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Some results in the variational theory of crack growth

Ph.D. Thesis

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Contents

Introduction

The present thesis is devoted to the study of some problems of fracture mechanics for *brittle materials*: the bodies we consider present a perfectly elastic behaviour outside the cracked region and no force is transmitted across the cracks. We are interested in *quasistatic evolutions*, i.e., motions which are performed so slowly that the system is in equilibrium at each instant. The time scale of external loads is longer than the intrinsic time scale of the process, so that in our analysis we do not "see" the oscillations of the system and we ignore dynamical effects.

The physical model relies on GRIFFITH's principle [28] that the propagation of a crack is the result of the competition between the elastic energy released when the crack opens and the energy spent to produce new crack. Griffith noticed that solutions to linear elasticity problems in brittle materials with cracks may develop singularities at crack tips. While studying surfaces with elliptic holes degenerating to lines, Griffith observed that around the crack tips the strain must assume high values tending to infinity.

Let us describe in detail the type of singularities observed by Griffith. We consider a cylinder, whose section is a smooth bounded open set $\Omega \subset \mathbb{R}^2$, subject to deformations of the type

$$\Omega \times \mathbb{R} \ni (x_1, x_2, x_3) \mapsto u(x_1, x_2, x_3) := (x_1, x_2, x_3 + v(x_1, x_2)) .$$

This is the case of *antiplane elasticity*. We assume that a cut is present on the domain, lying on a straight line $\Gamma_0 := \Omega \cap \{(x_1, 0) \colon x_1 \leq 0\}$ (of course, we suppose that $0 \in \Omega$). The elasticity equations for the displacement v take the form

$$\begin{cases} -\Delta v = f & \text{in } \Omega\,, \\ v = \psi & \text{on } \partial\Omega\,, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \Gamma_0\,, \end{cases} \tag{1}$$

where the external volume force f and the boundary condition ψ are given, while ν denotes the normal vector to Γ_0. The last line of the system says that there are no forces acting along the crack lips.

Fix a system of polar coordinates (r, θ) around the crack tip 0; then the variational solution $v \in H^1(\Omega \backslash \Gamma_0)$ to (1) can be written in the following form:

$$v = v^R + K\, r^{\frac{1}{2}} \sin \frac{\theta}{2}\,, \tag{2}$$

where $v^R \in H^2(U \backslash \Gamma_0)$ for every $U \subset\subset \Omega$ and $K \in \mathbb{R}$. This fact can be seen by writing the expansion of v in power series, in the simple case where Ω is a circle centred at 0 and $f = 0$; the complete proof requires some finer mathematical arguments, described, e.g., in Grisvard [29, 30]. Since the stress tensor σ is a linear function of ∇v, it is clear that $|\sigma| \to +\infty$ unless $K = 0$; hence, the multiplicative coefficient K is called *stress intensity factor*.

This phenomenon, appearing when the equations are linearized and a Neumann condition is prescribed on the crack, leads to a paradox from the physical point of view: a material subject to an infinite stress would immediately break up! Therefore, Griffith's remark permits excluding all models for crack growth based on an a priori bound on the stress intensity in the uncracked region, when the equations are linearized and homogeneous Neumann conditions are imposed on the crack path.

Nevertheless, Griffith proposed to keep the linearity of the problem and allow for the singularity it implies: then one may develop a model where the crack's stability does not depend on a bound on the stress, but it is connected to the energy balance. Indeed, his approach is based on an energy criterion: the stored elastic energy released by crack's increase is completely dissipated in the process of crack's formation; the crack stops growing if equilibrium is reached.

Griffith's criterion is based on the notion of *energy release rate*, that is the opposite of the derivative of the energy associated with the solution when the crack length varies. To be more precise, we define the increasing family of cracks $\Gamma_l := \Omega \cap \{(x_1, 0)\colon x_1 \le l\}$. For every $l \ge 0$ we consider the variational solution v_l of the problem

$$\begin{cases} -\Delta v_l = f & \text{in } \Omega\backslash\Gamma_l\,, \\ v_l = \psi & \text{on } \partial\Omega\,, \\ \frac{\partial v_l}{\partial \nu} = 0 & \text{on } \Gamma_l \end{cases} \tag{3}$$

and the associated elastic energy

$$\mathcal{E}^{\mathrm{el}}(l) := \frac{1}{2}\int_{\Omega\backslash\Gamma_l} |\nabla v_l(x)|^2 \,\mathrm{d}x - \int_{\Omega\backslash\Gamma_l} f(x)\, v_l(x)\,\mathrm{d}x\,. \tag{4}$$

Then the energy release rate is defined as $-\frac{\mathrm{d}\mathcal{E}^{\mathrm{el}}}{\mathrm{d}l}(0)$.

Assume now that the external force f and the boundary condition ψ vary in dependence on time, so that the energy becomes a function $\mathcal{E}^{\mathrm{el}}(t, l)$ of the instant and the crack length. In what follows, we assume, for such time dependence, all the regularity needed in order to derive the energy and the crack length. The fundamental contribution of Griffith is an energetic criterion to determine the crack length $l(t)$ during the evolution process; here, the energetic cost is related to the toughness $\kappa > 0$, a parameter depending on the material, which represents the energy needed to break atomic bonds along a line of length one.

According to Griffith's criterion, $l(t)$ must satisfy:

(a) $\dot{l}(t) \ge 0$, i.e, the crack growth is irreversible;

(b) $-\dfrac{\mathrm{d}\mathcal{E}^{\mathrm{el}}}{\mathrm{d}l}(t, l(t)) \le \kappa$, i.e., the rate cannot exceed the fracture toughness;

(c) $\left[\dfrac{\mathrm{d}\mathcal{E}^{\mathrm{el}}}{\mathrm{d}l}(t, l(t)) + \kappa\right] \dot{l}(t) = 0$, i.e., the crack grows only if the rate equals κ.

We have seen that near the crack tip the model introduces an infinite stress which is not present in the physical process, because of the error coming from linearization when the displacements are not small. However, the linearized system is still a good approximation away from the crack tip, while near the crack tip one may study the singularities and give them a precise physical interpretation when considering the problem from the energetic point of view. Indeed, IRWIN [31] observed that the energy release rate is connected to the stress intensity factor K appearing in (2) by the relation

$$-\frac{\mathrm{d}\mathcal{E}^{\mathrm{el}}}{\mathrm{d}l}(0) = \frac{\pi}{4}\,K^2\,; \tag{5}$$

we refer to [30, Theorem 6.4.1] for the proof.

Hence, Irwin's remark gives a physical meaning to the singularity of the solution. Moreover, the computation shows the double nature of the energy release rate: on the one hand, it can be expressed by a volume integral of a quantity depending on the elastic coefficients and on the deformation gradient; on the other hand, it is proportional to the stress intensity factor, which can be known from the solution in a neighbourhood of 0.

It is possible to see that Griffith's principle is equivalent to requiring the fundamental conditions of *unilateral stationarity* and *conservation* for the total energy

$$\mathcal{E}(t,l) := \mathcal{E}^{\mathrm{el}}(t,l) + \kappa\, l\,.$$

Notice that the total energy consists of the elastic energy and the energy spent to open the crack, which is proportional to its length (or area, in higher dimension): indeed, fracture is a macroscopic phenomenon due to debonding at atomic level. The stability property is *unilateral*, meaning that the configuration is compared only with competitors whose crack is larger. More precisely, it can be shown that Griffith's conditions (a–c) hold for the system (3) if and only if $l(t)$ satisfies:

- *Unilateral stability:* $l(t)$ is a stationary point for $\mathcal{E}(t,l)$ among all $l \geq l(t)$;
- *Irreversibility:* $\dot{l}(t) \geq 0$;
- *Energy balance:* if we set $E(t) := \mathcal{E}(t,l(t))$, we have

  $$\dot{E}(t) = \int_{\partial\Omega} \nabla v_{l(t)}(x)\cdot\dot{\psi}(t,x)\,\mathrm{d}x - \int_{\Omega\setminus\Gamma_{l(t)}} \dot{f}(t,x)\,v_{l(t)}(x)\,\mathrm{d}x\,,$$

  so the time derivative of the total energy is the power of the external forces.

We refer to [7, Proposition 2.1], where this fact is proven when the energy takes a more general form.

Stability and energy balance are the two fundamental features of the variational approach to *rate-independent processes* introduced by MIELKE (see [37] and the references therein). Therefore, crack growth fits in with a large class of phenomena, which are invariant under time rescaling.

In recent years, mathematicians developed variational methods to predict quasi-static crack growth in brittle materials. We focus our attention on the model proposed by FRANCFORT-MARIGO [23]: it is based on a procedure of time discretization giving rise to some incremental problems, solved through global minimization; the final evolution will be the limit of the solutions to approximate problems. This approach is a standard scheme in the treatment of rate-independent processes. Actually, one seeks solutions among the global minimizers of the energy and not among all possible critical points, because in this context stationarity is a difficult item to treat in a mathematical way. However, thanks to this approach the crack path Γ need not be prescribed a priori, but it is determined by the energy criterion.

Hence, the energy of the system depends on the pair of variables (v, Γ) and on time:

$$\mathcal{E}(t, v, \Gamma) := \frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla v(x)|^2 \, \mathrm{d}x + \mathcal{K}(\Gamma) - \mathcal{E}^{\mathrm{ext}}(t, v)\,,$$

where $\mathcal{K}(\Gamma)$ is the energy dissipated to open the crack Γ and $\mathcal{E}^{\mathrm{ext}}(t, v)$ is the potential of external forces. Notice that the problem is nonconvex, so we cannot expect uniqueness of solutions.

In the first existence results in the literature, Ω is contained in $\mathbb{R}^2$, the crack Γ is supposed to be a one-dimensional closed set, and the displacement v is represented by a Sobolev function on the domain $\Omega\setminus\Gamma$: this was studied by Dal Maso-Toader [18] for the antiplane case and by Chambolle [10] for the planar one. Instead, in the formulation of Francfort-Larsen [22], the functional setting for the displacement is the space of special functions of bounded variation $SBV(\Omega)$, introduced by De Giorgi-Ambrosio [19], while the crack is a rectifiable set containing the jump set $S(v)$: this allows them to treat the problem in arbitrary dimension ($\Omega \subset \mathbb{R}^n$) and to avoid some non-physical requirements on the number of connected components of the crack, which were present in the simplified formulation of [18, 10].

These results were generalized by Dal Maso-Francfort-Toader [14], who have considered, instead of $\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla v|^2 \, \mathrm{d}x$, a bulk energy term

$$\mathcal{W}(u) := \int_{\Omega} W(\nabla u(x)) \, \mathrm{d}x\,,$$

where the energy density W depends nonlinearly on the deformation $u(x) = x + v(x)$ through its gradient ∇u, according to the hypothesis of *hyperelasticity*. The function W is only assumed to be quasiconvex, with a condition of polynomial growth of the type $c\,|A|^p \leq W(A) \leq C\,|A|^p$ (here, $c, C > 0$, and $p > 1$). The framework of [14] is the space of generalized special functions of bounded variation $GSBV$, (i.e., functions whose truncates are SBV), which allows them to consider the vector case, under suitable coercivity hypotheses on the external forces $\mathcal{E}^{\mathrm{ext}}$.

In the present thesis we extend the theory of [18, 10, 22, 14] to the "physical" case of finite elasticity: these results, which are contained in [17, 35], are presented in Chapters 3 and 4, after the preliminary notions exposed in Chapters 1 and 2. In Chapter 5, where we present the results of [36], we come back to the bidimensional case for antiplane linearized elasticity and prove the existence of the stress intensity factor and its relation with the energy release rate under weak assumptions on the regularity of the cut and of the elasticity coefficients.

Quasistatic crack growth in finite elasticity

The first four chapters are devoted to study a model of quasistatic evolution in brittle materials, under hypotheses compatible with finite elasticity. As before, the elastic body is represented by a bounded open set $\Omega \subset \mathbb{R}^n$ and the state of the system is described by a pair of variables (u, Γ), where u is the deformation of Ω and Γ is the crack. The internal energy associated with a deformation u and a crack Γ is defined as

$$\mathcal{E}^{\mathrm{int}}(u, \Gamma) := \mathcal{W}(u) + \mathcal{K}(\Gamma)\,,$$

where $\mathcal{K}(\Gamma)$ is the energy spent to produce the crack Γ and $\mathcal{W}(u) = \int_\Omega W(\nabla u(x))\,\mathrm{d}x$ is the elastic energy stored in the body under the deformation u. The body is subjected to external forces, depending on time, with potential $\mathcal{E}^{\mathrm{ext}}(t, u)$. Hence the total energy is

$$\mathcal{E}(t, u, \Gamma) := \mathcal{E}^{\mathrm{int}}(u, \Gamma) - \mathcal{E}^{\mathrm{ext}}(t, u)\,. \tag{6}$$

Moreover, a time-dependent boundary condition $u = \psi(t)$ can be prescribed on a part of $\partial\Omega$.

The usual hypothesis in finite elasticity is that the strain energy diverges as the determinant of the deformation gradient vanishes:

$$\mathcal{W}(u) = +\infty \quad \text{if} \quad \det \nabla u \leq 0 \qquad \text{and} \qquad \mathcal{W}(u) \to +\infty \quad \text{if} \quad \det \nabla u \to 0^+\,. \tag{7}$$

This ensures the "physical" feature that the deformations with finite energy preserve orientation, i.e.,

$$\det \nabla u(x) > 0 \quad \text{for a.e. } x \in \Omega\,. \tag{8}$$

Unfortunately, (7) is incompatible with polynomial growth, which is a basic tool in the above mentioned articles [18, 10, 22, 14] for proving lower semicontinuity and controlling energy from above. In Chapters 3 and 4 we extend the previous results adopting some general assumptions compatible with finite elasticity, introduced in Ball [5], Francfort-Mielke [24], and Fusco-Leone-March-Verde [25]; the original work is contained in [17, 35].

We prove the existence of quasistatic evolutions $t \mapsto (u(t), \Gamma(t))$ minimizing (6) (*global stability*) and satisfying an *energy-dissipation balance* law, which states that the time derivative of the internal energy $\mathcal{E}^{\mathrm{int}}(u(t), \Gamma(t))$ equals the power of the external forces $\mathcal{E}^{\mathrm{ext}}(t, u(t))$. We work in spaces of SBV functions, thanks to the hypothesis that the body Ω is confined in a prescribed compact set $K \subset \mathbb{R}^n$ where all the deformations take place: indeed, this provides an automatic L^∞ bound which is crucial for the compactness theorem in SBV. The proof is based on the approximation by means of solutions to incremental minimum problems obtained by time discretization, as proposed in [23].

There are three main difficulties in passing from the polynomial growth condition to the context of finite elasticity:

- lower semicontinuity of the bulk energy,
- energy estimate,
- jump transfer.

When one has to prove the lower semicontinuity of the bulk energy, all theorems for quasiconvex functions require a polynomial growth, forbidden by (7); on the other hand, the convexity assumption is not compatible with finite elasticity, as showed in [5]. We overcome this difficulty by assuming the intermediate property of polyconvexity on $\mathcal{W}$: this allows us to apply a recent result [25], which requires only suitable bounds from below.

The key point for the energy estimates consists in replacing polynomial controls by a bound from above which is compatible with (7): namely, we suppose that for every $A \in GL_n^+$

$$\left|A^{\mathrm{T}} \mathrm{D}_A W(A)\right| \leq c_W^1 (W(A) + c_W^0)\,, \tag{9}$$

where $c_W^0 \geq 0$ and $c_W^1 > 0$ are two constants. The *multiplicative stress estimate* (9) is well known in mechanics [5] and holds in the case of Ogden materials [40, 41], a class of natural rubbers. We present in detail all the mechanical assumptions on the strain energy in Section 2.1.

The main difficulty in the approximation scheme consists in constructing a *joint recovery sequence*, in order to pass from the minimality of the approximating functions to the minimality of the limit in the proof of the global stability. The usual procedure for such a construction is called *jump transfer* and was performed for the first time in [22], using a reflection argument. Since reflections are forbidden by (7), we must modify the approach of [22]: so, reflections are replaced by dilations close to identity; the corresponding energy estimates are obtained thanks to (9).

In order to exploit (9), we use a method introduced in [24] and manipulate the solutions in a multiplicative way. More precisely, we look for minimizers to (6) of the form

$$u = \psi(t) \circ z\,, \tag{10}$$

where z coincides with the identity function on the Dirichlet part of $\partial\Omega$. This can be done provided that the boundary datum $\psi(t)$ is extended to a function defined on the whole set K (which contains Ω) and is a diffeomorphism of K onto itself.

The discrete energy inequality was obtained in [14] through an additive manipulation of the approximate solutions; moreover, the passage to the limit in this inequality was based on a lemma about the convergence of stresses, which requires polynomial growth. In our context, the discrete energy inequality relies on the multiplicative splitting method and requires a suitable continuity condition on the Kirchhoff stress; the passage to the limit is now obtained using a modification of the above mentioned lemma, proven in [24].

When studying quasistatic evolutions, one could be interested in the regularity of solutions with respect to time. On the contrary, our definition of quasistatic evolution does not guarantee even measurability in time! However, it is possible to select solutions so that they are measurable as functions with values in the Banach space SBV. This result relies on a previous theorem of Dal Maso-Giacomini-Ponsiglione [16] concerning the measurability of the solutions and their gradients regarded as functions with values in L^p.

In the exposition of Chapter 3, we focus on the new ideas and techniques used to avoid the polynomial growth condition, so we suppose that no forces are acting and the prescribed deformation $\psi(t)(x)$ is sufficiently smooth in both variables. This is not satisfactory for two reasons:

- the spatial smoothness of the boundary data is a strong requirement (whilst the solutions are only SBV);
- the class of boundary data considered in Chapter 3 is not invariant under Lipschitz reparametrizations of the time interval.

Therefore, we explore weaker hypotheses on the prescribed deformations in Chapter 4: in particular, we allow for boundary conditions which are Lipschitz in both variables, but not necessarily C^1. Hence, we can deal with a wider class of data, which is invariant under Lipschitz reparametrizations of time: the last feature is important from the point of view of rate-independent processes.

Due to the lack of regularity, even the chain rule is nontrivial when deriving the multiplicative splitting rule (10). Indeed, if z is SBV it may happen that the counterimage through z of the set of points of non-differentiability for $\psi(t)$ is a set of positive measure. It can be proven that this does not occur in our case because $\det \nabla z$ is a.e. positive.

In Chapter 4 we introduce also volume and surface forces ($\mathcal{E}^{\text{ext}}$ in (6)), studying the minimal regularity conditions. Also in this case, we find a class of functions which is invariant under Lipschitz reparametrizations of time; in addition, the assumptions hold in the case of *dead loads*.

In these results, we impose on the solutions a strong *non-interpenetration* requirement, the so-called Ciarlet-Nečas condition [12]: the deformations must preserve orientation as in (8) and be globally invertible, too. This property was studied in the SBV context by Giacomini-Ponsiglione [26], who proved a stability theorem under the weak* convergence in SBV; in Section 2.2 we present the definition and its consequences.

In addition, in Section 2.3 we discuss the physical meaning of this requirement and compare it with other possible notions of non-interpenetration, with some examples. In particular, we show that the *linearized self-contact condition* is not realistic in a nonlinear context; on the contrary, our arguments lead to the conclusion that the most desirable property from the physical point of view is the *progressive non-interpenetration.* A deformation satisfies the latter requirement if it can be reached by perturbing continuously the reference configuration, taking into account the Ciarlet-Nečas condition at each instant: our solutions satisfy this important property until the first discontinuity time.

Energy release rate and stress intensity factor in antiplane elasticity

In the last chapter, we study the bidimensional problem for antiplane linearized elasticity described in (1). In particular, we consider the case where the prescribed crack path Γ is a $C^{1,1}$ curve parametrized by a function $\gamma\colon [l_1, l_2] \to \overline{\Omega}$ (with $l_1 < 0 < l_2$). We consider the increasing familiy of cracks $\Gamma_l := \{\gamma(s)\colon l_1 \leq s \leq l\}$. We prove the existence of the stress intensity factor in this case and show its relation with the energy release rate; the results are contained in [36]. The basis of our arguments is the theory developed by Grisvard [29, 30], who studied the singularities of solutions to elliptic problems in polygonal domains.

The standard strategy for the computation of the derivative of the energy is to rewrite the integrals in (4) so that they are defined on a fixed domain. If the crack has a rectilinear path, it is easy to construct a diffeomorphism F_l which coincides with the identity in a neighbourhood of $\partial\Omega$ and transforms $\Omega_l := \Omega\backslash\Gamma_l$ into a fixed domain $\Omega_0 := \Omega\backslash\Gamma_0$. This procedure can be followed also if the crack is a curve of class C^2, defining F_l around 0 as the flow of a vector field tangent to Γ. However, this allows the computation of the energy release rate only if the second derivative of Γ exists at the crack tip.

We show a different method to calculate the derivative of the energy when the crack path Γ is only of class $C^{1,1}$, proving that the derivative exists at each point, even if the curve has not a second derivative. We reduce the problem to the rectilinear case, thanks to a diffeomorphism Φ which straightens the cut in a neighbourhood of 0; moreover, Φ transforms the elliptic coefficients so that the conormal vector is parallel to the normal. A similar procedure was performed by Mumford-Shah [39] for a slightly different variational problem. The change of variables Φ is used to show the existence of the stress intensity factor in this case, following the lines of a proof by Grisvard [29] for a pure Dirichlet problem. Our results have a natural generalization to elliptic operators with variable coefficients of class $C^{0,1}$.

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Notation

Throughout the thesis, $\mathbb{R}^n$ is endowed with the Euclidean scalar product $\cdot$ and the corresponding norm $|\cdot|$. The space of $m\times n$ real matrices is denoted by $\mathbb{M}^{m\times n}$; SO_n stands for the subset of orthogonal $n\times n$ matrices with determinant 1, while GL_n^+ stands for the subset of matrices with positive determinant; I is the identity matrix. The space $\mathbb{M}^{n\times n}$ is endowed with the scalar product $A:B := \operatorname{tr}\left(AB^{\mathrm{T}}\right)$, which coincides with the Euclidean scalar product in $\mathbb{R}^{n^2}$; we denote by $|\cdot|$ the corresponding norm. Given $A \in \mathbb{M}^{n\times n}$, we define $\operatorname{adj}_j A$ as the vector whose components are the minors of A of order j; its dimension is $\tau_j := \binom{n}{j}^2$.

As usual, $\mathcal{L}^n$ is the Lebesgue measure in $\mathbb{R}^n$, while $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. The expression *almost everywhere*, abbreviated as *a.e.*, always refers to $\mathcal{L}^n$, unless otherwise specified. Given two sets A and B in $\mathbb{R}^n$ we say that $A \,\tilde{\subset}\, B$ whenever $\mathcal{H}^{n-1}(A\backslash B) = 0$ and we say that $A \cong B$ whenever $\mathcal{H}^{n-1}(A \triangle B) = 0$, where $A \triangle B := (A\backslash B) \cup (B\backslash A)$ denotes the symmetric difference of A and B.

Chapter 1

Special functions of bounded variation

We present some preliminary notions about the space of SBV functions, introduced in [19]. We recall also some results concerning σ^p-convergence, due to [14].

1.1 Some notions in Geometric Measure Theory

We recall some well-known definitions and results of Geometric Measure Theory, referring to [4] for the details. In this section, $n \geq 2$ and $m \geq 1$ are two fixed integers and U is a bounded open subset of $\mathbb{R}^n$.

A set $\Gamma \subset \mathbb{R}^n$ is said to be $(\mathcal{H}^{n-1}, n-1)$-*rectifiable* if there is a sequence Γ_k of C^1-manifolds of dimension $n-1$ such that $\Gamma \cong \bigcup_k \Gamma_k$. Given a rectifiable set Γ, there is a *unit normal vector field* ν on Γ, i.e., an $\mathcal{H}^{n-1}$-measurable function $\nu\colon \Gamma \to \mathbb{R}^n$, such that $\nu(x) \in \mathbb{S}^{n-1}$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ and $\nu(x)$ is normal to Γ_k for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma_k$ and every k; this definition is independent of the decomposition of Γ.

Given a measurable function $u\colon U \to \mathbb{R}^m$, its *approximate limit* at a point $x \in U$ is an element $\tilde{u}(x) \in \mathbb{R}^m$ such that for every $\varepsilon > 0$

$$\lim_{r \to 0^+} \frac{1}{r} \int_{B_r(x)} |u(y) - \tilde{u}(x)| \, \mathrm{d}y = 0 \,, \tag{1.1}$$

where $B_r(x)$ is the open ball with centre x and radius r. The approximate limit, if it exists, is denoted by $\tilde{u}(x) = \operatorname{ap\,lim}_{y \to x} u(y)$. If $u(x) = \tilde{u}(x)$, the function u is said to be *approximately continuous* at x and the point x is called a *Lebesgue point* for u. Given $x \in U$ such that $\tilde{u}(x) = \operatorname{ap\,lim}_{y \to x} u(y)$ exists, we say that $\nabla u(x) \in \mathbb{M}^{m \times n}$ is the *approximate differential* of u at x if

$$\operatorname*{ap\,lim}_{y \to x} \frac{u(y) - \tilde{u}(x) - \nabla u(x)(y-x)}{|y-x|} = 0 \,. \tag{1.2}$$

On the contrary, $x \in U$ is an *approximate jump point* for u if there are $u^+(x) \neq u^-(x) \in \mathbb{R}^m$ and $\nu_u(x) \in \mathbb{S}^{n-1}$ such that

$$\lim_{r \to 0^+} \frac{1}{r} \int_{B_r^{\pm}(x, \nu_u(x))} \left| u(y) - u^{\pm}(x) \right| \, \mathrm{d}y = 0 \,, \tag{1.3}$$

where $B_r^{\pm}(x, \nu_u(x)) := \{ y \in B_r(x) \colon \pm (y-x) \cdot \nu_u(x) > 0 \}$. This determines the triplet $(u^+(x), u^-(x), \nu_u(x))$, up to a permutation of $(u^+(x), u^-(x))$ and a change of sign of $\nu_u(x)$. The *jump set* $S(u)$ of u is the set of all approximate jump points; it is Borel. The *jump* of u is the function $[u] := u^+ - u^-$ in $S(u)$, $[u] := 0$ elsewhere.

A function $u \in L^1(U;\mathbb{R}^m)$ is of *bounded variation* if its distributional gradient $\mathrm{D}u$ is a bounded Radon measure on U with values in $\mathbb{M}^{m\times n}$. The space of *functions of bounded variation* is denoted by $BV(U;\mathbb{R}^m)$. Given $u \in BV(U;\mathbb{R}^m)$, there is a canonical decomposition $\mathrm{D}u = \mathrm{D}^a u + \mathrm{D}^s u$, where $\mathrm{D}^a u$ is absolutely continuous with respect to $\mathcal{L}^n$ and $\mathrm{D}^s u$ is singular with respect to $\mathcal{L}^n$. Moreover, the approximate differential $\nabla u(x)$ exists for a.e. $x \in U$ and coincides a.e. with the density of $\mathrm{D}^a u$ with respect to $\mathcal{L}^n$. The jump set $S(u)$ is equal to the complement of the set of Lebesgue points for u, up to a set of $\mathcal{H}^{n-1}$-measure 0.

A function $u \in BV(U;\mathbb{R}^m)$ is a *special function of bounded variation* if $\mathrm{D}^s u$ is concentrated on $S(u)$, i.e., $|\mathrm{D}^s u|\,(U\setminus S(u)) = 0$. The subspace of such functions is denoted by $SBV(U;\mathbb{R}^m)$. If $u \in SBV(U;\mathbb{R}^m)$, we have

$$\mathrm{D}^s u = (u^+ - u^-) \otimes \nu_u\, \mathcal{H}^{n-1} \llcorner S(u)\,. \tag{1.4}$$

Fixed $p > 1$, we define the subspace

$$SBV^p(U;\mathbb{R}^m) := \left\{ u \in SBV(U;\mathbb{R}^m) \colon \nabla u \in L^p(U;\mathbb{M}^{m\times n}) \right\}. \tag{1.5}$$

It is endowed with the norm

$$\|u\|_{SBV^p(U;\mathbb{R}^m)} := \int_U |u|\, \mathrm{d}x + \left(\int_U |\nabla u|^p\, \mathrm{d}x \right)^{\frac{1}{p}} + |\mathrm{D}u|\,(U)\,, \tag{1.6}$$

which makes it a Banach space, as one can easily see. We provide a notion of convergence for sequences in $SBV^p(U;\mathbb{R}^m)$, usually called weak* convergence, in spite of the fact that it does not involve any predual space.

Definition 1.1.1. A sequence u_k *converges to* u *weakly* in* $SBV^p(U;\mathbb{R}^m)$ if

- $u_k, u \in SBV^p(U;\mathbb{R}^m)$;
- $u_k \to u$ in measure;
- $\|u_k\|_{L^\infty(U;\mathbb{R}^m)}$ is bounded uniformly with respect to k;
- $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(U;\mathbb{M}^{m\times n})$;
- $\mathcal{H}^{n-1}(S(u_k))$ is bounded uniformly with respect to k.

1.2 Semicontinuity and compactness

We provide some basic theorems of compactness and lower semicontinuity for functions in $SBV^p(U;\mathbb{R}^n)$, where $p \geq 2$ and $U \subset \mathbb{R}^n$ is bounded and open. We remark that from now on the dimension of the codomain is the same as that of the codomain, namely n: this will be the case of Chapters 3 and 4.

The next compactness property was proven in [2, Proposition 4.3] (see also [4, Theorem 4.8]).

Theorem 1.2.1 (Compactness). *Let u_k be a sequence in $SBV^p(U;\mathbb{R}^n)$ such that $\|u_k\|_{L^\infty(U;\mathbb{R}^n)}$, $\|\nabla u_k\|_{L^p(U;\mathbb{M}^{n\times n})}$, and $\mathcal{H}^{n-1}(S(u_k))$ are bounded uniformly with respect to k. Then there exists a subsequence which converges weakly* in $SBV^p(U;\mathbb{R}^n)$.*

We present a lower semicontinuity result for bulk energies of type

$$\mathcal{F}(u) := \int_U F(x, u(x), \nabla u(x))\,\mathrm{d}x\,, \tag{1.7}$$

where $u \in SBV(U;\mathbb{R}^n)$, and $F\colon U \times \mathbb{R}^n \times \mathbb{M}^{n\times n} \to [0,+\infty]$. The theorem relies on the property of *polyconvexity*, which is intermediate between convexity and quasiconvexity.

Definition 1.2.2. A function $F\colon U \times \mathbb{R}^n \times \mathbb{M}^{n\times n} \to [0,+\infty]$ is said to be *polyconvex* if there exists a function $\widetilde{F}\colon U \times \mathbb{R}^n \times \mathbb{R}^\tau \to [0,+\infty]$ such that $x \mapsto \widetilde{F}(x,y,\xi)$ is $\mathcal{L}^n$-measurable on U for every $(y,\xi) \in \mathbb{R}^n \times \mathbb{R}^\tau$, $(y,\xi) \mapsto \widetilde{F}(x,y,\xi)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^\tau$ for every $x \in U$, $\xi \mapsto \widetilde{F}(x,y,\xi)$ is convex on $\mathbb{R}^\tau$ for every $(x,y) \in U \times \mathbb{R}^n$, and

$$F(x,y,A) = \widetilde{F}(x,y,M(A)) \quad \text{for every } (x,y,A) \in U \times \mathbb{R}^n \times \mathbb{M}^{n\times n}\,,$$

where $M(A) := (\mathrm{adj}_1 A, \dots, \mathrm{adj}_n A)$ is the vector (of dimension $\tau := \tau_1 + \cdots + \tau_n$) composed of all minors of A.

The lower semicontinuity under weak* convergence in $SBV^p(U;\mathbb{R}^n)$ was proven in [25]. The original theorem assumes that F has values in $(0,+\infty)$; we adapt the proof to treat the case of functionals which may assume the value $+\infty$ (this result is contained in [17]).

Theorem 1.2.3 (Semicontinuity). *Let $\mathcal{F}$ be defined as in (1.7). Assume that F is polyconvex and there exist some constants $M \geq 0$, $\beta_0 \geq 0$, $\beta_1, \dots, \beta_n > 0$, and some exponents $p_1, p_2, \dots, p_n$, such that for every $(x,y) \in U \times \mathbb{R}^n$:*

1. *$F(x,x,I) \leq M$;*
2. *for every $A \in \mathbb{M}^{n\times n}$*

$$F(x,y,A) \geq \sum_{j=1}^n \beta_j \left|\mathrm{adj}_j A\right|^{p_j} - \beta_0\,,$$

with

$$p_1 := p \geq 2\,, \quad p_j \geq p' := \frac{p}{p-1} \text{ for } j = 2, \dots, n-1\,, \quad p_n > 1\,.$$

Let $u_k \rightharpoonup u_\infty$ weakly in $SBV^p(U;\mathbb{R}^n)$. Then*

$$\mathcal{F}(u_\infty) \leq \liminf_{k\to\infty} \mathcal{F}(u_k)\,. \tag{1.8}$$

Proof. We claim that there exists a nondecreasing sequence of everywhere finite functions F_j, polyconvex and converging pointwise to F. Let $\mathcal{F}_j$ be the corresponding integral functionals. By [25, Theorem 3.5] we have

$$\mathcal{F}_j(u_\infty) \leq \liminf_{k\to\infty} \mathcal{F}_j(u_k) \leq \liminf_{k\to\infty} \mathcal{F}(u_k)\,.$$

Passing to the limit with respect to j, we get (1.8).

It remains only to prove the claim. This will be done by constructing the sequence $\widetilde{F}_j$ associated to F_j by Definition 1.2.2. To this end we consider the convex conjugate $\widetilde{F}^*$ of $\widetilde{F}$ with respect to ξ, defined by

$$\widetilde{F}^*(x,y,\xi^*) := \sup_{\xi\in\mathbb{R}^\tau} \left[\xi^*\cdot\xi - \widetilde{F}(x,y,\xi)\right]\,.$$

By (1), we have $\widetilde{F}^*(x,y,\xi^*) > -\infty$ for every (x,y,ξ^*). Using (1) and (2), it is easy to see that for every $M>0$ there exists $R>0$ such that, if $|\xi^*|\leq M$, then

$$\widetilde{F}^*(x,y,\xi^*) = \sup_{|\xi|\leq R} \left[\xi^*\cdot\xi - \widetilde{F}(x,y,\xi)\right] \tag{1.9}$$

for every (x,y). By continuity, the supremum is attained, so that $\widetilde{F}^*(x,y,\xi^*) < +\infty$.

For every x, the function $(y,\xi^*) \mapsto \widetilde{F}^*(x,y,\xi^*)$ is lower semicontinuous, since the functions $(y,\xi^*) \mapsto \xi^*\cdot\xi - \widetilde{F}(x,y,\xi)$ are continuous for every ξ. To prove the continuity of $(y,\xi^*) \mapsto \widetilde{F}^*(x,y,\xi^*)$, it is enough to show that

$$\widetilde{F}^*(x,y_\infty,\xi^*_\infty) \geq \limsup_{k\to\infty} \widetilde{F}^*(x,y_k,\xi^*_k) \tag{1.10}$$

for every $(y_k,\xi^*_k) \to (y_\infty,\xi^*_\infty)$. Let $M>0$ be a constant such that $|\xi_k|\leq M$ for every k, let $R>0$ be a constant such that (1.9) is satisfied, and let ξ_k, with $|\xi_k|\leq R$, be a point where the supremum in (1.9) is attained for $(x,y,\xi^*) = (x,y_k,\xi^*_k)$. Passing to a subsequence, we may assume that $\xi_k \to \xi_\infty$, so that

$$\begin{aligned}\widetilde{F}^*(x,y_\infty,\xi^*_\infty) \;&\geq\; \xi^*_\infty\cdot\xi_\infty - \widetilde{F}(x,y_\infty,\xi_\infty) = \\ = \lim_{k\to\infty}&\left[\xi^*_k\cdot\xi_k - \widetilde{F}(x,y_k,\xi_k)\right] = \lim_{k\to\infty} \widetilde{F}^*(x,y_k,\xi^*_k),\end{aligned}$$

which shows (1.10) and concludes the proof of the continuity of $(y,\xi^*) \mapsto \widetilde{F}^*(x,y,\xi^*)$.

We now define

$$\widetilde{F}_j(x,y,\xi) := \max_{|\xi^*|\leq j} \left[\xi^*\cdot\xi - \widetilde{F}^*(x,y,\xi^*)\right]\,.$$

Arguing as before, it can be proven that $(y,\xi) \mapsto \widetilde{F}_j(x,y,\xi)$ is continuous. Moreover, $\xi \mapsto \widetilde{F}_j(x,y,\xi)$ is convex, being a supremum of affine functions. Finally, it is well known from Convex Analysis that

$$\widetilde{F}(x,y,\xi) := \sup_{\xi^*\in\mathbb{R}^\tau} \left[\xi^*\cdot\xi - \widetilde{F}^*(x,y,\xi^*)\right]\,.$$

This implies that $\widetilde{F}_j \nearrow \widetilde{F}$ and concludes the proof of the claim. □

Remark 1.2.4. When $p > n$, it suffices to suppose $F(x, y, A) \geq \beta_W^1 |A|^p$, instead of (2), thanks to [3, Corollary 4.9].

We will need also the following fact, which is proven in [25, Theorem 3.4] as an intermediate step to show Theorem 1.2.3; we recall that τ_j is the dimension of the vector $\mathrm{adj}_j A$ for $A \in \mathbb{M}^{n\times n}$.

Theorem 1.2.5. *Let u_k be a sequence in $SBV(U;\mathbb{R}^n)$, convergent in measure to a function $u_\infty \in SBV(U;\mathbb{R}^n)$. Suppose that, for $j = 1, \dots, n$, $\left\|\mathrm{adj}_j \nabla u_k\right\|_{L^{p_j}(U;\mathbb{R}^{\tau_j})}$ and $\mathcal{H}^{n-1}(S(u_k))$ are bounded uniformly with respect to k, where the exponents p_j satisfy the hypothesis of the previous theorem. Then, for $j = 1, \dots, n$, $\mathrm{adj}_j \nabla u_k \rightharpoonup \mathrm{adj}_j \nabla u_\infty$ weakly in $L^{p_j}(U;\mathbb{R}^{\tau_j})$.*

1.3 The σ^p-convergence

As for the cracks, we will employ a notion of convergence for sets, called σ^p-convergence, which was introduced in [14]. A sequence of sets σ^p-converges if they behave like the jump sets of a sequence of functions which converges weakly* in $SBV^p(U)$. In what follows, $p > 1$ is fixed and $U \subset \mathbb{R}^n$ is a bounded open set.

Definition 1.3.1. A sequence Γ_k *σ^p-converges to* Γ if $\Gamma_k, \Gamma \subset U$, $\mathcal{H}^{n-1}(\Gamma_k)$ is bounded uniformly with respect to k, and the following conditions are satisfied:

- if u_j converges weakly* to u in $SBV^p(U)$ and $S(u_j) \mathrel{\tilde{\subset}} \Gamma_{k_j}$ for some sequence $k_j \to \infty$, then $S(u) \mathrel{\tilde{\subset}} \Gamma$;
- there exist a function $u \in SBV^p(U)$ and a sequence u_k converging to u weakly* in $SBV^p(U)$ such that $S(u) \mathrel{\tilde{=}} \Gamma$ and $S(u_k) \mathrel{\tilde{\subset}} \Gamma_k$ for every k.

Now we present the basic properties of the σ^p-convergence. The compactness property was proven in [14, Theorems 4.3 and 4.7].

Theorem 1.3.2 (Compactness)**.** *Every sequence $\Gamma_k \subset U$ with $\mathcal{H}^{n-1}(\Gamma_k)$ uniformly bounded has a σ^p-convergent subsequence.*

We state a lower semicontinuity result for crack energies, with respect to the σ^p-convergence. The toughness of the material is represented by a locally bounded Borel function $\kappa\colon U \times \mathbb{R}^n \to \mathbb{R}$ such that

K1. for every $\varepsilon > 0$ there exists an open set V of 1-capacity $C_1(V) < \varepsilon$ such that $x \mapsto \kappa(x, \nu)$ is lower semicontinuous on $U \backslash V$ for every $\nu \in \mathbb{R}^n$,

K2. $\nu \mapsto \kappa(x, \nu)$ is a norm on $\mathbb{R}^n$ for every $x \in U$,

K3. $\kappa_1 |\nu| \leq \kappa(x, \nu) \leq \kappa_2 |\nu|$ for every $(x, \nu) \in U \times \mathbb{R}^n$,

for some constants $\kappa_1 > 0$ and $\kappa_2 > 0$. We recall that the 1-capacity of an open set $V \subset \mathbb{R}^n$ is defined as

$$C_1(V) := \left\{ \int_{\mathbb{R}^n} \nabla u(x)\,\mathrm{d}x \colon u \in W^{1,1}(\mathbb{R}^n)\,,\ u \geq 1 \text{ a.e. on } V \right\}.$$

Theorem 1.3.3 (Semicontinuity). *Let κ satisfy* (K1–3)*, let Γ_0, Γ_k, and Γ be countably $(\mathcal{H}^{n-1}, n-1)$- rectifiable subsets of U with $\mathcal{H}^{n-1}(\Gamma_0) < +\infty$, and let E be an $\mathcal{H}^{n-1}$-measurable set with $\mathcal{H}^{n-1}(E) < +\infty$. If Γ_k σ^p-converges to Γ, then*

$$\int_{(\Gamma\cup\Gamma_0)\setminus E} \kappa(x,\nu)\, \mathrm{d}\mathcal{H}^{n-1}(x) \leq \liminf_{k\to\infty} \int_{(\Gamma_k\cup\Gamma_0)\setminus E} \kappa(x,\nu_k)\, \mathrm{d}\mathcal{H}^{n-1}(x) \ , \tag{1.11}$$

where ν and ν_k are unit normal vector fields on $\Gamma \cup \Gamma_0$ and $\Gamma_k \cup \Gamma_0$, respectively.

The last result can be deduced from [1, Theorem 3.3], arguing as in [14, Theorems 2.8 and 4.3.].

Remark 1.3.4. Let E be an $\mathcal{H}^{n-1}$-measurable set with $\mathcal{H}^{n-1}(E) < +\infty$ and let u_k be a sequence converging to u weakly* in $SBV^p(U;\mathbb{R}^n)$. Applying Theorems 1.3.2 and 1.3.3 with $\Gamma_k = S(u_k)$ and $\Gamma_0 = \emptyset$, we can prove that, if $S(u_k) \tilde{\subset} E$ for every k, then $S(u) \tilde{\subset} E$.

In the following remark, we state some properties of the σ^p-convergence, referring to [14, Section 4] for the proofs.

Remark 1.3.5. If Γ_k σ^p-converges to Γ, then

- Γ is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable;
- $\mathcal{H}^{n-1}(\Gamma) < +\infty$;
- if in addition $\Gamma_k \tilde{\subset} \Gamma'_k$ and Γ'_k σ^p-converges to Γ', then $\Gamma \tilde{\subset} \Gamma'$;
- if C is relatively closed in U and $\Gamma_k \tilde{\subset} C$ for every k, then $\Gamma \tilde{\subset} C$; in particular, if Γ_k is rectifiable, so is Γ.

On the contrary, it can be shown that in general the inclusion $C \tilde{\subset} \Gamma_k$ for every k does not imply $C \tilde{\subset} \Gamma$, even if C is a compact subset of a $(n-1)$-dimensional manifold. This is because, when C is irregular, there is no $u \in SBV(U)$ with $S(u) \cong C$ (see [14]).

The following theorem, proven in [15, 14, Theorem 4.8], is the analogue of Helly's theorem for this set convergence.

Theorem 1.3.6 (Helly property). *Let $t \mapsto \Gamma_k(t)$ be a sequence of increasing set functions defined on an interval $I \subset \mathbb{R}$ with values in the class of rectifiable sets, i.e., $\Gamma_k(s) \tilde{\subset} \Gamma_k(t)$ for every $s, t \in I$ with $s < t$. Assume that the measures $\mathcal{H}^{n-1}(\Gamma_k(t))$ are bounded uniformly with respect to k and t. Then there exist a subsequence Γ_{k_j} and an increasing set function $t \mapsto \Gamma(t)$ on I such that $\Gamma_{k_j}(t)$ σ^p-converges to $\Gamma(t)$ for every $t \in I$.*

Chapter 2

Finite elasticity and non-interpenetration

In Chapters 3 and 4 we will consider bulk energies satisfying some "physical" hypotheses, which are compatible with local non-interpenetration: the deformations with finite energy are orientation-preserving, i.e., the Jacobian determinant is a.e. positive. These assumptions and their consequences are presented in Section 2.1. Moreover, we will consider a strong notion of non-interpenetration in the sense of Ciarlet-Nečas, described in Section 2.2: the deformations are required to be a.e.-injective, too. In Section 2.3 we discuss the reasons for the choice of the Ciarlet-Nečas condition and its physical motivation.

2.1 Finite elasticity

We consider the deformations of a *hyperelastic body*, which are represented by functions $u \in SBV(U;\mathbb{R}^n)$ defined on a bounded open set $U \in \mathbb{R}^n$. The *bulk energy* of a deformation u is

$$\mathcal{W}(u) := \int_U W(\nabla u(x))\,\mathrm{d}x\,, \tag{2.1}$$

where $W\colon \mathbb{M}^{n\times n} \to [0,+\infty]$ is continuous. We suppose that W is *frame indifferent*, i.e., for every $A \in \mathbb{M}^{n\times n}$

$$W(QA) = W(A) \text{ for every } Q \in SO_n\,. \tag{2.2}$$

In Chapters 3 and 4, W will have also an explicit dependence on x: this is irrelevant for the results contained in this section.

The standard hypothesis in *finite elasticity* is that

$$W(A) \to +\infty \text{ as } \det A \to 0^+ \quad \text{and} \quad W(A) = +\infty \text{ if } \det A \leq 0\,. \tag{2.3}$$

This prevents the deformations from reversing orientation: if $\mathcal{W}(u) < +\infty$, then $\det \nabla u(x) > 0$ for a.e. $x \in U$. Hence, (2.3) provides *local non-interpenetration* of matter; however, it does not ensure even local invertibility, since u is not C^1. A stronger notion of non-interpenetration, with a requirement of a.e.-injectivity, will be considered in the following section.

This assumption is not compatible with growth estimates of polynomial type, which are required in the most common theorems of the Calculus of Variations. Following [5], we consider instead two upper growth conditions which are compatible with (2.3):

$$\left|\mathrm{D}_A W(A)\, A^{\mathrm{T}}\right| \leq c_W^1 (W(A) + c_W^0) \tag{2.4}$$

and

$$\left|A^{\mathrm{T}}\,\mathrm{D}_A W(A)\right| \le c^1_W (W(A) + c^0_W)\,, \tag{2.5}$$

where $c^0_W \ge 0$ and $c^1_W > 0$ are two constants. To give sense to these inequalities, we suppose that W is finite and C^1 on GL^+_n; on the contrary, when $W(A) = +\infty$ they are automatically satisfied.

These estimates involve two stress tensors:

$$K(A) := \mathrm{D}_A\, W(A) A^{\mathrm{T}}\,, \tag{2.6}$$

sometimes called *Kirchhoff stress tensor*, and

$$L(A) := A^{\mathrm{T}}\,\mathrm{D}_A W(A)\,, \tag{2.7}$$

which appears in the expression of the so called *energy–momentum tensor*

$$W(A)\, I - A^{\mathrm{T}}\,\mathrm{D}_A W(A)\,. \tag{2.8}$$

In all these formulas, $\mathrm{D}_A W(A)$ denotes the matrix whose entries are the partial derivatives of W with respect to the corresponding entries of A.

In the next proposition, we prove that (2.5) implies (2.4) and show some consequences of these properties.

Proposition 2.1.1. *If W satisfies* (2.5)*, there exists a constant $\gamma \in (0,1)$ (depending only on n) such that, for every $A \in GL^+_n$ and every $B \in \mathbb{M}^{n\times n}$ with $|B - I| < \gamma$, we have $B \in GL^+_n$ and*

$$W(AB) + c^0_W \le \frac{n}{n-1}(W(A) + c^0_W)\,, \tag{2.9}$$

$$\left|A^{\mathrm{T}}\mathrm{D}_A W(AB)\right| \le \frac{n^2}{n-1}\, c^1_W (W(A) + c^0_W)\,. \tag{2.10}$$

If W satisfies also (2.2)*, then for every $A \in GL^+_n$*

$$\left|\mathrm{D}_A W(A) A^{\mathrm{T}}\right| \le \left|A^{\mathrm{T}}\mathrm{D}_A W(A)\right|\,,$$

so that (2.4) *holds.*

If W satisfies (2.4)*, there exists a constant, still denoted γ, such that, for every $A \in GL^+_n$ and every $B \in \mathbb{M}^{n\times n}$ with $|B - I| < \gamma$, we have $B \in GL^+_n$ and*

$$W(BA) + c^0_W \le \frac{n}{n-1}(W(A) + c^0_W)\,, \tag{2.11}$$

$$\left|\mathrm{D}_A W(BA)\, A^{\mathrm{T}}\right| \le \frac{n^2}{n-1}\, c^1_W (W(A) + c^0_W)\,. \tag{2.12}$$

Proof. The proof follows the lines of [5, Section 2.4]. Given $B \in \mathbb{M}^{n\times n}$, let $B_\lambda := (1-\lambda)I + \lambda\, B$ for $\lambda \in [0,1]$. Since $|I| = \sqrt{n} < n$, we can find $\gamma > 0$ such that for every B with $|B - I| < \gamma$ we have $B \in GL^+_n$ and $\left|B^{-1}_\lambda\right| \le n$. We may assume also $\gamma < 1/(n^2 c^1_W)$.

Given $A \in GL_n^+$ and $|B - I| < \gamma$, as $\frac{\mathrm{d}}{\mathrm{d}\lambda} W(AB_\lambda) = \mathrm{D}_A W(AB_\lambda) \colon [A(B_\lambda - I)]$ we have

$$\begin{aligned} W(AB) - W(A) &= \int_0^1 \left[(AB_\lambda)^{\mathrm{T}} \mathrm{D}_A W(AB_\lambda)\right] : \left[B_\lambda^{-1}(B_\lambda - I)\right] \mathrm{d}\lambda \leq \\ &\leq c_W^1 \int_0^1 (W(AB_\lambda) + c_W^0) \left|B_\lambda^{-1}\right| |B_\lambda - I| \, \mathrm{d}\lambda \leq \\ &\leq n \, \gamma \, c_W^1 \int_0^1 (W(AB_\lambda) + c_W^0) \, \mathrm{d}\lambda \, , \end{aligned}$$

where we have used (2.5) in the former inequality and the hypotheses on γ in the latter. Let now $M(A) := \sup_{|C - I| < \gamma} W(AC)$. Since $n \, \gamma \, c_W^1 < 1/n$, we get

$$W(AB) - W(A) \leq \tfrac{1}{n}(M(A) + c_W^0) \, ,$$

so that we have also

$$M(A) - W(A) \leq \tfrac{1}{n}(M(A) + c_W^0)$$

or equivalently

$$M(A) + c_W^0 \leq \tfrac{n}{n-1}(W(A) + c_W^0) \, ,$$

which proves (2.9). As for (2.10), using (2.5) and (2.9) we get

$$\left|A^{\mathrm{T}} \mathrm{D}_A W(AB)\right| \leq \left|B^{-1}\right| c_W^1 (W(AB) + c_W^0) \leq \tfrac{n^2}{n-1} \, c_W^1 (W(A) + c_W^0) \, ,$$

where we have used again the hypotheses on γ. If (2.4) holds, the proof of (2.11) and (2.12) is analogous.

Finally, if W is frame indifferent, the matrix $\mathrm{D}_A W(A) A^{\mathrm{T}}$ is symmetric for every $A \in GL_n^+$. Indeed, given $A \in GL_n^+$ we can consider its polar decomposition $A = Q\sqrt{A^{\mathrm{T}} A}$. Using (2.2) we can write $W(A) = \widehat{W}(A^{\mathrm{T}} A)$, where $\widehat{W}(B) := W(\sqrt{B})$. We have for every $B \in \mathbb{M}^{n \times n}$

$$\mathrm{D}_A W(A) A^{\mathrm{T}} \colon B = \mathrm{D}_A W(A) \colon BA = \mathrm{D}_A \widehat{W}(A^{\mathrm{T}} A) \colon \left[A^{\mathrm{T}}(B + B^{\mathrm{T}})A\right] \, ,$$

which gives the symmetry of $\mathrm{D}_A W(A) A^{\mathrm{T}}$. Hence,

$$\begin{aligned} \left|\mathrm{D}_A W(A) A^{\mathrm{T}}\right|^2 &= \left[\mathrm{D}_A W(A) A^{\mathrm{T}}\right] : \left[A \, \mathrm{D}_A W(A)^{\mathrm{T}}\right] = \\ &= \left[A^{\mathrm{T}} \mathrm{D}_A W(A)\right] : \left[A^{\mathrm{T}} \mathrm{D}_A W(A)\right]^{\mathrm{T}} \leq \left|A^{\mathrm{T}} \mathrm{D}_A W(A)\right|^2 \, , \end{aligned}$$

which implies (2.4). □

Remark 2.1.2. There are examples of functions satisfying (2.4) but not (2.5); instead, these properties are equivalent when the material is isotropic, i.e.,

$$W(A\,Q) = W(A) \text{ for every } Q \in SO_n \, . \tag{2.13}$$

If either (2.4) or (2.5) holds, there exists $c_W^2 > 0$ such that for every $A \in GL_n^+$

$$W(A) \leq c_W^2 \left(|A|^s + \left|A^{-1}\right|^s\right) \, , \tag{2.14}$$

where $s := n \, c_W^1$. All these properties can be found in [5].

Example 2.1.3 (Ogden Materials)**.** An important class of hyperelastic isotropic materials in dimension $n = 3$ was studied by Ogden in 1972 [40, 41] to describe the behaviour of natural rubbers. These materials provide a classical example in finite elasticity [11, Section 4.10]; the strain-energy associated with $A \in GL_3^+$ is given by

$$W(A) = \sum_{i=1}^{M} a_i \, |A|^{\gamma_i} + \sum_{j=1}^{N} b_j \, |\mathrm{cof} A|^{\delta_j} + h(\det A) \,,$$

where several material parameters appear: M, $N \geq 1$, a_i, $b_j > 0$, γ_i, $\delta_j \geq 1$. Moreover, $h\colon (0,\infty) \to \mathbb{R}$ is a convex function satisfying $h(t) \to +\infty$ as $t \to 0^+$. Here, $\mathrm{cof} A := (\det A) \; A^{-\mathrm{T}}$ stands for the cofactor matrix of A.

In general, the strain-energy considered in this example is polyconvex and satisfies the growth estimate (2) of Theorem 1.2.3 [11, 25]. Moreover, in [5] it is proven that W satisfies (2.4) and (2.5), provided that there is $C > 0$ such that

$$\left| t \, h'(t) \right| \leq C(h(t) + 1)$$

for every $t > 0$.

The Kirchhoff tensor $\mathrm{D}_A W(A) \, A^{\mathrm{T}}$ appearing in (2.4) is related with the "multiplicative increments" of type $W(BA) - W(A)$, because

$$\mathrm{D}_A W(A) \, A^{\mathrm{T}} : (B - I) = \mathrm{d}_A W(A)[BA - A] \,.$$

This suggests to write (2.4) without using derivatives.

Proposition 2.1.4. *Let W satisfy* (2.4)*. Then*

$$|W(BA) - W(A)| \leq \frac{n^2}{n-1} \, c_W^1 (W(A) + c_W^0) \, |B - I| \tag{2.15}$$

for every $A \in GL_n^+$ and every $B \in GL_n^+$ with $|B - I| < \gamma$, where γ is the constant introduced in the previous proposition.

Proof. Fixed A and B as in the statement, define for $\lambda \in [0,1]$ the function $w(\lambda) := W((1-\lambda)A + \lambda \, BA)$, whose derivative is $w'(\lambda) = \mathrm{D}_A W((1-\lambda)A + \lambda \, BA) \, A^{\mathrm{T}} : (B - I)$. We have $W(BA) - W(A) = \int_0^1 w'(\lambda) \, \mathrm{d}\lambda$. By (2.12), we get $|w'(\lambda)| \leq \frac{n^2}{n-1} \, c_W^1 (W(A) + c_W^0) \, |B - I|$, so we conclude. $\square$

In the next proposition, we present an estimate where multipliers need not to be near I.

Proposition 2.1.5. *Let W satisfy* (2.2) *and* (2.4)*. Then for every $M > 0$ there exists $c_M > 0$ such that*

$$W(BA) + c_W^0 \leq c_M (W(A) + c_W^0) \tag{2.16}$$

for every $A \in GL_n^+$ and every $B \in GL_n^+$ with $|B| < M$ and $\left| B^{-1} \right| < M$.

Proof. Let A, B, and M be as in the statement. Consider a decomposition $B = QC$ with $Q \in SO_n$ and C symmetric and positive definite (take $C := \sqrt{B^{\mathrm{T}}B}$). We can find an integer N such that

$$\left|C^{\frac{1}{N}} - I\right| < \gamma\,;$$

here, N depends only on the constant γ of Proposition 2.1.1 and on M, which controls $|B|$ and $\left|B^{-1}\right|$. We can apply (2.2) and (2.11) to get

$$W(BA) + c_W^0 = W\Big(\Big(C^{\frac{1}{N}}\Big)^N A\Big) + c_W^0 \le \left(\frac{n}{n-1}\right)^N (W(A) + c_W^0)\,.$$

This concludes the proof. □

We will use also the following consequence of (2.4).

Remark 2.1.6. By (2.4) we get

$$\left|\mathrm{D}_A W(BA)\, A^{\mathrm{T}}\right| \le c_W^1 (W(BA) + c_W^0)\left|B^{-1}\right| \tag{2.17}$$

for every $A, B \in GL_n^+$.

2.2 Non-interpenetration of matter in the sense of Ciarlet-Nečas

Injectivity is a nontrivial requirement for SBV deformations. We present a condition of non-interpenetration of matter which was developed first for Sobolev maps by Ciarlet and Nečas [12]; the generalization to SBV functions is due to [26]. As previously, $U \subset \mathbb{R}^n$ is a bounded open set.

Definition 2.2.1. A function $u \in SBV(U;\mathbb{R}^n)$ satisfies the *Ciarlet-Nečas non-interpenetration condition* if u *preserves orientation*, i.e.,

CN1. $$\det \nabla u(x) > 0 \quad \text{for a.e. } x \in U\,,$$

and u is *a.e.-injective*, i.e.,

CN2. there is $N \subset U$ such that $\mathcal{L}^n(N) = 0$ and u is injective on $U \backslash N$.

We state some consequences of (CN1), which will be fundamental in Chapter 4.

Remark 2.2.2. We recall from [27, Chapter 3, Section 1.5, Theorem 1] the following area formula: let $u\colon U \to \mathbb{R}^n$ be a.e.-approximately differentiable and $E \subset U$ be measurable; then

$$\int_E |\det \nabla u(x)|\, \mathrm{d}x = \int_{\mathbb{R}^n} m(u, y, E \cap U_D)\, \mathrm{d}y\,, \tag{2.18}$$

where U_D is the set of approximate differentiability points of u and

$$m(u, y, F) := \mathrm{card}\{x \in F \colon u(x) = y\}$$

is measurable as a function of y.

Given a function $u \in SBV(U;\mathbb{R}^n)$, we consider a particular representative, defined by

$$u_D(x) := \begin{cases} \tilde{u}(x) & \text{if } x \in U_D\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{2.19}$$

where $\tilde{u}(x)$ is the approximate limit of u at x and U_D is the set of approximate differentiability points of u, as before.

Following the arguments of [26, Sections 2 and 3], we prove a simpler area formula for u_D. First, we observe that u_D satisfies the so-called *N-property*, i.e., $\mathcal{L}^n(u_D(N)) = 0$ for every $N \subset U$ with $\mathcal{L}^n(N) = 0$. Indeed, by (2.18) we have

$$\mathcal{L}^n(u_D(N)) \leq \int_{\mathbb{R}^n} m(u_D, y, N \cap U_D)\,\mathrm{d}y = \int_N |\det \nabla u(x)|\,\,\mathrm{d}x = 0\,.$$

The area formula (2.18), together with the N-property, gives

$$\int_E |\det \nabla u(x)|\,\,\mathrm{d}x = \int_{\mathbb{R}^n} m(u_D, y, E)\,\mathrm{d}y \tag{2.20}$$

for every E measurable.

Remark 2.2.3. Let $u \in SBV(U;\mathbb{R}^n)$ satisfy (CN1). Then (CN2) holds if and only if, for any $E \subset \Omega$,

$$\int_E |\det \nabla u|\,\,\mathrm{d}x \leq \mathcal{L}^n(u_D(E))\,. \tag{2.21}$$

Indeed, (2.21) holds if and only if the set $F := \{y \in \mathbb{R}^n \colon m(u_D, y, U) \geq 2\}$ is negligible, thanks to (2.20). If (CN2) holds, i.e., u is injective on $U \backslash N$, F is negligible by the N-property, because $F \subset u_D(N)$ and $\mathcal{L}^n(N) = 0$. Viceversa, if F is negligible, setting $E = u_D^{-1}(F)$ in (2.20) and recalling (CN1), we see that u is a.e.-injective.

Actually (2.21) is an equality, because of (2.20).

Proposition 2.2.4. *Let $u \in SBV(U;\mathbb{R}^n)$ satisfy* (CN1)*. Then, for every $F \subset \mathbb{R}^n$ with $\mathcal{L}^n(F) = 0$, we have $\mathcal{L}^n(u^{-1}(F)) = 0$ (independently on the choice of the representative of u). As a consequence, given a measurable set M, the preimage $u^{-1}(M)$ is measurable.*

Proof. Let F be negligible and u_D be defined as in (2.19). By (CN1) and (2.20) with $E = u_D^{-1}(F)$, we get $\mathcal{L}^n(u_D^{-1}(F)) = 0$. If $\bar{u}$ is another representative of u, $\bar{u}^{-1}(F)$ differs from $u_D^{-1}(F)$ by a set of null measure, so $\mathcal{L}^n(\bar{u}^{-1}(F)) = 0$, too.

Moreover, given M measurable, we can write $M = B \cup M_0$, with B Borel and M_0 negligible; then $u^{-1}(B)$ is measurable and $u^{-1}(M_0)$ has null measure. This implies that $u^{-1}(M) = u^{-1}(B) \cup u^{-1}(M_0)$ is measurable. □

Finally, we recall from [26, Theorem 4.4] a stability property of the Ciarlet-Nečas non-interpenetration condition under weak* convergence in $SBV^p(U;\mathbb{R}^n)$: this will be a key point for the existence results in Chapter 3.

Theorem 2.2.5 (Stability of the Ciarlet-Nečas condition)**.** *Let u_k be a sequence converging to u weakly* in $SBV^p(U;\mathbb{R}^n)$. Suppose that every u_k satisfies* (CN1) *and* (CN2)*, u satisfies* (CN1)*, and* $\det \nabla u_k \rightharpoonup \det \nabla u$ *weakly in $L^1(U)$. Then u satisfies* (CN2)*.*

2.3 Comparison among various notions of non-interpenetration

Besides the Ciarlet-Nečas condition for cracked bodies, two other notions of non-interpenetration can be considered for a function $u \in SBV(U;\mathbb{R}^n)$:

(a) *Linearized self-contact condition:* for $\mathcal{H}^{n-1}$-a.e. $x \in S(u)$

$$[u(x)] \cdot \nu_u(x) \geq 0\,;$$

(b) *Progressive non-interpenetration:* there exists a “continuous” function $\lambda \mapsto u(\lambda)$, defined for $\lambda \in [0,1]$ and with values in $SBV(U;\mathbb{R}^n)$, such that $u(0)$ is the identity map, $u(1) = u$, and $u(\lambda)$ satisfies the Ciarlet-Nečas condition of Definition 2.2.1 for every $\lambda \in [0,1]$.

Condition (b) clearly depends on the choice of the notion of continuity: ideally, it should be selected so that $\lambda \mapsto u(\lambda)$ is continuous if and only if the associated motion can be realized by a physical process.

In [26, Section 6], Definition 2.2.1 and condition (a) have been compared, showing that neither property implies the other one. Moreover, if $u \in SBV^q(U;\mathbb{R}^n)$ for some $q > n$, it is proven in [26, Proposition 6.2] that (a) holds whenever the functions

$$u(\lambda, x) := x + \lambda\, v(x) \tag{2.22}$$

satisfy Definition 2.2.1 for every $\lambda \in [0,1]$, where $v(x) := u(x) - x$. Since this property usually holds when the displacement $v(x)$ is “small”, this result suggests that the linearized self-contact condition is natural for linearized elasticity. It also proves that (b) implies (a) in the special case where $u(\lambda)$ is given by (2.22).

The following examples, due to [17], show that in the general case the progressive non-interpenetration does not imply the linearized self-contact condition, even if $u(\lambda, x)$ is smooth out of the jump set. In both examples $n = 2$ and U is the open ball with centre 0 and radius 2.

Example 2.3.1. For every $\lambda \in [0,1]$ and $x \in U$, let

$$u(\lambda, x) := \begin{cases} x & \text{if } |x| \leq 1\,, \\ R_\lambda\, x & \text{if } |x| > 1\,, \end{cases}$$

where R_λ is the rotation of angle λ. Then for every λ the Ciarlet-Nečas condition is satisfied, the jump set $S(u(\lambda,\cdot))$ coincides with $\Gamma := \{|x| = \frac{1}{2}\}$, and

$$[u(\lambda, x)] \cdot \nu(x) = (R_\lambda\, x - x) \cdot x = \cos\lambda - 1 < 0 \quad \text{for every } x \in \Gamma\,.$$

In this case the lips of the crack in the deformed configuration remain in contact for every λ. However, we can obtain a similar example with an opening crack, defining

$$u(\lambda, x) := \begin{cases} x & \text{if } |x| \leq 1\,, \\ a_\lambda\, R_\lambda\, x & \text{if } |x| > 1\,, \end{cases}$$

where $\lambda \mapsto a_\lambda$ is continuous and $1 < a_\lambda < 1/\cos\lambda$ for $0 < \lambda < 1$.

In the previous example, the crack set in the reference configuration does not depend on λ, and $u(\lambda, x) = x$ on one of the regions determined by the crack set. The violation of (a) is obtained by exploiting the strict convexity of this region. Instead, the next example achieves the same result with a rectilinear crack.

Example 2.3.2. Let $\zeta \in C^\infty(\mathbb{R})$ be a nondecreasing function such that $\zeta(s) = 0$ for $s \leq 0$ and $\zeta(s) > 0$ if $s > 0$, and let $\Gamma := \{(x_1, 0) : 0 < x_1 < 1\}$. For every $\lambda \in [0,1]$ and every $x = (x_1, x_2) \in U \setminus \Gamma$ we define

$$u(\lambda, x) := \begin{cases} (x_1, x_2 + \lambda\, x_1^2) & \text{if } x_2 > 0\,, \\ (x_1, \lambda\, x_1^2) & \text{if } x_1 \leq 0 \text{ and } x_2 = 0\,, \\ (x_1[1 + \lambda\, \zeta(x_1)], x_2 + \lambda\, x_1^2[1 + \lambda\, \zeta(x_1)]) & \text{if } x_2 < 0\,. \end{cases}$$

First of all, we observe that $u(\lambda, \cdot)$ is injective in each of the three regions used for the definition (thanks to the monotonicity of ζ). To prove the injectivity on the whole domain, it is enough to show that these regions are mapped into pairwise disjoint sets. The image of $\{x_2 > 0\}$ lies strictly above the parabola $\Pi := \{(x_1, \lambda x_1^2) \colon x_1 \in \mathbb{R}\}$; the region $\{x_1 \leq 0\,, x_2 = 0\}$ is mapped into Π, while the image of the third region $\{x_2 < 0\}$ lies strictly below the curve $\{(x_1[1 + \lambda\, \zeta(x_1)], \lambda x_1^2[1 + \lambda\, \zeta(x_1)]) \colon x_1 \in \mathbb{R}\}$. The branch of this curve corresponding to $x_1 \leq 0$ is contained in Π, while the branch corresponding to $x_1 > 0$ lies strictly below Π for $\lambda > 0$, since $1 < 1 + \lambda\, \zeta(x_1)$. This shows that $u(\lambda, \cdot)$ is injective and that the crack lips in the deformed configuration overlap only at the crack tip $(0,0)$, except for $\lambda = 0$. Moreover, u belongs to $C^\infty([0,1] \times (U \backslash \Gamma))$ and all its partial derivatives have a finite limit on both sides of Γ. For every λ the jump set $S(u(\lambda, \cdot))$ coincides with Γ, and

$$[u(\lambda, x)] \cdot \nu(x) = -\lambda^2\, x_1^2\, \zeta(x_1) < 0 \quad \text{for every } x \in \Gamma\,.$$

In both cases condition (a) is violated not only by $u(1, \cdot)$, but also by $u(\lambda, \cdot)$ for every $\lambda > 0$. Hence, (a) may not hold even if the deformation satisfies (b) and is very close to the identity in a C^∞ sense. Notice that, if λ is interpreted as time, the function $u(\lambda, x)$ represents a physically admissible motion of the cracked body $U \backslash \Gamma$, starting from the undeformed configuration $u(0, x) = x$. Therefore, requiring (a) appears to be unnatural, unless one linearizes with respect to λ at $\lambda = 0$.

The previous discussion indicates that the correct notion of non-interpenetration in nonlinear fracture mechanics is condition (b), since it takes into account the fact that the deformation is always the result of a "continuous" evolution through non-interpenetrating intermediate states, starting from an initial condition, that may be taken as reference configuration. Unfortunately, up to now, there are no mathematical results concerning the stability of this property: this is the reason why we adopted instead the Ciarlet-Nečas condition.

However, let us consider a motion $t \mapsto u(t)$ such that the initial datum $u(0)$ satisfies (b), $u(t)$ satisfies the Ciarlet-Nečas condition at every time t, and $t \mapsto u(t)$ is continuous on some interval $[0, \tau]$ in the same sense chosen for (b): then $u(t)$ satisfies also the progressive non-interpenetration condition for every $t \in [0, \tau]$. This will be the case of incrementally-approximable quasistatic evolutions, which will be considered in the main results of this thesis (see Remark 3.2.18).

Chapter 3

Quasistatic crack growth in finite elasticity

Introduction

In this chapter we present an existence theorem for a quasistatic evolution problem for brittle cracks in hyperelastic bodies, in the context of finite elasticity; the results are contained in [17].

Following the lines of [23, 7], we develop a mathematical model, based on the variational approach to fracture mechanics that goes back to Griffith [28]. All existence results in the mathematical literature on this subject [18, 10, 22, 14] were obtained using energy densities with polynomial growth. This was not compatible with the standard assumption in finite elasticity that the strain energy tends to infinity as the determinant of the deformation gradient vanishes (2.3). Our model extends the previous results to a wide class of energy densities satisfying this property; moreover, it takes into account the non-interpenetration condition (Definition 2.2.1), which was not considered in the above mentioned papers.

Our definition of quasistatic evolution is based on the approximation by means of solutions to incremental minimum problems obtained by time discretization (Section 3.2.4). This approximation method was already used in the other mathematical papers on this subject, and is common in a large class of rate-independent problems. We prove an existence result (Theorem 3.2.11) and show also (Theorem 3.2.12) that our solutions satisfy the basic properties of the energy formulation presented in [37]:

- global stability,
- energy balance.

To simplify the functional framework, we impose a confinement condition: the deformed configuration is constrained to be contained in a prescribed compact set (Section 3.1.2). This allows us to formulate the problem in the space SBV of special functions of bounded variation [4], as in [22].

There are three main difficulties in passing from the polynomial growth condition to the context of finite elasticity:

- lower semicontinuity of the bulk energy,
- jump transfer,
- energy estimate.

As for the lower semicontinuity, the problem is that all theorems for quasiconvex functions require a polynomial growth, while the convexity assumption is not compatible with finite elasticity. We overcome this difficulty by assuming polyconvexity and applying a recent result [25], which requires only suitable bounds from below ((W4) in Section 3.1.3).

Jump transfer is a procedure introduced in [22] to prove global stability. One step of the original construction employs a reflection argument, which is forbidden by finite elasticity. We modify the jump transfer lemma, replacing the reflection argument by a suitable stretching argument (Section 3.4.1): the upper bounds needed in this step require a multiplicative stress estimate ((W5) in Section 3.1.3), already used in [5, 34].

The discrete energy inequality was obtained in [14] through an additive manipulation of the approximate solutions; moreover, the passage to the limit in this inequality was based on a lemma about the convergence of stresses, which requires a polynomial growth. In our new context, the discrete energy inequality relies on the multiplicative splitting introduced in [24], which requires a suitable continuity condition on the Kirchhoff stress ((W6) in Section 3.1.3); the passage to the limit is now obtained using a modification of the above mentioned lemma (Lemma 3.5.1), proven in [24].

The hypotheses introduced to overcome these difficulties ((W0-6) in Section 3.1.3) are compatible with finite elasticity and are satisfied, for instance, in the case of Ogden materials (Example 3.1.2). Since here we focus on the new ideas and techniques used to avoid the polynomial growth condition, we study a problem with no applied forces and with sufficiently smooth prescribed boundary conditions. The minimal regularity hypotheses on the boundary data, on the volume forces, and on the surface forces will be considered in the following chapter.

To deal with the non-interpenetration condition, we adopt a weak formulation for SBV functions (Definition 2.2.1), introduced in [26], and use a stability result (Theorem 2.2.5) with respect to weak* convergence in SBV proven in the same paper.

In Section 3.1 we present the hypotheses on the geometry of the body, on the strain energy, and on the prescribed deformations. In Section 3.2 we give the definition of quasistatic evolution and state the main theorems; first, we present their simplest form, using an auxiliary problem (Section 3.2.3) based on the multiplicative splitting method introduced in [24]; then, we formulate these results in the original setting. Section 3.3 contains the proof of the existence results, while Sections 3.4 and 3.5 are devoted to the proof of the global stability and of the energy balance; moreover, in Section 3.5.3 we show the convergence of the energies of the approximate solutions. Section 3.6 contains some results on the nontrivial problem of the measurability of solutions with respect to time. In Section 3.7 we sketch the extension to the case of applied volume forces with smooth potentials.

3.1 The mechanical assumptions

3.1.1 The body and its cracks

In this section we introduce a geometry modelling an elastic body with cracks, following [14]. The *reference configuration* of the body is the closure $\overline{\Omega}$ of a bounded open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$.

We will suppose that every deformation takes place in a *container* K, a compact set with Lipschitz boundary and with $\Omega \subset K$. We will assume also that every crack in the reference configuration is contained in the *brittle part* $\overline{\Omega}_B$ of $\overline{\Omega}$, and that $\overline{\Omega}_B$ is the closure of an open subset Ω_B of Ω with Lipschitz boundary.

We fix an open set Ω_D with Lipschitz boundary and with $\Omega \subset \Omega_D \subset K$, and define the *Dirichlet part* of the boundary of Ω as $\partial_D\Omega := \Omega_D \cap \partial\Omega$. The Dirichlet condition on $\partial_D\Omega$ is imposed by prescribing the deformation of $\Omega_D \backslash \Omega$, which may be considered as an *unbreakable body* in contact with Ω. The *Neumann part* of the boundary is the closed set $\partial_N\Omega := \partial\Omega \backslash \partial_D\Omega$. The case $\Omega_D = \Omega$ corresponds to a pure Neumann problem, while $\overline{\Omega} \subset \Omega_D$ corresponds to a pure Dirichlet problem (if so, it is not restrictive to take $\Omega_D = \operatorname{int} K$).

We suppose

$$\overline{\Omega}_B \cap \partial_D\Omega = \emptyset\,, \tag{3.1}$$

so that the boundary deformation acts on the brittle part Ω_B only through $\Omega \backslash \overline{\Omega}_B$, which can be regarded as a *layer of unbreakable material*. Notice that this condition does not imply that $\Omega_B \subset\subset \Omega_D$, but only that the brittle part $\overline{\Omega}_B$ does not meet the Dirichlet boundary $\partial_D\Omega = \Omega_D \cap \partial\Omega$. As a consequence, there cannot be interfacial cracks on $\partial_D\Omega$. We cannot avoid (3.1) for a technical reason, related to the non-interpenetration condition, that will appear in the proof of Lemma 3.4.1 about crack transfer.

A *crack* is represented in the reference configuration by a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set $\Gamma \tilde{\subset} \overline{\Omega}_B \cap \Omega_D$ with $\mathcal{H}^{n-1}(\Gamma) < +\infty$. The collection of *admissible cracks* is given by

$$\mathcal{R} := \left\{\Gamma\colon (\mathcal{H}^{n-1}, n-1)\text{-rectifiable, } \Gamma \tilde{\subset} \overline{\Omega}_B \cap \Omega_D\,,\ \mathcal{H}^{n-1}(\Gamma) < +\infty\right\}. \tag{3.2}$$

According to Griffith's theory, we assume that the *energy spent to produce the crack* $\Gamma \in \mathcal{R}$ is given by

$$\mathcal{K}(\Gamma) := \int_\Gamma \kappa(x, \nu_\Gamma(x))\, \mathrm{d}\mathcal{H}^{n-1}(x)\ , \tag{3.3}$$

where ν_Γ is a unit normal vector field on Γ and $\kappa\colon (\overline{\Omega}_B \cap \Omega_D) \times \mathbb{R}^n \to \mathbb{R}$ is a locally bounded Borel function $\kappa\colon U \times \mathbb{R}^n \to \mathbb{R}$ such that the hypotheses of Theorem 1.3.3 hold:

- K1. for every $\varepsilon > 0$ there exists an open set A of 1-capacity $C_1(A) < \varepsilon$ such that $x \mapsto \kappa(x, \nu)$ is lower semicontinuous on $\overline{\Omega}_B \cap \Omega_D \backslash A$ for every $\nu \in \mathbb{R}^n$,
- K2. $\nu \mapsto \kappa(x, \nu)$ is a norm on $\mathbb{R}^n$ for every $x \in \overline{\Omega}_B \cap \Omega_D$,
- K3. $\kappa_1\,|\nu| \le \kappa(x, \nu) \le \kappa_2\,|\nu|$ for every $(x, \nu) \in \overline{\Omega}_B \cap \Omega_D \times \mathbb{R}^n$,

for some constants $\kappa_1 > 0$ and $\kappa_2 > 0$. As a consequence, we have

$$\kappa_1 \mathcal{H}^{n-1}(\Gamma) \leq \mathcal{K}(\Gamma) \leq \kappa_2 \mathcal{H}^{n-1}(\Gamma) . \tag{3.4}$$

To simplify the exposition of auxiliary results, we extend κ to $\Omega_D \times \mathbb{R}^n$ by setting $\kappa(x,\nu) := \kappa_2 |\nu|$ if $x \in \Omega_D \backslash \overline{\Omega}_B$, and we define $\mathcal{K}(\Gamma)$ by (3.3) for every countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable subset Γ of $\mathbb{R}^n$.

3.1.2 Admissible deformations

A deformation of Ω_D is represented by a function u in $SBV(\Omega_D; K)$, which is defined as the set of functions $u \in SBV(\Omega_D; \mathbb{R}^n)$ such that $u(x) \in K$ for a.e. $x \in \Omega_D$. With this definition we are requiring that every deformation of the body remains in the container K. We assume that there is $\Gamma \in \mathcal{R}$ such that $S(u) \tilde{\subset} \Gamma$, so $S(u) \tilde{\subset} \overline{\Omega}_B \cap \Omega_D$.

The prescribed deformation of $\Omega_D \backslash \overline{\Omega}$ is given by a function $\psi \in W^{1,1}(\Omega_D \backslash \overline{\Omega}; K)$. The Dirichlet condition on u takes the form $u = \psi$ a.e. in $\Omega_D \backslash \overline{\Omega}$, i.e., we prescribe the deformation on the whole volume $\Omega_D \backslash \overline{\Omega}$ and not only on $\partial_D \Omega$. On the latter set the equality $u = \psi$ is satisfied in the sense of traces, because by (3.1) u is of class $W^{1,1}$ in the neighbourhood $\Omega_D \backslash \overline{\Omega}_B$ of $\partial_D \Omega$.

Then we define the set of *admissible deformations*, corresponding to a crack $\Gamma \in \mathcal{R}$ and a Dirichlet datum $\psi \in W^{1,1}(\Omega_D \backslash \overline{\Omega}; K)$, as

$$\begin{aligned} AD(\psi, \Gamma) := \Big\{ u \in SBV(\Omega_D; K) \colon & u \text{ satisfies (CN1), (CN2),} \\ & u|_{\Omega_D \backslash \overline{\Omega}} = \psi \,, \text{ and } S(u) \tilde{\subset} \Gamma \Big\} . \end{aligned} \tag{3.5}$$

If $AD(\psi, \Gamma) \neq \emptyset$, the equality $u|_{\Omega_D \backslash \overline{\Omega}} = \psi$ implies in particular that ψ satisfies (CN1) and (CN2) in $\Omega_D \backslash \overline{\Omega}$. Moreover, if $u \in AD(\psi, \Gamma)$ there exists $N \subset \Omega_D$ with $\mathcal{L}^n(N) = 0$ such that $u(\Omega \backslash N)$ does not intersect $\psi((\Omega_D \backslash \overline{\Omega}) \backslash N)$.

Remark 3.1.1. The first difference from the model of [14] is the non-interpenetration requirement for the admissible deformations; this suggests to formulate the boundary conditions in terms of the leading body $\Omega_D \backslash \Omega$. Furthermore, we introduce the confinement condition $u(x) \in K$, in order to simplify the functional framework (SBV instead of $GSBV$). Another relevant difference is given by the assumptions on the bulk energy, which will be stated in the next section.

3.1.3 Bulk energy

We present the assumptions on the bulk energy, which will allow us to deal with the case of finite elasticity. Hypotheses (W0), (W2), and (W5) were studied in [5] (see Section 2.1 here); (W1) and (W4) were presented in [25] (see Section 1.2 here); finally, (W6) was used in [24].

Given a crack $\Gamma \in \mathcal{R}$, we suppose that the uncracked part $\Omega \backslash \Gamma$ is hyperelastic and that the *bulk energy* on $\Omega \backslash \Gamma$ of any deformation $u \in SBV(\Omega_D; K)$ with $S(u) \tilde{\subset} \Gamma$ can be written as

$$\mathcal{W}(u) := \int_{\Omega \backslash \Gamma} W(x, \nabla u(x)) \, \mathrm{d}x = \int_{\Omega} W(x, \nabla u(x)) \, \mathrm{d}x \,, \tag{3.6}$$

where $W\colon \Omega \times \mathbb{M}^{n\times n} \to [0,+\infty]$ is independent of Γ and satisfies the following properties:

W0. *Frame indifference:* for every $(x,A) \in \Omega \times \mathbb{M}^{n\times n}$

$$W(x,QA) = W(x,A) \text{ for every } Q \in SO_n\,;$$

W1. *Polyconvexity:* there exists a function $\widetilde{W}\colon \Omega \times \mathbb{R}^\tau \to [0,+\infty]$ such that $x \mapsto \widetilde{W}(x,\xi)$ is $\mathcal{L}^n$-measurable on Ω for every $\xi \in \mathbb{R}^\tau$, $\xi \mapsto \widetilde{W}(x,\xi)$ is continuous and convex on $\mathbb{R}^\tau$ for every $x \in \Omega$, and

$$W(x,A) = \widetilde{W}(x,M(A)) \quad \text{for every } (x,A) \in \Omega \times \mathbb{M}^{n\times n}\,,$$

where $M(A) := (\mathrm{adj}_1 A, \dots, \mathrm{adj}_n A)$ is the vector (of dimension $\tau := \tau_1 + \dots + \tau_n$) composed of all minors of A;

W2. *Finiteness and regularity:* for every $x \in \Omega$ we have

$$W(x,A) < +\infty \quad \Longleftrightarrow \quad A \in GL_n^+$$

and $A \mapsto W(x,A)$ is of class C^1 on GL_n^+.

Furthermore, we require that there exist a function $c_W^0 \in L^1_+(\Omega)$, some constants $c_W^1 > 0$, $\beta_W^0 \geq 0$, $\beta_W^1, \dots, \beta_W^n > 0$, and some exponents $p_1, p_2, \dots, p_n$, such that for every $x \in \Omega$:

W3. *Bound at identity:* we have $W(x,I) \leq c_W^0(x)$;

W4. *Lower growth condition:* for every $A \in \mathbb{M}^{n\times n}$

$$W(x,A) \geq \sum_{j=1}^n \beta_W^j \left|\mathrm{adj}_j A\right|^{p_j} - \beta_W^0\,,$$

with

$$p_1 \geq 2\,, \quad p_j \geq p_1' := \frac{p_1}{p_1 - 1} \text{ for } j = 2, \dots, n-1\,, \quad p_n > 1\,;$$

W5. *Multiplicative stress estimate:* for every $A \in GL_n^+$

$$\left|A^{\mathrm{T}}\, \mathrm{D}_A W(x,A)\right| \leq c_W^1 (W(x,A) + c_W^0(x))\,;$$

W6. *Continuity of Kirchhoff stress:* for every $\varepsilon > 0$ there exists $\delta > 0$, independent of x, such that for every $A \in GL_n^+$ and $B \in GL_n^+$ with $|B - I| < \delta$

$$\left|\mathrm{D}_A W(x,BA)\,(BA)^{\mathrm{T}} - \mathrm{D}_A W(x,A)\,A^{\mathrm{T}}\right| \leq \varepsilon\,(W(x,A) + c_W^0(x))\,.$$

Henceforth, we will set $p := p_1$.

Thanks to Proposition 2.1.1, from (W5) we get that for every $(x,A) \in \Omega \times GL_n^+$

$$\left|\mathrm{D}_A W(x,A)\,A^{\mathrm{T}}\right| \leq c_W^1 (W(x,A) + c_W^0)\,; \tag{3.7}$$

moreover, there is a constant γ such that for every $B \in GL_n^+$ with $|B - I| < \gamma$,

$$W(x,AB) + c_W^0 \leq \frac{n}{n-1} \left(W(x,A) + c_W^0\right)\,. \tag{3.8}$$

Example 3.1.2 (Ogden Materials)**.** We show a simple example of Ogden material (see Example 2.1.3) satisfying all the properties we are requiring for W. A similar example is presented in [24], in the case $p > n$, where Ambrosio's result [3, Corollary 4.9] is sufficient to prove lower semicontinuity, so that one can take $\beta_W^1 = \cdots = \beta_W^{n-1} = 0$ in (W4). In our example $\beta_W^j > 0$ for every j, which allows us to consider the case $2 \le p \le n$, where Ambrosio's semicontinuity result cannot be applied. Another example can be found in [33].

Let $n = 3$ and take, for $A \in \mathbb{M}^{3\times 3}$,

$$W(A) := \begin{cases} \beta_W^1\,|A|^{p_1} + \beta_W^2\,|\mathrm{cof}A|^{p_2} + \beta_W^3\,|\det A|^{p_3} + \gamma\,|\det A|^{-q} & \text{if } \det A > 0\,, \\ +\infty & \text{otherwise,} \end{cases}$$

where $p_1 = p \ge 2$, $p_2 \ge p'$, $p_3 > 1$, and $\beta_W^j > 0$, as in (W4), and $q > 0$, $\gamma > 0$.

Let us verify that properties (W0–6) hold: polyconvexity (W1) and lower growth estimate (W4) are clear by construction; moreover, one can see that W satisfies frame indifference (W0), local non-interpenetration (W2), and isotropy (2.13). To check the other properties, we must compute the derivative of W for $A \in GL_3^+$; for this, we need the expression of the differential $\mathrm{d}_A \,\mathrm{cof}A$, considered as a linear map from $\mathbb{M}^{3\times 3}$ into $\mathbb{M}^{3\times 3}$:

$$\mathrm{d}_A \,\mathrm{cof}A\,[B] = \left[\mathrm{tr}\left(A^{-1}B\right)\, I - A^{-\mathrm{T}}B^{\mathrm{T}}\right] \mathrm{cof}A\,,$$

whence we conclude that $\mathrm{d}_A \,\mathrm{cof}A$ is symmetric, i.e.,

$$\mathrm{d}_A \,\mathrm{cof}A\,[B] : C = \mathrm{d}_A \,\mathrm{cof}A\,[C] : B\,. \tag{3.9}$$

Then we see that

$$(\mathrm{d}_A \,\mathrm{cof}A\,[\mathrm{cof}A])\, A^{\mathrm{T}} = |\mathrm{cof}A|^2\, I - \mathrm{cof}A\; \mathrm{cof}A^{\mathrm{T}}\,, \tag{3.10}$$

$$\mathrm{d}_A \,\mathrm{cof}A\,[CA] = \left(\mathrm{tr}\,(C)\; I - C^{\mathrm{T}}\right) \mathrm{cof}A\,. \tag{3.11}$$

Using (2.6), (3.9), and (3.10), we get

$$\begin{aligned} K(A) = {} & \beta_W^1 p_1\,|A|^{p_1-2}\,AA^{\mathrm{T}} + \beta_W^2 p_2 \left[|\mathrm{cof}A|^{p_2}\; I - |\mathrm{cof}A|^{p_2-2}\,\mathrm{cof}A\;\mathrm{cof}A^{\mathrm{T}}\right] + \\ & + \left(\beta_W^3 p_3\,|\det A|^{p_3} - \gamma q\,|\det A|^{-q}\right)\, I\,. \end{aligned}$$

This shows that (3.7) holds, so by Remark 2.1.2 (W5) is proven.

Finally, we compute the differential $\mathrm{d}_A K(A)$, considered as a linear map from $\mathbb{M}^{3\times 3}$ into $\mathbb{M}^{3\times 3}$. Using (3.11) we obtain

$$\begin{aligned} \mathrm{d}_A K(A)\,[CA] = {} & \quad \beta_W^1\, p_1(p_1-2)\,|A|^{p_1-4}\left(AA^{\mathrm{T}} : C\right) AA^{\mathrm{T}} + \\ & + \beta_W^1\, p_1\,|A|^{p_1-2}\,(CAA^{\mathrm{T}} + AA^{\mathrm{T}}C) + \\ & + \beta_W^2\, {p_2}^2 \left[|\mathrm{cof}A|^{p_2}\,\mathrm{tr}\,(C) - |\mathrm{cof}A|^{p_2-2}\left(\mathrm{cof}A^{\mathrm{T}}\;\mathrm{cof}A\right) : C\right] I + \\ & - \beta_W^2\, p_2(p_2-2)\,|\mathrm{cof}A|^{p_2-2}\,\mathrm{tr}\,(C)\;\mathrm{cof}A\;\mathrm{cof}A^{\mathrm{T}} + \\ & + \beta_W^2\, p_2(p_2-2)\,|\mathrm{cof}A|^{p_2-4}\left[\left(\mathrm{cof}A^{\mathrm{T}}\;\mathrm{cof}A\right) : C\right]\;\mathrm{cof}A\;\mathrm{cof}A^{\mathrm{T}} + \\ & - \beta_W^2\, p_2\,|\mathrm{cof}A|^{p_2-2}\left[\mathrm{tr}\,(C)\; I - C^{\mathrm{T}}\right]\mathrm{cof}A\;\mathrm{cof}A^{\mathrm{T}} + \\ & - \beta_W^2\, p_2\,|\mathrm{cof}A|^{p_2-2}\,\mathrm{cof}A\;\mathrm{cof}A^{\mathrm{T}}\left[\mathrm{tr}\,(C)\; I - C\right] + \\ & + \left[\beta_W^3\, p_3\,|\det A|^{p_3} - \gamma q\,|\det A|^{-q}\right]\mathrm{tr}\,(C)\; I\,. \end{aligned}$$

We deduce that there exists $c_W^3 > 0$ such that

$$|\mathrm{D}_A K_{ij}(x,A):(CA)| \leq c_W^3 (W(x,A) + c_W^0(x))\,|C| \tag{3.12}$$

for every $C \in \mathbb{M}^{n\times n}$ and $(x,A) \in \Omega \times GL_n^+$. In [24, Proposition 5.2] it is proven that this property guarantees (W6).

With the same procedure one can treat *Mooney-Rivlin materials* [11], where

$$W(A) := \begin{cases} a\,|A|^2 + b\,|\mathrm{cof}\,A|^2 + c\,|\det A|^2 - d\,\log\det A & \text{if } \det A > 0\,, \\ +\infty & \text{otherwise,} \end{cases}$$

where a, b, c, d are positive constants. Also in this case, because of the exponent $p = 2$, Ambrosio's result does not apply.

3.1.4 Prescribed deformations

We prescribe a time-dependent deformation of $\Omega_D \backslash \Omega$, requiring that $u(x) = \psi(t,x)$ for a.e. $x \in \Omega_D \backslash \Omega$, at every time $t \in [0,1]$. For technical reasons, we have to assume that $x \mapsto \psi(t,x)$ is defined for every $x \in K$, takes values in K, and has an inverse function on K, denoted by $y \mapsto \phi(t,y)$. This determines two functions

$$\psi,\ \phi\colon [0,1] \times K \to K\,.$$

With a small abuse of notation, the functions $x \mapsto \psi(t,x)$ and $y \mapsto \phi(t,y)$ are denoted by $\psi(t)\colon K \to K$ and $\phi(t)\colon K \to K$, respectively. At each time t they satisfy

BC1. $$\psi(t) \circ \phi(t) = I = \phi(t) \circ \psi(t)\,,$$

where I denotes the identical function in K.

We require that for every $i, j = 1, \dots n$

BC2. $\mathrm{D}_t\psi\,,\ \mathrm{D}_{x_i}\psi\,,\ \mathrm{D}_{x_i}\mathrm{D}_{x_j}\psi\,,\ \mathrm{D}_t\mathrm{D}_{x_i}\psi$ exist, continuous on $[0,1] \times K$

and

BC3. $\mathrm{D}_t\phi\,,\ \mathrm{D}_{y_i}\phi\,,\ \mathrm{D}_{y_i}\mathrm{D}_{y_j}\phi\,,\ \mathrm{D}_t\mathrm{D}_{y_i}\phi$ exist, continuous on $[0,1] \times K$.

This implies that the mixed derivative $\mathrm{D}_{x_i}\mathrm{D}_t\psi$ exists and coincides with $\mathrm{D}_t\mathrm{D}_{x_i}\psi$; the same is true for ϕ. We use the following notation: $\nabla\psi$ and $\nabla\phi$ are the Jacobian matrices with respect to x or y; moreover, $\dot\psi := \mathrm{D}_t\psi$, $\nabla\dot\psi := \nabla\mathrm{D}_t\psi = \mathrm{D}_t\nabla\psi$, and the same for ϕ.

We need a uniform bound on the energy of the prescribed deformation: we suppose that there exists a constant M such that

BC4. $$W(x, \nabla\psi(t,x)) \leq M$$

for every $(t,x) \in [0,1]\times\Omega$ (for example, this holds when $\psi(t) = I$). This assumption, together with (W2), gives

$$\det\nabla\psi(t,x) > 0 \text{ for a.e. } x \in K\,.$$

Since by (BC1) and (BC2) $\det \nabla\psi(t,x) \neq 0$ for every $t \in [0,1]$ and $x \in K$, by continuity one has

$$\det \nabla\psi(t,x) > 0 \text{ for every } x \in K\,, \tag{3.13}$$

which in turn implies

$$\det \nabla\phi(t,y) > 0 \text{ for every } y \in K\,. \tag{3.14}$$

Notice that (3.13) and the invertibility of $\psi(t)$ imply that $\psi(t)$ satisfies the Ciarlet-Nečas condition; as $S(\psi(t)) = \emptyset$, this implies that $\psi(t) \in AD(\psi(t),\Gamma)$ for every $\Gamma \in \mathcal{R}$.

3.2 Evolution of stable equilibria

Our aim is to study the evolution of stable equilibria for the physical system introduced in the previous section: an elastic body with cracks, subjected to a general strain energy, compatible with the non-interpenetration hypotheses (W2).

In the present section, we define the notion of *incrementally-approximable quasistatic evolution of global minimizers* for the total energy $\mathcal{E}$. The main results are the existence of such a quasistatic evolution with prescribed initial conditions (Theorem 3.2.16) and the analysis of its properties (Theorem 3.2.17); they appeared in [17].

3.2.1 Minimum energy configurations

We begin by discussing the notion of stable equilibrium, first considering only the bulk energy $\mathcal{W}$. For a fixed time $t \in [0,1]$ and a given crack $\Gamma \in \mathcal{R}$, a deformation u corresponding to an equilibrium is a critical point of the functional $\mathcal{W}$ on the set $AD(\psi(t),\Gamma)$ defined in (3.5). Among such critical points, we select the minimum points of the problem

$$\min_{u \in AD(\psi(t),\Gamma)} \mathcal{W}(u)\,, \tag{3.15}$$

which are called the *minimum energy deformations at time t with crack* Γ. Their existence is guaranteed by the following theorem, which will be proven in Section 3.3.1.

Theorem 3.2.1 (Minimization of the elastic energy)**.** *Assume that $\mathcal{W}$ satisfies* (W0–6)*. Consider the prescribed deformations defined in* (BC1–4)*. Then for every $t \in [0,1]$ and every $\Gamma \in \mathcal{R}$ the minimum problem* (3.15) *has a solution.*

Next, we define the total energy

$$\mathcal{E}(u,\Gamma) := \mathcal{W}(u) + \mathcal{K}(\Gamma)\,. \tag{3.16}$$

In Griffith's theory, an *equilibrium configuration* at a fixed time $t \in [0,1]$ is an admissible configuration $(u(t),\Gamma(t))$ which is a "critical point" of the functional $\mathcal{E}(u,\Gamma)$ on the set of configurations (u,Γ) with $\Gamma \in \mathcal{R}$, $\Gamma(t) \,\tilde{\subset}\, \Gamma$, and $u \in AD(\psi(t),\Gamma)$. Unfortunately, the definition of "critical point" in this context has never been made mathematically precise.

Following [23], among these equilibrium configurations we will consider only *minimum energy configurations*, which are defined as those admissible configurations $(u(t), \Gamma(t))$, with $\Gamma(t) \in \mathcal{R}$ and $u(t) \in AD(\psi(t), \Gamma(t))$, such that the unilateral minimality condition holds:

$$\mathcal{E}(u(t), \Gamma(t)) \leq \mathcal{E}(u, \Gamma) \tag{3.17}$$

for every $\Gamma \in \mathcal{R}$, with $\Gamma(t) \mathrel{\tilde{\subset}} \Gamma$, and every $u \in AD(\psi(t), \Gamma)$.

The next theorem ensures that for every $t \in [0, 1]$ and for every initial datum $\Gamma_0 \in \mathcal{R}$ there exists at least a minimum energy configuration $(u(t), \Gamma(t))$ such that $\Gamma_0 \mathrel{\tilde{\subset}} \Gamma(t)$; the proof is in Section 3.3.1.

Theorem 3.2.2 (Minimization of the total energy)**.** *Let $\mathcal{E}$ be the energy defined in* (3.16)*, where $\mathcal{W}$ satisfies* (W0–6) *and $\mathcal{K}$ satisfies* (K1–2)*. Consider the prescribed deformations defined in* (BC1–4)*. Then, for every $t \in [0, 1]$ and $\Gamma_0 \in \mathcal{R}$, the minimum problem*

$$\min \left\{ \mathcal{E}(u, \Gamma) \colon \Gamma \in \mathcal{R},\ \Gamma_0 \mathrel{\tilde{\subset}} \Gamma,\ u \in AD(\psi(t), \Gamma) \right\} \tag{3.18}$$

has a solution.

3.2.2 The discrete-time problems

To define a quasistatic evolution, we employ a standard method for rate-independent processes [37], developed in [23, 18, 14, 22] for problems in fracture mechanics: first, we consider a time-discretization of the problem and find some *incremental approximate solutions*; the desired *incrementally-approximable quasistatic evolution* will then be the limit of the discrete solutions.

Let us fix a sequence of subdivisions $\{t_k^i\}_{0 \leq i \leq k}$ of the interval $[0, 1]$, with

$$0 = t_k^0 < t_k^1 < \cdots < t_k^{k-1} < t_k^k = 1 \tag{3.19}$$

and

$$\lim_{k \to \infty} \max_{1 \leq i \leq k} (t_k^i - t_k^{i-1}) = 0\,. \tag{3.20}$$

We will call such a sequence a *time discretization*.

As a datum of the problem, we are given an initial condition (u_0, Γ_0), satisfying $\Gamma_0 \in \mathcal{R}$, $u_0 \in AD(\psi(0), \Gamma_0)$, and the unilateral minimality condition

$$\mathcal{E}(u_0, \Gamma_0) \leq \mathcal{E}(u, \Gamma) \tag{3.21}$$

for every $\Gamma \in \mathcal{R}$ with $\Gamma_0 \mathrel{\tilde{\subset}} \Gamma$ and every $u \in AD(\psi(0), \Gamma)$.

For every time subdivision, we define a corresponding incremental approximate solution, whose existence is guaranteed by Theorem 3.2.2.

Definition 3.2.3. Fix $k \in \mathbb{N}$. An *incremental approximate solution* for $\mathcal{E}$ corresponding to the time subdivision $\{t_k^i\}_{0 \leq i \leq k}$ with initial datum (u_0, Γ_0) is a function $t \mapsto (u_k(t), \Gamma_k(t))$, such that

(a) $(u_k(0), \Gamma_k(0)) = (u_0, \Gamma_0)$;

(b) $u_k(t) = u_k(t_k^i)$ and $\Gamma_k(t) = \Gamma_k(t_k^i)$ for $t \in [t_k^i, t_k^{i+1})$ and $i = 0, \dots, k-1$;

(c) for $i = 1, \dots, k$, $(u_k(t_k^i), \Gamma_k(t_k^i))$ is a solution of

$$\min \left\{ \mathcal{E}(u, \Gamma) \colon \Gamma \in \mathcal{R},\ \Gamma(t_k^{i-1}) \,\widetilde{\subset}\, \Gamma,\ u \in AD(\psi(t_k^i), \Gamma) \right\}. \tag{3.22}$$

Notice that, if $t \mapsto (u_k(t), \Gamma_k(t))$ is an incremental approximate solution, by the minimality and by (BC4) we have $\mathcal{E}(u_k(t), \Gamma_k(t)) < +\infty$ for every t, hence $u_k \in SBV^p(\Omega_D; K)$ by (W4), with $p = p_1$.

3.2.3 Formulation with time-independent prescribed deformations

Now we pass to an alternative formulation of the problem, where the Dirichlet conditions are time-independent, whilst the time-dependence is transferred to the energy terms; this approach is based on [24]. We look for a solution $u \in AD(\psi(t), \Gamma)$ to (3.18) in the form $u = \psi(t) \circ z$, with $z \in SBV(\Omega_D; K)$; this request implies $z \in AD(I, \Gamma)$. The chain rule in BV [4, Theorem 3.96] gives $\nabla u(x) = \nabla \psi(t, z(x)) \nabla z(x)$ for a.e. $x \in \Omega_D$, so that we define the auxiliary volume energy

$$\mathcal{V}(t, z) := \int_\Omega V(t, x, z(x), \nabla z(x)) \, \mathrm{d}x \,, \tag{3.23}$$

where

$$V(t, x, y, A) := W(x, \nabla \psi(t, y) A) \,. \tag{3.24}$$

Hence,

$$\mathcal{W}(u) = \mathcal{V}(t, \phi(t) \circ u) \,, \quad \mathcal{V}(t, z) = \mathcal{W}(\psi(t) \circ z) \,. \tag{3.25}$$

This leads to introduce a class of functions $V \colon [0,1] \times \Omega \times K \times \mathbb{M}^{n \times n} \to [0, +\infty]$ satisfying the following properties:

V1. *Polyconvexity:* there exists a function $\widetilde{V} \colon [0,1] \times \Omega \times K \times \mathbb{R}^\tau \to [0, +\infty]$ such that $x \mapsto \widetilde{V}(t, x, y, \xi)$ is $\mathcal{L}^n$-measurable on Ω for every $(t, y, \xi) \in [0,1] \times K \times \mathbb{R}^\tau$, $(t, y, \xi) \mapsto \widetilde{V}(t, x, y, \xi)$ is continuous on $[0,1] \times K \times \mathbb{R}^\tau$ for every $x \in \Omega$, $\xi \mapsto \widetilde{V}(t, x, y, \xi)$ is convex on $\mathbb{R}^\tau$ for every $(t, x, y) \in [0,1] \times \Omega \times K$, and

$$V(t, x, y, A) = \widetilde{V}(t, x, y, M(A)) \quad \text{for every } (t, x, y, A) \in [0,1] \times \Omega \times K \times \mathbb{M}^{n \times n} \,,$$

where $M(A)$ is defined as in (W1);

V2. *Finiteness and regularity:* for every $(t, x, y) \in [0,1] \times \Omega \times K$ we have

$$V(t, x, y, A) < +\infty \quad \Longleftrightarrow \quad A \in GL_n^+ \,,$$

and $(t, y, A) \mapsto V(t, x, y, A)$ is of class C^1 on $[0,1] \times K \times GL_n^+$ for every $x \in \Omega$;

furthermore, there exist a function $c_V^0 \in L^1_+(\Omega)$, some constants $c_V^1 > 0$, $\beta_V^0 \geq 0$, $\beta_V^1, \dots, \beta_V^n > 0$, and some exponents $p_1, p_2, \dots, p_n$, such that for every $(t, x, y) \in [0,1] \times \Omega \times K$:

V3. *Bound at identity:* we have $V(t,x,x,I) \leq c_V^0(x)$;

V4-5. *Dependence on the matricial term:* $A \mapsto V(t,x,y,A)$ satisfies (W4–5);

V6. *Estimate on the time derivative:* for every $A \in GL_n^+$

$$|\mathrm{D}_t V(t,x,y,A)| \leq c_V^1 (V(t,x,y,A) + c_V^0(x))\,;$$

V7. *Continuity of the time derivative:* for every $\varepsilon > 0$ there exists $\delta > 0$, independent of (t,x,y), such that for every $s \in [0,1]$ with $|t-s| < \delta$ and every $A \in GL_n^+$

$$|\mathrm{D}_t V(t,x,y,A) - \mathrm{D}_t V(s,x,y,A)| \leq \varepsilon\,(V(t,x,y,A) + c_V^0(x))\,;$$

V8. *Estimate on spatial derivatives:* for every $A \in GL_n^+$

$$|\mathrm{D}_y V(t,x,y,A)| \leq c_V^1 (V(t,x,y,A) + c_V^0(x))\,.$$

Proposition 3.2.4. *If* (W0–6) *and* (BC1–4) *hold, then the function V defined in* (3.24) *satisfies properties* (V1–8).

Proof. Properties (V1–2) are obvious.

Checking property (V4) reduces to estimate $\left|\mathrm{adj}_j \nabla\psi(t,y)\,A\right|$ from below in terms of $\left|\mathrm{adj}_j A\right|$, for given $t \in [0,1]$, $y \in K$, and $A \in GL_n^+$. Let $B \in GL_n^+$; then

$$\left|\mathrm{adj}_j(BA)\right| \leq \left|\mathrm{adj}_j B\right| \left|\mathrm{adj}_j A\right| \leq C_j \sup_{l,m} |b_{lm}|^j \; \left|\mathrm{adj}_j A\right| \leq C_j\,|B|^j \left|\mathrm{adj}_j A\right|\,,$$

where the first inequality is given by [13, Proposition 5.66], b_{lm} are the elements of B, and $C_j > 0$ depends only on n and j. This is equivalent to

$$\left|\mathrm{adj}_j(B^{-1}A)\right| \geq \frac{1}{C_j}\,|B|^{-j} \left|\mathrm{adj}_j A\right|\,.$$

For $B^{-1} = \nabla\psi(t,y)$, employing the hypotheses of boundedness (BC2), (BC3), and the invertibility condition (3.13), we conclude, modifying the constants properly.

We take

$$c_V^0 \geq c_W^0 \vee M\,, \quad c_V^1 \geq \max_{[0,1]\times K} \left\{ c_W^1,\ 1 + \left|\dot\psi\right|,\ |\nabla\psi|\,,\ c_W^1\,|\nabla\phi| \left|\nabla\dot\psi\right|,\ c_W^1\,|\nabla\phi| \left|\nabla^2\psi\right| \right\}\,,$$

where M is the constant of (BC4). Then (V3) comes from (BC4), while (V5), (V6), and (V8) follow from (V5), (BC2), and (BC3), using (2.17) (which holds because of (3.7)). Similarly, (V7) follows from (W6), thanks again to (3.7) and to the properties of ψ (see also [24, Lemma 5.5]). □

Remark 3.2.5. Frame indifference is not preserved under (3.24).

The previous proposition allows us to leave the setting introduced in Section 3.1 and consider the more general class of functions satisfying (V1–8). Here we underline some consequences of these properties.

Remark 3.2.6. Property (V5) implies (3.8) for V. Furthermore, (V6) gives, via the Gronwall Lemma,

$$V(t_2,x,y,A)+c_V^0(x)\le (V(t_1,x,y,A)+c_V^0(x))\,e^{c_V^1|t_2-t_1|} \tag{3.26}$$

for every $t_1,t_2\in[0,1]$ and $(x,y,A)\in\Omega\times K\times GL_n^+$, which ensures the uniform continuity of $t\mapsto V(t,x,y,A)$ on the sublevels of V. Analogously, (V8) implies

$$V(t,x,y_2,A)+c_V^0(x)\le (V(t,x,y_1,A)+c_V^0(x))\,e^{c_V^1|y_2-y_1|} \tag{3.27}$$

for every $y_1,y_2\in K$ and $(t,x,A)\in[0,1]\times\Omega\times GL_n^+$.

Estimate (3.26) has the following consequence: if $\mathcal{V}(t_0,z)<+\infty$ for a fixed time $t_0\in[0,1]$ and a function $z\in SBV(\Omega_D;K)$, then $\mathcal{V}(t,z)<+\infty$ for every $t\in[0,1]$; then, by (V6), $t\mapsto\mathcal{V}(t,z)$ is well defined and C^1 on [0,1], and its derivative $\mathrm{D}_t\mathcal{V}(t,z)$ is given by

$$\mathrm{D}_t\mathcal{V}(t,z)=\int_\Omega \mathrm{D}_tV(t,x,z(x),\nabla z(x))\,\mathrm{d}x\,. \tag{3.28}$$

We regard $\mathrm{D}_t\mathcal{V}(t,\cdot)$ as a functional defined on

$$\mathcal{U}_{\mathcal{V}}:=\big\{z\in SBV(\Omega_D;K)\colon \mathcal{V}(0,z)<+\infty\big\}\,. \tag{3.29}$$

Finally, we define

$$\mathcal{F}(t,z,\Gamma):=\mathcal{V}(t,z)+\mathcal{K}(\Gamma)\,. \tag{3.30}$$

Using the new formulation, (3.18) is equivalent to the auxiliary problem

$$\min\big\{\mathcal{F}(t,u,\Gamma)\colon \Gamma\in\mathcal{R},\ \Gamma_0\,\tilde{\subset}\,\Gamma,\ u\in AD(I,\Gamma)\big\}\,. \tag{3.31}$$

Also in this case, we provide two minimization results, proven in Section 3.3.1.

Theorem 3.2.7 (Minimization of the elastic energy)**.** *Let $\mathcal{V}$ satisfy* (V1–8). *Then for every $t\in[0,1]$ and every $\Gamma\in\mathcal{R}$ the minimum problem*

$$\min_{u\in AD(I,\Gamma)}\mathcal{V}(t,u) \tag{3.32}$$

has a solution.

Theorem 3.2.8 (Minimization of the total energy)**.** *Let $\mathcal{F}$ be the energy defined in* (3.30)*, where $\mathcal{V}$ satisfies* (V1–8) *and $\mathcal{K}$ satisfies* (K1–2)*. Then, for every $t\in[0,1]$ and $\Gamma_0\in\mathcal{R}$, the minimum problem* (3.31) *has a solution.*

3.2.4 Quasistatic evolution

Let us fix an initial condition (u_0,Γ_0). We suppose that it is a minimum energy configuration at time 0, i.e., $\Gamma_0\in\mathcal{R}$, $u_0\in AD(I,\Gamma_0)$, and

$$\mathcal{F}(0,u_0,\Gamma_0)\le\mathcal{F}(0,u,\Gamma) \tag{3.33}$$

for every $\Gamma\in\mathcal{R}$ with $\Gamma_0\,\tilde{\subset}\,\Gamma$ and every $u\in AD(I,\Gamma)$.

We define the notion of incremental approximate solution for $\mathcal{F}$, corresponding to a time subdivision $\big\{t_k^i\big\}_{0\le i\le k}$ (see (3.19) and (3.20)). The existence of such solutions is guaranteed by Theorem 3.2.8.

Definition 3.2.9. Fix $k \in \mathbb{N}$. An *incremental approximate solution* for $\mathcal{F}$ corresponding to the time subdivision $\left\{t_k^i\right\}_{0 \le i \le k}$ with initial datum (u_0, Γ_0) is a function $t \mapsto (u_k(t), \Gamma_k(t))$, such that

(a) $(u_k(0), \Gamma_k(0)) = (u_0, \Gamma_0)$;

(b) $u_k(t) = u_k(t_k^i)$ and $\Gamma_k(t) = \Gamma_k(t_k^i)$ for $t \in \left[t_k^i, t_k^{i+1}\right)$ and $i = 0, \dots, k-1$;

(c) for $i = 1, \dots, k$, $(u_k(t_k^i), \Gamma_k(t_k^i))$ is a solution of

$$\min\left\{\mathcal{F}(t_k^i, u, \Gamma) \colon \Gamma \in \mathcal{R},\ \Gamma_k^{i-1} \tilde{\subset} \Gamma,\ u \in AD(I, \Gamma)\right\}. \tag{3.34}$$

An incrementally-approximable quasistatic evolution for (3.31) is the limit of a sequence of incremental approximate solutions, as in the next definition.

Definition 3.2.10. A function $t \mapsto (u(t), \Gamma(t))$ from $[0,1]$ in $SBV^p(\Omega_D; K) \times \mathcal{R}$ is an *incrementally-approximable quasistatic evolution* of minimum energy configurations for problem (3.31) with initial datum (u_0, Γ_0), if there exist an increasing set function $t \mapsto \Gamma^*(t) \in \mathcal{R}$, a time discretization $\left\{t_k^i\right\}_{0 \le i \le k}$, and a corresponding sequence of incremental approximate solutions $t \mapsto (u_k(t), \Gamma_k(t))$ with the same initial datum, such that for every $t \in [0,1]$:

(a) $\Gamma_k(t)$ σ^p-converges to $\Gamma^*(t)$ and $\Gamma(t) = \Gamma^*(t) \cup \Gamma_0$;

(b) there is a subsequence $u_{k_j}(t)$, depending on t, such that $u_{k_j}(t) \rightharpoonup u(t)$ weakly* in $SBV^p(\Omega_D; K)$ and $\lim\limits_{k \to \infty} \theta_{k_j}(t) = \limsup\limits_{k \to \infty} \theta_k(t)$, where

$$\theta_k(t) := \mathrm{D}_t \mathcal{V}(t, u_k(t)). \tag{3.35}$$

We state the existence result for incrementally-approximable quasistatic evolutions, which will be proven in Section 3.3.2.

Theorem 3.2.11 (Existence of quasistatic evolutions)**.** *Let $\mathcal{F}$ be the functional defined in* (3.30)*, where $\mathcal{V}$ satisfies* (V1–8) *and $\mathcal{K}$ satisfies* (K1–2)*. Let (u_0, Γ_0) be a minimum energy configuration at time* 0 *as in* (3.33)*. Then there exists an incrementally-approximable quasistatic evolution $t \mapsto (u(t), \Gamma(t))$ with initial datum (u_0, Γ_0).*

Notice that in the definition of quasistatic evolution we make no measurability assumptions on the function $t \mapsto u(t)$. We will prove later, in Section 3.6, that there exists a quasistatic evolution such that the function $t \mapsto u(t)$ is strongly measurable, regarded as a function from $[0,1]$ into $SBV^p(\Omega_D; \mathbb{R}^n)$.

The next theorem guarantees that the definition of incrementally-approximable quasistatic evolution fits in with the general scheme of the energy formulation of *rate-independent processes* (see [37] and the references therein); for the proof, see Section 3.5.2.

Theorem 3.2.12 (Properties of quasistatic evolutions)**.** *For a given incrementally-approximable quasistatic evolution $t \mapsto (u(t), \Gamma(t))$ for $\mathcal{F}$, the following hold:*

1. Global stability: *for every* $t \in [0,1]$ *the pair* $(u(t), \Gamma(t))$ *is a minimum energy configuration at time* t, *i.e.*, $\Gamma(t) \in \mathcal{R}$, $u(t) \in AD(I, \Gamma(t))$, *and*

$$\mathcal{F}(t, u(t), \Gamma(t)) \leq \mathcal{F}(t, v, \Gamma) \tag{3.36}$$

for every $\Gamma \in \mathcal{R}$, *with* $\Gamma(t) \mathrel{\tilde{\subset}} \Gamma$, *and every* $v \in AD(I, \Gamma)$;

2. Energy balance: *the function* $F(t) := \mathcal{F}(t, u(t), \Gamma(t))$ *is absolutely continuous on* $[0,1]$ *and its time derivative satisfies*

$$\dot{F}(t) = \mathrm{D}_t \mathcal{V}(t, u(t), \Gamma(t)) \text{ for } \mathcal{L}^1\text{-a.e. } t \in [0,1]\,. \tag{3.37}$$

Remark 3.2.13. Notice that in these hypotheses $\mathcal{V}(u(t))$ is finite for every t, because I is a competitor in (3.36) and has finite energy by (V3).

In Section 3.5.3 we provide a further result about the convergence of the energy terms of the incremental approximate solutions: the elastic and the crack energy of an incrementally-approximable quasistatic evolution are the limit of the corresponding energies of the associated sequence of incremental approximate solutions; this holds for the whole sequence and not only for a subsequence.

In order to come back to the original energy $\mathcal{E}$, we compute the partial time derivative $\mathrm{D}_t \mathcal{V}$ when $\mathcal{V}$ is given by (3.24). The functionals will be defined on

$$\mathcal{U}_{\mathcal{W}} := \big\{ v \in SBV(\Omega_D; K) \colon \mathcal{W}(v) < +\infty \big\}\,. \tag{3.38}$$

Fix $t \in [0,1]$; if $u \in \mathcal{U}_{\mathcal{W}}$, then $z := \phi(t) \circ u \in \mathcal{U}_{\mathcal{V}}$, so by (3.24), (3.28), and Remark 3.2.6 $s \mapsto \mathcal{V}(s, z)$ is well defined and C^1 on $[0,1]$, with derivative

$$\mathrm{D}_t \mathcal{V}(s, z) = \int_\Omega \mathrm{D}_A W(x, \nabla(\psi(s) \circ z)) : \nabla\big(\dot{\psi}(s) \circ z\big)\, \mathrm{d}x\,.$$

For $s = t$, recalling that $u = \psi(t) \circ z$, we conclude that

$$\mathrm{D}_t \mathcal{V}(t, \phi(t) \circ u) = \mathcal{P}(t, u)\,, \tag{3.39}$$

where $\mathcal{P}$ represents the power of the system and is given by

$$\mathcal{P}(t, v) := \int_\Omega \mathrm{D}_A W(x, \nabla v) : \nabla\big(\dot{\psi}(t) \circ \phi(t) \circ v\big)\, \mathrm{d}x\,. \tag{3.40}$$

Remark 3.2.14. The integral appearing in the definition of $\mathcal{P}(t, v)$ is well defined for every v in $\mathcal{U}_{\mathcal{W}}$: indeed, it can be rewritten as

$$\int_\Omega \mathrm{D}_A W(x, \nabla v)(\nabla v)^{\mathrm{T}} : \nabla\big(\dot{\psi}(t) \circ \phi(t)\big)(v)\, \mathrm{d}x\,,$$

so that the existence of the integral can be deduced from (3.7), (BC2), (BC3), and (3.38).

Furthermore, if W, Ω, K, $u(t)$, and $\Gamma(t)$ are regular enough, we have

$$\mathcal{P}(t, u(t)) = \int_{\partial_D \Omega} \mathrm{D}_A W(x, \nabla u(t))\, \nu_\Omega(x) \cdot \dot{\psi}(t)\, \mathrm{d}x\,, \tag{3.41}$$

so that $\mathcal{P}(t, u(t))$ can be interpreted as the power of the surface forces acting on $\partial_D \Omega$ at time t.

To prove (3.41), one considers the Euler conditions of (3.18), taking into account the reaction forces generated by the confinement constraint K. Formula (3.41) is then obtained multiplying the Euler equations by $\dot{\psi}(t) \circ \phi(t) \circ u(t)$ and integrating by parts, as in [14, Section 3.8]. Indeed, the additional terms due to the reaction forces give no contribution, since they are orthogonal to $\partial\Omega_0$, while $\dot{\psi}(t) \circ \phi(t)$ is tangential at each point of $\partial\Omega_0$.

This discussion leads to the following definition of incrementally-approximable quasistatic evolution for $\mathcal{E}$ with initial condition (u_0, Γ_0), satisfying (3.21).

Definition 3.2.15. A function $t \mapsto (u(t), \Gamma(t))$ from $[0,1]$ in $SBV^p(\Omega_D; K) \times \mathcal{R}$ is an *incrementally-approximable quasistatic evolution* of minimum energy configurations for problem (3.18) with initial datum (u_0, Γ_0), if there exist an increasing set function $t \mapsto \Gamma^*(t)$, a time discretization $\{t_k^i\}_{0 \le i \le k}$, and a corresponding sequence of incremental approximate solutions $t \mapsto (u_k(t), \Gamma_k(t))$ with the same initial datum, such that for every $t \in [0,1]$:

(a) $\Gamma_k(t)$ σ^p-converges to $\Gamma^*(t)$ and $\Gamma(t) = \Gamma^*(t) \cup \Gamma_0$;

(b) there is a subsequence $u_{k_j}(t)$, depending on t, such that $u_{k_j} \rightharpoonup u(t)$ weakly* in $SBV^p(\Omega_D; K)$ and $\lim\limits_{k \to \infty} \eta_{k_j}(t) = \limsup\limits_{k \to \infty} \eta_k(t)$, where

$$\eta_k(t) := \mathcal{P}(t, u_k(t)) \, . \tag{3.42}$$

Theorems 3.2.11 and 3.2.12 have the following counterparts when dealing with $\mathcal{E}$; the proofs follow from (3.25) and (3.39).

Theorem 3.2.16 (Existence of quasistatic evolutions). *Let $\mathcal{E}$ be the functional defined in* (3.16)*, where $\mathcal{W}$ satisfies* (W0–6) *and $\mathcal{K}$ satisfies* (K1–2)*. Consider the prescribed deformations defined in* (BC1–4)*. Let (u_0, Γ_0) be a minimum energy configuration at time 0, i.e., assume $\Gamma_0 \in \mathcal{R}$, $u_0 \in AD(\psi(0), \Gamma_0)$, and* (3.21)*. Then there exists an incrementally-approximable quasistatic evolution $t \mapsto (u(t), \Gamma(t))$ with initial datum (u_0, Γ_0).*

Theorem 3.2.17 (Properties of quasistatic evolutions). *For a given incrementally-approximable quasistatic evolution $t \mapsto (u(t), \Gamma(t))$ for $\mathcal{E}$, the following hold:*

1. Global stability: *for every $t \in [0,1]$ the pair $(u(t), \Gamma(t))$ is a minimum energy configuration at time t, i.e., $\Gamma(t) \in \mathcal{R}$, $u(t) \in AD(\psi(t), \Gamma(t))$, and*

$$\mathcal{E}(u(t), \Gamma(t)) \le \mathcal{E}(v, \Gamma) \tag{3.43}$$

for every $\Gamma \in \mathcal{R}$, with $\Gamma(t) \tilde{\subset} \Gamma$, and every $v \in AD(\psi(t), \Gamma)$;

2. Energy balance: *the function $E(t) := \mathcal{E}(u(t), \Gamma(t))$ is absolutely continuous on $[0,1]$ and its time derivative satisfies, for $\mathcal{L}^1$-a.e. $t \in [0,1]$,*

$$\dot{E}(t) = \mathcal{P}(t, u(t)) \, , \tag{3.44}$$

where $\mathcal{P}$ is defined by (3.40).

Remark 3.2.18. Consider an incrementally-approximable quasistatic evolution $t \mapsto (u(t), \Gamma(t))$, according to Definition 3.2.15, such that the initial datum u_0 satisfies the progressive non-interpenetration condition (see Section 2.3). Assume in addition that $t \mapsto u(t)$ is continuous on some interval $[0, \tau]$, in the same sense chosen for the progressive non-interpenetration condition. Then, it follows immediately from the definition that $u(t)$ satisfies also the progressive non-interpenetration condition for every $t \in [0, \tau]$.

3.3 Existence results

This section is devoted to proving Theorem 3.2.11. Beforehand, we must show the existence of *minimum energy configurations*, in order to make rigorous Definition 3.2.10. For this, we will use the results presented in Chapter 1.

In particular, we employ Theorem 1.2.1 thanks to the following coercivity estimate for $\mathcal{V}$, which is a consequence of (V4):

$$\mathcal{V}(t, u) \geq \sum_{j=1}^{n} \beta_V^j \left\| \mathrm{adj}_j \nabla u \right\|_{L^{p_j}(\Omega_D; \mathbb{R}^{\tau_j})}^{p_j} - \beta_V^0 \, \mathcal{L}^n(\Omega_D) \tag{3.45}$$

for every $t \in [0, 1]$ and every $u \in SBV^p(\Omega_D; K)$. Moreover, by Theorem 1.2.3 and (3.26) we get

$$\mathcal{V}(t_\infty, u_\infty) \leq \liminf_{k \to \infty} \mathcal{V}(t_k, u_k) \tag{3.46}$$

for every $t_k \to t_\infty$ and every $u_k \rightharpoonup u_\infty$ weakly* in $SBV^p(\Omega_D; K)$.

3.3.1 Existence of minima

Now we can prove Theorems 3.2.7 and 3.2.8, adapting the arguments of [14, Theorem 3.9 and 3.10]. Theorems 3.2.1 and 3.2.2 are an immediate consequence of these results.

Proof of Theorem 3.2.7. Let us fix $t \in [0, 1]$ and $\Gamma \in \mathcal{R}$. Let u_k be a minimizing sequence of problem (3.32). The infimum in (3.32) is finite, because of (V3); then, a uniform bound holds for $\mathcal{V}(t, u_k)$ for k large enough, too.

Combining this bound with (3.45), we conclude that there exists $C > 0$ such that

$$\sum_{j=1}^{n} \beta_V^j \left\| \mathrm{adj}_j \nabla u_k \right\|_{L^{p_j}(\Omega_D; \mathbb{R}^{\tau_j})}^{p_j} \leq C \tag{3.47}$$

for k large; in particular, $u_k \in SBV^p(\Omega_D; K)$. Then, by the Compactness Theorem 1.2.1 there exists a subsequence, still denoted by u_k, which converges weakly* in $SBV^p(\Omega_D; K)$ to a function u. By Remark 1.3.4, we have $S(u) \,\tilde{\subset}\, \Gamma$; moreover, $u = I$ a.e. on $\Omega_D \backslash \Omega$.

By (3.46) we obtain

$$\mathcal{V}(t, u) \leq \liminf_{k \to \infty} \mathcal{V}(t, u_k) < +\infty \, . \tag{3.48}$$

Finally, we notice that u satisfies the orientation preserving condition (CN1): in $\Omega_D \backslash \Omega$ because $u = I$ a.e. on this set, in Ω because of (3.48) and (V2). Moreover, (3.47) and Theorem 1.2.5 imply that $\det \nabla u_k \rightharpoonup \det \nabla u$ weakly in $L^1(\Omega_D)$, hence Theorem 2.2.5 shows that u satisfies (CN2); then $u \in AD(I, \Gamma)$. The minimality follows from (3.48) and from the fact that u_k is a minimizing sequence. □

Proof of Theorem 3.2.8. Let us fix $t \in [0,1]$ and $\Gamma_0 \in \mathcal{R}$, and let (u_k, Γ_k) be a minimizing sequence of problem (3.31). Again, the infimum in (3.31) is finite by (V3). Moreover, by (3.4) and (3.45), there exists a constant $C \geq 0$ such that

$$\sum_{j=1}^{n} \beta_V^j \left\| \mathrm{adj}_j \nabla u_k \right\|^{p_j}_{L^{p_j}(\Omega_D;\mathbb{R}^{\tau_j})} + \mathcal{H}^{n-1}(\Gamma_k) \leq C$$

for every k, which implies that $u_k \in SBV^p(\Omega_D; K)$ and $\mathcal{H}^{n-1}(\Gamma_k)$ is uniformly bounded. By the Compactness Theorem 1.2.1 there exists a subsequence, still denoted by u_k, which converges weakly* in $SBV^p(\Omega_D; K)$ to a function u which satisfies $u = I$ a.e. on $\Omega_D \backslash \Omega$.

On the other hand, by the Compactness Theorem 1.3.2 and Remark 1.3.5, there exists a subsequence, still denoted by Γ_k, σ^p-converging to a set $\Gamma^* \in \mathcal{R}$. By Definition 1.3.1 we have $S(u) \tilde{\subset} \Gamma^*$. Finally, we take $\Gamma = \Gamma^* \cup \Gamma_0$, in order to get $\Gamma_0 \tilde{\subset} \Gamma$.

By Theorem 1.3.3 we have

$$\mathcal{K}(\Gamma) = \mathcal{K}(\Gamma^* \cup \Gamma_0) \leq \liminf_{k\to\infty} \mathcal{K}(\Gamma_k \cup \Gamma_0) = \liminf_{k\to\infty} \mathcal{K}(\Gamma_k)\,.$$

Arguing as in the proof of Theorem 3.2.7 we conclude that

$$\mathcal{F}(t, u, \Gamma) \leq \liminf_{k\to\infty} \mathcal{F}(t, u_k, \Gamma_k) < +\infty$$

and that u satisfies (CN1) and (CN2). Then we have $u \in AD(I, \Gamma)$, so that the last inequality implies that (u, Γ) is a minimum point of (3.31). □

3.3.2 Existence of quasistatic evolutions

The proof of Theorem 3.2.11 follows a scheme developed in [18, 14, 22, 24]: problem (3.31) is approximated via time discretization, then the existence result is obtained by passing to the limit as the time steps tend to zero.

First, we show that an incremental approximate solution satisfies an a-priori bound. Then, we will prove Theorem 3.2.11 as a consequence of Compactness Theorem 1.2.1 and Helly Theorem 1.3.6.

Henceforth, given a time discretization $\left\{t_k^i\right\}_{0\leq i\leq k}$ of $[0,1]$, we will use the following notation:

$$\tau_k(t) := t_k^i\,, \quad \mathcal{V}_k(t,\cdot) := \mathcal{V}(t_k^i,\cdot)\,, \quad \text{and} \quad \mathcal{F}_k(t,\cdot) := \mathcal{F}(t_k^i,\cdot) \quad \text{for } t \in \left[t_k^i, t_k^{i+1}\right)\,. \tag{3.49}$$

Proposition 3.3.1 (Discrete energy inequality). *Let $t \mapsto (u_k(t), \Gamma_k(t))$ be a sequence of incremental approximate solutions to* (3.31)*, corresponding to a time discretization $\{t_k^i\}_{0\le i\le k}$ of* $[0,1]$*. Let $\theta_k(t)$ be as in* (3.35)*, $\tau_k(t)$ and $\mathcal{F}_k(t)$ as in* (3.49)*. Then $\mathcal{H}^{n-1}(\Gamma_k(t))$, $\|\nabla u_k(t)\|_{L^p(\Omega_D;\mathbb{M}^{n\times n})}$, and $\theta_k(t)$ are bounded uniformly in k and t; in particular, $u_k(t) \in SBV^p(\Omega_D;K)$. Moreover, for every $t\in[0,1]$*

$$\mathcal{F}_k(t, u_k(t), \Gamma_k(t)) \le \mathcal{F}(0, u_0, \Gamma_0) + \int_0^{\tau_k(t)} \theta_k(s)\,\mathrm{d}s\,. \tag{3.50}$$

Proof. We recall the definition of $(u_k(t), \Gamma_k(t))$: for $i = 1,\dots,k$ the pair $(u_k^i, \Gamma_k^i) := (u_k(t_k^i), \Gamma_k(t_k^i))$ is a solution of (3.34); the definition is completed by setting $u_k(t) = u_k^i$ and $\Gamma_k(t) = \Gamma_k^i$ for $t \in [t_k^i, t_k^{i+1})$.

Taking $(u,\Gamma) = (I, \Gamma_k^{i-1})$ in (3.34), we get $\mathcal{V}(t_k^i, u_k^i) \le \mathcal{V}(t_k^i, I)$, thanks to the monotonicity of $\mathcal{K}$. Hence by (V3)

$$\mathcal{V}(t_k^i, u_k^i) < C\,, \tag{3.51}$$

for some constant C independent of k, i, and t, so that $\left\|\nabla u_k^i\right\|_{L^p(\Omega_D;\mathbb{M}^{n\times n})}$ is bounded uniformly in k and i by (3.45); in particular, $u_k \in SBV^p(\Omega_D;K)$.

Now we can compare (u_k^i, Γ_k^i) with $(u_k^{i-1}, \Gamma_k^{i-1})$: as $u_k^{i-1} \in AD(I, \Gamma_k^{i-1})$, by (3.34)

$$\mathcal{F}(t_k^i, u_k^i, \Gamma_k^i) \le \mathcal{F}(t_k^i, u_k^{i-1}, \Gamma_k^{i-1})\,. \tag{3.52}$$

Then, we rewrite the right-hand side in terms of $\mathcal{F}(t_k^{i-1}, u_k^{i-1}, \Gamma_k^{i-1})$. By (V6), (3.28), and (3.51) we get, modifying the value of C,

$$\left|\mathrm{D}_t\mathcal{V}(t, u_k^i)\right| \le C \tag{3.53}$$

so that $\theta_k(t)$ is bounded uniformly in k and t. Therefore, we have

$$\mathcal{V}(t_k^i, u_k^{i-1}) - \mathcal{V}(t_k^{i-1}, u_k^{i-1}) = \int_{t_k^{i-1}}^{t_k^i} \mathrm{D}_t\mathcal{V}(t, u_k^{i-1})\,\mathrm{d}t\,. \tag{3.54}$$

Summing up (3.52) and (3.54) and using (3.30), we obtain for every $t\in[0,1]$ the discrete energy inequality (3.50).

By (3.50) and (3.53), $\mathcal{F}_k(t, u_k(t), \Gamma_k(t))$ is bounded uniformly with respect to k and t. Hence the nonnegativity of V and (3.4) give a bound on $\mathcal{H}^{n-1}(\Gamma_k(t))$, uniform in k and t. ☐

Proof of Theorem 3.2.11. Take any time discretization $\{t_k^i\}_{0\le i\le k}$ of $[0,1]$ and consider the corresponding incremental approximate solutions. By Proposition 3.3.1, thanks to the uniform bound on $\mathcal{H}^{n-1}(\Gamma_k(t))$, we can use the Helly Theorem 1.3.6 to find a subsequence, still denoted Γ_k, and an increasing set function $t\mapsto \Gamma^*(t)\in\mathcal{R}$, such that $\Gamma_k(t)$ σ^p-converges to $\Gamma^*(t)$ for every $t\in[0,1]$; we define $\Gamma(t) := \Gamma^*(t)\cup\Gamma_0$. This determines the sequence $(u_k(t), \Gamma_k)$ and the set function $\Gamma(t)$ of Definition 3.2.10.

Consider the quantity $\theta_k(t)$ defined in (3.35). For every $t\in[0,1]$, we can extract a subsequence k_j, depending on t, such that

$$\limsup_{k\to\infty} \theta_k(t) = \lim_{j\to\infty} \theta_{k_j}(t)\,.$$

By Proposition 3.3.1 and the Compactness Theorem 1.2.1, there exists a further subsequence, still denoted by u_{k_j}, and a function $u(t)$ such that $u_{k_j}(t) \rightharpoonup u(t)$ weakly* in $SBV^p(\Omega_D; K)$. By Definition 1.3.1 we have $S(u(t)) \tilde{\subset} \Gamma^*(t)$. This determines the subsequence $u_{k_j}(t)$ and the function $u(t)$ of Definition 3.2.10; the proof is concluded. □

3.4 Stability of the limit process

In this section we obtain a stability result for the minimizers of problem (3.31) under the σ^p-convergence, stated in Theorem 3.4.3 and proven after the Crack Transfer Lemma 3.4.1. This allows us to prove property (1) in Theorem 3.2.12.

3.4.1 Crack transfer

An important tool in the proof of the stability result is the *Crack Transfer Lemma*. In the original version [22, Theorem 2.1], the jump set of a displacement u is modified ("transferred" into a fixed set) by replacing u with its reflection in some regions. In our framework, reflections are forbidden by non-interpenetration (see (3.5) and (V2)), so we adapt the proof using a suitable stretching as a substitute for the reflection. This result is contained in [17].

Lemma 3.4.1 (Crack Transfer). *Assume that $t_k \to t_\infty$ and $\Gamma_k \in \mathcal{R}$ σ^p-converges to $\Gamma^* \in \mathcal{R}$. Let $\Gamma \in \mathcal{R}$ with $\Gamma^* \tilde{\subset} \Gamma$. Assume that $\mathcal{V}$ satisfies* (V1–6) *and* (V8). *Let $v \in AD(I, \Gamma)$ be such that $\mathcal{V}(t_\infty, v) < +\infty$. Then there exist a sequence $\Gamma'_k \in \mathcal{R}$ with $\Gamma_k \tilde{\subset} \Gamma'_k$, a sequence $v_k \in AD(I, \Gamma'_k)$, and a sequence of closed sets $C_k \subset \Omega$ such that the following properties hold:*

(a) $\mathcal{L}^n(C_k) \to 0$;

(b) $v_k = v$ *a.e. in* $\Omega_D \backslash C_k$;

(c) $\int_{C_k} V(t_k, x, v_k(x), \nabla v_k(x))\,\mathrm{d}x \to 0$;

(d) $\mathcal{H}^{n-1}(\Gamma^* \backslash C_k) \to 0$;

(e) $(\Gamma'_k \backslash \Gamma_k) \backslash C_k \tilde{\subset} \Gamma \backslash C_k$;

(f) $\mathcal{H}^{n-1}((\Gamma'_k \backslash \Gamma_k) \cap C_k) \to 0$.

Proof. We modify the proof of [22, Theorem 2.1], with Ω and Ω' replaced by Ω_B and Ω_D (the fact that $\overline{\Omega}_B$ is not necessarily contained in Ω_D is irrelevant). According to Definition 1.3.1 there exist $u, u_k \in SBV^p(\Omega_D)$ such that $S(u) \tilde{=} \Gamma^*$, $S(u_k) \tilde{\subset} \Gamma_k$ for every k, and $u_k \rightharpoonup u$ weakly* in $SBV^p(\Omega_D)$; by Definition 1.1.1, u and u_k satisfy the hypotheses of [22], except possibly for the weak convergence of $|\nabla u_k|$ in $L^1(\Omega_D)$, replaced here by the equiintegrability, which is sufficient to obtain the results.

Let E_t be the set of the Lebesgue-density-one points for $\{x\colon u(x) > t\}$ and E_t^k the set of the Lebesgue-density-one points for $\{u_k > t\}$. It is possible to find a countable dense set $D \subset \mathbb{R}$ such that for every $t \in D$ the set E_t has finite perimeter and $\mathcal{L}^n(\{u = t\}) = 0$. Then

$$S(u) \cong G := \bigcup_{\substack{t_1, t_2 \in D \\ t_1 < t_2}} (\partial^* E_{t_1} \cap \partial^* E_{t_2}), \tag{3.55}$$

where ∂^* denotes the reduced boundary. For each $x \in G$, we can choose $t_1(x) < t_2(x)$ in D so that $x \in \partial^* E_{t_1(x)} \cap \partial^* E_{t_2(x)}$ and $t_2(x) - t_1(x) \geq \frac{1}{2} |[u](x)|$, where $[u]$ denotes the jump of u. It is possible to show that $\partial^* E_{t_1(x)}$ and $\partial^* E_{t_2(x)}$ have a common outward unit normal $\nu(x)$ at x. We refer to [22] for the details.

For every $x \in G$ and $r > 0$, we fix a closed cube $Q_r(x)$ centred at x, with side length $2r$, and with a face perpendicular to $\nu(x)$. We consider also the half-cubes

$$\begin{aligned} Q_r^+(x) &:= \{y \in Q_r(x) \colon (y - x) \cdot \nu(x) > 0\}, \\ Q_r^-(x) &:= \{y \in Q_r(x) \colon (y - x) \cdot \nu(x) < 0\} \end{aligned}$$

and the $(n-1)$-dimensional cubes

$$\begin{aligned} H_r(x) &:= \{y \in Q_r(x) \colon (y - x) \cdot \nu(x) = 0\}, \\ H_r(x, s) &:= \{y \in Q_r(x) \colon (y - x) \cdot \nu(x) = s\}. \end{aligned}$$

We fix a constant λ, with

$$1 < \lambda < \frac{1}{1 - \gamma}, \tag{3.56}$$

where γ is given by Proposition 2.1.1.

Let N be the set of points where $\partial\Omega_B$ is not differentiable; we set

$$G_j := \Big\{ x \in G \backslash N \colon \lim_{r \to 0} \frac{\mathcal{H}^{n-1}\big([S(u) \backslash \partial^* E_{t_1(x)}] \cap Q_r(x)\big)}{(2r)^{n-1}} = 0, \\ |[u](x)| > \frac{1}{j}, \quad \operatorname{dist}(x, \partial\Omega_D) > \frac{1}{j} \Big\},$$

so that $G_j \subset\subset \Omega_D$. As in [22], it can be proven that $G \cong \bigcup G_j$. Given $\varepsilon \in (0, \frac{\lambda - 1}{\lambda + 1})$, we fix $j = j(\varepsilon)$ such that

$$\mathcal{H}^{n-1}(G \backslash G_j) < \varepsilon. \tag{3.57}$$

Arguing as in [22], we consider a fine cover of $\mathcal{H}^{n-1}$-almost all of G_j, composed of a suitable collection of cubes $Q_r(x)$. Employing the Morse-Besicovitch Theorem [6, 21, 38], we can find $C > 0$, $m = m(\varepsilon) \in \mathbb{N}$, $k(\varepsilon) \in \mathbb{N}$, and, for $i = 1, \ldots, m$, $x_i \in \overline{\Omega}_B$, $r_i > 0$, and $t_i \in [t_1(x_i), t_2(x_i)]$, and, for every $k \geq k(\varepsilon)$, we can find $\delta_i^+, \delta_i^- > 0$, such that, setting $Q_i := Q_{r_i}(x_i)$, $Q_i^+ := Q_{r_i}^+(x_i)$, $Q_i^- := Q_{r_i}^-(x_i)$, $H_i := H_{r_i}(x_i)$, $H_i^+ := H_{r_i}(x_i, \delta_i^+)$, $H_i^- := H_{r_i}(x_i, -\delta_i^-)$, and R_i the open rectangle between H_i^+ and H_i^-, the following hold:

1. $\mathcal{L}^n(\bigcup_{i=1}^m Q_i) < \varepsilon$;

2. if $x_i \in \Omega_B$, then $Q_i \subset \Omega_B$; if $x_i \in \partial\Omega_B$, then $Q_i \subset \Omega$;

3. if $x_i \in \partial\Omega_B$, then $\partial\Omega_B \cap Q_i$ is a Lipschitz graph contained in R_i;
4. if $x_i \in \partial\Omega_B$, then $\mathcal{H}^{n-1}(\partial\Omega_B \cap Q_i) - (2r_i)^{n-1} < \varepsilon r_i^{n-1}$;
5. $\mathcal{H}^{n-1}(S(u) \cap \partial Q_i) = 0$;
6. $r_i^{n-1} \leq C\,\mathcal{H}^{n-1}(S(u) \cap Q_i)$;
7. $\mathcal{H}^{n-1}((S(v)\backslash S(u)) \cap Q_i) < \varepsilon r_i^{n-1}$;
8. $\sum_{i=1}^m \mathcal{H}^{n-1}\big((\partial^* E_{t_i}^k \cap Q_i)\backslash S(u_k)\big) < \varepsilon$ for $k \geq k(\varepsilon)$;
9. $\mathcal{L}^n((E_{t_i}^k \cap Q_i) \,\triangle\, Q_i^-) < \varepsilon(2r_i)^n$ for $k \geq k(\varepsilon)$;
10. $\mathcal{L}^n((E_{t_i} \cap Q_i) \,\triangle\, Q_i^-) < \varepsilon(2r_i)^n$;
11. $\mathcal{H}^{n-1}\big(H_i^\pm \cap E_{t_i}^k\big) < 8\varepsilon(2r_i)^{n-1}$ for $k \geq k(\varepsilon)$;
12. $\mathcal{H}^{n-1}\big(H_i^\pm \cap E_{t_i}\big) < 8\varepsilon(2r_i)^{n-1}$;
13. $\delta_i^\pm \in [\frac{\varepsilon}{2} r_i, \varepsilon r_i]$;
14. $\mathcal{H}^{n-1}(G_j \backslash (\bigcup_{i=1}^m R_i)) < C\varepsilon$.

In (3) by Lipschitz graph we mean that there exists a Lipschitz function $g_i \colon H_i \to \mathbb{R}$ such that $\partial\Omega_B \cap Q_i = \{x + g_i(x)\,\nu(x_i) \colon x \in H_i\}$.

Finally, we set

$$d_i^+ := \frac{\lambda\delta_i^+ + \delta_i^-}{\lambda - 1}\,, \quad d_i^- := \frac{\lambda\delta_i^- + \delta_i^+}{\lambda - 1}$$

and note that $\delta_i^\pm < d_i^\pm < r_i$, where the second inequality follows from (13) and the choice of ε.

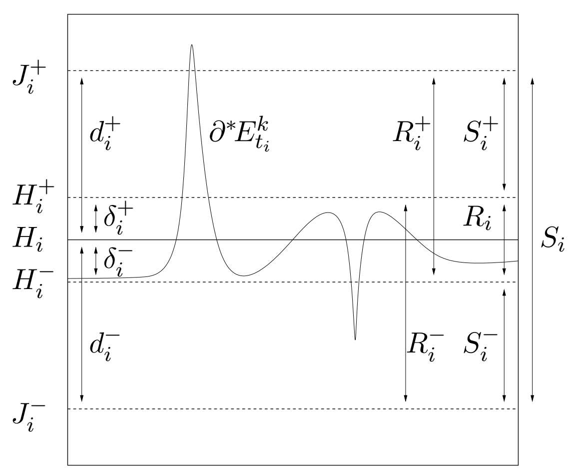

Figure 3.1: The cube Q_i.

We define the $(n-1)$-dimensional cubes $J_i^+ := H_{r_i}(x_i, d_i^+)$, $J_i^- := H_{r_i}(x_i, -d_i^-)$, and the following n-dimensional open rectangles: S_i between J_i^+ and J_i^-, S_i^+ between

J_i^+ and H_i^+, S_i^- between H_i^- and J_i^-, R_i^+ between J_i^+ and H_i^-, and R_i^- between H_i^+ and J_i^-, so that $R_i = R_i^+ \cap R_i^-$ (see Figure 3.1). We fix in Q_i an orthogonal system of coordinates (x', x_n) such that $H_i \subset \{x_n = 0\}$. The stretching $(x', x_n) \mapsto (x', \lambda(x_n - d_i^+) + d_i^+)$ maps S_i^+ into R_i^+; the stretching $(x', x_n) \mapsto (x', \lambda(x_n + d_i^-) - d_i^-)$ maps S_i^- into R_i^-.

Now we transfer the jump set $S(v)$ from $G_j \cap \bigcup_i Q_i$ to $\bigcup_i (\partial^* E^k_{t_i} \cap Q_i)$. For every i we consider the stretched version $v_i^\oplus$ of v, defined in R_i^+ by $v_i^\oplus(x', x_n) := v(x', \frac{1}{\lambda}(x_n - d_i^+) + d_i^+)$; analogously we consider the stretched version $v_i^\ominus$ of v, defined in R_i^- by $v_i^\ominus(x', x_n) := v(x', \frac{1}{\lambda}(x_n + d_i^-) - d_i^-)$. If $x_i \notin \partial\Omega_B$ we consider the functions v_k^ε defined in Q_i by

$$v_k^\varepsilon := \begin{cases} v & \text{in } Q_i \backslash S_i\,, \\ v_i^\oplus & \text{in } S_i^+ \cup (R_i \backslash E^k_{t_i})\,, \\ v_i^\ominus & \text{in } S_i^- \cup (R_i \cap E^k_{t_i})\,. \end{cases}$$

If $x_i \in \partial\Omega_B$, by (3) there are two cases: either $Q_i^+ \backslash R_i \subset \Omega_B$ or $Q_i^- \backslash R_i \subset \Omega_B$. In the former, we define v_k^ε on Q_i by

$$v_k^\varepsilon := \begin{cases} v & \text{in } Q_i \backslash S_i\,, \\ v_i^\oplus & \text{in } S_i^+ \cup (R_i \cap (\Omega_B \backslash E^k_{t_i}))\,, \\ v_i^\ominus & \text{in } S_i^- \cup (R_i \backslash (\Omega_B \backslash E^k_{t_i}))\,; \end{cases}$$

in the latter, we set

$$v_k^\varepsilon := \begin{cases} v & \text{in } Q_i \backslash S_i\,, \\ v_i^\oplus & \text{in } S_i^+ \cup (R_i \backslash (E^k_{t_i} \cap \Omega_B))\,, \\ v_i^\ominus & \text{in } S_i^- \cup (R_i \cap E^k_{t_i} \cap \Omega_B)\,. \end{cases}$$

We complete the definition of v_k^ε in Ω_D by $v_k^\varepsilon := v$ in $\Omega_D \backslash \bigcup_i Q_i$.

Now we fix an arbitrary decreasing sequence $\varepsilon_h \to 0$, with $\varepsilon_h < \frac{\lambda - 1}{\lambda + 1}$, and apply the previous construction with $\varepsilon = \varepsilon_h$. For $k \geq k(\varepsilon_1)$ we define v_k, j_k, and m_k by setting $v_k := v_k^{\varepsilon_h}$, $j_k := j(\varepsilon_h)$, and $m_k := m(\varepsilon_h)$ for $k \in [k(\varepsilon_h), k(\varepsilon_{h+1}))$. Moreover we define $\Gamma'_k := S(v_k) \cup \Gamma_k$ and $C_k := \bigcup_{i=1}^{m_k} \overline{S}_i$.

Let us prove that Γ'_k, v_k, and C_k satisfy the properties (a)–(f) required in the statement. By construction $\Gamma'_k \in \mathcal{R}$ and $\Gamma_k \mathrel{\widetilde{\subset}} \Gamma'_k$; moreover, as stretching preserves the non-interpenetration condition by (3.1), it is easy to see that $v_k \in AD(I, \Gamma'_k)$.

Condition (a) is a consequence of (1) and (b) is guaranteed by the definition of v_k. To prove (c), notice that in C_k we have $\nabla v_k(x) = \nabla v(x)\,\Lambda$, where Λ is the diagonal $n \times n$ matrix with entries $1, \ldots, 1$, and $\frac{1}{\lambda}$. As $|\Lambda - I| \leq \gamma$ by (3.56), in C_k we have

$$V(t_k, x, v_k(x), \nabla v_k(x)) + c_V^0(x) \leq \frac{n}{n-1}\left[V(t_k, x, v_k(x), \nabla v(x)) + c_V^0(x)\right]$$

thanks to (3.8). Moreover, by Remark 3.2.6 there exists a constant $C > 0$ (depending on the diameter of K) such that

$$V(t_k, x, v_k(x), \nabla v(x)) + c_V^0(x) \leq C\,\left[V(t_\infty, x, v(x), \nabla v(x)) + c_V^0(x)\right]\,.$$

As $\mathcal{V}(t_\infty, v) < +\infty$ and $\mathcal{L}^n(C_k) \to 0$, this shows (c).

Part (d) is a consequence of (3.55), (3.57), and (14), with $j = j_k$, while (e) follows from (b) and the definition of Γ'_k. To prove (f), it is enough to show that

$$\mathcal{H}^{n-1}\left((S(v_k)\backslash S(u_k)) \cap \bigcup_{i=1}^{m_k} \overline{S}_i \right) \to 0\,.$$

Arguing like in [22], we consider a partition $\overline{S}_i = P_i^1 \cup P_i^2 \cup P_i^3 \cup P_i^4 \cup P_i^5$, where

$$\begin{aligned}
P_i^1 &:= \overline{S}_i \cap \partial^* E^k_{t_i}\,,\\
P_i^2 &:= (S_i \cup J_i^+ \cup J_i^-)\backslash(H_i^+ \cup H_i^- \cup \partial\Omega_B \cup \partial^* E^k_{t_i})\,,\\
P_i^3 &:= (H_i^+ \cup H_i^-)\backslash \partial^* E^k_{t_i}\,,\\
P_i^4 &:= \partial S_i \backslash (J_i^+ \cup J_i^- \cup \partial^* E^k_{t_i})\,,\\
P_i^5 &:= (\partial\Omega_B \cap S_i)\backslash \partial^* E^k_{t_i}\,.
\end{aligned}$$

By (8) we have

$$\sum_i \mathcal{H}^{n-1}\big(P_i^1 \backslash S(u_k)\big) \to 0\,.$$

By the construction of v_k, we have

$$\mathcal{H}^{n-1}\big(P_i^2 \cap S(v_k)\big) \le \lambda\, \mathcal{H}^{n-1}\big(S(v) \cap (\overline{S}_i \backslash R_i)\big)\ ;$$

by (3.57) and (14)

$$\mathcal{H}^{n-1}\Big((S(v) \cap S(u)) \backslash \bigcup R_i \Big) \to 0\,,$$

while by (6) and (7)

$$\mathcal{H}^{n-1}\Big((S(v) \backslash S(u)) \cap \bigcup \overline{S}_i \Big) \to 0\,,$$

so that

$$\sum_i \mathcal{H}^{n-1}\big(P_i^2 \cap S(v_k)\big) \to 0\,.$$

As for P_i^3, the parts of $S(v_k)$ lying in $H_i^+ \backslash E^k_{t_i}$ and in $H_i^- \cap E^k_{t_i}$ can be controlled like those in P_i^2. Thanks to (11), the remaining parts $H_i^+ \cap E^k_{t_i}$ and $H_i^- \backslash E^k_{t_i}$ have $\mathcal{H}^{n-1}$-measure less than $C\varepsilon r_i^{n-1}$, hence by (6)

$$\sum_i \mathcal{H}^{n-1}\big(P_i^3 \cap S(v_k)\big) \to 0\,.$$

By (13) $d_i^\pm \le \frac{\lambda+1}{\lambda-1}\,\varepsilon r_i$, so that using again (6) we see that

$$\sum_i \mathcal{H}^{n-1}\big(P_i^4\big) \to 0\,.$$

Finally, we need a bound on P_i^5 when $x_i \in \partial\Omega_B$. Assume that $Q_i^- \backslash R_i \subset \Omega_B$ (the other possibility, $Q_i^+ \backslash R_i \subset \Omega_B$, is treated in the same way); then for the parts of

$S(v_k)$ lying in $(S_i \cap \partial\Omega_B)\backslash E^k_{t_i}$ we can argue like in the case of P^2_i. To estimate the jumps in $F := S_i \cap \partial\Omega_B \cap E^k_{t_i}$, we consider its partition $F = F^1_i \cup F^2_i$, with $F^1_i := \pi(\partial^* E^k_{t_i} \cap (Q_i\backslash\overline{\Omega}_B))$ and $F^2_i := F\backslash F^1_i$, where π is the projection of $Q_i\backslash\overline{\Omega}_B$ onto $\partial\Omega_B$, parallel to ν_i. If L denotes the Lipschitz constant of Ω_B (uniform with respect to k and i), we have

$$\mathcal{H}^{n-1}\big(F^1_i\big) \le \sqrt{1+L^2}\; \mathcal{H}^{n-1}\Big(\partial^* E^k_{t_i} \cap (Q_i\backslash\overline{\Omega}_B)\Big)\ ,$$

so that, using (8) and recalling that $S(u_k) \,\tilde{\subset}\, \overline{\Omega}_B$,

$$\sum_{x_i\in\partial\Omega_B} \mathcal{H}^{n-1}\big(F^1_i\big) \to 0\,.$$

As for F^2_i, let $\widetilde{F}^2_i := \pi^{-1}(F^2_i)$: by (3) and (13)

$$\mathcal{H}^{n-1}\big(F^2_i\big) \le \frac{\sqrt{1+L^2}}{r_i(1-\varepsilon_h)}\; \mathcal{L}^n\big(\widetilde{F}^2_i\big)\,.$$

As $Q^-_i\backslash R_i \subset \Omega_B$, by (3) and (13) we have $\mathcal{L}^n((Q_i\cap\Omega_B) \vartriangle Q^-_i) < (2r_i)^{n-1}\varepsilon$; by (9) and (6),

$$\sum_{x_i\in\partial\Omega_B} \frac{1}{r_i}\; \mathcal{L}^n(E^k_{t_i} \cap (Q_i\backslash\overline{\Omega}_B)) \to 0\,.$$

Now one can see that $\widetilde{F}^2_i \subset E^k_{t_i} \cap (Q_i\backslash\overline{\Omega}_B)$, except at most for a set of null Lebesgue-measure (for instance, apply Ambrosio's method of one-dimensional sections [4, Section 3.11]), hence

$$\sum_{x_i\in\partial\Omega_B} \mathcal{H}^{n-1}\big(F^2_i\big) \to 0\,.$$

We have shown

$$\sum_{x_i\in\partial\Omega_B} \mathcal{H}^{n-1}\Big(S_i \cap \partial\Omega_B \cap E^k_{t_i}\Big) \to 0\,,$$

so that

$$\sum_i \mathcal{H}^{n-1}\big(P^5_i \cap S(v_k)\big) \to 0\,.$$

Collecting the last results, we get (f) and complete the proof. □

The Crack Transfer Lemma implies the following consequences.

Corollary 3.4.2. *Let t_∞, t_k, Γ^*, Γ_k, Γ, Γ'_k, $\mathcal{V}$, v, v_k be as in Lemma 3.4.1. Moreover, let $\Gamma_0 \in \mathcal{R}$ such that $\Gamma_0 \,\tilde{\subset}\, \Gamma_k$ for every k; let $\Gamma_\infty := \Gamma^* \cup \Gamma_0 \in \mathcal{R}$. Then*

1. *$v_k \to v$ in measure;*
2. *$\nabla v_k \to \nabla v$ strongly in $L^p(\Omega_D;\mathbb{M}^{n\times n})$;*
3. *$\mathcal{V}(t_k, v_k) \to \mathcal{V}(t_\infty, v)$;*
4. *$\mathcal{H}^{n-1}((\Gamma'_k\backslash\Gamma_k)\backslash(\Gamma\backslash\Gamma_\infty)) \to 0$;*

5. $\limsup_{k\to\infty} \mathcal{K}(\Gamma'_k\backslash\Gamma_k) \le \mathcal{K}(\Gamma\backslash\Gamma_\infty)$.

Proof. Properties (1), (2), and (3) are given by the consequences (a)–(c) of the Lemma, with the aid of (V4). To get (4), use (f) for the part of $\Gamma'_k\backslash\Gamma_k$ contained in C_k; use (d) for the part contained in Γ_∞, recalling that $\Gamma_0 \mathrel{\tilde{\subset}} \Gamma_k$; use (e) for the remaining part. Employing (3.4) we see that $\mathcal{K}((\Gamma'_k\backslash\Gamma_k)\backslash(\Gamma\backslash\Gamma_\infty)) \to 0$, which implies (5). □

3.4.2 Stability of minimizers

Thanks to the Crack Transfer Lemma, we are now able to prove the stability of the minimizers of problem (3.18) with respect to the σ^p-convergence, adapting the arguments of [14, Theorem 5.5].

Theorem 3.4.3 (Stability of minimizers)**.** *Let $\mathcal{F}$ be the functional defined in* (3.30)*, where $\mathcal{V}$ satisfies* (V1–8) *and $\mathcal{K}$ satisfies* (K1–2)*. Let $t_k \to t_\infty \in [0,1]$. Let $\Gamma_k \in \mathcal{R}$ be a sequence such that Γ_k σ^p-converges to a set $\Gamma^* \in \mathcal{R}$; let $\Gamma_0 \in \mathcal{R}$ such that $\Gamma_0 \mathrel{\tilde{\subset}} \Gamma_k$ for every k. Let $u_k \in AD(I,\Gamma_k)$ be a sequence such that*

$$\mathcal{F}(t_k,u_k,\Gamma_k) \le \mathcal{F}(t_k,v,\Gamma) \tag{3.58}$$

for every $\Gamma \in \mathcal{R}$ with $\Gamma_k \mathrel{\tilde{\subset}} \Gamma$ and every $v \in AD(I,\Gamma)$. Assume that u_k converges to a function u_∞ weakly in $SBV^p(\Omega_D;K)$. Then $u_\infty \in AD(I,\Gamma_\infty)$, where $\Gamma_\infty := \Gamma^* \cup \Gamma_0 \in \mathcal{R}$; moreover*

$$\mathcal{F}(t_\infty,u_\infty,\Gamma_\infty) \le \mathcal{F}(t_\infty,v,\Gamma) \tag{3.59}$$

for every $\Gamma \in \mathcal{R}$, with $\Gamma_\infty \mathrel{\tilde{\subset}} \Gamma$, and every $v \in AD(I,\Gamma)$; in addition,

$$\mathcal{V}(t_k,u_k) \to \mathcal{V}(t_\infty,u_\infty)\,. \tag{3.60}$$

Proof. The fact that $u_k \in SBV^p(\Omega_D;K)$ comes from (3.58) with $\Gamma = \Gamma_k$ and $v = I$, with the aid of (V3) and (V4), recalling that $\mathcal{H}^{n-1}(\Gamma_k)$ is bounded by definition of σ^p-convergence. By Definition 1.3.1, we have $S(u) \mathrel{\tilde{\subset}} \Gamma_\infty$; moreover, by the weak* convergence in $SBV^p(\Omega_D;K)$ we get $u = I$ a.e. on $\Omega_D\backslash\Omega$.

To show the minimality property (3.59), let us fix $\Gamma \in \mathcal{R}$ with $\Gamma_\infty \mathrel{\tilde{\subset}} \Gamma$ and $v \in AD(I,\Gamma)$. By the Crack Transfer Lemma 3.4.1, we find a sequence $\Gamma'_k \in \mathcal{R}$ with $\Gamma_k \mathrel{\tilde{\subset}} \Gamma'_k$, a sequence $v_k \in AD(I,\Gamma'_k)$, and a sequence of closed sets $C_k \subset \Omega$ such that (a)–(f) hold. By the minimality condition (3.58) we have

$$\mathcal{V}(t_k,u_k) + \mathcal{K}(\Gamma_k) \le \mathcal{V}(t_k,v_k) + \mathcal{K}(\Gamma'_k)\,,$$

which implies

$$\mathcal{V}(t_k,u_k)) \le \mathcal{V}(t_k,v_k) + \mathcal{K}(\Gamma'_k\backslash\Gamma_k)\,.$$

Let $k \to \infty$: thanks to the weak* convergence in $SBV^p(\Omega_D;K)$ we get (3.46). In the right-hand side, we can pass to the lim sup by Corollary 3.4.2, obtaining

$$\limsup_{k\to\infty} \mathcal{V}(t_k,v_k) + \mathcal{K}(\Gamma'_k\backslash\Gamma_k) \le \mathcal{V}(t_\infty,v) + \mathcal{K}(\Gamma\backslash\Gamma_\infty)\,.$$

Hence we get (3.59), which in turn implies (CN1) for u_∞ (by (V2) and (V3)); arguing as in the proof of Theorem 3.2.7, we conclude that $u_\infty \in AD(I, \Gamma_\infty)$.

Repeating the construction with $v = u_\infty$ and $\Gamma = \Gamma_\infty$, we get (3.60). □

Remark 3.4.4. Let $t \mapsto (u(t), \Gamma(t))$ be an incrementally-approximable quasistatic evolution for $\mathcal{F}$. Definition 3.2.10 provides a sequence $t \mapsto (u_k(t), \Gamma_k(t))$ of incremental approximate solutions and, fixed t, a subsequence $(u_{k_j}(t), \Gamma_{k_j}(t))$ satisfying the hypotheses of Theorem 3.4.3 with $t_{k_j} = \tau_{k_j}(t)$ (see (3.49) and recall that $\Gamma_0 \mathrel{\tilde{\subset}} \Gamma_k(t)$). Hence the stability result guarantees that

$$\mathcal{V}(\tau_{k_j}(t), u_{k_j}(t)) \to \mathcal{V}(t, u(t)) \tag{3.61}$$

and that $(u(t), \Gamma(t))$ satisfies (3.36).

3.5 Energy balance

In this section we show property (2) of Theorem 3.2.12. The first step is passing to the limit in (3.50) to get the so called energy inequality, then the opposite inequality is obtained via a standard method based on stability. This procedure was developed in [18, 14, 22, 24].

3.5.1 The energy inequality

Let $t \mapsto (u(t), \Gamma(t))$ be an incrementally-approximable quasistatic evolution for $\mathcal{F}$ and let $t \mapsto (u_k(t), \Gamma_k(t))$ be an associated sequence of incremental approximate solutions as in Definition 3.2.10. Recall that $\Gamma_k(t)$ σ^p-converges to $\Gamma^*(t)$, $\Gamma_0 \mathrel{\tilde{\subset}} \Gamma_k(t)$, and $\Gamma(t) = \Gamma^*(t) \cup \Gamma_0$. Let $\theta_k(t)$ be as in (3.35), $\tau_k(t)$ and $\mathcal{F}_k(t)$ as in (3.49).

We have already seen in Proposition 3.3.1 that, for every sequence of incremental approximate solutions, $\mathcal{H}^{n-1}(\Gamma_k(t))$, $\|\nabla u_k(t)\|_{L^p(\Omega_D;\mathbb{M}^{n\times n})}$, and $\theta_k(t)$ are bounded uniformly in k and t. By Theorem 1.3.3 we have for every $t \in [0,1]$

$$\mathcal{K}(\Gamma(t)) = \mathcal{K}(\Gamma^*(t) \cup \Gamma_0) \le \liminf_{k\to\infty} \mathcal{K}(\Gamma_k(t) \cup \Gamma_0) = \liminf_{k\to\infty} \mathcal{K}(\Gamma_k(t))\,; \tag{3.62}$$

moreover, Fatou's lemma implies that the function

$$\theta_\infty(t) := \limsup_{k\to\infty} \theta_k(t) \tag{3.63}$$

belongs to $L^1([0,1])$ and

$$\limsup_{k\to\infty} \int_0^{\tau_k(t)} \theta_k(s)\,\mathrm{d}s \le \int_0^t \theta_\infty(s)\,\mathrm{d}s\,. \tag{3.64}$$

Fixed $s \in [0,1]$, by Definition 3.2.10 there is a subsequence $(u_{k_j}(s), \Gamma_{k_j}(s))$ such that

$$u_{k_j}(s) \rightharpoonup u(s) \text{ weakly}^* \text{ in } SBV^p(\Omega_D; K) \tag{3.65}$$

and

$$\theta_\infty(s) = \lim_{k\to\infty} \theta_{k_j}(s)\,. \tag{3.66}$$

By Remark 3.4.4 and (3.26) we have

$$\mathcal{V}(s, u_{k_j}(s)) \to \mathcal{V}(s, u(s))\,, \tag{3.67}$$

so that the function $s \mapsto \mathcal{V}(s, u(s))$ is measurable.

Now we would like to pass to the limit as $k_j \to \infty$ in (3.50): this is possible thanks to the following result. In our setting, hypothesis (3.68) is a consequence of (V7).

Lemma 3.5.1. *Let $\mathcal{V}\colon [0,1] \times SBV^p(\Omega_D; K) \to [0, +\infty]$ be a functional, differentiable in the first variable and lower semicontinuous with respect to the weak* convergence in $SBV^p(\Omega_D; K)$. Assume that for every $M > 0$ there is a modulus of continuity $\omega_M\colon [0,1] \to [0, +\infty)$ (i.e., a nondecreasing function of t, vanishing for $t \to 0$), such that*

$$|\mathrm{D}_t\mathcal{V}(t, u) - \mathrm{D}_t\mathcal{V}(s, u)| \le \omega_M(|t - s|) \tag{3.68}$$

for every $s, t \in [0,1]$ and every $u \in SBV^p(\Omega_D; K)$ such that $\mathcal{V}(0, u) \le M$. Fix $s \in [0,1]$ and let u_j be a sequence converging to u_∞ weakly in $SBV^p(\Omega_D; K)$. Assume that $\mathcal{V}(s, u_j) \to \mathcal{V}(s, u_\infty) < +\infty$. Then $\mathrm{D}_t\mathcal{V}(s, u_j) \to \mathrm{D}_t\mathcal{V}(s, u_\infty)$.*

Proof. See [24, Proposition 3.3]. □

Applying this lemma, from (3.65) and (3.67) we deduce that

$$\mathrm{D}_t\mathcal{V}(s, u_{k_j}(s)) \to \mathrm{D}_t\mathcal{V}(s, u(s))\,.$$

Hence, by (3.35) and (3.66), for every $s \in [0,1]$ we get

$$\theta_\infty(s) = \mathrm{D}_t\mathcal{V}(s, u(s))\,, \tag{3.69}$$

which is thus measurable.

By (3.30), (3.61), and (3.62) we have

$$\mathcal{F}(t, u(t), \Gamma(t)) \le \liminf_{j\to\infty} \mathcal{F}_{k_j}(t, u_{k_j}(t), \Gamma_{k_j}(t)) \le \limsup_{k\to\infty} \mathcal{F}_k(t, u_k(t), \Gamma_k(t))\,. \tag{3.70}$$

From (3.50), (3.63), (3.64), and (3.69) we obtain

$$\limsup_{k\to\infty} \mathcal{F}_k(t, u_k(t), \Gamma_k(t)) \le \mathcal{F}(0, u_0, \Gamma_0) + \int_0^t \mathrm{D}_t\mathcal{V}(s, u(s))\,\mathrm{d}s\,. \tag{3.71}$$

This leads to the energy inequality

$$\mathcal{F}(t, u(t), \Gamma(t)) \le \mathcal{F}(0, u_0, \Gamma_0) + \int_0^t \mathrm{D}_t\mathcal{V}(s, u(s))\,\mathrm{d}s\,. \tag{3.72}$$

3.5.2 The energy equality

The last point in the proof of Theorem 3.2.12 is the opposite of (3.72); we argue again by discretization and employ the stability property.

Proof of Theorem 3.2.12. Let $t \mapsto (u(t), \Gamma(t))$ be an incrementally-approximable quasistatic evolution for $\mathcal{F}$. Global stability property (1) has been proven in Remark 3.4.4.

Since a Lebesgue integral can be approximated by a suitable Riemann sum (see [32] and [14, Lemma 4.12]), there exists a sequence of subdivisions $\{s_k^i\}_{0\leq i\leq i_k}$, satisfying

$$0 = s_k^0 < s_k^1 < \cdots < s_k^{i_k-1} < s_k^{i_k} = t$$

and

$$\lim_{k\to\infty} \max_{1\leq i\leq i_k} (s_k^i - s_k^{i-1}) = 0\,,$$

such that

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \left| (s_k^i - s_k^{i-1}) \mathrm{D}_t \mathcal{V}(s_k^i, u(s_k^i)) - \int_{s_k^{i-1}}^{s_k^i} \mathrm{D}_t \mathcal{V}(s, u(s))\, \mathrm{d}s \right| = 0\,. \tag{3.73}$$

Comparing $(u(t), \Gamma(t))$ with $(I, \Gamma(t))$, by (3.36) and (V3) we find a uniform bound

$$\mathcal{V}(t, u(t)) < M\,. \tag{3.74}$$

For $i = 1, \dots, i_k$, we can compare $(u(s_k^{i-1}), \Gamma(s_k^{i-1}))$ with $(u(s_k^i), \Gamma(s_k^i))$: as $u(s_k^i) \in AD(I, \Gamma(s_k^i))$ and $\Gamma(s_k^{i-1}) \subset \Gamma(s_k^i)$, the stability result (3.36) guarantees that

$$\mathcal{F}(s_k^{i-1}, u(s(_k^{i-1}), \Gamma(s_k^{i-1})) \leq \mathcal{F}(s_k^{i-1}, u(s_k^i), \Gamma(s_k^i))\,.$$

Arguing as in Proposition 3.3.1, by (3.74) and (V6) we see that

$$\mathcal{F}(s_k^{i-1}, u(s_k^i), \Gamma(s_k^i)) = \mathcal{F}(s_k^i, u(s_k^i), \Gamma(s_k^i)) - \int_{s_k^{i-1}}^{s_k^i} \mathrm{D}_t \mathcal{V}(s, u(s_k^i))\, \mathrm{d}s\,.$$

Summing up,

$$\mathcal{F}(t, u(t), \Gamma(t)) \geq \mathcal{F}(0, u_0, \Gamma_0) + \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \mathrm{D}_t \mathcal{V}(s, u(s_k^i))\, \mathrm{d}s\,.$$

Finally,

$$\mathcal{F}(t, u(t), \Gamma(t)) \geq \mathcal{F}(0, u_0, \Gamma_0) + \sum_{i=1}^{i_k} (s_k^i - s_k^{i-1}) \mathrm{D}_t \mathcal{V}(s_k^i, u(s_k^i)) - \omega_k(t)\,,$$

where

$$\omega_k(t) := \sum_{i=1}^{i_k} \left| (s_k^i - s_k^{i-1}) \mathrm{D}_t \mathcal{V}(s_k^i, u(s_k^i)) - \int_{s_k^{i-1}}^{s_k^i} \mathrm{D}_t \mathcal{V}(s, u(s_k^i))\, \mathrm{d}s \right|\,.$$

By (V7) and (3.74) we have $\omega_k(t) \to 0$; hence, by (3.73) we find, recalling (3.72),

$$\mathcal{F}(t, u(t), \Gamma(t)) = \mathcal{F}(0, u_0, \Gamma_0) + \int_0^t \mathrm{D}_t \mathcal{V}(s, u(s))\, \mathrm{d}s\,, \tag{3.75}$$

which leads to the energy balance property (2). □

Remark 3.5.2. Let $(u(t), \Gamma(t))$ and $(u_{k_j}(t), \Gamma_{k_j}(t))$ be as in Definition 3.2.10; let $\mathcal{V}_k(t)$ and $\mathcal{F}_k(t)$ be as in (3.49). By (3.70), (3.71), and (3.75) we obtain

$$\mathcal{F}(t, u(t), \Gamma(t)) = \lim_{j\to\infty} \mathcal{F}_{k_j}(t, u_{k_j}(t), \Gamma_{k_j}(t))\,.$$

As by Remark 3.4.4

$$\mathcal{V}(t, u(t)) = \lim_{j\to\infty} \mathcal{V}_{k_j}(t, u_{k_j}(t))\,,$$

we get

$$\mathcal{K}(\Gamma(t)) = \lim_{j\to\infty} \mathcal{K}(\Gamma_{k_j}(t))\,.$$

3.5.3 Convergence of the discrete-time problems

In the last remark we have seen that the elastic energy and the crack energy of an incrementally-approximable quasistatic evolution are the limits of the corresponding energies for the associated subsequence of incremental approximate solutions. Now we show that the convergence holds for the whole sequence of incremental approximate solutions, adapting [14, Theorem 8.1].

Theorem 3.5.3 (Convergence of energies)**.** *Let $\mathcal{F}$ be the functional defined in* (3.30)*, where $\mathcal{V}$ satisfies* (V1–8) *and $\mathcal{K}$ satisfies* (K1–2)*. Let $(u(t), \Gamma(t))$, (u_0, Γ_0), $\Gamma^*(t)$, and $(u_k(t), \Gamma_k(t))$ be as in Definition 3.2.10; let $\mathcal{V}_k$ and $\mathcal{F}_k$ be as in* (3.49)*. Then for every $t \in [0, T]$*

$$\mathcal{V}(t, u(t)) = \lim_{k\to\infty} \mathcal{V}_k(t, u_k(t))\,, \tag{3.76}$$

$$\mathcal{K}(\Gamma(t)) = \lim_{k\to\infty} \mathcal{K}(\Gamma_k(t))\,. \tag{3.77}$$

Moreover, the functions $\theta_k(t)$ defined in (3.35) *satisfy*

$$\theta_k \to \theta_\infty \quad \textit{in } L^1([0, T])\,, \tag{3.78}$$

where $\theta_\infty(t)$ is given by (3.69)*.*

Proof. Let us fix $t \in [0, T]$ and let $u_{k_l}(t)$ be a subsequence of $u_k(t)$ such that

$$\lim_{l\to\infty} \mathcal{V}_{k_l}(t, u_{k_l}(t)) = \liminf_{k\to\infty} \mathcal{V}_k(t, u_k(t))\,.$$

By Proposition 3.3.1 and the Compactness Theorem 1.2.1, there exists a further subsequence, still denoted by u_{k_l}, and a function $u^*(t)$ such that $u_{k_l} \rightharpoonup u^*(t)$ weakly in $SBV^p(\Omega_D; K)$. Since $\Gamma_{k_l}(t)$ σ^p-converges to $\Gamma^*(t)$ and $\Gamma(t) = \Gamma^*(t) \cup \Gamma_0$, using

(3.34) we can apply Theorem 3.4.3 to $\Gamma_{k_l}(t)$, $u_{k_l}(t)$, and to the sequence $\tau_{k_l}(t)$ defined in (3.49). Therefore $u^*(t) \in AD(I, \Gamma(t))$,

$$\mathcal{V}(t, u^*(t)) = \lim_{l\to\infty} \mathcal{V}_{k_l}(u_{k_l}(t)) \,,$$

and

$$\mathcal{F}(t, u^*(t), \Gamma(t)) \leq \mathcal{F}(t, v, \Gamma)$$

for every $\Gamma \in \mathcal{R}$, with $\Gamma(t) \tilde{\subset} \Gamma$, and for every $v \in AD(I, \Gamma)$. Since $(u(t), \Gamma(t))$ satisfies the same minimality property by (3.36), we have

$$\mathcal{V}(t, u(t)) = \mathcal{V}(t, u^*(t)) \,.$$

Collecting these facts we get

$$\mathcal{V}(t, u(t)) = \liminf_{k\to\infty} \mathcal{V}_k(t, u_k(t)) \,, \tag{3.79}$$

so that by (3.62)

$$\mathcal{F}(t, u(t), \Gamma(t)) \leq \liminf_{k\to\infty} \mathcal{F}_k(t, u_k(t), \Gamma_k(t))$$

and from (3.71) and (3.75) we obtain

$$\mathcal{F}(t, u(t), \Gamma(t)) = \lim_{k\to\infty} \mathcal{F}_k(t, u_k(t), \Gamma_k(t)) \,. \tag{3.80}$$

Hence, (3.76) and (3.77) follow from (3.62), (3.79), and (3.80).

Moreover, by (3.50), (3.64), and (3.75) we get

$$\int_0^t \theta_\infty(s)\, \mathrm{d}s = \lim_{k\to\infty} \int_0^{\tau_k(t)} \theta_k(s)\, \mathrm{d}s$$

for every $t \in [0, T]$; in particular,

$$\int_0^1 \theta_\infty(t)\, \mathrm{d}t = \lim_{k\to\infty} \int_0^1 \theta_k(t)\, \mathrm{d}t \,.$$

By (3.63) $\theta_k \vee \theta_\infty$ converges to θ_∞ pointwise on $[0, T]$, so that $\theta_k \vee \theta_\infty$ converges to θ_∞ in $L^1([0, T])$ thanks to the uniform bound on $\theta_k(t)$ (see Proposition 3.3.1). Since $\theta_k + \theta_\infty = (\theta_k \vee \theta_\infty) + (\theta_k \wedge \theta_\infty)$, we conclude

$$\int_0^1 \theta_\infty(t)\, \mathrm{d}t = \lim_{k\to\infty} \int_0^1 (\theta_k \wedge \theta_\infty)(t)\, \mathrm{d}t \,.$$

As $\theta_k \wedge \theta_\infty \leq \theta_\infty$, this implies that $\theta_k \wedge \theta_\infty$ converges to θ_∞ in $L^1([0, T])$, which, together with the convergence of $\theta_k \vee \theta_\infty$, gives (3.78). □

3.6 Measurable evolutions

So far we have not taken care of the measurability properties of $t \mapsto u(t)$. The following result ensures that, during the limit process described in Section 3.3.2, it is possible to select an incrementally-approximable quasistatic evolution $(u(t), \Gamma(t))$ so that the function $t \mapsto u(t)$ is measurable from $[0,1]$ to $SBV^p(\Omega_D; \mathbb{R}^n)$, endowed with the norm (1.6).

Theorem 3.6.1 (Measurability of quasistatic evolutions)**.** *Let $\mathcal{F}$ be the functional defined in* (3.30)*, where $\mathcal{V}$ satisfies* (V1–8) *and $\mathcal{K}$ satisfies* (K1–2)*. Let (u_0, Γ_0) be a minimum energy configuration at time* 0 *as in* (3.33)*, let $(u_k(t), \Gamma_k(t))$ be a sequence of incremental approximate solutions with initial datum (u_0, Γ_0), such that $\Gamma_k(t)$ σ^p-converges to a set $\Gamma^*(t) \in \mathcal{R}$, and let $\Gamma(t) := \Gamma^*(t) \cup \Gamma_0$. Then there exists a measurable function $t \mapsto u(t)$ from $[0,1]$ to $SBV^p(\Omega_D; \mathbb{R}^n)$ such that $u(t)$ satisfies condition (b) of Definition 3.2.10.*

In view of the previous fact, repeating the proof of Theorem 3.2.11 we obtain an existence result for measurable evolutions.

Corollary 3.6.2 (Existence of measurable quasistatic evolutions)**.** *Let $\mathcal{F}$ be as before. Let (u_0, Γ_0) be a minimum energy configuration at time* 0 *as in* (3.33)*. Then there exists an incrementally-approximable quasistatic evolution $t \mapsto (u(t), \Gamma(t))$ with initial datum (u_0, Γ_0), such that $t \mapsto u(t)$ is measurable as a function from $[0,1]$ to $SBV^p(\Omega_D; \mathbb{R}^n)$.*

The first step in the proof of Theorem 3.6.1 is the measurability in L^p: the following lemma is an adaptation of [16, Theorem 3.5].

Lemma 3.6.3. *In the hypotheses of Theorem 3.6.1, there exists a function $t \mapsto u(t)$, satisfying condition (b) of Definition 3.2.10, such that the function $t \mapsto (\nabla u(t), u(t))$ is measurable from $[0,1]$ to $L^p(\Omega_D; \mathbb{M}^{n\times n}) \times L^p(\Omega_D; \mathbb{R}^n)$.*

Proof. Let $(u_k(t), \Gamma_k(t))$ be a sequence of incremental approximate solutions associated to $(u(t), \Gamma(t))$ as in Definition 3.2.10. Let $\theta_k(t)$ be as in (3.35) and $\theta_\infty(t)$ as in (3.63). For every $t \in [0,1]$, let us consider the sets

$$\begin{aligned}\mathcal{A}(t) := \big\{(\nabla u, u)\colon u \in SBV^p(\Omega_D; K) \text{ and there is a subsequence } k_j \text{ such that}\\ u_{k_j}(t) \rightharpoonup u \text{ weakly}^* \text{ in } SBV^p(\Omega_D; K) \text{ and } \theta_{k_j}(t) \to \theta_\infty(t)\big\}\,.\end{aligned}$$

By Definition 3.2.10, for any selection $t \mapsto (\nabla u(t), u(t))$ the function $t \mapsto (u(t), \Gamma(t))$ is an incrementally-approximable quasistatic evolution.

By the Dominated Convergence Theorem and the Compactness Theorem 1.2.1, $(\nabla u, u) \in \mathcal{A}(t)$ if and only if there is a subsequence k_j such that $\nabla u_{k_j}(t)$ converges to ∇u weakly in $L^p(\Omega_D; \mathbb{M}^{n\times n})$, $u_{k_j}(t)$ converges to u weakly in $L^p(\Omega_D; \mathbb{R}^n)$, and $\theta_{k_j}(t) \to \theta_\infty(t)$. Moreover, as the gradients $\nabla u_k(t)$ are bounded in $L^p(\Omega_D; \mathbb{M}^{n\times n})$ uniformly in k and t and the functions $u_k(t)$ take value in K, there exists a bounded closed convex set $B \subset L^p(\Omega_D; \mathbb{M}^{n\times n}) \times L^p(\Omega_D; \mathbb{R}^n)$ such that $(\nabla u_k(t), u_k(t)) \in B$ for every k and t. This leads to regard B as a compact metrizable space, endowed with the weak topology of $L^p(\Omega_D; \mathbb{M}^{n\times n}) \times L^p(\Omega_D; \mathbb{R}^n)$.

Thanks to [16, Lemma 3.6], the multifunction $t \mapsto \mathcal{A}(t)$ is measurable from $[0,1]$ to B. By the Aumann-von Neumann Selection Theorem [9, Theorem III.6], we can select $t \mapsto (\nabla u(t), u(t))$ in such a way that it is measurable from $[0,1]$ to $L^p(\Omega_D;\mathbb{M}^{n\times n}) \times L^p(\Omega_D;\mathbb{R}^n)$, endowed with the weak topology. The passage to the strong topology is an application of the Pettis Theorem [42, Chapter 5, Section 4]. □

Proof of Theorem 3.6.1. Consider the function $t \mapsto u(t)$ of the previous lemma; we want to show that it is measurable from $[0,1]$ to $SBV^p(\Omega_D;\mathbb{R}^n)$. Let $M_b(\Omega_D;\mathbb{M}^{n\times n})$ be the Banach space of all bounded $\mathbb{M}^{n\times n}$-valued Radon measures on Ω_D, endowed with the norm $\|\mu\|_{M_b(\Omega_D;\mathbb{M}^{n\times n})} := |\mu|(\Omega_D)$. Since $SBV^p(\Omega_D;\mathbb{R}^n)$ is isometric to a closed subspace of $L^1(\Omega_D;\mathbb{R}^n) \times L^p(\Omega_D;\mathbb{M}^{n\times n}) \times M_b(\Omega_D;\mathbb{M}^{n\times n})$ by (1.6), the measurability from $[0,1]$ to $SBV^p(\Omega_D;\mathbb{R}^n)$ is equivalent to requiring that

- $t \mapsto u(t)$ is measurable from $[0,1]$ to $L^1(\Omega_D;\mathbb{R}^n)$,
- $t \mapsto \nabla u(t)$ is measurable from $[0,1]$ to $L^p(\Omega_D;\mathbb{M}^{n\times n})$,
- $t \mapsto \mathrm{D}u(t)$ is measurable from $[0,1]$ to $M_b(\Omega_D;\mathbb{M}^{n\times n})$.

As $t \mapsto (\nabla u(t), u(t))$ is measurable from $[0,1]$ to $L^p(\Omega_D;\mathbb{M}^{n\times n}) \times L^p(\Omega_D;\mathbb{R}^n)$ and $S(u(t)) \mathrel{\tilde{\subset}} \Gamma(t)$, we must only prove the measurability of $t \mapsto [u(t)] \otimes \nu_{u(t)}\, \mathcal{H}^{n-1} \llcorner \Gamma(t)$, the jump part of $\mathrm{D}u(t)$, as a function from $[0,1]$ to $M_b(\Omega_D;\mathbb{M}^{n\times n})$. Notice that, by the monotonicity of $\Gamma(t)$, the unit normal vector $\nu_{u(t)}$ can be regarded as a time-independent term, equal to a prescribed unit normal ν to $\Gamma := \Gamma(1)$. Hence, $[u(t)] \otimes \nu_{u(t)}\, \mathcal{H}^{n-1} \llcorner \Gamma(t) = [u(t)] \otimes \nu\, \mathcal{H}^{n-1} \llcorner \Gamma$.

We are left to show the measurability of $t \mapsto [u(t)]\, \mathcal{H}^{n-1} \llcorner \Gamma$ from $[0,1]$ to $M_b(\Omega_D;\mathbb{R}^n)$. To this aim it is sufficient to prove that the function $t \mapsto [u(t)]$ is measurable from $[0,1]$ to $L^1_{\mathcal{H}^{n-1}}(\Gamma;\mathbb{R}^n)$. For every $r > 0$, we consider the bounded linear operator $\Phi_r : L^1(\Omega_D;\mathbb{R}^n) \to L^1_{\mathcal{H}^{n-1}}(\Gamma;\mathbb{R}^n)$ defined by

$$\Phi_r(u)(x) := \frac{2}{\mathcal{L}^n(B_r(x))} \Big(\int_{B_r^+(x)} u(y)\, \mathrm{d}y - \int_{B_r^-(x)} u(y)\, \mathrm{d}y \Big),$$

where $B_r^{\pm}(x)$ denotes the half-ball with centre x and radius r, oriented as $\pm\nu(x)$. Since $t \mapsto u(t)$ is measurable from $[0,1]$ to $L^1(\Omega_D;\mathbb{R}^n)$, the function $t \mapsto \Phi_r(u(t))$ is measurable from $[0,1]$ to $L^1_{\mathcal{H}^{n-1}}(\Gamma;\mathbb{R}^n)$ for every $r > 0$. As $u(t) \in BV(\Omega_D;\mathbb{R}^n) \cap L^\infty(\Omega_D;\mathbb{R}^n)$ for every t, we have $\Phi_r(u(t)) \to [u(t)]$ strongly in $L^1_{\mathcal{H}^{n-1}}(\Gamma;\mathbb{R}^n)$ as $r \to 0$. We conclude that $t \mapsto [u(t)]$ is measurable from $[0,1]$ to $L^1_{\mathcal{H}^{n-1}}(\Gamma;\mathbb{R}^n)$. □

Remark 3.6.4. We have proven the measurability in the sense of SBV^p as a consequence of the measurability in the sense of L^p. Viceversa, one can see that, for every measurable map $t \mapsto u(t)$ from $[0,1]$ to $SBV^p(\Omega_D;\mathbb{R}^n)$, the function $t \mapsto (\nabla u(t), u(t))$ is also measurable from $[0,1]$ to $L^p(\Omega_D;\mathbb{M}^{n\times n}) \times L^p(\Omega_D;\mathbb{R}^n)$, so that the conclusion of Lemma 3.6.3 follows from Theorem 3.6.1.

3.7 Extension to volume forces

For the sake of simplicity, we have treated the case without applied forces, where the time dependence is given only by the boundary data. Actually, with elementary modifications to the proofs presented here, it is possible to consider smooth volume forces, depending on time.

We assume that the applied forces are conservative, i.e., there exists a function $G\colon [0,1]\times\Omega\times K\to\mathbb{R}$ such that the force density per unit volume in the reference configuration corresponding to a deformation $u\in SBV(\Omega_D;K)$ is given by $\mathrm{D}_y G(t,x,u(x))$, where $\mathrm{D}_y G(t,x,y)$ denotes the partial gradient of G with respect to y. So, the work done by the body forces is given up to an additive constant by

$$\mathcal{G}(t,u) := \int_\Omega G(t,x,u(x))\,\mathrm{d}x\,. \tag{3.81}$$

We suppose that G satisfies the following properties:

- $x\mapsto G(t,x,y)$ is $\mathcal{L}^n$-measurable on Ω for every $(t,y)\in[0,1]\times K$;
- $(t,y)\mapsto G(t,x,y)$ is C^1 on $[0,1]\times K$ for every $x\in\Omega$;
- there exists a constant $a_G>0$ such that

  $$|G(t,x,y)|+|\mathrm{D}_t G(t,x,y)|+|\mathrm{D}_y G(t,x,y)|\le a_G$$

  for every $(t,x,y)\in[0,1]\times\Omega\times K$.

Under these assumptions, for any $u\in SBV(\Omega_D;K)$ the function $t\mapsto\mathcal{G}(t,u)$ is C^1 on $[0,1]$ and its derivative $\mathrm{D}_t\mathcal{G}(t,u)$ is given by

$$\mathrm{D}_t\mathcal{G}(t,u) = \int_\Omega \mathrm{D}_t G\left(t,x,u(x)\right)\,\mathrm{d}x\,. \tag{3.82}$$

Notice that the presence of the confinement hypothesis $u(x)\in K$ allows us to avoid the growth conditions with respect to y, required in [14].

We add the force term in (3.16) and redefine the total energy of the system, which now depends also on t:

$$\mathcal{E}(t,u,\Gamma) := \mathcal{W}(u)-\mathcal{G}(t,u)+\mathcal{K}(\Gamma)\,. \tag{3.83}$$

Following the technique of multiplicative splitting (see Section 3.2.3), we look for a solution $u\in AD(\psi(t),\Gamma)$ to (3.18) in the form $u=\psi(t)\circ z$, with $z\in SBV(\Omega_D;K)$. To treat the case of the volume forces, we substitute (3.24) with

$$V(t,x,y,A) := W(x,\nabla\psi(t,y)\,A)-G(t,x,\psi(t,y))+a_G\,. \tag{3.84}$$

The term a_G, which has no influence on the solution, has been added in order to get $V\ge 0$. As always, given $u\in SBV(\Omega_D;K)$, $\mathcal{V}(t,u)$ represents the integral of $V(t,x,u(x),\nabla u(x))$. We have

$$\mathcal{W}(u)-\mathcal{G}(t,u) = \mathcal{V}(t,\phi(t)\circ u)-b_G\,, \tag{3.85}$$

$$\mathcal{V}(t,z)-b_G = \mathcal{W}(\psi(t)\circ z)-\mathcal{G}(t,\psi(t)\circ z)\,, \tag{3.86}$$

where $b_G := a_G \mathcal{L}^n(\Omega)$. The last expression suggests that the minimal hypotheses on $\mathcal{G}$ depend on the assumptions on the prescribed deformation $\psi(t)$: they will be presented in the next chapter.

It is possible to prove that the new functional $\mathcal{V}$ satisfies the same properties (V1–8) stated in Section 3.2.3. Hence, the results concerning the existence and the main properties of quasistatic evolutions still hold. When coming back to the original formulation with time-dependent prescribed deformations, one should take into account the force term in the definition of the power of the system, which becomes

$$\begin{aligned} \mathcal{P}(t,u) := \quad & \int_\Omega \mathrm{D}_A W(x,\nabla u) : \nabla\big(\dot\psi(t)\circ\phi(t)\circ u\big)\,\mathrm{d}x \,+ \\ & - \int_\Omega \mathrm{D}_y G(t,x,u)\cdot\big(\dot\psi(t)\circ\phi(t)\circ v\big)\,\mathrm{d}x\,. \end{aligned} \tag{3.87}$$

The rule for the change of variables in the derivative of $\mathcal{V}$ is now

$$\mathrm{D}_t\mathcal{V}(t,\phi(t)\circ u) = \mathcal{P}(t,u) - \mathrm{D}_t\mathcal{G}(t,u)\,, \tag{3.88}$$

so that Definition 3.2.15 is modified by setting

$$\eta_k(t) := \mathcal{P}(t,u_k(t)) - \mathrm{D}_t\mathcal{G}(t,u_k(t))\,. \tag{3.89}$$

Finally, Theorems 3.2.16 and 3.2.17 also hold for the system with applied forces, with the energy balance law

$$\dot E(t) = \mathcal{P}(t,u(t)) - \mathrm{D}_t\mathcal{G}(t,u(t))\,, \tag{3.90}$$

where $E(t) := \mathcal{E}(t,u(t),\Gamma(t))$. We leave the details to the reader.

Chapter 4

The case of Lipschitz data

Introduction

Following [24], in Chapter 3 we have supposed that both the prescribed deformation $\psi(t)(x)$ and its spatial gradient $\nabla\psi(t)(x)$ are of class C^1 in (t,x), and the same for the inverse $\psi(t)^{-1}$. These hypotheses, which were made for the sake of simplicity, are not satisfactory for two reasons:

- the spatial smoothness of the boundary data is a strong requirement (whilst the solutions are only SBV);
- the class of boundary data is not invariant under Lipschitz time reparametrizations.

In this chapter, we assume that $\psi, \phi \in W^{1,\infty}([0,1]; W^{1,\infty}(K;K))$, which implies they are Lipschitz in both variables, but not necessarily C^1 (see Section 4.1.1 for the detailed definition of this space). Hence, the data are invariant under Lipschitz reparametrizations of time.

Due to the lack of regularity, even the chain rule is nontrivial when deriving the multiplicative splitting rule

$$u = \psi(t) \circ z\,. \tag{4.1}$$

Indeed, if z is SBV it may happen that the counterimage through z of the set of points of non-differentiability for $\psi(t)$ is a set of positive measure. This does not occur in our case because $\det \nabla z$ is a.e. positive, like $\det \nabla u$ is (see Remark 2.2.4 and Lemma 4.2.1 for the details). Notice that this property follows from the fact that u preserves orientation and does not require global invertibility.

Following [14], we introduce here also volume and surface forces, which were not present in the previous chapter. As we employ the multiplicative splitting (4.1), the minimal hypotheses on the external forces are strictly related with those on the boundary data. The assumptions we make (see Section 4.1.2) are compatible with Lipschitz reparametrizations of time; moreover, they hold in the case of *dead loads* (Example 4.1.7).

We show the existence of incrementally-approximable quasistatic evolutions, a generalization of Theorem 3.2.11. The proof of global minimality and energy balance requires some remarks about the consequences of the growth condition (2.5), stated in Section 2.1, and some results concerning the approximation of Lebesgue integrals with Riemann sums (Lemmas 4.3.7 and 4.3.8).

In Section 4.1 we explain the new hypotheses on the external forces and on the prescribed deformations; in Section 4.2 we present the auxiliary formulation with time-independent boundary data; Section 4.3 is devoted to redefine quasistatic evolutions and to prove their properties.

4.1 Setting of the problem

We refer to Section 3.1 for the definitions concerning the geometry of the body Ω, its admissible cracks and deformations, the crack energy $\mathcal{K}$, and the bulk energy $\mathcal{W}$. At each time $t \in [0,1]$, given $\psi \in W^{1,1}(\Omega_D \backslash \overline{\Omega}; K)$ and $\Gamma \in \mathbb{R}$, we look for deformations $u \in AD(\psi, \Gamma)$ minimizing the total energy

$$\mathcal{E}(t, u, \Gamma) := \mathcal{E}^{\mathrm{el}}(t, u) + \mathcal{K}(\Gamma)\,, \tag{4.2}$$

with

$$\mathcal{E}^{\mathrm{el}}(t, u) := \mathcal{W}(u) - \mathcal{G}(t, u) - \mathcal{S}(t, u)\,, \tag{4.3}$$

where $\mathcal{G}$ is the potential of the volume forces and $\mathcal{S}$ is the potential of the surface forces. Their properties are stated in Section 4.1.2.

First, we present the hypotheses on the prescribed deformations, which are weaker than the ones in Section 3.1.4. In what follows, we will call *modulus of continuity* a nondecreasing function $\omega\colon [0,1] \to [0,+\infty)$, such that $\omega(h) \to 0$ as $h \to 0$.

4.1.1 Prescribed deformations

At every time $t \in [0,1]$ we prescribe the deformation of $\Omega_D \backslash \Omega$, requiring that $u(x) = \psi(t,x)$ for a.e. $x \in \Omega_D \backslash \Omega$. As before, we suppose that $x \mapsto \psi(t,x)$ is defined for every $x \in K$, takes values in K, and has an inverse function on K, denoted by $y \mapsto \phi(t,y)$. This determines two functions

$$\psi,\ \phi\colon [0,1] \times K \to K\,,$$

satisfying, for every $(t,x) \in [0,1] \times K$,

BC1′. $$\psi(t, \phi(t,x)) = x = \phi(t, \psi(t,x))\,.$$

We weaken the hypotheses made in Chapter 3, assuming that

BC2′. $$\psi \in W^{1,\infty}([0,1]; W^{1,\infty}(K;K))$$

and

BC3′. $$\phi \in W^{1,\infty}([0,1]; W^{1,\infty}(K;K))\,.$$

According to [8, Appendix], this requirement means that

$$\psi, \phi \in C^0\left([0,1]; W^{1,\infty}(K;K)\right)$$

and there exist two functions

$$\dot{\psi}, \dot{\phi} \in L^\infty([0,1]; W^{1,\infty}(K;K))$$

such that for every $t \in [0,1]$

$$\psi(t) = \psi(0) + \int_0^t \dot{\psi}(s)\,\mathrm{d}s \quad \text{and} \quad \phi(t) = \phi(0) + \int_0^t \dot{\phi}(s)\,\mathrm{d}s\,, \tag{4.4}$$

where the integrals are defined in the sense of Bochner, with respect to the topology of $W^{1,\infty}(K;K)$ (endowed with the norm $\|u\|_{W^{1,\infty}(K;K)} = \sup_K |u| + \sup_K |\nabla u|$). In particular, one can define a.e. the Jacobian matrices $\nabla\psi$, $\nabla\phi$, $\nabla\dot{\psi}$, and $\nabla\dot{\phi}$.

Remark 4.1.1. In particular, these hypotheses imply that there exists $l > 0$ such that for every $t, t_1, t_2 \in [0,1]$

$$\|\psi(t,\cdot)\|_{W^{1,\infty}(K;K)} \leq l\,, \quad \|\phi(t,\cdot)\|_{W^{1,\infty}(K;K)} \leq l\,, \tag{4.5}$$

$$\|\psi(t_1) - \psi(t_2)\|_{W^{1,\infty}(K;K)} \leq l\,|t_1 - t_2|\;, \quad \|\phi(t_1) - \phi(t_2)\|_{W^{1,\infty}(K;K)} \leq l\,|t_1 - t_2|\;, \tag{4.6}$$

so $t \mapsto \psi(t)$ and $t \mapsto \phi(t)$ are Lipschitz into $W^{1,\infty}(K;K)$. Since the increment quotients are bounded by a constant depending on the maximum of the derivatives and on the measure of the domain, we can choose l so that

$$|\psi(t,y_1) - \psi(t,y_2)| \leq l\,|y_1 - y_2|\;, \tag{4.7}$$

$$|(\psi(t_1) - \psi(t_2))(y_1) - (\psi(t_1) - \psi(t_2))(y_2)| \leq l\,|t_1 - t_2|\,|y_1 - y_2| \tag{4.8}$$

for every $t, t_1, t_2 \in [0,1]$ and every $y_1, y_2 \in K$. Moreover, employing the Lebesgue Differentiation Theorem in (4.4), one gets the uniform convergence of the difference quotients to the derivative: for every $t \in [0,1]$ where $\dot{\psi}(t)$ exists, there is a modulus of continuity $\omega_t \colon [0,1] \to [0,+\infty)$ such that

$$\left\| \frac{\psi(t+h) - \psi(t)}{h} - \dot{\psi}(t) \right\|_{W^{1,\infty}(K;K)} \leq \omega_t(h) \tag{4.9}$$

for every $h \in [0,1]$. It is not restrictive to assume that $\omega_t(h)$ is uniformly bounded in both t and h: indeed, we can define

$$\omega_t(h) := \sup_{h' \leq h} \left\| \frac{\psi(t+h') - \psi(t)}{h'} - \dot{\psi}(t) \right\|_{W^{1,\infty}(K;K)}.$$

Remark 4.1.2. Let us discuss the definition of $W^{1,\infty}$ spaces and their relationship with Lipschitz spaces. Assume only that $\psi \in \mathrm{Lip}([0,1]; W^{1,\infty}(K;K))$, i.e., for every $t, s \in [0,1]$,

$$\|\psi(t) - \psi(s)\|_{W^{1,\infty}(K;K)} \leq L\,|t-s|$$

for a given constant $L > 0$. Regarding ψ as a function in $\mathrm{Lip}([0,1]; W^{1,r}(K;K))$ for r large enough, thanks to reflexivity we can find a derivative

$$\dot{\psi} \in L^\infty([0,1]; W^{1,r}(K;K))\,,$$

which is the $W^{1,r}$-limit of the difference quotients [8, Appendix]. Moreover, as the difference quotients are uniformly bounded in $W^{1,\infty}$, one concludes that

$$\dot{\psi} \in L^\infty([0,1]; W^{1,\infty}(K;K))\,.$$

Nevertheless, this derivative needs not to be the $W^{1,\infty}$-limit of the difference quotients, so that in general ψ is not in $W^{1,\infty}([0,1]; W^{1,\infty}(K;K))$.

For example, let $n = 1$ and

$$\psi(t,x) := \int_0^x |y-t| \,\mathrm{d}y\,.$$

Then, $\psi \in \mathrm{Lip}([0,1]; W^{1,\infty}([0,1]))$, but $\psi \notin W^{1,\infty}([0,1]; W^{1,\infty}([0,1]))$. Indeed, the difference quotients of the spatial derivative of ψ are continuous in x, while their pointwise limit is not continuous, so the convergence cannot be uniform, which contradicts (4.9).

We need also a uniform bound on the energy of the prescribed deformation: we suppose that there exists M such that for every $t \in [0,1]$

BC4$'$. $$\mathcal{W}(\psi(t)) < M\,.$$

Fixed t, (BC4$'$) and (W2) give

$$\det \nabla\psi(t,x) > 0 \text{ for a.e. } x \in K\,, \tag{4.10}$$

so that $\psi(t)$, being injective, satisfies the Ciarlet-Nečas condition; as $S(\psi(t)) = \emptyset$, this implies that $\psi(t) \in AD(\psi(t),\Gamma)$ for every $\Gamma \in \mathcal{R}$.

Remark 4.1.3. In these hypotheses, it is possible to find a negligible set $N \subset K$ containing ∂K, independent of t, such that, for every $t \in [0,1]$ and every $x \notin N$, $\psi(t,\cdot)$ is differentiable at x and $\det \nabla\psi(t,x) > 0$.

Indeed, let D be a countable dense subset of $[0,1]$; by (4.10) there is a set $N \subset K$ of null measure containing ∂K such that, when $t \in D$, $\psi(t,\cdot)$ is differentiable in $\Omega\backslash N$ and $\det\nabla\psi(t,x) > 0$ if $x \notin N$. Given $t_0 \in [0,1]$, let $t_k \in D$ such that $t_k \to t_0$; let $x_0 \notin N$. Since $\psi(t_k)$ is differentiable at x_0 and converges to $\psi(t_0)$ strongly in $W^{1,\infty}(K;K)$, $\psi(t_0)$ is also differentiable at x_0 and $\nabla\psi(t_k,x_0) \to \nabla\psi(t_0,x_0)$: this is guaranteed by Lemma 4.1.4, as stated below.

By convergence, we have $\det\nabla\psi(t_0,x_0) \geq 0$; suppose by contradiction that $\det\nabla\psi(t_0,x_0) = 0$; then, there is a vector ξ such that $\nabla\psi(t_0,x_0)\,\xi = 0$. Take $h \neq 0$ so small that $x_0 + h\,\xi \in K$; let $y_0 := \psi(t_0,x_0)$ and $y_h := \psi(t_0, x_0 + h\,\xi)$. By the hypothesis on ξ, we have, as $h \to 0$,

$$\frac{|y_h - y_0|}{|\phi(t_0,y_h) - \phi(t_0,y_0)|} = \frac{|\psi(t_0,x_0+h\,\xi) - \psi(t_0,x_0)|}{|h|} \to 0\,,$$

which is forbidden by the Lipschitz property of $\phi(t_0)$.

To conclude, we must only prove the following lemma.

Lemma 4.1.4. *Let v_k be a sequence converging to v strongly in $W^{1,\infty}(K;K)$. Let $x_0 \in \operatorname{int} K$ be such that v_k is differentiable at x_0 for every k. Then, v is differentiable at x_0 and $\nabla v_k(x_0) \to \nabla v(x_0)$.*

Proof. Fixed $\varepsilon > 0$, we have for every k and j large enough

$$|(v_k - v_j)(x) - (v_k - v_j)(x_0)| \leq \varepsilon\, |x - x_0| \tag{4.11}$$

for every $x \in K$; in fact, by convergence in $W^{1,\infty}(K;K)$, the function $v_k - v_j$ is Lipschitz with vanishing constant. Passing to the limit as $x \to x_0$, we get $|\nabla v_k(x_0) - \nabla v_j(x_0)| \leq \varepsilon$; then there exists $A_0 \in \mathbb{M}^{n\times n}$ such that, as $k \to \infty$, $\nabla v_k(x_0) \to A_0$. We deduce from (4.11) that for every $\varepsilon > 0$ there is k such that

$$\left| \frac{v_k(x) - v_k(x_0) - \nabla v_k(x_0)(x - x_0)}{|x - x_0|} - \frac{v(x) - v(x_0) - A_0(x - x_0)}{|x - x_0|} \right| \leq \varepsilon$$

for every $x \in K$. By differentiability, for every k, there is $\delta > 0$ such that for $|x - x_0| < \delta$

$$\frac{|v_k(x) - v_k(x_0) - \nabla v_k(x_0)(x - x_0)|}{|x - x_0|} \leq \varepsilon\,.$$

Hence, v is differentiable at x_0 with differential A_0. □

4.1.2 Forces

The body is subjected to a conservative volume force, depending on time, with potential $G\colon [0,1] \times \Omega \times K \to \mathbb{R}$. We suppose that, for every $t \in [0,1]$, $(x,y) \mapsto G(t,x,y)$ is $\mathcal{L}^n(\Omega)$-measurable in x and continuous in y, so that we can define the work of the body force under any deformation $u \in L^\infty(\Omega;K)$

$$\mathcal{G}(t,u) := \int_\Omega G(t,x,u(x))\,\mathrm{d}x\,. \tag{4.12}$$

We assume that there is an exponent $q \geq 1$ such that the following hold:

G1. there is a constant $c_G > 0$ such that for every $t \in [0,1]$, every $u \in L^\infty(\Omega;K)$, and every $v, w \in L^\infty(\Omega;\mathbb{R}^n)$ such that $u{+}v, u{+}w, u{+}v{+}w \in L^\infty(\Omega;K)$

$$|\mathcal{G}(t,u)| \leq c_G\,,$$
$$|\mathcal{G}(t,u{+}v){-}\mathcal{G}(t,u)| \leq c_G\,\|v\|_{L^q(\Omega;\mathbb{R}^n)}\,,$$
$$|\mathcal{G}(t,u{+}v{+}w){-}\mathcal{G}(t,u{+}v){-}\mathcal{G}(t,u{+}w){+}\mathcal{G}(t,u)| \leq c_G\,\|v\|_{L^q(\Omega;\mathbb{R}^n)}\,\|w\|_{L^q(\Omega;\mathbb{R}^n)}\,;$$

G2. there is a function $a_G \in L^1_+([0,1])$ such that for every $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and every $u \in L^\infty(\Omega;K)$

$$|\mathcal{G}(t_2,u){-}\mathcal{G}(t_1,u)| \leq \int_{t_1}^{t_2} a_G(s)\,\mathrm{d}s\,;$$

G3. there is a function $b_G \in L^1_+([0,1])$ such that for every $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and every $u_1, u_2 \in L^\infty(\Omega; K)$

$$|\mathcal{G}(t_2,u_1) - \mathcal{G}(t_1,u_1) - \mathcal{G}(t_2,u_2) + \mathcal{G}(t_1,u_2)| \leq \int_{t_1}^{t_2} b_G(s)\,\mathrm{d}s\,\|u_1 - u_2\|_{L^q(\Omega;\mathbb{R}^n)} .$$

Thanks to (G2), the function $t \mapsto \mathcal{G}(t,u)$ is absolutely continuous on $[0,1]$ for every $u \in L^\infty(\Omega; K)$, so that $\mathrm{D}_t\mathcal{G}(t,u)$ is defined $\mathcal{L}^1$-a.e. ; hence, (G3) is equivalent to requiring that for every $u_1, u_2 \in L^\infty(\Omega; K)$

$$|\mathrm{D}_t\mathcal{G}(t,u_1) - \mathrm{D}_t\mathcal{G}(t,u_2)| \leq b_G(t)\,\|u_1 - u_2\|_{L^q(\Omega;\mathbb{R}^n)} \quad \text{for a.e. } t \in [0,1] ,$$

where $b_G(t)$ denotes the approximate limit of b_G at the Lebesgue points. Analogously, (G1) provides estimates on $\mathrm{D}_u\mathcal{G}$ and $\mathrm{D}^2_u\mathcal{G}$, if they exist.

We introduce also a surface force, acting on a closed set $\partial_S\Omega \subset \partial_N\Omega$ such that

$$\overline{\Omega}_B \cap \partial_S\Omega = \emptyset ; \tag{4.13}$$

this is a technical requirement similar to (3.1), assumed also in [14]. The potential of the boundary force is given by a function $S\colon [0,1] \times \partial_S\Omega \times K \to \mathbb{R}$, $\mathcal{H}^{n-1}$-measurable in the second variable and continuous in the third; the work for a deformation $u \in L^1(\partial_S\Omega; K)$ is

$$\mathcal{S}(t,u) := \int_{\partial_S\Omega} S(t,x,u(x))\,\mathrm{d}\mathcal{H}^{n-1}(x) . \tag{4.14}$$

We impose these conditions on $\mathcal{S}$:

S1. there is a constant $c_S > 0$ such that for every $t \in [0,1]$, every $u \in L^\infty(\partial_S\Omega; K)$, and every $v, w \in L^\infty(\partial_S\Omega; \mathbb{R}^n)$ such that $u+v, u+w, u+v+w \in L^\infty(\partial_S\Omega; K)$

$$|\mathcal{S}(t,u)| \leq c_S ,$$
$$|\mathcal{S}(t,u+v) - \mathcal{S}(t,u)| \leq c_S\,\|v\|_{L^q(\partial_S\Omega;\mathbb{R}^n)} ,$$
$$|\mathcal{S}(t,u+v+w) - \mathcal{S}(t,u+v) - \mathcal{S}(t,u+w) + \mathcal{S}(t,u)| \leq c_S\,\|v\|_{L^q(\partial_S\Omega;\mathbb{R}^n)}\,\|w\|_{L^q(\partial_S\Omega;\mathbb{R}^n)} ;$$

S2. there is a function $a_S \in L^1_+([0,1])$ such that for every $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and every $u \in L^\infty(\partial_S\Omega; K)$

$$|\mathcal{S}(t_2,u) - \mathcal{S}(t_1,u)| \leq \int_{t_1}^{t_2} a_S(s)\,\mathrm{d}s ;$$

S3. there is a function $b_S \in L^1_+([0,1])$ such that for every $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and every $u_1, u_2 \in L^\infty(\partial_S\Omega; K)$

$$|\mathcal{S}(t_2,u_1) - \mathcal{S}(t_1,u_1) - \mathcal{S}(t_2,u_2) + \mathcal{S}(t_1,u_2)| \leq \int_{t_1}^{t_2} b_S(s)\,\mathrm{d}s\,\|u_1 - u_2\|_{L^q(\partial_S\Omega;\mathbb{R}^n)} .$$

Also in this case, the function $t \mapsto \mathcal{S}(t,u)$ is absolutely continuous on $[0,1]$ for every $u \in L^\infty(\partial_S\Omega; K)$, and the time derivative exists $\mathcal{L}^1$-a.e.

Notice that, if $u \in AD(\psi, \Gamma)$ for some $\psi \in W^{1,1}\big(\Omega_D \backslash \overline{\Omega}; K\big)$ and some $\Gamma \in \mathcal{R}$, since by (4.13) u is of class $W^{1,1}$ in the neighbourhood $\Omega_D \backslash \overline{\Omega}_B$ of $\partial_S\Omega$, one can define its trace on $\partial_S\Omega$. Moreover, by the confinement condition, the trace takes values in K, so that $\mathcal{S}(t,u)$ is well defined.

Remark 4.1.5. If (G1–3) and (S1–3) are satisfied for an exponent q, then they hold even substituting q with any $r \geq q$. So, the bigger is the exponent, the weaker are the assumptions.

Remark 4.1.6. In the case of a pure Neumann problem (or in the case of a Dirichlet problem with time-independent boundary conditions), the last estimate of (G1) can be avoided: see Section 4.2.3 for the details.

Example 4.1.7. These properties are compatible with the case of *dead loads*, where the density of the forces per unit volume in the reference configuration does not depend on the deformation. Let $r > 1$; if $g(t,\cdot) \in L^r(\Omega; \mathbb{R}^n)$ and $s(t,\cdot) \in L^r(\partial_S\Omega; \mathbb{R}^n)$ are the densities of the body and surface force at time t, we set $G(t,x,y) := g(t,x) \cdot y$ and $S(t,x,y) := s(t,x) \cdot y$. If we assume that $t \mapsto g(t,\cdot)$ and $t \mapsto s(t,\cdot)$ are absolutely continuous into $L^r(\Omega; \mathbb{R}^n)$ and $L^r(\partial_S\Omega; \mathbb{R}^n)$, respectively, then (G1–3) and (S1–3) are satisfied with $q = r' := \frac{r}{r-1}$.

Remark 4.1.8. We have seen that, by (G2), for every $u \in L^\infty(\Omega; K)$ there is a negligible set N_u such that $\mathrm{D}_t\mathcal{G}(t,u)$ exists for $t \notin N_u$. We would like to redefine this derivative in such a way that the exceptional set does not depend on u.

Fix a countable set D, dense in $L^\infty(\Omega; K)$ with respect to the norm of $L^q(\Omega; \mathbb{R}^n)$. Let $N_D := \big(\bigcup_{u \in D} N_u\big) \cup N_G$, where N_G is a negligible set such that each $t \notin N_G$ is a Lebesgue point for the function b_G of (G3). For $u \in D$, define

$$\mathrm{D}_t^*\mathcal{G}(t,u) := \begin{cases} \mathrm{D}_t\mathcal{G}(t,u) & \text{if } t \notin N_D\,, \\ 0 & \text{if } t \in N_D\,. \end{cases}$$

By (G3), we have for every $u_1, u_2 \in D$ and every t

$$|\mathrm{D}_t^*\mathcal{G}(t,u_1) - \mathrm{D}_t^*\mathcal{G}(t,u_2)| \leq b_G(t)\, \|u_1 - u_2\|_{L^q(\Omega;\mathbb{R}^n)}\ .$$

Then we can extend $\mathrm{D}_t^*\mathcal{G}(t,\cdot)$ to a $L^q(\Omega; \mathbb{R}^n)$-Lipschitz function on $L^\infty(\Omega; K)$.

Let $u \in L^\infty(\Omega; K)$ and $u_k \in D$ such that u_k converges to u in $L^q(\Omega; \mathbb{R}^n)$. If $t \notin N_u \cup N_D$, we have by (G3)

$$|\mathrm{D}_t\mathcal{G}(t,u) - \mathrm{D}_t^*\mathcal{G}(t,u_k)| \leq b_G(t)\, \|u - u_k\|_{L^q(\Omega;\mathbb{R}^n)}\ ,$$

so that, passing to the limit as $k \to \infty$, we get $\mathrm{D}_t^*\mathcal{G}(t,u) = \mathrm{D}_t\mathcal{G}(t,u)$. We have proven that for every $t \in [0,1]$ there exists a $L^q(\Omega; \mathbb{R}^n)$-Lipschitz function $\mathrm{D}_t^*\mathcal{G}(t,\cdot)$ such that for every $u \in L^\infty(\Omega; K)$ we have $\mathrm{D}_t^*\mathcal{G}(t,u) = \mathrm{D}_t\mathcal{G}(t,u)$ for $\mathcal{L}^1$-a.e. $t \in [0,1]$.

Arguing in the same way, we can find a function $\mathrm{D}_t^*\mathcal{S}(t,\cdot)$ with analogous properties. In the following integral formulas, we will identify $\mathrm{D}_t\mathcal{G}(t,\cdot)$ and $\mathrm{D}_t\mathcal{S}(t,\cdot)$ with $\mathrm{D}_t^*\mathcal{G}(t,\cdot)$ and $\mathrm{D}_t^*\mathcal{S}(t,\cdot)$, respectively.

4.1.3 Minimum energy configurations

As in Section 3.2, we consider the minimum problem

$$\min \left\{ \mathcal{E}(t,u,\Gamma)\colon \Gamma \in \mathcal{R},\ \Gamma_0 \,\tilde{\subset}\, \Gamma,\ u \in AD(\psi(t),\Gamma) \right\}, \tag{4.15}$$

where $t \in [0,1]$ and $\Gamma_0 \in \mathcal{R}$ are fixed. The next result is the counterpart of Theorem 3.2.2; the proof is analogous.

Theorem 4.1.9 (Minimization of the total energy). *Let $\mathcal{E}$ be the functional defined in* (4.2) *and* (4.3)*, where $\mathcal{W}$ satisfies* (W0–6)*, $\mathcal{G}$ satisfies* (G1–3)*, $\mathcal{S}$ satisfies* (S1–3)*, and $\mathcal{K}$ satisfies* (K1–3)*. Consider the prescribed deformations defined in* (BC1–4′)*. Then, for every $t \in [0,1]$ and $\Gamma_0 \in \mathcal{R}$, the minimum problem* (4.15) *has a solution.*

4.2 The auxiliary formulation

Following the scheme of Chapter 3, we study the evolution of (4.15) through the change of variables described in Section 3.2.3. Here we show the passage to the auxiliary formulation and state the properties of the new energy terms, which are weaker than the ones considered previously.

4.2.1 The multiplicative splitting method

Given $\psi \in W^{1,\infty}([0,1]; W^{1,\infty}(K;K))$ and $\Gamma \in \mathcal{R}$, we look for a solution $u \in AD(\psi(t),\Gamma)$ to (4.15) in the form $u = \psi(t) \circ z$, with $z \in SBV(\Omega_D;K)$ and $z \in AD(I,\Gamma)$. In order to express ∇u in terms of $\psi(t)$ and z, we have to check the chain rule for these functions: for this, we will exploit the non-interpenetration property of the solutions.

Lemma 4.2.1. *Let $v \in W^{1,\infty}(K;K)$. Assume that $z \in SBV(\Omega_D;K)$ is such that $\mathcal{L}^n(z^{-1}(F)) = 0$ whenever $\mathcal{L}^n(F) = 0$. Then $u := v \circ z \in SBV(\Omega_D;K)$ and $\nabla u(x) = \nabla v(z(x))\, \nabla z(x)$ for a.e. $x \in \Omega_D$.*

Proof. The proof is obtained by modifying the one of [4, Theorem 3.99]. By [4, Theorem 3.101] we get that $u = v \circ z \in SBV(\Omega_D;K)$ and $\mathrm{D}^j u = (v(z^+) - v(z^-)) \otimes \nu_z \mathcal{H}^{n-1} \llcorner S(z)$. It is possible to approximate v by mollification with a sequence v_k; let $u_k := v_k \circ z$. By [4, Theorem 3.96] we have $\nabla u_k = \nabla v_k(z)\, \nabla z$ and $\mathrm{D}^j u_k = (v_k(z^+) - v_k(z^-)) \otimes \nu_z \mathcal{H}^{n-1} \llcorner S(z)$. As u_k converges to u uniformly and $|\mathrm{D}u_k|(\Omega_D)$ is equibounded, we get that $\mathrm{D}u_k$ converges to $\mathrm{D}u$ weakly* in the sense of measures. As $\mathrm{D}^j u_k$ converges to $\mathrm{D}^j u$ strongly, ∇u_k converges to ∇u weakly* in the sense of measures. In order to see the convergence of $\nabla v_k(z)$, let F be the set of the points which are not Lebesgue for ∇v. As $\nabla v_k \to \nabla v$ pointwise on $\Omega_D \backslash F$ and $\mathcal{L}^n(z^{-1}(F)) = 0$, we obtain that $\nabla v_k(z)$ converges to $\nabla v(z)$ a.e. in Ω_D. The conclusion follows from the Dominated Convergence Theorem. □

Thanks to the non-interpenetration property (see Proposition 2.2.4), we get from the previous lemma $\nabla u(x) = \nabla \psi(t, z(x))\, \nabla z(x)$ for a.e. $x \in \Omega_D$.

Recall that, by Remark 4.1.3, there is a negligible set N containing ∂K such that, for every $t \in [0,1]$, $\psi(t,\cdot)$ is differentiable in $K \backslash N$, with $\det \nabla\psi(t,y) > 0$ at every $y \notin N$. This leads to define the auxiliary volume energy density imposing the chain rule where $\nabla\psi(t,y)$ exists:

$$V(t,x,y,A) := \begin{cases} W(x, \nabla\psi(t,y)\,A) & \text{if } y \notin N\,, \\ W(x,A) & \text{if } y \in N\,. \end{cases} \tag{4.16}$$

We consider the integral functional, defined for $z \in AD(I,\Gamma)$,

$$\mathcal{V}(t,z) := \int_\Omega V(t,x,z(x),\nabla z(x))\,\mathrm{d}x\,. \tag{4.17}$$

Notice that, in order to study $\mathcal{V}(t,z)$, we are free to choose any value for $V(t,x,y,A)$ when $y \in N$, because $z^{-1}(N)$ has null measure. For $u = \psi(t) \circ z$ we have

$$\mathcal{W}(u) = \mathcal{V}(t,\phi(t)\circ u)\,, \quad \mathcal{V}(t,z) = \mathcal{W}(\psi(t)\circ z)\,.$$

As for the external forces, we set

$$\mathcal{L}(t,z) := \mathcal{G}(t,\psi(t)\circ z)\,, \tag{4.18}$$

$$\mathcal{T}(t,z) := \mathcal{S}(t,\psi(t)\circ z)\,. \tag{4.19}$$

Finally, we define

$$\mathcal{F}^{\mathrm{el}}(t,z) := \mathcal{V}(t,z) - \mathcal{L}(t,z) - \mathcal{T}(t,z)\,, \tag{4.20}$$

$$\mathcal{F}(t,z,\Gamma) := \mathcal{F}^{\mathrm{el}}(t,z) + \mathcal{K}(\Gamma)\,. \tag{4.21}$$

Hence,

$$\mathcal{E}^{\mathrm{el}}(t,u) = \mathcal{F}^{\mathrm{el}}(t,\phi(t)\circ u)\,, \quad \mathcal{F}^{\mathrm{el}}(t,z) = \mathcal{E}^{\mathrm{el}}(\psi(t)\circ z)\,. \tag{4.22}$$

The properties of the auxiliary bulk energy and of the new force terms are stated in axiomatic form in the following sections.

4.2.2 Properties of the auxiliary volume energy

The previous discussion leads to introduce a class of functions $V\colon [0,1]\times\Omega\times K\times \mathbb{M}^{n\times n} \to [0,+\infty]$ satisfying the following requirements:

V1′. *Measurability:* for every $(t,A)\in[0,1]\times\mathbb{M}^{n\times n}$ the function $(x,y)\mapsto V(t,x,y,A)$ is $\mathcal{L}^n(\Omega)\otimes\mathcal{L}^n(K)$-measurable on $\Omega\times K$, and for every $(x,y)\in\Omega\times K$ the function $(t,A)\mapsto V(t,x,y,A)$ is continuous on $[0,1]\times\mathbb{M}^{n\times n}$.

V2′. *Finiteness:* for every $(t,x,y)\in[0,1]\times\Omega\times K$ we have $V(t,x,y,A) < +\infty$ if and only if $A \in GL_n^+$.

Thanks to Proposition 2.2.4, property (V1′) ensures, for every $z \in AD(I,\Gamma)$, the measurability of $V(t,x,z(x),\nabla z(x))$; hence, $\mathcal{V}(t,z)$ is well defined by (4.17). We require the following properties on this integral functional:

V3′. *Bound at identity:* there is a constant $M > 0$ such that $\mathcal{V}(t, I) \leq M$ for every $t \in [0, 1]$;

V4′. *Semicontinuity and coercivity:* if $t_k \to t$ and z_k converges to z weakly* in $SBV^p(\Omega_D; K)$,

$$\mathcal{V}(t, z) \leq \liminf_{k\to\infty} \mathcal{V}(t_k, z_k)\,;$$

moreover, there exist some constants $\beta_V^0, \ldots, \beta_V^n > 0$ such that, for every $t \in [0, 1]$ and every $z \in AD(I, \Gamma)$,

$$\mathcal{V}(t, z) \geq \sum_{j=1}^{n} \beta_V^j \left\|\mathrm{adj}_j \nabla u\right\|_{L^{p_j}(\Omega_D;\mathbb{R}^{\tau_j})}^{p_j} - \beta_V^0\,,$$

where $p_1 \geq 2$, $p_j \geq p_1' := \frac{p_1}{p_1-1}$ for $j = 2, \ldots, n-1$, $p_n > 1$, and τ_j is the dimension of $\mathrm{adj}_j \nabla u$.

Furthermore, we assume that there exist a constant $\gamma_V \in (0, 1)$, a function $c_V^0 \in L_+^1(\Omega)$ and a constant $c_V^1 > 0$, such that:

V5′. *Multiplicative stress estimate:* for every $(t, x, y, A) \in [0, 1] \times \Omega \times K \times GL_n^+$ and every $B \in GL_n^+$ with $|B - I| < \gamma_V$,

$$V(t, x, y, AB) + c_V^0(x) \leq c_V^1 (V(t, x, y, A) + c_V^0(x))\,;$$

V6′. *Estimate on time increments:* for every $(t_1, x, y, A) \in [0, 1] \times \Omega \times K \times GL_n^+$ and every $t_2 \in [0, 1]$ such that $|t_1 - t_2| < \gamma_V$,

$$|V(t_1, x, y, A) - V(t_2, x, y, A)| \leq c_V^1 (V(t_1, x, y, A) + c_V^0(x))\, |t_1 - t_2|\;;$$

V7′. *Estimate on the convergence of time increments:* for every $t \in [0, 1]$ there is a modulus of continuity $\omega_t \colon [0, 1] \to [0, +\infty)$ with $t \mapsto \omega_t(h)$ in $L^\infty([0, 1])$ for every $h \in [0, 1]$, such that, for every $(x, y, A) \in \Omega \times K \times GL_n^+$ where $\mathrm{D}_t V(t, x, y, A)$ is defined and every $h > 0$ with $t \pm h \in [0, 1]$,

$$\left|\mathrm{D}_t V(t, x, y, A) \mp \frac{V(t \pm h, x, y, A) - V(t, x, y, A)}{h}\right| \leq \omega_t(h)\, (V(t, x, y, A) + c_V^0(x))\,;$$

V8′. *Estimate on spatial increments:* for every $(t, x, y, A) \in [0, 1] \times \Omega \times K \times GL_n^+$ and every $y' \in K$,

$$V(t, x, y', A) + c_V^0(x) \leq c_V^1 (V(t, x, y, A) + c_V^0(x))\,.$$

Remark 4.2.2. Let $z \in SBV(\Omega_D; K)$. If for some $t_0 \in [0, 1]$ we have $\mathcal{V}(t_0, z) < +\infty$, then $\mathcal{V}(t, z) < +\infty$ for every $t \in [0, 1]$: indeed, by (V6′)

$$V(t, x, z(x), \nabla z(x)) + c_V^0(x) \leq (c_V^1 + 1)(V(t_0, x, z(x), \nabla z(x)) + c_V^0(x))\,. \tag{4.23}$$

Using again (V6′), one sees that $t \mapsto \mathcal{V}(t,z)$ is Lipschitz, with constant depending on $\mathcal{V}(t_0,z)$. Hence, $t \mapsto \mathcal{V}(t,z)$ has a derivative $\mathrm{D}_t\mathcal{V}(\cdot,z) \in L^\infty([0,1])$, defined in $[0,1]$ except for a negligible set depending on z, such that for every $t_1,t_2 \in [0,1]$

$$\mathcal{V}(t_2,z) - \mathcal{V}(t_1,z) = \int_{t_1}^{t_2} \mathrm{D}_t\mathcal{V}(t,z)\,\mathrm{d}t\,. \tag{4.24}$$

We are going to establish a representation formula for $\mathrm{D}_t\mathcal{V}(\cdot,z)$. Properties (V6′) and (4.23) imply that, for a.e. $x \in \Omega$, the function $t \mapsto V(t,x,z(x),\nabla z(x))$ is Lipschitz, with constant depending on z and x; then it is derivable out of a negligible set depending on z and x (so that, fixed z and x, (V7′) holds for $\mathcal{L}^1$-a.e. t). Let us define

$$\mathrm{D}_t^*V(t,x) = \begin{cases} \mathrm{D}_tV(t,x,z(x),\nabla z(x)) & \text{if the derivative exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Using (V6′), one can prove that this function is integrable on $[0,1]\times\Omega$; after exchanging the order of integration, we get

$$\mathcal{V}(t_2,z) - \mathcal{V}(t_1,z) = \int_{t_1}^{t_2}\int_\Omega \mathrm{D}_t^*V(t,x,z(x),\nabla z(x))\,\mathrm{d}x\,\mathrm{d}t\,.$$

Comparing the latter expression with (4.24), we obtain for $\mathcal{L}^1$-a.e. $t \in [0,1]$

$$\mathrm{D}_t\mathcal{V}(t,z) = \int_\Omega \mathrm{D}_t^*V(t,x,z(x),\nabla z(x))\,\mathrm{d}x\,.$$

We will identify D_t^*V and D_tV, so that the following expression makes sense:

$$\mathrm{D}_t\mathcal{V}(t,z) = \int_\Omega \mathrm{D}_tV(t,x,z(x),\nabla z(x))\,\mathrm{d}x\,. \tag{4.25}$$

Finally, we prove that the volume energy $\mathcal{V}$, obtained from $\mathcal{W}$ and ψ through the change of variable described in (4.16), satisfies the properties (V1–8′) stated above. We will employ (3.7), (3.8), (2.15), (2.16), and (2.17), which are consequences of (W5).

Proposition 4.2.3. *If* (W0–6) *and* (BC1–4′) *hold, then the functional* $\mathcal{V}$ *defined in* (4.16) *satisfies properties* (V1–8′).

Proof. Properties (V1–3′) are given by (W1–3). After a change of variables, one sees that (V4′) is a consequence of (W1) and (W4), thanks to the lower semicontinuity of $\mathcal{W}$ (see Theorem 1.2.3).

In what follows, we will take $c_V^0 := c_W^0$, $c_V^1 \geq \frac{n}{n-1}$, and $\gamma_V \leq \gamma$, where γ is the constant introduced in Proposition 2.1.1. Then (V5′) is implied by (3.8), because

$$W(x,\nabla\psi(t,y)AB) + c_W^0(x) \leq \frac{n}{n-1}\left(W(x,\nabla\psi(t,y)A) + c_W^0(x)\right)\,.$$

In order to see (V6′), take $\gamma_V \leq l^{-2}\gamma$, where l is the constant appearing in Remark 4.1.1. By (4.5) and (4.6), for a.e. $y \in K$ we have

$$|\nabla\psi(t_2,y)\nabla\phi(t_1,\psi(t_1,y)) - I| < \gamma$$

if $|t_1 - t_2| < \gamma_V$. Hence, we can apply (2.15) to get for every $A \in GL_n^+$

$$|W(x, \nabla\psi(t_2, y)A) - W(x, \nabla\psi(t_1, y)A)| \leq$$
$$\leq \frac{n^2}{n-1}\, l^2 c_W^1 (W(x, \nabla\psi(t_1, y)A) + c_W^0(x))\, |t_1 - t_2| \;;$$

then (V6′) follows for c_V^1 large enough.

By (4.16), property (V7′) is trivially satisfied when $y \in N$, where N is the negligible subset of K defined in Remark 4.1.3. Fixed $(x, y, A) \in \Omega \times K \times GL_n^+$ with $y \notin N$, $t \mapsto V(t, x, y, A)$ is $\mathcal{L}^1$-a.e. derivable, with

$$\mathrm{D}_t V(t, x, y, A) = \mathrm{D}_A W(x, \nabla\psi(t, y)A)\, A^{\mathrm{T}} : \nabla\dot\psi(t, y)\,.$$

Given $t \in [0, 1]$ where $\mathrm{D}_t V(t, x, y, A)$ exists and $h > 0$ small enough, using the Mean Value Theorem we can find a convex combination B_h of $\nabla\psi(t + h, y)$ and $\nabla\psi(t, y)$ such that

$$\left|\mathrm{D}_A W(\nabla\psi(t)A)\, A^{\mathrm{T}} : \nabla\dot\psi(t) - \frac{W(\nabla\psi(t+h)A) - W(\nabla\psi(t)A)}{h}\right| =$$
$$= \left|\mathrm{D}_A W(\nabla\psi(t)A)\, A^{\mathrm{T}} : \nabla\dot\psi(t) - \mathrm{D}_A W(B_h A)\, A^{\mathrm{T}} : \frac{\nabla\psi(t+h) - \nabla\psi(t)}{h}\right| \leq$$
$$\leq \left|\mathrm{D}_A W(\nabla\psi(t)A)\, A^{\mathrm{T}}\right| \left|\nabla\dot\psi(t) - \frac{\nabla\psi(t+h) - \nabla\psi(t)}{h}\right| +$$
$$+ \left|\mathrm{D}_A W(\nabla\psi(t)A)\, A^{\mathrm{T}} - \mathrm{D}_A W(B_h A)\, A^{\mathrm{T}}\right| \left|\frac{\nabla\psi(t+h) - \nabla\psi(t)}{h}\right|\,.$$

Here and henceforth, we omit the arguments x and y when they are obvious; it is understood that B_h is invertible for h small. Consider the first summand of the last expression; using (2.17) and (4.5) in the first factor and (4.9) in the second, we get

$$\left|\mathrm{D}_A W(\nabla\psi(t)A)\, A^{\mathrm{T}}\right| \left|\nabla\dot\psi(t) - \frac{\nabla\psi(t+h) - \nabla\psi(t)}{h}\right| \leq l\, c_W^1\, \omega_t(h)(W(\nabla\psi(t)A) + c_W^0)\,,$$

where ω_t is the modulus of continuity defined in Remark 4.1.1. As for the second summand, we can use (4.6) to control the last factor; the remaining part is

$$\left|\mathrm{D}_A W(B_h A)\, A^{\mathrm{T}} - \mathrm{D}_A W(\nabla\psi(t, y)A)\, A^{\mathrm{T}}\right| \leq$$
$$\leq \left|\mathrm{D}_A W(B_h' A')\, (B_h' A')^{\mathrm{T}} - \mathrm{D}_A W(A')\, A'^{\mathrm{T}}\right| \left|B_h^{-1}\right| +$$
$$+ \left|\mathrm{D}_A W(A')\, A'^{\mathrm{T}}\right| \left|B_h^{-1} - \nabla\phi(t, \psi(t, y))\right|\,,$$

where $B_h' := B_h \nabla\phi(t, \psi(t, y))$ and $A' := \nabla\psi(t, y)A$. The first term is estimated by (W6), since $\left|B_h^{-1}\right|$ is bounded by (4.5); as for the second one, we use (3.7), recalling that, if h is small enough, B_h is uniformly near to $\nabla\psi(t, y)$, being a convex combination of $\nabla\psi(t, y)$ and $\nabla\psi(t + h, y)$. Hence, there is a modulus of continuity $\omega\colon [0, 1] \to [0, +\infty)$ such that

$$\left|\mathrm{D}_A W(B_h A)\, A^{\mathrm{T}} - \mathrm{D}_A W(\nabla\psi(t, y)A)\, A^{\mathrm{T}}\right| \leq \omega(h)\, (W(\nabla\psi(t)A) + c_W^0)\,;$$

notice that, by (4.5) and (2.16), ω is bounded. This concludes the proof of (V7′) in the case of $t + h$; the case of $t - h$ is analogous.

Finally, (V8′) follows from (2.16), because $\nabla\psi(t, \cdot)$ and $\nabla\phi(t, \cdot)$ are uniformly bounded in $W^{1,\infty}(K; K)$ by (4.5). □

4.2.3 Properties of the force terms

The volume forces in the new formulation are given by a functional $\mathcal{L}(t, z)$, defined in $[0,1] \times AD(I, \Gamma)$, where $\Gamma \in \mathcal{R}$. We assume that there is an exponent $q \geq 1$ such that the following hold:

L1. there is a constant $c_L > 0$ such that for every $t \in [0,1]$ and every $z, z_1, z_2 \in L^\infty(\Omega; K)$

$$|\mathcal{L}(t, z)| \leq c_L \,,$$
$$|\mathcal{L}(t, z_1) - \mathcal{L}(t, z_2)| \leq c_L \left\| z_1 - z_2 \right\|_{L^q(\Omega;\mathbb{R}^n)} \;;$$

L2. there is a function $a_L \in L^1_+([0,1])$ such that for every $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and every $z \in L^\infty(\Omega; K)$

$$|\mathcal{L}(t_2, z) - \mathcal{L}(t_1, z)| \leq \int_{t_1}^{t_2} a_L(s) \, \mathrm{d}s \,;$$

L3. there is a function $b_L \in L^1_+([0,1])$ such that for every $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and every $z_1, z_2 \in L^\infty(\Omega; K)$

$$|\mathcal{L}(t_2, z_1) - \mathcal{L}(t_1, z_1) - \mathcal{L}(t_2, z_2) + \mathcal{L}(t_1, z_2)| \leq \int_{t_1}^{t_2} b_L(s) \, \mathrm{d}s \left\| z_1 - z_2 \right\|_{L^q(\Omega;\mathbb{R}^n)} \,.$$

As for the surface forces, they are given by a functional $\mathcal{T}(t, z)$, defined in $[0,1] \times AD(I, \Gamma)$. We suppose:

T1. there is a constant $c_T > 0$ such that for every $t \in [0,1]$ and every $z, z_1, z_2 \in L^\infty(\partial_S\Omega; K)$

$$|\mathcal{T}(t, z)| \leq c_T \,,$$
$$|\mathcal{T}(t, z_1) - \mathcal{T}(t, z_2)| \leq c_T \left\| z_1 - z_2 \right\|_{L^q(\partial_S\Omega;\mathbb{R}^n)} \;;$$

T2. there is a function $a_T \in L^1_+([0,1])$ such that for every $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and every $z \in L^\infty(\partial_S\Omega; K)$

$$|\mathcal{T}(t_2, z) - \mathcal{T}(t_1, z)| \leq \int_{t_1}^{t_2} a_T(s) \, \mathrm{d}s \,;$$

T3. there is a function $b_T \in L^1_+([0,1])$ such that for every $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and every $z_1, z_2 \in L^\infty(\partial_S\Omega; K)$

$$|\mathcal{T}(t_2, z_1) - \mathcal{T}(t_1, z_1) - \mathcal{T}(t_2, z_2) + \mathcal{T}(t_1, z_2)| \leq \int_{t_1}^{t_2} b_T(s) \, \mathrm{d}s \left\| z_1 - z_2 \right\|_{L^q(\partial_S\Omega;\mathbb{R}^n)} \,.$$

Thanks to (L2) and (T2), given any $z \in AD(I,\Gamma)$ the functions $t \mapsto \mathcal{L}(t,z)$ and $t \mapsto \mathcal{T}(t,z)$ are absolutely continuous on $[0,1]$, so that $\mathrm{D}_t\mathcal{L}(t,z)$ and $\mathrm{D}_t\mathcal{T}(t,z)$ exist $\mathcal{L}^1$-a.e. Arguing as in Remark 4.1.8, we may define for every $t \in [0,1]$ some $L^q(\Omega;\mathbb{R}^n)$-Lipschitz functions $\mathrm{D}_t^*\mathcal{L}(t,\cdot)$ and $\mathrm{D}_t^*\mathcal{T}(t,\cdot)$, such that for every $z \in AD(I,\Gamma)$ we have $\mathrm{D}_t^*\mathcal{L}(t,u) = \mathrm{D}_t\mathcal{L}(t,u)$ and $\mathrm{D}_t^*\mathcal{T}(t,u) = \mathrm{D}_t\mathcal{T}(t,u)$ for $\mathcal{L}^1$-a.e. $t \in [0,1]$. We identify $\mathrm{D}_t\mathcal{L}(t,\cdot)$ with $\mathrm{D}_t^*\mathcal{L}(t,\cdot)$ and $\mathrm{D}_t\mathcal{T}(t,\cdot)$ with $\mathrm{D}_t^*\mathcal{T}(t,\cdot)$; we set also

$$\mathrm{D}_t\mathcal{F}^{\mathrm{el}}(t,z) := \mathrm{D}_t\mathcal{V}(t,z) - \mathrm{D}_t\mathcal{L}(t,z) - \mathrm{D}_t\mathcal{T}(t,z). \tag{4.26}$$

We will use in particular these consequences of (L1–3) and (T1–3): given $z, z_k \in AD(I,\Gamma)$ such that $z_k \to z$ in measure,

$$\text{if } t_k \to t\,, \quad \mathcal{L}(t_k,z_k) \to \mathcal{L}(t,z) \text{ and } \mathcal{T}(t_k,z_k) \to \mathcal{T}(t,z)\,; \tag{4.27}$$

$$\text{for a.e. } t\,, \quad \mathrm{D}_t\mathcal{L}(t,z_k) \to \mathrm{D}_t\mathcal{L}(t,z) \text{ and } \mathrm{D}_t\mathcal{T}(t,z_k) \to \mathrm{D}_t\mathcal{T}(t,z)\,. \tag{4.28}$$

Finally, we prove that (L1–3) and (T1–3) are satisfied when $\mathcal{L}$ and $\mathcal{T}$ are given by (4.18) and (4.19).

Proposition 4.2.4. *If* (G1–3), (S1–3)*, and* (BC1–4′) *hold, then the functionals* $\mathcal{L}$ *and* $\mathcal{T}$ *defined in* (4.18) *and* (4.19) *satisfy properties* (L1–3) *and* (T1–3)*.*

Proof. We show (L1–3); the proof of (T1–3) is analogous.

Property (L1) comes immediately from (G1), taking $c_L := c_G(1 \vee l)$, where l is the constant of Remark 4.1.1.

Henceforth, we write $\psi_1 := \psi(t_1)$ and $\psi_2 := \psi(t_2)$. As for (L2), by (G1), (G2), and (4.6) we have

$$\begin{aligned} |\mathcal{L}(t_2,z) - \mathcal{L}(t_1,z)| &\leq |\mathcal{G}(t_2,\psi_1{\circ}z) - \mathcal{G}(t_1,\psi_1{\circ}z)| + |\mathcal{G}(t_1,\psi_1{\circ}z) - \mathcal{G}(t_1,\psi_2{\circ}z)| \leq \\ &\leq \int_{t_1}^{t_2} a_G(s)\,\mathrm{d}s + l\,c_G\,\mathcal{L}^n(\Omega)^{\frac{1}{q}}\,(t_2-t_1)\,, \end{aligned}$$

so we define $a_L(s) := a_G(s) + l\,c_G\,\mathcal{L}^n(\Omega)^{\frac{1}{q}}$.

To prove (L3), adding and subtracting we obtain

$$\begin{aligned} &|\mathcal{L}(t_2,z_1) - \mathcal{L}(t_1,z_1) - \mathcal{L}(t_2,z_2) + \mathcal{L}(t_1,z_2)| \leq \\ \leq\, &|\mathcal{G}(t_2,\psi_1{\circ}z_1) - \mathcal{G}(t_1,\psi_1{\circ}z_1) - \mathcal{G}(t_2,\psi_1{\circ}z_2) + \mathcal{G}(t_1,\psi_1{\circ}z_2)| + \\ &|\mathcal{G}(t_2,\psi_2{\circ}z_1) - \mathcal{G}(t_2,\psi_1{\circ}z_1) - \mathcal{G}(t_2,\psi_2{\circ}z_2) + \mathcal{G}(t_2,\psi_1{\circ}z_2)|\,. \end{aligned}$$

The first summand is controlled by $l\int_{t_1}^{t_2} b_G(s)\,\mathrm{d}s\,\|z_1-z_2\|_{L^q(\Omega;\mathbb{R}^n)}$ thanks to (G3) and (4.7). As for the second summand, we get from (G1), (4.7), and (4.6)

$$\begin{aligned} &|\mathcal{G}(t_2,\psi_1{\circ}z_1 + \psi_2{\circ}z_2 - \psi_1{\circ}z_2) - \mathcal{G}(t_2,\psi_1{\circ}z_1) - \mathcal{G}(t_2,\psi_2{\circ}z_2) + \mathcal{G}(t_2,\psi_1{\circ}z_2)| \leq \\ \leq\, &l^2\,c_G(t_2-t_1)\,\|z_1-z_2\|_{L^q(\Omega;\mathbb{R}^n)}\,. \end{aligned}$$

What remains is estimated with (G1) and (4.8):

$$|\mathcal{G}(t_2,\psi_1{\circ}z_1 + \psi_2{\circ}z_2 - \psi_1{\circ}z_2) - \mathcal{G}(t_2,\psi_2{\circ}z_1)| \leq l\,c_G(t_2-t_1)\,\|z_1-z_2\|_{L^q(\Omega;\mathbb{R}^n)}\,.$$

Then, we conclude taking $b_L(s) := l\,b_G(s) + l\,c_G + l^2\,c_G$. □

Remark 4.2.5. The time derivatives of the energies considered above have $\mathcal{L}^1$-a.e. the following form:

$$\begin{aligned}
\mathrm{D}_t\mathcal{V}(t,z) &= \int_\Omega \mathrm{D}_A W(x,\nabla(\psi(t)\circ z)) : \nabla\big(\dot\psi(t)\circ z\big)\,\mathrm{d}x\,,\\
\mathrm{D}_t\mathcal{L}(t,z) &= \int_\Omega \mathrm{D}_y G(t,x,\psi(t)\circ z)\cdot\big(\dot\psi(t)\circ z\big)\,\mathrm{d}x + \mathrm{D}_t\mathcal{G}(t,\psi(t)\circ z)\,,\\
\mathrm{D}_t\mathcal{T}(t,z) &= \int_{\partial_S\Omega} \mathrm{D}_y S(t,x,\psi(t)\circ z)\cdot\big(\dot\psi(t)\circ z\big)\,\mathrm{d}\mathcal{H}^{n-1}(x) + \mathrm{D}_t\mathcal{S}(t,\psi(t)\circ z)\,.
\end{aligned}$$

For $u=\psi(t)\circ z$, we define the power of the external forces

$$\begin{aligned}
\mathcal{P}(t,u) := &\int_\Omega \mathrm{D}_A W(x,\nabla u) : \nabla\big(\dot\psi(t)\circ\phi(t)\circ u\big)\,\mathrm{d}x\,+\\
&-\int_\Omega \mathrm{D}_y G(t,x,u)\cdot\big(\dot\psi(t)\circ\phi(t)\circ u\big)\,\mathrm{d}x\,+\\
&-\int_{\partial_S\Omega} \mathrm{D}_y S(t,x,u)\cdot\big(\dot\psi(t)\circ\phi(t)\circ u\big)\,\mathrm{d}\mathcal{H}^{n-1}(x)\;,
\end{aligned}$$

so that the time derivative of the total energy takes the form

$$\mathrm{D}_t\mathcal{F}^{\mathrm{el}}(t,\phi(t)\circ u) = \mathcal{P}(t,u) - \mathrm{D}_t\mathcal{G}(t,u) - \mathrm{D}_t\mathcal{S}(t,u)\,.$$

These formulas allow passing from the problem with fixed boundary data to the original one, as in the previous chapter.

4.3 Quasistatic evolution

Now we adapt the definitions and theorems of Section 3.2.4 to the context with external forces. As for the properties of global stability and energy balance, a different proof is needed, because of the weaker assumptions on the prescribed deformations.

Throughout the section, we adopt the formulation with time-independent boundary conditions, introduced in Section 4.2. All definitions and theorems presented here can be formulated in the framework with time-dependent boundary data of Section 4.1, using Remark 4.2.5 (see also the previous chapter).

4.3.1 Definitions and properties

We fix an initial condition (u_0,Γ_0), which is supposed to be a minimum energy configuration at time 0, i.e., $\Gamma_0\in\mathcal{R}$, $u_0\in AD(I,\Gamma_0)$, and

$$\mathcal{F}(0,u_0,\Gamma_0)\le\mathcal{F}(0,u,\Gamma) \tag{4.29}$$

for every $\Gamma\in\mathcal{R}$ with $\Gamma_0\,\tilde{\subset}\,\Gamma$ and every $u\in AD(I,\Gamma)$.

Given a *time discretization* $\{t_k^i\}_{0\le i\le k}$ satisfying (3.19) and (3.20), we define a corresponding incremental approximate solution.

Definition 4.3.1. Fix $k\in\mathbb{N}$. An *incremental approximate solution* for $\mathcal{F}$ corresponding to the time subdivision $\{t_k^i\}_{0\le i\le k}$ with initial datum (u_0,Γ_0) is a function $t\mapsto(u_k(t),\Gamma_k(t))$, such that

(a) $(u_k(0), \Gamma_k(0)) = (u_0, \Gamma_0)$;

(b) $u_k(t) = u_k(t_k^i)$ and $\Gamma_k(t) = \Gamma_k(t_k^i)$ for $t \in \left[t_k^i, t_k^{i+1}\right)$ and $i = 0, \ldots, k-1$;

(c) for $i = 1, \ldots, k$, $(u(t_k^i), \Gamma(t_k^i))$ is a solution of

$$\min \left\{ \mathcal{F}(t_k^i, u, \Gamma) \colon \Gamma \in \mathcal{R},\ \Gamma_k^{i-1} \,\tilde{\subset}\, \Gamma,\ u \in AD(I, \Gamma) \right\}. \tag{4.30}$$

The existence of incremental approximate solutions is guaranteed by the following theorem, whose proof can be obtained arguing as in Section 3.3.1.

Theorem 4.3.2 (Minimization of the total energy). *Let $\mathcal{F}$ be the functional defined in* (4.16)–(4.21)*, where $\mathcal{V}$ satisfies* (V1–8′)*, $\mathcal{L}$ satisfies* (L1–3)*, $\mathcal{T}$ satisfies* (T1–3)*, and $\mathcal{K}$ satisfies* (K1–3)*. Then, for every $t \in [0,1]$ and $\Gamma_0 \in \mathcal{R}$, the minimum problem*

$$\min \left\{ \mathcal{F}(t, u, \Gamma) \colon \Gamma \in \mathcal{R},\ \Gamma_0 \,\tilde{\subset}\, \Gamma,\ u \in AD(I, \Gamma) \right\} \tag{4.31}$$

has a solution.

To find an incrementally-approximable quasistatic evolution, we take a sequence of incremental approximate solutions and pass to the limit as the time step vanishes.

Definition 4.3.3. A function $t \mapsto (u(t), \Gamma(t))$ from $[0,1]$ in $SBV^p(\Omega_D; K) \times \mathcal{R}$ is an *incrementally-approximable quasistatic evolution* of minimum energy configurations with initial datum (u_0, Γ_0), if there exist an increasing set function $t \mapsto \Gamma^*(t) \in \mathcal{R}$, a time discretization $\left\{t_k^i\right\}_{0 \le i \le k}$, and a corresponding sequence of incremental approximate solutions $(u_k(t), \Gamma_k(t))$ with the same initial datum, such that for every $t \in [0,1]$

(a) $\Gamma_k(t)$ σ^p-converges to $\Gamma^*(t)$ and $\Gamma(t) = \Gamma^*(t) \cup \Gamma_0$;

(b) there is a subsequence $u_{k_j}(t)$, depending on t, such that $u_{k_j}(t) \rightharpoonup u(t)$ weakly* in $SBV^p(\Omega_D; K)$ and $\lim\limits_{k\to\infty} \theta_{k_j}(t) = \limsup\limits_{k\to\infty} \theta_k(t)$, where

$$\theta_k(t) := \mathrm{D}_t \mathcal{F}^{\mathrm{el}}(t, u_k(t)). \tag{4.32}$$

We state the existence result for measurable incrementally-approximable quasistatic evolutions; the proof can be done as in Sections 3.3.2 and 3.6, with minor modifications due to the presence of the forces (the semicontinuity property (3.46) is got combining Theorem 1.2.3 and (V6′)).

Theorem 4.3.4 (Existence of quasistatic evolutions). *Let $\mathcal{F}$ be the functional defined in* (4.16)–(4.21)*, where $\mathcal{V}$ satisfies* (V1–8′)*, $\mathcal{L}$ satisfies* (L1–3)*, $\mathcal{T}$ satisfies* (T1–3)*, and $\mathcal{K}$ satisfies* (K1–3)*. Let (u_0, Γ_0) be a minimum energy configuration at time* 0 *as in* (4.29)*. Then there exists an incrementally-approximable quasistatic evolution $t \mapsto (u(t), \Gamma(t))$ with initial datum (u_0, Γ_0), such that the function $t \mapsto u(t)$ is strongly measurable, regarded as a function from $[0,1]$ into $SBV^p(\Omega_D; \mathbb{R}^n)$.*

The properties of global stability and energy balance will be proven in the next section.

Theorem 4.3.5 (Properties of quasistatic evolutions)**.** *For a given incrementally-approximable quasistatic evolution $(u(t), \Gamma(t))$, the following hold:*

1. Global stability: *for every $t \in [0,1]$ the pair $(u(t), \Gamma(t))$ is a minimum energy configuration at time t, i.e., $\Gamma(t) \in \mathcal{R}$, $u(t) \in AD(I, \Gamma(t))$, and*
$$\mathcal{F}(t, u(t), \Gamma(t)) \leq \mathcal{F}(t, v, \Gamma) \tag{4.33}$$
for every $\Gamma \in \mathcal{R}$, with $\Gamma(t) \,\tilde{\subset}\, \Gamma$, and every $v \in AD(I, \Gamma)$;

2. Energy balance: *the function $F(t) := \mathcal{F}(t, u(t), \Gamma(t))$ is absolutely continuous on $[0,1]$ and its time derivative satisfies*
$$\dot{F}(t) = \mathrm{D}_t \mathcal{F}^{\mathrm{el}}(t, u(t), \Gamma(t)) \text{ for } \mathcal{L}^1\text{-a.e. } t \in [0,1]\,. \tag{4.34}$$

4.3.2 Proof of Theorem 4.3.5

The proof follows the scheme of Sections 3.4 and 3.5.

Let $(u(t), \Gamma(t))$ be an incrementally-approximable quasistatic evolution. Then there exist an increasing set function $t \mapsto \Gamma^*(t) \in \mathcal{R}$, a time discretization $\{t_k^i\}_{0 \leq i \leq k}$ such that (3.19) and (3.20) hold, and a sequence of incremental approximate solutions $(u_k(t), \Gamma_k(t))$ with the same initial datum (u_0, Γ_0), which fulfil properties (a) and (b) of Definition 4.3.3. Let $\theta_k(t)$ be as in (4.32),
$$\tau_k(t) := t_k^i\,, \quad \text{and} \quad \mathcal{F}_k(t, \cdot) := \mathcal{F}(t_k^i, \cdot) \quad \text{for } t \in \left[t_k^i, t_k^{i+1}\right)\,.$$

Global stability

The proof of (1) can be done as in Section 3.4, with obvious adaptations to treat the case where volume and surface forces are added. The properties of V presented before are sufficient to repeat the procedure of Section 3.4; in particular, the properties of Remark 3.2.6 used in the Crack Transfer Lemma 3.4.1 can be sostituted by the weaker ones (V6′) and (V8′) stated here.

Fixed $t \in [0,1]$, by Definition 4.3.3 there is a subsequence $u_{k_j}(t)$ converging to $u(t)$ weakly* in $SBV^p(\Omega_D; K)$. Arguing as in Remark 3.4.4, one can see that
$$\mathcal{V}(\tau_{k_j}(t), u_{k_j}(t)) \to \mathcal{V}(t, u(t))\,. \tag{4.35}$$

Discrete energy inequality

Let now $(u_k^i, \Gamma_k^i) := (u_k(t_k^i), \Gamma_k(t_k^i))$. Taking $(u, \Gamma) = (I, \Gamma_k^{i-1})$ in (4.30), we get $\mathcal{F}^{\mathrm{el}}(t_k^i, u_k^i) \leq \mathcal{F}^{\mathrm{el}}(t_k^i, I)$. Hence by (V3′), (L1), and (T1)
$$\mathcal{F}^{\mathrm{el}}(t_k^i, u_k^i) < M + c_L + c_T\,, \tag{4.36}$$
so that $\left\|\nabla u_k^i\right\|_{L^p(\Omega_D; \mathbb{M}^{n \times n})}$ is bounded uniformly in k and i by coercivity. As $u_k^{i-1} \in AD(I, \Gamma_k^{i-1})$, by (4.30) we have $\mathcal{F}(t_k^i, u_k^i, \Gamma_k^i) \leq \mathcal{F}(t_k^i, u_k^{i-1}, \Gamma_k^{i-1})$. By (V6′), (4.36), (L2), and (T2), the function $t \mapsto \mathcal{F}^{\mathrm{el}}(t, u_k^{i-1})$ is absolutely continuous; therefore,
$$\mathcal{F}^{\mathrm{el}}(t_k^i, u_k^{i-1}) - \mathcal{F}^{\mathrm{el}}(t_k^{i-1}, u_k^{i-1}) = \int_{t_k^{i-1}}^{t_k^i} \mathrm{D}_t \mathcal{F}^{\mathrm{el}}(t, u_k^{i-1})\,\mathrm{d}t\,.$$

Summing up, we obtain for every $t \in [0,1]$ the discrete energy inequality

$$\mathcal{F}_k(t, u_k(t), \Gamma_k(t)) \leq \mathcal{F}(0, u_0, \Gamma_0) + \int_0^{\tau_k(t)} \theta_k(s)\, \mathrm{d}s\,. \tag{4.37}$$

By (V6′), (4.36), (L2), (T2), and (4.37), $\mathcal{F}_k(t, u_k(t), \Gamma_k(t))$ is bounded uniformly with respect to k and t. The nonnegativity of V, (L1), (T1), and (3.4) give a bound also on $\mathcal{H}^{n-1}(\Gamma_k(t))$, uniform in k and t.

Energy inequality

By Fatou's lemma, the function

$$\theta_\infty(t) := \limsup_{k\to\infty} \theta_k(t) \tag{4.38}$$

belongs to $L^1([0,1])$ and

$$\limsup_{k\to\infty} \int_0^{\tau_k(t)} \theta_k(s)\, \mathrm{d}s \leq \int_0^t \theta_\infty(s)\, \mathrm{d}s\,. \tag{4.39}$$

Fixed $s \in [0,1]$, by Definition 4.3.3 there is a subsequence $(u_{k_j}(s), \Gamma_{k_j}(s))$ such that

$$u_{k_j}(s) \rightharpoonup u(s) \text{ weakly}^* \text{ in } SBV^p(\Omega_D; K) \tag{4.40}$$

and

$$\theta_\infty(s) = \lim_{k\to\infty} \theta_{k_j}(s)\,. \tag{4.41}$$

By (4.35), (V6′), and (4.27) we have

$$\mathcal{V}(s, u_{k_j}(s)) \to \mathcal{V}(s, u(s))\,,\ \mathcal{L}(s, u_{k_j}(s)) \to \mathcal{L}(s, u(s))\,,\ \mathcal{T}(s, u_{k_j}(s)) \to \mathcal{T}(s, u(s))\,. \tag{4.42}$$

In order to pass to the limit as $k_j \to \infty$ in (4.37), we argue as in Lemma 3.5.1. Following the proof in [24, Proposition 3.3], it is possible to see that (3.68) can be substituted with the following consequence of (V7′).

Remark 4.3.6. From (V7′) we deduce that for every $s \in [0,1]$ and $M > 0$ there exists a modulus of continuity $\omega_s^M \colon [0,1] \to [0,+\infty)$, with $s \mapsto \omega_s^M(h)$ in $L^\infty([0,1])$ for every $h \in [0,1]$, such that

$$\left| \mathrm{D}_t\mathcal{V}(s,v) \mp \frac{\mathcal{V}(s\pm h, v) - \mathcal{V}(s,v)}{h} \right| \leq \omega_s^M(h) \tag{4.43}$$

for every $v \in SBV^p(\Omega_D; K)$ such that $\mathcal{V}(0,v) \leq M$ and every $h > 0$ with $s \pm h \in [0,1]$, provided that $\mathrm{D}_t\mathcal{V}(s,v)$ is defined.

From the proof in [24, Proposition 3.3], it is clear that ω_t need not be uniform with respect to t. Then, the conclusion of Lemma 3.5.1 still holds under the weaker hypothesis (4.43). By (4.40) and (4.42), this implies that for $\mathcal{L}^1$-a.e. $s \in [0,1]$

$$\mathrm{D}_t\mathcal{V}(s, u_{k_j}(s)) \to \mathrm{D}_t\mathcal{V}(s, u(s))\,.$$

Notice that $\mathrm{D}_t\mathcal{V}(s,u_{k_j}(s))$ and $\mathrm{D}_t\mathcal{V}(s,u(s))$ are well defined for $\mathcal{L}^1$-a.e. $s\in[0,1]$ thanks to Remark 4.2.2.

The convergence of the derivatives of the force terms is given by (4.28). Hence, by (4.20), (4.32), and (4.41), we conclude that for $\mathcal{L}^1$-a.e. $s\in[0,1]$

$$\theta_\infty(s)=\mathrm{D}_t\mathcal{F}^{\mathrm{el}}(s,u(s))\,. \tag{4.44}$$

By (4.35), (4.27), and (3.62) we have

$$\mathcal{F}(t,u(t),\Gamma(t))\le\liminf_{j\to\infty}\mathcal{F}_{k_j}(t,u_{k_j}(t),\Gamma_{k_j}(t))\le\limsup_{k\to\infty}\mathcal{F}_k(t,u_k(t),\Gamma_k(t))\,.$$

From (4.37), (4.38), (4.39), and (4.44) we obtain

$$\limsup_{k\to\infty}\mathcal{F}_k(t,u_k(t),\Gamma_k(t))\le\mathcal{F}(0,u_0,\Gamma_0)+\int_0^t\mathrm{D}_t\mathcal{F}^{\mathrm{el}}(s,u(s))\,\mathrm{d}s\,.$$

Then we get the energy inequality

$$\mathcal{F}(t,u(t),\Gamma(t))\le\mathcal{F}(0,u_0,\Gamma_0)+\int_0^t\mathrm{D}_t\mathcal{F}^{\mathrm{el}}(s,u(s))\,\mathrm{d}s\,. \tag{4.45}$$

Finally, comparing $(u(t),\Gamma(t))$ with $(I,\Gamma(t))$, by (4.33), (V3′), (V4′), (L1), and (T1), we find a constant $C>0$ such that

$$\mathcal{V}(t,u(t))\le C\,,\quad\|\nabla u(t)\|_{L^p(\Omega_D;\mathbb{M}^{n\times n})}\le C\,,\quad\mathcal{H}^{n-1}(S(u(t)))\le C \tag{4.46}$$

uniformly in t.

Approximation with Riemann sums

For the next point, we will use the approximation of Lebesgue integrals with suitable Riemann sums [32]. Let $\mathcal{C}_1$ a countable subset of $L^\infty(\Omega;K)$, dense for the norm of $L^q(\Omega;\mathbb{R}^n)$, and $\mathcal{C}_2$ a countable subset of $L^\infty(\partial_S\Omega;K)$, dense for the norm of $L^q(\partial_S\Omega;\mathbb{R}^n)$. By [14, Lemma 4.12 and Remark 4.13], we can find a sequence of subdivisions $\{s_k^i\}_{0\le i\le i_k}$ satisfying:

$$0 = s_k^0 < s_k^1 < \cdots < s_k^{i_k-1} < s_k^{i_k} = t\,, \quad \lim_{k\to\infty} \max_{1\le i\le i_k} (s_k^i - s_k^{i-1}) = 0\,, \tag{4.47}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left|b_L(s_k^i) - b_L(s)\right| \,\mathrm{d}s = 0\,, \tag{4.48}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left|b_T(s_k^i) - b_T(s)\right| \,\mathrm{d}s = 0\,, \tag{4.49}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left|\mathrm{D}_t\mathcal{F}^{\mathrm{el}}(s_k^i, u(s_k^i)) - \mathrm{D}_t\mathcal{F}^{\mathrm{el}}(s, u(s))\right| \,\mathrm{d}s = 0\,, \tag{4.50}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left|\mathrm{D}_t\mathcal{L}(s_k^i, v) - \mathrm{D}_t\mathcal{L}(s, v)\right| \,\mathrm{d}s = 0 \quad \text{for every } v\in\mathcal{C}_1\,, \tag{4.51}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left|\mathrm{D}_t\mathcal{T}(s_k^i, v) - \mathrm{D}_t\mathcal{T}(s, v)\right| \,\mathrm{d}s = 0 \quad \text{for every } v\in\mathcal{C}_2\,, \tag{4.52}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left|\omega_{s_k^i}^C\left(\tfrac{1}{m}\right) - \omega_s^C\left(\tfrac{1}{m}\right)\right| \,\mathrm{d}s = 0 \quad \text{for every } m\in\mathbb{N}\,, \tag{4.53}$$

where ω_s^C is defined in Remark 4.3.6 and C is the constant of (4.46). In the previous formulas it is understood that all time derivatives are well defined at s_k^i. We can deduce the following lemma.

Lemma 4.3.7. *In the previous assumptions,*

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left|\mathrm{D}_t\mathcal{V}(s_k^i, u(s_k^i)) - \mathrm{D}_t\mathcal{V}(s, u(s_k^i))\right| \,\mathrm{d}s = 0\,. \tag{4.54}$$

Proof. Fixed $m\in\mathbb{N}$, we have $\max_i(s_k^i - s_k^{i-1}) \le 1/m$ for k large. Comparing the derivatives with the increment quotients and employing twice (4.43), we get

$$\int_{s_k^{i-1}}^{s_k^i} \left|\mathrm{D}_t\mathcal{V}(s_k^i, u(s_k^i)) - \mathrm{D}_t\mathcal{V}(s, u(s_k^i))\right| \,\mathrm{d}s \le \int_{s_k^{i-1}}^{s_k^i} \left[\omega_{s_k^i}^C\left(\tfrac{1}{m}\right) + \omega_s^C\left(\tfrac{1}{m}\right)\right] \mathrm{d}s$$

for every $s\in[s_k^{i-1}, s_k^i)$. We deduce that

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left|\mathrm{D}_t\mathcal{V}(s_k^i, u(s_k^i)) - \mathrm{D}_t\mathcal{V}(s, u(s_k^i))\right| \,\mathrm{d}s \le$$

$$\le \lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left[\omega_{s_k^i}^C\left(\tfrac{1}{m}\right) + \omega_s^C\left(\tfrac{1}{m}\right)\right] \mathrm{d}s \le 2\int_0^t \omega_s^C\left(\tfrac{1}{m}\right) \mathrm{d}s\,,$$

where in the last estimate we used (4.53). Passing to the limit as $m\to\infty$, we conclude by dominated convergence, thanks to the uniform bound on ω_s^C. ∎

As for the approximation of the force terms, we follow [14, Lemma 5.7].

Lemma 4.3.8. *In the previous assumptions,*

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left| \mathrm{D}_t \mathcal{L}(s_k^i, u(s_k^i)) - \mathrm{D}_t \mathcal{L}(s, u(s_k^i)) \right| \, \mathrm{d}s = 0 \,, \tag{4.55}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left| \mathrm{D}_t \mathcal{T}(s_k^i, u(s_k^i)) - \mathrm{D}_t \mathcal{T}(s, u(s_k^i)) \right| \, \mathrm{d}s = 0 \,. \tag{4.56}$$

Proof. Consider the set H of all functions $v \in SBV^p(\Omega_D; K)$ such that

$$\|\nabla v\|_{L^p(\Omega_D;\mathbb{M}^{n\times n})} \leq C \quad \text{and} \quad \mathcal{H}^{n-1}(S(v)) \leq C \,,$$

where C is the constant appearing in (4.46). By [4, Theorem 4.8], H is compact in $L^\infty(\Omega; K)$ with respect to the norm of $L^q(\Omega;\mathbb{R}^n)$. Fix $\varepsilon > 0$; there exists a finite number of functions $v_1, \dots, v_h \in \mathcal{C}_1$ such that for every $v \in H$ there exists j with $\|v - v_j\|_{L^q(\Omega;\mathbb{R}^n)} < \varepsilon$. By (L3), we have

$$|\mathrm{D}_t \mathcal{L}(s)(v) - \mathrm{D}_t \mathcal{L}(s)(v_j)| \leq \varepsilon \, b_L(s)$$

for $\mathcal{L}^1$-a.e. $s \in [0,1]$ (including the points s_k^i). Then,

$$\sum_{i=1}^{i_k} \sup_{v\in H} \int_{s_k^{i-1}}^{s_k^i} \left| \mathrm{D}_t \mathcal{L}(s_k^i, v) - \mathrm{D}_t \mathcal{L}(s, v) \right| \, \mathrm{d}s \leq$$
$$\leq \sum_{j=1}^{h} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left| \mathrm{D}_t \mathcal{L}(s_k^i, v_j) - \mathrm{D}_t \mathcal{L}(s, v_j) \right| \, \mathrm{d}s + \varepsilon \sum_{i=1}^{i_k} \int_0^t \left[b_L(s_k^i) + b_L(s) \right] \, \mathrm{d}s \,.$$

First we pass to the lim sup as $k \to \infty$, then we let $\varepsilon \to 0$; recalling (4.48) and (4.51) we find that the left hand side in the previous expression is vanishing. Hence, (4.55) follows. The proof of (4.56) is analogous. □

Summing up (4.50), (4.54), (4.55), and (4.56), we obtain

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left| \mathrm{D}_t \mathcal{F}^{\mathrm{el}}(s, u(s_k^i)) - \mathrm{D}_t \mathcal{F}^{\mathrm{el}}(s, u(s)) \right| \, \mathrm{d}s = 0 \,. \tag{4.57}$$

Energy equality

The converse of (4.45) is a consequence of the stability property, via a discretization argument.

For $i = 1, \dots, i_k$, $(u(s_k^{i-1}), \Gamma(s_k^{i-1}))$ and $(u(s_k^i), \Gamma(s_k^i))$ are competitors in (4.33): as $u(s_k^i) \in AD(I, \Gamma(s_k^i))$ and $\Gamma(s_k^{i-1}) \subset \Gamma(s_k^i)$, we get

$$\mathcal{F}(s_k^{i-1}, u(s(_k^{i-1}), \Gamma(s_k^{i-1})) \leq \mathcal{F}(s_k^{i-1}, u(s_k^i), \Gamma(s_k^i)) \,.$$

Arguing as in the proof of the discrete energy inequality, by (4.46), (V6′), (L2), and (T2) we obtain

$$\mathcal{F}(s_k^{i-1}, u(s_k^i), \Gamma(s_k^i)) = \mathcal{F}(s_k^i, u(s_k^i), \Gamma(s_k^i)) - \int_{s_k^{i-1}}^{s_k^i} \mathrm{D}_t \mathcal{F}^{\mathrm{el}}(s, u(s_k^i))\, \mathrm{d}s\,.$$

Summing up,

$$\mathcal{F}(t, u(t), \Gamma(t)) \geq \mathcal{F}(0, u_0, \Gamma_0) + \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \mathrm{D}_t \mathcal{F}^{\mathrm{el}}(s, u(s_k^i))\, \mathrm{d}s\,.$$

By (4.45) and (4.57) we have

$$\mathcal{F}(t, u(t), \Gamma(t)) = \mathcal{F}(0, u_0, \Gamma_0) + \int_0^t \mathrm{D}_t \mathcal{F}^{\mathrm{el}}(s, u(s))\, \mathrm{d}s\,,$$

which proves (2). The proof of Theorem 4.3.5 is concluded. □

Chapter 5

Energy release rate and stress intensity factor in antiplane elasticity

Introduction

In this last chapter, we study a bidimensional problem for antiplane linearized elasticity. In particular, we consider the case where the prescribed crack path Γ is a $C^{1,1}$ curve parametrized by a function $\gamma\colon [s_1, s_2] \to \overline{\Omega}$ (with $s_1 < 0 < s_2$). We consider the increasing family of cracks $\Gamma_s := \{\gamma(t)\colon s_1 \leq t \leq s\}$. We prove the existence of the stress intensity factor in this case and show its relation with the energy release rate; the results are contained in [36]. The basis of our arguments is the theory developed by Grisvard [29, 30], who studied the singularities of solutions to elliptic problems in polygonal domains.

The standard strategy for the computation of the derivative of the energy is to rewrite the energy integrals so that they are defined on a fixed domain. If the crack has a rectilinear path, it is easy to construct a diffeomorphism F_s which coincides with the identity in a neighbourhood of $\partial\Omega$ and transforms $\Omega_s := \Omega\backslash\Gamma_s$ into a fixed domain $\Omega_0 := \Omega\backslash\Gamma_0$. This procedure can be followed also if the crack is a curve of class C^2, defining F_s around $\gamma(0)$ as the flow of a vector field tangent to Γ. However, this allows the computation of the energy release rate only if the second derivative of Γ exists at the crack tip.

We show a different method to calculate the derivative of the energy when the crack path Γ is only of class $C^{1,1}$, proving that the derivative exists at all the points, even if the curve has not a second derivative. We reduce the problem to the rectilinear case, thanks to a diffeomorphism Φ which straightens the cut in a neighbourhood of $\gamma(0)$; moreover, Φ transforms the elliptic coefficients so that the conormal vector is parallel to the normal. A similar procedure was performed by Mumford-Shah [39] for a slightly different variational problem. The change of variables Φ is used to show the existence of the stress intensity factor in this case, following the lines of a proof by Grisvard [29] for a pure Dirichlet problem. Our results have a natural generalization to elliptic operators with variable coefficients of class $C^{0,1}$.

5.1 Singularities in elliptic equations

We will define the stress intensity factor in the case of elliptic operators with Lipschitz coefficients in domains with $C^{1,1}$ curvilinear cracks.

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, simply connected, with Lipschitz boundary. In Ω we consider a curve $\gamma\colon [s_1, s_2] \to \overline{\Omega}$ of class $C^{1,1}$, parametrized by arc length, without self-intersections; let $\Gamma := \gamma([s_1, s_2])$. We suppose that $s_1 < 0 < s_2$, $\gamma(s_1), \gamma(s_2) \in \partial\Omega$, and $\gamma(s) \in \Omega$ for $s \in (s_1, s_2)$; up to a rototranslation, we may suppose also that $\gamma(0) = 0 \in \Omega$ and the tangent vector $\dot\gamma(0)$ coincides with the first coordinate vector e_1. The set Ω is the section of an elastic body with a crack, represented by the portion of curve $\Gamma_0 := \gamma([s_1, 0])$.

Furthermore, we assume that $\Omega\backslash\Gamma$ has two connected components, both Lipschitz. In particular, the tangent vectors $\dot\gamma(s_1), \dot\gamma(s_2)$ are not parallel to $\partial\Omega$. This requirement is necessary to employ the Poincaré inequality in $\Omega\backslash\Gamma$.

We denote the two lips of Γ by Γ^+ and Γ^-: Γ^+ has the orientation given by the arc length parametrization, Γ^- the opposite, so that $\partial(\Omega\backslash\Gamma)$ is oriented as usual. Analogously, we denote by Γ_0^+ and Γ_0^- the two lips of Γ_0. We define the trace operators γ_Ω on $\partial\Omega$ and $\gamma^\pm$ on $\Gamma^\pm$. Finally, we denote the two connected components of $\Omega\backslash\Gamma$ by Ω^+ and Ω^-: the former is placed by the positive lip of Γ, the latter on the other side.

Consider an elliptic operator (with the only principal part, for the sake of simplicity)

$$\mathcal{A}u := -\sum_{i,j=1}^{2} \mathrm{D}_i\left(a_{ij}\mathrm{D}_j u\right) , \tag{5.1}$$

where the coefficients $a_{ij} = a_{ji} \in C^{0,1}(\overline{\Omega})$ are uniformly elliptic:

$$\sum_{i,j=1}^{2} a_{ij}\xi^i\xi^j \geq \alpha\left|\xi\right|^2 \quad \text{for every } x \in \Omega \text{ and every } \xi \in \mathbb{R}^2 ,$$

with $\alpha > 0$. Let A denote the coefficient matrix, $A(x) = (a_{ij}(x))_{ij}$.

Given $f \in L^2(\Omega\backslash\Gamma_0)$, we study the problem

$$\begin{cases} \mathcal{A}u = f & \text{in } \Omega\backslash\Gamma_0 , \\ \gamma_\Omega u = 0 & \text{on } \partial\Omega , \\ \gamma^\pm \frac{\partial}{\partial\nu_A^\pm} u = 0 & \text{on } \Gamma_0^\pm , \end{cases}$$

where $\nu_A^\pm := A\nu^\pm$ denotes the conormal vector to $\Gamma_0^\pm$.

We define the space of test functions, null in $\partial\Omega$,

$$H_0(\Omega\backslash\Gamma_0) := \left\{u \in H^1(\Omega\backslash\Gamma_0)\colon \gamma_\Omega u = 0 \text{ in } \partial\Omega\right\} .$$

Under these hypotheses, we have a result of existence and uniqueness for the variational solution: there is a unique function $u \in H_0(\Omega\backslash\Gamma_0)$ such that

$$\sum_{i,j=1}^{2} \int_{\Omega\backslash\Gamma_0} a_{ij}(x)\mathrm{D}_j u(x)\mathrm{D}_i w(x)\,\mathrm{d}x = \int_{\Omega\backslash\Gamma_0} f(x)w(x)\,\mathrm{d}x \tag{5.2}$$

for every $w \in H_0(\Omega\backslash\Gamma_0)$.

By the classical regularity theorems, we see that the variational solution u is H^2 inside $\Omega\backslash\Gamma_0$ and until the cut Γ_0, far from 0 and $\gamma(s_1)$ (where the boundary is not smooth).

Theorem 5.1.1. *Let u be the variational solution of* (5.2)*. Let V and W be two open sets such that $0 \in V \subset W \subset\subset \Omega$; let $\phi \in C_c^\infty(W\backslash V)$. Then $\phi u \in H^2(\Omega\backslash\Gamma_0)$.*

In the following section we characterize the singularity around the crack tip 0.

5.1.1 A diffeomorphism which straightens the crack

We construct a diffeomorphism which in a neighbourhood of the origin transforms the curve Γ into a segment and the elliptic operator $\mathcal{A}$ in an operator $\mathcal{B}$ with coefficients near to the Laplacian: this will allow us to reduce the problem to the one for the Laplacian with rectilinear crack, which was treated in [29, 30]. A similar construction was presented in [39, Appendix 1] for a slightly different variational problem.

First step. We define a diffeomorphism Φ_1 of class $C^{1,1}$ which induces an isometry from Γ to a segment, at least near the origin.

In a neighbourhood V of 0, we may write Γ as the graph of a cartesian curve $x_2 = \widetilde{\gamma}(x_1)$, defined for $-\delta \le x_1 \le \delta$. In V we set

$$\Phi_1(x_1, x_2) := \left(l(x_1, \widetilde{\gamma}(x_1)),\ x_2 - \widetilde{\gamma}(x_1)\right),$$

where $l(x_1, \widetilde{\gamma}(x_1)) := \int_0^{x_1} \sqrt{1 + \dot{\widetilde{\gamma}}(t)^2}\, \mathrm{d}t$ is the signed length of the part of the curve between $(x_1, \widetilde{\gamma}(x_1))$ and 0. Notice that $\Phi_1(0) = 0$ and $\Gamma \cap V$ is mapped in a segment on the line $\{x_2 = 0\}$.

The change of variables defined by Φ_1 transforms $\mathcal{A}$ in an operator $\mathcal{A}_1$ whose coefficient matrix is denoted by $A_1 = \left(a_{ij}^{(1)}\right)_{ij}$. We have $A_1(0) = A(0)$.

Second step. In the neighbourhood $W := \Phi_1(V)$ where the crack path is a segment we apply a diffeomorphism Φ_2 such that $\Phi_2(x_1, 0) = (x_1, 0)$ and the new coefficient matrix $A_2 = \frac{\nabla\Phi_2^{\mathrm{T}} A_1 \nabla\Phi_2}{|\det \nabla\Phi_2|} \circ \Phi_2^{-1}$ has the conormal vector proportional to the second coordinate vector e_2, i.e., $A_2(x_1, 0)e_2 = \lambda_2(x_1)e_2$.

For instance, we may take

$$\Phi_2(x_1, x_2) := \left(x_1 - \int_{x_1}^{x_1 + x_2} \frac{a_{12}^{(1)}(s, 0)}{a_{22}^{(1)}(s, 0)}\, \mathrm{d}s,\ x_2 \right),$$

with $\lambda_2(x_1) = a_{22}^{(1)}(x_1, 0)$. Notice that Φ_2 is well defined near 0 and of class $C^{1,1}$, since $a_{ij}^{(1)} \in C^{0,1}(\overline{W})$ and by uniform ellipticity $a_{22}^{(1)}$ is bounded away from 0.

It is easy to see that the matrix $A_2(0)$ is diagonal, with $a_{11}^{(2)}(0) = \frac{\det A(0)}{a_{22}(0)}$ and $a_{22}^{(2)}(0) = a_{22}(0)$.

Third step. Finally, we apply an affine transformation, so that the coefficient matrix of the resulting operator is a scalar multiple of the identity matrix in 0.

We define

$$\Phi_3(x_1, x_2) := \left(x_1,\ \frac{a_{11}^{(2)}(0)}{a_{22}^{(2)}(0)}\, x_2 \right) = \left(x_1,\ \frac{\det A(0)}{a_{22}(0)^2}\, x_2 \right) .$$

In this way the (signed) length of the piece of curve from the origin to the current point is preserved if this point belongs to a suitably small neighbourhood of the origin.

Moreover, the new coefficient matrix $A_3(0)$ is scalar: indeed, $A_3(0) = \frac{\det A(0)}{a_{22}(0)}\, I$.

We now consider the change of variables

$$\Phi := (1-\eta)\, I + \eta\, \Phi_3 \circ \Phi_2 \circ \Phi_1 ,$$

where η is a cut-off function equal to one near the origin and having support in V; let $\widetilde{\Omega} := \Phi(\Omega)$. The corresponding operator after the change of variables is

$$\frac{\det A(0)}{a_{22}(0)} \left(-\Delta v + \sum_{i,j=1}^{2} \mathrm{D}_i(b_{ij}\mathrm{D}_j v) \right) , \tag{5.3}$$

with coefficients b_{ij} of class $C^{0,1}$ until the boundary of $\widetilde{\Omega}$ and $b_{ij}(0) = 0$. Hence the problem becomes

$$\mathcal{B}v := -\Delta v + \sum_{i,j=1}^{2} \mathrm{D}_i(b_{ij}\mathrm{D}_j v) = g , \tag{5.4}$$

where $v(y) := u\left(\Phi^{-1}(y)\right)$ and $g(y) := \frac{a_{22}(0)}{\det A(0)} f\left(\Phi^{-1}(y)\right) \left|\det \nabla\Phi^{-1}(y)\right|$. We denote by $B := \left(\delta_{ij} - b_{ij}\right)_{ij}$ the new coefficient matrix (uniformly elliptic with a constant $\beta > 0$) and by $\nu_B := B\nu$ the conormal vector, which is proportional to ν near 0. We have a Dirichlet condition on $\partial\widetilde{\Omega}$ and a Neumann condition on the cut.

We point out the properties of the change of variables:

- Φ is a $C^{1,1}$-diffeomorphism,
- it coincides with the identity out of a neighbourhood of the origin,
- $\Phi(0) = 0 = \gamma(0)$,
- $\widetilde{\Gamma} := \Phi(\Gamma)$ is a segment on the axis $\{x_2 = 0\}$ in a neighbourhood of 0,
- the (signed) length of the piece of curve from the origin to the current point is preserved if this point belongs to a suitably small neighbourhood of the origin, i.e., for $|s|$ small enough we have $\mathcal{H}^1(\Phi \circ \gamma([0, s])) = s - 0$ if $s > 0$ and $\mathcal{H}^1(\Phi \circ \gamma([s, 0])) = -s$ if $s < 0$.

5.1.2 Fredholm property

Thanks to the change of variables Φ of the previous section, we can compare the problem with the case of the Laplacian with rectilinear crack, using the abstract theory of Fredholm operators. The Fredholm properties of the elliptic operator $\mathcal{B}$ introduced in (5.4) allow us to study the singularity of the solution at the crack tip. We adapt the methods of [29, Section 5.2].

For our purposes it is enough to restrict our study to a neighbourhood U of the crack tip 0, so we choose U to be an equilateral triangle centred at 0, with a vertex belonging to Γ_0. This choice allows us to employ Grisvard's theory [29, 30] for singularities in polygons: the angles are such that the only singularity appears in 0. We denote by $\Gamma_0^\pm$ the two lips of the crack Γ_0 lying in U, by $\gamma^\pm$ the trace operators on $\Gamma_0^\pm$, and by $\nu^\pm$ the normal vectors to $\Gamma_0^\pm$, which coincide with the conormal vectors $\nu_B^\pm := B\nu^\pm$. Moreover, γ_U is the trace operator on ∂U.

To restrict the problem to $U\backslash\Gamma_0$, we use a cut-off function equal to one near 0 and supported in $\overline{U}$. Changing the names of v and g, we are led to a problem with the same elliptic operator $\mathcal{B}$ defined in (5.3)–(5.4):

$$\begin{cases} \mathcal{B}v = g & \text{in } U\backslash\Gamma_0\,, \\ \gamma_U v = 0 & \text{on } \partial U\,, \\ \gamma^\pm \frac{\partial v}{\partial \nu^\pm} = 0 & \text{on } \Gamma_0^\pm\,. \end{cases}$$

The variational formulation is

$$\begin{cases} v \in H_0(U\backslash\Gamma_0)\,, \\ \int_{U\backslash\Gamma_0} \nabla v(x)\cdot B(x)\nabla w(x)\,\mathrm{d}x = \int_{U\backslash\Gamma_0} g(x)\,w(x)\,\mathrm{d}x \quad \text{for every } w \in H_0(U\backslash\Gamma_0)\,, \end{cases} \tag{5.5}$$

where the space of test functions is

$$H_0(U\backslash\Gamma_0) := \left\{ v \in H^1(U\backslash\Gamma_0) \colon \gamma_U v = 0 \text{ on } \partial U \right\}\,.$$

Furthermore, we consider the space of "strong solutions"

$$S^2(U\backslash\Gamma_0) := \left\{ v \in H^2(U\backslash\Gamma_0) \colon \gamma_U v = 0 \text{ on } \partial U\,,\ \gamma^\pm \frac{\partial v}{\partial \nu^\pm} = 0 \text{ on } \Gamma_0^\pm \right\}$$

and regard $\mathcal{B}$ as an operator which maps $S^2(U\backslash\Gamma_0)$ into $L^2(U\backslash\Gamma_0)$:

$$\mathcal{B}\colon S^2(U\backslash\Gamma_0) \to L^2(U\backslash\Gamma_0)\,.$$

We would like to extend the domain so that $\mathcal{B}$ becomes surjective: the first step is showing that $\operatorname{Rg}\mathcal{B}$ is closed, thanks to an a-priori bound; then we will compute its index.

We will use the following estimate on the Laplacian, which can be proven arguing as in [30, Theorem 2.2.3]: for every $v \in S^2(U\backslash\Gamma_0)$

$$\|v\|_{H^2(U\backslash\Gamma_0)} \le C_{U\backslash\Gamma_0} \|\Delta v\|_{L^2(U\backslash\Gamma_0)}\,, \tag{5.6}$$

where $C_{U\backslash\Gamma_0}$ is the Poincaré constant of $U\backslash\Gamma_0$. An analogous estimate holds for the operator $\mathcal{B}$.

Lemma 5.1.2. *There is a constant $C > 0$ (depending on U) such that*

$$\|v\|_{H^2(U\setminus\Gamma_0)} \leq C\left(\|\mathcal{B}v\|_{L^2(U\setminus\Gamma_0)} + \|v\|_{L^2(U\setminus\Gamma_0)}\right) \tag{5.7}$$

for every $v \in S^2(U\setminus\Gamma_0)$. In particular, $\mathcal{B}$ satisfies the Fredholm property, i.e., it is injective and $\operatorname{Rg}\mathcal{B}$ is closed.

Proof. We have for every $v \in S^2(U\setminus\Gamma_0)$

$$\begin{aligned}\|-\Delta v\|_{L^2(U\setminus\Gamma_0)} &= \left\|\mathcal{B}v - \sum_{i,j=1}^{2} \mathrm{D}_i(b_{ij}\mathrm{D}_j v)\right\|_{L^2(U\setminus\Gamma_0)} \leq \\ &\leq \|\mathcal{B}v\|_{L^2(U\setminus\Gamma_0)} + M_1\|v\|_{H^1(U\setminus\Gamma_0)} + 2M_0\|v\|_{H^2(U\setminus\Gamma_0)}\end{aligned}$$

where $M_0 := \max_{U\setminus\Gamma_0}|b_{ij}|$ and $M_1 := \max_{U\setminus\Gamma_0}|\nabla b_{ij}|$. Since $b_{ij} \to 0$ as $x \to 0$, we can rescale U so that $C_{U\setminus\Gamma_0}M_0 \leq \frac{1}{4}$; recalling (5.6), we find $C > 0$ such that for every $v \in S^2(U\setminus\Gamma_0)$

$$\|v\|_{H^2(U\setminus\Gamma_0)} \leq C\left(\|\mathcal{B}v\|_{L^2(U\setminus\Gamma_0)} + \|v\|_{H^1(U\setminus\Gamma_0)}\right).$$

To pass from $\|v\|_{H^1(U\setminus\Gamma_0)}$ to $\|v\|_{L^2(U\setminus\Gamma_0)}$, we integrate by parts, using the Dirichlet and Neumann conditions, and get

$$\left|\langle v, \mathcal{B}v\rangle_{L^2(U\setminus\Gamma_0)}\right| = \left|\int_{U\setminus\Gamma_0}\sum_{i,j=1}^{2} b_{ij}\mathrm{D}_i v\mathrm{D}_j v\,\mathrm{d}x\right| \geq \beta\|\nabla v\|^2_{L^2(U\setminus\Gamma_0)},$$

where we have used the uniform ellipticity of the coefficients. Thanks to the Poincaré inequality we obtain

$$\|v\|^2_{H^1(U\setminus\Gamma_0)} \leq \frac{C_{U\setminus\Gamma_0}}{\beta}\left|\langle v, \mathcal{B}v\rangle_{L^2(U\setminus\Gamma_0)}\right| \leq \frac{C_{U\setminus\Gamma_0}}{2\beta}\left(\|\mathcal{B}v\|_{L^2(U\setminus\Gamma_0)} + \|v\|_{L^2(U\setminus\Gamma_0)}\right)^2.$$

Hence we deduce (5.7), changing the value of C.

Finally, the injectivity is obvious, while the fact that $\operatorname{Rg}\mathcal{B}$ is closed descends from the compact immersion of H^2 in L^2, thanks to (5.7). □

The result about the index of $\mathcal{B}$, regarded as a Fredholm operator, follows.

Proposition 5.1.3. *We have* $\operatorname{codim}\operatorname{Rg}\mathcal{B} = 1$.

Proof. The theorem is an application of the Fredholm theory. By [30, Section 2.3] we deduce that $\operatorname{codim}\operatorname{Rg}(-\Delta) = 1$.

We compare $\mathcal{B}$ and $-\Delta$, so we consider the convex combinations between this two operators: for $\lambda \in [0,1]$ let $\mathcal{B}_\lambda = \lambda\mathcal{B} - (1-\lambda)\Delta$. Repeating the arguments of Lemma 5.1.2, we find for every $\lambda \in [0,1]$ a constant $C_\lambda > 0$ such that

$$\|v\|_{H^2(U\setminus\Gamma_0)} \leq C_\lambda\left(\|\mathcal{B}_\lambda v\|_{L^2(U\setminus\Gamma_0)} + \|v\|_{L^2(U\setminus\Gamma_0)}\right)$$

for every $v \in H^2(U\backslash\Gamma_0)$. Hence $\mathcal{B}_\lambda$ is a Fredholm operator (injective with closed range) for every $\lambda \in [0, 1]$.

As the index ι (i.e, the difference between the dimension of the kernel and the codimension of the range) is invariant under homotopy, we obtain $\iota(\mathcal{B}) = \iota(-\Delta) = -1$. By injectivity, $\dim\ker\mathcal{B} = \dim\ker(-\Delta) = 0$, so $\operatorname{codim}\operatorname{Rg}\mathcal{B} = \operatorname{codim}\operatorname{Rg}(-\Delta) = 1$. □

5.1.3 Singular solutions and stress intensity factor

We are now able to describe the singularities of a solution near 0. First, we argue in the case where the cut has been rectified by the diffeomorphism Φ of Section 5.1.1.

We introduce in U a system of polar coordinates (r, θ), where the straight part of the crack coincides with the discontinuity line of the angle. We define the singular solution

$$S := r^{\frac{1}{2}} \sin\frac{\theta}{2} \in H^1(U\backslash\Gamma_0) \setminus H^2(U\backslash\Gamma_0)\,. \tag{5.8}$$

Let η be a radial cut-off, equal to one around 0 and with support in U, and consider $\eta\, S$ and $F := \mathcal{B}(\eta\, S)$: uniqueness implies that $F \neq 0$, since $\eta\, S$ satisfies the Neumann and Dirichlet conditions being radial, and that $F \notin \operatorname{Rg}\mathcal{B}$, because $S \notin H^2$. Furthermore, from a direct computation of $-\Delta(\eta\, S)$ and recalling that the coefficients b_{ij} vanish near 0, we get that $F \in L^2$.

Since $\operatorname{Rg}\mathcal{B}$ is a closed subspace of $L^2(U\backslash\Gamma_0)$ with codimension one, we have the decomposition

$$L^2(U\backslash\Gamma_0) = \operatorname{Rg}\mathcal{B} \oplus \langle F\rangle\,. \tag{5.9}$$

Hence, given $g \in L^2(U\backslash\Gamma_0)$, there are a unique function $v^R \in S^2(U\backslash\Gamma_0)$ and a unique constant $K_0 \in \mathbb{R}$, such that

$$g = \mathcal{B}v^R + K_0\, F\,.$$

If $v \in H_0(U\backslash\Gamma_0)$ is the variational solution of (5.5), by uniqueness we obtain

$$v = v^R + K_0\, \eta\, S\,,$$

or equivalently

$$v - K_0\, S \in H^2(U\backslash\Gamma_0)\,,$$

as $K_0(1-\eta)S$ is regular.

To come back to the operator $\mathcal{A}$ defined in $\Omega\backslash\Gamma_0$, we apply the diffeomorphism Φ^{-1}; hence, recalling that $u = v \circ \Phi$ is the solution of (5.2) and setting $u^R := v^R \circ \Phi$, we get

$$u = u^R + K_0(\eta S) \circ \Phi\,,$$

so we have proven the following theorem.

Theorem 5.1.4. *Given $f \in L^2(\Omega\backslash\Gamma_0)$, let $u \in H^1(\Omega\backslash\Gamma_0)$ be the variational solution of* (5.2). *Then there exists a unique constant K_0, called* stress intensity factor*, such that*

$$u - K_0\; S \circ \Phi \in H^2(\Omega'\backslash\Gamma_0) \tag{5.10}$$

for every $\Omega' \subset\subset \Omega$.

The problem with a nonhomogeneous Dirichlet condition can be treated in the same way.

Remark 5.1.5. The stress intensity factor has been defined as the coefficient of the projection on $\langle F \rangle$ in the decomposition (5.9). Hence, the application which maps the force into the stress intensity factor of the associated solution is linear and continuous with respect to the convergence in L^2.

5.1.4 A simpler singular function

The singular solution in (5.10) must be computed after straightening the crack through the change of variables Φ described in Section 5.1.1. Here we give another singular function, which does not satisfy the boundary conditions, but whose computation is simpler.

First, we consider the case when $A(0) = I$, so that $\nabla\Phi(0) = I$. In Ω we fix a system of polar coordinates (ρ, ϑ), such that, at a point x, $\rho = |x|$ and ϑ is the determination of the angle between e_1 and $x - 0$, continuous in $\Omega \backslash \Gamma_0$ (see Figure 5.1).

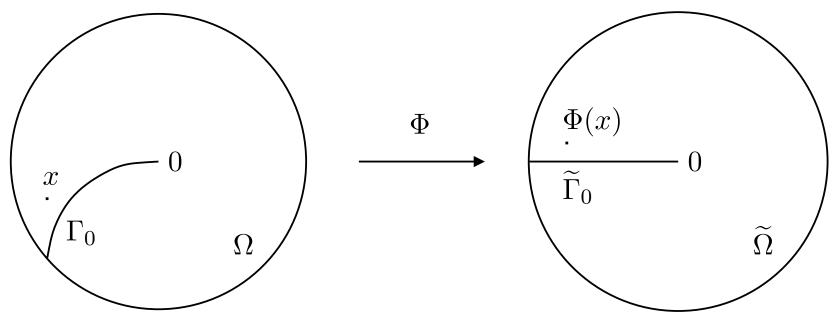

Figure 5.1: The angle ϑ is continuous in $\Omega\backslash\Gamma_0$, whilst θ is continuous in $\widetilde{\Omega}\backslash\widetilde{\Gamma}_0$. Hence, in the figure we have $\vartheta(x) > \pi$, $-\pi < \theta(x) < 0$, and $0 < \theta(\Phi(x)) < \pi$.

We define in $\Omega\backslash\Gamma_0$ the singular function

$$\widetilde{S} := \rho^{\frac{1}{2}} \sin\frac{\vartheta}{2}\,. \tag{5.11}$$

We show that $S \circ \Phi$ can be replaced by $\widetilde{S}$ in (5.10), because their difference is H^2.

Proposition 5.1.6. *For every $\Omega' \subset\subset \Omega$ we have*

$$\widetilde{S} - S \circ \Phi \in H^2(\Omega'\backslash\Gamma_0)\,. \tag{5.12}$$

Proof. As $S, \widetilde{S} \in H^1(\Omega'\backslash\Gamma_0)$, we have only to check the summability of the difference between the second derivatives in a neighbourhood of 0:

$$\mathrm{D}_{ij}(S\circ\Phi) - \mathrm{D}_{ij}\widetilde{S} = \mathrm{D}_{hk}S(\Phi)\,\mathrm{D}_i\Phi^h\,\mathrm{D}_j\Phi^k + \mathrm{D}_kS(\Phi)\,\mathrm{D}_{ij}\Phi^k - \delta_i^h\,\delta_j^k\,\mathrm{D}_{hk}\widetilde{S}\,.$$

Since $\mathrm{D}_kS(\Phi) \in L^2$ and $\mathrm{D}_{ij}\Phi^k \in L^\infty$, it is enough to estimate

$$\left|\mathrm{D}_{hk}S(\Phi)\,\mathrm{D}_i\Phi^h\,\mathrm{D}_j\Phi^k - \delta_i^h\,\delta_j^k\,\mathrm{D}_{hk}\widetilde{S}\right| \le \left|\mathrm{D}_{hk}\widetilde{S}\right|\left|\mathrm{D}_i\Phi^h\,\mathrm{D}_j\Phi^k - \delta_i^h\,\delta_j^k\right| + \\ + \left|\mathrm{D}_{hk}S(\Phi) - \mathrm{D}_{hk}\widetilde{S}\right|\left|\mathrm{D}_i\Phi^h\,\mathrm{D}_j\Phi^k\right|\,.$$

As for the first summand, we have $\left|\mathrm{D}_i\Phi^h - \delta_i^h\right| \leq L\,|x|$, where L is the Lipschitz constant of the derivatives of Φ, so

$$\left|\mathrm{D}_i\Phi^h\mathrm{D}_j\Phi^k - \delta_i^h\delta_j^k\right| \leq \left|\mathrm{D}_i\Phi^h - \delta_i^h\right|\left|\mathrm{D}_j\Phi^h\right| + \delta_i^h\left|\mathrm{D}_j\Phi^k - \delta_j^k\right| \leq C\,|x|$$

for some $C > 0$, whence

$$\left|\mathrm{D}_{hk}\widetilde{S}\right|\left|\mathrm{D}_i\Phi^h\mathrm{D}_j\Phi^k - \delta_i^h\delta_j^k\right| \leq C\,|x|^{-\frac{1}{2}} .$$

To estimate the second summand, we fix x such that $x \neq \Phi(x)$ (otherwise, the term is null); in particular, $x \neq 0$. We consider the segment $[x, \Phi(x)]$ between x and $\Phi(x)$; let d be its distance from 0.

Lemma 5.1.7. *If $|x|$ is sufficiently small, we have $d \geq \frac{1}{2}\,|x|$.*

Proof. As $\Phi \in C^{1,1}$, $\Phi(0) = 0$, and $\nabla\Phi(0) = I$, we get $|x - \Phi(x)| \leq \frac{L}{2}\,|x|^2$ (where L is the Lipschitz constant of the derivatives of Φ). Let $y \in [x, \Phi(x)]$ be the point of minimal distance from 0; we have

$$|x| \leq |y| + |x - y| \leq |y| + |x - \Phi(x)| \leq |y| + \frac{L}{2}\,|x|^2 ,$$

so $d \geq |x| - \frac{L}{2}\,|x|^2$. If $|x| \leq \frac{1}{L}$, we obtain $|x| - \frac{L}{2}\,|x|^2 \geq \frac{1}{2}\,|x|$. □

We compare S and $\widetilde{S}$, which are two different determinations of the multifunction $z \mapsto \operatorname{Im} z^{\frac{1}{2}}$. We fix two other determinations S^+ and S^- such that their common cut does not meet the segment $[x, \Phi(x)]$ (which passes far from 0 by the lemma): S^+ is chosen to be positive along $\{x_1 \leq 0,\, x_2 = 0\}$, S^- negative. Because of the definition of Φ we have

$$\widetilde{S}(x) = S^\pm(x) \quad \text{if and only if} \quad S(\Phi(x)) = S^\pm(\Phi(x))\,,$$

so we can replace both S and $\widetilde{S}$ writing either S^+ or S^-.

By the Mean Value Theorem we find $\overline{x} \in [x, \Phi(x)]$ such that

$$\left|\mathrm{D}_{hk}S^\pm(\Phi(x)) - \mathrm{D}_{hk}S^\pm(x)\right| \leq \left|\nabla\mathrm{D}_{hk}S^\pm(\overline{x})\right|\,|x - \Phi(x)| \;;$$

finally we control the third derivatives with $|\overline{x}|^{-\frac{5}{2}} \leq d^{-\frac{5}{2}} \leq C\,|x|^{-\frac{5}{2}}$ (by the lemma) and $|x - \Phi x|$ with $\frac{L}{2}\,|x|^2$, so the second summand is bounded by $C\,|x|^{-\frac{1}{2}}$ (for some $C > 0$). The proof is concluded. □

Remark 5.1.8. We have argued in the case when $A(0) = I$. For the general case, it suffices to apply an affine change of variables Ψ which transforms the coefficient matrix on the crack tip into the identity; then, one may take the singular function $\widetilde{S} \circ \Psi$.

In the next theorem we state the result just proven, for the particular case of the Laplacian.

Theorem 5.1.9. *Given $f \in L^2(\Omega\backslash\Gamma_0)$, let $u \in H^1(\Omega\backslash\Gamma_0)$ be the variational solution of the problem*

$$\begin{cases} -\Delta u = f \,, \\ \gamma_\Omega u = \gamma_\Omega \psi \quad \text{in } \partial\Omega\backslash\Gamma_0 \,, \\ \gamma^\pm \frac{\partial}{\partial \nu_A^\pm} u = 0 \quad \text{in } \Gamma_0 \,. \end{cases}$$

Let

$$\widetilde{S} := \rho^{\frac{1}{2}} \sin \frac{\vartheta}{2} \,,$$

where ρ and ϑ are polar coordinates such that ϑ is continuous in $\Omega\backslash\Gamma_0$. Then there exists a unique constant K_0 such that

$$u - K_0 \ \widetilde{S} \in H^2(\Omega'\backslash\Gamma_0) \tag{5.13}$$

for every $\Omega' \subset\subset \Omega$.

5.2 Computing the energy release rate in terms of the stress intensity factor

In this section we study the connection between the stress intensity factor and the energy release rate, that is the derivative of the energy with respect to crack length. The case of the Poisson equation in a domain with a rectilinear cut was treated in [20] and [30, Section 6.4]; our result is an extension to curvilinear cuts of class $C^{1,1}$ and operators with Lipschitz coefficients.

In the geometrical setting of Section 5.1, we define for $s \in (s_1, s_2)$ the increasing family of cracks

$$\Gamma_s := \{\gamma(t) \colon s_1 \le t \le s\} \,,$$

the cut domains

$$\Omega_s := \Omega\backslash\Gamma_s \,,$$

and the spaces of test functions

$$H_s := \{w \in H^1(\Omega_s) \colon \gamma_\Omega w = 0 \text{ in } \partial\Omega\} \,.$$

We consider the variational problem for the operator $\mathcal{A}$ defined in (5.1)

$$\begin{cases} u_s - \psi \in H_s \,, \\ \int_{\Omega_s} \nabla u_s(x) \cdot A(x) \nabla w(x) \, \mathrm{d}x = \int_{\Omega_s} f(x) \, w(x) \, \mathrm{d}x \quad \text{for every } w \in H_s \,, \end{cases} \tag{5.14}$$

where we assigned a force $f \in L^2(\Omega_0)$ and a boundary datum $\psi \in H^1(\Omega_0)$; without loss of generality we may assume that ψ is identically zero in a neighbourhood of 0.

By Theorem 5.1.4, the variational solution u_0 for $s = 0$ can be written as

$$u_0 = u_0^R + K_0 \ S \circ \Phi \,, \tag{5.15}$$

where $u_0^R \in H^2(U\backslash\Gamma_0)$ for every open set $U \subset\subset \Omega$, $K_0 \in \mathbb{R}$, $S = r^{\frac{1}{2}} \sin \frac{\theta}{2}$ (in polar coordinates around 0, with θ the angle measured from e_1), and Φ is the change of variable of Section 5.1.1.

Following the steps of [30, Theorem 6.4.1], we compute the derivative of the energy

$$\mathcal{E}(s) := \frac{1}{2}\int_{\Omega_s} \nabla u_s(x)\cdot A(x)\nabla u_s(x)\,\mathrm{d}x - \int_{\Omega_s} f(x)\,u_s(x)\,\mathrm{d}x\,.$$

Theorem 5.2.1. *$\mathcal{E}$ is differentiable at 0 and*

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}s}(0) = -\frac{\pi}{4}\frac{\det A(0)}{a_{22}(0)}\;K_0^2\,. \tag{5.16}$$

Proof. At a first stage we suppose that $\Gamma = \overline{\Omega}\cap\{x_2 = 0\}$, the conormal unit vector coincides with e_2 on Γ, $\gamma(0) = 0$, and $A(0) = I$. In this first part of the proof we assume also that the force is null in a neighbourhood of 0.

We consider a family of perturbations of the identical diffeomorphism

$$F_s := I + s\,V\,,$$

where V is a smooth vector field with compact support such that $V^1 \equiv 1$ around 0, $V^2 \equiv 0$, and

$$\mathrm{supp}\;\psi \cap \mathrm{supp}\;V = \emptyset = \mathrm{supp}\;f \cap \mathrm{supp}\;V\,. \tag{5.17}$$

We change variables through F_s and set $U_s := u_s \circ F_s$. By (5.14), for every $w \in H_s$ we have

$$\begin{aligned}\int_{\Omega_0} fW\,\mathrm{d}x = \int_{\Omega_s} fw\,\mathrm{d}x = \int_{\Omega_s} \nabla u_s\cdot A\nabla w\,\mathrm{d}x = \\ = \int_{\Omega_0} \nabla U_s\cdot\,\left[\nabla F_s^{-1}(F_s)\,A(F_s)\,(\nabla F_s^{-1}(F_s))^{\mathrm{T}}\;\det\nabla F_s\right]\nabla W\,\mathrm{d}x\,,\end{aligned}$$

with $W := w\circ F_s$. Hence we have recast the integral in (5.14) into an integral equation over a fixed domain, with operator

$$A(x,s) := \nabla F_s^{-1}(F_s(x))\,A(F_s(x))\,(\nabla F_s^{-1}(F_s(x)))^{\mathrm{T}}\;\det\nabla F_s(x)\,.$$

We need two simple lemmas about elliptic operators depending on a parameter. In what follows, H_0' is the dual space of H_0, endowed with the usual norm.

Lemma 5.2.2. *Let $s\mapsto a_{ij}(\cdot,s)\in L^\infty(\Omega_0)$ (for $i,j=1,2$) and $s\mapsto f_s\in H_0'$ be two functions, defined in a neighbourhood $(-\delta,\delta)$ of 0. Assume that*

- *$s\mapsto a_{ij}(x,s)$ is continuous for a.e. $x\in\Omega_0$,*
- *$s\mapsto f_s$ is continuous in H_0'.*

Furthermore, assume that there exist two constants $\alpha_0,\alpha_1>0$ such that

- *$\sum_{ij} a_{ij}(x,s)\,\xi^i\xi^j \geq \alpha_0\,|\xi|^2$ for every $\xi\in\mathbb{R}^2$, for every s, and a.e. x,*
- *$|a_{ij}(x,s)| \leq \alpha_1$ for every s and a.e. x.*

Given $\psi \in H^1(\Omega_0)$*, we consider the operator*

$$T\colon\ (-\delta,\delta) \to H_0+\psi \qquad s \mapsto u_s\,,$$

where u_s *solves*

$$\begin{cases} u_s-\psi \in H_0\,,\\ -\sum_{ij} \mathrm{D}_i(a_{ij}(x,s)\mathrm{D}_j u_s) = f_s \quad \text{in } H_0'\,. \end{cases} \tag{5.18}$$

Then T *is continuous.*

Proof. Let $s_k \to s$; then $a_{ij}(x,s_k) \to a_{ij}(x,s)$ for a.e. x. Let $u_k := T\,s_k$ and $u := T\,s$. By definition,

$$\sum_{i,j=1}^{2} \int_{\Omega_0} a_{ij}(x,s_k)\,\mathrm{D}_j u_k\,\mathrm{D}_i w\,\mathrm{d}x = \langle f_{s_k}, w\rangle \quad \text{for every } w \in H_0\,, \tag{5.19}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between H_0' and H_0.

Using $w := u_k - \psi$ as test function and recalling the uniform ellipticity of the coefficients and the Poincaré inequality in H_0, we find an a priori bound for $\|u_k-\psi\|_{H_0}$; hence, up to a subsequence, u_k converges to some u^* weakly in $H^1(\Omega_0)$.

The estimate from above for a_{ij} allows us to pass from the pointwise a.e.-convergence of $a_{ij}(x,s_k)\,\mathrm{D}_i w(x)$ to $a_{ij}(x,s)\,\mathrm{D}_i w(x)$, to the strong convergence in $L^2(\Omega_0)$. Therefore, passing to the limit in (5.19), by uniqueness $u^* = u$, and thus the whole sequence converges.

Moreover the convergence is also strong: taking $w = u_k - u \in H_0$ in (5.19), we have

$$\sum_{i,j=1}^{2} \int_{\Omega_0} [a_{ij}(x,s_k)\,\mathrm{D}_j u_k\,\mathrm{D}_i(u_k-u) - a_{ij}(x,s)\,\mathrm{D}_j u\,\mathrm{D}_i(u_k-u)]\,\mathrm{d}x =$$

$$= \langle f_{s_k} - f_s, u_k - u\rangle \to 0\,,$$

hence $\alpha_0 \int_{\Omega_0} |\nabla(u_k-u)|^2\,\mathrm{d}x \to 0$, by uniform ellipticity; as $u_k - u \in H_0$, the Poincaré inequality allows us to conclude that u_k converge to u strongly in $H^1(\Omega_0)$. □

Lemma 5.2.3. *Besides the hypotheses of Lemma 5.2.2 assume that:*

- $s \mapsto a_{ij}(x,s)$ *is differentiable in* 0 *for a.e.* $x \in \Omega_0$*,*
- $s \mapsto f_s$ *is differentiable in* H_0'*,*
- *there exists* $\alpha_2 > 0$ *such that* $|\mathrm{D}_s a_{ij}(x,0)| \le \alpha_2$ *for a.e.* x*.*

Then the partial derivative $\mathrm{D}_s u_0$ *exists, and it solves the equation obtained by deriving formally* (5.18)*. In particular, we have strong convergence for the incremental quotients:*

$$\frac{u_s-u_0}{s} \to \mathrm{D}_s u_0 \quad \text{in } H^1(\Omega_0) \quad \text{as } s \to 0\,.$$

Proof. By (5.18) the incremental quotient $\frac{u_s-u_0}{s}$ satisfies

$$\sum_{i,j=1}^{2}\int_{\Omega_0} a_{ij}(x,0)\,\mathrm{D}_j\left(\frac{u_s-u_0}{s}\right)\mathrm{D}_i w\,\mathrm{d}x = \\ = \left\langle \frac{f_s-f_0}{s}, w\right\rangle - \sum_{i,j=1}^{2}\int_{\Omega_0}\frac{a_{ij}(x,s)-a_{ij}(x,0)}{s}\,\mathrm{D}_j u_s\,\mathrm{D}_i w\,\mathrm{d}x$$

for every $w \in H_0$. We then define the element g_s of H_0' by

$$\langle g_s, w\rangle := \begin{cases} \left\langle \frac{f_s-f_0}{s}, w\right\rangle - \sum_{i,j=1}^{2}\int_{\Omega_0}\frac{a_{ij}(x,s)-a_{ij}(x,0)}{s}\,\mathrm{D}_j u_s\,\mathrm{D}_i w\,\mathrm{d}x & \text{if } s\neq 0\,,\\ \langle \mathrm{D}_s f_0, w\rangle - \sum_{i,j=1}^{2}\int_{\Omega_0}\mathrm{D}_s a_{ij}(x,0)\,\mathrm{D}_j u_0\,\mathrm{D}_i w\,\mathrm{d}x & \text{if } s=0\,. \end{cases}$$

In order to apply Lemma 5.2.2, we note that the coefficients of the left-hand side of the equation satisfy the assumptions, therefore it is enough to prove that $s \mapsto g_s$ is continuous. The continuity in the points $s \neq 0$ is obvious since, by Lemma 5.2.2, $s \mapsto \mathrm{D}_j u_s$ is continuous in $L^2(\Omega_0)$. In the point $s = 0$, it is enough to consider the second term in g_s: we have $\frac{a_{ij}(x,s)-a_{ij}(x,0)}{s} \mapsto \mathrm{D}_s a_{ij}(x,0)$ for a.e. x; using the costant α_2, we conclude by the generalized version of the Dominated Convergence Theorem.

We thus obtain that $\frac{u_s-u_0}{s}$ converges in $H^1(\Omega_0)$ to a function z which solves

$$\sum_{i,j=1}^{2}\int_{\Omega_0} a_{ij}(x,0)\,\mathrm{D}_j z\,\mathrm{D}_i w\,\mathrm{d}x = \langle \mathrm{D}_s f_0, w\rangle - \sum_{i,j=1}^{2}\int_{\Omega_0}\mathrm{D}_s a_{ij}(x,0)\,\mathrm{D}_j u_0\,\mathrm{D}_i w\,\mathrm{d}x\,.$$

This concludes the proof of the lemma. □

Since F_s is regular and the coefficients a_{ij} are Lipschitz continuous, the map $s \mapsto A(x,s)$ is continuous. Moreover, the derivative $D_s A(x,0)$ exists for a.e. $x \in \Omega$. Then we can apply Lemma 5.2.3 and conclude that the map $s \mapsto U_s$ has a derivative $\dot U_0$ in 0. In addition, since f and V have disjoint supports, for every $W \in H_0$ we have

$$\int_{\Omega_0}\nabla \dot U_0(x)\cdot A(x,0)\nabla W(x)\,\mathrm{d}x = -\int_{\Omega_0}\nabla u_0(x)\cdot \mathrm{D}_s A(x,0)\nabla W(x)\,\mathrm{d}x\,.$$

Computing $\mathrm{D}_s A(x,0)$ and substituting in the above equation we obtain that for every $W \in H_0$,

$$\int_{\Omega_0}\nabla \dot U_0\cdot A\nabla W\,\mathrm{d}x = \int_{\Omega_0}\big[\nabla u_0\cdot(\nabla V\,A)\nabla W + \nabla u_0\cdot(A\,\nabla V^{\mathrm{T}})\nabla W + \\ - \nabla u_0\cdot A\nabla W\ \mathrm{div}\ V - \nabla u_0\cdot \mathrm{D}_1 A\nabla W\,V^1\big]\,\mathrm{d}x\,,$$

where $\mathrm{D}_1 A$ indicates the matrix $(\mathrm{D}_1 a_{ij})_{ij}$.

Using $u_s - \psi$ as test function and recalling (5.17), we have

$$\begin{aligned}\mathcal{E}(s) =&\frac{1}{2}\int_{\Omega_s} \nabla u_s \cdot A\nabla\psi \,\mathrm{d}x - \frac{1}{2}\int_{\Omega_s} f(u_s+\psi)\,\mathrm{d}x = \\ =&\frac{1}{2}\int_{\Omega_0} \nabla U_s \cdot A\nabla\psi \,\mathrm{d}x - \frac{1}{2}\int_{\Omega_0} fU_s\,\mathrm{d}x - \frac{1}{2}\int_{\Omega_0} f\psi\,\mathrm{d}x\,;\end{aligned}$$

therefore, using $\dot{U}_0$ and $u_0 - \psi$ as test functions, we obtain that $\mathcal{E}$ is differentiable in 0 with derivative given by

$$\begin{aligned}\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}s}(0) =&\frac{1}{2}\int_{\Omega_0} \nabla\dot{U}_0 \cdot A\nabla\psi\,\mathrm{d}x - \frac{1}{2}\int_{\Omega_0} f\dot{U}_0\,\mathrm{d}x = \frac{1}{2}\int_{\Omega_0}\nabla\dot{U}_0\cdot A\nabla(\psi-u_0)\,\mathrm{d}x = \\ =&-\int_{\Omega_0}(\nabla u_0\nabla V)\cdot A(\nabla u_0)\,\mathrm{d}x + \frac{1}{2}\int_{\Omega_0}\nabla u_0\cdot A(\nabla u_0)\,\mathrm{div}\ V\,\mathrm{d}x+ \\ &+\frac{1}{2}\int_{\Omega_0}\nabla u_0\cdot \mathrm{D}_1 A\nabla u_0\, V^1\,\mathrm{d}x\,,\end{aligned}$$

since the terms containing the derivatives of ψ are null by (5.17). An explicit componentwise computation gives

$$\begin{aligned}\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}s}(0) =& -\int_{\Omega_0}\mathrm{D}_1u_0\,(a_{11}\mathrm{D}_1u_0 + a_{12}\mathrm{D}_2u_0)\,\mathrm{D}_1V^1\,\mathrm{d}x+ \\ &-\int_{\Omega_0}\mathrm{D}_1u_0\,(a_{12}\mathrm{D}_1u_0 + a_{22}\mathrm{D}_1u_0)\,\mathrm{D}_2V^1\,\mathrm{d}x+ \\ &+\frac{1}{2}\int_{\Omega_0}\sum_{i,j=1}^{2} a_{ij}\,\mathrm{D}_ju_0\,\mathrm{D}_iu_0\,\mathrm{D}_1V^1\,\mathrm{d}x+ \\ &+\frac{1}{2}\int_{\Omega_0}\sum_{i,j=1}^{2}\mathrm{D}_1a_{ij}\,\mathrm{D}_ju_0\,\mathrm{D}_iu_0\,V^1\,\mathrm{d}x = \\ =& -\int_{\Omega_0}\mathrm{D}_1V^1\frac{a_{11}(\mathrm{D}_1u_0)^2 - a_{22}(\mathrm{D}_2u_0)^2}{2}\,\mathrm{d}x+ \\ &-\int_{\Omega_0}\mathrm{D}_2V^1\left(a_{12}(\mathrm{D}_1u_0)^2 + a_{22}\mathrm{D}_1u_0\,\mathrm{D}_2u_0\right)\,\mathrm{d}x+ \\ &+\frac{1}{2}\int_{\Omega_0}V^1\sum_{i,j=1}^{2}\mathrm{D}_1a_{ij}\,\mathrm{D}_ju_0\,\mathrm{D}_iu_0\,\mathrm{d}x\,.\end{aligned}$$

As usual in this kind of computation [30], we first integrate on the subset $\Omega_0^\varepsilon := \Omega_0\backslash\overline{B_\varepsilon(0)}$, where ε is chosen so that $V^1 \equiv 1$ in $B_\varepsilon(0)$, and then we pass to the limit as $\varepsilon \to 0$. Then we integrate by parts the first two summands, taking into account the last term, containing the derivatives of a_{ij}. We obtain as volume integral

$$\int_{\Omega_0^\varepsilon} V^1\,\mathrm{D}_1u_0\sum_{i,j=1}^{2}\mathrm{D}_i\left(a_{ij}\mathrm{D}_ju_0\right)\,\mathrm{d}x = 0\,,$$

null because of (5.17). The contribution of $\partial\Omega$ is null, too, since supp V is compact, while on the cut we have $\nu^1 = 0$ and $(a_{12}\mathrm{D}_1u_0 + a_{22}\mathrm{D}_2u_0)\nu^2 = 0$ by the Neumann condition (here, ν denotes the normal to the cut). The only positive term is the one in ∂B_ε, where $V^1 \equiv 1$: we obtain

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}s}(0) = \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon}\left[\frac{a_{11}(\mathrm{D}_1u_0)^2 - a_{22}(\mathrm{D}_2u_0)^2}{2}\nu^1 + \mathrm{D}_1u_0\left(a_{12}\mathrm{D}_1u_0 + a_{22}\mathrm{D}_2u_0\right)\nu^2\right]\mathrm{d}\mathcal{H}^1\,,$$

where $(-\nu^1, -\nu^2) := (-\cos\theta, -\sin\theta)$ is the normal internal vector to B_ε.

Recalling (5.15), we get

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}(0) = \lim_{\varepsilon\to 0}\left(a_\varepsilon + b_\varepsilon + c_\varepsilon\right)\,,$$

where the first summand contains only quadratic terms in the derivatives of S,

$$a_\varepsilon = K_0^2\int_0^{2\pi}\left(\frac{a_{11}(\mathrm{D}_1S)^2 - a_{22}(\mathrm{D}_2S)^2}{2}\cos\theta + \mathrm{D}_1S\left(a_{12}\mathrm{D}_1S + a_{22}\mathrm{D}_2S\right)\sin\theta\right)\varepsilon\,\mathrm{d}\theta\,,$$

the second one contains mixed terms,

$$\begin{aligned}b_\varepsilon = K_0\int_{\partial B_\varepsilon}\Big[&\left(a_{11}\mathrm{D}_1u_0^R\,\mathrm{D}_1S - a_{22}\mathrm{D}_2u_0^R\,\mathrm{D}_2S\right)\cos\theta + \\ &+\left(2a_{12}\mathrm{D}_1u_0^R\,\mathrm{D}_1S + a_{22}\mathrm{D}_1u_0^R\,\mathrm{D}_2S + a_{22}\mathrm{D}_2\,u_0^R\mathrm{D}_1S\right)\sin\theta\Big]\mathrm{d}\mathcal{H}^1\,,\end{aligned}$$

and the third is given by the derivatives of u_0^R,

$$c_\varepsilon = \int_{\partial B_\varepsilon}\left[\frac{a_{11}(\mathrm{D}_1u_0^R)^2 - a_{22}(\mathrm{D}_2u_0^R)^2}{2}\cos\theta + \mathrm{D}_1u_0^R\left(a_{12}\mathrm{D}_1u_0^R + a_{22}\mathrm{D}_2u_0^R\right)\sin\theta\right]\mathrm{d}\mathcal{H}^1\,.$$

Now we show that b_ε and c_ε vanish as $\varepsilon \to 0$, so the only term for the derivative of the energy is a_ε. As for b_ε, as $|\mathrm{D}_kS| \le \frac{1}{2\sqrt{\varepsilon}}$ in ∂B_ε, using the Hölder inequality in L^2 we get

$$|b_\varepsilon| \le \frac{C_1}{\sqrt{\varepsilon}}\int_{\partial B_\varepsilon}\left|\nabla u_0^R\right|\,\mathrm{d}\mathcal{H}^1(x) \le \frac{C_1}{\sqrt{\varepsilon}}\left\|\nabla u_0^R\right\|_{L^2(\partial B_\varepsilon)}|\partial B_\varepsilon|^{\frac{1}{2}} = C_2\left\|\nabla u_0^R\right\|_{L^2(\partial B_\varepsilon)}\,.$$

On the other side, with the Hölder inequality in L^1 we obtain

$$|c_\varepsilon| \le C_3\int_{\partial B_\varepsilon}\left|\nabla u_0^R\right|^2\,\mathrm{d}\mathcal{H}^1(x) + C_3\int_{\partial B_\varepsilon}\left|\mathrm{D}_1u_0^R\right|\left|\mathrm{D}_2u_0^R\right|\,\mathrm{d}\mathcal{H}^1(x) \le C_4\left\|\nabla u_0^R\right\|^2_{L^2(\partial B_\varepsilon)}\,.$$

Hence, we are left to show that $\left\|\nabla u_0^R\right\|_{L^2(\partial B_\varepsilon)} \to 0$ as $\varepsilon \to 0$.

We employ the change of variables $y := \frac{x}{\varepsilon}$ and define $v(y) := u_0^R(\varepsilon y)$; thanks to the continuity of the trace operator, we have

$$\begin{aligned}\int_{\partial B_\varepsilon}\left|\nabla u_0^R\right|^2\,\mathrm{d}\mathcal{H}^1(x) &= \frac{1}{\varepsilon}\int_{\partial B_1}|\nabla v|^2\,\mathrm{d}\mathcal{H}^1(y) \le \\ &\le \frac{C}{\varepsilon}\int_{B_1}\left|\nabla^2 v\right|^2\,\mathrm{d}y + \frac{C}{\varepsilon}\int_{B_1}|\nabla v|^2\,\mathrm{d}y \le \\ &= C\varepsilon\int_{B_\varepsilon}\left|\nabla^2 u_0^R\right|^2\,\mathrm{d}x + \frac{C}{\varepsilon}\int_{B_\varepsilon}\left|\nabla u_0^R\right|^2\,\mathrm{d}x\,.\end{aligned}$$

The Hölder inequality in $L^{\frac{p}{2}}$, with $p > 1$, gives

$$\int_{\partial B_\varepsilon} \left|\nabla u_0^R\right|^2 \, \mathrm{d}\mathcal{H}^1(x) \leq C\varepsilon \left\|\nabla^2 u_0^R\right\|^2_{L^2(B_\varepsilon)} + \frac{C}{\varepsilon} \left\|\nabla u_0^R\right\|^2_{L^p(B_\varepsilon)} \left|B_\varepsilon\right|^{1-\frac{2}{p}} .$$

for $p = 4$, using the absolute continuity of integral we get

$$\int_{\partial B_\varepsilon} \left|\nabla u_0^R\right|^2 \, \mathrm{d}\mathcal{H}^1(x) \leq C\varepsilon \left\|\nabla^2 u_0^R\right\|^2_{L^2(B_\varepsilon)} + C' \left\|\nabla u_0^R\right\|^2_{L^4(B_\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0 .$$

Passing to the limit as $\varepsilon \to 0$ and recalling that $A(0) = I$, through a direct computation we find

$$\lim_{\varepsilon \to 0} a_\varepsilon = -\frac{\pi}{4} K_0^2 ,$$

so we conclude the proof in the case that $\Gamma = \overline{\Omega} \cap \{x_2 = 0\}$, the conormal unit vector coincides with e_2 on Γ, $A(0) = I$, and the force is null in a neighbourhood of 0.

If the domain and the operator have the general form, we employ the diffeomorphism Φ of Section 5.1.1, so we can apply the result just proven: since

$$\frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{\det A(0)}{a_{22}(0)} \, \mathcal{E} \right)(0) = -\frac{\pi}{4} K_0^2 ,$$

we have

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}s}(0) = -\frac{\pi}{4} \frac{\det A(0)}{a_{22}(0)} \, K_0^2 .$$

Finally, the case of a general force is treated by approximation in L^2 with a sequence of forces whose supports are disjoint from 0: indeed, the stress intensity factor is continuous with respect to the convergence of the force in L^2 (see Remark 5.1.5). □

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