# ASPECTS OF QUASI-RIEMANNIAN GEOMETRIES

Thesis submitted for the degree of "Doctor Philosophiae" at the International School for Advanced Studies, Trieste, Italy

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## ACKNOWLEDGEMENTS

I am very grateful to my supervisor Professor John Strathdee for his continuous help, assistance and encouragement and for many inspiring suggestions during my work on this thesis.

I wish also to express my gratitude to Professor C.J. Isham, for his helpful advice and for his kind hospitality at Imperial College in London, where part of this thesis was written.

Special thanks are due also to Professor G. Furlan, for his assistance and encouragement during the years I have spent in Trieste.

I have also benefited from some interesting discussions with Professor K. Barnes in Southampton and Professor G.F.R. Ellis in Trieste.

I would like to thank my colleagues and friends in Trieste and London, and especially Jan Sobczyk, David Wiltshire, Sunhil Gandhi and Dan Levy.

Finally, many thanks to Alessandra Bergamo for her help in the correction of the manuscript.

# CONTENTS

# INTRODUCTION

Soon after Einstein's discovery of General Relativity, a model for unifying gravitation and electromagnetism based on the assumption that the spacetime has more than four dimensions, was proposed by Kaluza [1]. He proved that the equations of general relativity in five dimensions, with a metric tensor independent from the fifth coordinate, can be reduced to those of four-dimensional General Relativity coupled to electromagnetism in such a way that the gauge invariance of electromagnetism can be seen as a part of the 5-dimensional coordinate invariance.

Kaluza's idea was developed by Klein [2] who, looking for a wave equation in a 5-dimensional space, noticed that the fifth dimension can be taken of constant size and then closed into a circle, so that the effective 4-dimensional charge and mass of the particles can assume only discrete values.

At that time, however, the significance of the unobserved extra dimension was not clear. The general opinion was to think about it as a mere mathematical artifice. The first who gave a consistent physical interpretation of the fifth dimension were Einstein and Bergmann [3]. They proposed to take seriously into account the possibility of considering it as a really physical dimension, and suggested that its unobservability is due to the fact that it is curled up into a circle of very small radius. As a consequence, the fields are periodic in the fifth dimension and only their average values over the fifth coordinate are observable. This can be considered the first proposal of the concept of compactification, which will become very important in the future developments of the theory.

Further investigations were performed by Thiry and Jordan, who considered the possibility that the size of the internal space depend on the ordinary spacetime coordinates, giving rise to a massless scalar in the effective 4-dimensional theory [4].

The idea was then almost abandoned for a long time. Only in more recent years was the interest renewed when it was realized that nonabelian symmetries also can be implemented in the theory, by considering a higher number of dimensions. Starting from a suggestion by De Witt [5], the idea was first developed by Rayski and Kerner [6], and then by Cho and others [7]. All these works considered the case of an internal manifold constituted by a group manifold G, which is invariant under the action of the group G itself, and showed that the the theory so constructed contains 4-dimensional general relativity coupled to Yang-Mills theory with gauge group G. Only later Luciani [8] and Witten [9] realized that it is sufficient to consider as internal space any homogeneous space G/H. In fact, any gravitational theory defined in 4+K dimensions, admitting a ground state in the form of a direct product of the flat 4-dimensional Minkowski space $M^4$ with a K-dimensional compact space $B^K$ with isometry group G, can be shown to contain 4-dimensional gravity and Yang-Mills theory with gauge group G.

Kaluza-Klein theories can then be viewed as theories where a special kind of spontaneous symmetry breaking occurs: the original coordinate invariance in 4+K dimensions is broken by the choice of a particular ground state to the smaller group $P^4 \times G$, with $P^4$ the Poincaré group and the gauge symmetries of the theory can be explained in terms of symmetries of the internal space. At this stage, however, it is not clear why the ground state should be of the form $M^4 \times B^K$ instead, for example, of the (4+K)-dimensional flat space.

A first step toward the solution of this problem was taken by Scherk and Schwarz [10], who in the context of dual models, took up the old idea of Einstein and Bergmann of a compact internal space noticing that it is consistent with the higher-dimensional Einstein equations and with a physical interpretation of the theory to have an N-torus as internal space.

But the most important achievement was the proposal by Cremmer and Scherk [11] of a mechanism of "spontaneous compactification". They showed in fact that some solutions of the field equations exist of the form $M^4 \times B^K$, where $B^K$ is a compact space with a non-abelian symmetry group, if matter fields are added to the higher-dimensional action.

This mechanism, if on the one hand explains why some of the dimensions are compactified, on the other hand loses the simplicity of the original Kaluza-Klein theories, which were based only on the geometry of the higher-dimensional spacetime.

Some alternative mechanisms of spontaneous compactification were then proposed. The most interesting are probably those based on supergravity. These theories, in fact, are formulated in a natural way in higher dimensions and contain some extra matter fields which can give rise to spontaneous compactification [12]. In particular 11-dimensional supergravity attracted much attention due to the possibility of obtaining solutions with the phenomenological invariance group SU(3)xSU(2)xU(1) [9,13].

Another interesting possibility is that of considering one-loop quantum corrections as a source for compactification [14]. Also modifications of the Einstein-Hilbert action in more than four dimensions were proposed to obtain compactification from pure gravity [15,16].

A different point of view is to consider the anisotropy between the physical and the internal space as a consequence of the cosmological evolution of the universe. Some solutions were found of the higher dimensional Einstein equations describing a universe which, starting from a phase where the 4-dimensional and the internal space are of the same size, evolves towards a phase where the internal dimensions compactify, while the observed ones continue to expand [17,18]. More refined models [19-21] showed the possibility that the shrinking of the internal dimension may cause an exponential growth of the physical space in the early epochs, thereby solving the well known problems of the homogeneity and flatness of the observed universe [22].

Many problems are still unsolved in the context of Kaluza-Klein theories. A first problem is that of establishing what is the true ground state of the theory. If on the one hand the stability against small perturbations of a given ground state can be studied [23] using the methods of the harmonic expansion on coset spaces, introduced by Salam and Strathdee [24], it is not possible to compare the energy of ground states exhibiting different topologies and it is then difficult to establish criteria of stability against semiclassical decay [25].

Another problem is constituted by the difficulty of obtaining in a natural way (i.e. without extremely accurate fine tuning of the parameters of the theory) a flat 4-dimensional space after compactification.

Finally, the impossibility of obtaining chiral fermions from dimensional reduction of standard higher-dimensional theories was proved by Witten [26]. Some proposals have been advanced in order to solve this problem. One of them is the introduction of elementary gauge fields in topologically nontrivial configurations in addition to the gravitational field [27,23]. Another possibility is to give up the requirement of compacteness of the internal space, allowing the introduction of noncompact spaces with small finite volume [28,29]. The smallness of the volume is sufficient to explain the unobservability of the extra dimensions, and under suitable boundary conditions physically acceptable models can be obtained.

Finally, one may think to abandon the Riemannian geometry. As has been pointed out by Weinberg [30], there is no compelling physical reason why the geometry of the higher-dimensional space-time should be the same as that of the 4-dimensional spacetime of general relativity. In some sense, one has again to take seriously the extra dimensions and then to consider the possibility that they carry alternative geometrical structures, which could give rise to more realistic models.

A first possibility is for example the introduction of torsion in the theory [31]. This can lead to an explanation of the vanishing of the 4-dimensional cosmological constant [32], but seems not to be successfull in solving the chirality problem [33].

A more drastic alternative was proposed by Weinberg [30]. He suggested the possibility that the tangent space invariance group of the higher dimensional spacetime is different from the orthogonal group of Riemannian geometry. Clearly, the freedom in the choice of the higher-dimensional tangent group is limited by the requirement that the theory be consistent with the observed Lorenz invariance of the 4-dimensional spacetime, but this request still leaves many possibilities of choice. It has been shown in fact that, in order to recover the usual 4-dimensional Lorentz invariance, it is necessary that the tangent space group have the structure $SO(1,N-1) \times G_T'$, where $G_T \subseteq GL(D-N)$ and $N \geq 4$ [34]. The resulting geometry has been called quasi-Riemannian.

Since then a few authors discussed some aspects of the theory, mainly in a Kaluza-Klein context, but the literature on the subject is at present very poor [35-42].

The aim of this thesis is to examine to some extent the physical implications of quasi-Riemannian theories, with particular attention to the modifications they introduce in higher dimensional theories of gravitation and to the applications to the Kaluza-Klein theories.

With the matter of investigation so vast, we preferred to study various features of the theory rather than specialize on one single subject. However, we limited ourselves to the case of a tangent space group of the kind $O(1,N-1) \times O(M)$. This is the simplest nontrivial choice but nevertheless it contains all the essential features of the quasi-Riemannian geometries.

The plan of the thesis is as follows.

In chapter I a short review of differential geometry and general relativity in higher dimensions is given, with special attention to the problems related to the choice of the tangent space group. The special features of the Riemannian geometry compared with more general structures are emphasized.

Chapter II contains a brief review of standard Kaluza-Klein theories. The problem of the fermion chirality is examined and some simple examples of spontaneous compactification are discussed.

In the next chapters quasi-Riemannian theories are studied in great detail. Most of the material contained there is based on my contributions to the subject either already published or being prepared for publication.

In particular, in chapter III the most general action for quasi-Riemannian theories with tangent space O(1,N-1)xO(M) compatible with some simple physical requirements is established. It is shown to depend on 9 independent parameters and is compared with the actions obtained from different approaches to the theory.

In chapter IV the stability of the flat space under small perturbations and the particle content of the theory are studied and it is shown that some very strong conditions must be imposed on the parameters of the theory in order to achieve stability. In fact, a gauge invariance must be introduced at the linear level, in order to avoid the appearance of ghost states.

In chapter V, the investigation is briefly extended to the classical fields defined on a quasi-Riemannian background and the definition of the metric and of the geodesics is discussed.

Chapter VI is devoted to the study of the solutions of the classical field equations stemming from the action introduced in chapter III. In particular, a quasi-Riemannian cosmological model is described and compared with its Riemannian limit. Also the possible generalizations of the Schwarzschild solution of general relativity are discussed.

Finally, chapter VII deals with the applications of quasi-Riemannian geometries to Kaluza-Klein theories. Some simple models exhibiting spontaneous compactification are introduced and the "zero-mode ansatz" is discussed. Unfortunately, the results are not satisfactory from a phenomenological point of view. A discussion on the possibility of obtaining more realistic models by using different tangent space groups concludes the chapter.

# I . DIFFERENTIAL GEOMETRY AND GENERAL RELATIVITY IN HIGHER DIMENSIONS

We start by giving a brief account of differential geometry and general relativity in higher dimensions, which permits us to introduce the basic framework which will be used in the following and to establish some notations (*). We shall try to be as general as possible, not specializing to Riemannian geometry, but including the most general case of any tangent space group [36].

## I .1 Covariant derivatives

Let us consider a D-dimensional space $B^D$ parametrized by a set of coordinates $z^M$. General relativity can be regarded as a field theory on this manifold based on two kinds of invariance: general coordinate transformations invariance and local tangent space invariance.

General coordinate transformations transform the coordinates $z^M$ which parametrize the space into arbitrary functions $z'^M$ of $z^M$:

$$z^M \rightarrow z'^M(z^M) \qquad (1$$

Covariant vectors $V_M$ and controvariant vectors $W^M$ are defined to transform as:

$$V_M \rightarrow \frac{\partial z^N}{\partial z'^M} V_N \qquad\qquad W^M \rightarrow \frac{\partial z'^M}{\partial z^N} W^N \qquad (2$$

In order to define a derivative operator which transforms like a vector, one must introduce an affine connection $\Gamma^L{}_{MN}$ which transforms as

$$\Gamma^L{}_{MN} \rightarrow \Gamma'^L{}_{MN} = \frac{\partial z^Q}{\partial z'^M} \frac{\partial z^R}{\partial z'^N} \frac{\partial z'^L}{\partial z^P} \Gamma^P{}_{QR} - \frac{\partial z^Q}{\partial z'^M} \frac{\partial z^R}{\partial z'^N} \frac{\partial^2 z'^L}{\partial z^Q \partial z^R} \qquad (3$$

(*) See also appendix A for the conventions adopted.

and which permits to define a covariant derivative, denoted by a semi-colon:

$$V_{N;M} = \partial_M V_N - \Gamma^L{}_{NM} V_L$$
$$W^N{}_{;M} = \partial_M W^N - \Gamma^N{}_{LM} W^L \quad (4$$

For the moment, we do not impose any symmetry on the indices of $\Gamma$, but simply note that the antisymmetric part of the affine connection

$$T^L{}_{NM} = \Gamma^L{}_{MN} - \Gamma^L{}_{NM} \quad (5$$

transforms as a tensor and is called torsion.

The other symmetry of the theory is the invariance under local linear transformations of the fields of the kind:

$$\psi^A(z) \to B^A{}_B [\gamma(z)] \psi^B(z) \quad (6$$

where $\gamma(z)$ is an element of a given Lie group $G_T$, called the tangent space group, and $B^A{}_B$ is a matrix belonging to a representation of $G_T$.

Also in this case one has to construct a derivative of the field $\psi$ which transforms as (6) under $G_T$-transformations. In order to do that one must introduce a new connection $\omega_M(z)$, belonging to the Lie algebra of the group $G_T$ and transforming under the action of an element $\gamma(z)$ of $G_T$ as

$$\omega_M(z) \to \gamma(z)\, \omega_M(z)\, \gamma^{-1}(z) + \gamma(z)\, \partial_M \gamma^{-1}(z) \quad (7$$

One can then define a covariant derivative:

$$\nabla_M \psi^A = \partial_M \psi^A + d^A{}_B(\omega_M)\, \psi^B \quad (8$$

where $d^A{}_B(\omega_M)$ is in the representation of the Lie algebra of $G_T$ corresponding to the representation $D^A{}_B$ of the group. The connection $\omega_M$ is usually called "spin connection", because it must be introduced in order to define the covariant derivative for fermions.

Till now the group $G_T$ has nothing to do with the geometry of the manifold. In order to establish a relation, one must assume that $G_T$ has a faithful D-dimensional "defining" representation $\Lambda^A{}_B$ and that exists at every point a nonsingular matrix $e^A{}_M$ , called vielbein, which transforms under $G_T$ as a member of the defining representation, and under coordinate transformations as a covariant vector, namely:

$$e^A{}_M(z) \rightarrow e'^A{}_M(z') = \Lambda^A{}_B[\gamma(z)] \frac{\partial z^N}{\partial z'^M} e^B{}_N \tag{9}$$

By means of the vielbein one can convert coordinate indices into tangent group ones:

$$\psi^A(z) = e^A{}_M(z)\, \psi^M(z) \tag{10}$$

The vielbein can then be interpreted as a basis for vectors in the tangent space to the manifold at any point.

One can furthermore transform tangent space indices into covariant ones by means of the inverse vielbein $e_A{}^M(z)$, defined as the inverse matrix of $e^A{}_M$.

Another basic assumption is the so called "metricity postulate", which states that the totally covariant derivative $D_M$ of the vielbein is vanishing (*):

$$D_M e^A{}_N = \partial_M e^A{}_N + \omega^A{}_{BM}\, e^A{}_N - \Gamma^L{}_{MN}\, e^A{}_L = 0 \tag{11}$$

where $\omega^A{}_{BM} = \lambda^A{}_B\, [\omega_M]$ is the matrix element of $\omega_M$ in the defining representation of the Lie algebra of $G_T$.

This property establishes a simple relation between the totally covariant derivative calculated in the coordinate and in the tangent space basis:

$$D_M\, \psi^A = e^A{}_N\, D_M\, \psi^N \tag{12}$$

## I .2 Metrics

A way to characterize the tangent space group $G_T$ is to consider the

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(*) By totally covariant, we mean covariant under both coordinate and $G_T$ transformations.

constant tensors which are invariant under $G_T$.

For each of them one can construct a "metric" by contracting the tangent space indices with the vielbeins or their inverses. For example, from an invariant $G_T$-tensor $\eta_{AB}$, one can construct the metric

$$g_{MN} = \eta_{AB}\, e^A{}_M\, e^B{}_N \qquad (13$$

By the definition of $\eta_{AB}$ and the metricity condition (11) one can easily deduce that $g_{MN}$ has vanishing covariant derivative:

$$g_{MN;L} = 0 \qquad (14$$

and moreover transforms as a tensor under coordinate transformations.

Let us consider some examples.

In Riemannian geometry $G_T = O(D)$, apart from the signature. O(D) can be defined as the set of real DxD matrices $\Lambda^A{}_B$ which leave invariant some nonsingular symmetric matrix $\eta_{AB}$:

$$\eta_{AB} = \Lambda^C{}_A\, \Lambda^D{}_B\, \eta_{CD} \qquad (15$$

In this case we have only one invariant tensor $\eta_{AB}$ and then only one metric can be defined as in (13).

For $G_T$ = Sp (D,R), the situation is similar, but the matrix $\eta_{AB}$ is now antisymmetric, so that one can construct a unique metric $g_{MN}$, which is antisymmetric. The geometry so defined is called symplectic.

Finally, for $G_T = U(^D/_2)$, it can be shown that two real constant $G_T$-invariant tensors exist, of which one is symmetric and the other antisymmetric, and then two metrics can be constructed. A manifold with this structure is called a Kaehler manifold.

## I .3 Connections

Eqn. (11) permits to establish a relation between the affine and spin connections and the derivatives of the vielbein, but it is not sufficient to determine one of them in terms of the others unless some information is added. In the case of Riemann geometry and vanishing torsion one can determine both $\Gamma$ and $\omega$ as functions of the vielbein, as is well known from general relativity:

$$\Gamma^L{}_{MN} = \tfrac{1}{2}\, g^{LP}( \partial_N\, g_{PM} + \partial_M\, g_{PN} + \partial_P\, g_{MN} ) \tag{16}$$

$$\omega^A{}_{BN} = e^A{}_L\, e_B{}^M\, \Gamma^L{}_{MN} - e_B{}^M\, \partial_N\, e^A{}_M \tag{17}$$

But this is not possible in the general case. It can be shown, in fact, that a unique torsion free solution does not exist unless dim $G_T \leq \frac{1}{2}D(D-1)$. On the other hand, if dim $G_T < \frac{1}{2}D(D-1)$, no torsion free solution exists, unless some constraints are imposed on the vielbeins and their derivatives [30,35].

A different approach to the problem is to give up any a priori condition on the torsion and require simply that $\omega$ and $\Gamma$ are linear combinations of the first derivatives of the vielbein. In the case of tangent group O(D) this gives rise again to the unique solution (16,17), whereas if $G_T \subseteq O(D)$ it is always possible to construct at least one solution to the equations (10). No solution is known if $G_T \not\subset O(D)$, and it has been conjectured that it does not in fact exist [35].

A third possibility which we shall consider in the following is to regard the vielbein and the spin connection as independent variables and to determine the relations between them by means of the field equations obtained from a suitable action. Given e and $\omega$, $\Gamma$ is then determined uniquely by (11).

## I .4 Geodesics

A geometrical object $\psi(z)$ is said to be parallel transported along a curve $z^L(s)$ if $D\psi=0$ along the curve, or more precisely

$$\frac{D\psi}{Ds} = \frac{dz^L(s)}{ds}\, D_L\psi = 0 \tag{18}$$

A geodesics is defined by the request that its tangent vector $\frac{dz^L(s)}{ds}$ is parallel transported along the curve, i.e.

$$\frac{D}{Ds}\left[\frac{dz^L(s)}{ds}\right] \equiv \frac{d^2z^L}{ds^2} + \Gamma^L{}_{MN}\frac{dz^M}{ds}\frac{dz^N}{ds} = 0 \qquad (19$$

It is easy to see that in Riemannian geometry with vanishing torsion free particles move along geodesics. In fact, their action is usually defined as

$$S = -m\int ds = -m\int\sqrt{g_{MN}\frac{dz^M}{ds}\frac{dz^N}{ds}}\,ds \qquad (20$$

By varying this expression with respect to $g_{MN}$, one obtains the equation

$$\frac{d^2z^L}{ds^2} + \{^L{}_{MN}\}\frac{dz^M}{ds}\frac{dz^N}{ds} = 0 \qquad (21$$

where

$$\{^L{}_{MN}\} = ½\, g^{LP}(\,\partial_N g_{PM} + \partial_M g_{PN} + \partial_P g_{MN}\,) \qquad (22$$

which coincides with the expression (16) for $\Gamma^L{}_{MN}$. In more general cases, however, the affine connection may have a different form and the free particles do not follow geodesics.

## I .5 Curvature and torsion

From the connection and the vielbein and their first derivatives one can construct two covariant objects, which are called curvature and torsion and are defined respectively as:

$$R^A{}_{BMN} = \partial_M\, \omega^A{}_{BN} + \omega^A{}_{CM}\, \omega^C{}_{BN} - (M\leftrightarrow N)$$
$$T^A{}_{MN} = \partial_M\, e^A{}_N + \omega^A{}_{BM}\, e^B{}_N - (M\leftrightarrow N) \qquad (23$$

It is easy to see that this definition of torsion is equivalent to the one given previously.

To perform the calculations it is useful to write them in a tangent space basis [31], by introducing the so called anholonomity of the vielbein defined by:

$$c^A{}_{BC} = -c^A{}_{CB} = e_B{}^M e_C{}^N ( \partial_M e^A{}_N - \partial_N e^A{}_M ) \tag{24}$$

The name is due to the fact that the tangent space basis is anholonomic and

$$[\partial_A, \partial_B] = c^C{}_{AB}\, \partial_C \tag{25}$$

In this basis, the definitions of curvature and torsion read:

$$\begin{aligned} R^A{}_{BCD} \equiv e_C{}^M e_D{}^N R^A{}_{BMN} &= \partial_C\, \omega^A{}_{BD} - \partial_D\, \omega^A{}_{BC} \\ &+ \omega^A{}_{EC}\, \omega^E{}_{BD} - \omega^A{}_{ED}\, \omega^E{}_{BC} + \omega^A{}_{BE}\, c^E{}_{CD} \end{aligned} \tag{26}$$

$$T^A{}_{BC} \equiv e_B{}^M e_C{}^N T^A{}_{MN} = \omega^A{}_{CB} - \omega^A{}_{BC} + c^A{}_{BC} \tag{27}$$

where $\omega^A{}_{BC} \equiv e_C{}^M \omega^A{}_{BM}$.

The above tensors are clearly antisymmetric under exchange of the last two indices. Furthermore, from the definition (23) follows that $R^A{}_{BMN}$ can be seen as a matrix in the two indices A and B belonging to the algebra of $G_T$. Then, if $G_T$ is a subgroup of O(D), $R_{ABCD} = - R_{BACD}$. For more general groups different symmetries may hold between the first two indices.

Other relations can be found between the curvature and the torsion and their covariant derivatives:

$$\begin{aligned} \sum_{(BCD)} R^A{}_{BCD} &= \sum_{(BCD)} \nabla_D\, T^A{}_{BC} + T^A{}_{BE}\, T^E{}_{CD} \\ \sum_{(ECD)} \nabla_E\, R^A{}_{BCD} &= \sum_{(ECD)} R^A{}_{BFE}\, T^F{}_{CD} \end{aligned} \tag{28}$$

The second one is the well-known Bianchi identity.

If $G_T$ is a subgroup of O(D), another quantity can be introduced which will be useful in the following. It is the so-called contortion $K_{ABC}$ which is defined as

$$K_{ABC} = \omega_{ABC} - L_{ABC} \tag{29}$$

where

$$L_{ABC} = - \tfrac{1}{2} (c_{CAB} - c_{ABC} - c_{BCA}) \tag{30}$$

and $\omega_{ABC} = \eta_{AD}\, \omega^D{}_{BC}$ , $\eta_{AB}$ being the unique O(D) invariant constant tensor defined in (15).

$L_{ABC}$ is the classical expression for the Riemannian spin connection for vanishing torsion, and it can be obtained by solving (27) for T=0. It follows that

$$T_{ABC} = K_{ACB} - K_{ABC} \tag{31}$$

Conversely, the contortion can be expressed in terms of the torsion as

$$K_{ABC} = \tfrac{1}{2} (T_{CAB} - T_{ABC} - T_{BCA}) \tag{32}$$

The above expressions can be written in a more concise way by considering $e^A$ and $\omega^A{}_B$ as 1-forms on $B^D$. One can then define the 2-forms:

$$\begin{aligned} R^A{}_B &= d\, \omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B \\ T^A &= de^A + \omega^A{}_B \wedge e^B \end{aligned} \tag{33}$$

which are related to the previous quantities by:

$$\begin{aligned} R^A{}_B &= \tfrac{1}{2} R^A{}_{BMN}\, dx^M \wedge dx^N \\ T^A &= \tfrac{1}{2} T^A{}_{MN}\, dx^M \wedge dx^N \end{aligned} \tag{34}$$

The anholonomity coefficients are then defined by the relation:

$$de^A + \tfrac{1}{2} c^A{}_{BC}\, e^B \wedge e^C = 0 \qquad (35$$

## I .6 Action

We pass now to discuss the dynamics of a gravitational theory on a D-dimensional manifold, by establishing an action functional from which the field equations can be derived. For the moment, we shall limit ourselves to the Riemannian case.

In General Relativity, one can proceed in two ways in order to construct an action functional. The usual way is to consider the vielbeins as the only independent fields, imposing a priori the form of the connection as a function of the vielbein, by requiring the vanishing of the torsion (second order formalism).

In this case the action invariant under coordinate transformations and tangent group rotations can be determined uniquely by requiring that the field equations obtained by its variation be second order differential equations for the vielbein [43]. The action so defined is the well-known Einstein-Hilbert action with an arbitrary cosmological constant $\lambda$:

$$S = -\frac{1}{\kappa^2}\int e_D\, d^Dz\, ( R + \lambda ) \qquad (36$$

where R is the Ricci scalar $R = R_{ABBA}$ calculated for $\omega_{ABC} = L_{ABC}$ and $e_D = |\det e^A{}_M|$.

A more natural approach is the Einstein-Cartan (first order) formalism, according to which one can regard the vielbein and the spin connection as independent fields and vary the action with respect to both of them [44].

In the case in which the field equations for the connection reduce to algebraic ones (i.e. do not contain derivatives), one can be reconduced to the second order formalism by means of the so-called Palatini formalism which consists in solving the field equations for the connection in terms of the vielbein, and then substitute them back into the original action.

This procedure is equivalent to the previous one only if no matter fields are present which can act as sources for the connection field equations, otherwise the action obtained by means of the Palatini formalism may differ from the Einstein-Hilbert one by some contact terms.

At this point one can ask what is the most general action which gives rise to second order differential equations for the vielbein in Einstein-Cartan theory [45]. It must clearly be a scalar under coordinate and $G_T$-transformations and must be a function of the vielbein, the connection and their first derivatives. As we have seen, the only covariant objects that can be constructed from the vielbein, the connection and their first derivatives are curvature and torsion. It is easy to see that in order to have at most second derivatives of the vielbein in the field equations the action can contain curvature only linearly and torsion only quadratically.

Contracting the indices in all possible inequivalent ways one obtains the general result

$$S = -\frac{1}{\kappa^2}\int e_D\, d^Dz\,(\,\alpha R + \lambda + a_1\, T_{ABC}\, T_{ABC} + \\ + a_2\, T_{ABC}\, T_{BAC} + a_3\, T_{AAC}\, T_{BBC}\,) \qquad (37$$

Varying with respect to the connection yields, in the absence of matter fields:

$$(\alpha - 2a_2)\, T_{CAB} + (4a_1 - 2a_2)\, T_{[AB]C} - 2\,(\alpha + a_2)\,\eta_{C[A|}\, T_{DD|B]} = 0 \qquad (38$$

This equation has the general solution

$$T_{ABC} = 0 \qquad (39$$

i.e.

$$\omega_{ABC} = L_{ABC} \qquad (40$$

Substituting back into S, one recovers the Einstein-Hilbert action:

$$S = -\frac{1}{\kappa^2}\int e_D\, d^Dz\,[\,\alpha R\,(\,\omega = L\,) + \lambda\,] \qquad (41$$

It is interesting to notice that for $\alpha \neq 0$ the same solution is obtained for any choice of $a_1$, $a_2$ and $a_3$, so that they are in some sense redundant parameters. This is due to the fact that in the Einstein-Cartan theory with the action (37) the connection is not really a dynamical variable, i.e. it does not propagate.

For future reference, we write here the Einstein-Hilbert action in terms of the anholonomity coefficients, as can be easily obtained substituting (40) in (37):

$$S = -\frac{1}{\kappa^2}\int e_D\, d^Dz\, [\alpha(-2\,\partial_A c_{BBA} + \frac{1}{4} c_{ABC}\, c_{ABC} - \frac{1}{2} c_{ABC}\, c_{BCA} + c_{AAC}\, c_{BBC}) + \lambda] \quad (42$$

$$= -\frac{1}{\kappa^2}\int e_D\, d^Dx\, [\alpha(\frac{1}{4} c_{ABC}\, c_{ABC} - \frac{1}{2} c_{ABC}\, c_{BCA} - c_{AAC}\, c_{BBC}) + \lambda]$$

where the second expression is obtained after integration by parts.

We conclude by noting that the Einstein-Hilbert action can be generalized in more than four dimensions if one permits that the field equations contain at most second derivatives of the vielbein, but not necessarily in a linear way [43,46]. In this case the so-called Gauss-Bonnet terms can be added to the action. They are defined as:

$$S_p = \int e_D\, d^Dz\, \varepsilon^{A_1 \dots\dots A_p}\, \varepsilon_{B_1 \dots\dots B_p}\, R_{A_1A_2}{}^{B_1B_2} \dots R_{A_{p-1}A_p}{}^{B_{p-1}B_p} \quad (43$$

where the $\varepsilon$ are totally antisymmetric tensors and p is even and smaller than D. The simplest examples are given by $S_2$, which is proportional to the Einstein Hilbert action, and by

$$S_4 = \int e_D\, d^Dz\, (R_{ABCD}\, R_{ABCD} - 4\, R_{AB}R_{AB} + R^2) \quad (44$$

where $R_{AB}$ is the Ricci tensor $R_{ACCB}$.

An important property of these action functionals is that they do not introduce ghosts or tachyons at the linearized level, contrary to what usually happens for actions containing powers of the curvature tensor [46,47]. For this reason they have been extensively studied in connection with Kaluza-Klein and string theories [16,48].

## I .7 Symmetries

Any specific solution of the field equations necessarily breaks the group of symmetries constituted by coordinate and $G_T$ transformations down to a much smaller group.

As we discussed above, the vielbein and the connection transform under coordinates and $G_T$ transformations as

$$e^A{}_M(z) \rightarrow e'^A{}_M(z') = \Lambda^A{}_B(z)\, e^B{}_N \frac{\partial z^M}{\partial z'^N}$$

$$\omega_M(z) \rightarrow \omega'_M(z') = [\Lambda \omega_M \Lambda^{-1} + \Lambda^{-1}\partial_M \Lambda] \frac{\partial z^M}{\partial z'^N} \qquad (45$$

Given a manifold described by a vielbein $e^A{}_M$ and a spin connection $\omega_M$, one can define its isometry group G as the subgroup of coordinates and $G_T$ transformations which keeps invariant the form of e and $\omega$ ,i.e.

$$e^A{}_M(x) = e'^A{}_M(x)$$

$$\omega_M(x) = \omega'_M(x) \qquad (46$$

An infinitesimal isometry with parameter $\varepsilon$ can be written as:

$$x^M \rightarrow x'^M(x) = x^M + \varepsilon K^M(x)$$

$$\Lambda(x) \rightarrow 1 + \varepsilon\, \Theta(x) \qquad (47$$

where the vector field $K^M$ is called Killing vector, and the matrix $\Theta$, belonging to the Lie algebra of $G_T$ is called Killing angle. From (35) and (36), they satisfy the equations [36]:

$$K^M{}_{;N} = T^M{}_{LN} K^N + e_A{}^M (\Theta - K^M \omega_M)^A{}_B\, e^B{}_N$$

$$K^M{}_{;N;P} = R^M{}_{NQP} K^Q + (T^M{}_{NQ} K^Q)_{;P} \qquad (48$$

From these relations it is possible to derive an important property of the isometry group: G is always finite-dimensional and dim G $\leq$ D + dim $G_T$.

Another important group which describes the symmetries of a manifold is the isotropy group at some point $x_0$, $H(x_0)$. It is defined as the subgroup of G which leaves the point $x_0$ fixed, i.e.

$$x'^M ( x_0^M ) = x_0^M \qquad (49$$

For infinitesimal isotropies this can be written as

$$K^M( x_0) = 0 \qquad (50$$

It can be shown that the isotropy group is always a subgroup of the tangent space group:

$$H ( x_0 ) \subseteq G_T \qquad (51$$

If the isometry group G acts transitively, that is to say, if any point x can be carried into any point x' by some element of G, the space is called homogeneous. For these spaces the structure of $H( x_0)$ is independent of G. This permits to identify the homogeneous space with the coset G/H.

# II . KALUZA-KLEIN THEORIES

In this chapter, we discuss some general features of Kaluza-Klein theories. More extended reviews can be found in [49,50], and in [13,51] in the context of supergravity. There are also two books containing excellent articles on the subject [52,53].

## II .1 The Kaluza theory (*)

The simplest example of unification of gravitational and gauge interaction by means of extra spacetime dimensions is provided by the original Kaluza theory [1].

Let us consider a five dimensional spacetime with coordinates $x^M = (x^\mu, x^5)$ , $\mu = 1,2,3,4$ , metric $\gamma_{MN}$ and signature (-++++) , and assume that the fifth dimension is curled up in a circle so small as to render it undetectable [3]. The physical fields will then be periodic in the fifth coordinate and can be expanded in Fourier series [2]. For example,

$$\gamma_{MN} = \sum_{-\infty}^{\infty}{}_n \, \gamma_{MN}^{(n)} \, e^{-\frac{inx^5}{2\pi a}} \qquad (1$$

Let us however assume for the moment that the metric $\gamma_{MN}$ is independent from the fifth coordinate, i.e.

$$\partial_5 \gamma_{MN} = 0 \qquad (2$$

In this way we discard the massive modes from the theory and keep only the massless ones, corresponding to $n = 0$ (zero-mode ansatz).

For simplicity we also assume that

$$\partial_\mu \gamma_{55} = 0 \qquad (3$$

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(*) In this section we use different conventions from the rest of the thesis.

This condition means that the fifth coordinate has a constant radius. If we give up this condition the theory will contain a massless scalar besides the graviton and the photon [4]. We point out that strictly speaking the condition (3) is not compatible with the equations of motion of the theory [54], but this is not important for our considerations.

A special coordinate system can now be found where

$$\gamma_{5\mu} = a_\mu(x^\mu) \qquad \gamma_{55} = 1 \tag{4}$$

Furthermore, one can define a new metric

$$g_{\mu\nu} = \gamma_{\mu\nu} - a_\mu a_\nu \qquad g_{5\mu} = \gamma_{5\mu} = a_\mu \tag{5}$$

The ground state of the theory, due to its particular topology, is not invariant under the original group of 5-dimensional coordinate transformations, but only under the subgroup

$$x^\mu \rightarrow x'^\mu(x^\mu) \qquad x^5 \rightarrow x^5 \tag{6}$$

$$x^\mu \rightarrow x^\mu \qquad x^5 \rightarrow x^5 + f(x^\mu) \tag{7}$$

It is easy to see that under (6) $a_M$ and $g_{MN}$ behave like 4-dimensional tensors, while under (7), they behave in the following manner:

$$g_{\mu\nu} \rightarrow g_{\mu\nu} \qquad a_\mu \rightarrow a_\mu - \partial_\mu f(x) \tag{8}$$

i.e. $a_M$ behave as a gauge field under (7) . We can therefore identify (6) with 4-dimensional coordinate transformations and (7) with gauge transformations of the electromagnetic field $a_M$.

Furthermore, the 5-dimensional Einstein-Hilbert action

$$S = -\frac{1}{\kappa^2}\int d^5x \sqrt{|\det g_{MN}|}\; R^{MN}{}_{NM} \tag{9}$$

can be written in terms of the 5-dimensional quantities as

$$S = -\int d^4x \sqrt{|\det g_{\mu\nu}|}\left[\frac{2\pi a}{\kappa^2} R^{\mu\nu}{}_{\mu\nu} + \frac{1}{4} F^{\mu\nu}F_{\mu\nu}\right] \tag{10}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $A_\mu = \sqrt{\frac{2\pi a}{\kappa^2}}\, a_\mu$. The constant a is the radius of the Klein circle. If one puts

$$\frac{2\pi a}{\kappa^2} = \frac{1}{16\pi G_N} \tag{11}$$

where $G_N$ is the Newton constant, one recovers the action of 4-dimensional gravity coupled to electromagnetism (*).

Moreover, projecting on the 4-manifold the 5-dimensional geodesic equation

$$\frac{d^2x^L}{ds^2} + \Gamma^L{}_{MN} \frac{dx^M}{ds}\frac{dx^N}{ds} = 0 \tag{12}$$

one obtains

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda{}_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + l_P\, Q\, F^\lambda{}_\nu \frac{dx^\nu}{ds} = 0 \tag{13}$$

where

$$Q = \frac{dx^5}{ds} = \text{constant} \tag{14}$$

and $l_P = (16\pi G_N)^{1/2} = 10^{-31}$cm is the Planck length. if we put $Q = e\,(M l_P)^{-1}$ with e the electric charge and M the mass of the particle, this is the equation of motion for a charged particle in general relativity. We may then identify the electric charge with the component of the momentum along the fifth dimension in suitable units. Hence particles follow geodesic trajectories in five dimensions , but in four dimensions particles with different electric charge follow different trajectories because their "momentum" in the fifth dimensions is different.

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(*) If a timelike signature had been chosen for the fifth dimension, the wrong sign would have been obtained for the Maxwell action.

Let us now consider a scalar field coupled to gravity in five dimensions. It obeys the Klein equation [2]:

$$\nabla^M \nabla_M \, \phi = 0 \tag{15}$$

Expanding $\phi$ as in (1) and substituting in (15) one can obtain the field equations obeyed by the harmonics:

$$\left[ \left( \nabla_\mu + iA_\mu \frac{n l_P}{2\pi a} \right)^2 - \left( \frac{n}{2\pi a} \right)^2 \right] \phi^{(n)} = 0 \tag{16}$$

Then the nth harmonic has a four dimensional mass $^n/2\pi a$ and an electric charge $^{n l}P/2\pi a$ and both electric charge and mass are quantized. If one identifies $^l P/2\pi a$ with the elementary electric charge **e**, he can obtain for the Klein radius the value $a = 10^{-32}$ cm, corresponding to masses of order $10^{-19}$ GeV for the charged particles, which are clearly unobservable at present.

We stress, however, that the Klein radius cannot be determined directly from the equations of motion, at least classically.

To obtain the complete spectrum generated by the theory one must compute the part of the action bilinear in the fluctuations around the ground state $M^4 \times S^1$. In order to do this, one expands the metric as

$$g_{MN} = \eta_{MN} + \kappa h_{MN} \tag{17}$$

and substitute into the action obtaining:

$$S = -\int d^5x \left[ \frac{1}{4} \partial_L h_{MN} \partial_L h_{MN} - \frac{1}{4} \partial_L h_{MM} \partial_L h_{NN} + \right.$$
$$\left. - \frac{1}{2} \partial_M h_{ML} \partial_N h_{NL} + \frac{1}{2} \partial_M h_{MN} \partial_N h_{LL} \right] \tag{18}$$

At this point it is necessary to impose a gauge condition. The most useful choice is the light-cone gauge, since in this gauge only the physical degrees of freedom appear in the spectrum. The gauge condition is given by:

$$h_{A-} \equiv 2^{-1/2}(h_{A4} - h_{A1}) = 0 \tag{19}$$

In this gauge the action becomes

$$S = \int d^5x \Big[\frac{1}{4} h^t{}_{ij} (\partial_\mu{}^2 + \partial_5{}^2) h^t{}_{ij} + \frac{1}{2} h_{j5} (\partial_\mu{}^2 + \partial_5{}^2) h_{j5} + \frac{3}{8} h_{55} (\partial_\mu{}^2 + \partial_5{}^2) h_{55}\Big] \tag{20}$$

where

$$h^t{}_{ij} = h_{ij} - \tfrac{1}{2} h_{kk} \eta_{ij} \tag{21}$$

and i,j =2,3.

By Fourier expanding the $h_{MN}$ as in (1) it is then easy to realize that the spectrum of massless states, corresponding to n = 0, is composed by a particle of helicity 2 (graviton), one of helicity 1 (photon) and one of helicity 0 (Brans-Dicke scalar). The massive spectrum is constituted by a tower of spin 2 particles with masses $^n/2\pi a$ , obtained combining the modes with the same mass and helicity 2,1 and 0.

## II .2 Non abelian gauge symmetries from extra dimensions

The Kaluza-Klein mechanism can be generalized to non abelian groups by going to higher dimensions [5-8]. As in the 5-dimensional case one considers a gravitational theory with an action (usually the Einstein-Hilbert one) invariant under D-dimensional coordinate and tangent group transformations, and assumes that the ground state of the theory is given by a spacetime with topology $M^4 \times B^K$, where $M^4$ is the flat Minkowski space and $B^K$ is a compact space with K = D-4 dimensions admitting an isometry group G generated by a set of Killing vectors $K_{\hat{A}}{}^{\check{M}}(y)$ with A = 1,...., dim G. By definition, the Killing vectors satisfy the commutation relations (*)

(*) For the conventions adopted, see appendix A.

$$K_{\hat{A}}^{\tilde{M}}\,\partial_{\tilde{M}} K_{\hat{B}}^{\tilde{N}} - K_{\hat{B}}^{\tilde{M}}\,\partial_{\tilde{M}} K_{\hat{A}}^{\tilde{N}} = -\frac{1}{a} f^{\hat{C}}_{\hat{A}\hat{B}} K_{\hat{C}}^{\tilde{N}} \qquad (22$$

where $f^C{}_{AB}$ are the structure constants of the group G and a is the length scale of the internal space. A simple example of such a space is a group manifold G, which is invariant under the action of G itself by right and left multiplication. A more interesting example are the coset spaces G/H, which will be discussed in detail later on.

In any case, the choice of the ground state breaks the original invariance of the theory, which reduces to the subgroup of the original transformations which leaves the ground state invariant, namely $P^4 \times G$, with $P^4$ the Poincaré group.

In analogy with the 5-dimensional case, one can now state a "zero-mode ansatz" which relates the higher-dimensional vielbein $e^A{}_M$ to the 4-dimensional vielbein $e^A{}_M$ and to the Yang-Mills field $A^B{}_M$ of the gauge group G:

$$e^{A}_{\ M} = \begin{pmatrix} e^{\dot{A}}_{\ \dot{M}}(x) & -a\,A^{\hat{B}}_{\ \dot{M}}(x)\,K_{\hat{B}}^{\tilde{N}}(y)\,e^{\tilde{A}}_{\ \tilde{N}}(y) \\ 0 & e^{\tilde{A}}_{\ \tilde{M}}(y) \end{pmatrix} \qquad (23$$

The ansatz (23) is in general invariant under 4-dimensional coordinate transformations and x-dependent G transformations of the internal space, which can be written infinitesimally as:

$$y^{\tilde{M}} \rightarrow y^{\tilde{M}} + \varepsilon^{\hat{A}}(x)\,K_{\hat{A}}^{\tilde{M}} \qquad (24$$

with $\varepsilon^{\hat{A}}$ arbitrary. From this condition it is easy to obtain that $A^{\hat{A}}{}_{\dot{M}}$ must transform under (24) as

$$\delta A^{\hat{A}}_{\ \tilde{M}} = \partial_{\tilde{M}}\,\varepsilon^{\hat{A}}(x) - f^{\hat{C}}_{\ \hat{A}\hat{B}}\,A^{\hat{A}}_{\ \tilde{M}}\,\varepsilon^{\hat{B}} \qquad (25$$

which is precisely the transformation rule of the Yang-Mills fields with gauge group G.

If one now inserts the ansatz into the Einstein-Hilbert action, one obtains

$$-\frac{1}{\kappa^2}\int d^D z\, e_D\, R_D$$
$$= -\frac{1}{\kappa^2}\int d^4x\, d^K y\, e_4\, e_K\, \left( R_4 + R_K + \frac{a^2}{4} K_{\hat{A}}^{\tilde{C}} K_{\hat{B}}^{\tilde{C}} F^{\hat{A}}_{\dot{M}\dot{N}} F^{\hat{B}}_{\dot{M}\dot{N}} \right) \tag{26}$$

where $F^A{}_{MN}$ is the Yang-Mills field strength:

$$F^{\hat{A}}_{\dot{M}\dot{N}} = \partial_{\dot{M}} A^{\hat{A}}_{\dot{N}} - \partial_{\dot{N}} A^{\hat{A}}_{\dot{M}} - f^{\hat{A}}_{\hat{B}\hat{C}} A^{\hat{B}}_{\dot{M}} A^{\hat{C}}_{\dot{N}} \tag{27}$$

and $R_D$, $R_4$, $R_K$ are the Ricci tensors calculated in the subspace to which the subscripts refer, and analougously for the determinant of the vielbein $e_D$, $e_4$, $e_K$.

It is always possible to choose the Killing vectors so that [55]:

$$\int d^K y\, e_K\, K_{\hat{A}}^{\tilde{C}} K_{\hat{B}}^{\tilde{C}} = q\, V_K\, \delta_{\hat{A}\hat{B}} \tag{28}$$

where $V_K$ is the volume of the internal space and q is a constant depending on the particular model (q=1 for a group manifold and q=(dim G/H)(dim G)$^{-1}$ for a coset space).

Substituting in (26) one obtains:

$$S = -\frac{V_N}{\kappa^2}\int d^4x\, e_4\, \left( R_4 + \frac{a^2}{4} q\, F^2 \right) \tag{29}$$

which is the Einstein-Yang-Mills action, provided one puts

$$\frac{V_N}{\kappa^2} = \frac{1}{16\pi G_N} \qquad\qquad \frac{1}{g^2} = \frac{V_N}{\kappa^2} q\, a^2 \tag{30}$$

where g is the Yang-Mills coupling constant. A more precise definition of the scale length a of the compact space and its relation with the Yang-Mills coupling constant is given in [55].

## II .3 The geometry of the coset spaces

As we have mentioned before, the most interesting case of compactification is that on coset spaces, due to the fact that, for a given invariance group G, they provide the space with the maximal number of dimensions which is invariant under G.

Let us consider a group G which acts on a space Y [24,56]. Given any $g \in G$ and $y \in Y$, we denote this action by $g(y) = g\,y$. For any point $y_0 \in Y$ one can consider the subgroup H of elements of G which leave $y_0$ fixed ( $h\,y_0 = y_0$ for any $h \in H$). H is called the isotropy subgroup of G. Then for each $g \in G$, we can define an equivalence class [g] by $[g] = \{ g' \in G:\ g' = g\,h$ for some $h \in H \}$. The set of all distinct equivalence classes is denoted by G/H and is called a coset space.

Let us assume that G acts transitively on Y (i.e. any two points of Y can be related by the action of some element of G). Then any $y \in Y$ can be written as $y = g\,y_0$. But $(g\,h)\,y_0 = g\,(h\,y_0) = g\,y_0$ for any $h \in H$. Therefore, any two elements belonging to the same equivalence class correspond to the same point in Y, and there is a natural identification between the points in y and the elements of the coset space.

The simplest examples of coset spaces are the N-spheres $S^N$, which can be identified with the coset space SO(N+1)/SO(N). Another important example are the complex projective spaces $CP^N = SU(N+1)/U(N)$.

It is easy to see that the dimension of G/H is given by dim G - dim H.

Let the Lie algebra $\mathfrak{G}$ of G be generated by a set of generators $Q_{\hat{A}}$, obeying the commutation relations

$$[\,Q_{\hat{A}},\,Q_{\hat{B}}] = f^{\hat{C}}{}_{\hat{A}\hat{B}}\,Q_{\hat{C}} \qquad (31$$

Let us denote by $Q_{\bar{A}}$ the subset of the $Q_{\hat{A}}$ which generates the subalgebra $\mathfrak{H}$, corresponding to the subgroup H:

$$[Q_{\bar{A}}, Q_{\bar{B}}] = f^{\bar{C}}{}_{\bar{A}\bar{B}} Q_{\bar{C}} \tag{32}$$

The remaining generators, which we shall denote $Q_{\tilde{A}}$, generate the coset space G/H. Clearly, $f^{\tilde{C}}{}_{\bar{A}\bar{B}} = 0$, because $\mathfrak{H}$ is a subalgebra and we can write $\mathfrak{G} = \mathfrak{H} + \mathfrak{L}$ ,where $\mathfrak{L} = \{Q_{\tilde{A}}\}$.

The coset space is called reductive if $[\mathfrak{H}, \mathfrak{L}] \subset \mathfrak{L}$, which corresponds to the conditions on the structure constants:

$$f^{\bar{C}}{}_{\bar{A}\tilde{B}} = 0 \tag{33}$$

An important theorem states that if H is compact, G/H is reductive. This is always the case with Kaluza-Klein theories.

If, in addition, $[\mathfrak{L}, \mathfrak{L}] \subset \mathfrak{H}$, G/H is called a symmetric space. This condition is equivalent to the condition on the structure constants

$$f^{\tilde{C}}{}_{\tilde{A}\tilde{B}} = 0 \tag{34}$$

The matrices of the adjoint representation of G are defined by

$$g^{-1} Q_{\hat{A}}\, g = D_{\hat{A}}{}^{\hat{B}}(g)\, Q_{\hat{B}} \tag{35}$$

For reductive algebras the matrices corresponding to elements of $\mathfrak{H}$ split in a block-diagonal form:

$$D_{\tilde{A}}{}^{\bar{B}}(h) = D_{\bar{A}}{}^{\tilde{B}}(h) = 0 \tag{36}$$

Let the points of the coset space be labeled by the coordinates $y^{\tilde{M}}$. For each point $y^{\tilde{M}}$ one can choose a representative element $L_y$ from the corresponding equivalence class of elements of G. (For example one may choose $L_y = \exp(y^{\hat{A}} Q_{\hat{A}})$. Under left multiplication by an element of G one obtains from $L_y$ a new element of G/H which in general lies in a different equivalence class. One can write

$$g\,L_y = L_{y'}\,h \tag{37}$$

for some $h \in H$.

To define a covariant basis on G/H one can consider the algebra valued 1-form

$$e(y) = L_y^{-1}\, dL_y \tag{38}$$

Since it belongs to the algebra of G, it can be expressed as

$$e\,(y) = e^{\hat{A}}(y)\, Q_{\hat{A}} = dy^{\tilde{M}}\, e^{\hat{A}}{}_{\tilde{M}}(y)\, Q_{\hat{A}} \tag{39}$$

From (37) and (38) one can deduce the behaviour of e(y) under the action of G. The result is:

$$e^{\hat{A}}(y') = e^{\hat{B}}(y)\, D_{\hat{B}}{}^{\hat{A}}(h^{-1}) + (h\, dh^{-1})^{\hat{A}} - (g\, dg^{-1})^{\hat{B}}\, D_{\hat{B}}{}^{\hat{A}}(L_y\, h^{-1}) \tag{40}$$

From this formula, making use of eqn. (36), one can obtain after a long calculation the transformation rules for the components of $e^{\hat{A}}{}_{\tilde{M}}$. They are:

$$e^{\tilde{A}}{}_{\tilde{M}}(y') = \frac{\partial y^{\tilde{N}}}{\partial y^{\tilde{M}}}\, e^{\tilde{B}}{}_{\tilde{N}}\, D_{\tilde{B}}{}^{\tilde{A}}(h^{-1}) \tag{41}$$

$$\begin{aligned} e^{\bar{A}}{}_{\tilde{M}}(y') &= \frac{\partial y^{\tilde{N}}}{\partial y^{\tilde{M}}}\, e^{\bar{B}}{}_{\tilde{N}}\, D_{\bar{B}}{}^{\bar{A}}(h^{-1}) + (h\, \partial_{\tilde{N}} h^{-1})^{\bar{A}} = \\ &= \frac{\partial y^{\tilde{N}}}{\partial y^{\tilde{M}}}\, [\, h^{-1} e_{\tilde{N}}\, h + h\, \partial_{\tilde{N}} h^{-1}]^{\bar{A}} \end{aligned} \tag{42}$$

These transformation rules show that if $D^{\tilde{A}}{}_{\tilde{B}}(h)$ is a matrix belonging to the defining representation of $G_T$, then the metric is G invariant (because the G-transformations are compensated by tangent group ones), while the $e^A$ behaves as a connection on G/H.

This requirement fixes the embedding of H into $G_T$. In particular, if $G_T$ is the K-dimensional orthogonal group or one of its subgroups, one has for infinitesimal transformations:

$$D_{\tilde{A}}^{\ \tilde{B}}(h) = \delta_{\tilde{A}}^{\ \tilde{B}} + \delta h^{\bar{C}} f^{\tilde{B}}_{\ \tilde{A}\bar{C}} = \delta_{\tilde{A}}^{\ \tilde{B}} + \varepsilon_{\tilde{A}}^{\ \tilde{B}} \tag{43}$$

where $\varepsilon_{AB} = -\varepsilon_{BA}$. Viewed as an SO(K) transformation with generators $\frac{1}{2}\Sigma^{AB}$, this reads

$$\frac{1}{2}\varepsilon_{\tilde{A}\tilde{B}}\Sigma^{\tilde{A}\tilde{B}} = \delta h^{\bar{C}} Q_{\bar{C}} \tag{44}$$

which yields:

$$Q_{\bar{C}} = -\frac{1}{2}\Sigma^{\tilde{A}\tilde{B}} f_{\tilde{A}\tilde{B}\bar{C}} \tag{45}$$

From (40) it is also possible to obtain an explicit expression for the Killing vectors in terms of the adjoint representation of G:

$$K_{\hat{B}}^{\ \tilde{M}} = D_{\hat{B}}^{\ \tilde{A}} e_{\tilde{A}}^{\ \tilde{M}} \tag{46}$$

The transformation rules obtained above can be used to discuss the invariance of the zero-mode ansatz (23) in the case of compactification to a coset space. It is easy to see that the ansatz is invariant under the subgroup of the D-dimensional symmetries given by the 4-dimensional coordinate transformations

$$x^{\dot{M}} \rightarrow x'^{\dot{M}}(x') \tag{47}$$

and G-transformations (i.e. x-dependent left translations of the coset space):

$$y^{\tilde{M}} \rightarrow y'^{\tilde{M}}(x,y) \tag{48}$$

with associated frame rotations $D_{\tilde{B}}{}^{\tilde{A}}(h)$. In fact, under these symmetries, the components $e^A{}_M$ transform according to (40) and

$$e'^{\dot{A}}{}_{\dot{M}} = \frac{\partial x^{\dot{M}}}{\partial x'^{\dot{N}}} e^{\dot{A}}{}_{\dot{N}} \tag{49}$$

$$e'^{\tilde{A}}{}_{\dot{M}} = \left[ \frac{\partial x^{\dot{M}}}{\partial x'^{\dot{N}}} e^{\tilde{B}}{}_{\dot{N}} + \frac{\partial y^{\tilde{M}}}{\partial x'^{\tilde{N}}} e^{\tilde{B}}{}_{\tilde{N}} \right] D_{\tilde{B}}{}^{\tilde{A}}(h^{-1}) \tag{50}$$

From (50) and (23) the transformation rules of the gauge field can be deduced:

$$A'^{\hat{B}}{}_{\dot{A}} = A^{\hat{C}}{}_{\dot{M}} D_{\hat{C}}{}^{\hat{B}}(g^{-1}) - e_{\dot{A}}{}^{\dot{N}} (g\, \partial_{\dot{N}}\, g^{-1})^{\hat{B}} \tag{51}$$

or equivalentely

$$A'_{\dot{M}}(x) = [\, g\, A_{\dot{M}}\, g^{-1} - g\, \partial_{\dot{M}}\, g^{-1} ] \tag{52}$$

with $A_{\dot{M}} = A^{\hat{B}}{}_{\dot{M}} Q_{\hat{B}}$. It is then evident that $A_{\dot{M}}$ transforms as a Yang-Mills field for the group .

We stress again that, in order for an action constructed out of this ansatz to be invariant, the embedding of H in $G_T$ discussed above is a necessary and sufficient condition.

The curvature and the torsion of a coset space can be obtained in terms of the structure constants of G. The result, however, is not unique, because of the freedom in the choice of the torsion.

Deriving (38) one finds

$$de(y) = dL_y{}^{-1} \wedge dL_y = - e(y) \wedge e(y) \tag{53}$$

This is called the Maurer-Cartan equation. It can be written, using (39) and the commutation relations of G

$$de^{\hat{A}} = - \frac{1}{2} e^{\hat{B}} \wedge e^{\hat{C}} f^{\hat{A}}{}_{\hat{B}\hat{C}} \tag{54}$$

Combining (54) with the definition of torsion

$$T^{\tilde A} = d\, e^{\tilde A} + \omega^{\tilde A}{}_{\tilde B} \wedge e^{\tilde B} \qquad (55$$

one obtains

$$T^{\tilde A} = -\frac{1}{2} f^{\tilde A}{}_{\hat B \hat C}\, e^{\hat B} \wedge e^{\hat C} + \omega^{\tilde A}{}_{\tilde B} \wedge e^{\tilde B} \qquad (56$$

In general, one can solve (56) by putting

$$T^{\tilde A} = -\frac{\mu}{2} f^{\tilde A}{}_{\tilde B \tilde C}\, e^{\tilde B} \wedge e^{\tilde C} \qquad (57$$

and

$$\omega^{\tilde A}{}_{\tilde B} = -f^{\tilde A}{}_{\tilde B \bar C}\, e^{\bar C} - \frac{1-\mu}{2} f^{\tilde A}{}_{\tilde B \tilde C}\, e^{\tilde C} \qquad (58$$

An important case occurs when $\mu = 0$ . In this case the torsion vanishes and

$$\omega^{\tilde A}{}_{\tilde B} = -f^{\tilde A}{}_{\tilde B \bar C}\, e^{\bar C} - \frac{1}{2} f^{\tilde A}{}_{\tilde B \tilde C}\, e^{\tilde C} \qquad (59$$

and consequently

$$R^{\tilde A}{}_{\tilde B \tilde C \tilde D} = f^{\tilde A}{}_{\tilde B \bar E} f^{\bar E}{}_{\tilde C \tilde D} + \frac{1}{2} f^{\tilde A}{}_{\tilde B \tilde E} f^{\tilde E}{}_{\tilde C \tilde D} - \frac{1}{2} f^{\tilde A}{}_{\tilde C \tilde E} f^{\tilde E}{}_{\tilde B \tilde D} \qquad (60$$

A more natural choice is to put $\mu = 1$. In this case one has:

$$T^{\tilde A}{}_{\tilde B \tilde C} = -f^{\tilde A}{}_{\tilde B \tilde C} \qquad (61$$

$$\omega^{\tilde A}{}_{\tilde B} = -f^{\tilde A}{}_{\tilde B \bar C}\, e^{\bar C} \qquad (62$$

$$R^{\tilde{A}}_{\tilde{B}\tilde{C}\tilde{D}} = f^{\tilde{A}}_{\tilde{B}\bar{E}} f^{\bar{E}}_{\tilde{C}\tilde{D}} \tag{63}$$

For a group manifold (H=1) this choice gives vanishing curvature.

If the coset space is symmetric ( $f^{\tilde{A}}_{\tilde{B}\tilde{C}} = 0$ ) the two choices are equivalent.

Finally, we notice that in some cases more general results can be obtained by rescaling the vielbeins [57].

## II .4 Harmonic expansion on coset spaces

In order to calculate the effective 4-dimensional theory, it is very useful to consider the harmonic expansion of fields on the internal space [24,49]. This is essential if one wants to calculate the spectrum of the massive modes.

It is well known that, given a function $\phi$ on a group G, it can be expanded as

$$\phi(g) = \sum_{n} \sum_{p,q} \sqrt{d_n}\; D^n{}_{pq}(g)\; \phi^n{}_{pq} \tag{64}$$

where $D^n{}_{pq}$ are unitary matrices of dimension $d_n$ and the sum is performed on all the matrix elements of all the unitary irreducible representations $g \rightarrow D^n{}_{pq}(g)$.

To expand functions on a coset space G/H, one must consider the representations $D^n{}_{pq}$ of G constrained by the requirement that they should contain a representation $\mathbb{D}^k{}_{ij}$ of the group H, namely

$$D^n(hg) = \mathbb{D}^k(h)\, D^n(g) \tag{65}$$

with

$$\phi_i(hg) = \mathbb{D}^k{}_{ij}(h)\;\phi_j(g) \tag{66}$$

Every function belonging to an irreducible representation of H, $\mathbb{D}^k(h)$ of dimension $d_k$ can then be expanded as

$$\phi_i(L_y) = \sum_n \sum_{\zeta,q} \sqrt{\frac{d_N}{d_K}}\, D^n{}_{i\zeta,q}(L_y)\ \phi^n{}_{pq} \qquad (67$$

where the notation $D^n{}_{i\zeta,q}$ means that from the matrix $D^n$ only the rows that correspond to the subspace which carries $\mathbb{D}(h)$ must be taken and the index $\zeta$ is needed if $\mathbb{D}^k(h)$ is contained more than once in $D^n$.

A generic function $\phi_i(x,y)$ on $M^4 \times G/H$, belonging to an irreducible representation of H, can thus be written as

$$\phi_i(x,y) = \sum_n \sum_{\zeta,q} \sqrt{\frac{d_N}{d_K}}\, D^n{}_{i\zeta,q}(L_y)\ \phi^n{}_{pq}(x) \qquad (68$$

The $D^n$ are normalized so that

$$\int_{G/H} d\mu\, D^n{}_{q,i\zeta}\, D^{n'}{}_{i\zeta',q'} = \frac{d_k}{d_n} V_{G/H}\ \delta_{nn'}\,\delta_{qq'}\,\delta_{\zeta\zeta'} \qquad (69$$

where $d\mu$ is the invariant measure of G and $V_{G/H}$ is the volume of the coset space.

An important property of the harmonics is their simple behaviour under covariant differentiation. It can be shown, in fact, [49] that using the canonical connection (62)

$$\nabla_{\tilde{A}} D^n{}_{ip}(L_y) = -\frac{1}{a} D^n{}_{iq}(Q_{\tilde{A}})\, D^n{}_{pq}(L_y) \qquad (70$$

This formula permits to reduce all differential operations on G/H to algebraic ones. For example

$$\nabla_{\tilde{A}}\nabla_{\tilde{A}} D^n{}_{ip}(L_y) = \frac{1}{a^2} D^n{}_{iq}(Q_{\tilde{A}} Q_{\tilde{A}} L_y) = \frac{1}{a^2}[\,C_G(D^n) - C_H(\mathbb{D}^k)]\, D^n{}_{iq}(L_y) \qquad (71$$

where $C_G(D^n)$ and $C_H(\mathbb{D}^k)$ denote the values of the Casimir operators of G and H in the representations $D^n$ and $\mathbb{D}^k$ respectively.

As an example of application of the formalism, let us consider the case of a 2-sphere $S^2 = SU(2)/U(1)$.

The tangent space group O(2) coincides with H = U(1), so that the embedding is trivial. Any 2-vector $V_A$ reduces into helicity $\lambda = \pm 1$ combinations $V_\pm$:

$$V_\pm = \frac{1}{\sqrt{2}} ( V_1 \pm i V_2 ) \tag{72}$$

A function $\phi$ of helicity $\lambda$ integer or half-integer can be expanded as:

$$\phi_\lambda (x,y) = \sum_{j \geq \lambda} \sqrt{2j+1} \sum_{j}^{-j} {}_m D^j_{\lambda m} (L_y) \, \phi^j_{\lambda m} (x) \tag{73}$$

where j takes the values $|\lambda|, |\lambda|+1, \ldots$ and $D^j$ denotes the (2j+1)-dimensional representation of SU(2).

The covariant derivatives can then be expanded as

$$\nabla_\pm D^j_{\lambda m}(L_y) = - \frac{1}{a} D^j_{\lambda m} (Q_\pm L_y) = \frac{i}{a} \sqrt{\frac{(j \pm \lambda)(j \pm \lambda - 1)}{2}} \, D^j_{\lambda \pm 1, m} (L_y) \tag{74}$$

where $Q_\pm$ is the combination of the generators $Q_i$ of SU(2) given by:

$$Q_\pm = \frac{1}{\sqrt{2}} ( Q_1 \pm i Q_2 ) \tag{75}$$

## II .5 Spontaneous compactification and stability

As we have seen, generalized Kaluza-Klein theories may provide an interesting framework to deal with the unification of gravitational and gauge forces. Unfortunately, however, some problems arise when one tries to obtain a realistic theory.

The first question one must answer is why one has to choose a seemingly arbitrary ground state of the kind $M^4 x B^K$ instead of the more obvious background provided by the flat D-dimensional space.

A first possible answer to this problem was given by Cremmer and Scherk [11]. They proposed that the ground state $M^4 x B^K$ should be a solution of the field equations of the D-dimensional theory, which breaks the original invariance of the action, whence the name of "spontaneous compactification" given to this mechanism. Unfortunately, pure Riemannian gravity with the Einstein-Hilbert action does not admit a solution of the kind we are looking for.

In fact, let us consider the equations of motion of D-dimensional general relativity in absence of matter fields [58]. They can be written as

$$R_{\dot{A}\dot{B}} = -\frac{\lambda}{D-2}\eta_{\dot{A}\dot{B}}$$

(76

$$R_{\tilde{A}\tilde{B}} = -\frac{\lambda}{D-2}\eta_{\tilde{A}\tilde{B}}$$

For $\lambda = 0$ they admit a solution consisting in the product of the flat Minkowski space with a Ricci-flat space. But compact Ricci-flat spaces cannot admit non-abelian symmetries and then are not useful to unify gravitation and Yang-Mills fields.

One could be less restrictive and let the 4-dimensional spacetime to be deSitter or anti-deSitter (i.e. $R_{MN} = -3h\,\eta_{MN}$, with $h>0$, $h<0$ respectively). This can be achieved for nonvanishing $\lambda$. From (76) it is however evident that the internal space must be an Einstein space with negative curvature if the 4-dimensional spacetime is deSitter and with positive curvature if it is anti-deSitter. In the first case, one gets in trouble because deSitter space is not a suitable ground state (a positive energy theorem cannot be defined on it). In the second case one has an antideSitter spacetime, which is less problematic, but the internal space must be an Einstein space with positive curvature and hence, by a theorem of differential geometry, cannot possess any nonabelian symmetry [58].

An even more serious problem is that, according to eqns. (76), both the physical and the internal space should have the same length scale. But this is wrong by 120 orders of magnitude! The internal space, in fact, should be of the size of the Planck length to fit with the Kaluza-Klein unification, while the observed 4-dimensional cosmological constant is very near to zero.

One must then look for a different mechanism of compactification, by adding an energy momentum tensor to the field equations (76).

Many solutions have been proposed to this problem: to stay in the context of pure gravity, one could add to the lagrangian terms of higher order in the curvature, like those discussed in section I.6 [16,48], or consider Casimir-like quantum effects as a source for compactification [14].

Another possibility is to couple elementary matter fields to D-dimensional gravity. Several models have been studied with couplings to Maxwell or Yang-Mills fields [11,23], scalar fields [59], fermions [60,31], or three index antisymmetric fields like those appearing in 11-dimensional supergravity [12]. In particular, many solutions of 11-dimensional supergravity exhibiting the phenomenological invariance group SU(3)xSU(2)xU(1) have been discussed [13,61].

Finally, one could overcome the difficulties by relaxing the condition that the internal space must be compact [28,29]. As discussed in more detail in section 8, physically acceptable models can be obtained in this way.

Unfortunately, most of the mechanisms proposed give rise to a vanishing effective 4-dimensional cosmological constant only after fine-tuning of the parameters.

Once a solution of a specific model has been found, one must check its stability. This is usually done by studying the spectrum of fluctuations around the ground state and showing that it does not contain tachyons nor ghosts [23]. In order to perform the calculations, the harmonic expansion technique introduced in the previous section is very useful. A variety of models have been studied in this way [62,63], but no general criterion for stability is known.

Unfortunately, the stability against small fluctuations is not sufficient for a solution to be the real ground state of the theory, because one should consider also the stability against quantum tunnelling effects [64].

This is a very difficult problem, because in general relativity it is not possible to compare the energy of solutions having different topologies, since the definition of energy depends on the asymptotic behaviour of the spacetime [25]. However, it has been explicitely shown that if changes of topology are admitted, the 5-dimensional Kaluza-Klein theory is unstable against semiclassical decay [25].

## II .6 Fermions

In order to construct a theory which describes the observed world, one has to introduce fermions. One may think of fermions as originating from supergravity, which contais spin $^1/_2$ and $^3/_2$ multiplets [65], or simply as independent fundamental fields, minimally coupled to gravity in higher dimensions.

In order for spinors to be defined in a D-dimensional space, the tangent group must obviously be an orthogonal or pseudoorthogonal group SO(D',D"). The physically relevant case is SO(1,D-1). The spinor $\psi(z)$ has in this case $2^{[\frac{1}{2}D]}$ components which transform as scalars under coordinate transformations and as spinors under SO(1,D-1):

$$\psi(z) \to \psi'(z) = D\,[\,\gamma(z)]\,\psi(z) \qquad (77$$

with D belonging to the spinor representation of SO(1,D-1).

Infinitesimally

$$\delta\,\psi(z) = \tfrac{1}{2}\,\varepsilon_{AB}\,\Sigma^{AB} \qquad (78$$

where $\frac{1}{2}\Sigma^{AB}$ are the generators of the algebra of O(1,D-1) in the spinorial representation:

$$\Sigma^{AB} = \tfrac{1}{4}\,[\,\Gamma^A\,,\,\Gamma^B\,] \qquad (79$$

and $\Gamma^A$ are the D-dimensional Dirac matrices, obeying the anticommutation relations:

$$\{\,\Gamma^A\,,\,\Gamma^B\,\} = -2\,\eta^{AB} \qquad (80$$

The covariant derivative of a spinor is defined as:

$$\nabla_M\,\psi = (\partial_M + \omega_M)\,\psi \qquad (81$$

where $\omega_M$ belongs to the algebra of O(1,D-1). and can be written as:

$$\omega_M = -\frac{1}{2}\,\omega_{ABM}\,\Sigma^{AB} = -\frac{1}{8}\,\omega_{ABM}\,[\Gamma^A,\,\Gamma^B] \qquad (82$$

The straightforward generalization to D dimensions of the 4-dimensional lagrangian invariant under coordinate and $G_T$ transformations is given by

$$\mathcal{L}_T = \frac{i}{2}\bar{\psi}\,\Gamma^A e_A{}^M \nabla_M \psi + \text{h.c.} \qquad (83$$

One may now substitute the zero-mode ansatz (23) into (83) and obtain the effective 4-dimensional lagrangian. This is done, for example, in [24]. The result is the usual 4-dimensional lagrangian for a spinor minimally coupled to gravitation and to the Yang-Mills fields, containing in general also a mass term and an additional "Pauli momentum term", i.e. a nonminimal coupling between spinor and gauge fields.

What is more interesting, however, is to study the presence of massless spinors in the spectrum. This is due essentially to the fact that the observed spinors have very light masses with respect to the Planck mass and must then be identified with the massless modes of the dimensionally reduced lagrangian. Their mass will be explained by other mechanisms, at a much lower mass scale.

Witten [26] has shown that the analysis of the massless fermions can be done by purely topological methods.

First of all, one can split the Dirac equation stemming from (83) in two terms:

$$i\Gamma^A\nabla_A\psi = i\Gamma^{\dot{A}}\nabla_{\dot{A}}\psi + i\Gamma^{\tilde{A}}\nabla_{\tilde{A}}\psi = 0 \qquad (84$$

where the first term is the usual 4-dimensional Dirac operator, while the second gives rise to an effective mass term. One can then classify the massless states by studying the zero-modes of the Dirac operator on the internal space.

A theorem by Lichnerowicz [66] gives a strong limit to the possibility of having zero-modes in the case of a compact space. The theorem states that the Dirac operator has no zero eigenvalues on a Riemannian manifold with vanishing torsion and Ricci scalar negative everywhere. This can be seen as follows [26]: consider the Dirac operator $\Gamma^A\nabla_A$ on a Riemannian manifold with positive signature and take its square:

$$\Gamma^A\nabla_A\Gamma^B\nabla_B = \nabla_A\nabla_A + \frac{1}{4}[\Gamma^A,\Gamma^B][\nabla_A, \nabla_B] =$$
$$= \nabla^2 + \frac{1}{32}[\Gamma^A,\Gamma^B][\Gamma^C,\Gamma^D]\, R_{ABCD} = \nabla^2 + \frac{R}{4} \quad (85$$

where the commutation relation

$$[\nabla_A, \nabla_B] = R_{ABCD}\,\Sigma^{CD} \quad (86$$

has been used.

Since $\nabla^2$ is negative definite on a compact manifold, if R<0 everywhere, the Dirac operator cannot have zero-modes at all.

Another constraint imposed by the phenomenology on the spectrum of fermions is that they should be chiral, i.e. should belong to a complex representation of the gauge group G. This means that right-handed and left-handed fermions transform differently under G.

Some considerations on the spinorial representations of O(1,D-1) limit the possibility of obtaining chiral fermions from dimensional reduction.

Let us consider a (4+K)-dimensional spacetime. For odd K, the O(K) group has only one spinor representation. Likewise, O(1,3+K), has only one spinor representation which transforms under O(1,3)xO(K) as the product of a Dirac spinor of O(4) with the spinor of O(K). Therefore, right and left handed spinors in 4 dimensions transform in the same way under the internal space tangent group.

In even dimensions the situation is more complicated. In fact, in this case the operator $\Gamma \equiv \Gamma_1 \ldots \Gamma_K$ commutes with the generators of O(K) and the group has then two inequivalent spinor representations, labeled by the eigenvalues of $\Gamma$.

In 4+K dimensions, one can define:

$$\Gamma = \Gamma^1 \ldots \Gamma^{4+K} \qquad \Gamma_e = \Gamma^1 \ldots \Gamma^4 \qquad \Gamma_i = \Gamma^5 \ldots \Gamma^{4+K} \quad (87$$

For fixed $\Gamma$, the 4-dimensional and internal chiralities are related, since

$$\Gamma = \Gamma_e\,\Gamma_i \quad (88$$

For example, for $\Gamma=1$, fermions have the same 4-dimensional and internal chirality, and then right-handed fermions transform differently from left-handed under the internal group. Therefore they obey different Dirac equations and their zero-modes may have different quantum numbers.

This however is not true if K is divisible by 4. In this case, in fact, $\Gamma^2=-1$ and its eigenvalues $\pm i$ are complex conjugates and are then correlated by CPT, which requires that an equal number of fields with $\Gamma=i$ and $\Gamma=-i$ are present. Therefore, to any left-handed fermion corresponds a right-handed one which transforms in the same way under the internal group.

In 4k+2 dimensions, instead, $\Gamma^2=1$ and $\Gamma$ has eigenvalues $\pm 1$ which are not related by CPT. One is then free to consider fermions belonging to one specific eigenvalue of $\Gamma$ , which have then a definite correlation between internal and external chirality.

A more drastic limitation on the possibility of obtaining chiral fermions was obtained by Witten [26], who proved that there are no chiral massless fermions in the spectrum of a theory with a compact internal space invariant under a nonabelian group.

This can be shown by using a theorem of Lawson and Yau [68], which states that in any compact manifold admitting a nonabelian isotropy group G, exists a G-invariant metric with negative scalar curvature and then, by Lichnerowicz theorem, no zero eigenvalues for the Dirac operator. Now, the difference between left and right-handed zero eigenvalues of the Dirac equation in a given representation of the isometry group is a topological invariant and then does not depend on the metric one chooses. Hence it follows that the eigenvalues of the Dirac operator are always coupled in pairs of opposite chirality (*).

For what concerns spin $^3/_2$ fields, a weaker theorem states that it is not possible to obtain chiral spin $^1/_2$ fields by dimensional reduction if the internal space is homogeneous [26].

Some ways of escaping the theorem are still possible: one of them is to introduce in the theory gauge fields in topologically non trivial configurations [27,23]. Another is to consider noncompact internal manifolds [29]. Finally, one may change the geometrical structure of the theory by introducing quasi-Riemannian geometries [30].

We shall briefly discuss the first two possibilities in the next sections

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(*) This theorem can be extended to the case in which torsion is non-vanishing, even if in that case Lichnerowicz theorem is not valid [33].

and will examine the last one in more extent in chapter VII.

## II .7 Compactification with gauge fields

A possible solutions to the problems of Kaluza-Klein theories is given by the introduction of elementary gauge fields in addition to gravitation. This is contrary to the spirit of the original Kalüza theory, but can be justified for example in supergravity, where some bosonic fields arise naturally in the supersymmetric multiplets.

We describe here a simple 6-dimensional model, consisting in pure gravity coupled to a Maxwell field [23]. This model admits a solution of the form $M^4 x S^2$, provided that the Maxwell field assumes a monopole configuration in the internal manifold. Moreover, if a fermionic field is added to the model, massless chiral fermions can be obtained by dimensional reduction.

This model can be generalized to Yang-Mills fields. It can be shown, in fact, that topologically non-trivial solutions of the Einstein-Yang-Mills equations of the form $M^4 x$ G/H always exist if the Yang-Mills gauge group contains H [69].

The 6-dimensional action is given by:

$$- \int d^6 z \, e_6 \left[ \frac{1}{\kappa^2} (R + \lambda) + \frac{1}{4} F_{AB} F_{AB} \right] \qquad (89$$

where $F_{AB} = \partial_A A_B - \partial_B A_A$.

The ensuing equations of motion are:

$$R_{AB} - \frac{1}{2} \eta_{AB} R = - \frac{\kappa^2}{2} (F_{AC} F_{BC} - \frac{1}{4} \eta_{AB} F^2) - \frac{\lambda}{2} \eta_{AB}$$

$$\nabla_A F_{AB} = 0 \qquad (90$$

These equations admit a solution consisting in the product of a maximally symmetric 4-dimensional space, with effective cosmological constant $\Lambda$, and a 2-sphere, provided $A_M$ assumes a monopole configuration in the internal space:

$$A_{\dot{M}}\, dx^{\dot{M}} = 0$$

$$A_{\tilde{M}}\, dy^{\tilde{M}} = \frac{n}{2e}(\cos\theta \pm 1)\, d\,\varphi \quad (91$$

with n an integer. The metric of $S^2$ is taken to be

$$g_{\tilde{M}\tilde{N}}\, dy^{\tilde{M}} dy^{\tilde{N}} = a^2(d\theta^{\,2} + \sin^2\theta\, d\varphi) \quad (92$$

The parameters a and $\Lambda$ must obey the equations:

$$-\Lambda = -\frac{n^2}{16\, e^2 a^4} + \frac{\lambda}{2} \qquad \frac{2}{a^2\kappa^2} = \frac{3n^2}{16\, e^2 a^4} + \frac{\lambda}{2} \quad (93$$

They admit a solution for $\Lambda$=0 if the parameters of the theory satisfy the relation:

$$e^2 = \frac{n^2}{8}\kappa^4\lambda \quad (94$$

which yields

$$a^2 = \frac{n^2}{8}\frac{\kappa^2}{e^2} \quad (95$$

The gauge invariance of the effective 4-dimensional theory corresponding to the ground state described above is given by $P^4$xSU(2)xU(1), where $P^4$ is the Poincaré group and SU(2) ~ SO(3) is the isometry group of $S^2$, while U(1) comes from the original Maxwell invariance.

An explicit calculation of the spectrum of the theory shows that it is stable, and that the massless excitations are given by the graviton and the gauge bosons of SU(2)xU(1) [23].

It is also possible, by using the harmonic expansion technique, to show that this background admits chiral fermions [23]. This can also be proved by the following argument [26].

As discussed in the previous section, in 6 dimensions the external and internal chirality are correlated. It is then sufficient to show that the number of zero-modes of the Dirac operator corresponding to left-handed and right-handed spinors on $S^2$ is different.

Let us call $\lambda$ the U(1) quantum number of a spinor on $S^2$. It is well known that in the harmonic expansion of a particle of "helicity" $\lambda$ on $S^2$ will appear once all states with $j = |\lambda|, |\lambda|+1, \ldots$, where j labels the SU(2) representation. Now, $\lambda$ takes the value $+\frac{1}{2}$ for states of positive chirality and $-\frac{1}{2}$ for states of negative chirality (*), so that spinors of different chirality have the same absolute value for $\lambda$ and there is a complete matching between their SU(2) representations.

But if a monopole of strength $e = \frac{1}{2} n$ with n integer is placed at the center of the sphere, the effective helicity $\lambda$ acquires an extra piece, $\lambda \rightarrow \lambda + e$, so that a fermion of chirality $\frac{1}{2}$ has effective helicity $e + \frac{1}{2}$ and a fermion of opposite chirality has effective helicity $e - \frac{1}{2}$. If, for example, $e > 0$, the allowed values of j are $j = e + \frac{1}{2}, e + \frac{3}{2}, \ldots$ for $\Gamma = +1$ and $j = e - \frac{1}{2}, e + \frac{1}{2}, e + \frac{3}{2} \ldots$ for $\Gamma = -1$. There is then one more state with $\Gamma = -1$. This state must be annihilated by the Dirac operator, since otherwise the Dirac operator should carry the states with $\Gamma = -1$, $j = e - \frac{1}{2}$ into states with $\Gamma = +1$, $j = e - \frac{1}{2}$ which are not present in the spectrum, and corresponds then to a zero mode.

This explicit example shows that massless chiral fermions can be present in the 4-dimensional spectrum if gauge fields in non-trivial topological configurations are present. In more general cases, some topological theorems exist that permit to know the difference between right- and left-handed zero modes of the Dirac operator in terms of the configuration of the gauge fields on any manifold [70].

## II .8 Non-compact internal spaces

As we have already mentioned, another possibility of escaping Witten theorem on chiral fermions is to consider noncompact internal manifolds with finite volume admitting a compact isometry group [29,71]. The

(*) We define the chirality of a spinor on $S^2$ as the eigenvalue of the operator $\Gamma = i\Gamma_1$, with $\Gamma_i$ defined in (87).

finiteness of the volume is necessary to obtain finite gauge couplings and to justify the unobservability of the internal space.

To see how this can happen, let us consider a simple model [71]. Let the internal space be a 2-dimensional manifold admitting a U(1) isometry group. Let us parametrize the manifold by the coordinates r and $\varphi$ ,with $0<r<\infty$ and $0\leq \varphi \leq 2\pi$

The most general U(1) invariant metric is given by:

$$ds^2 = f^2(r)\,(dr^2 + r^2\, d\theta^2) \qquad (96$$

Depending on f(r) this metric can describe a compact or a non-compact manifold. For example, if

$$f(r) = \frac{2}{1+r^2} \qquad (97$$

one obtains the metric of a 2-sphere.

The volume of the space is given by:

$$V = 2\pi \int_0^\infty dr\, r\, f^2(r) \qquad (98$$

which is finite if f(r) decreases faster than $r^{-1}$ for large r.

In two dimensions the spinor field have dimension 2. We choose the basis:

$$\Gamma^1 = i\,\sigma^1 \qquad\qquad \Gamma^2 = i\,\sigma^2 \qquad (99$$

We can then define $\Gamma^3 = i\,\Gamma^1\Gamma^2$ and $\Sigma^{12} = -\frac{1}{4}\,[\Gamma^1, \Gamma^2]$ , or explicitly:

$$\Gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad\qquad \Sigma^{12} = \frac{i}{2}\,\Gamma^3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad (100$$

A bidimensional Dirac spinor can then be written

$$\psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \qquad (101$$

where $\psi^+$ and $\psi^-$ are Weyl spinors, i.e. eigenstates of $\Gamma^3$.

We can now expand the spinors in terms of the harmonics, i.e. eigenstates of the charge operator Q of the U(1) isotropy group, which in a suitable basis takes the form $Q = -i\partial_\varphi$:

$$\psi^+_n = \exp[\, i\,(n + \tfrac{1}{2})\,\varphi\,]\,\chi^+_n$$
$$\psi^-_n = \exp[\, i\,(n + \tfrac{1}{2})\,\varphi\,]\,\chi^-_n \qquad (102$$

$$Q\,\psi^\pm_n = (n + \tfrac{1}{2})\;\psi^\pm_n \qquad (103$$

with n integer.

The zero-modes $\Gamma^A \nabla_A$ obey the equations:

$$\left(\partial_r + \frac{1}{2} f^{-1}\partial_r f - \frac{n}{2}\right)\chi^+_n = 0 \qquad (104$$

$$\left(\partial_r + \frac{1}{2} f^{-1}\partial_r f + \frac{n+1}{2}\right)\chi^-_n = 0 \qquad (105$$

By charge conjugation, every zero-mode in $\psi^+$ with charge Q corresponds to a zero-mode in $\psi^-$ with charge -Q.

Four-dimensional chirality will then result if the harmonic expansion of $\psi^+$ contains normalizable zero-modes with charge Q, but no normalizable zero-mode with charge -Q. This gives rise in 4 dimensions to two massless left-handed spinors with charge Q, one from $\psi^+$ and the other from $\psi^-$.

Eqn. (104) can be easily solved:

$$\chi^+_n = a^+ \, r^n \, f(r)^{-1/2} \qquad (106$$

Normalizability of zero-modes requires

$$\int_0^\infty dr\, r\, f^2(r)|\chi|^2 = \int_0^\infty dr\, r^{2n+1}\, f(r) < \infty \qquad (107$$

which, for f(r) going to infinity as $r^{-N}$ and to a constant for r=0, is satisfied if N>2 and $0 \le n < \frac{1}{2}(N-2)$.

For the sphere N=2 and no zero-mode exist, in accordance to Lichnerowicz theorem. For N=3, on the contrary, one has a non-compact internal space with finite volume and two chiral fermions with charge $Q = \frac{1}{2}$.

Another important property of noncompact internal spaces admitting a compact group of isometries is that they are solutions of the Einstein equations with arbitrary 4-dimensional cosmological constant, which can then be put to zero [28,72]. In general, however, these solutions have not a direct product structure, but are of the form:

$$g_{MN} = \begin{bmatrix} \sigma^2(r)\,\eta_{\dot{M}\dot{N}} & 0 \\ 0 & f^2(r)\,\eta_{\tilde{M}\tilde{N}} \end{bmatrix} \qquad (108$$

The "warp factor" $\sigma(r)$ is nevertheless compatible with the Poincaré invariance of the ground state.

It is also important to point out that, contrary to what one could expect, the spectrum of masses arising from this compactification is in general not continuous, and can admit a finite mass gap between the zero-modes and the first excited state [73].

# III . QUASI-RIEMANNIAN THEORIES OF GRAVITATION

Quasi-Riemannian geometries were introduced by Weinberg in 1983 as a possible solution to the problems which plagued the orthodox Kaluza-Klein theories [30]. He observed that there is no physical reason why the higher-dimensional manifold of Kaluza-Klein theories should have the same Riemannian structure as the four dimensional spacetime of general relativity. In particular he proposed to consider the case of a tangent space group different from the usual D-dimensional orthogonal group.

However, some conditions must be imposed in order to obtain a sensible 4-dimensional effective theory [30]. First of all, the D-dimensional tangent space group must contain the 4-dimensional Lorentz group O(1,3), in such a way that the defining representation of $G_T$ breaks up under O(1,3) into a simple 4-vector and (D-4) 4-scalars. Moreover, the tangent group should admit spinorial representations.

It can be shown [36] that the only choice compatible with these requirements is a group $O(N_1, N_2) \times G_T'$ with $G_T' \subseteq GL(D-N_1-N_2)$ and $N_1 \geq 1$, $N_2 \geq 3$. In order to avoid closed time-like paths, which can lead to violations of causality, one must further require that $N_1 = 1$, so that the only physically accettable structure for the tangent space group is $G_T = O(1,N-1) \times G_T'$, with $N \geq 4$ and $G_T' \subseteq GL(D-N)$.

In [34] was also observed that a possible justification for the choice of these seemingly arbitrary groups may come from supergravity: as is well known, in fact, supergravity theories can be constructed starting from a Lie superalgebra. But the bosonic part of the superalgebra has always a direct sum structure [74], corresponding to a group of the kind discussed above.

In the following we shall concentrate our attention on the simplest case of non-trivial tangent group, namely $G_T = O(1,N-1) \times O(M)$ with $M = D-N$.

We shall denote a D-dimensional manifold with such tangent group as $Q^{N,M}$.

## III .1 The action

The group $O(1,N-1) \times O(M)$ can be defined as the group of D x D matrices

$\Lambda^A{}_B$ which leave invariant the two matrices (*)

$$\eta^{I}{}_{AB} = \begin{bmatrix} \eta_{\alpha\beta} & 0 \\ 0 & 0 \end{bmatrix} \qquad \eta^{II}{}_{AB} = \begin{bmatrix} 0 & 0 \\ 0 & \eta_{ab} \end{bmatrix} \tag{1}$$

where $\eta_{\alpha\beta}$ and $\eta_{ab}$ are two arbitrary real symmetric non-singular matrices of dimension NxN and MxM respectively, with signature (-1,1,...,1) and (1,...,1). This means that

$$\Lambda^A{}_C \ \Lambda^B{}_D \ \eta^{I(II)}{}_{AB} = \eta^{I(II)}{}_{CD} \tag{2}$$

For definiteness, in the following we shall put $\eta_{\alpha\beta}$ = diag (-1,1,...,1) and $\eta_{ab}$ = diag (1,...,1). Moreover, we shall rise and lower the indices by means of the metric $\eta_{AB} = \eta^{I}{}_{AB} + \eta^{II}{}_{AB}$, which is obviously $G_T$-invariant being a linear combination of $\eta^{I}$ and $\eta^{II}$.

From the definition it is evident that $G_T$ is the subgroup of the pseudoorthogonal group O (1,D-1) formed by block diagonal matrices with $\Lambda^\alpha{}_a = \Lambda^a{}_\alpha = 0$.

By consequence, the Lie algebra of $G_T$ is formed by two sets of antisymmetric matrices, which respectively mix the greek indices and the roman ones.

It follows that the components of the spin connection with mixed indices are vanishing:

$$\omega^\alpha{}_a = \omega^a{}_\alpha = 0 \tag{3}$$

A consequence of this fact is that some of the components of the torsion are by definition functions only of the vielbeins, i.e.

$$\begin{array}{ll} T_{a\beta\gamma} = c_{a\beta\gamma} & T_{\alpha bc} = c_{\alpha bc} \\ T_{(\alpha\beta)c} = c_{(\alpha\beta)c} & T_{(ab)\gamma} = c_{(ab)\gamma} \end{array} \tag{4}$$

Therefore, if one imposes vanishing torsion, some non-trivial constraints are imposed on the vielbeins and their derivatives, in accordance with the

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(*) For the notation see appendix A.

general discussion of section I.3.

For this reason and also because, as shown in [34,37] and in appendix B, torsion is necessary for a Kaluza-Klein interpretation of the theory, we shall consider the more general case of nonvanishing torsion.

Our purpose is now to write down a suitable action for a gravitational theory on a quasi-Riemannian manifold with tangent group $G_T = O(1,N-1) \times O(M)$.

As in the Riemannian case, we demand that the action be a function of the vielbein and the connection, invariant under coordinate transfomations on the manifold:

$$e^A{}_M \rightarrow \frac{\partial z^N}{\partial z'^M} e^A{}_N \qquad\qquad \omega^A{}_{BM} \rightarrow \frac{\partial z^N}{\partial z'^M} \omega^A{}_{BN} \tag{5}$$

and under local tangent group transformations:

$$\begin{aligned} e^\alpha{}_M &\rightarrow \Lambda^\alpha{}_\beta e^\beta{}_M & \omega^\alpha{}_{\beta M} &\rightarrow \Lambda^\alpha{}_\gamma \omega^\gamma{}_{\delta M} (\Lambda^{-1})^\delta{}_\beta + \Lambda^\alpha{}_\gamma \partial_M (\Lambda^{-1})^\gamma{}_\beta \\ e^a{}_M &\rightarrow \Lambda^a{}_b e^b{}_M & \omega^a{}_{bM} &\rightarrow \Lambda^a{}_c \omega^c{}_{dM} (\Lambda^{-1})^d{}_b + \Lambda^a{}_c \partial_M (\Lambda^{-1})^c{}_b \end{aligned} \tag{6}$$

Furthermore, we require that the field equations be second order in the vielbein. In order to achieve this, we proceed as in the Palatini formalism of general relativity (see section I.6) and consider an action linear in the curvature and quadratic in the torsion. In this case, however, due to the peculiar structure of the tangent space, one has two possible invariant tensors by which he can contract the indices, and the resulting action has the more complicate form [41]:

$$S = -\frac{1}{\kappa^2}\int e_D\, d^Dz\, [\, \alpha R_{\alpha\beta\beta\alpha} + \beta R_{abba} + \lambda$$

$$+ a_1 T_{\alpha\beta\gamma} T_{\alpha\beta\gamma} + a_2 T_{\alpha\beta\gamma} T_{\beta\alpha\gamma} + a_3 T_{\alpha\alpha\gamma} T_{\beta\beta\gamma}$$

$$+ b_1 T_{abc} T_{abc} + b_2 T_{abc} T_{bac} + b_3 T_{aac} T_{bbc}$$

$$+ c_1 T_{\alpha\beta c} T_{\alpha\beta c} + c_2 T_{\alpha\beta c} T_{\beta\alpha c} + c_3 T_{\alpha\alpha c} T_{\beta\beta c} \quad (7$$

$$+ d_1 T_{ab\gamma} T_{ab\gamma} + d_2 T_{ab\gamma} T_{ba\gamma} + d_3 T_{aa\gamma} T_{bb\gamma}$$

$$+ e_1 T_{a\beta\gamma} T_{a\beta\gamma} + e_2 T_{a\beta\gamma} T_{\beta a\gamma} + e_3 T_{aa\gamma} T_{\beta\beta\gamma}$$

$$+ f_1 T_{\alpha bc} T_{\alpha bc} + f_2 T_{\alpha bc} T_{b\alpha c} + f_3 T_{\alpha\alpha c} T_{bbc}\, ]$$

where $e_D = |\det e^A{}_M|$, $\kappa^2$ is a dimensional constant proportional to the d-dimensional gravitational constant, and $\alpha, \beta, \lambda, a_1, \ldots, f_3$ are 21 free parameters.

Contrary to what one may think, the density $e_D$ is uniquely determined [35]. In fact, the most general expression one can choose is

$$e_D = |\det e^\alpha{}_M + \eta\, e^a{}_M| \quad (8$$

where $\eta$ is an arbitrary constant parameter. One has however:

$$|\det [\, e_B{}^M ( e^\alpha{}_M + \eta\, e^a{}_M )]\,| = |\det e^B{}_M|^{-1}\, |\det ( e^\alpha{}_M + \eta\, e^a{}_M)| =$$

$$= |\det ( \delta^\alpha{}_\beta + \eta\, \delta^a{}_b)| = |\eta|^M \quad (9$$

which yields

$$|\det e^\alpha{}_M + \eta\, e^a{}_M| = |\eta|^M\, |\det e^A{}_M| \quad (10$$

Then, apart from an ininfluent constant, $e_D$ is unique.

As one may suspect by examining equations (4), this is not the most economical action one can write, in the sense that some of the parameters are in fact redundant. This can be seen after varying with respect to the connection components to obtain the field equations in absence of spinning sources:

$$(\alpha-2a_2)\, T_{\gamma\alpha\beta} + (4a_1-2a_2)\, T_{[\alpha\beta]\gamma} - 2(\alpha+a_3)\, \eta_{\gamma[\alpha|}T_{\delta\delta|\beta]}$$

$$- (e_3+2\alpha)\, \eta_{\gamma[\alpha|}T_{dd|\beta]} = 0$$

$$(\beta-2b_2)\, T_{cab} + (4b_1-2b_2)\, T_{[ab]c} - 2(\beta+b_3)\, \eta_{c[a|}T_{dd|b]}$$

$$- (f_3+2\beta)\, \eta_{c[a|}T_{\delta\delta|b]} = 0$$

(11

$$(\alpha-e_2)\, T_{c\alpha\beta} + 2(c_1-c_2)\, T_{[\alpha\beta]c} = 0$$

$$(\beta-f_2)\, T_{\gamma ab} + 2(d_1-d_2)\, T_{[ab]\gamma} = 0$$

They admit the solution:

$$\omega_{\alpha\beta\gamma} = L_{\alpha\beta\gamma} + \chi\, \eta_{\gamma[\alpha|}C_{dd|\beta]} \qquad \omega_{abc} = L_{abc} + \chi'\, \eta_{c[a|}C_{\delta\delta|b]}$$

(12

$$\omega_{\alpha\beta c} = \frac{\alpha-e_2}{2(c_1-c_2)}\, C_{c\alpha\beta} + C_{[\alpha\beta]c} \qquad \omega_{ab\gamma} = \frac{\beta-f_2}{2(d_1-d_2)}\, C_{\gamma ab} + C_{[ab]\gamma}$$

where

$$\chi = -\frac{2\alpha + e_3}{(N-2)\,\alpha + 2a_1 + a_2 + (N-1)\, a_3}$$

(13

$$\chi' = -\frac{2\beta + f_3}{(M-2)\,\beta + 2b_1 + b_2 + (M-1)\, b_3}$$

and $L_{ABC}$ is defined in ( I.30).

By means of (12) one can now express the torsion in terms of the vielbeins:

$$T_{\alpha\beta\gamma} = \chi\, \eta_{\alpha[\beta|}c_{dd|\gamma]} \qquad T_{abc} = \chi'\, \eta_{a[b|}c_{\delta\delta|c]}$$

$$T_{\alpha\beta c} = c_{(\alpha\beta)c} - \frac{\alpha - e_2}{2(c_1 - c_2)}\, c_{c\alpha\beta} \qquad T_{ab\gamma} = c_{(ab)\gamma} - \frac{\beta - f_2}{2(d_1 - d_2)}\, c_{\gamma ab} \tag{14}$$

The solutions (12,14) can now be substituted back into the action, together with (4) to read

$$\begin{aligned} S = -\frac{1}{\kappa^2}\int e_D\, d^Dz\, \{ & A\,[-2\partial_\alpha c_{\beta\beta\alpha} + \tfrac{1}{4}\, c_{\alpha\beta\gamma}c_{\alpha\beta\gamma} - \tfrac{1}{2}\, c_{\alpha\beta\gamma}c_{\beta\gamma\alpha} + c_{\alpha\alpha\gamma}c_{\beta\beta\gamma} \\ & - c_{a\beta\gamma}c_{\beta\gamma a}] + C\, c_{(\alpha\beta)c}\, c_{(\alpha\beta)c} + \tfrac{1}{4} E\, c_{a\beta\gamma}c_{a\beta\gamma} + G\, c_{\alpha\alpha c}\, c_{\alpha\alpha c} \\ & + B\,[-2\,\partial_a c_{bba} + \tfrac{1}{4}\, c_{abc}c_{abc} - \tfrac{1}{2}\, c_{abc}c_{bca} + c_{aac}c_{bbc} \\ & - c_{\alpha bc}c_{bc\alpha}] + D\, c_{(ab)\gamma}c_{(ab)\gamma} + \tfrac{1}{4} F\, c_{\alpha bc}c_{\alpha bc} + H\, c_{aa\gamma}c_{aa\gamma} + \lambda\} \end{aligned} \tag{15}$$

where

$$A = \alpha \qquad B = \beta$$

$$C = c_1 + c_2 \qquad D = d_1 + d_2$$

$$E = 4e_1 - \frac{(\alpha - e_2)^2}{(c_1 - c_2)} \qquad F = 4f_1 - \frac{(\beta - f_2)^2}{(d_1 - d_2)} \tag{16}$$

$$G = c_3 - \frac{M-1}{4}\,\frac{(2\beta + f_3)^2}{(M-2)\,\beta + 2b_1 + b_2 + (M-1)b_3}$$

$$H = d_3 - \frac{N-1}{4}\,\frac{(2\alpha + e_3)^2}{(N-2)\,\alpha + 2a_1 + a_2 + (N-1)\,a_3}$$

The action depends now only on 9 parameters instead of the original 21. This is reminiscent of what happens in general relativity (see section I.6), where the addition to the Einstein-Hilbert action of terms quadratic in the torsion has no influence on the dynamics of the theory, so that some of the parameters present in the original action are redundant.

After integration by parts, eqn. (15) can be written as

$$S = -\frac{1}{\kappa^2}\int e_D\, d^Dz\, \{A\, [\, \tfrac{1}{4}\, c_{\alpha\beta\gamma}\, c_{\alpha\beta\gamma} - \tfrac{1}{2}\, c_{\alpha\beta\gamma}\, c_{\beta\gamma\alpha} - c_{\alpha\alpha\gamma}\, c_{\beta\beta\gamma} - 2\, c_{a a\gamma}\, c_{\beta\beta\gamma}$$

$$- c_{a\beta\gamma}\, c_{\beta\gamma a}\, ] + C\, c_{(\alpha\beta)c}\, c_{(\alpha\beta)c} + \tfrac{1}{4}\, E\, c_{a\beta\gamma}\, c_{a\beta\gamma} + G\, c_{\alpha\alpha c}\, c_{\alpha\alpha c}$$

$$+ B\, [\, \tfrac{1}{4}\, c_{abc}\, c_{abc} - \tfrac{1}{2}\, c_{abc}\, c_{bca} - c_{aac}\, c_{bbc} - 2\, c_{\alpha\alpha c}\, c_{bbc} \qquad (17$$

$$- c_{\alpha bc}\, c_{bc\alpha}\, ] + D\, c_{(ab)\gamma}\, c_{(ab)\gamma} + \tfrac{1}{4}\, F\, c_{\alpha bc}\, c_{\alpha bc} + H\, c_{aa\gamma}\, c_{aa\gamma} + \lambda\}$$

Comparing this form of the action with ( I.42) consents to check easily that the Riemannian limit is obtained if $A = B = C = D = E = F = -G = -H$.

### III .2 Alternative formulations.

An alternative way of constructing an action for our theory was proposed by de Alwis and Randjbar-Daemi [38] and in an equivalent way, but in tensorial language, by Weinberg [35]. In these formulations, only the vielbein is considered as an independent variable, while the connection is defined a priori as a function of the vielbein. It can be shown [35] that in this way one obtains the most general action formulated in terms of the vielbein which satisfies the general requirements of last section. Nevertheless it is less general than ours, because it imposes some unnecessary constraints on the form of the connection. However, the two formulations are equivalent at second order (i.e. when written in terms of the vielbeins only) at least if no matter coupling is present.

The authors of ref. [38] start by considering a D-dimensional Riemannian manifold with tangent group O(1,D-1) and connection $\Omega^A{}_B$. The connection $\Omega^A{}_B$ is by definition a one-form which takes values in the Lie algebra of O(1,D-1) and can be written as

$$\Omega^A{}_B = \omega^A{}_B + \varpi^A{}_B \qquad (18$$

were $\omega^A{}_B$ is in the Lie algebra of $G_T = O(1,N-1) \times O(M)$ and $\varpi^A{}_B$ is in the complementary subspace. This means that $\omega^a{}_\alpha = \omega^\alpha{}_a = 0$, whereas $\varpi^\alpha{}_\beta = \varpi^a{}_b = 0$.

Under $G_T$, $\omega$ transforms as a connection, whereas $\varpi$ transforms covariantly.

In order to obtain an expression for $\omega$ and $\varpi$ in terms of the vielbein, one imposes that $\Omega$, as an $O(1,D-1)$ connection, is torsion free, i.e.

$$de^A + \Omega^A{}_B \wedge e^B = 0 \tag{19}$$

This gives the usual expression for $\Omega_{AB} = \Omega_{ABC} \wedge e^C$:

$$\Omega_{ABC} = L_{ABC} \tag{20}$$

Eqns. (17) and (19) permit to express $\omega$ and $\varpi$ as functions of the vielbeins. The $G_T$ connection $\omega$ possesses a nonvanishing torsion. This is evident if one writes (19) as

$$T^A = de^A + \omega^A{}_B \wedge e^B = - \varpi^A{}_B \wedge e^B \tag{21}$$

It is now easy to see that in this formalism, the most general action invariant under coordinate and $G_T$ transformations which leads to second order differential equations for the vielbeins can be written as

$$\begin{aligned} S = -\frac{1}{\kappa^2}\int e_D\, d^Dz\, [\, & \lambda_1 R_{\alpha\beta\beta\alpha} + \lambda_2 R_{abba} + \lambda \\ & + \lambda_3 \varpi_{a[\beta\gamma]} \varpi_{a[\beta\gamma]} + \lambda_4 \varpi_{a(\beta\gamma)} \varpi_{a(\beta\gamma)} + \lambda_5 \varpi_{a\beta\beta} \varpi_{a\gamma\gamma} \\ & + \lambda_6 \varpi_{\alpha[bc]} \varpi_{\alpha[bc]} + \lambda_7 \varpi_{\alpha(bc)} \varpi_{\alpha(bc)} + \lambda_8 \varpi_{\alpha bb} \varpi_{\alpha cc} \,] \end{aligned} \tag{22}$$

where $\lambda_1,\ldots,\lambda_8, \lambda$ are nine arbitrary constants.

If one now substitutes in (22) the expressions (20) for $\omega$ and $\varpi$ as functions of the vielbein, one recovers the equation (15), where now:

$$A = \lambda_1 \qquad B = \lambda_2$$

$$C = \lambda_3 \qquad D = \lambda_7 \tag{23}$$

$$E = \lambda_3 + 2\lambda_1 \qquad F = \lambda_6 + 2\lambda_2$$

$$G = \lambda_5 \qquad H = \lambda_8$$

A refined version of this formalism was proposed by Viswanathan and Wong [39]. They adopted an action very similar to that of ref. [35]:

$$S = -\frac{1}{\kappa^2}\int e_D\, d^Dz\, [\, \lambda_1 R_{\alpha\beta\beta\alpha} + \lambda_2 R_{abba} + \lambda_9\, r_{\alpha b\alpha b} + \lambda$$

$$+ \lambda_3\, \varpi_{a[\beta\gamma]}\, \varpi_{a[\beta\gamma]} + \lambda_4\, \varpi_{a(\beta\gamma)}\, \varpi_{a(\beta\gamma)} + \lambda_5\, \varpi_{a\beta\beta}\, \varpi_{a\gamma\gamma} \tag{24}$$

$$+ \lambda_6\, \varpi_{\alpha[bc]}\, \varpi_{\alpha[bc]} + \lambda_7\, \varpi_{\alpha(bc)}\, \varpi_{\alpha(bc)} + \lambda_8\, \varpi_{\alpha bb}\, \varpi_{\alpha cc}\, ]$$

but, instead of imposing a priori the form of $\omega$ and $\varpi$, they considered e, $\omega$ and $\varpi$ as independent variables. The field $\varpi$ is now introduced in the theory as an auxiliary variable which replaces the "missing" components of the connection. The term $r_{\alpha b\alpha b}$ with

$$r_{\alpha b\gamma d} = e_\alpha{}^M\, e_b{}^M\, [\, (\partial_M\, \varpi^\gamma{}_{dN} + \omega^\gamma{}_{\varepsilon M}\, \varpi^\varepsilon{}_{dN} - \omega^e{}_{dM}\, \varpi^\gamma{}_{eN}) - (M \leftrightarrow N)\, ] \tag{25}$$

is the field strength of the auxiliary field $\varpi$. It is not present in (22) because in that case, after integration by parts it can be reduced, by means of (21), to a term of the kind $\varpi^2$.

By varying the action with respect to $\omega$ and $\varpi$, one can obtain as usual the expression for these fields as functions of the vielbein and then substitute them back into the action obtaining again an expression like (15).

The fact that (15) can be obtained from many different theories induces to suspect that it is independent from the model chosen. It is in fact possible to check explicitly that it is the most general expression invariant under coordinate and local $G_T$ transformations containing at most two derivatives of the vielbein.

This can be seen as follows: first of all, the most general expression containing the first derivatives of the vielbein $\partial_M e_N$ must be anti-symmetric in the two world indices, in order to be covariant under coordinate transformations. It must then be constructed in terms of the anholonomity coefficients $c_{ABC}$, which can appear as squares $c_{ABC}\, c_{DEF}$, or as derivatives $\partial_A c_{BCD}$. By contracting in all possible ways these expressions with the two invariant $G_T$-tensors one obtains a list of possible terms. One must then check their invariance under local $G_T$-transformations, for which

$$\delta\, c_{ABC} = \varepsilon_{AD}(z)\, c_{DBC} + \varepsilon_{BD}(z)\, c_{ADC} + \varepsilon_{CD}(z)\, c_{ABD} + \partial_C\, \varepsilon_{AB}(z) - \partial_B\, \varepsilon_{AC}(z) \qquad (26$$

with $\varepsilon_{\alpha\beta} = \varepsilon_{\beta\alpha}$ ; $\varepsilon_{ab} = \varepsilon_{ba}$ ; $\varepsilon_{\alpha a} = \varepsilon_{a\alpha} = 0$ . It turns out that the only invariant combinations are (modulo integrations by parts) those present in the action (15).

### III .3 The field equations

Varying the action (12) with respect to the vielbein one can now obtain the field equations for the theory. For convenience, we have added to the action a matter part with lagrangian $\mathfrak{M}$ which contributes to the field equations with a term $t_{MN}$ defined as (*):

$$t_{MN} \equiv e^{-1} e_N{}^{\Pi} \frac{\delta\, e\,\mathfrak{M}}{\delta e^M{}_{\Pi}} \qquad (27$$

The ensuing field equations are very cumbersome. They can be written in the following way:

---

(*) In order to render more redeable the field equations, in this section we use the indices M,N (and m,n,$\mu$,$\nu$) as tangent space indices and $\Pi$ as a world index.

$$A\,\mathcal{T}^A{}_{MN} + B\,\mathcal{T}^B{}_{MN} + C\,\mathcal{T}^C{}_{MN} + D\,\mathcal{T}^D{}_{MN} + E\,\mathcal{T}^E{}_{MN}$$

$$+ F\,\mathcal{T}^F{}_{MN} + G\,\mathcal{T}^G{}_{MN} + H\,\mathcal{T}^H{}_{MN} + \eta_{MN}\,\lambda = \kappa^2\,t_{MN} \qquad (28$$

where for example $\mathcal{T}^A{}_{MN}$ is obtained varying as in (27) the terms of the gravitational lagrangian proportional to the parameter A. We report here the components of the tensors $\mathcal{T}_{MN}$ :

$$\begin{aligned}\mathcal{T}^A{}_{\mu\nu} = {} & (-\partial_\beta + c_{\alpha\alpha\beta} + c_{aa\beta})(c_{\beta\mu\nu} - 2\,c_{(\mu\nu)\beta}) + 2\,\partial_\mu\, c_{\beta\beta\nu} \\ & + (-\partial_b + c_{\alpha\alpha b} + c_{aab})\, c_{b\mu\nu} - 2\,(-\partial_\mu + c_{aa\mu})\, c_{bb\nu} \\ & - 2\, c_{(\alpha\beta)\mu}\, c_{(\alpha\beta)\nu} - c_{a\beta\mu}\, c_{\beta a\nu} - c_{\beta a\mu}\, c_{a\beta\nu} + \tfrac{1}{2}\, c_{\mu\alpha\beta}\, c_{\nu\alpha\beta} \\ & + \eta_{\mu\nu}\ [\, -2\,\partial_\beta (c_{\alpha\alpha\beta} + c_{aa\beta}) + \tfrac{1}{4}\, c_{\alpha\beta\gamma}\, c_{\alpha\beta\gamma} - \tfrac{1}{2}\, c_{\alpha\beta\gamma}\, c_{\beta\gamma\alpha} \\ & + c_{\alpha\alpha\gamma}\, c_{\beta\beta\gamma} + 2\, c_{aa\gamma}\, c_{bb\gamma} + 2\, c_{aa\gamma}\, c_{\beta\beta\gamma} - c_{a\beta\gamma}\, c_{\beta\gamma a}\, ]\end{aligned}$$

$$\begin{aligned}\mathcal{T}^B{}_{\mu\nu} = {} & \eta_{\mu\nu}\,(-2\,\partial_a\, c_{bba} + \tfrac{1}{4}\, c_{abc}\, c_{abc} - \tfrac{1}{2}\, c_{abc}\, c_{bca} + c_{aac}\, c_{bbc} \\ & - c_{\alpha bc}\, c_{bc\alpha})\end{aligned}$$

$$\begin{aligned}\mathcal{T}^C{}_{\mu\nu} = {} & -2\,(-\partial_b + c_{\alpha\alpha b} + c_{aab})\, c_{(\mu\nu)b} + c_{\mu\beta c}\, c_{\nu\beta c} - c_{\alpha b\mu}\, c_{\alpha b\nu} \\ & + \eta_{\mu\nu}\, c_{(\alpha\beta)c}\, c_{(\alpha\beta)c}\end{aligned}$$

$$\mathcal{T}^D{}_{\mu\nu} = -2\, c_{(ab)\mu}\, c_{(ab)\nu} + \eta_{\mu\nu}\, c_{(ab)\gamma}\, c_{(ab)\gamma}$$

$$\mathcal{T}^E{}_{\mu\nu} = -\, c_{a\beta\mu}\, c_{a\beta\nu} + \eta_{\mu\nu}\,(\tfrac{1}{4}\, c_{a\beta\gamma}\, c_{a\beta\gamma})$$

$$\mathcal{T}^F{}_{\mu\nu} = \tfrac{1}{2}\, c_{\mu bc}\, c_{\nu bc} + \eta_{\mu\nu}\,(\tfrac{1}{4}\, c_{\alpha bc}\, c_{\alpha bc})$$

$$\mathcal{T}^G{}_{\mu\nu} = -\,\eta_{\mu\nu}\,(-2\,\partial_c + 2\, c_{aac} + c_{\alpha\alpha c})\, c_{\beta\beta c}$$

$$\mathcal{T}^H{}_{\mu\nu} = -2\, c_{aa\mu}\, c_{bb\nu} + \eta_{\mu\nu}\, c_{aa\gamma}\, c_{bb\gamma}$$

$$\mathfrak{T}^{A}{}_{\mu n} = -(-\partial_\beta + c_{\alpha\alpha\beta} + 3\, c_{aa\beta})\, c_{n\mu\beta} - c_{a\beta\mu}\, c_{na\beta} + \tfrac{1}{2}\, c_{\mu\alpha\beta}\, c_{n\alpha\beta}$$

$$\mathfrak{T}^{B}{}_{\mu n} = -(-\partial_b - c_{\alpha\alpha b} + c_{aab})\, c_{n\mu b} + (-\partial_b + c_{\alpha\alpha b} + c_{aab})\, c_{b\mu n} + 2\,\partial_\mu\, c_{aan}$$
$$+ 2\, c_{aa\mu}\, c_{\beta\beta n} - 2\, c_{(ab)\mu}\, c_{(ab)\nu} - c_{a\beta\mu}\, c_{\beta an} - c_{\beta a\mu}(c_{a\beta n} + c_{n\beta a})$$

$$\mathfrak{T}^{C}{}_{\mu n} = (-\partial_\beta + c_{\alpha\alpha\beta} + c_{aa\beta})(c_{\beta\mu n} - c_{\mu n\beta}) + c_{\mu\alpha b}\, c_{n\alpha b} - c_{\beta a\mu}\, c_{n\beta a}$$
$$- 2\, c_{(\alpha\beta)\mu}\, c_{(\alpha\beta)n}$$

$$\mathfrak{T}^{D}{}_{\mu n} = c_{a\beta\mu}\, c_{na\beta} - c_{a\beta\mu}\, c_{a\beta n}$$

$$\mathfrak{T}^{E}{}_{\mu n} = 0$$

$$\mathfrak{T}^{F}{}_{\mu n} = -(-\partial_b + c_{\alpha\alpha b} + c_{aab})\, c_{\mu nb} + \tfrac{1}{2}\, c_{\mu ab}\, c_{nab} - c_{\alpha b\mu}\, c_{\alpha bn}$$

$$\mathfrak{T}^{G}{}_{\mu n} = 2\,[(-\partial_\mu + c_{aa\mu})\, c_{\beta\beta n} + c_{\alpha\alpha b}\, c_{n\mu b}]$$

$$\mathfrak{T}^{H}{}_{\mu n} = -2\, c_{aa\beta}\, c_{n\mu\beta}$$

$$\mathfrak{T}^{A}{}_{mn} = \eta_{mn}(-2\,\partial_\alpha\, c_{\beta\beta\alpha} + \tfrac{1}{4}\, c_{\alpha\beta\gamma}\, c_{\alpha\beta\gamma} - \tfrac{1}{2}\, c_{\alpha\beta\gamma}\, c_{\beta\gamma\alpha} + c_{\alpha\alpha\gamma}\, c_{\beta\beta\gamma} - c_{a\beta\gamma}\, c_{\beta\gamma a})$$

$$\mathfrak{T}^{B}{}_{mn} = (-\partial_b + c_{\alpha\alpha b} + c_{aab})(c_{bmn} - 2\, c_{(mn)b}) + 2\,\partial_m\, c_{aan}$$
$$+ (-\partial_\beta + c_{\alpha\alpha\beta} + c_{aa\beta})\, c_{\beta mn} - 2\,(-\partial_m + c_{\alpha\alpha m})\, c_{\beta\beta n}$$
$$- 2\, c_{(ab)m}\, c_{(ab)n} - c_{\beta am}\, c_{a\beta n} - c_{a\beta m}\, c_{\beta an} + \tfrac{1}{2}\, c_{mab}\, c_{nab}$$
$$+ \eta_{mn}[-2\,\partial_a(c_{bba} + c_{\beta\beta a}) + \tfrac{1}{4}\, c_{abc}\, c_{abc} - \tfrac{1}{2}\, c_{abc}\, c_{bca}$$
$$+ c_{aac}\, c_{bbc} + 2\, c_{\alpha\alpha c}\, c_{bbc} + 2\, c_{\alpha\alpha c}\, c_{\beta\beta c} - c_{\alpha bc}\, c_{bc\alpha}]$$

$$\mathfrak{T}^{C}{}_{mn} = -2\, c_{(\alpha\beta)m}\, c_{(\alpha\beta)n} + \eta_{mn}\, c_{(\alpha\beta)c}\, c_{(\alpha\beta)c}$$

$$\mathfrak{T}^{D}{}_{mn} = -\,2\,(-\,\partial_\beta + c_{\alpha\alpha\beta} + c_{aa\beta})\,c_{(mn)\beta} + c_{mb\alpha}\,c_{nb\alpha} - c_{a\beta m}\,c_{a\beta n} + \eta_{mn}\,c_{(ab)\gamma}\,c_{(ab)\gamma}$$

$$\mathfrak{T}^{E}{}_{mn} = \tfrac{1}{2}\,c_{m\beta\gamma}\,c_{n\beta\gamma} + \eta_{mn}\,(\tfrac{1}{4}\ c_{a\beta\gamma}\,c_{a\beta\gamma})$$

$$\mathfrak{T}^{F}{}_{mn} = -\,c_{\alpha bm}\,c_{\alpha bn} + \eta_{mn}\,(\tfrac{1}{4}\,c_{\alpha bc}\,c_{\alpha bc})$$

$$\mathfrak{T}^{G}{}_{mn} - 2\,c_{\alpha\alpha m}\,c_{\beta\beta n} + \eta_{mn}\,c_{\alpha\alpha c}\,c_{\beta\beta c}$$

$$\mathfrak{T}^{H}{}_{mn} = -\,\eta_{mn}\,(-\,2\,\partial_\beta + 2\,c_{\alpha\alpha\beta} + c_{aa\beta})\,c_{cc\beta}$$

$$\mathfrak{T}^{A}{}_{m\nu} = (-\,\partial_\beta + c_{\alpha\alpha\beta} - c_{aa\beta})\,c_{\nu m\beta} + (-\,\partial_\beta + c_{\alpha\alpha\beta} + c_{aa\beta})\,c_{\beta m\nu} + 2\,\partial_m\,c_{\alpha\alpha\nu} + 2\,c_{\alpha\alpha m}\,c_{bb\nu} - 2\,c_{(\alpha\beta)m}\,c_{(\alpha\beta)\nu} - c_{a\beta m}\,c_{\beta a\nu} - c_{\beta am}\,c_{a\beta\nu} + c_{a\beta m}\,c_{\nu a\beta}$$

$$\mathfrak{T}^{B}{}_{m\nu} = (-\,\partial_b + 3\,c_{\alpha\alpha b} + c_{aab})\,c_{\nu mc} - c_{\beta am}\,c_{\nu\beta a} + \tfrac{1}{2}\,c_{mab}\,c_{\nu ab}$$

$$\mathfrak{T}^{C}{}_{m\nu} = +\,c_{\alpha bm}\,c_{\nu\alpha b} - c_{\alpha bm}\,c_{\alpha b\nu}$$

$$\mathfrak{T}^{D}{}_{m\nu} = (-\,\partial_b + c_{\alpha\alpha b} + c_{aab})\,(c_{bm\nu} - c_{m\nu b}) + c_{mb\alpha}\,c_{\nu b\alpha} - c_{a\beta m}\,c_{na\beta} - 2\,c_{(ab)m}\,c_{(ab)\nu}$$

$$\mathfrak{T}^{E}{}_{m\nu} = -\,(-\,\partial_\beta + c_{\alpha\alpha\beta} + c_{aa\beta})\,c_{m\nu\beta} + \tfrac{1}{2}\,c_{m\alpha\beta}\,c_{\nu\alpha\beta} - c_{a\beta m}\,c_{a\beta\nu}$$

$$\mathfrak{T}^{F}{}_{m\nu} = 0$$

$$\mathfrak{T}^{G}{}_{m\nu} = -\,2\,c_{\alpha\alpha b}\,c_{\nu mb}$$

$$\mathfrak{T}^H{}_{m\nu} = 2 [ ( - 2 \partial_m + c_{\alpha\alpha m} ) c_{bb\nu} + c_{aa\beta} c_{\nu m\beta} ]$$

It is important to remark that the $\mathfrak{T}_{\mu n}$ are different from the $\mathfrak{T}_{n\mu}$. This asymmetry is characteristic of quasi-Riemannian theories and is a consequence of the structure of the tangent group.

An alternative procedure to obtain the field equations of the theory, which has the advantage of being explicitly covariant, is to adopt a first order formalism, and consider the system consisting in the equations (10), obtained by varying the original action (7) with respect to the connections, and the equations obtained by varying (7) with respect to the vielbein. As usual, the two procedures are equivalent as long as no source term is present in the field equations (10) for the connection. This second set of equations is reported here:

$$\begin{aligned}
&\alpha R_{\nu\alpha\alpha\mu} \\
&+ a_1 [ 2 (-\nabla_\beta + T_{\alpha\alpha\beta} + T_{aa\beta} ) T_{\mu\nu\beta} + 2 T_{\alpha\beta\mu} T_{\alpha\beta\nu} - T_{\mu\alpha\beta} T_{\nu\alpha\beta} ] \\
&+ a_2 [ 2 (-\nabla_\beta + T_{\alpha\alpha\beta} + T_{aa\beta} ) (T_{\nu\mu\beta} - T_{\beta\mu\nu} ) + T_{\alpha\beta\mu} T_{\beta\alpha\nu} ] \\
&- a_3 (-\nabla_\mu + T_{aa\mu} ) T_{\beta\beta\nu} \\
&+ c_1 [ (-\nabla_b + T_{\alpha\alpha b} + T_{aab} ) T_{\mu\nu b} + T_{\alpha b\mu} T_{\alpha b\nu} - T_{\mu\alpha b} T_{\nu\alpha b} ] \\
&+ c_2 (-\nabla_b + T_{\alpha\alpha b} + T_{aab} ) T_{\nu\mu b} \\
&+ d_1 T_{ab\mu} T_{ab\nu} + d_2 T_{ab\mu} T_{ba\nu} + d_1 T_{aa\mu} T_{bb\nu} \\
&+ 2 e_1 T_{a\beta\mu} T_{a\beta\nu} \\
&- \tfrac{1}{2} e_2 [ (-\nabla_b + T_{\alpha\alpha b} + T_{aab} ) T_{b\mu\nu} - T_{b\alpha\mu} T_{\alpha b\nu} - T_{\alpha b\mu} T_{b\alpha\nu} ] \\
&- \tfrac{1}{2} e_3 [ -\nabla_\mu T_{aa\nu} + T_{aa\mu} ( T_{bb\nu} - T_{\beta\beta\nu} ) ] \\
&- f_1 T_{\mu ab} T_{\nu ab} \\
&- \tfrac{1}{2} \mathfrak{Z} \eta_{\mu\nu} = - \tfrac{1}{2} \kappa^2 t_{\mu\nu}
\end{aligned} \qquad (29$$

where $t_{\mu\nu}$ is defined as before, and

$$
\begin{aligned}
\mathfrak{T} = {} & \alpha R_{\alpha\beta\beta\alpha} + \beta R_{abba} + \lambda \\
& + a_1 T_{\alpha\beta\gamma}T_{\alpha\beta\gamma} + a_2 T_{\alpha\beta\gamma}T_{\beta\alpha\gamma} - a_3 (-2\nabla_\gamma + T_{\alpha\alpha\gamma} + 2 T_{aa\gamma}) T_{\beta\beta\gamma} \\
& + b_1 T_{abc}T_{abc} + b_2 T_{abc}T_{bac} + b_3 T_{aac}T_{bbc} \\
& + c_1 T_{\alpha\beta c}T_{\alpha\beta c} + c_2 T_{\alpha\beta c}T_{\beta\alpha c} - c_3 (-2\nabla_c + T_{\alpha\alpha c} + 2 T_{aac}) T_{\beta\beta c} \\
& + d_1 T_{ab\gamma}T_{ab\gamma} + d_2 T_{ab\gamma}T_{ba\gamma} + d_3 T_{aa\gamma}T_{bb\gamma} \\
& + e_1 T_{a\beta\gamma}T_{a\beta\gamma} + e_2 T_{a\beta\gamma}T_{\beta a\gamma} - e_3 (\nabla_\gamma + T_{aa\gamma}) T_{bb\gamma} \\
& + f_1 T_{\alpha bc}T_{\alpha bc} + f_2 T_{\alpha bc}T_{b\alpha c} - f_3 (-\nabla_c + T_{aac}) T_{bbc}
\end{aligned} \tag{30}
$$

The equations referring to the mixed components of the energy-momentum tensor are given by:

$$
\begin{aligned}
& - a_1 T_{\mu\beta\gamma}T_{n\beta\gamma} + a_2 T_{\beta\gamma\mu}T_{n\beta\gamma} - a_3 T_{\beta\beta\gamma}T_{n\mu\gamma} \\
& + c_1 [ (-\nabla_\beta + T_{\alpha\alpha\beta} + T_{aa\beta}) T_{\mu n\beta} + T_{\alpha\beta\mu}T_{\alpha\beta n} - T_{\mu\alpha b}T_{n\alpha b} ] \\
& - c_2 [ (-\nabla_\beta + T_{\alpha\alpha\beta} + T_{aa\beta}) T_{\beta\mu n} - T_{\alpha\beta\mu}T_{\beta\alpha n} - T_{\beta a\mu} T_{n\beta a} ] \\
& - c_3 [ (-\nabla_\mu + T_{aa\mu}) T_{\beta\beta n} + T_{\alpha\alpha b} T_{n\mu b} ] \\
& + d_1 T_{a\mu\beta}T_{an\beta} - d_2 T_{a\beta\mu}T_{na\beta} + d_3 T_{n\mu\beta}T_{aa\beta} \\
& + \tfrac{1}{2} e_2 [ (-\nabla_\beta + T_{\alpha\alpha\beta} + T_{aa\beta}) T_{n\mu\beta} + T_{a\beta\mu}T_{na\beta} - T_{\alpha\beta\mu}T_{n\alpha\beta} ] \\
& + \tfrac{1}{2} e_3 ( T_{\alpha\alpha\beta} - T_{aa\beta} ) T_{n\mu\beta} \\
& + f_1 [ 2 (-\nabla_b + T_{\alpha\alpha b} + T_{aab}) T_{\mu nb} + 2 T_{\alpha b\mu}T_{\alpha bn} - T_{\mu ab}T_{nab} ] \\
& + \tfrac{1}{2} f_2 [ (-\nabla_b + T_{\alpha\alpha b} + T_{aab}) (T_{n\mu b} + T_{b\mu n}) + T_{\alpha b\mu} (T_{b\alpha n} - T_{n\alpha b}) \\
& \qquad + T_{b\alpha\mu} T_{\alpha bn} + T_{ab\mu} T_{nab} ) ] \\
& - \tfrac{1}{2} f_3 [ -\nabla_\mu T_{aan} + T_{aa\mu} ( T_{bbn} - T_{\beta\beta n} ) + T_{n\mu a} ( T_{bba} - T_{\beta\beta a} ) ] \\
& = - \tfrac{1}{2} \kappa^2 t_{\mu n}
\end{aligned} \tag{31}
$$

The other two field equations can be obtained by exchanging greek and latin indices and at the same time exchanging the parameters as follows:

$$\alpha \leftrightarrow \beta \qquad a_i \leftrightarrow b_i$$
$$c_i \leftrightarrow d_i \qquad e_i \leftrightarrow f_i \tag{32}$$

Also in this case the equations for $t_{\mu n}$ are different from those for $t_{n\mu}$.

It may be interesting to compare these equations with the analogous ones obtained in the standard Einstein-Cartan theory, which can be found for example in [75].

# IV . STABILITY OF THE FLAT SPACE IN QUASI-RIEMANNIAN THEORIES

We undertake now the study of the linearized theory [41]. This is essential in order to check the stability against small fluctuations and to obtain the particle content of the theory around a particular ground state [23]. We shall consider in particular the case of a flat background, and find some non-trivial conditions that the parameters of the theory must satisfy so that the flat space should be a stable ground state.

## IV .1 The linearized theory

We consider the expansion of the action (III.15) in small fluctuations around a background constituted by the flat D-dimensional Minkowski space. It can be easily verified that this is a solution of the field equations of section III.3 for $\lambda=0$. We shall therefore assume in the following that the cosmological constant $\lambda$ is vanishing.

We put:

$$e^A{}_M = \delta^A{}_M + \kappa h^A{}_M \qquad (1$$

and define

$$h_{AB} = \eta_{AC}\, \delta_B{}^M\, h^C{}_M \qquad (2$$

In general, $h_{AB} \neq h_{BA}$. We can now substitute this expression into the action. The terms linear in $h_{AB}$ vanish, while the bilinears govern the propagation of the weak disturbances. The bilinear action then reads:

$$S_{(2)} = -\int d^Dz\, [\, A\, \mathfrak{L}_A + B\, \mathfrak{L}_B + C\, \mathfrak{L}_C + D\, \mathfrak{L}_D + E\, \mathfrak{L}_E + F\, \mathfrak{L}_F + G\, \mathfrak{L}_G + H\, \mathfrak{L}_H - t_{AB}\, h_{AB}\, ] \qquad (3$$

where:

$$\mathcal{L}_A = \partial_\alpha h_{(\beta\gamma)} \partial_\alpha h_{(\beta\gamma)} - 2 \partial_\alpha h_{(\alpha\gamma)} \partial_\beta h_{(\beta\gamma)} - \partial_\alpha h_{\beta\beta} \partial_\alpha h_{\gamma\gamma}$$
$$+ 2 \partial_\gamma h_{\alpha\alpha} \partial_\beta h_{(\beta\gamma)} + \partial_\gamma h_{a\beta} \partial_\gamma h_{\beta a} - \partial_\beta h_{a\beta} \partial_\gamma h_{\gamma a}$$
$$- 2 \partial_a h_{a\gamma} ( \partial_\beta h_{(\beta\gamma)} - \partial_\gamma h_{\beta\beta} ) + 2 \partial_\gamma h_{aa} ( \partial_\beta h_{(\beta\gamma)} - \partial_\gamma h_{\beta\beta} )$$

$$\mathcal{L}_B = \partial_a h_{(bc)} \partial_a h_{(bc)} - 2 \partial_a h_{(ac)} \partial_b h_{(bc)} - \partial_a h_{bb} \partial_a h_{cc}$$
$$+ 2 \partial_c h_{aa} \partial_b h_{(bc)} + \partial_c h_{\alpha b} \partial_c h_{b\alpha} - \partial_b h_{\alpha b} \partial_c h_{c\alpha}$$
$$- 2 \partial_\alpha h_{\alpha c} ( \partial_b h_{(bc)} - \partial_c h_{bb} ) + 2 \partial_c h_{\alpha\alpha} ( \partial_b h_{(bc)} - \partial_c h_{bb} )$$

$$\mathcal{L}_C = \partial_a h_{(\beta\gamma)} \partial_a h_{(\beta\gamma)} + \tfrac{1}{2} \partial_\gamma h_{\beta a} \partial_\gamma h_{\beta a} + \tfrac{1}{2} \partial_\beta h_{\beta a} \partial_\gamma h_{\gamma a}$$
$$- 2 \partial_\beta h_{(\beta\gamma)} \partial_a h_{\gamma a}$$

$$\mathcal{L}_D = \partial_\alpha h_{(bc)} \partial_\alpha h_{(bc)} + \tfrac{1}{2} \partial_c h_{b\alpha} \partial_c h_{b\alpha} + \tfrac{1}{2} \partial_b h_{b\alpha} \partial_c h_{c\alpha}$$
$$- 2 \partial_b h_{(bc)} \partial_\alpha h_{c\alpha}$$

$$\mathcal{L}_E = \tfrac{1}{2} \partial_\gamma h_{a\beta} \partial_\gamma h_{a\beta} - \tfrac{1}{2} \partial_\beta h_{a\beta} \partial_\gamma h_{a\gamma}$$

$$\mathcal{L}_F = \tfrac{1}{2} \partial_c h_{\alpha b} \partial_c h_{\alpha b} - \tfrac{1}{2} \partial_b h_{\alpha b} \partial_c h_{\alpha c}$$

$$\mathcal{L}_G = \partial_a h_{\beta\beta} \partial_a h_{\gamma\gamma} + \partial_\beta h_{\beta a} \partial_\gamma h_{\gamma a} - 2 \partial_a h_{\beta\beta} \partial_\gamma h_{\gamma a}$$

$$\mathcal{L}_H = \partial_\alpha h_{bb} \partial_\alpha h_{cc} + \partial_b h_{b\alpha} \partial_c h_{c\alpha} - 2 \partial_\alpha h_{bb} \partial_c h_{c\alpha}$$

and the parameters A,...,H have been defined above. We have added to the action a source term $h_{AB} t_{AB}$, which will be useful in the following calculations.

The coordinate and $G_T$ transformations invariance of the original action is reflected in the bilinear expansion as invariance under the transformations

$$\delta h_{AB} = \partial_B \xi_A + \varepsilon_{AB} \qquad (4$$

where $\varepsilon_{\alpha\beta} = \varepsilon_{\beta\alpha}$ ; $\varepsilon_{ab} = \varepsilon_{ba}$ ; $\varepsilon_{\alpha a} = \varepsilon_{a\alpha} = 0$.

One must then impose some gauge conditions. For what concerns the tangent space invariance, the most natural choice of gauge is given by the ½ N(N-1)+½ M(M-1) conditions:

$$h_{[\alpha\beta]} = h_{[ab]} = 0 \qquad (5$$

With respect to coordinate transformations invariance is convenient to impose the so called light-cone gauge [76]:

$$h_{A-} = 0 \qquad (6$$

where we have defined $h_{A\pm} = 2^{-\frac{1}{2}} (h_{A,N-1} \pm h_{A,0})$ . We stress that combining (5) with (6) one has $h_{\alpha-} = h_{-\alpha} = 0$ and $h_{a-} = 0$, but $h_{-a} \neq 0$, in contrast to what happens in the Riemannian case , when all fields carrying a - index vanish.

A useful property of the light-cone gauge is that it permits to obtain directly the physical states from the propagators, without recurring to the conservation laws for the source terms. These can be obtained by requiring the invariance of the source terms under the same symmetries as the rest of the action and are:

$$t_{[\alpha\beta]} = t_{[ab]} = 0 \qquad (7$$

$$\partial_B t_{AB} = 0 \qquad (8$$

We notice that contrary to the Riemann case one cannot exchange the indices of $t_{AB}$ in (8).

In order to study the spectrum of quasi-Riemannian theories, it is important to observe that they admit a natural interpretation in terms of dimensional reduction. The reason is the following: in ordinary field theory the particle states are classified in terms of their spin or helicity and of their rest mass (i.e. the square of their momentum). This is due to the fact that these quantities permit to classify the representations of the D-dimensional Poincaré group $P^D$, to which the particle states belong. In quasi-Riemannian theories with $G_T = O(1,N-1) \times O(M)$, the invariance group of the flat space is not anymore $P^D$, but a direct product $P^N \times P^M$, so that one has to classify the particle states by giving their "spin" and "mass" with respect to both $P^N$ and $P^M$. This is evident, for example, from the fact that in this theory the squares of the momentum $k_\alpha{}^2$ and $k_a{}^2$ are two independent quantities that can appear with different coefficients in the linearized action. Thus, in order to give a physical interpretation of the theory, one is forced to consider the N-dimensional momentum and spin as the "physical" ones, and the M-dimensional quantities as "internal" quantum numbers. But this is just the same point of view one adopts when performing a dimensional reduction of an ordinary (i.e. Riemannian) theory. The analogy is obvious if one considers that a dimensionally reduced Riemannian theory has in fact a quasi-Riemannian effective invariance group. We shall then adopt the language of Kaluza-Klein theories to discuss the spectrum of the theory.

To perform the calculations it is useful to Fourier-transform the action to the momentum space. Furthermore, one can choose a particular reference frame. We choose the one in which only the components $k_+$, $k_-$ and $k_N$ of the momentum are not vanishing (by $k_N$ we denote the first component of the internal momentum $k_a$).

By varying (3) with respect to $h_{AB}$ one can obtain the equations of motion satisfied by the irreducible components of $h_{AB}$ under the group $O(N-2) \times O(M)$, which classifies the massless excitations.

In the particular system of coordinates we have chosen, they split into six non-interacting sectors, corresponding to different values of the "physical" and "internal" spin of the states. We report them in the massless case ($k_a = 0$):

Sector 1)

$$2Ak^2\, h^t{}_{\alpha\beta} = t^t{}_{\alpha\beta}$$

Sector 2)

$$k^2 \begin{bmatrix} C & A \\ A & E \end{bmatrix} \begin{bmatrix} h_{\alpha a} \\ h_{a\alpha} \end{bmatrix} = \begin{bmatrix} t_{\alpha a} \\ t_{a\alpha} \end{bmatrix}$$

Sector 3)

$$-4Ak_-{}^2\, h_{\alpha+} = t_{\alpha-}$$

Sector 4)

$$2Dk^2 h^t{}_{ab} = t^t{}_{ab}$$

Sector 5)

$$\begin{bmatrix} (C+2G)k_+{}^2 & (3C+2G)k_+k_- & Ak_+k_- \\ (3C+2G)k_+k_- & (C+2G)k_-{}^2 & -Ak_-{}^2 \\ Ak_+k_- & -Ak_-{}^2 & Ek_-{}^2 \end{bmatrix} \begin{bmatrix} h_{-a} \\ h_{+a} \\ h_{a+} \end{bmatrix} = \begin{bmatrix} t_{+a} \\ t_{-a} \\ t_{a-} \end{bmatrix}$$

Sector 6)

$$2\begin{bmatrix} 0 & -Ak_-{}^2 & -Ak_-{}^2 \\ -Ak_-{}^2 & A\frac{N-3}{N-2}k^2 & Ak^2 \\ -Ak_-{}^2 & Ak^2 & -(H+\frac{D}{M})k^2 \end{bmatrix} \begin{bmatrix} h_{++} \\ h_{\alpha\alpha} \\ h_{aa} \end{bmatrix} = \begin{bmatrix} t_{--} \\ t_{\alpha\alpha} \\ t_{aa} \end{bmatrix}$$

where greek indices run from 1 to N-2 and we have put $k^2 = 2k_+k_-$. Moreover:

$$h^t{}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{N-2} h_{\gamma\gamma}\eta_{\alpha\beta}$$

$$h^t{}_{ab} = h_{ab} - \frac{1}{M} h_{cc}\,\eta_{ab} \qquad (9$$

One can now solve the equations of motion with respect to the components of $h_{AB}$, and substitute them back into (3). In this way one obtains the propagator of the physical excitations. The resulting propagators are:

Sector 1) $$I_1 = \frac{1}{4}\,\frac{|t^t{}_{\alpha\beta}|^2}{Ak^2}$$

Sector 2) $$I_2 = \frac{1}{2}\left[\frac{C|t_{a\alpha}-\frac{A}{C}t_{\alpha a}|^2}{EC-A^2} + \frac{|t_{\alpha a}|^2}{C}\right]$$

Sector 3) $$I_3 = -\frac{1}{8}\frac{|t_{\alpha -}|^2}{Ak_-^2}$$

Sector 4) $$I_4 = \frac{1}{4}\frac{|t^t{}_{ab}|^2}{Dk^2}$$

Sector 5)

$$I_5 = \frac{1}{16(A^2-EC)(C+G)}\left\{[A^2+E(C+2G)]\left[\frac{|t_{+a}|^2}{k_+^2} + \frac{|t_{-a}|^2}{k_-^2}\right] + 8C(C+G)\frac{|t_{a-}|^2}{k_-^2}\right.$$

$$\left. +2[A^2-E(3C+2G)]\frac{\mathrm{Re}(t^*_{+a}t_{-a})}{k_+k_-} + 8A(C+G)\left[\frac{\mathrm{Re}(t^*_{+a}t_{a-})}{k_+k_-} - \frac{\mathrm{Re}(t^*_{-a}t_{a-})}{k_-^2}\right]\right\}$$

Sector 6) $$I_6 = \frac{1}{4}\frac{|t_{\alpha\alpha}-t_{aa}|^2}{\left[A\frac{N-1}{N-2} + H + \frac{D}{M}\right]k^2}$$

We recall that in the light-cone gauge the $x_+$ coordinate is regarded as "time" and only the equations involving the derivatives $\partial_+$ are dynamical. The propagators of the physical particles have then poles in $k_+$. In order to have stability these poles must be simple and their residua positive (absence of ghosts). Moreover, the massive states must have positive mass squares (absence of tachyons) [23,77].

It is easily seen that the first condition is not satisfied in our case. In fact, contrary to the Riemann case, in which the fields carrying a + or − index do not propagate and can be eliminated, in this case some of them are dynamical and must be carefully considered. From the propagator it is in fact evident that in sector 5 a double pole is present. In order to cancel the double pole one must impose the condition

$$A^2 + E ( C + 2G ) = 0 \tag{10}$$

Unfortunately, this is not sufficient to assure stability, because the residue at the simple pole is not positive definite. One can calculate it from $I_5$:

$$\mathrm{Res}\, I_5 \Big|_{k_+=0} = \frac{2[A^2 - E(3C+2G)]\mathrm{Re}(t^*_{+a}t_{-a}) + 8A(C+G)\,\mathrm{Re}(t^*_{+a}t_{a-})}{8\,(C+G)\,(A^2 - EC)} \tag{11}$$

Clearly, this is not a positive definite expression, because the source terms do not appear as squares. One must then require

$$\begin{aligned} A^2 - E ( 3C + 2G ) &= 0 \\ A ( C + G ) &= 0 \end{aligned} \tag{12}$$

It is easy to see that the conditions (10) and (12) imply that

$$A^2 - EC = 0 \tag{13}$$

If in addition one requires that A is nonvanishing (which is a necessary condition to have a propagating graviton), the solution of eqns. (10) and (12) is unique and is given by (13) and

$$C + G = 0 \tag{14}$$

We shall discuss in the following section the consequences of these conditions.

We want just to stress that in the Riemannian case these poles are not present because of the source constraint (valid for $k_a = 0$) $k_\alpha t_{\alpha a} = k_\alpha t_{a\alpha} = 0$. In our case, instead, $k_\alpha t_{\alpha a}$ cannot be deduced from (8) because $t_{a\alpha}$ is not anymore symmetric in the two indices, and therefore the double poles cannot be suppressed. This fact can be seen more easily in the covariant gauge $k_\alpha h_{a\alpha}=0$. In this gauge it is necessary to use the conservation laws (7,8) in order to determine the physical states.

The propagator of the spin 1 sector (corresponding to sectors 2 and 5 in light-cone gauge) is given by

$$I = \frac{1}{2k^2}\left[\frac{C}{EC-A^2}\,|\,t^T_{a\alpha} - \frac{A}{C}\,t^T_{\alpha a}|^2 + \frac{1}{C}\,|t^T_{\alpha a}|^2 + \frac{1}{2(C+G)}\,|t^L_{\alpha a}|^2\right] \tag{15}$$

where

$$t^T_{a\alpha} = \left(\eta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right) t_{a\beta}$$
$$t^L_{a\alpha} = \frac{k_\alpha k_\beta}{k^2}\, t_{a\beta} \tag{16}$$

and analogously for $t_{\alpha a}$

Now, by use of the source constraint

$$k_\alpha t_{a\alpha} = 0 \tag{17}$$

one can eliminate the dangerous terms from

$$\frac{|t^T_{a\alpha}|^2}{k^2} = \frac{|t_{a\alpha}|^2}{k^2} - \frac{|k_\alpha t_{a\alpha}|^2}{k^4} \tag{18}$$

but this is not possible for the terms containing $t^T{}_{\alpha a}$ and $t^L{}_{\alpha a}$ since, contrary to the Riemannian case, they do not satisfy any source constraint, whence the double poles.

We summarize here the spectrum of massless states:

| Sector | O(N-2) helicity | O(M) helicity | Number of multiplets |
|---|---|---|---|
| 1 | 2 | 0 | 1 |
| 2 | 1 | 1 | 2 |
| 3 | 1 | 0 | does not propagate |
| 4 | 0 | 2 | 1 |
| 5 | 0 | 1 | 2 ghosts |
| 6 | 0 | 0 | 1 |

For completeness, we report also the massive spectrum.

| O(N-1) spin | O(M-1) spin | $(\text{mass})^2$ |
|---|---|---|
| 2 | 0 | $\frac{C}{A} k_a^2$ |
| 1 | 1 | $\frac{2AB-CD-EF \pm \sqrt{(CD-EF)^2 + A^2BF + B^2CE - 4AB(CD+EF)}}{2(A^2-EC)} k_a^2$ |
| 1 | 0 | $-\frac{2C(D+H)}{A^2-EC} k_a^2$ |
| 0 | 1 | $-\frac{B^2-DF}{2D(C+G)} k_a^2$ |
| 0 | 2 | $\frac{B}{D} k_a^2$ |
| 0 | 0 | ? |

## IV.2 Stability

The conditions (13) and (14) seem to introduce some singularity in the propagator. This is the signal that a new gauge invariance has been introduced in the massless sector of the bilinear action [78]. In fact the action is now invariant under the local transformations

$$\delta h_{a\alpha} = \xi_{a\alpha}(x) \qquad\qquad \delta h_{\alpha a} = -\frac{A}{C}\xi_{a\alpha}(x) \qquad (19$$

which constitute an NM-parameters abelian group. This invariance is due to the fact that the fields $h_{a\alpha}$ and $h_{\alpha a}$ are now present in the action only in the linear combination $h_{\alpha a} + {}^{A}/_{C}\, h_{a\alpha}$.

For consistency, it is necessary to extend this gauge invariance to the whole bilinear action, comprising also the massive sector $k_a \neq 0$. This is achieved by imposing some further condition on the parameters:

$$D = -H = \frac{AB}{C} \qquad\qquad F = \frac{BC}{A} \qquad (20$$

One remains then with only three independent parameters.

The new invariance yields a new source constraint:

$$t_{a\alpha} - \frac{A}{C}\, t_{\alpha a} = 0 \qquad (21$$

which, combined with (7), gives $k_\alpha\, t_{\alpha a} = 0$.

The new gauge constraints for the fields arise naturally:

$$h_{a\alpha} - \frac{A}{C}\, h_{\alpha a} = 0 \qquad (22$$

Because of the new gauge invariance, one must once again perform the calculations from the beginning, imposing the new gauge condition (22). For convenience, we define the field

$$h^g_{\alpha a} \equiv h^g_{a\alpha} = h_{\alpha a} + \frac{A}{C} h_{a\alpha} \qquad (23$$

With this definition, the bilinear lagrangian can be written as follows:

$$\begin{aligned}
\mathfrak{L} = {} & A\,[\partial_\alpha h_{(\beta\gamma)}\, \partial_\alpha h_{(\beta\gamma)} - 2\, \partial_\alpha h_{(\alpha\gamma)}\, \partial_\beta h_{(\beta\gamma)} - \partial_\alpha h_{\beta\beta}\, \partial_\alpha h_{\gamma\gamma} \\
& + 2\, \partial_\gamma h_{\alpha\alpha}\, \partial_\beta h_{(\beta\gamma)} + 2\, \partial_\gamma h_{aa}\ (\partial_\beta h_{(\beta\gamma)} - \partial_\gamma h_{\beta\beta})] \\
& + B\,[\, \partial_a h_{(bc)}\, \partial_a h_{(bc)} - 2\, \partial_a h_{(ac)}\, \partial_b h_{(bc)} - \partial_a h_{bb}\, \partial_a h_{cc} \\
& + 2\, \partial_c h_{aa}\, \partial_b h_{(bc)} + 2\, \partial_c h_{\alpha\alpha}\ (\partial_b h_{(bc)} - \partial_c h_{bb}) \\
& - 2\, \partial_\alpha h^g_{\alpha c}\, (\partial_b h_{(bc)} - \partial_c h_{bb})\,] \qquad (24 \\
& + C\,[\,(\partial_a h_{(\beta\gamma)}\, \partial_a h_{(\beta\gamma)} - \partial_a h_{\beta\beta}\, \partial_a h_{\gamma\gamma}) + \tfrac{1}{2}\,(\partial_\gamma h^g_{\beta a}\, \partial_\gamma h^g_{\beta a} \\
& - \partial_\beta h^g_{\beta a}\, \partial_\gamma h^g_{\gamma a}) - 2\, \partial_a h^g_{\gamma a}\, (\partial_\beta h_{(\beta\gamma)} - \partial_\gamma h_{\beta\beta})\,] \\
& ABC^{-1}\,[\, \partial_\alpha h_{(bc)}\, \partial_\alpha h_{(bc)} - \partial_\alpha h_{bb}\, \partial_\alpha h_{cc}\,] + \\
& \tfrac{1}{2}\, CBA^{-1}\,[\, \partial_c h^g_{\alpha b}\, \partial_c h^g_{\alpha b} - \partial_b h^g_{\alpha b}\, \partial_c h^g_{\alpha c}\,]
\end{aligned}$$

We can now impose the light-cone gauge $h_{A-} = 0$. Taking into account the other gauge conditions, this can be written as:

$$\begin{aligned} h_{\alpha-} &= h_{-\alpha} = 0 \\ h_{a-} &= h_{-a} = 0 \end{aligned} \qquad (25$$

The field equations are now (*):

Sector 1)

$$2Ak^2\, h^t_{\alpha\beta} = t^t_{\alpha\beta}$$

Sector 2)

$$Ck^2\, h^g_{\alpha a} = t_{\alpha a}$$

Sector 3)

$$-4Ak_-^2\, h_{\alpha+} = t_{\alpha-}$$

---

(*) In the following, greek indices run from 1 to N-2.

Sector 4)

$$2\,\frac{AB}{C}\,h^{t}{}_{ab} = t^{t}{}_{ab}$$

Sector 5)

$$-2Ck_{-}{}^{2}\,h^{g}{}_{+a} = t_{a-}$$

Sector 6)

$$2\begin{bmatrix} 0 & -Ak_{-}{}^{2} & -Ak_{-}{}^{2} \\ -Ak_{-}{}^{2} & A\frac{N-3}{N-2}k^{2} & Ak^{2} \\ -Ak_{-}{}^{2} & Ak^{2} & \frac{AB}{C}\frac{M-1}{M}k^{2} \end{bmatrix}\begin{bmatrix} h_{++} \\ h_{\alpha\alpha} \\ h_{aa} \end{bmatrix} = \begin{bmatrix} t_{--} \\ t_{\alpha\alpha} \\ t_{aa} \end{bmatrix}$$

They give rise to the propagators:

$$I = \frac{1}{4}\left[\frac{|t_{\alpha\beta}|^{2}}{Ak^{2}} + \frac{|t_{\alpha a}|^{2}}{\frac{C}{2}k^{2}} + \frac{|t_{ab}|^{2}}{\frac{AB}{C}k^{2}} + \frac{|t_{\alpha\alpha}-t_{aa}|^{2}}{\left[A\,\frac{N-1}{N-2} - \frac{B}{C}\frac{M-1}{M}\right]k^{2}}\right] \qquad (26$$

It is easy to see from this expression that ghosts are absent if

$$A,\ B,\ C > 0$$

$$\frac{B}{C} < \frac{M(N-1)}{(M-1)(N-2)} \qquad (27$$

The spectrum of the propagating massless states is the following:

| Sector | O(N-2) helicity | O(M) helicity | Number of multiplets |
|---|---|---|---|
| 1 | 2 | 0 | 1 |
| 2 | 1 | 1 | 1 |
| 4 | 0 | 2 | 1 |
| 6 | 0 | 0 | 1 |

It is the same spectrum one would obtain by dimensional reduction in the Riemannian case.

In the massive case ($k_a \neq 0$), the nonvanishing momentum component $k_N$ distinguishes a direction in the "internal" space. The field must then be decomposed by separating the Nth component, and rearranged into representations of the group $O(N-1) \times O(M-1)$, which classifies the massive representations. Also in this case the linearized field equations split up in six sectors (*):

Sector 1)

$$2\,(Ak^2 + Ck_N^2)\, h^t{}_{\alpha\beta} = t^t{}_{\alpha\beta}$$

Sector 2)

$$C\,(k^2 + \frac{B}{A} k_N^2)\, h^g{}_{\alpha a} = t_{\alpha a}$$

Sector 3)

$$\begin{bmatrix} Ck^2 & -2Ck_-k_N \\ -2Ck_-k_N & -4Ak_-^2 \end{bmatrix} \begin{bmatrix} h^g{}_{\alpha N} \\ h_{\alpha +} \end{bmatrix} = \begin{bmatrix} t_{\alpha N} \\ t_{\alpha -} \end{bmatrix}$$

Sector 4)

$$\begin{bmatrix} \frac{4AB}{C} k^2 & -2Bk_-k_N \\ -2Bk_-k_N & -Ck_-^2 \end{bmatrix} \begin{bmatrix} h_{aN} \\ h^g{}_{a+} \end{bmatrix} = \begin{bmatrix} t_{aN} \\ t_{a-} \end{bmatrix}$$

---

(*) Latin indices run now from N+1 to N+M-1.

Sector 5)

$$2\left(\frac{AB}{C}k^2 + B\,k_N{}^2\right) h^t{}_{ab} = t^t{}_{ab}$$

Sector 6)

$$\begin{bmatrix} \frac{N-3}{N-2}(Ak^2+Ck_N{}^2) & Ak^2+Bk_N{}^2 & Ak^2 & -Ck_Nk_- & -Ak_-{}^2 \\ Ak^2+Bk_N{}^2 & \frac{M-2}{M-1}\frac{B}{C}(Ak^2+Ck_N{}^2) & \frac{AB}{C}k^2 & -Bk_Nk_- & -Ak_-{}^2 \\ Ak^2 & \frac{AB}{C}k^2 & 0 & 0 & -Ak_-{}^2 \\ -Ck_Nk_- & -Bk_Nk_- & 0 & \frac{C}{2}k_-{}^2 & 0 \\ -Ak_-{}^2 & -Ak_-{}^2 & -Ak_-{}^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_{\alpha\alpha} \\ h_{aa} \\ h_{NN} \\ h^g{}_{N+} \\ h_{++} \end{bmatrix} = \begin{bmatrix} t_{\alpha\alpha} \\ t_{aa} \\ t_{NN} \\ t^g{}_{N-} \\ t_{--} \end{bmatrix}$$

As usual, one can now calculate the propagators of the massive sectors. They are:

Sector 1)

$$I_1 = \frac{1}{4}\frac{|t^t_{\alpha\beta}|^2}{Ak^2+Ck_N^2}$$

Sector 2)

$$I_2 = \frac{1}{4}\frac{A}{C}\frac{|t^g_{a\alpha}|^2}{Ak^2+Bk_N^2}$$

Sector 3)

$$I_3 = \frac{1}{2}\frac{A}{C}\frac{|t^g_{N\alpha}|^2}{Ak^2+Ck_N^2}$$

Sector 4)

$$I_4 = \frac{1}{4}\frac{C}{B}\frac{|t^t_{ab}|^2}{Ak^2+Ck_N^2}$$

Sector 5)

$$I_5 = \frac{1}{8}\frac{C}{B}\frac{|t_{Na}|^2}{Ak^2 + Bk_N^2}$$

Sector 6)

$$I_6 = \frac{|t_{aa} - \frac{N-2}{N-1}\frac{B}{C}t_{\alpha\alpha}|^2}{4\left[\left(\frac{M}{M-1}\frac{AB}{C} - \frac{AB^2}{C^2}\frac{N-2}{N-1}\right)k^2 + \left(\frac{N}{N-1}\frac{B^2}{C} - \frac{M-2}{M-1}B\right)k_N^2\right]} +$$

$$+ \frac{N-2}{N-1}\frac{|t_{aa}|^2}{4(Ak^2 + Ck_N^2)}$$

One can then deduce the spectrum of the massive states:

| O(N-1) spin | O(M-1) spin | $(\text{mass})^2$ |
|---|---|---|
| 2 | 0 | $\frac{C}{A}k_N^2$ |
| 1 | 1 | $\frac{B}{A}k_N^2$ |
| 0 | 2 | $\frac{C}{A}k_N^2$ |
| 0 | 2 | $\frac{C}{A}\frac{\frac{N}{N-1}B - \frac{M-2}{M-1}C}{\frac{M}{M-1}C - \frac{N-2}{N-1}B}k_N^2$ |

The conditions for the absence of ghosts and tachyons are trivially obtained by requiring the positivity of the squares of the masses and of the residua at the poles. They are:

$$A, B, C > 0$$

$$\frac{(N-1)(M-2)}{N(M-1)} < \frac{B}{C} < \frac{M(N-1)}{(M-1)(N-2)} \tag{28}$$

Clearly, we have obtained a continuous spectrum, since $k_N^2$ can assume any positive value. This is because no compactification was assumed to occur in our model. We shall consider later on the case of compactification on a torus, which can be easily derived from the results obtained here. We also observe that in the Riemannian limit one obtains the same spectrum, but with all the values of the masses degenerate.

We have then established what conditions one must impose on the action in order to obtain a spectrum free of ghosts and tachyons. The eight parameters originally present in the action are constrained by the five relations:

$$E = \frac{A^2}{C} \qquad G = -C$$

$$F = \frac{BC}{A} \qquad D = -H = \frac{AB}{C} \tag{29}$$

and reduce to only three independent parameters, which must also satisfy the inequalities (28).

It seems likely that these conditions are inherent in the structure of the theory and do not depend on the particular background chosen. It would however be interesting to check this explicitly, by considering some models which exhibit spontaneous compactification. Another important problem which is still open is how the gauge invariance we have introduced can be extended to the full (non-linear) action, i.e. one should establish what transformations of the vielbein leave invariant the action obeying the conditions (29).

We finally notice that in order to have a consistent theory, it seems to be essential to have a particle spectrum containing the same states as the Riemannian one.

# V . CLASSICAL ASPECTS OF QUASI-RIEMANNIAN THEORIES

In this chapter we study some classical features of quasi-Riemannian theories and in particular the generalization of the action for classical fields and the definition of the metric and of the geodesics on quasi-Riemannian manifolds with tangent space group SO(1,N-1)xSO(M).

## V. 1 Classical fields

First of all we consider the electromagnetic field, defined by means of a vector potential $A_M$ and require that the theory is invariant under the abelian gauge transformation $A_M \rightarrow A_M + \partial_M \Lambda$.

The gauge invariant field strength is given by

$$F_{MN} = \partial_M A_N - \partial_N A_M = A_{N;M} - A_{M;N} + T^L{}_{NM} A_L \qquad (1$$

From this expression it is easy to see that if torsion is nonvanishing, some complications arise in the definition of a covariant field strength [44]. We shall then consider first the case of vanishing torsion.

In order to construct a $G_T$-invariant action, it is natural to use tangent space indices, defining

$$F_{AB} = e_A{}^M e_B{}^M F_{MN} = \nabla_A A_B - \nabla_B A_A \qquad (2$$

On quasi-Riemannian manifolds, the Maxwell action can be generalized by contracting in all possible ways $F_{AB} F_{CD}$ with the two $G_T$-invariant tensors $\eta_{\alpha\beta}$ and $\eta_{ab}$. The resulting action is:

$$S = \int e_D \, d^D z \, ( \mathfrak{L} - j_A A_A ) \qquad (3$$

with

$$\mathfrak{L} = - \tfrac{1}{4} (a F_{\alpha\beta} F_{\alpha\beta} + b F_{ab} F_{ab} + 2c F_{\alpha b} F_{\alpha b} ) \qquad (4$$

where a, b, c are some constants and $j_A$ is a source term, which satisfies the conservation law

$$\nabla_A j_A = 0 \qquad (5$$

Varying (4) one can obtain the field equations:

$$a \nabla_\alpha F_{\alpha\beta} + c \nabla_a F_{a\beta} = j_\beta$$
$$(6$$
$$c \nabla_\alpha F_{\alpha b} + b \nabla_a F_{ab} = j_b$$

while the other set of Maxwell equations can be obtained as usual directly from the definition of $F_{MN}$:

$$F_{[MN;L]} = \partial_{[L} F_{MN]} = 0 \qquad (7$$

or, in an orthogonal basis, $\nabla_{[A} F_{BC]} = 0$.

One can also calculate the components of the energy-momentum tensor, defined as

$$t_{AB} = \partial_A A_C \frac{\delta \mathfrak{L}}{\delta(\partial_B A_C)} - \eta_{AB} \mathfrak{L} \qquad (8$$

which satisfies the coservation law $\nabla_B t_{AB} = 0$. They are

$$t_{\alpha\beta} = -( a F_{\alpha\gamma} F_{\alpha\gamma} + c F_{\alpha c} F_{\beta c} - \eta_{\alpha\beta} \mathfrak{L} )$$
$$t_{ab} = -( b F_{ac} F_{bc} + c F_{a\gamma} F_{b\gamma} - \eta_{ab} \mathfrak{L} )$$
$$(9$$
$$t_{\alpha b} = -( b F_{\alpha c} F_{bc} + c F_{\alpha\gamma} F_{b\gamma} )$$
$$t_{a\beta} = -( a F_{ac} F_{\beta c} + c F_{a\gamma} F_{\beta\gamma} )$$

As is apparent, the components $t_{\alpha a}$ and $t_{a\alpha}$ of the energy-momentum tensor are different. This fact is characteristic of quasi-Riemannian theories, and is a consequence of the peculiar structure of the tangent space group.

As we have seen, if torsion is present, in order to preserve gauge invariance, one must renounce to have a minimal coupling of the electromagnetic field with gravity, because the torsion appears in the definition of the gauge invariant field strength. In this case the action can be defined as in (4), with

$$F_{AB} = e_A{}^M e_B{}^M F_{MN} = \nabla_A A_B - \nabla_B A_A - T_{CAB} A_C \qquad (10$$

The simplest (but not explicitly covariant) way to write the field equations in this case is:

$$\partial_{[L} \mathfrak{F}_{MN]} = 0$$

$$a\, \partial_\alpha \mathfrak{F}_{\alpha\beta} + c\ \partial_a \mathfrak{F}_{a\beta} = \mathfrak{J}_\beta \qquad (11$$

$$c\, \partial_\alpha \mathfrak{F}_{\alpha b} + b\, \partial_a \mathfrak{F}_{ab} = \mathfrak{J}_b$$

with $\mathfrak{F}_{AB} = e\, F_{AB}$ and $\mathfrak{J}_A = e\, j_A$.

In this case also the conservation laws assume a more general form [44]:

$$\nabla_B t_{AB} + T_{CCB}\, t_{AB} + 2\, T_{CBA}\, t_{CB} = 0$$

$$\partial_A \mathfrak{J}_A = 0 \qquad (12$$

In the case of a scalar field, the covariant derivatives coincide with the partial derivatives and the generalized lagrangian density is

$$\mathfrak{L} = h_1 \nabla_\alpha \phi \nabla_\alpha \phi + h_2 \nabla_a \phi \nabla_a \phi - m^2 \phi^2 \qquad (13$$

which yields the field equations:

$$h_1 \nabla_\alpha \nabla_\alpha \phi + h_2 \nabla_a \nabla_a \phi + m^2 \phi \qquad (14$$

and the conserved energy momentum tensor:

$$t_{\alpha\beta} = h_1 \nabla_\alpha \phi \nabla_\beta \phi - \eta_{\alpha\beta} \mathfrak{L} \qquad t_{a\beta} = h_1 \nabla_a \phi \nabla_\beta \phi$$

$$t_{ab} = h_2 \nabla_a \phi \nabla_b \phi - \eta_{ab} \mathfrak{L} \qquad t_{\alpha b} = h_2 \nabla_\alpha \phi \nabla_b \phi \qquad (15$$

The lagrangian can be generalized for a complex scalar field coupled to the Maxwell field by replacing $\nabla_A \to \nabla_A + iA_A$. The new lagrangian is invariant under the gauge transformation:

$$\phi \to e^{-i\lambda(z)}\,\phi \qquad\qquad A_A \to A_A + \partial_A\,\lambda(z) \tag{16}$$

and the electromagnetic current is given by:

$$\begin{aligned} j_\alpha &= h_1\,(\phi^* \nabla_\alpha \phi - \phi\,\nabla_\alpha \phi^*) \\ j_a &= h_2\,(\phi^* \nabla_a \phi - \phi\,\nabla_a \phi^*) \end{aligned} \tag{17}$$

Finally, we consider the spinor fields, which will be treated in more detail later on. We define a spinor as a field which transforms accordingly to the spinor representation of O(1,N-1). It can however transform in any representation of O(M). For definiteness, however, we assume here that it is a spinor also under O(M).

The lagrangian density can be written in general as:

$$\mathfrak{L} = i\,\bar{\psi}\,(g_1 \Gamma^\alpha \nabla_\alpha + g_2\,\Gamma^a \nabla_a - m)\,\psi \tag{18}$$

where $\Gamma^\alpha$ and $\Gamma^a$ are a set of Dirac matrices satisfying $\{\Gamma^\alpha, \Gamma^\beta\} = 2\,\eta^{\alpha\beta}$, $\{\Gamma^a, \Gamma^b\} = 2\,\eta^{ab}$ and $[\Gamma^\alpha, \Gamma^a] = 0$. (The $\Gamma^\alpha$ are the unit matrix in the a, b space and viceversa).

The field equations are:

$$i\,(g_1\,\Gamma^\alpha \nabla_\alpha + g_2\,\Gamma^a \nabla_a)\,\psi = 0 \tag{19}$$

and the conserved energy momentum is given by:

$$\begin{aligned} t_{A\beta} &= \tfrac{1}{2}\, i\, g_1 (\bar{\psi}\,\Gamma_A \nabla_\beta \psi - \nabla_\beta \bar{\psi}\,\Gamma_A\,\psi) \\ t_{Ab} &= \tfrac{1}{2}\, i g_2 (\bar{\psi}\,\Gamma_A \nabla_b \psi - \nabla_b \bar{\psi}\,\Gamma_A\,\psi) \end{aligned} \tag{20}$$

Also in this case the gauge invariant coupling to the Maxwell field is obtained by the substitution $\nabla_A \to \nabla_A + iA_A$ which renders the lagrangian invariant under $\psi \to e^{i\Lambda}\psi$; $A_A \to A_A + \partial_A \Lambda$. In this case, the electromagnetic current is given by:

$$j_\alpha = g_1 \overline{\psi} \Gamma_\alpha \psi \qquad\qquad j_a = g_2 \overline{\psi} \Gamma_a \psi \tag{21}$$

It must be pointed out that the spinorial fields, being minimally coupled to gravity, give rise to a source term in the field equations for the connection of the form

$$\tfrac{1}{4}\, \overline{\psi}\, \Gamma^{[A}\, \Gamma^{B}\, \Gamma^{C]}\, \psi \tag{22}$$

By consequence, if one adopts the Palatini formalism, the connection $\omega_{ABC}$ will contain terms proportional to (22), and some contact terms proportional to $\psi^3$ shall appear in the field equations for the spinor.

## V. 2 Perturbative expansion

In analogy to what we have done for the gravitational action, we consider here the particle content of the electromagnetic action, by studying the fluctuations of the quasi-Riemannian electromagnetic action (3) around the background $A_A = 0$, on a flat space. Also in this case the most convenient choice of gauge is given by the the light-cone gauge $A_- = 0$. In this gauge, the linearized action reads:

$$S_2 = -\frac{1}{2}\int d^D z\,\big[(a\partial_\alpha^2 + c\,\partial_a^2)\,A_\beta^2 - a\,\partial_-^2\,A_+^2 + (c\,\partial_\alpha^2 + b\,\partial_a^2)\,A_b^2$$
$$-b\,(\partial_a A_a)^2 - 2\,c\,\partial_-\,A_+\,\partial_a A_a\big] \tag{23}$$

As in the gravitational case, the linearized action can now be inverted by means of the field equations and Fourier-transformed, in order to obtain the propagators. The propagator of the massless excitations is given by:

$$I = \frac{1}{2}\left[\frac{|j_\alpha|^2}{ak^2} + \frac{|j_a|^2}{ck^2}\right] \quad (24$$

where $\alpha = 1,...,N-1$ and $a = N,..., N+M-1$ and $k^2$ is the square of the N-dimensional momentum. The propagator describes a helicity 1 state (photon) and a M-plet of scalars.

The propagator of the massive states is given by:

$$I = \frac{1}{2}\left[\frac{|j_\alpha|^2 + \frac{a}{c}|j_N|^2}{ak^2 + ck_N^2} + \frac{|j_a|^2}{ck^2 + bk_N^2}\right] \quad (25$$

where now $a = N+1,...,N+M-1$ and the reference frame is chosen so that $j_N$ is the only nonvanishing component of the internal momentum.

The massive states are a tower of spin 1 particles with $(\text{mass})^2 = {}^c/_a\, k_N^2$ and a tower of spin 0, $(\text{mass})^2 = {}^b/_c\, k_N^2$ particles. As usual, the values assumed by $k_N^2$ depend on the structure of the internal space.

It is evident that ghosts and tachyons are absent if $a, b, c > 0$, since in this case the poles have positive residua and the masses squared are positive.

We also notice that from dimensional reduction of the Riemannian action, one would obtain the same spectrum, except that both the spin 1 and spin 0 particles have the same $(\text{mass})^2 = k_N^2$.

As an example of application of the formalism given above, we give here the Feynman rules for the quantum electrodynamics on a $Q^{N,M}$ background.

As we have seen, the higher-dimensional electromagnetic field contains a N-vector $A_\alpha$ and a multiplet of N-scalars $A_a$. We normalize them by the redefinition

$$A_\alpha \rightarrow a^{-1/2} A_\alpha \qquad\qquad A_a \rightarrow c^{-1/2} A_a \quad (26$$

Moreover, we redefine the parameters of the spinor lagrangian (18), putting $g_1 = 1$, $g_2 = g$ and $m = 0$.

The propagators become:

$$A_\alpha : \qquad \frac{\delta_{\alpha\beta}}{k_\alpha{}^2 + \frac{c}{a} k_a{}^2}$$

$$A_a : \qquad \frac{\delta_{ab}}{k_\alpha{}^2 + \frac{b}{c} k_a{}^2} \tag{27}$$

$$\psi : \qquad \frac{1}{\Gamma^\alpha k_\alpha + g\,\Gamma^a k_a}$$

If $k_a = 0$, the spinor propagator describes a massless spinor, while for $k_a \neq 0$, the spinor carries a $(\mathrm{mass})^2 = g^2 k_a{}^2$.

The interaction vertices are proportional to

$$\psi\psi A_\alpha : \qquad -i\, a^{-1/2}\, \Gamma^\alpha\, \delta(\Sigma\, k_A)$$

$$\psi\psi A_a : \qquad -i\, g\, a^{-1/2}\, \Gamma^a\, \delta(\Sigma\, k_A) \tag{28}$$

The electromagnetic coupling constant is then given by $a^{-1/2}$, while the coupling constant for the scalar-fermion interaction is given by $g\, a^{-1/2}$.

We stress that also in the limit of vanishing masses some corrections to the usual electromagnetic interaction are given by the diagrams involving the scalars.

## V .3 Equations of state

In view of the applications to cosmology, it is interesting to obtain the equations of state for a perfect fluid in higher dimensions.

As shown for example in [79], the energy momentum tensor for a gas of non-interacting particles is given by

$$T_{AB} = \sum_n \frac{k_A^{(n)} k_B^{(n)}}{|k_0^{(n)}|} \tag{29}$$

where $k_A{}^{(n)}$ are the components of the momentum of the n-th particle.

For an isotropic fluid on a D-dimensional Riemannian manifold, considerations of symmetry require that the only nonvanishing components of the energy-momentum tensor are given by:

$$T_{oo} = \rho \qquad\qquad T_{ij} = p\,\eta_{ij} \tag{30}$$

with i,j = 1,...,D-1.

If the field is massles (pure radiation), one has from (29):

$$T_{AA} = \sum_n \frac{k_A^{(n)} k_A^{(n)}}{|k_0^{(n)}|} = 0 \tag{31}$$

since $k_A{}^2 = 0$ for a massless particle. Comparing with (30) one gets:

$$\rho + (D-1)\,p = 0 \tag{32}$$

and then the equation of state:

$$p = \frac{\rho}{D-1} \tag{33}$$

In the quasi-Riemannian case the energy-momentum tensor (30) can have the more general form :

$$T_{oo} = \rho \qquad\qquad T_{ij} = p\,\eta_{ij} \qquad\qquad T_{ab} = q\,\eta_{ab} \tag{34}$$

with i =1,...,N-1 and a = N,...,N+M-1. Hence, if $k_A{}^2 = 0$, eqn. (31) yields:

$$\rho + (N-1)\,p + Mq = 0 \tag{35}$$

If one puts q = zp , with z a suitable constant depending on the model considered, one can deduce from (35) that

$$q = zp = \frac{z\rho}{N-1+zM} \tag{36}$$

However, a consistent definition of massless particle in quasi-Riemannian theories seems to require that both $k_\alpha{}^2$ and $k_a{}^2$ be vanishing (compare with the discussion in section IV.1). This corresponds to put z=0 in (36) and the equation of state reduces to that of ordinary N-dimensional radiation.

Similar conclusions can be obtained calculating the equation of state of electromagnetic radiation starting from the energy-momentum tensor (9) of the generalized Maxwell lagrangian.

## IV .4 Geodesics

In order to define the notion of a test particle in a quasi-Riemannian manifold, one has to establish an action for it. By varying the action with respect to the metric (or the vielbein), one can then determine the trajectory. We assume that the action is identical to that of general relativity:

$$S = -m \int ds = -m \int \sqrt{g_{MN} \frac{dz^M}{ds} \frac{dz^N}{ds}}\, ds \tag{37}$$

This definition is however ambiguous, because in quasi-Riemannian theories the metric is not defined in a unique way: as we have seen, in fact, one can define several metrics which are covariantly constant. In our case, for example, one can define two linearly independent metrics $g^I{}_{MN} = \eta_{\alpha\beta}\, e^\alpha{}_M e^\beta{}_N$ and $g^{II}{}_{MN} = \eta_{ab}\, e^a{}_M e^b{}_N$, which are symmetric in N and M. Any linear combination of these two metrics can be interpreted as the physical one. This freedom is related to the fact that the standards of length in the N-dimensional and M-dimensional subspaces are independent. The choice of a particular metric can then be seen as a fixing of gauge in the theory. We make for the physical metric the obvious choice

$$g_{MN} = g^I{}_{MN} + g^{II}{}_{MN} = \eta_{AB}\, e^A{}_M e^B{}_N \tag{38}$$

This definition of the physical metric will always be adopted in the following, where classical solutions of the field equations will be discussed. However, we stress that in quasi-Riemannian theories the primary object is not the metric but the vielbein, even if a definition of the metric is necessary in order to give a physical interpretation of the theory.

Anyway, given any metric $g_{MN}$ , the equations of motion for spinless particles can be easily worked out by varying (37) with respect to $g_{MN}$, in the same way as in general relativity. The result is:

$$\frac{d^2z^L}{ds^2} + \{^L{}_{MN}\} \frac{dz^M}{ds} \frac{dz^N}{ds} = 0 \qquad (39$$

with

$$\{^L{}_{MN}\} = \tfrac{1}{2}\, g^{LP}( \partial_N g_{PM} + \partial_M g_{PN} + \partial_P g_{MN} ) \qquad (40$$

It is important to realize that in general the Christoffel symbol $\{^L{}_{MN}\}$ is different from the affine connection $\Gamma^L{}_{MN}$ defined by ( I.10), which defines the parallel transport on our manifold (*). This is usual in all the theories with non vanishing torsion [44], as the quasi-Riemannian usually are.

We consider now the coupling of a test particle of charge **e** with the electromagnetic field. This is obtained by adding to the free action (37) a term

$$S' = - \int e \frac{dz^M}{ds} A_M \, ds \qquad (41$$

The equations of motion then read:

$$\frac{d^2z^L}{ds^2} + \{^L{}_{MN}\} \frac{dz^M}{ds} \frac{dz^N}{ds} = \frac{e}{m} F^L{}_M \frac{dz^M}{ds} \qquad (42$$

The form of the coupling (41) is dictated by the gauge invariance

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(*) For example, for $\omega$ defined as in (III.18-20), one has [35]:

$$\Gamma^L{}_{MN} = \Pi^L{}_P \{^P{}_{MQ}\} \Pi^Q{}_N + \Sigma^L{}_P \{^P{}_{MQ}\} \Sigma^Q{}_N + \Pi^L{}_P \partial_M \Pi^P{}_N + \Sigma^L{}_P \partial_M \Sigma^P{}_N$$

with $\Pi^P{}_Q = e^P{}_\alpha e^\alpha{}_Q$ and $\Sigma^P{}_Q = e^P{}_a e^a{}_Q$ .

$A_M \rightarrow A_M + \partial_M \Lambda$ and the continuity equation

$$\left[ e \, \frac{dz^M}{ds} \right]_{;M} = 0 \qquad (43$$

which rule out couplings of the kind

$$A^N \, \frac{dz^M}{ds} \left( e_1 g^{I}_{MN} + e_2 g^{II}_{MN} \right) \qquad (44$$

# VI . CLASSICAL SOLUTIONS OF THE FIELD EQUATIONS

We turn now to the study of the classical solutions of quasi-Riemannian field equations. As in general relativity, one can find a large variety of solutions, depending on the boundary conditions one imposes. In this chapter we study two classes of problems: the cosmological solutions, and the generalization of the Schwarzschild metric. In the next chapter we shall investigate the possibility of obtaining solutions giving rise to spontaneous compactification of the higher-dimensional space.

In general, the smallness of the tangent space in quasi-Riemannian theories, imposes strong constraints on the kind of solutions one can find. In fact, as discussed in chapter I, the isotropy group of the solution must be a subgroup of the tangent space group $G_T$, so that on the quasi-Riemannian spaces $Q^{N,M}$ the isotropy group of the solution can be at most $O(1,N-1) \times O(M)$.

The only exception to this rule is the degenerate case of the flat space, which, as we have seen, is a solution of the field equations for $\lambda = 0$.

## VI . 1 Kaluza-Klein cosmology.

An important class of solutions of the higher-dimensional Einstein equations are those describing cosmological models of the evolution of the universe. Before discussing how these solutions are modified in the quasi-Riemannian case, we shall give a brief account of their main features.

In the standard Robertson-Walker cosmology, the scale factor of the universe grows with time. A realistic higher-dimensional cosmology should be approximated by a four-dimensional cosmology for the late evolution of the universe, but at some early time the Robertson-Walker scale factor must have been of the same order of magnitude as the length scale of the internal space. The aim of the Kaluza-Klein cosmology is then to explain why the two scales are now so different as a consequence of the dynamical evolution of the universe.

The first proposal for a Kaluza-Klein cosmological model was done by Chodos and Detweiler [17], who generalized to five dimensions the Kasner solution of general relativity equations, obtaining a model with the three spatial dimensions expanding and the internal one shrinking. A generalization of this model to higher dimensions was proposed by Freund [18], who established on general grounds the field equations of the higher dimensional cosmology. The physical implications of these models were studied by many authors. In particular, it was pointed out that the contraction of the internal space can give rise to entropy production in 4 dimensions [80] and to an inflationary expansion of the internal space [19-21]. This possibility is very attractive since, as is well known, inflationary models for the evolution of the early universe were very successful in ordinary cosmology to solve some important problems, like the so-called horizon and flatness puzzles [22]. In the original inflationary models, the universe suffers an exponential growth, caused by a phase transition of the matter fields, which produces a large amount of entropy. In higher dimensional models, instead, the total entropy is conserved, but its four-dimensional effective value is increased by the shrinking of the internal dimensions.

To make the discussion more concrete, let us examine in some detail the higher dimensional model of ref. [20].

The metric is a generalization of the Robertson-Walker line element, where the higher-dimensional space is the direct product of two maximally symmetric spaces, of dimension (N-1) and M and scale factors r(t) and R(t) respectively :

$$ds^2 = - dt^2 + r^2(t) \frac{(dx^i)^2}{\left[1+\frac{hx^ix^i}{4}\right]^2} + R^2(t) \frac{(dx^a)^2}{\left[1+\frac{kx^ax^a}{4}\right]^2} \qquad (1$$

where i runs over the set 1,.....,N-1 and a over N,.....,N+M-1.

As usual, h and k can assume the values -1, 0 and 1 , corresponding to positive curvature, flat , and negative curvature spaces.

The physical case is N=4. We observe however that there is no preferred way in the Riemannian theory to split the original D dimensions in N "external" and M "internal", and the ansatz contains therefore a large amount of arbitrariness.

Substituting the ansatz into the Einstein equations, one gets:

$$\frac{1}{2}(N-1)(N-2)\frac{\dot{r}^2+h}{r^2}+\frac{1}{2}M(M-1)\frac{\dot{R}^2+k}{R^2}+M(N-1)\frac{\dot{r}\dot{R}}{rR}=\frac{\kappa^2}{2}\varrho$$

$$(N-2)\left[\frac{\ddot{r}}{r}+\frac{N-3}{2}\frac{\dot{r}^2+h}{r^2}\right]+M\left[\frac{\ddot{R}}{R}+\frac{M-1}{2}\frac{\dot{R}^2+k}{R^2}\right]+M(N-2)\frac{\dot{r}\dot{R}}{rR}=-\frac{\kappa^2}{2}p \qquad (2$$

$$(M-1)\left[\frac{\ddot{R}}{R}+\frac{M-2}{2}\frac{\dot{R}^2+k}{R^2}\right]+(N-1)\left[\frac{\ddot{r}}{r}+\frac{N-2}{2}\frac{\dot{r}^2+h}{r^2}\right]+(N-1)(M-1)\frac{\dot{r}\dot{R}}{rR}=-\frac{\kappa^2}{2}q$$

where a dot denotes a derivative with respect to the time and we have used the most general form for the energy momentum tensor of a perfect fluid in higher dimension compatible with the ansatz (1) for the metric:

$$t_{oo}=\varrho \qquad t_{ij}=p\,\eta_{ij} \qquad t_{ab}=q\,\eta_{ab} \qquad (3$$

It satisfies the conservation law $\nabla_B t_{AB}=0$ , i.e. :

$$\dot{\varrho}+(N-1)(p+\varrho)\frac{\dot{r}}{r}+M(q+\varrho)\frac{\dot{R}}{R}=0 \qquad (4$$

To solve the field equations, one must now establish an equation of state for the matter. We consider the equation for "pure radiation" in higher dimensions:

$$p=q=\frac{\varrho}{M+N-1} \qquad (5$$

One can now substitute (5) in (4) and integrate. The result is

$$\varrho\,(r^{N-1}R^{M})^{\frac{N+M}{N+M-1}}=L \qquad (6$$

with L a suitable constant.

This equation can now be inserted into (2) to solve the system.

Unfortunately this is quite difficult and can be solved analytically only in a few special cases [19,81], but numerical solutions can be obtained [19,20]. The resulting scenario is the following: the two radii start expanding until a time $t_0$ , when the internal one starts to collapse. During the collapse, the external radius grows very rapidly (inflation). If no new mechanism stops the collapse, the internal radius will shrink to zero at some finite time $t_0$, while the external one expands to infinity. In order to avoid the new singularity at $t_0$, one must assume that some new mechanism (presumably due to quantum effects), stops the collapse of the internal radius near the Planck length at a time $t_2 < t_0$ , so that after $t_2$ the internal space decouples from the ordinary spacetime, which continues to expand following the usual Robertson-Walker solution.

The behaviour of the two solutions near the two singularities at t=0 and $t=t_0$ can be studied by a power series expansion [20]. Let us consider first the case t=0. We assume that the initial conditions at t=0 are r(t)=R(t)=0, and both radii behave as powers of the time:

$$r(t) = A\ t^{\alpha} + \dots \qquad\qquad R(t) = B\ t^{\beta} + \dots \qquad\qquad (7$$

Substituting (7) into the equations (2) one obtains

$$\alpha = \beta = \frac{2}{N+M} \qquad\qquad (8$$

so that short after the big bang the two radii expand with the same slope.

To obtain the behavior of the solutions near $t_0$, one can proceed in the same manner, defining $\tau = t_0 - t$ , and expanding:

$$r(t) = C\ t^{\gamma} + \dots \qquad\qquad R(t) = D\ t^{\beta} + \dots \qquad\qquad (9$$

From the field equations, one obtains:

$$\gamma = \frac{1 - \sqrt{\frac{M}{N-1}(M+N-2)}}{M+N-1} \qquad\qquad \delta = \frac{1 + \sqrt{\frac{N-1}{M}(M+N-2)}}{M+N-1} \qquad\qquad (10$$

The knowledge of the coefficients $\gamma$ and $\delta$ is important, because, as shown in [20], at first order near $t_0$

$$Q = \frac{r(t_2)}{r(t_1)} \sim \left[\frac{R(t_1)}{R(t_2)}\right]^{-\frac{\gamma}{\delta}} \qquad (11$$

so that one can roughly calculate the amount of inflation suffered by the physical space during the collapse of the internal one, provided one can estimate the rate $r(t_2)/r(t_1)$. This is usually believed to be of the order of $10^2$, while, in order to solve the horizon problem, Q is required to be of the order of $10^{-29}$ [20]. One must then have $-\gamma/\delta \approx -15$, which can be achieved only by postulating a great number of extra dimensions (>40).

An alternative solution to this problem without recurring to a large number of dimensions may be provided by the addition to the action of higher order terms in the curvature [82].

## VI .2 Quasi-Riemannian cosmologies

We wish now to discuss how this scenario is modified in the quasi-Riemannian case [42]. One of the advantages of the quasi-Riemannian cosmological models is that, due to the peculiar structure of the tangent space, the M internal dimensions are distinguished ab initio from the N "physical" ones, so that, contrary to the Riemannian case, there is no arbitrariness in the splitting of the higher-dimensional space. Consequently, the generalization to $Q^{N,M}$ of the Robertson-Walker line element must necessarily have the form (1). More precisely, the vielbeins are:

$$e^0 = dx^0 \equiv dt \qquad e^i = \frac{dx^i}{1 + \frac{hx_i^2}{4}} \qquad e^a = \frac{dx^a}{1 + \frac{kx_a^2}{4}} \qquad (12$$

where the indices 0 and $i = 1,\dots\dots,N-1$ belong to the subgroup $O(1,N-1)$ of $G_T$ and the indices $a = N,\dots\dots,N+M-1$ to the subgroup $O(M)$.

The nonvanishing components of the anholonomity tensor are then:

$$c_{ijo} = -\frac{\dot{r}}{r}\delta_{ij} \qquad c_{ijk} = \frac{h}{2r}(x_k\,\delta_{ij} - x_j\,\delta_{ik})$$

$$c_{abo} = -\frac{\dot{R}}{R}\,\delta_{ab} \qquad c_{abc} = \frac{k}{2R}(x_c\,\delta_{ab} - x_b\,\delta_{ac}) \tag{13}$$

Substituting in (III.28), one obtains the field equations:

$$\frac{A}{2}(N-1)(N-2)\frac{\dot{r}^2+h}{r^2} - M\,\frac{HM+D}{2}\,\frac{\dot{R}^2}{R^2} + \frac{B}{2}M(M-1)\frac{k}{R^2} + AM(N-1)\frac{\dot{r}\,\dot{R}}{r\,R} = \frac{\kappa^2}{2}\rho$$

$$A(N-2)\left[\frac{\ddot{r}}{r} + \frac{N-3}{2}\,\frac{\dot{r}^2+h}{r^2}\right] + AM\,\frac{\ddot{R}}{R} + \frac{M}{2}\left[\,(2A+H)\,M+(D-2A)\,\right]\frac{\dot{R}^2}{R^2} +$$
$$+\frac{B}{2}M(M-1)\,\frac{k}{R^2} + AM(N-2)\,\frac{\dot{r}\,\dot{R}}{r\,R} = -\frac{\kappa^2}{2}\,p \tag{14}$$

$$-(HM+D)\,\frac{\ddot{R}}{R} + \frac{1}{2}\left[\,HM^2+(D-2H)M - 2D\,\right]\frac{\dot{R}^2}{R^2} + \frac{B}{2}(M-1)(M-2)\,\frac{k}{R^2} +$$
$$+A(N-1)\left[\frac{\ddot{r}}{r} + \frac{N-2}{2}\,\frac{\dot{r}^2+h}{r^2}\right] - (N-1)\,(HM+D)\,\frac{\dot{r}\,\dot{R}}{r\,R} = -\frac{\kappa^2}{2}\,q$$

where the general form (3) of the energy momentum tensor has been used.

We shall assume in the following that the conditions (IV.28,29), necessary to have a stable flat space, are valid in our case. Under this assumption, the action depends on three parameters only. For convenience, we shall take as independent parameters A, B and D, which must satisfy the conditions:

$$A, B, D > 0$$

$$\frac{(N-1)(M-2)}{N(M-1)} < \frac{D}{A} < \frac{(N-1)M}{(N-2)(M-1)} \tag{15}$$

The equations of motion are now:

$$\frac{A}{2}(N-1)(N-2)\frac{\dot{r}^2+h}{r^2} + \frac{M(M-1)}{2}\left[D\frac{\dot{R}^2}{R^2} + B\frac{k}{R^2}\right] + AM(N-1)\frac{\dot{r}\dot{R}}{rR} = \frac{\kappa^2}{2}\rho \tag{16}$$

$$\frac{A}{2}(N-2)\left[\frac{\ddot{r}}{r} + \frac{N-3}{2}\frac{\dot{r}^2+h}{r^2}\right] + AM\frac{\ddot{R}}{R} + \frac{M(M-1)}{2}\left[(2A-D)\frac{\dot{R}^2}{R^2} + B\frac{k}{R^2}\right] + \\ +AM(N-2)\frac{\dot{r}\dot{R}}{rR} = -\frac{\kappa^2}{2}p \tag{17}$$

$$D(M-1)\frac{\ddot{R}}{R} + \frac{1}{2}(M-1)(M-2)\left[D\frac{\dot{R}^2}{R^2} + B\frac{k}{R^2}\right] + A(N-1)\left[\frac{\ddot{r}}{r} + \frac{N-2}{2}\frac{\dot{r}^2+h}{r^2}\right] + \\ + D(N-1)(M-1)\frac{\dot{r}\dot{R}}{rR} = -\frac{\kappa^2}{2}q \tag{18}$$

Deriving (18) and comparing with (16) and (17) it is possible to verify that the usual conservation law

$$\dot{\rho} + (N-1)(p+\rho)\frac{\dot{r}}{r} + M(q+\rho)\frac{\dot{R}}{R} = 0 \tag{19}$$

is still valid in our model.

We must now choose the equation of state for the matter. In particular, we are interested in the case of pure radiation. Since, as discussed in chapter V, the definition of an equation of state for radiation on quasi-Riemannian spaces is not obvious, we shall simply adopt that of ordinary matter, and assume that the breakdown of higher dimensional Lorentz invariance resides only in the gravitational interactions. In particular, we shall consider the case of pure radiation, which satisfies the equation of state (5). As in the Riemannian case, one can then obtain the relation (*):

$$\rho\,(r^{N-1}R^{M})^{\frac{N+M}{N+M-1}} = L \qquad (20$$

with L a constant. In order to study the solutions of (16-18), we shall assume for simplicity a flat physical space (h=0) and suppose that the universe begins with a big bang with both the internal and external radii set to zero. The relevant equations can be obtained by substituting (20) in (16-18) and combining them to obtain:

$$a(N-1)\frac{\ddot r}{r} + bM\frac{\ddot R}{R} + (a-b)(N-1)M\left[\frac{\dot R^2}{R^2} - \frac{\dot r\,\dot R}{r\,R}\right] = -\frac{\kappa^2}{2}L\left[r^{N-1}R^{M}\right]^{-\frac{N+M}{N+M-1}}$$

$$(21$$

$$a(N-1)\left[\frac{\ddot r}{r} + (N-2)\frac{\dot r^2}{r^2}\right] + bM\left[\frac{\ddot R}{R} + (M-1)\frac{\dot R^2}{R^2}\right] + (a+b)M(N-1)\frac{\dot r\,\dot R}{r\,R} =$$

$$= \frac{\kappa^2}{2}L\left[r^{N-1}R^{M}\right]^{-\frac{N+M}{N+M-1}} - BM(M-1)\frac{k}{R^2}$$

where $a = (N+M-2)A$ and $b = (N-1)A + (M-1)D$.

As we shall see, the behaviour of the solutions of the field equations is qualitatively analogous to the one obtained in the case of Riemannian geometry, at least for a certain range of values of $D/A$ : the two radii expand until a time $t_1$ when the internal radius starts to shrink, while the

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(*) Had one chosen the quasi-Riemannian equation of state ( V.36 ) for the radiation, one would have obtained:

$$\rho\,(r^{(N-1)(N+zM)}R^{M(N-1+zM-z)})^{\frac{1}{N+zM-1}} = \text{constant}$$

physical space grows hugely. For $t \to t_0$, $r \to \infty$ and $R \to 0$, but quantum effects are supposed to come into play at a time $t_2$, when R reaches the size of the Planck length, stopping the collapse and stabilizing the internal space, while the physical one continues to evolve as in the Robertson-Walker model. However, some differences are present in the details of the two models. This can be seen by studying the behaviour of the solutions near the singularities. Let us therefore expand r(t) and R(t) in powers of t, as in (7) and substitute into the equations of motion (21). It is easy to see that in this limit the term proportional to $B/R^2$ can be neglected and the following relations hold:

$$(N-1)\alpha + M\beta = 2\,\frac{N+M-1}{N+M} \qquad (22$$

$$a(N-1)\alpha(\alpha-1) + bM\beta(\beta-1) + (a-b)(N-1)M\beta(\beta-\alpha) = -\frac{\kappa^2}{2}L\left[A^{N-1}B^{M}\right]^{-\frac{N+M}{N+M+1}}$$

$$[a(N-1)\alpha + bM\beta][(N-1)\alpha + M\beta - 1] + (a+b)(N-1)M\alpha\beta = \frac{\kappa^2}{2}L\left[A^{N-1}B^{M}\right]^{-\frac{N+M}{N+M-1}}$$

The system admits two solutions for $\alpha$ and $\beta$:

$$\alpha_1 = \beta_1 = \frac{2}{N+M}$$

$$\alpha_2 = \frac{2}{N+M}\,\frac{(N+M-1)[AM-D(M-1)]}{AMN-D(N-1)(M-1)} \qquad \beta_2 = \frac{2}{N+M}\,\frac{(N+M-1)A}{AMN-D(N-1)(M-1)} \qquad (23$$

The first one is the same as in the Riemannian case. The second is more interesting since the two radii have a different behaviour. For example, for N=3, M=6, one can choose, according to (15), $D=3/5A$, which gives $\alpha = 9/25$, $\beta = 3/25$ and hence $r(t) \approx R(t)^3$, so that the internal space grows slower than the physical one. It must be noticed, however, that for $0 > D/A > M/M-1$ either $\alpha_2$ or $\beta_2$ changes sign and becomes incompatible with the initial conditions $r(0) = R(0) = 0$. We shall then restrict our considerations to:

$$0 < \frac{D}{A} < \frac{M}{M-1} \qquad (24$$

To study the behaviour of the solution near the second singularity at $t=t_0$, one can expand r and R as in (9) and substitute in (21). In this limit the r.h.s. of (21) can be neglected, and one gets:

$$(N-1)\gamma + M\delta = 1$$

$$a(N-1)\gamma(\gamma-1) + bM\delta(\delta-1) + (a-b)(N-1)M\delta(\delta-\gamma) = 0 \qquad (25$$

$$[a(N-1)\gamma + bM\delta][(N-1)\gamma + M\delta - 1] + (a+b)(N-1)M\gamma\delta = 0$$

The solution of the system is given by:

$$\gamma = \frac{AM - D(M-1) - \sqrt{\frac{M}{N-1}[AM(N-1) - D(M-1)(N-2)]}}{AMN - D(N-1)(M-1)}$$

$$(26$$

$$\delta = \frac{A + \sqrt{\frac{N-1}{M}[AM(N-1) - D(M-1)(N-2)]}}{AMN - D(N-1)(M-1)}$$

In the range of values given by (24), $\delta$ is always positive and $\gamma$ negative, corresponding to the physical space expanding to infinity and the internal one going to zero at $t_0$. With the same values of the constants as before, one has for example $\gamma = -0.13$ and $\delta = 0.23$, to be compared with the Riemannian values $\gamma = -\delta = -0.33$ .

As discussed in the previous section, in order to obtain a sufficient amount of inflation for the physical space, one should have for $-\gamma/\delta$ a value close to 15. Unfortunately, as in the Riemannian case, this can be achieved only by postulating a large number of extra dimensions, because the parameters A and D are strongly constrained by the conditions (15).

It seems then that, in spite of the differences in the geometric structure, the breaking of the higher dimensional Lorentz invariance does not introduce a substantial difference in the physical behaviour of the cosmological solutions of the gravitational field equations. However, any attempt to obtain a realistic estimate of the physical parameters involved in this model, needs a more careful numerical evaluation of the solutions of (21).

## VI. 3 The generalization of the Schwarzschild solution

When one tries to extend to quasi-Riemannian geometries the Schwarzschild solution of general relativity, some difficulties arise. From the discussion at the beginning of this chapter it is in fact evident that it is not possible to construct on $Q^{N,M}$ a solution which is spherically symmetric, where by spherically symmetric we mean a solution with isometry group O(D-1). One can at most find a solution with isometry group O(N-1) x O(M+1). On the other hand, for N=4, this is the physically relevant solution from a Kaluza-Klein point of view, because it describes a space which is "spherically symmetric" in 4 dimensions, and exhibits maximal symmetry in the internal space (*). Unfortunately, no solution of this kind is known for ordinary Riemannian geometry, except in the special case of 11 dimensions [86] or in the case of a flat internal space [87]. In the last case a class of solutions is known, of which the simplest one is the direct product of the N-dimensional Schwarzschild solution with a M-dimensional Ricci-flat space with metric $\tilde{g}_{ab}(y)$:

$$ds^2 = -\left(1 - \frac{2m}{r^{N-1}}\right) dt^2 + \left(1 - \frac{2m}{r^{N-1}}\right)^{-1} dr^2 + r^2 d\Omega^2_{N-2} + R^2 \tilde{g}_{ab}\, dy_a\, dy_b \qquad (27$$

with R constant. We denote by $d\,\Omega_K{}^2$ the metric of the K-sphere, which in appropriate coordinates can be taken to be:

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(*) We do not consider here solutions containing electric or magnetic monopoles of the kind discussed in [83-84]. "Spherically symmetric" solutions in D dimension are discussed in [85].

$$d\Omega_K^2 = \sum_{i=1}^{K} \frac{dx^i dx^i}{\left[1+\frac{x^i x^i}{4}\right]^2} \tag{28}$$

In view of a generalization to quasi-Riemannian geometries, we are however interested in a static solution with O(N-1) x O(M+1) isometry group [88]. The most general ansatz compatible with this symmetry is:

$$ds^2 = -e^{2\nu(r)} dt^2 + e^{2\mu(r)} dr^2 + r^2 d\Omega^2_{N-2} + e^{2\phi(r)} d\Omega^2_M \tag{29}$$

This line element describes a space which is the direct product of a spherically symmetric N-dimensional space and an internal space constituted by a sphere whose radius $e^{2\phi}$ depends on the distance from the origin of the physical space.

The nonvanishing components of the Riemann tensor calculated from (29) are:

$$R_{0101} = e^{-2\mu}(\nu'' + \nu'^2 - \nu'\mu') \tag{30}$$

$$R_{oioj} = \frac{\nu'}{r} e^{-2\mu} \delta_{ij} \qquad R_{1i1j} = \frac{\mu'}{r} e^{-2\mu} \delta_{ij}$$

$$R_{oaob} = e^{-2\mu} \phi' \nu' \delta_{ab} \qquad R_{1a1b} = -e^{-2\mu}(\phi'' + \phi'^2 - \phi'\mu')\delta_{ab}$$

$$R_{ijkl} = \frac{1-e^{-2\mu}}{r^2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \qquad R_{abcd} = (e^{-2\phi} - \phi'^2 e^{-2\mu})(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})$$

$$R_{iajb} = -e^{-2\mu}\frac{\phi'}{r}\delta_{ij}\delta_{ab}$$

where a prime denotes a derivative with respect to r and i,j run from 2 to N-1 while a,b from N to N+M-1.

The ensuing Einstein equations are:

$$e^{-2\mu}[\nu'' + \nu'^2 - \nu'\mu' + \frac{N-2}{r}\nu' + M\phi'\nu'] = 0 \tag{31}$$

$$- e^{-2\mu} [ \nu'' + \nu'^2 - \nu'\mu' - \frac{N-2}{r} \mu' + M ( \phi'' + \phi'^2 - \phi'\mu' )] = 0 \qquad (32$$

$$(N-3) \frac{1 - e^{-2\mu}}{r^2} + \frac{e^{-2\mu}}{r} ( \mu' - \nu' - M \phi' ) = 0 \qquad (33$$

$$(M-1) e^{-2\phi} - e^{-2\mu} [ \phi'' + \phi' (M \phi' + \nu' - \mu' ) + \frac{N-2}{r} \phi' ] = 0 \qquad (34$$

Summing the first two equations one obtains:

$$- \frac{N-2}{r} ( \mu' + \nu' ) + M [ \phi'' + \phi'^2 - \phi' ( \mu' + \nu' )] = 0 \qquad (35$$

which can be solved using the simple ansatz

$$\nu' + \mu' = 0 \qquad (36$$

from which follows the equation

$$\phi'' + \phi'^2 = 0 \qquad (37$$

It admits a trivial solution $\phi = 0$, corresponding to the Ricci flat case discussed above, and a more interesting one given by:

$$\phi = \ln (r-c) - ½ d \qquad (38$$

with c and d two integration constants. A solution to the system (31-34) can be obtained if c=0. In this case, substituting (36) and (38) in (33), and solving the differential equations, one obtains:

$$e^{-2\mu} = e^{2\nu} = \frac{N-3}{N+M-3} + \frac{b}{r^{N+M-3}} \qquad (39$$

with b an integration constant. Finally, comparing (33) with (34) one can fix the value of d:

$$d = \frac{N-3}{M-1} \tag{40}$$

By rescaling r and t and defining a new integration constant m out of b, one can finally write the solution as (*):

$$ds^2 = -\left(1-\frac{2m}{r^{N+M-3}}\right)dt^2 + \left(1-\frac{2m}{r^{N+M-3}}\right)^{-1}dr^2$$
$$+\frac{N-3}{M+N-3}\,r^2\,d\Omega^2_{N-2} + \frac{M-1}{M+N-3}\,r^2\,d\Omega^2_{M} \tag{41}$$

An explicit computation of the Riemann tensor shows that the metric has only one singularity at r=0. Moreover, it is evident that an event horizon is present at $r^{D-3}= 2m$ and that the metric has a good behaviour at spatial infinity. The solution describes an (N-1)-dimensional black hole with an unusual topology. Unfortunately, however, it is not suitable for a Kaluza-Klein interpretation, essentially because the size of the internal space diverges at spatial infinity.

To show this, let us consider for a moment the physical case N=4. As it is well known [89], after dimensional reduction, one is left with an effective metric ${}^4ds^2$, obtained by Weyl rescaling and projecting the original one, and a scalar field $\Phi$. In our case they are:

$${}^4ds^2 = -r^M\left(1-\frac{2m}{r^{M+1}}\right)dt^2 + r^M\left(1-\frac{2m}{r^{M+1}}\right)dr^2 + \frac{r^{M+2}}{M+1}\,d\Omega^2_2$$
$$\Phi = \sqrt{\tfrac{1}{2}M(M+1)}\;\ln r \tag{42}$$

The effective metric so obtained is not acceptable as a 4-dimensional black hole, because it is not asymptotically flat, due to the factors $r^M$. Moreover, also the scalar field $\Phi$ diverges at spatial infinity, and is then unphysical.

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(*) This solution has been obtained also in [86] in the 11-dimensional case.

We turn now to the generalization to quasi-Riemannian theories [88]. We are interested in a metric of the form (29). The appropriate ansatz for the vielbein is then:

$$e^0 = e^{\nu(r)}\, dt \qquad\qquad e^1 = e^{\mu(r)}\, dr$$

$$e^i = r\, \frac{dx^i}{1+\frac{x^k x^k}{4}} \qquad\qquad e^a = e^{\phi(r)}\, \frac{dx^a}{1+\frac{x^c x^c}{4}} \tag{43}$$

with 0,1 and $i = 2,\ldots,N-1$ belonging to the $O(1,N-1)$ part and $a = N,\ldots,N+M-1$ to the $O(M)$ part of $G_T$.

The nonvanishing components of the anholonomity of the vielbeins (43) are:

$$c_{001} = -\nu'\, e^{-2\mu}$$

$$c_{ij1} = -\frac{e^{-2\mu}}{r}\,\delta_{ij} \qquad\qquad c_{ijk} = \frac{1}{2r}\,(x_k\delta_{ij} - x_j\delta_{ik}) \tag{44}$$

$$c_{ab1} = -\phi'\, e^{-2\mu}\,\delta_{ab} \qquad\qquad c_{abc} = \tfrac{1}{2}\, e^{-2\phi}\,(x_c\delta_{ab} - x_b\delta_{ac})$$

The field equations can now be derived from (III.28):

$$e^{-2\mu}\left\{A\left[-\frac{N-2}{r}\,(-\mu' + M\phi') - M\phi'' + M\phi'\mu'\right] - \left[(2A+H)M+D\right]M\,\frac{\phi'^2}{2}\right\} + \\ + \frac{B}{2}M(M-1)e^{-2\phi} - \frac{A}{2}\,\frac{(N-2)(N-3)}{r^2}\,(e^{-2\mu}-1) = 0$$

(45

$$e^{-2\mu}\left\{A\left[-\frac{N-2}{r}\,(\nu' + M\phi') - M\phi'\mu'\right] - (HM+D)M\,\frac{\phi'^2}{2}\right\} + \\ + \frac{B}{2}M(M-1)e^{-2\phi} - \frac{A}{2}\,\frac{(N-2)(N-3)}{r^2}\,(e^{-2\mu}-1) = 0$$

$$e^{-2\mu}\Big\{A\Big[-\frac{N-3}{r}(\nu'-\mu'+M\phi')-M\phi''-M\phi'(\nu'-\mu')-\nu''-\nu'(\nu'-\mu')\Big]-$$

$$-[(2A+H)M+D]M\frac{\phi'^2}{2}\Big\}+\frac{B}{2}M(M-1)e^{-2\phi}-\frac{A}{2}\frac{(N-3)(N-4)}{r^2}(e^{-2\mu}-1)=0$$

(45

$$e^{-2\mu}\Big\{-(HM+D)\Big[-\frac{N-2}{r}\phi'-\phi''-\phi'(\nu'-\mu')-\phi'(\nu'-\mu')-M\frac{\phi'^2}{2}\Big]+A\Big[-\nu''-\nu'(\nu'-\mu')-$$

$$-\frac{N-2}{r}(\nu'-\mu')\Big]\Big\}+\frac{B}{2}(M-1)(M-2)e^{-2\phi}-\frac{A}{2}\frac{(N-2)(N-3)}{r^2}(e^{-2\mu}-1)=0$$

As usual, we assume that the conditions (15) hold. In this case, eqns. (45) can be arranged to read:

$$\nu''+\nu'(\nu'-\mu'+M\phi')+\frac{N-2}{r}\nu'+\Theta[\phi''+\phi'(\nu'-\mu'+M\phi')+\frac{N-2}{r}\phi']=0$$

$$\nu''+\nu'(\nu'-\mu')-\frac{N-2}{r}\mu'+(\Theta+M)[\phi''+\phi'(\phi'-\mu')]+$$

$$+\Theta[\frac{N-2}{r}\phi'+\phi'(\nu'-(N-1)\phi')+\frac{N-2}{r}\phi']=0$$

(46

$$(\nu'-\mu'+M\phi')\frac{e^{-2\mu}}{r}+\frac{N-3}{r^2}(e^{-2\mu}-1)+$$

$$\Theta[\phi''+\phi'(\nu'-\mu'+M\phi')+\frac{N-2}{r}\phi']e^{-2\mu}=0$$

$$(M-1)e^{-2\phi}-\Omega e^{-2\mu}[\phi''+\phi'(M\phi'+\nu'-\mu')+\frac{N-2}{r}\phi']=0$$

where

$$\Theta=\frac{(D-A)M(M-1)}{A(N+M-2)}\qquad\qquad \Omega=\frac{A(N-1)M-D(N-2)(M-1)}{B(N+M-2)}\qquad (47$$

The Riemannian solution suggests to consider the ansatz:

$$e^{-2\phi} = d\, r^{-2} \tag{48}$$

Substituting in (46) one obtains three independent equations:

$$\frac{\nu' + \mu'}{r} + \frac{\Theta}{r^2} = 0 \tag{49}$$

$$(\Theta + 1)\left[\frac{\nu' - \mu'}{r} + \frac{M+N-3}{r^2}\right] e^{-2\mu} - \frac{N-3}{r^2} = 0 \tag{50}$$

$$\Omega\left[\frac{\nu' - \mu'}{r} + \frac{M+N-3}{r^2}\right] e^{-2\mu} - d\,\frac{M-1}{r^2} = 0 \tag{51}$$

Substituting (49) in (50) and integrating yields:

$$e^{-2\mu} = \frac{N-3}{(1+\Theta)(N+M-3-\Theta)} - \frac{b}{r^{N+M-3-\Theta}} \tag{52}$$

and then

$$e^{2\nu} = \frac{k}{r^{2\Theta}} e^{-2\mu} = \frac{k}{r^{2\Theta}}\left[\frac{N-3}{(1+\Theta)(N+M-3-\Theta)} - \frac{b}{r^{N+M-3-\Theta}}\right] \tag{53}$$

where b and k are integration constants.
Finally, comparing (50) and (51) one can determine d:

$$d = \frac{N-3}{M-1}\,\frac{\Omega}{1+\Theta} \tag{54}$$

After rescaling t and r, the metric can be put in the form

$$ds^2 = -\frac{k'}{R^{2\Theta}}\left(1 - \frac{2m}{R^{\Delta}}\right) dt^2 + \left(1 - \frac{2m}{R^{\Delta}}\right)^{-1} dr^2 + \frac{N-3}{\Delta(1+\Theta)}\, r^2 d\Omega^2_{N-2} + \frac{M-1}{\Delta\Omega}\, r^2 d\Omega^2_M$$

where

$$\Delta = N + M - 3 - \Theta \tag{55}$$

In order to study the singularities of the solution, it is useful to use the formalism of de Alwis et al. [38].

One can consider the invariants built up by means of the nonvanishing components of the connection and of the contorsion, which are

$$R_{0101} = [\, m\,\Delta\,(\Delta+1+3\Theta) - \Theta\,(\Theta+1)\,(1-2mr^{-\Delta})\,]\, r^{-2}$$

$$R_{0i0j} = [\, \Theta - m\,(\Delta+2\Theta)\, r^{-\Delta})\,]\, r^{-2}\,\delta_{ij}$$

$$R_{1i1j} = -\, m\,\Delta\, r^{-\Delta-2}\,\delta_{ij} \tag{56}$$

$$R_{ijkl} = \left[\frac{(1+\Theta)\Delta}{N-3} - 1 + 2m\, r^{-\Delta}\right] r^{-2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

$$R_{abcd} = \left[\frac{\Omega\,\Delta}{M-1} - 1 + 2m\, r^{-\Delta}\right] r^{-2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})$$

$$\varpi_{1ab} = -\,(1 - 2mr^{-\Delta})^{-1/2}\, r^{-1}\,\delta_{ij}$$

Also in this case a unique singularity is present at $r=0$ , and it is hidden by an event horizon at $r^{\Delta} = 2m$. If $\Theta \neq 0$, however, the metric seems to suffer from some pathologies. For $\Theta > 0$ it diverges at infinity, while for $\Theta < 0$ the spatial infinity is a null surface. Investigations are in progress in order to understand the physical significance of this behaviour.

In any case, the solution is genuinely higher-dimensional, in the sense that it is difficult to make sense of its 4-dimensional projection for reasons analogous to the ones discussed in the Riemannian limit.

We notice, however, that (27) is also a solution of the quasi-Riemannian field equations, and is the suitable one if one looks for a metric which reduces to the Schwarzschild solution after dimensional reduction.

To conclude we notice that both the Riemannian and quasi-Riemannian solutions we have discussed above might be a special case of a more general class, obtained by generalizing the ansätze (36) and (48), but we were not able to find any solution of this more general kind.

# VII . SPONTANEOUS COMPACTIFICATION IN QUASI-RIEMANNIAN THEORIES

In this chapter, we consider the possible applications of quasi-Riemannian theories to the Kaluza-Klein unification of gravity and gauge interactions.

The most interesting feature of quasi-Riemannian theories in this context is the possibility of obtaining chiral fermions by dimensional reduction of the purely gravitational higher-dimensional theory [30]. We shall discuss this possibility in a simple case. Another interesting property is the existence of solutions of the field equations which exhibit spontaneous compactification to the product of a maximally symmetric 4-dimensional spacetime with a compact internal space. We shall illustrate a model of this kind, which unfortunately is only partially satisfactory, and discuss the problems which arise when one looks for more realistic solutions.

## VII .1 Compactification on tori and spheres

First of all, we consider the simplest case of compactification on an M-torus $T^M$. It is easy to see that if the cosmological constant $\lambda$ is set to zero, the field equations admit a solution $M^N x T^M$, with $M^N$ the flat Minkowski space. The physical situation arises for N=4.

Most of the results necessary to discuss this solution have already been obtained in the discussion of chapter IV on the stability of the flat space. The spectrum of masses is extracted from that obtained there by simply substituting for the values of the square of the "internal" momentum $k_a{}^2$ the eigenvalues of the Laplace operator on the torus, which are given by

$$\sum_{i=1}^{M} \left(\frac{n_i}{2\pi a_i}\right)^2 \qquad (1$$

where $a_i$ is the radius of the kth coordinate and $n_i$ are M integers labeling the representations of the group $[U(1)]^M$, which is the isometry group of the torus.

From the discussion of chapter IV it is easy to see that if one does not impose any constraints on the free parameters of the action, some massless vectors are present besides the ones one would expect as gauge bosons of the group $[U(1)]^M$. These have longitudinal components which give rise to ghosts. We shall examine in more detail this fact in the next section, in the simple case of the 5-dimensional theory.

If $N>4$, one can consider a compactification to $M^4 \times T^{N+M-4}$. In this case the spectrum becomes a bit more complicated, but no qualitatively new features appear, and in particular the conditions for stability are unchanged.

A more interesting kind of compactification is that in which the internal space is a coset space, since it leads to nonabelian gauge symmetries. It is then interesting to check if quasi-Riemannian theories admit spontaneous compactification to $M^4 \times G/H$.

The simplest case occurs when $N=4$. In this case the $O(1,N-1)$ indices refer to the physical spacetime, and the $O(M)$ to the internal space. By consequence, the field equations factorize. It is easy to see that the solutions must have vanishing torsion and satisfy the equations:

$$A\, R_{\alpha\gamma\gamma\beta} = -\frac{\lambda}{N+M-2}\,\eta_{\alpha\beta} \qquad\qquad B\, R_{accb} = -\frac{\lambda}{N+M-2}\,\eta_{ab} \qquad (2$$

They are very similar to the Riemannian ones. For simplicity let us consider the case in which the ground state is the product of two maximally symmetric spaces $M^N \times M^M$. This is the most symmetric solution compatible with the structure of $Q^{N,M}$. The curvature tensor is given by:

$$R_{\alpha\beta\gamma\delta} = \frac{h}{a^2}(\eta_{\alpha\gamma}\eta_{\beta\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma}) \qquad\qquad R_{abcd} = \frac{k}{b^2}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) \qquad (3$$

where $h>0$ corresponds to de Sitter space and $h<0$ to Anti-de Sitter space.

Substituting in (2) one has:

$$\frac{Ah}{a^2}(N-1) = \frac{Bk}{b^2}(M-1) = \frac{\lambda}{N+M-2} \qquad (4$$

It is evident that one obtains a flat N-space only if $\lambda = 0$ , but in this case also the internal space must be flat. As discussed in section II.5, the other physically relevant possibility is Anti-deSitter physical space and compact internal space with negative curvature. This is a solution if A and B have opposite sign, but this presumably leads to the appearence of ghost states in the 4-dimensional spectrum. Moreover, if one requires b to be of the order of the Planck scale and a to be compatible with the observed value of the cosmological constant, the ratio $B/A$ should be of the order of $10^{-120}$. Another reason for discarding this kind of compactification is that it does not give rise to chiral fermions.

We are then led to consider the case N>4. The most symmetric solution compatible with $G_T$ is now a direct product of three maximally symmetric spaces $M^4 \times M^{N-4} \times M^M$. In this case the field equations are:

$$\frac{3Ah}{a^2} = \frac{Ak}{b^2}(N-5) = \frac{Bl}{c^2}(M-1) = \frac{\lambda}{N+M-2} \qquad (5$$

where a, b and c are the radii of $M^4$ , $M^{N-4}$ and $M^M$ respectively. If k>0 (compact internal space), the physical spacetime must be deSitter, and moreover the sizes of $M^4$ and $M^{N-4}$ are compatible. This situation extends also to more complicated cases of compactification to $M^4 \times G/H$. We shall discuss in section 5 some possible solutions of this problem.

## VII .2 The five-dimensional case

In order to illustrate in a simple case the differences between compactification in ordinary and quasi-Riemannian Kaluza-Klein theories, we consider the case of a five dimensional theory with tangent space group O(1,3), compactified to $M^4 \times S^1$.

First of all, we specialize to five dimensions some basic results obtained in chapter III and IV for the general case.

The most general invariant action on $Q^{4,1}$ is given by

$$S = -\frac{1}{\kappa^2}\int e_5\, d^5z \{ A [ \tfrac{1}{4} c_{\alpha\beta\gamma} c_{\alpha\beta\gamma} - \tfrac{1}{2} c_{\alpha\beta\gamma} c_{\beta\gamma\alpha} - c_{\alpha\alpha\gamma} c_{\beta\beta\gamma}$$
$$- 2 c_{55\gamma} c_{\beta\beta\gamma} - c_{5\beta\gamma} c_{\beta\gamma 5} ] + C\, c_{(\alpha\beta)5} c_{(\alpha\beta)5} \qquad (6$$
$$+ \tfrac{1}{4} E\, c_{5\beta\gamma} c_{5\beta\gamma} + G\, c_{\alpha\alpha 5} c_{\beta\beta 5} + H\, c_{55\gamma} c_{55\gamma} \}$$

were we have put the cosmological constant $\lambda = 0$. The action depends on five parameters only, because M=1 is a degenerate case.

We assume the ground state to be $M^4 \times S^1$, where $S^1$ is a circle of radius a. The background vielbein is then:

$$\bar{e}^A{}_M = \begin{pmatrix} \delta^\alpha{}_\mu & 0 \\ 0 & a \end{pmatrix} \qquad (7$$

Expanding around this ground state, one can obtain the bilinear action. We are mainly interested in the massless sector ($\partial_5=0$), for which it takes the form:

$$S_2 = \int d^5z \{ A [ h_{(\alpha\beta)} \partial^2 h_{(\alpha\beta)} - 2 h_{(\alpha\beta)} \partial_\beta \partial_\gamma h_{(\alpha\gamma)} - h_{\alpha\alpha} \partial^2 h_{\beta\beta} + 2 h_{\alpha\alpha} \partial_\beta \partial_\gamma h_{(\beta\gamma)}$$
$$- h_{5\gamma} \partial^2 h_{\gamma 5} + h_{5\alpha} \partial_\alpha \partial_\beta h_{\beta 5} + 2 h_{55} \partial^2 h_{\gamma\gamma} - 2 h_{55} \partial_\alpha \partial_\beta h_{(\alpha\beta)} ]$$
$$- \tfrac{1}{2} C ( h_{\gamma 5} \partial^2 h_{\gamma 5} + h_{\alpha 5} \partial_\alpha \partial_\beta h_{\beta 5} ) - \tfrac{1}{2} E ( h_{5\gamma} \partial^2 h_{5\gamma} - h_{5\alpha} \partial_\alpha \partial_\beta h_{5\beta} ) \qquad (8$$
$$- G\, h_{\alpha 5} \partial_\alpha \partial_\beta h_{\beta 5} - H\, h_{55} \partial^2 h_{55} \}$$

One can then calculate the spectrum as in the higher-dimensional case. The massless states are the graviton, two vectors coming from a mixing of the $h_{\alpha 5}$ and $h_{5\alpha}$ excitations, and two scalars corresponding to the longitudinal part of $h_{5\alpha}$, i.e. $h_{5+}$ and $h_{5-}$. These scalars are ghostly and, in order to eliminate them one must impose some strong conditions on the parameters:

$$A^2 - EC = 0 \qquad G = -C \qquad H = 0 \tag{9}$$

These conditions render the action invariant under the gauge transformations:

$$\delta h_{5\alpha} = \varepsilon_{5\alpha} \qquad \delta h_{\alpha 5} = -\frac{A}{C}\varepsilon_{5\alpha} \tag{10}$$

Under these constraints, the massless spectrum of the theory changes drastically, and one is left, as in the Riemannian case, with one graviton, one photon and one scalar.

The massive spectrum is given by a tower of spin 2 particles with

$$M^2 = \frac{C}{A} k_5^2 = \frac{C}{A}\left[\frac{n}{2\pi a}\right]^2 \tag{11}$$

and integer n, to be compared with the Riemannian case where $M^2 = ({}^{n}/_{2\pi a})^2$.

We wish now to investigate what kind of zero-mode ansatz is appropriate for the (unconstrained) 5-dimensional theory, to better understand the origin of the doubling of the massless vectors in the spectrum [88]. It is interesting to note that, contrary to the standard Kaluza- Klein theory the new massless vectors do not seem to be related to any local invariance.

In ordinary Kaluza-Klein theory, the zero-mode ansatz takes the form

$$e^A{}_M = \begin{pmatrix} e^\alpha{}_\mu(x^\mu) & A_\mu(x^\mu) \\ 0 & \phi(x^\mu) \end{pmatrix} \tag{12}$$

It is important to notice, however, that, as pointed out by Cremmer and Julia long time ago [90], this ansatz is only possible because when the original O(1,4) invariance of the theory is broken to O(1,3) by the choice of the ground state, one can still profit from the original invariance, choosing a gauge in which the lower left hand corner of the matrix is vanishing. In our case, on the contrary, this is not possible, because the original invariance of the theory is too small (*).

One is then forced to consider the more general ansatz

$$e^{A}{}_{M} = \begin{pmatrix} e^{\alpha}{}_{\mu}(x^{\mu}) & A_{\mu}(x^{\mu}) \\ B^{\alpha}(x^{\mu}) & \phi(x^{\mu}) \end{pmatrix} \tag{13}$$

This accounts for the presence of two massless vector fields in the spectrum of the theory, corresponding to $A_{\mu}$ and $B_{\mu} = e^{\alpha}{}_{\mu} B^{\alpha}$. We also point out that it is not consistent with the field equations to put $B_{\mu} = 0$ and $A_{\mu} \neq 0$. This can be seen from the equations obtained by substituting the ansatz (12) into the field equations stemming from the action (6). It is easy to see by comparing the $\alpha 5$ and $5\alpha$ components of these equations that they require $F_{\mu\nu} = 0$. This is analogous to the well-known fact that in ordinary Kaluza-Klein theory the scalar field $\phi$ cannot be consistently taken to be constant [54].

In analogy with the Riemannian case, the ansatz (13) is invariant under the transformations:

$$x^{\mu} \rightarrow x'^{\mu} = x'^{\mu}(x^{\mu}) \tag{14}$$

$$x^{5} \rightarrow x'^{5} = x^{5} + \lambda(x^{\mu}) \tag{15}$$

As usual, (14) can be identified with the 4-dimensional coordinate transformations, and (15) with gauge transformations. In this case, however, the gauge transformations assume a more complicated form:

$$\begin{aligned} e^{\alpha}{}_{\mu}(x^{\mu}) &\rightarrow e'^{\alpha}{}_{\mu}(x^{\mu}) = e^{\alpha}{}_{\mu}(x^{\mu}) + \partial_{\mu}\lambda(x^{\mu})\, B^{\alpha}(x^{\mu}) \\ A_{\mu}(x^{\mu}) &\rightarrow A'_{\mu}(x^{\mu}) = A_{\mu}(x^{\mu}) + \partial_{\mu}\lambda(x^{\mu})\, \phi(x^{\mu}) \\ B^{\alpha}(x^{\mu}) &\rightarrow B'^{\alpha}(x^{\mu}) = B^{\alpha}(x^{\mu}) \\ \phi(x^{\mu}) &\rightarrow \phi'(x^{\mu}) = \phi(x^{\mu}) \end{aligned} \tag{16}$$

(*) This argument can easily be generalized to higher dimensions.

In particular, the fact that the vielbein is affected by gauge transformations, renders non-trivial to check the gauge invariance of the effective action. This can be obtained from (13) by calculating the anholonomity coefficients:

$$\hat{c}_{\alpha\beta\gamma} = c_{\alpha\beta\gamma} + \frac{A_\beta\,(\,\partial_\gamma B_\alpha - c_{\alpha\gamma\delta}B_\delta\,) - (\,\beta \leftrightarrow \gamma)}{\phi - A_\delta B_\delta}$$

$$\hat{c}_{\alpha\beta 5} = \frac{\partial_\beta B_\alpha - c_{\alpha\beta\gamma}B_\gamma}{\phi - A_\delta B_\delta}$$

(17

$$\hat{c}_{5\beta\gamma} = F_{\beta\gamma} + \frac{A_\beta B_\delta F_{\delta\gamma} + A_\beta \partial_\gamma \phi - (\beta \leftrightarrow \gamma)}{\phi - A_\alpha B_\alpha}$$

$$\hat{c}_{55\gamma} = -\frac{B_\beta F_{\beta\gamma} + \partial_\gamma \phi}{\phi - A_\alpha B_\alpha}$$

where the hat refers to 5-dimensional quantities, and the determinant

$$e_5 = e_4\,(\,\phi - A_\alpha B_\alpha\,) \qquad (18$$

Substituting in (6) one has:

$$S = -\frac{2\pi a}{\kappa^2}\int \frac{e_4\, d^4x}{\phi - A_\alpha B_\alpha}\Big\{A\,[\,(\phi - A_\alpha B_\alpha)^2 R_4 - F_{\beta\gamma}\nabla_\beta B_\gamma\,] + C\,G_{\beta\alpha}G_{\beta\alpha}$$

$$+ G\,G_{\alpha\alpha}\,G_{\beta\beta} + \frac{E}{4}\,(\phi - A_\alpha B_\alpha)^2 F_{\beta\gamma}F_{\beta\gamma} + H\,\partial_\gamma\phi\,\partial_\gamma\phi \qquad (19$$

$$+ \text{higher order terms}\Big\}$$

where $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$ and $G_{\alpha\beta} = \nabla_\alpha B_\beta + \nabla_\beta B_\alpha$, with $\nabla_\alpha$ the usual (torsionless) Riemannian covariant derivative of the 4-dimensional manifold and $R_4$ the corresponding Ricci scalar.

In order to obtain the physical action, one should now Weyl rescale the metric, so that a constant coefficient appears in front of $R_4$ [90].

We do not pursue this in the general case. We rather notice that, when the conditions (9) are imposed, a new gauge invariance is introduced into the linearized theory. If this invariance can be extended to the full action, it may be exploited to eliminate the field $B_\alpha$ and hence to use the standard ansatz (7). In this case, after Weyl rescaling, one has

$$S = -\frac{2\pi a}{\kappa^2}\int e_4\, d^4x\, [\, A\, (\, R_4 + \frac{1}{6}\partial_\alpha\phi\, \partial_\alpha\phi\, ) + \frac{E}{4}\phi^2\, F_{\alpha\beta}F_{\alpha\beta}\, ] \qquad (20$$

In the ground state $\phi = a$, and one can relate the parameters of the theory to the observed physical constants:

$$\frac{2\pi a A}{\kappa^2} = \frac{1}{16\pi G_N} \qquad\qquad e^2 = \frac{\kappa^2}{2\pi a^3 E} = \frac{16\pi G_N}{a^2}\,\frac{A}{E} \qquad (21$$

and then

$$\frac{1}{a^2} = \frac{E}{A}\,\frac{e^2}{16\pi G_N} \qquad (22$$

This relation suggests the possibility that if $E \ll A$, the compactification may require a scale for the internal space larger than the Planck one.

## VII .3 Compactification on coset spaces and zero-mode ansatz

We can now generalize the previous discussion to the case of compactification to coset spaces in higher dimensions. We consider the case in which N>4.

The most important observation is that not every coset space can be chosen as internal space [39]. As was discussed in section II.5, in fact, in order to have invariance under G transformations, the isotropy group H must be embedded in $G_T$ in such a way that the matrices $D_\alpha{}^\beta(h^{-1})$ from the adjoint representation of G can be considered as $G_T$-transformations of the internal space. In our case the matrices of $G_T$ are block-diagonal and then one must have

$$D_{\tilde{a}}{}^{\tilde{\beta}}(h^{-1}) = D_{\tilde{\alpha}}{}^{\tilde{b}}(h^{-1}) = 0 \tag{23}$$

Infinitesimally, these conditions can be written as

$$f_{\tilde{\alpha}\tilde{b}\bar{C}} = f_{\tilde{b}\tilde{\alpha}\bar{C}} = 0 \tag{24}$$

where $f_{\hat{A}\hat{B}\hat{C}}$ are the structure constants of G.

In another way, one can say that the embedding of H in $G_T$ via the usual formula

$$Q_{\bar{C}} = \tfrac{1}{2} f_{\tilde{\alpha}\tilde{\beta}\bar{C}} \Sigma^{\tilde{\alpha}\tilde{\beta}} + \tfrac{1}{2} f_{\tilde{a}\tilde{b}\bar{C}} \Sigma^{\tilde{a}\tilde{b}} \tag{25}$$

succeeds only if (24) is satisfied.

An important consequence of this is that the generators $Q_{\tilde{\alpha}}$ and $Q_{\tilde{a}}$ carry two independent representations of H, since

$$[Q_{\tilde{\alpha}}, Q_{\bar{B}}] \subset \{Q_{\tilde{\alpha}}\} \qquad [Q_{\tilde{a}}, Q_{\bar{B}}] \subset \{Q_{\tilde{a}}\} \tag{26}$$

We assume here that the gauge invariance introduced in chapter IV at the linearized level can be extended to the nonlinear theory, and that it is then possible to write the zero-mode ansatz in the usual form:

$$\begin{pmatrix} e^{\dot{A}}{}_{\dot{M}}(x) & -a\, D_{\hat{B}}{}^{\tilde{A}}(y)\, A^{\hat{B}}{}_{\dot{M}}(x) \\ 0 & e^{\tilde{A}}{}_{\tilde{M}}(y) \end{pmatrix} \tag{27}$$

where a is a constant related to the length scale of the coset space.

As in the Riemannian case, if the condition (23) is satisfied, the ansatz is invariant under:

$$x^{\dot M} \to x'^{\dot M}(x^{\dot M})$$

$$y^{\tilde M} \to y'^{\tilde M}(x^{\dot M}, y^{\tilde N}) \qquad (28$$

representing 4-dimensional coordinate and G-transformations.

In this case it is easy to calculate the effective action by means of the anholonomity coefficients:

$$\hat{C}^{\dot A}{}_{\dot B \dot C} = C^{\dot A}{}_{\dot B \dot C} \qquad \hat{C}^{\tilde A}{}_{\tilde B \tilde C} = C^{\tilde A}{}_{\tilde B \tilde C}$$

$$\hat{C}^{\dot A}{}_{\dot B \tilde C} = 0 \qquad \hat{C}^{\tilde A}{}_{\tilde B \dot C} = a\, A^{\hat D}{}_{\dot C}\,(D_{\hat D}{}^{\bar F} - e^{\bar F}{}_{\tilde M}\, e_{\tilde E}{}^{\tilde M} D_{\hat D}{}^{\tilde E})\, f^{\tilde A}{}_{\tilde B \bar F} \qquad (29$$

$$\hat{C}^{\dot A}{}_{\tilde B \tilde C} = 0 \qquad \hat{C}^{\tilde A}{}_{\dot B \dot C} = a\, D_{\hat D}{}^{\tilde A}\, F^{\hat D}{}_{\dot B \dot C}$$

where

$$F^{\hat A}{}_{\dot B \dot C} = \partial_{\dot B} A^{\hat A}{}_{\dot C} - \partial_{\dot C} A^{\hat A}{}_{\dot B} - f^{\hat A}{}_{\hat B \hat C}\, A^{\hat B}{}_{\dot B}\, A^{\hat C}{}_{\dot C} \qquad (30$$

Substituting into the action (III. 15) one obtains [39]:

$$-\frac{1}{\kappa^2}\int_{G/H} e_4 d^4x\, d\mu \left[ A R_4 + \frac{a^2}{4}(D_{\hat B}{}^{\tilde\alpha} D_{\hat C}{}^{\tilde\alpha} + E\, D_{\hat B}{}^{\tilde a} D_{\hat C}{}^{\tilde a})\, F^{\hat B}{}_{\dot\beta\dot\gamma} F^{\hat C}{}_{\dot\beta\dot\gamma} \right] \qquad (31$$

where $d\mu = e_K\, d^K y$ is the G-invariant measure on G/H and $R_4$ is the usual Ricci scalar of general relativtity.

The integration can be performed by using the methods of harmonic expansion on coset spaces [24,49] and exploiting (26) to show that $D_{\hat B}{}^{\tilde\alpha} D_{\hat C}{}^{\tilde\alpha}$ and $D_{\hat B}{}^{\tilde a} D_{\hat C}{}^{\tilde a}$ are covariant G-tensors. The result is

$$S = -\frac{V_K}{\kappa^2}\int e_4\, d^4x \left[ A R_4 + \frac{a^2}{4} q\, F^{\hat A}{}_{\dot\beta\dot\gamma} F^{\hat A}{}_{\dot\beta\dot\gamma} \right] \qquad (32$$

where

$$q = \frac{A\,(N-4)}{\dim G} + \frac{EM}{\dim G} \tag{33}$$

## VII .4 Compactification to $M^4 \times SU(2)xU(1)/U(1)$

A simple model exhibiting the main features of the compactification in quasi-Riemannin theories and in particular showing the possibility of obtaining chiral fermions after dimensional reduction, was suggested by Weinberg in his original paper on quasi-Riemannian geometries [30]. However, he did not discuss the possibility of obtaining it as a solution of the theory. We want to discuss it in some detail and to show that it is in fact a solution of the field equations of section III. 3 [88].

We consider a 7-dimensional space with tangent group O(1,5) instead of the usual O(1,6). The internal part of the tangent space group is then simply $G'_T = O(2)$. We assume that the internal space is maximally symmetric, so that H has an isotropy group $O(2) \sim U(1)$ and an isometry group $O(3)xO(2) \sim SU(2)xU(1)$. The generator of H is taken to be a linear combination of the generator of the U(1) part of G and of one of the generators of SU(2), namely:

$$Q_{\bar{0}} = \frac{Q_{\hat{3}} + k\, Q_{\hat{0}}}{\sqrt{1+k^2}} \tag{34}$$

where k can be chosen integer by an appropriate normalization of $Q_{\hat{0}}$. Here $Q_{\hat{0}}$ is the generator of the U(1) factor of G, while $Q_1$, $Q_2$ and $Q_3$ are the generators of SU(2). Finally, $Q_0$ generates H.

We use the following generators for the coset space:

$$Q_{\tilde{1}} = Q_{\hat{1}} \qquad\qquad Q_{\tilde{2}} = Q_{\hat{2}}$$

$$Q_{\tilde{3}} = \mu\, \frac{Q_{\hat{0}} - k\, Q_{\hat{3}}}{\sqrt{1+k^2}} \tag{35}$$

We identify 1,2 with the O(1,5) (greek) indices of the internal space, while 3 is left invariant by O(1,5) transformations (and is then a latin index). The coefficient $\mu$ is related to the freedom in the rescaling of the vielbeins [57].

Inverting (34) and (35) one obtains

$$Q_{\hat{0}} = \frac{1}{\sqrt{1+k^2}}\left(Q_{\bar{0}} - k\,\frac{Q_{\tilde{3}}}{\mu}\right)$$

$$Q_{\hat{3}} = \frac{1}{\sqrt{1+k^2}}\left(k\,Q_{\bar{0}} + \frac{Q_{\tilde{3}}}{\mu}\right) \qquad (36$$

The nonvanishing structure constants of G, defined as usual by

$$[\,Q_{\hat{A}}\,,\,Q_{\hat{B}}\,] = f_{\hat{C}\hat{A}\hat{B}}\,Q_{\hat{C}} \qquad (37$$

are given in this basis by:

$$f_{\bar{0}\tilde{\alpha}\tilde{\beta}} = \frac{1}{\sqrt{1+k^2}}\,\varepsilon_{\tilde{\alpha}\tilde{\beta}} \qquad f_{\tilde{\alpha}\tilde{\beta}\bar{0}} = \frac{1}{\sqrt{1+k^2}}\,\varepsilon_{\tilde{\alpha}\tilde{\beta}}$$

$$f_{\tilde{3}\tilde{\alpha}\tilde{\beta}} = \frac{-k\mu^{-1}}{\sqrt{1+k^2}}\,\varepsilon_{\tilde{\alpha}\tilde{\beta}} \qquad f_{\tilde{\alpha}\tilde{\beta}\tilde{3}} = \frac{-k\mu}{\sqrt{1+k^2}}\,\varepsilon_{\tilde{\alpha}\tilde{\beta}} \qquad (38$$

where $\tilde{\alpha}, \tilde{\beta}$ run over 1,2.

It is very easy to see that the condition (24) is satisfied, since

$$f_{\tilde{a}\tilde{3}\bar{0}} = f_{\tilde{3}\tilde{a}\bar{0}} = 0 \qquad (39$$

Moreover, from the canonical embedding (25) of H in $G'_T$, one easily sees that $G'_T$ is generated by

$$\frac{1}{2}\Sigma_{\tilde{1}\tilde{2}} = \sqrt{1+k^2}\, Q_{\bar{0}} = Q_{\hat{3}} + k\, Q_{\hat{0}} \tag{40}$$

As shown in section II.3 a vielbein for the coset space can be constructed, for example by taking

$$L_{\theta\phi\psi} = e^{\psi Q_{\bar{0}}}\, e^{-\phi Q_{\tilde{3}}}\, e^{-\theta Q_{\tilde{2}}} \tag{41}$$

The components of the vielbein $L^{-1}dL$ are then

$$e^{\tilde{1}} = -d\theta \qquad\qquad e^{\tilde{2}} = \sin\theta\, d\phi$$
$$e^{\tilde{3}} = \frac{d\psi + k\cos\theta\, d\phi}{\mu\sqrt{1+k^2}} \tag{42}$$

and the H-connection is given by

$$e^{\bar{0}} = k\mu e^{\tilde{3}} - \sqrt{1+k^2}\, \mathrm{cotg}\,\theta\, e^{\tilde{2}} = \frac{k\, d\psi - \cos\theta\, d\phi}{\sqrt{1+k^2}} \tag{43}$$

From (42) one can then calculate the metric:

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2 + \frac{1}{\mu^2(1+k^2)}(d\psi + k\cos\theta\, d\phi)^2 \tag{44}$$

In order to avoid essential singularities, if k is an integer $\psi$ must have period $4\pi$ [91]. This is because the metric (44) is singular at $\theta = 0$ and $\theta = \pi$. It is then necessary to make a coordinate transformation which renders the metric regular at these points. This can be achieved at $\theta = 0$ by defining for $\theta < \pi$ a new coordinate $\psi' = \psi + k\,\phi$ and at $\theta = \pi$ defining for $\theta > 0$ a coordinate $\psi'' = \psi - k\,\phi$. These two transformations must be compatible in the overlap region, e.g. for $\theta = ½\,\pi$. But $d\,\psi'' = d\,\psi' - 2\,k\,d\,\phi$ and a line integral at fixed $\psi'$ around the equator changes $\psi''$ by $-4\pi k$.

and since k has been taken integer, $\psi''$ and hence $\psi$ must have a period $4\pi$.

The spaces constructed in this way can be described as nontrivial U(1) bundles over $S^2$. Some special cases arise for $k = 0$, which is $S^2 \times S^1$ and $k = 1$ which is $S^3$. The parameter $\mu$ gives the length scale of the bundle. For $k = 1$ and $\mu \neq 1$, one has a "squashed" 3-sphere.

As discussed in section II.5, in general one has a certain freedom in the choice of the spin connection in a coset space:

$$\omega^{\tilde{A}}{}_{\tilde{B}} = -f^{\tilde{A}}{}_{\tilde{B}\bar{C}}\, e^{\bar{C}} - \frac{1-\rho}{2} f^{\tilde{A}}{}_{\tilde{B}\tilde{C}}\, e^{\tilde{C}} \tag{45}$$

However, in quasi-Riemannian theories, the mixed components of the connection $\omega^{\alpha}{}_{b}$ must vanish and therefore, unless $f_{\tilde{a}\tilde{\beta}\tilde{C}} = f_{\tilde{\alpha}\tilde{b}\tilde{C}} = 0$, one is forced to choose the canonical form of the connection, corresponding to $\rho = 1$, and then to have nonvanishing torsion (see also appendix B).

In our case $f_{\tilde{3}\tilde{\alpha}\tilde{\beta}} \neq 0$ and therefore we must choose:

$$\omega_{\tilde{1}\tilde{2}} = -f_{\tilde{1}\tilde{2}\bar{0}}\, e^{\bar{0}} = -\frac{k\lambda}{\sqrt{1+k^2}}\, e^{\tilde{3}} + \operatorname{cotg}\theta\; e^{\tilde{2}} \tag{46}$$

The torsion and the curvature can now be calculated from (42) and (46). Their nonvanishing components are:

$$T_{\tilde{1}\tilde{2}\tilde{3}} = \frac{k\mu}{\sqrt{1+k^2}} \qquad T_{\tilde{3}\tilde{1}\tilde{2}} = \frac{k}{\mu\sqrt{1+k^2}}$$

$$R_{\tilde{1}\tilde{2}\tilde{1}\tilde{2}} = \frac{1}{1+k^2} \tag{47}$$

in accordance with the general formulae ( II.59, II.60 ).

In order to solve the field equations it is convenient to use the Palatini formalism discussed in chapter III.

When the equations (44) are substituted in the field equations for the connection, only one nontrivial relation arises:

$$\frac{k}{\sqrt{1+k^2}}\left[\frac{\alpha-e_2}{\mu} + 2(c_1 - c_2)\mu\right] = 0 \qquad (48$$

which fixes the value of $\mu$:

$$\mu^2 = \frac{e_2 - \alpha}{2(c_1 - c_2)} \qquad (49$$

Substituting (47) in (III.30) one can obtain the field equations for the vielbein:

$$\alpha R_{\dot\nu\dot\alpha\dot\alpha\dot\mu} - \tfrac{1}{2}\eta_{\dot\mu\dot\nu}\,\mathfrak{L} = 0$$

$$\alpha R_{\nu\alpha\alpha\mu} + c_1\,(T_{\alpha b\mu}T_{\alpha b\nu} - T_{\mu\alpha b}T_{\nu\alpha b}) + 2\,e_1\,T_{a\beta\mu}T_{a\beta\nu}$$

$$+ \tfrac{1}{2}e_2\,(T_{a\beta\mu}T_{\beta a\nu} + T_{\beta a\mu}T_{a\beta\nu}) - \tfrac{1}{2}\eta_{\mu\nu}\,\mathfrak{L} = 0 \qquad (50$$

$$c_1\,T_{\alpha\beta m}T_{\alpha\beta n} - c_2\,T_{\alpha\beta m}T_{\beta\alpha n} - e_1\,T_{m\alpha\beta}T_{n\alpha\beta} - \tfrac{1}{2}\eta_{\mu\nu}\,\mathfrak{L} = 0$$

with

$$\mathfrak{L} = \alpha(R_{\dot\alpha\dot\beta\dot\beta\dot\alpha} + R_{\alpha\beta\beta\alpha}) + c_1\,T_{\alpha\beta c}T_{\alpha\beta c} + c_2\,T_{\alpha\beta c}T_{\beta\alpha c}$$

$$+ e_1\,T_{a\beta\gamma}T_{a\beta\gamma} + e_2\,T_{a\beta\gamma}T_{\beta a\gamma} + \lambda \qquad (51$$

where for convenience we have omitted the ~ on the coset space indices and have assumed that the physical spacetime is maximally symmetric.

After some algebra and making use of (49), the field equations can be written as:

$$\frac{3Ah}{a^2} + \frac{A}{b^2} - \frac{E}{c^2} - \frac{\lambda}{2} = 0$$

$$\frac{6Ah}{a^2} + \frac{E}{c^2} - \frac{\lambda}{2} = 0 \qquad (52$$

$$\frac{6Ah}{a^2} + \frac{A}{b^2} - \frac{3E}{c^2} - \frac{\lambda}{2} = 0$$

where A and E are defined as usual and a and b are the scale factors of the 4-dimensional space and of the internal one respectively. Moreover we have defined:

$$c^2 = \mu^2 b^2 \frac{1+k^2}{k^2} \qquad (53$$

The equations (52) admit a solution for h = 1 and:

$$\frac{1}{a^2} = \frac{\lambda}{15A} \qquad \frac{1}{b^2} = \frac{2\lambda}{5A} \qquad \frac{1}{c^2} = \frac{\lambda}{10E} \qquad (54$$

Furthermore, eqns. (53) and (54) yield

$$\mu^2 = \frac{k^2}{1+k^2} \frac{4E}{A} \qquad (55$$

which, compared with (49) fixes the value of k:

$$\frac{k^2}{1+k^2} = \frac{A}{4E} \frac{e_2 - \alpha}{2(c_1 - c_2)} = \frac{\alpha - e_2}{\alpha} - \frac{2(c_1 - c_2)}{\alpha - e_2} \qquad (56$$

It seems then that also the topology of the solution is fixed by the field equations. It is interesting to notice, for example, that for $e_1 = e_2 = 0$, k must vanish, and the internal space must be $S^2 x S^1$.

Unfortunately, as it is clear from (54), it is not possible to obtain a flat 4-dimensional spacetime. This seems to be a general feature of any model of this kind. We shall discuss some possible ways out in section 6.

## VII .5 The fermion spectrum on SU(2)xU(1)/U(1)

We pass now to consider the dimensional reduction of fermions in the model considered in the previous section [30].

Fermions are defined to transform as spinors under the tangent group SO(1,5). This group admits two inequivalent spinor representations, which are not correlated by CPT [26]. One can then construct a theory containing spinors belonging to only one of the two representations, i.e. with definite six-dimensional chirality. This implies that to a given 4-dimensional chirality corresponds a specific representation of $G'_T \sim H$.

For example, for $\psi$ in 4 , the parts of the spinor field with $\gamma_5 = \pm 1$ are in the representation of $G'_T$ with "helicity" $\pm½$ .

The representations of G are labelled by the SU(2) isospin j and the U(1) hypercharge q. From (40) it can be deduced that for a given q, the H quantum number of a spinor of helicity $\pm ½$ is $\pm ½ - kq$ , and then the harmonic expansion of the $\gamma_5 = \pm 1$ parts will contain once all the representations (j,q), with $j = |\pm½-kq|$ , $|\pm½-kq|+1,\ldots$, with kq integer or half integer. There is then a complete matching between the representations for $\gamma_5 = -1$ and $\gamma_5 = +1$ , except that for $\gamma_5 = +1$ there is one extra with $j = kq-½$ for each q with kq positive , while for $\gamma_5 = -1$ there is one extra with $j = |kq|-½$ for each q with qk negative. These appear in 4 dimensions as an infinite tower of massless fermions. We wish to show more explicitly this fact [88].

Let the O(1,5) spinor algebra be generated by the 8-dimensional matrices $\Gamma_\alpha$ which satisfy

$$\{ \Gamma_\alpha , \Gamma_\beta \} = 2\, \eta_{\alpha\beta} \qquad (57$$

with $\alpha = 0,1,2,3,\tilde{1},\tilde{2}$.

We consider the particular realization of the algebra given by:

$$\Gamma_{\dot\alpha} = \gamma_{\dot\alpha} \times \sigma_1 \qquad\qquad \Gamma_{\tilde 1} = \gamma_5 \times \sigma_1 \qquad\qquad \Gamma_{\tilde 2} = 1 \times \sigma_2 \qquad (58$$

where $\gamma_\alpha$ and $\sigma_i$ are the usual Dirac and Pauli matrices.

The generators of O(1,5) are given by $½\, \Sigma_{\alpha\beta}$ , where:

$$½\, \Sigma_{\alpha\beta} = -\, ¼\, [ \Gamma_\alpha , \Gamma_\beta ] \qquad (59$$

which commute with $\Gamma_7 = 1 \times \sigma_3$, which is the O(1,5) equivalent of the $\gamma_5$ matrix. The fundamental representations of O(1,5) correspond to the eigenvalues 1 and -1 of $\Gamma_7$.

Let us assume that

$$\Gamma_7\psi = \sigma_3\psi = 1 \tag{60}$$

From (58) and (59) it follows that the group $G'_T$ of rotations of the internal space is generated by

$$\frac{1}{2}\Sigma_{\tilde{1}\tilde{2}} = -\frac{i}{2}\gamma_5 \times \sigma_3 \tag{61}$$

Acting on the chiral components of $\psi$, $\psi_L$ and $\psi_R$, corresponding to $\gamma_5$=-1 and $\gamma_5$ =+1 respectively, one has

$$\frac{1}{2}\Sigma_{\tilde{1}\tilde{2}}\psi_L = -i\,\lambda_L\psi_L = \frac{i}{2}\psi_L$$

$$\frac{1}{2}\Sigma_{\tilde{1}\tilde{2}}\psi_R = -i\,\lambda_R\psi_R = -\frac{i}{2}\psi_R \tag{62}$$

This shows that $\psi_L$ and $\psi_R$ belong to different representations of $G'_T$.
Consider now the Dirac operator:

$$\Gamma_{\dot\alpha}\nabla_{\dot\alpha} = (\gamma_{\dot\alpha}\sigma_1\nabla_{\dot\alpha} + \gamma_5\sigma_1\nabla_1 + \sigma_2\nabla_2)\psi =$$

$$\sigma_1(\gamma_{\dot\alpha}\nabla_{\dot\alpha} + \gamma_5\nabla_1 + i\nabla_2)\psi = \sigma_1\left(\gamma_{\dot\alpha}\nabla_{\dot\alpha} + \frac{1+\gamma_5}{\sqrt{2}}\nabla_- - \frac{1-\gamma_5}{\sqrt{2}}\nabla_+\right)\psi \tag{63}$$

where we have used $\sigma_3\psi = \psi$ and we have defined

$$\nabla_\pm = \frac{1}{\sqrt{2}}(\nabla_+ \pm i\,\nabla_-) \tag{64}$$

The spinor lagrangian takes the form

$$\bar{\psi}\gamma_{\dot\alpha}\nabla_{\dot\alpha}\psi + \sqrt{2}\,(\bar{\psi}_L \nabla_- \psi_R - \bar{\psi}_R \nabla_+ \psi_L) \tag{65}$$

As explained in section III.2, in order to evaluate the mass term it is useful to introduce the harmonic expansion [23]:

$$\psi_\lambda(x,y) = \sum_{j\geq m(\lambda,q)} \sqrt{2j+1} \sum_{n,q} D^{j,q}_{mn}(L_y)\,\psi^{j,q}_{\lambda n} \tag{66}$$

where $\lambda$ labels the representations of $G'_T$ and by (60) is $\pm½$ for $\psi_{R,L}$ respectively, while $D^{j,q}$ are the unitary representations of the group G = SU(2)xU(1), labeled by the SU(2) isospin j and the U(1) "hypercharge" q. The index m refers to the representation of the subgroup H of G corresponding to the representation of $G'_T$ labeled by $\lambda$, according to the embedding (25). It is then easy to see from (40) that $\lambda = m + kq$ , i.e.

$$m = \lambda - kq \tag{67}$$

Making use of the results of section II.5 one has:

$$\nabla_\pm D^j{}_{mn}(L_y) = -\frac{1}{\sqrt{2}\,b} D^j{}_{mn}[(Q_1 \pm iQ_2)L_y] = \frac{i}{b}\sqrt{\frac{(j\mp m)(j\pm m-1)}{2}}\, D^j{}_{m\pm 1,n}(L_y)$$

$$= \frac{i}{b}\sqrt{\frac{(j\mp\lambda\pm kq)\,(j\pm\lambda\mp kq+1)}{2}}\, D^j{}_{m\pm 1,n}(L_y) \tag{68}$$

After substitution in (63) the mass term becomes

$$\sum_{j,q} \frac{i}{b}\sqrt{(j+½)^2 - k^2q^2}\,(\bar{\psi}_L\psi_R - \bar{\psi}_R\psi_L) \tag{69}$$

From this equation one can recover the result given above: a massless chiral multiplet exists for any $j = -½ + |kq|$ . For kq positive the massless mode is a right-handed multiplet while for kq negative it is left-handed.

An undesiderable feature of this model is that it contains an infinite number of massless fermions since kq can take any integer or half integer value. This fact is in disagreement with the observations. However, as we shall discuss in the next section, some examples of a quasi-Riemannian theory containing a finite number of massless fermions can be constructed [92].

We also mention the fact that in principle one could add to the lagrangian a term of the type $\bar{\psi}\partial_3\psi$, which would give rise to a bare mass, but terms of this kind are ruled out if one considers fermions of definite O(1,5) chirality, since in this case any term of the kind $\bar{\psi}\psi$ vanishes.

We finally notice the similarity existing between this model and the 6-dimensional Einstein-Maxwell theory of section II.7. In that case the appearance of chiral fermions was due to the monopole configuration of the Maxwell field. In our case the rôle of the Maxwell field is taken by the extra dimension, and by consequence an infinite number of fermions arises, instead of the unique multiplet which was obtained in the Maxwell case for a given monopole charge.

## VII .6 Concluding remarks

The example given in the previous sections can be generalized to higher dimensions. A 11-dimensional model with tangent group O(1,9) and internal space SU(3)xSU(2)xU(1)/SU(2)xU(1)xU(1) is now under study. It presents essentially the same features of the previous one and chiral fermions can be obtained by the same mechanism.

It is instead very difficult to obtain a compactification with more general tangent space groups of the kind O(1,N-1)xO(M) with M>1, because in this case it seems unlikely to find an ansatz for the torsion which permits to satisfy all the field equations. In fact, these essentially require the vanishing of the components of the torsion with all the indices of the same kind, but not of those with mixed greek and latin indices and these conditions are compatible with the Maurer-Cartan equations only in very special cases.

In any case, no compactification with flat 4-dimensional spacetime seems to be possible in the context of purely gravitational quasi-Riemannian theories with tangent space O(1,N-1)xO(M).

It seems then that in order to obtain realistic models of compactification also in quasi-Riemannian theories one is forced to introduce additional matter fields.

The most attractive possibility, due to the natural appearance of torsion in the theory, is perhaps a mechanism analogous to that proposed in the Riemannian case by the authors of ref. [60], where a fermionic condensate provides a source term for the connection equations, which permits more freedom in the choice of the torsion of the internal space.

Another possibility [40] is to add to the gravitational action terms quadratic in the curvature, possibly in a suitable generalization of the Gauss-Bonnet terms [46], or of one of the other ghost-free actions of the Einstein-Cartan theory [93].

Finally, a more drastic but probably more interesting possibility is to study more general tangent spaces, of the kind $G_T = O(1,3) \times G'_T$, with $G_T \subset O(M)$, as for example $G_T = O(1,3) \times U(L)$. In this case additional terms can arise in the gravitational action, which can act as sources for compactification. In fact, it has been shown by de Alwis and others [37] that, when these more general tangent spaces are considered, it is possible to obtain a solution in the form of a product of a flat spacetime with a nonsymmetric coset space, as for example SU(6)/U(3). Unfortunately, the number of terms permitted in the action grows very rapidly with the complexity of the tangent group and the theory becomes almost untractable, even if in principle one can reduce the allowed terms by requiring stability, absence of anomalies, etc.

Another important property of the spaces with tangent group O(1,3)xU(L) is that they may admit a finite number of chiral fermions. An explicit example of a 10-dimensional theory with $G_T = O(1,3) \times U(3)$ and internal space $CP^3 = SU(4)/U(3)$ which admits a finite number of chiral fermions has been given by Witten [92].

To summarize, it seems that quasi-Riemannian theories may provide a promising field of research in the context of a Kaluza-Klein program of unification of gauge and gravitational interactions, in particular for what concerns the problem of the chirality of fermions.

However, many problems are still unsolved, first of all that of the large freedom in the choice of the tangent group and of the great number of free parameters which appear in the action. If the first problem can probably be solved only by comparing the predictions of the theory with the observation, the second can be faced from a theoretical point of view, by imposing some natural requirements on the theory, like absence of ghosts and tachyons, which, as we have seen in the special case of the tangent group O(1,N-1)xO(M), can provide strong constraints on the parameters of the theory.

Other constraint can be imposed for example by requiring the absence of anomalies in the quantum theory. This would require a study of the perturbative expansion of the theory to higher orders, which is still missing.

Also an extension to the non-linear theory of the gauge invariance we had to introduce at the linearized level in order to avoid instabilities deserves further investigations.

# APPENDIX A

One of the most difficult problems, when dealing with Kaluza-Klein theories is to choose a notation which is clear and consistent. Unfortunately there is absolutely no agreement between various authors about this matter, so that it is always very annoying to translate from one notation to another. The situation is also worst in quasi-Riemannian theories, because of the need of distinguishing between indices belonging to different subgroups of the tangent space group.

In this thesis we are mainly concerned with D-dimensional spaces with tangent group O(1,N-1)xO(M), where M = D-N. These spaces will be denoted by $Q^{N,M}$. The conventions adopted (if not differently specified) are the following: the signature of the flat metric is (-,+, .....,+). Early letters from the alphabet denote tangent space indices. Because of the structure of the tangent space they naturally split into two sets: 0,...,N-1 and N,...,D-1. We put A,B,C... = 0,...,D-1 ; $\alpha,\beta,\gamma$...= 0,...,N-1 and a,b,c = N,...,D-1.

Analougous conventions are used for the world indices, which are denoted by middle letters from the alphabet: M,N,P... = 0,...,D-1; $\mu,\nu$,... = 0,...,N-1 and m,n,... = N,...,D-1.

When compactification to coset spaces G/H is considered, indices from the algebra of G will be denoted by a hat: $\hat{A},\hat{B},\hat{C}$. They split into H indices $\bar{A},\bar{B},\bar{C}$,... and coset (tangent space) indices $\tilde{A},\tilde{B},\tilde{C}$,..., which run from 1 to K = D-4 . Finally, the four dimensional indices $\dot{A},\dot{B},\dot{C}$,... are denoted by a dot.

We also usually denote by $z^M$ the coordinates of the full space, by $x^{\dot{M}}$ the four dimensional ones and by $y^{\tilde{M}}$ those of the internal space.

The Ricci tensor of Riemannian geometry is defined as

$$R_{AB} = R_{ACCB}$$

and the Ricci scalar as

$$R = R_{ABBA}$$

With these conventions the sphere has negative curvature.

# APPENDIX B

In this appendix we prove that in order to obtain an effective 4-dimensional theory containing gauge bosons by dimensional reduction of a quasi-Riemannian gravitational theory, one must have nonvanishing torsion [28].

In fact, let us assume that the usual zero-mode ansatz is valid in our case:

$$\begin{pmatrix} e^{\dot{A}}{}_{\dot{M}}(x) & -D_{\hat{B}}{}^{\tilde{A}}(y)\, A^{\hat{B}}{}_{\dot{M}}(x) \\ 0 & e^{\tilde{A}}{}_{\tilde{M}}(y) \end{pmatrix} \qquad (1$$

A crucial observation is that in order to arrange suitably the gauge field $A_M$, one cannot have a block-diagonal vielbein.

Consider now the holonomy group of the manifold $B^D$, defined as the subgroup of $G_T$ consisting of all the rotations which a geometric object can suffer under parallel transport on any closed path through a point. A theorem by Berger [94] states that in D dimensions with $G_T \subset O(D)$, the vielbein is that of a symmetric space unless the holonomy group is one of the following:

$$\begin{gathered} SO(D)\ ;\ U(D/2)\ [D\ \text{even}]\ ;\ SU(D/2)\ [D\ \text{even}]\ ; \\ Sp(D/2)\times Sp(2)\ [D=4n]\ ;\ Sp(D/2)\ [D=4n]\ ; \\ Spin(9)\ [D=18]\ ;\ Spin(7)\ [D=8]\ ;\ G_2\ [D=7] \end{gathered} \qquad (2$$

Now, symmetric spaces cannot admit a vielbein of the form (1), nor the groups (2) can fit into tangent spaces of the kind $O(N) \times G_T'$ with nontrivial $G_T'$. Finally, if the space is a direct product, the vielbein takes a direct product form and cannot provide gauge fields as in (1).

In the case of a O(1,N-1)xO(M) tangent group, a more direct proof can be obtained by an explicit calculation [31].

The ansatz (1) yields, for nonvanishing torsion:

$$\omega_{\tilde{A}\dot{B}\dot{C}} = \frac{1}{2} F^{\hat{D}}{}_{\dot{B}\dot{C}}(x)\, D_{\hat{D}}{}^{\tilde{A}}(y) \qquad (3$$

$$\omega_{\tilde{A}\tilde{B}\dot{C}} = A^{\hat{D}}{}_{\dot{C}} \left( D_{\hat{D}}{}^{\tilde{E}} e_{\tilde{E}}{}^{\tilde{M}} e_{\tilde{M}}{}^{\bar{F}} - D_{\hat{D}}{}^{\bar{F}} \right) f_{\tilde{A}\bar{F}\tilde{B}} \qquad (4$$

$$\omega_{\tilde{A}\tilde{B}\tilde{C}} = \frac{1}{2} f_{\tilde{A}\tilde{B}\tilde{C}} + e_{\tilde{A}}{}^{\tilde{M}} e_{\tilde{M}}{}^{\bar{D}} f_{\tilde{C}\bar{D}\tilde{B}} \qquad (5$$

But, because of the quasi-Riemannian structure,

$$\omega_{\alpha bc} = \omega_{a\beta c} = 0 \qquad (6$$

One has then from (3):

$$F^{\hat{D}}{}_{\dot{\beta}\dot{\gamma}}\, D_{\hat{D}}{}^{\tilde{a}}(y) = 0 \qquad (7$$

and from (4) and (5):

$$f^{\tilde{\gamma}}{}_{\tilde{a}\hat{B}} = f^{\tilde{c}}{}_{\tilde{\alpha}\hat{B}} = 0 \qquad (8$$

The Lie algebra of G must then contain two subalgebras $\mathfrak{G}$ and $\mathfrak{G}'$ spanned by:

$$\{ Q_{\bar{A}}, Q_{\tilde{\alpha}} \} \qquad \text{and} \qquad \{ Q_{\bar{A}}, Q_{\tilde{a}} \} \qquad (9$$

with

$$[ Q_{\tilde{a}}, Q_{\tilde{\alpha}} ] = 0 \qquad (10$$

Eqn. (8) implies the vanishing of $D_{\tilde{\alpha}}{}^{\tilde{a}}(y)$ and therefore (7) can be written as

$$F^{\tilde{a}}{}_{\dot{\beta}\dot{\gamma}}\, D_{\tilde{a}}{}^{\tilde{b}}(y) + F^{\bar{A}}{}_{\dot{\beta}\dot{\gamma}}\, D_{\bar{A}}{}^{\tilde{b}}(y) = 0 \qquad (11$$

Since in general $D_a{}^b(y)$ and $D_{\bar{A}}{}^b(y)$ are different from zero, $F^{\tilde{a}}{}_{\dot{\beta}\dot{\gamma}}$ and $F^{\bar{A}}{}_{\dot{\beta}\dot{\gamma}}$ vanish and the gauge fields of G are absent from the spectrum.

The situation is different if one introduces torsion. In this case the connection and the torsion can always be chosen such that

$$\omega_{\tilde{a}\dot{\beta}\dot{\gamma}} = 0 \qquad (12$$

Moreover, if one chooses the canonical connection for the coset space:

$$\omega_{\tilde{A}\tilde{B}\tilde{C}} = e_{\tilde{A}}{}^{\tilde{M}}\, e_{\tilde{M}}{}^{\bar{D}}\, f_{\tilde{C}\tilde{B}\bar{D}} \qquad (13$$

the conditions (7) are satisfied provided

$$f^{\tilde{\gamma}}{}_{\tilde{a}\bar{B}} = f^{\tilde{C}}{}_{\tilde{\alpha}\bar{B}} = 0 \qquad (14$$

This was shown in chapter VII to be a necessary condition for the coset space to admit a quasi-Riemannian structure.

## REFERENCES


[1] T. Kaluza, Sitz. Preuss. Akad. Wiss. , page 966 (1921)

[2] O. Klein, Z. Phys. 37, 895 (1926); 46, 188 (1928)

[3] A. Einstein, P. Bergmann, Ann. Math. 39, 683 (1938)

[4] A. Lichnerowicz, P. Thiry, C. R. Acad. Sci. Paris 224, 529 (1947)
P. Thiry, C. R. Acad. Sci. Paris 226, 216 (1948)
P. Jordan, Ann. der Phys. 1, 219 (1947)

[5] B.S. De Witt in "Dynamical Theory of Groups and Fields", Gordon and Breach 1965;

[6] R. Kerner, Ann. Inst. Poincaré 9, 143 (1968);
J. Rayski, Acta Phys. Pol. 27, 947 ; 28, 87 (1965)

[7] Y.M. Cho, J. Math. Phys. 16, 2029 (1975)
Y.M. Cho, P.G.O. Freund, Phys. Rev. D12, 1711 (1975)
Y.M. Cho, P.S. Jang, Phys. Rev. D12, 3789 (1975)

[8] J.F. Luciani, Nucl. Phys. B135, 111 (1978)

[9] E. Witten, Nucl. Phys. B186, 412 (1981)

[10] J.Scherk, J. Schwarz, Phys. Lett. 57B, 463 (1975);
E. Cremmer, J. Scherk, Nucl. Phys. B103, 399 (1976)

[11] E. Cremmer, J. Scherk, Nucl. Phys. B108, 409 (1976); B118, 61 (1977)

[12] P.G.O. Freund, M.A. Rubin Phys. Lett. 97B, 233 (1980)
F. Englert, Phys. Lett. 119B, 339 (1982)

[13] M.J. Duff, B.E.W. Nilsson, C.N. Pope, Phys. Rep. 130, 1 (1986)

[14] T. Appelquist, M. Chodos, Phys. Rev. Lett. 50, 141 (1983); Phys. Rev. D28, 772 (1983);
P. Candelas, S. Weinberg, Nucl. Phys. B237, 397 (1984)

[15] C. Wetterich, Phys. Lett. 113B, 337 (1982)

[16] J. Madore, Phys. Lett. 110A, 289 (1985)
F. Möller-Hoissen, Phys. Lett. 163B, 106 (1985)

[17] A. Chodos, S. Detweiler, Phys. Rev. D21, 2167 (1980)

[18] P.G.O. Freund, Nucl. Phys. B209, 146 (1982)

[19] D. Sahdev, Phys. Lett. 137B, 155 (1984); Phys. Rev. D30, 2495 (1984)

[20] R.B. Abbott, S.M. Barr, S.D. Ellis, Phys. Rev. D30, 720 (1984); D31, 673 (1985)

[21] E.W. Kolb, D. Lindsey, D. Seckel, Phys. Rev. D30, 1205 (1984)

[22] A. Guth, Phys. Rev. D23, 347 (1981)
A.D. Linde, Phys. Lett. 108B, 309 (1982)

[23] S. Randjbar-Daemi, A. Salam, J. Strathdee, Nucl. Phys. B214, 491 (1983)

[24] A. Salam, J. Strathdee, Ann. Phys. 141, 316 (1982)

[25] E. Witten, Nucl. Phys. B195, 481 (1982) and in "Particle and Fields 2", A.Z. Capri, A. Kamal eds., Plenum 1983

[26] E. Witten, in "Shelter Island II", R. Jackiw, N.N. Khuri, S. Weinberg, E. Witten eds., MIT press 1985

[27] Z. Horvath, L. Palla, E. Cremmer, J. Scherk, Nucl. Phys. B127, 57(1977)
Z. Horvath, L. Palla, Nucl.Phys. B 142, 327 (1978)

[28] V.A. Rubakov, M.E. Shaposnikov, Phys. Lett. 125B, 139 (1983)

[29] C. Wetterich, Nucl. Phys. B242, 473 (1984); B244, 359 (1984); B253, 366 (1985)

[30] S. Weinberg, Phys. Lett. 138 B, 47 (1984)

[31] C. Destri, C.A. Orzalesi, C. Rossi, Ann. Phys. 147, 321 (1983)

[32] C.A. Orzalesi, M. Pauri, Phys. Lett. 107B, 186 (1981)

[33] Y.S. Wu, A. Zee, J. Math. Phys. 25, 2696 (1984)

[34] S. Weinberg, in "Fifth Workshop on Grand Unification", K. Kang, H. Fried, P. Frampton, eds., World Scientific 1984

[35] S. Weinberg, in "From SU(3) to gravity", E. Gotsman, G. Tauber eds. Cambridge University Press 1985

[36] S. Weinberg, in [53]

[37] S. P. de Alwis, I.G. Koh, Phys. Lett. 148B, 428 (1984)

[38] S. P. de Alwis, S. Randjbar-Daemi, Phys. Rev. D32, 1345 (1985)

[39] K.S. Viswanathan, B. Wong, Phys. Lett. 151B, 407 (1985); Phys. Rev. D 32, 3108 (1985)

[40] M. Gasperini, Phys. Rev. D33, 3594 (1986)

[41] S. Mignemi, SISSA preprint 88/87/E.P. (1987), to be published in Gen. Rel. Grav.

[42] S. Mignemi, Phys. Lett. 209B, 203 (1988)

[43] D. Lovelock, J. Math. Phys. 12, 498 (1971)

[44] F.W. Hehl, P. van der Heyde, G.D. Kerlick, J.M. Nester, Rev. Mod. Phys. 48, 393 (1976)

[45] P. van der Heyde, Phys. Lett. 51A, 381 (1978)

[46] B. Zwiebach, Phys. Lett. 156B, 315 (1985);
B. Zumino, Phys. Rep. 137, 109 (1986)

[47] K.S. Stelle, Phys. Rev. D16, 953 (1974); Gen. Rel. Grav. 9, 353 (1974)

[48] S. Mignemi, Mod. Phys. Lett. A1, 337 (1986)

[49] J. Strathdee, Int. Jour. Mod. Phys. A1,1 (1986)

[50] D. Bailin, A. Love, Rep. Progr. Phys. 50, 1087 (1987)

[51] P. van Nieuwenhuzen, in "Relativity, Groups and Topology II",
B.S. De Witt, R. Stora eds., Elsevier 1984

[52] H.C. Lee, ed. "An Introduction to Kaluza-Klein theories", World Scientific 1984

[53] T. Piran, S. Weinberg, eds., "Physics in higner dimensions", World Scientific 1986

[54] M.J. Duff, B.E.W. Nilsson, C.N. Pope, N.P. Warner, Phys. Lett. 149B, 90 (1984)

[55] S. Weinberg, Phys. Lett. 125B, 265 (1983)

[56] D.J. Toms in [52]

[57] L. Castellani, C.J. Romans, N.P. Warner, Ann. Phys. 157, 394 (1984)

[58] M.J. Duff in [52]

[59] C. Omero, R. Percacci, Nucl. Phys. B165, 351 (1980)
M. Gellmann, B. Zwiebach, Phys. Lett. 141B, 333 (1984)

[60] M.J. Duff, C. Orzalesi, Phys. Lett. 122B, 37 (1983)

[61] L. Castellani, R. D'Auria, P. Fré, Nucl. Phys. B239, 610 (1984)
L. Castellani, C.J. Romans, N.P. Warner, Nucl. Phys. B241, 429 (1984)

[62] S. Randjbar-Daemi, A. Salam, J. Strathdee, Phys. Lett. 124B, 345 (1983); Nucl. Phys. B242, 447 (1984)

[63] A. Schellekens, Nucl. Phys. B248, 706 (1984); B262, 661 (1986)

[64] S. Coleman, Phys. Rev. D15, 2929 (1977)

[65] P. van Nieuwenhuzen, Phys. Rep. 68, 189 (1981)

[66] A. Lichnerowicz, C. R. Acad. Sci. Paris 257, 7 (1963)

[67] C. Wetterich, Nucl. Phys. B211,177; B222, 20; B223, 109 (1983)

[68] B. Lawson, S.T. Yau, Comm. Math. Helv. 49, 232 (1974)

[69] S. Randjbar-Daemi, R. Percacci, Phys. Lett. 117B, 41 (1982)

[70] T.E. Eguchi, P.B. Gilkey, A.J. Hanson, Phys. Rep. 66, 213 (1980)

[71] C. Wetterich, in [53]

[72] S. Randjbar-Daemi, C. Wetterich, Phys. Lett. 166B, 165 (1986)

[73] H. Nicolai, C. Wetterich, Phys. Lett. 150B, 347 (1985)

[74] W. Nahm, Nucl. Phys. B135, 149 (1978)

[75] K. Hayashi, T. Shirafuji, Progr. Theor. Phys. 64, 866 (1980);
F. Möller-Hoissen, J. Nitsch, Phys. Rev. D28, 718 (1983)

[76] S. Randjbar-Daemi, A. Salam, J. Strathdee, N. Cim. 84B, 167 (1984)

[77] M. Reuter, C. Wetterich, Nucl. Phys. B289, 757 (1987)

[78] P. van Nieuwenhuzen, Nucl. Phys. B60, 478 (1973)

[79] S. Weinberg, Gravitation and cosmology, Wiley 1972

[80] E. Alvarez, M. Ben Gavela, Phys. Rev. Lett. 51, 931 (1983)
S.M. Barr, L.S.B. Brown, Phys. Rev. D29, 2779 (1984)

[81] Y. Tosa, Phys. Rev. D30, 2054 (1984)

[82] Q.S. Shafi, C. Wetterich, Phys. Lett. 129B, 387 (1983)
C. Wetterich, Nucl. Phys. B252, 309 (1985)

[83] A. Chodos, S. Detweiler, Gen. Rel. Grav. 14, 879 (1982)
D.J. Gross, M.J. Perry, Nucl. Phys. B226, 29 (1983)
R.D. Sorkin, Phys. Rev. Lett. 51, 87 (1983)

[84] I.G. Angus, Nucl. Phys. B264,349 (1986)
G.W. Gibbons, D.L. Wiltshire, Ann. Phys. 167, 201 (1986)

[85] R.C. Myers, M.J. Perry, Ann. Phys. 172, 304 (1986)

[86] P. van Baal, F.N. Bais, P. van Nieuwenhuizen, Nucl. Phys. B233, 477 (1984)

[87] M. Yoshimura, Phys. Rev. D34, 1021 (1986)

[88] S. Mignemi, in preparation

[89] J. Scherk, J. Schwarz, Nucl. Phys. B153, 61 (1979)

[90] E. Cremmer, B. Julia, Nucl. Phys. B159, 141 (1979)

[91] D.N. Page, C.N. Pope, Phys. Lett. 145B, 337 (1984)

[92] E. Witten, unpublished

[93] E. Sezgin, P. van Nieuwenhuzen, Phys. Rev. D21, 3269 (1980)

[94] M. Berger, Bull. Soc. Math. France, 23, 279 (1955); C. R. Acad. S. Paris A262, 1318 (1966)