+1+1$ split defined by u_a and n_a. This time-like slices admit a smooth limit to causal hoorizon when $N \to 0$, with

$$\begin{aligned} N\,u^a &\to \chi^a \\ N\,n^a &\to \chi^a \end{aligned} \tag{3.23}$$

where χ^a will be the null generator of the horizon.

From a mathematical point of view, the dynamics of the time-like congruence comprising the stretched horizon is consistently described by fluid mechanics. In this sense, one can formally identify the stretched horizon system with a “fluid”, the Killing vector ξ^a defining the horizon fluid *rest frame*. In particular, one can introduce a moving frame for the fluid, by means of a boost in the $x^i \equiv (x, y)$ directions

$$\begin{aligned} \hat{\tau} &= \gamma(\tau - \beta^i x_i) \\ \hat{x}^i &= \gamma(x^i - \beta^i \tau), \end{aligned} \tag{3.24}$$

The (boosted) Rindler horizon has fixed area and the entropy is unchanging.

In this equilibrium state we expect the fluid is described by a surface stress tensor in the perfect fluid form

$$T_S^{ab} = (\epsilon + P)u^a u^b + P\gamma^{ab} \tag{3.25}$$

where $\gamma_{ab} = g_{ab} - n_a n_b$ and the superscript S indicates this is a surface tensor. Just like the entropy density s, the surface energy density ϵ and pressure P are formally divergent quantities that may depend on the number and nature of fields in the thermal atmosphere.

As usual one deals with this by introducing a UV cutoff ℓ_c, whose value is initially unknown. The stretched horizon boundary metric

$$ds^2 \approx g_{ab} dx^a dx^b = -\kappa^2 \xi^2 d\tau^2 + d\xi^2 + dy^2 + dz^2. \tag{3.26}$$

is flat and invariant under translations in time and space. These local translational symmetries in the boundary imply the surface stress tensor is conserved. In particular, by using the thermodynamic relations $\epsilon + P = sT$, $d\epsilon = T ds$, and $dP = s dT$ one can show that the entropy density current

$$\partial_\mu (s\, u^a) = 0 \tag{3.27}$$

is conserved, as expected.

In fact, the stretched horizon system and the equilibrium fluid do not agree in general: the fluid velocity u^a is not proportional to ξ^a except as $\tau \to \infty$ at $x = y = 0$. We have chosen this point because in the limit $N \to 0$, it approaches the bifurcation point at p. This supplies the notion of local equilibrium in the general fluid.

When we move beyond the equilibrium, the fluid description gives a more complete description of the dynamics. For the non-equilibrium horizon fluid entropy is created externally via heat flux from the outside of the system, but also *internally* from the friction of expansions and shears. As before, this implies the horizon area is not fixed, the entropy current in Eq. (3.27) is not conserved, and the spacetime can no longer be exactly flat. However, in this case, we can study the local geometry perturbation via hydrodynamics. To parameterize the near horizon curved metric we follow the construction used by [Bhattacharyya 2008a] to study perturbations of black brane metrics and assume the previously constant κ, in Eq. (3.26), and boost parameter β^i, in Eq. (3.24), as functions of the stretched horizon coordinates $x^a \equiv (\tau, x, y)$. Therefore,

$$u^a = N^{-1}\gamma(x^a)\left(\frac{\partial}{\partial\tau} + \beta^i(x^a)\frac{\partial}{\partial x^i}\right), \tag{3.28}$$

with $\kappa(x^a)$ and the boost parameter $\beta^i(x^a)$ approaching constant values at $(\infty, 0, 0)$, where there is no entropy production and the expansion and shear must vanish.

For hydrodynamics to be an applicable description, the horizon gradients $\nabla_b \ln \kappa$ (or equivalently of $\ln T$) and $\nabla_b \beta^i(x^a)$ (or of u^a) in the local Rindler coordinates need to be small compared to the inverse of the fluid mean free path at $(\infty, 0, 0)$.

By dimensional analysis the inverse mean free path of this thermal state is position dependent and goes like $g^2 T/\hbar$, where g^2 is an unknown dimensionless parameter. The gradients and the inverse mean free path are divergent as we approach the true causal horizon $N \to 0$, but their ratios are finite. The horizon gradient of the local temperature is $\nabla_b \ln T \sim N^{-1}\nabla_b \ln \kappa(x^a)$, while Eq. (3.23) implies that the gradient has the form

$$\nabla_b u^a = N^{-1}\nabla_b \chi^a. \tag{3.29}$$

Thus, we need $\nabla_b \ln \kappa, \nabla_b \chi^a \ll g^2 T_0/\hbar \sim \kappa g^2$ where now $x^a = (v, x^i)$ for horizon Killing time v. This criterion is clearly satisfied for derivatives in v. This can be seen because the local equilibration time for the system goes like $g^{-2}\kappa^{-1}$, while the process is assumed to occur for an infinite amount of Killing time before terminating in the equilibrium state [Eling 2006].

Furthermore, in introducing the LIF, there was no requirement on the size of the changes in x^i directions. This implies no scale restriction for the horizon fluid. The stretched horizon cross-section at $\tau \to \infty$ can be tuned so that the changes in $\beta(x^i)$ and $\ln \kappa(x^i)$ are $\ll \kappa g^2$ near p. Thus, there is no obstruction to working in the hydrodynamic regime and therefore an order by order expansion in derivatives is justified. In the next subsection we will use the equations of hydrodynamics and the properties of stretched horizons to generalize the equilibrium relation (3.8) to a near-equilibrium entropy balance law.

3.2.3 Entropy Balance Law and Vacuum Viscosity

Following the above review of hydrodynamics, we can proceed to add a dissipative part to the perfect fluid stress tensor (3.25) and expand it in derivatives of the flow velocity. Using conservation of the stress tensor in the stretched horizon, the thermodynamic relations $\epsilon + P = sT$, $d\epsilon = Tds$, $dP = sdT$, and within the gauge choice $\Pi_{\mu\nu}u^\mu = 0$, it follows [Landau 2000] that the entropy balance law for the horizon fluid is

$$\partial_a(s\,u^a) = \frac{\delta Q}{T} + \frac{2\eta}{T}\sigma_{ab}\sigma^{ab} + \frac{\zeta}{T}\theta^2, \tag{3.30}$$

where $\sigma^{ab} = \nabla_{(a}u_{b)} - (1/2)\,\theta\,\gamma_{ab}$, $\theta = \nabla_b u^b$ and the Clausius term is just the flux of bulk matter energy into the fluid, as heat. We will see below that the entropy change on the left hand side of this equation is a finite quantity; the ratios of the divergent quantities on the right hand side will be finite.

Integrating over a volume in the horizon fluid we find

$$\int \partial_\mu(s\,u^\mu)N d\tau d^{D-2}x = N\,\frac{2\pi}{\hbar\kappa}\int \left[T_{ab}u^a n^b + 2\eta\sigma_{ab}\sigma^{ab} + \zeta\theta^2\right] d\tau d^{D-2}x \tag{3.31}$$

Using Stokes theorem on the left hand side and then taking the limit $N \to 0$ along with (3.23) yields

$$\delta S(v) = \frac{2\pi}{\hbar\kappa}\int \left[T_{ab}\chi^a\chi^b + 2\eta\hat{\sigma}_{ab}\hat{\sigma}^{ab} + \zeta\hat{\theta}^2\right] dv d^{D-2}x, \tag{3.32}$$

where the $\hat{\sigma}^{\mu\nu}$ and $\hat{\theta}$ are now the expansion and shear of the null horizon generator χ^μ. Notice how the N dependence has also canceled out of the right hand side and the relativistic entropy balance law (3.31) has been reduced to a non-relativistic form in the horizon limit, with the left hand side just a change in total entropy in Killing time. This result agrees with the equation for the "long-time" evolution of black hole entropy in the membrane paradigm [Thorne 1986; Damour 1979], if we identify $\hbar\kappa/2\pi$ as a

Hawking temperature. This approach then provides a new conceptual picture of the entropy balance law as a consequence of relativistic hydrodynamics. This is not present in the Damour–Price–Thorne membrane paradigm because no hydrodynamic limit was identified. Thus, η and ζ are not just analogous to viscosities; in our framework it is consistent to identify them as the shear and bulk viscosity of the horizon fluid.

Working with the bifurcation point parameterized as $v = \infty$ is not convenient; therefore we change to the affine parameter $\lambda = -\kappa^{-1}e^{-\kappa v}$ so that p is at the origin: $\lambda = 0$ and $x = y = 0$. Using the relations $\chi^a = (d\lambda/dv)k^a$, $\hat{\theta} = (d\lambda/dv)\theta$, $\hat{\sigma}^{ab} = (d\lambda/dv)\sigma^{ab}$, yields

$$\delta S(v) = -\frac{2\pi}{\hbar}\int \left[T_{ab}k^a k^b + 2\eta\sigma_{ab}\sigma^{ab} + \zeta\theta^2\right]\lambda d\lambda d^{D-2}x \tag{3.33}$$

which is consistent with the form of the entropy balance law written in terms of the optical scalars of the null bundle of geodetics definig the boundary of the past of $\mathcal{B}$ with tangent vector k^a (see next chapter, eq. (4.3)). Therefore, in the entropy balance the viscous terms provide an extra *internal production* entropy d_iS.

Beyond the formal identification, the fluid description of the local stretched horizon has an interesting physical interpretation. Indeed, the Minkowski vacuum, a thermal state once localized in the Rindler wedge, obeys the holographic principle. The area-scaling behavior of the wedge fields entanglement entropy suggests that the degrees of freedom in the vacuum thermal state are encoded into the $2+1$ stretched horizon boundary of the wedge. In this sense, the hydrodynamical perturbations of the vacuum should be manifest in the dynamics of the stretched horizon fluid.

In the next chapter we will show how gravitational dynamics can be derived, within this setting, from the hydrodynamical reformulation of the entropy balance (3.8). Afterwards, in the following chapters, we will consider the possibility to find an effective connection between such hydrodynamical derivation and the gauge/gravity duality arising in the string theory framework.

Chapter 4

The Einstein Equation of State

Starting from the *entropy balance principle* introduced in the previous chapter Jacobson [Jacobson 1995] was able to derive the Einstein equations as equilibrium constitutive equations for spacetime, starting from the thermodynamical properties of local causal horizons and the thermal nature of the Minkowski vacuum. Here we essentially review the details of the original derivation, strongly characterized by the equilibrium formulation. Thereby, we consider the logic extension to non-equilibrium associated with the hydrodynamical setting [Chirco 2010b].

4.1 The Einstein Equation of State

We reconsider the situation of a small change of the stationary local Rindler frame, corresponding to a small deformation of the approximated Killing horizon. From the fundamental assumption

$$dS = \alpha \delta A, \tag{4.1}$$

the system entropy variation is obtained just by applying the definition for the change of the horizon area

$$\delta A = \int_H \widetilde{\boldsymbol{\epsilon}}\, \theta \, d\lambda. \tag{4.2}$$

Thereby, by using the Raychaudhury equation, the change of the horizon area is expressed in terms of the expansion rate of the null geodesics comprising it. It follows, up to $\mathcal{O}(\lambda^2)$,

$$dS = \alpha \int_H \widetilde{\boldsymbol{\epsilon}}\, d\lambda \left[\theta - \lambda \left(\frac{1}{2}\theta^2 + ||\sigma||^2 + R_{ab}k^a k^b \right) \right]_p . \tag{4.3}$$

The corresponding matter stress tensor perturbation comes at first order in λ, namely

$$\delta Q = \int_H \widetilde{\boldsymbol{\epsilon}}\, d\lambda\, (-\lambda\kappa)\, T_{ab} k^a k^b\,. \tag{4.4}$$

Therefore, by asking for relation $\delta S = \delta Q/T$ to hold for all null vectors k^a, one can equate the $\mathcal{O}(\lambda)$ integrands in (4.3) and (4.4). At the zeroth order in λ, the value of heat flux at p is zero, hence one necessarily gets $\theta_p = 0$. Then, to the first order,

$$\frac{2\pi}{\hbar\alpha}\, T_{ab}\ell^a\ell^b = (\|\sigma\|^2 + R_{ab}\ell^a\ell^b)_p\,. \tag{4.5}$$

In fact, in a stationary picture, associated somehow to the adiabatic process considered, the first order nature of the matter perturbation suggest that the metric perturbation itself, and therefore θ and σ will be of first order. In this sense the terms θ^2 and σ^2 can be generally be neglected in such a stationary formulation. In this sense, Eq. (4.5) reduces to

$$\frac{2\pi}{\hbar\alpha}\, T_{ab} = R_{ab} + \Phi\, g_{ab}\,, \tag{4.6}$$

where Φ is an undetermined integration function.

Eventually, by assuming the local energy conservation, that is $\nabla^b T_{ab} = 0$, applying the divergence operator on both sides of (4.6), and using the contracted Bianchi identity $\nabla^b R_{ab} = \frac{1}{2}\nabla_a R$, one finally gets $\Phi = -\frac{1}{2}R - \Lambda$, hence

$$\frac{2\pi}{\hbar\alpha}\, T_{ab} = R_{ab} - \frac{1}{2} R\, g_{ab} - \Lambda\, g_{ab}\,, \tag{4.7}$$

where Λ is some arbitrary integration constant. Once the condition

$$\alpha = \frac{1}{4\hbar G} \tag{4.8}$$

is imposed, one can easily recognize the familiar Einstein equations. Noticeably, Eq.(4.8) implies that the entropy density of the local Rindler horizon is the same as the one of a black hole.

This fundamental result states that, given the entropy and energy conservation for the system, the local thermodynamical equilibrium condition is, in fact, equivalent to the Einstein equation for the local thermal spacetime. Furthermore, the EP implies that the above construction can be done at *any* spacetime point p, and hence that equation (4.7) holds everywhere in spacetime.

However, such a thermodynamical approach does not have a detailed control over the degrees of freedom of the resulting gravity theory. In particular, on the way they are effectively involved in the perturbation of the spacetime causal structure. Indeed, by allowing for some shear at p, the local equilibrium condition is formally split in two parts: a k^a dependent part, which leads to the Einstein equation, and a ∂k^a dependent part, related to the shear term, which remains unexplained. Indeed, getting rid of σ^2 terms implies an assumption on the equilibrium value of the shear, i.e. $\sigma_p = 0$. This assumption would be in any case arbitrary without a dynamical constraint. In this sense the local equilibrium condition is broken by the unavoidable presence of non-local terms, ∂k^a.

From a geometrical point of view, the equilibrium thermodynamics can give a suitable description only under the assumption that the affine congruence orthogonal to B has zero expansion and shear at p. However, we saw that this is equivalent to require that the chosen B (and hence its associated null congruence) is less general than the one allowed by the assumed entropy-area relation (or alternatively by the SEP). Such an ansatz seems too restrictive for considering (4.7) as a general result.

4.2 Non-equilibrium Thermodynamics

In fact, a non zero affine shear at p may change the way in which the equilibrium is approached by the system in the Killing frame. Given the relations

$$\hat{\theta} = \left(\frac{d\lambda}{dv}\right)\theta = -\kappa\lambda\,\theta \quad \text{and} \quad \hat{\sigma} = \left(\frac{d\lambda}{dv}\right)\sigma = -\kappa\lambda\,\sigma, \tag{4.9}$$

one realizes that the Killing shear falls off to zero at p as $\hat{\sigma} \sim e^{-2\kappa v}$ when σ vanishes, while only as $\hat{\sigma} \sim e^{-\kappa v}$ when σ is non vanishing. In this sense, for a non vanishing affine shear, the equilibrium approach can be considered slow enough for the system to be in a near-equilibrium regime. This was firstly realized in [Eling 2006].

The limits of the derivation above seem to be overcome if one allows for some shear in p. As discussed in the previous section, the LRF is not sensitive to the exact value of the affine expansion and shear at p. Therefore, setting $\sigma_p = 0$ is an unjustified arbitrary choice.

This argument was used in [Eling 2006] to recast the thermodynamical derivation in a non-equilibrium setting, where the hydrodynamical description of the horizon dynamics provides the evidence for

$$dS > \delta Q/T. \tag{4.10}$$

In this new context, the Clausius law is replaced by the entropy balance law,

$$dS(v) = -\frac{2\pi}{\hbar}\int \left[T_{ab}k^a k^b + 2\eta\sigma_{ab}\sigma^{ab} + \zeta\theta^2\right]\lambda d\lambda d^{D-2}x \tag{4.11}$$

or, equivalently

$$dS = \delta Q/T + d_i S, \tag{4.12}$$

The extra shear term in (4.5) is then associated with the internal entropy production $d_i S$, generated by the system out of equilibrium. The internal entropy contribution, $\mathcal{O}(\lambda)$, has the form

$$d_i S = -\frac{4\pi\eta}{\hbar}\int_H \widetilde{\boldsymbol{\epsilon}}\,\lambda\, d\lambda\, ||\sigma||_p^2 \tag{4.13}$$

and it is interpreted as an internal entropy production term due to some internal spacetime viscosity, with $\eta = \hbar\alpha/4\pi$ [1]. Furthermore, it is also noticed in [Eling 2006] that by using (4.8), one gets $\eta = 1/(16\pi G)$ in agreement with the value obtained for the shear viscosity of the stretched horizon of a black hole in the so called membrane paradigm [Price 1986; Thorne 1986; Damour 1979]. This result concludes the review of the argument described in [Jacobson 1995; Eling 2006; Eling 2008].

In some way, the shear contribution in (4.5) brings into the entropy balance process a new degree of freedom, which is not fixed by the Ricci tensor and so has nothing to do with the local matter energy sources. Actually, the surface shear is srelated to the Weyl tensor and usually associated with the distortion on the geodesics congruence due to a gravitational perturbation.

In fact, this argument opens an issue about the absence so far of any role for gravitational fluxes in the system energy perturbation mechanism. Due to their non-local nature, the gravitational energy fluxes cannot be taken into account with a proper stress-energy tensor (SET). However, allowing for non-local terms, as the one in $\|\sigma\|^2$ in (4.5), seems at odds with neglecting the role of these non-local energy contributions.

On the other hand, the interpretation of the internal entropy contribution as a by-product of some sort of viscous work on the system, given in [Eling 2006; Eling 2008], is very reasonable, because the term (4.13) is actually related to some mechanical deformation due to the presence of shear in the null congruence generating the horizon.

In this sense, such a spacetime viscosity seems naturally related to the distorsive effect of a gravitational flux, to be intended as a local curvature perturbation which is independent from the Einstein equation. This suggests that gravitational energy fluxes

[1] The internal entropy production terms originating from the squared gradients of state variables is a universal property of systems with viscosity in non-equilibrium thermodynamics [de Groot 1962],

can possibly play a role into the total entropy balance of the system without entering into the Einstein equilibrium relation.

Starting from these remarks, we are led to reconsider the thermodynamical argument of this section in a fully non-equilibrium setting. In particular, for the motivations given above, we show that the internal entropy production, such as (4.13), actually indicates the presence of dissipative (irreversible) processes, to be related to the conformal components of the spacetime curvature.

4.3 Internal Entropy Production

In classical non-equilibrium thermodynamics, the rate of change of the entropy is generally written as the sum of two contributions:

$$dS = d_e S + d_i S, \tag{4.14}$$

where $d_e S$ is the rate of entropy exchange with the surroundings while $d_i S$ comes from the process occurring inside the system and is a non-negative quantity, accordingly to the second law of thermodynamics. In particular, $d_i S$ is zero for reversible (quasi-static) processes and positive for irreversible processes.

The Clausius relation used for the equilibrium approach in section III, is actually equivalent to the Clausius definition of entropy for the equilibrium system, that is

$$d_e S = \delta Q / T \quad \text{and} \quad d_i S = 0, \tag{4.15}$$

as, in that case, the horizon perturbation is effectively described as a quasi-static process occurring in continuous equilibrium with the surrounding. However, this definition does not hold true any more as irreversible processes come into play.

Actually, in the non-equilibrium thermodynamical setting, the Clausius definition of entropy is generalized to the expression

$$dS = \frac{\delta Q}{T} + \delta N, \tag{4.16}$$

where δQ is classically referred to as *compensated heat*, that is the heat transferred between the system and its surroundings, while δN, the so called *uncompensated heat*, indicates the amount of entropy associated with the heat which is intrinsic to the system when it undergoes an irreversible process.

Let us stress that the above definition is very general, as it does not require either an a priori specification of the nature of the non-equilibrium variable, or the nature of the process involved. It generalizes the notion of local equilibrium by extending the entropy balance to the unbalanced contributions related to the irreversible processes, like dissipation (see e.g. [D. Jou 2001]).

The generalized Clausius relation (4.16) is helpful in order to clarify the nature of the equilibrium and non-equilibrium contributions defining our system entropy. In fact, by using the definition of the non-equilibrium entropy given in (4.14), we can write

$$d_e S + d_i S = \frac{\delta Q}{T} + \delta N \,, \tag{4.17}$$

and identify the external and internal entropy in terms of the compensated and uncompensated heat, respectively

- $d_e S = \delta Q/T$, at the reversible level,
- $d_i S = \delta N$, at the irreversible level.

With this approach, the argument described in the previous section acquires a straightforward interpretation. Indeed, the extra contribution (4.13) introduced by the non vanishing horizon shear is an internal entropy production term allowed by the most general choice of the null congruence associated with $\mathcal{B}$ compatible with the area-entropy relation for GR (that we linked to the choice of the EP formulation). Therefore, it has to be seen as a by-product of the presence of internal/purely gravitational degrees of freedom of the theory which can be responsible for irreversible dissipative processes.

However, in order to physically identify an internal entropy contribution $d_i S$ into the general expression for the horizon entropy given in (4.3), one needs a clear understanding of the relation between non-equilibrium forces and intrinsic spacetime properties involved.

Since all the thermal information of the Rindler wedge vacuum is recorded on the horizon boundary [Eling 2008], the internal spacetime variables involved in the non-equilibrium process should be related to the null geodesic congruence kinematics around p. In this sense, a possible way to capture non-equilibrium features of the thermal system is to use the analogy between the congruence bundle comprising the horizon and a classical fluid.

4.4 GR from Non-equilibrium Spacetime Thermodynamics

We now have a clear way to relate the non-equilibrium features, arising in the thermodynamical derivation of the Einstein equations, to those kinematical degrees of freedom of the horizon congruence which are turned on by the local spacetime curvature. The horizon kinematics actually defines the intrinsic spacetime properties involved in the irreversible processes.

We can then reproduce the thermodynamical derivation of the Einstein equations within the hydrodynamical setting introduced in the previous chapter, simply by starting, in a quite general way, from a generic spacelike 2-surface patch $\mathcal{B}$ at p, with non vanishing θ_p and σ_p.

Since we are now dealing with a non-equilibrium setting, we expect that the entropy can be expressed as a sum of two different contributions $dS = d_eS + d_iS$. Moreover, for the argument given in section (4.2), we are able to identify the form of the non-equilibrium, unbalanced, entropy terms. Therefore, we can split (4.3) as

$$d_eS = \alpha \int_H \widetilde{\boldsymbol{\epsilon}}\, d\lambda\, (\theta - \lambda R_{ab} k^a k^b)_p \tag{4.18}$$

$$d_iS = -\alpha \int_H \widetilde{\boldsymbol{\epsilon}}\, d\lambda\, \lambda \left(\frac{1}{2}\theta^2 + \|\sigma\|^2 \right)_p , \tag{4.19}$$

and separate, as previously argued, the reversible and irreversible levels

- (4.18) $= \delta Q/T$, at the reversible level,
- (4.19) $= \delta N$, at the irreversible level.

From the first expression above, one has

$$\begin{aligned} d_eS &= \alpha \int_H \widetilde{\boldsymbol{\epsilon}}\, d\lambda\, \left(\theta - \lambda R_{ab} k^a k^b\right)_p \quad = \\ &= -\frac{2\pi}{\hbar} \int_H \widetilde{\boldsymbol{\epsilon}}\, d\lambda\, \lambda\, T_{ab} k^a k^b \quad = \quad \frac{\delta Q}{T}, \end{aligned} \tag{4.20}$$

where the heat flux is still defined by the expression in (4.4). Even for the non-equilibrium setting the reversible heat will vanish at $\lambda = 0$. Thereby, at the zero order in λ, one deduces again $\theta_p = 0$, while, at the first order, the relation $R_{ab} + \Phi g_{ab} = (2\pi/\hbar\alpha) T_{ab}$, is recovered for all null vectors ℓ^a. Following the previous discussion this implies, together with the conservation of the matter stress-energy tensor, the Einstein equations if $\alpha = (4\hbar G)^{-1}$.

On the other hand, for the irreversible level, we have, in accordance with (4.13),

$$\delta N \,=\, d_i S = \,-\,\alpha \int_H \widetilde{\boldsymbol{\epsilon}}\, d\lambda\, \lambda\, \|\sigma\|^2_p\,. \tag{4.21}$$

This again identifies the shear contribution as an internal entropy term, associating it to some irreversible dissipative process occurring in the thermal Rindler wedge.

To get a physical interpretation of δN with respect to the thermal properties of the Rindler wedge, it is helpful to express equation (4.21) in terms of the Killing horizon parameters. In the new frame,

$$\delta N \,=\, d_i S = \frac{\alpha}{\kappa} \int_H \widetilde{\boldsymbol{\epsilon}}\, dv\, \|\hat{\sigma}\|^2_p \geq 0, \tag{4.22}$$

in accordance with the second law of thermodynamics.

By a comparison with expression (4.12), one can actually interpret the expression in (4.22) as the standard entropy production term for a fluid with shear viscosity η, defined by

$$\frac{2\eta}{T} = \frac{\alpha}{\kappa}, \tag{4.23}$$

that is $\eta = \hbar\alpha/4\pi$, in agreement with the universal relation for the viscosity to entropy density ratio found in the AdS/CFT context [Maldacena 1998].

4.4.0.1 Tensorial Degrees of Freedom and Gravitational Dissipation

While the previous discussion shows that the spacetime thermodynamics nicely fits into a non-equilibrium hydrodynamical setting, we now want to take this arguments a step further and ask whether the expression in (4.22) can be effectively related to some gravitational energy flux.

The expression for the *uncompensated heat* given in (4.22) quantifies the energy of the system which is effectively dissipated by the viscous process,

$$T\,\delta N = \frac{\alpha\, T}{\kappa} \int_H \widetilde{\boldsymbol{\epsilon}}\, dv\, \|\hat{\sigma}\|^2_p. \tag{4.24}$$

Then, by substituting $\alpha = (4\hbar G)^{-1}$, from the reversible sector of the thermodynamical approach, the quantity in (4.24) reads

$$T\,\delta N = \frac{1}{8\pi G} \int_H \widetilde{\boldsymbol{\epsilon}}\, dv\, \|\hat{\sigma}\|^2_p, \tag{4.25}$$

which coincides with the Hartle–Hawking formula for the tidal heating of a classical black hole [Teukolsky 1974; Hawking 1972; Chandrasekhar 1983; Poisson 2004; Poisson 2005].[1]

This is a striking result as it defines the internal entropy production as a purely gravitational effect. Indeed, it can be associated with the work done on the horizon by the perturbative tidal field which is described by the electric part of the Weyl curvature tensor[2]. Furthermore, relation (4.23) suggests that such a work has to be seen as acted upon the internal/microscopic degrees of freedom of the theory rather than on macroscopic quantities (in this sense (4.25) cannot be interpreted as a standard/reversible work term). The horizon viscosity implies that such a work will be converted into internal heat. Hence, the presence of the internal entropy term can then be directly related to the process of dissipation via gravitational/internal degrees of freedom. In this sense, the irreversible sector contains the information about the possible activation/propagation of such degrees of freedom of the theory.

4.5 Summary and Discussion

In a non-equilibrium hydrodynamical description of the horizon dynamics the viscous dissipative effects appear to be naturally associated with purely gravitational energy fluxes. Their association with the irreversible/dissipative sector of the theory strongly suggests an interpretation of their nature as, non-local, internal heat flows associated with the internal spacetime degrees of freedom and clarifies why in GR a local, background independent, description of gravitational waves is precluded.

Noticeably, in order to recover the field equations one always needs to effectively isolate these dissipative contributions by neatly separating the reversible and irreversible sectors of the constitutive equation. The analogy between the stretched horizon membrane and the fluid allows to recognize the natural terms related to the irreversible sector of the entropy balance. This effective separation of the reversible and irreversible regimes is further supported, at least in GR, by the fact that the energy contributions occurring in the equilibrium constitutive relations have a local nature, being always related to the gravity field sources (Ricci curvature), whereas the non-equilibrium terms

[1] Note that both in [Teukolsky 1974] and in [Chandrasekhar 1983] the Hartle–Hawking formula expressing the relation between the horizon area variation and the horizon shear is utilized with a definition of the surface gravity ϵ which is half of that used in [Hawking 1972].

[2] Even though the magnetic part of the Weyl tensor is actually necessary in order to define the Weyl curvature and the equations governing its propagation (Bianchi identities), this part does not play any direct role in determining the time derivative (evolution) of the congruence kinematic quantities, as it is just related to their spatial gradients.

are intrinsically non-local and related to those curvature components which are independent from the sources distribution (Weyl curvature). This actually shows that the thermodynamical derivation of the gravitational field equations is very general as it is sensitive to the whole spacetime curvature.

However, a different issue is the interpretation of the internal entropy production terms related to dissipation with respect to a particular spacetime solution. While the association between internal entropy and allowed form of gravitational fluxes seems quite clear (e.g. we showed that the energy dissipated in GR *coincides* exactly the Hartle-Hawking tidal heating term), it might seem however puzzling that the arbitrariness in the choice of B allows for non-zero shear and expansion of the null congruence (and hence for internal entropy production terms) even, for example, if one imagine to have performed the local Rindler wedge construction in a Minkowski spacetime.

In fact, the thermodynamical approach is providing us just with the constitutive equations of the thermal system associated with local Rindler wedge, not of the spacetime in which the latter is constructed. The arbitrariness of the choice of $\mathcal{B}$ (and hence of the thermal system properties) implies that such equations will at most characterize the structure of the gravitational theory selected by the entropy-area relation (the EP formulation). In this sense they will not be associated with physical fluxes or curvatures of the spacetime as a whole. Hence, the possible presence of internal entropy terms, even when the local Rindler wedge is constructed in flat spacetime, does not imply that the latter can be seen as a system in a non-equilibrium state.

Of course, one might take an alternative point of view, and claim that the above discussion actually shows an intrinsic limitation of the standard construction adopted here, as in [Jacobson 1995; Eling 2006]. In addressing this issue, a possibility could consist in a further characterization of the local Rindler wedge construction. In fact, one might choose to construct the 2-surface $\mathcal{B}$ in such a way that it will be sensitive to the local curvature at p and reduce to a plane in the flat spacetime case (i.e. $\mathcal{B}$ would lead generically to a non-zero θ_p and σ_p but would also reduce to the standard bifurcation surface at p for a Rindler wedge, whose orthogonal null congruence has $\theta_p = \sigma_p = 0$, in the flat spacetime limit). For example, this could be achieved by constructing $\mathcal{B}$ as a totally geodesic 2-dimensional spacelike sub-manifold of the spacetime passing through p [O'Neill 1983]. (That is, any geodesic passing through p and there tangent to $\mathcal{B}$ would have to be completely contained in $\mathcal{B}$.) Within this alternative approach, while all the formula would still pertain to the thermodynamical behavior of the local Rindler wedge at p [1], they would now be able to specialize to a specific spacetime choice and hence

[1]E.g. one can talk about dissipation only with reference to the local Rindler wedge as spacetime

link the dynamical behavior of the wedge to the actual local matter-curvature content of the chosen spacetime.

We do not see at the moment a decisive argument to go in one sense or the other. All in all, the whole point of the thermodynamical approach is not to provide an instrument able to reconstruct the kind of spacetime one is living in. Rather it is aimed to put in evidence the thermodynamical structure and the internal degrees of freedom of gravitational theories. In this sense the traditional construction, with an arbitrary B, seems sufficient. We plan however to further explore this issue in future work.

The extension to non-equilibrium suggests that the Einstein equations effectively arise from the hydrodynamics of the local vacuum. Remarkably, this argument also fixes the entropy density and shear viscosity of the vacuum such that their ratio is $\hbar/4\pi$.

This picture seems to imply that microscopic dynamics (which could include quantum gravity below the cutoff) leads to (semi-)classical Einstein gravity as collective hydrodynamic behavior at low energies. The interesting point [Eling 2006] consists in the fact that some hydrodynamic properties turn out to be universal although we initially allowed for the properties of the horizon fluid to depend on the number and nature of the quantum fields and treated the viscosities as being purely phenomenological. Once the value of the the UV cutoff scale ℓ_c is fixed to be roughly a Planck length, the entropy density associated with all local Rindler horizons is the Bekenstein–Hawking entropy density and η/s is universally $\hbar/4\pi$. All the dependence on the number and nature of the quantum fields is apparently absorbed into the low energy Newton constant G_N. This in accord with arguments that the Bekenstein–Hawking entropy is dependent implicitly on the nature of quantum fields through the renormalization of the gravitational constant and is either partly or wholly the entanglement entropy of the thermal atmosphere. Low energy physics (the balance law) and this one observation turn out to be enough to determine the entropy density and the shear viscosity of the fluid. However,the bulk viscosity is not fixed by the balance law.Though somehow required by the fluid analogy, a straightforward physical interpretation for the bulk viscosity is missing. We shall consider this issue in the next chapter.

Finally, the value $\hbar/4\pi$ also appears in the AdS/CFT literature as the universal value of the shear viscosity to entropy density ratio of gauge theories with an Einstein gravity dual. This naturally leads to the question whether a connection between this gauge/gravity duality result and the hydrodynamic derivation should be expected.

as a whole as to be seen as a conservative system.

First, in both cases holography is crucial: we postulated the thermal vacuum state is holographic, while AdS/CFT is a precise realization of the equivalence of a higher dimensional gravity theory to a lower dimensional non-gravitational theory on a boundary. Furthermore, in the duality, d dimensional gauge theories in high temperature deconfining phases are dual to large black hole or black brane spacetimes in $d+1$ AdS [Witten 1998b]. Therefore one can use classical perturbations of the large black hole or black brane spacetimes (see [Son 2007] for a review) to perform analytical computations of the hydrodynamic transport coefficients. According to the AdS/CFT dictionary the notion of viscosity is meaningful in the infrared regime of the gauge theory, which corresponds to the near horizon limit of the translationally invariant black object. In this sense these black objects have viscosities, just like the viscosity we found for local stretched horizons. In both cases the hydrodynamics of a flat spacetime system is manifested in the dynamics of a horizon boundary. However, since $\hbar/4\pi$ holds for all local acceleration horizons it seems more fundamental than the AdS/CFT results for large black holes and black branes in AdS spacetimes [Eling 2006]. These considerations will become central in the last two chapters.

Chapter 5

Generalized Gravity Theories from Thermodynamics

A crucial assumption in the previous derivation was the validity of the SEP which allowed to consider the entropy density α as a constant. One might wonder what are the consequences of relaxing such an assumption in favor of the less restrictive Einstein Equivalence Principle (EEP). In this case, one might generically expect that the entropy density is promoted to a spacetime function (basically because the EEP is consistent with a spacetime dependent Newton constant). However, in the definition for the entanglement entropy of the Rindler wedge, this implies a possibly very complicated spacetime dependence for the UV cut-off. Furthermore, the specific form of such a cut-off is not uniquely fixed by the EEP correspondingly to the fact that the latter generically allows for many generalized theories of gravity.

5.1 Thermodynamical Derivation of F(R) Gravity

In order to make the argument as simple as possible, following [Eling 2006], we start by considering the specific case of $F(R)$ gravity, which is known to be equivalent to a single field scalar-tensor theory (more precisely a Brans-Dicke theory with $\omega = 0$ and a specific potential for the scalar field [Sotiriou 2010])[1] In this case, the UV cut-off is known to be proportional to some function of the curvature $f(R) \equiv F'(R)$ (where the prime indicates the derivative with respect to R), playing the role of the inverse of the

[1] In a similar thermodynamical approach, $f(R)$ gravity is alternatively considered in [Akbar 2007; Elizalde 2008]. See also [Paranjape 2006] for a further extension of the thermodynamical perspective with a Lanczos-Lovelock gravity.

gravitational coupling. In this case, the area entropy relation is known to be given by

$$S = \alpha\, f(R)\, \widetilde{\boldsymbol{\epsilon}} \tag{5.1}$$

where α is still a constant (albeit *a priori* different from the one considered in the previous section).

It is easy to see that in this case the entropy variation along the null congruence will be

$$\frac{dS}{d\lambda} = \alpha \left(\frac{df}{d\lambda}\, \widetilde{\boldsymbol{\epsilon}} + f\, \frac{d\widetilde{\boldsymbol{\epsilon}}}{d\lambda} \right), \tag{5.2}$$

where, by definition $\widetilde{\boldsymbol{\epsilon}}^{-1} d\widetilde{\boldsymbol{\epsilon}}/d\lambda = \theta$.

Consequently, the entropy change along the horizon will read [Eling 2006]

$$dS = \alpha \int_H \widetilde{\boldsymbol{\epsilon}}\, d\lambda\, (\dot{f} + f\,\theta), \tag{5.3}$$

therefore acquiring, with respect to the previous argument, an extra contribution $\dot{f}$ coupled to the dynamics of the scalar function f. (Here the dot stays for differentiation with respect to λ.)

For this reason, in order to set the instantaneous stationarity condition at p, that is $dS = 0$, the affine expansion is no more a good dynamical variable. In this sense, it is helpful to define the quantity $\tilde{\theta} \equiv (\theta\, f + \dot{f})$ as a sort of effective expansion for the congruence [Chirco 2011b]. Consequently, the equilibrium surface for the system will be fixed by the condition

$$\tilde{\theta}_p = 0\,, \tag{5.4}$$

that is $\theta_p = -\dot{f}/f$, where $\dot{f} = f'(R)\, k^a\, R_{,a}$ is generally nonzero. In particular, this actually provides an example of LRF equilibrium surface, for which θ_p is always non-vanishing, apart from the trivial case where f is constant, for which the theory will be equivalent to GR.

From the discussion in the previous chapter, we could already expect that the presence of the non-vanishing affine expansion would produce a non-equilibrium contribution to the system entropy. In order to get a quantitative expression for the entropy change in the neighborhood of p, one again can consider an infinitesimal deviation of the entropy from its equilibrium value.

Let us then Taylor expand the integrand in (5.3) around p up to the first order in λ, that is

$$\tilde{\theta} = \tilde{\theta}_p + \lambda\, \dot{\tilde{\theta}}\Big|_p + \mathcal{O}(\lambda^2), \tag{5.5}$$

where

$$\dot{\tilde{\theta}}_p = (\ddot{f} - f^{-1}\dot{f}^2 + f\,\dot{\theta})_p. \tag{5.6}$$

One can use the Raychaudhuri equation and the geodesic equation $k^a k^b{}_{;a} = 0$, to obtain the $\mathcal{O}(\lambda)$ expression for the entropy change

$$dS = \alpha \int_H \widetilde{\epsilon}\, d\lambda\, \lambda \left[(f_{;ab} - f\,R_{ab})\, k^a k^b - \right. \\ \left. - 3/2\, f\, \theta^2 - f\, ||\sigma||^2 \right]_p, \tag{5.7}$$

where relation (5.4) is used to substitute $f^{-2}\dot{f}^2 = \theta^2$ at p. Now, keeping the expression in (4.4) for the heat flux, one can finally reproduce the same approach shown in the previous chapter (see section 4.4).

At the reversible level, the generalized Clausius relation gives

$$f\,R_{ab} - f_{;ab} + \Psi\, g_{ab} = (2\pi/\hbar\alpha)\, T_{ab} \tag{5.8}$$

where Ψ is an undetermined function. Following the original argument given in [Jacobson 1995; Eling 2006], one then requires the conservation of the matter stress-energy tensor and use the contracted Bianchi identity to write the commutator of the covariant derivative as $2v^c{}_{;[ab]} = R_{abd}{}^c\, v^d$. In this way, one finds

$$(f\,R_{ab} - f_{;ab})^{;a} = \left(\frac{1}{2}\, f - \Box f\right)_{,b}, \tag{5.9}$$

and thereby

$$\Psi = \left(\Box f - \frac{1}{2}\, f\right). \tag{5.10}$$

Eventually, equation (5.10), together with (5.8) exactly leads, as expected, to the field equations of $f(R)$ gravity

$$f\,R_{ab} - f_{;ab} + \left(\Box f - \frac{1}{2}\, f\right) g_{ab} = \frac{2\pi}{\hbar\alpha}\, T_{ab}. \tag{5.11}$$

with the identification $\alpha = (4\hbar G)^{-1}$. In [Eling 2006], the same result was obtained starting from the entropy balance relation $dS = \delta Q/T$, assuming $\sigma = 0$, and then identifying the extra entropy term in θ in the second line of (5.8) with a suitable internal entropy term. There, it was also shown that the alternative route of keeping the θ^2 in equation (5.9) is not compatible with the conservation of the matter energy-momentum tensor.

Indeed, following the previous discussion, the above term is expected (together with a shear dependent term) as an unavoidable contribution related to the irreversible sector of the generalized Clausius relation $\delta S = \delta Q/T + \delta N$,

$$\delta N = -\int_H \widetilde{\boldsymbol{\epsilon}}\, d\lambda\, \lambda\, (\alpha\, f)\, \left[\frac{3}{2}\,\theta^2\, +\, ||\sigma||^2\right]_p, \tag{5.12}$$

which identifies the internal entropy production terms of the system.

As expected, the internal entropy in (5.12) now shows contribution both from scalar and tensorial degrees of freedom. Indeed, by using the same argument as in the GR case, we again have a natural interpretation for the expression in (5.12) as the dissipative function of the system.

The shear squared contribution is equivalent to the one found for GR, with a shear viscosity coefficient which now takes a factor f,

$$\eta = \frac{\hbar\, \alpha\, f}{4\pi}, \tag{5.13}$$

as a consequence of the UV cut-off chosen for the area entropy relation.

On the other hand, the internal entropy contribution due to the scalar degree of freedom is now given by

$$d_i S_\theta = -\int_H \widetilde{\boldsymbol{\epsilon}}\, d\lambda\, \lambda\, (\alpha\, f)\, \frac{3}{2}\, \theta_p^2. \tag{5.14}$$

By making use of a kinematical analogy, and by expressing the above equation in the Killing frame, one is naturally led to define the bulk viscosity ζ as

$$\frac{\zeta}{T} = \frac{3\,\alpha\, f}{2\kappa}, \tag{5.15}$$

that is $\zeta = 3\,\hbar\,\alpha\, f/4\pi$, as already found in [Eling 2006].

5.1.1 Gravitational Dissipation in Scalar-Tensor Gravity

In order to give a physical interpretation to (5.14), one can use the equivalence between $f(R)$ and scalar-tensor gravity, thereby interpreting f as an effective massive dilaton [Chirco 2010b; Chirco 2011b].

The action for $f(R)$ gravity is given by

$$\mathcal{A} = \frac{\hbar\alpha}{4\pi} \int d^4x \sqrt{-g}\, f(R) \,+\, \mathcal{A}_{mat}. \tag{5.16}$$

By introducing an auxiliary field[1] $\varphi \equiv \mathbf{f}(R)$ and assuming $f''(R) \neq 0$ for all R, one can take $V(\varphi)$ as the Legendre transform of $f(R)$ so that $R = V'(\varphi)$, thereby rewriting the expression in (5.16) as

$$\mathcal{A} = \frac{\hbar\alpha}{4\pi} \int d^4x \sqrt{-g}\, [\varphi\, R \,+\, V(\varphi)] \,+ \mathcal{A}_{mat}. \tag{5.17}$$

The Euler-Lagrange equations, in the Jordan frame, take the form

$$\varphi \left(R_{ab} - \frac{1}{2} g_{ab} R \right) \,+\, (g_{ab}\Box - \nabla_a \nabla_b)\, \varphi \,+ \\ +\, \frac{1}{2} g_{ab} V(\varphi) = \frac{2\pi}{\hbar\alpha}\, T_{ab}, \tag{5.18}$$

equivalent to field equations given in (5.11).

In this frame, by using the relation (5.4), one can express the dissipated energy coupled to the bulk and shear viscosity in (5.14), in terms of the auxiliary scalar field φ

$$T\delta N = -\int_H \widetilde{\boldsymbol{\epsilon}}\, d\lambda\, \lambda\, (\alpha\, \varphi)\, T\, \left[\frac{3}{2}\, \varphi^{-2}\, \dot{\varphi}^2 + ||\sigma||^2 \right]_p. \tag{5.19}$$

Similarly to the GR case, one expect that relation (5.19) expresses some purely gravitational energy loss for the system, this time involving both scalar and tensorial fluxes. The interpretation of the term related to the shear is straightforward as it is clearly the generalization to a scalar-tensor theory of the tidal heating already obtained for the GR case [Will 1981].

More problematic is the interpretation of the bulk viscosity (purely scalar) contribution as no equivalent derivation as that for the tidal heating term in GR has been performed (to our knowledge) for scalar-tensor theories of gravity.

The entropy production terms associated with the quadratic expansion appear as an extra purely scalar dissipative contribution. However, the scalar dissipative contribution seems to be formally local, as a derivative of a scalar field at a point. Being local, this term would be frame independent, thereby it would exist for any observer

[1] Note that the fact that in $f(R)$ gravity the associated scalar field is not a generic spacetime function but rather just of R, makes it possible to derive a closed system of equations without having to assume the equations of motion of the scalar field separately.

(accelerated or inertial) in the local patch of spacetime and would always end up describing the dynamics of the global spacetime. Therefore, an interpretation of this term as a dissipative contribution would imply waves of the scalar field would be dissipative in the spacetime. This is inconsistent with that fact that classical gravitational theories are time reversal invariant.

In order to address this problem, we start in this work by extending the thermodynamical approach, previously applied to metric $F(R)$ gravity, and work in the Palatini formalism, where the connection is a priori an independent variable from the metric.

5.2 Palatini $F(\mathcal{R})$ Gravity

In the Palatini formalism, one treats the connection as an a priori independent variable. The Riemann tensor $\mathcal{R}_{abrl}$, constructed out of this connection, is now therefore also independent of the metric.

Our starting point will be an entropy density which is an arbitrary function of the independent Ricci scalar

$$S = \alpha \int \sqrt{h} f(\mathcal{R}) d^2x. \tag{5.20}$$

If the thermodynamic approach works in this case, we expect now *two* equations, which are the equations of motion following from the variation of the Lagrangian

$$I_P = \frac{\alpha}{4\pi} \int \sqrt{-g}(F(\mathcal{R}) + L_{matt}(g_{ab}, \psi)) \tag{5.21}$$

with respect to g^{ab} and $\Gamma^{\lambda}{}_{ab}$. Note that in the Palatini formalism the matter part of the action is assumed not to depend on the independent connection.

Palatini $F(\mathcal{R})$ gravity theory has been discussed extensively over the past decade as an alternative theory of gravity [Sotiriou 2010]. Here we pause briefly to review the properties of this theory. Defining $f = dF/dR$ as before, the equations of motion are

$$f(\mathcal{R})\mathcal{R}_{ab} - \frac{1}{2}F(\mathcal{R})g_{ab} = \left(\frac{2\pi}{\alpha}\right) T_{ab} \tag{5.22}$$

$$\bar{\nabla}_s(\sqrt{-g} f(\mathcal{R}) g^{s(a} d^{b)}{}_l - \bar{\nabla}_l(\sqrt{-g} f(\mathcal{R}) g^{ab}) = 0, \tag{5.23}$$

where $\bar{\nabla}$ represents the covariant derivative defined with respect to the independent connection. The connection equation is equivalent to the more compact condition

$$\bar{\nabla}_l(\sqrt{-g} f(\mathcal{R}) g^{ab}) = 0. \tag{5.24}$$

Note that when f is equal to a constant this equation reduces to the usual metric compatibility condition for g^{ab}. Therefore we see the textbook equivalence of the Palatini and metric formalisms of GR. In general however, (5.24) implies the conformally related metric

$$\bar{g}_{ab} = f(\mathcal{R})g_{ab} \tag{5.25}$$

is compatible with the connection. Imposing this condition, one can relate the Ricci tensor and scalar constructed from $\bar{g}_{ab}$ to the metric quantities

$$\begin{aligned}\mathcal{R}_{ab} &= R_{ab} + \frac{3}{2}\frac{1}{f^2}(\nabla_a f')(\nabla_b f) \\ &\quad -\frac{1}{f}(\nabla_a\nabla_b - \frac{1}{2}g_{ab}\Box)f \qquad (5.26)\\ \mathcal{R} &= R + \frac{3}{2}\frac{1}{f^2}(\nabla_a f)(\nabla^a f) + \frac{3}{f}\Box f. \qquad (5.27)\end{aligned}$$

These equations can be substituted into (5.22) to yield [Sotiriou 2010]

$$\begin{aligned}fG_{ab} &= \left(\frac{2\pi}{\alpha}\right)T_{ab} - \frac{1}{2}g_{ab}\left(f\mathcal{R} - F(\mathcal{R})\right) \\ &\quad +\frac{1}{f}(\nabla_a\nabla_b - g_{ab}\Box)f - \frac{3}{2}\frac{1}{f^2}[(\nabla_a f)(\nabla_b f) \\ &\quad -\frac{1}{2}g_{ab}(\nabla f)^2]. \end{aligned} \tag{5.28}$$

Solving the trace of (5.22) for $\mathcal{R}$ in terms of T, one can completely eliminate the connection as an independent variable and reduce the system to one equation of motion that looks like GR with a modified source. Following the reasoning of [Chirco 2010b] one then should expect that no bulk viscosity term appears in this case, as no-extra dynamical gravitational degree of freedom with respect to the metric appears in Palatini $F(\mathcal{R})$ gravity.

5.2.1 Thermodynamic Formalism

With the connection now an independent variable and the metric no longer a priori compatible with it, we want to consider the effect (if any) on the basics of the thermodynamics of spacetime formalism introduced in Section II. This is not only worth doing for checking the validity of the above discussed expectation (no-bulk viscosity associated with $F(\mathcal{R})$) but also as a first step towards the generalization of the spacetime thermodynamics formalism to the broader class of metric-affine theories of gravity

[Chirco 2011b].

In the neighborhood of each point, spacetime is still locally flat and we still can construct the boost Killing vector χ^a since the Killing equation $\mathcal{L}_\chi g_{ab} = 0$ does not depend on the connection. On the local horizon χ^a is still a null generator. However, in the presence of an independent connection, there is a priori an ambiguity in whether χ^a is a geodesic with respect to the independent connection or the metric one. We have either

$$\chi^b \bar{\nabla}_b \chi^a = \bar{\kappa}\chi^a \tag{5.29}$$

or

$$\chi^b \nabla_b \chi^a = \kappa\chi^a, \tag{5.30}$$

and the corresponding choices for the affine parametrization

$$\chi^a = -\bar{\kappa}\bar{\lambda}k^a \tag{5.31}$$

$$\chi^a = -\kappa\lambda k^a. \tag{5.32}$$

Each vector above is affinely parameterized with respect to either the independent or the Levi-Civita connection: $k^b\bar{\nabla}_b k^a = 0$ or $k^b\nabla_b k^a = 0$. Therefore, in the entropy change δS, one must consider changes with respect to either affine parameter. A priori the different Clausius relations could yield two different equations of motion. In the next two subsections we will consider variations with respect to $\bar{\lambda}$ and λ in turn.

5.2.1.1 Variation using Independent Connection

First we consider the heat flux. We express the Killing field in terms of the affine k^a using the formula $\chi^a = -\bar{\lambda}k^a$, where we scale the $\bar{\kappa} = 1$ as usual.[1] Therefore,

$$\frac{\delta Q}{T} = 2\pi \int T^M{}_{ab} k^a k^b (-\bar{\lambda}) d\bar{\lambda} \sqrt{h} d^2x. \tag{5.33}$$

In the expression above, differently from Eq. (4.4), we use the affine null vector k^a with respect to the independent connection and the null geodesic bundle comprising the horizon is now parametrized by $\bar{\lambda}$. On the other hand, as the matter only feels the metric g_{ab}, the relevant volume element is still given by $\sqrt{g}$, reducing to $\sqrt{h}$ on the horizon. In this sense, we can write the relevant horizon volume element as $d\Sigma^b = k^a\sqrt{h}d^2x d\bar{\lambda}$.

[1] It is worth noticing here that there is an ambiguity about which κ (barred or unbarred) would actually appear in the Tolman–Unruh temperature T (see Chapter 2). If the unbarred κ is chosen, due to the coupling of matter fields only to metric, then we assume the ratio of the two surface gravities can be scaled to unity without loss of generality.

Along the same parameter $\bar{\lambda}$, we now consider a general variation of the entropy. This has the same form as (4.3), but we express it in a slightly different way,

$$\delta S = \int \sqrt{h} f \bar{\theta} d^2 x, \tag{5.34}$$

where

$$\bar{\theta} = \frac{1}{\sqrt{h}} \frac{d\sqrt{h}}{d\bar{\lambda}} + \frac{1}{f} \frac{df}{d\lambda} = \frac{d \ln(f\sqrt{h})}{d\bar{\lambda}}. \tag{5.35}$$

This suggests that the new expansion measuring the product of f and of the transverse element is the relevant one. Thus, we make the transformation

$$\sqrt{\bar{h}} = f(\mathcal{R})\sqrt{h}. \tag{5.36}$$

so that in terms of this new variable, the entropy is an area entropy

$$S = \alpha \int \sqrt{\bar{h}} d^2 x. \tag{5.37}$$

Imposing the entropy balance equation and matching order by order in $\bar{\lambda}$ we find the zeroth order equilibrium condition, which can be expressed as

$$\bar{\theta} = 0 \rightarrow k^a \bar{\nabla}_a(\sqrt{\bar{h}}) = k^a \bar{\nabla}_a(f'\sqrt{h}) = 0. \tag{5.38}$$

Note that we could have performed the above conformal transformation also in metric $F(R)$ gravity, but in that case the equilibrium condition involves the Levi–Civita connection. On the other hand, the above formula in terms of the independent connection is reminiscent of the metric compatibility condition (5.24).

First, note that the vanishing of the expansion $\bar{\theta}$ can be expressed as

$$\frac{d}{d\bar{\lambda}} \sqrt{det(\bar{g}_{ab} e^a_A e^b_B)} = \frac{d}{d\bar{\lambda}} \sqrt{det(\bar{g}_{ab}) det(e^a_A e^b_B)} = 0. \tag{5.39}$$

The quantities e^a_A are basis vectors in the cross-section of the horizon and the index A runs over the two transverse directions. Since these basis vectors by construction are Lie transported along the horizon, we must have that

$$\frac{d}{d\bar{\lambda}} \sqrt{-\bar{g}} = 0. \tag{5.40}$$

But since

$$\frac{d}{d\bar{\lambda}}\sqrt{-\bar{g}} = \frac{1}{2}\bar{g}_{ab}\frac{d}{d\bar{\lambda}}\bar{g}^{ab} = 0, \tag{5.41}$$

this condition implies that

$$\frac{d}{d\bar{\lambda}}(\sqrt{-\bar{g}}\bar{g}^{ab}) = 0. \tag{5.42}$$

Demanding this equation hold for all null vectors k^a, at each point p, and using the fact that (5.36) implies

$$\bar{g}_{ab} = f g_{ab}, \tag{5.43}$$

we arrive at the metric compatibility equation (5.24),

$$\bar{\nabla}_\lambda(f\sqrt{-g}g^{ab}), \tag{5.44}$$

as the equivalennt of the auxiliary equilibrium condition in the metric formalism.

Now we go back to the entropy balance law and continue to next order in $\bar{\lambda}$. We find

$$\delta S = \alpha \int \sqrt{\bar{h}}\frac{d\bar{\theta}}{d\bar{\lambda}}\bar{\lambda}d^2x. \tag{5.45}$$

Using the Raychaudhuri equation this can be re-expressed as

$$\delta S = -\alpha \int \sqrt{\bar{h}}\mathcal{R}_{ab}k^a k^b \bar{\lambda}d^2x, \tag{5.46}$$

in terms of the Ricci tensor constructed from $\bar{g}_{ab}$. Note the appearance of $\sqrt{h}$ as opposed to the $\sqrt{\bar{h}} = f\sqrt{h}$. Imposing the Clausius relation, and matching both sides, we find

$$f\mathcal{R}_{ab} + \Phi g_{ab} = \left(\frac{2\pi}{\alpha}\right) T^M_{ab}. \tag{5.47}$$

The f in front of the Ricci tensor has reappeared to account for the mismatch between the effective volume element and the metric volume element felt by matter flux.

In the Palatini formalism, $\bar{\nabla}^a T_{ab} \neq 0$, but the usual metric conservation law $\nabla^a T_{ab} = 0$ holds. Imposing this condition yields

$$\nabla_b \Phi = -f\nabla^a \mathcal{R}_{ab} - \mathcal{R}_{ab}\nabla^a f. \tag{5.48}$$

The main problem is to calculate the g covariant divergence of the $\bar{g}$ Ricci tensor. We know that $\mathcal{R}_{ab}$ is related to R_{ab} via a conformal transformation with conformal factor $f^{1/2}$. The connection relating the covariant derivatives with respect to the two metrics has the form

$$\Gamma^{\sigma}{}_{ab} = -\delta^{\sigma}_{(a}\bar{\nabla}_{b)} \ln f' + \frac{1}{2}\bar{g}_{ab}\bar{g}^{\sigma\delta}\bar{\nabla}_{\delta} \ln f'. \tag{5.49}$$

Using the formula (see e.g. [Koivisto 2006])

$$\nabla^a \mathcal{G}_{ab} = -\nabla^a \ln f \mathcal{R}_{ab} \tag{5.50}$$

and defining $\mathcal{G}_{ab} = \mathcal{R}_{ab} - \frac{1}{2}g_{ab}\mathcal{R}$, we ultimately find that

$$\nabla_b \Phi = -\frac{1}{2} f \bar{\nabla}_b \mathcal{R} = -\frac{1}{2}\bar{\nabla}_b F, \tag{5.51}$$

so that $\Phi = -1/2F + const.$ as we expect. Therefore in the irreversible sector, there is no need for a bulk viscosity term, and the shear viscosity remains the same as in GR and metric $F(R)$ gravity.

5.2.1.2 Variation using Levi–Civita Connection

Now we re-consider the same problem, but working instead with quantities defined with respect to the Levi–Civita connection. Hence we consider the affinely parameterized tangent to be k^a. The representation of the heat flow and the entropy change is exactly the same as for the metric $F(R)$ case in section 5.16. As a result, the analysis shows that we have an equation

$$f R_{ab} - \nabla_a \nabla_b f + \frac{3}{2f}\nabla_a f \nabla_b f + \Phi g_{ab} = \left(\frac{2\pi}{\alpha}\right) T^{M}{}_{ab}, \tag{5.52}$$

but instead of the fully metric-derived object $f(R)$, now we have $f(\mathcal{R})$. The Ricci tensor that appears explicitly is constructed from the metric and comes from the Raychaudhuri equation in terms of metric compatible variables.

We need to solve for the unknown function Φ. In the metric theory this lead to a contradiction and one had to cancel the kinetic $\nabla_a f \nabla^a f$ by introducing a bulk viscosity (or equivalently, move it into the irreversible sector), but here we have to consider the presence of two different curvature tensors. Taking a covariant divergence of (5.52), we find that

$$\begin{aligned}\nabla_b \Phi = &-(\nabla^a f) R_{ab} - f \nabla^a R_{ab} + \nabla^a \nabla_a \nabla_b f \\ &+ \frac{3}{2f^2}\nabla_a f \nabla^a f \nabla_b f - \frac{3}{2f}\Box f \nabla_b f \\ &- \frac{3}{2f}\nabla^a f \nabla_a \nabla_b f.\end{aligned} \tag{5.53}$$

Next, we use the Bianchi identity, $\nabla^a R_{ab} = \frac{1}{2}\nabla_b R$ and a contracted version of the commutator of covariant derivatives

$$\nabla^a \nabla_b V_a - \nabla_b \nabla^a V_a = R_b^\tau V_\tau, \tag{5.54}$$

where $V_a \equiv \nabla_a f$ to re-express the second and third terms on the right hand side above. In addition, note that the last term can be re-expressed as

$$\begin{aligned} &-\frac{3}{2f}\nabla^a f \nabla_a \nabla_b f = -\frac{3}{4f}\nabla_b(\nabla^a f \nabla_a f) = \\ &-\nabla_b \left(\frac{3}{4f}\nabla^a f \nabla_a f\right) - \frac{3}{4f^2}\nabla_a f \nabla^a f \nabla_b f. \end{aligned} \tag{5.55}$$

Combining these results, we obtain

$$\begin{aligned} \nabla_b \Phi = \nabla_b \left(\Box f - \frac{3}{4f}\nabla_a f \nabla^a f\right) - \frac{1}{2} f(\mathfrak{R}) \nabla_b R \\ - \left(\frac{3}{4f^2}\nabla_a f \nabla^a f + \frac{3}{2f}\Box f\right)\nabla_b f. \end{aligned} \tag{5.56}$$

Here, the key term is the $-1/2f(R)\nabla_b R$. We can introduce the following ansatz

$$R = \mathfrak{R} - Y, \tag{5.57}$$

where Y is some unknown function. Then this term can be manipulated into total derivative terms plus a term multiplying $\nabla_b f$:

$$-\frac{1}{2}f(R)\nabla_b R = -\frac{1}{2}\nabla_b f + \frac{1}{2}\nabla_b(fY) + \frac{1}{2}Y\nabla_b f. \tag{5.58}$$

As total derivatives, the first two pieces contribute to the solution for Φ, which now agrees with the set of terms proportional to g_{ab} in the single equation of motion (5.28). The last term above combines with the terms proportional to $\nabla_b f$ in (5.56). Demanding that term be zero as a type of consistency or integrability condition implies

$$Y = \mathfrak{R} - R = \frac{3}{2f^2}\nabla_a f \nabla^a f + \frac{3}{f}\Box f, \tag{5.59}$$

which is exactly the relationship between the two Ricci scalars in (5.27) derived from the conformal transformation.

We have derived (albeit somewhat indirectly) the equations of motion for Palatini $F(\mathfrak{R})$ gravity when the connection is eliminated as a independent variable. The fact that equations of motion derived in Sections 5.2.1.1 and 5.2.1.2 are equivalent can be seen a posteriori from the conformal relationship between $\bar{g}_{ab}$ and g_{ab}. This implies that the Killing vector χ^a is geodesic with respect to both the independent and metric connections. Therefore, we have shown that the thermodynamic approach can be extended to encompass the Palatini formalism and Palatini $F(\mathcal{R})$ gravity. In this case no additional bulk viscosity term is needed in the analysis.

5.3 Scalar-Tensor Representations

It is well known that both the metric and Palatini versions of $F(R)$ gravity are equivalent to particular scalar-tensor theories. First consider the metric $F(R)$ action. One can treat $f(R) \equiv \frac{\partial F}{\partial R}$ as an auxiliary field φ and assume $F''(R) \neq 0$ for all R. Then one can take the potential $V(\varphi)$ as the Legendre transform of $F(R)$ so that $R = V'(\varphi)$. Therefore one can rewrite the action in the equivalent form

$$I_{\omega=0} = \frac{\alpha}{4\pi} \int d^4x \sqrt{-g} (\varphi R + V(\varphi) + L_{matt}). \tag{5.60}$$

This is the Jordan frame representation of a Brans-Dicke scalar-tensor theory with the Dicke coupling constant set to $\omega = 0$. The corresponding equations of motion are

$$R = V'(\varphi) \tag{5.61}$$

$$\varphi G_{ab} = \nabla_a \nabla_b \varphi + \left(\frac{2\pi}{\alpha}\right) T^M_{ab} - g_{ab} \Box \varphi - \frac{1}{2} g_{ab} V(\varphi). \tag{5.62}$$

These equations also simply follow from the metric equation of motion

$$f R_{ab} - \nabla_a \nabla_b f + \Phi g_{ab} = \left(\frac{2\pi}{\alpha}\right) T^M{}_{ab}, \tag{5.63}$$

with the identification $f \equiv \varphi$. Hence in the scalar-tensor representation the "bulk viscosity" term has the form [Chirco 2010b]

$$\delta N_{bulk} = \alpha \int d^2x d\lambda \lambda \sqrt{h} \frac{3}{2\varphi} k^a k^b \nabla_a \varphi \nabla_b \varphi. \tag{5.64}$$

The procedure is the same for the Palatini action (5.21) and one finds

$$I_{pal} = \frac{\alpha}{4\pi} \int d^4x\sqrt{-g}(\varphi R + V(\varphi) + L_{matt}). \tag{5.65}$$

Using the relationship between R and $\mathcal{R}$ found earlier in (5.27) we can express this action (up to surface terms) as

$$I_{\omega=-3/2} = \frac{\alpha}{4\pi} \int d^4x\sqrt{-g}(\varphi R + \frac{3}{2\varphi}\nabla_a\varphi\nabla^a\varphi + V(\varphi) + L_{matt}), \tag{5.66}$$

which is the Brans–Dicke theory with $\omega = -3/2$.

To apply the thermodynamic formalism, much of the previous analysis can be carried over, just with the identification $\varphi = f$. However, it is initially unclear how the equations of motion for the scalar field can emerge out of this analysis. To start, we assume the holographic entropy has the form

$$S = \alpha \int \sqrt{h}\varphi d^2x. \tag{5.67}$$

Suppose we follow [Eling 2006] and cancel out the expansion term by treating it as a part of the irreversible sector. Then we arrive at

$$\varphi R_{ab} - \nabla_a\nabla_b\varphi + \Phi g_{ab} = \left(\frac{2\pi}{\alpha}\right) T^M{}_{ab}, \tag{5.68}$$

for some undetermined function Φ. To determine Φ we can demand the local conservation of matter-energy as usual. Imagine that we know the action for the matter fields present. It is a functional of $I_{matt}(g_{ab}, \psi)$, where ψ represents some arbitrary matter. Using the diffeomorphism invariance of this action and assuming the matter fields satisfy their equation of motion $\delta I_{matt}/\delta\psi = 0$, one can easily show the following conservation equation holds[1]

$$\nabla^a T^M{}_{ab} = 0. \tag{5.69}$$

Imposing this equation we find the following equation for Φ,

$$\nabla_b\Phi = -(\nabla^a\varphi)R_{ab} - \varphi\nabla^a R_{ab} + \nabla^a\nabla_a\nabla_b\varphi. \tag{5.70}$$

Using the contracted Bianchi identity and the commutator of covariant derivatives, we

[1] In general, the matter part of the action can also depend on the scalar field: $I_{matt}(g, \psi, \varphi)$. Then the matter stress tensor is not conserved: $\nabla^a T^M{}_{ab} = \frac{1}{2}T_\varphi\nabla_b\varphi$, where $T_\varphi = (\sqrt{-g})^{-1}\delta I_{matt}/\delta\varphi$.

are left with

$$\nabla_b \Phi = \nabla_b(\Box\varphi) - \frac{1}{2}\nabla_b(\varphi R) + \frac{1}{2}R\nabla_b\varphi. \tag{5.71}$$

Now in order to solve this equation, we must impose the following "integrability condition" on the last term of the r.h.s of the previous equation. Namely, we assume it can be expressed as the derivative of some function, chosen to have the form $\frac{1}{2}\ V(\varphi)$, i.e. $\frac{1}{2}R\nabla_b\varphi = \frac{1}{2}\ \nabla_b V(\varphi)$. Therefore, we have the condition that

$$\frac{dV}{d\varphi} = R. \tag{5.72}$$

Meanwhile, the solution for Φ is

$$\Phi = \Box\varphi - \frac{1}{2}\varphi R + \frac{1}{2}V(\varphi) + \Lambda, \tag{5.73}$$

and the reversible equation becomes

$$\begin{aligned} \varphi R_{ab} - \nabla_a\nabla_b\varphi - \frac{1}{2}\varphi R g_{ab} + g_{ab}\Box\varphi \\ +\frac{1}{2}g_{ab}V(\varphi) = \left(\frac{2\pi}{\alpha}\right) T^M{}_{ab}, \end{aligned} \tag{5.74}$$

where we have absorbed the cosmological constant Λ into the potential $V(\varphi)$. This is exactly the set of field equations for the $\omega = 0$ theory. The scalar equation of motion is an integrability condition we must impose for consistency with the conservation of local energy-momentum.

Suppose, on the other hand, that we do not introduce a bulk viscosity term. Then equation describing reversible changes is

$$\varphi R_{ab} - \nabla_a\nabla_b\varphi + \frac{3}{2\varphi}\nabla_a\varphi\nabla_b\varphi + \Phi g_{ab} = \left(\frac{2\pi}{\alpha}\right) T^M{}_{ab}. \tag{5.75}$$

This has the same form as the $F(R)$ equations of motion

$$fR_{ab} - \nabla_a\nabla_b f + \frac{3}{2f}\nabla_a f\nabla_b f + \Phi g_{ab} = \left(\frac{2\pi}{\alpha}\right) T^M{}_{ab}, \tag{5.76}$$

but now that φ is an independent field, we can repeat the analysis above to solve for the unknown Φ function and find the scalar equation of motion as an integrability condition.

Taking a covariant divergence of (5.75), we find that

$$\nabla_b\Phi = -(\nabla^a\varphi)R_{ab} - \varphi\nabla^a R_{ab} + \nabla^a\nabla_a\nabla_b\varphi + \frac{3}{2\varphi^2}\nabla_a\varphi\nabla^a\varphi\nabla_b\varphi \\ -\frac{3}{2\varphi}\Box\varphi\nabla_b\varphi - \frac{3}{2\varphi}\nabla^a\varphi\nabla_a\nabla_b\varphi. \tag{5.77}$$

As before, we can use the Bianchi identity and the commutator of covariant derivatives (5.54) along with the formula (5.55) in Section (5.2.1.2). Combining these results, we obtain

$$\nabla_b\Phi = \nabla_b\left(\Box\varphi - \frac{1}{2}\varphi R - \frac{3}{4\varphi}\nabla_a\varphi\nabla^a\varphi\right) \\ + \left(\frac{1}{2}R - \frac{3}{4\varphi^2}\nabla_a\varphi\nabla^a\varphi - \frac{3}{2\varphi}\Box\varphi\right)\nabla_b\varphi. \tag{5.78}$$

We now impose the integrability condition on the second term as before

$$\frac{dV}{d\varphi} = R + \frac{3}{2\varphi^2}\nabla_a\varphi\nabla^a\varphi - \frac{3}{\varphi}\Box\varphi, \tag{5.79}$$

which allows us to solve for Φ. The resulting metric field equation is

$$\varphi R_{ab} - \frac{1}{2}\varphi R g_{ab} - \nabla_a\nabla_b\varphi + \frac{3}{2\varphi}\nabla_a\varphi\nabla_b\varphi + \Box\varphi g_{ab} - \frac{3}{4\varphi}\nabla_a\varphi\nabla^a\varphi g_{ab} \\ +\frac{1}{2}V(\varphi)g_{ab} = \left(\frac{2\pi}{\alpha}\right)T^M{}_{ab}. \tag{5.80}$$

Therefore we arrive at the equations of motion for Palatini $F(R)$ theory in $\omega = -3/2$ scalar-tensor representation.

5.4 General Brans–Dicke Theories and "Bulk Viscosity" as a Heat Flux

In the above section we showed how the thermodynamic approach can be used to derive the field equations for both metric and Palatini $F(R)$ gravity purely in their scalar-tensor representations. However, the entropy functional (5.67) holds also for a general Brans–Dicke theory, which has the action

$$I_{gen} = \int \sqrt{-g}\, d^4x \left[\frac{\alpha}{4\pi}\left(\varphi R - \frac{\omega}{\varphi}\nabla_a\varphi\nabla^a\varphi + V(\varphi)\right) + L_{matt}\right]. \tag{5.81}$$

Previously we were only able to derive the equations of motion for the special cases $\omega = 0$ and $\omega = -3/2$, depending on whether "bulk viscosity" term is needed. In particular, for the $\omega = -3/2$ case equivalent to Palatini, no such term was needed to complete the analysis.

Whether or not we need an additional term appears to be directly related to the existence of an additional propagating scalar degree of freedom in $F(R)$ and scalar-tensor gravity, as was first hypothesized in [Chirco 2010b]. Since in Palatini $F(R)$ the connection is only an auxiliary field, one would not identify it with any additional propagating degree of freedom. It is possible to more clearly show this distinction between an auxiliary field and dynamical propagating one in the scalar-tensor representation. Consider, for example, the $\omega = 0$ theory (any general ω will do). The trace of the metric field equation (5.74) and the scalar integrability condition (5.72) yield

$$3\Box\varphi + 2V(\varphi) - \varphi\frac{dV}{d\varphi} = \left(\frac{2\pi}{\alpha}\right) T^{M\,a}_{\ \ a}, \tag{5.82}$$

so the propagation of the scalar is determined by the matter sources, as usual. On the other hand, in the special case where $\omega = -3/2$ the same procedure yields

$$2V(\varphi) - \varphi\frac{dV}{d\varphi} = \left(\frac{2\pi}{\alpha}\right) T^{M\,a}_{\ \ a}. \tag{5.83}$$

Therefore in this case the scalar field is algebraically related to the matter sources and does not propagate.

In Chapter 3 we considered the possibility that an additional scalar degree of freedom could be associated with an effective new "gravitational" channel available for dissipating energy (e.g. for relaxing horizon perturbations). As far as the shear squared term, the additional tensorial degree of freedom was thought of as indicating a channel for horizon dissipation. In GR, this channel is sourced by a flux of gravitational perturbations across the horizon (specifically, a perturbation of the electric part of the Weyl tensor) and gives rise to the Hartle-Hawking tidal heating term [Hartle 1976; Chandrasekhar 1983; Poisson 2004; Poisson 2005]

$$\delta N_{shear} = \frac{1}{8\pi G_N}\int \hat{\sigma}_{ab}\hat{\sigma}^{ab} dv\sqrt{h}d^2x. \tag{5.84}$$

Indeed, the above expression generalizes in metric $F(R)$ in a similar way, acquiring only an overall $f(R)$ factor.

Note that this does not mean gravitational waves are dissipative. While the spacetime dynamics is completely conservative, dissipation only exists for the thermodynamical horizon system. Furthermore, in the thermodynamical argument, the shear at p only depends on the warping of $\mathcal{B}$ and therefore is completely independent of the spacetime geometry[1].

On the other hand, it is unclear if also the term (5.64) related to the additional scalar degree of freedom should be associated with some irreversible branch of the thermodynamic equations. The problem with this interpretation can be realized by analyzing the form of the "bulk viscosity" term itself given in (5.64): Unlike the shear squared term, it does not depend on derivatives of k^a and both the integral over the horizon and the arbitrary k^a vectors can be peeled off. Hence, like the other terms with this structure, it has a local interpretation, which is consistent with fact that a scalar field has a local stress tensor. After the k^a vectors are peeled off, local terms at p are frame independent. They exist for any observer (accelerated or inertial) in the local patch of spacetime and always end up describing the dynamics of the global spacetime. Indeed, the expansion is no longer an arbitrary quantity defining the local horizon system, but instead fundamentally linked to the derivative of the scalar field on the spacetime. Therefore, if we insist that this term is irreversible, in this case it would imply waves of the scalar field would be effectively dissipative in the spacetime. This is inconsistent with the fact that classical gravitational theories are time reversal invariant.

Indeed, one can effectively interpret this term as a contribution to the heat flux δQ of reversible thermodynamics [Chirco 2011b]. Let us return to the beginning of the argument and the entropy

$$S = \int d^4x \sqrt{h} \varphi(x). \tag{5.85}$$

Here we have promoted the entropy density to be an independent field in the spacetime, with dimensions of $[L]^{-2}$. In order to be consistent with the principle of background independence, this field should not be a fixed structure, it must be varied like other fields. It also must contribute to the total Lagrangian of the theory, i.e.

$$L_{matt}(g_{ab}, \psi) + L_{scalar}(g_{ab}, \varphi) \tag{5.86}$$

where ψ represents "ordinary" matter fields. Upon varying with respect to the metric,

[1]Gravitational waves have no local stress tensor, and correspondingly the Hartle-Hawking term has a non-local character as the integral over the horizon of an object constructed out of derivatives of the null normal k^a. This term is consistent with an irreversible flux, which in non-equilibrium thermodynamics [de Groot 1962; Landau 2000] is positive definite and constructed out of derivatives of the state functionals of the system.

we have the usual stress tensor of the various matter fields which do not contribute to the horizon entropy, plus a stress tensor for the scalar field. The components relevant for the heat flux across the horizon are given by contraction with the null vectors

$$\delta Q \sim k^a k^b (T^M_{ab} + T^{\varphi}_{ab}). \tag{5.87}$$

Generally, the Lagrangian for a scalar field consists of possible interaction terms, e.g. a mass squared term, φ^4 term, etc. These can be represented as part of a generic scalar potential $V(\varphi)$. However, in the stress tensor, this term's contribution is proportional to g_{ab}, so it does not appear in the heat flux. On the other hand, kinetic terms in the scalar field action must contribute. We assume that the action is constructed out of first derivatives of the scalar field. This eliminates non-minimally coupled k-essence models [Armendariz-Picon 2000], where the scalar fields have non-canonical kinetic terms.

Based on dimensional analysis, the most general contribution of the scalar to the heat flux has the basic form

$$\delta Q_{scalar} \sim \frac{\Omega(\varphi)}{\varphi} k^a k^b \nabla_a \varphi \nabla_b \varphi \tag{5.88}$$

where Ω is some dimensionless function. Since φ is dimensionfull, one would have to introduce a new length scale in order to construct a non-trivial Ω. Therefore we take Ω to be an arbitrary constant. The general form of the heat flux is now

$$\frac{\delta Q}{T} = -\int d^4x \sqrt{h}\lambda \left[2\pi T^M_{ab} k^a k^b + \left(\frac{\Omega}{\varphi}\right) k^a k^b \nabla_a \varphi \nabla_b \varphi \right]. \tag{5.89}$$

Following the analysis as before, the equation of motion the reversible sector is now (5.75), but with the extra heat flux contribution

$$\varphi R_{ab} - \nabla_a \nabla_b \varphi + \left(\frac{3/2 - \Omega}{\varphi}\right) \nabla_a \varphi \nabla_b \varphi + \Phi g_{ab} = 2\pi {T^M}_{ab}, \tag{5.90}$$

which captures any Brans-Dicke theory if we set the Dicke constant $\omega = \Omega - 3/2$.

In the special case when $\omega = -3/2$, $\Omega = 0$ and there is no additional contribution to heat flux from the scalar. This is consistent with the fact that the φ field is non-propagating (has no kinetic term) in this particular case. In addition, note that when $\omega < -3/2$ the scalar field flux comes in as a "ghost" with a negative sign and the change in the black hole entropy can no longer be positive definite due to the violation of the null energy condition. This is consistent with the study of the classical second law for Brans–Dicke theory done in [Kang 1996] and the numerical results of the gravitational collapse of scalar matter pulses [Hwang 2010], which indicated a violation of weak cosmic censorship when $\omega < -3/2$.

It is also interesting to make the transformation to the so-called Einstein frame of the Brans–Dicke theory. One makes a conformal transformation and a redefinition of the scalar field

$$\begin{aligned} \tilde{g}_{ab} &= \varphi g_{ab} \\ d\tilde{\varphi} &= \frac{d\varphi}{\varphi} \end{aligned} \qquad (5.91)$$

in the action (5.81). The action now has the form

$$I_{Ein} = \int \sqrt{-\tilde{g}} d^4x \, [\tilde{R} - (\omega + 3/2)\tilde{\nabla}_a\tilde{\varphi}\tilde{\nabla}^a\tilde{\varphi} + \exp(-2\tilde{\varphi})L_{matt}(\tilde{g})], \qquad (5.92)$$

which is just Einstein gravity with the scalar field as a matter field minimally coupled to gravity, but universally coupled to the other matter fields[1].

In the thermodynamic approach, this transformation returns the entropy to just an area in the new conformally related metric. Working in terms of this new metric, we arrive at the just the Einstein equations

$$\tilde{G}_{ab}k^a k^b = 2\pi T^{total}_{ab} k^a k^b, \qquad (5.93)$$

where

$$T^{total}_{ab} \sim \frac{\delta(I_{scalar}(\tilde{g}, \tilde{\varphi}) + I_{matt}(\tilde{g}, \tilde{\varphi}, \psi))}{\delta\tilde{g}^{ab}}. \qquad (5.94)$$

In this case there is no need for a bulk viscosity term, but in order to be consistent with the equations of motion we now must explicitly include a scalar flux as a part of the heat flow due to the matter fields. This has the form

$$\delta Q_{scalar} = (\omega + 3/2) \int d^4x \sqrt{\tilde{h}} k^a k^b \tilde{\nabla}_a\tilde{\varphi}\tilde{\nabla}_b\tilde{\varphi}. \qquad (5.95)$$

Rewriting this term in the Jordan frame using (5.91), we find it is exactly the scalar field flux we argued for in (5.89). Therefore the new interpretation of the scalar field

[1] Note that our redefinition for the scalar field differs from the standard one used in [Faraoni 2004]. In that case there is an overall factor of $(\omega+3/2)^{-1/2}$ in (5.91), which normalizes the scalar kinetic term to the canonical unity value in the Einstein frame. Hence, in this ansatz one cannot go to the Einstein frame in the $\omega < -3/2$ regime, where the theory is likely to be sick as we discussed earlier. Also in our case one can clearly see that something goes wrong in the same region of parameter space: Eq. (5.92) shows in fact that for $\omega < -3/2$ the kinetic term for the field $\tilde{\varphi}$ changes sign leading effectively to a ghost field

contribution as a heat flux ultimately does not depend on the choice of conformal frame. We can work in a frame where the scalar is purely matter, with no contribution to the entropy, or in a frame where it is another gravitational field.

5.5 Discussion

In this chapter we have extended the thermodynamics of spacetime formalism to Palatini gravity, where the connection is a priori an independent variable from the metric. We applied this procedure to Palatini $F(R)$ gravity and derived the field equations as a consequence of enforcing an entropy balance law on the local Rindler wedge system. Unlike the metric $F(R)$ case studied previously in [Eling 2006], no "bulk viscosity" term was required in order to have equations consistent with the conservation of local energy-momentum. Motivated by the fact that both versions of $F(R)$ gravity are equivalent at the classical level to particular Brans–Dicke theories, we considered an entropy density that is a scalar spacetime function $\varphi(x)$. This amounts to promoting the inverse of the Newton constant to be an independent scalar field. We showed how the thermodynamic derivation in this case can capture both the field equations of the metric and the scalar field. As a key part of our analysis, we recognized that previous interpretations introducing an irreversible bulk viscosity were incorrect. Instead, we argue that the heat flux δQ naturally contains a separate contribution from the scalar field. This description also is consistent when one works *ab initio* in the Einstein conformal frame of the scalar-tensor theory. In these theories, it seems the bulk viscosity ξ_B should actually be zero.

It is worth noting that if one works a priori the metric $F(R)$ theory, as was done in [Eling 2006], interpreting the extra term needed for consistency with local energy-momentum conservation as a separate heat flux is not as clear. In this representation, there isn't a distinct scalar field we need to endow with its own dynamics, only the metric and the general function $f(R)$. Of course, it is well-known that there is an extra dynamical scalar degree of freedom in a theory of gravity which is fourth order in metric derivatives. For example, the trace of the equation of motion (5.63) gives a wave equation relating $\Box f$ to the trace of the matter stress tensor T^M; there is no longer just an algebraic link between scalar curvature and T^M as in GR. Therefore, one can argue that f needs its own dynamics, but it appears there is no way this can be done a priori starting only with an entropy functional $\int \sqrt{h} d^2x f(R)$. Instead, the warning that there is an extra degree of freedom (effectively, an extra flux) is given by the fact that local energy-momentum conservation fails.

These results naturally raise the question whether other diffeomorphism invariant theories of gravity admit a thermodynamic interpretation. In particular, what kind of heat fluxes and viscosities appear in different theories? For example, generalized Lovelock gravities are of particular interest and have been studied in a different thermodynamic picture of gravity [Paranjape 2006]. Some other interesting examples are the generalized Palatini gravities discussed in [Vitagliano 2010] and the "metric-affine" theories [Sotiriou 2007b; Vitagliano 2011], where the matter is now coupled to the independent connection and not just the metric. One can also consider theories with a non-zero torsion, either as a dynamical propagating field [Carroll 1994] or algebraically determined by spin, as in Einstein–Cartan theory [Hehl 1976]. How do these types of geometrical structures get mapped into thermodynamics?

Finally, note that while we argued for generalizations of the entanglement entropy density by appealing to less restrictive formulations of the equivalence principle, our choices were always consistent with [Vollick 2007; Faraoni 2010] Wald's Noether charge entropy formula [Wald 1993]. Since the field equations are an assumption in the derivation of the Noether charge entropy, one may worry our approach is just a consistency check.

Chapter 6

Rindler Horizon Viscosity from Entanglement

In Chapter 3 we showed that the Einstein equations effectively arise from the hydrodynamics of the local vacuum. Remarkably, the non equilibrium thermodynamical derivation of gravitational dynamics also fixed the entropy density and shear viscosity of the vacuum such that their ratio is $\hbar/4\pi$. Given that the value $\hbar/4\pi$ also appears in the AdS/CFT literature, as the universal value of the shear viscosity to entropy density ratio of gauge theories with an Einstein gravity dual, we raised the question whether a connection between this gauge/gravity duality result and the hydrodynamic derivation may exist. In this chapter, we will consider some results suggesting that the answer may be in the affirmative.

6.1 Horizon Transport Coefficients from Vacuum Fluctuations

The relationship between the dynamics of a fluid and the dynamics of any black hole event horizon is just an analogy. The reason is that hydrodynamics is only a valid effective theory of many-body systems on long spatial and time scales [Forster 1995; Lifshitz 2000]. In order for hydrodynamics to be a valid description, the characteristic wavelength and time scale of perturbations to the system must be much larger than the microscopic scale set by a correlation length (or mean free path). This basic criterion cannot be fulfilled even in the familiar example of a spherically symmetric Schwarzschild horizon. This is the reason why the membrane paradigm relates the black hole horizon to a fictitious fluid with unphysical negative bulk viscosity [Eling 2009; Eling 2010].

However, there are black hole spacetimes where a large scale hydrodynamic limit exists. Important examples are black holes and branes in asymptotically Anti-de Sitter (AdS) spacetimes. These have been extensively studied in the literature over the past decade due to their role in the celebrated AdS/conformal field theory (CFT) correspondence [Maldacena 1998; Aharony 2000]. The correspondence relates (quantum) gravity in D dimensional asymptotically AdS spacetimes to certain conformal field theories on the $D-1$ dimensional AdS boundary. In the duality, a classical black hole in AdS spacetime corresponds to a strongly coupled thermal CFT on the boundary at the Hawking temperature. The large scale dynamics of the black hole therefore is dual to the hydrodynamics of the thermal gauge theory [Bhattacharyya 2008a].

Hydrodynamic transport coefficients such as viscosities are calculated from a microscopic theory using "Kubo formulas", which involve finite temperature Green's functions of conserved currents. This is not an easy calculation even at weak coupling (see for example, [Jeon 1995]), and seems to be extremely hard at strong coupling. However, the duality picture allows one to determine the transport coefficients of these strongly coupled theories in a fairly straightforward way by mapping the calculation of Green's functions into a classical boundary value problem in the bulk spacetime [Son 2007]. An application of this mapping is that the transport coefficients of the dual gauge theory can be calculated directly at the black hole horizon from the membrane paradigm [Kovtun 2003a; Saremi 2007; Fujita 2008; Starinets 2009; Iqbal 2009]. A key early result that emerged from this work is that, in the limit of infinite coupling, any (not necessarily conformal) gauge theory with an Einstein gravity dual has a shear viscosity to entropy density ratio of $\eta/s = \hbar/4\pi k_B$. This value was conjectured by Kovtun, Son, and Starinets (KSS) to be a universal lower bound [Kovtun 2005]. Using the membrane formalism, general formulas have been recently developed which characterize the shear viscosity of gauge theories with generalized gravity duals in terms of an effective coupling of gravitons at the horizon [Brustein 2009b; Banerjee 2009; Myers 2009a; Paulos 2010].

Although the universal KSS ratio seems to be rooted in gravitational physics, curiously it does not depend on the Newton constant G_N. Furthermore, the ratio also appears to be saturated even for a Rindler acceleration horizon in flat Minkowski spacetime [Eling 2006; Eling 2008; Chirco 2010b], where gravity is absent. Indeed, one can assume that, like a black hole, the Rindler causal horizon can be endowed with a finite area entropy density s. Although there is no holographic duality like AdS/CFT in this case, the hydrodynamic limit exists and a shear viscosity of $\hbar s/4\pi k_B$ emerges when one studies the dynamics of the horizon using the membrane paradigm [Eling 2009; Eling 2010].

However, in the absence of a clear holographic duality, the interpretation of this shear viscosity to entropy density ratio seems to be unclear. For example, what is the underlying fluid system that is being probed by these calculations?

The fluctuation-dissipation theorem links viscous dissipation to fluctuations of a thermal equilibrium state. An attempt to interpret the viscous dissipation rate of a horizon in terms of the quantized gravitational fluctuations of the horizon shear was already developed many years ago [Candelas 1977]. Here we take a different approach, based on the notion of quantum entanglement together with the properties of vacuum fluctuations.

It is well-known that observables restricted to the Rindler "wedge" of the global spacetime perceive the Minkowski vacuum to be a mixed thermal state at the Tolman–Unruh temperature [Unruh 1976]. In addition, there is a corresponding statistical entanglement entropy for matter fields in the Rindler wedge. This quantity is quadratically ultraviolet (UV) divergent, due to the infinite redshift/blueshift at the horizon. When a cut-off is introduced, the entropy scales not with volume of the wedge, but instead like the area of the horizon boundary. Hence the Rindler wedge is equipped with thermodynamic properties, which seem to be naturally encoded into a "pre-holographic" lower dimensional description associated with the horizon boundary.

On large scales this thermal vacuum state should behave as a fluid, with hydrodynamics as an effective description. In this regime, we expect to find a holographic "entanglement viscosity" which, when similarly cut off, scales exactly with the entanglement entropy so that the KSS ratio is satisfied universally. To test this hypothesis, we propose a microscopic Kubo-like formula for the shear viscosity associated with the fluid description of the vacuum thermal state. The Kubo formula is constructed from the Green's functions of the energy-momentum stress tensor for the matter fields in the wedge. All quantum fields in nature must contribute to the vacuum fluctuations and therefore to the entanglement entropy and viscosity. For simplicity, we start by considering a free, minimally coupled scalar field theory. Remarkably, we show that the ratio of our shear viscosity to the entanglement entropy density is exactly the KSS ratio. This suggests that the KSS ratio may be a fundamental holographic property of spacetime (rather than just of the aforementioned AdS black hole solutions).

In the following of this chapter we discuss two examples where a shear viscosity emerges from classical hydrodynamics applied to the Rindler thermal state. This serves as a motivation for the Kubo formula developed in section 7.2. Section 6.4 contains our calculations for the free, non-minimally coupled scalar field. We conclude in section 6.5 with the possible implications of our result, a discussion of the relationship between entanglement entropy and black hole entropy, and extensions to higher dimensions.

6.2 Rindler Wedge Viscosity

We want to consider the perturbation of a global Rindler spacetime on a large scale, with the dynamics governed by Einstein equation [Eling 2009; Eling 2010].

We start with a general D dimensional flat manifold in the Rindler coordinates $Y^a = (\tau, \xi, x^i)$, where $i = 1..d$ (we will use $D = d+2$, to identify d with the number of transverse spatial dimensions),

$$ds^2 = g_{ab}dY^a dY^b = \kappa^2\,\xi^2\,d\tau^2 - d\xi^2 - \sum_{i=1}^{d} dx^i dx_i, \tag{6.1}$$

obtained by a coordinate transformation of the usual Minkowski inertial coordinates X^a. To work conveniently at the horizon we rewrite the Rindler metric in Eddington–Finkelstein like coordinates with the following parameterization

$$\begin{aligned} v &= \tau + (2\kappa)^{-1}\ln(r) \\ r &= \kappa\,\xi^2 \\ \tilde{x}^i &= \kappa^{-1}x^i, \end{aligned} \tag{6.2}$$

so that the metric has the form

$$ds^2 = \kappa r dv^2 - dvdr - \kappa^2 \sum_{i=1}^{d} d\tilde{x}^i d\tilde{x}_i. \tag{6.3}$$

Consider a uniform boost of the Rindler spacetime (6.3) in $\tilde{x}^i$ directions, which is an isometry of the vacuum state. The result is a boosted metric

$$ds^2 = \kappa r u_\mu u_\nu d\tilde{x}^\mu d\tilde{x}^\nu - u_\mu d\tilde{x}^\mu dr - \kappa^2 P_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu, \tag{6.4}$$

where the now $d+1$ dimensional set of coordinates is $\tilde{x}^\mu = (v, \tilde{x}^i)$, the $d+1$ dimensional vector $u^\mu = (\gamma, \gamma v^i)$ (i.e. $u^\xi = 0$), and the projection tensor $P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu$. One can think of this bulk spacetime as describing a general flow of the thermal state with velocity v^i with respect to the frame of a static observer.

Now imagine, for example, gravitational waves are impinging on the system. As described in Chapter 3, though in a slightly different setting, to parameterize the perturbations, we can take $u^a(\tilde{x}^a)$ and $\kappa(\tilde{x}^\mu)$ (thereby associating a scale with κ), so that the temperature and $d+1$ velocity of the flow are slowly varying functions of the $\tilde{x}^\mu$ coordinates. In particular, the hydrodynamic limit requires the scale L of the perturbations to satisfy $L \gg \kappa^{-1}$. The metric

$$ds^2 = \kappa(\tilde{x})r\ u_\mu(\tilde{x})u_\nu(\tilde{x})d\tilde{x}^\mu d\tilde{x}^\nu - u_\mu(\tilde{x})d\tilde{x}^\mu dr - \kappa^2(\tilde{x})P_{\mu\nu}d\tilde{x}^\mu d\tilde{x}^\nu, \tag{6.5}$$

is no longer flat and hence does not satisfy $R_{ab} = 0$. However we can obtain a solution (at least in principle) to the vacuum Einstein equations working order by order in a derivative expansion. We take $u(\lambda\tilde{x}^\mu)$ and $\kappa(\lambda\tilde{x}^\mu)$ where λ is a book keeping factor to keep track of derivatives of temperature and velocity (see also Chapter 7). For example, at lowest order there should be solution to the equations $R_{ab} = 0 + O(\lambda^2)$ of the form

$$g_{ab} = g_{ab}^{(0)} + \lambda g_{ab}^{(1)}(\partial u, \partial\kappa), \tag{6.6}$$

where $g_{ab}^{(1)}$ is a $O(\lambda)$ correction to the metric (6.5).

In the membrane paradigm, we want to consider the subset of $(d+1)$ vacuum Einstein equations projected into the Rindler horizon

$$R_{\mu\nu}k^\nu = 0, \tag{6.7}$$

where k^μ is the null normal to the horizon. At lowest order, $k^\mu = u^\mu$. Note that u^μ is unit normalized with respect to the flat metric $\eta_{\mu\nu}$, but is null on the horizon $(r = 0)$ of the full bulk metric. Using the horizon Gauss–Codazzi equations and the membrane paradigm, this set of Einstein equations can be expressed solely in terms of horizon geometrical variables - i.e. the extrinsic curvature components (the horizon shear, expansion, surface gravity) and intrinsic metric of the horizon surface. At the lowest orders in λ, it is sufficient to calculate these quantities directly from the metric (6.5), the near-horizon data, and a choice of gauge. For example, the horizon shear is just the fluid shear, which is given by the symmetric, trace-free transverse part of $\partial_\mu u_\nu$

$$\tilde{\sigma}_{\mu\nu} = P_\mu^\rho P_\nu^\sigma(\partial_\rho u_\sigma + \partial_\sigma u_\rho - 2/d\ \eta_{\rho\sigma}\partial_\delta u^\delta), \tag{6.8}$$

and the horizon expansion is

$$\tilde{\theta} = \partial_\mu u^\mu + du^\mu\partial_\mu \ln\kappa. \tag{6.9}$$

Remarkably, up to $O(\lambda^2)$ the Einstein equations (6.7) imply

$$R_{\mu\nu}k^{\nu} = \partial_{\nu}{T_{(F)}}^{\nu}{}_{\mu} = 0, \tag{6.10}$$

where $\partial_{\nu}{T_{(F)}}^{\nu}{}_{\mu} = 0$ are the hydrodynamic equations of a viscous conformal fluid living on a flat Minkowski metric in one less dimension. In general, a viscous fluid stress tensor has the form of a perfect fluid, plus shear and expansion terms that are first order in λ

$$T_{(F)}{}^{\mu}_{\nu} = \epsilon u^{\mu}u_{\nu} + P(\delta^{\mu}_{\nu} + u^{\mu}u_{\nu}) - 2\eta\sigma^{\mu}_{\nu} - \xi_B(\partial_{\rho}u^{\rho})\delta^{\mu}_{\nu}. \tag{6.11}$$

Here $\xi_B = 0$, consistent with the conformal condition that $T^{\mu}{}_{\mu} = 0$, while the shear viscosity is $\eta = v/16\pi G_N$ [Eling 2010], where v is a scalar area density associated with the horizon. Assuming a Bekenstein-Hawking area entropy density $v/4G_N$, the shear viscosity to entropy density ratio turns out to be precisely the KSS ratio.

This picture is consistent with what seen in the non equilibrium thermodynamical derivation described in Chapter 3, where the equation describing reversible changes matched the Einstein equation, with Newton's constant determined by the entropy density s,

$$G_N = \frac{1}{4s}, \tag{6.12}$$

or conversely, $s = 1/4G_N = 1/4L_P^2$. In that case, the shear viscosity is $1/16\pi G_N$ and that the dissipative term

$$\delta N = \frac{s}{\kappa}\int \hat{\sigma}_{ab}\hat{\sigma}^{ab} d\tau d^2A \tag{6.13}$$

can be exactly identified with the well-known Hartle-Hawking formula for the tidal heating of a classical black hole [Hawking 1972; Teukolsky 1974; Chandrasekhar 1983; Poisson 2004; Poisson 2005].

It seems that once we demand a finite area entropy density for Rindler horizons, an entropy balance law can naturally imply gravity. This would more generally indicate that any Lorentz invariant quantum field theory with a UV cutoff (and therefore a finite entropy and a large, but finite number of degrees of freedom) must have gravity. Interestingly, this sort of induced gravity is consistent with the AdS/CFT correspondence. In the usual formulation, the CFT on the boundary has no cutoff and infinite entanglement entropy. This corresponds to the case where $G_N^{(d+1)} = 0$ and the CFT on the boundary is not coupled to gravity. Introducing a cutoff to the CFT corresponds to a brane in the AdS bulk that cuts off the region from some radial coordinate r_0 to infinity. The dual CFT on the brane is coupled to gravity and has a finite entanglement entropy that seems to match the BH entropy [Hawking 2001; Ryu 2006; Brustein 2006; Emparan 2006].

6.3 Microscopic Description and Kubo Formula

Together the two examples above provide a mutually consistent picture of a shear viscosity coefficient emerging from large scale perturbations of the Rindler thermal state. Typically, in classical hydrodynamics the viscosities are phenomenological coefficients, either measured directly in the laboratory or calculated by matching to a microscopic description of the fluid system. However, in the above examples, our classical calculations require both the entropy density and the viscosity to have a trivial relation to the observed low energy Newton constant. All the dependence on the number and nature of the quantum fields is apparently absorbed into this quantity. In order to explore this unexpected universality further, we would like to find a microscopic description for the shear viscosity in terms of the fluctuations of a thermal state in a finite temperature quantum theory.

6.3.1 Linear Response Theory in the AdS/CFT Framework

First, it is instructive to consider calculations of viscosity in the AdS/CFT correspondence. In this case, η and s are the viscosity and entropy density of an infinitely strongly coupled $(d+1)$-dimensional finite temperature gauge theory with a dual gravitational description in terms of a black hole or brane in AdS spacetime. The conformal theory lives in flat Minkowski spacetime and is thought of as being on the hologram at the AdS boundary. In the AdS/CFT prescription, a massless field φ in the bulk spacetime is dual to an operator $\mathcal{O}$ in the boundary field theory. In particular, perturbations of the bulk field act as sources for the field theory operators on the boundary via the coupling

$$\int \varphi_0 \mathcal{O} \; d^{d+1}x, \tag{6.14}$$

where φ_0 is the boundary value. For small perturbations, determining the change of the expectation value of $\mathcal{O}$ is a well-known problem in time dependent perturbation theory. In Fourier space $(k^0, \vec{k})$ the result is [Kapusta 2006]

$$\langle \delta\mathcal{O}(k^0, \vec{k}) \rangle = G_R(k^0, \vec{k}) \varphi_0(k^0, \vec{k}), \tag{6.15}$$

where G_R is the retarded two point thermal Green's function (the brackets represent a thermal average) of $\mathcal{O}$,

$$G_R(k^0, \vec{k}) = \int dt d^d x e^{ik^0 t} e^{-i\vec{k}\cdot\vec{x}} \theta(t) \langle [\mathcal{O}(x), \mathcal{O}(0)] \rangle. \tag{6.16}$$

On the other hand, linear response theory [de Groot 1962] implies that in the large scale limit $k^0, \vec{k} \to 0$

$$\langle \delta\mathcal{O}(k^0, \vec{k}) \rangle = \chi \partial_t \varphi_0, \tag{6.17}$$

where χ is some generic phenomenological transport coefficient. Matching these two descriptions, one finds the Kubo formula

$$\chi = \lim_{k^0 \to 0} \frac{1}{k^0} Im G_R(k^0, \vec{k} = 0). \tag{6.18}$$

Therefore, generic dissipative transport phenomena are described by fluctuations about the thermal equilibrium state. In the case of shear viscosity, the relevant field operator $\mathcal{O}$ is the stress tensor T^{xy} (or, in general, the trace-free spatial parts of $T^{\mu\nu}$, see (6.11)), while the classical source φ is identified with corresponding transverse metric perturbations, for example h_{xy}.

The prescription for computing the retarded Green's function is to first solve the perturbation equations for $h_{\mu\nu}$, subject to the Dirichlet condition at the asymptotic boundary and requiring at the horizon the field be purely ingoing [Son 2002]. From the on-shell action, one can derive [Iqbal 2009]

$$\chi = \lim_{k^a \to 0} \lim_{r \to \infty} \frac{\Pi(r, k^0, \vec{k})}{ik^0 \varphi(r, k^0, \vec{k})}, \tag{6.19}$$

where Π is the radial canonical momentum conjugate to the field. In the low frequency limit it turns out that the radial evolution of Π is trivial. Essentially all the relevant physics is at the horizon and this is the natural place to evaluate the above quantity. In the near-horizon limit the geometry of a black hole solution dual to a gauge theory thermal state reduces to the Rindler metric. Furthermore, in the membrane paradigm, the condition that fields be regular at the horizon immediately fixes the shear viscosity in terms of the coupling constant for transverse gravitons. In Einstein gravity, the result is simply the universal gravitational coupling $\eta = (16\pi G_N)^{-1}$ (which matches the results discussed in Section II), while in higher derivative theories one can derive a formula for η in terms of horizon quantities similar to Wald's Noether charge formula for the entropy [Brustein 2009b; Banerjee 2009; Myers 2009a; Paulos 2010].

6.3.2 Rindler Horizon Pre-holography

In flat Rindler space, there is no holographic duality of the AdS/CFT type, i.e. no string theoretic mapping between classical bulk fields and operators in a strongly coupled theory and no time-like boundary surface at infinity capable of supporting a dual holographic theory. Therefore the type of constructions reviewed above for calculating Green's functions do not appear to be available to us. However, as we have seen, there is a type of holography at the horizon due to entanglement when observables in the vacuum state are restricted to a subregion. For example, the entropy of fields in the Rindler wedge is naturally associated with the horizon boundary. Since the degrees of freedom in the wedge are packed into this membrane surface, the physics of the bulk spacetime can be effectively reduced to a lower dimensional description associated with a "stretched horizon" boundary. Hence, the shear viscosity associated with the Rindler horizon must be induced by the matter fields in the quantum vacuum state, just like the entanglement entropy.

The dual lower dimensional description of the vacuum state and the near-horizon degrees of freedom are characterized by the stress-energy tensor (6.11) and as such can be associated to a strongly coupled thermal CFT living effectively on the flat Minkowski metric $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = d\tau^2 - \sum_i dx_i dx^i$. In addition, we expect the total energy-momentum in the bulk Rindler space should be the total energy-momentum of the dual description.

In Rindler space the explicit translational symmetry in the z (or ξ) direction is broken. However, the symmetry in the other directions remains, so that the Lagrangian of a field theory must be invariant under

$$x^\mu \to x^\mu + a^\mu \tag{6.20}$$

Using the Noether theorem we can write a canonical energy-momentum tensor for the bulk fields in the Rindler spacetime

$$T_{(R)}{}^\mu_\nu = \frac{\partial L_R}{\partial(\partial_\mu \psi)}\partial_\nu \psi - \delta^\mu_\nu L_R, \tag{6.21}$$

where ψ represents a generic matter field. This stress tensor is conserved quantity in the flat spacetime sense: $\partial_\mu T_R{}^\mu_\nu = 0$. Note that the Lagrangian density is $L_R = \sqrt{-g}L_{\text{Mink}}$ (where L_{Mink} is the field Lagrangian in Minkowski spacetime) and evaluates to $L_R = \kappa\xi L_{\text{Mink}}$. Therefore, the canonical energy-momentum tensor for the Rindler wedge is $\kappa\xi$ times the $(\mu\nu)$ components of usual Minkowski space stress tensor, T^a_b.

On large scales, the holographic state must be described by a conserved lower dimensional stress tensor operator $\langle \hat{T}^{(d+1)\mu\nu} \rangle$,

$$\partial_\mu \langle \hat{T}^{(d+1)\mu\nu} \rangle = 0. \tag{6.22}$$

Here the brackets represent a thermal average $Z^{-1} Tr(\rho \hat{T}^{(d+1)\mu}{}_\nu)$ at the Tolman-Unruh temperature, which is equivalent to the Minkowski vacuum expectation value $\langle 0 | \hat{T}^{(d+1)\mu}{}_\nu | 0 \rangle$. As a simple ansatz we assume

$$\langle \hat{T}^{(d+1)\mu}{}_\nu \rangle = \int_{\ell_c}^{\infty} d\xi \, \langle \hat{T}_{(R)}{}^\mu_\nu \rangle = \int_{\ell_c}^{\infty} d\xi \, \kappa\xi \langle \hat{T}^\mu_\nu \rangle, \tag{6.23}$$

that the energy-momentum density in the lower dimensional description is a radial integral of the bulk quantities, which as usual must be cut off at a stretched horizon located at proper distance ℓ_c from the true horizon in order to be rendered finite.

This prescription is consistent with the literature on thermodynamic quantities in Rindler wedge. The Minkowski vacuum expectation value $\langle 0 | \hat{T}^a_b | 0 \rangle$ for free spin-0, spin-1/2 and spin-1 fields in the Rindler wedge was calculated long ago [Sciama 1981]. To regularize the stress tensor operator, one can impose a Fulling–Rindler subtraction

$$\langle F | \hat{T}^a_b | F \rangle = 0. \tag{6.24}$$

As expected, one finds that the Minkowski vacuum expectation value has the form of a perfect fluid stress tensor. For example, in four spacetime dimensions the bulk energy density for a scalar field has the Planckian form

$$\epsilon(\xi) = \frac{\pi^2 T^4}{30} = \frac{1}{480\pi^2 \xi^4}. \tag{6.25}$$

From our ansatz (6.23), we find an energy density that appropriately scales like the area of the horizon boundary [Dowker 1994]

$$\epsilon^{2+1} = \frac{\kappa}{960\pi^2 \ell_c^2}. \tag{6.26}$$

Using the Gibbs relation $\epsilon + P = sT$, and equation of state $\epsilon = 3P$ for the massless bulk scalar field, we find the entropy density s obeys

$$s = \frac{2\pi^3}{45}T^3 = \frac{1}{180\pi\xi^3}. \tag{6.27}$$

Integrating over ξ from ℓ_c to ∞ to find the effective area entropy yields

$$s = \frac{1}{360\pi\ell_c^2}, \tag{6.28}$$

which agrees with standard results in the literature for the brick wall/entanglement entropy ['t Hooft 1985; Susskind 1994; Dowker 1994].

If we apply the formalism of viscous hydrodynamics to this system, the shear viscosity should be given by the Kubo formula (6.18) in terms of the effective stress tensor of the lower dimensional theory associated with the horizon

$$\eta = \lim_{\omega\to 0}\frac{1}{\omega}\int d\tau d^d x e^{i\omega\tau}\theta(\tau)\langle[T_{xy}^{d+1}(\tau,x,y), T_{xy}^{d+1}(0)]\rangle, \tag{6.29}$$

where ω is a Rindler frequency. Using our ansatz that the lower dimensional densities are radial integrals of the bulk matter stress-tensor, we arrive at the following formula

$$\eta = \lim_{\omega\to 0}\frac{1}{\omega}\int_{\ell_c}^{\infty} d\xi' \int_{\ell_c}^{\infty} d\xi \int d\tau d^d x e^{i\omega\tau}\theta(\tau)\kappa^2\xi\xi'\langle[T_{xy}(\tau,x,y,\xi), T_{xy}(0,\xi')]\rangle. \tag{6.30}$$

Since we have translational invariance in (τ, x, y), we can safely choose one of the points to be at $\tau = x = y = 0$, so that the most general expression is a function $G^R_{xy,xy}(\tau, x, y, \xi, \xi')$. This type of expression is similar to those developed in [Yarom 2005; Brustein 2005; Brustein 2004]. The authors showed that correlation functions of certain operators expressed as an integral of a density over a sub-volume of Minkowski are UV divergent and scale like the horizon/boundary area. As an example, they found the heat capacity due to entanglement in the Rindler wedge.

As a first test case of our viscosity formula, we consider the thermal state to consist of a free, minimally coupled scalar field in a four dimensional Rindler spacetime. One apparent problem with this choice is that the shear viscosity in a free field theory is typically ill-defined. In physical terms, shear viscosity measures the rate of transverse momentum diffusion between the elements of a fluid. Although the quasi-particle description in kinetic theory is not a good one in a strongly coupled system, we can gain some guidance by thinking of shear viscosity as a diffusion process. One can show that $\eta \sim \epsilon l_{\rm mfp}$, where $l_{\rm mfp}$ is the mean free path of the fluid. Since in a free field theory the mean free path diverges, η diverges as well. This is just a consequence of the breakdown of the effective hydrodynamic theory.

On the other hand, in our case the equivalence principle implies a field theory in Rindler space can be thought of as being in a constant gravitational field. As we argued in section 6.2, imposing a UV cutoff on this system seems to introduce gravitational dynamics. If the cutoff is placed near the Planck length (as we suspect) the gravitational dynamics is strongly coupled there. The idea is that the dominant effect in the relaxation of the vacuum thermal state is the strongly coupled gravitational interaction. This also seems to explain how there can be universality in the result for η. In principle, all quantum matter fields should be present in the vacuum state. However, the ratio η/s should be $1/4\pi$ regardless of the type of quantum fields in the wedge or the dimension of the spacetime. Since gravity interacts with all fields in the same way, it should not make a difference whether we consider the soup of fields to be made up of a free scalar field, free fermions, or some type of interacting fields.

6.4 Universal Viscosity to Entropy Ratio from Entanglement

Since the thermal average is at the Tolman-Unruh temperature T_0, it is equivalent to an ordinary Minkowski vacuum expectation value

$$\langle 0|[T_{xy}(\tau, x, y, \xi), T_{xy}(0, \xi')]|0\rangle \tag{6.31}$$

which makes calculations much simpler. One can compute the correlator in the Minkowski vacuum state, change from inertial coordinates X^A to Rindler coordinates Y^A and then perform the Fourier transform. The Minkowski stress tensor for a free, massless scalar field has the form

$$T^a_b = \frac{\partial L}{\partial(\partial_a \varphi)} \partial_b \varphi - \delta^a_b L, \tag{6.32}$$

where $L = g^{AB}\partial_A\varphi\partial_B\varphi$. One can insert this in Eqn. (6.30) which is in terms of the retarded Green's function, but it is also possible to write the Kubo formula in terms of different types of Green's functions. Since the thermal Green's functions satisfy the relation [Son 2002]

$$G^1(\omega, \boldsymbol{p}) = -\coth\left(\frac{\omega}{2T}\right) ImG^R(\omega, \boldsymbol{p}), \tag{6.33}$$

where the G's represent any bosonic operator, one can also work with the symmetrized Schrodinger-Hadamard correlator of the stress tensor $G^1(\omega, \boldsymbol{p})$. So we have, for example

$$\eta = \frac{1}{2T_0} \lim_{\omega \to 0} \int_{\ell_c}^{\infty} d\xi' \int_{\ell_c}^{\infty} d\xi \int e^{i\omega\tau} dt \int d^2x \, \kappa^2 \xi \xi' G^1_{xy,xy}(\tau, x, y, \xi, \xi'). \tag{6.34}$$

Furthermore, in the hydrodynamic limit ($\omega, \boldsymbol{k} \ll \hbar^{-1} T_0$) the symmetrized correlator is not different from the Wightman correlator

$$G^+_{xy,xy} = \langle \hat{T}_{xy}(\tau, x, y, \xi, \xi') \hat{T}_{xy}(0, 0, 0, \xi') \rangle. \tag{6.35}$$

At the quantum level the difference between the correlators in frequency space is smaller than the correlators themselves by the factor ω/T, and the hydrodynamic limit here is exactly where $\omega \ll T$ [Kovtun 2003b].

In practice, we found it was easiest to work with the Wightman correlator. We first expand the scalar field operator into the usual set of normal mode solutions to the Klein-Gordon field equation

$$\hat{\varphi}(t, \boldsymbol{x}) = \int \frac{d^{d+1}p}{(2\pi)^{d+1}\sqrt{2\omega}} \left[a(\boldsymbol{p}) e^{i\boldsymbol{p}\cdot\boldsymbol{x} - i\omega t} + a^\dagger(\boldsymbol{p}) e^{-i\boldsymbol{p}\cdot\boldsymbol{x} + i\omega t} \right], \tag{6.36}$$

where $\omega = |\boldsymbol{p}|$ and $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$ are creation and annihilation operators. Inserting this into the Wightman function, we find

$$G^{+}{}_{xy,xy}(t, x, y, z, z') = \int \frac{d^3p d^3q d^3p' d^3q'}{4(2\pi)^{12}\sqrt{pp'qq'}} p_x q_y p'_x q'_y \langle 0 | \cdots | 0 \rangle \tag{6.37}$$

where the $\cdots$ represent sixteen terms involving combinations of four creation and annihilation operators and exponentials of the momenta. However, the only two terms that contribute are

$$\begin{aligned} &\langle 0 | a(\boldsymbol{p}) a^\dagger(\boldsymbol{q}) a(\boldsymbol{p'}) a^\dagger(\boldsymbol{q'}) | 0 \rangle e^{-i(P_a - Q_a)x^a} e^{-i(P'_a - Q'_a)x'^a} \\ &+ \langle 0 | a(\boldsymbol{p}) a(\boldsymbol{q}) a^\dagger(\boldsymbol{p'}) a^\dagger(\boldsymbol{q'}) | 0 \rangle e^{-i(P_a + Q_a)x^a} e^{i(P'_a + Q'_a)x'^a}, \end{aligned} \tag{6.38}$$

where $P_a = (|\boldsymbol{p}|, \boldsymbol{p})$ and $x'^a = (0, 0, 0, z')$. Using the commutation relation

$$[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p'})] = (2\pi)^{d+1} \delta^{d+1}(\boldsymbol{p} - \boldsymbol{p'}), \tag{6.39}$$

we find that

$$\langle 0 | a(\boldsymbol{p}) a^\dagger(\boldsymbol{q}) a(\boldsymbol{p'}) a^\dagger(\boldsymbol{q'}) | 0 \rangle = (2\pi)^6 \delta^3(\boldsymbol{p'} - \boldsymbol{q'}) \delta^3(\boldsymbol{p} - \boldsymbol{q}) \tag{6.40}$$

and

$$\langle 0|a(\boldsymbol{p})a(\boldsymbol{q})a^{\dagger}(\boldsymbol{p'})a^{\dagger}(\boldsymbol{q'})|0\rangle = (2\pi)^6 \left(\delta^3(\boldsymbol{p}-\boldsymbol{p'})\delta^3(\boldsymbol{q}-\boldsymbol{q'}) + \delta^3(\boldsymbol{q}-\boldsymbol{p'})\delta^3(\boldsymbol{p}-\boldsymbol{q'})\right). \tag{6.41}$$

Putting this together and integrating gives

$$G^{+}{}_{xy,xy}(t,x,y,z,z') = \int \frac{d^3p d^3q}{4(2\pi)^6}\frac{1}{pq}\left[(p_x^2 q_y^2 + p_x p_y q_x q_y)e^{-i(P^a+Q^a)(x_a - x'_a)} + p_x p_y q_x q_y\right]. \tag{6.42}$$

The last term, which comes from the first piece in (6.38) seems to give an infinite contribution in general. It is associated with the summation over the zero point modes and would be absent if we had followed the usual prescription of normal ordering the stress tensor operator so that its expectation value is set to zero. Instead, if we use the Fulling-Rindler subtraction (6.24) these Casimir type terms must be present. In the present case, however, the integration over this term over the momentum space gives zero identically. This is consistent with the perfect fluid form of expectation value of $\hat{T}_{\mu\nu}$, whose (xy) components are zero in the equilibrium rest frame.

In order to deal with the remaining term, note that the Wightman function for the scalar field operator is

$$G^{+}(t,x,y,z,z') = \langle 0|\varphi(t,x,y,z)\varphi(0,0,0,z')|0\rangle = \int \frac{d^3p}{2p(2\pi)^3}e^{-iP_a(x^a - x'^a)}. \tag{6.43}$$

Therefore the Wightman function of the stress tensor can be expressed in terms of derivatives of the scalar field Wightman function

$$\begin{aligned} G^{+}{}_{xy,xy}(t,x,y,z,z') &= (\partial_x^2 G^{+}(t,x,y,z,z'))(\partial_y^2 G^{+}(t,x,y,z,z')) \\ &+ (\partial_x\partial_y G^{+}(t,x,y,z,z'))(\partial_x\partial_y G^{+}(t,x,y,z,z')). \end{aligned} \tag{6.44}$$

The scalar Wightman function for a massless field has the form [Bogoliubov 1980]

$$\frac{-1}{4\pi^2}\frac{1}{\lambda + \epsilon(t)i\epsilon} \tag{6.45}$$

where $\lambda = -t^2 + x^2 + y^2 + (z-z')^2$ is the spacetime interval between the two points and $\epsilon(t)$ is sign function (+1 if $t > 0$, -1 if $t < 0$). The $i\epsilon$ prescription for dealing with the singularity here is interpreted as

$$\lim_{\epsilon\to 0}\frac{1}{\lambda \pm i\epsilon} = P/\lambda \mp i\pi\delta(\lambda), \tag{6.46}$$

where P represents the Cauchy principal value. Using (6.44) we find the Wightman function for the stress tensor is

$$G^{+}_{xy,xy}(\tau,x,y,\xi,\xi') = \frac{1}{16\pi^4}\left(\frac{128x^2y^2}{(\lambda+i\epsilon)^6} - \frac{16x^2}{(\lambda+i\epsilon)^5} - \frac{16y^2}{(\lambda+i\epsilon)^5} + \frac{4}{(\lambda+i\epsilon)^4}\right) \quad (6.47)$$

where we re-express the interval in Rindler coordinates: $\lambda = \xi^2 - 2\xi\xi'\cosh(\kappa\tau) + \xi'^2 + x^2 + y^2$.

We want to calculate the Fourier transform into Rindler frequency and momentum (taking the zero momentum limit)

$$\tilde{G}^{+}_{xy,xy}(\omega,\xi,\xi') = \int_{\ell_c}^{\infty} d\xi' \int_{\ell_c}^{\infty} d\xi \int_{-\infty}^{\infty} e^{i\omega\tau} d\tau \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\ \kappa^2\xi\xi'\ G^{+}_{xy,xy}(\tau,x,y,\xi,\xi'). \quad (6.48)$$

We first make a coordinate change to

$$x = \rho\cos(\theta) \quad (6.49)$$

$$y = \rho\sin(\theta) \quad (6.50)$$

so that the integrations over the x and y directions become

$$\int_0^{\infty} \rho d\rho \int_0^{2\pi} d\theta. \quad (6.51)$$

After integrating over the angular direction (6.48) becomes

$$\begin{aligned}\tilde{G}^{+}_{xy,xy}(\omega,\xi,\xi') = & \int_{\ell_c}^{\infty} d\xi' \int_{\ell_c}^{\infty} d\xi \int_{-\infty}^{\infty} e^{i\omega\tau} d\tau \int_0^{\infty} d\rho\kappa^2\xi\xi' \left(\frac{2\rho^5}{\pi^3(\rho^2+\alpha)^6}\right. \\ & \left. - \frac{2\rho^3}{\pi^3(\rho^2+\alpha)^5} + \frac{\rho}{2\pi^3(\rho^2+\alpha)^4}\right), \end{aligned} \quad (6.52)$$

where $\alpha = \xi^2 + \xi'^2 - 2\xi\xi'\cosh(\kappa\tau) + i\epsilon$. Since α is complex valued, the integration of ρ over the real axis is well-defined and yields

$$\tilde{G}^{+}_{xy,xy}(\omega,\xi,\xi') = \int_{\ell_c}^{\infty} d\xi' \int_{\ell_c}^{\infty} d\xi \int_{-\infty}^{\infty} e^{i\omega\tau} d\tau \frac{1}{30\pi^2}\ \frac{\kappa^2\xi\xi'}{(\xi^2+\xi'^2-2\xi\xi'\cosh(\kappa\tau))^3}. \quad (6.53)$$

This function has a periodicity in the τ coordinate due to the cosh function. We need a prescription for dealing with the poles, which are always on the real axis at

$$\tau_0 = \pm\kappa^{-1}\ln(\xi/\xi'). \quad (6.54)$$

The usual way for handling these types of integrations is to assume τ is a complex variable and that the contour for the integration in the complex τ plane should be rectangular. One horizontal piece is along the real axis (the part we want), the other in the opposite direction at $\tau = i2\pi/\kappa$ to keep the cosh function invariant. Because of this fact there are also poles at

$$\tau_0 = \pm\kappa^{-1}\ln(\xi/\xi') + 2\pi i/\kappa. \tag{6.55}$$

The vertical parts are at $\tau = i\infty$ and do not contribute. The result will be the sum of the residues enclosed in the contour,

$$I = 2\pi i(1 - e^{-2\pi\omega/\kappa})^{-1}\ \Sigma(res), \tag{6.56}$$

where I is the integral in (6.53).

There are multiple choices we can make for this contour depending on which poles we choose to enclose. However, it turns out we have to include an even number of the poles (two or all four) in order to preserve the symmetry of the integrand under the interchange of ξ and ξ'. In the case of the Wightman function, we have the explicit $i\epsilon$ prescription, which is to include both poles on the real axis in the contour (by pushing them up), while leaving out the ones at $2\pi i/\kappa$. Computing the residues, and taking the $\omega \to 0$ limit, we find

$$\tilde{G}^{+}_{xy,xy}(0,\xi,\xi') = \frac{\xi\xi'\kappa}{30\pi^3}\,\frac{-3(\xi^4 - \xi'^4) + 2\xi^4\ln(\xi/\xi') + 8\xi^2\xi'^2\ln(\xi/\xi') + 2\xi'^4\ln(\xi/\xi')}{(\xi^2 - \xi'^2)^5}. \tag{6.57}$$

Next, we must perform the radial integrations over ξ and ξ'. The first integration of (6.57) over ξ gives

$$\tilde{G}^{+}_{xy,xy}(0,\xi') = \frac{\kappa}{240\pi^2}\frac{\xi'^4 + 4\ell_c^2\xi'^2 - 5\ell_c^4 + 4\ell_c^4\ln(\ell_c/\xi') + 8\ell_c^2\xi'^2\ln(\ell_c/\xi')}{(\ell_c^2 - \xi'^2)^4}. \tag{6.58}$$

Integrating this expression over ξ' and multiplying by the overall $(2T_0)^{-1} = \pi/\kappa$ in (6.34), we ultimately arrive at

$$\eta = \frac{1}{1440\pi^2\ell_c^2}, \tag{6.59}$$

which, as expected, is divergent in the limit $\ell_c \to 0$ and scales in ℓ_c as a $2+1$ quantity.

The final task is to compare this result with the entanglement entropy density for the wedge. Comparing with our η in (6.59) with the entanglement entropy density calculated previously (6.28), we find

$$\eta/s = 1/4\pi. \tag{6.60}$$

The UV cutoff length cancels out and we are left with exactly the KSS ratio.

6.5 Summary and Discussion

In this chapter we have argued that the universal shear viscosity to entropy density ratio of $1/4\pi$ is also associated with a Rindler causal horizon in a flat (either globally or locally) spacetime. Its appearance in this case is mysterious since there is no gravity and the familiar formalism of AdS/CFT holography is completely absent. In order to provide a microscopic basis for this result, we have turned to the properties of quantum entanglement and vacuum fluctuations. Namely, when a quantum state is restricted to a sub-region of the spacetime (in this case Minkowski vacuum state in the Rindler wedge), quantum fluctuations of this state have a dual, thermal description associated with the horizon boundary. An effective description of the large-scale dynamics of this vacuum thermal state is always provided by hydrodynamics. To this end, we have developed a simple Kubo-like formula for the viscosity induced on the horizon in terms of a two point stress-energy tensor correlation function for the quantum fields in the Rindler wedge. We calculated this quantity in the simplest case of a free massless scalar field in a four dimensional spacetime and found the ratio of our η to the entanglement entropy s is exactly $1/4\pi$[1].

The results found suggest that the $1/4\pi$ ratio might be a fundamental property of quantum entanglement and its associated holography. It also provides support for the hypothesis that semi-classical gravity on macroscopic scales is induced or emergent as an effective theory of some lower dimensional, strongly coupled quantum system with a large number of degrees of freedom. In this picture, the $1/4\pi$ ratio is saturated in gauge theories with a Einstein gravity dual because 1) they have an area (BH) entropy and 2) as we mentioned at the end of section 7.2, in the large N limit the number of degrees of freedom diverges and gravity is turned off as the Newton constant goes to zero.

It would be useful to understand if our results can be extended to more general quantum field theories and to higher dimensional spacetimes. Since all fields in nature

[1]Note that, strictly speaking, we have not proven our result is independent of the regularization scheme used on η and s. However, we do not expect the choice of regularization to matter since both divergences arise in the same radial integration over the local energy-momentum density.

contribute in principle to the vacuum fluctuations, our hypothesis is that the η/s ratio is $1/4\pi$ universally for any matter field. Also, the arguments of section 7.2 can be extended to any dimension; since the BH entropy density is $(4G_N^D)^{-1}$ and $\eta = (16\pi G_N^D)^{-1}$ for a general spacetime dimension D, the ratio should not depend on the number of dimensions.

As a simple first check of a different field theory, we considered a massless, but now non-minimally coupled scalar field given by the action

$$I_s = \frac{1}{2}\int \sqrt{-g}(\nabla_a\varphi\nabla^a\varphi - \xi_C R\varphi^2). \tag{6.61}$$

In the flat spacetime limit, the stress tensor reduces to

$$T_{ab} = \partial_a\varphi\partial_b\varphi - \frac{1}{2}\eta_{ab}(\partial\varphi)^2 - 2\xi_c\partial_a(\varphi\partial_b\varphi) + 2\xi_c\eta_{ab}\nabla_c(\varphi\nabla^c\varphi). \tag{6.62}$$

Repeating the steps at the beginning of Section IV, we arrive at the following for the Wightman function of the stress tensor $G^+_{xy,xy}(\tau,x,y,\xi,\xi')$ in terms of the scalar field Wightman function $G(\tau,x,y,\xi,\xi)$

$$\begin{aligned} G^+_{xy,xy}(\tau,x,y,\xi,\xi') &= (1-2\xi_C)^2(\partial_x^2G^+)(\partial_y^2G^+) - 4\xi(1-2\xi)(\partial_xG^+)(\partial_x\partial_y^2G^+) \\ &\quad -4\xi(1-2\xi)(\partial_yG^+)(\partial_x^2\partial_yG^+) + 4\xi_C^2G^+(\partial_x^2\partial_y^2G^+) \\ &\quad +(1-4\xi_C+8\xi_C^2)(\partial_x\partial_yG^+)(\partial_x\partial_yG^+). \end{aligned} \tag{6.63}$$

Inserting in the form of the Wightman function (6.45), we can calculate the Fourier transform in (6.48). Integrating over x and y as before, we find (7.45) again. The dependence on the coupling to the scalar curvature ξ_C vanishes in the low momentum regime, and therefore η is not changed.

There are different results in the literature for the entropy density s of a non-minimally coupled scalar field. In [Demers 1995], the authors worked in the brick-wall approach, calculating the density of states for a thermal field outside the horizon. In this case, since the scalar curvature on the background spacetime is always zero, the ξ_C dependence drops out of the scalar field equation and the entropy density is unchanged. This is consistent with our calculation and, if we use this result, the KSS ratio is preserved. However, there is an important difficulty here that cannot be overlooked. The divergence in the entropy density found by [Demers 1995] cannot be absorbed into the renormalization of the Newton constant, which is ξ_C dependent. This is a problem since we have argued both the entanglement entropy and the viscosity are proportional to the (renormalized) Newton constant.

On the other hand, the entropy can also be calculated in an Euclidean functional integral approach from the one-loop effective action. In this case, one works off-shell and includes the contributions of manifolds where $\beta \neq 2\pi/\kappa$. When the solution is not the Hartle–Hawking instanton ($\beta = 2\pi/\kappa$), the manifolds have a conical singularity and therefore the scalar curvature coupling contributes a delta function term to the partition function. The resulting entropy is ξ_C dependent and can in fact be reabsorbed into the renormalized Newton constant [Solodukhin 1995; Larsen 1996]. However, there is an interpretational issue with this result. When $\xi_c > 1/6$ the statistical mechanical contribution to the entropy seems to be negative, while $S_{ent} = -Tr\hat{\rho}\ln\hat{\rho}$ must be positive definite [Hotta 1997].

The reason for this unusual behavior is rooted in the fact that the black hole entropy has in this case an additional non-statistical term proportional to the integral of φ^2 over the horizon

$$S_N = 2\pi\xi_c \int_H \varphi^2 \sqrt{-h} d^2x, \tag{6.64}$$

which can be thought of as a Noether charge correction term [Frolov 1997b]. Indeed, if one considers a non-minimal field coupled to gravity

$$I_{grav} = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G} + \frac{1}{2}(\nabla\varphi)^2 - \frac{\xi_C}{2}R\varphi^2\right) \tag{6.65}$$

the resulting theory is a scalar-tensor theory of gravity, whose classical Wald Noether charge entropy [Wald 1993] includes the correction (6.64). Note that this kind of correction is not limited to non-minimally coupled scalar fields. It also appears in generic vector field theories [Frolov 1998]. The black hole entropy is generally composed of three contributions: a statistical entanglement entropy, the non-statistical "bare" gravitational entropy and the Noether charge term. Induced gravity models remove the need for the bare gravitational entropy, but they currently cannot fully explain the existence of the Noether charge term from a statistical point of view [Fursaev 2005].

Hence our preliminary investigation of the viscosity to entropy density ratio in different field theories has lead us to a key issue. Namely, while the η/s ratio seems to remain $1/4\pi$ if we compare our entanglement viscosity only to the statistical entanglement entropy, in general the relevant quantities are the black hole (Wald) entropy and likely a corresponding general definition of viscosity. The problem is that the Wald entropy in a general diffeomorphism invariant theory of gravity does not just depend on the horizon area. This does not seem to fit with the induced gravity scenario implied by the thermodynamics of spacetime argument, reviewed in Chapter 3, where the horizon entropy is purely due to entanglement.

In this sense, the investigations [Eling 2006; Eling 2008; Chirco 2010b] seem to lend some insight towards a possible resolution. In particular, different formulations of the equivalence principle and their role in determining the characteristics of a gravitational theory may be the key point. The only theory of gravity known to be consistent with the *strong* equivalence principle is Einstein gravity. The strong equivalence principle implies that gravity is purely geometrical. Physics (gravity included) is the same in any locally flat region of spacetime, which means G_N is a universal constant and there are no extra gravitational fields. Under these conditions the UV cutoff ℓ_c should be a constant. However, in a general theory of gravity (such as scalar-tensor theories), the strong equivalence principle is not satisfied. Consequently, it is reasonable to assume the UV cutoff to be dependent on the spacetime location. In this case it is necessary to promote it to a spacetime field, which will have to be a dynamical one in order to assure the background independence of the resulting gravitational theory. This is exactly what is naturally suggested by the extensions of the spacetime thermodynamics approach beyond General Relativity [Eling 2006; Eling 2008; Chirco 2010b]. If this is true, it may be always possible to re-express the Wald entropy in the form of an entanglement entropy by suitably characterizing the spacetime dependence of ℓ_c[1].

[1]A related proposal can be found in [Brustein 2009a; Elizalde 2008], where the authors found that Wald entropy evaluated on static, spherically symmetric black hole solutions in generalized theories of gravity can be expressed as $A/4G_{\text{eff}}$, where G_{eff} is an effective gravitational coupling at the horizon.

Chapter 7

The Rindler Horizon Fluid: An Effective Duality

Beyond the connection between the classical Navier–Stokes equations and a classical geometry, some recent works have introduced a novel formalism to describe a holographic fluid theory defined on an arbitrary time-like surface in a general spacetime with a causal horizon [Bredberg 2011b]. On this surface, one fixes the boundary condition that the induced metric is flat, and in the spirit of the Wilsonian approach to the renormalization, the asymptotic physics outside this surface plays no role. Moving this surface between the horizon and the asymptotic boundary can be thought of as a renormalization group flow between a boundary fluid and a horizon fluid. In [Bredberg 2011a] the authors considered the specific case of perturbations about a Rindler metric, taking the time-like surface to be one of the family of hyperbolas associated with the worldlines of an accelerated observer. Working in the non-relativistic hydrodynamic expansion, the authors presented a geometry that is a solution to the Einstein equations if the data on surfaces of r_c satisfy the incompressible Navier–Stokes equations. Alternatively, one can consider the physically inequivalent near-horizon expansion in small r_c and obtain the same results.

These works actually suggested the possibility of an underlying holographic duality relating a theory on fixed r_c to the interior bulk of the Rindler spacetime. A first step toward a detailed study of the behavior of this dual system was taken in [Compere 2011], with the introduction of an algorithm for constructing the geometry and the explicit expression for the viscous transport coefficients to second order in the hydrodynamic expansion.

The independence of the fluid/Rindler holographic duality from the asymptotic geometry makes this correspondence interesting beyond the AdS/CFT context. Indeed, the Rindler metric is associated with an accelerated observer in the locally flat surroundings of any point in spacetime. Therefore, one can ask whether the flat spacetime duality can be applied locally and then possibly used to patch together a holographic description of any spacetime [Compere 2011; Chirco 2011a].

In this chapter, we will study the approach given in [Bredberg 2011a] and try to provide a further probe to the duality by asking what effect higher curvature terms in the dual gravitational theory have on the transport coefficients of the fluid dual to the Rindler geometry. In the AdS/CFT correspondence, such terms are associated with quantum corrections or other deformations, which modify the values of the transport coefficients. Remarkably, we show here that the shear viscosity of the Rindler fluid is not modified if higher curvature terms are introduced. Equivalently, at lowest orders in the non-relativistic expansion, the dual metric solution has the property of being a solution to GR and to any higher curvature theory of gravity. The first place the higher curvature corrections appear is in the second order transport coefficients of the fluid. Working in the case where the higher curvature theory is Einstein–Gauss–Bonnet gravity, we calculate some of these coefficients [Chirco 2011a]. We then conclude with a discussion of the implications of these results and their possible connection to approaches using the local Rindler geometry as a tool for a thermodynamical derivation of gravitational dynamics.

7.1 General Setup

We want to construct a Lorentzian geometry that acts as the holographic dual description of a fluid flow in $d+1$ dimensions. Based on the holographic principle, we expect the fluid is defined on a $d+1$ dimensional time-like surface S_c embedded in a $D = d+2$ dimensional bulk spacetime. We choose the time-like surface to be defined by fixed bulk radial coordinate, $r = r_c$. We also specialize to the case where the fluid moves on a flat background. In this case, the induced metric on S_c should be flat as well, e.g.

$$\gamma_{\mu\nu}dx^\mu dx^\nu = -\Phi(r_c)dt^2 + e^{2\Psi(r_c)}dx_i dx^i, \tag{7.1}$$

where Φ and Ψ are some functions of r. We use the notation that coordinates on the hypersurface S_c are $x^\mu = (t, x^i)$, where $i = 1...d$. The $d+2$ dimensional bulk coordinates are defined with the notation $x^a = (t, x^i, r)$. The final requirement is that the bulk spacetime must contain a regular, stationary causal horizon. The bulk spacetime therefore has a time-like Killing vector field, which becomes null on the horizon. The full bulk metric therefore has the general form [Bredberg 2011b],

$$ds^2 = -\Phi(r)dt^2 + 2dtdr + e^{2\Psi(r)}dx_i dx^i, . \tag{7.2}$$

where at some radius $r = r_h$ there is a horizon where $\Phi(r) = 0$ and the time-like Killing vector $\chi^a = (\partial_t)^a$ becomes null. If one considers quantum field theory on the background (7.2), one finds equilibrium thermal states associated with the presence of the horizon. For example, one can compute the Hawking temperature (in units where $\hbar = c = 1$)

$$T_H = \frac{\kappa}{2\pi} = \frac{\Phi'(r_h)}{4\pi}, \tag{7.3}$$

where the surface gravity κ can be defined via $\chi^b \nabla_b \chi^a = \kappa \chi^a$. Dividing by the redshift factor at r_c, $\sqrt{-g_{tt}} = \sqrt{\Phi(r_c)}$ yields the local Tolman temperature

$$T_{loc} = \frac{\Phi'(r_h)}{4\pi\sqrt{\Phi(r_c)}}. \tag{7.4}$$

There is also an associated Bekenstein–Hawking entropy proportional to the cross-sectional area of the horizon

$$S_{BH} = 4\pi e^{d\Psi(r_h)}, \tag{7.5}$$

where here and throughout we use units such that $16\pi G = 1$. We want to identify these thermodynamical properties with the thermodynamical properties of the dual fluid in $d+1$ dimensions. Therefore, the general metric can be thought of as the dual geometrical description of an equilibrium thermal state associated with some lower dimensional theory defined on the surface $r = r_c$.

The metric (7.2) can describe many different black hole solutions. Here we will focus on the special case of a region of flat $(d+2)$ dimensional Minkowski spacetime in "ingoing Rindler" coordinates

$$ds^2 = -rdt^2 + 2dtdr + dx_i dx^i, \tag{7.6}$$

where in terms of the above parametrization, $\Phi(r) = r$ and $\Psi(r) = 0$. The null surface $r = 0$ acts as a horizon to accelerated observers, whose worldlines correspond to surfaces of constant $r = r_c$.

Although the Rindler metric is just a patch of flat spacetime, a general horizon thermodynamics allows to define a local Unruh temperature, associated with surfaces of $r = r_c$

$$T = \frac{1}{4\pi\sqrt{r_c}}. \tag{7.7}$$

and to assign the Rindler horizon a Bekenstein–Hawking entropy based on the holographic principle, or, more concretely, via the thermal entanglement entropy of the quantum fields in Rindler wedge, i.e.

$$s = 4\pi. \tag{7.8}$$

Given the existence of an equilibrium Unruh temperature and a Bekenstein–Hawking entropy density, the metric (7.6) can be thought of as a dual geometrical description of a perfect fluid in one lower dimension. This duality can be formalized by considering the Brown–York stress energy tensor [Brown 1993], which in GR takes the form,

$$T^{BY}_{\mu\nu} = 2(K\gamma_{\mu\nu} - K_{\mu\nu}), \tag{7.9}$$

where $K_{\mu\nu} = \frac{1}{2}\mathcal{L}_N\gamma_{\mu\nu}$ and $\mathcal{L}_N$ is the Lie derivative along the normal to the slice N^A. One can show that $T^{BY}_{\mu\nu}$ (and its generalization for higher curvature gravity) is indeed equivalent to the stress energy tensor of the perfect fluid with a rest frame energy density ρ and pressure P. In this case

$$\rho = 0, \quad p = \frac{1}{\sqrt{r_c}}. \tag{7.10}$$

7.2 Equivalence of Viscous Hydrodynamics in Einstein and Higher Curvature Gravities

7.2.1 The Seed Metric

In this section we will argue that the first order viscous hydrodynamics of the fluid defined on S_c is independent of whether the dual gravitational theory is Einstein or some higher curvature generalization. In order to study the hydrodynamics of this fluid, we must perturb the background Rindler geometry. To start, we review the formalism for perturbing the Rindler metric developed in [Compere 2011]. The first step is to make a set of coordinate transformations to obtain a new metric (or class of metrics). These transformations should keep the induced metric at r_c flat. The transformed metric should also preserve a perfect fluid form of the stress energy tensor associated with the slice, as well as the time-like Killing vector and the homogeneity in the x^i direction. It was shown in [Compere 2011] that these set of conditions uniquely identify two diffeomorphisms, namely a boost and the translation.

The boost of the metric takes the form,

$$\sqrt{r_c}t \to \sqrt{r_c}t - \gamma\beta_i x^i, \quad x^i \to x^i - \gamma\beta^i\sqrt{r_c}t + (\gamma - 1)\frac{\beta_i\beta_j}{\beta^2}x^j, \tag{7.11}$$

where $\gamma = (1-\beta^2)^{-1/2}$ and $\beta_i = r_c^{-1/2}v_i$ is the boost parameter. The linear shift of the radial coordinate and re-scaling of t, which moves the horizon from $r = 0$ to an $r = r_h < r_c$, is instead

$$r \to r - r_h, \quad t \to (1 - r_h/r_c)^{-1/2}t. \tag{7.12}$$

The resulting metric for the flat spacetime is

$$\begin{aligned} ds^2 = &\frac{dt^2}{1 - v^2/r_c}\left(v^2 - \frac{r - r_h}{1 - r_h/r_c}\right) + \frac{2\gamma}{\sqrt{1 - r_h/r_c}}dtdr - \frac{2\gamma v_i}{r_c\sqrt{1 - r_h/r_c}}dx^i dr \\ &+ \frac{2v_i}{1 - v^2/r_c}\left(\frac{r - r_c}{r_c - r_h}\right)dx^i dt + \left(\delta_{ij} - \frac{v_i v_j}{r_c^2(1 - v^2/r_c)}\left(\frac{r - r_c}{1 - r_h/r_c}\right)\right)dx^i dx^j. \end{aligned} \tag{7.13}$$

We now want to investigate the hydrodynamic system dual to the above metric. To do that, we need to consider the dynamics of the metric perturbations within a hydrodynamic limit. One can perturb (7.13) by promoting the spatial velocity and horizon radius to be functions of space and time: $v^i(t, x^i)$ and $r_h(t, x^i)$. Now the metric is no longer flat and no longer a solution of the vacuum Einstein equation. However, one can introduce a particular non-relativistic hydrodynamical expansion [Fouxon 2008; Bhattacharyya 2009] in terms of a small parameter ϵ,

$$v^i \sim \epsilon v^i(\epsilon x^i, \epsilon^2 t) \quad P \sim \epsilon^2 P(\epsilon x^i, \epsilon^2 t), \tag{7.14}$$

where the *non-relativistic* pressure $P(t, x^i)$ is defined in the following way as a small perturbation of the horizon radius, [1]

$$r_h = 0 + 2P + O(\epsilon^4). \tag{7.15}$$

[1]Note that the ϵ expansion is performed in such a way that at zeroth order $v^i = r_h = 0$ so that the standard Rindler metric (7.6) is recovered. Also, there is no scaling of bulk radial derivatives.

Using (7.14) one scales down the amplitudes (ϵ can be thought of as the inverse of the speed of light), while at the same time scaling to large times t and spatial distances x^i. This corresponds to looking at small perturbations in the hydrodynamic limit.

Expanding the metric (7.13) out to $O(\epsilon^2)$ in this manner yields the "seed metric" solution originally found by Bredberg, Keeler, Lysov and Strominger in [Bredberg 2011a],

$$\begin{aligned} ds^2 &= -r dt^2 + 2 dt dr + dx_i dx^i \\ &\quad - 2\left(1 - \frac{r}{r_c}\right) v_i dx^i dt - \frac{2v_i}{r_c} dx^i dr \\ &\quad + \left(1 - \frac{r}{r_c}\right)\left[(v^2 + 2P)dt^2 + \frac{v_i v_j}{r_c} dx^i dx^j\right] + \left(\frac{v^2}{r_c} + \frac{2P}{r_c}\right) dt dr. \end{aligned} \tag{7.16}$$

The seed metric is the unique singularity-free solution to the vacuum Einstein equations up to $O(\epsilon^3)$, provided $\partial_i v^i = 0$. As required, the induced metric on the slice $r = r_c$ is flat.

In GR, the momentum constraint equations on the surface S_c can be expressed in terms of the Brown-York stress tensor

$$R_{\mu a} N^a = \partial^\nu T^{BY}_{\mu\nu} = 0. \tag{7.17}$$

At second and third order in ϵ, momentum constraint equations are

$$R^{(2,3)}_{\mu a} N^a = r_c^{-1/2} R^{(2,3)}_{t\mu} + r_c^{1/2} R^{(2,3)}_{r\mu} = 0, \tag{7.18}$$

while the Brown-York stress-tensor for the seed metric is given by [Bredberg 2011a]

$$T^{BY}_{\mu\nu} dx^\mu dx^\nu = \frac{d\vec{x}^2}{\sqrt{r_c}} - \frac{2v_i}{\sqrt{r_c}} dx^i dt + \frac{v^2}{\sqrt{r_c}} dt^2 + r_c^{-3/2}\Big[P\delta_{ij} + v_i v_j - 2r_c \partial_i v_j\Big] dx^i dx^j + O(\epsilon^3)\,. \tag{7.19}$$

Then, at second order, using the expression in (7.19), the momentum constraint equations (7.17) reduce to the incompressibility condition $\partial_i v^i = 0$ we discussed above. At third order one finds the Navier-Stokes equations with a particular kinematic viscosity

$$\partial_t v_i + v^j \partial_j v_i + \partial_i P - r_c \partial^2 v_i = 0. \tag{7.20}$$

Therefore, imposing the the incompressible Navier-Stokes equations on the fluid variables guarantees the dual metric is a solution to the field equations.

Noticeably, these results can be obtained as a non-relativistic expansion of a relativistic viscous fluid stress tensor. To see this, we work in the relativistic hydrodynamic expansion in derivatives of the fluid velocity and pressure: ∂u and ∂p. Then, at first order, the relativistic viscous fluid stress tensor has the form,

$$T^{\text{fluid}}_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} - 2\eta K_{\mu\nu} - \xi h_{\mu\nu}(\partial_\lambda u^\lambda). \tag{7.21}$$

Here $h_{\mu\nu} = \gamma_{\mu\nu} + u_\mu u_\nu$, while $K_{\mu\nu} = h^\lambda_\mu h^\sigma_\nu \partial_{(\lambda} u_{\sigma)}$ is the fluid shear, η the shear viscosity, and ξ the bulk viscosity.

The viscous terms above are written in the Landau or transverse frame [Landau 2000], which can be defined as a condition on the first order part of the stress tensor

$$T^{\text{fluid }(1)}_{\mu\sigma} u^\sigma = 0. \tag{7.22}$$

This frame is constructed so that the viscous fluid velocity is defined as the velocity of energy transport. The seed stress tensor in (7.19) follows from the ϵ expansion of (7.21), if we identify

$$u^\mu = \frac{1}{\sqrt{r_c - v^2}}(r_c, v^i), \quad \rho = 0 + O(\epsilon^3), \quad p = \frac{1}{\sqrt{r_c}} + \frac{P}{r_c^{3/2}}, \quad \eta = 1. \tag{7.23}$$

This is consistent with the earlier equilibrium calculation of ρ and p in (7.10). Note also that the bulk viscosity term in (7.21) actually drops out and bulk viscosity is not an independent transport coefficient. This is due to the fact that at viscous order we can impose the ideal order equation $\partial_\mu u^\mu = 0$, which follows from $\rho = 0$ and continuity.

7.2.2 Higher Curvature Gravity

Now we want to study how the hydrodynamics of the fluid is modified when the gravity theory is not GR, but instead some theory with higher curvature terms. The first question is whether we need a new, modified seed metric in a higher curvature theory of gravity. Interestingly, we can show that the seed metric (7.16) and its $O(\epsilon^3)$ correction is a solution to a wide class of higher curvature gravity theories at lowest orders in the ϵ expansion.

We start by noting that the flat, equilibrium Rindler metric at zeroth order is a vacuum solution to both Einstein and higher curvature gravity theories. The higher curvature terms could be thought of as modified gravity theories in their own right or they can be seen as quantum corrections to Einstein gravity in an effective field theory picture. Here we will not consider exotic theories involving inverse powers of curvature invariants.

As a first example of a higher curvature theory we consider Einstein-Gauss-Bonnet gravity (in the absence of a cosmological constant), defined by the action

$$I_{GB} = \int d^{d+2}x\sqrt{-g}\left[R + \alpha\left(R^2 - 4R_{cd}R^{cd} + R_{cdef}R^{cdef}\right)\right], \tag{7.24}$$

where α is the Gauss–Bonnet coupling constant. We consider $d \geq 3$ since for $d < 3$ the Gauss–Bonnet term is topological and does not affect the field equations. The interest in looking at a Gauss–Bonnet term is twofold. Such a term arises in the low energy limit of string theories. Secondly, Einstein–Gauss–Bonnet gravity is notable because even though the action is higher order in the curvature, for the unique combination of curvature invariants in the second term of (7.24), the field equations remain second order in derivatives of the metric.

Varying this action with respect to the metric yields the field equations,

$$G_{ab} + 2\alpha H_{ab} = 0, \tag{7.25}$$

where the Lovelock tensor H_{AB} is

$$\begin{aligned} H_{ab} = {} & RR_{ab} - 2R_{ac}R^c_b - 2R^{cd}R_{acbd} + R_a{}^{cde}R_{bcde} \\ & -\frac{1}{4}g_{ab}\left(R^2 - 4R_{cd}R^{cd} + R_{cdef}R^{cdef}\right). \end{aligned} \tag{7.26}$$

Now, using the seed metric, the first non-zero components of the Riemann tensor $R_{abc}{}^d$ are at $O(\epsilon^2)$. If we examine the Lovelock tensor, (7.26), it is clear that the first contributions from the Gauss-Bonnet terms can only appear at $O(\epsilon^4)$ at the lowest. A similar conclusion obviously holds for Lovelock gravities [Lovelock 1971], which are the extension of the action (7.24) including contributions with higher powers of the curvature but still yielding 2nd order field equations.

The field equations of other higher curvature theories of gravity generally involve covariant derivatives of the Riemann tensor and its contractions. These are no longer second order in metric derivatives. At second order in the curvature the gravitational action has the form

$$I = \int d^{d+2}x\sqrt{-g}\left(R + \beta_1 R^2 + \beta_2 R_{ab}R^{ab} + \beta_3 R_{abcd}R^{abcd}\right). \tag{7.27}$$

The field equations can be expressed in the form $G_{ab} = S^{eff}_{ab}$, where

$$S^{eff}_{ab} = \beta_1 \left(RR_{ab} - \nabla_a \nabla_b R + g_{ab}(\Box R - \frac{1}{2}R^2) \right) +$$
$$\beta_2 \left(g_{ab} R_{cd} R^{cd} + 4\nabla_c \nabla_b R^c_a - 2\Box R_{ab} - g_{ab}\Box R - 4R^c_a R_{cb} \right)$$
$$+ \beta_3 \left(g_{ab} R_{abcd} R^{abcd} - 4R_{acde} {R_b}^{cde} - 8\Box R_{ab} + 4\nabla_b \nabla_a R \right.$$
$$\left. + 8R^c_a R_{cb} - 8R^{cd} R_{acbd} \right). \tag{7.28}$$

Let's consider the possible terms that can appear at the lowest orders in ϵ. First, the second covariant derivative terms of R could in principle contribute β_i corrections at $O(\epsilon^2)$. However, the Ricci scalar $R = g^{ab} R_{ab}$ can be expanded out as follows,

$$R = g^{tt} R_{tt} + 2g^{rt} R_{tr} + 2g^{ti} R_{ti} + 2g^{ri} R_{ri} + g^{rr} R_{rr} + g^{ij} R_{ij}. \tag{7.29}$$

Before imposing incompressibility, one can show that the only non-zero component of R_{ab} at $O(\epsilon^2)$ is

$$R_{tt} = \tfrac{1}{2} \partial_i v^i. \tag{7.30}$$

However, for the background Rindler metric (7.6), the zeroth order ${g^{tt}}_{(0)}$ is zero, so the Ricci scalar R is in fact higher order. Since one cannot form a scalar constructed from v^i, P, ∂_t, and ∂_i with odd powers of ϵ, we expect R is of $O(\epsilon^4)$. For instance, the spatial vector R_{ti} is $O(\epsilon^3)$, but this multiplies g^{ti}, which is $O(\epsilon)$. Therefore, R is $O(\epsilon^4)$ and its covariant derivatives are of the same order or higher.

The remaining terms of interest are the $\Box R_{ab}$ and $\nabla_c \nabla_b R^c_a$ terms proportional to β_2 and β_3. We know that R_{ab} a priori has non-zero components at $O(\epsilon^2)$ and $O(\epsilon^3)$. The question is whether the radial derivatives and background connection for the Rindler metric (7.6) allow the above two terms to also contribute at these orders in ϵ thereby affecting the hydrodynamics at these orders. This we checked with an explicit calculation. The result is again negative.

Thus, as a general principle, higher curvature corrections to the Einstein equations come in at $O(\epsilon^4)$, at least when we perturb the fluid dual to the flat Rindler spacetime geometry. Terms of even higher order in the curvature (schematically $\sim R^n$, where $n > 2$) will typically appear at even higher orders. This includes the often studied case of $f(R)$ theories, when f can be expanded around the Hilbert term: $f = R + R^2 + R^3 + \cdots$.

As a result, the solution to the higher curvature theories at the lowest orders $O(\epsilon^2)$ and $O(\epsilon^3)$ is the same as the GR solution found previously [Bredberg 2011a;

Compere 2011]. Since all the higher curvature quantities vanish at the lowest orders, this solution has the property of being approximately *strongly universal* [Coley 2008] . The explicit solution at $O(\epsilon^3)$ can be constructed from the algorithm for Einstein gravity given in [Compere 2011], which we will expand upon and generalize to Einstein–Gauss–Bonnet in the next section. At the present, we note that the equivalence of the solutions to $O(\epsilon^4)$ implies that the 1st order viscous hydrodynamics of the dual fluid is the same both in Einstein gravity and its higher curvature generalizations. In particular, the incompressible Navier–Stokes equations (7.20) are the same in any theory, with the kinematic viscosity fixed to be r_c. Furthermore, as Compère, et. al. pointed out, it is clear that the non-relativistic ϵ expansion is capturing the non-relativistic limit of a relativistic fluid theory whose full structure is unknown. Nevertheless, the ϵ expansion seems to be able to capture some of the transport properties of this fluid theory. In particular, the shear viscosity of the relativistic fluid, η, is apparently fixed to be 1 (or $(16\pi G)^{-1}$ if we restore the gravitational constant).

One may worry about using the non-relativistic limit to draw conclusions about the properties of the relativistic parent fluid. However, we can show that our analysis of higher curvature terms can be extended to the relativistic hydrodynamics. The first step is write the metric (7.13) in a manifestly boost covariant form. This metric turns out to be

$$ds^2 = -(1 + p^2(r - r_c))u_\mu u_\nu dx^\mu dx^\nu - 2pu_\mu dx^\mu dr + h_{\mu\nu} dx^\mu dx^\nu. \tag{7.31}$$

In this line element we have replaced r_h with the relativistic pressure p using the general formula

$$p = \frac{1}{\sqrt{r_c - r_h}}. \tag{7.32}$$

Expanding u^a and p in terms of v^i and P using (7.23), the metric (7.31) reproduces the seed metric up to $O(\epsilon^2)$. In addition, if we compute the Brown–York stress tensor at $r = r_c$ for this metric (7.31), we find directly

$$T_{\mu\nu} dx^\mu dx^\nu = p h_{\mu\nu} dx^\mu dx^\nu, \tag{7.33}$$

which is the ideal part of (7.21) with $\rho = 0$.

To perturb in this case, we now treat $u^\mu(x^\mu)$ and $p(x^\mu)$, but leave r_c fixed. The metric is no longer a solution to the vacuum Einstein equations, but one can expand and work order by order in derivatives of u^μ and p as discussed earlier. This follows the standard approach used in the fluid-gravity correspondence [Bhattacharyya 2008a].

We now have that (7.31) is a zeroth order solution, i.e. $R_{ab} = 0 + O(\lambda)$, where the parameter λ counts derivatives of u^μ and p. Therefore, $R_{abc}{}^d \sim O(\lambda)$ and the curvature squared terms in (7.28) must appear at $O(\lambda^2)$. The other terms involve the covariant derivatives of the Ricci scalar and tensor. The generalization of (7.29) is

$$R^{(1)} = g^{rr} R^{(1)}_{rr} + 2g^{r\mu} R^{(1)}_{r\mu} + g^{\mu\nu} R^{(1)}_{\mu\nu}. \tag{7.34}$$

From (7.31) we find

$$\begin{aligned} R^{(1)}_{rr} &= 0 \\ R^{(1)}_{r\mu} &= 0 \\ R^{(1)}_{\mu\nu} &= \partial_{(\mu} p u_{\nu)} + Dp\, u_\mu u_\nu + \frac{1}{2} p(\partial_\lambda u^\lambda) u_\mu u_\nu + p u_{(\mu} a_{\nu)}, \end{aligned} \tag{7.35}$$

where we have defined $D = u^\mu \partial_\mu$ and $a_\mu = u^\lambda \partial_\lambda u_\mu$. Since $g^{\mu\nu} = h^{\mu\nu}$, which projects orthogonal to u^μ, $R^{(1)} = 0$. Finally, the fact that the remaining terms $\Box R_{ab}$ and $\nabla_c \nabla_b R^c_a$ are also of $O(\lambda^2)$ can be shown by explicit calculation as before.

Therefore, we conclude again that the higher curvature terms affect only the second order viscous hydrodynamics. The equilbrium stress tensor will be given by Eqn. (7.33) in any higher curvature theory of gravity. This follows just from the fact that the zeroth order metric (7.31) is a solution in any theory. Computing the $O(\lambda)$ corrections to (7.31) and (7.33) confirms that $\eta = 1$ and the bulk viscosity is not a transport coefficient, but we will save the details for another paper [Chirco].

In higher curvature theories, the entropy is given by the Wald formula [Wald 1993]. In general, Bekenstein–Hawking area entropy will be modified by the higher curvature terms, leading to an expression that can depend on both the intrinsic and extrinsic geometry of horizon. However, since we are working with a Rindler horizon in flat spacetime, all these corrections vanish and the equilibrium entropy density s remains 4π. The ratio $\eta/s = 1/4\pi$ was first derived in the context of the AdS/CFT correspondence [Policastro 2001]. It was shown that the ratio goes to this value for any infinitely strongly coupled holographic gauge theory fluid with an Einstein gravity dual [Buchel 2004]. On the gauge theory side, the number of colors $N \to \infty$ and the 't Hooft coupling $\lambda \to \infty$. This is essentially a classical limit; quantum corrections to the η/s ratio at finite N and λ, which can be calculated in specific string theory realizations [Myers 2009b], correspond to specific higher derivative corrections to the dual gravitational theory. Another approach is to work outside the context of particular string theories and consider a generic higher curvature gravity action of the form given in

(7.27). In this case, it has been shown [Brigante 2008; Kats 2009; Cai 2011] that ratio changes to

$$\frac{\eta}{s} = \frac{1}{4\pi}(1 - 8\beta_3). \tag{7.36}$$

This result holds in five spacetime dimensions and to linear order in the β_i, which are effectively suppressed by powers of the Planck length. It is also important to note that while the ratio is unchanged when $\beta_3 = 0$, both η and s do depend on $\beta_{1,2}$. Finally, in the special case of Einstein–Gauss–Bonnet, (7.24), $\beta_3 = \alpha$. Given the nice properties of this theory (linked to the field equations remaining 2nd order in derivatives), one can work non-perturbatively and consider finite α corrections which allow the ratio to approach zero.

It is then remarkable that in the case of a flat Rindler background there is no higher curvature correction to the ratio or to the viscosity itself. The viscosity is protected against quantum corrections or other deformations to the dual theory. At a technical level, the difference is that the result (7.36) follows by considering perturbations around a background asymptotically AdS black brane solution in the higher curvature gravity theory. In Einstein–Gauss–Bonnet gravity with negative cosmological constant, this solution is [Cai 2002]

$$ds^2 = N^2 f(r) - \frac{1}{f(r)} dr^2 + r^2 dx_i dx^i \tag{7.37}$$

where N is some constant and

$$f(r) = \frac{r^2}{4\alpha}\left(1 - \sqrt{1 - 8\alpha\left(1 - \frac{r_h^4}{r^4}\right)}\right), \tag{7.38}$$

with r_h the value of the horizon radius. In this solution, thermodynamic quantities such as the temperature and entropy density depend explicitly on α, which translates into the calculations of the entropy and shear viscosity.

In contrast, in the Rindler case the metric does not depend on α and the Unruh temperature and entanglement entropy are kinematical quantities in the sense that they are independent of the underlying gravitational theory. The shear viscosity seems to have the same behavior since it is also unaffected by the choice of gravitational dynamics. This is further evidence for the picture of $\eta/s = 1/4\pi$ as a kinematical property associated with entanglement in Rindler spacetime [Chirco 2010a], as discussed in the previous chapter.

Before moving on, we want to point out the interesting duality here between the relativistic λ expansion in derivatives in the holographic theory and an effective field theory expansion of the bulk gravitational theory. First note that the λ expansion is equivalent to an expansion in small dimensionless *Knudsen number*, which is defined as $Kn = \frac{\ell_{\rm mfp}}{L}$, where $\ell_{\rm mfp}$ is the mean free path associated with the microscopic system and L is the characteristic size of the perturbations to the system. Secondly, although the bulk gravity theory is non-renormalizable, it is still valid as an effective theory when the dimensionless ratio of the Planck length to the radius of curvature, $\frac{L_{\rm planck}}{R_{\rm curv}}$, is small. The effective action is given as an expansion in this ratio. At zeroth order there is some cosmological constant, at first order, the Hilbert term, and then the pieces higher order in curvature invariants.

In the duality, the scale of perturbations L in the system on $r = r_c$ is linked to the scale $R_{\rm curv}$ of perturbations to the flat bulk spacetime. As we have seen, a flat spacetime is dual to the fluid in equilibrium, Einstein gravity dual to the viscous hydrodynamics characterized by a shear viscosity, and second order transport coefficients linked to curvature squared terms. It is tempting to associate the universality of the shear viscosity with the universality of the Hilbert action at low energies and take $\ell_{\rm mfp} \sim L_{\rm planck}$, the scale at which gravity is strongly coupled. This line of reasoning also suggests it may be interesting to consider a seed metric constructed from the region of a de Sitter spacetime where there is also a causal "observer dependent" horizon and the associated thermodynamics. What effect does a non-zero cosmological constant have on the dual fluid?

7.3 Second Order Transport Coefficients

Now let's consider the hydrodynamic expansion at higher order in derivatives. Here we expect the gravitational dynamics to affect the hydrodynamics of the dual fluid. To second order, $O(\lambda^2)$, the general stress tensor for a relativistic fluid with zero energy density (hence incompressible) has the form [Compere 2011]

$$\begin{aligned} T^{\rm fluid}_{\mu\nu} &= \rho u_\mu u_\nu + p h_{\mu\nu} - 2\eta K_{\mu\nu} \\ &\quad + c_1 K_\mu^\lambda K_{\lambda\nu} + c_2 K^\lambda_{(\mu}\Omega_{|\lambda|\nu)} + c_3 \Omega_\mu{}^\lambda \Omega_{\lambda\nu} + c_4 P_\mu^\lambda P_\nu^\sigma D_\lambda D_\sigma \ln p \\ &\quad + c_5 \sigma_{\mu\nu} D \ln p + c_6 D^\perp_\mu \ln p \, D^\perp_\nu \ln p, \end{aligned} \tag{7.39}$$

where $D = u^\mu \partial_\mu$, $D^\perp_\mu = P^\nu_\mu \partial_\nu$, and $\Omega_{\mu\nu} = P^\lambda_\mu P^\sigma_\nu \partial_{[\lambda} u_{\sigma]}$. There are also viscous corrections to the energy density ρ at this order, which can be parameterized as

$$\rho = b_1 K_{\mu\nu} K^{\mu\nu} + b_2 \Omega_{\mu\nu}\Omega^{\mu\nu} + b_3 D \ln p \, D \ln p + b_4 D^2 \ln p + b_5 D^\perp_\mu \ln p \, D^{\perp\mu} \ln p. \quad (7.40)$$

The c_i, $i = 1..6$, and b_j, $j = 1..5$, are the possible new transport coefficients. When one expands these expressions in powers of ϵ, many of the second order transport coefficients appear at $O(\epsilon^4)$ in a general non-relativistic fluid stress tensor,

$$\begin{aligned} T^{\text{fluid }(4)}_{\mu\nu} dx^\mu dx^\nu = r_c^{-3/2}\Big[& v^2(v^2+P) - \eta r_c \sigma_{ij} v^i v^j + \frac{b_1 r_c^{3/2}}{2}\sigma_{ij}\sigma^{ij} + \frac{b_2 r_c^{3/2}}{2}\omega_{ij}\omega^{ij}\Big] dt^2 \\ & + r_c^{-5/2}\Big[v_i v_j (v^2+P) + 2\eta r_c v_{(i}\partial_{j)} P + c_4 r_c^{3/2}\partial_i \partial_j P + \frac{c_1}{4} r_c^{3/2} \sigma_{ik}\sigma^k{}_j \\ & + \frac{c_3}{4} r_c^{3/2}\omega_{ik}\omega^k{}_j - \frac{c_2}{4} r_c^{3/2}\sigma_{k(i}\omega_{j)}{}^k - 2\eta r_c^2 v_{(i}\partial^2 v_{j)} - \eta r_c v_{(i}\partial_{j)} v^2 \\ & - \frac{r_c}{2}\eta\sigma_{ij} v^2 \Big] dx^i dx^j. \end{aligned} \quad (7.41)$$

Here $\sigma_{ij} = 2\partial_{(i} v_{j)}$ and $\omega_{ij} = 2\partial_{[i} v_{j]}$. Only c_5, c_6 and b_3, b_4, and b_5 are absent at this order in the ϵ expansion.

We argued that $O(\epsilon^4)$ is the first to receive corrections from any higher curvature terms in the gravity theory. In the next section, we will solve for the fourth order (non-relativistic) metric in five dimensional Einstein-Gauss-Bonnet gravity. With this result in hand, we will use the corresponding Brown-York stress tensor to read-off various second order transport coefficients for the dual fluid.

7.3.1 Constructing the Einstein–Gauss–Bonnet Solution

We first outline the construction due to [Compere 2011], where one starts with the metric solution at $O(\epsilon^{n-1})$. In practice, the first n is 3 , i.e. one starts the process with the seed metric solution (7.16). We then want to add to the metric a new piece g^ν_{ab} that solves the field equations to $O(\epsilon^{n+1})$. Since radial derivatives carry no powers of ϵ, the addition of g^ν_{ab} produces a change in the bulk curvature tensors at the same order. This is effectively a perturbation around the zeroth order background Rindler metric (7.6). We work in the gauge where

$$g^\nu_{ra} = 0, \quad (7.42)$$

for all the contributions with $n \geq 3$. With this choice, we find that changes in the Einstein tensor $\delta G_{ab} = \delta R_{ab} - \frac{1}{2} g^{(0)}_{ab} \delta R$ have the form

$$
\begin{aligned}
\delta G^{\nu}_{rr} &= -\frac{1}{2}\partial_r^2 g^{\nu}_{ii}, \\
\delta G^{\nu}_{ij} &= -\frac{1}{2}\partial_r(r\partial_r g^{\nu}_{ij}) - \frac{1}{2}\delta_{ij}\left(\partial_r^2 g^{\nu}_{tt} - \partial_r(r\partial_r g^{\nu}_{ij})\right), \\
\delta G^{\nu}_{ti} &= -r\delta G^{\nu}_{ri} = -\frac{r}{2}\partial_r^2 g^{\nu}_{ti}, \\
\delta G^{\nu}_{tt} &= -r\delta G^{\nu}_{rt} = -\frac{r}{4}\left(2r\partial_r^2 g^{\nu}_{ii} + \partial_r g^{\nu}_{ii}\right).
\end{aligned}
\tag{7.43}
$$

We define $g^{\nu}_{ii} \equiv \delta^{ij} g^{\nu}_{ij}$ and $\delta G^{\nu}_{ii} \equiv \delta^{ij}\delta G^{\nu}_{ij}$. In contrast, there is no change to the Lovelock tensor (7.26) at the same order n since the curvature of the Rindler background is zero and any term in the variation would contain some factor of curvature at zero order.

We want to find the g^{ν}_{ab} that cancels out the $O(\epsilon^n)$ part of the field equations arising from the pre-existing solution at $O(\epsilon^{n-1})$. That is, we require

$$
\delta G^{\nu}_{ab} + \hat{G}^{\nu}_{ab} + 2\alpha \hat{H}^{\nu}_{ab} = 0 \tag{7.44}
$$

where the hat denotes the parts of the curvature arising from the pre-existing solution. In order for this set of equations to be consistent, one must impose the integrability conditions

$$
\hat{E}^{\nu}_{tt} + r\hat{E}^{\nu}_{tr} = 0 \tag{7.45}
$$

$$
\hat{E}^{\nu}_{ti} + r\hat{E}^{\nu}_{ri} = 0 \tag{7.46}
$$

$$
\partial_r(\hat{E}^{\nu}_{tr} + r\hat{E}^{\nu}_{rr}) + (1/2)\hat{E}^{\nu}_{rr} = 0 \tag{7.47}
$$

where we have defined $\hat{E}^{\nu}_{ab} = \hat{G}^{\nu}_{ab} + 2\alpha\hat{H}^{\nu}_{ab}$. These are consistent with the Bianchi identity and (7.45) follows from the conservation of the Brown-York stress tensor extended to Gauss-Bonnet gravity [Davis 2003], i.e.

$$
(G_{a\nu} + 2\alpha H_{a\nu})N^a = \partial^{\mu} T_{\mu\nu} = 0, \tag{7.48}
$$

where

$$
T_{\mu\nu} = 2(K\gamma_{\mu\nu} - K_{\mu\nu}) + 4\alpha(J\gamma_{\mu\nu} - 3J_{\mu\nu} - 2\hat{P}_{\mu\rho\nu\sigma}K^{\rho\sigma}). \tag{7.49}
$$

The symbol $\hat{P}_{\mu\rho\nu\sigma} = \hat{R}_{\mu\rho\nu\sigma} + 2\hat{R}_{\rho[\nu}\gamma_{\sigma]\mu} - 2\hat{R}_{\mu[\nu}\gamma_{\sigma]\rho} + \hat{R}\gamma_{\mu[\nu}\gamma_{\sigma]\rho}$ is the divergence free part of the induced Riemann tensor and can be neglected here because we work with a flat induced metric, while

$$J_{\mu\nu} = \frac{1}{3}(2KK_{\mu\sigma}K^{\sigma}_{\nu} + K_{\sigma\lambda}K^{\sigma\lambda}K_{\mu\nu} - 2K_{\mu\sigma}K^{\sigma\lambda}K_{\lambda\nu} - K^2K_{\mu\nu}). \tag{7.50}$$

Using (7.43), one can solve the differential equations subject to two conditions: (i) that $g^{\nu}_{ab} = 0$ at $r = r_c$ (the metric on S_c remains flat) and (ii) that there is no singularity at $r = 0$. The resulting solution is

$$g^{\nu}_{tt} = (1 - r/r_c)F^{\nu}_t(t, x^i) + \int_r^{r_c} dr' \int_{r'}^{r_c} dr'' \frac{2}{3}\left(\hat{E}^{\nu}_{ii} - 4\hat{E}^{\nu}_{tr} - 2r\hat{E}^{\nu}_{rr}\right) \tag{7.51}$$

$$g^{\nu}_{ti} = (1 - r/r_c)F^{\nu}_i(t, x^i) - 2\int_r^{r_c} dr' \int_{r'}^{r_c} dr'' \hat{E}^{\nu}_{ti} \tag{7.52}$$

$$g^{\nu}_{ij} = -2\int_r^{r_c} dr' \frac{1}{r}\int_0^{r'} dr'' \left(\hat{R}^{\nu}_{ij} + 2\alpha(\hat{H}^{\nu}_{ij} - \frac{1}{3}\hat{H}^{\nu}_{kk})\right) \tag{7.53}$$

where $F^{\nu}_t(t, x^i)$ and $F^{\nu}_i(t, x^i)$ are arbitrary functions.

These two remaining functions can be fixed by imposing gauge choices on the Brown-York stress tensor of the fluid (7.49). The addition of the new metric piece at $O(\epsilon^n)$ has the following effect on the extrinsic curvature at the same order

$$\delta K^{\nu}_{\mu\nu} = \tfrac{1}{2}\sqrt{r_c}\partial_r g^{\nu}_{\mu\nu}\big|_{S_c} \tag{7.54}$$

implying that

$$\begin{aligned}
\delta K^{\nu}_{tt} &= -\frac{F^{\nu}_t(t, x^i)}{2\sqrt{r_c}}, \qquad \delta K^{\nu}_{ti} = -\frac{F^{\nu}_i(t, x^i)}{2\sqrt{r_c}}, \\
\delta K^{\nu}_{ij} &= +\frac{1}{\sqrt{r_c}}\int_0^{r_c} dr' \left(\hat{R}^{\nu}_{ij} + 2\alpha(\hat{H}^{\nu}_{ij} - \frac{1}{3}\delta_{ij}\hat{H}^{\nu}_{kk})\right).
\end{aligned} \tag{7.55}$$

By explicit calculation, we verified that there is no corresponding $O(\epsilon^n)$ variation of the $J_{\mu\nu}$ part of the stress tensor. Thus, the variation $\delta T^{\nu}_{\mu\nu}$ comes only from the linear part in the extrinsic curvature:

$$\begin{aligned}
\delta T^{\nu}_{tt} &= -\sqrt{r_c}\int_0^{r_c} 2R^{\nu}_{ii}, \qquad \delta T^{\nu}_{ti} = \frac{F^{\nu}_i(t, x^i)}{\sqrt{c}} \\
\delta T^{\nu}_{ij} &= \delta_{ij}\left(\frac{F^{\nu}_t(t, x^i)}{r_c^{3/2}} + \frac{2}{\sqrt{r_c}}\int_0^{r_c} dr'(R^{\nu}_{kk} + \frac{2\alpha}{3}\hat{H}^{\nu}_{kk})\right) - \frac{2}{\sqrt{r_c}}\int_0^{r_c} dr'(\hat{R}^{\nu}_{ij} + 2\alpha\hat{H}^{\nu}_{ij}).
\end{aligned}$$

The complete stress-tensor has the form

$$T^{\nu}_{\mu\nu} = \delta T^{\nu}_{\mu\nu} + 2\left(\hat{K}^{\nu}\gamma_{\mu\nu} - \hat{K}^{\nu}_{\mu\nu}\right) + 4\alpha\left(\hat{J}^{\nu}\gamma_{\mu\nu} - 3\hat{J}^{\nu}_{\mu\nu}\right), \tag{7.57}$$

where as before, the hat notation indicates the part of the stress-tensor originating from the solution at $O(\epsilon^{n-1})$. The function $F^{\nu}_{i}(t, x^i)$ is fixed by imposing the Landau gauge condition (7.22) order by order in the non-relativistic expansion. This plays a role only at odd orders in ϵ. The other function $F^{\nu}_{t}(t, x^i)$, which appears at even orders, is fixed by requiring that there are no higher order corrections to the definition of the non-relativistic pressure, i.e. the isotropic part of T_{ij} is

$$T^{iso}_{ij} = \left(\frac{1}{\sqrt{r_c}} + \frac{P}{r_c^{3/2}}\right)\delta_{ij} \tag{7.58}$$

at all orders.

7.3.2 Solution to $O(\epsilon^5)$

We now apply the algorithm to solve for the metric to $O(\epsilon^5)$. One first starts with the seed metric solution (7.16) and constructs the solution at $O(\epsilon^3)$. As we argued earlier, the corrections due to the Gauss-Bonnet coupling constant arise at $O(\epsilon^4)$. Therefore, the Gauss-Bonnet terms do not contribute and the solution reduces to the GR one found previously in [Compere 2011], where the only non-vanishing component is

$$g^{(3)}_{ti} = \frac{r - r_c}{2r_c}\left[\left(v^2 + 2P\right)\frac{2v_i}{r_c} + 4\partial_i P - (r + r_c)\partial^2 v_i\right]. \tag{7.59}$$

The next step is to compute the $\hat{R}^{(4)}_{AB}$ and $\hat{H}^{(4)}_{AB}$ using this metric. Via direct calculation of the Lovelock tensor (7.26), we find that

$$H^{(4)}_{ij} = \frac{3}{4r_c^2}\left(\omega_{ik}\omega^{k}{}_{j} + \tfrac{1}{2}\delta_{ij}\omega_{kl}\omega^{kl}\right) \tag{7.60}$$

with all other components of $H^{(4)}_{ab}$ equal to zero. At even order in ϵ, $R_{ti} = 0$ and as a result $g^{(4)}_{ti} = 0$. The remaining components to compute are $R^{(4)}_{tt}$, $R^{(4)}_{rr}$, $R^{(4)}_{rt}$, and $R^{(4)}_{ij}$, which we will not display explicitly here.

Using (7.60), the solution for $g^{(4)}_{tt}$ in (7.51) reduces to

$$g^{(4)}_{tt} = (1 - r/r_c)F^{(4)}_{t}(t, x^i) + \int_r^{r_c} dr' \int_{r'}^{r_c} dr'' \left(\hat{R}^{\nu}_{ii} + \frac{4}{3}\alpha\hat{H}^{(4)}_{ii} - 2\hat{R}^{(4)}_{rt} - r\hat{R}^{(4)}_{rr}\right) \tag{7.61}$$

and we find that

$$g_{tt}^{(4)} = (1 - r/r_c)F_t^{(4)}(t, x^i) + \frac{(r - r_c)^2}{8r_c}\left(8v_k\partial^2 v^k - \sigma_{kl}\sigma^{kl}\right) + \frac{(r - r_c)^2(r - r_c + 2\alpha)}{8r_c}\omega_{kl}\omega^{kl}. \tag{7.62}$$

The gauge condition on the stress tensor (7.58) fixes

$$\begin{aligned} F_t^{(4)}(t, x^i) &= \frac{9}{8r_c}v^4 + \frac{5}{2r_c}Pv^2 + \frac{P^2}{r_c} - 2r_c v_i\partial^2 v^i - \left(\frac{r_c + \alpha}{2}\right)\sigma_{kl}\sigma^{kl} - \frac{\alpha}{2}\omega_{kl}\omega^{kl} \\ &- 2\partial_t P + 2v^k\partial_k P. \end{aligned} \tag{7.63}$$

Note that in these expressions we have imposed incompressibility $\partial_i v^i = 0$, used the Navier-Stokes equation (7.20) to eliminate time derivatives of v_i, and imposed

$$\partial^2 P = -\partial_i v_j \partial^j v^i, \tag{7.64}$$

which also follows from (the divergence of) Navier-Stokes. Meanwhile Eqn. (7.53), yields

$$\begin{aligned} g_{ij}^{(4)} =& (1 - \frac{r}{r_c})\left[\frac{1}{r_c^2}v_i v_j(v^2 + 2P) + \frac{2}{r_c}v_{(i}\partial_{j)}P - 4\partial_i\partial_j P - \frac{1}{2}\sigma_{ik}\sigma^k{}_j + \frac{r - 5r_c + 12\alpha}{4r_c}\omega_{ik}\omega^k{}_j \right.\\ &+ \sigma_{k(i}\omega_{j)}{}^k - \frac{r + r_c}{r_c}v_{(i}\partial^2 v_{j)} + \frac{r + 5r_c}{4}\partial^2\sigma_{ij} \\ &\left. - \frac{1}{r_c}v_{(i}\partial_{j)}v^2 - \frac{1}{2r_c}\sigma_{ij}(v^2 + 2P) + \frac{\alpha}{r_c}\delta_{ij}\omega_{kl}\omega^{kl}\right]. \end{aligned} \tag{7.65}$$

We now use (7.57) and (7.56) to find the stress tensor components $T_{tt}^{(4)}$ and $T_{ij}^{(4)}$. The non-zero components of the $J_{\mu\nu}^{(4)}$ tensor are

$$J_{tt}^{(4)} = -\frac{1}{24r_c^{1/2}}\sigma_{ij}\sigma^{ij}, \quad J_{ij}^{(4)} = \frac{1}{12r_c^{3/2}}\sigma_{ik}\sigma^k{}_j \tag{7.66}$$

Using this result, we find

$$T_{tt}^{(4)} = r_c^{-3/2}\left[v^2(v^2 + P) - \frac{r_c^2}{2}\sigma_{ij}\sigma^{ij} - r_c\sigma_{ij}v^i v^j\right] \tag{7.67}$$

and

$$T^{(4)}_{ij} = r_c^{-5/2}\Bigg[v_i v_j(v^2+P) + 2r_c v_{(i}\partial_{j)}P - 4r_c^2\partial_i\partial_j P - \frac{r_c^2}{2}\left(1+\frac{2\alpha}{r_c}\right)\sigma_{ik}\sigma^k{}_j$$
$$+ -r_c^2\left(1+\frac{3\alpha}{r_c}\right)\omega_{ik}\omega^k{}_j r_c^2\sigma_{k(i}\omega_{j)}{}^k - 2r_c^2 v_{(i}\partial^2 v_{j)} + \frac{3r_c^3}{2}\partial^2\sigma_{ij}$$
$$+ -r_c v_{(i}\partial_{j)}v^2 - \frac{r_c}{2}\sigma_{ij}v^2\Bigg]. \qquad (7.68)$$

Note that the $T^{(4)}_{tt}$ has no α corrections. They cancel out and the energy density $T_{\mu\nu}u^\mu u^\nu$ is not affected by α at fourth order. Comparing with the general form of the fluid stress tensor (7.41) we read off that

$$b_1 = -\sqrt{r_c}\,,\ b_2 = 0\,,\ c_1 = -2\sqrt{r_c}\left(1+\frac{2\alpha}{r_c}\right),\ c_3 = -4\sqrt{r_c}\left(1+\frac{3\alpha}{r_c}\right),\ c_2 = c_4 = -4\sqrt{r_c} \qquad (7.69)$$

as expected, there is no change in the value of $\eta = 1$. However, the Gauss-Bonnet term does modify the two transport coefficients c_1 and c_3 from their purely GR values.

7.4 Discussion

We have argued that higher curvature corrections to the Einstein equations always come in at $O(\epsilon^4)$ in the non-relativistic hydrodynamic expansion and at $O(\lambda^2)$ in the relativistic Knudsen number expansion, at least when we perturb the fluid dual to the flat Rindler spacetime geometry. Hence, the solution to the higher curvature theories at the lowest orders is the same as the GR solution found previously [Bredberg 2011a; Compere 2011]. Working in the specific case where the higher curvature theory is Einstein–Gauss–Bonnet gravity, we then showed explicitly that the 1st order viscous hydrodynamics of the dual fluid is the same both in Einstein gravity and its higher curvature generalization, while the effect of the higher curvature corrections shows up in the second order transport coefficients of the fluid. We calculated some of these transport coefficients and found that two of them depend on the Gauss–Bonnet coupling constant. It would be interesting to complete the relativistic calculation outlined in section **??** in order to find all the second order transport coefficients in both the Einstein and Einstein–Gauss–Bonnet examples.

The approximate strong universality [Coley 2008] of the seed solution about which the hydrodynamic expansion is made is an interesting result. The lack of a higher curvature correction to the viscosity implies that it is protected against quantum corrections or other deformations to the dual theory. One way of thinking about these results is to note that shear viscosity and entropy density typically scale like T^d, where T is the equilibrium temperature of the thermal system. In AdS/CFT, this temperature is given by the Hawking temperature T_H of the black brane solution, which would depend in this case on the Gauss–Bonnet coupling constant, due to the non-trivial curvature of the background solution. In contrast, the shear viscosity and entropy density are constants independent of the temperature in the Rindler case. This suggests the two types of holographic duality are different.

Given the independence of the fluid/Rindler holographic duality from the asymptotic geometry, one is naturally lead to consider the case of the Rindler metric is associated with an accelerated observer in the locally flat surroundings of any point in spacetime. In this sense, one can ask whether the flat spacetime duality can be applied locally and then possibly used to patch together a holographic description of any spacetime [Compere 2011].

In the previous chapter, we showed how the idea that gravity may emerge from the holographic hydrodynamics of some microscopic, quantum system can be effectively developed in a local Rindler system [Jacobson 1995; Eling 2006; Eling 2008; Chirco 2010b]. We saw how non equilibrium thermodynamical derivation, associated with the hydrodynamics of the local horizon, yields the GR Einstein equation and fixes the shear viscosity to entropy density ratio to be $1/4\pi$. Also, we proved that the bulk viscosity does not appear as an independent transport coefficient [Eling 2008; Chirco 2011b], which is strikingly similar to the viscous hydrodynamics of the global Rindler fluid.

On the other hand, extensions of this type of derivation to $f(R)$ gravities [Eling 2006; Chirco 2010b; Chirco 2011b] require the horizon entropy to depend on the curvature, which inevitably leads to the same behavior in the shear viscosity. Ultimately, while the metric around any point is flat, the curvature itself does not vanish at the point. This fact means that one cannot simply import results pertaining to perturbations of the globally flat Rindler solution into the locally flat patch. However, there is some evidence that the inherent fuzziness in the local Killing vector, which is associated with the local notion of thermal equilibrium, may be of the same order of magnitude as higher curvature corrections [Eling 2006]. If this is the case, the approximate notion of a local fluid would not be affected by these corrections. It would be interesting to investigate further the relationship between the fluid/Rindler correspondence and these ideas of emergent gravitational dynamics.

Chapter 8

Conclusions

In this thesis we considered the possibility to reproduce a scenario where gravity emerges from the holographic hydrodynamics of some microscopic, quantum system by starting from the "synthetic" setting of an accelerating observer. Effectively, we took the local Rindler horizon as the basic structure to study the nature of gravitational dynamics.

Starting from the notion of local causal horizon we reviewed the arguments leading to a localization of black hole thermodynamics toward the construction of a general horizon thermodynamics, where the entropy of the local Rindler horizon is holographically identified with the finite area entanglement entropy of the local Minkowski vacuum state. Perturbations to the horizon system are then described in terms of the hydrodynamics of a local stretched horizon membrane and assumed to obey an entropy balance law, relating a change in the entropy to the "heat" associated with a flux of matter, plus an internal entropy production term from shear viscosity. Gravitational dynamics is therefore derived from the horizon hydrodynamics, as the equivalent of an equation of state.

We then concentrated on the possibility to provide a physical interpretation for the relation between gravity and the fluid description of the local stretched horizon dynamics in our local setting. We started from the assumption that the Minkowski vacuum, a thermal state once localized in the Rindler wedge, obeys the holographic principle. The area-scaling behavior of the wedge fields entanglement entropy was then interpreted as an evidence of the fact that the degrees of freedom in the vacuum thermal state are encoded into the $2+1$ stretched horizon boundary of the wedge. In this sense, the hydrodynamical perturbations of the vacuum are expected to be manifested in the dynamics of the stretched horizon fluid.

The natural development of this line of thought consisted in identifying an effective dimensionally reduced local quantum field theory leaving on the $2+1$ stretched surface, hoping to find a consistent vocabulary to realize a local Rindler holographic duality. We then argued about the possibility to think of this duality as a local realization of a fluid/gravity correspondence. Indeed, very interestingly, such a framework could provide a consistent picture of gravity as an effective theory which emerges from the holographic hydrodynamics of some microscopic, quantum system. In this picture, we then argue that the effective, thermodynamical/hydrodynamical level of description would naturally arise in a co-dimension one holographic screen, while gravity would be extracted in the "bulk" via an entropy balance law gluing the fluid dynamics at the different slices, in reminiscence of a renormalization group flow in the fluid.

Along this view, the holographic principle, which we described as a consequence of gravity, then becomes a premise for gravitational dynamics. As a conclusive argument, in this sense, we want to reconsider the holographic derivation of gravity as an entropic force, starting again from the spacetime neighborhood of a point.

8.1 Horizons, Holographic Screens and Entropic Force

The holographic characterization of the local, observer dependent Rindler horizon is very close to the concept of general holographic screen which plays a central role in the interpretation of gravity as an entropic force given in [Verlinde 2011].

Here, starting from a static background with a global time like Killing vector ξ^a, holographic screens are defined by surfaces at constant redshift $e^{\varphi} = \sqrt{-\xi^a \xi_a}$. The redshift perpendicular to the screen is then understood microscopically as originating from an entropy gradient [1]. It is useful here to reproduce part of the argument given in [Verlinde 2011]. One can start by considering the force that acts on a particle of mass m. In a general relativistic setting, one can give an invariant meaning to the concept of force by using the time-like Killing vector [Wald 1992]. The four velocity u^a of the particle and its acceleration $a^b \equiv u^a \nabla_a u^b$ can be expressed in terms of the Killing vector ξ^b as

$$u^b = e^{-\varphi} \xi^b, \quad a^b = e^{-2\varphi} \xi^a \nabla_a \xi^b \tag{8.1}$$

Then, by making use of the Killing equation $\nabla_a \xi_b + \nabla_b \xi_a = 0$ and the definition of φ, one can express the acceleration in the simple form

$$a^b = -\nabla^b \varphi \tag{8.2}$$

[1] Note the notation in which c and k_B are put equal to one, while G and $\hbar$ are kept explicit.

Therefore, like in the non relativistic situation, the acceleration is perpendicular to a holographic screen $\mathcal{S}$. In particular, one can turn it into a scalar quantity by contracting it with a unit outward pointing vector n^b normal to the screen $\mathcal{S}$ and to ξ^b.

The local temperature T on the screen is defined by

$$T = \frac{\hbar}{2\pi} e^{\varphi} n^b \nabla_b \varphi, \tag{8.3}$$

as measured with respect to the reference point at infinity. To find the force one assumes that the change of entropy at the screen is 2π for a displacement by one Compton wavelength normal to the screen. Hence,

$$\nabla_a S = -2\pi \frac{m}{\hbar} n_a \tag{8.4}$$

where the minus sign comes from the fact that the entropy increases when we cross from the outside to the inside. The entropic force now follows from (8.3)

$$F_a = T \nabla_a S = -m e^{\varphi} \nabla_a \varphi. \tag{8.5}$$

This is indeed the correct gravitational force that is required to keep a particle at fixed position near the screen, as measured from the reference point at infinity. It is the relativistic analogue of Newton's law of inertia $F = ma$. The additional factor e^{φ} is due to the redshift. Note that $\hbar$ has again dropped out.

The force equation (8.5) can be rewritten in a microcanonical form. Let $S(E; x^a)$ be the total entropy associated with a system with total energy E that contains a particle with mass m at position x^a. Here E also includes the energy of the particle.

From a statistical point of view, a general system with many degrees of freedom shows the tendency to sample the microscopic states in an unbiased way, that is to increase its entropy. As a consequence, configurations corresponding to a larger number of micro states than others are favored. Then, a change in the phase space volume, due to some excluded volume effect, corresponds to a gradient in the entropy of the system which yields a macroscopic dynamics that prefers configurations that are denser in terms of micro-states. This configurational interaction is called *entropic force*. Entropic interactions are not mediated by forces acting at micro scales. Entropic forces exist even in systems that at a microscopic level posses no energies other than the kinetic one. Interestingly, yet, at a macroscopic level such a system is subject to configurational accelerations.

Now, an entropic force can be determined micro-canonically by adding by hand an external force term, and impose that the entropy is extremal. For this situation this condition looks like

$$\frac{d}{dx^a} S(E + e^{\varphi} m, x^a) = 0 \tag{8.6}$$

One easily verifies that this leads to the same equation (8.5) and fixes the equilibrium point the one at which the external force, parametrized by $\varphi(x)$ and the entropic force statistically balance each other.

Equation (8.6) tells us that the entropy remains constant if we move the particle and simultaneously reduce its energy by the redshift factor. This also means that redshift function $\varphi(x)$ is entirely fixed by the other matter in the system.

The fundamental idea in [Verlinde 2011] is that Eq. (8.6) can effectively be obtained by starting from the microscopics and defining the space dependent concepts in terms of them. In this line of thought, the redshift must be seen as a consequence of the entropy gradient and not the other way around. The equivalence principle tells us that redshifts can be interpreted in the emergent spacetime as either due to a gravitational field or due to the fact that one considers an accelerated frame. Both views are equivalent in the relativistic setting, but neither view is microscopic. Acceleration and gravity are both emergent phenomena.

8.2 Future Perspectives

Our choice to study the nature of gravitational dynamics within a local Rindler horizon setting has proven to be conceptually very convenient. With such a basic geometro-dynamical structure, one can account for the fundamental physics features at the root of the holographic conjecture and the fluid/gravity duality. Moreover, for its local nature, this system provides an ideal tool for the study of gravity as an emergent phenomenon. The local Rindler horizon system, in fact, plays the role of a spacetime microscope.

In particular, the most remarkable advantage of going local actually consists in the possibility to deal with non-equilibrium effects, which, at the global level, would be killed by the symmetries of the specific gravitational solution considered.

Now, if one supports the possibility to derive gravity as an effective theory from statistical thermodynamics, then the most exiting perspective is associated with find a consistent conceptual extension to the non-equilibrium sector of such an approach. Indeed, being effectively characterized by the specific properties of the microscopic level of description, the non-equilibrium setting would necessarily provide fundamental insights on the nature of the microscopic degrees of freedom of spacetime.

In our semiclassical approximation of the microscopic fundamental system, non-equilibrium is expressly associated with the random fluctuations of the vacuum fields. Fluctuations are related to dissipation, which on macroscopic scale give raise to entropy production and account for the dissipative character of the horizons. Dissipation, at the microscopic level, turns out to be naturally associated with the non local propagation of the tensorial gravitational degrees of freedom, at the macroscopic level.

In the same line, in the AdS/CFT correspondence, the non-equilibrium transport coefficients actually characterize the nature of the dual lower dimensional field theory. Also, in the fluid/gravity duality Einsteins equations in the presence of a regular event horizon are conceptually thought of as the strong coupling analogue of the Boltzmann transport equations.

In the original entropic gravity approach this non-equilibium framework is apparently not taken into account. This is due to the fact that the derivation is conceived as a direct offspring of the holographic principle. In fact, the covariant entropy bound conjecture, at the root of the holographic principle, is based on a time reversal invariant argument [Bousso 1999a]. Hence, its origin cannot be thermodynamic, but must be statistical. In this sense, the gravitational entropic force is described as a reversible force, arising from the changes in the phase space of the microscopic theory degrees of freedom, via a purely statistical argument.

Now, in general, the reversibility of the entropic force at the macroscopic level depends on the properties of the heat bath. In particular, reversibility, and the consequent conservative character of the force, is again effectively associated with a microscopic thermal equilibrium condition, or equivalently, to the presence of an infinite thermal heat bath for the microscopic system. For the case of gravity the speed of light determines the size of the heat bath, since its energy content is given by $E = Mc^2$. So in the non relativistic limit the heat bath is infinite. Indeed, Newton's laws are perfectly conservative. When one includes relativistic effects, the heat bath is no longer infinite. Here one should expect some irreversibility [Verlinde 2011].

From a classical statistical mechanical point of view, the irreversible nature of thermodynamics is associated with the presence of "instability" at the microscopic level [Nicolis 1977]. In our work, we provided some evidence for the fact that, within a thermodynamical derivation of gravity, the equilibrium setting cannot account for several fundamental features of gravitational dynamics. In this sense, gravity actually seems to require for some "irreversibility" or "instability" at the microscopic dynamic level.

Intriguingly, a deep understanding of irreversibility, in a thermodynamical emergent scenario, would then shed some light on the nature of causal horizons in gravity.

This thesis work can be considered as an attempt to put together the viable hints in this direction, coming from different approaches, into a general, though oversimplified, setting. Much needs to be done. However, the recent advances here described seem to suggest that, at last, some light can be shed on the fundamental nature of space-time and gravity. It is up to us now to take over the challenge.

References

[Adler 1982] S. L. Adler. *Einstein gravity as a symmetry-breaking effect in quantum field theory.* Rev. Mod. Phys., vol. 54, no. 3, pages 729–766, Jul 1982. 6

[Aharony 2000] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz. *Large N field theories, string theory and gravity.* Phys. Rept., vol. 323, pages 183–386, 2000. 7, 97

[Akbar 2007] M. Akbar and Rong-Gen Cai. *Thermodynamic Behavior of Field Equations for f(R) Gravity.* Phys. Lett., vol. B648, pages 243–248, 2007. 74

[Armendariz-Picon 2000] C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt. *A dynamical solution to the problem of a small cosmological constant and late-time cosmic acceleration.* Phys. Rev. Lett., vol. 85, pages 4438–4441, 2000. 92

[Bañados 1994] M. Bañados, C. Teitelboim and J. Zanelli. *Black hole entropy and the dimensional continuation of the Gauss-Bonnet theorem.* Phys. Rev. Lett., vol. 72, no. 7, pages 957–960, Feb 1994. 2

[Balasubramanian 1999a] V. Balasubramanian and P. Kraus. *A stress tensor for anti-de Sitter gravity.* Commun. Math. Phys., vol. 208, pages 413–428, 1999. 46

[Balasubramanian 1999b] V. Balasubramanian and P. Kraus. *Space-time and the holographic renormalization group.* Phys.Rev.Lett., vol. 83, pages 3605–3608, 1999. 42

[Banerjee 2009] N. Banerjee and S. Dutta. *Higher Derivative Corrections to Shear Viscosity from Graviton's Effective Coupling.* JHEP, vol. 03, page 116, 2009. 97, 104

[Barceló 2005] S.-Visser M. Barceló C. Liberati. *Analogue Gravity*. Living Reviews in Relativity, vol. 8, no. 12, 2005. 6

[Bardeen 1973] J. M. Bardeen, B. Carter and S. W. Hawking. *The Four laws of black hole mechanics*. Commun. Math. Phys., vol. 31, pages 161–170, 1973. 2, 25, 49

[Bekenstein 1973] J. D. Bekenstein. *Black holes and entropy*. Phys. Rev., vol. D7, pages 2333–2346, 1973. 2, 25, 26

[Bekenstein 1975] J. D. Bekenstein. *Statistical Black Hole Thermodynamics*. Phys. Rev., vol. D12, pages 3077–3085, 1975. 2

[Bekenstein 1981] J. D. Bekenstein. *Energy Cost of Information Transfer*. Phys. Rev. Lett., vol. 46, pages 623–626, 1981. 37

[Bhattacharyya 2008a] S. Bhattacharyya, V. E Hubeny, S. Minwalla and M. Rangamani. *Nonlinear Fluid Dynamics from Gravity*. JHEP, vol. 02, page 045, 2008. 7, 43, 45, 59, 97, 125

[Bhattacharyya 2008b] S. Bhattacharyya, S. Lahiri, R. Loganayagam and S. Minwalla. *Large rotating AdS black holes from fluid mechanics*. JHEP, vol. 09, page 054, 2008. 45

[Bhattacharyya 2009] S. Bhattacharyya, S. Minwalla and S. R. Wadia. *The Incompressible Non-Relativistic Navier-Stokes Equation from Gravity*. JHEP, vol. 08, page 059, 2009. 43, 120

[Birrell 1982] N.D. Birrell and P.C.W. Davies. Quantum fields in curved space. Cambridge University Press, 1982. 35

[Bisognano 1975] J. J Bisognano and E. H. Wichmann. *On the Duality Condition for a Hermitian Scalar Field*. J. Math. Phys., vol. 16, pages 985–1007, 1975. 23

[Bisognano 1976] J. J Bisognano and E. H. Wichmann. *On the Duality Condition for Quantum Fields*. J. Math. Phys., vol. 17, pages 303–321, 1976. 23

[Bogoliubov 1980] N. N. Bogoliubov and D. V. Shirkov. Introduction to the theory of the quantized field. John Wiley and Sons, 1980. 109

[Bombelli 1986] L. Bombelli, R. K. Koul, J. Lee and R. D. Sorkin. *A Quantum Source of Entropy for Black Holes*. Phys. Rev., vol. D34, pages 373–383, 1986. 1, 27, 28, 52

[Bondi 1957] H. Bondi. *Negative Mass in General Relativity.* Rev. Mod. Phys., vol. 29, no. 3, pages 423–428, Jul 1957. 13

[Bousso 1999a] R. Bousso. *A Covariant Entropy Conjecture.* JHEP, vol. 07, page 004, 1999. 140

[Bousso 1999b] R. Bousso. *Holography in general space-times.* JHEP, vol. 06, page 028, 1999. 43

[Bousso 2002] R. Bousso. *The holographic principle.* Rev. Mod. Phys., vol. 74, pages 825–874, 2002. 36, 38, 39, 40, 41

[Bousso 2004] R. Bousso and J. Polchinski. *The string theory landscape.* Sci. Am., vol. 291, pages 60–69, 2004. 6

[Bredberg 2011a] I. Bredberg, C. Keeler, V. Lysov and A. Strominger. *From Navier-Stokes To Einstein.* 2011. 10, 116, 117, 121, 124, 134

[Bredberg 2011b] I. Bredberg, C. Keeler, V. Lysov and A. Strominger. *Wilsonian Approach to Fluid/Gravity Duality.* JHEP, vol. 03, page 141, 2011. 10, 116, 118

[Brigante 2008] M. Brigante, H. Liu, R. C. Myers, S. Shenker and S. Yaida. *Viscosity Bound Violation in Higher Derivative Gravity.* Phys. Rev., vol. D77, page 126006, 2008. 127

[Brown 1993] J. D. Brown and J. W. York Jr. *Quasilocal energy and conserved charges derived from the gravitational action.* Phys. Rev., vol. D47, pages 1407–1419, 1993. 119

[Brustein 2004] R. Brustein and A. Yarom. *Thermodynamics and area in Minkowski space: Heat capacity of entanglement.* Phys. Rev., vol. D69, page 064013, 2004. 106

[Brustein 2005] R. Brustein and A. Yarom. *Holographic dimensional reduction from entanglement in Minkowski space.* JHEP, vol. 01, page 046, 2005. 106

[Brustein 2006] R. Brustein, M. B. Einhorn and A. Yarom. *Entanglement interpretation of black hole entropy in string theory.* JHEP, vol. 01, page 098, 2006. 102

[Brustein 2009a] R. Brustein, D. Gorbonos and M. Hadad. *Wald's entropy is equal to a quarter of the horizon area in units of the effective gravitational coupling*. Phys. Rev., vol. D79, page 044025, 2009. 115

[Brustein 2009b] R. Brustein and A. J. M. Medved. *The ratio of shear viscosity to entropy density in generalized theories of gravity*. Phys. Rev., vol. D79, page 021901, 2009. 97, 104

[Buchel 2004] A. Buchel and J. T. Liu. *Universality of the shear viscosity in supergravity*. Phys. Rev. Lett., vol. 93, page 090602, 2004. 126

[Cai 2002] Rong-Gen Cai. *Gauss-Bonnet black holes in AdS spaces*. Phys. Rev., vol. D65, page 084014, 2002. 127

[Cai 2005] Rong-Gen Cai and S. P. Kim. *First law of thermodynamics and Friedmann equations of Friedmann-Robertson-Walker universe*. JHEP, vol. 02, page 050, 2005. 3

[Cai 2008] Rong-Gen Cai, Li-Ming Cao, Ya-Peng Hu and Sang Pyo Kim. *Generalized Vaidya spacetime in Lovelock gravity and thermodynamics on the apparent horizon*. Phys. Rev. D, vol. 78, no. 12, page 124012, Dec 2008. 3

[Cai 2010] Rong-Gen Cai and N. Ohta. *Horizon thermodynamics and gravitational field equations in Hořava-Lifshitz gravity*. Phys. Rev. D, vol. 81, no. 8, page 084061, 2010. 3

[Cai 2011] Rong-Gen Cai, L. Li and Yun-Long Zhang. *Non-Relativistic Fluid Dual to Asymptotically AdS Gravity at Finite Cutoff Surface*. 2011. 127

[Candelas 1977] P. Candelas and D. W. Sciama. *Irreversible Thermodynamics of Black Holes*. Phys. Rev. Lett., vol. 38, pages 1372–1375, 1977. 10, 29, 98

[Carlip 2001] S. Carlip. *Quantum gravity: A progress report*. Rep. Prog. Phys., vol. 64, pages 885–942, 2001. 6

[Carroll 1994] S. M. Carroll and G. B. Field. *Consequences of propagating torsion in connection dynamic theories of gravity*. Phys. Rev., vol. D50, pages 3867–3873, 1994. 95

[Carter 1973] B. Carter. Black holes. Gordon and Breach, New York, 1973. 2, 24, 25, 49

[Chandrasekhar 1983] S. Chandrasekhar. The mathematical theory of black holes. Clarendon Press, Oxford, 1983. 70, 90, 101

[Chirco] G. Chirco, Eling C. and S. Liberati. *Work in progress.* 126

[Chirco 2010a] G. Chirco, C. Eling and S. Liberati. *The universal viscosity to entropy density ratio from entanglement.* Phys. Rev., vol. D82, page 024010, 2010. 10, 127

[Chirco 2010b] G. Chirco and S. Liberati. *Non-equilibrium Thermodynamics of Spacetime: the Role of Gravitational Dissipation.* Phys. Rev., vol. D81, page 024016, 2010. 5, 47, 62, 77, 80, 86, 90, 97, 115, 135

[Chirco 2011a] Eling C.-Liberati S. Chirco G. *Higher Curvature Gravity and the Holographic fluid dual to flat spacetime.* JHEP, vol. 1108, page 009, 2011. 117

[Chirco 2011b] G. Chirco, C. Eling and S. Liberati. *Reversible and Irreversible Spacetime Thermodynamics for General Brans-Dicke Theories.* Phys. Rev., vol. D83, page 024032, 2011. 5, 75, 77, 81, 91, 135

[Christodoulou 1971] D. Christodoulou and R. Ruffini. *Reversible transformations of a charged black hole.* Phys. Rev., vol. D4, pages 3552–3555, 1971. 2

[Coley 2008] A. A. Coley, G. W. Gibbons, S. Hervik and C. N. Pope. *Metrics With Vanishing Quantum Corrections.* Class. Quant. Grav., vol. 25, page 145017, 2008. 125, 134

[Compere 2011] G. Compere, P. McFadden, K. Skenderis and M. Taylor. *The holographic fluid dual to vacuum Einstein gravity.* 2011. 10, 116, 117, 119, 120, 125, 128, 129, 132, 134, 135

[D. Jou 2001] G. Lebon D. Jou J. Casas-Vázquez. Extended irreversible thermodynamics. Springer-Verlag, Berlin Heidelberg, 3rd édition, 2001. 67

[Damour 1979] T. Damour. *Quelques propriétés mécaniques, électromagnétiques, thermodynamiques et quantiques des trous noirs.* 1979. Thèse de Doctorat d'Etat, Université Pierre et Marie Curie, Paris VI. 4, 60, 65

[Das 2000] Sumit R. Das and Samir D. Mathur. *The Quantum Physics of Black Holes: Results from String Theory.* Annual Review of Nuclear and Particle Science, vol. 50, no. 1, pages 153–206, 2000. 2

[Davis 2003] S. C. Davis. *Generalised Israel junction conditions for a Gauss-Bonnet brane world*. Phys. Rev., vol. D67, page 024030, 2003. 130

[de Groot 1962] S. R. de Groot and P. Mazur. Non-equilibrium thermodynamics. North-Holland, 1962. 65, 91, 103

[Demers 1995] Jean-Guy Demers, R. Lafrance and R. C. Myers. *Black hole entropy without brick walls*. Phys. Rev., vol. D52, pages 2245–2253, 1995. 113

[Deruelle 2004] N. Deruelle, J. Katz and S. Ogushi. *Conserved charges in Einstein Gauss-Bonnet theory*. Class. Quant. Grav., vol. 21, page 1971, 2004. 32

[Dowker 1994] J. S. Dowker. *Remarks on geometric entropy*. Class. Quant. Grav., vol. 11, pages L55–L60, 1994. 105, 106

[Einstein 1907] A. Einstein. *Über das Relativitätsprinzip und die aus demselben gezogene Folgerungen*. Jahrb. Radioakt, vol. 4, page 411, 1907. 1, 14

[Eling 2006] C. Eling, R. Guedens and T. Jacobson. *Non-equilibrium Thermodynamics of Spacetime*. Phys. Rev. Lett., vol. 96, page 121301, 2006. 5, 47, 54, 59, 64, 65, 71, 72, 73, 74, 75, 76, 77, 87, 94, 97, 115, 135

[Eling 2008] C. Eling. *Hydrodynamics of spacetime and vacuum viscosity*. JHEP, vol. 11, page 048, 2008. 5, 47, 52, 65, 67, 97, 115, 135

[Eling 2009] C. Eling, I. Fouxon and Y. Oz. *The Incompressible Navier-Stokes Equations From Membrane Dynamics*. Phys. Lett., vol. B680, pages 496–499, 2009. 96, 97, 99

[Eling 2010] C. Eling and Y. Oz. *Relativistic CFT Hydrodynamics from the Membrane Paradigm*. JHEP, vol. 02, page 069, 2010. 96, 97, 99, 101

[Elizalde 2008] E. Elizalde and P. J. Silva. *F(R) gravity equation of state*. Phys. Rev., vol. D78, page 061501, 2008. 74, 115

[Emparan 2006] R. Emparan. *Black hole entropy as entanglement entropy: A holographic derivation*. JHEP, vol. 06, page 012, 2006. 102

[Faraoni 2004] V. Faraoni. *Cosmology in scalar tensor gravity*. 2004. Dordrecht, Netherlands: Kluwer Academic. 93

[Faraoni 2010] V. Faraoni. *Black hole entropy in scalar-tensor and f(R) gravity: an overview*. 2010. 95

[Forster 1995] D. Forster. Hydrodynamic fluctuations, broken symmetry, and correlation functions. Westview Press, 1995. 5, 96

[Fouxon 2008] I. Fouxon and Y. Oz. *Conformal Field Theory as Microscopic Dynamics of Incompressible Euler and Navier-Stokes Equations*. Phys. Rev. Lett., vol. 101, page 261602, 2008. 120

[Freidel 2005] L. Freidel. *Group field theory: An Overview*. Int.J.Theor.Phys., vol. 44, pages 1769–1783, 2005. 6

[Friedan 2003] D. Friedan. *A tentative theory of large distance physics*. J. High Energy Phys., vol. 2003, no. 10, 2003. 6

[Frolov 1996] V. P. Frolov, D. V. Fursaev and A. I. Zelnikov. *Black Hole Entropy: Off-Shell vs On-Shell*. PHYS.REV.D, vol. 54, page 2711, 1996. 31

[Frolov 1997a] V. P. Frolov and D. V. Fursaev. *Mechanism of the generation of black hole entropy in Sakharov's induced gravity*. Phys. Rev. D, vol. 56, no. 4, pages 2212–2225, Aug 1997. 2

[Frolov 1997b] V. P. Frolov, D. V. Fursaev and A. I. Zelnikov. *Statistical origin of black hole entropy in induced gravity*. Nucl. Phys. B, vol. 486, pages 339–352, 1997. 2, 114

[Frolov 1998] V. P. Frolov and D. Fursaev. *Black hole entropy in induced gravity: Reduction to 2D quantum field theory on the horizon*. Phys. Rev., vol. D58, page 124009, 1998. 114

[Fujita 2008] M. Fujita. *Non-equilibrium thermodynamics near the horizon and holography*. JHEP, vol. 10, page 031, 2008. 97

[Fulling 1973] S. A. Fulling. *Nonuniqueness of canonical field quantization in Riemannian space-time*. Phys. Rev., vol. D7, pages 2850–2862, 1973. 19

[Fursaev 2004] D. V Fursaev. *Can One Understand Black Hole Entropy without Knowing Much about Quantum Gravity?* page 38, 2004. arxiv/gr-qc/0404038. 6

[Fursaev 2005] D. V. Fursaev. *Can one understand black hole entropy without knowing much about quantum gravity?* Phys. Part. Nucl., vol. 36, pages 81–99, 2005. 114

[Gerard 2007] Jean-Marc Gerard. *The strong equivalence principle from gravitational gauge structure*. Class. Quant. Grav., vol. 24, pages 1867–1878, 2007. 55

[Gerlach 1976] U. H. Gerlach. *Why is a black hole hot?* Phys. Rev. D, vol. 14, no. 12, pages 3290–3293, Dec 1976. 2

[Gibbons 1977] G. W. Gibbons and S. W. Hawking. *Cosmological Event Horizons, Thermodynamics, and Particle Creation*. Phys. Rev., vol. D15, pages 2738–2751, 1977. 16, 29

[Gubser 1998] S. S. Gubser, I. R. Klebanov and A. M. Polyakov. *Gauge theory correlators from non-critical string theory*. Phys. Lett., vol. B428, pages 105–114, 1998. 7, 40

[Haag 1977] R. Haag and E. Trych-Pohlmeyer. *Stability properties of equilibrium states*. Communications in Mathematical Physics, vol. 56, pages 213–224, 1977. 51

[Haag 1992] R. Haag. Local quantum physics. Springer-Verlag, Berlin, 1992. 19, 51

[Hartle 1976] J. B. Hartle and S. W. Hawking. *Path Integral Derivation of Black Hole Radiance*. Phys. Rev., vol. D13, pages 2188–2203, 1976. 90

[Hawking 1971] S. W. Hawking. *Gravitational radiation from colliding black holes*. Phys. Rev. Lett., vol. 26, pages 1344–1346, 1971. 2, 24

[Hawking 1972] S. W. Hawking and J. B. Hartle. *Energy and angular momentum flow into a black hole*. Commun. Math. Phys., vol. 27, pages 283–290, 1972. 10, 70, 101

[Hawking 1975a] S. W. Hawking. *Particle Creation by Black Holes*. Commun. Math. Phys., vol. 43, pages 199–220, 1975. 1, 2, 25

[Hawking 1975b] Stephen W. Hawking and G. F. R. Ellis. The large scale structure of space-time (cambridge monographs on mathematical physics). Cambridge University Press, 1975. 41

[Hawking 1976] S. W. Hawking. *Black Holes and Thermodynamics*. Phys. Rev. D, vol. 13, pages 191–197, 1976. 2

[Hawking 2001] S. Hawking, J. M. Maldacena and A. Strominger. *DeSitter entropy, quantum entanglement and AdS/CFT*. JHEP, vol. 05, page 001, 2001. 102

[Hehl 1976] F. W. Hehl, P. Von Der Heyde, G. D. Kerlick and J. M. Nester. *General Relativity with Spin and Torsion: Foundations and Prospects*. Rev. Mod. Phys., vol. 48, pages 393–416, 1976. 95

[Hotta 1997] M. Hotta, T. Kato and K. Nagata. *A comment on geometric entropy and conical space*. Class. Quant. Grav., vol. 14, pages 1917–1925, 1997. 114

[Hubeny 2011] V. E. Hubeny, S. Minwalla and M. Rangamani. *The fluid/gravity correspondence*. 2011. arXiv/hep-th/1107.5780. 7, 43

[Hwang 2010] Dong-il Hwang and Dong-han Yeom. *Responses of the Brans-Dicke field due to gravitational collapses*. Class. Quant. Grav., vol. 27, page 205002, 2010. 93

[Iqbal 2009] N. Iqbal and H. Liu. *Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm*. Phys. Rev. D, vol. 79, page 025023, 2009. 7, 97, 103

[Israel 1967] W. Israel. *Event Horizons in Static Vacuum Space-Times*. Phys. Rev., vol. 164, no. 5, pages 1776–1779, Dec 1967. 2, 24

[Israel 1986] W. Israel. *Third Law of Black-Hole Dynamics: A Formulation and Proof*. Phys. Rev. Lett., vol. 57, no. 4, pages 397–399, Jul 1986. 25

[Jacobson 1994a] T. Jacobson. *A Note on Hartle-Hawking vacua*. Phys.Rev., vol. D50, pages 6031–6032, 1994. 23

[Jacobson 1994b] T. Jacobson. *Black hole entropy and induced gravity*. 1994. arXiv/gr-qc/9404039. 28, 35

[Jacobson 1994c] T. Jacobson, G. Kang and R. C. Myers. *On black hole entropy*. Phys. Rev. D, vol. 49, no. 12, pages 6587–6598, Jun 1994. 2

[Jacobson 1995] T. Jacobson. *Thermodynamics of space-time: The Einstein equation of state*. Phys. Rev. Lett., vol. 75, pages 1260–1263, 1995. iii, 4, 9, 47, 54, 62, 65, 71, 76, 135

[Jacobson 2003] T. Jacobson and R. Parentani. *Horizon Entropy*. Found. Phys., vol. 33, pages 323–348, 2003. 3, 35

[Jacobson 2005] T. Jacobson. Introductory Lectures on Black Hole Thermodynamics. 2005. www.physics.umd.edu/grt/taj/776b/lectures.pdf. 19, 29

[Jamil 2009] M. Jamil and M. Akbar. *Wormhole Thermodynamics at Apparent Horizons*. 2009. 3

[Janik 2006] R. A. Janik and R. B. Peschanski. *Asymptotic perfect fluid dynamics as a consequence of AdS/CFT*. Phys. Rev., vol. D73, page 045013, 2006. 45

[Jeon 1995] S. Jeon. *Hydrodynamic transport coefficients in relativistic scalar field theory*. Phys. Rev., vol. D52, pages 3591–3642, 1995. 97

[Kang 1996] G. Kang. *On black hole area in Brans-Dicke theory*. Phys. Rev., vol. D54, pages 7483–7489, 1996. 92

[Kapusta 2006] J. I. Kapusta and C. Gale. Finite-temperature field theory: Principles and applications. 2006. 102

[Kats 2009] Y. Kats and P. Petrov. *Effect of curvature squared corrections in AdS on the viscosity of the dual gauge theory*. JHEP, vol. 01, page 044, 2009. 127

[Koivisto 2006] T. Koivisto. *Covariant conservation of energy momentum in modified gravities*. Class. Quant. Grav., vol. 23, pages 4289–4296, 2006. 84

[Kothawala] D. Kothawala, S. Sarkar and T. Padmanabhan. 3

[Kothawala 2009] D. Kothawala and T. Padmanabhan. *Thermodynamic structure of Lanczos-Lovelock field equations from near-horizon symmetries*. Phys. Rev. D, vol. 79, no. 10, page 104020, 2009. 3

[Kovtun 2003a] P. Kovtun, D. T. Son and A. O. Starinets. *Holography and hydrodynamics: Diffusion on stretched horizons*. JHEP, vol. 10, page 064, 2003. 7, 97

[Kovtun 2003b] P. Kovtun and L. G. Yaffe. *Hydrodynamic fluctuations, long-time tails, and supersymmetry*. Phys. Rev., vol. D68, page 025007, 2003. 108

[Kovtun 2005] P. Kovtun, D. T. Son and A. O. Starinets. *Viscosity in strongly interacting quantum field theories from black hole physics*. Phys. Rev. Lett., vol. 94, page 111601, 2005. 7, 97

[Laflamme 1989] R. Laflamme. *Geometry and Thermofields*. Nucl.Phys.B, vol. 324, page 233, 1989. 23

[Landau 2000] L. D. Landau and E. M. Lifshitz. Fluid mechanics. Butterworth-Heinemann, 2000. 60, 91, 122

[Larsen 1996] F. Larsen and F. Wilczek. *Renormalization of black hole entropy and of the gravitational coupling constant.* Nucl. Phys., vol. B458, pages 249–266, 1996. 114

[Lifshitz 2000] E. M. Lifshitz and L. P. Pitaevsky. Physical kinetics. Butterworth-Heinemann, 2000. 5, 96

[Lopes Cardoso 2000] G. Lopes Cardoso, B. de Wit and T. Mohaupt. *Deviations from the area law for supersymmetric black holes.* Fortsch. Phys., vol. 48, pages 49–64, 2000. 32, 35

[Lovelock 1971] D. Lovelock. *The Einstein tensor and its generalizations.* J. Math. Phys., vol. 12, pages 498–501, 1971. 123

[Maldacena 1998] J. M. Maldacena. *The large N limit of superconformal field theories and supergravity.* Adv. Theor. Math. Phys., vol. 2, pages 231–252, 1998. 7, 40, 69, 97

[Mukhopadhyay 2006] A. Mukhopadhyay and T. Padmanabhan. *Holography of gravitational action functionals.* Phys. Rev. D, vol. 74, no. 12, page 124023, Dec 2006. 3

[Myers 2009a] R. C. Myers, M. F. Paulos and A. Sinha. *Holographic Hydrodynamics with a Chemical Potential.* JHEP, vol. 06, page 006, 2009. 97, 104

[Myers 2009b] R. C. Myers, M. F. Paulos and A. Sinha. *Quantum corrections to η/s.* Phys. Rev., vol. D79, page 041901, 2009. 126

[Nicolis 1977] Prigogine I. Nicolis G. Self organization in nonequilibrium systems: From dissipative structures to order through fluctuations. John Wiley & Sons, Inc, 1977. 140

[Nordström 1913] G. Nordström. *Zur Theorie der Gravitation vom Standpunkt des Relativitätsprinzips.* Annalen der Physik, vol. 347, no. 13, pages 533–554, 1913. 15

[Novozhilov 1991] Y. V. Novozhilov and D. V. Vassilevich. *Induced gravity.* 1991. In *Moscow 1991, Proceedings, Sakharov memorial lectures in physics, vol. 1* 73-81. 6

[O'Neill 1983] B. O'Neill. Semi-riemannian geometry: with applications to relativity. Academic Press, San Diego, California, 1983. 71

[Oriti 2005] D. Oriti. *Quantum gravity as a quantum field theory of simplicial geometry.* 2005. arXiv/gr-qc/0512103. 6

[Padmanabhan 2002] T. Padmanabhan. *Classical and quantum thermodynamics of horizons in spherically symmetric spacetimes.* Class. Quan. Grav., vol. 19, page 5387, 2002. 3, 4

[Padmanabhan 2010] T. Padmanabhan. *Thermodynamical Aspects of Gravity: New insights.* Rept. Prog. Phys., vol. 73, page 046901, 2010. 3, 31, 57

[Paranjape 2006] A. Paranjape, S. Sarkar and T. Padmanabhan. *Thermodynamic route to field equations in Lancos-Lovelock gravity.* Phys. Rev., vol. D74, page 104015, 2006. 3, 74, 95

[Parikh 2009] M. K. Parikh and S. Sarkar. *Beyond the Einstein Equation of State: Wald Entropy and Thermodynamical Gravity.* 2009. 4

[Paulos 2010] M. F. Paulos. *Transport coefficients, membrane couplings and universality at extremality.* JHEP, vol. 02, page 067, 2010. 97, 104

[Peet 1999] A. W. Peet and J. Polchinski. *UV/IR relations in AdS dynamics.* Phys. Rev., vol. D59, page 065011, 1999. 42

[Penrose 1971] R. Penrose and R. M. Floyd. *Extraction of rotational energy from a black hole.* Nature, vol. 229, pages 177–179, 1971. 2

[Poisson 2004] E. Poisson. *Absorption of mass and angular momentum by a black hole: Time-domain formalisms for gravitational perturbations, and the small-hole / slow-motion approximation.* Phys. Rev., vol. D70, page 084044, 2004. 70, 90, 101

[Poisson 2005] E. Poisson. *Metric of a tidally distorted, nonrotating black hole.* Phys. Rev. Lett., vol. 94, page 161103, 2005. 70, 90, 101

[Policastro 2001] G. Policastro, D. T. Son and A. O. Starinets. *The shear viscosity of strongly coupled $N = 4$ supersymmetric Yang-Mills plasma.* Phys. Rev. Lett., vol. 87, page 081601, 2001. 45, 126

[Price 1986] R. H. Price and K. S. Thorne. *Membrane View-point on Black Holes: Properties and Evolution of the Stretched Horizon.* Phys. Rev., vol. D33, pages 915–941, 1986. 65

[Racz 1992] I. Racz and R. M. Wald. *Extension of space-times with Killing horizon.* Class. Quant. Grav., vol. 9, pages 2643–2656, 1992. 25, 49

[Racz 1996] I. Racz and R. M. Wald. *Global extensions of space-times describing asymptotic final states of black holes.* Class. Quant. Grav., vol. 13, pages 539–553, 1996. 49

[Rangamani 2009] M. Rangamani. *Gravity and Hydrodynamics: Lectures on the fluid-gravity correspondence.* Class. Quant. Grav., vol. 26, page 224003, 2009. 7, 45

[Rovelli 2004] C. Rovelli. Quantum gravity. Cambridge University Press, 2004. 2, 6

[Ryu 2006] S. Ryu and T. Takayanagi. *Holographic derivation of entanglement entropy from AdS/CFT.* Phys. Rev. Lett., vol. 96, page 181602, 2006. 102

[Sakharov 1968] A.D. Sakharov. *Vacuum quantum fluctuations in curved space and the theory of gravitation.* Sov. Phys. Dokl., vol. 12, pages 1040–1041, 1968. 6

[Saremi 2007] O. Saremi. *Shear waves, sound waves on a shimmering horizon.* 2007. arXiv/hep-th/0703170. 7, 97

[Sciama 1981] D. W. Sciama, P. Candelas and D. Deutsch. *Quantum Field Theory, Horizons and Thermodynamics.* Adv. Phys., vol. 30, pages 327–366, 1981. 29, 105

[Sewell 1982] G. L. Sewell. *Quantum fields on manifolds: PCT and gravitationally induced thermal states.* Annals Phys., vol. 141, pages 201–224, 1982. 1, 23

[Sewell 1986] G. L. Sewell. Quantum Theory of Collective Phenomena. Oxford University Press, Oxford, 1986. 19

[Smolin 2003] L. Smolin. *How far are we from the quantum theory of gravity?*, 2003. arXiv/hep-th/0303185. 6

[Solodukhin 1995] S. N. Solodukhin. *One loop renormalization of black hole entropy due to nonminimally coupled matter.* Phys. Rev., vol. D52, pages 7046–7052, 1995. 114

[Son 2002] D. T. Son and A. O. Starinets. *Minkowski-space correlators in AdS/CFT correspondence: Recipe and applications.* JHEP, vol. 09, page 042, 2002. 103, 107

[Son 2007] D. T. Son and A. O. Starinets. *Viscosity, Black Holes, and Quantum Field Theory*. Ann. Rev. Nucl. Part. Sci., vol. 57, pages 95–118, 2007. 7, 73, 97

[Sorkin 1986] R. D. Sorkin. *Toward a proof of entropy increase in the presence of quantum black holes*. Phys. Rev. Lett., vol. 56, pages 1885–1888, 1986. 28

[Sotiriou 2007a] T. P. Sotiriou. *Modified Actions for Gravity: Theory and Phenomenology*. 2007. Ph.D. Thesis. 15

[Sotiriou 2007b] T. P. Sotiriou and S. Liberati. *Metric-affine $f(R)$ theories of gravity*. Annals Phys., vol. 322, pages 935–966, 2007. 95

[Sotiriou 2010] T. P. Sotiriou and V. Faraoni. *f(R) Theories of Gravity*. Rev. Mod. Phys., vol. 82, pages 451–497, 2010. 74, 79, 80

[Srednicki 1993] M. Srednicki. *Entropy and Area*. Phys. Rev. Lett., vol. 71, pages 666–669, 1993. 1, 27

[Starinets 2009] A. O. Starinets. *Quasinormal spectrum and the black hole membrane paradigm*. Phys. Lett., vol. B670, pages 442–445, 2009. 7, 97

[Susskind 1994] L. Susskind and J. Uglum. *Black hole entropy in canonical quantum gravity and superstring theory*. Phys. Rev., vol. D50, pages 2700–2711, 1994. 2, 28, 106

[Susskind 1995] L. Susskind. *The World as a hologram*. J. Math. Phys., vol. 36, pages 6377–6396, 1995. 6, 35, 39

[Susskind 1998] L. Susskind and E. Witten. *The Holographic bound in anti-de Sitter space*. 1998. arXiv/hep-th/9805114. 41, 42

['t Hooft 1985] G. 't Hooft. *On the Quantum Structure of a Black Hole*. Nucl. Phys. B, vol. 256, page 727, 1985. 1, 26, 52, 106

['t Hooft 1990] G. 't Hooft. *The black hole interpretation of string theory*. Nucl. Phys. B, vol. 335, no. 1, page 138, 1990. 2

['t Hooft 1993] G. 't Hooft. Dimensional reduction in quantum gravity. Abdus Salam Festschrift: A Collection of Talks, World Scientific, Singapore, 1993. 6, 35, 39

['t Hooft 1999] G. 't Hooft. *The holographic principle: Opening lecture*. 1999. arXiv/hep-th/0003004. 40

['t Hooft 2001] G. 't Hooft. *Quantum mechanics and determinism*. 2001. arXiv/hep-th/0105105. 40

['t Hooft 2003] G. 't Hooft. *Determinism in free bosons*. Int. J. Theor. Phys., vol. 42, pages 355–361, 2003. 40

[Teukolsky 1974] S. A. Teukolsky and W. H. Press. *Perturbations of a rotating black hole. III - Interaction of the hole with gravitational and electromagnet ic radiation*. Astrophys. J., vol. 193, pages 443–461, 1974. 70, 101

[Thorne 1986] R. H. Thorne K. S.and Price and D. A. MacDonald. Black holes: the membrane paradigm. Yale University Press, 1986. 4, 60, 65

[Unruh 1976] W. G. Unruh. *Notes on black hole evaporation*. Phys. Rev. D, vol. 14, page 870, 1976. 1, 19, 98

[Unruh 1982] W. G. Unruh and R. M. Wald. *Acceleration Radiation and Generalized Second Law of Thermodynamics*. Phys. Rev., vol. D25, pages 942–958, 1982. 37

[Verlinde 2011] E. Verlinde. *On the origin of gravity and the laws of Newton*. Journal of High Energy Physics, vol. 2011, pages 1–27, 2011. 10.1007/JHEP04(2011)029. 8, 137, 139, 140

[Visser 1993] M. Visser. *Dirty black holes: Entropy as a surface term*. Phys. Rev. D, vol. 48, no. 12, pages 5697–5705, Dec 1993. 2

[Visser 2002] M. Visser. *Sakharov's induced gravity: A modern perspective*. Mod. Phys. Lett. A, vol. 17, pages 977–992, 2002. 6

[Vitagliano 2010] V. Vitagliano, T. P. Sotiriou and S. Liberati. *The dynamics of generalized Palatini Theories of Gravity*. Phys. Rev., vol. D82, page 084007, 2010. 95

[Vitagliano 2011] V. Vitagliano, T. P. Sotiriou and S. Liberati. *The dynamics of metric-affine gravity*. Annals Phys., vol. 326, pages 1259–1273, 2011. 95

[Vollick 2007] D. N. Vollick. *Noether Charge and Black Hole Entropy in Modified Theories of Gravity*. Phys. Rev., vol. D76, page 124001, 2007. 95

[Wald 1984] R. M. Wald. General relativity. University of Chicago Press, 1984. 22, 24

[Wald 1992] R. M. Wald. Quantum field theory in curved spacetime and black hole thermodynamics. University of Chicago Press, 1992. 3, 24, 31, 35, 137

[Wald 1993] Robert M. Wald. *Black hole entropy is the Noether charge.* Phys. Rev., vol. D48, pages 3427–3431, 1993. 31, 95, 114, 126

[Will 1981] C. M. Will. Theory and experiment in gravitational physics. Cambridge University Press, New York, 1981. 14, 78

[Will 2005] C. M. Will. *The confrontation between general relativity and experiment.* Living Rev. Rel., vol. 9, page 3, 2005. 55

[Witten 1998a] E. Witten. *Anti-de Sitter space and holography.* Adv. Theor. Math. Phys., vol. 2, pages 253–291, 1998. 40

[Witten 1998b] E. Witten. *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories.* Adv. Theor. Math. Phys., vol. 2, pages 505–532, 1998. 73

[Yarom 2005] A. Yarom and R. Brustein. *Area-scaling of quantum fluctuations.* Nucl. Phys., vol. B709, pages 391–408, 2005. 106

[Zurek 1985] W. H. Zurek and K. S. Thorne. *Statistical mechanical origin of the entropy of a rotating, charged black hole.* Phys. Rev. Lett., vol. 54, page 2171, 1985. 26