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Free Boundary Constant Mean Curvature Hypersurfaces

by

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Chapter 1

Introduction

1.1 Overview

1.1.1 Free boundary CMC hypersurfaces

Letting Ω be a bounded domain in a Riemannian manifold $\mathcal{M}$, we call $\Sigma \subset \mathcal{M}$ a *free boundary Constant Mean Curvature* (CMC for short) hypersurface if it has a non empty boundary with $\partial\Sigma \subset \partial\Omega$ and which intersects $\partial\Omega$ at a constant angle $\gamma \in (0, \pi)$. Such hypersurfaces, called also *stationary capillary* hypersurfaces, are critical points of an energy functional under volume constraint. The energy functional is defined as follows. The surface Σ separates Ω into two parts and consider among these two parts, the one inside which the angle γ is measured, and call Ω' the part of its boundary that lies on $\partial\Omega$. The energy functional is then

$$\Sigma \mapsto \mathcal{E}(\Sigma) := \text{Area}(\Sigma \cap \Omega) - \cos\gamma\, \text{Area}(\Omega').$$

From the physical point of view, when two fluids (at least one a liquid) are adjacent, the free surface of their interface is called a capillary surface. The most interesting questions, then, in stationary capillary problems is the regularity, location and the shape of the surface. In this thesis, we study stationary capillary problems, in which neither fluid is flowing. The study of capillary surfaces is very classical (see R. Finn's book [34]"Equilibrium Capillary Surfaces" for some historical comments) but is still far from settled. The quantity $\cos(\gamma)\,\Omega'$ is interpreted as the *wetting energy* and γ the *contact angle* while $\cos(\gamma)$ is the *relative adhesion coefficient* between the fluid bounded by Σ and Ω'. We have been interested in a configuration in the absence of gravity. A more general setting including the gravitational energy and works on capillary surfaces can be found also in the book by R. Finn.

The geometric problem (referred also as the *partitioning problem* in the literature when $\gamma = \frac{\pi}{2}$) we study here and derived from the Euler-Lagrange associated to the above functional reads as follows: for a given real number H and an angle $\gamma \in (0, \pi)$, find a hypersurface Σ (with prescribed topology) satisfying the following conditions:

$$(GMP)\begin{cases} H_\Sigma \equiv H & \text{in } \Sigma, \\ \partial\Sigma \subset \partial\Omega, \\ \langle N_\Sigma, N_{\partial\Omega}\rangle = \cos\gamma & \text{on } \partial\Sigma, \end{cases}$$

where H_Σ is the mean curvature of Σ and N_Σ (resp. $N_{\partial\Omega}$) is the outer unit normal of Σ (resp. $\partial\Omega$).

The above problem and the boundary-less case have been studied by several authors. For reasons of exposition, in this section we shall cite those whose their work are closer to what is done here. The isoperimetric problem (see the survey of A. Ros [76]) which consists of minimizing the energy functional $\mathcal{E}$ allows to distinguish some special stationary surfaces like, spheres (in $\mathbb{R}^{m+1}$); hemispheres (in the half space $\mathbb{R}^{m+1}_+$), half-spheres, cylinders and unduloids, (in slabs $\mathbb{R}^m \times [0,1]$); hyperplanes through the origin or spherical caps (in a Ball), etc ... On the other hand there are various CMC surfaces satisfying (GMP) which are not necessarily minimizers and not even embedded. Some well know examples are Wente's torus, [92], and also the Delauney surfaces [22]. One can also see the work of Kapouleas [51], Mahmoudi-Pacard-Mazzeo [56, 64], Jeleli [49], Struwe [86, 88, 90], Grüter-Jost [41, 50], etc... Few information on the solutions of the isoperimetric problem are available even though some progress has been done when we are in curved spaces. So, one is led to build CMC surfaces with as a reference model the euclidean space. In the early 90's, motivated by possible applications in general relativity, R. Ye proved the existence of constant mean curvature spheres in Riemannian manifolds concentrating at non-degenerate critical points of the scalar curvature of $\mathcal{M}$. This is (some how) extended recently by Pacard and Xu [73] to possibly degenerate critical points of the scalar curvature by incorporating a variational argument. A new phenomenon was then discovered and analyzed in the pioneering work of Malchiodi and Montenegro [59], namely the existence of solutions to singularly perturbed partial differential equations concentrating along minimal submanifolds. The approach of Malchiodi and Montenegro was also used from a geometric context; to construct (Delauney-type) CMC hypersurfaces condensing along minimal submanifolds by F. Mahmoudi, R. Mazzeo, F. Pacard [56] in the boundary less case.

In this thesis, we build various solutions to the above problem with non-trivial topology by studying (GMP) in a geometric and PDE point of view.

The Free boundary Plateau Problem for H-surfaces

Here we consider surfaces realized as a mapping u over a given domain. Along with its advantages, the definition of a surface as a mapping has certain drawbacks: there is an a priory restriction on the topological complexity and the natural topology lacks compactness properties due to invariance under non-compact class of diffeomorphisms. As a concrete problem, we have the well known non-linear Partial Differential Equations (which is related to (GMP) with $\gamma = \frac{\pi}{2}$) called the Free Boundary Plateau Problem (FBPP) for H-surfaces over the unit disc B of $\mathbb{R}^2$. Namely, if we suppose that $\Sigma \subset \mathbb{R}^3$ is parametrized in isothermal coordinates by a map $u \in C^2(B;\mathbb{R}^3) \cap C^1(\bar{B};\mathbb{R}^3)$ over the unit disc B, then (GMP) becomes the FBPP for H-surfaces:

$$\begin{cases} \Delta u = 2H u_x \wedge u_y & \text{in } B, \\ |u_x|^2 - |u_y|^2 = 0 = u_x \cdot u_y & \text{in } B, \end{cases} \tag{1.1}$$

$$\begin{cases} u(\partial B) \subset \partial\Omega, \\ \frac{\partial u}{\partial n}(\sigma) \perp T_{u(\sigma)}\partial\Omega & \forall \sigma \in \partial B. \end{cases} \tag{1.2}$$

The main features in studying this problem are the functional setting and the invariance by (non-compact) group of conformal transformations of the unit disc. The above system and its parabolic counterpart have been the subject of several works, see for instance the paper [13] by Bürger-Kuwert and also [88] by M.Struwe. The latter generalizes the existence result in [90] and in some sense extends Hildebrandt's work [47] for the Plateau problem for H-surfaces, namely (3.1) with the following boundary condition instead of (1.2)

$$u_{|\partial B} : \partial B \to \Gamma \text{ is a parametrization of a given Jordan curve } \Gamma \subset \mathbb{R}^3. \tag{1.3}$$

For $H = 0$ (1.1), (1.3) constitute the classical Plateau problem for minimal surfaces solved by J.Douglas [25] and T.Radò [74]. Generalizations for $H \neq 0$ were obtained in [47], where the existence of a stable solution was proved. For "small" H, Brezis-Coron [12], K.Steffen [83] and M.Struwe [87], found the existence of unstable solutions as well. These results were

extended in [86] where the following result was established: for $H \neq 0$, there is always an unstable solution of (1.1),(1.3) provided there is a stable solution.
By analogy, since in the free boundary problem stable solutions (trivial solutions) always exist for any H, one could expect unstable solutions to exist for any $H \neq 0$. Furthermore it is not hard to see that the closer a CMC surface is (say, in the Hausdorff metric) to a point, the larger its mean curvature must be. In other words, the mean curvatures of the elements of a condensing family of CMC hypersurfaces must tend to infinity.

Motivated by these facts, we study here the above system for H arbitrarily large. Indeed taking into advantage the variational characterization of this problem, we were able to reduce the problem to finding critical points of some function F_H defined on $\partial\Omega$. Therefore the Lusternik-Schnierelman theory allows then to obtain existence of solutions for any large H. Furthermore for H large, F_H admits an asymptotic expansion involving the mean curvature of $\partial\Omega$. Applying topological degree argument, stable solutions of the mean curvature of $\partial\Omega$ give rise to the existence of H-surfaces.

The H-surfaces u_H found here are embeddings and yield CMC surfaces solving (GMP). Moreover they are similar to hemispheres and concentrating to a point as H becomes large. A natural question is therefore, what about higher dimensional concentrations?

Concentration on minimal submanifolds

We study here also the existence of cylindrical type hypersurfaces in $\Omega \subset \mathbb{R}^{m+1}$. If K is a k-dimensional smooth submanifold of $\partial\Omega$, we consider the "half"-geodesic tube contained in Ω around K of radius 1:

$$S_\varepsilon(K) := \{q \in \bar{\Omega} : \quad d(q, K) = \varepsilon\},$$

with

$$d(q, K) := \sqrt{|\mathrm{dist}^{\partial\Omega}(\tilde{q}, K)|^2 + |q - \tilde{q}|^2}$$

where $\tilde{q}$ is the projection of q on $\partial\Omega$ and

$$\mathrm{dist}^{\partial\Omega}(\tilde{q}, K) = \inf \big\{ \mathrm{length}(\gamma) \quad : \quad \gamma \in C^1([0,1]) \text{ is a geodesic in } \partial\Omega;\ \gamma(0) \in K;\ \gamma(1$$

By the smoothness of $\partial\Omega$ and K, the tube is a smooth, (possibly) immersed, hypersurface provided ε is sufficiently small. This tube by construction satisfies almost (GMP),

$$\begin{cases} H_{S_\varepsilon(K)} &= \frac{m-k}{\varepsilon} + \mathcal{O}(1) \quad \text{in } S_\varepsilon(K), \\ \partial S_\varepsilon(K) &\subset \quad \partial\Omega, \\ \langle N_{S_\varepsilon(K)}, N_{\partial\Omega}\rangle &= \quad 0 \qquad \text{on } \partial S_\varepsilon(K), \end{cases}$$

Hence one naturally expect to be plausible, under some rather mild assumptions on K, that it might be possible to perturb this tube to satisfy the system (GMP). We were able to achieve this only for some special values of ε and provided K is a non-degenerate minimal submanifold. According to our argument, if one were to compare this result with the concentration at points p, then the assumption on p being critical point for the mean curvature is now replaced by the fact that K has to be a minimal submanifold.

Even thought the main ingredients in treating this question is contained in the one dimensional concentrations, here some new bifurcation phenomena appear which prevent to carry out a construction for any small values of ε. This is related to some resonance phenomena peculiar to concentration on positive dimensional sets and it appears in the study of several classes of (geometric) non-linear PDEs.

Minimal disc-type surfaces

This thesis also aims to find capillary minimal surfaces inside some tubular neighborhood Ω_ρ of a given curve Γ in $\mathcal{M}$. First of all we will need to define the special domains Ω_ρ we are working with. We consider the parametric curve $[a, b] \ni s \to (\kappa(s)\,, \phi(s)) \in \mathbb{R}^2$ and the surface of revolution in $\mathbb{R}^{m+1}$, $m \geq 2$ using the standard parametrization

$$S(s, z) = (\kappa(s)\,, \phi(s)\,\Theta(z))\,,$$

where $z \mapsto \Theta(z) \in S^{m-1}$, $\phi(s) \neq 0 \quad \forall s \in [a, b]$. Assuming that the rotating curve is parametrized by arc length namely

$$(\phi'(s))^2 + (\kappa'(s))^2 = 1,$$

clearly the disc $\mathscr{D}_{s,1}$ centered at $(\kappa(s)\,, 0)$ (on the axis of rotation) with radius $\phi(s)$ parametrized by

$$B_1^m \ni x \mapsto (\kappa(s)\,, \phi(s)\,x)\,,$$

solves (GMP) with $H = 0$ and $\gamma = \arccos\,\phi'(s)$.

To extend these definitions of surface of revolution in a Riemannian setting, we let Γ be an embedded curve parametrized by a map $\gamma : [0,1] \to \mathcal{M}$. We consider a local parallel orthonormal frame $E_1, \cdots, E_m$ of $N\Gamma$ along Γ. This determines a coordinate system by

$$[0,1] \times \mathbb{R}^m \ni (x_0, y) \mapsto \overline{F}(x_0, y) := \exp_{\gamma(x_0)}(y^i E_i) \in \mathcal{M}.$$

For a small parameter $\rho > 0$, consider the *Riemannian surface of revolution* $\mathscr{C}^\rho$ around Γ in $\mathcal{M}$ parametrized by

$$(s, z) \longrightarrow \overline{f}(\rho\, S(s, z)) = \overline{F}(\rho\, \kappa(s)\,,\, \rho\, \phi(s)\Theta(z)) = \exp_{\gamma(\rho\,\kappa(s))}(\rho\, \phi(s)\Theta^i(z)\, E_i),$$

where $z \mapsto \Theta(z) \in S^{m-1}$, and call its interior $\Omega_\rho := \operatorname{int} \mathscr{C}^\rho$ which is nothing but a tubular neighborhood for Γ if ρ is small enough. Here we are assuming always that $\phi(s) \neq 0$ and that $(\phi'(s))^2 + (\kappa'(s))^2 = 1$. For any $s \in [a, b]$, if we consider the following set

$$D_{s,\rho} := \overline{F}(\rho\, \kappa(s)\,,\, \rho\, \phi(s)\, B_1^m)$$

then is clear that

$$\begin{cases} H_{D_{s,\rho}} = \mathcal{O}(\rho) & \text{in } D_{s,\rho}, \\ \partial D_{s,\rho} \subset \partial\Omega_\rho, & \\ \langle N_{D_{s,\rho}}, N_{\partial\Omega_\rho} \rangle = \phi'(s) + \mathcal{O}(\rho) & \text{on } \partial D_{s,\rho}. \end{cases}$$

Our aim is to perturb $D_{s,\rho}$ to a capillary minimal submanifold, $\mathscr{D}_{s,\rho}$, of Ω_ρ centered on Γ with contact angle $\arccos \phi'(s)$ along $\partial\mathscr{D}_{s,\rho} \subset \mathscr{C}^\rho$, as it happens in $\mathbb{R}^{m+1}$. We have shown that this is the case when $\phi(s_0)\phi''(s_0) > 0$ and ρ small. Moreover, we have obtained that smaller open sets $O_\rho \subset \Omega_\rho$ can be foliated locally by such minimal disc $\mathscr{D}_{s,\rho}$, in a neighborhood of s_0. Furthermore when we consider the case where Ω_ρ is a geodesic tube ($\phi \equiv 1$ and $\kappa =$Id), in this situation (recall that in this case the angle of contact is $\frac{\pi}{2}$) it is the geometry of the manifold to determine the position of the discs. More precisely, due to invariance by translations along the axis of rotation, we reduced our problem of finding minimal surfaces to a finite-dimensional one where the main term is determined by the Riemann tensor along Γ.

1.1.2 Perimeter minimizing sets

For any measurable subset E of $\mathcal{M}$, we let $\mathcal{P}_g(E,\Omega)$ be the *De Girogi perimeter* (see the book of E. Giusti [36]) of E relative to Ω, defined as

$$\mathcal{P}_g(E,\Omega) := \sup\left\{ \int_E \mathrm{div}_g Y\, dv_g \quad : \quad \langle Y, Y\rangle \leq 1 \right\},$$

where Y is a smooth vector-field on $\mathcal{M}$ with compact support in Ω. Notice that if a set E is smooth then the Gauss-Green formula yields $\mathcal{P}_g(E,\Omega) = \mathrm{Area}(\partial E \cap \Omega)$.

We have been interested by the *isoperimetric profile* of a domain Ω in a Riemannian manifold, namely the mapping

$$v \mapsto I_\Omega(v) := \min_{E\subset\Omega, |E|_g = v} \mathcal{P}_g(E,\Omega).$$

Much of the information concerning the partitioning problem (problem (GMP) with $\gamma = \frac{\pi}{2}$) is contained in the functional I_Ω. Explicit lower bounds for the profile I_Ω are very important in applications and are called *geometric isoperimetric inequalities* for instance see [18] and [19]. If Ω is bounded, the direct methods of the calculus of variation imply that minimizers always exist for any v, their boundaries are smooth and have constant mean curvatures up to a closed set of singularities with high Hausdorff co-dimension 7. Moreover when $\partial\Sigma \cap \partial\Omega \neq \emptyset$ then Σ will meet orthogonally $\partial\Omega$ on $\partial\Sigma \cap \partial\Omega$. Actually up to now the complete description of minimizers has been achieved only in some special cases, one can see for example the survey of A.Ros [76] and the examples cited above.

Perimeter minimizing sets in $\mathcal{M}$ enclosing small volumes have been studied by F. Morgan and D.L. Johnson [69]. They proved that if v is small enough, minimizers of $I_{\mathcal{M}}(v)$ are "smooth" and look asymptotically like spheres. Namely up to scaling, they converges smoothly to spheres (without singularities). Recently, Narduli [72], in his thesis weakened the minimizing property. Moreover he showed that minimizers are located near strict-maxima of the scalar curvature of $\mathcal{M}$.

In 1982, Bérard-Meyer, motivated by the study of nodal domains for Dirichlet eigenvalues, have shown that, in the infinitesimal level, the isoperimetric profile of a compact Riemannian manifold $\mathcal{M}^{m+1}$ approaches that of $\mathbb{R}^{m+1}$. Namely they established that $I_{\mathcal{M}}(v) \sim I_{\mathbb{R}^{m+1}}(v)$ as $v \to 0$. This was adapted by Bayle and Rosales [8] for the relative profile $I_\Omega(v) \sim I_{\mathbb{R}^{m+1}_+}(v)$

as $v \to 0$. The former result has been refined by Druet [27] who gave the first coefficient in the Taylor expansion of $I_{\mathcal{M}}$

$$I_{\mathcal{M}}(v) \sim \left(1 - \alpha_m \max_{p \in \partial\mathcal{M}} S(p)\, v^{\frac{2}{m+1}} + O\left(v^{\frac{4}{m+1}}\right)\right) I_{\mathbb{R}^{m+1}}(v),$$

where α_m is a constant depending only on m and S is the scalar curvature of $\mathcal{M}$. Some applications of this result to the expansion of the Faber-Krahn and Cheeger isoperimetric profile have been recently derived by Druet [26] and the author [32]. We have to mention also that Bayle and Rosales showed, under local convexity assumption of $\partial\Omega$, that $I_\Omega(v) < I_{\mathbb{R}^{m+1}_+}(v)$ for small v.

In this thesis we also study regularity and location of minimizers for $I_\Omega(v)$ with small volumes v. It turns out that the solutions to the isoperimetric problem are smooth up to the (free) boundary and they are located near the strict maxima of the mean curvature of $\partial\Omega$. Our regularity result allows us to derive a Taylor expansion of the relative profile I_Ω given by

$$I_\Omega(v) \sim \left(1 - \beta_m \max_{p \in \partial\Omega} H_{\partial\Omega}(p)\, v^{\frac{1}{m+1}} + O\left(v^{\frac{2}{m+1}}\right)\right) I_{\mathbb{R}^{m+1}_+}(v),$$

where β_m is a constant depending only on m and $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$. From this we derive, as corollaries, some local isoperimetric inequalities involving only the mean curvatures of the obstacle $\partial\Omega$ weakening the convexity of the afore mentioned result to domains Ω with positive boundary mean curvature.

Chapter 2

Preliminary and Notations

In this manuscript, manifolds $(\mathcal{M}^{m+1}, g)$ are assumed to be orientable and complete with metric g and dimension $m+1$ and connection ∇. If there is no confusion, we will use the notation $g(\cdot\,,\,\cdot) = \langle\cdot\,,\,\cdot\rangle$.

Referring to the books of Do Carmo [23] [24], we first, rquickly, recall the definition of the mean curvature for hypersurfaces.

Let Σ be an orientable smooth hypersurfaces of $\mathcal{M}$. For a point $p \in \Sigma$, we let N_Σ a unit vector in $T_pM^\perp$. For X in $T_p\mathcal{M}$, we define the linear mapping $h_\Sigma(X) := \nabla_X N_\Sigma$. The *second fundamental form* of Σ at p is given by $\mathit{II}_p(X) := \langle h_\Sigma(X), X\rangle$ for all $X \in T_p\Sigma$. The operator operator h_Σ is symmetric from $T_p\Sigma \to T_p\Sigma$ hence there exists an orthonormal basis $\{E_1; \ldots ; E_m\}$ of real eigenvectors $k_1, \cdots, k_m$. Notice that N_Σ is uniquely determined if we require that both $\{E_1; \ldots ; E_m\}$ is a basis in the orientation of $T_p\Sigma$, and $\{E_1; \ldots ; E_m; N_\Sigma\}$ is a basis in the orientation of T_pM. The symmetric function of $k_1, \cdots, k_m$ are invariants under immersions representing Σ and are called *principal directions*. The (normalized) *mean curvature* at p of Σ is given by $H_\Sigma(p) := \frac{1}{m}\sum_{i=1}^m \langle h_\Sigma(E_i), E_i\rangle$. Clearly, the sign of H_Σ depends on the choice of the orientation. In this thesis, we will specify, during the computations of mean curvatures of various hypersurfaces, the orientations chosen and also if they are normalized.

2.1 First and second variation of area for capillary hypersurfaces

Letting Ω be a bounded domain in an $(m+1)$-Riemannian manifold $\mathcal{M}$, we call $\Sigma \subset \mathcal{M}$ a *free boundary Constant Mean Curvature* (CMC for short) hypersurface if it has a non empty boundary with $\partial\Sigma \subset \partial\Omega$ and which intersects $\partial\Omega$ at a constant angle $\gamma \in (0, \pi)$. Such hypersurfaces, called also *stationary capillary* hypersurfaces, are critical points of an energy functional under volume constraint. The energy functional is defined as follows. The surface Σ separates Ω into two parts, and consider among these two parts, the one inside which the angle γ is measured, and call Ω'

the part of its boundary that lies on $\partial\Omega$. The energy functional is then

$$\Sigma \mapsto \mathcal{E}(\Sigma) := \text{Area}(\Sigma \cap \Omega) - \cos\gamma \,\text{Area}(\Omega').$$

Since $\Sigma \subset \Omega$separates Ω into two parts, we will call Λ the boundary of one of these parts in $\partial\Omega$. We now recall the first and second variation of the energy $\mathcal{E}$.

2.1.1 First variation of area

Let F_t be a variation of Σ with variation vector field

$$\zeta(p) = \frac{\partial F_t}{\partial t}(p)_{|t=0} \quad \text{for every } p \in \Sigma.$$

A variation is called *admissible* if both $F_t(int\,\Sigma) \subset \Omega$ and $F_t(\partial\Sigma) \subset \partial\Omega$. Let N_Σ be a unit outer normal vector along Σ; H_Σ its *mean curvature* and υ (respectively $\bar{\upsilon}$) be the unit exterior normal vector along $\partial\Sigma$ in Σ (respectively in Λ).

An admissible variation induces hypersurfaces Σ_t and Λ_t. Let $A(t)$ (respectively $T(t)$) be the volume of Σ_t (respectively Λ_t) and $V(t)$ the signed volume bounded by Σ and Σ_t. For a given angle $\gamma \in (0, \pi)$, we consider the total energy

$$\mathcal{E}(t) := A(t) - \cos(\gamma)\, T(t). \tag{2.1}$$

It is well known (see for example [78]) that

$$\mathcal{E}'(0) = -\int_\Sigma mH_\Sigma \langle \zeta, N_\Sigma \rangle dA + \oint_{\partial\Sigma} \langle \zeta, \upsilon - \cos(\gamma)\,\bar{\upsilon} \rangle ds \tag{2.2}$$

and

$$V'(0) = \int_\Sigma \langle \zeta, N_\Sigma \rangle dA. \tag{2.3}$$

A variation is called *volume-preserving* if $V(t) = 0$ for every t. Σ is called *capillary hypersurface* if Σ is stationary for the total energy ($\mathcal{E}'(0) = 0$) for any volume-preserving admissible variation. Consequently if Σ is capillary, it has a constant mean curvature and intersects $\partial\Omega$ with the angle γ in the sense that the angle between the normals of υ and $\bar{\upsilon}$ is γ or equivalently the angle between N_Σ and $N_{\partial\Omega}$ is γ, where $N_{\partial\Omega}$ is the unit outer normal field along $\partial\Omega$.
Physically, in the tree-phase system the quantity $\cos(\gamma)\,T(0)$ is interpreted as the *wetting energy* and γ the *contact angle* while $\cos(\gamma)$ is the *relative adhesion coefficient* between the fluid bounded by Σ and Λ and the walls $\partial\Omega$. Here we are interested in a configuration in the absence of gravity.

2.1.2 The Jacobi operator of Σ

We denote by II_Σ and $\mathit{II}_{\partial\Omega}$ the second fundamental forms of Σ and $\partial\Omega$ respectively. Assume that Σ is a capillary hypersurface. Recall that the Jacobi operator (the linearized mean curvature operator about Σ) is given by the second variation of the total energy functional $\mathcal{E}$. For any volume-preserving admissible variation, we have (see [78] Appendix for the proof)

$$\mathcal{E}''(0) = -\int_\Sigma \left(\omega\Delta_\Sigma\omega + |\mathit{II}_\Sigma|^2\omega^2 + Ric_g(N_\Sigma, N_\Sigma)\omega^2\right) dA + \oint_{\partial\Sigma} \left(\omega\frac{\partial\omega}{\partial\upsilon} - q\,\omega^2\right) ds, \tag{2.4}$$

where Ric_g is the Ricci curvature of $\mathcal{M}$,

$$\omega = \langle\zeta,\, N_\Sigma\rangle \qquad \text{and} \qquad q = \frac{1}{\sin(\gamma)}\,\mathit{II}_{\partial\Omega}(\bar{\upsilon}) + \cot(\gamma)\,\mathit{II}_\Sigma(\upsilon).$$

Here Δ_Σ is Laplace-Beltrami on Σ while Ric_g is the Ricci tensor of $\mathcal{M}$. Since for any smooth ω with $\int_\Sigma \omega dA = 0$ there exits an admissible, volume-preserving variation with variation vector field $\omega\, N_\Sigma$ as a normal part (Barbosa Do Carmo [7]), we have now the Jacobi operator of Σ that we define by duality as

$$\langle \mathfrak{L}_{\Sigma,N_\Sigma}\,\omega, \omega'\rangle := \int_\Sigma \left\{\nabla\omega\nabla\omega' - \left(|\mathit{II}_\Sigma|^2 + Ric_g(N_\Sigma, N_\Sigma)\right)\omega\,\omega'\right\} dA + \oint_{\partial\Sigma} q\,\omega\,\omega' ds.$$

2.1.1. EXAMPLE. *In $\Omega = \mathbb{R}^{m+1}_+$, with $\partial\Omega = \mathbb{R}^m \times \{0\} = \mathbb{R}^n \times \mathbb{R}^k \times \{0\}$. We refer to Section 2.3 below for notations. Let $S^n(\gamma)$ be the n-dimensional spherical cap centered on $\partial\Omega$ and making an angle γ with it. The Jacobi operator of the Capillary cylindrical cup $\mathcal{C}_\gamma := S^n(\gamma) \times \mathbb{R}^k$ around $K := \mathbb{R}^k$ is the following*

$$\langle \mathfrak{L}_{\mathcal{C}_\gamma}\,\omega, \omega'\rangle = -\int_{S^n(\gamma)\times K} \left(\Delta_K\omega + \Delta_{S^n(\gamma)}\omega + n\omega\right)\omega'\, dA + \oint_{\partial S^n(\gamma)\times K} \left(\frac{\partial\omega}{\partial\eta(\gamma)} - \cot(\gamma)\omega\right)$$

2.1.1. REMARK. *Let us observe that any smooth transverse vector field $\hat{N}_\Sigma$ along Σ induces an admissible volume preserving variation. The linearized mean curvature operators $\mathfrak{L}_{\Sigma,N_\Sigma}$ and $\mathfrak{L}_{\Sigma,\hat{N}_\Sigma}$ are linked by*

$$\mathfrak{L}_{\Sigma,\hat{N}_\Sigma}\,\hat{\omega} = \mathfrak{L}_{\Sigma,N_\Sigma}\left(\langle N_\Sigma, \hat{N}_\Sigma\rangle\,\hat{\omega}\right) + m\,\hat{N}^T_\Sigma(H_\Sigma)\,\hat{\omega},$$

where $\hat{N}^T_\Sigma$ is the orthogonal projection of $\hat{N}_\Sigma$ on $T\Sigma$. This shows that $\mathfrak{L}_{\Sigma,\hat{N}_\Sigma}$ is self-adjoint with respect to the inner product

$$\int_\Sigma \hat{\omega}\,\hat{\omega}'\,\langle N_\Sigma, \hat{N}_\Sigma\rangle\, dA.$$

2.2 The free boundary Plateau problem

In particular for regular surfaces in $(\mathcal{M}^{m+1}, g) = (R^3, \cdot)$, we consider Σ parametrized by a mapping $u : \mathcal{U} \to \Sigma$ over an open smooth domain $\mathcal{U} \subset \mathbb{R}^2$. The expression of the metric on Σ is given by

$$E\,dxdx + 2F\,dxdy + G\,dydy,$$

where

$$E = |u_x|^2 = u_x \cdot u_x, \quad F = u_x \cdot u_y, \quad E = |u_y|^2 = u_y \cdot u_y.$$

Note that, denoting by $\wedge$ the exterior product in $\mathbb{R}^3$, one has $u_x \wedge u_y \neq 0$ on $\mathcal{U}$, and hence

$$N_\Sigma = \frac{u_x \wedge u_y}{|u_x \wedge u_y|} \tag{2.5}$$

defines a unit normal vector at $u(x, y)$.

At any point $p = u(x, y)$, the differential $dN_\Sigma\Big|_p : T_p\Sigma \to T_p\Sigma$ defines a symmetric operator. Setting

$$e = u_{xx} \cdot N_\Sigma, \quad f = u_{xy} \cdot N_\Sigma, \quad g = u_{yy} \cdot N_\Sigma,$$

the expression of second fundamental form in the basis $\{u_x, u_y\}$ is

$$e\,dxdx + 2f\,dxdy + g\,dydy.$$

In terms of the first and second fundamental form, the mean curvature is given by

$$2H_\Sigma = \frac{eE - 2fF + gG}{EG - F^2}. \tag{2.6}$$

In problems concerning mean curvatures for parametric surfaces, it is convenient to use conformal parametrizations, since this leads to an equation for the mean curvature that can be handled with powerful tools in functional analysis.

2.2.1. DEFINITION. *Let Σ be a 2-dimensional regular surface in $\mathbb{R}^3$ and let $u : \mathcal{U} \to \Sigma$ be parametrization. Then u is said to be conformal if and only if for every $z \in \mathcal{U}$, the linear map $du(z) : \mathbb{R}^2 \to T_{u(z)}\Sigma$ preserves angles, that is there exists $\lambda(z) > 0$ such that*

$$\langle du(z)[v], du(z)[w]\rangle = \lambda(z)\langle u, w\rangle \quad \textit{for every } u, w \in \mathbb{R}^2. \tag{2.7}$$

Note also that the conformality condition (2.7) can be equivalently written as:

$$|u_x|^2 - |u_y|^2 = 0 = u_x \cdot u_y \qquad \text{in } \mathcal{U}. \tag{2.8}$$

This is equivalent to $E = G$ and $F = 0$ so that

$$2H_\Sigma = \frac{\Delta u \cdot N_\Sigma}{|u_x|^2}. \tag{2.9}$$

On the other hand, differentiating (2.8) with respect to x and y, we can deduce that Δu is orthogonal to both u_x and u_y. Hence, recalling the expression (2.5), we infer that Δu and N_Σ are parallel. Moreover by (2.8), $|u_x \wedge u_y| = |u_x|^2 = |u_y|^2$, and then, from (2.9) it follows that

$$\Delta u = 2H_\Sigma(u)\, u_x \wedge u_y.$$

If moreover we assume that that $u(\partial\mathcal{U}) \subset \partial\Omega$ and intersecting it perpendicularly (in the sense that the outer unit normal of $u(\partial\mathcal{U})$ in $u(\mathcal{U})$ and the outer unit normal of $u(\partial\mathcal{U})$ in $\partial\Omega$ makes an angle equal $\frac{\pi}{2}$, see (2.1.1)), then the tangential derivative of $\frac{\partial u}{\partial t}(\sigma)$, in the direction $t(\sigma) \in T_{u(\sigma)}u(\partial\mathcal{U})$, along $u(\partial\mathcal{U})$ and the normal of $u(\partial\mathcal{U})$ in $\partial\Omega$ form a basis in $\partial\Omega$ and are orthogonal because $\frac{\partial u}{\partial t}(\sigma) \in T_{u(\sigma)}u(\partial\mathcal{U})$. Since u is conformal, we deduce that $\frac{\partial u}{\partial n}(\sigma)$, which is also tangent to Σ is orthogonal to $\frac{\partial u}{\partial t}(\sigma)$. Hence, we finally obtain the free boundary Plateau problem for H-surfaces.

$$\begin{cases} \Delta u = 2H_\Sigma(u) u_x \wedge u_y & \text{in } \mathcal{U}, \\ |u_x|^2 - |u_y|^2 = 0 = u_x \cdot u_y & \text{in } \mathcal{U}, \end{cases} \tag{2.10}$$

$$\begin{cases} u(\partial\mathcal{U}) \subset \partial\Omega, & \\ \frac{\partial u}{\partial n}(\sigma) \perp T_{u(\sigma)}\partial\Omega & \forall \sigma \in \partial\mathcal{U}. \end{cases} \tag{2.11}$$

2.3 The stereographic projection

We will denote by $\mathbf{p} : \mathbb{R}^n \to S^n$ the inverse of the stereographic projection from the south pole. $\mathbf{p} = \left(\mathbf{p}^1, \ldots, \mathbf{p}^n, \mathbf{p}^{n+1}\right)$ is a conformal parametrization of S^n and for any $z = (z^1, \ldots, z^n) \in \mathbb{R}^n$,

$$\begin{aligned} \mathbf{p}(z) &= (z, 1)\,\mu(z) - E_{n+1} \\ &= \left(\frac{2\,z^1}{1+|z|^2}, \ldots, \frac{2\,z^n}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2} \right) \end{aligned}$$

with conformal factor given by

$$\mu(z) := \frac{2}{1+|z|^2}. \tag{2.12}$$

We often use the projection of $\mathbf{p}$ on $\mathbb{R}^n$ and denote it by

$$\tilde{\mathbf{p}}(z) := (z,0)\,\mu(z). \tag{2.13}$$

We collect in the following lemma some properties of the function $\mathbf{p}$ which will be useful later on, we omit here the proof which can be obtained rather easily just using simple computations

2.3.1. Lemma. *For every $i, j, l = 1, \ldots, n$, there hold*

$$\langle \mathbf{p}_i, \mathbf{p}_j \rangle = \mu^2\,\delta_{ij}; \qquad \mathbf{p}_i^{n+1} = -\mu\,\mathbf{p}^i; \qquad \tilde{\mathbf{p}}_i = -\mathbf{p}^i\,\tilde{\mathbf{p}} + \mu\,E_i;$$

$$\langle \mathbf{p}_{ii}, \mathbf{p}_l \rangle = \mu^2\,\mathbf{p}^l - 2\mu^2\,\mathbf{p}^i\,\delta_{il}.$$

Here $\mathbf{p}_i$ and $\mathbf{p}_{ij}$ stand for $\frac{\partial \mathbf{p}}{\partial z^i}$ and $\frac{\partial^2 \mathbf{p}}{\partial z^i \partial z^j}$ respectively.

Recall that the Laplace operator on S^n (embedded in $\mathbb{R}^{n+1}$) can be expressed in terms of the Euclidean one by the formula

$$\Delta_{S^n} = \frac{1}{\mu^2}\left(\Delta_{\mathbb{R}^n} - \mu^2(n-2)\mathbf{p}^k\partial_k\right).$$

Moreover, it is easy to verify that

$$\Delta_{S^n}\mathbf{p} + n\mathbf{p} = 0.$$

It is clear that for any $0 < r \leq 1$ the restriction of $\mathbf{p}$ on B_r^n parametrizes a spherical cap $S^n(r)$, where B_r^n is the ball centered at 0 with radius r. Given $\gamma \in (0,\pi)$, if we let $r^2 = \frac{1-\cos(\gamma)}{1+\cos(\gamma)}$, the image by $\mathbf{p}$ of B_r^n is the spherical cap $S^n(\gamma)$ which intersects the horizontal plane $\mathbb{R}^n + \cos(\gamma)\,E_{n+1}$ and makes an angle γ with it. In particular we denote (henceforth define)

$$\Theta(\gamma) := \mathbf{p}\Big|_{B^n_{r(\gamma)}} - \cos(\gamma)\,E_{n+1}; \qquad \Theta := \Theta(\frac{\pi}{2})$$

$$S_+^n := S^n(\frac{\pi}{2}) = \left\{x = (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} \quad : \quad |x| = 1 \text{ and } x^{n+1} > 0\right\}.$$

For any $0 < r \leq 1$, denote by τ_r the unit outer normal vector of ∂B_r^n, the normal field (not unitary) of $\partial S^n(r)$ in $S^n(r)$ expressed as follows

$$\frac{\partial \mathbf{p}}{\partial \tau_r}\Bigg|_{\partial B_r^n} = \mu\,|\tilde{\mathbf{p}}|\left(\mathbf{p}^{n+1}\frac{\tilde{\mathbf{p}}}{|\tilde{\mathbf{p}}|^2} - E_{n+1}\right)\Bigg|_{\partial B_r^n}.$$

Now when $r^2 = \frac{1-\cos(\gamma)}{1+\cos(\gamma)}$, the unit normal in $S^n(\gamma)$ of $\partial S^n(\gamma)$ is given and denoted by

$$\eta(\gamma) = \cot(\gamma)\,\tilde{\Theta}(\gamma) - \sin(\gamma)\,E_{n+1}, \qquad \text{in particular} \qquad \eta := \eta(\frac{\pi}{2}) = -E_{n+1} \tag{2.14}$$

while the unit normal of $\partial S^n(\gamma)$ in the plane $\mathbb{R}^n + \cos(\gamma)\,E_{n+1}$ is $\frac{\tilde{\Theta}(\gamma)}{|\tilde{\Theta}(\gamma)|}|_{\partial B^n_r}$. Observe that the angle between the two normals $\frac{\tilde{\Theta}(\gamma)}{|\tilde{\Theta}(\gamma)|}$ and $\eta(\gamma)$ is γ along $\partial S^n(\gamma)$, namely since $|\tilde{\Theta}(\gamma)| = \sin(\gamma)$ on ∂B^n_r,

$$\langle \frac{\tilde{\Theta}(\gamma)}{|\tilde{\Theta}(\gamma)|}, \eta(\gamma) \rangle = cos(\gamma) \qquad \text{on } \partial S^n(\gamma).$$

Consider the eigenvalue problem, $u : S^n(\gamma) \to \mathbb{R}$,

$$\begin{cases} \Delta_{S^n(\gamma)} u + nu &= 0 \qquad S^n(\gamma); \\ \dfrac{\partial u}{\partial \eta(\gamma)} &= \cot(\gamma)\, u \qquad \partial S^n(\gamma). \end{cases}$$

It is well known that the only solutions to the interior equation are the degree one homogeneous polynomials on S^n_+, spanned by the $n+1$ components of $\mathbf{p}$. By (2.14) the boundary condition is satisfied only by $\Theta^i(\gamma)$, $i = 1, \cdots, n$. For $\gamma = \frac{\pi}{2}$, consider the eigenvalue problem

$$\begin{aligned} \Delta_{S^n_+} u &= \lambda u \qquad S^n_+, \\ \frac{\partial u}{\partial \eta} &= 0 \qquad \partial S^n_+. \end{aligned}$$

Letting

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$$

be the eigenvalues, up to a reflection, it is well known that $\lambda_k = (n+k-1)$ and the eigenspaces corresponding to $\lambda_0 = 0$ and $\lambda_1 = n$ are

$$\text{span}\,\{1\} \quad \text{and} \quad \text{span}\,\{\Theta^1, \cdots, \Theta^n\} \tag{2.15}$$

respectively. We denote by Π_0 and Π_1 the L^2 projections onto these spaces respectively and we define

$$\Pi := \text{Id} - \Pi_1 - \Pi_0 \qquad \text{and} \qquad \Pi_1^\perp := \Pi_0 + \Pi.$$

We collect some useful properties of the map Θ in the following lemma in $\mathbb{R}^2$. The proof is just simple computations.

2.3.1. LEMMA. *If we denote by $n = (x, y)$ (resp. $t = (-y, x)$) the outer unit normal (resp. tangent) vector of the unit disc B of $\mathbb{R}^2$ then from the notation above, Θ satisfies*

1. $\Delta\Theta = 2\Theta_x \wedge \Theta_y = -2\mu^2\Theta$;
2. $\Theta(\sigma) = (\sigma, 0)$, $\frac{\partial\Theta}{\partial n}(\sigma) = -e_3$, $\frac{\partial\Theta}{\partial t}(\sigma) = (t, 0) = (-y, x, 0)$ $\forall\sigma = (x, y) \in \partial B$;
3. $\frac{1}{2}|\nabla\Theta|^2 = |\Theta_x|^2 = |\Theta_y|^2 = |\Theta_x \wedge \Theta_y| = \mu^2$;
4. $\Theta_x \wedge \Theta = \Theta_y$, $\Theta_y \wedge \Theta = -\Theta_x$;
5. $\int_B \Theta \cdot [f_x \wedge \Theta_y + \Theta_x \wedge f_y] = -\int_B \nabla\Theta \cdot \nabla f \qquad \forall \quad f \in H^{1,2}(B, \mathbb{R}^3)$;
6. $\int_B |\nabla\Theta|^2 = 4\pi$.

2.4 Notations

- Unless otherwise stated, Ω is an open bounded domain of $\mathcal{M}^{m+1}$ with boundary $\partial\Omega$. If $\mathcal{M}^{m+1} = \mathbb{R}^{m+1}$, we some times denote by $S = \partial\Omega$ also $S_\varepsilon = \frac{1}{\varepsilon}S$ and $\Omega_\varepsilon = \frac{1}{\varepsilon}\Omega$ for $\varepsilon > 0$.
- For $1 \le p \le \infty$, $k \in \mathbb{N}$, and $\alpha \in (0, 1)$ let $L^p(B, \mathbb{R}^n)$, $H^{k,p}(B, \mathbb{R}^n)$, $\mathcal{C}^{k,\alpha}(\overline{B}, \mathbb{R}^n)$ denote the usual Lebesgue- Sobolev-Hölder spaces with norms $\|\cdot\|_p$, $\|\cdot\|_{k,p}$, $\|\cdot\|_{2,\alpha}$. In particular we will write $\|\cdot\|_2 = \|\cdot\|$.
- For every $u = (u^1, \ldots, u^n), v = (v^1, \ldots, v^n) \in H^{k,p}(B, \mathbb{R}^n)$, we define

$$u \cdot v = \sum_{i=1}^{n} u^i v^i, \quad \nabla u \cdot \nabla v = u_x \cdot v_x + u_y \cdot v_y.$$

  Also we will write $|u|^2 = u \cdot u$ and $|\nabla u|^2 = u_x \cdot u_x + u_y \cdot u_y$.
- Let $\mathcal{U}$ be smooth domain of $\mathbb{R}^2$ and $k \ge 1$ an integer. Since $\partial\mathcal{U}$ is of class C^∞, covering $\partial\mathcal{U}$ by coordinate charts, one can define the Sobolev spaces $H^{k,p}(\partial\mathcal{U}, \mathbb{R}^n)$ (see [1]; paragraph 7.51) as well as the fractional Sobolev spaces, for any ($s \in \mathbb{R}$), $k < s < k + 1$ and $1 \le p < \infty$, by

$$H^{s,p}(\partial\mathcal{U}, \mathbb{R}^n) = \left\{ u \in H^{k,p}(\partial\mathcal{U}, \mathbb{R}^n) \ : \ \frac{|u(\sigma) - u(\sigma')|}{|\sigma - \sigma'|^{s+\frac{1}{p}}} \in H^{k,p}(\partial\mathcal{U} \times \partial\mathcal{U}, \mathbb{R}^n) \right\},$$

  endowed with the natural norm.
  Now if $1 < p < \infty$, $u \in H^{k,p}(\mathcal{U}, \mathbb{R}^n)$ then the trace of u, $u_{\partial\mathcal{U}}$ belongs

to $H^{k-\frac{1}{p},p}(\partial\mathcal{U},\mathbb{R}^n)$. As a consequence of the trace theorem there exists a constant $C_1 > 0$ depending only on $\mathcal{U}$ such that

$$\|u_{\partial\mathcal{U}}\|_{H^{k-\frac{1}{p},p}} \leq C_1\|u\|_{k,p},$$

and conversely if $v \in H^{k-\frac{1}{p},p}(\partial\mathcal{U},\mathbb{R}^n)$, there exists $u \in H^{k,p}(\mathcal{U},\mathbb{R}^n)$ such that $u_{\mathcal{U}} = v$ on $\mathcal{U}$ and

$$\|u\|_{k,p} \leq C_2\|v\|_{H^{k-\frac{1}{p},p}}$$

for some $C_2 > 0$ depending only on $\mathcal{U}$ (see [1]; paragraph 7.56). For brevity in the sequel we will simply write $u(\sigma)$ instead of $u_{\partial\mathcal{U}}(\sigma)$ for a.e. $\sigma \in \partial\mathcal{U}$ if $u \in H^{k,p}(\mathcal{U},\mathbb{R}^n)$.

Chapter 3

Free boundary Plateau problem for large H-surfaces

Let Ω be a bounded open set of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. This chapter is devoted to the proof of existence of $H-$surfaces supported by $\partial\Omega$ for very large $H \in \mathbb{R}$. Here, by an H-surface parametrized by u and supported by $\partial\Omega$, we mean a map $u \in C^2(B; \mathbb{R}^3) \cap C^1(\bar{B}; \mathbb{R}^3)$ of the unit disc

$$B := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

into $\mathbb{R}^3$ satisfying the following conditions:

$$\begin{cases} \Delta u = 2H u_x \wedge u_y & \text{in } B, \\ |u_x|^2 - |u_y|^2 = 0 = u_x \cdot u_y & \text{in } B, \end{cases} \tag{3.1}$$

$$\begin{cases} u(\partial B) \subset \partial\Omega, \\ \frac{\partial u}{\partial n}(\sigma) \perp T_{u(\sigma)}\partial\Omega & \forall \sigma \in \partial B. \end{cases} \tag{3.2}$$

The main result in this chapter is the following:

3.0.1. Theorem. *Suppose $\Omega \subset \mathbb{R}^3$, is a smooth domain. Suppose $Q_0 \in \partial\Omega$ is a local strict maximum or minimum, or a non-degenerate critical point of the mean curvature of $\partial\Omega$. Then there exists a family u_ε of $\frac{1}{\varepsilon}$-surface supported by $\partial\Omega$ such that u^ε is an embedding from B into Ω. Moreover $\frac{1}{\varepsilon}u^\varepsilon$, suitably translated, converges smoohtly to a hemisphere of radius* 1.

Our next result concerns multiplicity of solutions depending on the topology of $\partial\Omega$, with no assumptions on the mean curvature of the boundary of Ω. Given any smooth function F defined on $\partial\Omega$, we denote by $\lambda_{\partial\Omega}(F)$ the number of critical points of F . Recall that $cat(\partial\Omega)$, the Lusternik-Schnierelman category of $\partial\Omega$, is defined to be the minimal value of $\lambda_{\partial\Omega}(F)$ as $F \in C^\infty(\partial\Omega)$ varies. We refer to [5].

3.0.2. Theorem. *Under the assumption of Theorem 4.0.1, there exists at least $cat(\partial\Omega)$ geometrically distinct $\frac{1}{\varepsilon}$-surfaces supported by $\partial\Omega$.*

3.0.3. Remark. *1. It is worth noticing that, comparing our result with the one of M.Struwe [88], no assumptions on Ω are made. Furthermore our result is "complementary" to Struwe's one in the sense that* *his admissible mean curvatures are bounded while ours are arbitrarily large.*

2. *We believe that it should be possible to extend the result to higher-dimensional $H-$ systems (in this direction see [28]).*

Since we look for solutions with a given asymptotic profile, it is convenient to scale the problem by a factor $\frac{1}{\varepsilon}$: letting $S_\varepsilon = \frac{1}{\varepsilon}\partial\Omega = \frac{1}{\varepsilon}S$, we consider the equivalent problem

$$\begin{cases} \Delta u = 2u_x \wedge u_y & \text{in } B, \\ |u_x|^2 - |u_y|^2 = 0 = u_x \cdot u_y & \text{in } B, \\ u(\partial B) \subset S_\varepsilon, \\ \frac{\partial u}{\partial n}(\sigma) \perp T_{u(\sigma)}S_\varepsilon & \forall \sigma \in \partial B. \end{cases} \tag{3.3}$$

At first glance, as $\varepsilon \to 0$, in the limit we get a plane as a supporting surface, so one is led to consider the *limit* problem

$$\begin{cases} \Delta u = 2u_x \wedge u_y & \text{in } B, \\ |u_x|^2 - |u_y|^2 = 0 = u_x \cdot u_y & \text{in } B, \\ u(\partial B) \subset \mathbb{R}^2 \times \{0\}, \\ \frac{\partial u}{\partial n}(\sigma) \perp \mathbb{R}^2 \times \{0\} & \forall \sigma \in \partial B. \end{cases} \tag{3.4}$$

The latter problem admits a solution Θ, the inverse of the stereographic projection (from the south pole) restricted on B (see (2.3)), and a family of solutions of the form $\Theta \circ g + p$, where $p \in \mathbb{R}^2 \times \{0\}$ and g is any conformal diffeomorphism of the unit disc. It turns out that this set of solutions defines a manifold $\tilde{Z}$ of critical points of the Euler functional I_0 associated to (3.4). It is clear that $\tilde{Z} = G \times \mathbb{R}^2$, where G is the group of Möbius transformations of dimension 3 (see (3.11)).
Thanks to some results already known in the literature (see [17], [63], [48]), we are able to prove that $\tilde{Z}$ is a non-degenerate manifold; that is the tangent space $T_z\tilde{Z}$ of $\tilde{Z}$ at any $z \in \tilde{Z}$ coincides with the kernel of $d^2I_0(z)$. Hence by the Fredholm theorem we can solve (3.3) if we are suitably perpendicular to $T_\Theta\tilde{Z}$ in a suitable sense, see Lemma 3.1.4. This is the key step for a finite dimensional reduction of our problem (see [3], [[4] Section

2.4], [15], [17], [33], [63], [48], [16], [93] for related methods).
As in [88], we take advantage of the variational structure of (3.3). While in [88] it was necessary to impose a topological condition on Ω (in order to define an extension operator on a subclass of Sobolev functions, see Section 3.2) we can *localize* the variational formulation using the smallness of ε, see Lemma 3.2.2.
Because of the free boundary condition in (3.3), a natural set to study the problem are maps of B into $\mathbb{R}^3$ of class $H^{1,2}$ such that ∂B is sent into S_ε (which we call *admissible functions*). The subset of admissible functions with $H^{2,2}$ regularity is a Hilbert manifold, dense in the above set. Looking for solutions close to Θ, reasoning as for the flat case, we impose suitable constraints on the tangent plane of the Hilbert manifold, in order to guarantee a (partial) invertibility of the linearized equation as remarked before. Once we have this, we fully solve the equation with a finite-dimensional reduction.
To begin the procedure, we construct approximate solutions, which are nothing but suitable perturbations of hemispheres which intersect $\partial\Omega$ almost orthogonally. The reduction is done transforming the problem into finding critical points of a functional F_ε defined on S_ε, see Proposition 3.2.11. For ε small, F_ε admits the asymptotic expansions in (3.62), where we see the role played by the mean curvature of $\partial\Omega$.

A similar technique was used by R.Ye [93] to find constant mean curvature surfaces in manifolds, and the approximate solutions were perturbations of geodesic spheres. These surfaces concentrate near non-degenerate critical points of the scalar curvature, see the Remark 3.3.3 for related comments.
One of the main features of performing the Lyapunov-Schmidt reduction for our problem is the action of the Möbius group, which generates some *extra* dimensions in the kernel of the linearized equation. To deal with this problem, we use the invariance of the functional under this action, and show that the gradient of the functional has basically no component in the subspace $T_{Id}G$ of $T_\Theta\tilde{Z}$. Another issue is the regularity of admissible functions: while the variational approach settles naturally in $H^{1,2}$ (where we have coercivity, Fredholm properties, etc...), it is from other points of view convenient to work in $H^{2,2}$ since we have stronger embeddings and the functionals involved are more regular. To handle this, we crucially use

the smallness of ε, the smoothness of $\partial\Omega$ and elliptic regularity estimates, see Lemma 3.2.9.

3.1 Preliminary results

Through this chapter we will identify $\mathbb{R}^2$ by $\mathbb{R}^2 \times \{0\}$ as a subspace of $\mathbb{R}^3$. As anticipted in the previous section, we shall consider the unperturbed problem:

$$\begin{cases} \Delta u = 2u_x \wedge u_y & \text{in } B, \\ |u_x|^2 - |u_y|^2 = 0 = u_x \cdot u_y & \text{in } B, \\ u(\partial B) \subset \mathbb{R}^2, & \\ \frac{\partial u}{\partial n}(\sigma) \perp \mathbb{R}^2 & \forall \sigma \in \partial B. \end{cases} \tag{3.5}$$

We define the Hilbert subspace $\mathcal{H}$ of $H^{1,2}(B;\mathbb{R}^3)$ as

$$\mathcal{H} = \{u \in H^{1,2}(B;\mathbb{R}^3) \ : \ u(\sigma) \in \mathbb{R}^2 \ \text{ for a.e. } \sigma \in \partial B\} = H^{1,2}(B,\mathbb{R}^2) \times H_0^{1,2}(B,\mathbb{R}).$$

For every $u \in \mathcal{H} \cap H^{2,2}(B,\mathbb{R}^3)$, we define the functional:

$$I_0(u) = \frac{1}{2}\int_B |\nabla u|^2 + 2V(u), \tag{3.6}$$

where the volume term V is defined for every $u \in H^{2,2}(B,\mathbb{R}^3)$ by

$$V(u) = \frac{1}{3}\int_B u \cdot (u_x \wedge u_y). \tag{3.7}$$

It turns out that (3.5) is the Euler-Lagrange equation of the functional I_0, namely

3.1.1. Lemma.

$u \in \mathcal{H} \cap H^{2,2}(B,\mathbb{R}^3)$ solves problem (3.5) iff $\langle dI_0(u), v\rangle = 0 \quad \forall v \in \mathcal{H}$.

Proof. We have, integrating by parts

$$\langle dV(u), v\rangle = \int_B (u_x \wedge u_y)\cdot v + \frac{1}{3}\int_{\partial B} (\frac{\partial u}{\partial t} \wedge u)\cdot v, \qquad \forall u \in H^{2,2}(B;\mathbb{R}^3), \ \forall v \in H^{1,2}(B;\mathbb{R}^3) \tag{3.8}$$

where $t(x,y)=(-y,x)$ is the tangent vector at $(x,y)\in\partial B$ to ∂B and $\frac{\partial u}{\partial t}$ is the tangential derivative of u. When $u\in\mathcal{H}\cap H^{2,2}(B,\mathbb{R}^3)$ and $v\in\mathcal{H}$, one has $(\frac{\partial u}{\partial t}\wedge u)\cdot v=0$ a.e. on ∂B since $\frac{\partial u}{\partial t}(\sigma),u(\sigma),v(\sigma)\in\mathbb{R}^2$, for a.e. $\sigma\in\partial B$, so it turns out that for every $u\in\mathcal{H}\cap H^{2,2}(B,\mathbb{R}^3)$

$$\begin{aligned}\langle dI_0(u),v\rangle &= \int_B \nabla u\cdot\nabla v+2\int_B (u_x\wedge u_y)\cdot v \qquad \forall v\in\mathcal{H}\\ &= \int_B[-\Delta u+2u_x\wedge u_y]\cdot v+\int_{\partial B}\frac{\partial u}{\partial n}\cdot v.\end{aligned}$$

Since $H^1_0(B,\mathbb{R}^3)\subset\mathcal{H}$, it follows that a critical point $u\in\mathcal{H}\cap H^{2,2}(B,\mathbb{R}^3)$ of I_0 satisfies the first equation of (3.5) and then $\frac{\partial u}{\partial n}(\sigma)\perp\mathbb{R}^2$ for a.e. $\sigma\in\partial B$ so that

$$\frac{\partial u}{\partial n}(\sigma)\perp\frac{\partial u}{\partial t}(\sigma)\quad\text{ for a.e. }\sigma\in\partial B. \tag{3.9}$$

Now, setting

$$\Phi(x,y)=\left(\left|\frac{\partial u}{\partial n}\right|^2-\left|\frac{\partial u}{\partial t}\right|^2\right)-2i\frac{\partial u}{\partial n}\cdot\frac{\partial u}{\partial t}\quad\text{ for every } n=(x,y),\ t=(-y,x)\in B$$

we see that Φ is holomorphic and by (3.9) is real on ∂B. Therefore by the Cauchy-Riemann equations Φ is constant in $\bar{B}$ but since $\Phi(0,0)=0$, u is conformal. Boundary regularity and strong orthogonality follow from standard elliptic theory, we refer to [35]. ■

Now for every $u,w\in\mathcal{H}\cap H^{2,2}(B,\mathbb{R}^3)$ and $v\in\mathcal{H}$ by similar argument, we have

$$\begin{aligned}3\langle d^2V(u)w,v\rangle &= \int_B w\cdot(v_x\wedge u_y+u_x\wedge v_y)+\int_B v\cdot(w_x\wedge u_y+u_x\wedge w_y)+\int_B u\cdot(v_x\wedge w\\ &= 3\int_B(u_x\wedge w_y+w_x\wedge u_y)\cdot v+\int_{\partial B}(\frac{\partial u}{\partial t}\wedge w+\frac{\partial w}{\partial t}\wedge u)\cdot v\\ &= 3\int_B(u_x\wedge w_y+w_x\wedge u_y)\cdot v\end{aligned}$$

and thus by density for every $u\in\mathcal{H}\cap H^{2,2}(B,\mathbb{R}^3)$ there hold

$$\begin{aligned}\langle d^2I_0(u)w,v\rangle &= \int_B\nabla w\cdot\nabla v+2\int_B v\cdot(w_x\wedge u_y+u_x\wedge w_y)\qquad\forall w,v\in\mathcal{H};\\ \langle d^2I_0(u)v,v\rangle &= \int_B|\nabla v|^2+4\int_B u\cdot(v_x\wedge v_y)\qquad\forall v\in\mathcal{H}.\end{aligned}\tag{3.10}$$

Note that equation (3.5) is invariant under the action of the group of Möbius transformation of the unit disc and by translation in the direction of vectors in the plane. Following [12], up to a reflection with respect to the plane, I_0 has a manifold of critical points generated by the inverse of the stereographic projection Θ from the south pole restricted on B. Namely if we set

$$G = \left\{ g_{\theta,a}(X) = e^{i\theta}\frac{X-a}{1-\bar{a}X}, \quad \theta \in [-\pi,\pi), \quad a=(a_1,a_2)\in B \right\}, \quad (3.11)$$

where in complex notations, $X=(x,y)=x+iy$, then the manifold of critical points is

$$\tilde{Z} = \left\{ \Theta \circ g + \tilde{p}, \qquad g\in G, \quad \tilde{p}\in\mathbb{R}^2 \right\}.$$

We prove that the manifold $\tilde{Z}$ is *non-degenerate*, namely that $T_z\tilde{Z} = Ker\ d^2I_0(z)$ for all $z\in\tilde{Z}$ where $T_z\tilde{Z}$ denotes the tangent space of $\tilde{Z}$ at z. We first characterize explicitly $T_\Theta\tilde{Z}$.

3.1.2. LEMMA. *In the above notations we have*

$$T_\Theta\tilde{Z} = span\left\{ \Theta\wedge e_3;\ (e_1\cdot\Theta)\Theta;\ (e_2\cdot\Theta)\Theta;\ e_1;\ e_2 \right\}.$$

PROOF. By easy computations one finds

$$\begin{aligned}
\frac{\partial\Theta\circ g_{\theta,(0,0)}}{\partial\theta}_{|\theta=0} &= e_3\wedge\Theta,\\
\frac{1}{2}\frac{\partial\Theta\circ g_{0,(a_1,0)}}{\partial a_1}_{|a_1=0} &= (e_1\cdot\Theta)\Theta - e_1,\\
\frac{1}{2}\frac{\partial\Theta\circ g_{0,(0,a_2)}}{\partial a_2}_{|a_2=0} &= (e_2\cdot\Theta)\Theta - e_2,\\
\frac{\partial(\Theta+\tilde{p})}{\partial\tilde{p}_i}_{|p_i=0} &= e_i, \quad i=1,2.
\end{aligned}$$

The lemma then follows immediately. ∎

We fix the following notations

$$\begin{gathered}
E_1 = e_3\wedge\Theta, \quad E_2 = (e_1\cdot\Theta)\Theta, \quad E_3 = (e_2\cdot\Theta)\Theta\\
G_\Theta = span\ \{E_1;\ E_2;\ E_3\}.
\end{gathered} \quad (3.12)$$

The above result can be restated in the following way

$$T_\Theta\tilde{Z} = \left\{ \begin{pmatrix} c_1\\ c_2\\ 0\end{pmatrix} + \begin{pmatrix} 0\\ 0\\ b_3\end{pmatrix}\wedge\Theta + \left(\begin{pmatrix} a_1\\ a_2\\ 0\end{pmatrix}\cdot\Theta\right)\Theta, \qquad c_i,a_i,b_3\in\mathbb{R},\ i=1,2 \right\}. \quad (3.13)$$

We are now ready to prove the non-degeneracy condition which plays here a key role, we shall state the following

3.1.3. Lemma. *The following equality holds*

$$T_\Theta \tilde{Z} = Ker\ d^2 I_0(\Theta).$$

Proof. It is enough to show that $T_\Theta \tilde{Z} \supseteq Ker\ d^2 I_0(\Theta)$ since the reverse inclusion always holds true. Let us first emphasize that, in view of (3.10), by partial integration $w \in ker d^2 I_0(\Theta)$ if and only if it satisfies the following equation

$$\begin{cases} \Delta w = 2(w_x \wedge \Theta_y + \Theta_x \wedge w_y) & \text{in } B, \\ w(\partial B) \subset \mathbb{R}^2, \\ \frac{\partial w}{\partial n}(\sigma) \perp \mathbb{R}^2 & \forall \sigma \in \partial B. \end{cases} \tag{3.14}$$

Equivalently after inverse of the stereographic projection on the sphere S^2_+, the first equation in (3.14) becomes

$$\Delta_{g_0} w = \frac{2}{\sin\phi}(w_\phi \wedge \Theta_\theta + \Theta_\phi \wedge w_\theta) \quad \text{in } S^2_+, \tag{3.15}$$

where $0 \le \theta \le 2\pi$, $0 \le \phi \le \frac{\pi}{2}$ are the spherical coordinates on the half sphere S^2_+ and Δ_{g_0} is the Laplacian with respect to the standard metric on S^2_+.

We shall extend Θ and w to the whole sphere S^2. We may write $\Theta(\phi,\theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) = (x_1, x_2, x_3)$, $\phi \in [0, \frac{\pi}{2})$ and define:

$$\begin{aligned} \tilde{\Theta}(x_1,x_2,x_3) &= (x_1,x_2,x_3) \quad \text{if } 0 \le \phi \le \frac{\pi}{2}, \\ \tilde{\Theta}(x_1,x_2,x_3) &= (x_1,x_2,-x_3((\pi-\phi),\theta)) \quad \text{if } \frac{\pi}{2} \le \phi \le \pi. \end{aligned}$$

$\tilde{\Theta}$ is nothing but the inverse of the stereographic projection. Similarly we also extend $w(x_1,x_2,x_3) = (w^1(x_1,x_2,x_3), w^2(x_1,x_2,x_3), w^3(x_1,x_2,x_3))$ on S^2 by

$$\begin{aligned} \tilde{w} &= (w^1(x_1,x_2,x_3), w^2(x_1,x_2,x_3), w^3(x_1,x_2,x_3)) \quad \text{if } 0 \le \phi \le \frac{\pi}{2}, \\ \tilde{w} &= (w^1(x_1,x_2,-x_3), w^2(x_1,x_2,-x_3), -w^3(x_1,x_2,-x_3)) \quad \text{if } \frac{\pi}{2} \le \phi \le \pi. \end{aligned}$$

Clearly $\tilde{w} \in H^{1,2}(S^2)$ and satisfies

$$\Delta_{g_0}\tilde{w} = \frac{2}{\sin\phi}(\tilde{\Theta}_\phi \wedge \tilde{w}_\theta + \tilde{w}_\phi \wedge \tilde{\Theta}_\theta) \quad \text{on } S^2. \tag{3.16}$$

Now by a result in [17] Lemma 9.2 or [63] Proposition 3.1

$$\tilde{w} = c + b \wedge \tilde{\Theta} + (a \cdot \tilde{\Theta})\tilde{\Theta}, \qquad \text{for some } a, b, c \in \mathbb{R}^3.$$

Now since $\tilde{w} = w$ on S^2_+, returning on the plane, we infer that

$$w = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \wedge \Theta + \left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \Theta \right) \Theta, \qquad \text{on } B.$$

The fact that $w \in \mathcal{H}$, implies that $c_3 = b_1 = b_2 = 0$, as well as the orthogonality condition in (3.14) implies that $a_3 = 0$. From this we see that w is of the form as in (3.13). ■

As mentioned before, equation (3.1)-(3.2) is invariant under the non-compact group of conformal transformations of the unit disc and therefore it is impossible for the Palais-Smale condition to be satisfied. A convenient way to factor out the symmetry group could be to impose a three-point-condition on admissible functions, for instance see [89]. In our case the boundary data are allowed to vary freely on $\partial\Omega$, so we shall normalize the admissible functions by imposing integral constraints, restricting ourselves to the following Hilbert space

$$\mathcal{H}_n = \left\{ u \in H^{1,2}(B; \mathbb{R}^3) \; : \; \int_B \nabla u \cdot \nabla E_i = 0, \quad i = 1, 2, 3 \right\}.$$

Now let $Z = \tilde{Z} \cap \mathcal{H}_n = \left\{ \Theta + \tilde{p}, \quad \tilde{p} \in \mathbb{R}^2 \right\}$ and also letting

$$(T_\Theta Z)^\perp = \left\{ v \in \mathcal{H} \; : \; \langle v, e_i \rangle_{1,2} = \int_B v^i = 0, \quad i = 1, 2 \right\}; \qquad W_\Theta = (T_\Theta Z)^\perp \cap \mathcal{H}_n, \tag{3.17}$$

we see that $\mathcal{H}$ is decomposed as

$$\mathcal{H} = T_\Theta Z \oplus (T_\Theta Z)^\perp = T_\Theta Z \oplus W_\Theta \oplus G_\Theta. \tag{3.18}$$

Since every $v \in (T_\Theta Z)^\perp \subset \mathcal{H}$ satisfies $v_3 \in H^{1,2}_0(B, \mathbb{R})$ by Poincaré inequality, the space W_Θ endowed with the norm $\|\nabla v\|$ is Hilbert and moreover if we impose orthogonality to Θ, $d^2 I_0(\Theta)$ becomes coercive on W_Θ, namely the following result holds true (the proof is similar to the one in [48], Lemma 5.5).

3.1.4. Lemma. *There exists a constant $C > 0$ such that*

$$\begin{aligned} \langle d^2 I_0(\Theta) v, v \rangle &\geq C \|\nabla v\|^2 \qquad \forall v \in W_\Theta \quad \textit{with} \int_B \nabla v \cdot \nabla \Theta = 0, \\ \langle d^2 I_0(\Theta)\Theta, \Theta \rangle &= -4\pi. \end{aligned}$$

3.2 The abstract method

We start with some preliminaries. Let us consider the (signed) distance function defined by

$$d(\tilde{X}) := \begin{cases} \mathrm{dist}(\tilde{X}, S) & \text{if } \tilde{X} \in \Omega, \\ -\mathrm{dist}(\tilde{X}, S) & \text{if } \tilde{X} \in \mathbb{R}^3 \setminus \Omega. \end{cases}$$

For some small $r_0 > 0$ depending on $S = \partial\Omega$, it is well known (for instance see [35] 14.6 Appendix) that d is as smooth as S where

$$\Sigma_{r_0} := \left\{ \tilde{X} \in \mathbb{R}^3 \ : \ |d(\tilde{X})| < 2r_0 \right\}.$$

If $q \in S$, then up to a rotation (depending on q), we may assume that T_qS coincides with $\mathbb{R}^2$ and e_3 with the inner unit normal at q. Moreover letting $B_{r_0}(q) = r_0 B + q$, we can assume that $S \cap B_{r_0}(q) - q$ is the graph of some smooth function φ^q satisfying $\varphi^q(0,0) = 0$ and $d\varphi^q(0,0) = 0$, with Taylor expansion

$$\varphi^q(X) = \frac{1}{2}\langle h_q X, X\rangle + O(|X|^3) \qquad \forall X = (x,y) \text{ with } |X| < r_0.$$

Here h_q (the second fundamental form of S at q) is the Hessian matrix of φ^q at $(0,0)$. Similarly, one also has (see [35] 14.6 in the Appendix)

$$d(\tilde{X}) = e_3 \cdot \tilde{X} + \frac{1}{2}\langle \tilde{h}_q \tilde{X}, \tilde{X}\rangle + O(|\tilde{X}|^3) \qquad \forall \tilde{X} \in B_{r_0},$$

where

$$\tilde{h}_q = \begin{pmatrix} -h_q & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.19}$$

The mean curvature of $\partial\Omega$ at q is given by $H_{\partial\Omega}(q) = \frac{1}{2}\mathrm{tr} h_q$.

Let $\varphi^{\varepsilon,q}(X) = \frac{1}{\varepsilon}\varphi^q(\varepsilon X)$, so $S_\varepsilon \cap B_{\frac{r_0}{\varepsilon}}(p) - p$ is the graph of $\varphi^{\varepsilon,q}$, with $p = \frac{1}{\varepsilon}q$ and $d^\varepsilon(\tilde{X}) = \frac{1}{\varepsilon}d(\varepsilon\tilde{X})$. Then we have

$$\varphi^{\varepsilon,q}(X) = \frac{\varepsilon}{2}\langle h_q X, X\rangle + \varepsilon^2 O(|X|^3) \qquad \forall X = (x,y) \text{ with } |X| < \frac{r_0}{\varepsilon}, \tag{3.20}$$

and moreover

$$d^\varepsilon(\tilde{X}) = e_3 \cdot \tilde{X} + \frac{\varepsilon}{2}\langle \tilde{h}_q \tilde{X}, \tilde{X}\rangle + \varepsilon^2 O(|\tilde{X}|^3) \qquad \forall \tilde{X} \in B_{\frac{r_0}{\varepsilon}}. \tag{3.21}$$

The inner normal of $S_\varepsilon = \frac{1}{\varepsilon}S$ at the point $p + (X, \varphi^{\varepsilon,q}(X))$ has the following expansions:

$$N_\varepsilon(X) = (-\nabla\varphi^{\varepsilon,q}(X), 1) = (-\varepsilon h_q X, 1) + \varepsilon^2 O(|X|^2). \tag{3.22}$$

3.2.1 Functional setting

Admissible functions. The class on which we will study problem (3.3) is

$$\mathcal{M}(S_\varepsilon) = \left\{u \in H^{1,2}(B, \mathbb{R}^3) \ : \ u(\partial B) \subset S_\varepsilon \text{ a.e.}\right\}.$$

For $u \in \mathcal{M}(S_\varepsilon)$, we will also define the Hilbert subspace of $H^{1,2}(B, \mathbb{R}^3)$,

$$\mathcal{M}_u(S_\varepsilon) = \left\{v \in H^{1,2}(B, \mathbb{R}^3) \ : \ v(\sigma) \in T_{u(\sigma)} S_\varepsilon \text{ a.e. } \sigma \in \partial B\right\}.$$

Note that the subclass of $\mathcal{M}(S_\varepsilon)$ defined by

$$\mathcal{M}_2(S_\varepsilon) = \mathcal{M}(S_\varepsilon) \cap H^{2,2}(B, \mathbb{R}^3)$$

is dense in $\mathcal{M}(S_\varepsilon)$ and it is a Hilbert manifold (while $\mathcal{M}(S_\varepsilon)$ is not) with tangent space at $u \in \mathcal{M}_2(S_\varepsilon)$ given by

$$T_u\mathcal{M}_2(S_\varepsilon) = \left\{v \in H^{2,2}(B, \mathbb{R}^3) \ : \ v(\sigma) \in T_{u(\sigma)} S_\varepsilon \quad \forall \sigma \in \partial B\right\} = \mathcal{M}_u(S_\varepsilon) \cap H^{2,2}(B, \mathbb{R}^3)$$

which is also dense in $\mathcal{M}_u$ by [80].
Since we are dealing with free boundary surfaces, in order to have a functional whose Euler-Lagrange equations are (3.3), following [88] one can correct the term $V(u)$ by subtracting the volume of some surface $\tilde{u}$ contained in S_ε and depending on u. First of all we define

$$\tilde{\mathcal{M}}(S_\varepsilon) = \{\tilde{u} \in \mathcal{M}(S_\varepsilon) \ : \ \tilde{u}(B) \subset S_\varepsilon \text{ a.e. }\} \quad \text{and} \quad \tilde{\mathcal{M}}_2(S_\varepsilon) = \tilde{\mathcal{M}}(S_\varepsilon) \cap H^{2,2}(B, \mathbb{R}^3)$$

with

$$T_{\tilde{u}}\tilde{\mathcal{M}}_2(S_\varepsilon) = \left\{\tilde{v} \in H^{2,2}(B, \mathbb{R}^3) \ : \ \tilde{v}(X) \in T_{\tilde{u}(X)}\tilde{\mathcal{M}}_2(S_\varepsilon) \quad \forall X \in \bar{B}\right\}.$$

Recall that an *extension* of $u \in \mathcal{M}(S_\varepsilon)$ is a map $\tilde{u} \in \tilde{\mathcal{M}}(S_\varepsilon)$ such that $u = \tilde{u}$ on ∂B and an *extension operator* is a map $\eta_\varepsilon : \mathcal{D}(\eta_\varepsilon) \subset \mathcal{M}(S_\varepsilon) \to \tilde{\mathcal{M}}(S_\varepsilon)$ with open domain $\mathcal{D}(\eta_\varepsilon)$ such that $\eta_\varepsilon(u)$ is an extension of u for all $u \in \mathcal{D}(\eta_\varepsilon)$ and smooth restriction $\eta_\varepsilon : \mathcal{D}(\eta_\varepsilon) \cap \mathcal{M}_2(S_\varepsilon) \to \tilde{\mathcal{M}}_2(S_\varepsilon)$.

3.2.1. LEMMA. *Let $\bar{u} \in H^{1,2}(B, \mathbb{R}^3)$ be an harmonic map. There holds*

$$\sup_{X \in B} \mathit{dist}\,(\bar{u}(X), \bar{u}(\partial B)) \leq \frac{1}{\sqrt{\pi}} \|\nabla \bar{u}\|.$$

PROOF. It will be enough to prove

$$\operatorname*{ess\,inf}_{\sigma \in \partial B} |\bar{u}(0) - \bar{u}(\sigma)| \leq \frac{1}{\sqrt{\pi}} \|\nabla \bar{u}\| \tag{3.23}$$

because for every $X \in B$, there exists a conformal diffeomorphism (in G, see (3.11)) $g : B \to B$ such that $g(X) = 0$ and $g(\partial B) = \partial B$, therefore we may replace $\bar{u}$ with $\bar{u} \circ g^{-1}$ thanks to the conformal invariance of the Laplace equation.

By the mean value property of harmonic functions we have

$$\bar{u}(0) = \frac{1}{2\pi} \int_{\partial B} \bar{u}(\sigma) d\sigma,$$

by Hölder inequality and again by the mean value property

$$\begin{aligned} \operatorname{ess} \inf_{\sigma' \in \partial B} |\bar{u}(0) - \bar{u}(\sigma')| &\leq \frac{1}{2\pi} \int_{\partial B} |\bar{u}(0) - \bar{u}(\sigma')| d\sigma' \\ &\leq \frac{1}{(2\pi)^2} \int_{\partial B} \int_{\partial B} |\bar{u}(\sigma) - \bar{u}(\sigma')| d\sigma d\sigma' \\ &\leq \frac{1}{(2\pi)^2} \left(\int_{\partial B} \int_{\partial B} |\sigma - \sigma'|^2 d\sigma d\sigma' \right)^{\frac{1}{2}} \left(\int_{\partial B} \int_{\partial B} \frac{|\bar{u}(\sigma) - \bar{u}(\sigma')|^2}{|\sigma - \sigma'|^2} \right. \end{aligned}$$

Since

$$\int_{\partial B} \int_{\partial B} |\sigma - \sigma'|^2 d\sigma d\sigma' = 8\pi^2 \text{ and } \int_{\partial B} \int_{\partial B} \frac{|\bar{u}(\sigma) - \bar{u}(\sigma')|^2}{|\sigma - \sigma'|^2} d\sigma d\sigma' \leq 2\pi \int_B |\nabla \bar{u}|^2,$$

we get the result. ■

We define

$$\bar{\mathcal{M}}(S_\varepsilon) = \left\{ u \in \mathcal{M}(S_\varepsilon) \ : \ \|\nabla u\| < \frac{r_0}{\varepsilon} \sqrt{\pi} \right\},$$

and also

$$\bar{\mathcal{M}}_2(S_\varepsilon) = \bar{\mathcal{M}}(S_\varepsilon) \cap H^{2,2}(B, S_\varepsilon).$$

We now state the following result which is in some sense a localized version of Lemma 2.1 in [88].

3.2.2. LEMMA. *For every $\varepsilon > 0$, there exists an extension operator η_ε with domain $\mathcal{D}(\eta_\varepsilon) = \bar{\mathcal{M}}(S_\varepsilon)$.*

PROOF. Let $u \in \bar{\mathcal{M}}(S_\varepsilon)$ and $\bar{u}$ denote its harmonic extension of u,

$$\begin{cases} \Delta \bar{u} = 0 & \text{in } B, \\ \bar{u} = u & \text{on } \partial B. \end{cases}$$

By standard elliptic regularity, $\bar{u}$ is as smooth as u and depends smoothly on u in the $H^{1,2}$-topology (in fact it is linear in u). Let $X \in B$ then

$$\operatorname{dist}(\bar{u}(X), S_\varepsilon) \leq |\bar{u}(X) - \bar{u}(\sigma')| + \operatorname{dist}(\bar{u}(\sigma'), S_\varepsilon) \qquad \text{for a.e. } \sigma' \in \partial B.$$

The second term of the right hand side is zero since $\bar{u}(\partial B) \subset S_\varepsilon$, hence from the above lemma and the fact that $\|\nabla \bar{u}\| \leq \|\nabla u\|$ we get

$$\operatorname{dist}(\bar{u}(X), S_\varepsilon) \leq \operatorname{ess}\inf_{\sigma \in \partial B} |\bar{u}(X) - \bar{u}(\sigma)| \leq \frac{1}{\sqrt{\pi}} \|\nabla u\| \leq \frac{r_0}{\varepsilon}.$$

Consequently, by the regularity of S_ε, we can project (pointwise) $\bar{u}$ on S_ε to obtain a unique extension $\tilde{u}$ defined by the following implicit equation: for every $X \in B$,

$$\tilde{u}(X) = \bar{u}(X) - \nu^\varepsilon(\tilde{u}(X)) d^\varepsilon(\bar{u}(X)), \tag{3.24}$$

where $\nu^\varepsilon(p)$ is the inner unit normal of S_ε at a point $p \in S_\varepsilon$. Moreover the mapping $u \to \bar{u} \to \tilde{u}$ defines an extension operator η_ε with domain $\mathcal{D}(\eta_\varepsilon) = \bar{\mathcal{M}}(S_\varepsilon)$. ∎

We notice that, in fact, η_ε is defined on $\bar{\mathcal{M}}(S_\varepsilon) + H_0^{1,2}(B, \mathbb{R}^3)$ and

$$\eta_\varepsilon(u + \varphi) = \eta_\varepsilon(u) \qquad \forall \varphi \in H_0^{1,2}(B, \mathbb{R}^3).$$

Since $\eta_\varepsilon(u(\sigma)) = u(\sigma)$ for all $\sigma \in \partial B$, one has that

$$\langle d\eta_\varepsilon(u(\sigma)), \tilde{v}(\sigma) \rangle = \tilde{v}(\sigma) \ \forall \sigma \in \partial B, \quad \forall \tilde{v} \in T_{\tilde{u}} \tilde{\mathcal{M}}_2(S_\varepsilon).$$

Moreover since $u = \eta_\varepsilon(u) = \tilde{u}$ on ∂B, $\frac{\partial u}{\partial t} = \frac{\partial \tilde{u}}{\partial t}$ a.e. on ∂B and so by integration by parts, for every $u \in \bar{\mathcal{M}}_2(S_\varepsilon)$ we have

$$\langle dV \circ \eta_\varepsilon(u), v \rangle = \frac{1}{3} \int_{\partial B} (\frac{\partial u}{\partial t} \wedge u) \cdot v ds, \qquad \forall v \in T_u \bar{\mathcal{M}}_2(S_\varepsilon) = H_0^{1,2}(B, \mathbb{R}^3) \cap H^{2,2}(B, \mathbb{R}^3) + \tag{3.25}$$

Now we define for every $u \in \bar{\mathcal{M}}_2(S_\varepsilon)$,

$$I_\varepsilon(u) = \frac{1}{2} \int_B |\nabla u|^2 + 2[V(u) - V \circ \eta_\varepsilon(u)]. \tag{3.26}$$

3.2.3. Remark. *For u smooth, it is clear that the term $[V(u) - V \circ \eta_\varepsilon(u)]$ represents the volume of the set bounded by the image u and ∂S_ε. As already explained in the introduction, M.Struwe in [88] needed to impose some conditions on Ω to define the extension η. In our case instead, since we look for solutions with bounded energy as $\varepsilon \to 0$, no restriction on Ω is needed.*

Note that by (3.25) and (3.8), the differential of I_ε at a point $u \in \bar{\mathcal{M}}_2(S_\varepsilon)$ is independent of η_ε, namely we have by density

$$\langle dI_\varepsilon(u), v\rangle = \int_B \nabla u \cdot \nabla v + 2\int_B (u_x \wedge u_y)\cdot v \qquad \forall v \in \mathcal{M}_u(S_\varepsilon)$$

and

$$\langle d^2 I_\varepsilon(u)w, v\rangle = \int_B \nabla w \cdot \nabla v + 2\int_B (w_x \wedge u_y + u_x \wedge w_y)\cdot v \qquad \forall v, w \in \mathcal{M}_u(S_\varepsilon).$$

Hence I_ε is smoothly defined on $\bar{\mathcal{M}}_2(S_\varepsilon)$ moreover (see [88] Lemma 2.2) it easily follows the

3.2.4. Lemma. *Let $u \in \bar{\mathcal{M}}_2(S_\varepsilon)$, then*

$$\langle dI_\varepsilon(u), v\rangle = 0, \quad \forall v \in T_u\bar{\mathcal{M}}_2(S_\varepsilon) \quad \textit{iff} \quad u \textit{ solves problem (3.3).}$$

3.2.2 Construction of approximate solutions

We start by proving the following technical lemma.

3.2.5. Lemma. *Let $T = (T_{ij})$ be a 2×2 symmetric matrix, and consider the following problem*

$$\begin{cases} L\omega = -L(\frac{1}{2}\langle TX, X\rangle e_3), & \textit{in } B, \\ \omega(\partial B) \subset \mathbb{R}^2, \\ \frac{\partial \omega}{\partial n}(X) = (TX, 0) & X \in \partial B, \end{cases} \tag{3.27}$$

where L is the operator

$$Lu = -\Delta u + 2[u_x \wedge \Theta_y + \Theta_x \wedge u_y].$$

Then (3.27) admits a solution $\omega_T \in \mathcal{H} \cap C^\infty(\bar{B}, \mathbb{R}^3)$ which satisfies

$$\|\omega_T\|_{2,2} \le C|T|_\infty, \tag{3.28}$$

where C is a fixed positive constant.

Proof. Problem (3.27) can be reformulated as

$$d^2 I_0(\Theta)[\omega] = g_T, \tag{3.29}$$

where $g_T \in \mathcal{H}$ is defined by duality as

$$\langle g_T, v\rangle = \int_B (tr\ Te_3 - 2[(T_1X)e_3 \wedge \Theta_y + (T_2X)\Theta_x \wedge e_3])\cdot v - \int_{\partial B}(TX, 0)\cdot v \quad \forall v \in \mathcal{H},$$

where T_i, $i = 1, 2$, denote the rows of the matrix T. Observing that the operator $d^2 I_0(\Theta)$ is of the form "Identity+compact". Thanks to Fredholm theorem and Lemma 3.1.3 problem (3.29) is solvable if and only if g_T is orthogonal to the vectors e_i, and E_j for $i = 1, 2$, $j = 1, 2, 3$. We have, by symmetry of T,

$$\begin{aligned} -2[(T_1 X)e_3 \wedge \Theta_y + (T_2 X)\Theta_x \wedge e_3] &= \mu^2 \left(x(1 + x^2 - y^2)T_{11} + y(1 + 3x^2 - y^2)T_{12} + \right. \\ &+ \mu^2 \left(y(1 + y^2 - x^2)T_{22} + x(1 + 3y^2 - x^2)T_{12} + \right. \end{aligned}$$

Since $E_1 = x\mu\Theta$ and $E_2 = y\mu\Theta$, by oddness, we have that $\langle h_T, e_i\rangle = \langle h_T, E_j\rangle = 0$ for $i, j = 1, 2$. Now writing $E_1 = \mu(y, -x, 0)$, and since $\langle E_1, e_3\rangle = 0$, by oddness and symmetry of T, we get

$$\langle h_T, E_1\rangle = T_{12} \int_B \mu^2 \left(x^2 - y^2 + y^4 - x^4\right) - T_{12} \int_{\partial B} \mu(x^2 - y^2).$$

hence by antisymmetry of the integrands, we infer that $\langle g_T, E_1\rangle = 0$. Now the estimate (3.28) follows by standard elliptic regularity. ■

We will denote by $\omega_q = (\omega_q^1, \omega_q^2, \omega_q^3)$ the solution ω_{A^q} of (3.27) for every $q \in \partial\Omega$ and $\omega'_q = (\omega_q^1, \omega_q^2)$, where h_q, defined in Section 3.2, is the second fundamental form of $S = \partial\Omega$ at q. As we will see later in Lemma 3.2.7 below, the role of ω_q is to make the approximate solutions more accurate. From now on it will be understood that $O_q(X)$ (resp. $O_q(\sigma)$) denotes a smooth function depending on $X \in B$ (resp. $\sigma \in \partial B$) and maybe on ε, uniformly bounded together with its derivatives in q as $\varepsilon \to 0$ for every $X \in B$ (resp. $\sigma \in \partial B$) and $q \in S$.

We define our approximate solutions to be

$$z^{\varepsilon,p}(X) = \Theta(X) + p + \varepsilon\omega_q(X) + \varphi^{\varepsilon,q}(X + \varepsilon\omega'_q(X))e_3 \quad \text{for every } X \in B$$

with $q = \varepsilon p$, and let

$$\begin{aligned} \Psi^{\varepsilon,q}(X) &= \quad \varepsilon\omega_q(X) + \varphi^{\varepsilon,q}(X + \varepsilon\omega'_q(X))e_3 \\ &= \varepsilon\omega_q(X) + \tfrac{\varepsilon}{2}\langle h_q X, X\rangle e_3 + \varepsilon^2 O_q(X), \end{aligned} \tag{3.30}$$

so that $z^{\varepsilon,p} = \Theta + \Psi^{\varepsilon,q}$. Then if ε is small, by construction of ω_q, $z^{\varepsilon,p}$ has the following properties

$$z^{\varepsilon,p} \in \bar{\mathcal{M}}_2(S_\varepsilon),$$

$$\begin{aligned}\frac{\partial z^{\varepsilon,p}}{\partial n}(\sigma) &= (\varepsilon h_q \sigma, -1) + \varepsilon O_q(\sigma) e_3 \\ &= -N_\varepsilon(\sigma + \varepsilon \omega_q'(\sigma)) + \varepsilon O_q(\sigma) e_3 + \varepsilon^2 O_q(\sigma) \qquad \text{on } \partial B,\end{aligned} \tag{3.31}$$

where $N_\varepsilon(\sigma + \varepsilon\omega_q'(\sigma))$ is the normal of S_ε at the point $z^{\varepsilon,p}(\sigma) = p + (\sigma + \varepsilon\omega_q'(\sigma), \varphi^{\varepsilon,q}(\sigma + \varepsilon\omega_q'(\sigma)))$ having the expansion given by (3.22). Moreover since

$$z^{\varepsilon,p} = \Theta + p + \varepsilon O_q(X), \tag{3.32}$$

by (3.30), we have that

$$\frac{\partial z^{\varepsilon,p}}{\partial p_i} = e_i + \varepsilon^2 O_q(X) \qquad i = 1, 2, \tag{3.33}$$

where $\frac{\partial z^{\varepsilon,p}}{\partial p_i}$, $i = 1, 2$ are derivatives in the directions $(1, 0, \varphi_x^{\varepsilon,q}) = \partial_x(X, \varphi^{\varepsilon,q}(X))$ and $(0, 1, \varphi_y^{\varepsilon,q}) = \partial_y(X, \varphi^{\varepsilon,q}(X))$ respectively.

Recalling the expressions of E_i, see (3.12) and of Möbius group G, we set

$$\begin{aligned} E_1^{\varepsilon,q} &= \frac{\partial z^{\varepsilon,p} \circ g_{\theta,(0,0)}}{\partial \theta}_{|\theta=0} &&= E_1 + \frac{\partial \Psi^{\varepsilon,q} \circ g_{\theta,(0,0)}}{\partial \theta}_{|\theta=0}, \\ E_2^{\varepsilon,q} &= \frac{\partial z^{\varepsilon,p} \circ g_{0,(a_1,0)}}{\partial a_1}_{|a_1=0} + 2e_1 &&= 2E_2 + \frac{\partial \Psi^{\varepsilon,q} \circ g_{0,(a_1,0)}}{\partial a_1}_{|a_1=0}, \\ E_3^{\varepsilon,q} &= \frac{\partial z^{\varepsilon,p} \circ g_{0,(0,a_2)}}{\partial a_2}_{|a_2=0} + 2e_2 &&= 2E_3 + \frac{\partial \Psi^{\varepsilon,q} \circ g_{0,(0,a_2)}}{\partial a_2}_{|a_2=0} \end{aligned} \tag{3.34}$$

and we define

$$G_{z^{\varepsilon,p}} = span\ \{E_1^{\varepsilon,q};\ E_2^{\varepsilon,q};\ E_3^{\varepsilon,q}\}.$$

Now having the approximate solutions $z^{\varepsilon,p}$, we define the sub-manifold of $\bar{\mathcal{M}}_2(S_\varepsilon)$ by

$$Z_\varepsilon = \{z^{\varepsilon,p}\ :\ p \in S_\varepsilon\} \tag{3.35}$$

with tangent space at $z^{\varepsilon,p}$

$$T_{z^{\varepsilon,p}} Z_\varepsilon = span\ \left\{\frac{\partial z^{\varepsilon,p}}{\partial p_i}, \quad i = 1, 2\right\}.$$

We let

$$(T_{z^{\varepsilon,p}} Z_\varepsilon)^\perp = \left\{v \in \mathcal{M}_{z^{\varepsilon,p}}(S_\varepsilon)\ :\ \langle v, \frac{\partial z^{\varepsilon,p}}{\partial p_i}\rangle_{1,2} = 0 \quad i = 1, 2\right\}$$

so that $\mathcal{M}_{z^{\varepsilon,p}}(S_\varepsilon) = T_{z^{\varepsilon,p}} Z_\varepsilon \oplus (T_{z^{\varepsilon,p}} Z_\varepsilon)^\perp$, where $\langle \cdot, \cdot \rangle_{1,2}$ is the scalar product in $H^{1,2}(B, \mathbb{R}^3)$.

3.2.6. Remark. *Let $v \in \mathcal{M}_{z^{\varepsilon,p}}(S_\varepsilon)$. Since for every $\sigma \in \partial B$, $T_{z^{\varepsilon,p}(\sigma)}S_\varepsilon$ is spanned by the vectors $(1, 0, \varphi_x^{\varepsilon,q}(\sigma + \varepsilon\omega_q'(\sigma)))$ and $(0, 1, \varphi_y^{\varepsilon,q}(\sigma + \varepsilon\omega_q'(\sigma)))$ so for a.e. $\sigma \in \partial B$ we have*

$$v^3(\sigma) = v^1(\sigma)\varphi_x^{\varepsilon,q}(\sigma + \varepsilon\omega_q'(\sigma)) + v^2(\sigma)\varphi_y^{\varepsilon,q}(\sigma + \varepsilon\omega_q'(\sigma))$$

and hence by the trace theorem

$$\|v^3\|_{H^{\frac{1}{2},2}(\partial B)} \leq C\varepsilon(\|v^1\|_{1,2} + \|v^2\|_{1,2}) \leq C\varepsilon\|v\|_{1,2}. \tag{3.36}$$

Secondly we observe that there exists $C > 0$ depending only on Ω such that for every $\varepsilon \ll 1$

$$\int_B |v|^2 \leq C\int_B |\nabla v|^2, \qquad \forall v \in (T_{z^{\varepsilon,p}}Z_\varepsilon)^\perp. \tag{3.37}$$

In fact on the one hand letting $v \in (T_{z^{\varepsilon,p}}Z_\varepsilon)^\perp$ we have by (3.33), $\left|\int_B v^i\right| \leq \varepsilon\|v\|_{1,2}$ and by Poincaré inequality we have

$$\|v^i\| \leq C\left(\|\nabla v^i\| + \left|\int_B v^i\right|\right) \qquad i = 1, 2.$$

On the other hand by (3.36)

$$\|v^3\|_{L^2(\partial B)} \leq \varepsilon\|v\|_{1,2},$$

so using the following inequality (see [85], Theorem A.9),

$$\|v^3\| \leq C\left(\|\nabla v^3\| + \|v^3\|_{L^2(\partial B)}\right),$$

we obtain

$$\|v\| \leq C\|\nabla v\| + \varepsilon\|v\|_{1,2}.$$

3.2.3 The finite-dimensional reduction

We define

$$\mathcal{W}_{z^{\varepsilon,p}} = \left\{v \in (T_{z^{\varepsilon,p}}Z_\varepsilon)^\perp \ : \ \int_B \nabla v \cdot \nabla E_i^{\varepsilon,q} = 0 \ \ i = 1, 2, 3\right\}, \tag{3.38}$$

so that by (3.37) we may assume that the following decompositions hold

$$\mathcal{M}_{z^{\varepsilon,p}}(S_\varepsilon) = T_{z^{\varepsilon,p}}Z_\varepsilon \oplus (T_{z^{\varepsilon,p}}Z_\varepsilon)^\perp = T_{z^{\varepsilon,p}}Z_\varepsilon \oplus \mathcal{W}_{z^{\varepsilon,p}} \oplus G_{z^{\varepsilon,p}}. \tag{3.39}$$

On the other hand by the regularity of the approximate solutions, we have $T_{z^{\varepsilon,p}}Z_\varepsilon$ and $G_{z^{\varepsilon,p}}$ are subspaces of $T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon)$, so we may also assume the following splitting:

$$T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon) = T_{z^{\varepsilon,p}}Z_\varepsilon \oplus \mathcal{W}_{z^{\varepsilon,p}} \cap T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon) \oplus G_{z^{\varepsilon,p}}. \tag{3.40}$$

As explained in the first section, we want to reduce the problem of finding critical points of I_ε on $\bar{\mathcal{M}}_2(S_\varepsilon)$ to a finite dimensional one. A sub-manifold $\tilde{Z}_\varepsilon$ of $\bar{\mathcal{M}}_2(S_\varepsilon)$ is said to be a *natural constraint* for I_ε if

$$\langle dI_{\varepsilon|\tilde{Z}_\varepsilon}(u), \phi\rangle = 0 \quad \forall \phi \in T_u\tilde{Z}_\varepsilon \Longrightarrow \langle dI_\varepsilon(u), \phi\rangle = 0 \quad \forall \phi \in T_u\bar{\mathcal{M}}_2(S_\varepsilon).$$

Our aim is to perturb the sub-manifold of approximate solutions Z_ε to a natural constraint. This will be done by finding solutions of the form $u = \exp_z(w)$ with $z \in Z_\varepsilon$ and $w \in \mathcal{W}_z \cap T_z\bar{\mathcal{M}}_2(S_\varepsilon)$ such that $\Xi_z dI_\varepsilon(u) \in T_zZ_\varepsilon \oplus G_z$. Here $(z, w) \mapsto \exp_z(w)$ denote the exponential map of $\bar{\mathcal{M}}_2(S_\varepsilon)$, $\Xi_z : H^{2,2}(B, \mathbb{R}^3) \to T_z\bar{\mathcal{M}}_2(S_\varepsilon)$ is the orthogonal projection onto $T_z\bar{\mathcal{M}}_2(S_\varepsilon)$ and $u \mapsto dI_\varepsilon(u) \in T_u\bar{\mathcal{M}}_2(S_\varepsilon)$ is the gradient vector-field of I_ε. If we denote by $P_z : T_z\bar{\mathcal{M}}_2(S_\varepsilon) \to \mathcal{W}_z \cap T_z\bar{\mathcal{M}}_2(S_\varepsilon)$ the restriction of the orthogonal projection $\mathcal{M}_z \to \mathcal{W}_z$, our problem becomes equivalent to the system

$$\begin{cases} P_z\Xi_z dI_\varepsilon(\exp_z(w)) = 0, \\ P_z w = w. \end{cases}$$

By a Taylor expansion this is equivalent to solve the following fixed point problem

$$w = -{L_z}^{-1}\{P_z dI_\varepsilon(z) + P_z\Xi_z\mathcal{N}_z(w)\},$$

where $L_z = P_z d^2I_\varepsilon(z)$ and $\mathcal{N}_z$ is quadratic in w. We will solve it in a small ball of $\mathcal{W}_z \cap T_z\bar{\mathcal{M}}_2(S_\varepsilon)$ to get $w(\varepsilon, z)$. To verify that $\tilde{Z}_\varepsilon = \{\exp_z(w(\varepsilon, z)) : z \in Z_\varepsilon\}$ is a natural constraint for I_ε, we will use the argument of [[4] section 8.4] and [16].

The remaining of this section is devoted to carry out this program. We need first to show that $\|dI_\varepsilon(z^{\varepsilon,p})\|$ is small, and that $d^2I_\varepsilon(z^{\varepsilon,p})$ is uniformly invertible on $\mathcal{W}_{z^{\varepsilon,p}} \cap T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon)$. From now on, we will assume that $p = 0$ corresponds to the origin by replacing S_ε with $S_\varepsilon - p$.

3.2.7. Lemma. *There exists constant $C_1 > 0$ such that for every $\varepsilon > 0$,*

$$|\langle dI_\varepsilon(z^{\varepsilon,p}), v\rangle| \le C_1\varepsilon^2\|v\|_{1,2} \qquad \forall v \in \mathcal{M}_{z^{\varepsilon,p}}(S_\varepsilon).$$

PROOF. First of all, thanks to (3.31) and (3.36), for every $v \in \mathcal{M}_{z^{\varepsilon,p}}(S_\varepsilon)$ one has

$$\int_{\partial B} \frac{\partial z^{\varepsilon,p}}{\partial n} \cdot v ds = \int_{\partial B} N_\varepsilon(\sigma + \varepsilon\omega_q'(\sigma)) \cdot v(\sigma) ds + \|v\|_{1,2} O(\varepsilon^2).$$

Moreover since $N_\varepsilon(\sigma + \varepsilon\omega_q'(\sigma))$ is normal to $T_{z^{\varepsilon,p}(\sigma)} S_\varepsilon$, it follows that

$$\left| \int_{\partial B} \frac{\partial z^{\varepsilon,p}}{\partial n} \cdot v ds \right| \le C\varepsilon^2 \|v\|_{1,2}. \tag{3.41}$$

By Lemma 3.2.5, one has that $L(\omega^q + \frac{1}{2}\langle h_q X, X\rangle) = 0$ and then by (3.30)

$$|\langle L\Psi^{\varepsilon,q}, v\rangle| \le C\varepsilon^2 \|v\|_{1,2}. \tag{3.42}$$

If $v \in \mathcal{M}_{z^{\varepsilon,p}}(S_\varepsilon)$ then by integration by parts,

$$\begin{aligned} \langle dI_\varepsilon(z^{\varepsilon,p}), v\rangle &= \int_B \nabla z^{\varepsilon,p} \cdot \nabla v + 2\int_B (z_x^{\varepsilon,p} \wedge z_y^{\varepsilon,p}) \cdot v \\ &= -\int_B (\Delta\Theta - 2\Theta_x \wedge \Theta_y) \cdot v - \int_B \left(\Delta\Psi^{\varepsilon,q} - 2\Psi_x^{\varepsilon,q} \wedge \Psi_y^{\varepsilon,q}\right) \cdot v \\ &\quad + \int_{\partial B} \frac{\partial z^{\varepsilon,p}}{\partial n} \cdot v ds + 2\int_B (\Psi_x^{\varepsilon,q} \wedge \Psi_y^{\varepsilon,q}) \cdot v. \end{aligned}$$

Now we use *1.* in Lemma 2.3.1 and the Hölder inequality to have

$$|\langle dI_\varepsilon(z^{\varepsilon,p}), v\rangle| \le |\langle L\Psi^{\varepsilon,q}, v\rangle| + \left| \int_{\partial B} \frac{\partial z^{\varepsilon,p}}{\partial n} \cdot v ds \right| + 2\|\nabla\Psi^{\varepsilon,q}\|^2 \|v\|,$$

hence from (3.41) and (3.42) the lemma follows. ∎

3.2.8. PROPOSITION. *There exists a constant $C_2 > 0$ such that for all $\varepsilon > 0$ small,*

$$\begin{aligned} \langle d^2 I_\varepsilon(z^{\varepsilon,p}) v, v\rangle &\ge C_2 \|\nabla v\|^2 \qquad \forall v \in \mathcal{W}_{z^{\varepsilon,p}} \text{ with } \int_B \nabla v \cdot \nabla z^{\varepsilon,p} = 0, \\ \langle d^2 I_\varepsilon(z^{\varepsilon,p}) z^{\varepsilon,p}, z^{\varepsilon,p}\rangle &= -4\pi + O(\varepsilon). \end{aligned}$$

PROOF. Since $d^2 I_\varepsilon(z^{\varepsilon,p}) \simeq d^2 I_0(\Theta)$, we may rely on Lemma 3.1.4.
By (3.32) and (3.33) and recalling that $W_\Theta = (T_\Theta Z)^\perp \cap \mathcal{H}_n$ (see (3.17)), it is enough to consider those $v \in \mathcal{M}_{z^{\varepsilon,p}}(S_\varepsilon) \cap \mathcal{H}_n$ satisfying

$$\int_B v^i = 0, \; i = 1, 2 \quad \text{with} \int_B \nabla v \cdot \nabla\Theta = 0.$$

For such v, we define

$$\begin{cases} \Delta\bar{v}^3 = 0 & \text{in } B, \\ \bar{v}^3 = v^3 & \text{on } \partial B. \end{cases} \tag{3.43}$$

By (3.36),

$$\|\bar{v}^3\|_{1,2} = O(\varepsilon)\|\nabla v\|. \tag{3.44}$$

Let $\check{v}^3 = v^3 - \bar{v}^3 \in H_0^1(B;\mathbb{R})$. We then have $v = \check{v} + \bar{v}$ where $\check{v} = (v^1, v^2, \check{v}^3)$ and $\bar{v} = (0,0,\bar{v}^3)$; $\|\nabla v\|^2 = \|\nabla \check{v}\|^2 + \|\nabla \bar{v}\|^2$ and $\check{v} \in W_\Theta$. Clearly

$$\langle d^2 I_0(\Theta)v, v\rangle = \langle d^2 I_0(\Theta)\check{v}, \check{v}\rangle + 2\langle d^2 I_0(\Theta)\check{v}, \bar{v}\rangle + \langle d^2 I_0(\Theta)\bar{v}, \bar{v}\rangle.$$

Since

$$\langle d^2 I_0(\Theta)\check{v}, \bar{v}\rangle = \int_B \nabla\check{v}\cdot\nabla\bar{v} + \frac{1}{3}\int_B \check{v}\cdot(\bar{v}_x\wedge\Theta_y + \Theta_x\wedge\bar{v}_y) + \frac{1}{3}\int_B \bar{v}\cdot(\check{v}_x\wedge\Theta_y + \Theta_x\wedge\check{v}_y) + \frac{1}{3}\int_B \Theta\cdot$$

by (3.44) we have

$$\langle d^2 I_0(\Theta)\check{v}, \bar{v}\rangle = O(\varepsilon)\|\nabla v\|^2.$$

It is easy to verify that

$$\langle d^2 I_0(\Theta)\bar{v}, \bar{v}\rangle = \|\nabla\bar{v}\|^2,$$

so we get

$$\langle d^2 I_0(\Theta)v, v\rangle = \langle d^2 I_0(\Theta)\check{v}, \check{v}\rangle + O(\varepsilon)\|v\|_{1,2}^2 + \|\nabla\bar{v}\|^2. \tag{3.45}$$

Let us estimate $\langle d^2 I_0(\Theta)\check{v}, \check{v}\rangle$. We define $\underline{v} = \check{v} + \phi$, where

$$\phi = \left(0, 0, \frac{\int_B \nabla\bar{v}^3\cdot\nabla\Theta^3}{\|\nabla\Theta^3\|^2}\Theta^3\right).$$

Clearly $\phi \in H_0^1(B,\mathbb{R}^3)$ and $\phi = O(\varepsilon)\|\nabla v\|e_3$, moreover $\underline{v} \in W_\Theta$ and satisfies $\int_B \nabla\underline{v}\cdot\nabla\Theta = 0$. Furthermore $\|\nabla\underline{v}\|^2 = \|\nabla\check{v}\|^2 + \|\nabla\phi\|^2 + O(\varepsilon)\|\nabla v\|^2$ hence by Lemma 3.1.4

$$\langle d^2 I_0(\Theta)\underline{v}, \underline{v}\rangle \geq C\|\nabla\underline{v}\|^2 = C\|\nabla\check{v}\|^2 + O(\varepsilon)\|\nabla v\|^2.$$

Now we have

$$\langle d^2 I_0(\Theta)\check{v}, \check{v}\rangle = \langle d^2 I_0(\Theta)\underline{v}, \underline{v}\rangle - 2\langle d^2 I_0(\Theta)\underline{v}, \phi\rangle + \langle d^2 I_0(\Theta)\phi, \phi\rangle$$

and by Hölder inequality

$$\langle d^2 I_0(\Theta)\underline{v}, \phi\rangle = O(\varepsilon)\|\nabla\underline{v}\|\|\nabla v\|, \qquad \langle d^2 I_0(\Theta)\phi, \phi\rangle = \|\nabla\phi\|^2 = O(\varepsilon)\|\nabla v\|^2,$$

thus

$$\langle d^2 I_0(\Theta)\check{v}, \check{v}\rangle \geq C\|\nabla\check{v}\|^2 + O(\varepsilon)\|\nabla v\|^2. \tag{3.46}$$

Therefore by (3.45) and (3.46), we conclude that

$$\langle d^2 I_0(\Theta)v, v\rangle \geq \bar{C}_2\|\nabla v\|^2 + O(\varepsilon)\|\nabla v\|^2 \qquad \forall v \in \mathcal{W}_{z^{\varepsilon,p}}, \quad \int_B \nabla v\cdot\nabla z^{\varepsilon,p} = 0.$$

This ends the proof. ■

We will also need the following result.

3.2.9. Lemma. *Let $f \in L^2(B, \mathbb{R}^3)$, and $u = (u^1, u^2, u^3) \in \mathcal{M}_{z^{\varepsilon,p}}(S_\varepsilon)$ satisfy*

$$\int_B \nabla u \cdot \nabla v = \int_B f \cdot v \qquad \forall v \in \mathcal{M}_{z^{\varepsilon,p}}(S_\varepsilon). \tag{3.47}$$

Then $u \in H^{2,2}(B, \mathbb{R}^3)$ and there exists a constant $C > 0$ such that for every $\varepsilon \ll 1$,

$$\|u\|_{2,2} \leq C\|f\|.$$

Proof. Without loss of generality, we may assume that $z^{\varepsilon,p}(x,y) = (x, y, \varphi^{\varepsilon,q}(x,y))$ for every $(x,y) \in \partial B$. Notice that $H_0^{1,2}(B, \mathbb{R}^3) \subset \mathcal{M}_{z^{\varepsilon,p}}(S_\varepsilon)$ so u satisfies $\Delta u = f$ a.e. in B and thus $u \in H^{2,2}_{loc}(B, \mathbb{R}^3)$.
Let $\psi \in H^{1,2}(B, \mathbb{R})$, then considering the test function $v = (1, 0, \varphi_x^{\varepsilon,q})\psi \in \mathcal{M}_{z^{\varepsilon,p}}(S_\varepsilon)$ and using the fact that

$$\begin{aligned}\nabla u \cdot \nabla v &= \nabla u^1 \cdot \nabla \psi + \varphi_x^{\varepsilon,q}(\nabla u^3 \cdot \nabla \psi) + (\nabla u^3 \cdot \nabla \varphi_x^{\varepsilon,q})\psi \\ &= \nabla u^1 \cdot \nabla \psi + \nabla(\varphi_x^{\varepsilon,q} u^3) \cdot \nabla \psi - u^3 \nabla \varphi_x^{\varepsilon,q} \cdot \nabla \psi + (\nabla u^3 \cdot \nabla \varphi_x^{\varepsilon,q})\psi\end{aligned}$$

we have

$$\int_B \nabla(u^1 + \varphi_x^{\varepsilon,q} u^3) \cdot \nabla \psi - \int_B u^3 \nabla \varphi_x^{\varepsilon,q} \cdot \nabla \psi = \int_B \tilde{f}\psi,$$

where $\tilde{f} = f^1 + f^3 \varphi_x^{\varepsilon,q} - \nabla u^3 \cdot \nabla \varphi_x^{\varepsilon,q}$. By the Gauss-Green formula

$$\int_B u^3 \nabla \varphi_x^{\varepsilon,q} \cdot \nabla \psi = \int_B div(u^3 \nabla \varphi_x^{\varepsilon,q})\psi - \int_{\partial B} (\nabla \varphi_x^{\varepsilon,q} \cdot n) u^3 \psi ds.$$

Setting $w = u^1 + \varphi_x^{\varepsilon,q} u^3 \in H^{1,2}(B, \mathbb{R})$, $g = \tilde{f} + div(u^3 \nabla \varphi_x^{\varepsilon,q}) \in L^2(B, \mathbb{R})$ and $\phi = (\nabla \varphi_x^{\varepsilon,q} \cdot n) u^3 \in H^{\frac{1}{2},2}(\partial B, \mathbb{R})$ then w satisfies

$$\int_B \nabla w \cdot \nabla \psi = -\int_{\partial B} \phi \psi ds + \int_B g\psi, \qquad \forall \psi \in H^{1,2}(B, \mathbb{R}).$$

It follows that w is a weak solution of the problem

$$\begin{cases} \Delta w = -g & \text{in } B; \\ \frac{\partial w}{\partial n} = -\phi & \text{on } \partial B, \end{cases}$$

hence the following properties hold

$$u^1 + \varphi_x^{\varepsilon,q} u^3 \in H^{2,2}(B, \mathbb{R}); \tag{3.48}$$

$$\|u^1 + \varphi_x^{\varepsilon,q} u^3\|_{2,2} \leq C(\|f\| + \|\varphi^{\varepsilon,q}\|_{C^2(\bar{B})} \|u^3\|_{1,2}). \tag{3.49}$$

By a similar argument testing on $v = (0, 1, \varphi_y^{\varepsilon,q})\psi$, we have

$$u^2 + \varphi_y^{\varepsilon,q} u^3 \in H^{2,2}(B, \mathbb{R}); \tag{3.50}$$

$$\|u^2 + \varphi_y^{\varepsilon,q} u^3\|_{2,2} \le C(\|f\| + \|\varphi^{\varepsilon,q}\|_{C^2(\bar{B})}\|u^3\|_{1,2}). \tag{3.51}$$

Since $u \in \mathcal{M}_{z^{\varepsilon,p}}(S_\varepsilon)$,

$$u^3(\sigma) = u^1(\sigma)\varphi_x^{\varepsilon,q}(\sigma) + u^2(\sigma)\varphi_y^{\varepsilon,q}(\sigma) \text{ for a.e. } \sigma \in \partial B. \tag{3.52}$$

We multiply equation (3.48) by $\varphi_x^{\varepsilon,q}$ and (3.50) by $\varphi_y^{\varepsilon,q}$, and take the sum to have

$$u^3(1 + |\nabla\varphi^{\varepsilon,q}|^2) \in H^{\frac{3}{2},2}(\partial B, \mathbb{R})$$

thus thanks to (3.48) and (3.50), $u \in H^{\frac{3}{2},2}(\partial B, \mathbb{R}^3)$ and hence $u \in H^{2,2}(B, \mathbb{R}^3)$. Now let us estimate the $H^{2,2}$-norm of u^3. We write $u = \check{u} + \bar{u}$ where $\bar{u} = \bar{u}^3 e_3$ and $\bar{u}^3$ is the harmonic extension of u^3 as in the proof of Proposition 3.2.8. We have by (3.52) and the trace theorem that

$$\|\bar{u}^3\|_{2,2} \le C\|u^3\|_{\frac{3}{2},2} \le \tilde{C}\varepsilon\|u\|_{2,2}.$$

If we consider the test functions in the form $v = \psi e_3 = (0, 0, \psi) \in \mathcal{M}_{z^{\varepsilon,p}}(S_\varepsilon)$ for every $\psi \in H_0^{1,2}(B, \mathbb{R})$, it follows that

$$\|\check{u}^3\|_{2,2} \le C\|f\|.$$

The two previous inequalities give

$$\|u^3\|_{2,2} \le C(\|f\| + \tilde{C}\varepsilon\|u\|_{2,2}).$$

Hence using (3.49) and (3.51) we obtain

$$\|u\|_{2,2} \le C(\|f\| + \bar{C}\varepsilon\|u\|_{2,2}),$$

so we have the result. ■

We let $P(= P_{\varepsilon,p}) : \mathcal{M}_{z^{\varepsilon,p}}(S_\varepsilon) \to \mathcal{W}_{z^{\varepsilon,p}}$ be the projection onto $\mathcal{W}_{z^{\varepsilon,p}}$. By (3.40) the restriction of P on the tangent space $T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon)$ satisfies:

$$P : T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon) \to \mathcal{W}_{z^{\varepsilon,p}} \cap T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon).$$

Via duality, we will be considering $d^2 I_\varepsilon(z^{\varepsilon,p}) : T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon) \to T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon)$ and we define

$$L_{\varepsilon,p} = P \circ d^2 I_\varepsilon(z^{\varepsilon,p}) : \mathcal{W}_{z^{\varepsilon,p}} \cap T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon) \to \mathcal{W}_{z^{\varepsilon,p}} \cap T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon).$$

From Proposition 3.2.8 and Lemma 3.2.9, we deduce the

3.2.10. Corollary. *For for every $\varepsilon \ll 1$, and every $p \in S_\varepsilon$, the operator $L_{\varepsilon,p}$ is invertible on $\mathcal{W}_{z^{\varepsilon,p}} \cap T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon)$ and there exists a constant $\bar{C}_2 > 0$ such that*

$$\|L^{-1}_{\varepsilon,p} v\|_{2,2} \le \bar{C}_2 \|v\|_{1,2} \qquad \forall v \in \mathcal{W}_{z^{\varepsilon,p}} \cap T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon),\ p \in S_\varepsilon.$$

Proof. Since $\int_B \nabla z^{\varepsilon,p} \cdot \nabla \frac{\partial z^{\varepsilon,p}}{\partial p_i} = O(\varepsilon)$, one has

$$\|\nabla z^{\varepsilon,p} - \nabla P z^{\varepsilon,p}\| = O(\varepsilon) \tag{3.53}$$

and from Proposition 3.2.8,

$$L_{\varepsilon,p} z^{\varepsilon,p} = -4\pi P z^{\varepsilon,p} + O(\varepsilon).$$

Now following [4]-(section 8.4) and according to Remark 3.2.6-(3.37), setting

$$V_1 = \mathbb{R} P z^{\varepsilon,p}; \quad V_2 = \left\{ v \in \mathcal{M}_{z^{\varepsilon,p}}(S_\varepsilon) \ : \ \int_B \nabla v \cdot \nabla z^{\varepsilon,p} = 0 \right\},$$

thanks to (3.53), we may assume that $V_1 \perp V_2$. We decompose $\mathcal{W}_{z^{\varepsilon,p}} = V_1 \oplus V_2$ then in matrix form with respect to V_1 and V_2, $L_{\varepsilon,p}$ can be written as

$$\begin{pmatrix} -4\pi Id + O(\varepsilon) & O(\varepsilon) \\ O(\varepsilon) & B_{\varepsilon,p} \end{pmatrix},$$

where $B_{\varepsilon,p}$ satisfies, by Proposition 3.2.8, $\|B_{\varepsilon,p} v\|_{1,2} \ge C_2 \|v\|^2_{1,2}$ for every $v \in V_2 \cap T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon)$. Hence there exists $\tilde{C}_2 > 0$ such that for $\varepsilon \ll 1$

$$\|L^{-1}_{\varepsilon,p} v\|_{1,2} \le \tilde{C}_2 \|v\|_{1,2} \qquad \forall v \in \mathcal{W}_{z^{\varepsilon,p}} \cap T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon). \tag{3.54}$$

We set $u = L^{-1}_{\varepsilon,p} v$ so that $P d^2 I_\varepsilon(z^{\varepsilon,p})[u] = Pv$, or equivalently $d^2 I_\varepsilon(z^{\varepsilon,p})[u] - v \in T_{z^{\varepsilon,p}} Z_\varepsilon \oplus G_{z^{\varepsilon,p}}$ by (3.40). Then, there exist α_i, $i = 1,2$ and β_j, $j = 1,2,3$ such that for any $\phi \in T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon)$,

$$\langle d^2 I_\varepsilon(z^{\varepsilon,p}) u, \phi\rangle - \int_B v \cdot \phi = \sum_{i=1}^{2} \alpha_i \int_B \left(\nabla \frac{\partial z^{\varepsilon,p}}{\partial p_i} \cdot \nabla \phi + \frac{\partial z^{\varepsilon,p}}{\partial p_i} \cdot \phi \right) + \sum_{j=1}^{3} \beta_j \int_B \nabla E_j^{\varepsilon,q} \cdot \nabla \phi.$$

We first estimate α_i and β_j. By equation (3.33) and (3.34), we may write

$$\begin{aligned} \frac{\partial z^{\varepsilon,p}}{\partial p_i} &= e_i + \varepsilon O_q^i(X) \qquad i = 1,2, \\ E_j^{\varepsilon,q} &= E_j + \varepsilon O_q^j(X) \qquad j = 1,2,3, \end{aligned}$$

therefore for every $\phi \in T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon)$

$$\sum_{i=1}^{2}\alpha_i\int_B e_i\cdot\phi+\sum_{j=1}^{3}\beta_j\int_B \nabla E_j\cdot\nabla\phi+\varepsilon\sum_{i=1}^{2}\alpha_i\int_B O_q^i\cdot\phi+\varepsilon\sum_{j=1}^{3}\beta_j\int_B O_q^j\cdot\phi = \langle d^2I_\varepsilon(z^{\varepsilon,p})u,\phi\rangle-$$

thus by the mutual orthogonality of e_i, E_j for $i = 1, 2,\ j = 1, 2, 3$ it follows by (3.54) that

$$\sum_{i=1}^{2}|\alpha_i|+\bar{C}\sum_{j=1}^{3}|\beta_j|-\varepsilon\tilde{C}\left(\sum_{i=1}^{2}|\alpha_i|+\sum_{j=1}^{3}|\beta_j|\right)\leq C(\|u\|_{1,2}+\|v\|_{1,2})\leq C\|v\|_{1,2}.$$

Finally observe that u satisfies the following:

$$\int_B\nabla\left(u-\sum_{i=1}^{2}\alpha_i\frac{\partial z^{\varepsilon,p}}{\partial p_i}-\sum_{j=1}^{3}\beta_i E_j^{\varepsilon,q}\right)\cdot\nabla\phi=-2\int_B\left(u_x\wedge z_y^{\varepsilon,p}+z_x^{\varepsilon,p}\wedge u_y\right)\cdot\phi+\int_B v\cdot\phi+\sum$$

for all $\phi \in T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon)$. By density of $T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon)$ in $\mathcal{M}_{z^{\varepsilon,p}}(S_\varepsilon)$, Lemma 3.2.9 and (3.54) we get

$$\|u\|_{2,2}\leq\bar{C}_2(\|u\|_{1,2}+\|v\|_{1,2})\leq\bar{C}_2\|v\|_{1,2}.$$

This concludes the proof. ∎

We consider the projection $\Xi\,(=\Xi_{\varepsilon,p}) : H^{2,2}(B,\mathbb{R}^3)\to T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon)$ which is well defined since $T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon)$ is a closed subspace of $H^{2,2}(B,\mathbb{R}^3)$ by the compact embedding of $C(\bar{B},\mathbb{R}^3)$ into $H^{2,2}(B,\mathbb{R}^3)$. This projection varies differentially in p by the regularity of $\bar{\mathcal{M}}_2(S_\varepsilon)$, see also [90].
With an abuse of notation, we denote $u\mapsto dI_\varepsilon(u)\in T_u\bar{\mathcal{M}}_2(S_\varepsilon)$ the gradient vector-field of I_ε.
The following proposition shows that, by Lemma 3.2.7 and Corollary 3.2.10, the manifold of approximate solution Z_ε can be perturbed to a natural constraint $\tilde{Z}_\varepsilon$ for I_ε.

3.2.11. PROPOSITION. *Let I_ε be the functional defined in (3.26) and $\mathcal{W}_{z^{\varepsilon,p}}$ in (3.38). Then for $\varepsilon > 0$ small and $p \in S_\varepsilon$, there exists a unique $w = w(\varepsilon,p)\in\mathcal{W}_{z^{\varepsilon,p}}\cap T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon)$ such that $\Xi_{\varepsilon,p}dI_\varepsilon(\exp_{z^{\varepsilon,p}}(w))\in T_{z^{\varepsilon,p}}Z_\varepsilon\oplus G_{z^{\varepsilon,p}}$. The function $S_\varepsilon\ni p\mapsto w(\varepsilon,p)$ is of class C^1. Moreover, the function $F_\varepsilon(p)=I_\varepsilon(\exp_{z^{\varepsilon,p}}(w(\varepsilon,p)))$ is of class C^1 in p and satisfies*

$$F'_\varepsilon(p)=0 \quad\Longrightarrow\quad \langle dI_\varepsilon(\exp_{z^{\varepsilon,p}}(w(\varepsilon,p))),\phi\rangle=0 \qquad \forall\phi\in T_{\exp_{z^{\varepsilon,p}}(w(\varepsilon,p))}\bar{\mathcal{M}}_2(S_\varepsilon).$$

PROOF.

Our aim is to solve the problem:

$$\begin{cases} P\Xi dI_\varepsilon(\exp_{z^{\varepsilon,p}}(w)) = 0, \\ Pw = w. \end{cases} \tag{3.55}$$

We make the Taylor expansion for the mapping $w \mapsto \exp_{z^{\varepsilon,p}}(w) \mapsto dI_\varepsilon(\exp_{z^{\varepsilon,p}}(w))$ form $T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon)$ into $T_{\exp_{z^{\varepsilon,p}(w)}}\bar{\mathcal{M}}_2(S_\varepsilon)$ in terms of w:

$$dI_\varepsilon(\exp_{z^{\varepsilon,p}}(w)) = dI_\varepsilon(z^{\varepsilon,p}) + d^2 I_\varepsilon(z^{\varepsilon,p})[w] + \mathcal{N}_{\varepsilon,p}(w)$$

with $\frac{\|\mathcal{N}(w)\|_{2,2}}{\|w\|_{2,2}} \to 0$ as $\|w\|_{2,2} \to 0$ uniformly in ε and p. Observe that with this expansion, (3.55) is equivalent to find $w \in \mathcal{W}_{z^{\varepsilon,p}} \cap T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon)$ such that

$$-Pd^2 I_\varepsilon(z^{\varepsilon,p})[w] = PdI_\varepsilon(z^{\varepsilon,p}) + P\Xi\mathcal{N}_{\varepsilon,p}(w)$$

because $dI_\varepsilon(z^{\varepsilon,p})$ and $d^2 I_\varepsilon(z^{\varepsilon,p})[w]$ belong to $T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon)$ and thus we are led to the following fixed point problem

$$w = -{L_{\varepsilon,p}}^{-1}\{PdI_\varepsilon(z^{\varepsilon,p}) + P\Xi\mathcal{N}(w)\}.$$

We define the map $T_{\varepsilon,p}(w) = -{L_{\varepsilon,p}}^{-1}\{PdI_\varepsilon(z^{\varepsilon,p}) + P\Xi\mathcal{N}(w)\}$. By Corollary 3.2.10, $T_{\varepsilon,p}$ is defined from $\mathcal{W}_{z^{\varepsilon,p}} \cap T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon)$ into itself. We are going to find a fixed point of $T_{\varepsilon,p}$ in the ball on $T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon)$ defined as

$$B_R = \left\{w \in \mathcal{W}_{z^{\varepsilon,p}} \cap T_{z^{\varepsilon,p}}\bar{\mathcal{M}}_2(S_\varepsilon) \ : \ ||w||_{2,2} \le \varepsilon^2 R\right\}$$

where $R > 0$ will be determined later. Let $w \in B_R$. Thanks to Corollary 3.2.10 and Lemma 3.2.7 one has that

$$||T_{\varepsilon,p}(w)||_{2,2} \le \bar{C}_2\{\varepsilon^2 C_1 + C\varepsilon^4 R^2\}$$

and thus, if we choose R sufficiently large and ε small respectively, $T_{\varepsilon,p}(w) \in B_R$. Now since $\mathcal{N}(w)$ is quadratic, if ε is small enough, the map $T_{\varepsilon,p}$ is a contraction in B_R, yielding the existence.
Following [6] Proposition 4.3, we can deduce the C^1 regularity of $p \mapsto w(\varepsilon,p)$ and hence of F_ε with

$$\left\|\frac{\partial w(\varepsilon,p)}{\partial p}\right\|_{2,2} = O(\varepsilon^2). \tag{3.56}$$

To prove the last assertion, we set $\tilde{Z}_\varepsilon = \left\{\exp_{z^{\varepsilon,p}}(w(\varepsilon,p)) \ : \ p \in S_\varepsilon = \frac{1}{\varepsilon}\partial\Omega\right\}$. With an abuse of notation, we write $z = z^{\varepsilon,p}$, $w = w(\varepsilon,p)$ and let $u =$

$\exp_z(w)$. Call G_u the subspace of $T_u\bar{\mathcal{M}}_2(S_\varepsilon)$ spanned by the following vectors

$$\begin{aligned} E_1^u &= \frac{\partial u \circ g_{\theta,(0,0)}}{\partial \theta}_{|\theta=0}, \\ E_2^u &= \frac{\partial u \circ g_{0,(a_1,0)}}{\partial a_1}_{|a_1=0} + 2e_1, \\ E_3^u &= \frac{\partial u \circ g_{0,(0,a_2)}}{\partial a_2}_{|a_2=0} + 2e_2. \end{aligned} \tag{3.57}$$

By the smallness of w, it is not difficult to see that $E_j^u = E_j^{\varepsilon,q} + O(\varepsilon)$ with $q = \varepsilon p$ and the $E_j^{\varepsilon,q}$'s are defined in (3.34). Moreover we may assume that $T_u\bar{\mathcal{M}}_2(S_\varepsilon)$ splits, as in (3.40), in the following way:

$$T_u\bar{\mathcal{M}}_2(S_\varepsilon) = T_u\tilde{Z}_\varepsilon \oplus \mathcal{W}_u \cap T_u\bar{\mathcal{M}}_2(S_\varepsilon) \oplus G_u$$

because the result of Remark 3.2.6-(3.37) holds true also for u (in the place of $z^{\varepsilon,p}$) by the smallness of w and (3.56).

We claim that $\tilde{Z}_\varepsilon$ is a natural constraint for I_ε. In fact, suppose that p is a critical point of F_ε thus $u = \exp_{z^{\varepsilon,p}}(w(\varepsilon,p))$ is a critical point of $I_\varepsilon|_{\tilde{Z}_\varepsilon}$ meaning that $dI_\varepsilon(u)$ is perpendicular to $T_u\tilde{Z}_\varepsilon$ on the one hand. On the other hand, since also $o(\varepsilon) = w \in \mathcal{W}_z \simeq \mathcal{W}_u$ solves (3.55), we may assume that $dI_\varepsilon(u)$ is perpendicular to $\mathcal{W}_u \cap T_u\bar{\mathcal{M}}_2(S_\varepsilon)$ hence it remains only to check if

$$0 \neq dI_\varepsilon(u) \in G_u.$$

Following ([16] section 2), let us show that the latter case cannot happen. Indeed, suppose there exist $\tilde{\beta}_k$ such that

$$\langle dI_\varepsilon(u), \phi\rangle = \sum_{k=1}^{3} \tilde{\beta}_k \int_B \nabla E_k^u \cdot \nabla \phi, \tag{3.58}$$

for every $\phi \in T_u\bar{\mathcal{M}}_2(S_\varepsilon)$.
We use the invariance of I_ε with the group, that is $I_\varepsilon(u \circ g) = I_\varepsilon(u)$ for

every $g \in G$. We have, thanks to (3.57) and (3.58),

$$\begin{aligned} 0 &= \frac{d}{d\theta} I_\varepsilon(u \circ g_{\theta,(0,0)})_{|\theta=0} = \langle dI_\varepsilon(u), \frac{\partial u \circ g_{\theta,(0,0)}}{\partial \theta}_{|\theta=0} \rangle \\ &= \sum_{k=1}^{3} \tilde{\beta}_k \int_B \nabla E_k^u \cdot \nabla E_1^u \\ &= \sum_{k=1}^{3} (\delta_{1k} + O(\varepsilon)) \tilde{\beta}_k. \end{aligned}$$

This shows that $\tilde{\beta}_1 = 0$ if ε is sufficiently small. Applying the same argument using the curves $a_1 \to g_{0,(a_1,0)}$ and $a_2 \to g_{0,(0,a_2)}$, we can see that $\tilde{\beta}_k = 0$ for every $k \in \{2,3\}$. In conclusion we have

$$\langle dI_\varepsilon(u), \phi \rangle = 0 \qquad \forall \phi \in T_u \bar{\mathcal{M}}_2(S_\varepsilon) = T_u \tilde{Z}_\varepsilon \oplus \mathcal{W}_u \cap T_u \bar{\mathcal{M}}_2(S_\varepsilon) \oplus G_u$$

which end the proof. ■

3.2.4 Embedded solutions to the partitioning problem

Let us show that the image of solutions of our problem which are given by Proposition 3.2.11 are embedded if ε is small enough. We do this by showing that, up to translations, the solutions are C^1 closed to a hemisphere. Let $\varepsilon \to p_\varepsilon \in S_\varepsilon = \frac{1}{\varepsilon}\partial\Omega$ be a curve of critical points of F_ε such that $\varepsilon p_\varepsilon \to q \in S$. In view of our construction there hold

$$\|exp_{z^{\varepsilon,p_\varepsilon}}(w(\varepsilon,p_\varepsilon)) - \big(\Theta - \varepsilon^{-1}q\big)\|_{C(\bar{B},\mathbb{R}^3)} \leq C\|exp_{z^{\varepsilon,p_\varepsilon}}(w(\varepsilon,p_\varepsilon)) - \big(\Theta - \varepsilon^{-1}q\big)\|_{2,2} \to 0 \text{ a} \tag{3.59}$$

by the Sobolev embedding.

Letting $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon) = exp_{z^{\varepsilon,p_\varepsilon}}(w(\varepsilon,p_\varepsilon))$ then by construction we can write $u^\varepsilon = \Theta + f^\varepsilon$ with $\|f^\varepsilon\|_{2,2} \to 0$ as $\varepsilon \to 0$. Because of the embedding of $H^{1,2}(B,\mathbb{R}^3)$ into $L^h(B,\mathbb{R}^3)$ for $1 \leq h < \infty$ we have $\|\nabla f^\varepsilon\|_{L^h(B,\mathbb{R}^3)} \to 0$ when ε tends to zero for $1 \leq h < \infty$. This shows that by Hölder inequality,

$$\|u_x^\varepsilon \wedge u_y^\varepsilon - \Theta_x \wedge \Theta_y\|_{L^h(B,\mathbb{R}^3)} = \|f_x^\varepsilon \wedge \Theta_y^\varepsilon + \Theta_x \wedge f_y^\varepsilon + f_x^\varepsilon \wedge f_y^\varepsilon\|_{L^h(B,\mathbb{R}^3)} \to 0 \text{ as } \varepsilon \to 0. \tag{3.60}$$

Observing that from Lemma 2.3.1, u^ε satisfies

$$\Delta(u^\varepsilon - \Theta) = 2\left(u_x^\varepsilon \wedge u_y^\varepsilon - \Theta_x \wedge \Theta_y\right) \qquad \text{in } B,$$

we deduce from (3.60) that

$$\|u^\varepsilon - (\Theta + \varepsilon^{-1} q)\|_{C^{1,\beta}(B,\mathbb{R}^3)} \leq C\|u^\varepsilon - (\Theta + \varepsilon^{-1} q)\|_{H^{2,h}_{loc}(B,\mathbb{R}^3)} \to 0 \text{ as } \varepsilon \to 0$$

for $h > 2$ by regularity theory and Sobolev embedding. We conclude that $u^\varepsilon(B)$ is embedded (in particular has no interior branch points).

We now show that $u^\varepsilon(B)$ is a solution to the partitioning problem. In the orientation chosen in Section 3.2 by assuming that $q = (0,0,0)$ is the origin and the tangent plane, spanned by $(1, 0, \varphi^{\varepsilon,q}_x)$ and $(0, 1, \varphi^{\varepsilon,q}_y)$, corresponds to $\mathbb{R}^2 \times \{0\}$, we have that $u^\varepsilon(\bar{B})$ is contained in a ball of $\mathbb{R}^3$ centered at the origin (with radius 2 for example) by (3.59). Moreover it is evident that the boundary conditions is equivalent to $u^\varepsilon_3 = \varphi^{\varepsilon,q}(u^\varepsilon_1, u^\varepsilon_2)$ on ∂B while $\langle \frac{\partial u^\varepsilon}{\partial n}, (1, 0, \varphi^{\varepsilon,q}_x(u^\varepsilon_1, u^\varepsilon_2))\rangle = \langle \frac{\partial u^\varepsilon}{\partial n}, (0, 1, \varphi^{\varepsilon,q}_y(u^\varepsilon_1, u^\varepsilon_2))\rangle = 0$ on ∂B. This together with Lemma 2.3.1-2, imply that $u^\varepsilon - \Theta$ satisfies:

$$\begin{cases} \Delta(u^\varepsilon - \Theta) = 2\left(u^\varepsilon_x \wedge u^\varepsilon_y - \Theta_x \wedge \Theta_y\right) =: g^\varepsilon & \text{in } B, \\ u^\varepsilon_3 - \Theta^3 = \varphi^{\varepsilon,q}(u^\varepsilon_1, u^\varepsilon_2) & \text{on } \partial B, \\ \dfrac{\partial(u^\varepsilon_1 - \Theta^1)}{\partial n} = -\varphi^{\varepsilon,q}_x(u^\varepsilon_1, u^\varepsilon_2)\dfrac{\partial u^\varepsilon_3}{\partial n} & \text{on } \partial B, \\ \dfrac{\partial(u^\varepsilon_2 - \Theta^2)}{\partial n} = -\varphi^{\varepsilon,q}_y(u^\varepsilon_1, u^\varepsilon_2)\dfrac{\partial u^\varepsilon_3}{\partial n} & \text{on } \partial B. \end{cases}$$

Therefore by elliptic regularity theory and (3.60), we get

$$\|u^\varepsilon_3 - \Theta^3\|_{2,h} \leq C\|g^\varepsilon\|_h + C\varepsilon\left(\|u^\varepsilon_1\|_{2,h} + \|u^\varepsilon_2\|_{2,h}\right).$$

On the other hand we have, thanks to Sobolev embeddings and (3.59), for $i = 1, 2$ and $h > 2$ there hold

$$\|u^\varepsilon_i - \Theta^i\|_{2,h} \leq C\|g^\varepsilon\|_h + \tilde{C}\varepsilon\left(\|u^\varepsilon_3 - \Theta^3\|_{2,h} + \|\Theta^3\|_{2,h}\right)$$

It follows from the previous inequalities and again from Sobolev embeddings that for $h > 2$ we have

$$\|u^\varepsilon - (\Theta + \varepsilon^{-1} q)\|_{C^{1,\beta}(\bar{B},\mathbb{R}^3)} \leq C\|u^\varepsilon - (\Theta + \varepsilon^{-1} q)\|_{2,h} \to 0 \text{ as } \varepsilon \to 0. \quad (3.61)$$

Finally we show that $\varepsilon u^\varepsilon(\bar{B})$ is contained in $\bar{\Omega}$ when ε is small enough. Since $u^\varepsilon(\partial B) \subset S_\varepsilon = \frac{1}{\varepsilon}\partial\Omega$, it follows that $d^\varepsilon(u^\varepsilon(\sigma)) = 0$ for every $\sigma \in \partial B$, where d^ε is the distance function defined in Section 3.2. Observe that by (3.61) we have

$$d^\varepsilon(u^\varepsilon(X)) = \Theta^3 + O(\varepsilon) = \mu(X) - 1 + \varepsilon O_{p_\varepsilon}(X) \quad \forall X \in B; \qquad \lim_{t \to 0^+} \frac{d^\varepsilon(u^\varepsilon(\sigma - tn))}{t} = 1 + \varepsilon$$

where μ (see Lemma 2.3.1) satisfies $1 < \mu < 2$ in B and $\|O_{p_\varepsilon}\|_{C(\bar{B})} \leq C$ for any ε positive small by (3.61). This shows that if ε is small enough $d^\varepsilon(u^\varepsilon(X)) > 0$ for every $X \in B$. We conclude that $\varepsilon u^\varepsilon = \varepsilon \exp_{z^{\varepsilon,p_\varepsilon}}(w(\varepsilon, p_\varepsilon))$ is an embedding of B into Ω if ε is sufficiently small without neither interior nor boundary branch points by (3.61).

3.3 Proof of Theorem 4.0.1 and Theorem 3.0.2

In view of Proposition 3.2.11, we can obtain existence of solutions to (3.3) by finding critical points of the functional $F_\varepsilon(p)$. The following lemmas are devoted to the expansions of F_ε with respect to p and ε.

For $i \in \{1, \ldots, 4\}$, the mapping $G_i : \partial\Omega \to \mathbb{R}$ is a smooth function, maybe depending on ε and uniformly bounded together with its derivative as $\varepsilon \to 0$.

3.3.1. Lemma. *For ε small one has*

$$V \circ \eta_\varepsilon(z^{\varepsilon,p}) = V(\tilde{z}^{\varepsilon,p}) = -\frac{\pi}{12}\varepsilon H_{\partial\Omega}(q) + \varepsilon^2 G_1(q)$$

with $q = \varepsilon p$.

Proof. We first need to provide the expansion of the extension $\tilde{z}^{\varepsilon,p}$ of $z^{\varepsilon,p}$. For simplicity, we assume that p is the origin of $\mathbb{R}^3$ and we write $z = z^{\varepsilon,p}$ also $\tilde{z}$ (resp. $\bar{z}$) will mean the extension of $\tilde{z}^{\varepsilon,p}$ (resp. the harmonic extension) of $z^{\varepsilon,p}$.
We recall form (3.24) that

$$\tilde{z}(X) = \bar{z}(X) - \nu^\varepsilon(\tilde{z}(X))d^\varepsilon(\bar{z}(X)),$$

and the expansion of the interior normal ν^ε, given by (3.22), at the point $\tilde{z}(X) \in S_\varepsilon$:

$$\nu^\varepsilon(\tilde{z}(X)) = e_3 - \varepsilon(h_q\tilde{z}'(X), 1) + \varepsilon^2 O_q(X),$$

where $\tilde{z}'$ stands for the first two components of $\tilde{z}$: $\tilde{z}' = (\tilde{z}^1, \tilde{z}^2)$. Moreover

$$d^\varepsilon(\bar{z}(X)) = e_3 \cdot \bar{z}(X) + \frac{\varepsilon}{2}\langle \tilde{h}_q \bar{z}(X), \bar{z}(X)\rangle + \varepsilon^2 O_q(X).$$

Using (3.19), the fact that the harmonic extension of Θ is $(X, 0)$ and that $\bar{\omega}_q^3 = 0$, we have

$$\bar{z}(X) = (X, 0) + \bar{\varphi}^{\varepsilon,q}(X, \varepsilon\omega_q'(X))e_3 + \varepsilon^2 O_q(X),$$

where $\bar{\varphi}^{\varepsilon,q}(X,\varepsilon\omega_q'(X))$ denotes the harmonic extension of the composition of mappings $X \mapsto \big(X,\varepsilon\omega_q'(X)\big) \mapsto \varphi^{\varepsilon,q}(X,\varepsilon\omega_q'(X))$. It follows that

$$d^\varepsilon(\bar{z}(X)) = \bar{\varphi}^{\varepsilon,q}(X) - \frac{\varepsilon}{2}\langle h_q X, X\rangle + \varepsilon^2 O_q(X).$$

Hence we obtain

$$\tilde{z}^{\varepsilon,p}(X) = (X + \varepsilon\bar{\omega}_q'(X), \frac{\varepsilon}{2}\langle h_q X, X\rangle) + \varepsilon^2 O_q(X),$$

and thus

$$\begin{aligned}(\tilde{z}^{\varepsilon,p}_x \wedge \tilde{z}^{\varepsilon,p}_y)\cdot\tilde{z}^{\varepsilon,p} &= -\varepsilon(X+\varepsilon\bar{\omega}_q')\cdot(h_q X) + \frac{\varepsilon}{2}\langle h_q X, X\rangle + \varepsilon^2 O_q(X)\\ &= -\frac{\varepsilon}{2}\langle h_q X, X\rangle + \varepsilon^2 O_q(X).\end{aligned}$$

We conclude that

$$V(\tilde{z}^{\varepsilon,p}) = -\frac{\varepsilon}{6}\int_B \langle h_q X, X\rangle + \frac{\varepsilon^2}{3}\int_B O_q(X) = -\frac{\varepsilon\pi}{12}H_{\partial\Omega}(q) + \varepsilon^2 G_1(q).$$

■

3.3.2. Lemma. *For ε small, the following expansions holds*

$$I_\varepsilon(z^{\varepsilon,p}) = \frac{2\pi}{3} - \frac{\pi}{3}\varepsilon H_{\partial\Omega}(q) + \varepsilon^2 G_4(q),$$

where $q = \varepsilon p$.

Proof. We recall that $z^{\varepsilon,p} = \Theta + \Psi^{\varepsilon,q}$ and

$$I_\varepsilon(z^{\varepsilon,p}) = \frac{1}{2}\int_B |\nabla z^{\varepsilon,p}|^2 + \frac{2}{3}\int_B z^{\varepsilon,p}\cdot(z^{\varepsilon,p}_x \wedge z^{\varepsilon,p}_y) - 2V(\tilde{z}^{\varepsilon,p}).$$

Let us expand term by term the right hand side of the above equality

$$\frac{1}{2}\int_B |\nabla z^{\varepsilon,p}|^2 = \frac{1}{2}\int_B |\nabla\Theta|^2 + \frac{1}{2}\int_B |\nabla\Psi^{\varepsilon,q}|^2 + \int_B \nabla\Theta\cdot\nabla\Psi^{\varepsilon,q};$$

$$\begin{aligned}\int_B z^{\varepsilon,p}\cdot(z^{\varepsilon,p}_x \wedge z^{\varepsilon,p}_y) &= \int_B z^{\varepsilon,p}\cdot\big(\Theta_x\wedge\Theta_y + \Theta_x\wedge\Psi^{\varepsilon,q}_y + \Psi^{\varepsilon,q}_x\wedge\Theta_y\big) + \int_B z^{\varepsilon,p}\cdot(\Psi^{\varepsilon,q}_x \wedge \Psi^{\varepsilon}_y\\ &= \int_B \Theta\cdot(\Theta_x\wedge\Theta_y) + \int_B \Theta\cdot\big(\Theta_x\wedge\Psi^{\varepsilon,q}_y + \Psi^{\varepsilon,q}_x\wedge\Theta_y\big) + \int_B \Psi^{\varepsilon,q}\cdot(\Theta_x\\ &\quad + \int_B \Psi^{\varepsilon,q}\cdot\big(\Theta_x\wedge\Psi^{\varepsilon,q}_y + \Psi^{\varepsilon,q}_x\wedge\Theta_y\big) + \int_B z^{\varepsilon,p}\cdot(\Psi^{\varepsilon,q}_x\wedge\Psi^{\varepsilon,q}_y).\end{aligned}$$

Using *1.* and *5.* of Lemma 2.3.1 and an integration by parts, we have

$$\int_B z^{\varepsilon,p}\cdot(z_x^{\varepsilon,p}\wedge z_y^{\varepsilon,p}) = -\frac{1}{2}\int_B |\nabla\Theta|^2 - \int_B \nabla\Theta\cdot\nabla\Psi^{\varepsilon,q} - \frac{1}{2}\int_B \nabla\Theta\cdot\nabla\Psi^{\varepsilon,q} + \frac{1}{2}\int_{\partial B}\frac{\partial\Theta}{\partial n}$$
$$+2\int_B \Theta\cdot(\Psi_x^{\varepsilon,q}\wedge\Psi_y^{\varepsilon,q}) - \int_{\partial B}\Theta\cdot(\Psi^{\varepsilon,q}\wedge\frac{\partial\Psi^{\varepsilon,q}}{\partial t})ds + \int_B z^{\varepsilon,p}\cdot($$

Hence adding up, we conclude that

$$I_\varepsilon(z^{\varepsilon,p}) = \frac{1}{6}\int_B |\nabla\Theta|^2 - \frac{1}{3}\int_{\partial B}\varphi^{\varepsilon,q}ds - 2V(\tilde{z}^{\varepsilon,p}) + \varepsilon^2 G_2(q),$$

where

$$\varepsilon^2 G_2(q) = \frac{1}{2}\int_B |\nabla\Psi^{\varepsilon,q}|^2 + 2\int_B \Theta\cdot(\Psi_x^{\varepsilon,q}\wedge\Psi_y^{\varepsilon,q}) - \frac{2}{3}\int_{\partial B}\Theta\cdot(\Psi^{\varepsilon,q}\wedge\frac{\partial\Psi^{\varepsilon,q}}{\partial t})ds + \int_B \Psi^{\varepsilon,q}\cdot(\Psi_x^{\varepsilon,q}\wedge$$

Now by property *6.* of Lemma 2.3.1 and the following computations

$$\int_{\partial B}\varphi^{\varepsilon,q}ds = \frac{\varepsilon}{2}\int_{\partial B}\langle h_q X, X\rangle ds + \varepsilon^2\int_{\partial B}\langle h_q X, \omega_q'\rangle ds + \varepsilon^3\int_{\partial B} O_q(\sigma)ds = \pi\varepsilon H_{\partial\Omega}(q) + \varepsilon^2 G$$

we obtain

$$I_\varepsilon(z^{\varepsilon,p}) = \frac{2\pi}{3} - \frac{\pi}{3}\varepsilon H_{\partial\Omega}(q) - 2V\circ\eta_\varepsilon(z^{\varepsilon,p}) + \varepsilon^2 G_2(q) + \varepsilon^2 G_3(q).$$

Therefore the conclusion follows from Lemma 3.3.1. ∎

Proof of Theorem 4.0.1 and Theorem 3.0.2

First of all we have

$$F_\varepsilon(p) = I_\varepsilon(\exp_{z^{\varepsilon,p}}(w(\varepsilon,p))) = I_\varepsilon(z^{\varepsilon,p}) + \langle dI_\varepsilon(z^{\varepsilon,p}), w(\varepsilon,p)\rangle + \mathcal{N}(w(\varepsilon,p)).$$

Using Lemma 3.2.7 and the fact that $\|w(\varepsilon,p)\|_{2,2}\le R\varepsilon^2$ we infer that

$$F_\varepsilon(p) = I_\varepsilon(z^{\varepsilon,p}) + O(\varepsilon^4).$$

Hence Lemma 3.3.2 yields

$$F_\varepsilon(p) = \frac{2\pi}{3} - \frac{\pi}{6}\varepsilon H_{\partial\Omega}(\varepsilon p) + \varepsilon^2 G(\varepsilon p) + O(\varepsilon^4). \tag{3.62}$$

It follows that if Q_0 is a strict local maximum or minimum of $H_{\partial\Omega}$, F_ε will have critical points p_ε for which $\varepsilon p_\varepsilon \to Q_0$ as $\varepsilon\to 0$. Furthermore we have by (3.56) that

$$\|F_\varepsilon - \frac{2\pi}{3} + \frac{\pi}{6}\varepsilon H_{\partial\Omega}\circ\rho_\varepsilon\|_{C^1(S_\varepsilon)} = O(\varepsilon^2), \tag{3.63}$$

where $\rho_\varepsilon(p) = \varepsilon p$. If Q_0 is a non-degenerate critical point of $H_{\partial\Omega}$ then the implicit function theorem yields also a curve $\varepsilon \to p_\varepsilon$ of critical point of F_ε with $\varepsilon p_\varepsilon \to Q_0$.
The proof of Theorem 3.0.2 follows immediately by Proposition 3.2.11 and the fact that F_ε is C^1 so it has at least $cat(\partial\Omega)$ critical points. We refer to [5]. ∎

3.3.3. Remark. *As mentioned in Section earlier, we can compare our result to the one of [93]. The expansions of the mean curvature of a geodesic sphere of radius* ε *contains only terms of order* ε^2 *and higher, see [93], equation (1.4). If we perform our construction in a manifold, by (3.62) it is evident that the boundary mean curvature would determine the main terms in the Lyapunov-Schmidt reduction. Other geometric quantities, with respect to the scalar curvature as the second fundamental form of the boundary could be relevant for the location of solutions only when the mean curvature is constant, see § 7.3.*

Chapter 4

Free boundary CMC hypersurfaces condensing along a sub-manifold

We let $\Omega \subset \mathbb{R}^{m+1}$ and K a k-dimensional smooth submanifold of $\partial\Omega$. We let $n := m - k$ be the dimension of the normal bundle of K in $\partial\Omega$. We define $\Omega_\varepsilon := \varepsilon^{-1}\Omega$ and $K_\varepsilon := \varepsilon^{-1} K$.

Recall that our aim is to find solutions of (GMP). Consider the "half"-geodesic tube contained in Ω_ε around K_ε of radius 1

$$\bar{S}_\varepsilon(K_\varepsilon) := \{q \in \bar{\Omega}_\varepsilon : \quad d(q, K_\varepsilon) = 1\},$$

with

$$d(q, K_\varepsilon) := \sqrt{|\mathrm{dist}^{\partial\Omega_\varepsilon}(\tilde{q}, K_\varepsilon)|^2 + |q - \tilde{q}|^2}$$

where $\tilde{q}$ is the projection of q on $\partial\Omega_\varepsilon$ and

$$\mathrm{dist}^{\partial\Omega_\varepsilon}(\tilde{q}, K_\varepsilon) = \inf \big\{ \mathrm{length}(\gamma) \quad : \quad \gamma \in C^1([0,1]) \text{ is a geodesic in } \partial\Omega_\varepsilon;\ \gamma(0) \in K_\varepsilon;\ \gamma$$

By the smoothness of $\partial\Omega$ and K, the tube is a smooth, possibly immersed, hypersurface provided ε is sufficiently small. This tube by construction meets $\partial\Omega_\varepsilon$ perpendicularly. Furthermore the mean curvature of this tube satisfies (see also § 4.2)

$$mH(\bar{S}_\varepsilon(K_\varepsilon)) = n + \mathcal{O}(\varepsilon) \tag{4.1}$$

as ε tends to zero and hence it is plausible under some rather mild assumptions on K that we might be able to perturb this tube to satisfy (GMP) with mean curvature $\frac{n}{m}$. It turns out that this is not known to be possible for every (small) $\varepsilon > 0$ but we prove the following theorem :

4.0.1. THEOREM. *Let Ω be a smooth bounded domain of $\mathbb{R}^{m+1}$, $m \geq 2$. Suppose that K is a non-degenerate minimal submanifold of $\partial\Omega$. Then, there exist a sequence of intervals $I_i = (\varepsilon_i^-, \varepsilon_i^+)$, with $\varepsilon_i^- < \varepsilon_i^+$ and $\lim_{i\to+\infty} \varepsilon_i^+ = 0$ such that, for all $\varepsilon \in I := \cup_i I_i$ the "half" geodesic tube $\varepsilon\, \bar{S}_\varepsilon(K_\varepsilon)$ may be perturbed to a hypersurface $\varepsilon S_\varepsilon$ satisfying (GMP) with mean curvature $H_{\varepsilon S_\varepsilon} \equiv \frac{n}{m}\, \varepsilon^{-1}$ and contact angle $\frac{\pi}{2}$. Namely there exists a family of embedded constant mean curvature hypersurfaces in Ω with boundary on $\partial\Omega$ and intersecting it perpendicularly.*

4.0.1. Remark. • *We emphasize that our argument provides also a stationary area separating of* $\mathbb{R}^{m+1} \setminus \bar{\Omega}$ *when considering the lower hemisphere parameterized by the stereographic projection from the north pole over the unit ball, see Section 4.2.*

- *Notice that the surfaces we obtained might have interesting topology. In fact as far as* ε *tends to zero, our solutions concentrate along* K *hence inherit its topological structure. Furthermore we cite that some existence result of various minimal immersions were obtained in [55] and [81].*
  We believe that the minimality condition on K *should also be necessary to obtain a result in spirit of Theorem 4.0.1, see the last paragraph of [64]. The non-degeneracy condition might fail in some interesting situations, for example when a symmetry is present. In this case however, one can take advantage of it working in a subclass of invariant functions: this might also guarantee existence for all small* ε*, see [64] Section 5.*

- *The hypersurface* S_ε *is a small perturbation of* $\bar{S}_\varepsilon(K_\varepsilon)$ *in the sense that it is the normal graph (for some function whose* L^∞ *norm is bounded by a constant times* ε*) over a small translate of* K_ε *in* $\partial\Omega_\varepsilon$ *(by some translation whose* L^∞ *norm is bounded by a constant), we refer to Section 4.3 for the precise formulation of the construction of* S_ε*.*

- *This result also remains true for the existence of capillary hypersurfaces in* Ω *namely those with stationary area which intersect* $\partial\Omega$ *in a constant angle* $\gamma \in (0, \pi)$ *along there boundaries. For more precise comments see Remark 4.5.1.*

To prove the latter theorem, following [56], [64] and [93], we parameterize all surfaces nearby $\bar{S}_\varepsilon(K_\varepsilon)$ having boundaries in $\partial\Omega_\varepsilon$ by two parametric functions $\Phi : K \to \mathbb{R}^n$ and $w : S^n_+ \times \varepsilon^{-1}K \to \mathbb{R}$. Here

$$S^n_+ := \left\{ x = (x^1, \cdots, x^{n+1}) \in \mathbb{R}^{n+1} \quad : \quad |x| = 1 \text{ and } x^{n+1} > 0 \right\}.$$

This yields a perturbed tube $S_\varepsilon(w, \Phi)$. A standard computations show that the mean curvature $H(w, \Phi)$ of $S_\varepsilon(w, \Phi)$ is constant, with the right boundary conditions, is equivalent to solve a system of nonlinear partial differential equations where the principal part is the Jacobi operator about a hypersurface close to $\bar{S}_\varepsilon(K_\varepsilon)$. The solvability is based on the invertibility

of this linear operator depending on ε (small parameter). As we will see later, it turns out that this is possible only for some values of ε tending to zero. Once we have the invertibility our problem becomes readily a fixed point problem that we can solve provided our approximate solution is accurate enough. Our method here is similar in spirit to the one in [56]. It goes back to Malchiodi-Montenegro in [59] (see also [57], [58] and [60], for related issues).
To begin the procedure, we construct first an approximate solution in the following way: let $(y^1, y^2 \dots, y^k) \in \mathbb{R}^k$ (resp. $(z^1, z^2 \dots, z^n) \in B_1^n$) be the local coordinate variables on K_ε (resp. on S^n_+). Letting $\Phi : K \to \mathbb{R}^n$ and $w : B_1^n \times K_\varepsilon \to \mathbb{R}$, consider

$$S_0 \ : (y, z) \mapsto y \times \varepsilon^{-1}\Phi(\varepsilon y) \ + \ (1 + w(y, z))\, \Theta(z).$$

The surfaces nearby $\bar{S}_\varepsilon(K_\varepsilon)$ are parameterized (locally) by

$$G(y, z) : (y, z) \longrightarrow S_0(y, z) \longrightarrow F^\varepsilon(S_0(y, z))$$

where $F^\varepsilon : \mathbb{R}^k \times \mathbb{R}^{n+1} \to \bar{\Omega}$ is defined in (4.10) is "an almost isometry" which parameterize a neighborhood of K_ε in Ω_ε, B_1^n is the unit ball centered at the origin and $\Theta = \left(\Theta^1, \dots, \Theta^n, \Theta^{n+1}\right)$ is the stereographic projection from the south pole. Call the image of this map $S_\varepsilon(w, \Phi)$, so in particular

$$S_\varepsilon(0, 0) = \bar{S}_\varepsilon(K_\varepsilon).$$

Notice that since $\Theta^{n+1}\Big|_{\partial B_1^n} = 0$, it follows that all these surfaces close to $S_\varepsilon(K_\varepsilon)$ parameterized in this way have boundaries on $\partial\Omega_\varepsilon$.

One of the main features of this work is to we compute the mean curvature of $S_\varepsilon(w, \Phi)$, in § 4.2 which can be done following [56] but in contrast with that paper, we have to gather some new linear and quadratic terms involving Φ which will be relevant for the solvability. The linearized mean curvature operator about $\bar{S}_\varepsilon(K_\varepsilon)$ splits into some linear operators on w and Φ, given by

$$-\mathcal{L}_\varepsilon\, w - \varepsilon\, \langle \mathfrak{J}\, \Phi, \tilde{\Theta}\rangle + \varepsilon \mathcal{L}^1 w + \varepsilon \mathcal{J}^1(\Phi) + \varepsilon^2\, L(w, \Phi), \tag{4.2}$$

where $\mathfrak{J}$ is the Jacobi operator about K in the supporting surface $\partial\Omega$, see § 4.1.2;

$\mathcal{L}_\varepsilon := \varepsilon^2\, \Delta_K + \Delta_{S^n_+} + n; \qquad \mathcal{J}^1\Phi := -(3n+1)\, \Theta^{n+1} h(\tilde{\Theta})^a \langle \Phi_{\bar{a}}, \tilde{\Theta}\rangle + \Theta^{n+1} h(\Phi_{\bar{a}})^a + 2\Theta^{n+1} h$

and $\mathcal{L}^1$, $L(w,\Phi)$ are second order differential operators, see § 4.1.3, here h (resp. Γ) is the second fundamental form of $\partial\Omega$ (resp. K) and $h : \Gamma = h_{ab}\Gamma_{ab}$, where summation over repeated indices is understood. The quadratic part of the mean curvature is given by

$$\frac{n}{2}(\varepsilon w_{\bar a} + \langle \Phi_{\bar a}, \tilde\Theta\rangle)^2 - \varepsilon\langle \Phi_{\bar a}, \nabla_{S^n} w_{\bar a}\rangle - 2\varepsilon^2 \nabla_K^2 w : \Gamma(\Phi)$$
$$+\frac{n+2}{6}\langle R(\Phi,\tilde\Theta)\Phi\,,\,\tilde\Theta\rangle - \frac{1}{3}\langle R(\Phi,E_i)\Phi\,,\,E_i\rangle + Q(w) + \varepsilon\, Q(w,\Phi), \tag{4.3}$$

where $\tilde\Theta = \big(\Theta^1,\ldots,\Theta^n,0\big)$. Finally the boundary condition reads

$$\langle N, \mathcal{V}^\varepsilon\rangle = (-1+w)\,\frac{\partial w}{\partial \eta} + \bar{\mathcal{O}}(\varepsilon^2) + \varepsilon^2\,\bar L(w,\Phi) + \varepsilon\,\bar Q(w,\Phi) \qquad \text{on } \partial S^n_+ \times K,$$

where $\eta = -E_{n+1}$ is the normal vector field of ∂S^n_+ in S^n_+.

As we will explain later, the Jacobi operator about $\bar S_\varepsilon(K_\varepsilon)$ (very closed to the operator (4.2)) has inverse norm which blows-up at rate $\frac{1}{\varepsilon^R}$ for some $R>0$ and then one do not hope to apply a fixed point argument at this state.

However, we can adjust the tube $\bar S_\varepsilon(K_\varepsilon)$ as accurate as possible, to a tube $S_\varepsilon(\hat w^{(r)}, \hat\Phi^{(r)})$ satisfying (4.4) below. For that, letting $r \geq 1$ be an integer and setting

$$\hat w^{(r)} = \sum_{d=1}^{r} \varepsilon^d w^{(d)} \quad \text{and} \quad \hat\Phi^{(r)} = \sum_{d=1}^{r-1} \varepsilon^d \Phi^{(d)},$$

we have solved

$$\begin{array}{rcll} m\,H(\hat w^{(r)}, \hat\Phi^{(r)}) &=& n + \mathcal{O}(\varepsilon^{r+1}) & \text{in} \quad S_\varepsilon(\hat w^{(r)}, \hat\Phi^{(r)}), \\ \langle N, \mathcal{V}^\varepsilon\rangle &=& \bar{\mathcal{O}}(\varepsilon^{r+2}) & \text{on} \quad \partial S_\varepsilon(\hat w^{(r)}, \hat\Phi^{(r)}). \end{array} \tag{4.4}$$

This leads to an iterative scheme see § 4.3. The term of order $\mathcal{O}(\varepsilon)$ appearing in the expansion of the mean curvature (§ 4.2) depends linearly on the tangential curvature of K which is in the kernel of $\Delta_{S^n_+} + n$ (spanned by Θ^i with $i = 1,\ldots,n$) and normal curvature K which is perpendicular to this kernel. Consequently by Fredholm theorem, we can kill these terms by $w^{(1)}$ provided K is minimal.

Now to annihilate the higher order terms with suitable couples $(w^{(d)}, \Phi^{(d-1)})$, $d \geq 2$, if we project on the kernel of $\Delta_{S^n_+} + n$, there appears only $\mathfrak{J}$ (the Jacobi operator about K) acting on $\Phi^{(d-1)}$ because when we project, the term $\mathcal{J}^1\Phi^{(d-1)}$ disappear by oddness. Moreover neither the nonlinear terms appearing in the expansion of $H(w,\Phi)$ nor the perpendicularity condition will influence the iteration as well. Therefore the non-degeneracy of K is sufficient for this procedure at each step of the iterative scheme. In this way for any integer $r \geq 1$ we will be able to have (4.4) yielding good approximate solutions. We notice that it is more convenient to use the operator $\Delta_{S^n_+} + n + \langle \mathfrak{J}, \tilde{\Theta}\rangle$ to accomplish this task because it is invertible in $L^2(S^n_+ \times K)$. Unfortunately one cannot use it for the full solvability of the problem because w may not gain regularity. We refer to Section 4.3 for more details.

The final step (see § 4.4) is more delicate and consists of the invertibility of the Jacobi operator about $S_\varepsilon(\hat{w}^{(r)}, \hat{\Phi}^{(r)})$ which we call $\mathbb{L}_{\varepsilon,r}$. Let us mention that at this level all terms in the expansion depend on r except the model operator $-\mathcal{L}_\varepsilon\, w - \varepsilon\,\langle \mathfrak{J}\,\Phi, \tilde{\Theta}\rangle$. At first glance one sees that the operator $\mathbb{L}_{\varepsilon,r}$ is not so close to the model one in the usual Sobolev norms because of the competition between the operators $\langle \mathfrak{J}\,\Phi, \tilde{\Theta}\rangle$ and $\mathcal{L}^1_r$. This is due to fact that if one consider a tube of radius ε in a manifold $\mathcal{M}$ with boundary sitting on $\partial\mathcal{M}$, the mean curvature expansion makes appear terms of order ε depending on the second fundamental form of $\partial\mathcal{M}$. On the contrary, dealing with manifolds without boundary, as in [56], it turns out that in this case the first error terms are of order ε^2 and thus also in the expansion of the mean curvature of there perturbed tube, there cannot appear terms like εL, see [56] Proposition 4.1. Having bigger error terms than those in [56], we need more accurate approximate solutions and different spaces for the spectral analysis. Since our operator $\mathbb{L}_{\varepsilon,r}$ acts on the couple (w,Φ) almost separately, to tackle this it is natural to adjust the norms used for w and Φ. For any $v \in L^2(S^n_+ \times K)$ we decompose it as $v = \varepsilon^{1-2s}\, w + \langle \Phi, \tilde{\Theta}\rangle$ where Φ^i, $i = 1,\dots,n$ are the components of the projection of v onto the Kernel of $\Delta_{S^n_+} + n$ for some $s \in (0, 1/2)$. With this decomposition, in a suitable weighted Hilbert subspace of $L^2(S^n_+ \times K)$ we can see $\mathbb{L}_{\varepsilon,r}$ as a perturbation of the model one, see Proposition 4.4.1.

As mentioned above the existence of families of constant mean curvature surfaces holds only for a suitable sequence of intervals with length decreasing to zero and not the whole ε is related to a resonance phenomenon peculiar to concentration on positive dimensional sets and it appears in the study of several class of (geometric) non-linear PDE's. Concentration along sets of dimension $k = 1, \dots, n-1$ has been proved here, and analogous spectral properties hold true. By the Weyl's asymptotic formula, if solutions concentrate along a set of dimension d the average distance between those close to zero is of order ε^d. The resonance phenomenon was taken care of using a theorem by T. Kato, see [52], page 445, which allows to differentiate eigenvalues with respect to ε. In the aforementioned papers it was shown that, when varying the parameter ε, the spectral gaps near zero almost do not shrink, and invertibility can be obtained for a large family of epsilon's. The case of one dimensional limit sets can be handled using a more direct method based on a Lyapunov-Schmidt reduction, indeed in this case the distance between two consecutive small eigenvalues, candidates to be resonant, is sufficiently large and working away from resonant modes one can perform a contraction mapping argument quite easily. Here instead the average distance between two consecutive eigenvalues becomes denser and denser, to overcome this problem one needs to apply Kato's Theorem constructing first good approximate eigenfunctions, we refer to Section 4.4. And finally following [56], one can estimate the size of the spectral gaps, which determine the size of the norm of the inverse of $\mathbb{L}_{\varepsilon,r}$. For suitable values of ε the norm of the inverse of $\mathbb{L}_{\varepsilon,r}$ is of order $O(\frac{1}{\varepsilon^R})$ with a fixed $R > 0$ independent of r. Now as far as r can be chosen arbitrary large, our fixed point problem can be merely solved. This program is carried out in the last section.

4.1 Geometric backgroung

Let K be a k-dimensional submanifold of $(\partial\Omega, \overline{g})$ $(1 \le k \le m-1)$ and set $n = m - k$. We choose along K a local orthonormal frame field $((E_a)_{a=1,\cdots k}, (E_i)_{i=1,\cdots,n})$ which is oriented and call $N_{\partial\Omega}$ the interior normal field along $\partial\Omega$ and $N_{\partial\Omega}\Big|_K = E_{n+1}$. At points of K, $\mathbb{R}^{m+1}$ splits naturally as $T\partial\Omega \oplus \mathbb{R}E_{n+1}$ with $T\partial\Omega = TK \oplus NK$, where TK is the tangent space to K and $NK := NK^{\partial\Omega}$ represents the normal bundle in $\partial\Omega$, which are spanned respectively by $(E_a)_a$ and $(E_j)_j$.

4.1.1 Fermi coordinates on $\partial\Omega$ near K

Denote by $\bar{\nabla}$ the connection induced by the metric $\overline{g}$ and by $\bar{\nabla}^{\perp}$ the corresponding normal connection on the normal bundle. Given $q \in K$, we use some geodesic coordinates $\overline{y}$ centered at q defined by

$$f: \ \overline{y} \longrightarrow \exp_q^K(\overline{y}^a E_a). \tag{4.5}$$

This yields the coordinate vector fields $\overline{X}_a := f_*(\partial_{\overline{y}^a})$. We also assume that at q the normal vectors $(E_i)_i$, $i = 1, \ldots, n$, are transported parallely (with respect to $\bar{\nabla}^{\perp}$) through geodesics from q, so in particular

$$\overline{g}\left(\bar{\nabla}_{E_a} E_j\,, E_i\right) = 0 \quad \text{ at } q, \qquad\qquad i,j = 1, \ldots, n, a = 1, \ldots, k. \tag{4.6}$$

In a neighborhood of q, we choose *Fermi coordinates* $(\overline{y}, \zeta)$ on $\partial\Omega$ defined by

$$\overline{F}: \ (y, \zeta) \longrightarrow \exp^{\partial\Omega}_{f(\overline{y})}(\sum_{i=1}^{n} \zeta^i\, E_i); \qquad\qquad (\overline{y}, \zeta) = \left((\overline{y}^a)_a, (\zeta^i)_i\right). \tag{4.7}$$

Hence we have the coordinate vector fields

$$\overline{X}_i := \bar{F}_*(\partial_{\zeta^i}) \qquad \textit{and} \qquad \overline{X}_a := \bar{F}_*(\partial_{\overline{y}^a}).$$

By our choice of coordinates, on K the metric $\overline{g}_{\alpha,\beta} := \langle \overline{X}_\alpha, \overline{X}_\beta \rangle$ splits in the following way

$$\overline{g}(q) = \overline{g}_{ab}(q)\, d\overline{y}^a \otimes d\overline{y}^b + \overline{g}_{ij}(q)\, d\zeta^i \otimes d\zeta^j; \qquad\qquad q \in K. \tag{4.8}$$

We denote by $\Gamma_a^b(\cdot)$ the 1-forms defined on the normal bundle of K by

$$\Gamma_a^b(E_i) = \overline{g}(\nabla_{E_a} E_b, E_i). \tag{4.9}$$

The submanifold K is said to be *minimal* if the trace $\Gamma_a^a(\cdot) = 0$.

When we consider the metric coefficients in a neighborhood of K, we obtain a deviation from formula (4.8), which is expressed by the next lemma, see Proposition 2.1 in [56] for the proof. Denote by r the distance function from K.

4.1.1. LEMMA. *In the above coordinates $(\overline{y}, \zeta)$, for any $a = 1, ..., k$ and any $i, j = 1, ..., n$, we have*

$$\overline{g}_{ij}(0, \zeta) \ = \delta_{ij} + \tfrac{1}{3}\, \langle R(E_i, E_s)E_t, E_j \rangle\, \zeta^s\, \zeta^t \ + \ \mathcal{O}(r^3);$$

$$\overline{g}_{aj}(0, \zeta) \ = \mathcal{O}(r^2);$$

$$\overline{g}_{ab}(0, \zeta) \ = \delta_{ab} - 2\, \Gamma_a^b(E_i)\, \zeta^i + \left[\langle R(E_s, E_a)E_b, E_l \rangle + \Gamma_a^c(E_s)\, \Gamma_c^b(E_l)\right] \zeta^s \zeta^l + \mathcal{O}(r^3).$$

Here R_{istj} are computed at the point q of K parameterized by $(0,0)$.

The boundary of the scaled domain $\partial\Omega_\varepsilon := \frac{1}{\varepsilon}\partial\Omega$ is parameterized, in a neighborhood of $\varepsilon^{-1}q \in K_\varepsilon := \varepsilon^{-1}K$ by

$$\bar{F}^\varepsilon(y,x') := \frac{1}{\varepsilon}\bar{F}(\varepsilon y, \varepsilon x') \qquad \text{with } x' := (x^i, \cdots, x^n).$$

Hence we have the induced coordinate vector fields

$$X_i := \bar{F}^\varepsilon_*(\partial_{x^i}) \qquad \text{and} \qquad X_a := \bar{F}^\varepsilon_*(\partial_{y^a}).$$

By construction, $X_{\alpha|\varepsilon^{-1}q} = E_\alpha$ and $\mathcal{V}^\varepsilon(\varepsilon^{-1}q) = E_{n+1}$. From Lemma 4.1.1 it is evident that the metric g on $(\partial\Omega_\varepsilon, g)$ has the expansion given by the

4.1.2. LEMMA. *In a neighborhood of K_ε the following estimates hold*

$$\begin{aligned}
g_{ij}(0,x) &= \delta_{ij} + \tfrac{\varepsilon}{3}\langle R(E_i,E_s)E_t,E_j\rangle\, x^s\, x^t \;+\; \mathcal{O}(\varepsilon^2 r^3);\\
g_{aj}(0,x) &= \mathcal{O}(\varepsilon r^2);\\
g_{ab}(0,x) &= \delta_{ab} - 2\,\Gamma^b_a(E_i)\,x^i + \varepsilon\,\left[\langle R(E_s,E_a)E_b,E_l\rangle + \Gamma^c_a(E_s)\,\Gamma^b_c(E_l)\right] x^s x^l + \mathcal{O}(\varepsilon^2 r^3)
\end{aligned}$$

We can now parameterize tubular neighborhood of K_ε in Ω_ε,

$$F^\varepsilon(y,x',x^{n+1}) = \frac{1}{\varepsilon}\bar{F}(\varepsilon y,\varepsilon x') + x^{n+1}\mathcal{V}^\varepsilon(y,x'), \tag{4.10}$$

where $\mathcal{V}^\varepsilon(y,x') := N_{\partial\Omega}(\frac{1}{\varepsilon}\bar{F}(\varepsilon y,\varepsilon x'))$. We denote by h the second fundamental form of $\partial\Omega$ so that:

$$\langle d\mathcal{V}^\varepsilon(p)[X_\alpha], X_\beta\rangle = \varepsilon\, h_{\alpha,\beta}(q) \tag{4.11}$$

when $q = \bar{F}^\varepsilon(p)$.

4.1.2 The Jacobi operator about K

The linearized mean curvature operator about K is given by

$$\mathfrak{J} := \Delta^\perp - \mathcal{R}^\perp + \mathcal{B}, \tag{4.12}$$

with the normal Laplacian $\Delta^\perp$ is defined as

$$\Delta^\perp := \bar{\nabla}^\perp_{E_a}\,\bar{\nabla}^\perp_{E_a} - \bar{\nabla}^\perp_{\bar{\nabla}^T_{E_a}E_a},$$

$\bar{\nabla}^\perp$ denoting the connection on the normal bundle of K in $\partial\Omega$ while $\mathcal{B}$ is a symmetric operator defined by

$$\bar{g}(\mathcal{B}(X),Y) = \Gamma^b_a(X)\,\Gamma^a_b(Y) \qquad \text{for all } X,Y \in NK,$$

Γ is defined in (4.9) and $\mathcal{R}^\perp : N_pK \longrightarrow N_pK$ is given by

$$\mathcal{R}^\perp := (R(E_a, \cdot)\, E_a)^\perp ,$$

and $(\cdot)^\perp$ denotes the orthogonal projection on NK. Finally, we recall that the Ricci tensor is defined by

$$\mathrm{Ric}(X, Y) = -\bar{g}(R(X, E_\gamma)\, Y, E_\gamma) \qquad \text{for all } X, Y \in T_p\partial\Omega.$$

Finally, we recall that submanifold K is said to be *non-degenerate* if the Jacobi operator $\mathfrak{J}$ is invertible, or equivalently if the equation $\mathfrak{J}\Phi = 0$ has only the trivial solution among the sections in NK.

4.1.3 Notations for error terms

In the following, expressions of the form $L(w, \Phi)$ denote linear operators, in the functions w and Φ^j as well as their derivatives with respect to the vector fields $\varepsilon\, X_a$ and X_i up to second order, the coefficients of which are smooth functions on $S^n(\gamma) \times K$ bounded by a constant independent of ε in the $\mathcal{C}^\infty$ topology (where derivatives are taken using the vector fields $X_{\bar{a}}$ and X_i). Also $\bar{L}(w, \Phi)$ are restrictions of expressions like $L(w, \Phi)$ on $\partial S^n(\gamma) \times K$ with $L(w, \Phi)$ contains only one derivative of w or Φ with respect to the vector fields $\varepsilon\, X_a$ and X_i.

Similarly, expressions of the form $Q(w, \Phi)$ denote nonlinear operators, in the functions w and Φ^j as well as their derivatives with respect to the vector fields $\varepsilon\, X_a$ and X_i still up to second order, whose coefficients of the Taylor expansion are smooth functions on $S^n(\gamma) \times K$ which are bounded by a constant independent of ε in $\mathcal{C}^\infty$ topology (where derivatives are taken using the vector fields X_a and X_i). Moreover, Q vanishes quadratically in the pair (w, Φ) at 0 (that is, its Taylor expansion does not involve any constant nor any linear term). Also $\bar{Q}(w, \Phi)$ are restrictions of expressions like $Q(w, \Phi)$ on $\partial S^n(\gamma) \times K$ with $Q(w, \Phi)$ contains only one derivative of w or Φ with respect to the vector fields $\varepsilon\, X_a$ and X_i.

Finally, terms denoted $\mathcal{O}(\varepsilon^d)$ are smooth functions on $S^n(\gamma) \times K_\varepsilon$ which are bounded by a constant times ε^d in $\mathcal{C}^\infty$ topology (where derivatives are taken using the vector fields X_a and X_i). Also expressions like $\bar{\mathcal{O}}(\varepsilon^d)$ are restrictions of $\mathcal{O}(\varepsilon^d)$ on $\partial S^n(\gamma) \times K$.

4.2 Geometry of tubes

We derive expansions as ε tends to 0 for the metric, second fundamental form and mean curvature of $\bar{S}_\varepsilon(K_\varepsilon)$ and their perturbations.

Perturbed tubes

We now describe a suitable class of deformations of the geodesic tubes (in the metric induced by F^ε on $\mathbb{R}^{m+1}$) $\bar{S}_\varepsilon(K_\varepsilon)$, depending on a section Φ of $NK_\varepsilon := S^n_+ \times K_\varepsilon$ and a scalar function w on the spherical normal bundle $(SNK_\varepsilon)_+$ in $\partial\Omega_\varepsilon$.

We recall that $(y^1, y^2 \ldots, y^k) \in \mathbb{R}^k$ (resp. $(z^1, z^2 \ldots, z^n) \in B^n_1$) are the local coordinate variables on K_ε (resp. on S^n_+). Letting $\Phi : K \to \mathbb{R}^n$ and $w : B^n_1 \times K_\varepsilon \to \mathbb{R}$, consider

$$S_0 \,: (y, z) \mapsto y \times \varepsilon^{-1}\Phi(\varepsilon y) \,+\, (1 + w(y, z))\,\Theta(z).$$

The nearby surfaces of $\bar{S}_\varepsilon(K_\varepsilon)$ is parameterized (locally) by

$$G(y, z) : (y, z) \longrightarrow S_0(y, z) \longrightarrow F^\varepsilon(S_0(y, z))$$

namely

$$G(y, z) := F^\varepsilon\left(y, \frac{1}{\varepsilon}\Phi(\varepsilon y) + (1 + w(y, z))\tilde{\Theta}(z), (1 + w(y, z))\Theta^{n+1}(z)\right).$$

Since $\Theta^{n+1}\Big|_{\partial B^n_1} = 0$, it follows

$$G(y, z)\Big|_{\partial B^n_1} \in \partial\Omega_\varepsilon \qquad \text{for any } y.$$

The image of this map will be called $S_\varepsilon(w, \Phi)$. In particular

$$S_\varepsilon(0, 0) = \bar{S}_\varepsilon(K_\varepsilon).$$

It will be understood that for any fixed point $p = F^\varepsilon(y, 0) \in K_\varepsilon$, $\Phi(\varepsilon\, y) \in NK_\varepsilon \subset T_p\partial\Omega_\varepsilon$ and $\Theta(z) \in S^n_+ \subset NK_\varepsilon \oplus \mathbb{R}E_{n+1}$ are in the tangent space at p of $\mathbb{R}^{m+1}$ endowed with the metric induced by F^ε. For more convenience we introduce the following notations

Notation: On K_ε we will consider

$$\Phi := \Phi^j\, E_j \qquad\qquad \Phi_a := \partial_{y^a}\, \Phi^j\, E_j \qquad\qquad \Phi_{ab} := \partial_{y^a}\partial_{y^b}\, \Phi^j\, E_j$$

$\Theta := \Theta^j\, E_j + \Theta^{n+1} E_{n+1} = \tilde{\Theta} + \Theta^{n+1} E_{n+1} \qquad \Theta_i := \partial_{z^i}\Theta^j\, E_j + \partial_{z^i}\Theta^{n+1} E_{n+1} = \tilde{\Theta}_i + \Theta_i^{n+1} E$

For simplicity, we will write

$$w_j := \partial_{z^j} w; \quad w_a := \partial_{y^a} w; \qquad w_{ij} := \partial_{z^i}\, \partial_{z^j} w; \quad w_{ab} := \partial_{y^a}\, \partial_{y^b} w; \quad w_{aj} := \partial_{y^a}\, \partial_{z^j} w;$$

It is easy to see that the tangent space to $S_\varepsilon(w, \Phi)$ is spanned by the vector fields

$$\begin{aligned} Z_a &= G_*(\partial_{y^a}) = X_a + w_a\, \Upsilon + \Psi_a + (1+w)\Theta^{n+1} D_a \mathcal{V}^\varepsilon, \qquad a = 1, \ldots, k \\ Z_j &= G_*(\partial_{z^j}) = (1+w)\, \Upsilon_j + w_j\, \Upsilon + (1+w)\Theta^{n+1} D_j \mathcal{V}^\varepsilon, \qquad j = 1, \ldots, n, \end{aligned} \tag{4.13}$$

where

$$\Psi := \Phi^j\, X_j; \qquad \Psi_a := \partial_{y^a}\, \Phi^j\, X_j;$$

$$\Upsilon := \Theta^j\, X_j + \Theta^{n+1} \mathcal{V}^\varepsilon; \qquad \Upsilon_i := \partial_{z^i}\Theta^j\, X_j + \partial_{z^i}\Theta^{n+1} \mathcal{V}^\varepsilon$$

and

$$\begin{aligned} D_a \mathcal{V}^\varepsilon(y, (1+w(y,z))\tilde{\Theta} + \varepsilon^{-1}\Phi(\varepsilon y)) &= \varepsilon\left(h_{a\alpha} + (w_a \Theta^l + \Phi^l_a) h_{l\alpha}\right) X_\alpha; \\ D_j \mathcal{V}^\varepsilon(y, (1+w(y,z))\tilde{\Theta} + \varepsilon^{-1}\Phi(\varepsilon y)) &= \varepsilon\left(w_j \Theta^l + (1+w)\Theta^l_j\right) h_{l\alpha} X_\alpha. \end{aligned} \tag{4.14}$$

The first fundamental form

In this subsection we expand the coefficients of the first fundamental form of $S_\varepsilon(w, \Phi)$. Using the expansions in Lemma 4.1.2, one can easily get

$$\begin{aligned} \langle X_a, X_b \rangle &= \delta_{ab} - 2\,\varepsilon\, \Gamma_a^b(\Theta) - 2\, \Gamma_a^b\,(\Phi) + \mathcal{O}(\varepsilon^2) + \varepsilon\, L(w, \Phi) + Q(w, \Phi) \\ \langle X_i, X_j \rangle &= \delta_{ij} + \tfrac{\varepsilon}{3}\big(\langle R(\Theta, E_i)\, \Phi, E_j\rangle + \langle R(\Phi, E_i)\, \Theta, E_j\rangle\big) + \tfrac{1}{3}\langle R(\Phi, E_i)\, \Phi, E_j\rangle \\ &+ \mathcal{O}(\varepsilon^2) + \varepsilon^2\, L(w, \Phi) + \varepsilon\, Q(w, \Phi) \\ \langle X_i, X_a \rangle &= \mathcal{O}(\varepsilon^2) + \varepsilon\, L(w, \Phi) + Q(w, \Phi). \end{aligned} \tag{4.15}$$

These together with the fact that $R(\tilde{\Theta}, \tilde{\Theta}) = 0$ imply

$$\langle \Upsilon, \Upsilon_j \rangle = \frac{\varepsilon}{3}\, \langle R(\Phi, \tilde{\Theta})\, \tilde{\Theta}, \tilde{\Theta}_j \rangle + \frac{1}{3}\, \langle R(\Phi, \tilde{\Theta})\, \Phi, \tilde{\Theta}_j \rangle + \mathcal{O}(\varepsilon^2) + \varepsilon^2\, L(w, \Phi) + \varepsilon\, Q(w, \Phi) \tag{4.16}$$

Using similar arguments, and the fact that $\langle \Upsilon, \Upsilon \rangle = 1$ on K_ε yields

$$\langle \Upsilon, \Upsilon \rangle = 1 + \frac{1}{3}\, \langle R(\Phi, \tilde{\Theta})\, \Phi, \tilde{\Theta} \rangle + \mathcal{O}(\varepsilon^2) + \varepsilon^2\, L(w, \Phi) + Q(w, \Phi) \tag{4.17}$$

Moreover

$$\begin{aligned}\langle \Upsilon_i, \Upsilon_j\rangle &= \langle \Theta_i, \Theta_j\rangle + \frac{1}{3}\Big(\langle R(\Phi,\tilde\Theta_i)\,\tilde\Theta,\tilde\Theta_j\rangle + \langle R(\Phi,\tilde\Theta_j)\,\tilde\Theta,\tilde\Theta_i\rangle\Big) \qquad (4.18)\\ &+ \frac{1}{3}\langle R(\Phi,\tilde\Theta_i)\,\Phi,\tilde\Theta_j\rangle + \mathcal{O}(\varepsilon^2) + \varepsilon^2\, L(w,\Phi) + Q(w,\Phi).\end{aligned}$$

Now, by (4.14) we have that

$$\langle D_j \mathcal{V}^\varepsilon, \Upsilon\rangle = \varepsilon(1+w)\langle h(\tilde\Theta),\tilde\Theta_j\rangle + \varepsilon w_j \langle h(\tilde\Theta),\tilde\Theta\rangle + \mathcal{O}(\varepsilon^2) + \varepsilon^2\, L(w,\Phi) + \varepsilon Q(w,\Phi) \qquad (4.19)$$

and

$$\langle D_j \mathcal{V}^\varepsilon, \Upsilon_i\rangle = \varepsilon(1+w)\langle h(\tilde\Theta_i),\tilde\Theta_j\rangle + \varepsilon w_j \langle h(\tilde\Theta),\tilde\Theta_i\rangle + \mathcal{O}(\varepsilon^2) + \varepsilon^2\, L(w,\Phi) + \varepsilon Q(w,\Phi) \qquad (4.20)$$

We are now in position to expand the coefficients of the first fundamental form of $S_\varepsilon(w,\Phi)$. We have

4.2.1. Proposition. *For any $a, b \in \{1, \cdots, k\}$ and $i, j \in \{1, \cdots, n\}$, we have that*

$$\langle Z_a, Z_b\rangle = \delta_{ab} + 2\varepsilon\Theta^{n+1} h_{ab} - 2\varepsilon\Gamma_a^b(\tilde\Theta) - 2\,\Gamma_a^b(\Phi) + \mathcal{O}(\varepsilon^2) + \varepsilon\, L(w,\Phi) + Q(w,\Phi) \qquad (4.21)$$

$$\langle Z_a, Z_j\rangle = 2\varepsilon\Theta^{n+1} h(\tilde\Theta_j)^a + \langle \Phi_{\bar a},\tilde\Theta_j\rangle + \mathcal{O}(\varepsilon^2) + \varepsilon L(w,\Phi) + Q(w,\Phi) \qquad (4.22)$$

$$\begin{aligned}\langle Z_i, Z_j\rangle &= \langle \Theta_i,\Theta_j\rangle\,(1+2w) + 2\varepsilon(1+3w)\Theta^{n+1}\langle h(\tilde\Theta_i),\tilde\Theta_j\rangle\\ &+ 2\varepsilon\Theta^{n+1}\Big(\langle h(\tilde\Theta_i),\tilde\Theta\rangle w_j + \langle h(\tilde\Theta_j),\tilde\Theta\rangle w_i\Big)\\ &+ \frac{\varepsilon}{3}\Big(\langle R(\tilde\Theta,\tilde\Theta_i)\,\Phi,\tilde\Theta_j\rangle + \langle R(\tilde\Theta,\tilde\Theta_j)\,\Phi,\tilde\Theta_i\rangle\Big) + w_i w_j + \langle \Theta_i,\Theta_j\rangle w^2 \qquad (4.23)\\ &+ \frac{1}{3}\langle R(\Phi,\tilde\Theta_i)\,\Phi,\tilde\Theta_j\rangle + \mathcal{O}(\varepsilon^2) + \varepsilon^2\, L(w,\Phi) + \varepsilon Q(w,\Phi).\end{aligned}$$

The normal vector field

In this subsection we expand the unit normal to $S_\varepsilon(w,\Phi)$. Define the vector field

$$\tilde N := -\,\Upsilon + \alpha^j\, Z_j + \beta^c\, Z_c,$$

it is the outer normal field along $S_\varepsilon(w,\Phi)$ if we can determine α^j and β^c so that $\tilde N$ is orthogonal to all of the Z_b and Z_i. This leads to a linear system for α^j and β^a.

We have the following expansions

$$\langle \Upsilon, Z_a\rangle = w_a + \langle \Phi_{\bar a}, \tilde\Theta\rangle + \varepsilon \Theta^{n+1} \, (h(\tilde\Theta))^a + \varepsilon^2 \, L(w,\Phi) + \varepsilon \, Q(w,\Phi); \quad (4.24)$$

$$\begin{aligned}\langle \Upsilon, Z_j\rangle &= w_j + \varepsilon(1+2w)\Theta^{n+1}\langle h(\tilde\Theta), \tilde\Theta_j\rangle + 2\varepsilon\Theta^{n+1} w_j \langle h(\tilde\Theta), \tilde\Theta\rangle \\ &+ \frac{\varepsilon}{3}\, \langle R(\Phi,\tilde\Theta)\,\tilde\Theta, \tilde\Theta_j\rangle + \frac{1}{3}\, \langle R(\Phi,\tilde\Theta)\,\Phi, \tilde\Theta_j\rangle + \mathcal{O}(\varepsilon^2) + \varepsilon^2\, L(w,\Phi) + \varepsilon\, Q(w,\Phi). \quad (4.25)\end{aligned}$$

These follow from (4.15) together with the fact that $\langle \Upsilon, Z_a\rangle = 0$ and $\langle \Upsilon, Z_j\rangle = 0$ on K_ε.

Using Proposition 4.2.1, and some algebraic calculations, one can obtain

$$\beta^c = w_c + \langle \Phi_c, \tilde\Theta\rangle + \varepsilon\Theta^{n+1} h(\tilde\Theta)^c + \mathcal{O}(\varepsilon^2) + \varepsilon\, L(w,\Phi) + Q(w,\Phi). \quad (4.26)$$

and

$$\begin{aligned}\alpha^j \, \langle \Theta_j, \Theta_i\rangle &= w_i + \varepsilon\Theta^{n+1}\langle h(\tilde\Theta), \tilde\Theta_i\rangle + \varepsilon\Theta^{n+1}\langle h(\tilde\Theta), \tilde\Theta\rangle w_i \\ &- 2\varepsilon\Theta^{n+1}\Big(\langle h(\tilde\Theta_l), \tilde\Theta_i\rangle w_l + h(\tilde\Theta_i)^a w_a + h(\tilde\Theta_i)^a\langle \Phi_a, \tilde\Theta\rangle\Big) \\ &+ \frac{1}{3}\,\varepsilon\langle R(\Phi,\tilde\Theta)\,\tilde\Theta, \tilde\Theta_i\rangle - \varepsilon\Theta^{n+1} h(\tilde\Theta)^a\langle \Phi_a, \tilde\Theta_i\rangle \quad (4.27) \\ &- 2ww_i - w_a\langle \Phi_a, \tilde\Theta_i\rangle - \langle \Phi_a, \tilde\Theta\rangle\langle \Phi_a, \tilde\Theta_i\rangle + \frac{1}{3}\,\langle R(\Phi,\tilde\Theta)\,\Phi, \tilde\Theta_i\rangle \\ &+ \mathcal{O}(\varepsilon^2) + \varepsilon^2\, L(w,\Phi) + \varepsilon Q(w,\Phi).\end{aligned}$$

Using these and the fact that $\langle \Theta_j, \Theta_i\rangle = \mu^2\delta_{ij}$, a straightforward computations imply

$$\begin{aligned}|\tilde N|^{-1} &= 1 + \varepsilon\Theta^{n+1}\left(\frac{1}{\mu^2}\langle h(\tilde\Theta), \tilde\Theta_i\rangle w_i + h(\tilde\Theta)^c w_c + h(\tilde\Theta)^c\langle \Phi_c, \tilde\Theta\rangle\right) + \frac{1}{6}\,\langle R(\Phi,\tilde\Theta)\,\Phi, \tilde\Theta\rangle \\ &+ \frac{1}{2}\left(w_c^2 + \frac{1}{\mu^2} w_j^2 + 2w_c\langle \Phi_c, \tilde\Theta\rangle + \langle \Phi_c, \tilde\Theta\rangle^2\right) + \mathcal{O}(\varepsilon^2) + \varepsilon^2\, L(w,\Phi) + \varepsilon Q(w,\Phi).\end{aligned}$$

The unit normal to the perturbed geodesic tube is then given simply by $N = \frac{\tilde N}{|\tilde N|}$. We summarize this in the following lemma

4.2.2. Proposition. *The normal vector field N to $S_\varepsilon(w,\Phi)$ is given by $N = \frac{\tilde N}{|\tilde N|}$ where*

$$\tilde N := -\,\Upsilon + \alpha^j\, Z_j + \beta^c\, Z_c \quad (4.28)$$

and where the coefficients α^j and β^c are given by formulas (7.9) *and* (4.26).

Using the fact that $\Theta^{n+1}\Big|_{\partial B_1^n} = 0$ we can easily deduce

4.2.1. Lemma. *The perpendicularity condition is given by*

$$\langle N, \mathcal{V}^\varepsilon \rangle = (-1+w)\, w_j z^j + \bar{\mathcal{O}}(\varepsilon^2) + \varepsilon^2\, \bar{L}(w,\Phi) + \varepsilon\, \bar{Q}(w,\Phi) \qquad \text{on } \partial(SNK)_+,$$

Proof. Since $\Theta^{n+1}\Big|_{\partial B_1^n} = 0$ it follows that $\langle \mathcal{V}^\varepsilon, -\Upsilon + \beta^c\, Z_c \rangle = 0$ on ∂B_1^n on the other hand using the fact that $R(E_i, E_i) = 0$ with $\frac{\partial \tilde{\Theta}}{\partial \tau}\Big|_{\partial B_1^n} = 0$ (see §2.3) we get

$$\langle \alpha^j\, Z_j, \mathcal{V}^\varepsilon \rangle = (-1+w)\, w_j \Theta_j^{n+1} + \bar{\mathcal{O}}(\varepsilon^2) + \varepsilon^2\, \bar{L}(w,\Phi) + \varepsilon\, \bar{Q}(w,\Phi) \qquad \text{on } \partial(SNK)_+.$$

The lemma now follows since $\Theta_j^{n+1} = -\mu\Theta^j = -\mu^2 z^j$ and $\mu\Big|_{\partial B_1^n} = 1$. ∎

The second fundamental form

In this subsection we expand the coefficients of the second fundamental form. Recall that $\bar{\nabla}$ is the Levi-Civita connection on $\partial\Omega$ and h its second fundamental form, the derivation for vector fields on $\partial\Omega$ yields

$$\frac{\partial}{\partial z^i} X_\alpha(y, (1+w(y,z))\tilde{\Theta} + \varepsilon^{-1}\Phi(\varepsilon y)) = \varepsilon (w_i \Theta^l + (1+w)\Theta_i^l) \left(\bar{\nabla}_{X_l} X_\alpha - h_{l\alpha} \mathcal{V}^\varepsilon \right),$$

$$\frac{\partial}{\partial y^a} X_\alpha(y, (1+w(y,z))\tilde{\Theta} + \varepsilon^{-1}\Phi(\varepsilon y)) = \varepsilon \delta_{ab} \left(\bar{\nabla}_{X_b} X_\alpha - h_{b\alpha} \mathcal{V}^\varepsilon \right) + \varepsilon \left(w_a \Theta^l + \Phi_a^l \right) \left(\bar{\nabla}_{X_l} X_\alpha \right.$$

4.2.3. Proposition. *The following expansions hold*

$$\begin{aligned} \langle N, \frac{\partial}{\partial y^a} Z_a \rangle \;&=\; -\varepsilon \Gamma_a^a(\tilde{\Theta}) + \varepsilon \Theta^{n+1} h_{aa} - w_{aa} - \varepsilon\, \langle \Phi_{aa}, \tilde{\Theta} \rangle - \varepsilon\, \langle R(\Phi, E_a)\, E_a, \tilde{\Theta} \rangle \\ &+\; \varepsilon\, \Gamma_a^c(\tilde{\Theta})\, \Gamma_c^a(\Phi) - 2\varepsilon \Theta^{n+1} w_a h(\tilde{\Theta})^a + \frac{\varepsilon}{\mu^2} w_l \left(\Gamma_a^a(\tilde{\Theta}_l) - h_{aa} \Theta_l^{n+1} \right) \qquad (4.29) \\ &+\; \mathcal{O}(\varepsilon^2) + \varepsilon^2\, L(w,\Phi) + \varepsilon Q(w,\Phi); \end{aligned}$$

$$
\begin{aligned}
\langle N, \frac{\partial}{\partial z^j} Z_j \rangle &= \mu^2(1+w) - w_{jj} - \varepsilon \Theta^{n+1} \langle h(\tilde\Theta), \tilde\Theta \rangle w_{jj} - 2\varepsilon \Theta_j^{n+1} \langle h(\tilde\Theta), \tilde\Theta \rangle w_j \\
&+ \varepsilon (1+2w) \Big(\Theta^{n+1} \langle h(\tilde\Theta_j), \tilde\Theta_j \rangle - 2\Theta_j^{n+1} \langle h(\tilde\Theta), \tilde\Theta_j \rangle - \Theta^{n+1} \langle h(\tilde\Theta), \tilde\Theta_{jj} \rangle \Big) \\
&+ \frac{\varepsilon}{\mu^2} w_k \Big(2\Theta^{n+1} \langle h(\tilde\Theta_k), \tilde\Theta_{ii} \rangle + 2\Theta_i^{n+1} \langle h(\tilde\Theta_k), \tilde\Theta_i \rangle + \Theta_k^{n+1} \langle h(\tilde\Theta_i), \tilde\Theta_i \rangle \Big) \\
&+ \frac{2}{3} \varepsilon \, \langle R(\Phi, \tilde\Theta_j) \, \tilde\Theta, \tilde\Theta_j \rangle - \frac{\varepsilon}{3} \, \langle R(\Phi, \tilde\Theta) \, \tilde\Theta, \tilde\Theta_{jj} \rangle + 2\varepsilon w_c \Big(\Theta_j^{n+1} h(\tilde\Theta_j)^c + \Theta^{n+1} \\
&+ 2\varepsilon \langle \Phi_{\bar c}, \tilde\Theta \rangle \Big(\Theta_j^{n+1} h(\tilde\Theta_j)^c + \Theta^{n+1} h(\tilde\Theta_{jj})^c \Big) + \varepsilon \Theta^{n+1} h(\tilde\Theta)^c \Big(\langle \Phi_c, \tilde\Theta_{jj} \rangle + \mu^2 \\
&+ \varepsilon \Theta^{n+1} h(\tilde\Theta)^c \Big(w_c \langle \tilde\Theta, \tilde\Theta_{jj} \rangle + \mu^2 w_c \Big) - \frac{1}{6} \mu^2 \langle R(\Phi, \tilde\Theta) \, \Phi, \tilde\Theta \rangle - \frac{1}{3} \langle R(\Phi, \tilde\Theta) \, \Phi \\
&- \frac{1}{2} \mu^2 w_c^2 + \frac{1}{2} \mu^2 |\langle \Phi_c, \tilde\Theta \rangle|^2 - \frac{1}{2} w_k^2 + 2 w_j^2 + \langle \Phi_c, \tilde\Theta_{jj} \rangle w_c + \langle \Phi_c, \tilde\Theta \rangle \langle \Phi_c, \tilde\Theta_{jj} \rangle \\
&+ (1+2w) \alpha^k \langle \Theta_{jj}, \Theta_k \rangle + \mathcal{O}(\varepsilon^2) + \varepsilon^2 \, L(w, \Phi) + \varepsilon Q(w, \Phi);
\end{aligned}
$$

$$
\begin{aligned}
\langle N, \frac{\partial}{\partial y^a} Z_b \rangle &= -\Gamma_a^b(\tilde\Theta) + \varepsilon \Theta^{n+1} h_{ab} - w_{ab} + \mathcal{O}(\varepsilon^2) + \varepsilon L(w, \Phi) + Q(w, \Phi)) \quad a \neq b; \\
\langle N, \frac{\partial}{\partial y_a} Z_j \rangle &= \varepsilon \Theta_j^{n+1} h(\tilde\Theta)^a + \varepsilon \Theta^{n+1} h(\tilde\Theta_j)^a - w_{aj} + \mathcal{O}(\varepsilon^2) + \varepsilon L(w, \Phi) + Q(w, \Phi); \\
\langle N, \frac{\partial}{\partial z_i} Z_j \rangle &= -w_{ij} - \varepsilon \Theta_i^{n+1} \langle h(\tilde\Theta), \tilde\Theta_j \rangle - \varepsilon \Theta_j^{n+1} \langle h(\tilde\Theta), \tilde\Theta_i \rangle + \varepsilon \Theta^{n+1} \langle h(\tilde\Theta_i), \tilde\Theta_j \rangle \\
&- \varepsilon \Theta^{n+1} \langle h(\tilde\Theta), \tilde\Theta_{ij} \rangle + \alpha^k \langle \Theta_{ij}, \Theta_k \rangle + \mathcal{O}(\varepsilon^2) + \varepsilon L(w, \Phi) + Q(w, \Phi), \quad i \neq
\end{aligned}
$$

Proof. The proof is similar in spirit to the one of Proposition 3.3 in [56]. So we will be sketchy here referring to the aforementioned paper for more details. We have that

$$
\begin{aligned}
\frac{\partial}{\partial y^a} Z_a &= \varepsilon \left(\bar\nabla_{X_a} X_a - h_{aa} \mathcal{V}^\varepsilon \right) + w_{aa} \Upsilon + 2\Theta^{n+1} w_a D_a \mathcal{V}^\varepsilon + \varepsilon \Phi_{aa}^l X_l + \Theta^{n+1} D_a D_a \mathcal{V}^\varepsilon \\
&+ \left(\mathcal{O}(\varepsilon^2) + \varepsilon^2 L(w, \Phi) + \varepsilon Q(w, \Phi) \right) X_\alpha + \left(\mathcal{O}(\varepsilon^2) + \varepsilon^2 L(w, \Phi) + \varepsilon Q(w, \Phi) \right) \mathcal{V}^\varepsilon
\end{aligned}
$$

and for $a \neq b$

$$
\begin{aligned}
\frac{\partial}{\partial y^a} Z_b &= \varepsilon \left(\bar\nabla_{X_b} X_a - h_{ab} \mathcal{V}^\varepsilon \right) + w_{ab} \Upsilon \\
&+ \left(\mathcal{O}(\varepsilon^2) + \varepsilon L(w, \Phi) + Q(w, \Phi) \right) X_\alpha + \left(\mathcal{O}(\varepsilon^2) + \varepsilon^2 L(w, \Phi) + \varepsilon Q(w, \Phi) \right) \mathcal{V}^\varepsilon;
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial z^i} Z_i &= w_{ii}\Upsilon + 2\, w_i \Upsilon_i + 2\varepsilon \Theta^l \Theta_i^s w_i \left(\bar\nabla_{X_s} X_l - h_{sl}\mathcal{V}^\varepsilon\right) + 2\Theta^{n+1} D_i \mathcal{V}^\varepsilon\, w_i + (1+w)\Upsilon_{ii} \\
&+ (1+w)\left(2\Theta_i^{n+1} D_i \mathcal{V}^\varepsilon + \Theta^{n+1} D_i D_i \mathcal{V}\right) + \varepsilon(1+2w)\Theta_i^l \Theta_i^s \left(\bar\nabla_{X_s} X_l - h_{sl}\mathcal{V}^\varepsilon\right) \\
&+ \left(\mathcal{O}(\varepsilon^2) + \varepsilon^2 L(w,\Phi) + \varepsilon Q(w,\Phi)\right) X_\alpha + \left(\mathcal{O}(\varepsilon^2) + \varepsilon^2 L(w,\Phi) + \varepsilon Q(w,\Phi)\right) \mathcal{V}^\varepsilon
\end{aligned}
$$

and for $i \neq j$

$$
\begin{aligned}
\frac{\partial}{\partial z^i} Z_j &= w_{ij}\Upsilon + w_i\Upsilon_j + w_j \Upsilon_i + \Theta_i^{n+1} D_j \mathcal{V}^\varepsilon + \Theta_j^{n+1} D_i \mathcal{V}^\varepsilon + (1+w)\Upsilon_{ij} + \varepsilon \Theta_i^l \Theta_j^s \left(\bar\nabla_X\right. \\
&+ \Theta^{n+1} D_i D_j \mathcal{V}^\varepsilon + \left(\mathcal{O}(\varepsilon^2) + \varepsilon L(w,\Phi) + Q(w,\Phi)\right) X_\alpha + \left(\mathcal{O}(\varepsilon^2) + \varepsilon^2 L(w,\Phi) +\right.
\end{aligned}
$$

Finally

$$
\begin{aligned}
\frac{\partial}{\partial y^a} Z_j = \frac{\partial}{\partial z_j} Z_a &= \varepsilon \Theta_j^s \left(\bar\nabla_{X_s} X_a - h_{as}\mathcal{V}^\varepsilon\right) + w_{aj}\Upsilon + w_a \Upsilon_j + \Theta_j^{n+1} D_a \mathcal{V}^\varepsilon \\
&+ \left(\mathcal{O}(\varepsilon^2) + \varepsilon^2 L(w,\Phi) + \varepsilon Q(w,\Phi)\right) X_\alpha + \left(\mathcal{O}(\varepsilon^2) + \varepsilon^2 L(w,\Phi) + \varepsilon Q\right.
\end{aligned}
$$

Recalling the expansions, see Lemma 2.1 in [56].

$$
\begin{aligned}
\bar\nabla_{X_i} X_j &= (\mathcal{O}(\varepsilon) + L(w,\Phi) + Q(w,\Phi))\, X_\gamma, \\
\bar\nabla_{X_a} X_i &= -\Gamma_a^b(E_i)\, X_b + (\mathcal{O}(\varepsilon) + L(w,\Phi) + Q(w,\Phi))\, X_\gamma,
\end{aligned}
\tag{4.31}
$$

We will also need the following expansion which follows from the result of Lemma 2.2 in [56] (with obvious modifications).

$$
\begin{aligned}
\bar\nabla_{X_a} X_b &= \Gamma_a^b(E_j)\, X_j - \langle R(\varepsilon\,\tilde\Theta + \Phi, E_a)\, E_j, E_b\rangle\, X_j \\
&+ \frac{1}{2}\Big(\langle R(E_a, E_b)\,(\varepsilon\,\tilde\Theta + \Phi), E_j\rangle - \Gamma_a^c(\varepsilon\,\tilde\Theta + \Phi)\,\Gamma_c^b(E_j) - \Gamma_c^b(\varepsilon\,\tilde\Theta + \Phi)\,\Gamma_a^c(E_j)\Big. \\
&+ (\mathcal{O}(\varepsilon) + L(w,\Phi) + Q(w,\Phi))\, X_c + (\mathcal{O}(\varepsilon^2) + \varepsilon\, L(w,\Phi) + Q(w,\Phi))\, X_j.
\end{aligned}
$$

These imply in particular

$$
\begin{aligned}
\langle \Upsilon, \bar\nabla_{X_a} X_a\rangle &= \Theta^l \Gamma_a^a(E_i)\left(\partial_{li} + 2\varepsilon\Theta^{n+1} h_{li}\right) - \varepsilon\langle R(\tilde\Theta, E_a)\tilde\Theta, E_a\rangle - \langle R(\tilde\Theta, E_a)\Phi, E_a\rangle \\
&- \varepsilon \Gamma_a^c(\tilde\Theta)\Gamma_c^a(\tilde\Theta) - \Gamma_a^c(\tilde\Theta)\Gamma_c^a(\Phi) + \mathcal{O}(\varepsilon^2) + \varepsilon L(w,\Phi) + Q(w,\Phi).
\end{aligned}
$$

On the other hand we have that

$$
D_a D_a \mathcal{V}^\varepsilon = \varepsilon w_{aa} h(\tilde\Theta)^\alpha X_\alpha + \left(\mathcal{O}(\varepsilon^2) + \varepsilon^2 L(w,\Phi) + \varepsilon Q(w,\Phi)\right) X_\beta + \left(\mathcal{O}(\varepsilon^2) + \varepsilon^2 L(w,\Phi)\right.
$$

which implies

$$
\langle D_a D_a \mathcal{V}^\varepsilon, \Upsilon\rangle = \varepsilon w_{aa}\langle h(\tilde\Theta), \tilde\Theta\rangle + \mathcal{O}(\varepsilon^2) + \varepsilon^2 L(w,\Phi) + \varepsilon Q(w,\Phi). \tag{4.33}
$$

Using these together with (4.26), (7.9) and Lemma 4.1.2, the first estimate follows at once. For the other estimates one can proceed similarly.■

The mean curvature of perturbed tubes

Collecting the estimates of the last subsection we obtain the expansion of the mean curvature of the hypersurface $S_\varepsilon(w,\Phi)$. In the coordinate system defined in the previous sections, we get

$$\begin{aligned}
m\,H(w,\Phi) \;&=\; n-\varepsilon\,\Gamma_a^a(\tilde\Theta)+\varepsilon\,\Theta^{n+1}\,h_{aa}+\varepsilon\,\Theta^{n+1}\Big[(n+3)\langle h(\tilde\Theta),\tilde\Theta\rangle-h_{jj}\Big]+\mathcal{O}(\varepsilon^2)\\
&-\;\big(\Delta_{K_\varepsilon}w+\Delta_{S^n}w+nw\big)-\varepsilon\Big(\langle\,\Delta_K\Phi+R(\Phi,E_a)\,E_a,\tilde\Theta\,\rangle-\Gamma_a^c(\Phi)\,\Gamma_c^a(\tilde\Theta)\Big)\\
&-\;\varepsilon\Theta^{n+1}\langle h(\tilde\Theta),\tilde\Theta\rangle\,\Delta_{S^n}w-2\varepsilon(n+3)\,\Theta^{n+1}\langle h(\tilde\Theta),\nabla_{S^n}w\rangle+2\varepsilon\Theta^{n+1}\,\nabla^2_{S^n}w\\
&-\;\varepsilon\Big(\langle h(\tilde\Theta),\tilde\Theta\rangle+h_{jj}+h_{aa}\Big)\,\langle\nabla_{S^n}w,E_{n+1}\rangle-(1+3n)\varepsilon\Theta^{n+1}h(\tilde\Theta)^a\,w_a\\
&-\;2\varepsilon\Theta^{n+1}h(\nabla_{S^n}w_a)^a+\varepsilon\,\Gamma_a^a(\nabla_{S^n}w)-2\varepsilon\nabla^2_{K_\varepsilon}w:\Gamma(\tilde\Theta)+2\varepsilon\Theta^{n+1}h_{aa}w_{aa}\\
&-\;(3n+1)\varepsilon\,\Theta^{n+1}h(\tilde\Theta)^a\langle\Phi_{\bar a},\tilde\Theta\rangle+\varepsilon\,\Theta^{n+1}h(\Phi_{\bar a})^a+2\varepsilon\Theta^{n+1}h:\Gamma(\Phi)\\
&+\;n\,w^2+\tfrac{2-n}{2}\,|\nabla_{S^n}w|^2+2\,w\,\Delta_{S^n}w-\tfrac{n}{2}(w_a+\langle\Phi_{\bar a},\tilde\Theta\rangle)^2\\
&-\;\langle\Phi_{\bar a},\nabla_{S^n}w_a\rangle-2\nabla^2_{K_\varepsilon}w:\Gamma(\Phi)+\tfrac{n+2}{6}\langle R(\Phi,\tilde\Theta)\Phi\,,\,\tilde\Theta\rangle-\tfrac{1}{3}\langle R(\Phi,E_i)\Phi\,,\,E\\
&+\;\mathcal{O}(\varepsilon^2)+\varepsilon^2\,L(w,\Phi)+\varepsilon\,Q(w,\Phi).
\end{aligned}$$

Here we have used the formulas in Lemma 2.3.1, the fact that

$$\Delta_{S^n}=\frac{1}{\mu^2}\,(\Delta_{\mathbb{R}^n}-\langle\Theta_{ii},\Theta_k\rangle\partial_k)\,.$$

and the notation $A:B=A_{st}B_{st}$ for two linear operators A and B. Here summation over repeated indices is understood.

Let us emphasize the use of the variables $y_{\bar a}=\varepsilon y_a$ on K. With an abuse of notation, we call w the function $\bar w(\bar y)=w(y)=w(\varepsilon^{-1}\bar y)$ defined on K so that $\varepsilon w_{\bar a}=w_a$ and $\varepsilon^2 w_{\bar a\bar a}=w_{aa}$. We first define the following operators appearing in the above expansion

$$\begin{aligned}
\mathcal{L}^1(w):\;&=\;-\langle h(\tilde\Theta),\tilde\Theta\rangle\,\Delta_{S^n}w-2(n+3)\,\Theta^{n+1}\langle h(\tilde\Theta),\nabla_{S^n}w\rangle+2\Theta^{n+1}\,\nabla^2_{S^n}w:h\\
&-\;\Big(\langle h(\tilde\Theta),\tilde\Theta\rangle+h_{jj}+h_{aa}\Big)\,\langle\nabla_{S^n}w,E_{n+1}\rangle\\
&-\;\varepsilon(1+3n)\Theta^{n+1}\langle h(\tilde\Theta),\nabla_K w\rangle+\varepsilon\Theta^{n+1}h(\nabla_{S^n}w_{\bar a})^a-2\varepsilon^2\nabla^2_K w:\Gamma(\tilde\Theta)+2\varepsilon^2\Theta^{n}
\end{aligned}$$

$$\mathcal{J}^1\Phi := -(3n+1)\,\Theta^{n+1} h(\tilde\Theta)^a \langle \Phi_{\bar a}, \tilde\Theta\rangle + \Theta^{n+1} h(\Phi_{\bar a})^a + 2\Theta^{n+1} h : \Gamma(\Phi), \tag{4.35}$$

and the quadratic term

$$\begin{aligned} \mathcal{Q}^1(w,\Phi) : \;&=\; n\,w^2 + \frac{2-n}{2}\,|\nabla_{S^n} w|^2 + 2\,w\,\Delta_{S^n} w - \frac{n}{2}(\varepsilon w_{\bar a} + \langle \Phi_{\bar a}, \tilde\Theta\rangle)^2 - \varepsilon\langle \Phi_{\bar a}, \nabla_{S^n} w \\ &-\; 2\varepsilon^2 \nabla^2_K w : \Gamma(\Phi) + \frac{n+2}{6}\langle R(\Phi,\tilde\Theta)\Phi\,,\,\tilde\Theta\rangle - \frac{1}{3}\langle R(\Phi, E_i)\Phi\,,\,E_i\rangle. \qquad (4.3 \end{aligned}$$

Next, we define

$$\mathcal{L}_\varepsilon := \varepsilon^2\,\Delta_K + \Delta_{S^n} + n, \qquad \mathcal{L}_0 := \Delta_{S^n} + n$$

and the Jacobi operator about K in $(\partial\Omega, \bar g)$, see § 4.1.2

$$\mathfrak{J} := \Delta^\perp - \mathcal{R}^\perp + \mathcal{B}.$$

Recall that (see § 2.3) the outer unit normal to the boundary of ∂S^n_+ in S^n_+ is $\eta = -E_{n+1}$,

$$\frac{\partial w}{\partial \eta} = -\langle \nabla_{S^n_+} w, E_{n+1}\rangle.$$

Using these definitions, we obtain the following result :

4.2.4. Proposition. *Assume that K is a minimal submanifold, then the mean curvature of $S_\varepsilon(w,\Phi)$ can be expanded as*

$$\begin{aligned} m\,H(w,\Phi) \;&=\; n + \varepsilon\,\Theta^{n+1} h_{aa} + \varepsilon\,\Theta^{n+1}\Big[(n+3)\langle h(\tilde\Theta),\tilde\Theta\rangle - h_{jj}\Big] + \mathcal{O}(\varepsilon^2) \\ &-\; \mathcal{L}_\varepsilon\, w - \varepsilon\,\langle \mathfrak{J}\,\Phi, \tilde\Theta\rangle + \varepsilon\mathcal{L}^1 w + \varepsilon\mathcal{J}^1(\Phi) + \mathcal{Q}^1(w,\Phi) \\ &+\; \varepsilon^2\, L(w,\Phi) + \varepsilon\, Q(w,\Phi) \qquad\qquad \textit{in } S_\varepsilon(w,\Phi), \end{aligned}$$

where $\mathcal{L}^1$ is defined in (4.34)*, $\mathcal{J}^1$ is given in* (4.35)*, while $\mathcal{Q}^1$ is a quadratic term defined in* (4.36)*. Moreover, the orthogonality condition is equivalent to the following boundary condition on the function w:*

$$\langle N, \mathcal{V}^\varepsilon\rangle = -\frac{\partial w}{\partial\eta} + w\,\frac{\partial w}{\partial\eta} + \bar{\mathcal{O}}(\varepsilon^2) + \varepsilon^2\,\bar L(w,\Phi) + \varepsilon\,\bar Q(w,\Phi) \qquad \textit{on } \partial S_\varepsilon(w,\Phi). \tag{4.37}$$

Proof. The expression of the mean curvature can be obtained rather easily taking into account the above definitions (with obvious modifications) and the minimality of K which implies

$$\Gamma^a_a = 0.$$

■

With these notations finding w and Φ such that the equation $m\,H = n$ and $\langle N, \mathcal{V}^\varepsilon\rangle = 0$ hold is equivalent to solve

$$\begin{cases} \mathcal{L}_\varepsilon\, w + \varepsilon\,\langle \mathfrak{J}\,\Phi, \tilde{\Theta}\rangle &= \varepsilon\,\Theta^{n+1}\,h_{aa} + \varepsilon\,\Theta^{n+1}\left[(n+3)\langle h(\tilde{\Theta}), \tilde{\Theta}\rangle - h_{jj}\right] + \mathcal{O}(\varepsilon^2) \\ &+ \;\varepsilon\,\mathcal{J}^1(\Phi) + \varepsilon\mathcal{L}^1 w + \mathcal{Q}^1(w,\Phi) + \varepsilon^2\,L(w,\Phi) + \varepsilon\,Q(w,\Phi) \quad \text{in } S^n_+ \\ \dfrac{\partial w}{\partial \eta} &= w\,\dfrac{\partial w}{\partial \eta} + \bar{\mathcal{O}}(\varepsilon^2) + \varepsilon^2\,\bar{L}(w,\Phi) + \varepsilon\,\bar{Q}(w,\Phi) \quad \text{on } \partial S^n_+ \end{cases} \tag{4.38}$$

4.3 Adjusting the tube $\bar{S}_\varepsilon(K_\varepsilon)$

In this section we annihilate the error terms ($\mathcal{O}(\varepsilon)$) appearing in (4.38) at any given order. The non-degeneracy of the submanifold K will play a crucial role in such a construction. We denote by Π_1 the L^2 projection on the subspace spanned by the Θ^i, $i = 1, \cdots, n$ and set $(SNK)_+ := S^n_+ \times K$. We set

$$\hat{w}^{(r)} = \sum_{d=1}^{r} \varepsilon^d w^{(d)} \qquad \text{and} \qquad \hat{\Phi}^r = \sum_{d=1}^{r-1} \varepsilon^d \Phi^{(d)}.$$

Construction of $w^{(1)}$: We first want to kill the term $\mathcal{O}(\varepsilon)$. This is equivalent to have

$$\begin{cases} m\,H(\hat{w}^{(r)}, \hat{\Phi}^{(r)}) = n + \mathcal{O}(\varepsilon^2) & \text{in } S_\varepsilon(\hat{w}^{(r)}, \hat{\Phi}^{(r)}), \\ \langle N, \mathcal{V}^\varepsilon\rangle = \bar{\mathcal{O}}(\varepsilon^2) & \text{on } \partial S_\varepsilon(\hat{w}^{(r)}, \hat{\Phi}^{(r)}). \end{cases}$$

This gives the following equation in $w^{(1)}$

$$\mathcal{L}_0 w^{(1)} = \Theta^{n+1}\,h_{aa} + \Theta^{n+1}\left[(n+3)\langle h(\tilde{\Theta}), \tilde{\Theta}\rangle - h_{jj}\right] \quad \text{in } (SNK)_+;$$
$$\frac{\partial w^{(1)}}{\partial \eta} = 0 \quad \text{on } \partial(SNK)_+.$$

By the result from § 2.3 (with $\gamma = \frac{\pi}{2}$) and Fredholm alternative theorem, the solvability of the above system is possible provided

$$\int_{S^n_+} \left(\Theta^{n+1}\,h_{aa} + \Theta^{n+1}\left[(n+3)\langle h(\tilde{\Theta}), \tilde{\Theta}\rangle - h_{jj}\right]\right)\,\Theta^i\,d\theta = 0 \qquad \text{for all } i = 1, \cdots, n$$

which is the case by oddness, here $d\theta$ denotes the volume element on S^n_+. Notice that the variable $\bar{y}$ is being considered as a parameter so that $w^{(1)}$ is as smooth as the right hand side in this variable.

Constructing $w^{(2)}$ **:** We turn now to the term of order ε^2. We have

$$\begin{cases} m\, H(\hat{w}^{(r)}, \hat{\Phi}^{(r)}) = n + \mathcal{O}(\varepsilon^3) & \text{in } S_\varepsilon(\hat{w}^{(r)}, \hat{\Phi}^{(r)}), \\ \langle N, \mathcal{V}^\varepsilon \rangle = \bar{\mathcal{O}}(\varepsilon^2) & \text{on } \partial S_\varepsilon(\hat{w}^{(r)}, \hat{\Phi}^{(r)}). \end{cases}$$

Since the terms involving Φ in $\mathcal{Q}^1(\varepsilon w^{(1)}, \varepsilon\Phi^{(1)})$ are of the form $\varepsilon^3 L(\Phi^{(1)})$ and $Q(\hat{\Phi}^{(r)}, \hat{\Phi}^{(r)})$, we are led to a system in $w^{(2)}$ and $\Phi^{(1)}$ given by

$$\begin{aligned} \mathcal{L}_0 w^{(2)} &= \langle \mathfrak{J}\Phi^{(1)}, \tilde{\Theta} \rangle + \mathcal{O}(1) + \mathcal{L}^1 w^{(1)} + \mathcal{J}^1(\Phi^{(1)}) + Q(\hat{\Phi}^{(r)}, \hat{\Phi}^{(r)}) \quad \text{in } (SNK)_+ \\ \frac{\partial w^{(2)}}{\partial \eta} &= \bar{\mathcal{O}}(1) \quad \text{on } \partial(SNK)_+. \end{aligned}$$

Note that $\Pi_1\, \mathcal{J}^1 = 0$ and $\Pi_1\, Q(\Phi^{(1)}, \Phi^{(1)}) = 0$ so the above problem is solvable if and only if

$$\int_{S^n_+} \langle \mathfrak{J}\Phi^{(1)}, \tilde{\Theta} \rangle\, \Theta^i\, d\theta + \int_{S^n_+} \Big(\mathcal{O}(1) + \mathcal{L}^1 w^{(1)} \Big)\, \Theta^i\, d\theta + \oint_{\partial S^n_+} \bar{\mathcal{O}}(1)\, \Theta^i\, d\bar{\theta} = 0 \quad \text{for all } i = 1 \cdot$$

where $d\theta$ and $d\bar{\theta}$ are the volume elements on S^n_+ and ∂S^n_+ respectively. This gives an equation on $\Phi^{(1)}$ which can be solved using the non degeneracy of the submanifold K because in this case $\mathfrak{J}$ is invertible. Once this is done, we obtain readily $w^{(2)}$.

Constructing $w^{(r)}$**:** We want to construct an approximate solution as accurate as possible, and to do so we will use an iterative scheme. Suppose the couple $(w^{(r-1)}, \Phi^{(r-2)})$ is already determined. To find $(w^{(r)}, \Phi^{(r-1)})$, it suffices to check that when we project on the Kernel of $\mathcal{L}_0$, the operator involving $\Phi^{(r-1)}$ should be only the invertible Jacobi operator $\mathfrak{J}$. This is the case since the only term that can bring $\Phi^{(r-1)}$ at this iteration step is $\mathcal{Q}^1_{r-1}(w, \Phi)$ which gives only terms of the form $\varepsilon^2\Phi$ and $Q(\hat{\Phi}^{(r)}, \hat{\Phi}^{(r)})$ moreover $\Pi_1\, \mathcal{J}^1_{r-1}(\Phi^{(r-1)}) = \Pi_1\, Q(\hat{\Phi}^{(r)}, \hat{\Phi}^{(r)}) = 0$.

The index r appearing in the linear and quadratic terms means that they depend on the iteration step while the operator $\mathcal{J}^1_r$ keep its same properties because it is influenced only by the even quadratic terms in $Q(\hat{\Phi}^{(r)} + \Phi, \hat{\Phi}^{(r)} + \Phi)$ appearing in $Q^1(\hat{w}^{(r)} + w, \hat{\Phi}^{(r)} + \Phi)$.

By induction, in the same argument, for every $r \in \mathbb{N}$, we can find $(w^{(d)}, \Phi^{(d)})$, $d = 1, \cdots, r$ smooth such that

$$\hat{w}^{(r)} = \sum_{d=1}^{r} \varepsilon^d w^{(d)} = \mathcal{O}(\varepsilon) \quad \text{and} \quad \hat{\Phi}^{(r)} = \sum_{d=1}^{r-1} \varepsilon^d \Phi^{(d)} = \mathcal{O}(\varepsilon) \qquad (4.39)$$

and that

$$m\, H(\hat{w}^{(r)}, \hat{\Phi}^{(r)}) = n + \mathcal{O}(\varepsilon^{r+1}) \quad \text{in} \quad S_\varepsilon(\hat{w}^{(r)}, \hat{\Phi}^{(r)}), \qquad \langle N, \mathcal{V}^\varepsilon \rangle = \bar{\mathcal{O}}(\varepsilon^{r+2}) \quad \text{on} \quad \partial S_\varepsilon$$

4.3.1. Remark. *Notice that as in [57] we omitted the terms involving derivatives with respect to $\bar{y}$ of the function w (by considering $\mathcal{L}_0$ instead of $\mathcal{L}_\varepsilon$), this is due to the fact that since w is slow dependent on y_a, when differentiating with respect to $y_{\bar{a}}$ we pick up an ε at each differentiation, this gives us smaller terms. However, when applying elliptic regularity theorems we might loose two derivatives at each iteration. This indeed is not a problem since one needs just a finite number of iterations. We refer the reader to [57], where a more explanation is given.*

We are left to find w and Φ such that

$$\begin{aligned} m\, H(\hat{w}^{(r)} + w, \hat{\Phi}^r + \Phi) &= n \quad \text{in} \quad S_\varepsilon(\hat{w}^{(r)} + w, \hat{\Phi}^r + \Phi), \\ \langle N, \mathcal{V}^\varepsilon \rangle &= 0 \quad \text{on} \quad \partial S_\varepsilon(\hat{w}^{(r)} + w, \hat{\Phi}^r + \Phi). \end{aligned} \qquad (4.40)$$

We define the linearized mean curvature operator about $S_\varepsilon(\hat{w}^r, \hat{\Phi}^r)$

$$\mathbb{L}_{\varepsilon,r}(w, \Phi) = \frac{1}{\varepsilon}\left(\mathcal{L}_\varepsilon\, w + \varepsilon \mathcal{L}^1_r(w)\right) + \langle \mathfrak{J}\Phi, \tilde{\Theta} \rangle + \mathcal{J}^1_r(\Phi) + \varepsilon L_r(w, \Phi).$$

The index r appearing in the constant, linear and quadratic terms means that they depend on the iteration step but keep there properties.
We Notice that $\mathbb{L}_{\varepsilon,r}$ is not precisely the usual Jacobi operator because we are parameterizing this hypersurface as a graph over $S_\varepsilon(\hat{w}^r, \hat{\Phi}^r)$ using the vector field $-\Upsilon$ rather than the unit normal N.

Using Remark 2.1.1 (with $\gamma = \frac{\pi}{2}$), suppose that $\Sigma = S_\varepsilon(\hat{w}^r, \hat{\Phi}^r)$ and $\hat{N} = -\Upsilon$. From (4.39) and Proposition 4.2.2 we have

$$\langle N, -\Upsilon \rangle = 1 + \mathcal{O}(\varepsilon^2).$$

Furthermore, from Proposition 4.2.1 and (4.39), the volume forms of the tubes $S_\varepsilon(\hat{w}^r, \hat{\Phi}^r)$ and $(SNK)_+$ are related by

$$dvol_{S_\varepsilon(\hat{w}^r, \hat{\Phi}^r)} = (1 + \mathcal{O}(\varepsilon))\, dvol_{(SNK)_+}.$$

We define $\delta_{\varepsilon,r} > 0$ by

$$\langle N, -\Upsilon \rangle \, dvol_{S_\varepsilon(\hat{w}^r, \hat{\Phi}^r)} = \delta_{\varepsilon,r} \, dvol_{(SNK)_+}. \tag{4.41}$$

Multiplying by $\delta_{\varepsilon,r}$, the system (4.40) will change the terms $\mathcal{L}_r^1$, L_r, $\bar{L}_r$, the constant and quadratic terms will keep there properties and there will be a new linear operator $\bar{\mathcal{L}}_r^1(w)$ on the boundary. We keep the same notations for these terms and call $\mathbb{L}_{\varepsilon,r}$ the new selfadjoint operator $\delta_{\varepsilon,r} \, \mathbb{L}_{\varepsilon,r}$ with respect to the standard $L^2(SNK)_+$-inner product.

Now since $\bar{L}_r(w, \Phi)$ and $\bar{\mathcal{L}}_r^1(w)$ involves only terms of the form w, $\partial_{z^i} w$, we may extend $\bar{L}_r(w, \Phi)$, $\bar{\mathcal{L}}_r^1(w)$ and $\bar{\mathcal{O}}_r(\varepsilon^{r+1})$ in $(SNK)_+$ and this will just add some terms in $L_r(w, \Phi)$, $\mathcal{L}_r^1(w)$ and $\mathcal{O}_r(\varepsilon^r)$ respectively which will maintain there properties.

Without loss of generality we may replace the solvability of (4.40) with the following equation.

$$\begin{aligned} \mathbb{L}_{\varepsilon,r}(w, \Phi) &= \tfrac{1}{\varepsilon} Q_r(w, \Phi) + \mathcal{O}_r(\varepsilon^r) \quad && \text{in } (SNK)_+, \\ \frac{\partial w}{\partial \eta} &= \frac{1}{\varepsilon} \bar{Q}_r(w, \Phi) && \text{on } \partial(SNK)_+. \end{aligned} \tag{4.42}$$

We will try to invert the linear operator on the left hand side and this will lead us to study the spectrum of the operator by selfadjointness.

4.4 Spectral analysis

Function space: Fix $\frac{1}{2} > s > 0$. For any $v \in L^2(SNK)_+ := L^2(S_+^n \times K)$, set

$$\langle \Phi, \tilde{\Theta} \rangle := \Pi_1 \, v, \qquad \varepsilon^{-1+2s} \, w := \Pi_1^\perp v,$$

so that

$$v = \varepsilon^{1-2s} \, w + \langle \Phi, \tilde{\Theta} \rangle. \tag{4.43}$$

It will be understood that Φ^i for $i = 1, \cdots, n$ are the components of $\Pi_1 \, v$ on NK. Conversely if couple a $(w, \Phi) \in \Pi_1^\perp \, L^2(SNK)_+ \times L^2(K, NK)$ is given, we associate to it a function v as in (4.43).

Later we will often decompose

$$w = w_0 + w_1 \tag{4.44}$$

where w_0 is a function on K and w_1 has zero mean value with respect to the angular integrals.

The volume element of $(SNK)_+ = S^n_+ \times K$ will be denoted by $d\theta\, d\bar{y}$. As it will be apparent later, we will consider the following weighted Hilbert subspaces of $L^2(SNK)_+$

$$L^2_\varepsilon := \Big\{ v = \varepsilon^{1-2s}\, w + \langle \Phi, \tilde{\Theta} \rangle \in L^2(SNK)_+ \quad : \quad \varepsilon^{-2s} \int_{(SNK)_+} |w|^2 \, d\theta \, d\bar{y} + \int_K |\Phi|^2 \, d\bar{y} <$$

with corresponding norm

$$\|v\|^2_{L^2_\varepsilon} := \varepsilon^{-2s} \int_{(SNK)_+} |w|^2 \, d\bar{\varepsilon} \, d\bar{y} + \int_K |\Phi|^2 \, d\bar{y}.$$

We also define

$$H^1_\varepsilon := \Big\{ v \in L^2_\varepsilon \quad : \quad \varepsilon^{-2s} \int_{(SNK)_+} (\varepsilon^2 \, |\nabla_K w|^2 + |\nabla_{S^n_+} w|^2 + |w|^2) \, d\theta \, d\bar{y} + \int_K (|\nabla_K \Phi|^2 +$$

with corresponding norm

$$\|v\|^2_{H^1_\varepsilon} := \varepsilon^{-2s} \int_{(SNK)_+} (\varepsilon^2 \, |\nabla_K w|^2 + |\nabla_{S^n_+} w|^2 + |w|^2) \, d\theta \, d\bar{y} + \int_K (|\nabla_K \Phi|^2 + |\Phi|^2) \, d\bar{y}.$$

Let $|S^n_+|$ denote the volume of S^n_+. Notice that

$$\int_{S^n_+} (\Theta^i)^2 \, d\theta = \frac{|S^n_+|}{n+1} \qquad \text{for all } i = 1 \cdots n.$$

We define $\varrho_n := \frac{|S^n_+|}{n+1}$.

With these definitions in mind we redefine $\mathbb{L}_{\varepsilon,r}$ by duality as follows

$$\int_{(SNK)_+} v \, \mathbb{L}_{\varepsilon,r} \, v' \, d\bar{\varepsilon} \, d\bar{y} :=$$

$$- \;\; \varepsilon^{-2s} \int_{(SNK)_+} \varepsilon^2 w' \; \Delta_K w \, d\bar{\varepsilon} \, d\bar{y} + \varepsilon^{-2s} \int_{(SNK)_+} (\nabla_{S^n_+} w \, \nabla_{S^n_+} w' - n \, w \, w') \, d\theta \, d\bar{y}$$

$$+ \;\; \varrho_n \int_K \langle \mathfrak{J} \Phi, \Phi' \rangle \, d\bar{y} + \int_{(SNK)_+} (\mathcal{J}^1_r(\Phi) + \mathcal{L}^1_r(w) + \varepsilon L_r(w, \Phi)) \, (\varepsilon^{1-2s} \, w' + \langle \Phi', \tilde{\Theta} \rangle) \, d\theta \, d$$

We associate to $\mathbb{L}_{\varepsilon,r}$ its quadratic bi-linear form

$$\mathcal{C}_{\varepsilon,r}(v, v') := \int_{(SNK)_+} v \, \mathbb{L}_{\varepsilon,r} \, v' \, d\theta \, d\bar{y},$$

and the associated quadratic form $\mathcal{Q}_{\varepsilon,r}(v) := \mathcal{C}_{\varepsilon,r}(v, v)$.
As mentioned in the first section, following [57], we want to find the values of ε for which the operator $\mathbb{L}_{\varepsilon,r}$ is invertible. By selfadjointness this leads to

find the values of ε for which the eigenvalues of the form $\mathcal{Q}_{\varepsilon,r}$ are bounded away from zero. Such techniques requires first that our form should be very close to a model one that we can characterize its spectrum (just the small eigenvalues). Secondly, to understand the behavior of small eigenvalues seeing as "set" valued functions in ε. We will estimate the Morse index of $\mathcal{Q}_{\varepsilon,r}$ and prove the monotonicity of its small eigenvalues. The former can be done using Weyl's asymptotic formula and the latter can be obtained by applying a result by Kato. We shall do this in the remaining of this section.

We define the model form, by duality, as

$$\begin{aligned}\mathcal{C}_0(v,v') &:= -\varepsilon^{-2s}\int_{(SNK)_+}\varepsilon^2 w'\,\Delta_K w\,d\theta\,d\bar{y} + \varepsilon^{-2s}\int_{(SNK)_+}(\nabla_{S^n_+} w\,\nabla_{S^n_+} w' - n\,w\,w')\,d\\ &+\ \varrho_n\int_K \langle\mathfrak{J}\Phi,\Phi'\rangle\,d\bar{y}\end{aligned}$$

and the associated quadratic form $\mathcal{Q}_0(v) := \mathcal{C}_0(v,v)$.

4.4.1. Proposition. *There exists a constant $c>0$ (independent of r) such that*

$$|\mathcal{C}_{\varepsilon,r}(v,v') - \mathcal{C}_0(v,v')| \le c\,\varepsilon^s\,\|v\|_{H^1_\varepsilon}\,\|v'\|_{H^1_\varepsilon}. \tag{4.45}$$

Proof. First of all we notice that in $\mathcal{L}^1_r(w)$ their may appear expressions of the forms w, $\varepsilon\partial_{\bar{y}^a}w$, $\varepsilon^2\,\partial_{\bar{y}^a}\partial_{\bar{y}^b}w$, $\partial_{z^j}w$, $\partial_{z^j}\partial_{z^{j'}}w$. Nevertheless after integrating by parts and using Hölder inequality

$$\left|\int_{(SNK)_+}\varepsilon^{1-2s}w'\,\mathcal{L}^1_r(w)\,d\theta\,d\bar{y}\right| \le \varepsilon c\|v\|_{H^1_\varepsilon}\,\|v'\|_{H^1_\varepsilon},$$

and by definition of the H^1_ε norm

$$\begin{aligned}\left|\int_{(SNK)_+}\langle\Phi',\tilde{\Theta}\rangle\,\mathcal{L}^1_r(w)\,d\theta\,d\bar{y}\right| &\le c\varepsilon^s\|\varepsilon^{1-2s}w\|_{H^1_\varepsilon}\,\|\Phi'\|_{L^2(K,NK)}\\ &\le c\varepsilon^s\|v\|_{H^1_\varepsilon}\,\|v'\|_{H^1_\varepsilon}.\end{aligned}$$

Furthermore $\Pi_1\mathcal{J}^1(\Phi)=0$. Now it is clear that even if $\mathcal{J}^1_r(\Phi)+L_r(w,\Phi)$ involves terms of the form w, $\varepsilon\partial_{\bar{y}^a}w$, $\varepsilon\,\partial_{\bar{y}^a}\partial_{\bar{y}^b}w$, $\partial_{z^j}w$, $\partial_{z^j}\partial_{z^{j'}}w$ and also Φ^j, $\partial_{\bar{y}^a}\Phi^j$ and $\partial_{\bar{y}^a}\partial_{\bar{y}^b}\,\Phi^j$, in any case after integration by parts and using Hölder inequality we get

$$\left|\int_{(SNK)_+}(\varepsilon^{-1}\,\mathcal{J}^1_r(\Phi)+L_r(w,\Phi))\,(\varepsilon^{1-2s}\,w'+\langle\Phi',\tilde{\Theta}\rangle)\;d\theta\,d\bar{y}\right| \le c\|v\|_{H^1_\varepsilon}\,\|v'\|_{H^1_\varepsilon}.$$

The result follows at once. ∎

The Morse index of $\mathcal{Q}_{\varepsilon,r}$: Define the two quadratic forms

$$\mathcal{Q}^{\pm}(v) := \mathcal{Q}_0(v) \pm \gamma\, \varepsilon^s\, \|v\|^2_{H^1_\varepsilon}.$$

From (4.45), if $\gamma > 0$ is sufficiently large and ε small enough, then

$$\mathcal{Q}^- \leq \mathcal{Q}_{\varepsilon,r} \leq \mathcal{Q}^+,$$

so that the index of $\mathcal{Q}_{\varepsilon,r}$ is bounded by those of $\mathcal{Q}^+$ and $\mathcal{Q}^-$.

Given any function w defined on $(SNK)_+$, we set

$$D_0^{\pm}(w) := (1 \pm \gamma\, \varepsilon^s) \int_K \varepsilon^2\, |\nabla_K w|^2\, d\bar{y} - (n \mp \gamma\, \varepsilon^s) \int_K |w|^2\, d\bar{y},$$

$$D_1^{\pm}(w) := (1 \pm \gamma\, \varepsilon^s) \int_{(SNK)_+} (\varepsilon^2\, |\nabla_K w|^2 + |\nabla_{S^n_+} w|^2)\, d\theta\, d\bar{y} - (n \mp \gamma\, \varepsilon^s) \int_{(SNK)_+} |w|^2\, d\theta\, d\bar{y},$$

and finally,

$$D^{\pm}(\Phi) := -(1 \pm \gamma\, \varepsilon^s) \int_K \langle \mathfrak{J}\, \Phi, \Phi \rangle\, d\bar{y}.$$

With these definitions in mind, we have

$$\mathcal{Q}^{\pm}(v) = (n+1)\varrho_n\, \varepsilon^{-2s}\, D_0^{\pm}(w_0) + \varepsilon^{-2s}\, D_1^{\pm}(w_1) + \varrho_n\, D^{\pm}(\Phi),$$

if we decompose $v = \varepsilon^{1-2s}\, w + \langle \Phi, \tilde{\Theta} \rangle$ and further decompose $w = w_0 + w_1$ as usual. Following Section 6.3 in [56] it is easy to see that if $(1 \pm \gamma\, \varepsilon^s) > 0$ then the index of $D^{\pm}$ is the index of K. Moreover the index of $D_1^{\pm}$ is equal to zero if $2\,(n+1)\,(1 - \gamma\, \varepsilon^s) - (n + \gamma\, \varepsilon^s) > 0$ because

$$\Pi_1\, w_1 = 0 \qquad \text{and} \qquad \int_{S^n_+} w_1\, d\bar{\varepsilon} = 0$$

hence

$$\int_{S^n_+} |\nabla_{S^n_+} w_1|^2\, d\theta \geq 2\,(n+1) \int_{S^n_+} |w_1|^2\, d\theta.$$

This shows that the asymptotic behavior of the index of $\mathcal{Q}_{\varepsilon,r}$ should be determined by $D_0^{\pm}$. It is the case since its index is given by

$$\sharp\{j \quad : \quad (1 \pm \gamma \varepsilon^s)\lambda_j < (n \mp \gamma \varepsilon^s)\},$$

where λ_j are the eigenvalues of $-\varepsilon^2 \Delta_K$ counted with multiplicities. Now using Weyl's formula one obtain its index,

$$\text{Ind}\, D_0^{\pm} \sim c_K \left(\frac{n}{\varepsilon^2} \right)^{\frac{k}{2}}.$$

Collecting these estimates, one obtains the following

4.4.1. Lemma. *The Morse index of $\mathcal{Q}_{\varepsilon,r}$ is asymptotic to $c\varepsilon^{-k}$ when ε tends to zero, where c depends only on m and K.*

Approximate eigenfunctions: In order to apply Kato's theorem [52] we need to characterize the eigenfunctions (eigenspaces) corresponding to small eigenvalues. We prove

4.4.2. Lemma. *Let σ be an eigenvalue of $\mathbb{L}_{\varepsilon,r}$ and $v = \varepsilon^{1-2s}\, w + \langle \Phi, \tilde\Theta\rangle$ a corresponding eigenfunction and $\varepsilon^{1-2s}\, w_0 = \int_{S^n_+} v\, d\theta$ is the decomposition from (4.44). There exist constants $c, c_0 > 0$ such that if $|\sigma| \leq c_0$, then*

$$\|v - \varepsilon^{1-2s}\, w_0\|^2_{H^1_\varepsilon} \leq c\,\varepsilon^s\, \|v\|^2_{H^1_\varepsilon},$$

for all $\varepsilon > 0$ small enough.

Proof. For any $v' = \varepsilon^{1-2s}\, w' + \langle \Phi', \Theta\rangle$, we have

$$\begin{aligned}\mathcal{C}_{\varepsilon,r}(v,v') &= \sigma \int_{(SNK)_+} (\varepsilon^{2-4s} w\, w' + \langle \Phi, \Theta\rangle\langle \Phi', \Theta\rangle)\, d\theta\, d\bar y \\ &= \sigma \int_{(SNK)_+} \varepsilon^{2-4s} w\, w'\, d\theta\, d\bar y + \sigma\, \varrho_n \int_K \langle \Phi, \Phi'\rangle\, d\bar y.\end{aligned}$$

In addition, (4.45) gives

$$\begin{aligned}\Bigg| \int_{(SNK)_+} \varepsilon^{-2s}(\varepsilon^2\, \nabla_K w\, \nabla_K w' + \nabla_{S^n_+} w \nabla_{S^n_+} w' - (n + \sigma\, \varepsilon^{2-4s})\, w\, w')\, d\theta\, d\bar y \\ + \varrho_n \int_K (\langle \mathfrak{J}\Phi, \Phi'\rangle - \sigma\, \langle \Phi, \Phi'\rangle)\, d\bar y \Bigg| \leq c\,\varepsilon^s\, \|v\|_{H^1_\varepsilon}\, \|v'\|_{H^1_\varepsilon}.\end{aligned} \tag{4.46}$$

Step 1 : Let $\Phi' = 0$ and $w' = w_1$ to get

$$\left| \int_{(SNK)_+} \varepsilon^{-2s}(\varepsilon^2\, |\nabla_K w_1|^2 + |\nabla_{S^n_+} w_1|^2 - (n - \sigma\, \varepsilon^{2-4s})\, |w_1|^2)\, d\theta\, d\bar y \right| \leq c\,\varepsilon^s\, \|v\|_{H^1_\varepsilon}\, \|\varepsilon^{1-2s}\, w$$

However, since

$$\Pi_1\, w_1 = 0 \qquad \text{and} \qquad \int_{S^n_+} w_1\, d\theta = 0,$$

we have

$$\int_{S^n_+} |\nabla_{S^n_+} w_1|^2\, d\bar\varepsilon \geq 2\,(n+1) \int_{S^n_+} |w_1|^2\, d\theta,$$

hence

$$\left| \int_{(SNK)_+} \varepsilon^{-2s}(\varepsilon^2\, |\nabla_K w_1|^2 + \frac{1}{2}\, |\nabla_{S^n_+} w_1|^2 + (1 - |\sigma|\, \varepsilon^{2-4s})\, |w_1|^2)\, d\theta\, d\bar y \right| \leq c\,\varepsilon^s\, \|v\|^2_{H^1_\varepsilon}.$$

This implies that

$$\|\varepsilon^{1-2s} w_1\|^2_{H^1_\varepsilon} \le c\,\varepsilon^s \|v\|^2_{H^1_\varepsilon},$$

for all $\varepsilon \in (0,1)$, provided $|\sigma| \le 1/2$.

Step 2: Now let $w' = 0$ and $\Phi' = \Phi^+$ (resp. $\Phi' = \Phi^-$) in (4.46), where Φ^+ (resp. Φ^-) is the L^2 projection of Φ over the space of eigenfunctions of $\mathfrak{J}$ associated to positive (resp. negative) eigenvalues. This yields

$$\left|\int_K (\langle \mathfrak{J}\Phi, \Phi^\pm\rangle - \sigma\,\langle \Phi, \Phi^\pm\rangle)\,d\bar{y}\right| \le c\,\varepsilon^s \|v\|_{H^1_\varepsilon} \|\langle \Phi^\pm, \tilde{\Theta}\rangle\|_{H^1_\varepsilon}.$$

Since $\mathfrak{J}$ is invertible, there exists $c_1 > 0$ such that

$$c_1 \|\langle \Phi^\pm, \tilde{\Theta}\rangle\|^2_{H^1_\varepsilon} \le \left|\int_K \langle \mathfrak{J}\Phi, \Phi^\pm\rangle\,d\bar{y}\right|.$$

Hence

$$(c_1 - |\sigma|)\,\|\langle \Phi^\pm, \tilde{\Theta}\rangle\|^2_{H^1_\varepsilon} \le c\,\varepsilon^s \|v\|^2_{H^1_\varepsilon}.$$

This conclude the proof with $c_0 := \min\{1/2, c_1/2\}$. ∎

4.4.1. Remark. *If v is an eigenspace corresponding to an eigenvalue given by the above lemma, then it satisfies*

$$\left|\int_{(SNK)_+} \varepsilon^{-2s}(\varepsilon^2 |\nabla_K w|^2 + |\nabla_{S^n_+} w|^2 - (n + \sigma\varepsilon^{2-4s})\,|w|^2)\,d\theta\,d\bar{y} \right.$$
$$\left. + \varrho_n \int_K (\,\langle \mathfrak{J}\Phi, \Phi\rangle - \sigma\,\langle \Phi, \Phi\rangle)\,d\bar{y}\right| \le c\,\varepsilon^s \|v\|^2_{H^1_\varepsilon},$$

and

$$\left|\int_{(SNK)_+} \varepsilon^{-2s}(\varepsilon^2 |\nabla_K w|^2 + |\nabla_{S^n_+} w|^2 - n\,|w|^2)\,d\theta\,d\bar{y}\right| \le c\,\varepsilon^s \|v\|^2_{H^1_\varepsilon}. \quad (4.47)$$

Notice that $\nabla_{S^n_+} w = \nabla_{S^n_+} w_1$ if w is decomposed as $w = w_0 + w_1$ one has

$$\left|\int_{(SNK)_+} \varepsilon^{-2s}(\varepsilon^2 |\nabla_K w|^2 - n\,|w|^2)\,d\theta\,d\bar{y}\right| \le c\,\varepsilon^s \|v\|^2_{H^1_\varepsilon},$$

so that

$$\varepsilon^{-2s}\int_{(SNK)_+} \varepsilon^2 |\nabla_K w|^2\,d\theta\,d\bar{y} \le c\,\varepsilon^s \|v\|^2_{H^1_\varepsilon} + n\varepsilon^{-2s}\int_{(SNK)_+} |w|^2\,d\theta\,d\bar{y}.$$

In particular we have

$$\|v\|_{H^1_\varepsilon} \le c\|v\|_{L^2_\varepsilon}.$$

Variation of small eigenvalues with respect to ε: To understand the behavior of small eigenvalues of the symmetric quadratic form $\mathcal{Q}_{\varepsilon,r}$, we need to apply a result by Kato, see [52]. Considering the eigenvalues $\sigma(\varepsilon)$ as differentiable multivalued function in ε, the result states that

$$\partial_\varepsilon \sigma \in \left\{ \int_{(SNK)_+} v\,(\partial_\varepsilon \mathbb{L}_{\varepsilon,r})\, v\, d\theta\, d\bar{y} \qquad : \qquad \mathbb{L}_{\varepsilon,r} v = \sigma\, v, \qquad \|v\|_{L^2} = 1 \right\}. \tag{4.48}$$

An good estimate of a bound for the set on the right of (4.48) allows one to estimate the spectral gaps of the linearized operator when the parameter ε is small, see [56] § 6.3.

This is indeed given in the following lemma.

4.4.3. LEMMA. *There exist constants $c_1, c > 0$ such that, if σ is an eigenvalue of $\mathbb{L}_{\varepsilon,r}$ with $|\sigma| < c_1$, then*

$$\varepsilon\, \partial_\varepsilon \sigma \geq 2\, n - c\, \varepsilon^s,$$

provided ε is small enough.

PROOF. We have just to provide bounds for the set on the right of (4.48) using the above remark.

Assume that $\mathbb{L}_{\varepsilon,r} v = \sigma\, v$, but rather than normalizing the function v by $\|v\|_{L^2} = 1$, assume instead that $\|v\|_{L^2_\varepsilon} = 1$. In order to compute $\partial_\varepsilon \mathbb{L}_{\varepsilon,r}$, recall that

$$w = \varepsilon^{-1+2s}\, \Pi_1^\perp v \qquad \text{and that} \qquad \langle \mathfrak{J}\Phi, \tilde{\Theta} \rangle = \Pi_1\, v,$$

so we can write

$$\begin{aligned} \mathbb{L}_{\varepsilon,r}\, v \;\; = \;\; & -\varepsilon^{2s} \Delta_K\, (\Pi_1^\perp\, v) + \frac{1}{\varepsilon^{2-2s}}\, \mathcal{L}_0\, (\Pi_1^\perp\, v) + \Pi_1\, v + \frac{1}{\varepsilon^{1-2s}}\, \mathcal{L}_r^1\, (\Pi_1^\perp\, v) \\ & + \mathcal{J}^1\, (\mathfrak{J}_r^{-1} \Pi_1\, v) + \varepsilon\, L_r(\varepsilon^{-1+2s}\, \Pi_1^\perp v, \mathfrak{J}^{-1} \Pi_1\, v). \end{aligned}$$

Since Π_1 and $\Pi_1^\perp$ are independent of ε, we have

$$\begin{aligned} \partial_\varepsilon \mathbb{L}_{\varepsilon,r} v \;\; = \;\; & -2s\varepsilon^{-1+2s} \Delta_K\, (\Pi_1^\perp\, v) + (-2+2s)\varepsilon^{-3+2s}\, \mathcal{L}_0\, (\Pi_1^\perp v) + (-1+2s)\varepsilon^{-2+2s}\, \mathcal{L}_r^1\, (\Pi \\ + \;\; & \tilde{L}_r(\varepsilon^{-1+2s}\, \Pi_1^\perp v, \mathfrak{J}^{-1} \Pi_1\, v), \end{aligned}$$

where the operator $\tilde{L}_r$ varies from line to line but satisfies the usual assumptions. This now gives

$$\begin{aligned}\left|\int_{(SNK)_+} v\,(\partial_\varepsilon \mathbb{L}_{\varepsilon,r})\,v\,d\theta\,d\bar{y}\right. &- 2\varepsilon^{-1-2s}\int_{(SNK)_+} \varepsilon^2|\nabla_K w|^2\,d\theta\,d\bar{y}\\ &+ \frac{(2-2s)}{\varepsilon}\,\varepsilon^{-2s}\int_{(SNK)_+}(\varepsilon^2|\nabla_K w|^2+|\nabla_{S^n_+}w|^2-n\,|w|^2)\,d\theta\,d\bar{y}\\ &\leq c\,\|v\|^2_{H^1_\varepsilon}+\left|\frac{1-2s}{\varepsilon}\int_{(SNK)_+}\langle\Phi,\tilde{\Theta}\rangle\mathcal{L}^1_r(w)\,d\theta\,d\bar{y}\right|\\ &\leq \frac{c}{\varepsilon^{1-s}}\,\|v\|^2_{H^1_\varepsilon}.\end{aligned}$$

Consequently if v is an eigenfunction of $\mathbb{L}_{\varepsilon,r}$ with corresponding eigenvalue $|\sigma|\leq c_0$, where c_0 is given in the previous lemma, by the inequality (4.47), see the above remark, we have

$$\left|\int_{(SNK)_+} v\,(\partial_\varepsilon \mathbb{L}_{\varepsilon,r})\,v\,d\theta\,d\bar{y}-2\varepsilon^{-1-2s}\int_{(SNK)_+}\varepsilon^2|\nabla_K w|^2\,d\theta\,d\bar{y}\right|\leq\frac{c}{\varepsilon^{1-s}}\,\|v\|^2_{H^1_\varepsilon}. \tag{4.49}$$

Again from the above remark, one gets

$$\varepsilon^{-1-2s}\int_{(SNK)_+}\varepsilon^2|\nabla_K w|^2\,d\theta\,d\bar{y}\leq c\,\varepsilon^{-1+s}\,\|v\|^2_{H^1_\varepsilon}+n\,\varepsilon^{-1-2s}\int_{(SNK)_+}|w|^2\,d\theta\,d\bar{y}.$$

If we normalize v by $\|v\|_{L^2_\varepsilon}=1$ then inserting this into (4.49) we get

$$\left|\int_{(SNK)_+} v\,(\partial_\varepsilon \mathbb{L}_{\varepsilon,r})\,v\,d\theta\,d\bar{y}-\frac{2}{\varepsilon}\,n\right|\leq\frac{c}{\varepsilon^{1-s}} \tag{4.50}$$

for all eigenfunction v such that $\mathbb{L}_{\varepsilon,r}v=\sigma\,v$ which is normalized by $\|v\|_{L^2_\varepsilon}=1$.

Now since $||v||_{L^2}\leq\|v\|_{L^2_\varepsilon}$, we conclude that

$$\inf_{\substack{\mathbb{L}_\varepsilon v=\sigma\,v\\ \|v\|_{L^2}=1}}\int_{(SNK)_+} v\,(\partial_\varepsilon\mathbb{L}_\varepsilon)\,v\,d\theta\,d\bar{y}\quad\geq\inf_{\substack{\mathbb{L}_\varepsilon v=\sigma\,v\\ \|v\|_{L^2_\varepsilon}=1}}\int_{(SNK)_+} v\,(\partial_\varepsilon\mathbb{L}_\varepsilon)\,v\,d\theta\,d\bar{y},$$

and (4.50) implies that

$$\partial_\varepsilon\sigma\geq\frac{2}{\varepsilon}\,n-\frac{c}{\varepsilon^{1-s}}.$$

This completes the proof of the result. ∎

4.5 Proof of Theorem 4.0.1

Using Lemma 4.4.1 and Lemma 4.4.3, reasoning as for the proof of Lemma 6.3 in [56] we can find a sequence of open interval I_i, $i \in \mathbb{N}$ such that the smallest eigenvalue of $\mathbb{L}_{\varepsilon,r}$ is bounded away from zero for any $\varepsilon \in \cup_i I_i$. More precisely we have

4.5.1. LEMMA. *Fix any $q \geq 2$. Then there exists a sequence of disjoint nonempty open intervals $I_i = (\varepsilon_i^-, \varepsilon_i^+)$, $\varepsilon_i^\pm \to 0$ and a constant $c_q > 0$ such that when $\varepsilon \in I^q := \cup_i I_i$, the operator $\mathbb{L}_{\varepsilon,r}$ is invertible and*

$$(\mathbb{L}_{\varepsilon,r})^{-1} : L^2_\varepsilon \longrightarrow L^2_\varepsilon,$$

has norm bounded by $c_q\,\varepsilon^{-k-q+1}$, uniformly in $\varepsilon \in I$. Furthermore, $I^q := \cup_i I_i$ satisfies

$$\left|\mathcal{H}^1((0,\varepsilon) \cap I^q) - \varepsilon\right| \leq c\,\varepsilon^q, \qquad \varepsilon \searrow 0.$$

For $p \in \mathbb{N}$ and $0 < \alpha < 1$, we denote by $\mathcal{C}^{p,\alpha}$ the usual Hölder spaces on the closure of $(SNK)_+$.

4.5.2. LEMMA. *Let $f \in \mathcal{C}^{0,\alpha}$ and v satisfy*

$$\mathbb{L}_{\varepsilon,r}\, v = f.$$

Then there exit a constant $c > 0$ (independent of ε but may depend on r) and $R > 0$ depending only on q, α, s and k such that

$$\|v\|_{\mathcal{C}^{2,\alpha}} \leq c\,\varepsilon^{-R}\,\|f\|_{\mathcal{C}^{0,\alpha}}$$

for any $\varepsilon \in I^q$.

PROOF. Fix $q \geq 2$. Observe that by definition of the weighted norm of L^2_ε, from Lemma 4.5.1 we have

$$\|v\|_{L^2} \leq c_q\,\varepsilon^{-k-q+1-s}\,\|f\|_{L^2}.$$

By standard elliptic regularity theory, there exists $c > 0$ (may be depending on r) such that the following Hölder estimate holds

$$\varepsilon^{2+\alpha}\|v\|_{\mathcal{C}^{2,\alpha}} \leq c\,\varepsilon^2\,\|f\|_{\mathcal{C}^{0,\alpha}} + c\,\varepsilon^{-\frac{k}{2}}\,\|v\|_{L^2}.$$

From these last two inequalities, we can choose $R > \frac{3k}{2} + q + \alpha + 1 + s$. ∎

We end the proof of the main theorem by finding a fixed point for the mapping

$$T_{\varepsilon,r}(v) := -(\mathbb{L}_{\varepsilon,r})^{-1}\left\{\mathcal{O}_r(\varepsilon^r)+\mathcal{N}_{\varepsilon,r}(v)\right\},$$

where

$$\begin{aligned}\int_{(SNK)_+}\mathcal{N}_{\varepsilon,r}(v)\,v'\,d\theta\,d\bar{y} \;:=\;& \int_{(SNK)_+}\varepsilon^{-1}\,Q_r(\varepsilon^{-1+2s}\,\Pi_1^{\perp}\,v,\Pi_1\,v)\,v'\,d\theta\,d\bar{y}\\ &+\oint_{\partial(SNK)_+}\varepsilon^{-1}\,\bar{Q}_r(\varepsilon^{-1+2s}\,\Pi_1^{\perp}\,v,\Pi_1\,v)\,v'\,d\bar{\theta}\,d\bar{y}.\end{aligned}$$

Since by definition, Q_r and $\bar{Q}_r$ are (at least) quadratic we have

$$\begin{aligned}\|\mathcal{N}_{\varepsilon,r}(v)\|_{\mathcal{C}^{0,\alpha}} &= \varepsilon^{-2+2s}\,O(\|v\|_{\mathcal{C}^{2,\alpha}})\,\|v\|^2_{\mathcal{C}^{2,\alpha}};\\ \|\mathcal{N}_{\varepsilon,r}(v_1)-\mathcal{N}_{\varepsilon,r}(v_2)\|_{\mathcal{C}^{0,\alpha}} &= \varepsilon^{-2+2s}\,O(\|v_1\|_{\mathcal{C}^{2,\alpha}},\|v_2\|_{\mathcal{C}^{2,\alpha}})\|v_1-v_2\|_{\mathcal{C}^{2,\alpha}}.\end{aligned}$$

Now we fix $r>2\,R+2-2\,s$. By Lemma 4.5.2 and the above inequalities, for every $\varepsilon\in I^q$, $T_{\varepsilon,r}(v)$ maps the ball

$$\{v\in\mathcal{C}^{2,\alpha}\quad:\quad\|v\|_{\mathcal{C}^{2,\alpha}}\le C\,\varepsilon^{r+1-R}\}$$

into itself moreover it is a contraction. Therefore it has a unique fixed point $v=\varepsilon^{1-2s}\,w+\langle\Phi,\tilde{\Theta}\rangle$ in the ball yielding

$$\begin{aligned}m\,H(\hat{w}^{(r)}+w,\hat{\Phi}^r+\Phi) &= n &&\text{in}\quad S_\varepsilon(\hat{w}^{(r)}+w,\hat{\Phi}^r+\Phi)\subset\Omega_\varepsilon,\\ \langle N,\mathcal{V}^\varepsilon\rangle &= 0 &&\text{on}\quad \partial S_\varepsilon(\hat{w}^{(r)}+w,\hat{\Phi}^r+\Phi)\subset\partial\Omega_\varepsilon.\end{aligned}$$

If $\varepsilon\in I^q$ is sufficiently small then rescaling back, the tube $\varepsilon\,S_\varepsilon(\hat{w}^{(r)}+w,\hat{\Phi}^r+\Phi)$, is an embedded hypersurface of Ω (because the $\mathcal{C}^{1,\alpha}$-norm of $\hat{w}^{(r)}+w$ tends to zero as $\varepsilon\to 0$) with constant mean curvature equal to $\frac{n}{m}\varepsilon^{-1}$ and intersecting the boundary of Ω perpendicularly along its boundary.

4.5.1. Remark. ***Existence of stationary Capillary hypersurfaces***.

Letting $\gamma\in(0,\pi)$ be an angle, recall from § 4.1.1 that $(y^1,y^2\,\dots,y^k)\in\mathbb{R}^k$ (resp. $(z^1,z^2\,\dots,z^n)\in B^n_{r(\gamma)}$) are the local coordinate variables on K_ε (resp. on $S^n(\gamma)$), where $r(\gamma):=\frac{1-\cos\gamma}{1+\cos(\gamma)}$ (see § 2.3) and

$$\Theta(\gamma):=\mathbf{p}\Big|_{B^n_{r(\gamma)}}-\cos(\gamma)\,E_{n+1}$$

parameterizes the spherical cap $S^n(\gamma)$ *which intersects the horizontal plane* $\mathbb{R}^m$ *with angle* γ.
As in the case where $\gamma = \frac{\pi}{2}$, *we can use the same class of deformations letting* $\Phi : K \to NK_\varepsilon$ *and* $w : B^n_\gamma \times K_\varepsilon \to \mathbb{R}$, *consider*

$$S_\gamma \ : (y,z) \mapsto y \times \varepsilon^{-1}\Phi(\varepsilon y) \ + \ (1 + w(y,z))\, \Theta(\gamma).$$

The surfaces nearby a geodesic tube around K_ε *which make an angle almost equal to* γ *with* $\partial\Omega_\varepsilon$ *can be parameterized (locally) by*

$$G_\gamma(y,z) : (y,z) \longrightarrow S_\gamma(y,z) \longrightarrow F^\varepsilon(S_\gamma(y,z)),$$

namely

$$G_\gamma(y,z) := F^\varepsilon \left(y, \frac{1}{\varepsilon}\Phi(\varepsilon y) + (1 + w(y,z))\tilde{\Theta}(\gamma), (1 + w(y,z))\Theta^{n+1}(\gamma) \right).$$

Notice that $\Theta^{n+1}(\gamma)\Big|_{\partial B^n_{r(\gamma)}} = 0$, *so*

$$G_\gamma(y,z)\Big|_{\partial B^n_{r(\gamma)}} \in \partial\Omega_\varepsilon \qquad \text{for any } y$$

The image of this map will be called $S^\gamma_\varepsilon(w,\Phi)$.

Observe that the hypersurfaces close to $S^\gamma_\varepsilon(0,0)$ *are parameterized using the vector field* $-\Upsilon(\gamma) = \Theta^j(\gamma)\, X_j + \Theta^{n+1}(\gamma)\mathcal{V}^\varepsilon$ *rather than the normal* $\Xi := \mathbf{p}^j\, X_j + \mathbf{p}^{n+1}\mathcal{V}^\varepsilon$ *because it is more reasonable if we want the boundary of* $S^\gamma_\varepsilon(w,\Phi)$ *to be on* $\partial\Omega_\varepsilon$ *without imposing simultaneously a Neumann and Dirichlet boundary condition on* w. *Suppose* $Z_j(\gamma), Z_a(\gamma)$ *span the tangent space of* $S^\gamma_\varepsilon(w,\Phi)$ *as in § 7.2.1, we can obtain the normal fields* $N(\gamma)$ *by finding* $\alpha^j(\gamma)$ *and* $\beta^a(\gamma)$ *so that*

$$N(\gamma) = -\Xi + \alpha^j(\gamma)Z_j(\gamma) + \beta^a(\gamma)Z_a(\gamma).$$

As we did in Section 4.2, the mean curvature at every point of $S^\gamma_\varepsilon(w,\Phi)$

can be obtained:

$$
\begin{aligned}
m\, H(w,\Phi) \;&=\; n - \varepsilon\Big(\Gamma_a^a(\tilde{\mathbf{p}}) + \mathbf{p}^{n+1}\, h_{aa} + \mathbf{p}^{n+1}\,[3\langle h(\tilde{\mathbf{p}}),\tilde{\mathbf{p}}\rangle - h_{jj}] + n\,\Theta^{n+1}(\gamma)\langle h(\tilde{\mathbf{p}}),\tilde{\mathbf{p}} \\
&-\; \Big(\varepsilon^2\Delta_K\left(\langle\Theta(\gamma),\mathbf{p}\rangle w\right) + \Delta_{S^n}\left(\langle\Theta(\gamma),\mathbf{p}\rangle w\right) + n\left(\langle\Theta(\gamma),\mathbf{p}\rangle w\right)\Big) \\
&-\; \varepsilon\Big(\langle\, \Delta_K\Phi + R(\Phi,E_a)\,E_a\,,\,\tilde{\mathbf{p}}\,\rangle - \Gamma_a^c(\Phi)\,\Gamma_c^a(\tilde{\mathbf{p}})\Big) \\
&-\; \varepsilon\Big((3n+1)\,\Theta^{n+1}(\gamma)h(\tilde{\mathbf{p}})^a\langle\Phi_{\bar a},\tilde{\mathbf{p}}\rangle +\, \mathbf{p}^{n+1}h(\Phi_{\bar a})^a + 2\mathbf{p}^{n+1}h : \Gamma(\Phi)\Big) \\
&-\; \tfrac{n}{2}(\varepsilon w_{\bar a} + \langle\Phi_{\bar a},\tilde{\mathbf{p}}\rangle)^2 - \langle\Phi_{\bar a},\varepsilon\nabla_{S^n}w_{\bar a}\rangle - 2\varepsilon^2\nabla_K^2 w : \Gamma(\Phi) \\
&+\; \tfrac{n+2}{6}\langle R(\Phi,\tilde{\mathbf{p}})\Phi\,,\,\tilde{\mathbf{p}}\rangle - \tfrac{1}{3}\langle R(\Phi,E_i)\Phi\,,\,E_i\rangle \\
&+\; \varepsilon\, L(w) + \varepsilon^2\, L(w,\Phi) + Q(w) + \varepsilon\, Q(w,\Phi).
\end{aligned}
$$

Moreover (recall that $\mathcal{V}^\varepsilon$ *is the interior normal of* $\partial\Omega_\varepsilon$*) using the fact that* $\Theta^{n+1}(\gamma)\Big|_{\partial B^n_{r(\gamma)}} = 0$*, the equation* $\langle -\mathcal{V}^\varepsilon, N\rangle = \cos(\gamma)$ *is equivalent to*

$$
\langle\Theta(\gamma),\mathbf{p}\rangle(1-w)\frac{\partial w}{\partial\eta(\gamma)} = \bar{\mathcal{O}}(\varepsilon^2) + \varepsilon^2\,\bar{L}(w,\Phi) + \bar{Q}^1(w,\Phi) + \varepsilon\,\bar{Q}(w,\Phi) \qquad \text{on } \partial S^n(\gamma) \times
$$

which is again equivalent to

$$
\begin{aligned}
\frac{\partial(\langle\Theta(\gamma),\mathbf{p}\rangle w)}{\partial\eta(\gamma)} \;&=\; w\frac{\partial\langle\Theta(\gamma),\mathbf{p}\rangle}{\partial\eta(\gamma)} + \bar{\mathcal{O}}(\varepsilon^2) + \varepsilon^2\,\bar{L}(w,\Phi) + \bar{Q}^1(w,\Phi) + \bar{Q}(w) \\
&\quad +\varepsilon\,\bar{Q}(w,\Phi) \qquad\qquad \text{on } \partial S^n(\gamma)\times K \\
&=\; w\,\cot(\gamma) + \bar{\mathcal{O}}(\varepsilon^2) + \varepsilon^2\,\bar{L}(w,\Phi) + \bar{Q}^1(w,\Phi) \\
&\quad +\bar{Q}(w) + \varepsilon\,\bar{Q}(w,\Phi) \qquad\qquad \text{on } \partial S^n(\gamma)\times K,
\end{aligned}
$$

where

$$
\bar{Q}^1(w,\Phi) := \cot(\gamma)\left(\varepsilon w_{\bar a}\langle\Phi_{\bar a},\tilde{\mathbf{p}}\rangle + \langle\Phi_{\bar a},\tilde{\mathbf{p}}\rangle\langle\Phi_{\bar a},\tilde{\mathbf{p}}\rangle - \frac{1}{3}\,\langle R(\Phi,\tilde{\mathbf{p}})\,\Phi,\tilde{\mathbf{p}}\rangle\right).
$$

Using the results from § 2.3 and from § 4.3, one can adjust the tube to $S_\varepsilon^\gamma(\hat{w}^{(r)},\hat{\Phi}^{(r)})$ *accurately. Moreover with the decomposition of the functions* $v = \varepsilon^{1-2s}\,w + \langle\Phi,\tilde{\mathbf{p}}\rangle \in L^2(S^n(\gamma)\times K)$ *as in (4.43) we conclude that the spectral analysis of the linearized mean curvature operator over* $S_\varepsilon^\gamma(\hat{w}^{(r)},\hat{\Phi}^{(r)})$ *carried out as we obtain in Section 4.4 in the new weighted Hilbert sub-*

spaces of $L^2(S^n(\gamma)\times K)$

$$L^2_{\varepsilon,\gamma} := \Big\{ v=\varepsilon^{1-2s}\, w+\langle \Phi,\tilde{\mathbf{p}}\rangle \in L^2(S^n(\gamma)\times K) \quad : \quad \varepsilon^{-2s}\int_{S^n(\gamma)\times K}\langle \Theta(\gamma),\mathbf{p}\rangle |w|^2\, d\theta(\gamma)\, d\bar y+\int_K |\Phi|^2\, d\bar y<\infty\Big\}$$

$$\Big\{ v\in L^2_{\varepsilon,\gamma} \quad : \quad \varepsilon^{-2s}\int_{S^n(\gamma)\times K}\langle \Theta(\gamma),\mathbf{p}\rangle (\varepsilon^2\,|\nabla_K w|^2+|\nabla_{S^n(\gamma)} w|^2+|w|^2)\, d\theta(\gamma)\, d\bar y + \int_K (|\nabla_K\Phi|^2+|\Phi|^2)\, d\bar y<\infty\Big\}.$$

Under the usual assumptions on K*, if* $\varepsilon\in I^q$ *is sufficiently small then rescaling back, we can find a couple* (w,Φ) *so that the tube* $\varepsilon\, S^\gamma_\varepsilon(\hat w^{(r)}+w,\hat{\Phi}^r+\Phi)$*, is an embedded hypersurface of* Ω *with constant mean curvature* $\frac{n}{m}\varepsilon^{-1}$ *and intersecting* $\partial\Omega$ *with and angle* γ*. This yields a set of stationary Capillary hypersurfaces in* Ω *with constant "contact angle"* γ *and condensing to the submanifold* K*.*

Chapter 5

Capillary minimal surfaces in Riemannian manifolds

This Chapter deals with minimal surfaces sloving (GMP) in Riemannian manifolds. Minimal surfaces are surfaces with mean curvature vanishing everywhere. These include, but are not limited to, surfaces of minimal area subject to various constraints.
In this chapter we are interested in minimal surfaces which intersect a given hypersurface with a constant angle. We prove existence results of capillary surfaces with prescribed topology in Riemannian manifolds. Roughly speaking, we first show the existence of a class of capilary (minimal) disc-type surfaces embedded in a Riemannian surface of revolution (see below). In particular, shrinking enough the thickness of the surface of revolution, this class constitutes a foliation. Secondly we have existence of minimal disc-type surfaces embedded in a geodesic tube of a curve which intersect perpendicularly the boundary of the tube.
Before stating the main results, we need to define what we mean by Riemannian surface of revolution.

A surface of revolution, is a surface created by rotating a parametric curve $[a, b] \ni s \to (\kappa(s)\,,\,\phi(s)) \in \mathbb{R}^2$ lying on some plane around a straight line (the axis of rotation) in the same plane.
The resulting surface $\mathscr{C}^1$ therefore always has azimuthal symmetry. Examples of surfaces of revolution include cylinder (excluding the ends), hyperboloid, paraboloid, sphere, torus, etc.
In more generality one can obtain surfaces of revolution in $\mathbb{R}^{m+1}$, $m \geq 2$ using the standard parametrization

$$S(s, z) = (\kappa(s)\,,\,\phi(s)\,\Theta(z))\,,$$

where $z \mapsto \Theta(z) \in S^{m-1}$, $\phi(s) \neq 0 \quad \forall s \in [a, b]$.
Assuming that the rotating curve is parameterized by arc length namely

$$(\phi'(s))^2 + (\kappa'(s))^2 = 1,$$

clearly the disc $\mathscr{D}_{s,1}$ centered at $(\kappa(s)\,,\,0)$ (on the axis of rotation) with radius $\phi(s)$ parameterized by

$$B_1^m \ni x \mapsto (\kappa(s)\,,\, \phi(s)\,x)\,,$$

has zero mean curvature and intersects the above surface of revolution with a constant angle equal to arccos $\phi'(s)$, where B_1^m stands for the unit ball of $\mathbb{R}^m$ centered at the origin, namely $\mathscr{D}_{s,1}$ is a capillary surface.
Motivated by capillarity problems, for questions of stability, see [34], it is not restrictive to assume that the angle of contact is in $(0,\pi)$, namely $\phi'(s) \in (-1,1)$ or equivalently

$$\kappa'(s) \neq 0. \tag{5.1}$$

We shall extend these definitions of surface of revolution in a Riemannian setting.

Let $(\mathcal{M}^{m+1}, g)$ be Riemannian manifold, and Γ an embedded curve parameterized by a map $\gamma : [0,1] \to \mathcal{M}$. As in Section 4.1, we consider a local parallel orthogonal frame $E_1, \cdots, E_m$ of $N\Gamma$ along Γ. This determines a coordinate system by

$$[0,1] \times \mathbb{R}^m \ni (x_0, \zeta) \mapsto (x_0, \zeta) := \exp_{\gamma(x_0)}(\zeta^i E_i) \,\in\, \mathcal{M}.$$

For a small parameter $\rho > 0$, consider the *Riemannian surface of revolution* $\mathscr{C}^\rho$ around Γ in $\mathcal{M}$ parameterized by

$$(s,z) \longrightarrow \overline{F}(\rho\, S(s,z)) = \overline{F}(\rho\,\kappa(s)\,,\, \rho\,\phi(s)\Theta(z)) = \exp_{\gamma(\rho\,\kappa(s))}(\rho\,\phi(s)\Theta^i(z)\,E_i),$$

where $z \mapsto \Theta(z) \in S^{m-1}$, and call its interior $\Omega_\rho := \operatorname{int}\mathscr{C}^\rho$ which is nothing but a tubular neighborhood for Γ if ρ is small enough. Here we are assuming always that $\phi(s) \neq 0$ and that $(\phi'(s))^2 + (\kappa'(s))^2 = 1$.

For any $s \in [a,b]$, we consider the following set

$$D_{s,\rho} := \overline{F}(\rho\,\kappa(s)\,,\, \rho\,\phi(s)\,B_1^m),$$

it is clear that $\partial D_{s,\rho} \subset \mathscr{C}^\rho$ and we have that the mean curvature $H_{D_{s,\rho}}$ of $D_{s,\rho}$, see § 5.3.1, satisfies

$$H_{D_{s,\rho}} = \mathcal{O}(\rho) \qquad \text{in } D_{s,\rho} \tag{5.2}$$

while the angle between the unit outer normals (see also § 5.3.2) can be expanded as

$$\langle N_{D_{s,\rho}}, N_{\mathscr{C}^\rho}\rangle = \phi'(s) + \mathcal{O}(\rho) \qquad \text{on } \partial D_{s,\rho}. \tag{5.3}$$

Our aim is to perturb $D_{s,\rho}$ to a capillary minimal submanifold, $\mathscr{D}_{s,\rho}$, of Ω_ρ centered on Γ with contact angle $\arccos \phi'(s)$ along $\partial\mathscr{D}_{s,\rho} \subset \mathscr{C}^\rho$, as it happens in $\mathbb{R}^{m+1}$.

5.0.1. THEOREM. *Suppose we are in the situation described above. Let $[a', b'] \subset [a, b]$ be such that $\phi(s)\phi''(s) > 0$ for every $s \in [a', b']$. Then there exists $\rho_0 > 0$ such that for any $s \in [a', b']$ and $\rho \in (0, \rho_0)$, there exists an embedded minimal disc $\mathscr{D}_{s,\rho} \subset \Omega_\rho$, intersecting $\mathscr{C}^\rho$ by an angle equal to $\phi'(s)$ along its boundary. Moreover $\mathscr{D}_{s,\rho}$ is a normal graph over the set $D_{s,\rho}$ for which the norm (in the $\mathcal{C}^{2,\alpha}$-topology) of this function defining the graph tends to zero uniformly as ρ tends to zero.*
Furthermore there exists a tubular neighborhood O_ρ of $\gamma([a', b'])$ foliated by such minimal discs for which each leaf intersects ∂O_ρ transversally along its boundary.

5.0.2. REMARK.

- *When we parameterize in particular $\mathscr{C}^\rho$ with $\kappa(s) = s$, and if we require the capillary discs to be perpendicular to $\mathscr{C}^\rho$, we obtain the conditions $\phi' = 0$ and $\phi'' \neq 0$. This means that non-degenerate extrema of the width ϕ determine the location of such surfaces.*
- *An example is the hyperboloid, $\phi(s) = \cosh s$ and $\kappa(s) = \sinh s$. Here one may see $\mathcal{M}$ as a Lorentzian manifold modeled on the Minkowski space $\mathbb{R}^m_1$. Letting $q \in \mathcal{M}$ and E_0 a unit time-like vector of $T_q\mathcal{M}$ and $\gamma(x_0) = \exp_q(x_0 E_0)$ so one can see $\mathscr{D}_{0,\rho}$ as a space-like minimal disc in the geodesic sphere of radius ρ.*

An interesting particular case which is not covered by Theorem 5.0.1 is when $\phi \equiv 1$ and $\kappa =$Id, namely when we deal with geodesic tubes. In this situation (recall that in this case the angle of contact is $\frac{\pi}{2}$) it is the geometry of the manifold to determine the position of the discs. More precisely, we have that $\mathscr{C}^\rho$ is the geodesic tube of radius $\rho > 0$ around Γ,

$$\mathscr{C}^\rho = \{q \in \mathcal{M} \quad : \quad \mathrm{dist}_g(q, \Gamma) = \rho\},$$

and its interior is nothing but

$$\Omega_\rho := \{q \in \mathcal{M} \quad : \quad \mathrm{dist}_g(q, \Gamma) < \rho\}.$$

In this case due to invariance by translations along the axis of rotation, we reduced our problem of finding minimal surfaces to a finite-dimensional one. Namely we have obtained the following

5.0.3. THEOREM. *There exists a smooth function $\psi_\rho : [a,b] \to \mathbb{R}$ such that, for ρ small, if s_0 is a critical point of ψ_ρ the set $D_{s_0,\rho}$ can be smoothly perturbed to an embedded minimal hyper-surface $\mathscr{D}_{s_0,\rho} \subset \Omega_\rho$ intersecting $\mathscr{C}^\rho$ perpendicularly along its boundary. Furthermore, for any integer k, there exists a constant c_k (independant on ρ) such that*

$$\|\psi_\rho - \sum_{i,j}^{n} \langle R_p(E_j, E_i)E_j, E_i\rangle\|_{\mathcal{C}^k[a,b]} \leq c_k \rho^2,$$

where R_p is the Riemann tensor of $\mathcal{M}$ at $p = \gamma(\rho\, s)$.

Some remarks are due: let $\Gamma \ni p \to \Psi(p) = \sum_{i,j}^{m} \langle R_p(E_j, E_i)E_j, E_i\rangle$ any strict maxima or minima of Ψ imply the existence of minimal surfaces. In particular suppose at some point $p_0 = \gamma(\rho\, s_0)$ interior to Γ, there hold

$$d\Psi(p_0)[\dot\gamma(\rho\, s_0)] = 0 \qquad \text{and} \qquad \left|d^2\Psi(p_0)[\dot\gamma(\rho\, s_0), \dot\gamma(\rho\, s_0)]\right| > c,$$

for some constant c independent on ρ. By the implicit function theorem, there exits a curve $(0, \rho_0) \ni \rho \mapsto s_\rho$ with $s_\rho \to s_0$ such that s_ρ is a critical point of ψ_ρ. Hence for every $\rho \in (0, \rho_0)$, there exits an embedded minimal disc $\mathscr{D}_{s_\rho,\rho}$, centered at $\gamma(\rho\, s_\rho)$, contained in Ω_ρ that intersects $\partial\Omega_\rho$ perpendicularly along its boundary.

5.0.4. REMARK.

- *We have that*

$$\Psi(p) = \sum_{i,j}^{m} \langle R_p(E_j, E_i)E_j, E_i\rangle = \boldsymbol{S}(p) + 2\, Ric_p(\dot\gamma(\rho\, s), \dot\gamma(\rho\, s)),$$

where

$$\boldsymbol{S}(p) = \sum_{\alpha,\beta=0}^{m} \langle R_p(E_\alpha, E_\beta)E_\alpha, E_\beta\rangle$$

is the scalar curvature of $\mathcal{M}$ at $p = \gamma(\rho\, s)$, $E_0 = \dot\gamma(\rho\, s)$ and Ric_p is the Ricci tensor of $\mathcal{M}$ at p. From Theorem 5.0.3, we have that if $s \mapsto Ric_p(\dot\gamma(\rho\, s), \dot\gamma(\rho\, s))$ is constant along Γ then stable critical points the scalar curvature yields existence of minimal discs.

- *Recall that if $(\mathcal{M}_1^{m_1}, g_1)$ and $(\mathcal{M}_2^{m_2}, g_2)$ are two manifolds, the Riemann tensor R of the (Riemannian) Cartesian product $\mathcal{M}^{m_1+m_2} := (\mathcal{M}_1 \times \mathcal{M}_2, g_1 \oplus g_2)$ decomposes as $R = R^1 \oplus R^2$ since the connection*

∇ is given by $\nabla_{X_1+X_2}(Y_1+Y_2) = \nabla^1_{X_1} Y_1 + \nabla^2_{X_2} Y_2$ for any X_1, Y_1 (resp. X_2, Y_2) vector fields of $\mathcal{M}_1$ (resp. $\mathcal{M}_2$), where ∇^i is the connection of $\mathcal{M}_i$. Clearly for any $p_2 \in \mathcal{M}_2$, the set $(\mathcal{M}_1)(p_2) := \{(p_1, p_2) \in \mathcal{M} \quad : \quad p_1 \in \mathcal{M}_1\}$ is a submanifold of $\mathcal{M}$, diffeomorphic to $\mathcal{M}_1$.
In particular if $m_1 = 1$, $R^1 = 0$, by Theorem 5.0.3 we obtain that stable critical points of the mapping $\boldsymbol{S}\Big|_{(\mathcal{M}_1)(p_2)}$ yield existence of minimal discs inside (small) geodesic tubes around the curve $(\mathcal{M}_1)(p_2)$, where as before $\boldsymbol{S}$ is the scalar curvature of $\mathcal{M}$.

- *As a simple byproduct of our analysis, we find that if Γ is a closed curve, we have at least 2 (equal to the Lusternik-Schnierelman category of Γ, see [5]) solutions (without any assumptions on the curvature of $\mathcal{M}$).*

- *We believe that this result might be generalized to higher codimensions namely if N^ℓ, $1 < \ell < m$, is an ℓ-dimensional submanifold of $\mathcal{M}^{m+1}$ and considering the following surface of revolution with axis of rotation $\mathbb{R}^\ell$*
$$S(s, z) = (\kappa^1(s), \dots, \kappa^\ell(s), \phi(s)\, \Theta(z)),$$
where $z \mapsto \Theta(z) \in S^{m-\ell}$, one could obtain $(m-\ell+1)$-dimensional minimal disc-type submanifolds of $\mathcal{M}$ centered on N^ℓ.

Let us describe the proof of the theorems above. We first recall, see [78], that Capillary hypersurfaces with constant contact angle $\arccos \phi'(s)$ are stationary for the energy functional

$$\mathcal{E}(D) = \mathrm{Area}(D \cap \Omega_\rho) - \phi'(s)\, \mathrm{Area}(\Omega'_\rho), \tag{5.4}$$

among (orientable smooth) surfaces $D \subset \Omega_\rho$ with $\partial D \subset \partial\Omega_\rho$ and $\Omega'_\rho \subset \partial\Omega_\rho$ is the part (on one side of D) for which the angle is measured. Moreover the Euler-Lagrange equations is nothing but

$$\begin{aligned} H_D \quad &= \quad 0 \qquad \text{in } D, \\ \langle N_D, N_{\partial\Omega_\rho} \rangle \quad &= \quad \phi'(s) \qquad \text{on } \partial D. \end{aligned} \tag{5.5}$$

Here H_D is the mean curvature of D while N_D and $N_{\partial\Omega_\rho}$ are outer unit normals of D and $\partial\Omega_\rho$ respectively. Since we look for stationary surfaces with a given profile for this energy functional, clearly by (5.2)-(5.3) a manifold of approximate solutions is given by $Z_\rho := \{D_{s,\rho} \quad : \quad s \in [a, b]\}$.

For any given hyper-surface $D_{s,\rho} \in Z_\rho$, we parametrize (locally) a neighborhood of $D_{s,\rho}$ (in the manifold in $\mathcal{M}$) by a mapping $F^s : \mathbb{R} \times B_1^m \to \mathcal{M}$ for which $F^s(t, \partial B_1^m) \subset \partial\Omega_\rho$, for every t, while the direction $F^s_*(\partial_t)$ is nearly normal to $D_{s,\rho}$, and moreover $D_{s,\rho} = F^s(0, B_1^m)$, see (5.9). This allows to parametrize any set $\mathscr{D}$ nearby $D_{s,\rho}$ satisfying $\partial\mathscr{D} \subset \partial\Omega_\rho$ by a function $w : B_1^m \to \mathbb{R}$ such that $\mathscr{D}(w) = F^s(w, B_1^m)$. We call $\mathcal{H}(s,\rho,w)$ the mean curvature of $\mathscr{D}(w)$ and $\mathcal{B}(s,\rho,w)$ the angle between the normals $N_{\partial\mathscr{D}(w)}$ and $N_{\mathscr{C}^\rho}$ of $\partial\mathscr{D}(w)$ and $\partial\Omega_\rho$ respectively.
One of the main features in this work in the (technical) Sections 5.3.1, § 5.3.2 is to calculate $\mathcal{H}(s,\rho,w)$ as a nonlinear elliptic partial differential operator, depending on ρ and s acting on w coupled with the mixed boundary operator which we denote by $\mathcal{B}(s,\rho,w)$. In these calculations it is important to gather various different types of error terms, some of which depend linearly and some nonlinearly on w, and some of which are inhomogeneous terms vanishing to some order in ρ. It turns out to be helpful to rescale the local coordinates y by $\varepsilon(s) = \rho\phi(s)$ which is the radius of the discs. The final expression, Proposition 5.3.5, for the mean curvature of $\mathscr{D}(w)$ then is

$$-\frac{\phi}{\kappa'}\mathcal{H}(s,\rho,w) = -\mathbb{L}_{\rho,s}(w) + \mathcal{O}(\rho^2) + \rho\, Q(w) \qquad \text{in } \mathscr{D}(w),$$

where $\mathbb{L}_{\rho,s}$ is the linearized mean curvature operator about $\mathscr{D}(0) = D_{s,\rho}$:

$$\mathbb{L}_{\rho,s}(w) = -\Delta\, w + \rho L_s(w) \qquad \text{in } \mathscr{D}(w);$$

also the angle between the normals satisfies (see Proposition 5.3.6)

$$\rho^{-1}\ (\mathcal{B}(s,\rho,w) - \phi'(s)) = \mathbb{B}_{\rho,s}(w) + \mathcal{O}(1) + \rho\,\bar{Q}(w),$$

where

$$\mathbb{B}_{\rho,s}(w) = \left((\kappa'(s))^2 \frac{\partial w}{\partial \eta} + \phi\phi'' w \right) + \rho\, \bar{L}_s(w) \qquad \text{on } \partial\mathscr{D}(w).$$

Here L_s (resp. $\bar{L}_s$) is a second order (resp. first order) differential operator and $Q(w)$, $\bar{Q}(w)$ are quadratic in w, see also the end of Section 5.1 for more precise definitions.

It turns out that the problem of finding w such that $\mathscr{D}(w)$ solves (5.5) namely

$$\mathcal{H}(s,\rho,w) = 0 \qquad \text{in } \mathscr{D}(w),$$
$$\mathcal{B}(s,\rho,w) = \phi'(s) \qquad \text{on } \partial\mathscr{D}(w),$$

can be transformed to a fixed point problem for which the solvability is based on the invertibility of $\mathbb{L}_{\rho,s}$ on a suitable space of functions w such that $\mathbb{B}_{\rho,s}(w)=0$. If $\phi\phi''>0$, the operator $\mathbb{L}_{\rho,s}$ (resp. $-\mathbb{L}_{\rho,s}$) is invertible by means of usual Sobolev inequalities. Hence after suitable adjustment of the disc $\mathscr{D}(w)$, we readily prove the first theorem. This program is carried out in § 5.4.1. Now in the situation where $\phi\equiv 1$ and $\kappa =$Id, it is clear that the linearized mean curvature $\mathbb{L}_{\rho,s}$ about any $D\in Z_\rho$ may have small (possibly zero) eigenvalues on the space of functions for which $\mathbb{B}_{\rho,s}(w)=\frac{\partial w}{\partial\eta}+\rho\,\bar{L}_s(w)=0$. This is related to the invariance by translations along the axis of rotation in the "flat" case. Hence $\mathbb{L}_{\rho,s}$ may not be invertible on such space. However restricting again ourselves on space of function orthogonal to the constant function 1, we can perturb Z_ρ to a manifold $\mathcal{Z}_\rho$ (constituted by sets having constant and small mean curvatures, see § 5.4.3) which turns out to be a *natural constraint* for $\mathcal{E}$ namely critical point of $\mathcal{E}\Big|_{\mathcal{Z}_\rho}$ is also stationary for $\mathcal{E}$. For that we use an argument from Kapouleas in [51] which was successfully employed in [73]. We will follow the argument of the latter, we refer to § 5.4.3.

It is worth noticing that this method is also closely related to variational-perturbative methods introduced by Ambrosetti and Badiale in [3] and subsequently used with success to get existence and multiplicity results for a wide class of variational problems in some perturbative setting we refer to the book by Ambrosetti Malchiodi [4] for more details and related applications.

5.1 Preliminaries and notations

We consider $(\phi,\,\kappa)\;:\;[a,b]\to\mathbb{R}^2$ smooth with $\kappa'(s)$, $\phi(s)\neq 0$ for every $s\in[a,b]$ moreover we assume that s is the arc length of the rotating curve $s\mapsto(\phi(s),\kappa(s))$ precisely

$$(\phi'(s))^2+(\kappa'(s))^2=1 \qquad \forall s\in[a,b].$$

We also assume that x_0 is the arc length of γ, and we will let $E_0:=\gamma'$. We choose a parallel (local) orthonormal frame $E_1,\cdots,E_m$ of $N\Gamma$ along Γ.

This determines a coordinate system by defining

$$\overline{F}(x_0,\zeta) := \exp_{\gamma(x_0)}(\zeta^i E_i) \qquad \text{for } \zeta = (\zeta^1, \cdots, \zeta^m)$$

which therefore defines coordinates vector fields :

$$Y_0 := \overline{F}_*(\partial_{x_0}), \qquad Y_i := \overline{F}_*(\partial_{z_i}).$$

We will adopt the convention that the indices $i, j, k, \cdots \in \{1, \dots, m\}$ while $\alpha, \beta, \cdots \in \{0, \dots, m\}$ with $Y_\alpha = Y_0$ when $\alpha = 0$.
By construction, $\nabla_{X_i} Y_0\Big|_\Gamma \in T\Gamma$ so that we can define

$$\langle \nabla_{X_i} Y_0, Y_0 \rangle\Big|_\Gamma = -\Gamma_0^0(E_i).$$

There also holds

$$\nabla_{Y_i} Y_j(\zeta) = O(|\zeta|)_\gamma Y_\gamma. \tag{5.6}$$

If $q = \overline{F}(x_0, \zeta) \in \mathcal{M}$ near the point $p = \overline{F}(x_0, 0) \in \Gamma$, we can expand the metric $g_{\alpha\beta}(q) = \langle Y_\alpha, Y_\beta \rangle$ in ζ, more accurately than in Lemma 5.1.1 by looking at $\mathcal{M}$ here as Ω. (See for instance [56], Proposition 2.1 for the proof).

5.1.1. LEMMA. *In the above coordinates (x_0, ζ), for any $i, j = 1, ..., m$, we have*

$$\begin{aligned}
g_{ij}(q) &= \delta_{ij} + \tfrac{1}{3} \langle R_p(Y, E_i) Y, E_j \rangle + \tfrac{1}{6} \langle \nabla_Y R_p(Y, E_i) Y, E_j \rangle + O_p(|\zeta|^4);\\
g_{0j}(q) &= \tfrac{2}{3} \langle R_p(Y, E_0) Y, E_j \rangle + O_p(|\zeta|^3);\\
g_{00}(q) &= 1 - 2\Gamma_0^0(Y) + \langle R_p(Y, E_0) Y, E_0 \rangle + O_p(|\zeta|^3),
\end{aligned}$$

where $Y := \zeta^i E_i$.

Notation for error terms: Any expression of the form $L(\omega)$ (resp. $\bar{L}(\omega)$) denotes a linear combination of the function ω together with its derivatives with respect to the vector fields Y_i up to order 2 (resp. order 1). The coefficients of L or $\bar{L}$ might depend on ρ and s but, for all $k \in \mathbb{N}$, there exists a constant $c > 0$ independent of $\rho \in (0,1)$ and $s \in [a, b]$ such that

$$\|L(\omega)\|_{\mathcal{C}^{k,\alpha}(\overline{B_1^m})} \leq c \, \|\omega\|_{\mathcal{C}^{k+2,\alpha}(\overline{B_1^m})},$$

$$\|\bar{L}_s(\omega)\|_{\mathcal{C}^{k,\alpha}(\overline{B_1^m})} \leq c \, \|\omega\|_{\mathcal{C}^{k+1,\alpha}(\overline{B_+^m})}.$$

Similarly, any expression of the form $Q(\omega)$ (resp $\bar{Q}(\omega)$) denotes a nonlinear operator in the function ω together with its derivatives with respect to the vector fields Y_i up to order 2 (resp. 1). The coefficients of the Taylor expansion of $Q^a(\omega)$ in powers of ω and its partial derivatives might depend on ρ and s and, given $k \in \mathbb{N}$, there exists a constant $c > 0$ independent of $\rho \in (0,1)$ and $s \in [a,b]$ such that

$$\|Q(\omega_1)-Q(\omega_2)\|_{\mathcal{C}^{k,\alpha}(\overline{B_1^m})} \leq c \left(\|\omega_1\|_{\mathcal{C}^{k+2,\alpha}(\overline{B_1^m})} + \|\omega_2\|_{\mathcal{C}^{k+2,\alpha}(\overline{B_1^m})} \right) \|\omega_1-\omega_2\|_{\mathcal{C}^{k+2,\alpha}(\overline{B_1^m})},$$

provided $\|\omega_i\|_{\mathcal{C}^{k+2,\alpha}(\overline{B_1^m})} \leq 1$, $i = 1,2$. Also

$$\|\bar{Q}(\omega_1)-\bar{Q}(\omega_2)\|_{\mathcal{C}^{k,\alpha}(\overline{B_1^m})} \leq c \left(\|\omega_1\|_{\mathcal{C}^{k+1,\alpha}(\overline{B_1^m})} + \|\omega_2\|_{\mathcal{C}^{k+1,\alpha}(\overline{B_1^m})} \right) \|\omega_1-\omega_2\|_{\mathcal{C}^{k+1,\alpha}(\overline{B_1^m})},$$

provided $\|\omega_i\|_{\mathcal{C}^{k+2,\alpha}(\overline{B_1^m})} \leq 1$. We also agree that any term denoted by $\mathcal{O}(r^d)$ (with $r \in \mathbb{R}$ may depend on s) is a smooth function on B_1^m that might depend on s but satisfies

$$\|\frac{\mathcal{O}(r^d)}{|r|^d}\|_{\mathcal{C}^{k,\alpha}(\overline{B_1^m})} \leq c$$

for a constant c independent of s.

5.2 On the surface of revolution around Γ

We start by fixing the following notations which will be useful later.

Notations:

Through the following of this chapter,

$$\varepsilon(s) = \rho\,\phi(s) \qquad \text{and} \qquad \varepsilon_1(s) = \rho\,\kappa(s) \qquad \text{for every } s \in [a,b].$$

In terms of cylindrical coordinates, letting $\Theta(z) : \mathbb{R}^{m-1} \to S^{m-1}$, the surface of revolution $\mathscr{C}^\rho$ around Γ can be parameterized by

$$\mathcal{C}^\rho(s,z) := \overline{F}(\varepsilon_1(s), \varepsilon(s)\Theta(z)) = \exp_{\gamma(\varepsilon_1(s))}(\varepsilon(s)\Theta^i(z)\,E_i).$$

The tangent plane is spanned by the vector fields

$$\begin{aligned} Z_0^c &= \mathcal{C}_*^\rho(\partial_{x_0}) = \varepsilon_1' Y_0 + \varepsilon'\,\Upsilon, \\ Z_j^c &= \mathcal{C}_*^\rho(\partial_{z^j}) = \varepsilon\,\Upsilon_j, \qquad j = 1, \cdots, m, \end{aligned}$$

where

$$\Upsilon = \Theta^i Y_i.$$

We recall also from [56]

5.2.1. Lemma. *Let $q = \mathcal{C}^\rho(s,z) \in \mathscr{C}^\rho$, there hold*

$$\begin{aligned}\langle \Upsilon, \Upsilon \rangle_q &= 1,\\ \langle \Upsilon, Y_0 \rangle_q &= 0,\\ \langle \Upsilon, \Upsilon_j \rangle_q &= 0.\end{aligned}$$

5.2.2. Lemma. *In the notations above, the first fundamental form of $\mathscr{C}^\rho$ has the following expansions*

$$\begin{aligned}\langle Z_0^c, Z_0^c \rangle &= \frac{\varepsilon^2}{\phi^2} - 2\varepsilon|\varepsilon_1'|^2\Gamma_0^0(\Theta) + \varepsilon^2|\varepsilon_1'|^2\langle R(\Theta,E_0)\Theta, E_0\rangle + \mathcal{O}(\varepsilon^5),\\ \langle Z_l^c, Z_k^c \rangle &= \varepsilon^2\langle \Theta_l, \Theta_k\rangle + \mathcal{O}(\varepsilon^4),\\ \langle Z_0^c, Z_k^c \rangle &= \mathcal{O}(\varepsilon^4).\end{aligned}$$

Proof. Recalling that

$$|\varepsilon_1'|^2 + |\varepsilon'|^2 = \frac{\varepsilon^2}{\phi^2}$$

we obtain, using also the Lemmas 5.2.1, 5.1.1, that

$$\begin{aligned}\langle Z_0^c, Z_0^c \rangle &= |\varepsilon_1'|^2\left(1 - 2\varepsilon\Gamma_0^0(\Theta) + \varepsilon^2\langle R(\Theta,E_0)\Theta, E_0\rangle + \mathcal{O}(\varepsilon^3)\right) + |\varepsilon'|^2\\ &= \frac{\varepsilon^2}{\phi^2} - 2\varepsilon|\varepsilon_1'|^2\Gamma_0^0(\Theta) + \varepsilon^2|\varepsilon_1'|^2\langle R(\Theta,E_0)\Theta, E_0\rangle + \mathcal{O}(\varepsilon^5)\end{aligned}$$

The other expansions are easy consequences of the Lemmas 5.2.1 5.1.1. ∎

5.2.1 The unit normal field to the surface of revolution

Call

$$M := \varepsilon' X_0 - \varepsilon_1' \Upsilon$$

and set

$$\tilde{N}_c(s,z) = M + \alpha_0 Z_0^c + \alpha_k Z_k^c.$$

Note that this vector filed is normal (not necessary unitary) to the surface whenever we can determined α_k so that $\langle \tilde{N}_c, Z_k^c\rangle = \langle \tilde{N}_c, Z_0^c \rangle = 0$ for all

$k = 1, \ldots, m$. This therefore leads to solving a linear system. Observe that

$$\begin{aligned} \langle M, Z_0^c \rangle &= \varepsilon_1' \varepsilon' \langle Y_0, Y_0 \rangle - \varepsilon_1' \varepsilon' \langle \Upsilon, \Upsilon \rangle \\ &= \varepsilon_1' \varepsilon' \left(1 - 2\varepsilon \Gamma_0^0(\Theta) + \varepsilon^2 \langle R(\Theta, E_0)\Theta, E_0 \rangle - 1 + \mathcal{O}(\varepsilon^3) \right), \end{aligned}$$

hence

$$\langle M, Z_0^c \rangle = -2\varepsilon_1' \varepsilon' \varepsilon \Gamma_0^0(\Theta) + \varepsilon_1' \varepsilon' \varepsilon^2 \langle R(\Theta, E_0)\Theta, E_0 \rangle + \mathcal{O}(\varepsilon^5).$$

Also we have

$$\langle M, Z_k^c \rangle = \varepsilon \varepsilon' \langle X_0, Y_k \rangle - \varepsilon_1' \varepsilon \langle \Upsilon, Y_k \rangle = \mathcal{O}(\varepsilon^4).$$

If we use Lemma 5.2.2, we have

$$\begin{aligned} \alpha_0 \langle Z_0^c, Z_0^c \rangle &= -\alpha_k \langle Z_k^c, Z_0^c \rangle - \langle M, Z_0^c \rangle \\ &= \alpha_k \mathcal{O}(\varepsilon^4) + 2\varepsilon \varepsilon_1' \varepsilon' \Gamma_0^0(\Theta) - \varepsilon^2 \varepsilon_1' \varepsilon' \langle R(\Theta, E_0)\Theta, E_0 \rangle + \mathcal{O}(\varepsilon^5) \end{aligned}$$

so

$$\alpha_0 = \frac{\varepsilon' \varepsilon_1'}{\varepsilon^2} \phi^2 \left(2\varepsilon \Gamma_0^0(\Theta) - |\varepsilon_1'|^2 \langle R(\Theta, E_0)\Theta, E_0 \rangle \right) + 4\phi^2 \frac{|\varepsilon_1'|^3}{\varepsilon^2} \varepsilon' \Gamma_0^0(\Theta) \Gamma_0^0(\Theta) + \mathcal{O}(\varepsilon^3) + \alpha_k \mathcal{O}(\varepsilon^4) \tag{5.7}$$

Since

$$\alpha_k \langle Z_k^c, Z_l^c \rangle + \alpha_0 \langle Z_0^c, Z_l^c \rangle = -\langle M, Z_l^c \rangle$$

and using (5.7)

$$\alpha_k \langle Z_k^c, Z_l^c \rangle + \alpha_k \mathcal{O}(\varepsilon^6) + \mathcal{O}(\varepsilon^5) = \mathcal{O}(\varepsilon^4)$$

we get

$$\alpha_k \left(\varepsilon^2 \langle \Theta_l, \Theta_k \rangle + \mathcal{O}(\varepsilon^4) \right) = \mathcal{O}(\varepsilon^4),$$

therefore

$$\alpha_k = \mathcal{O}(\varepsilon^2).$$

Recalling that $\varepsilon = \rho\, \phi$ while $\varepsilon_1 = \rho\, \kappa$ we define $\bar{\alpha}_0$ by the relation

$$\alpha_0 = \phi' \bar{\alpha}_0 + \mathcal{O}(\varepsilon^3).$$

Namely

$$\bar{\alpha}_0 = 2\varepsilon_1' \phi \Gamma_0^0(\Theta) - \varepsilon' \varepsilon_1' \phi \langle R(\Theta, E_0)\Theta, E_0 \rangle + 4\varepsilon_1' \kappa' \varepsilon \Gamma_0^0(\Theta) \Gamma_0^0(\Theta).$$

Now let us compute the norm of this normal vector field. Since

$$\tilde N_c(s,z) := M + \alpha_0 Z_0^c + \alpha_k Z_k^c$$

we have by construction

$$\langle \tilde N_c, \tilde N_c\rangle = \langle M, M\rangle + a_0^2\langle Z_0^c, Z_0^c\rangle + \alpha_k\alpha_l\langle Z_k^c, Z_l^c\rangle + 2\alpha_0\langle M, Z_0^c\rangle + 2\alpha_k\langle M, Z_k^c\rangle + 2\alpha_k\alpha_0\langle Z_k^c, Z$$

Notice that

$$\begin{aligned}\alpha_0\langle Z_0^c, Z_0^c\rangle &= -\langle M, Z_0^c\rangle - \alpha_k\langle Z_k^c, Z_0^c\rangle\\ &= -\langle M, Z_0^c\rangle + \mathcal{O}(\varepsilon^6)\end{aligned}$$

and

$$\alpha_0^2\langle Z_0^c, Z_0^c\rangle = -\alpha_0\langle M, Z_0^c\rangle + \mathcal{O}(\varepsilon^6),$$

hence

$$\langle \tilde N_c, \tilde N_c\rangle = \langle M, M\rangle + \alpha_0\langle M, Z_0^c\rangle + \mathcal{O}(\varepsilon^6).$$

Now observe that

$$\begin{aligned}\langle M, M\rangle &= |\varepsilon'|^2\langle X_0, X_0\rangle + |\varepsilon_1'|^2\langle \Upsilon, \Upsilon\rangle\\ &= \frac{\varepsilon^2}{\phi^2} + |\varepsilon'|^2\left(-2\varepsilon\Gamma_0^0(\Theta) + \varepsilon^2\langle R(\Theta, E_0)\Theta, E_0\rangle + \mathcal{O}(\varepsilon^3)\right)\end{aligned}$$

and

$$\alpha_0\langle M, Z_0^c\rangle = -2\varepsilon\varepsilon_1'\varepsilon'\Gamma_0^0(\Theta)\alpha_0 + \mathcal{O}(\varepsilon^4)\alpha_0 = -4\frac{\varepsilon^2}{\rho^2}|\varepsilon'|^2|\varepsilon_1'|^2\Gamma_0^0(\Theta)\Gamma_0^0(\Theta) + \mathcal{O}(\varepsilon^5).$$

So we have

$$\tfrac{\phi^2}{\varepsilon^2}\langle \tilde N_c, \tilde N_c\rangle = 1 + \tfrac{|\varepsilon'|^2}{\varepsilon^2}\phi^2\left(-2\varepsilon\Gamma_0^0(\Theta) + \varepsilon^2\langle R(\Theta, E_0)\Theta, E_0\rangle\right) - 4\tfrac{\phi^4}{\varepsilon^2}|\varepsilon'|^2|\varepsilon_1'|^2\Gamma_0^0(\Theta)\Gamma_0^0(\Theta$$

Finally we conclude that

$$\tfrac{\varepsilon}{\phi}|\tilde N_c|^{-1} = 1 + \tfrac{|\varepsilon'|^2}{\varepsilon}\phi^2\Gamma_0^0(\Theta) + \left(3|\tfrac{\varepsilon'}{\varepsilon}|^4\phi^4\varepsilon^2 + 2|\tfrac{\varepsilon_1'}{\varepsilon}|^2|\varepsilon'|^2\phi^4\right)\Gamma_0^0(\Theta)\Gamma_0^0(\Theta) - \frac{|\varepsilon'|^2}{2}\langle R(\Theta, E$$

and, setting

$$H_c(\Theta,\Theta) := \left(3\frac{|\varepsilon'|^2}{\varepsilon^2} + 2\frac{|\varepsilon_1'|^2}{\varepsilon^2}\right)\phi^2\Gamma_0^0(\Theta)\Gamma_0^0(\Theta) - \frac{1}{2}\langle R(\Theta, E_0)\Theta, E_0\rangle,$$

we can simply write

$$\frac{\varepsilon}{\phi}|\tilde N_c|^{-1} = 1 + |\varepsilon'|^2\phi^2\left[\frac{1}{\varepsilon}\Gamma_0^0(\Theta) + H_c(\Theta,\Theta)\right] + \mathcal{O}(\varepsilon^3).$$

We collect all these in the following

5.2.3. PROPOSITION. *There exists an interior (non unit) normal vector field of $\mathscr{C}^\rho$ which has the following expansions*

$$\tilde{N}_{\mathscr{C}^\rho}(s,z) = -\varepsilon_1'\Upsilon + \varepsilon' Y_0 + (\phi'\bar{\alpha}_0 + \mathcal{O}(\varepsilon^3))Z_0^c + \alpha_k Z_k^c,$$

where

$$\bar{\alpha}_0 = 2\varepsilon_1'\phi\Gamma_0^0(\Theta) - \varepsilon'\varepsilon_1'\phi\langle R(\Theta,E_0)\Theta,E_0\rangle + 4\varepsilon_1'\kappa'\varepsilon\Gamma_0^0(\Theta)\Gamma_0^0(\Theta);$$

$$\alpha_k = \mathcal{O}(\varepsilon^2).$$

Moreover

$$\rho\left|\tilde{N}_{\mathscr{C}^\rho}\right|^{-1} = 1 + |\phi'|^2\left(\varepsilon\Gamma_0^0(\Theta) + \varepsilon^2 H_c(\Theta,\Theta)\right) + \mathcal{O}(\varepsilon^3),$$

where

$$H_c(\Theta,\Theta) = \left(|\phi'|^2 + 2\phi^4\right)\Gamma_0^0(\Theta)\Gamma_0^0(\Theta) - \frac{1}{2}\langle R(\Theta,E_0)\Theta,E_0\rangle + \mathcal{O}(\varepsilon^3).$$

5.3 Discs centered on Γ with boundary on $\mathscr{C}^\rho$

For $\delta > 0$, B_δ^m will denote the ball of $\mathbb{R}^m$ with radius δ centered at the origin. For any s, we consider the disc $\mathscr{D}^{s,\varepsilon}$ of radius ε centered at $\gamma(\varepsilon(s))$ given by

$$\mathscr{D}^{s,\varepsilon} := \overline{F}(\varepsilon_1(s)\,,\,\varepsilon(s)\,B_1^m),$$

parameterized by

$$B_1^m \ni x \mapsto \mathcal{D}^{s,\varepsilon}(x) = \overline{F}(\varepsilon_1,\,\varepsilon\,x).$$

Notations

$$\bar{\varepsilon}(s,t) := \varepsilon(s+\varepsilon(s)t) \qquad \bar{\varepsilon}_1(s,t) := \varepsilon_1(s+\varepsilon(s)t);$$

$$\bar{\varepsilon}'(s,t) := \partial_t\varepsilon(s+\varepsilon(s)t) = \varepsilon(s)\varepsilon'(s+\varepsilon(s)t) \qquad \bar{\varepsilon}_1'(s,t) := \partial_t\varepsilon_1(s+\varepsilon(s)t) = \varepsilon(s)\varepsilon_1'(s+$$

Notice that

$$\bar{\varepsilon}'(s,t) = (\varepsilon' + \varepsilon\varepsilon''\,t + \varepsilon^3\mathcal{O}(t^2))\varepsilon, \qquad \bar{\varepsilon}_1'(s,t) = (\varepsilon_1' + \varepsilon\varepsilon_1''\,t + \varepsilon^3\mathcal{O}(t^2))\varepsilon. \quad (5.8)$$

A parametrization of the neighborhood of the disc (in $\mathcal{M}$) centered at $p = \gamma(\varepsilon_1(s)) \in \Gamma$ with radius ε (ρ small) can be defined by

$$F^s(t,x) := \overline{F}(\bar{\varepsilon}_1(s,t)\,,\,\bar{\varepsilon}(s,t)\,x) \qquad \forall x \in B_1^m,\ |t| \ll 1. \qquad (5.9)$$

Note that by construction,

$$F^s(t\,,\,\partial B_1^m) \subset \mathscr{C}^\rho, \qquad \forall |t| \ll 1$$

and more precisely, for every $|t| \ll 1$

$$F^s(t,x) = \mathcal{C}^\rho(s+\varepsilon\,t, \varepsilon(s+\varepsilon\,t)\,x) \qquad \forall x \in \partial B_1^m = S^{m-1}.$$

We consider the following vector fields induced by F^s

$$\begin{aligned} \varepsilon\,T_0 &:= F_*(\partial_t) &= \bar{\varepsilon}_1'\,Y_0(t) + \bar{\varepsilon}'\,X(t),\\ \bar{\varepsilon}\,T_j &:= F_*(\partial_{y^j}) &= \bar{\varepsilon}\,Y_j(t). \end{aligned}$$

Here $X = x^i E_i$.

5.3.1. Lemma. *At $q = F^s(t,x)$,we have*

$$\begin{aligned} \langle Y_i(t), Y_j(t)\rangle_q &= \delta_{ij} + \tfrac{\varepsilon^2}{3}\langle R_p(X,E_i)X, E_j\rangle + \tfrac{\varepsilon^3}{6}\langle \nabla_X R_p(X,E_i)X, E_j\rangle + \mathcal{O}_p(\varepsilon^4) + \varepsilon^3\mathcal{O}\\ \langle Y_0(t), Y_j(t)\rangle_q &= \tfrac{2\varepsilon^2}{3}\langle R_p(X,E_0)X, E_j\rangle + \mathcal{O}_p(\varepsilon^3) + \varepsilon^3\mathcal{O}_p(t) + \varepsilon^4\mathcal{O}_p(t^2);\\ \langle Y_0(t), Y_0(t)\rangle_q &= 1 - 2\bar{\varepsilon}\Gamma_0^0(X) + 2\varepsilon^2 U_0^0(X)t + \varepsilon^2\langle R_p(X,E_0)X, E_0\rangle + \mathcal{O}_p(\varepsilon^3) + \varepsilon^3\mathcal{O}_p \end{aligned}$$

where $p = \gamma(\varepsilon_1(s)) \in \Gamma$ and

$$\varepsilon U_0^0(X) = \varepsilon_1'\Gamma_{00}^0 - \varepsilon'\Gamma_0^0(X).$$

Proof. There holds

$$\begin{aligned} \frac{d}{dt}\langle Y_0(t), Y_0(t)\rangle_q\Big|_{t=0} &= 2\langle \nabla_{\varepsilon T_0} Y_0, Y_0\rangle\Big|_{t=0}\\ &= 2\bar{\varepsilon}_1'\langle \nabla_{Y_0} Y_0, Y_0\rangle\Big|_{t=0} + 2\bar{\varepsilon}'\langle \nabla_X Y_0, Y_0\rangle\Big|_{t=0}\\ &= 2\varepsilon(\varepsilon_1'\Gamma_{00}^0 - \varepsilon'\Gamma_0^0(X)) + \mathcal{O}_p(\varepsilon^3). \end{aligned}$$

In this formula we have used (5.8), hence the last expansion follows. On the other hand one has

$$\begin{aligned} \frac{d}{dt}\langle Y_0(t), Y_i(t)\rangle_q\Big|_{t=0} &= \langle \nabla_{\varepsilon T_0} Y_0, Y_i\rangle\Big|_{t=0} + \langle Y_0, \nabla_{\varepsilon T_0} Y_i\rangle\Big|_{t=0}\\ &= \bar{\varepsilon}_1'\langle \nabla_{Y_0} Y_0, Y_i\rangle\Big|_{t=0} + 2\bar{\varepsilon}'\langle \nabla_X Y_0, Y_i\rangle\Big|_{t=0} + \bar{\varepsilon}_1'\langle Y_0, \nabla_{Y_0} Y_i\rangle\Big|_{t=0} + 2\bar{\varepsilon} \end{aligned}$$

Since by construction $\langle \nabla_{E_0} E_0, E_i\rangle + \langle E_0, \nabla_{E_0} E_i\rangle = 0$ and also $\langle \nabla_X E_0, E_i\rangle + \langle E_0, \nabla_X E_i\rangle = 0$ on Γ, we infer that

$$\frac{d}{dt}\langle Y_0(t), Y_i(t)\rangle_q\Big|_{t=0} = \mathcal{O}_p(\varepsilon^3).$$

In the same way, the first expansions follows similarly. ■

Using the above lemma, and (5.8) we get

5.3.2. Lemma. *The following expansions hold*

$$
\begin{aligned}
\langle T_i, T_j\rangle &= \delta_{ij} + \tfrac{\varepsilon^2}{3}\langle R_p(X,E_i)X,E_j\rangle + \tfrac{\varepsilon^3}{6}\langle \nabla_X R_p(X,E_i)X,E_j\rangle + \mathcal{O}_p(\varepsilon^4) + \varepsilon^3\mathcal{O}(t) + \varepsilon \\
\langle T_0, T_j\rangle &= \varepsilon' x^i + \mathcal{O}_p(\varepsilon^3) + \varepsilon^4\mathcal{O}_p(t) + \varepsilon^5\mathcal{O}_p(t^2); \\
\langle T_0, T_0\rangle &= \tfrac{\varepsilon^2}{\phi^2} + |\varepsilon_1'|^2\left(-2\bar\varepsilon\Gamma_0^0(X) + 2\varepsilon^2 U_0^0(X)t\right) + \mathcal{O}_p(\varepsilon^5) + \varepsilon^5\mathcal{O}_p(t) + \varepsilon^6\mathcal{O}_p(t^2),
\end{aligned}
$$

where $p = \gamma(\varepsilon_1(s)) \in \Gamma$.

Observe that all disc-type surfaces nearby $\mathscr{D}_{s,\rho}$ with boundary contained in $\mathscr{C}^\rho$ can be parameterized by

$$G^s(x) := F^s(w(x)\,,\,x), \tag{5.10}$$

for some smooth function $w : B_1^m \to \mathbb{R}$. We will call $\mathscr{D}^{s,\varepsilon(s)}(w) = G^s(B_1^m)$.

5.3.1 Mean curvature of Perturbed disc $\mathscr{D}(w)$

It is not difficult to see that the tangent plane of $\mathscr{D}(w) = \mathscr{D}^{s,\varepsilon(s)}(w)$ is spanned by the vector fields

$$Z_j \;=\; G_*^s(\partial_{x^j}) \;= \varepsilon\, w_{x^j} T_0 + \bar\varepsilon(s,w)\, T_j.$$

From Lemma 5.3.2, it is clear that at the point $q = G^s(x) = F^s(w(x)\,,\,x)$ there hold

$$
\begin{aligned}
\langle T_i, T_j\rangle_q &= \delta_{ij} + \tfrac{\varepsilon^2}{3}\langle R_p(X,E_i)X,E_j\rangle + \tfrac{\varepsilon^3}{6}\langle \nabla_X R_p(X,E_i)X,E_j\rangle + \mathcal{O}(\varepsilon^4) + \varepsilon^3 L(w) + \varepsilon \\
\langle T_0, T_j\rangle_q &= \varepsilon' x^j + \mathcal{O}(\varepsilon^3) + \varepsilon^4 L(w) + \varepsilon^5 Q(w); \\
\langle T_0, T_0\rangle_q &= \tfrac{\varepsilon^2}{\phi^2} + |\varepsilon_1'|^2\left(-2\bar\varepsilon\Gamma_0^0(X) + 2\varepsilon^2 U_0^0(X)w\right) + \mathcal{O}_p(\varepsilon^5) + \varepsilon^5 L(w) + \varepsilon^6 Q(w).
\end{aligned}
\tag{5.11}
$$

Observing that $\bar\varepsilon(s,w) = \varepsilon + \varepsilon\varepsilon'\,w + \varepsilon^3 Q(w)$ and using (5.11), we get the first fundamental form $h_{ij} := \langle Z_i, Z_j\rangle$,

$$
\begin{aligned}
\varepsilon^{-2}h_{ij} \;&=\; (1+2\varepsilon'\,w)\,\delta_{ij} + \varepsilon'(w_{x^i}x^j + w_{x^j}x^i) + \tfrac{\varepsilon^2}{3}\langle R_p(X,E_i)X,E_j\rangle + \tfrac{\varepsilon^3}{6}\langle \nabla_X R_p(X,E_i \\
&+\; \mathcal{O}(\varepsilon^4) + \varepsilon^3 L(w) + \varepsilon^2 Q(w).
\end{aligned}
\tag{5.12}
$$

The normal vector field

Consider the vector field

$$\tilde{N}_{\mathscr{D}} = Y_0 + a_k Z_k.$$

Observe that it is normal (not necessary unitary) to the disc whenever we can find a_k such that $\langle \tilde{N}_{\mathscr{D}}, Z_k \rangle = 0$ for any $k = 1, \dots, m$. Namely a_k satisfies

$$\mathscr{D}^{s,\varepsilon(s)}(w)\, a_k\, h_{ik} = -\langle Y_0, Z_i \rangle. \tag{5.13}$$

Since

$$\langle Y_0, Z_i \rangle = \bar{\varepsilon}_1' w_{x^i} \langle Y_0, Y_0 \rangle + \bar{\varepsilon}' w_{x^i} \langle Y_0, X \rangle + \bar{\varepsilon} \langle Y_0, Y_i \rangle$$

then from (5.12) and (5.8), we get the formula

$$\varepsilon^2 a_k = \varepsilon \varepsilon_1' w_{x^k} \langle Y_0, Y_0 \rangle + \varepsilon \langle Y_0, Y_k \rangle + \mathcal{O}(\varepsilon^4) + \varepsilon^4 L(w) + \varepsilon^3 Q(w). \tag{5.14}$$

And also since $\bar{\varepsilon} = \varepsilon + \varepsilon^2 L(w)$, $\bar{\varepsilon}_1 = \varepsilon_1 + \varepsilon^2 L(w)$, we get

$$\varepsilon^2\, a_k = -\varepsilon \varepsilon_1' (1 - 2\varepsilon \Gamma_0^0(X)) w_{x^k} - \frac{2\varepsilon^3}{3} \langle R(X, E_0)X, E_k \rangle + \mathcal{O}(\varepsilon^4) + \varepsilon^4 L(w) + \varepsilon^3 Q(w)$$

and thus

$$a_k = -\frac{\varepsilon_1'}{\varepsilon} (1 - 2\varepsilon \Gamma_0^0(X)) w_{x^i} - \frac{2\varepsilon}{3} \langle R(X, E_0)X, E_i \rangle + \mathcal{O}(\varepsilon^2) + \varepsilon^2 L(w) + \varepsilon Q(w).$$

Moreover using also (5.13) we have

$$\begin{aligned}
\langle \tilde{N}_{\mathscr{D}}, \tilde{N}_{\mathscr{D}} \rangle &= \langle Y_0, Y_0 \rangle - a_k a_l\, h_{kl} \\
&= \langle Y_0, Y_0 \rangle - a_k (\mathcal{O}(\varepsilon^3) + \varepsilon^2 L(w) + \varepsilon^4 Q(w)) \\
&= \langle Y_0, Y_0 \rangle - \big(\mathcal{O}(\varepsilon) + L(w) + \varepsilon^2 Q(w)\big) \big(\mathcal{O}(\varepsilon^3) + \varepsilon^2 L(w) + \varepsilon^2 Q(w)\big) \\
&= \langle Y_0, Y_0 \rangle + \mathcal{O}(\varepsilon^4) + \varepsilon^3 L(w) + \varepsilon^2 Q(w).
\end{aligned}$$

Hence

$$\left| \langle \tilde{N}_{\mathscr{D}}, \tilde{N}_{\mathscr{D}} \rangle \right|^{-1} = |\langle Y_0, Y_0 \rangle|^{-1} + \mathcal{O}(\varepsilon^4) + \varepsilon^3 L(w) + \varepsilon^2 Q(w) \tag{5.15}$$

Therefore

$$\left| \langle \tilde{N}_{\mathscr{D}}, \tilde{N}_{\mathscr{D}} \rangle \right|^{-1} = 1 + \bar{\varepsilon} \Gamma_0^0(X) + \mathcal{O}(\varepsilon^2) + \varepsilon^2 L(w) + \varepsilon^2 Q(w). \tag{5.16}$$

We then conclude that the unit normal has the following expansions:

$$\begin{aligned} N_{\mathscr{D}} &= \quad \left(1+\varepsilon\Gamma_0^0(X)+\mathcal{O}(\varepsilon^2)+\varepsilon^2 L(w)+\varepsilon^2 Q(w)\right) Y_0 \\ &+ \left(-\frac{\varepsilon_1'}{\varepsilon}(1+\varepsilon\Gamma_0^0(X))w_{x^k}-\frac{2\varepsilon}{3}\left\langle R(X,E_0)X,E_k\right\rangle+\mathcal{O}(\varepsilon^2)+\varepsilon^2 L(w)+\varepsilon Q(w)\right) Z_k. \end{aligned} \tag{5.17}$$

Sometimes we will simply need to write $N_{\mathscr{D}}$ in the more compact form

$$N_{\mathscr{D}} = Y_0 + \left(\mathcal{O}(\varepsilon^2)+\varepsilon L(w)+\varepsilon^2 Q(w)\right)_\alpha Y_\alpha.$$

The Second Fundamental Form

Observe that in the scaled variables $\zeta = \varepsilon\, x$, since the functions $\mathcal{O}(\varepsilon^d)$, $L(w)$ and $Q(w)$ are depending on x whereas the vector fields Y_α depend on $\zeta = \varepsilon\, x$, we have for any integers $1 \le m$ and $d \ge 1$

$$E_i(\mathcal{O}(\varepsilon^d)) = \mathcal{O}(\varepsilon^{d-1}), \qquad E_i(\varepsilon^d L(w)) = \varepsilon^{d-1}L(w), \qquad E_i(\varepsilon^d Q(w)) = \varepsilon^{d-1}Q(w).$$

Having this in mind, we state the following

5.3.3. Lemma. *There holds*

$$\langle T_0, \nabla_{Z_i} N_{\mathscr{D}}\rangle = \mathcal{O}(\varepsilon^3)+\varepsilon L(w)+\varepsilon^2 Q(w).$$

Proof. Using (5.17) and recall that $T_0 = \varepsilon_1' Y_0 + \varepsilon' X + \varepsilon L(w)_\alpha Y_\alpha$, we have

$$\begin{aligned} \langle T_0, \nabla_{Z_i} N_{\mathscr{D}}\rangle\Big|_{w=0} &= \langle T_0, \nabla_{\varepsilon Y_i}(1+\varepsilon\Gamma_0^0(X))Y_0\rangle + \mathcal{O}(\varepsilon^3) \\ &= \varepsilon\Gamma_0^0(E_i)\langle T_0, Y_0\rangle + \varepsilon(1+\varepsilon\Gamma_0^0(X))\langle T_0, \nabla_{Y_i}Y_0\rangle + \mathcal{O}(\varepsilon^3) \\ &= \varepsilon\varepsilon_1'\Gamma_0^0(E_i)\langle Y_0, Y_0\rangle + \varepsilon\varepsilon_1'(1+\varepsilon\Gamma_0^0(X))\langle Y_0, \nabla_{Y_i}Y_0\rangle + \mathcal{O}(\varepsilon^3) \\ &= \varepsilon\varepsilon_1'\Gamma_0^0(E_i) - \varepsilon\varepsilon_1'\Gamma_0^0(E_i) + \mathcal{O}(\varepsilon^3). \end{aligned}$$

Hence we get the result. ∎

Let us now estimate the second fundamental form of $\mathscr{D}(w)$.

5.3.4. Lemma. *The following expansion holds.*

$$\begin{aligned} \langle \nabla_{Z_i} Z_j, N_{\mathscr{D}}\rangle &= \varepsilon\left(1-\varepsilon\Gamma_0^0(E_l)x^l\right) w_{x^i x^j} + \varepsilon^2\langle \nabla_{Y^i} Y_j, Y_0\rangle + \mathcal{O}(\varepsilon^4) \\ &- \varepsilon^2\left(w_{x^j}\Gamma_0^0(E_i) + w_{x^i}\Gamma_0^0(E_j)\right) + \varepsilon^3 L(w) + \varepsilon^3 Q(w). \end{aligned}$$

Proof. We have

$$\langle \nabla_{Z_i} Z_j, N_{\mathscr{D}}\rangle = \varepsilon\langle \nabla_{Z_i}(w_{x^j}T_0), N\rangle + \langle \nabla_{Z_i}(\bar\varepsilon T_j), N_{\mathscr{D}}\rangle.$$

We first estimate $\langle \nabla_{Z_i}(w_{x^j}T_0), N_{\mathscr{D}}\rangle$**.**
Observe that

$$\frac{\partial}{\partial x^i}\langle w_{x^j}T_0, N_{\mathscr{D}}\rangle = \langle \nabla_{Z_i}(w_{x^j}T_0), N_{\mathscr{D}}\rangle + \langle w_{x^j}T_0, \nabla_{Z_i}N_{\mathscr{D}}\rangle,$$

which implies that

$$\langle \nabla_{Z_i}(w_{x^j}T_0), N_{\mathscr{D}}\rangle = \frac{\partial}{\partial x^i}\langle w_{x^j}T_0, N_{\mathscr{D}}\rangle - w_{x^j}\langle T_0, \nabla_{Z_i}N_{\mathscr{D}}\rangle.$$

The formula (5.13) shows that

$$\langle Y_0, \tilde N_{\mathscr{D}}\rangle = \langle Y_0, Y_0\rangle + a_k\langle Z_k, Y_0\rangle = \langle Y_0, Y_0\rangle - a_k a_l\langle Z_k, Z_l\rangle = \left|\tilde N_{\mathscr{D}}\right|^2$$

and then

$$\langle Y_0, N_{\mathscr{D}}\rangle = \left|\tilde N_{\mathscr{D}}\right|.$$

From the fact that $\langle Y_0, X\rangle = 0$ when $w = 0$ and that

$$\langle Z_k, X\rangle = \varepsilon x^k + \mathcal{O}(\varepsilon^4) + \varepsilon^2 L(w) + \varepsilon^5 Q(w)$$

we obtain $a_k\langle Z_k, X\rangle = \mathcal{O}(\varepsilon^2) + \varepsilon L(w) + \varepsilon^2 Q(w)$, from which the following hold

$$\begin{aligned}
\langle \varepsilon T_0, N_{\mathscr{D}}\rangle &= \bar\varepsilon_1'\langle Y_0, N_{\mathscr{D}}\rangle + \bar\varepsilon'\langle X, N_{\mathscr{D}}\rangle \\
&= \bar\varepsilon_1'\left|\tilde N_{\mathscr{D}}\right| + \bar\varepsilon'(\varepsilon^2 + \varepsilon L(w) + \varepsilon^2 Q(w)) \\
&= \varepsilon\varepsilon_1'(1 - \varepsilon\Gamma_0^0(X)) + \mathcal{O}(\varepsilon^4) + \varepsilon^3 L(w) + \varepsilon^4 Q(w).
\end{aligned}$$

From this, we deduce that

$$\frac{\partial}{\partial x^i}\langle w_{x^j}T_0, N_{\mathscr{D}}\rangle = \varepsilon_1'\left(1 - \varepsilon\Gamma_0^0(X)\right) w_{x^ix^j} - \varepsilon\varepsilon_1'\Gamma_0^0(E_i)w_{x^j} + \varepsilon^3 L(w) + \varepsilon^2 Q(w).$$

We conclude using also Lemma 5.3.3 that

$$\langle \nabla_{Z_i}(w_{x^j}T_0), N_{\mathscr{D}}\rangle = \varepsilon_1'\left(1 - \varepsilon\Gamma_0^0(X)\right) w_{x^ix^j} - \varepsilon\varepsilon_1'\Gamma_0^0(E_i)w_{x^j} + \varepsilon^3 L(w) + \varepsilon^2 Q(w). \tag{5.18}$$

It remains the term $\langle \nabla_{Z_i}(\bar\varepsilon T_j), N_{\mathscr{D}}\rangle = \varepsilon w_{x^i}\langle \nabla_{T_0}(\bar\varepsilon T_j), N_{\mathscr{D}}\rangle + \bar\varepsilon\langle \nabla_{T_i}(\bar\varepsilon T_j), N_{\mathscr{D}}\rangle$.
Since

$$\bar\varepsilon(s, w) = \varepsilon(s) + \varepsilon L(w) + \varepsilon^3 Q(w), \tag{5.19}$$

we can write

$$\langle \nabla_{T_i}(\bar{\varepsilon} T_j), N_{\mathscr{D}} \rangle \;=\; \varepsilon \langle \nabla_{T_i} T_j, N \rangle + \langle \nabla_{T_i}((\varepsilon^2 L + \varepsilon^3 Q) T_j), N_{\mathscr{D}} \rangle.$$

Recalling that $\nabla_{Y_i} Y_j = (\mathcal{O}(\varepsilon) + \varepsilon L + \varepsilon^2 Q)_\alpha Y_\alpha$ also $\langle T_i, N_{\mathscr{D}} \rangle = \varepsilon^2 + \varepsilon L + \varepsilon^2 Q$ thus

$$\langle \nabla_{T_i}(\bar{\varepsilon} T_j), N_{\mathscr{D}} \rangle \;=\; \varepsilon \langle \nabla_{Y_i} Y_j, Y_0 \rangle \Big|_{w=0} + \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^3 Q(w).$$

Moreover (5.8) and (5.19) yield

$$\langle \nabla_{T_0}(\bar{\varepsilon} T_j), N_{\mathscr{D}} \rangle \Big|_{w=0} \;=\; \varepsilon\varepsilon\varepsilon_1' \langle \nabla_{Y_0} Y_j, N_{\mathscr{D}} \rangle + \varepsilon\varepsilon\varepsilon' \langle \nabla_X Y_j, N_{\mathscr{D}} \rangle = -\varepsilon\varepsilon\varepsilon_1' \Gamma_0^0 + \mathcal{O}(\varepsilon^4).$$

This implies that

$$\langle \nabla_{T_0}(\bar{\varepsilon} T_j), N_{\mathscr{D}} \rangle = -\varepsilon\varepsilon\varepsilon_1' \Gamma_0^0 + \mathcal{O}(\varepsilon^4) + \varepsilon^2 L(w) + \varepsilon^3 Q(w).$$

Finally, collecting these and using (5.19) it turns out that

$$\langle \nabla_{Z_i}(\bar{\varepsilon} T_j), N_{\mathscr{D}} \rangle = \varepsilon\varepsilon \langle \nabla_{Y_i} Y_j, Y_0 \rangle \Big|_{w=0} + \mathcal{O}(\varepsilon^4) - \varepsilon\varepsilon\varepsilon_1' \Gamma_0^0(E_j) w_{x^i} + \varepsilon^4 L(w) + \varepsilon^3 Q(w). \tag{5.20}$$

The result follows from (5.18) and (5.20). ∎

We need also to expand more precisely $\langle \nabla_{Y_i} Y_j, Y_0 \rangle \Big|_{w=0}$. By construction it vanish on Γ and

$$Y_l \langle \nabla_{Y_i} Y_j, Y_0 \rangle = \langle \nabla_{Y_l} \nabla_{Y_i} Y_j, Y_0 \rangle + \langle \nabla_{Y_i} Y_j, \nabla_{Y_l} Y_0 \rangle.$$

Furthermore by (5.6) and since (see for instance [38] Lemma 9.20)

$$\nabla_{Y_l} \nabla_{Y_i} Y_j \Big|_{\gamma(x^0)} = -\frac{1}{3} \left(R(E_l, E_i) E_j + R(E_l, E_j) E_i \right),$$

it follows that

$$\langle \nabla_{Y_i} Y_j, Y_0 \rangle = -\frac{\varepsilon}{3} (\langle R(X, E_i) E_j, E_0 \rangle + \langle R(X, E_j) E_i, E_0 \rangle) + \mathcal{O}(\varepsilon^2).$$

We conclude that from Lemma 5.3.4 that the Second fundamental form $\mathrm{II}_{ij} = \langle \nabla_{Z_i} Z_j, N^{\mathscr{D}} \rangle$ of the perturbed disc $\mathscr{D}^{s,\varepsilon(s)}(w)$ centered at the point $\gamma(\varepsilon(s))$ with radius $\varepsilon(s)$ is given by

$$\begin{aligned} \mathrm{II}_{ij} \;=\; & \varepsilon\varepsilon_1' \left(1 - \varepsilon \Gamma_0^0(E_l) x^l\right) w_{x^j x^i} - \tfrac{\varepsilon^3}{3} \left(\langle R(X, E_i) E_j, E_0 \rangle + \langle R(X, E_j) E_i, E_0 \rangle \right) \\ & + \mathcal{O}(\varepsilon^4) - \varepsilon^2 \varepsilon_1' \left(w_{x^j} \Gamma_0^0(E_i) + w_{x^i} \Gamma_0^0(E_j) \right) + \varepsilon^4 L(w) + \varepsilon^3 Q(w). \end{aligned} \tag{5.21}$$

We recall that if E_α is an orthogonal basis of $T_p\mathcal{M}$, then

$$\mathrm{Ric}_p(X,Y) = -\langle R_p(X,E_\alpha)Y, E_\alpha\rangle \qquad \forall X,Y \in T_p\mathcal{M}.$$

Finally we obtain

$$\begin{aligned} \mathrm{II}_{ij}\, h^{ij} &= \tfrac{\varepsilon_1'}{\varepsilon}\left(1-\varepsilon\Gamma_0^0(X)\right)\,\Delta w - \tfrac{2\varepsilon}{3}\mathrm{Ric}_p(X,E_0) \\ &\quad + \mathcal{O}(\varepsilon^2) - 2\varepsilon_1'\,\Gamma_0^0(\nabla w) + \varepsilon^2 L(w) + \varepsilon Q(w), \end{aligned}$$

where $X = x^l E_l$.

5.3.5. Proposition. *In the above notation, the mean curvature $\mathcal{H}(s,\rho,w)$ of $\mathscr{D}_{s,\rho}(w)$ has the following expansions*

$$\frac{\phi}{\kappa'}\mathcal{H}(s,\rho,w) = \Delta w - \frac{2\rho}{3}\frac{\phi^2}{\kappa'}\,Ric_p(X,E_0) + \mathcal{O}(\rho^2) + \rho L(w) + \rho Q(w).$$

In particular if Γ is a geodesic, $\Gamma_0^0 = 0$ then

$$\frac{\phi}{\kappa'}\mathcal{H}(s,\rho,w) = \Delta w - \frac{2\rho}{3}\frac{\phi^2}{\kappa'}\,Ric_p(X,E_0) + \mathcal{O}(\rho^2) + \rho^2 L(w) + \rho Q(w).$$

5.3.2 Angle between the normals

By construction, at $q = G^s(x)$ we have $F^s(w(x),x) = \mathcal{C}^\varepsilon(s+\varepsilon w(x), \varepsilon(s+\varepsilon w(x)))$, for every $x \in \partial B_1^m$. Recall from § 5.2.1 and 5.3.1 that

$$\tilde{N}_{\mathscr{C}^\rho}(s,z) = -\varepsilon_1'(s)X + \varepsilon'(s)Y_0 + \alpha_0 Z_0^c + \alpha_k Z_k^c,$$

where $\alpha_0 = \phi'\bar{\alpha}_0 + \mathcal{O}(\varepsilon^3)$ and $\alpha_k = \mathcal{O}(\varepsilon^2)$ also

$$\tilde{N}_{\mathscr{D}} = Y_0 + a_k Z_k.$$

One easily verifies that

$$\begin{aligned} \langle \tilde{N}_{\mathscr{D}}, \rho^{-1}\tilde{N}_{\mathscr{C}^\varepsilon}(s+\varepsilon w)\rangle_q &= -\kappa'\langle X,Y_0\rangle_q + \phi'\langle Y_0,Y_0\rangle_q + \frac{\alpha_0}{\rho}\langle Z_0^c,Y_0\rangle_q + \frac{\alpha_k}{\rho}\langle Z_k^c,Y_0\rangle_q \\ &- \kappa' a_k\langle X,Z_k\rangle_q + \phi' a_k\langle Z_k,Y_0\rangle_q + \frac{a_k}{\rho}\alpha_0\langle Z_0^c,Z_k\rangle_q + \frac{\alpha_k}{\rho}a_l\langle Z_k, \end{aligned}$$

We have to expand

$$\kappa'(s+\varepsilon w) = \kappa'(s) + \varepsilon\kappa''(s)w + \varepsilon^2 Q(w) \qquad \phi'(s+\varepsilon w) = \phi'(s) + \varepsilon\phi''(s)w + \varepsilon^2 Q(w).$$

We will also need the following result which uses just the expansions of the metric Lemma 5.3.1

$$\begin{aligned} \langle Z_0^c, Z_k\rangle &= \mathcal{O}(\varepsilon^2) + \varepsilon^2 L(w) + \varepsilon^3 Q(w), \\ \langle Z_l^c, Z_k\rangle &= \mathcal{O}(\varepsilon^2) + \varepsilon^2 L(w) + \varepsilon^3 Q(w), \\ \langle X, Z_k\rangle_q &= (\varepsilon + \varepsilon'\varepsilon w)x^k + \varepsilon^3 L(w) + \varepsilon^5 Q(w). \end{aligned} \tag{5.22}$$

We use the fact that $a_k\langle Z_l, Z_k\rangle = -\langle Y_0, Z_k\rangle$ to have

$$\begin{aligned} \rho^{-1}\langle \tilde N_{\mathscr{D}}, \tilde N_{\mathscr{C}^\varepsilon}(s+\varepsilon w)\rangle_q &= (\phi' + \varepsilon w\phi'')\langle Y_0, Y_0\rangle_q + (\kappa' + \varepsilon\,\kappa'' w)\alpha_0\langle Y_0, Y_0\rangle_q + \frac{\alpha_k}{\rho}\langle Z_k^c, Y_0 \\ &- (\kappa' + \varepsilon\kappa'' w)a_k\langle X, Z_k\rangle_q - \phi' a_k a_l\langle Z_k, Z_l\rangle_q + \frac{a_k}{\rho}\alpha_0\langle Z_0^c, Z_k\rangle_q + \\ &+ \varepsilon^3 L(w) + \varepsilon^2 Q(w). \end{aligned}$$

Now from (5.22) we get

$$-\phi' a_k a_l\langle Z_k, Z_l\rangle_q + \frac{\alpha_0}{\rho}a_k\langle Z_0^c, Z_k\rangle_q + \frac{\alpha_k}{\rho}a_l\langle Z_k, Z_l^c\rangle_q = \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^2 Q(w)$$

and also since

$$\langle Z_k^c, Y_0\rangle_q = \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^4 Q(w),$$

one has

$$\begin{aligned} \rho^{-1}\langle \tilde N_{\mathscr{D}}, \tilde N_{\mathscr{C}^\varepsilon}(s+\varepsilon w)\rangle_q &= (\phi' + \varepsilon w\phi'')\langle Y_0, Y_0\rangle_q + (\kappa' + \varepsilon w\kappa'')\bar\alpha_0\langle Y_0, Y_0\rangle_q \\ &- (\kappa' + \varepsilon\kappa'' w)a_k\langle X, Z_k\rangle_q + \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^2 Q(w). \end{aligned}$$

From (5.22) and recalling the formula for a_k in (5.14) we get

$$\begin{aligned} a_k\langle X, Z_k\rangle_q &= -\varepsilon_1'\frac{\partial w}{\partial\eta}\langle Y_0, Y_0\rangle_q - (1+\varepsilon' w)\langle Y_0, X\rangle_q + \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^2 Q(w) \\ &= -\varepsilon_1'\frac{\partial w}{\partial\eta}\langle Y_0, Y_0\rangle_q + \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^2 Q(w) \end{aligned}$$

and then we deduce that

$$\begin{aligned} \rho^{-1}\langle \tilde N_{\mathscr{D}}, \tilde N_{\mathscr{C}^\varepsilon}(s+\varepsilon w)\rangle_q &= \phi'\left(1+\kappa'\bar\alpha_0\right)\langle Y_0, Y_0\rangle_q + (\varepsilon w\phi'' + \kappa'\varepsilon_1')\frac{\partial w}{\partial\eta}\langle Y_0, Y_0\rangle_q + 2\varepsilon^2\kappa''\kappa \\ &+ \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^2 Q(w). \end{aligned}$$

Using (5.15), we have that

$$\begin{aligned} \rho^{-1}\langle N_{\mathscr{D}}, \tilde N_{\mathscr{C}^\varepsilon}(s+\varepsilon w)\rangle_q &= \phi'\left(1+\kappa'\bar\alpha_0\right)|Y_0|_q + (\varepsilon w\phi'' + \kappa'\varepsilon_1')\frac{\partial w}{\partial\eta}|Y_0|_q + 2\varepsilon^2\kappa''\kappa' w\Gamma_0^0(X \\ &+ \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^2 Q(w). \end{aligned}$$

Since $\alpha_0 = \phi'(s+\varepsilon w)\bar{\alpha}_0 + \mathcal{O}(\varepsilon^3)$, one has

$$\alpha_0 = (\phi'(s) + \phi''(s)\varepsilon(s)w)\bar{\alpha}_0 + \varepsilon^2 Q(w)\bar{\alpha}_0$$

so that

$$\alpha_0 = \phi'(s)\bar{\alpha}_0 - 2\phi''(s)\varepsilon_1'\varepsilon w\Gamma_0^0(X) + \varepsilon^3 L(w) + \varepsilon^2 Q(w).$$

Moreover notice that

$$\Gamma_0^0(X)\Big|_q = \Gamma_0^0(X) + \varepsilon^2 L(w) + \varepsilon^3 Q(w)$$

and also

$$|\phi'(s+\varepsilon w)|^2 = (\phi' + 2\varepsilon w\phi'')\phi' + \varepsilon^2 Q(w),$$

we have that

$$\rho\left|\tilde{N}_{\mathscr{C}^\rho}(s+\varepsilon w)\right|^{-1} = 1 + (\phi' + 2\varepsilon\phi'' w + \varepsilon' w)\varepsilon\phi'\Gamma_0^0(X) + |\phi'(s)|^2\varepsilon^2 H_c(X,X) + \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(\cdots$$

from which we deduce that

$$\begin{aligned}\rho\left|\tilde{N}_{\mathscr{C}^\rho}(s+\varepsilon w)\right|_q^{-1}|Y_0|_q &= 1 - (\kappa')^2\varepsilon\Gamma_0^0(X) + \varepsilon^2\left(3 - (\phi')^2 + (\phi')^4 + 2(\phi')^2\phi^4\right)\Gamma_0^0(X)\cdots\\ &+ \frac{2-(\phi')^2}{2}\langle R_p(X,E_0)X,E_0\rangle + \varepsilon w\left(-\varepsilon' + 2\varepsilon\phi\phi'' + \varepsilon'\phi'\right)\Gamma_0^0(\cdots\\ &+ \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^2 Q(w).\end{aligned}$$

Consequently we may expand the angle as

$$\begin{aligned}\langle N_{\mathscr{D}}, N_{\mathscr{C}^\varepsilon}(s+\varepsilon w)\rangle_q &= \rho\phi'\left(1 + \kappa'\bar{\alpha}_0\right)|Y_0|_q\left|\tilde{N}_{\mathscr{C}^\rho}(s+\varepsilon w)\right|_q^{-1} + 2\varepsilon^2\kappa''\kappa' w\Gamma_0^0(X)\\ &+ \left(1 - (\kappa')^2\varepsilon\Gamma_0^0(X)\right)\left(\varepsilon w\phi'' + \kappa'\varepsilon_1'\frac{\partial w}{\partial\eta}\right)\\ &+ \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^2 Q(w).\end{aligned}$$

Hence we get

$$\begin{aligned}\langle N_{\mathscr{D}}, N_{\mathscr{C}^\varepsilon}(s+\varepsilon w)\rangle_q &= \phi'(s)\Big(1 + (\kappa')^2\varepsilon\Gamma_0^0(X)\Big) + \Big(1 - (\kappa')^2\varepsilon\Gamma_0^0(X)\Big)\Big(\varepsilon w\phi'' + \kappa'\varepsilon_1'\cdots\\ &+ \phi'\varepsilon^2\Big(3 - 2(\kappa')^4 - (\phi')^2 + (\phi')^4 + 2(\phi')^2\phi^4\Big)\Gamma_0^0(X)\Gamma_0^0(X)\\ &+ \tfrac{2-(\phi')^2}{2}\phi'\varepsilon^2\langle R_p(X,E_0)X,E_0\rangle + \varepsilon\phi' w\Big(-\varepsilon' + 2\varepsilon\kappa''\kappa' + 2\varepsilon\phi\phi''\cdots\\ &+ 2\varepsilon^2 U_0^0(X)w + \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^2 Q(w).\end{aligned}$$

We define

$$\mathcal{B}(s,\rho,w) := \langle N_{\mathscr{D}(w)}, N_{\mathscr{C}^\varepsilon}(s+\varepsilon w)\rangle_q.$$

Now we conclude this section by collecting all these in the following

5.3.6. PROPOSITION. *In the above notations,*

$$\begin{aligned}\mathcal{B}(s,\rho,w) &= \phi'(s)\left(1+(\kappa')^2\rho\phi\Gamma_0^0(X)\right)+\rho\left((\kappa')^2\frac{\partial w}{\partial\eta}+\phi\phi''\,w\right)\\ &+ \mathcal{O}(\rho^2)+\rho^2\bar{L}(w)+\rho^2\bar{Q}(w),\end{aligned}$$

while if $\phi'(s)=0$, one has

$$\mathcal{B}(s,\rho,w)=\rho(\kappa')^2\frac{\partial w}{\partial\eta}+\mathcal{O}(\rho^3)+\rho^3\bar{L}(w)+\rho^2\bar{Q}(w).$$

In particular if Γ is a geodesic, we get precisely

$$\begin{aligned}\mathcal{B}(s,\rho,w) &= \phi'(s)\left(1+\frac{1+(\kappa')^2}{2}\rho^2\phi^2\langle R_p(X,E_0)X,E_0\rangle\right)+\rho\left((\kappa')^2\frac{\partial w}{\partial\eta}+\phi\phi''\,w\right)\\ &+ \mathcal{O}(\rho^3)+\rho^3\bar{L}(w)+\rho^2\bar{Q}(w).\end{aligned}$$

5.4 Existence of capillary minimal submanifolds

5.4.1 Case where $\phi(s_0)\phi''(s_0)>0$

We may assume that $\phi(s)\phi''(s)>0$ for all $s\in I_{s_0}(\delta):=[s_0-\delta,s_0+\delta]$ for some $\delta>0$ small.

We define the following operator $\mathcal{L}_s$ by

$$(\mathcal{L}_s w,v):=\int_{B_1^m}\nabla w\nabla v\,dx+\frac{\phi\phi''}{(\kappa')^2}\oint_{\partial B_1^m} wv\,d\sigma.$$

It is clear from the inequality (see [85], Theorem A.9)

$$\int_{B_1^m} w^2\,dx \le C(m)\left(\int_{B_1^m}|\nabla w|^2\,dx+\oint_{\partial B_1^m} w^2\,d\sigma\right),\qquad \forall w\in H^1, \tag{5.23}$$

that the operator $\mathcal{L}_s$ is coercive if ρ is small. We call $w_1^{s,\rho}$ the unique solution to the equation

$$(\mathcal{L}_s w_1,v):=-\phi\oint_{\partial B_1^m}\Gamma_0^0(X)v\,d\sigma.$$

Namely $w_1^{s,\rho}$ solves the problem

$$\begin{aligned}-\Delta w_1 &= 0 && \text{in } B_1^m,\\ \frac{\partial w_1}{\partial\eta}+\frac{\phi\phi''}{(\kappa')^2}w_1 &= -\phi\,\Gamma_0^0(X) && \text{on } \partial B_1^m.\end{aligned}$$

By elliptic regularity theory, there exist a constant $c > 0$ (independent of ρ and s) such that

$$\|w_1^{s,\rho}\|_{\mathcal{C}^{2,\alpha}} \leq c \qquad \forall s \in I_{s_0}(\delta).$$

Moreover we have that for all $k \geq 0$

$$\|\frac{\partial^k w_1^{s,\rho}}{\partial s^k}\|_{\mathcal{C}^{2,\alpha}} \leq c_k \qquad \forall s \in I_{s_0}(\delta),$$

for some constant c_k which does not depend on s nor on ρ small.
Clearly by construction

$$\begin{cases} \mathcal{H}(s,\rho,w_1^{s,\rho}) & = & \mathcal{O}(\rho) & \text{in} \quad \mathscr{D}_{s,\rho}(w_1^{s,\rho}), \\ \mathcal{B}(s,\rho,w_1^{s,\rho}) & = & \phi'(s) + \mathcal{O}(\rho^2) & \text{on} \quad \partial\mathscr{D}_{s,\rho}(w_1^{s,\rho}). \end{cases}$$

We define the space

$$\begin{aligned} \mathcal{C}^{2,\alpha}_{s,\rho} &:= \left\{ w \in \mathcal{C}^{2,\alpha}(\overline{B_1^m}) \quad : \quad \frac{\partial}{\partial v}\mathcal{B}(s,\rho,w_1^{s,\rho}+v)\Big|_{v=0}[w] = 0 \right\} \\ &= \left\{ w \in \mathcal{C}^{2,\alpha}(\overline{B_1^m}) \quad : \quad \frac{\partial w}{\partial \eta} + \frac{\phi\phi''}{(\kappa')^2} w + \rho \bar{L}_s(w) = 0 \right\}. \end{aligned}$$

We consider the linearized mean curvature operator about $\mathscr{D}^{s,\rho}(w_1^{s,\rho})$ (see Proposition 5.3.5), $\mathbb{L}_{\rho,s}(w) : \mathcal{C}^{2,\alpha}(\overline{B_1^m}) \to \mathcal{C}^{0,\alpha}(\overline{B_1^m})$ defined by

$$\mathbb{L}_{\rho,s}(w) := -\frac{\phi}{\kappa'}\frac{\partial}{\partial v}\mathcal{H}(s,\rho,w_1^{s,\rho}+v)\Big|_{v=0}[w] = -\Delta\, w + \rho L_s(w).$$

We define also $\Phi(s,\rho,x),\ \mathcal{Q}_{s,\rho}(w) \in \mathcal{C}^{0,\alpha}(\overline{B_1^m})$ by duality as

$$(\Phi(s,\rho,x),w') := -\frac{\phi}{\kappa'}\int_{B_1^m}\mathcal{H}(s,\rho,w_1^{s,\rho})w'\,dx + \rho^{-1}\oint_{\partial B_1^m}\left(\mathcal{B}(s,\rho,w_1^{s,\rho}) - \phi'(s)\right)w'\,ds$$

and for every $w \in \mathcal{C}^{2,\alpha}$

$$(\mathcal{Q}_{s,\rho}(w),w') := \int_{B_1^m} Q(w)w'\,dx + \oint_{\partial B_1^m}\bar{Q}(w)w'\,ds, \qquad \forall w' \in L^2.$$

Clearly the solvability of the system

$$\begin{cases} \mathcal{H}(s,\rho,w_1^{s,\rho}+w) & = & 0 & \text{in} \quad \mathscr{D}_{s,\rho}(w_1^{s,\rho}+w), \\ \mathcal{B}(s,\rho,w_1^{s,\rho}+w) & = & \phi'(s) & \text{on} \quad \partial\mathscr{D}_{s,\rho}(w_1^{s,\rho}+w) \end{cases} \tag{5.24}$$

is equivalent to the fixed point problem

$$w = -\left(\mathbb{L}_{\rho,s}\Big|_{\mathcal{C}^{2,\alpha}_{s,\rho}}\right)^{-1}\left\{\Phi(s,\rho,x) + \rho\,\mathcal{Q}_{s,\rho}(w)\right\}. \tag{5.25}$$

Furthermore one has

$$\begin{aligned}\|\mathcal{Q}_{s,\rho}(w)\|_{\mathcal{C}^{0,\alpha}} &= O(\|w\|_{\mathcal{C}^{2,\alpha}})\,\|w\|^2_{\mathcal{C}^{2,\alpha}};\\ \|\mathcal{Q}_{s,\rho}(w_1) - \mathcal{Q}_{s,\rho}(w_2)\|_{\mathcal{C}^{0,\alpha}} &= O(\|w_1\|_{\mathcal{C}^{2,\alpha}}, \|w_2\|_{\mathcal{C}^{2,\alpha}})\|w_1 - w_2\|_{\mathcal{C}^{2,\alpha}},\end{aligned}$$

also by construction, there exist a constant $c > 0$ (independent of ρ and s) such that

$$\|\Phi(s,\rho,\cdot)\|_{\mathcal{C}^{0,\alpha}} \le c\rho \qquad \forall s \in I_{s_0}(\delta).$$

By (5.23) the operator $\mathbb{L}_{\rho,s}$ is coercive on $\mathcal{C}^{2,\alpha}_{s,\rho}$ if ρ is small enough and also by elliptic regularity theory, $\mathbb{L}_{\rho,s}$ is an isomorphism from $\mathcal{C}^{2,\alpha}_{s,\rho}$ into $\mathcal{C}^{0,\alpha}(\overline{B^m_1})$ therefore we can solve the fixed point problem (5.25) in a ball of $\mathcal{C}^{2,\alpha}_{s,\rho}$ with radius $C\rho$ for some $C > 0$ which does not depend neither on ρ small nor s. And thus for ρ small and $s \in I_{s_0}(\delta)$ there exists a function $w^{s,\rho} \in \mathcal{C}^{2,\alpha}_{s,\rho}$, with $\|w^{s,\rho}\|_{\mathcal{C}^{2,\alpha}} \le C\rho$ such that

$$\begin{cases}\mathcal{H}(s,\rho,w_1^{s,\rho} + w^{s,\rho}) = 0 & \text{in} \quad \mathscr{D}_{s,\rho}(w_1^{s,\rho} + w^{s,\rho}),\\ \mathcal{B}(s,\rho,w_1^{s,\rho} + w^{s,\rho}) = \phi'(s) & \text{on} \quad \partial\mathscr{D}_{s,\rho}(w_1^{s,\rho} + w^{s,\rho}).\end{cases}$$

Namely $\mathscr{D}_{s,\rho}(w_1^{s,\rho} + w^{s,\rho})$ is a capillary submanifold of Ω_ρ with constant contact angle arccos $\phi'(s)$ if ρ is small enough by $\mathcal{C}^{2,\alpha}$ bound up to the boundary of $\tilde{w}^{s,\rho} = w_1^{s,\rho} + w^{s,\rho}$. Furthermore it follows from the construction that, for all $k \ge 0$

$$\|\frac{\partial^k \tilde{w}^{s,\rho}}{\partial s^k}\|_{\mathcal{C}^{2,\alpha}} \le c_k\rho \qquad \forall s \in I_{s_0}(\delta), \tag{5.26}$$

for some constant c_k which does not depend on s nor on ρ small.

5.4.2 Foliation by minimal discs

Call $\tilde{w}^{s,\rho} = w_1^{s,\rho} + w^{s,\rho}$. From (5.9), Lemma 5.3.1 and (5.26) the mapping

$$I_{s_0}(\delta) \times B_1^m \ni (s,x) \xrightarrow{\Psi_\rho} F^s(\tilde{w}^{s,\rho}(x), x) = \overline{F}(\bar{\varepsilon}_1(s, \tilde{w}^{s,\rho}(x))\,,\, \bar{\varepsilon}(s, \tilde{w}^{s,\rho}(x))\,x)$$

has Jacobian determinant which expands as

$$\rho^{2m+2}\left(\left(1 - |x|^2(\phi')^2\right)\,\phi^{2m} + \mathcal{O}_s(\rho)\right)$$

and hence since $(\phi')^2 \in (0,1)$ (see (5.1)), Ψ_ρ is a local homeomorphism if ρ is small enough. In particular it is a homeomorphism of a neighborhood of $(s_0, 0)$ which implies that there exist $0 < \delta' < \delta$ and $\varrho > 0$ such that

$$\Psi_\rho(s\,,\,B^m_\varrho) \cap \Psi_\rho(s'\,,\,B^m_\varrho) = \emptyset \qquad \forall s \neq s' \in I_{s_0}(\delta'),$$

for every ρ sufficiently small.
In this way the family of discs $\mathscr{D}_{s,\rho\varrho}(\tilde{w}^{s,\rho})$, $s \in I_{s_0}(\delta')$ with radius $\rho\varrho\,\phi(s)$ centered at $\gamma(\rho\,\kappa(s))$ constitutes a foliation of a neighborhood of $\gamma(\rho\,\kappa(s_0))$ for which each leaf $\mathscr{D}_{s,\rho\varrho}(\tilde{w}^{s,\rho})$ is a minimal disc intersecting $\mathscr{C}^{\rho\varrho}$ transversely along its boundary (the angle of contact may not be equal to $\arccos\phi'(s)$).

5.4.3 $\phi \equiv 1$ and $\kappa =$Id, $\mathscr{C}^\rho$ is the geodesic tube around Γ

In this situation,

$$\mathscr{C}^\rho = \{q \in \mathcal{M} \quad : \quad \mathrm{dist}_g(q, \Gamma) = \rho\}$$

and its interior is

$$\Omega_\rho = \{q \in \mathcal{M} \quad : \quad \mathrm{dist}_g(q, \Gamma) < \rho\}.$$

By [78], it is well known that (smooth) minimal surfaces $D \subset \Omega_\rho$ with $\partial D \subset \mathscr{C}^\rho$ are stationary for the area functional relative to $\mathscr{C}^\rho$ which is $D \mapsto \mathrm{Area}(D \cap \Omega_\rho)$ under variations $\Psi_t : D \to \mathcal{M}$ such that $\partial\Psi_t(D) \subset \mathscr{C}^\rho$ moreover the Euler-Lagrange equations are given by

$$\begin{aligned} H_D &= 0 \qquad \text{in } D, \\ \langle N_D, N_{\mathscr{C}^\rho}\rangle &= 0 \qquad \text{on } \partial D. \end{aligned} \tag{5.27}$$

A finite-dimensional reduction

For every $s \in [a,b]$ and $X = x^i E_i$, we let $w_1^{s,\rho}$ be the solution of the following problem:

$$\begin{aligned} -\Delta w_1 &= -\tfrac{2}{3}\,\mathrm{Ric}_p(X, E_0) \qquad && \text{in } B^m_1, \\ \frac{\partial w_1}{\partial \eta} &= 0 && \text{on } \partial B^m_1, \end{aligned}$$

where $p = \gamma(\rho\,\kappa(s))$.
By elliptic regularity theory, there exist a constant $c > 0$ (independent of ρ and s) such that

$$\|w_1^{s,\rho}\|_{\mathcal{C}^{2,\alpha}} \leq c \qquad \forall s \in [a,b]. \tag{5.28}$$

As in § 5.4.1, we let

$$\begin{aligned}\mathcal{C}^{2,\alpha}_{s,\rho} &:= \left\{ w \in \mathcal{C}^{2,\alpha}(\overline{B^m_1}) \quad : \quad \frac{\partial}{\partial v}\mathcal{B}(s,\rho,\rho w_1^{s,\rho}+v)\Big|_{v=0}[w]=0 \right\} \\ &= \left\{ w \in \mathcal{C}^{2,\alpha}(\overline{B^m_1}) \quad : \quad \frac{\partial w}{\partial \eta}+\rho \bar{L}(w)=0 \right\}.\end{aligned}$$

As explained in the first section, the linearized mean curvature operator about $\mathscr{D}^{s,\rho}(\rho\, w_1^{s,\rho})$ restricted on $\mathcal{C}^{2,\alpha}_{s,\rho}$ defined by

$$\mathbb{L}_{\rho,s}(w) := -\frac{\partial}{\partial v}\mathcal{H}(s,\rho,\rho\, w_1^{s,\rho}+v)\Big|_{v=0}[w] = -\Delta\, w + \rho L_s(w).$$

may have small (possibly zero) eigenvalues hence it may not be invertible on $\mathcal{C}^{2,\alpha}_{s,\rho}$. However instead of solving (5.27), we will prove that there exists a constant $\lambda_{s,\rho}\in\mathbb{R}$ and a function $w^{s,\rho}\in\mathcal{C}^{2,\alpha}_{s,\rho}$ such that

$$\begin{cases} \mathcal{H}(s,\rho,w^{s,\rho}) = \lambda_{s,\rho} & \text{in} \quad \mathscr{D}_{s,\rho}(w^{s,\rho}), \\ \mathcal{B}(s,\rho,w^{s,\rho}) = 0 & \text{on} \quad \partial\mathscr{D}_{s,\rho}(w^{s,\rho}). \end{cases} \tag{5.29}$$

To achieve this we let P be the L^2 projection on the space of functions $w\in L^2$ which are orthogonal to the constant function 1, $\int_{B^m_1} w\,dx = 0$. Now if ρ is small enough, the Poincare inequality implies together with elliptic regularity theory that the operator $P\circ\mathbb{L}_{s,\rho}$ is an isomorphism from $P\mathcal{C}^{2,\alpha}_{s,\rho}$ into $P\mathcal{C}^{0,\alpha}(\overline{B^m_1})$. Here letting

$$(\Phi(s,\rho,x),w') := -\int_{B^m_1}\mathcal{H}(s,\rho,\rho w_1^{s,\rho})w'\,dx + \oint_{\partial B^m_1}\mathcal{B}(s,\rho,\rho w_1^{s,\rho})w'\,ds$$

one has

$$\|\Phi(s,\rho,\cdot)\|_{\mathcal{C}^{0,\alpha}} \le c\rho^2 \qquad \forall s\in[a,b].$$

Consequently for ρ small, our fixed point problem

$$w = \left(P\circ\mathbb{L}_{\rho,s}\Big|_{\mathcal{C}^{2,\alpha}_{s,\rho}}\right)^{-1}\{P\circ\Phi(s,\rho,x)+\rho P\circ\mathcal{Q}_{s,\rho}(w)\}$$

admits a unique solution $w^{s,\rho}\in P\mathcal{C}^{2,\alpha}_{s,\rho}$, in a ball of radius $c\,\rho^2$ of $P\mathcal{C}^{2,\alpha}_{s,\rho}$. More precisely

$$\int_{B^m_1} w^{s,\rho}\,dx = 0 \qquad \text{and} \qquad \|w^{s,\rho}\|_{\mathcal{C}^{2,\alpha}(\overline{B^m_1})}\le c\rho^2 \qquad \forall s\in[a,b]. \tag{5.30}$$

Furthermore it follows from the construction that, for all $k\ge 0$

$$\|\frac{\partial^k w^{s,\rho}}{\partial s^k}\|_{\mathcal{C}^{2,\alpha}(\overline{B^m_1})}\le c_k\rho^2 \qquad \forall s\in[a,b],$$

for some constant c_k which does not depend on s nor on ρ small. We then conclude that $P \circ \mathcal{H}(s, \rho, \rho\, w_1^{s,\rho} + w^{s,\rho}) = 0$ hence the existence of a real number $\lambda_{s,\rho} \sim \rho^2$ such that (5.29) is satisfied.

We have to mention that by (5.30), provided ρ is small, the corresponding disc $\mathscr{D}_{s,\rho} := \mathscr{D}_{s,\rho}(\tilde{w}^{s,\rho})$ with $\tilde{w}^{s,\rho} = \rho\, w_1^{s,\rho} + w^{s,\rho}$ is embedded into Ω_ρ. This defines a one dimensional manifold of sets satisfying (5.29):

$$\mathcal{Z}_\rho := \{\mathscr{D}_{s,\rho} \subset \Omega_\rho,\ \partial\mathscr{D}_{s,\rho} \subset \partial\Omega_\rho \quad : \quad s \in [a,b]\}.$$

5.4.1. Remark. *We notice that, in section 5.4.1, the same argument as above implies that whenever $\phi''(s_0) = 0$, there will be a capillary disc centered at $\gamma(\rho\, s_0)$ with constant and small mean curvature.*

Variational argument:

We will show that in fact problem (5.27) can be reduced to a finite dimensional one. We now define the *reduced functional* $\varphi_\rho : [a,b] \to \mathbb{R}$ by

$$\varphi_\rho(s) := \mathrm{Area}(\mathscr{D}_{s,\rho})$$

for any $\mathscr{D}_{s,\rho} \in \mathcal{Z}_\rho$. We have to show the following

5.4.2. Lemma. *There exists ρ_0 small such that for any $\rho \in (0,\rho_0)$ if s is a critical point of φ_ρ then $\lambda_{s,\rho} = 0$.*

Proof. Let $\lambda \in \mathbb{R}$ and let $q = \gamma(\rho(s + \lambda\, t))$. Then provided t is small, it is clear that the hyper-surface $\mathscr{D}_{q,\rho}$ can be written as a normal graph over $\mathscr{D}_{p,\rho}$, $p = \gamma(\rho s)$ by a smooth function $g_{p,\rho,t,\lambda}$. This defines the variation vector field

$$\zeta_{p,\rho,\lambda} = \frac{\partial g_{p,\rho,t,\lambda}}{\partial t}\Big|_{t=0} N_{\mathscr{D}_{p,\rho}}.$$

Letting Z be the parallel transport of λE_0 along geodesics issued from $p = \gamma(\rho\, s)$. Then, we can easily get the estimates:

$$\|\zeta - Z\| \leq c\rho|\lambda|.$$

Assume that s is a critical point of φ_ρ then from the first variation of area see § 2.1.1,

$$\begin{aligned} 0 &= \frac{d\varphi_\rho(\rho(s+\lambda t))}{dt}\Big|_{t=0} = \lambda\rho\varphi_\rho'(s) \\ &= m\int_{\mathscr{D}_{s,\rho}} H_{\mathscr{D}_{s,\rho}}\langle \zeta, N_{\mathscr{D}_{s,\rho}}\rangle\, ds + \oint_{\partial\mathscr{D}_{s,\rho}} \langle \zeta, N_{\partial\mathscr{D}_{s,\rho}}^{\mathscr{D}s,\rho}\rangle, \end{aligned}$$

where $N_{\partial \mathscr{D}_{s,\rho}}^{\mathscr{D}s,\rho} \in T\mathscr{D}_{s,\rho}$ stands for the normal of $\partial\mathscr{D}_{s,\rho}$ in $\mathscr{D}_{s,\rho}$. Therefore by construction one has

$$0 = \lambda_{s,\rho} \int_{\mathscr{D}_{s,\rho}} \langle \zeta, N_{\mathscr{D}_{s,\rho}} \rangle \, ds. \tag{5.31}$$

Notice that

$$\langle \zeta, N_{\mathscr{D}_{s,\rho}} \rangle - \langle Z, Y_0 \rangle = \langle \zeta - Z, N_{\mathscr{D}_{s,\rho}} \rangle + \langle Z, N_{\mathscr{D}_{s,\rho}} - Y_0 \rangle,$$

so using the fact that $N_{\mathscr{D}_{s,\rho}} = Y_0 + \mathcal{O}(\rho)$, see § 5.3.1, we have

$$\left| \langle \zeta, N_{\mathscr{D}_{s,\rho}} \rangle - \lambda \right| \leq c\rho |\lambda|.$$

Inserting this into (5.31), we get

$$-\lambda \, \lambda_{s,\rho} \, \mathrm{Area}(\mathscr{D}_{s,\rho}) \leq c\rho \, |\lambda_{s,\rho}| \, |\lambda| \, \mathrm{Area}(\mathscr{D}_{s,\rho})$$

but since $\mathrm{Area}(\mathscr{D}_{s,\rho}) = \mathrm{Area}(\rho B_1^m) + O_s(\rho^{2+m})$ by (5.12); (5.28) and (5.30), it follows that

$$-\lambda \, \lambda_{s,\rho} \leq c\rho \, |\lambda_{s,\rho}| \, |\lambda| \, .$$

Therefore taking $\lambda = -\lambda_{s,\rho}$, we see that $|\lambda_{s,\rho}|^2 \leq c\rho \, |\lambda_{s,\rho}|^2$ and this implies that $\lambda_{s,\rho=0}$. ■

We shall end the proof of the Theorem 5.0.3 by giving the expansion of φ_ρ. From (5.30) the first fundamental form h_{ij} of a disc $\mathscr{D}_{s,\rho}$ expands as

$$\rho^{-2} \, h_{ij} = \delta_{ij} + \frac{\rho^2}{3} \langle R_p(X, E_i)X, E_j \rangle + \frac{\rho^3}{6} \langle \nabla_X R_p(X, E_i)X, E_j \rangle + \mathcal{O}_s(\rho^4),$$

where $p = \gamma(\rho s) \in \Gamma$.
From the formula

$$\sqrt{\det(I + A)} = 1 + \frac{1}{2}\mathrm{tr}(A) + O(|A|^2),$$

we obtain the volume form:

$$\rho^{-m} \sqrt{\det(h)} = 1 - \frac{\rho^2}{6} \langle R_p(X, E_i)X, E_i \rangle + \frac{\rho^3}{12} \langle \nabla_X R_p(X, E_i)X, E_i \rangle + \mathcal{O}_s(\rho^4)$$

and since by oddness $\int_{B_1^m} \langle \nabla_X R_p(X, E_i)X, E_j \rangle \, dx = 0$ we deduce that

$$\varphi_\rho(s) = \mathrm{Area}(\mathscr{D}_{s,\rho}) = \mathrm{Area}(B_\rho^m) \left(1 - \frac{\rho^2}{6m} \sum_{i,j=1}^{m} \langle R_p(E_j, E_i)E_j, E_i \rangle + O_s(\rho^4) \right).$$

Thus setting

$$\psi_\rho(s) := \frac{6m}{\rho^2}\left(1 - \frac{\varphi_\rho(s)}{\mathrm{Area}(B^m_\rho)}\right) = \sum_{i,j=1}^{m} \langle R_p(E_j, E_i)E_j, E_i\rangle + O_s(\rho^4)$$

we get the result.

Chapter 6

Perimeter minimizing sets enclosing small volumes

Let $\mathcal{M}$ be a complete Riemannian manifold and Ω a smooth bounded domain of $\mathcal{M}$. We recall that the *De Girogi perimeter* is defined as

$$\mathcal{P}_g(E,\Omega) := \sup\left\{\int_E \mathrm{div}_g Y\, dv_g \quad : \quad \langle Y,Y\rangle \leq 1\right\},$$

where Y is a smooth vectorfield on $\mathcal{M}$ with compact support in Ω. Here we are counting only the part of E inside Ω. Notice that if a set E is smooth then the Gauss-Green formula yields $\mathcal{P}_g(E,\Omega) = \mathrm{Area}(\partial E \cap \Omega)$. The *relative isoperimetric profile* of Ω is the mapping

$$v \mapsto I_\Omega(v) := \min_{E\subset\Omega, |E|_g = v} \mathcal{P}_g(E,\Omega).$$

By combination of the results of Almgren [2], Grüter [39], Gonzalez, Massari, Tamanini [37] we obtain the following fundamental existence and regularity theorem (see also Morgan [67]).

6.0.3. Proposition. *Let Ω be a smooth bounded domain in a Riemannian manifold $(\mathcal{M}^{m+1}, g)$. For any $v \in (0, |\Omega|_g)$ there is an open set $E \subset \Omega$ which minimizes the perimeter $\mathcal{P}_g(\cdot,\Omega)$ for any volume v . The boundary $\Sigma := \partial E \cap \Omega$ can be written as a disjoint union $\Sigma_1 \cup \Sigma_0$, where Σ_1 is the regular part of Σ and Σ_0 is the set of singularities. Precisely, we have*

1. *Σ_1 is a smooth, embedded hypersurface with constant mean curvature.*
2. *If $p \in \Sigma_1 \cap \partial\Omega$, then Σ_1 is a smooth, embedded hypersurface with boundary contained in in a neighborhood of p; in this neighborhood, Σ_1 has constant mean curvature and meets $\partial\Omega$ orthogonally.*
3. *Σ_0 is a closed set of Hausdorff dimension less than or equal to $m-7$.*
4. *At every point $q \in \Sigma_0$, there is a tangent minimal cone $C \subset T_q\mathcal{M}$ different from a hyperplane. The square sum of the principal curvatures of Σ_1 tends to ∞ when we approach q from Σ_1.*

It is clear then that the most important questions is to undertand the topology and geometric properties of minimizers. This has been achieved only in some special cases, one can see for example [13], [75], [21], [77], [53], [84], etc... In particular perimeter minimizing sets in $\mathcal{M}$ trapping small volumes have been studied by F. Morgan and D.L. Johnson. The authors prove that if v is small enouh, minimizers of $I_{\mathcal{M}}(v)$ are "smooth" spheres. Namely up to scaling, they converge smoothly to spheres (no presence of singularities). Recently, Narduli, in his Phd thesis has weakened the minimizing property. Moreover he shwod that minimizers are located near strict-maxima of the scalar curvature of $\mathcal{M}$.

In 1982, Bérard-Meyer, motivated by the study of nodal domains for Dirichlet eigenvalues, have shown that, in the infenitesimal level, the isoperimetric profile of a compact Reimannian manifold $\mathcal{M}^{m+1}$ approaches that of $\mathbb{R}^{n+1}$. Namely they establish that $I_{\mathcal{M}}(v) \sim I_{\mathbb{R}^{m+1}}(v)$ as $v \to 0$. This was adapted by Bayle and Rosales for the relative profile $I_{\Omega}(v) \sim I_{\mathbb{R}^{m+1}_+}(v)$ as $v \to 0$. The former result has been refined by Druet (2002) who gave the first coefficient in the Taylor expansion of $I_{\mathcal{M}}$

$$I_{\mathcal{M}}(v) \sim \left(1 - \alpha_m \max_{p \in \partial\mathcal{M}} S(p)\, v^{\frac{2}{m+1}} + O\left(v^{\frac{4}{m+1}}\right)\right) I_{\mathbb{R}^{m+1}}(v),$$

where α_m is a constant depeding only on m and S is the scalar curvature of $\mathcal{M}$. Our main goal in this chapter is the location of minimal area separating hyper-surfaces of Ω enclosing a small volume.

6.0.4. THEOREM. *Isoperimetric regions with small volume in Ω are hemispheres centred near stricly global maxima of the mean curvature of $\partial\Omega$.*

To prove the above theorem, we first show a regularity result which generalizes Theorem 2.2 in [69] see Lemma 6.1.2. We notice that the proof of Theorem 2.2 in [69] highlights that the diameter of an isoperimetric region E_v tends to zero as the volume v tends to zero. Moreover as pointed out by Bayle and Rosales [8], this set must touch the boundary $\partial\Omega$ if v is small enough (E_v is not compactly contained in Ω). From this one sees that E_v is contained in a geodesic sphere centered at some point $p \in \partial\Omega$ for v small. Hence using results in [69] Theorem 2.2, one has that the hyper-surface $\Sigma_v = \partial E_v \cap \Omega$ can be written, after suitable scaling, as a graph over a round hemisphere and the function which defines the graph tends to zero. This also shows that $\partial\Sigma_v \subset \partial\Omega$. But, according to our argument we need a convergence up to the free boundary. We achieve this, following [40], by proving a monotonicity result for the area of Σ_v in a tubular neighborhood of $\partial\Omega$. This allow us to get a bound for the area of $\partial\Sigma_v$ and hence, by compactness, to have a weak convergence up to the free boundary and smoothly by [41].

The second step is to reduce the isoperimetric problem to a finite dimensional variational one, see Lemma 6.1.7 by adopting a variant of the method in [72]. To this end, by means of the implicit function theorem we construct, for any fixed v sufficiently small, a manifold of sets having a volume v that we call $\mathfrak{C}_v$ which is diffeomorphic to $\partial\Omega$, see Lemma 6.1.6. A set $E \in \mathfrak{C}_v$ is a *pseudo-half-ball* (see Definition 6.1.5) which is uniquely determined by its center of mass $p \in \partial\Omega$ while its boundary, $\partial E = \Sigma_{p,\omega^{p,v}}$, is a normal graph over a geodesic sphere centered at p with $\omega^{p,v}$ (defining the graph) tends to zero as $v \to 0$.
Finally we show that an isoperimetric region with small volume v must belong to $\mathfrak{C}_v$ so looking for the minimum of the perimeter among sets in Ω with volume v is equivalent to take the minimum among sets in $\mathfrak{C}_v$. Taking advantage of the role of the mean curvature in the expansion of the area of normal graphs centered at the free boundary $\partial\Omega$ Appendix A, the theorem then follows.

From the reduction of the isoperimetric problem to a finite dimensional one, Lemma 6.1.7, we can determine the first coefficient of the asymptotic expansions of the profile of Ω near *zero*.
Letting $v = \left|rB_+^{m+1}\right|$ in Lemma 6.1.7-6.1.8. We have obtained that

$$I_\Omega(v) = \min_{p\in\partial\Omega} \left\{ I_{\mathbb{R}_+^{m+1}}(v) - \frac{m}{m+2}\frac{|B^m|}{\left|B_+^{m+1}\right|} H_{\partial\Omega}(p)\, v + O_p\left(v^{\frac{m+2}{m+1}}\right) \right\},$$

where $H_{\partial\Omega}(p)$ is the mean curvature of $\partial\Omega$ at p and $O_p(\rho)$ is a smooth function in p and ρ tending to zero uniformly with respect to p as ρ tends to zero. Hence we have the corollary

6.0.5. Corollary. *There holds*

$$I_\Omega(v) \sim \left(1 - \beta_m \max_{p\in\partial\Omega} H_{\partial\Omega}(p)\, v^{\frac{1}{m+1}} + O\left(v^{\frac{2}{m+1}}\right)\right) I_{\mathbb{R}_+^{m+1}}(v),$$

where $\beta_m = \frac{m}{(m+1)(m+2)} \frac{|B^m|}{\left|B_+^{m+1}\right|^{\frac{m+2}{m+1}}}$.

Let us also mention that in [21], the authors have shown that an isoperimetric region outside a convex domain, in euclidean space, has no less perimeter than the area of a hemisphere provided it encloses the volume of half ball. Futhermore in [8]-Proposition 5.1, the authors show, under convexity assumption of Ω in a Riemannian manifold, that $I_\Omega(v) < I_{\mathbb{R}^{m+1}_+}(v)$ for small v. Here, from Corollary 6.0.5, we can weaken the convexity by strictly H-convex domain (a domain with non-negative mean curvature) under small volume constarints.

6.0.6. Corollary. *If Ω is a strictly H-convex smooth bounded domain of $\mathbb{R}^{m+1}$ then provided v small enough*

$$I_{\mathbb{R}^{m+1}\setminus\Omega}(v) > I_{\mathbb{R}^{m+1}_+}(v).$$

As a final result, we have the following geometric comparion which is also a direct consequence of Corollary 6.0.5.

6.0.7. Corollary. *Suppose Ω is a bounded smooth domain in a Riemannian manifold $(\mathcal{M}^{m+1}, g)$ let also Ω_0 be a bounded smooth domain in any other Riemannian manifold $(\mathcal{M}_0^{m+1}, g_0)$ with mean curvatures satisfying* $\max_{p\in\partial\Omega} H_{\partial\Omega}(p) < \max_{p\in\partial\Omega_0} H_{\partial\Omega_0}$. *Then if v is small enough,*

$$I_\Omega(v) > I_{\Omega_0}(v).$$

6.1 Proof Theorem 6.0.4 and expansions of the isoperimetric profile I_Ω

P.Berard and D.Meyer ([9], Appendix C) have shown by a localization argument that the isoperimetric profile of a compact Riemannian manifold asymptotically approaches that of $\mathbb{R}^{m+1}$ while, V.Bayle and C.Rosales ([8], proposition 2.1) proved that the relative isoperimetric profile of a domain Ω of a Riemannian manifold behaves like the profile of the half space $\mathbb{R}^{m+1}_+$. Precisely setting

$$I(r) := I_\Omega\left(\left|rB^{m+1}_+\right|\right) = \min_{E\subset\Omega,\, |E|_g=\left|rB^{m+1}_+\right|} \mathcal{P}_g(E,\Omega)$$

and

$$I_+(r) = I_{\mathbb{R}^{m+1}_+}\left(\left|rB^{m+1}_+\right|\right) = \mathcal{P}(rB^{m+1}, \mathbb{R}^{m+1}_+),$$

they proved that

6.1.1. PROPOSITION. *For any $\varepsilon > 0$, there exists $r_0(\varepsilon) > 0$ such that*

$$(1-\varepsilon)I_+(r) \le I(r) \le (1+\varepsilon)I_+(r), \qquad \textit{whenever } r \le r_0.$$

Notice that from this upper bound, an isoperimetric region with small volume must touch the boundary (perpendicularly) because otherwise it would contradict the lower bound in [9] Appendix C. Moreover this upper bound will help after suitable scaling together with the Heintze-Karcher inequality to obtain a uniform bound for the mean curvature of the minimizing hyper-surface trapping a small volume, see[69] § 2 .
We start by proving the following regularity result which was obtained in [69] and under weaker assumptions in [70] for compact Riemannian manifolds.

6.1.2. LEMMA. *There exits $r_0 > 0$ such that if $r \in (0, r_0)$ any set $E \subset \Omega$ satisfying $\mathcal{P}_g(E, \Omega) = I(r)$, there exist $p \in \partial\Omega$ and $\omega^{p,r} : S^m_+ \to \mathbb{R}$ such that*

$$\overline{\partial E \cap \Omega} = F^p(r(1 + \omega^{p,r}\, \overline{S^m_+}))$$

with $\|\omega^{p,r}\|_{\mathcal{C}^{2,\alpha}(S^m_+)} + \|\omega^{p,r}\|_{\mathcal{C}^{1,\alpha}(\overline{S^m_+})} \to 0$ as $r \to 0$ and F^p is a local parametrization of a neiborhood, in $\mathcal{M}$, of $p \in \partial\Omega$ defined in (7.1).

PROOF. We let $E_j \subset \Omega$ such that $\mathcal{P}_g(E_j, \Omega) = I(r_j)$, $r_j \to 0$ as $j \to +\infty$. Call $\Omega_j = \frac{1}{r_j}\,\Omega$ and $E'_j = \frac{1}{r_j}\,E_j$ so that $|E'_j|_{g_j} = \left|B^{m+1}_+\right|$ and $\mathcal{P}_{g_j}(E'_j, \Omega_j) = \frac{1}{r_j^m}\,\mathcal{P}_g(E_j, \Omega) \le c'\,I_+(1)$.

Following [69] § 2 with the help of Proposition 6.1.1, we may assume that there exists a constant $R > 0$ such that

$$\mathrm{diam}_{g_j}(E'_j) \le R$$

and since $\partial E'_j$ intersects $\partial\Omega_j$, then

$$\sup_{e \in \partial E_j} \mathrm{dist}_{g_j}(e, \partial\Omega_j) \le \mathrm{diam}_{g_j}(E'_j) \le R.$$

We can let $p_j \in \partial\Omega_j$ and $U_j \subset \mathbb{R}^{m+1}_+$ be such that $E'_j = F_j(U_j)$, where $F_j : \gamma_j B^{m+1}_+ \to \Omega_j$ is defined by $F_j(\cdot) := \frac{1}{r_j} F^{p_j}(r_j(\cdot))$ and $\gamma_j \to \infty$ as $j \to \infty$.

For j fixed and sufficiently large, let $h_j := (F_j)_*(g_j)$ be the metric induced by F_j on $\mathbb{R}^{m+1}_+$, one has that U_j minimizes the perimeter $\mathcal{P}_{h_j}(\cdot, \gamma_j B^{m+1}_+)$ in $(\gamma_j B^{m+1}_+, h_j)$ among sets enclosing its volume $|U_j|_{h_j} = |B^{m+1}_+|$ and also intersects perpendicularly $\partial\mathbb{R}^{m+1}_+ = \mathbb{R}^m \times \{0\}$.

Now since h_j is converging to the euclidean metric, we get $\mathrm{diam}(U_j) \leq c$ for every large j. And so we have $\mathcal{P}_{h_j}(U_j, \mathbb{R}^{n+1}_+) \leq c$ which implies that $\mathcal{P}(U_j, \mathbb{R}^{m+1}_+) \leq c$. Hence by compactness there exist $U \subset \mathbb{R}^{m+1}_+$ such that $D1_{U_j} \overset{*}{\rightharpoonup} D1_U$. Furthermore by the trace theorem, $1_{U_j}\Big|_{\mathbb{R}^m \times \{0\}} \overset{L^1}{\to} 1_U\Big|_{\mathbb{R}^m \times \{0\}}$. Now to see that U is a minimizer, we let $V \subset \mathbb{R}^{m+1}$ such that $|V| = |B^{m+1}_+|$ and define $c_j \to 1$ such that $c_j |V|_{h_j} = |B^{m+1}_+|$ (this is possible since also h_j converges to the euclidean metric) but then we have

$$\mathcal{P}_{h_j}(U_j, \mathbb{R}^{m+1}_+) \leq c_j^{\frac{m}{m+1}} \mathcal{P}_{h_j}(V, \mathbb{R}^{m+1}_+)$$

and this implies together with the semi-continuity of the perimeter that

$$\mathcal{P}(U, \mathbb{R}^{m+1}_+) \leq \mathcal{P}(V, \mathbb{R}^{m+1}_+).$$

We conclude that U is a minimizer in $\mathbb{R}^{m+1}_+$ among sets that enclose the volume $|B^{m+1}_+|$, namely $U = B^{m+1}_+$. Finally, again by [69] § 2, we have a smooth convergence because mean curvatures are bounded. Hence we may assume that there exists $\omega^{p_j, r_j} \in \mathcal{C}^{2,\alpha}(S^m_+)$ such that

$$\Sigma_j := \partial U_j \cap \mathbb{R}^{m+1}_+ = (1 + \omega^{p_j, r_j})\, S^m_+ \tag{6.1}$$

with $\|\omega^{p_j, r_j}\|_{\mathcal{C}^{2,\alpha}(S^m_+)}$ tending to zero as $j \to \infty$.

We now estimate the free boundary, $\mathcal{H}^m(\partial \Sigma_j)$, by slicing with hyperplanes $\mathbb{R}^m \times \{0\} + hN_{\partial\Omega}$ with $h \in \mathbb{R}$. For terminology, we refer the reader to [68]. In the following, with an abuse of notation, we will call Σ_j the integer ($\mathcal{H}^{m+1}(\Sigma_j) + \mathcal{H}^m(\partial \Sigma_j) \leq \infty$ if j is sufficiently large) rectifiable current associated to the set Σ_j. We define $\mu_j(h)$ by

$$\mu_j(h) := \mathcal{H}^m(\Sigma_j \cap \{d < h\}) = \mathcal{P}(U_j, \{d_j < h\}) \text{ for } 0 < h < \frac{1}{2},$$

where $\mathbb{R}^{m+1} \ni x \mapsto d(x) = x^{m+1}$ (is the distance function from $\partial \mathbb{R}^{m+1}_+ = \mathbb{R}^m \times \{0\}$). For $h \geq 0$ we consider the slice

$$\langle \Sigma_j, d, h_+ \rangle := (\partial \Sigma_j) \llcorner \{d > h\} - \partial(\Sigma_j \llcorner \{d > h\}).$$

Clearly we deduce that

$$\langle \Sigma_j, d, 0_+ \rangle = \partial \Sigma_j.$$

From [68], § 4.11, (3) we get

$$\mathcal{H}^{m-1}(\langle \Sigma_j, d, 0_+ \rangle) \leq Lip(d) \liminf_{h \searrow 0} \frac{\mu_j(h)}{h}$$

and from $Lip(d) = 1$ it follows that

$$\mathcal{H}^{m-1}(\langle \Sigma_j, d, 0_+\rangle) \leq \liminf_{h \searrow 0} \frac{\mu_j(h)}{h}.$$

Since $\mu_j(h)$ is increasing the same argument yields for $\mathscr{L}^1$ a.e. $h > 0$

$$\mathcal{H}^{m-1}(\langle \Sigma_j, d, h_+\rangle) \leq \mu_j'(h).$$

Observe that by (6.1) and Lemma 7.3.2

$$\begin{aligned}\mu_j(h) = \mathcal{P}(U_j, \{d < h\}) &= \mathcal{H}^m((1 + \omega^{p_j, r_j})\, S^m_+ \cap \{d < h\}) \\ &\leq h\,(1 + O(r_j))\,\mathcal{H}^{n-1}(\langle \Sigma_j, d, h_+\rangle) \\ &\leq h\,(1 + O(r_j))\,\mu_j'(h).\end{aligned}$$

Hence we get

$$\mu_j(h) \leq 2\,h\,\mu_j'(h),$$

which is equivalent to

$$\left(\frac{\mu_j(h)}{h} + 2\,\mu_j(h)\right)' \geq 0$$

for every $\mathscr{L}^1$ a.e. $\frac{1}{2} > h > 0$. From this and the fact that μ_j is increasing we conclude that

$$\mathcal{H}^{m-1}(\partial \Sigma_j) \leq \mathcal{H}^m(\Sigma_j \llcorner \{d < h\})\,(\frac{1}{h} + 2)$$

for every $h \in (0, \frac{1}{2})$.

From this together with Lemma 7.3.2 we have

$$\mathcal{H}^{m-1}(\partial \Sigma_j) \leq c\mathcal{H}^m(\Sigma_j) \leq \tilde{c} \quad \text{for any large } j.$$

Consequently, Σ_j is an integral current and moreover by compactness ([68], 5.5) $\partial \Sigma_j$ converges weakly to ∂S^m_+. Since mean curvatures of Σ_j are bounded (see also [69] (2.4)), $\mathcal{C}^{1,\alpha}$ convergence up to the free boundary follows by Gruter-Jost [41]. Hence finally we can assume that $\omega^{p_j, r_j} \in \mathcal{C}^{1,\alpha}(\overline{S^n_+})$ if j is sufficiently large with

$$\partial E_j' \cap \overline{\Omega}_j = \frac{1}{r_j} F^{p_j}(r_j(1 + \omega^{p_j, r_j})\,\overline{S^m_+}),$$

and $\|\omega^{p_j, r_j}\|_{\mathcal{C}^{2,\alpha}(S^m_+)} + \|\omega^{p_j, r_j}\|_{\mathcal{C}^{1,\alpha}(\overline{S^m_+})} \to 0$ as $r \to 0$. ■

6.1.3. Remark. *Observe that when applying the first compaccity result namely* $D1_{U_j} \overset{*}{\rightharpoonup} D1_U$ *we also have (by Rellich theorem) that* $1_{U_j} \overset{L^1}{\to} 1_U$*. Since* $U = B_+^{m+1}$*, by [70],* $\partial U_j \cap \mathbb{R}_+^{m+1}$ *can be written as a normal graph over* S_+^m *by a smooth function* ω^{p_j,r_j} *for which* $\|\omega^{p_j,r_j}\|_{\mathcal{C}^{2,\alpha}(S_+^m)} \to 0$ *as* $r \to 0$*. We also notice that* $\mathcal{C}^{k,\alpha}$ *regularity and estimates of* ω^{p_j,r_j} *can be obtained by a boot-strap argument using Proposition 7.2.2 as in [69], [70].*

The following lemma shows the smoothness of the *center of mass* $c(r,p,\omega) \in \partial\Omega$ of the hyper-surface $\Sigma_{p,r,\omega} := F^p(r(1+\omega)\,\overline{S_+^m}))$ as a function in r, p and ω. The proof can be obtained, with slight modifications, from [72] Lemma 1.3-1.4.

6.1.4. Lemma. *There exists a smooth map* $c : \mathbb{R} \times \partial\Omega \times \mathcal{C}^{2,\alpha}(\overline{S_+^m}) \to \partial\Omega$ *such that*

$$\int_{\Sigma_{p,r,\omega}} (F^c)^{-1}(z)\, dvol_{\Sigma_{p,r,\omega}} = 0.$$

Moreover there exists a smooth vector field $X_{p,r,\omega}$ *on* $T_p\partial\Omega$ *such that*

$$c(r,p,\omega) = exp_p^{\partial\Omega}(r\, X_{p,r,\omega})$$

with

$$X_{p,0,\omega} = \frac{\int_{S_+^m} (1+\omega)^m\, \tilde{\Theta}\, \sqrt{\|d\omega\|^2 + (1+\omega)^2}\, d\sigma}{\int_{S_+^m} (1+\omega)^{m-1}\, \sqrt{\|d\omega\|^2 + (1+\omega)^2}\, d\sigma},$$

where $d\omega$ *is the differential of* ω*.*

According to Proposition 7.2.2, with $H(p,r,\omega)$ being the mean curvature of $\Sigma_{p,r,\omega}$, we define $T(p,r,\cdot) : \mathcal{C}^{2,\alpha}(\overline{S_+^m}) \to \mathcal{C}^{0,\alpha}(\overline{S_+^m})$

$$\int_{S_+^m} T(p,r,\omega)\omega'\, d\sigma := \int_{S_+^m} r\, H(p,r,\omega)\omega'\, d\sigma - \oint_{\partial S_+^m} \langle N_{\partial\Sigma_{p,r,\omega}}^{\partial\Omega}, N_{\partial\Sigma_{p,r,\omega}}^{\Sigma_{p,r,\omega}} \rangle \omega' ds, \tag{6.2}$$

for every $\omega' \in H^1$.

We recall that $E_{p,r,\omega}$ is the set bounded by the hyper-surface $\Sigma_{p,r,\omega}$ and $\partial\Omega$.

6.1.5. Definition. *A set* $E_{p,r,\omega}$ *is called a pseudo-half-ball if*

$$\Pi \circ T(p,r,\omega) \equiv 0,$$

which is equivalent to

$$\Pi_1{}^\perp \circ T(p,r,\omega) \equiv Const \in \mathbb{R},$$

where $T(p,r,\omega)$ is defined in (6.2).

Observe that letting $\Xi \in T_p\partial\Omega$ be such that $\Pi_1\,\omega = \langle \Xi, \tilde\Theta\rangle$, by Lemma 6.1.4, we get

$$c(r,p,\omega) = p + \frac{|S^m_+|}{m+1}\, r\,\Xi + r^2\{L_p(\omega) + \mathcal{O}(r) + Q_p(\omega)\}^\alpha E_\alpha. \qquad (6.3)$$

On the other hand from the expansion of the volume of the sets $E_{p,r,\omega}$, Lemma 7.3.3, we define

$$\begin{aligned}\Phi(p,r,\omega) &:= r^{-m-1}\,|E_{p,r,\omega}|_g - \left|B^{m+1}_+\right| \\ &= \int_{S^m_+} \omega\, d\sigma + \mathcal{O}(r) + \int_{S^m_+}\Big(\mathcal{O}(r)\,\omega + \hat Q_p(\omega)\Big)\, d\sigma.\end{aligned}$$

It turns out that

$$\Phi(p,0,0) = 0, \qquad \frac{\partial \Phi(p,0,0)}{\partial\omega}[u] = \Pi_0\, u.$$

Now for any hyper-surface $\Sigma_{p,r,\omega}$, we can associate to it the smooth mapping

$$\Psi : \partial\Omega \times (0,1) \times \mathcal{C}^{2,\alpha}(\overline{S^m_+}) \to T_p\partial\Omega \times \Pi\mathcal{C}^{0,\alpha}(\overline{S^m_+}) \times \mathbb{R}$$

by

$$\Psi(p,r,\omega) := \left(\frac{m+1}{|S^m_+|}\, X_{p,r,\omega},\ \Pi\circ T(p,r,\omega),\ -m\,\Phi(p,r,\omega)\right).$$

6.1.6. Lemma. *There exist $r_0 > 0$ and $c_0 > 0$ such that for any $p \in \partial\Omega$ and $r \in (0, r_0)$, there exists a unique smooth $\omega^{p,r} \in \mathcal{C}^{2,\alpha}(\overline{S^m_+})$ with*

$$\|\omega^{p,r}\|_{\mathcal{C}^{2,\alpha}(\overline{S^m_+})} \leq c_0 r_0$$

such that $\Psi(p,r,\omega^{p,r}) = (0,0,0)$, namely

$$c(r,p,\omega^{p,r}) = p; \qquad \Pi_1{}^\perp \circ T(p,r,\omega^{p,r}) \in \mathbb{R} \qquad \text{and} \qquad |E_{p,r,\omega^{p,r}}|_g = \left|rB^{m+1}_+\right|.$$

for every $r \in (0, r_0)$.

Proof. We make the following identification: $\mathcal{C}^{k,\alpha}(\overline{S^m_+}) \equiv T_p\partial\Omega \times \Pi\mathcal{C}^{k,\alpha}(\overline{S^m_+}) \times \mathbb{R}$ and that for any $u \in \mathcal{C}^{k,\alpha}(\overline{S^m_+})$, we decompose it as $u = \langle \Xi, \tilde\Theta\rangle + \Pi\, u + u_0 = \langle \Xi, \tilde\Theta\rangle + w$.

It is easy to see that $\Psi(p,0,0)=(0,0,0)$ while

$$\langle\frac{\partial\Psi}{\partial\omega}(p,0,0)[u],u'\rangle=\int_{S^m_+}\nabla_{S^m_+}w\nabla_{S^m_+}w'-m\,ww'\,d\sigma+\int_{S^m_+}\langle\Xi,\tilde\Theta\rangle\langle\Xi',\tilde\Theta\rangle\,d\sigma.$$

Since $\frac{\partial\Psi}{\partial\omega}(p,0,0)$ is an isomorphism from $\mathcal{C}^{2,\alpha}(\overline{S^m_+})$ in to $\mathcal{C}^{0,\alpha}(\overline{S^m_+})$, the lemma then follows by the implicit function theorem. ∎

By choosing r_0 small enough in Lemma 6.1.6, we may assume that the hyper-surfaces $\Sigma_{p,r,\omega^{p,r}}$ are embedded into Ω for any $r\in(0,r_0)$ since $\|\omega^{p,r}\|_{\mathcal{C}^{1,\alpha}(\overline{S^m_+})}\to 0$ as $r_0\to 0$. For simplicity, we will call $E_{p,r}:=E_{p,r,\omega^{p,r}}$ the sets bounded by $\Sigma_{p,r,\omega^{p,r}}$ and $\partial\Omega$.

Remark that the above lemma yields, for any fixed $r\in(0,r_0)$, a manifold of pseudo-half-ball diffeomorphic to $\partial\Omega$ having volume $\left|rB^{m+1}_+\right|$ defined by

$$\mathfrak{C}_r:=\{E_{p,r,\omega^{p,r}}\subset\Omega\quad:\Psi(p,r,\omega^{p,r})=(0,0,0),\quad\|\omega^{p,r}\|_{\mathcal{C}^{2,\alpha}(\overline{S^m_+})}\le c_0r_0,\quad p\in\partial\Omega\}.$$

We can now prove the following result

6.1.7. Lemma. *If* $r\ll1$, *then*

$$I(r)=\inf_{E\in\mathfrak{C}_r}\mathcal{P}_g(E,\Omega)=\inf_{p\in\partial\Omega}\mathcal{P}_g(E_{p,r},\Omega),$$

where $E_{p,r}$, $p\in\partial\Omega$, *denote the elements of* $\mathfrak{C}_r$.

Proof. We have to check that a solution to the isoperimetric problem with volume $\left|rB^{m+1}_+\right|$ belongs to $\mathfrak{C}_r$ if r is small enough.

Let E be a solution to the isoperimetric problem with $|E|_g=\left|rB^{m+1}_+\right|$, then if $r\ll1$, Lemma 6.1.2 implies that $\overline{\partial E\cap\Omega}=F^q(r(1+u^{q,r})\overline{S^m_+})$ for some $q\in\partial\Omega$ and $\|u^{q,r}\|_{\mathcal{C}^{2,\alpha}(\overline{S^m_+})}\to0$ as $r\to0$.
Letting $p\in\partial\Omega$ be the center of mass of ∂E then by (6.3), $\mathrm{dist}_g(p,q)\le c\left(r^2+r\,\|u^{q,r}\|_{\mathcal{C}^{2,\alpha}(\overline{S^n_+})}\right)$ so if $r\ll1$, we can find $v(p,r)$ with $\|v(p,r)\|_{\mathcal{C}^{2,\alpha}(\overline{S^m_+})}\to 0$ as $r\to0$ such that $\overline{\partial E\cap\Omega}=\overline{\Sigma}_{p,r,v(p,r)}$. Clearly since p is the center of mass, it follows that $X_{p,r,v(p,r)}=0$. From the mean curvature expansions, we get $\Pi\circ T(p,r,v(p,r))=0$ because the mean curvature of ∂E is constant and ∂E intersect $\partial\Omega$ perpendicularly. Consequently $\Psi(p,r,v(p,r))=(0,0,0)$. We conclude that if r is small enough then $E\in\mathfrak{C}_r$. ∎

6.1.8. LEMMA. *For any $E_{p,r} \in \mathfrak{C}_r$, we have*

$$\mathcal{P}_g\left(E_{p,r},\Omega\right) = \mathcal{P}\left(rB^{m+1},\mathbb{R}^{m+1}_+\right) - \frac{m}{m+2}\frac{|B^m|}{\left|B^{m+1}_+\right|}H_{\partial\Omega}(p)\left|rB^{m+1}_+\right| + O_p\left(\left|rB^{m+1}_+\right|^{\frac{m+2}{m+1}}\right),$$

where $O_p(\rho)$ is smooth and tends to zero as $\rho \to 0$ uniformly in p.

PROOF. Let $E_{p,r} \in \mathfrak{C}_r$: differentiating the expression $\Phi(p,r,\omega^{p,r}) = 0$ with respect to r, we can deduce that

$$\Pi_0\omega^{p,r} = \int_{S^m_+}\omega^{p,r}\,d\sigma = -\frac{r}{m+2}\langle S_p(E_i),E_i\rangle\int_{S^m_+}\Theta^{m+1}d\sigma + O_p(r^2).$$

This together with Lemma 7.3.2 we get

$$\begin{aligned} r^{-m}\mathcal{P}_g(E_{p,r},\Omega) &= \mathcal{P}(B^{m+1},\mathbb{R}^{m+1}_+) + r\int_{S^m_+}\left(\langle S_p(E_i),E_i\rangle - \langle S_p(\tilde\Theta),\tilde\Theta\rangle\right)\Theta^{m+1}d\sigma + mr \\ &+ O_p(r^2) \\ &= \mathcal{P}(B^{m+1},\mathbb{R}^{m+1}_+) + \frac{2r}{m+2}\langle S_p(E_i),E_i\rangle\int_{S^m_+}\Theta^{m+1}d\sigma - \int_{S^m_+}\langle S_p(\tilde\Theta),\tilde\Theta\rangle \\ &+ O_p(r^2). \end{aligned}$$

Recall that $H_{\partial\Omega}(p) = -\frac{1}{m}\langle S_p(E_i),E_i\rangle$. Moreover since

$$\int_{S^m_+}\langle S_p(\tilde\Theta),\tilde\Theta\rangle\Theta^{m+1}d\sigma = \langle S_p(E_i),E_j\rangle\int_{S^m_+}\Theta^i\Theta^j\Theta^{m+1}d\sigma$$

and observing that

$$\int_{S^m_+}\Theta^i\Theta^j\Theta^{m+1}d\sigma = 0 \qquad \text{if } i \neq j,$$

we deduce that

$$\mathcal{P}_g(E_{p,r},\Omega) = \mathcal{P}(rB^{m+1},\mathbb{R}^{m+1}_+) - c_m\,H_{\partial\Omega}(p)\left|rB^{m+1}_+\right| + O_p\left(\left|rB^{m+1}_+\right|^{\frac{m+2}{m+1}}\right)$$

with

$$\begin{aligned} c_m &= \frac{m}{\left|B^{m+1}_+\right|}\int_{S^m_+}\left(\frac{2}{m+2} - (\Theta^1)^2\right)\Theta^{m+1}d\sigma \\ &= \frac{m}{m+2}\frac{|B^m|}{\left|B^{m+1}_+\right|}. \end{aligned}$$

We have used the fact that

$$\int_{S^m_+}\Theta^{m+1}d\sigma = \frac{\text{Area}(S^{m-1})}{m} = |B^m|, \qquad \int_{S^m_+}(\Theta^i)^2\Theta^{m+1}d\sigma = \frac{\text{Area}(S^{m-1})}{m(m+2)}. \tag{6.4}$$

■

The proof of Theorem 6.0.4 is finalized by the following

6.1.9. LEMMA. *Let r_k be a sequence tending to 0 and $E_k \subset \Omega$ satisfy $|E_k|_g = |r_k B_+^{m+1}|$ and $\mathcal{P}_g(E_k, \Omega) = I(r_k)$. Let $p_k \in \partial\Omega$ be the center of mass of ∂E_k converging to a point $p \in \partial\Omega$. Then*

$$H_{\partial\Omega}(p) = \max_{q \in \partial\Omega} H_{\partial\Omega}(q).$$

PROOF.

If k is large enough, $E_k = E_{p_k, r_k} \in \mathfrak{C}_{r_k}$ and also by Lemma 6.1.7 we have that

$$\mathcal{P}_g(E_{p_k,r_k}, \Omega) = I(r_k) = \min_{q \in \partial\Omega} \mathcal{P}_g(E_{q,r_k}, \Omega)$$

where E_{p,r_k}, $p \in \partial\Omega$, denote the elements of $\mathfrak{C}_{r_k}$. Now by Lemma 6.1.8, we have

$$-H_{\partial\Omega}(p_k) + O(p_k, r_k) = \min_{q \in \partial\Omega} (-H_{\partial\Omega}(q) + O(q, r_k))$$

with $|O(p_k, r_k)| \to 0$ and $\sup_{q \in \partial\Omega} |O(q, r_k)| \to 0$ when k tends to infinity. The lemma then follows taking k to infinity. ■

Chapter 7

Appendix

7.1 Preliminaries and notations

Throughout this chapter, Ω is a smooth domain of an $(m+1)$-Riemannian $(\mathcal{M}, g)$. We denote by $N_{\partial\Omega}$ the unit interior normal vector field along $\partial\Omega$. We consider $((E_i)_{i=1,\dots,m}, N_{\partial\Omega})$ be an (oriented) orthonormal frame of $\mathcal{M}$ along $\partial\Omega$. Recalling from the first chapter, the mean curvature of $\partial\Omega$ at p is given by the trace of $h(= h_{\partial\Omega})$ the second fundamental form of $\partial\Omega$. Namely $H_{\partial\Omega}(p) := -\frac{1}{m}\langle h_p(E_i), E_i\rangle$. We first introduce geodesic normal coordinates in a neighborhood (in $\partial\Omega$) of a point $p \in \partial\Omega$ with coordinates $x' = (x^1, \dots, x^m) \in \mathbb{R}^m$. We set

$$f^p(x') := exp_p^{\partial\Omega}(x^i E_i).$$

This choice of coordinates induces coordinate vector-fields on $\partial\Omega$:

$$Y_i(x') = f_*(\partial_{x^i}) \quad \text{for } i = 1, \dots, m.$$

Now consider a local parametrization of a neighborhood of p in $\mathcal{M}$ by

$$F^p(x) := exp^{\mathcal{M}}_{f^p(x')}(x^{m+1} N_{\partial\Omega}), \qquad x = (x', x^{m+1}) \in \mathbb{R}^{m+1}. \tag{7.1}$$

This yields the coordinate vector fields in $\mathcal{M}$,

$$\begin{aligned} X_i(x) &:= F^p_*(\partial_{x^i}) \qquad i = 1, \dots, m; \\ X_{m+1}(x) &:= F^p_*(\partial_{x^{m+1}}). \end{aligned}$$

7.1.1. Lemma. *Near the point $F^p(x', 0) = f^p(x')$*

$$X_i = Y_i + x^{m+1} h(Y_i) + \frac{(x^{m+1})^2}{2} R_p^{\mathcal{M}}(N_{\partial\Omega}, Y_i) N_{\partial\Omega} + O(|x^{m+1}|^3).$$

Moreover near $p = F^p(0)$ we have

$$\langle Y_i, Y_j\rangle = \delta_{ij} + \frac{1}{3}\langle R_p(E_k, E_i)E_l, E_j\rangle x^k x^l + O(|x|^3).$$

Where R_p (resp. $R_p^{\mathcal{M}}$) is the Riemannian tensor of $\partial\Omega$ (resp. $\mathcal{M}$).

PROOF. By construction we have

$$\nabla^k_{X_{m+1}} X_{m+1}\Big|_{f(x')} = 0 \qquad \text{for any integer } k \geq 1.$$

By definition, $\nabla_{X_{m+1}} X_i\Big|_{f(x')} = \nabla_{X_i} X_{m+1}\Big|_{f(x')} = h(Y_i)$ and $X_{m+1}\Big|_{f(x')} = N_{\partial\Omega}$. We also have that

$$\nabla^2_{X_{m+1}} X_i\Big|_{f(x')} = \nabla_{X_{m+1}}(\nabla_{X_i} X_{m+1})\Big|_{f(x')} = R^{\mathcal{M}}_p(N_{\partial\Omega}, Y_i)N_{\partial\Omega} + \nabla_{X_i}\nabla_{X_{m+1}} X_{m+1}\Big|_{f(x')} =$$

Finally, the proof of the last expansions follows from Lemma 4.1.1. ■

From the above lemma, we have the following proposition which gives the expansions of the metric $g_{\alpha\beta} := \langle X_\alpha, X_\beta\rangle$ in a neighborhood, of $p \in \partial\Omega$ in $\mathcal{M}$, with $\alpha, \beta \in \{1, \ldots, m, m+1\}$.

7.1.2. PROPOSITION. *In a neighborhood of* p,

$$\begin{aligned} g_{ij} &= \delta_{ij} + 2\langle h(Y_i), Y_j\rangle x^{m+1} + \frac{1}{3}\langle R_p(E_k, E_i)E_l, E_j\rangle x^k x^l \\ &\quad + r^2\left(\langle R^{\mathcal{M}}_p(N_{\partial\Omega}, E_i)N_{\partial\Omega}, E_j\rangle + \langle h(Y_i), h(Y_j)\rangle\right)(x^{m+1})^2 + O(|x|^3); \\ g_{im+1} &= O(|x|^3); \\ g_{m+1m+1} &= 1. \end{aligned}$$

Where R_p *(resp.* $R^{\mathcal{M}}_p$*) is the Riemann tensor of* $\partial\Omega$ *(resp.* $\mathcal{M}$*) at* p.

Observe that all hypersurfaces nearby a geodesic sphere centered at $p \in \partial\Omega$ with radius r can be parametrized by a mapping $G : B^m \to \mathcal{M}$ defined by

$$G(z) := F^p\left(r(1+\omega)\tilde{\Theta}(z), r(1+\omega)\Theta^{m+1}(z)\right), \tag{7.2}$$

for some $p \in \partial\Omega$ and $\omega : S^m_+ \to \mathbb{R}$. Notice that by construction, since $\Theta^{m+1} = 0 \quad \text{on } \partial S^m_+$,

$$\partial\Sigma_{p,r,\omega} \subset \partial\Omega.$$

Given $p \in \partial\Omega$ and $\omega : S^m_+ \to \mathbb{R}$, throught this chapter, the expression $\Sigma_{p,r,\omega}$ will denote the hyper-surface $F^p(r(1+\omega)S^m_+)$ while $E_{p,r,\omega}$ will denote the set bounded by $\Sigma_{p,r,\omega}$ and $\partial\Omega$.

Notation: Any expression of the form $L_p(\omega)$ (resp. $\bar{L}_p(\omega)$) denotes a linear combination of the function ω together with its derivatives with respect to the vector fields Θ_i up to order 2 (resp. order 1). The coefficients of L_p or $\bar{L}_p$ might depend on r and p but, for all $k \in \mathbb{N}$, there exists a constant $c > 0$ independent of $r \in (0,1)$ and $p \in \partial\Omega$ such that

$$\|L_p(\omega)\|_{\mathcal{C}^{k,\alpha}(\overline{S^m_+})} \le c\,\|\omega\|_{\mathcal{C}^{k+2,\alpha}(\overline{S^m_+})},$$
$$\|\bar{L}_p(\omega)\|_{\mathcal{C}^{k,\alpha}(\overline{S^m_+})} \le c\,\|\omega\|_{\mathcal{C}^{k+1,\alpha}(\overline{S^m_+})}.$$

Similarly, given $a \in \mathbb{N}$, any expression of the form $Q^a_p(\omega)$ (resp. $\bar{Q}^a_p(\omega)$) denotes a nonlinear operator in the function ω together with its derivatives with respect to the vector fields Θ_i up to order 2 (resp. 1). The coefficients of the Taylor expansion of $Q^a_p(\omega)$ in powers of ω and its partial derivatives might depend on r and p and, given $k \in \mathbb{N}$, there exists a constant $c > 0$ independent of $r \in (0,1)$ and $p \in \mathcal{M}$ such that $Q^a_p(0) = 0$ and

$$\|Q^a_p(\omega_1) - Q^a_p(\omega_2)\|_{\mathcal{C}^{k,\alpha}(\overline{S^m_+})} \le c\left(\|\omega_1\|_{\mathcal{C}^{k+2,\alpha}(\overline{S^m_+})} + \|\omega_2\|_{\mathcal{C}^{k+2,\alpha}(\overline{S^m_+})}\right)^{a-1} \|\omega_1 - \omega_2\|_{\mathcal{C}^{k+2,\alpha}(\overline{S^m_+})},$$

provided $\|\omega_i\|_{\mathcal{C}^{k+2,\alpha}(\overline{S^m_+})} \le 1$. Also

$$\|\bar{Q}^a_p(\omega_1) - \bar{Q}^a_p(\omega_2)\|_{\mathcal{C}^{k,\alpha}(\overline{S^m_+})} \le c\left(\|\omega_1\|_{\mathcal{C}^{k+1,\alpha}(\overline{S^m_+})} + \|\omega_2\|_{\mathcal{C}^{k+1,\alpha}(\overline{S^m_+})}\right)^{a-1} \|\omega_1 - \omega_2\|_{\mathcal{C}^{k+1,\alpha}(\overline{S^m_+})},$$

provided $\|\omega_i\|_{\mathcal{C}^{k+2,\alpha}(\overline{S^m_+})} \le 1$. We also agree that any term denoted by $\mathcal{O}_p(r^d)$ is a smooth function on S^m_+ that might depend on p but satisfies

$$\|\mathcal{O}_p(r^d)\|_{\mathcal{C}^{k,\alpha}(\overline{S^m_+})} \le c\,r^d,$$

for a constant c independent of p.

The tangent space of $\Sigma_{p,r,\omega}$ is spanned by the vector-fields

$$Z_j \;=\; G_*(\partial_{z^j}) \;=\; r(1+\omega)\,\Upsilon_j + r\omega_j\,\Upsilon, \qquad j = 1,\dots,m. \tag{7.3}$$

It is not difficult to see that at the point $G(z)$, one has by Proposition 7.1.2 that

$$\begin{aligned}\langle X_i, X_j\rangle = \delta_{ij} \;&+\; 2r(1+\omega)\langle h(\tilde{\Upsilon}_i), \tilde{\Upsilon}_j\rangle\Theta^{m+1} + \tfrac{r^2(1+\omega)^2}{3}\langle R_p(\tilde{\Theta}, \tilde{\Theta}_i)\tilde{\Theta}, \tilde{\Theta}_j\rangle\\ &+\; r^2\left(\langle R^{\mathcal{M}}_p(N_{\partial\Omega}, \tilde{\Theta}_i)N_{\partial\Omega}, \tilde{\Theta}_j\rangle + \langle h(\tilde{\Upsilon}_i), h(\tilde{\Upsilon}_j)\rangle\right)(\Theta^{m+1})^2\\ &+\; \mathcal{O}(r^3) + r^3L(\omega) + r^3Q(\omega);\end{aligned} \tag{7.4}$$

$$\langle X_i, X_{m+1}\rangle \;=\; \mathcal{O}(r^3) + r^3L(\omega) + r^3Q(\omega); \tag{7.5}$$

$$\langle X_{m+1}, X_{m+1}\rangle \;=\; 1. \tag{7.6}$$

Letting $g^{\Sigma_{p,r,\omega}}_{ij} := \langle Z_i, Z_j\rangle$, from the above, we get the first fundamental form of $\Sigma_{p,r,\omega}$.

7.1.3. PROPOSITION.

$$(1+\omega)^{-2}r^{-2}g^{\Sigma_{p,r,\omega}}_{ij} = \mu^2 + \Big(2\omega_j\langle h(\tilde{\Upsilon}),\tilde{\Upsilon}_i\rangle + 2\omega_i\langle h(\tilde{\Upsilon}),\tilde{\Upsilon}_j\rangle + 2(1+\omega)\langle h(\tilde{\Upsilon}_j),\tilde{\Upsilon}_i\rangle\Big)$$

$$+ \ \omega_i\omega_j + \frac{1}{3}\langle R_p(\tilde{\Theta},\tilde{\Theta}_i)\tilde{\Theta},\tilde{\Theta}_j\rangle r^2 + \langle h(\tilde{\Upsilon}_j),h(\tilde{\Upsilon}_i)\rangle r^2(\Theta^{m+1})^2$$

$$+ \ (\Theta^{m+1})^2\langle R^{\mathcal{M}}_p(N_{\partial\Omega},\tilde{\Theta}_j)N_{\partial\Omega},\tilde{\Theta}_i\rangle r^2 + \mathcal{O}(r^3) + r^2L(\omega) + rQ^2(\omega)$$

7.2 Mean curvature expansion of $\Sigma_{p,r,\omega}$

This section is devoted to give the expansion of the mean curvature $H(p,r,\omega)$ of a hyper-surface $\Sigma_{p,r,\omega}$ in terms of r and ω. The proof is similar to the one in Chapter 4 so we will give a sketch here for the reader's convenience.

Let $z \mapsto G(z)$ parametrizes $\Sigma_{p,r,\omega}$ as defined in (7.2).

Notation: With an abuse of notations, at the point p, we let

$$\Theta := \Theta^j\, E_j + \Theta^{m+1}N_{\partial\Omega} = \tilde{\Theta} + \Theta^{m+1}N_{\partial\Omega}, \qquad \Theta_i := \partial_{z^i}\Theta^j\, E_j + \partial_{z^i}\Theta^{m+1}N_{\partial\Omega},$$

while at the point $G(z)$, we define the vector fields

$$\Upsilon := \Theta^j\, X_j + \Theta^{m+1}X_{m+1} = \tilde{\Upsilon} + \Theta^{m+1}X_{m+1}, \qquad \Upsilon_i := \partial_{z^i}\Theta^j\, X_j + \partial_{z_j}\Theta^{m+1}X_{m+1}.$$

We also set

$$\omega_j := \partial_{z^j}\omega \qquad \omega_{ij} := \partial_{z^i}\,\partial_{z^j}\omega.$$

From the above notations, it is clear that the tangent space of $\Sigma_{p,r,\omega}$ is spanned by the vector fields

$$Z_j = G_*(\partial_{z^j}) = r(1+\omega)\,\Upsilon_j + r\omega_j\,\Upsilon, \qquad j = 1,\dots,m. \tag{7.7}$$

Letting $g^{\Sigma}_{ij} := \langle Z_i, Z_j\rangle$ be the first fundamental form of $\Sigma_{p,r,\omega}(=\Sigma)$, using Proposition 7.1.3 one has

$$(1+\omega)^{-2}r^{-2}g^{\Sigma}_{ij} = \mu^2 + 2\,r\,\langle S(\tilde{\Theta}_j),\tilde{\Theta}_i\rangle\Theta^{m+1} + \mathcal{O}(r^2) + rL(\omega) + Q(\omega). \tag{7.8}$$

7.2.1 The normal vector field

In this subsection we expand the unit normal to $\Sigma_{p,r,\omega}$. The vector field

$$\tilde{N}_\Sigma := -r\,\Upsilon + \alpha^j\, Z_j.$$

is the outer normal field (not necessarily unitary) along $\Sigma_{p,r,\omega}$ if we can determine α^j so that its tangential components $\langle \tilde{N}_\Sigma, Z_j\rangle$ vanish. This leads to a linear system for α^j .

We have the following expansion

$$\langle \Upsilon, Z_j\rangle \;=\; r\omega_j + 2r^2\Theta^{m+1}\langle S(\tilde\Theta), \tilde\Theta_j\rangle + \mathcal{O}(r^3) + r^2\, L(\omega) + r^2 Q(\omega),$$

which follows from (7.4)-(7.6).

Using (7.8), and some algebraic calculations, one obtains

$$\begin{aligned}\alpha^j\,\langle Z_j, Z_i\rangle \;&=\; r\langle \Upsilon, Z_i\rangle \qquad (7.9)\\ &=\; r^2\left(\omega_i + 2r\Theta^{m+1}\langle S(\tilde\Theta), \tilde\Theta_i\rangle + \mathcal{O}(r^2) + r\, L(\omega) + Q(\omega)\right),\end{aligned}$$

hence straightforward computations imply that

$$\alpha^k\langle \Theta_i, \Theta_k\rangle = \omega_i + 2r\Theta^{m+1}\langle S(\tilde\Theta), \tilde\Theta_i\rangle + \mathcal{O}(r^2) + r\, L(\omega) + Q(\omega).$$

Now we have, using also (7.9) that

$$\begin{aligned}\langle \tilde N_\Sigma, \tilde N_\Sigma\rangle \;&=\; r^2\langle \Upsilon, \Upsilon\rangle - 2r\alpha^k\langle Z_k, \Upsilon\rangle + \alpha_l\alpha_k\langle Z_k, Z_l\rangle\\ &=\; r^2(1 + 2r\Theta^{m+1}\langle S(\tilde\Theta), \tilde\Theta\rangle + \mathcal{O}(r^2) + rL(\omega) + Q(\omega)) - \alpha_l\alpha_k\langle Z_k, Z_l\rangle\\ &=\; r^2(1 + 2r\Theta^{m+1}\langle S(\tilde\Theta), \tilde\Theta\rangle + \mathcal{O}(r^2) + rL(\omega) + Q(\omega)).\end{aligned}$$

From this we deduce that

$$|\tilde N_\Sigma|^{-1} \;=\; r^{-1}\left(1 - r\Theta^{m+1}\langle S(\tilde\Theta), \tilde\Theta\rangle + \mathcal{O}(r^2) + r\, L(\omega) + Q(\omega)\right).$$

Therefore the unit normal can be expanded as

$$N_\Sigma = \frac{\tilde N_\Sigma}{|\tilde N_\Sigma|} = -\left(1 - r\Theta^{m+1}\langle S(\tilde\Theta), \tilde\Theta\rangle\right)\Upsilon + \alpha_k\, Z_k + \big(\mathcal{O}(r^2) + r\, L(\omega) + Q(\omega)\big)_\alpha\, X_\alpha.$$

7.2.2 The second fundamental form

In this subsection we expand the coefficients of the second fundamental form.

Noticing that by definition, $\nabla_{Z_i} Z_j \simeq \frac{DZ_j}{dz^i}$, we can readily get the following expansions:

$$\begin{aligned}r^{-1}\nabla_{Z_i} Z_j \;&=\; \omega_{ij}\Upsilon + \omega_j\Upsilon_i + \omega_i\Upsilon_j + (1+\omega)\Upsilon_{ij} + r\Theta_i^\alpha\Theta_j^\beta\nabla_{X_\alpha}X_\beta\\ &+\; \big(O(r^2) + rL(\omega) + Q(\omega)\big)_\alpha\, X_\alpha,\end{aligned}$$

so using (7.4)-(7.6), we get

$$
\begin{aligned}
r^{-1}\langle N_\Sigma, \nabla_{Z_i} Z_j\rangle &= -(1+\omega)\left(1 - r\Theta^{m+1}\langle S(\tilde\Theta), \tilde\Theta\rangle\right)\langle \Upsilon_{ij}, \Upsilon\rangle - \omega_{ij} + \alpha^k \langle \Upsilon_{ij}, \Upsilon_k\rangle \\
&+ r\left(\Theta^{m+1}\langle S(\tilde\Theta_i), \tilde\Theta_j\rangle - \Theta_i^{m+1}\langle S(\tilde\Theta), \tilde\Theta_j\rangle - \Theta_j^{n+1}\langle S(\tilde\Theta), \tilde\Theta_i\rangle\right) \\
&+ \mathcal{O}(r^2) + r\, L(\omega) + Q(\omega).
\end{aligned}
$$

Observing that

$$
\langle \Upsilon_{ij}, \Upsilon\rangle = \langle \Theta_{ij}, \Theta\rangle + 2r\Theta^{m+1}\langle S(\tilde\Theta), \tilde\Theta_{ij}\rangle + \mathcal{O}(r^2) + r\, L(\omega) + Q(\omega)
$$

and also

$$
\langle \Upsilon_{ij}, \Upsilon_k\rangle = \langle \Theta_{ij}, \Theta_k\rangle + \mathcal{O}(r) + r\, L(\omega) + Q(\omega),
$$

we obtain with a little work the

7.2.1. PROPOSITION. *The second fundamental form of the $\Sigma_{p,r,\omega}$ has the following expansion*

$$
\begin{aligned}
r^{-1}\langle N_\Sigma, \nabla_{Z_i} Z_j\rangle &= - \left(1 + \omega - r\Theta^{m+1}\langle S(\tilde\Theta), \tilde\Theta\rangle\right)\langle \Theta_{ij}, \Theta\rangle \\
&- \left(\omega_{ij} + 2r\Theta^{m+1}\langle S(\tilde\Theta), \tilde\Theta_{ij}\rangle\right) + \alpha^k\langle \Theta_{ij}, \Theta_k\rangle \\
&+ r\left(\Theta^{m+1}\langle S(\tilde\Theta_i), \tilde\Theta_j\rangle - \Theta_i^{n+1}\langle S(\tilde\Theta), \tilde\Theta_j\rangle - \Theta_j^{m+1}\langle S(\tilde\Theta), \tilde\Theta_i\rangle\right) \\
&+ \mathcal{O}(r^2) + r\, L(\omega) + Q(\omega).
\end{aligned}
$$

Let $H(p, r, \omega)$ be the mean curvature of the hyper-surface $\Sigma_{p,r,\omega}$, contracting with the metric, (7.8), and using also Lemma 2.3.1, we have the

7.2.2. PROPOSITION. *In the above notations there hold*

$$
\begin{aligned}
rH(p, r, \omega) = m &- \left(\Delta_{S^m_+}\omega + m\omega\right) + r\Theta^{m+1}\left((m+3)\langle S(\tilde\Theta), \tilde\Theta\rangle - \langle S(E_i), E_i\rangle\right) \\
&+ \mathcal{O}(r^2) + r\, L(\omega) + Q(\omega) \qquad \textit{in } \Sigma_{p,r,\omega}; \\
\langle N^{\Sigma_{p,r,\omega}}_{\partial\Sigma_{p,r,\omega}}, N^{\partial\Omega}_{\partial\Sigma_{p,r,\omega}}\rangle &= - \frac{\partial\omega}{\partial\eta} + r^2\, \bar L(\omega) + \bar Q(\omega) \qquad \textit{on } \partial\Sigma_{p,r,\omega},
\end{aligned}
$$

where $\eta \equiv -N_{\partial\Omega}$ is the outer unit normal to ∂S^m_+ and N^A_B stands for the normal of B in A.

PROOF.

We first determine $N^{\partial\Omega}_{\partial\Sigma_{p,r,\omega}}$. Let $s \mapsto \bar\Theta(s) \in S^{m-1} = \partial B^m$ a parametrization of the unit sphere. Noting that $\Theta(\bar\Theta(s)) = \bar\Theta(s)$, the mapping

$$s \mapsto \bar G(s) := f^p(r(1+\omega)\bar\Theta(s)) = F^p(r(1+\omega)\bar\Theta(s), 0)$$

parametrizes $\partial\Sigma_{p,r,\omega} \subset \partial\Omega$ and hence its tangent space is spanned by

$$\bar Z_i = r(1+\omega)\,\bar\Upsilon_i + r\partial_{s^i}\omega\,\bar\Upsilon \qquad i = 1, \dots, m-1,$$

where

$$\bar\Upsilon := \bar\Theta^j\, Y_j, \qquad\qquad \bar\Upsilon_i := \partial_{s^i}\bar\Theta^j\, Y_j.$$

Hence setting

$$\tilde N^{\partial\Omega}_{\partial\Sigma_{p,r,\omega}} = -r\,\bar\Upsilon + \beta^k \bar Z_k, \tag{7.10}$$

we need only to find β^k, $k = 1, \dots, m$ so that it is orthogonal to $\bar Z_i$. But, this can be found in [73] Lemma 2.1 and one has

$$\beta^k \langle \bar Z_k, \bar Z_i\rangle = r\partial_{s^i}\omega, \tag{7.11}$$

while $r^{-2}\langle \bar Z_k, \bar Z_i\rangle = \langle \bar\Theta_k, \bar\Theta_i\rangle\,\big(1 + \mathcal{O}(r^2) + r^2 L(\omega) + Q(\omega)\big)$ and also $\left|\tilde N^{\partial\Omega}_{\partial\Sigma_{p,r,\omega}}\right|^{-1} = r^{-1}\,(1 + Q(\omega))$.

We now determine $N^{\Sigma_{p,r,\omega}}_{\partial\Sigma_{p,r,\omega}}$. To this aim, we denote by ν the unit outer normal to the unit disc B^m and similarly as we have expanded $\tilde N^{\partial\Omega}_{\partial\Sigma_{p,r,\omega}}$, we let

$$\begin{aligned}
\tilde N^{\Sigma}_{\partial\Sigma} &= G_*(\partial_\nu)\Big|_{\partial B^m} + \gamma^k \bar Z_k \\
&= -r\,(1+\omega)N_{\partial\Omega} + r\,\partial_\nu\omega\tilde\Upsilon\Big|_{\partial B^m} + \gamma^k \bar Z_k \\
&= -r\,(1+\omega)N_{\partial\Omega} + r\,\partial_\nu\omega\bar\Upsilon + \gamma^k \bar Z_k,
\end{aligned}$$

we have use the fact that $\partial_\nu\tilde\Theta\Big|_{\partial B^m} = 0$ and $\partial_\nu\Theta^{m+1}\Big|_{\partial B^m} = -1$. Noting that $\langle N_{\partial\Omega}, \bar Z_j\rangle$=0 and $\langle\bar\Upsilon, \bar Z_j\rangle = r\,\omega_j$, then $\tilde N^{\Sigma}_{\partial\Sigma} \in T\Sigma_{p,r,\omega}$ is normal to $\partial\Sigma_{p,r,\omega}$ if

$$\gamma^k\langle \bar Z_k, \bar Z_j\rangle = r^2\omega_j\partial_\nu\omega.$$

Moreover we can deduce that $\left|\tilde N^{\Sigma}_{\partial\Sigma}\right|^{-1} = r^{-1}\,(1 + Q(\omega))$.

Collecting these with the fact that $\langle N^{\Sigma}_{\partial\Sigma}, N^{\partial\Omega}_{\partial\Sigma}\rangle = 0$ when $\omega = 0$, we have that

$$\langle N^{\Sigma}_{\partial\Sigma}, N^{\partial\Omega}_{\partial\Sigma}\rangle = -\frac{\partial\omega}{\partial\eta} + r^2\,\bar L(\omega) + \bar Q^2(\omega) \qquad \text{on } \partial\Sigma_{p,r,\omega}$$

because $\partial_\nu \omega = \frac{\partial \omega}{\partial \eta}$ which holds true since $\mu = 1$. ■

7.3 Area and volume expansion of geodesic hemispheres

In this section, we give expansions of the area and enclosed volume of hyper-surfaces $\partial E_{p,r,\omega} = \Sigma_{p,r,\omega}$.
Using Proposition 7.1.3 we can deduce the expansion of the volume form.

7.3.1. Lemma. *The volume form expands as*

$$\begin{aligned} r^m \sqrt{\det g^{\Sigma_{p,r,\omega}}} &= \mu^m + r\Theta^{m+1}\mu^{m-2}\langle h(\tilde{\Upsilon}_i), \tilde{\Upsilon}_i\rangle + m\omega\mu^m \\ &+ r\Theta^{m+1}\mu^{m-2}\left(3(m+1)\omega\langle h(\tilde{\Upsilon}_i), \tilde{\Upsilon}_i\rangle + 2\omega_i\langle \tilde{\Upsilon}, \tilde{\Upsilon}\rangle\right) \\ &+ \frac{r^2}{2}(\Theta^{m+1})^2\mu^{m-2}\left(\langle h(\tilde{\Upsilon}_i), h(\tilde{\Upsilon}_i)\rangle + \mu^{-2}|\langle h(\tilde{\Upsilon}_i), \tilde{\Upsilon}_i\rangle|^2 - 2\mu^{-2}|\langle h(\tilde{\Upsilon}_i) \right. \\ &+ \frac{r^2}{6}\mu^{m-2}\langle R_p(\tilde{\Theta}, \tilde{\Theta}_i)\tilde{\Theta}, \tilde{\Theta}_i\rangle + \frac{r^2}{2}(\Theta^{m+1})^2\mu^{m-2}\langle R_p^{\mathcal{M}}(N_{\partial\Omega}, \tilde{\Theta}_i)N_{\partial\Omega}, \tilde{\Theta}_i\rangle \\ &+ \frac{\mu^{m-2}}{2}\left(\omega_i^2 + m(m-1)\mu^2\omega^2\right) + \mathcal{O}(r^3) + r^2L(\omega) + rQ^2(\omega) + Q^3(\omega). \end{aligned}$$

Observe that

$$\langle h(Y_k), Y_l\rangle\Big|_{f^p(r(1+\omega)\tilde{\Theta})} = \langle h(E_k), E_l\rangle + \langle T(\tilde{\Theta}, E_k), E_l\rangle + \mathcal{O}(r^2) + rL(\omega) + Q(\omega), \tag{7.12}$$

where $T(Y_i, Y_k) = \nabla_{Y_i}\nabla_{Y_k}N_{\partial\Omega}$. In fact we have

$$Y_i\langle h(Y_k), Y_l\rangle = \langle T(Y_i, Y_k), Y_l\rangle + \langle \nabla_{Y_k}N_{\partial\Omega}, \nabla_{Y_i}Y_l\rangle.$$

By the parallel transport of the vector-fields Y_j with respect to the connection $\nabla^{\partial\Omega}$ of $\partial\Omega$, we have

$$\nabla^{\partial\Omega}_{Y_i}Y_l\Big|_p = 0.$$

Since

$$\nabla_{Y_i}Y_l = \nabla^{\partial\Omega}_{Y_i}Y_l - \langle h(Y_i), Y_l\rangle N_{\partial\Omega},$$

it follows that

$$\langle \nabla_{Y_k}N_{\partial\Omega}, \nabla_{Y_i}Y_l\rangle\Big|_p = 0.$$

Therefore by odness, we readly deduce the following

7.3.2. LEMMA. *The area of the hyper-surface $\Sigma_{p,r,\omega}$ has the following expansion*

$$
\begin{aligned}
r^{-m}A(\Sigma_{p,r,\omega}) &= \mathcal{P}(B^{m+1},\mathbb{R}^{m+1}_+) + r\int_{S^m_+}\Big(\langle h(E_i),E_i\rangle - \langle h(\tilde\Theta),\tilde\Theta\rangle\Big)\Theta^{m+1}d\sigma + m\int_{S^m_+}\omega \\
&+ 3r(m+1)\int_{S^m_+}\Big(\langle h(E_i),E_i\rangle - \langle h(\tilde\Theta),\tilde\Theta\rangle\Big)\Theta^{m+1}\omega d\sigma + 2r\int_{S^m_+}\Theta^{m+1}\langle h(\\
&+ \frac{r^2}{2}\int_{S^m_+}\Big(\Big(\langle h(E_i),h(E_i)\rangle - \langle h(\tilde\Theta),h(\tilde\Theta)\rangle\Big) + \Big(\langle h(E_i),E_i\rangle - \langle h(\tilde\Theta),\tilde\Theta\rangle \\
&- \frac{r^2}{2}\int_{S^m_+}\Big(|\langle h(\tilde\Theta),\tilde\Theta\rangle|^2 - 2|\langle h(E_i),E_i\rangle|^2 + |\langle h(E_j),h(E_i)\rangle|^2\Big)d\sigma - \frac{r^2}{6}\int_{S} \\
&- \frac{r^2}{2}\int_{S^m_+}\Big(Ric^{\mathcal{M}}_p(\tilde\Theta,\tilde\Theta) + \langle R^{\mathcal{M}}_p(N_{\partial\Omega},\tilde\Theta)N_{\partial\Omega},\tilde\Theta\rangle\Big)(\Theta^{m+1})^2 d\sigma \\
&+ \frac{1}{2}\int_{S^m_+}\left(|\nabla_{S^m}\omega|^2 + m(m-1)\omega^2\right)d\sigma + O_p(r^3) + \int_{S^m_+}\left(r^2L(\omega) + rQ^2(\omega\right.
\end{aligned}
$$

Use Proposition 7.1.2 and (7.12) to have the volume form of $B^{\mathcal{M}}(p,\rho)$ in $\mathcal{M}$ for ρ small,

$$
\begin{aligned}
\rho^{-m}\sqrt{\det g_{ij}} &= 1 + \rho\Theta^{m+1}\Big(\langle h(E_i),E_i\rangle + \rho\langle T(\tilde\Theta,E_i),E_i\rangle\Big) + \frac{\rho^2}{2}\langle R^{\mathcal{M}}_p(N_{\partial\Omega},E_i)N_{\partial\Omega},E \\
&+ \frac{\rho^2}{6}\langle R_p(\tilde\Theta,E_i)\tilde\Theta,E_i\rangle + \rho^2(\Theta^{m+1})^2\langle R^{\mathcal{M}}_p(N_{\partial\Omega},E_i)N_{\partial\Omega},E_i\rangle + \frac{\rho^2}{2}(\Theta^{m+1})^2(\\
&+ \frac{\rho^2}{8}(\Theta^{m+1})^2|\langle h(E_i),E_j\rangle|^2 - \frac{\rho^2}{4}(\Theta^{m+1})^2|\langle h(E_i),E_k\rangle|^2 + \mathcal{O}(\rho^3).
\end{aligned}
$$

Integration over the set $\rho \le r(1+\omega)$ gives the expansion of the volume bounded by $\Sigma_{p,\omega,r}$ and $\partial\Omega$.

7.3.3. LEMMA. *The following expansion holds*

$$\begin{aligned} r^{-m-1}|E(p,r,\omega)|_g &= \frac{1}{m+1}\mathcal{P}(B^{m+1},\mathbb{R}^{m+1}_+) + \frac{r}{m+2}\langle h(E_i),E_i\rangle \int_{S^m_+} \Theta^{m+1} d\sigma + \int_{S^m_+} \omega \\ &+ \frac{r^2}{m+3}\left(\frac{1}{8}|\langle h(E_i),E_i\rangle|^2 + \frac{1}{2}\langle h(E_i),h(E_i)\rangle - \frac{1}{4}|\langle h(E_i),E_j\rangle|^2\right) \int \\ &+ -\frac{r^2}{6(m+3)}\int_{S^m_+} Ric_p(\tilde\Theta,\tilde\Theta)d\sigma - \frac{r^2}{2(m+3)}Ric^{\mathcal{M}}_p(N_{\partial\Omega},N_{\partial\Omega})\int_{S^m_+} \\ &+ r\langle h(E_i),E_i\rangle \int_{S^m_+}\Theta^{m+1}\omega d\sigma + \frac{m}{2}\int_{S^m_+}\omega^2 d\sigma + O_p(r^3) \\ &+ \int_{S^m_+}\left(\mathcal{O}(r^2)\omega + \mathcal{O}(r)\hat{Q}^2(\omega) + \hat{Q}^3(\omega)\right) d\sigma, \end{aligned}$$

where $\hat{Q}^a(\omega)$ is a polynomial in w, at least of order a, with smooth coefficients depending on p, Θ and maybe on r but uniformly bounded by a constant depending only on Ω.

7.4 CMC hemispheres in Riemannian manifolds

Let E be an open smooth subset of Ω and $\Sigma := \partial E \cap \Omega$. Assume that the boundary of $\partial\Sigma$ is nonempty and is contained in $\partial\Omega$. From the first variation of area, see for instance [77], E is a critical point for the perimeter functional under variations that keep the volume invariant if and only if

$$mH_\Sigma \equiv const. \quad \text{in } \Sigma \qquad \text{and} \qquad \langle N^\Sigma_{\partial\Sigma}, N^{\partial\Omega}_{\partial\Sigma}\rangle_g = 0 \quad \text{in } \partial\Sigma,$$

where for $B \subset A$, the expression N^A_B denotes the unit outer normal of B in A while H_Σ is the mean curvature of Σ.

We have seen in Section 6.1 that solutions E_r to the isoperimetric problem trapping a volume $\left|rB^{m+1}_+\right|$ have mean curvatures $H_{\partial E_r}$ blowing up and in fact $H_{\partial E_r} \sim \frac{m}{r}$. Moreover their boundaries are normal graph over a hemisphere centered at some point in $\partial\Omega$. Also in Chapter 3, we prove the existence of $\frac{1}{\varepsilon}$-surfaces u_ε concentrating at critical points of the mean curvature of $\partial\Omega$ while the image of these maps solve GMP. The result of this section, in geometric point of view, is a generalization of the afore mentioned result. Our aim in this section is to prove the

7.4.1. PROPOSITION. *There exist $r_0 > 0$ and a smooth function $f : (0, r_0) \times \partial\Omega \to \mathbb{R}$ such that for every $r \in (0, r_0)$, if p is a critical point of $f(r, \cdot)$ then (GMP) admits a solution $\Sigma_{p,r}$ which is a normal graph over $F^p(rS^m_+)$. Furthermore*

$$\|f(r, \cdot) - H_{\partial\Omega}\|_{\mathcal{C}^1(\partial\Omega)} \leq c\, r,$$

for some positive constant c.

Let us describe the proof of this. We have to recall that we look for stationary sets with a given profile for the total energy functional

$$\mathcal{E}_r(E) = \mathcal{P}_g(E, \Omega) + \frac{m}{r}|E|_g.$$

We have that the set $Z_r := \{F^p(r\, B^{m+1}_+), \quad p \in \partial\Omega\}$ is a manifold of approximate solutions for $\mathcal{E}_r$. Namely

$$r^{-m}\, \frac{\partial}{\partial\omega}\mathcal{E}_r(F^p(r(1+\omega)\, B^{m+1}_+))\Big|_{\omega=0} = \mathcal{O}(r),$$

see Lemma 7.3.2 and Lemma 7.3.3. Moreover the linearized mean curvature operator together with the orthogonality conditions (see Proposition 7.2.2) may has small (possibly zero) eigenvalues, so we cannot invert it to apply fixed point argument to solve the problem. However we will perturb Z_r to a manifold $\tilde{Z}_r$ of critical point for $\mathcal{E}$ modulo m "Lagrange-multipliers". This is related to the invariance by translations when $\partial\Omega = \mathbb{R}^n$ is "flat". In this case we have an m dimensional kernel for the Jacobi operator about S^m_+ which is $-\Delta_{S^m_+} + m$. (In contrast with the the Free Boundary Plateau Problem, the invariance by the Möbius group is not taken into account here because it essentially give the same geometric object.)

The second step is to show that in fact $\tilde{Z}_r$ is a *natural constraint* for $\mathcal{E}$ namely critical point of $\mathcal{E}\Big|_{\tilde{Z}_r}$ is also stationary for $\mathcal{E}$. For that we use an argument from Kapouleas in [51] which were successfully employed by [73] to obtain constant mean curvature spheres in Riemannian manifolds. We will closely follow the argument of the latter.
It is worth noticing that this method is also closely related to variational-perturbative methods introduced by Ambrosetti and Badiale in [3] which we also adapt in Chapeter 3 for the Free Boundary Plateau Problem (FBPP). In contrast with the the FBPP, when working with the geometric object $F^p(r\, B^{m+1}_+)$ instead of paramterisation, the invariance by Möbius group, rotation are not taken into account here because they essentially

give the same object. This is exactly what we had in mind in order to carry out the proof of Proposition 3.2.11.

7.5 Existence of CMC hemispheres

We first recall the mean curvature expansion in Proposition 7.2.2,

$$\begin{aligned} rH(p,r,\omega) &= m - \left(\Delta_{S^m_+}\omega + m\omega\right) + r\Theta^{m+1}\left((m+3)\langle h(\tilde\Theta),\tilde\Theta\rangle - \langle h(E_i),E_i\rangle\right) \\ &\quad + \mathcal{O}(r^2) + r\,L(\omega) + Q(\omega) \qquad \text{in } \Sigma_{p,r,\omega}; \\ \langle N^{\Sigma_{p,r,\omega}}_{\partial\Sigma_{p,r,\omega}}, N^{\partial\Omega}_{\partial\Sigma_{p,r,\omega}}\rangle &= -\frac{\partial\omega}{\partial\eta} + r^2\,\bar L(\omega) + \bar Q(\omega) \qquad \text{on } \partial\Sigma_{p,r,\omega}, \end{aligned}$$

where $\eta \equiv -N_{\partial\Omega}$ is the outer unit normal to ∂S^m_+. Define

$$\langle \mathcal{L}_0(u), u'\rangle := \int_{S^m_+} (\nabla_{S^m_+} u \nabla_{S^m_+} u' - m\,uu')\,d\sigma,$$

since the Kernel of this operator is Λ_1, see (2.15), by the Fredholm theorem there exists a unique $\bar\omega^p \in \mathcal{C}^{2,\alpha}(S^m_+)$ such that

$$\begin{cases} \mathcal{L}_0\bar\omega = \Theta^{m+1}\left((m+3)\langle h(\tilde\Theta),\tilde\Theta\rangle - \langle h(E_i),E_i\rangle\right) & \text{in } S^m_+; \\ \dfrac{\partial\bar\omega}{\partial\eta} = 0 & \text{on } \partial S^m_+ \end{cases}$$

because of the evenness of the right hand side. Moreover $\bar\omega^p$ satisfies

$$m\int_{S^m_+} \bar\omega^p d\sigma = \int_{S^m_+} \Theta^{m+1}\left((m+3)\langle h(\tilde\Theta),\tilde\Theta\rangle - \langle h(E_i),E_i\rangle\right)d\sigma. \qquad (7.13)$$

Fixed point argument:

7.5.1. LEMMA. *for every $p \in \partial\Omega$ and r small, there exit a unique $\hat\omega^{p,r}$ and a vector field $\Xi_{p,r}$ on $T_p\partial\Omega$ such that*

$$\begin{aligned} rH(p,r,r\bar\omega^p + \hat\omega) &= m & &\textit{in } S^m_+; \\ \langle N^{\Sigma}_{\partial\Sigma}, N^{\partial\Omega}_{\partial\Sigma}\rangle &= \langle \Xi_{p,r},\tilde\Theta\rangle & &\textit{on } S^{m-1}. \end{aligned} \qquad (7.14)$$

PROOF.

We recall that Π_1 is the L^2 projection on Λ_1, the space spanned by Θ^i, $i = 1\cdots m$. For any $v \in L^2(S^m_+)$, we decompose it as

$$v = \hat\omega + \langle \Xi,\tilde\Theta\rangle = \omega_1 + \omega_0 + \langle \Xi,\tilde\Theta\rangle,$$

where $\Pi_0\omega = \omega_0 = \int_{S^m_+} \omega \, d\sigma$ and $\hat{\omega} = {\Pi_1}^{\perp} v$. Recalling the definition of T in (6.2), we define $\mathcal{L}_{p,r} : \mathcal{C}^{2,\alpha}(\overline{S^m_+}) \to \mathcal{C}^{0,\alpha}(\overline{S^m_+})$ by

$$\langle \mathcal{L}_{p,r} v, v' \rangle := \langle \frac{\partial T(r,p,r\bar{\omega}^p + \omega)}{\partial \omega}(p,r,0)[v], v' \rangle - \oint_{S^{m-1}} \langle \Xi, \tilde{\Theta} \rangle \langle \Xi', \tilde{\Theta} \rangle ds, \quad \forall v' \in L^2.$$

Namely

$$(\mathcal{L}_{p,r} v, v') = \int_{S^m_+} (\nabla_{S^m} \hat{\omega} \nabla_{S^m} \hat{\omega}' - m \hat{\omega} \hat{\omega}') d\sigma - \oint_{S^{m-1}} \langle \Xi, \tilde{\Theta} \rangle \langle \Xi', \tilde{\Theta} \rangle ds + r \int_{S^m_+} v' L(\hat{\omega}) d\sigma + r \oint_{S^{m-1}}$$

Since $\int_{S^m_+} |\nabla \omega_1|^2 \, d\sigma \geq 2(m+1) \int_{S^m_+} |\omega_1|^2 \, d\sigma$, it is easy to see that

$$\begin{aligned} \Pi \circ \mathcal{L}_{p,r} &\geq \frac{1}{2} + o_r(1); \\ \Pi_0 \circ \mathcal{L}_{p,r} &\leq -m + o_r(1); \\ \Pi_1 \circ \mathcal{L}_{p,r} &\leq -\frac{|S^m_+|}{m+1} + o_r(1), \end{aligned}$$

where $o_r(1)$ is a function in r (maybe depending on p) which tends to zero (uniformly in p) as $r \to 0$. From this, we deduce that $\mathcal{L}_{p,r}$ is uniformly invertible and there exists a constant independent on p and r such that

$$||\mathcal{L}_{p,r}^{-1}||_{L^2} \leq C \text{ for any } p \in \partial\Omega, \quad r \ll 1.$$

Now the system (7.14) is equivalent to the fixed point equation

$$v = (\mathcal{L}_{p,r})^{-1} \left\{ \mathcal{O}_p(r^2) + Q_p(\hat{\omega}) \right\},$$

where $Q^2(\hat{\omega})$ is the quadratic part of the mapping T defined in (6.2). By elliptic regularity theory, in a small ball of radius cr^2 in ${\Pi_1}^{\perp}\mathcal{C}^{2,\alpha}(\overline{S^m_+}) \times T_p\partial\Omega$ the above equation has a unique solution $(\hat{\omega}^{p,r}, \Xi_{p,r})$ such that (7.14) is satisfied.

■

We notice that since also the implicit function theorem applies, one has the smoothness of $p \mapsto \omega^{p,r}$ and $p \mapsto \Xi_{p,r} \in T_p\partial\Omega$. Moreover differentiating the mean curvature equation in p, using standard elliptic regularity theory, we can deduce that

$$\|\omega^{(\cdot),r}\|_{C^{2,\alpha} \times \mathcal{C}^1(\partial\Omega)} + \|\Xi_{(\cdot),r}\|_{\mathcal{C}^1(\partial\Omega)} \leq c\, r^2$$

for some constant $c > 0$ independent of r.

Variational argument:
By Lemma 7.5.1, fixing $r >$ small, for any $p \in \partial\Omega$, we have a unique hyper-surface $\Sigma_{p,r} := \Sigma_{p,r,\omega^{p,r}}$ which is embedded if r is small because the $\mathcal{C}^{1,\alpha}$ bound (up to the boundary) of $\omega^{p,r} := r\bar{\omega}^p + \hat{\omega}^{p,r}$ tends to zero as $r \to 0$. This now yields for fixed $r > 0$ a manifold $\tilde{Z}_r$ of sets $E^{p,r} \subset \Omega$, $p \in \partial\Omega$, bounded by $\Sigma_{p,r}$ and $\partial\Omega$ which is homeomorphic to $\partial\Omega$. We have to show that $\tilde{Z}_r$ is a natural constraint for $\mathcal{E}$. For that we define the reduced functional $\varphi_r : \partial\Omega \to \mathbb{R}$ by

$$\varphi_r(p) = \mathcal{E}(E^{p,r}) = \mathcal{P}_g(E^{p,r}, \Omega) - \frac{m}{r}\left|E^{p,r}\right|_g, \tag{7.15}$$

for any $E^{p,r} \in \tilde{Z}_r$. We have to prove the following

7.5.2. Lemma. *Let φ_r given by (7.15). Suppose that p is a critical point of φ_r then $\Xi_{p,r} = 0$.*

Proof. Let p be a critical point of φ_r. Then for any vector field Ξ on $T_p\partial\Omega$,

$$d\varphi_r(p)[\Xi] = 0.$$

If $q := exp_p^{\partial\Omega}(t\Xi)$, then for t sufficiently small the surface $\Sigma_{q,r}$ is a graph over $\Sigma_{p,r}$ for some smooth function $w_{p,r,\Xi,t}$ with variation vector field $\zeta_{p,r,\Xi}$ in $T_p\mathcal{M}$ satisfying

$$\zeta_{p,r,\Xi} \;:=\; \frac{\partial}{\partial t} w_{p,r,\Xi,t}{}_{|t=0} N^{\partial\Omega}_{\partial\Sigma} \qquad \text{on } \partial\Sigma_{p,r} \subset \partial\Omega,$$

where $N^{\partial\Omega}_{\partial\Sigma}$ is the normal of $\partial\Sigma_{p,r}$ in $\partial\Omega$.
It is easy to see that for any parallel transport (in $\partial\Omega$) Z along geodesics issued from p of Ξ we have the estimate

$$\|\zeta_{p,r,\Xi} - Z\| \leq cr\|\Xi\| \qquad \text{on} \quad \partial\Sigma_{p,r}.$$

Now the first variation of area and volume yield

$$\begin{aligned} 0 &= d\varphi_r(p)[\Xi] \\ 0 &= \int_{\Sigma_{p,r}} \left(H_{\Sigma_{p,r}} - \frac{m}{r}\right) \langle \zeta_{p,r,\Xi}, N^{\Sigma}_{\partial\Sigma}\rangle d\sigma + \oint_{\partial\Sigma_{p,r}} \langle \zeta_{p,r,\Xi}, N^{\Sigma}_{\partial\Sigma}\rangle ds. \end{aligned}$$

By construction,

$$H_{\Sigma_{p,r}} = \frac{m}{r} \quad \text{in} \quad \Sigma_{p,r} \qquad \text{and} \qquad \langle N^{\Sigma}_{\partial\Sigma}, N^{\partial\Omega}_{\partial\Sigma}\rangle = \langle \Xi_{p,r}, \tilde{\Theta}\rangle \quad \text{on} \quad \partial\Sigma_{p,r}$$

thus

$$\oint_{\partial\Sigma_{p,r}} \langle \zeta_{p,r,\Xi}, N^{\partial\Omega}_{\partial\Sigma} \rangle \langle \Xi_{p,r}, \tilde{\Theta} \rangle ds = 0.$$

We have

$$\langle \zeta_{p,r,\Xi}, N^{\partial\Omega}_{\partial\Sigma} \rangle = -\langle Z, \tilde{\Upsilon} \rangle + \langle \zeta_{p,r,\Xi} - Z, N^{\partial\Omega}_{\partial\Sigma} \rangle + \langle Z, N^{\partial\Omega}_{\partial\Sigma} + \tilde{\Upsilon} \rangle \qquad \text{on} \quad \partial\Sigma_{p,r}.$$

The expansions of the metric together with the normal $N^{\partial\Omega}_{\partial\Sigma}$ (see (7.10) and (7.11)) show that

$$N^{\partial\Omega}_{\partial\Sigma} + \tilde{\Upsilon} = O(r) \qquad \text{while} \qquad \tilde{\Upsilon} = \tilde{\Theta}(1 + O(r)).$$

Therefore we have the estimates

$$|\langle \zeta_{p,r,\Xi}, N^{\partial\Omega}_{\partial\Sigma} \rangle + \langle \Xi, \tilde{\Theta} \rangle| \leq cr\|\Xi\|.$$

This implies, also by Hölder inequality, that

$$\begin{aligned} \oint_{\partial\Sigma_{p,r}} \langle \Xi_{p,r}, \tilde{\Theta} \rangle \langle \Xi, \tilde{\Theta} \rangle ds &\leq cr\|\Xi\| \oint_{\partial\Sigma_{p,r}} \langle \Xi_{p,r}, \tilde{\Theta} \rangle ds \\ &\leq cr\|\Xi\| \left(\oint_{\partial\Sigma_{p,r}} ds \right)^{\frac{1}{2}} \left(\oint_{\partial\Sigma_{p,r}} |\langle \Xi_{p,r}, \tilde{\Theta} \rangle|^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Using the expansion of the metric of small perturbed geodesic sphere (see [73] Lemma 2.1) we find that

$$\oint_{\partial\Sigma_{p,r}} \langle \Xi_{p,r}, \tilde{\Theta} \rangle \langle \Xi, \tilde{\Theta} \rangle ds \leq cr\|\Xi\| r^{\frac{m-1}{2}} \left(\oint_{\partial\Sigma_{p,r}} |\langle \Xi_{p,r}, \tilde{\Theta} \rangle|^2 ds \right)^{\frac{1}{2}},$$

while

$$\frac{1}{2}\text{Area}(S^{m-1}) r^{m-1} \|\Xi\|^2 \leq m \oint_{\partial\Sigma_{p,r}} |\langle \Xi, \tilde{\Theta} \rangle|^2 ds.$$

Hence we have

$$\oint_{\partial\Sigma_{p,r}} \langle \Xi_{p,r}, \tilde{\Theta} \rangle \langle \Xi, \tilde{\Theta} \rangle ds \leq cr \left(\oint_{\partial\Sigma_{p,r}} |\langle \Xi, \tilde{\Theta} \rangle|^2 ds \right)^{\frac{1}{2}} \left(\oint_{\partial\Sigma_{p,r}} |\langle \Xi_{p,r}, \tilde{\Theta} \rangle|^2 ds \right)^{\frac{1}{2}}.$$

And, finally setting $\Xi = \Xi_{p,r}$, we obtain

$$\oint_{\partial\Sigma_{p,r}} |\langle \Xi_{p,r}, \tilde{\Theta} \rangle|^2 ds \leq cr \oint_{\partial\Sigma_{p,r}} |\langle \Xi_{p,r}, \tilde{\Theta} \rangle|^2 ds.$$

Consequently it must be $\Xi_{p,r} = 0$ for r small. ∎

Using Lemmas 7.3.2 and 7.3.3, we get

$$r^{-m}\mathcal{P}_g\left(E_{p,r},\Omega\right) = \mathcal{P}\left(B^{m+1},\mathbb{R}^{m+1}_+\right) + r\int_{S^m_+}\left(\langle h(E_i),E_i\rangle - \langle h(\tilde\Theta),\tilde\Theta\rangle\right)\Theta^{m+1}d\sigma + m$$
$$+ \ O_p(r^2);$$

$$r^{-1-m}|E_{p,r}|_g = \frac{1}{m+1}\mathcal{P}\left(B^{m+1},\mathbb{R}^{m+1}_+\right) + \frac{r}{m+2}\langle h(E_i),E_i\rangle\int_{S^m_+}\Theta^{m+1}d\sigma + \int_{S^m_+}\omega d\sigma$$

This now give (recalling (6.4))

$$\begin{aligned} r^{-m}\,\varphi_r(p) &= r^{-m}\,\mathcal{E}_r(E^{p,r}) \\ &= \frac{1}{m+1}\mathcal{P}\left(B^{m+1},\mathbb{R}^{m+1}_+\right) + r\int_{S^m_+}\left(\frac{2}{m+2}\langle h(E_i),E_i\rangle - \langle h(\tilde\Theta),\tilde\Theta\rangle\right)\Theta^{m+1} \\ &= \frac{1}{m+1}\mathcal{P}\left(B^{m+1},\mathbb{R}^{m+1}_+\right) - \frac{m\,|B^m|}{(m+2)}\,r\,H_{\partial\Omega}(p) + O_p(r^2). \end{aligned}$$

We end the proof of Proposition 7.4.1 by setting

$$f(r,p) := \frac{-(m+2)}{rm\,|B^m|}\left(r^{-m}\varphi(p) - \frac{1}{m+1}\mathcal{P}\left(B^{m+1},\mathbb{R}^{m+1}_+\right)\right) = H_{\partial\Omega}(p) + O_p(r).$$

7.6 Area and Volume expansion of CMC hemispheres

Using (7.13), we also get precise expansions of the area of the of constant mean curvature hypersurfaces as well as the volume of the domain they enclose.

7.6.1. Corollary. *For any $E^{p,r}\in\tilde Z_r$, there hold*

$$\begin{aligned} r^{-m}\,\mathcal{P}_g(E^{p,r},\Omega) &= \mathcal{P}(B^{m+1},\mathbb{R}^{m+1}_+) - m\,|B^m|\,r\,H_{\partial\Omega}(p) + O_p(r^2); \\ r^{-m-1}\,\left|E^{p,r}\right|_g &= \left|B^{m+1}_+\right| - \frac{m+1}{m+2}\,|B^m|\;r\,H_{\partial\Omega}(p) + O_p(r^2). \end{aligned}$$

Chapter 8

Concluding remarks and open problems

The study of (GMP) is clearly interesting in itself because it involves different fields of mathematics like spectral theory, partial differential equations, differential geometry, calculus of variations, asymptotic analysis,... Our interest and involvement in the study of the afore mentioned problem, allows us to give a (little) contribution in the "study CMC hypersurfaces" based on perturbative methods and critical point theory and also to be able to set some open questions related to what we have done in this thesis. We first notice how our the results of Chapters 3,4,6 are parallel to those of the singularly perturbed problem:

$$\varepsilon^2 \Delta u - u + u^p = 0 \quad \text{in } \Omega \subset \mathbb{R}^{m+1}, \qquad \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial\Omega, \tag{8.1}$$

where $u : \Omega \to \mathbb{R}$ and satisfies $u > 0$ in Ω and η is the unit outer normal to $\partial\Omega$. This problem arises in several contexts, as the nonlinear Schrödinger equation or the modeling reaction-diffusion systems. Solutions with multiple concentration at stable critical points of the mean curvature of $\partial\Omega$, as ε tends to zero have been proved to exist in [46]. Moreover in [71], Ni-Takagi proved that least-energy solutions u_ε, ε small, has only one local maximum point p_ε with $H_{\partial\Omega}(p_\varepsilon) \to \max_{p\in\partial\Omega} H_{\partial\Omega}(p)$ as $\varepsilon \to 0$. Moreover Mahmoudi-Malchiodi [57] provided a sequence of solutions u_{ε_m}, $\varepsilon_m \to 0$ as $m \to \infty$, which concentrate along non-degenerate minimal submanifolds of $\partial\Omega$. If now we consider

$$\varepsilon^2 \Delta u - u + u^p = 0 \quad \text{in } \mathcal{M}, \qquad u > 0 \text{ in } \mathcal{M}, \tag{8.2}$$

with $(\mathcal{M}, g)$ a compact Riemannian manifold with metric g, it turns out that the role of the mean curvature in (8.1) is now played by the scalar curvature of g. This is obtained by Micheletti-Pistoia [62] and Byeon-Park [14]. These results parallel the one of Ye [93], Pacard-Xu[73] and Nardulli [72] in the study of (GMP).

- According to our knowledge in the literature, the counterpart of the result in [56] (solutions of (8.2) which concentrate along minimal submanifolds of $\mathcal{M}$) has not yet been treated.

- Recently, under generic assumptions with $n = 1$, Wei-Yang established the existence of a sequence of solutions which concentrate along a curve which intersect transversely Ω and meet $\partial\Omega$ perpendicularly. One can investigating the counterpart of the latter result to the problem (GMP).
  On the other hand, one can attache an infinite CMC cylinder outside a bounded domain Ω and intersecting $\partial\Omega$ perpendicularly.

- If $\phi'' \equiv 0$, namely when $\phi(s) = cs + d$, S parametrizes the cone, we were not able to conclude. However we notice that the proof of Theorem 5.0.3 highlights that near a point $\gamma(\rho\, s_0)$ for which $\phi''(s_0) = 0$, there will be capillary surfaces with constant and small mean curvatures, see Remark 5.4.1.

- Recalling the notation of Chapter 5, an interesting question is also to perturb the set
  $$\exp_{\gamma(\rho\,\kappa(s))}(\rho\,\phi(s) S^{m-1})$$
  to a closed minimal submanifold of $\mathscr{C}^\rho$. One can see also the work by S.Secchi [82].

- Another problem can be set as follows. Let $\mathcal{U}$ be a smooth bounded domain of $\mathcal{M}$, and $\Gamma \hookrightarrow \partial\mathcal{U}$ be a smooth curve. We let $\tilde{y} = (y^1, \dots, y^m)$ and $N_{\partial\mathcal{U}}$ be a unit interior normal field along $\partial\mathcal{U}$. Choosing an oriented orthogonal frame $(E_1 \dots, E_{m-1})$ along Γ in $\partial\mathcal{U}$, one obtains a coordinate system by letting, for any $y = (\tilde{y}, y^m) = (y^1, \dots, y^{m-1}, y^m)$,
  $$F(x_0, \tilde{y}, y^m) := \exp^{\mathcal{M}}_{\exp^{\partial\mathcal{U}}_{\gamma(x_0)}(y^i E_i)}(y^m\, N_{\partial\mathcal{U}}).$$
  Now consider the set
  $$F(\rho\,\kappa(s), \rho\,\phi(s) B^m_+),$$
  where $B^m_+ = \{x = (x^1, \dots, x^{m-1}, x^m) \in \mathbb{R}^m \quad : \quad |x| = 1, \quad x^m > 0\}$. One may be tempted to perturb the set above into capillary minimal surfaces that meet the "half"-surface of revolution
  $$F(\rho\,\kappa(s), \rho\,\phi(s) S^{m-1}_+)$$
  by an angle equal to $\arccos \phi'$. In this case, as we believe, a result like Theorem 5.0.1 would carry over. On the other hand one would need maybe to impose some conditions on the principal curvature of $\partial\mathcal{U}$ along Γ in order to obtain a variant of Theorem 5.0.3.

- In Chapter 6, we showed that perimeter minimizing sets enclosing small volumes are centered near strict maxima of the mean curvature of the supporting surface, in particular are critical points of the mean curvature. It should be then interesting to ask the following question. Let Σ_H, a family of CMC solving (GMP) and concentrating to a point $p \in \partial\Omega$ as $H \to \infty$, does this force p to be a critical point of the mean curvature of $\partial\Omega$? Results supporting the affirmative of this question can be found in [64].

Bibliography

[1] Adams R., Sobolev Spaces. Academic Press, New York-San Francisco-London (1975).

[2] Almgren F.J., Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints, Mem. AMS 165 (1976).

[3] Ambrosetti A. and Badiale M., Variational perturbative methods and bifurcation of bound states from the essential spectrum, Proc. Roy. Soc. Edinburgh Sect. A 128, (1998), 1131-1161.

[4] Ambrosetti A. and Malchiodi A., Perturbation Methods and Semilinear Elliptic Problems on R^n. Progress in Mathematics, Birkhäuser Verlag, Basel-Boston-Berlin (2005).

[5] A. Ambrosetti and A. Malchiodi, Nonlinear Analysis and Semilinear Elliptic Problems, Cambridge Univ. Press, Cambridge Studies in Advanced Mathematics, No.104 (2007).

[6] A. Ambrosetti, A. Malchiodi, W.M. Ni, Singularly Perturbed Elliptic Equations with Symmetry: Existence of Solutions Concentrating on Spheres, Part I, Comm. Math. Phys., 235 (2003), 427-466.

[7] Barbosa J. L.; do Carmo M., Stability of hypersurfaces with constant mean curvature. Math. Z. 185 (1984), no. 3, 339-353.

[8] Bayle V. and Rosales C., Some isoperimetric comparison theorems for convex bodies in Riemannian manifolds, Indiana Univ. Math. J. 54 (2005), 1371-1394.

[9] Berard P. and Meyer D., Inegalites isoperimetriques et applications. Ann. Sci. Ec. Norm. Super., IV. Ser. 15, (1982) 513-541.

[10] Bethuel F., Caldiroli P. and Guida M., Rend. Sem. Mat. Univ. Pol. Torino, Vol. 60, 4 (2002). Turin Lectures.

[11] Bethuel F. and Roy O., Multiple solutions to the Plateau problem for non constant mean curvature, Duke Math. J. 73 (1994), 593-646.

[12] Brezis H. and Coron J.M., Convergence of solutions of H-systems or how to blow bubbles, Arch. Rational Mech. Anal. 89, (1985), 21-56.

[13] Bürger W. and Kuwert E., Area-minimizing disks with free boundary and prescribed enclosed volume. J. Reine Angew. Math. 621 (2008), 1-27..

[14] Byeon J. and Park J., Singularly perturbed nonlinear elliptic problems on manifolds (English summary). Calc. Var. Partial Differ. Equ. 24(4), 459-477 (2005).

[15] Caldiroli P., H-bubbles with prescribed large mean curvature, Manuscripta Math. 113 (2004), 125-142.

[16] Caldiroli P. and Musina R., H-bubbles in a perturbative setting: The finite-dimensional reduction method, Duke Math. Journal, V. 122, No3, 2004.

[17] Chanillo S. and Malchiodi A., Asymptotic Morse theory for the equation $\Delta v = 2v_x \wedge v_y$, Comm. Anal. Geom., V.13, no. 1, (2005), 187-251.

[18] Chavel I. , Eigenvalues in Riemannian geometry, Academic Press, 1984.

[19] Chavel I., Isoperimetric Inequalities: Differential Geometric and Analytic Perspectives, Cambridge Tracts in Mathematics 145, Cambridge University Press, Cambridge, 2001.

[20] Cheeger J., A lower bound for the smallest eigenvalue of the Laplacian, in: Problems in Analysis, A Symposium in Honor of Salomon Bochner, R.C. Gunning, Ed., Princeton Univ. Press, 1970, pp. 195199.

[21] Choe J., Ghomi M., Ritor M., The relative isoperimetric inequality outside a convex domain in R^n, Cal. Var. Partial Differential Equations 29 no. 4 (2007) 421-429.

[22] Delaunay C., Sur la surface de revolution dont la courbure moyenne est constante, Jour. de Mathématique, 6, (1841) 309-320.

[23] Do Carmo M. P., Riemannian geometry, Mathematics: Theory and Applications. Birkhäuser Boston, Inc., Boston, MA, 1992.

[24] Do Carmo M. P., Differential geometry of curves and surfaces. Translated Inc., Englewood Cliffs, N.J., 1976.

[25] Douglas J., Solution of the Problem of Plateau, Trans. AMS 33 (1931), 263-321.

[26] Druet O., Asymptotic expansion of the FaberKrahn prole of a compact Riemannian mani- fold, C. R. Acad. Sci. Paris, Ser. I 346 (2008).

[27] Druet O., Sharp local isoperimetric inequalities involving the scalar curvature, Proceedings of the American Mathematical Society, 130, 8, 2351- 2361, (2002).

[28] Duzaar F. and Grotowski J. F., Existence and regularity for higher-dimensional H-systems Duke Math. J. 101, no. 3 (2000), 459-485.

[29] Fall M.M., Embedded disc-type surfaces with large constant mean curvature and free boundaries, preprint SISSA 2007 (to appear in Commun. Contemp.Math.).

[30] Fall M.M. and Mahmoudi F.,Hyper-surfaces with free boundary and large constant mean curvature: concentration along sub-manifolds,. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) Vol. VII (2008), 1-40.

[31] Fall M.M., Area minimizing regions with small volume in Riemannian manifolds with boundary. SISSA preprint 2008. To appear in Pacific Journal.

[32] Fall M.M., Some local eigenvalue estimates involving curvature. Preprint SISSA 2008. To appear in Cal.Var. and PDEs. DOI: 10.1007/s00526-009-0236-3.

[33] Felli V., A note on the existence of H-bubbles via perturbation methods. Rev. Mat. Iberoamericana 21 (2005), no. 1, 163-178.

[34] Finn R., *Equilibrium Capillary Surfaces*, Springer-Verlag, New York, 1986

[35] Gilbarg D. and Trudinger N. S., Elliptic partial differential equations of second order. 2^{nd} edition, Grundlehren 224, Springer, Berlin-Heidelberg-New York-Tokyo (1983).

[36] Giusti E., Minimal surfaces and functions of bounded variation. Monographs in Mathematics, 80. Birkhuser Verlag, Basel, 1984.

[37] Gonzalez E., Massari E. and Tamanini I., On the regularity of boundaries of sets minimizing perimeter with a volume constraint, Indiana Univ. Math. J. 32 (1983), 25-37.

[38] Gray A., Tubes, AddisonWesley, Advanced Book Program, Redwood City, CA, 1990.

[39] Gruter M., Boundary regularity for solutions of a partioning problem, Arch. Rat. Mech. Anal. 97 (1987), 261-270.

[40] Grüter M., Regularity resuts for minimizing currents with free boundary, J. Reine Angew. Math., 375/376 (1987) 307-325.

[41] Grüter M. and Jost J., On embedded minimal discs in convex bodies, Annales de l'institut Henri Poincaré (C) Analyse non linéaire, 3, no. 5, (1986), 345-390.

[42] Grüter M., Über die Regularität schwakher Lösungen des Systems $\Delta x = 2H(x)x_u \wedge x_v$, Dissertation, Düsseldorf (1979).

[43] Grüter M., regularity of weak H-surfaces, J. Reine Angew. Math.329 (1981), 1-15.

[44] Grüter M., . Hildebrandt S and Nitsche J.C.C., Regularity for surfaces of constant mean curvature with free boundaries, Acta. Math. 156, (1986), 119-152.

[45] Grüter M. and Jost J., On embedded minimal discs in convex bodies, Annales de l'institut Henri Poincaré (C) Analyse non linaire, 3, no. 5, (1986), 345-390.

[46] Gui C., Multipeak solutions for a semilinear Neumann problem, Duke Math. J. 84 (1996) 739-769.

[47] Hildebrandt S., On the Plateau problem for surfaces of constant mean curvature, Comm. Pure Appl. Math. 23, (1970), 97-114.

[48] Isobe T., On the asymptotic analysis of H-systems. I. Asymptotic behavior of large solutions. Adv. Differential Equations 6 (2001), no. 5, 513-546.

[49] Jleli M., Phd thesis, Universite de Paris 12, (2004).

[50] Jost J., Existence results for embedded minimal surfaces of controlled topological type I, Ann. Sci. Sup. Pisa. Classe di Scienze Sr. 4, 13 no. 1 (1986), p. 15-50.

[51] Kapouleas N., Compact constant mean curvature surfaces in Euclidean three-space. J. Differ. Geom. 33, No.3, (1991), 683-715.

[52] Kato T., *Perturbation theory for linear operators*, GMW 132, Springer-Verlag (1976).

[53] Kohn R.; Sternberg P., Local minimisers and singular perturbations. Proc. Roy. Soc. Edinburgh Sect. A 111 (1989), no. 1-2, 69-84.

[54] Lawson H.B., Complete minimal surfaces in S^3. Ann. of Math. (2) 92 (1970) 335-374.

[55] Lawson H.B., *Lectures on minimal submanifolds*, Vol.I. Second edition. Mathematics Lecture Series, 9. Pulish or Perish, Wimington, Del., 1980.

[56] Mahmoudi, F., Mazzeo, R., Pacard, F., constant mean curvature hypersurfaces condensing along a submanifold, Geom. funct. anal. Vol. 16 (2006) 924-958.

[57] Mahmoudi, F., Malchiodi, A., Concentration on minimal submanifolds for a singularly perturbed Neumann problem, Adv. in Math. 209 (2007) 460-525.

[58] Malchiodi, A.: Concentration at curves for a singularly perturbed Neumann problem in three-dimensional domains, G.A.F.A., 15-6 (2005), 1162-1222.

[59] Malchiodi, A., Montenegro, M., Boundary concentration phenomena for a singularly perturbed elliptic problem, Comm. Pure Appl. Math. 55 (2002), no. 12, 1507-1568.

[60] Malchiodi, A., Montenegro, M., Multidimensional Boundary-layers for a singularly perturbed Neumann problem, Duke Math. J. 124 (2004), no. 1, 105-143.

[61] Massari U., Esistenza e regolaritá delle ipersurfici di curvature media assegnata in $\mathbb{R}^n$, Arch. Rat. Mech. Anal. 55, (1974), 357-382.

[62] Micheletti A.M. and Pistoia A., The role of the scalar curvature in a nonlinear elliptic problem on Riemannian manifolds. Calc. Var. (2009) 34 233-265.

[63] Musina R., The role of the spectrum of the Laplace operator on S^2 in the H-bubble problem, J. Anal. Math. 94, (2004), 265-291.

[64] Mazzeo, R., Pacard, F., Foliations by constant mean curvature tubes, Comm. Anal. Geom. 13 (2005), no. 4, 633-670

[65] Morgan F., Geometric measure theory: Abiginner's Guide, Academic Press, 1988.

[66] Morgan F. and Johnson D.L. Some sharp isoperimetric theorems for riemannian manifolds. Indiana Univ. Math. J., (2000) 49 (2).

[67] Morgan F., Regularity of isoperimetric hypersurfaces in Riemannian manifolds, Trans. AMS. Volume 355, Number 12, 5041-5052.

[68] Morgan F.,Geometric measure theory: Abiginner's Guide, Academic Press, 1988.

[69] Morgan F. and Johnson D.L. Some sharp isoperimetric theorems for riemannian manifolds. Indiana Univ. Math. J., (2000) 49 (2).

[70] Nardulli S. Régularité des solutions du probl'eme isoperimetrique proches de varietes lisses. Preprint Universite de Paris Sud, Octobre 2006.

[71] Ni W-N. and Takagi I., Locating the peaks of least-energy solutions to a semilinear Neumann problem, Duke Math. J. Volume 70, Number 2 (1993), 247-281.

[72] Nardulli S. The isoperimetric profile of a compact Riemannian manifold for small volumes. Preprint University of Paris Sud (Orsay), 2006.

[73] Pacard F. and Xu X., Constant mean curvature spheres in Riemannian manifolds, preprint (2005). Manuscripta Math. 128 (2009), no. 3, 275–295.

[74] Rado T.,On Plateau's problem, Ann. of Math. 31, (1930), 457-469.

[75] Ritoré M., and Rosales C., Existence and charaterization of regions minimizing perimeter under a volume constraint inside Euclidean cones, Trans. Amer. Math. Soc., V. 356, no. 11, 4601-4622.

[76] Ros. A, The isoperimetric problem, Lecture series given during the Caley Mathematics Institute Summer School on the Global Theory of Minimal Surfaces at the MSRI, Berkley, California (2001).

[77] Ros A. and Vergasta E., Satability for hypersurfaces of constant mean curvature with free boundary, Geom. Dedicata 56 (1995), no. 1, 19-33.

[78] Ros A. and Souam R., On stability of capillary surfaces in a ball, Pacific J. Math. 178 (1997) 345- 361.

[79] Sacks J. and Uhlenbeck K., the existence of minimal immersions of 2-spheres, Ann. Math. 113 (1981), 1-24.

[80] Schoen R., Uhlenbeck K., A regularity theory for harmonic maps, J. Diff. Geom. 17 (1982), 307-335.

[81] Schoen R. and Yau S. T., Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature. Ann. of Math. (2) 110 (1979), no. 1, 127-142.

[82] Secchi S., A note on closed geodesics for a class of non-compact Riemannian manifolds. Adv. Nonlinear Stud. 1 (2001), no. 1, 132–142.

[83] Steffen K., On the non-uniqueness of surfaces with prescribed constant mean curvature spanning a given contour, Arch. Rat. Mech. 94, (1986), 101-122.

[84] Sternberg P. and Zumbrun K., Connectivity of phase boundaries in strictly convex domains, Arch. Rational. Mech. Anal., 141 (1998), no. 4, 375-400.

[85] Struwe M., Variational methods: applications to nonlinear partial differential equations and Hamiltonian systems, third edition, Springer (Berlin) 2000.

[86] Struwe M., Large H-surface via the mountain-pass-lemma, Math. Ann. 270, (1985), 441-459.

[87] Struwe M., Non-uniqueness in the Plateau problem for surfaces of constant mean curvature, Arch. Rat. Mech. Anal. 93, (1986), 135-157.

[88] Struwe M., The existence of surfaces of constant mean curvature with free boundaries, Acta Math. 160 (1988), no. 1-2, 19-64.

[89] Struwe M., Plateau's problem and the calculus of variations, Mathematical Notes, 35. Princeton University Press, Princeton, NJ, 1988.

[90] Struwe M., On a free boundary problem for minimal surfaces, Inv. Math. 75 (1984), 547-560.

[91] Taylor J. E., Boundary regularity for solutions to various capillarity and free boundary problems, Comm. PDE, 2, (1977), 323-357.

[92] Wente H., An existence theorem for surfaces of constant mean curvature, J. Math. Anal. Appl. 26 (1969), 318344.

[93] Ye R., Foliation by constant mean curvature spheres, Pacific J. Math. 147 (1991), no. 2, 381–396.