Scuola Internazionale Superiore di Studi Avanzati - Trieste

Parametrized Curves in Lagrange Grassmannians and Sub-Riemannian Geometry

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Contents

Chapter 0

Introduction

The thesis is devoted to Differential Geometry of parametrized curves in Lagrange Grassmannians and its applications to Optimal Control Problems and Hamiltonian Dynamics, especially to Sub-Riemannian Geometry.

0.1 Jacobi curves

About a decade ago A. Agrachev proposed the program of studying Differential Geometry of geometric structures on manifolds via Differential Geometry of curves in Lagrangian Grassmannians ([6]). He introduced the notion of the Jacobi curve of an extremal of the optimal control problems, naturally associated with the geometric structure. The Jacobi curve is a curve in a Lagrangian Grassmannian defined up to a symplectic transformation and containing all information about the solutions of the Jacobi equations along this extremal. The reason to introduce Jacobi curves was two-fold. On one hand, it can be used to construct curvature-type invariants of geometric structures (state-feedback invariants of control systems), namely, any symplectic invariant of Jacobi curve, i.e. an invariant with respect to the action of the linear symplectic group on the Lagrange Grassmannian, produces an invariant of the original geometric structure. On the other hand, the Jacobi curve contains all information about conjugate points along the extremals.

In more detail, by a smooth geometric structure on a manifold we mean any submanifold $\mathfrak{A} \subset TM$, transversal to the fibers. Let $\mathfrak{A}_q = \mathfrak{A} \cap T_qM$. For example, if $\mathfrak{A}_q$ is an intersection of an ellipsoid centered at the origin with a linear subspace $\mathcal{D}_q$ in T_qM (where both the ellipsoids and the subspaces $\mathcal{D}_q$ depend smoothly on q), then $\mathfrak{A}$ is called a *sub-Riemannian structure on M with underlying distribution $\mathcal{D}$*. In this case $\mathfrak{A}_q$ is the unit sphere w.r.t. the unique Euclidean norm $||\cdot||_q$ on $\mathcal{D}_q$, i.e. fixing an ellipsoid in $\mathcal{D}_q$ is equivalent to fixing an Euclidean norm on $\mathcal{D}_q$ for any $q \in M$. This reformulation justifies the term "sub-Riemannian". In particular, it defines in the obvious way the length of any curve tangent to the underlying distribution. If in the constructions above we replace the ellipsoids by the boundaries of strongly convex bodies in T_qM containing the origin in their interior (sometimes also assumed to be symmetric w.r.t. the origin) we will get a sub-Finslerian structure on M. Note also that, if the underlying distribution $\mathcal{D} = TM$, we get just a Riemannian (a Finslerian) structure on M.

Actually one can associate with the geometric structure $\mathfrak{A}$ certain control system on M: the set $\mathfrak{A}_q$ defines the set of all admissible velocities of motions from the point q. A Lipschitzian curve $\gamma : [0, T] \longrightarrow M$ is said to be *admissible*, if $\dot\gamma(t) \in \mathfrak{A}_{\gamma(t)}$ for a.e. $t \in [0, T]$. Now we can consider the time-optimal problem on $\mathfrak{A}$: given two points q_0 and q_1 to find an admissible curve, steering from q_0 to q_1 in a minimal time. The extremals of this optimal-control problem are obtained by Pontryagin Maximum Principle of Optimal Control Theory([22]).

Here for simplicity let us assume that the maximized Hamiltonian of the Maximum Principle

$$h(p,q) = \max_{v\in\mathfrak{A}_q} p\cdot v, \quad q \in M, p \in T_q^*M \tag{0.1.1}$$

is well defined and smooth in an open domain $O \subset T^*M$ and for some $c > 0$ (and therefore for any $c > 0$ by homogeneity of h on each fiber of T^*M) the corresponding level set

$$\mathcal{H}_c = \{\lambda \in O : h(\lambda) = c\}$$

is nonempty and consists of regular points of h.

Now let $\pi : T^*M \longrightarrow M$ be the canonical projection. For any $\lambda \in T^*M$, $\lambda = (p,q)$, $q \in M$, $p \in T_q^*M$, let $\varsigma(\lambda)(\cdot) = p(\pi_*\cdot)$ be the *tautological Liouville form* and $\sigma = -d\varsigma$ be the *standard symplectic structure on* T^*M. Consider the Hamiltonian vector field $\vec{h}$ on $\mathcal{H}_c$, corresponding to the Hamiltonian h, i.e. the vector field satisfying $i_{\vec{h}}\sigma = dh$. The integral curves of this Hamiltonian system are normal Pontryagin extremals of the time-optimal problem, associated with geometric structure $\mathfrak{A}$, or, shortly, normal extremals of $\mathfrak{A}$. For example, if $\mathfrak{A}$ is a sub-Riemannian structure with underlying distribution $\mathcal{D}$, then the maximized Hamiltonian satisfies

$$h(p,q) = ||p|_{\mathcal{D}_q}||_q, \tag{0.1.2}$$

i.e. $h(p,q)$ is equal to the norm of the restriction of the functional $p \in T_q^*M$ on $\mathcal{D}_q$ w.r.t. the Euclidean norm $||\cdot||_q$ on $\mathcal{D}_q$; $O = T^*M \backslash \mathcal{D}^\perp$, where $\mathcal{D}^\perp$ is the annihilator of $\mathcal{D}$,

$$\mathcal{D}^\perp = \{(p,q) \in T^*M : p\cdot v = 0,\ \forall v \in \mathcal{D}_q\},\ \mathcal{D}_q^\perp = \mathcal{D}^\perp \cap T_q^*M. \tag{0.1.3}$$

The projections of the trajectories of the corresponding Hamiltonian systems to the base manifold M are normal sub-Riemannian geodesics. If $\mathcal{D} = TM$, then they are exactly the Riemannian geodesics of the corresponding Riemannian structure.

Further let $\mathcal{H}_c(q) = \mathcal{H}_c \cap T_q^*M$. Then $\mathcal{H}_c(q)$ is a codimension 1 submanifold of T_q^*M. For any $\lambda \in \mathcal{H}_c$ denote $\Pi_\lambda = T_\lambda\big(\mathcal{H}_c(\pi(\lambda))\big)$, where $\pi : T^*M \longrightarrow M$ is the canonical projection. Actually Π_λ is the vertical subspace of $T_\lambda\mathcal{H}_c$,

$$\Pi_\lambda = \{\xi \in T_\lambda\mathcal{H}_c : \pi_*\xi = 0\}. \tag{0.1.4}$$

Now with any integral curve of $\vec{h}$ one can associate a curve in a Lagrange Grassmannian, which describes the dynamics of the vertical subspaces Π_λ along this integral curve w.r.t. the flow $e^{t\vec{h}}$, generated by $\vec{h}$. For this let

$$t \mapsto \mathfrak{J}_\lambda(t) \triangleq e_*^{-t\vec{h}}\big(\Pi_{e^{t\vec{h}}\lambda}\big)\big/\{\mathbb{R}\vec{h}(\lambda)\}. \tag{0.1.5}$$

The curve $\mathfrak{J}_\lambda(t)$ is a curve in the Lagrange Grassmannian of the linear symplectic space

$$W_\lambda = T_\lambda\mathcal{H}_c/\{\mathbb{R}\vec{h}(\lambda)\}$$

(endowed with the symplectic form σ_λ induced in the obvious way by the canonical symplectic form σ of T^*M). It is called *the Jacobi curve of the curve* $e^{t\vec{h}}\lambda$ *attached at the point* λ. Note also that if $\bar{\lambda} = e^{\bar{t}\vec{h}}\lambda$ and $\Phi : W_\lambda \longrightarrow W_{\bar{\lambda}}$ is a symplectic transformation induced in the natural way by a linear mapping $e_*^{\bar{t}\vec{h}} : T_\lambda\mathcal{H}_c \longrightarrow T_{\bar{\lambda}}\mathcal{H}_c$, then by (0.1.5) we have

$$\mathfrak{J}_{\bar{\lambda}}(t) = \Phi\big(\mathfrak{J}_\lambda(t-\bar{t})\big). \tag{0.1.6}$$

In other words, the Jacobi curves of the same integral curve of $\vec{h}$ attached at two different points of this curve are the same, up to symplectic transformation between the corresponding ambient linear symplectic spaces and the corresponding shift of the parameterizations. Therefore, any symplectic invariant of the Jacobi curve produces the function on the manifold $\mathcal{H}_c$, intrinsically related to the geometric structure $\mathfrak{A}$. The value of this function at $\lambda \in \mathcal{H}_c$ is equal to the value of the chosen symplectic invariant of the curve $\mathfrak{J}_\lambda(t)$ at $t = 0$. In this way the problem of finding differential invariants of the geometric structures can be essentially reduced to the much more treatable problem of finding symplectic invariants of certain curves in a Lagrange Grassmannian.

Jacobi curves of integral curves of $\vec{h}$ are not arbitrary curves of Lagrangian Grassmannian but they inherit special features of the geometric structure $\mathfrak{A}$. To specify these features recall that the tangent space $T_\Lambda L(W)$ to the Lagrangian Grassmannian $L(W)$ at the point Λ can be naturally identified with the space $\mathrm{Quad}(\Lambda)$ of all quadratic forms on linear space $\Lambda \subset W$. Namely, given $\mathfrak{V} \in T_\Lambda L(W)$ take a curve $\Lambda(t) \in L(W)$ with $\Lambda(0) = \Lambda$ and $\dot{\Lambda} = \mathfrak{V}$. Given some vector $l \in \Lambda$, take a curve $\ell(\cdot)$ in W such that $\ell(t) \in \Lambda(t)$ for all t and $\ell(0) = l$. Define the quadratic form

$$Q_{\mathfrak{V}}(l) = \omega(l, \frac{d}{dt}\ell(0)). \tag{0.1.7}$$

Using the fact that the spaces $\Lambda(t)$ are Lagrangian, it is easy to see that $Q_{\mathfrak{V}}(l)$ does not depend on the choice of the curves ℓ and $\Lambda(t)$ with the above properties, but depends only on $\mathfrak{V}$. So, we have the linear mapping from $T_\Lambda L(W)$ to the spaces $\mathrm{Quad}(\Lambda)$, $\mathfrak{V} \mapsto Q_{\mathfrak{V}}$. A simple counting of dimensions shows that this mapping is a bijection and it defines the required identification. A curve $\Lambda(\cdot)$ in a Lagrange Grassmannian is called *regular at a point* τ, if its velocity at τ is a nondegenerated quadratic form, and *nonregular at* τ otherwise. The rank of the velocity $\dot{\Lambda}(\tau)$ of a curve $\Lambda(\cdot)$ at a point τ is called shortly the *rank of* $\Lambda(\cdot)$ at τ. A curve $\Lambda(\cdot)$ is called *monotonically nondecreasing (nonincreasing)* if the velocity is nonnegative (nonpositive) definite at any point. We also will call such curves *monotonic.*

It turns out (see, for example, [3, Proposition 1]) that the velocity of the Jacobi curve $J_\lambda(t)$ at $t = 0$ is equal to the restriction of the Hessian of h to the tangent space to $\mathcal{H}_{h(\lambda)}$ at λ. This together with (0.1.6) implies easily ([3]) that the rank of the Jacobi curve $J_\lambda(t)$ at $t = \tau$ is not greater then $\dim \mathfrak{A}_{\pi(e^{\tau\vec{h}}\lambda)}$. For sub-Riemannian structures the rank of Jacobi curves at any point is equal to $\operatorname{rank}\mathcal{D} - 1$, where $\mathcal{D}$ is the underlying distribution, i.e. except the case $\mathcal{D} = TM$ (corresponding to a Riemannian structure), the Jacobi curves appearing in sub-Riemannian structures are *nonregular at any point.* Besides, if h is the maximized Hamiltonian, the corresponding Jacobi curves are monotonic.

The Jacobi curve contains all information about conjugate points along the extremals. Recall that time t_0 is called conjugate to 0 if

$$e_*^{t_0\vec{h}}\Pi_\lambda \cap \Pi_{e^{t_0\vec{h}}\lambda} \neq 0. \tag{0.1.8}$$

and the dimension of this intersection is called the multiplicity of t_0. The curve $\pi(\lambda(\cdot))|_{[0,t]}$ is W^1_∞-optimal (and even C-optimal) if there is no conjugate point in $(0,t)$ and is not optimal otherwise. Note that (0.1.8) can be rewritten as: $e_*^{-t_0\vec{h}}\Pi_{e^{t_0\vec{h}}\lambda} \cap \Pi_\lambda \neq 0$, which is equivalent to

$$\mathfrak{J}_\lambda(t_0) \cap \mathfrak{J}_\lambda(0) \neq 0.$$

Remark 0.1. Jacobi curves can be constructed in more general situation, when the maximized Hamiltonian is not defined (for example, for sub-pseudo-Riemannian structures, defined by a distribution $\mathcal{D}$ and pseudo-Euclidean norms on each space $\mathcal{D}(q)$). Assume that for some open subset $O \subset T^*M$ there exists a smooth map $u : O \to \mathfrak{A}$ such that for any $\lambda = (p, q) \in O$

the point $u(\lambda)$ is a critical point of a function $h_\lambda : \mathfrak{A}_q \to \mathbb{R}$, where $h_\lambda(v) \triangleq p(v)$. Define $\widetilde{h}(\lambda) = p\big(u(\lambda)\big)$. The function $\widetilde{h}$ is called a *critical Hamiltonian* associated with the geometric structure $\mathfrak{A}$ and one can make the same constructions as above with any critical Hamiltonian.

0.2 Description of main problems and results

The following general questions arise naturally in the context of the previous constructions:

1. How to construct a complete system of symplectic invariants of curves in Lagrangian Grassmannians?
2. Given a geometric structure on a manifold how to calculate invariants coming from Jacobi curves of its extremals in terms of the geometric structure itself?
3. How do the symplectic invariants effect the appearance of the conjugate points along extremals of optimal control problems (and therefore the optimality properties of them) and other qualitative properties of the flow of extremals (e.g. hyperbolicity).

Regarding the first question, the basic characteristic of a curve in a Lagrange Grassmannian is its Young diagram. The rank of the curve is the number of boxes in its first column. The number of boxes in its kth column is equal to the rank of the kth derivative of the curve (which is an appropriately defined linear mapping) at a generic point. The complete system of symplectic invariants for curves in Lagrangian Grassmanians was previously constructed only in the following two cases: for regular curves (or, equivalently, when the Young diagram consists of one column, which corresponds to the Jacobi curves of extremals of Riemannian or Finslerian structures ([6]), and for rank 1 curves (or, equivalently, when the Young diagram consists of one row ([3], [4], and in the final form in [23]), which corresponds to the Jacobi curves of extremals in optimal control problems with scalar input, in particular, in sub-Riemannian structures on rank 2 distributions. Also, the notion of cross-ratio of four points in Lagrange Grassmannians was used in [3] in order to construct some basic symplectic invariants of curves (both parametrized and unparametrized) in Lagrange Grassmannians of any rank.

In the first chapter of the thesis we give the answer to the first question in full generality. We construct the canonical bundle of moving frames and the complete system of symplectic invariants for parametrized monotonic curves in Lagrange Grassmannians with any given Young diagram and for non-monotonic curve, satisfying certain generic assumption (condition (G), see subsection 1.2.3) with any given Young diagram. As a consequence, for a very wide class of geometric structures and control systems on a manifold M (including sub-Riemannian and sub-Finslerian structures) one has the canonical (in general, non-linear) connection on an open subset of the cotangent bundle, the canonical splitting of the tangent spaces to the fibers of the cotangent bundle T^*M and the tuple of maps, called curvature maps, between the subspaces of the splitting intrinsically related to the geometric structure or the control system. Besides, the structural equation for a canonical moving frame of the Jacobi curve of an extremal can be interpreted as the normal form for the Jacobi equation along this extremal and the curvature maps can be seen as the "coefficients" of this normal form.

Regarding the second question, we restrict ourselves to sub-Riemannian structures. Note that in the case of a Riemannian metric there is only one curvature map and it is naturally related to the Riemannian sectional curvature. However, for the proper sub-Riemannian structures (i.e. when $D \neq TM$), very little is known about the curvature maps, except that they depend algebraically on points of fibers of T^*M. The curvature maps were explicitly calculated before only in the case of contact distributions of three-dimensional manifolds (in the unpublished notes of A. Agrachev and I. Zelenko and then in [7]) and the calculations used coordinates. In order to understand better the curvature maps, we suggest to study them for a special class of sub-Riemannian structures on distributions $\mathcal{D}$ having sufficiently many infinitesimal symmetries which span integrable distribution transversal to $\mathcal{D}$. In this case, at least locally, we can make a factorization of M by the foliation of the integral manifolds of this transversal distribution and the sub-Riemannian structure induces a Riemannian metric on the reduced manifold. Such sub-Riemannian structures appear naturally on principal connections of principal bundles over Riemannian manifolds (including Yang-Mills fields as a particular case): the sub-Riemannian structure is given by a pull-back (with respect to the canonical projection) of the Riemannian metric of the base manifold to the distribution defining the connection. How the above-mentioned curvature maps are expressed in terms of the Riemannian curvature tensor of the base manifold and the curvature form of the principal connection? In chapter 2 we answer this question in the case when principal bundles have one-dimensional fibers. It is well known that such geometric structures describe magnetic fields on Riemannian manifolds, where the connection form is seen as the magnetic potential. We also develop the language, which allows to implement all calculations in the coordinate-free way. We believe that this coordinate-free language will be useful in the treatment of the more general situations mentioned above. We also estimate the number of conjugate points along the sub-Riemannian extremals in terms of the bounds for the Riemannian curvature tensor of the base manifold and the magnetic field in the case of a uniform magnetic field, giving the partial answer to the third question above. Note that before the estimation of conjugate points (Comparison Theorems) in terms of symplectic invariants were obtain only in the following two cases: in [6] for regular curves and for rank 1 curves in the Lagrangian Grassmannians of four dimensional symplectic space in [4], appearing as Jacobi curves of extremals of sub-Riemannian structures on rank 2 distributions of three-dimensional manifolds.

Finally, in chapter 3 we apply our technique of the calculations of the symplectic invariants to the study of hyperbolicity of sub-Riemannian geodesic flows. We consider sub-Riemannian structures, appearing naturally on principal connections of principal bundles over Riemannian manifolds, when the structure group of the bundle is commutative. In this case one can proceed with the *Poission (symplectic) reduction* to obtain the reduced flows of the sub-Riemannian geodesic flows (on the common level set of all integrals in the cotangent bundle). We give sufficient conditions for this flow to be hyperbolic in terms of the Riemannian curvature tensor of the base manifold and the curvature form of the principal connection by applying the criteria of [8] for the hyperbolicity of Hamiltonian flows. This result is the generalization of results of [14] on Anosov magnetic flows (corresponding to bundles with one-dimensional fibers).

Chapter 1

Differential geometry of curves in Lagrange Grassmannian with Given Young diagram

We will describe the construction of the canonical bundle of moving frames and the complete system of symplectic invariants, called curvature maps, for parametrized curves in Lagrange Grassmannians satisfying with very general assumptions. It allows to develop in a unified way local differential geometry of very wide classes of geometric structures on manifolds, including both classical geometric structures such as Riemannian and Finslerian structures and less classical ones such as sub-Riemannian and sub-Finslerian structures, defined on nonholonomic distributions. The results of this chapter are published in [25] and [24].

1.1 The main results on curves in Lagrangian Grassmannians

Let W be a $2m$-dimensional linear space endowed with a symplectic form ω. Recall that an m-dimensional subspace Λ of W is called *Lagrangian*, if $\omega|_\Lambda = 0$. *Lagrange Grassmannian* $L(W)$ of W is the set of all Lagrangian subspaces of W. It has a structure of smooth manifold ([5]). The linear symplectic group (the set of all linear maps preserving the symplectic form) acts naturally on $L(W)$. Invariants of curves in a Lagrange Grassmannian w.r.t. this action are called *symplectic*.

1.1.1 The flag and the Young diagrams associated with a curve

With any curve $\Lambda(\cdot)$ in Grassmannian $G_k(W)$ of k-dimensional subspaces of a linear space W one can associate a curve of flags of subspaces in W. For this let $\mathfrak{S}(\Lambda)$ be the set of all smooth curves $\ell(t)$ in W such that $\ell(t) \in \Lambda(t)$ for all t. Denote

$$\Lambda^{(i)}(\tau) = \operatorname{span}\Big\{\frac{d^j}{d\tau^j}\ell(\tau) : \ell \in \mathfrak{S}(\Lambda),\, 0 \leq j \leq i\Big\}. \tag{1.1.1}$$

The subspaces $\Lambda^{(i)}(\tau)$ are called *the ith extension* of the curve $\Lambda(\cdot)$ at the point τ. Recall that the tangent space $T_\Lambda G_k(W)$ to any subspace $\Lambda \in G_k(W)$ can be identified with the space $\operatorname{Hom}(\Lambda, W/\Lambda)$ of linear mappings from Λ to W/Λ. Using this identification, if $P : \Lambda \longrightarrow W/\Lambda$ is the canonical projection to the factor, then $\Lambda^{(1)}(\tau) = P^{(-1)}(\operatorname{Im}\dot{\Lambda}(\tau))$, which implies that $\dim \Lambda^{(1)}(\tau) - \dim \Lambda(\tau) = \operatorname{rank}\dot{\Lambda}(\tau)$. By construction $\Lambda^{(i-1)}(\tau) \subseteq \Lambda^{(i)}(\tau)$. The flag

$$\Lambda(\tau) \subseteq \Lambda^{(1)}(\tau) \subseteq \Lambda^{(2)}(\tau) \subseteq \dots \tag{1.1.2}$$

is called the *associated (right) flag of the curve* $\Lambda(\cdot)$ *at the point* t.

¿From now on we suppose that dimensions of all subspaces $\Lambda^{(i)}(t)$ (and therefore of $\Lambda_{(i)}(t)$) are independent of t. In this case from (1.1.1) it is easy to obtain that the following inequalities hold

$$\dim\Lambda^{(i+1)} - \dim\Lambda^{(i)} \leq \dim\Lambda^{(i)} - \dim\Lambda^{(i-1)}. \tag{1.1.3}$$

Using inequalities (1.1.3), to any curve $\Lambda(\cdot)$ we can assign the Young diagram in the following way: the number of boxes in the ith column of this Young diagram is equal to $\dim\Lambda^{(i)} - \dim\Lambda^{(i-1)}$. It will be called the *Young diagram of the curve* $\Lambda(\cdot)$. In particular, the number of boxes in the first column is equal to the rank of the curve.

Now suppose that W is an even-dimensional linear space endowed with a symplectic structure ω and the curve $\Lambda(\cdot)$ is a curve in the Lagrangian Grassmannian $L(W)$.

Remark 1.1. Without loss of generality, we will suppose that there exists an integer p such that $\Lambda^{(p)}(t) = W$. Otherwise, if $\Lambda^{(p+1)}(t) = \Lambda^{(p)}(t) \subsetneq W$, then the subspace $\Lambda^{(p)}(t)$ does not depend on t. Set $V = \Lambda^{(p)}(t)$. Then $V^\angle \subset \Lambda(t)$ for any t and all information about the original curve $\Lambda(\cdot)$ is contained in the curve $\Lambda(\cdot)/V^\angle$, which is the curve of Lagrangian subspaces in the symplectic space $V/V^\angle$, and the pth extension of the curve $\Lambda(\cdot)/V^\angle$ is equal to $V/V^\angle$. So, we can work with the curve $\Lambda(\cdot)/V^\angle$ and the symplectic space $V/V^\angle$ instead of the curve $\Lambda(\cdot)$ and the symplectic space W. □

1.1.2 The normal moving frame.

The Young diagram is a basic invariant of the curve in Lagrange Grassmannians. As indices of vectors in our Darboux moving frames we will take the boxes of the Young diagram instead of the natural numbers. We found it extremely useful both for formulation of our results and their proofs.

First note that using the flag, to any $\Lambda(\cdot)$ we can assign the Young diagram in the following way: the number of boxes of the ith column is equal to $\dim\ \Lambda^{(i)}(t) - \dim\ \Lambda^{(i-1)}(t)$. Assume that the length of the rows of D be p_1 repeated r_1 times, p_2 repeated r_2 times, . . ., p_d repeated r_d times with $p_1 > p_2 > ... > p_d$. In this case, the Young diagram D is the union of d rectangular diagrams of size $r_i \times p_i, 1 \leq i \leq d$. Denote them by $D_i, 1 \leq i \leq d$. For our convenience, we also assign a "smaller"Young diagram Δ, consisting of d rows such that the ith row has p_i boxes. It will be called the reduced diagram or the *reduction of the diagram D.* In order to distinguish between boxes and rows of the diagram D and its reduction Δ, the boxes of Δ will be called *superboxes* and the rows of Δ will be called *levels.* To the jth superbox a of the ith level of Δ one can assign the jth column of the rectangular subdiagram D_i of D and the integer number r_i (equal to the number of boxes in this subcolumn), called the *size* of the superbox a.

As usual, by $\Delta \times \Delta$ we will mean the set of pairs of superboxes of Δ. Also denote by Mat the set of matrices of all sizes. The mapping $R : \Delta \times \Delta \longrightarrow$ Mat is called *compatible with the Young diagram D*, if to any pair (a, b) of superboxes of sizes s_1 and s_2 respectively the matrix $R(a, b)$ is of the size $s_2 \times s_1$. The compatible mapping R is called *symmetric* if for any pair (a, b) of superboxes the following identity holds

$$R(b, a) = R(a, b)^T. \tag{1.1.4}$$

Denote by Υ_i the ith level of Δ.

Also denote by a_i and σ_i the first and the last superboxes of the ith level Υ_i respectively and by $r : \Delta\backslash\{\sigma_i\}_{i=1}^d \longrightarrow \Delta$ the right shift on the diagram Δ. The last superbox of any level will be called *special.* For any pair of integers (i, j) such that $1 \leq j < i \leq d$ consider the following tuple of pairs of superboxes

$$\begin{aligned}&(a_j,a_i),\ (a_j,r(a_i)),\ (r(a_j),r(a_i)),\ (r(a_j),r^2(a_i)),\dots,(r^{p_i-1}(a_j),r^{p_i-1}(a_i)),\\&(r^{p_i}(a_j),r^{p_i-1}(a_i)),\dots,(r^{p_j-1}(a_j),r^{p_i-1}(a_i)).\end{aligned}\tag{1.1.5}$$

Actually the tuple (1.1.5) is obtained as follows: the first pair consists of the last two superboxes of the considered levels, then until the superbox of the ith level will not become special, each next even pair is obtained from the previous pair of the tuple by the right shift of the superbox of the ith level in the previous pair and each next odd pair is obtained from the previous pair of the tuple by the right shift of the superbox of the jth level in the previous pair. When the superbox of the ith level become special, each next pair is obtained from the previous pair of the tuple by the right shift of the superbox of the jth level.

Now we are ready to introduce two crucial notions, which will be very useful in the formulation of our main theorem:

Definition 1.1. *A symmetric compatible mapping* $R:\Delta\times\Delta\longrightarrow \mathrm{Mat}$ *is called* *quasi-normal* *if the following two conditions hold:*

1. *Among all matrices* $\mathcal{R}(a,b)$*, where the superbox* b *is not higher than the superbox* a *in the diagram* Δ*, the only possible nonzero matrices are the following: the matrices* $\mathcal{R}(a,a)$ *for all* $a\in\Delta$*, the matrices* $R\big(a,r(a)\big)$*,* $R\big(r(a),a\big)$ *for all nonspecial boxes, and the matrices, corresponding to the pairs, which appear in the tuples* (1.1.5)*, for all* $1\le j<i\le d$*;*

2. *The matrix* $R\big(a,r(a)\big)$ *is antisymmetric for any nonspecial superbox* a*.*

Definition 1.2. *A quasi-normal mapping* $R:\Delta\times\Delta\longrightarrow\mathrm{Mat}$ *is called* *normal* *if it satisfies the following condition: for any* $1\le j<i\le d$*, the matrices, corresponding to the first* (p_j-p_i-1) *pairs of the tuple* (1.1.5)*, are equal to zero.*

Now let us fix some terminology about the frames in W, indexed by the boxes of the Young diagram D. A frame $\big(\{e_\alpha\}_{\alpha\in D},\{f_\alpha\}_{\alpha\in D}\big)$ of W is called *Darboux* or *symplectic*, if for any $\alpha,\beta\in D$ the following relations hold

$$\omega(e_\alpha,e_\beta)=\omega(f_\alpha,f_\beta)=\omega(e_\alpha,f_\beta)-\delta_{\alpha,\beta}=0,\tag{1.1.6}$$

where $\delta_{\alpha,\beta}$ is the analogue of the Kronecker index defined on $D\times D$. In the sequel it will be convenient to divide a moving frame $\big(\{e_\alpha(t)\}_{\alpha\in D},\{f_\alpha(t)\}_{\alpha\in D}\big)$ of W indexed by the boxes of the Young diagram D into the tuples of vectors indexed by the superboxes of the reduction Δ of D, according to the correspondence between the superboxes of Δ and the subcolumns of D. More precisely, given a superbox a in Δ of size s, take all boxes $\alpha_1,\dots,\alpha_s$ of the corresponding subcolumn in D in the order from the top to the bottom and denote

$$E_a(t)=\big(e_{\alpha_1}(t),\dots,e_{\alpha_s}(t)\big),\quad F_a(t)=\big(f_{\alpha_1}(t),\dots,f_{\alpha_s}(t)\big).$$

In what follows we will suppose that the curve $\Lambda(t)$ is monotonically nondecreasing, i.e. the velocity $\dot\Lambda(t)$ is a nonnegative definite quadratic form for any t. The case of monotonically nonincreasing curve can be treated then by reversing of time. We restrict ourselves to the monotonic curves just in order to avoid technicalities both in the formulation and the proof of our main result (Theorem 1.1 below). The similar result with essentially the same proof is valid also for nonmonotonic curves under additional generic assumptions, which will be introduced in Subsection 1.2.3 (see condition (G) there). In Section 1.3 we point out what changes one has to make in Theorem 1.1 in nonmonotonic situation (see Theorem 1.3 below). Note also that Jacobi curves in sub-Riemannian and, more generally, in sub-Finslerian geometry are monotonic, because the corresponding maximized Hamiltonians are convex on the fibers of T^*M (see the Introduction).

Remark 1.2. *In [25] we used alternative language of graded spaces for description of the results and of the condition (G), inspired by discussions with Pierre Deligne. This language allows to describe the results on monotonic and nonmonotonic case in more short and unified way, but it less elementary and requires from the reader more efforts to understand the results.*

Definition 1.3. *The moving Darboux frame* $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$ *is called the normal (quasi-normal) moving frame of a monotonically nondecreasing curve* $\Lambda(t)$ *with the Young diagram* D*, if*

$$\Lambda(t) = \operatorname{span}\{E_a(t)\}_{a\in\Delta}$$

for any t *and there exists an one-parametric family of normal (quasi-normal) mappings* $R_t : \Delta\times\Delta \longrightarrow \mathrm{Mat}$ *such that the moving frame* $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$ *satisfies the following structural equation:*

$$\begin{cases} E_a'(t) = E_{l(a)}(t) & \text{if } a \in \Delta\setminus \mathcal{F}_1 \\ E_a'(t) = F_a(t) & \text{if } a \in \mathcal{F}_1 \\ F_a'(t) = -\sum\limits_{b\in\Delta} E_b(t)R_t(a,b) - F_{r(a)}(t) & \text{if } a \in \Delta\setminus \mathcal{S} \\ F_a'(t) = -\sum\limits_{b\in\Delta} E_b(t)R_t(a,b) & \text{if } a \in \mathcal{S} \end{cases}, \tag{1.1.7}$$

where $\mathcal{F}_1$ *is the first column of the diagram* Δ*,* $\mathcal{S}$ *is the set of all its special superboxes, and* $l : \Delta\setminus\mathcal{F}_1 \longrightarrow \Delta$*,* $r : \Delta\setminus \mathcal{S} \longrightarrow \Delta$ *are the left and right shifts on the diagram* Δ*. The mapping* R_t*, appearing in (1.1.7), is called the normal (quasi-normal) mapping, associated with the normal moving frame* $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$*.*

With all this terminology we are ready to formulate our main theorem:

Theorem 1.1. *For any monotonically nondecreasing curve* $\Lambda(t)$ *with the Young diagram* D *in the Lagrange Grassmannian there exists a normal moving frame* $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$*. A moving frame*

$$(\{\widetilde{E}_a(t)\}_{a\in\Delta}, \{\widetilde{F}_a(t)\}_{a\in\Delta})$$

is a normal moving frame of the curve $\Lambda(\cdot)$ *if and only if for any* $1 \le i \le d$ *there exists a constant orthogonal matrix* U_i *of size* $r_i \times r_i$ *such that for all* t

$$\widetilde{E}_a(t) = E_a(t)U_i, \quad \widetilde{F}_a(t) = F_a(t)U_i, \quad \forall\, a \in \Upsilon_i. \tag{1.1.8}$$

Actually, the second statement of this theorem means that if for any $\bar{t}$ one collects all possible Darboux frame $(\{E_a\}_{a\in\Delta}, \{F_a)\}_{a\in\Delta})$ in W such that there exists a normal moving frame, which coincides with $(\{E_a\}_{a\in\Delta}, \{F_a)\}_{a\in\Delta})$ at $t = \bar{t}$, then one gets the principle $O(r_1)\times\ldots\times O(r_d)$ bundle over the curve $\Lambda(t)$ endowed with the canonical principal connection in the following way: the normal moving frames are horizontal curves w.r.t. this connection.

1.1.3 The canonical splitting and curvature maps

Before proving Theorem 1.1 let us discuss it a little bit. Take some normal moving frame

$$(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta}).$$

Relations (1.1.8) imply that for any superbox $a \in \Delta$ of size s the following s-dimensional subspace

$$V_a(t) = \operatorname{span}\{E_a(t)\} \tag{1.1.9}$$

of $\Lambda(t)$ does not depend on the choice of the normal moving frame. The subspace V_a will be called the *subspace, associated with the superbox a*. So, there exists the *canonical splitting of the subspace* $\Lambda(t)$:

$$\Lambda(t) = \bigoplus_{a\in\Delta} V_a(t). \tag{1.1.10}$$

Moreover, each subspace $V_a(t)$ is endowed with the *canonical Euclidean structure* such that the tuple of vectors E_a constitute an orthonormal frame w.r.t. to it. Note that the canonical splitting is obtained in one of the first steps of the normalization procedure in the proof of Theorem 1.1 (see subsection 1.2.3).

Another very important consequence of (1.1.8) is that the following subspace

$$\Lambda^{\text{trans}}(t) = \bigoplus_{a\in\Delta} \text{span}\{F_a(t)\} \tag{1.1.11}$$

does not depend on the choice of the normal moving frame. By construction, $W = \Lambda(t) \oplus \Lambda^{\text{trans}}(t)$ for any t. The curve $\Lambda^{\text{trans}}(t)$ will be called the *canonical complementary curve of the curve* $\Lambda(\cdot)$. As we will see in Section 1.4 this notion is crucial for the construction of the canonical (non-linear) connection for sub-Riemannian and, more generally, sub-Finsler structures.

Remark 1.3. Note also that the canonical complementary curve is different in general from the so-called derivative curve $\Lambda^0(\cdot)$, constructed in [3], which is also intrinsically related to $\Lambda(\cdot)$ such that the space $\Lambda^0(t)$ is transversal to $\Lambda(t)$ for any t. The main disadvantage of the derivative curve $\Lambda^0(\cdot)$, comparing to the curve $\Lambda^{\text{trans}}(\cdot)$, constructed here, is that if one uses it for the construction of the moving frames intrinsically related to the curve $\Lambda(\cdot)$, as was done in [3] and [4] (see also [5]), then it is very hard to analyze their structural equations and to distinguish a complete system of invariants from it (in the mentioned papers it was partially done only in the case of curves of rank 1), while in the present paper we construct the normal moving frame step by step according to the heuristic rule that the matrix of its structural equation should be as simple as possible (should contain as much zeros as possible), which gives the complete system of invariants automatically.□

Further, we say that a pair (a, b) of superboxes is *essential* if $R(a, b)$ is not necessarily zero for a normal mapping $R : \Delta \times \Delta \longrightarrow \text{Mat}$. Note that this notion depends only on the mutual locations of the superboxes a and b in the diagram Δ, except the case of consecutive superboxes a and b in the same level of Δ. In the last case it depends on the size of the superboxes. Namely, the pair $(a, r(a))$ is essential if and only if the size of a is greater than 1 (see condition (1) of Lemma 1.6).

Assume that $R_t : \Delta \times \Delta \longrightarrow \text{Mat}$ and $\widetilde{R}_t : \Delta \times \Delta \longrightarrow \text{Mat}$ are the normal mappings, associated with normal moving frames $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$ and $(\{\widetilde{E}_a(t)\}_{a\in\Delta}, \{\widetilde{F}_a(t)\}_{a\in\Delta})$, which are related by (1.1.8). Then from (1.1.7) and (1.1.8) it follows immediately that

$$\widetilde{R}_t(a, b) = U_j^{-1} R_t(a, b) U_i, \quad a \in \Upsilon_i, b \in \Upsilon_j. \tag{1.1.12}$$

The last relation means actually that for any essential pair (a, b) of superboxes the linear mapping $\mathfrak{R}_t(a, b) : V_a \longrightarrow V_b$, having the matrix $R_t(a, b)$ w.r.t. the bases E_a and E_b of V_a and V_b respectively, does not depend on the choice of a normal moving frame.[1] The linear mapping

[1]Here we restrict ourselves to essential pairs, because for nonessential pairs such linear mappings are zeros automatically.

$\mathfrak{R}_t(a,b)$ will be called the *(a,b)-curvature map of the curve* $\Lambda(\cdot)$. Finally, all (a,b)-curvature maps form the canonical map $\mathfrak{R}_t : \Lambda(t) \to \Lambda(t)$ as follows:

$$R_t v_a = \sum_{b \in \Delta} R_t(a,b) v_a, \forall v_a \in V_a(t), a \in \Delta. \tag{1.1.13}$$

The map $\mathfrak{R}_t$ is called the *big curvature map of the curve* $\Lambda(\cdot)$ *at time* t.

The only nontrivial blocks in the matrix of the structural equations for the normal moving frames correspond to (a,b)-curvature mappings. Hence the tuple of all (a,b)-curvature maps constitute a kind of complete system of symplectic invariants of the curve. For precise formulation of this statement it is convenient to use the notion of quivers and their representations ([13]). Recall that a quiver is an oriented graph, where loops and multiple arrows between two vertices are allowed. A representation of a quiver assigns a vector space X_α to each vertex α of the quiver and a linear mapping from X_α to X_β to each arrow of the quiver, connecting a vertex α with a vertex β.

Take the quiver $\mathfrak{Q}_D$ such that its vertices are levels of the diagram Δ and the set of arrows from the level Υ_i to the level Υ_j is parametrized by essential pairs $(a,b) \in \Upsilon_i \times \Upsilon_j$. A representation of the quiver $\mathfrak{Q}_D$ will be called *compatible with the Young diagram* D if for any $1 \le i \le d$ the space of the representation corresponding to the vertex Υ_i is a r_i-dimensional Euclidean space and the linear mappings $\mathcal{R}(a,b)$ of the representation corresponding to the arrows (a,b) satisfy the following relations: $\mathcal{R}(a,b)^* = \mathcal{R}(b,a)$ and $\mathcal{R}(a, r(a))$ are antisymmetric w.r.t. the corresponding Euclidean structure.

The subspaces $V_a(t)$ for any t and any $a \in \Upsilon_i$ are naturally identified together with the canonical Euclidean structure on them ($V_{a_1}(t_1) \sim V_{a_2}(t_2)$ by sending $E_{a_1}(t_1)$ to $E_{a_2}(t_2)$). Therefore, we can identify all these spaces with one Euclidean space, which will be denoted by $\mathcal{X}_i$. The tuple of spaces $\mathcal{X}_i$ and the (a,b)-curvature maps of the curve $\Lambda(t)$, considered as elements of $\mathrm{Hom}(\mathcal{X}_i, \mathcal{X}_j)$ for $(a,b) \in \Upsilon_i \times \Upsilon_j$, define the one-parametric family $\mathfrak{R}_t$ of compatible representations of the quiver $\mathfrak{Q}_D$. This family will be called the *quiver of curvatures of the curve* $\Lambda(t)$. Here the linear mappings corresponding to the arrows of the quiver depend on t, while the linear spaces, corresponding to its vertices, are independent of t. In the sequel we will consider only this type of one-parametric families of representations of quivers. Two families $\Xi_1(t)$ and $\Xi_2(t)$ of compatible representations of the quiver $\mathfrak{Q}_D$ are called isomorphic, if there exists a tuple of isometries (independent of t) between the corresponding spaces of the representations, conjugating all corresponding linear mappings. If the sizes of all superboxes in Δ are equal to 1, then the normal moving frames of the curve are defined up to the discrete group (U_i in (1.1.8) are scalars, which are equal to 1 or -1) and all (a,b)-curvature maps are determined by scalar functions of t, which are symplectic invariants of the curve. These scalar functions will be called, for short, *(a,b)-curvatures*. Besides, the compatible representations of the quiver $\mathfrak{Q}_D$ is in one-to-one correspondence with tuples of numbers parametrized by the essential pairs of Δ (which is equal to D in the considered case). The following theorem is the direct consequence of the structural equations for normal moving frames and Theorem 1.1:

Theorem 1.2. *For the given one-parametric family $\Xi(t)$ of representations of the quiver $\mathfrak{Q}_D$ compatible with the Young diagram D with $|D|$ boxes there exists the unique, up to a symplectic transformation, monotonically nondecreasing curve $\Lambda(t)$ in the Lagrange Grassmannian of $2|D|$-dimensional symplectic space with the Young diagram D such that the quiver of curvatures of $\Lambda(t)$ is isomorphic to $\Xi(t)$. If, in addition, all rows of D have different length, then given a tuple of smooth functions $\{\rho_{a,b}(t) : (a,b) \in \Delta \times \Delta, (a,b)$ is an essential pair$\}$ there exists the unique, up to a symplectic transformation, monotonically nondecreasing curve $\Lambda(t)$ in the Lagrange Grassmannian of $2|D|$-dimensional symplectic space with the Young diagram D such that for any essential pair $(a,b) \in \Delta \times \Delta$ and any t its (a,b)-curvature map at t coincides with $\rho_{a,b}(t)$.*

Finally note that rank 1 curves in Lagrange Grassmannians, considered in [23], have the Young diagrams, consisting of just one row, and the main results of the mentioned paper (Theorems 2 and 3 there) are very particular cases of Theorems 1.1 and 1.2 here. In this case the pair (a, b) of superboxes is essential if and only if $a = b$.

1.2 Proof of Theorem 1

The proof consists of several steps.

1.2.1 Contractions of the curve $\Lambda(\cdot)$

We start with some general constructions for curves in Grassmannians. Given a curve $\Lambda(\cdot)$ in the Grassmannian $G_k(W)$, for any τ we will construct a monotonic sequence of subspaces of $\Lambda(\tau)$ in addition to the extensions $\Lambda^{(i)}$. For this let $\Lambda_{(0)}(t) = \Lambda(t)$ and recursively

$$\Lambda_{(i)}(\tau) = \left\{ v \in \Lambda_{(i-1)}(\tau) : \begin{array}{c} \exists \ell \in \mathfrak{S}(\Lambda_{(i-1)}) \text{ with } \ell(\lambda) = v \\ \text{such that } \ell'(\tau) \in \Lambda_{(i-1)}(\tau) \end{array} \right\} \tag{1.2.1}$$

where, by analogy with above, $\mathfrak{S}(\Lambda_{(i)})$, $i \geq 0$, is the set of all smooth curves $\ell(t)$ in W such that $\ell(t) \in \Lambda_{(i-1)}(t)$ for any t. The subspaces $\Lambda_{(i)}(\tau)$ are called *the ith contraction* of the curve $\Lambda(\cdot)$ at the point τ. Under the identification $T_\Lambda G_k(W) \sim \text{Hom}\,(\Lambda, W/\Lambda)$ the first contraction $\Lambda_{(1)}(\tau)$ is exactly the kernel of the velocity $\dot{\Lambda}(\tau)$, $\Lambda_{(1)}(\tau) = \text{Ker}\,\dot{\Lambda}(\tau)$. Moreover, it implies that

$$\dim \Lambda^{(1)}(\tau) - \dim \Lambda(\tau) = \dim \Lambda(\tau) - \dim \Lambda_{(1)}(\tau). \tag{1.2.2}$$

Indeed, for the velocity $\dot{\Lambda}(\tau)$ we assign a unique $B_\Lambda(\tau) \in \text{Hom}\,(\Lambda, W/\Lambda)$ by $B(v) = -[\ell'(0)]$ for a curve $\ell(\cdot) : \ell(0) = v$. It is evident that $B(v)$ does not depend on the choice of the curve $\ell(\cdot)$. As a consequence, we see that the left-hand side of (1.2.2) is equal to $\dim(\text{Im}\, B_\Lambda(\tau))$, while the righthand side is equal to $\dim \Lambda(\tau) - \dim(\text{Ker}\, B_\Lambda(\tau))$.

Note also that in (1.2.1) one can replace the quantor $\exists$ by $\forall$, because the existence of a curve $\ell \in \mathfrak{S}(\Lambda_{(i-1)})$ with $\ell(\tau) = v$ and $\ell'(\tau) \in \Lambda_{(i-1)}(\tau)$ implies that any smooth curve $\tilde{\ell} \in \mathfrak{S}(\Lambda_{(i-1)})$ with $\tilde{\ell}(\tau) = v$ satisfies $\tilde{\ell}'(\tau) \in \Lambda_{(i-1)}(\tau)$. Note that the following relations follow directly from the definitions

$$\left(\Lambda_{(i)}(\tau)\right)_{(1)} = \Lambda_{(i+1)}(\tau), \quad \left(\Lambda_{(i)}(\tau)\right)^{(1)} \subseteq \Lambda_{(i-1)}(\tau) \tag{1.2.3}$$

If we suppose that $\Lambda(\cdot)$ is a curve in Lagrange Grassmannian of the symplectic space W, then the symplectic structure gives an additional relation between the ith extension and the ith contraction. Namely, given a subspace $L \subset W$ denote by $L^\angle$ its skew-symmetric complement, i.e. $L^\angle = \{v \in W : \omega(v, l) = 0\ \forall l \in L\}$.

Lemma 1.1. *The subspaces $\Lambda_{(i)}(\tau)$ is a skew-symmetric complement of the subspace $\Lambda^{(i)}(\tau)$ for any τ, namely*

$$\Lambda_{(i)}(\tau) = \left(\Lambda^{(i)}(\tau)\right)^\angle, \quad \forall \tau. \tag{1.2.4}$$

Proof. We proceed the proof by induction on i. For $i = 0$ there is nothing to prove, because $\Lambda(\tau)$ ($= \Lambda^0(\tau) = \Lambda_0(\tau)$ by definition) is a Lagrangian subspace. Assume that (1.2.4) is valid for $i = \bar{i} - 1$ and prove it for $i = \bar{i}$, $\bar{i} \geq 1$. Indeed, if $v \in \Lambda_{(\bar{i})}(\tau)$, then by definition there exists a regular curve of vectors $v(t)$ such that $v(t) \in \Lambda_{(\bar{i}-1)}(t)$ for any t close to τ, $v(\tau) = v$ and $v'(\tau) \in \Lambda_{(\bar{i}-1)}(\tau)$. Let us prove that $v \in \left(\Lambda^{(\bar{i})}(\tau)\right)^\angle$. For this take $v_1 \in \Lambda^{(\bar{i})}(\tau)$. Then by definition there exist a curve of vectors $w(t)$ in W such that $w(t) \in \Lambda^{(\bar{i}-1)}(t)$ for any t close to

τ and $w'(\tau) = v_1$. By induction hypothesis $\omega\big(v(t), w(t)\big) = 0$. Differentiating the last identity at $t = \tau$ we get

$$\omega(v, v_1) = -\omega\big(v'(\tau), w(\tau)\big) = 0. \tag{1.2.5}$$

(the last equality holds because of the relations $v'(\tau) \in \Lambda_{(\bar{i}-1)}(\tau)$, $w(\tau) \in \Lambda^{(\bar{i}-1)}(\tau)$ and the induction hypothesis). Since (1.2.5) holds for any $v_1 \in \Lambda^{(\bar{i})}(\tau)$, we get that $v \in \big(\Lambda^{(\bar{i})}(\tau)\big)^{\angle}$. So, we have proved that $\Lambda_{(i)}(\tau) \subset \Big(\Lambda^{(i)}(\tau)\Big)^{\angle}$.

Now let us prove the inclusion in the opposite direction. Suppose that $v \in \Big(\Lambda^{(\bar{i})}(\tau)\Big)^{\angle}$. Take any $w \in \Lambda^{(\bar{i}-1)}(\tau)$ and a curve of vectors $w(t)$ in W such that $w(t) \in \Lambda^{(\bar{i}-1)}(t)$ for any t close to τ and $w(\tau) = w$. Then by definition $w'(\tau) \in \Lambda^{(\bar{i})}(\tau)$ and by our assumptions

$$\omega\big(v, w'(\tau)\big) = 0. \tag{1.2.6}$$

On the other hand, since $\Lambda^{(\bar{i}-1)}(\tau) \subset \Lambda^{(\bar{i})}(\tau)$, then $\Big(\Lambda^{(\bar{i})}(\tau)\Big)^{\angle} \subset \Big(\Lambda^{(\bar{i}-1)}(\tau)\Big)^{\angle} = \Lambda_{(\bar{i}-1)}(\tau)$ (the last equality is our induction hypothesis). So, $v \in \Lambda_{(\bar{i}-1)}(\tau)$. Take a curve of vectors $v(t)$ in W such that $v(t) \in \Lambda_{(\bar{i}-1)}(t)$ for any t close to τ and $v(\tau) = v$. Then by induction hypothesis $\omega(v(t), w(t)) = 0$ for any t close to τ. Differentiating the last identity at $t = \tau$ and using (1.2.6) we get that $\omega\big(v'(\tau), w\big) = 0$. Since the last identity holds for any $w \in \Lambda^{(\bar{i}-1)}(\tau)$, then $v'(\tau) \in \Big(\Lambda^{(\bar{i}-1)}(\tau)\Big)^{\angle} = \Lambda_{(\bar{i}-1)}(\tau)$ (the last equality is our induction hypothesis). So, $v \in \Lambda_{(\bar{i})}(\tau)$, which implies the inclusion $\Big(\Lambda^{(\bar{i})}(\tau)\Big)^{\angle} \subset \Lambda_{(\bar{i})}(\tau)$. The proof of the lemma is completed. □

1.2.2 Filling the Young diagram D by bases of $\Lambda(t)$

As before, assume that the reduced diagram Δ of the curve consists of d level, the number of superboxes in the ith level of the diagram Δ is equal to p_i, and their sizes are equal to r_i. By our assumptions $\Lambda^{(p_1)}(t) = W$, which together with (1.2.4) implies that

$$\Lambda_{(p_1)}(t) = 0, \quad \dim \Lambda_{(p_1-1)}(t) = r_1. \tag{1.2.7}$$

Denote also by σ_i the special (i.e. the last) superbox of the ith level of Δ. ¿From the second relation of (1.2.3) it follows that

$$\Big(\Lambda_{(p_i)}\Big)^{(1)}(t) \subseteq \Lambda_{(p_i-1)}(t), \quad \forall 1 \leq i \leq q \tag{1.2.8}$$

For any $1 \leq i \leq d$ choose a complement $\widetilde{V}_{\sigma_i}(t)$ of the subspace $\Big(\Lambda_{(p_i)}\Big)^{(1)}(t)$ in the space $\Lambda_{(p_i-1)}(t)$ (smoothly w.r.t. t):

$$\Lambda_{(p_i-1)} = \Big(\Lambda_{(p_i)}\Big)^{(1)}(t) \oplus \widetilde{V}_{\sigma_i}(t). \tag{1.2.9}$$

Note that from (1.2.7) it follows that $\widetilde{V}_{\sigma_1}(t) = \Lambda_{(p_1-1)}(t)$. Let $\widetilde{\Delta}$ be the diagram, obtained from Δ by joining to Δ one more column from the left, having the same length as the first column of Δ. The boxes of $\widetilde{\Delta}$ will be called superboxes as well. For any $1 \leq i \leq d$ take a tuple of vectors $E_{\sigma_i}(t)$, constituting a basis of $\widetilde{V}_{\sigma_i}(t)$ (smoothly in t). Then to any superbox of $\widetilde{\Delta}$ we will assign a tuple of vectors in the following way

$$E_{l^j(\sigma_i)}(t) \stackrel{def}{=} E_{\sigma_i}^{(j)}(t), \quad \forall 0 \leq j \leq p_i, \tag{1.2.10}$$

where l is the left shift on the diagram $\widetilde{\Delta}$.

Lemma 1.2. *Assume that a superbox $a \in \widetilde{\Delta}$ lies in the $j(a)$th column and $i(a)$th level of the diagram $\widetilde{\Delta}$ and let Ov_a be the set of all superboxes, lying over a in the column of a. Then the following relations hold*

$$\begin{gathered}\{E_a(t)\}\bigcap\Big(\Big(\bigoplus_{b\in \mathrm{Ov}_a}\mathrm{span}\{E_b(t)\}\Big)\oplus\Lambda_{(j(a)-1)}(t)\Big)=0,\\ \dim\,\mathrm{span}\{E_a(t)\}=\dim\,\mathrm{span}\{E_{\sigma_{i(a)}}(t)\}=r_{i(a)}.\end{gathered}\tag{1.2.11}$$

Proof. Let $\prec$ be the order on the set of superboxes of the diagram $\widetilde{\Delta}$, defined as follows: $b_1\prec b_2$ if either b_1 is higher than b_2 in $\widetilde{\Delta}$ or they are on the same level, but b_1 is located from the right to b_2 (or, equivalently, either $i(b_1)<i(b_2)$ or $i(b_1)=i(b_2)$, but $j(b_1)>j(b_2)$). Let us prove (1.2.11) by induction on the set of superboxes of the diagram $\widetilde{\Delta}$ with the introduced order $\prec$. For $a=\sigma_1$ relations (1.2.11) follow immediately from (1.2.7). Now assume that (1.2.11) is true for any superbox $a\in\widetilde{\Delta}$ such that $a\prec\sigma$ and prove it for $a=\sigma$. We have the following two cases:

1. The superbox σ is special. In this case by induction hypothesis it is easy to show that

$$\Big(\bigoplus_{b\in \mathrm{Ov}_\sigma}\mathrm{span}\{E_b(t)\}\Big)\oplus\Lambda_{(p_{i(\sigma)})}(t)=\big(\Lambda_{(p_{i(\sigma)})}\big)^{(1)}(t)\tag{1.2.12}$$

This together with (1.2.9) and the definitions of the numbers r_i implies (1.2.11) for $a=\sigma$.

2. The superbox σ is not special. Using our induction assumptions we can choose a subspace $C(t)$ of $\Lambda_{(j(\sigma)-1)}(t)$ smoothly w.r.t. t such that

$$\Lambda_{(j(\sigma)-1)}(t)=\Big(\bigoplus_{b\in \mathrm{Ov}_{r(\sigma)}}\mathrm{span}\{E_b(t)\}\Big)\oplus\mathrm{span}\{E_{r(\sigma)}(t)\}\oplus\Lambda_{(j(\sigma))}(t)\oplus C(t),\tag{1.2.13}$$

where as before $r(\sigma)$ is the superbox, located from the right to σ in $\widetilde{\Delta}$.

¿From (1.2.2), the first relation of (1.2.3), and (1.2.13) it follows that

$$\begin{gathered}\dim(\Lambda_{(j(\sigma)-1)})^{(1)}(t)-\dim\Lambda_{(j(\sigma)-1)}(t)=\dim\Lambda_{(j(\sigma)-1)}(t)-\dim\Lambda_{(j(\sigma))}(t)=\\ \sum_{b\in \mathrm{Ov}_{r(\sigma)}\cup r(\sigma)}\dim\mathrm{span}\{E_b(t)\}+\dim C(t)=\sum_{k=1}^{i(\sigma)}r_k+\dim C(t).\end{gathered}\tag{1.2.14}$$

On the other hand, using (1.2.10), (1.2.13), one gets easily that

$$\begin{gathered}\dim(\Lambda_{(j(\sigma)-1)})^{(1)}(t)-\dim\Lambda_{(j(\sigma)-1)}(t)\leq\sum_{k=1}^{i(\sigma)-1}r_k+\big(\dim\mathrm{span}\{E_{r(\sigma)}(t),E_\sigma(t)\}-\\ \dim\mathrm{span}\{E_{r(\sigma)}(t)\}\big)+\big(\dim C^{(1)}(t)-\dim C(t)\big)\leq\sum_{k=1}^{i(\sigma)}r_k+\dim C(t).\end{gathered}\tag{1.2.15}$$

If for $a=\sigma$ one of the identities in (1.2.11) does not hold, then in the chain of the inequalities (1.2.15) there is at least one strict inequality, which is in the contradiction with (1.2.14). So, the identities (1.2.11) are valid for $a=\sigma$, which completes the proof of (1.2.11) by induction. □

Let $\mathcal{F}_k$ be the kth column of the diagram Δ. From Lemma 1.2 it follows easily the following

Corollary 1.1. *The following splittings hold for any $0 \le j \le p_1$*

$$\Lambda_{(j)}(t) = \bigoplus_{a \in \bigcup_{s=j+1}^{p_1} \mathcal{F}_s} \operatorname{span}\{E_a(t)\}, \quad \left(\Lambda_{(j)}\right)^{(1)}(t) = \bigoplus_{a \in \bigcup_{s=j+1}^{p_1} \mathcal{F}_s \cup l(F_{j+1})} \operatorname{span}\{E_a(t)\} \tag{1.2.16}$$

In particular, $\Lambda(t) = \bigoplus_{a \in \Delta} \operatorname{span}\{E_a(t)\}$.

One can imagine that we fill the diagram Δ (or the original diagram D) by columns $E_a(t)^T$ by choosing bases of the subspaces $\widetilde{V}_{\sigma_i}$, satisfying (1.2.9), and by differentiating these bases as in (1.2.10). Tuples $\{E_a(t)\}_{a \in \Delta}$, obtained in this way, will be called *fillings of the Young diagram D, associated with the curve* $\Lambda(\cdot)$. The flag $0 = \Lambda_{(p_1)}(t) \subset \Lambda_{(p_1-1)}(t) \ldots \subset \Lambda_{(0)}(t) = \Lambda(t)$ can be recovered from this filling by the first relation of (1.2.16). In particular, this flag (and therefore the curve $\Lambda(\cdot)$ itself) can be recovered from the curves $t \longrightarrow V_{\sigma_i}(t)$, $1 \le i \le d$ by taking the corresponding extensions of them.

1.2.3 The canonical complement of $\left(\Lambda_{(p_i)}\right)^{(1)}(t)$ in $\Lambda_{(p_i-1)}(t)$ and the canonical Euclidean structure on it

In the present subsection we will show that the complement $\widetilde{V}_{\sigma_i}$ (as in (1.2.9)) can be chosen canonically if the following condition holds.

Condition (G) *For any $1 \le i \le d-1$ and any t the rank of the restriction of the quadratic form $\dot{\Lambda}(t)$ to the subspace $\left(\Lambda_{(p_i-1)}\right)^{(p_i-1)}(t)$ is equal to* $\sum_{k=1}^{i} r_k$,

$$\forall 1 \le i \le d-1 \text{ and } \forall t: \quad \operatorname{rank}\left(\dot{\Lambda}(t)|_{(\Lambda_{(p_i-1)})^{(p_i-1)}(t)}\right) = \sum_{k=1}^{i} r_k. \tag{1.2.17}$$

Since $\operatorname{Ker}\dot{\Lambda}(t) = \Lambda_{(1)}(t)$ and $(\Lambda_{(p_i-1)})^{(p_i-2)}(t) \subset \Lambda_{(1)}(t)$ (as a consequence of (1.2.3)), any curve $\Lambda(t)$ with the Young diagram D satisfies: $\operatorname{rank}\left(\dot{\Lambda}(t)|_{(\Lambda_{(p_i-1)})^{(p_i-1)}(t)}\right) \le \sum_{k=1}^{i} r_k$ for any $1 \le i \le d$. It implies easily that germs of curves, satisfying condition (G), are generic among all germs of curves with given Young diagram D. Besides, it is clear that curves with rectangular Young diagram satisfy condition (G) automatically (condition (G) is void in this case).

Lemma 1.3. *Any monotonic curve $\Lambda(t)$ with the Young diagram D satisfies condition (G).*

Proof. For definiteness, let the curve $\Lambda(t)$ be monotonically nondecreasing. Take a filling $\{E_a(t)\}_{a \in \Delta}$ of the Young diagram D, associated with the curve $\Lambda(\cdot)$. Let

$$Z_i(t) = \operatorname{span}\{E_{\sigma_k}^{(p_k-1)}(t)\}_{k=1}^{i}, \quad 1 \le i \le q. \tag{1.2.18}$$

It is clear that $\{Z_i(t)\}_{i=1}^{d}$ is a monotonically increasing (by inclusion) sequence of subspaces for any t. As a consequence of Lemma 1.2, we have

$$\dim Z_i(t) = \sum_{k=1}^{i} r_k, \tag{1.2.19}$$

$$(\Lambda_{(p_i-1)})^{(p_i-1)}(t) = \left((\Lambda_{(p_i-1)})^{(p_i-1)}(t) \cap \Lambda_{(1)}(t)\right) \oplus Z_i(t) \tag{1.2.20}$$

Since $\operatorname{Ker}\dot\Lambda(t) = \Lambda_{(1)}(t)$, we get from (1.2.20) that

$$\operatorname{rank}\left(\dot\Lambda(t)|_{(\Lambda_{(p_i-1)})^{(p_i-1)}(t)}\right) = \operatorname{rank}\left(\dot\Lambda(t)|_{Z_i(t)}\right). \tag{1.2.21}$$

Besides, from monotonicity the quadratic form $\dot\Lambda(t)|_{Z_d(t)}$ is positive definite. Hence, the quadratic forms $\dot\Lambda(t)|_{Z_i(t)}$ are positive definite as well. Then the lemma follows form relations (1.2.19) and (1.2.21). □

Now define the following subspaces of the ambient symplectic space W:

$$W_i(t) = (\Lambda_{(p_1-1)}(t))^{(2p_1-1)} + (\Lambda_{(p_2-1)}(t))^{(2p_2-1)} + \ldots + (\Lambda_{(p_i-1)}(t))^{(2p_i-1)}. \tag{1.2.22}$$

Lemma 1.4. *If a curve $\Lambda(t)$ with the Young diagram D satisfies condition (G), then for any $1 \leq i \leq d$ the restriction of the symplectic form ω to the subspace $W_i(t)$ is nondegenerated and $dim W_i = 2\sum_{k=1}^{i} p_k r_k$.*

Proof. The proof of the lemma is by induction w.r.t. i. First let us introduce some notations. Let $\overline{\Delta}$ be the diagram obtained from Δ by the reflection w.r.t. its left edge. We will work with the diagram $\Delta \cup \overline{\Delta}$, which is symmetric w.r.t. the left edge of the diagram Δ. Similar to above, we will denote by l the left shift on the diagram $\Delta \cup \overline{\Delta}$. If S is a subset of the diagram Δ, we will denote by $\bar{S}$ the subset of $\overline{\Delta}$, obtained by the reflection of S w.r.t. the left edge of Δ. Also in the sequel, given two tuples of vectors $V_1 = (v_{11}, \ldots, v_{1n_1})$ and $V_2 = (v_{21}, \ldots, v_{2n_2})$ by $\omega(V_1, V_2)$ we will mean the $n_1 \times n_2$-matrix with the (i,j)-entry equal to $\omega(v_{1i}, v_{2j})$. Take a filling $\{E_a(t)\}_{a\in\Delta}$ of the Young diagram D, associated with the curve $\Lambda(\cdot)$. Define tuples E_a also for $a \in \bar{\Delta}$ in the following way: $E_{l^j(a_i)} = E_{a_i}^{(j)}(t)$, $1 \leq j \leq p_i$, where, as before, a_i is the first superbox in the ith level Υ_i of Δ. By definition $W_i(t) = \operatorname{span}\{E_a(t)\}_{a\in\cup_{k=1}^{i}\Upsilon_k\cup\bar\Upsilon_k}$.

1. Let us prove the lemma for $i = 1$. By condition (G) the matrix $\omega\big(E_{\sigma_1}^{(p_1-1)}(t), E_{\sigma_1}^{(p_1)}(t)\big)$ is nonsingular. On the other hand, since $\Lambda_{(1)}(t) = \left(\Lambda^{(1)}(t)\right)^{\angle}$, one has $\omega\big(E_{\sigma_1}^{(p_1)}(t), E_{\sigma_1}^{(p_1-2)}(t)\big) \equiv 0$. Differentiating the last identity, we get $\omega\big(E_{\sigma_1}^{(p_1+1)}(t), E_{\sigma_1}^{(p_1-2)}(t)\big) = -\omega\big(E_{\sigma_1}^{(p_1)}(t), E_{\sigma_1}^{(p_1-1)}(t)\big)$. In the same way, using (1.2.4), it is easy to obtain that

$$\omega\big(E_{\sigma_1}^{(p_1+i)}(t), E_{\sigma_1}^{(p_1-i-1)}(t)\big) = (-1)^i\omega\big(E_{\sigma_1}^{(p_1)}(t), E_{\sigma_1}^{(p_1-1)}(t)\big).$$

In particular, all matrices $\omega\big(E_{\sigma_1}^{(p_1+i)}(t), E_{\sigma_1}^{(p_1-i-1)}(t)\big)$ are nonsingular. Therefore the matrix with the entries, which are equal to the value of the form ω on all pairs of vectors from the tuple $\{E_a(t)\}_{a\in\Upsilon_1\cup\bar\Upsilon_1}$, is block-triangular w.r.t. the nonprincipal diagonal with nonsingular blocks on the nonprincipal diagonal. This implies that the tuple $\{E_a(t)\}_{a\in\Upsilon_1\cup\bar\Upsilon_1}$ constitutes the basis of W_1 and the form $\omega|_{W_1}$ is nondegenerated, which completes the proof of the statement of the lemma in the case $i = 1$.

2. Now assume that the statement of the lemma holds for $i = i_0 - 1$ and prove it for $i = i_0$. Let Δ_i be the subdiagram of Δ, consisting of the first i rows of Δ, $\Delta_i = \bigcup_{k=1}^{i}\Upsilon_k$. Divide the diagram $\Delta_{i_0} \cup \bar\Delta_{i_0}$ on four parts $\{A_k\}_{k=1}^4$: A_1 is a union of the last $p_1 - p_{i_0}$ columns of the diagram Δ_{i_0}, A_2 is obtained by the reflection of A_1 w.r.t. the left edge of Δ_{i_0}, i.e. $A_2 = \bar{A}_1$, $A_3 = \Delta_{i_0-1}\backslash(A_1 \cup A_2)$, and $A_4 = \Upsilon_{i_0}$.

Set $C_k(t) = \operatorname{span}\{E_a(t)\}_{a\in A_k}$, $k = 1, \ldots, 4$. Note that from (1.2.16) it follows that $C_1(t) = \Lambda_{(p_{i_0})}(t)$. By constructions $W_{i_0}(t) = C_1(t) + C_2(t) + C_3(t) + C_4(t)$ and $W_{i_0-1} = C_1(t) + C_2(t) +$

$C_3(t)$. Moreover, by comparison of dimensions,

$$W_{i_0-1}(t) = C_1(t) \oplus C_2(t) \oplus C_3(t) \tag{1.2.23}$$
$$C_1(t)^{\angle} \cap W_{i_0-1}(t) = C_1(t) \oplus C_3(t). \tag{1.2.24}$$

Besides, using (1.2.4), one has also that

$$C_1 + C_3 + C_4 \subset C_1(t)^{\angle}. \tag{1.2.25}$$

Assume that $x \in \operatorname{Ker} \omega|_{W_{i_0}(t)}$, $x = \sum_{k=1}^{4} x_k$, where $x_k \in C_k(t)$. Then (1.2.25) implies that $\omega(v, x) = \omega(v, x_2) = 0$ for any $v \in C_1(t)$. This together with (1.2.23) and (1.2.24) yields that $x_2 = 0$.

Further, by the same arguments as in the proof of the case $i = 1$, applied for the tuple $\{E_a\}_{a \in \mathcal{F}_{p_{i_0}} \cap \Delta_{i_0}}$ instead of the tuple E_{σ_1}, one obtains from (1.2.17) for $i = i_0$ that $\omega|_{C_3(t)+C_4(t)}$ is nondegenerated and $\dim(C_3(t)+C_4(t)) = 2p_{i_0}\sum_{k=1}^{i_0} r_k$. The latter implies that $C_3(t) \cap C_4(t) = 0$. Besides, from (1.2.25) it follows that $\omega(v, x) = \omega(v, x_3 + x_4) = 0$ for any $v \in C_3(t) + C_4(t)$, which together with two previous sentences implies that $x_3 = x_4 = 0$. Therefore $x \in C_1(t) \subset W_{i_0-1}(t)$, which implies that $x = x_1 = 0$ by induction hypothesis. This yields that the form $\omega|_{W_{i_0}(t)}$ is nondegenerated. Moreover, from the same arguments it follows that the condition $\sum_{k=1}^{4} x_k = 0$ implies that $x_k = 0$ for any $1 \le k \le 4$. Hence $W_{i_0}(t) = C_1(t) \oplus C_2(t) \oplus C_3(t) \oplus C_4(t)$ and the statement of the lemma about the dimension of $W_{i_0}(t)$ holds. The proof of the lemma is completed. □

Finally, let

$$V_i(t) = \Lambda_{(p_i-1)}(t) \cap W_{i-1}(t)^{\angle} \tag{1.2.26}$$

As a direct consequence of Lemma 1.4, we get that the subspace $V_i(t)$ is complementary to $\left(\Lambda_{(p_i)}\right)^{(1)}(t)$ in $\Lambda_{(p_i-1)}(t)$,

$$\Lambda_{(p_i-1)}(t) = \left(\Lambda_{(p_i)}\right)^{(1)}(t) \oplus V_i(t). \tag{1.2.27}$$

The subspaces $V_i(t)$, defined by (1.2.26) will be called the *canonical complement of* $\left(\Lambda_{(p_i)}\right)^{(1)}(t)$ *in* $\Lambda_{(p_i-1)}(t)$. The following equivalent description of the subspaces $V_i(t)$ will be very useful in the sequel:

Lemma 1.5. *A sequence of subspaces* $\{\widetilde{V}_{\sigma_i}(t)\}_{i=1}^{d}$, *satisfying* (1.2.9), *consists of the canonical complements of* $\left(\Lambda_{(p_i)}\right)^{(1)}(t)$ *in* $\Lambda_{(p_i-1)}(t)$ *for any* $1 \le i \le d$ *if and only if smooth (w.r.t. t) tuples of vectors* $E_{\sigma_i}(t)$, *constituting bases of* $\widetilde{V}_{\sigma_i}(t)$, *satisfy:*

$$\forall 1 \le j < i \le d \text{ and } \forall 1 \le k \le p_j - p_i + 1: \quad \omega\big(E_{\sigma_i}^{(p_i-1)}(t), E_{\sigma_j}^{(p_j-1+k)}(t)\big) = 0 \tag{1.2.28}$$

or, equivalently, taking into account notations in (1.2.10),

$$\forall 1 \le j < i \le d \text{ and } \forall 1 \le k \le p_j - p_i + 1: \quad \omega\big(E_{a_i}(t), E_{a_j}^{(k)}(t)\big) = 0. \tag{1.2.29}$$

The lemma can be easily proved by rewriting identity (1.2.26) in terms of bases $E_{\sigma_i}(t)$ and appropriate differentiations.

Further, it turns out that on each canonical complement $V_i(t)$ one can define the canonical quadratic form. Indeed, given a vector $v \in V_i(t)$ take a smooth curve $\varepsilon(t)$ in W such that

1. $\varepsilon(\tau) = v$;

2. $\varepsilon(t) \in V_i(t)$ for any t close to τ.

Then by our constructions it is easy to see that for any $0 \le j \le p_i - 1$

$$\varepsilon^{(j)}(\tau) \in \Lambda_{(p_i-1-j)}(\tau), \tag{1.2.30a}$$

$$\varepsilon^{(j+1)}(\tau) \notin \Lambda_{(p_i-1-j)}(\tau), \quad \text{if } v \neq 0, \tag{1.2.30b}$$

$$\varepsilon^{(j+1)}(\tau) \in \Lambda_{(p_i-1-j)}(\tau), \quad \text{if } v = 0 \tag{1.2.30c}$$

For this take a basis $E_{\sigma_i(t)}$ of $V_i(t)$, depending smoothly on t, expand our curve $\varepsilon(t)$ w.r.t. this basis, and use the fact that for any $0 \le j \le p_i - 1$

$$\bigoplus_{s=0}^{j} \text{span}\,\{E_{\sigma_i}^{(s)}(t)\} \subset \Lambda_{(p_i-1-j)}(\tau), \quad \text{span}\,\{E_{\sigma_i}^{(j+1)}(t)\} \cap \Lambda_{(p_i-1-j)}(\tau) = 0, \tag{1.2.31}$$

which is a direct consequence of Lemma 1.2. From (1.2.30a), (1.2.30c), the fact that $\Lambda(t)$ is the curve of Lagrangian subspaces,

$$Q_{i,\tau}(v) = \omega\big(\varepsilon^{(p_i-1)}(\tau), \varepsilon^{(p_i)}(\tau)\big) \tag{1.2.32}$$

is a well defined quadratic form on $V_i(\tau)$, which does not depend on the choice of the curve $\varepsilon(\tau)$ satisfying conditions (1) and (2) above. The form $Q_{i,\tau}(v)$ will be called the *canonical quadratic form on* $V_i(\tau)$. Moreover, the quadratic forms $Q_{i,\tau}(v)$ are nondegenerated for any $1 \le i \le d$. Indeed, if tuples $E_{\sigma_i(t)}$ constitute bases of $V_i(t)$ for any $1 \le i \le d$ and $Z_d(t)$ is as in (1.2.18), then from Lemma 1.5 it follows that the matrix of the quadratic form $\dot\Lambda(\tau)|_{Z_d(\tau)}$ in the basis $\{E_{\sigma_k}^{(p_k-1)}(\tau)\}_{k=1}^{d}$ is block-diagonal and the diagonal blocks are exactly the matrices of the forms $Q_{i,\tau}(v)$ in the bases $E_{\sigma_i(t)}$. Then the nondegenericity of the form $Q_{i,\tau}(v)$ follows from condition (G) and (1.2.21). Moreover, if the curve $\Lambda(t)$ is monotonically nondecreasing, then the forms $Q_{i,\tau}$ are positive definite. In this case the Euclidean structure on $V_{\sigma_i}(\tau)$, corresponding to the form $Q_{i,\tau}$ will be called the *canonical Euclidean structure on* $V_i(\tau)$.

From now on for simplicity of presentation we will assume that the curve $\Lambda(t)$ is monotonically nondecreasing. All necessary changes in the formulation of the results for nonmonotonic curves, satisfying condition (G), will be indicated in Section 1.3. For any $1 \le i \le d$, let $\mathfrak{B}_i$ be a fiber bundle over the curve $\Lambda(t)$ such that the fiber of $\mathfrak{B}_i$ over the point $\Lambda(t)$ consists of all orthonormal bases of the space $V_i(t)$ w.r.t. the canonical Euclidean structure on $V_i(t)$. Note that $\mathfrak{B}_i$ is the principle bundle with the structure group $O(r_i)$.

1.2.4 The canonical connections on the bundles $\mathfrak{B}_i$.

Now let us prove the following

Proposition 1.1. *Each bundle* $\mathfrak{B}_i$ *is endowed with the* canonical principal connection *uniquely characterized by the following condition: the section* $E_{\sigma_i}(t)$ *of* $\mathfrak{B}_i$ *is horizontal w.r.t. this connection if and only if* span$\{E_{\sigma_i}^{(p_i)}(t)\}$ *are isotropic subspaces of* W *for any* t. *Given any two horizontal sections* $E_{\sigma_i}(t)$ *and* $\widetilde{E}_{\sigma_i}(t)$ *of* $\mathfrak{B}_i$ *there exists a constant orthogonal matrix* U_i *such that*

$$\widetilde{E}_{\sigma_i}(t) = E_{\sigma_i}(t)U_i. \tag{1.2.33}$$

Proof. As in the proof of Lemma 1.4, given two tuples of vectors $V_1 = (v_{11}, \dots, v_{1n_1})$ and $V_2 = (v_{21}, \dots, v_{2n_2})$ by $\omega(V_1, V_2)$ we will mean the $n_1 \times n_2$-matrix with the (i,j)-entry equal to $\omega(v_{1i}, v_{2j})$. With this notation, it is obvious that if $V_i = \operatorname{span}\{\widetilde{E}_{\sigma_i}\}$, then the subspace $\operatorname{span}\{\widetilde{E}_{\sigma_i}^{(p_i)}(t)\}$ is isotropic if and only if

$$\omega\big(\widetilde{E}_{\sigma_i}^{(p_i)}(t), \widetilde{E}_{\sigma_i}^{(p_i)}(t)\big) = 0. \tag{1.2.34}$$

Note also that from definition of the canonical Euclidean structure it follows immediately that for any section $E_{\sigma_i}(t)$ of the bundle $\mathfrak{B}_i$ the following identity holds

$$\omega\big(E_{\sigma_i}^{(p_i-1)}(t), E_{\sigma_i}^{(p_i)}(t)\big) = \mathrm{Id}. \tag{1.2.35}$$

Take any two section $E_{\sigma_i}(t)$ and $\widetilde{E}_{\sigma_i}(t)$ of the bundle $\mathfrak{B}_i$. Then there exists a curve $U_i(t)$ of orthonormal matrices such that $\widetilde{E}_{\sigma_i}(t) = E_{\sigma_i}(t)U_i(t)$. Using relation $\Lambda_{(1)}(t) = \big(\Lambda^{(1)}(t)\big)^{\angle}$ and formula (1.2.35), it is easy to get that

$$\omega\big(\widetilde{E}_{\sigma_i}^{(p_i)}(t), \widetilde{E}_{\sigma_i}^{(p_i)}(t)\big) = U(t)^T\left(-2p_iU'(t) + \omega\big(E_{\sigma_i}^{(p_i)}(t), E_{\sigma_i}^{(p_i)}(t)\big)U(t)\right).$$

So, relation (1.2.34) holds if and only the matrix $U(t)$ satisfies the following differential equation

$$-2p_iU'(t) + \omega\big(E_{\sigma_i}^{(p_i)}(t), E_{\sigma_i}^{(p_i)}(t)\big)U(t) = 0. \tag{1.2.36}$$

Note that the matrix $\omega\big(E_{\sigma_i}^{(p_i)}(t), E_{\sigma_i}^{(p_i)}(t)\big)$ is antisymmetric. So, equation (1.2.36) has solutions in $O(r_i)$, which are defined up to the right translation there. This completes the proof of the proposition. □

Now, if for any $1 \le i \le d$ we take a horizontal section $E_{\sigma_i}(t)$ of the bundle $\mathfrak{B}_i$ and set , as before, $E_{l^j(\sigma_i)}(t) = E_{\sigma_i}^{(j)}(t)$ for $0 \le j \le p_i - 1$, then from (1.2.33) it follows that for any superbox a the subspaces $V_a(t) = \operatorname{span}\{E_a(t)\}$ do not depend on the choice of a horizontal sections $E_{\sigma_i}(t)$. Moreover, from this and Lemma 1.2 we get the *canonical splitting* $\Lambda(t) = \bigoplus_{a\in\Delta} V_a(t)$ *of the subspaces* $\Lambda(t)$.

1.2.5 The completion of horizontal sections to quasi-normal moving frames.

In the sequel it will be more convenient to use the following obviously equivalent description of quasi-normal mappings:

Lemma 1.6. *A symmetric compatible mapping* $R : \Delta \times \Delta \longrightarrow \mathrm{Mat}$ *is quasi-normal if and only if the following four conditions hold:*

1. *If a and b are two consecutive superboxes in the same level of Δ, then the matrix $R(a,b)$ is antisymmetric;*

2. *If both superboxes a and b are not special and do not lie in the same or adjacent columns, then $R(a,b) = 0$;*

3. *If both superboxes a and b are not special, lie in the adjacent (but not the same) columns and one of the superboxes is located from below and from the left w.r.t. the other, then $R(a,b) = 0$;*

4. *If a superbox a is special, a superbox b is not special and b is located from the left to a, but not in the adjacent column, then $R(a,b) = 0$.*

Further, for all $1 \le i \le d$, fix a horizontal section $E_{\sigma_i}(t)$ of the bundle $\mathfrak{B}_i$ and complete it to the moving basis $\{E_a(t)\}_{a\in\Delta}$ of $\Lambda(t)$ by setting, as before, $E_{l^j(\sigma_i)}(t) = E^{(j)}_{\sigma_i}(t)$ for $0 \le j \le p_i - 1$. Also let

$$F_{a_i}(t) = E'_{a_i}(t). \tag{1.2.37}$$

¿From the definition of the canonical Euclidean structure it follows that $\omega(E_{a_i}(t), F_{a_i}(t)) = Id$. From the normalization conditions (1.2.29) with $k = 1$ it follows that $\omega(F_{a_i}(t), E_{a_j}(t)) = 0$ for any $i \ne j$. Further, by definition of the horizontal section of the bundle $\mathfrak{B}_i$ one has $\omega(F_{a_i}(t), F_{a_i}(t)) = 0$. Finally, from the normalization conditions (1.2.29) with $k = 2$ it follows that $\omega(F_{a_i}(t), F_{a_j}(t)) = 0$ for $i \ne j$ as well. Combining all these identities with the fact that the subspaces $\Lambda(t)$ are Lagrangian and the relation $\Lambda_{(1)}(t) = \big(\Lambda^{(1)}(t)\big)^{\angle}$, we get that the tuple $\big(\{E_a\}_{a\in\Delta}, \{F_b(t)\}_{b\in\mathcal{F}_1}\big)$, where, as before, $\mathcal{F}_1$ denotes the first column of Δ, does not contradict the relations for a Darboux frame. Besides, by our constructions it satisfies first two equations of (1.1.7). In this subsection we prove the following

Proposition 1.2. *The tuple $\big(\{E_a\}_{a\in\Delta}, \{F_b(t)\}_{b\in\mathcal{F}_1}\big)$ can be uniquely completed to a quasi-normal moving frame of the curve $\Lambda(t)$.*

Proof. Take a tuple $\{F_b(t)\}_{b\in\Delta\setminus\mathcal{F}_1}$, which completes the tuple $\big(\{E_a\}_{a\in\Delta}, \{F_b(t)\}_{b\in\mathcal{F}_1}\big)$ to a moving Darboux frame in W. Then from the definition of Darboux frame and the first two equations of (1.1.7) it follows that this moving Darboux frame have the structural equation (1.1.7) for some symmetric mappings $R_t : \Delta \times \Delta \longrightarrow \mathrm{Mat}$ compatible with the Young diagram D. As before, denote by $\mathcal{F}_j$ the jth column of Δ, $1 \le j \le p_1$. Our proposition will follow from the following

Statement 1. *For any $1 \le k \le p_1$ there exists a unique tuple of columns of vectors*

$$\{F_b(t) : b \in \bigcup_{j=1}^{k} \mathcal{F}_j\}$$

such that the tuple $\big(\{E_a\}_{a\in\Delta}, \{F_b(t) : b \in \bigcup_{j=1}^{k} \mathcal{F}_j\}\big)$ can be completed to a moving Darboux frame

$$\big(\{E_a\}_{a\in\Delta}, \{F_b(t)\}_{b\in\Delta}\big)$$

such that if the mapping $R_t : \Delta \times \Delta \longrightarrow \mathrm{Mat}$ appears in the structural equation (1.1.7) for this moving frame, then the mapping R_t satisfies conditions (1)-(4) of Lemma 1.6 for any pair (a, b) with at least one superbox belonging to the first $(k-1)$ columns of Δ.

Indeed, our proposition is just Statement 1 in the case $k = p_1$ (the only pair of superboxes, which is not covered by Statement 1, is (σ_1, σ_1), where, as before, σ_1 is the special (the last) superbox of the first level, but this pair does not satisfy any of conditions (1)-(4) of Lemma 1.6).

We will prove Statement 1 by induction w.r.t. k. For $k = 1$ there is nothing to prove, because the tuple $\{F_c\}_{c\in\mathcal{F}_1}$ is uniquely determined by the second line of (1.1.7) (which together with the first line of (1.1.7) is equivalent to (1.2.37)), while the Statement 1 for $k = 1$ does not impose any conditions on the symmetric compatible mapping R_t, appearing in (1.1.7).

Now suppose that Statement 1 is proved for some $k = \bar{k}$, where $1 \le \bar{k} \le p_1 - 1$, and prove it for $k = \bar{k} + 1$. Let $\{F_b(t) : b \in \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j\}$ be the tuple, satisfying Statement 1 for $k = \bar{k}$. Take a tuple $\{F_b(t) : b \in \Delta \setminus \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j\}$, which completes the tuple $\left(\{E_a\}_{a\in\Delta}, \{F_b(t) : b \in \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j\}\right)$ to a moving Darboux frame in W and assume that $R_t : \Delta \times \Delta \longrightarrow \mathrm{Mat}$ is the mapping, appearing in the structural equation for this frame. If $\{\widehat{F}_b(t) : b \in \Delta \setminus \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j\}$ is another tuple, completing the tuple $\left(\{E_a\}_{a\in\Delta}, \{F_b(t)\}_{b\in\mathcal{F}_1}\right)$ to a moving Darboux's frame in W, then there exists a symmetric mapping $\Gamma_t : (\Delta \setminus \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j) \times (\Delta \setminus \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j) \longrightarrow \mathrm{Mat}$, compatible with the diagram, obtained from D by erasing the first $\bar{k}$ column, such that

$$\forall a \in \Delta \setminus \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j \quad F_a(t) = \widehat{F}_a(t) + \sum_{b\in\Delta\setminus \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j} E_b(t)\Gamma_t(a,b). \tag{1.2.38}$$

Suppose that $\widehat{R}_t : \Delta \times \Delta \longrightarrow \mathrm{Mat}$ is the symmetric mapping compatible with the Young diagram D such that similarly to last two equations of (1.1.7) one has

$$\begin{cases} F_a'(t) = -\sum_{b\in\Delta} E_b R_t(a,b) - F_{r(a)} & \text{if } a \in \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j \\ \widehat{F}_a'(t) = -\sum_{b\in\Delta} E_b \widehat{R}_t(a,b) - \widehat{F}_{r(a)} & \text{if } \ a \in \Delta \setminus (\bigcup_{j=1}^{\bar{k}} \mathcal{F}_j \cup \mathcal{S}) \\ \widehat{F}_a'(t) = -\sum_{b\in\Delta} E_b R_t(a,b) & \text{if } \ a \in \mathcal{S}, \end{cases} \tag{1.2.39}$$

(note that from the first line of (1.2.39), one has $\widehat{R}_t(a,b) = R_t(a,b)$, if at least one of the superboxes (a,b) belongs to the first $\bar{k}$ columns of Δ). Let us extend the mappings $\Gamma_t : (\Delta \setminus \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j) \times (\Delta \setminus \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j) \longrightarrow \mathrm{Mat}$ to the symmetric mapping, still denoted by Γ_t, from $\Delta \times \Delta$ to Mat compatible with the diagram D, by setting

$$\Gamma_t(a,b) = \Gamma_t(b,a)^T = 0, \quad \forall b \in \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j, a \in \Delta. \tag{1.2.40}$$

Then, substituting (1.2.38) into two last lines of (1.2.39) and using (1.1.7), one can easily obtain

$$\widehat{R}_t(a,b) = R_t(a,b) + \frac{d}{dt}\Gamma_t(a,b) + \Gamma_t\big(a, r(b)\big) + \Gamma_t\big(r(a), b\big), \tag{1.2.41}$$

where the term $\Gamma_t(a, r(b))$ is omitted, if b is special, and the term $\Gamma_t(r(a), b)$ is omitted, if a is special. Using transformation rule (1.2.41), we will prove the following

Statement 2. *There exists the unique choice of matrices $\Gamma_t(\tilde{a}, \tilde{b})$ with at least one of the superboxes belonging to the $(\bar{k}+1)$th column of Δ and the other one lying from the right to the $\bar{k}$th column of Δ such that the matrix $\widehat{R}_t(a,b)$ satisfies all conditions (1)-(4) of Lemma 1.6 for any pairs (a,b) with at least one of the superboxes belonging to the $\bar{k}$th column of Δ and the other one lies from the right to the $(\bar{k}-1)$th column of Δ*

It is clear that Statement 2, relation (1.2.38), and the induction hypothesis will imply Statement 1 for $k = \bar{k}+1$. Let us prove statement 2. Suppose that $a \in \mathcal{F}_{\bar{k}}$. Then from (1.2.40) it follows that $\frac{d}{dt}\Gamma_t(a,b) = 0$ and $\Gamma_t\big(a, r(b)\big) = 0$. So, relations (1.2.41) in this case have a form

$$\widehat{R}_t(a,b) = R_t(a,b) + \Gamma_t\big(r(a), b\big), \tag{1.2.42}$$

where the term $\Gamma_t(r(a), b)$ is omitted, if a is special. Therefore, according to (1.2.42), if a is special or $b \in \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j$ we have $\widehat{R}_t(a,b) = R_t(a,b)$, i.e. the matrix $R_t(a,b)$ is already independent of the choice of the complement of

$$(\{E_{\tilde{a}}\}_{\tilde{a}\in\Delta}, \{F_{\tilde{b}}(t) : \tilde{b} \in \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j\})$$

to a moving Darboux frame.

Now assume that a is not special and $b \notin \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j$. Then there are the following three cases:

a) $b \notin \bigcup_{j=1}^{\bar{k}+1} \mathcal{F}_j$, i.e. b is not in the first $\bar{k}+1$ columns of Δ. Then the matrix $\Gamma_t(r(a), b)$ appears only once in all relations,

$$\widehat{R}_t(\tilde{a}, \tilde{b}) = R_t(\tilde{a}, \tilde{b}) + \Gamma_t(r(\tilde{a}), \tilde{b}), \tag{1.2.43}$$

where $\tilde{a}$ runs over the whole $\bar{k}$th column $\mathcal{F}_{\bar{k}}$ of Δ. Putting

$$\Gamma_t(r(a), b) = -R_t(a,b), \tag{1.2.44}$$

we get $\widehat{R}_t(a,b) = 0$ for any $a \in \mathcal{F}_{\bar{k}}$, which corresponds to conditions (2) and (4) of Lemma 1.6, if b is not from the left to a. Obviously, the choice of $\Gamma_t(r(a), b)$ as in (1.2.44) is the unique one with these properties.

b) $b \in \mathcal{F}_{\bar{k}+1}$, but $b \neq r(a)$, i.e. b lies in the $(\bar{k}+1)$th column of Δ, but it is not in the same row with a. Let $a_1 = l(b)$. Then from the symmetricity of the mapping Γ_t (i.e. the relation $\Gamma_t(a,b) = (\Gamma_t(b,a))^T$) it follows that the matrix $\Gamma_t(r(a_1), r(a))$ appears twice in all relations (1.2.43), where $\tilde{a}$ runs over the $\bar{k}$th column $\mathcal{F}_{\bar{k}}$ of Δ and $\tilde{b}$ runs over the $(\bar{k}+1)$th column $\mathcal{F}_{\bar{k}+1}$ of Δ. Namely, substituting $(\tilde{a}, \tilde{b}) = (r(a), a_1)$ into (1.2.43) and using the symmetricity of the mapping Γ_t we will get the following relation in addition to (1.2.42) (with $b = r(a_1)$):

$$\widehat{R}_t(a_1, r(a)) = R_t(a_1, r(a)) + \Gamma_t(r(a), r(a_1))^T. \tag{1.2.45}$$

Hence, from symmetricity again we have

$$\widehat{R}_t(a, r(a_1)) - \widehat{R}_t(r(a), a_1) = R_t(a, r(a_1)) - R_t(r(a), a_1),$$

i.e. the matrix $R_t(a, r(a_1)) - R_t(r(a), a_1)$ does not depend on the choice of the complement of

$$(\{E_{\tilde{a}}\}_{\tilde{a}\in\Delta}, \{F_{\tilde{b}}(t) : \tilde{b} \in \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j\})$$

to a moving Darboux frame. Besides, for any pair of superboxes (a, a_1), $a \neq a_1$ in the $\bar{k}$th column $\mathcal{F}_{\bar{k}}$ by an appropriate choice of $\Gamma_t(r(a), r(a_1))$ we cannot "kill" both matrices $R_t(r(a), a_1)$ and $R_t(a, r(a_1))$, but only one of them. We choose the following normalization: $\widehat{R}(a, r(a_1)) = 0$, if a_1 is higher than a. We can do it by putting $\Gamma_t(r(a), r(a_1))) = -R_t(a, r(a_1))$. This normalization corresponds to conditions (3) of Lemma 1.6. Obviously, such choice of $\Gamma_t(r(a), r(a_1))$ is the unique one with these properties.

c) $b = r(a)$. Then the matrix $\Gamma_t\big(r(a), r(a)\big)$ appears only once in all relations (1.2.43) where $\tilde{a}$ runs over the whole $\bar{k}$th column $\mathcal{F}_{\bar{k}}$ of Δ, namely

$$\widehat{R}_t(a, r(a)) = R_t(a, r(a)) + \Gamma_t\big(r(a), r(a)\big). \tag{1.2.46}$$

On the other hand, by our assumptions $\Gamma_t\big(r(a), r(a)\big)$ should be symmetric. Therefore, using (1.2.46), we cannot "kill" the whole matrix $R_t(a, r(a))$, but only its symmetric part (by putting $\Gamma_t\big(r(a), r(a)\big) = -\frac{1}{2}\big(R_t(a, r(a)) + R_t(a, r(a))^T\big)$). It corresponds to conditions (1) of Lemma 1.6 with $a \in \mathcal{F}_{\bar{k}}$. Obviously, such choice of $\Gamma_t\big(r(a), r(a)\big)$ is the unique one with these properties.

In this way we have found uniquely all matrices $\Gamma_t(\tilde{a}, \tilde{b})$ with $\tilde{a} \in \mathcal{F}_{\bar{k}+1}$, $\tilde{b} \notin \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j$ such that the matrix $\widehat{R}_t(a, b)$ satisfies all conditions (1)-(4) of Lemma 1.6 for any pairs (a, b), where $a \in \mathcal{F}_{\bar{k}}$, $b \notin \bigcup_{j=1}^{\bar{k}-1} \mathcal{F}_j$. Taking $\Gamma_t(\tilde{b}, \tilde{a}) = \Gamma_t(\tilde{a}, \tilde{b})^T$, we will have the same properties for $\widehat{R}_t(b, a)$ with a and b as in the previous sentence. This completes the proof of Statement 2, therefore also the proof of the Statement 1 for $k = \bar{k} + 1$, and then by induction the proof of Proposition 1.2. $\square$

1.2.6 Normality of the obtained quasi-normal moving frames

The normalization conditions (1.2.29) with $k \geq 3$, which is not used before, will ensure the normality of the obtained quasi-normal moving frame. As before, we denote by d the number of levels in the diagram Δ, by p_i the number of superboxes in the ith level, and by a_i the first superbox in the ith level. The normality of the constructed quasinormal frame will obviously follow from the following

Proposition 1.3. *A quasi-normal moving frame* $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$ *is normal if and only if conditions (1.2.29) hold for any* $1 \leq j < i \leq d$ *and* $3 \leq k \leq p_j - p_i + 1$.

Proposition 1.3 will follow by induction from the following

Statement 3. *Fix* $s \in \mathbb{N}$ *and let* $R_t : \Delta \times \Delta \to \mathrm{Mat}$ *be a quasi-normal mapping, satisfying the following condition: for any* i *and* j, $1 \leq j < i \leq d$, *the matrix* $R_t(a, b) \equiv 0$ *for all first* $\min\{s-1, p_j - p_i - 1\}$ *pairs* (a, b) *in the tuple* (1.1.5). *Then for any* i *and* j, $1 \leq j < i \leq d$, *such that* $1 \leq s \leq p_j - p_i$, *the* s*th pair* $(\bar{a}_i^s, \bar{a}_j^s)$ *of the tuple* (1.1.5) *satisfies*

$$R_t(\bar{a}_i^s, \bar{a}_j^s) = \pm\omega\big(E_{a_j}^{(s+2)}(t), E_{a_i}(t)\big). \tag{1.2.47}$$

Before proving Statement 3, let us introduce some notations. As in the proof of Lemma 1.4, let $\overline{\Delta}$ be the diagram obtained from Δ by the reflection w.r.t. its left edge. In the sequel we will work with the diagram $\Delta \cup \overline{\Delta}$. The boxes of this diagram will be also called superboxes. Similar to above, we will denote by l and r the left and the right shifts on the diagram $\Delta \cup \overline{\Delta}$, respectively.

Definition 1.4. *A (finite) sequence* $\eta = \{b_0, \ldots, b_n\}$ *of superboxes of the diagram* $\Delta \cup \overline{\Delta}$ *is called an admissible path in this diagram, if the following two conditions hold:*

1. *If* $b_i \in \Delta$ *then* $b_{i+1} \in \{b_i, l(b_i)\}$*;*
2. *If* $b_i \in \overline{\Delta}$ *then* $b_{i+1} \in \{b_i, l(b_i)\} \cup \Delta$

(see an example on Figure 1). *The superboxes from the admissible path* η *will be called the* vertices *of the path. We will distinguish three types of vertices: the vertex* b_m, $0 \leq m < n$, *will be called* walking, *if* $b_{m+1} = l(b_m)$, *it will be called* sleeping, *if* $b_{m+1} = b_m$, *and it will be called* jumping, *if* $b_m \in \overline{\Delta}$ *and* $b_{m+1} \in \Delta$.

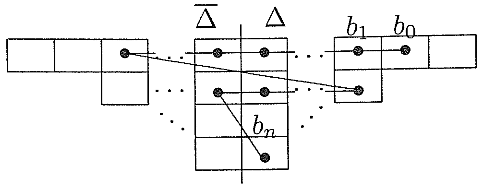

Figure 1.

Further, given any superbox x of Δ we will denote by $\bar{x}$ the superbox in $\overline{\Delta}$, obtained from x by the reflection of x w.r.t. the left edge of the diagram Δ; Similarly, given any superbox x of $\overline{\Delta}$ we will also denote by $\bar{x}$ the superbox , obtained from x by the reflection of x w.r.t. the right edge of the diagram $\overline{\Delta}$.

¿From the definition of Darboux frame it follows that the quantity $\omega(E_{a_i}, E_{a_j}^{(s+2)})$, we are interested in, is equal to the coefficient near F_{a_i} of the expansion of $E_{a_j}^{(s+2)}$ into linear combination w.r.t. the frame $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$, satisfying the structural equation (1.1.7). Admissible paths in the diagram $\Delta\cup\overline{\Delta}$ help to describe the coefficients of such expansions. For this to any admissible path $\eta = \{b_0, \dots, b_n\}$ we will assign a curve of size$(b_n)\times$size(b_0)-matrices $P_\eta(\cdot)$. If η consists of only one vertex, $\eta = \{b_0\}$, we set $P_\eta(t)$ to be the identity matrix for any t. Further for the path $\eta = \{b_0, \dots, b_{n-1}, b_n\} (n \geq 1)$ the curve of matrices $P_\eta(\cdot)$ is obtained from the curve of matrices $P_{\{b_0,\dots,b_{n-1}\}}$ by the following recursive formula:

$$P_{\{b_0,\dots,b_{n-1},b_n\}}(t) = \begin{cases} P_{\{b_0,\dots,b_{n-1}\}}(t) & \text{if } b_n = l(b_{n-1}),\ b_{n-1}\in\Delta, \\ -P_{\{b_0,\dots,b_{n-1}\}}(t) & \text{if } b_n = l(b_{n-1}),\ b_{n-1}\in\overline{\Delta}, \\ P'_{\{b_0,\dots,b_{n-1}\}}(t) & \text{if } b_n = b_{n-1}, \\ R_t(\bar{b}_{n-1}, b_n)P_{\{b_0,\dots,b_{n-1}\}}(t) & \text{if } b_{n-1}\in\overline{\Delta},\ b_n\in\Delta \end{cases} \tag{1.2.48}$$

Given $\{a,b\}\subset\Delta\cup\overline{\Delta}$ and $n\in\mathbb{N}\cup\{0\}$ denote by $\Omega(a,b,n)$ the set of all admissible paths in the diagram $\Delta\cup\overline{\Delta}$, starting at a, ending at b, and consisting of $n+1$ vertices. Then from structural equation (1.1.7), definition (1.2.48) of matrices P_η, and elementary rules of differentiations it follows that

$$\omega(E_{a_i}, E_{a_j}^{(s+2)}) = \sum_{\eta\in\Omega(a_j,\bar{a}_i,s+2)} P_\eta \tag{1.2.49}$$

Remark 1.4. It is clear from the last line of the recursive formula (1.2.48) that if $P_\eta(t) \neq 0$, then

$$R_t(\bar{b}_m, b_{m+1}) \neq 0$$

for any jumping vertex b_m of η. □

Further, it is convenient to enumerate the columns of the diagram $\Delta\cup\overline{\Delta}$ by integers in the following way: to the jth column (from the left) of Δ we assign the same number j while to the jth column from the right of $\overline{\Delta}$ we assign the number $1-j$. Given a superbox $a\in\Delta\cup\bar{\Delta}$, denote by $c(a)$ the number of the column, according to the rule described in the previous sentence. The following simple lemma will be useful in the sequel

Lemma 1.7. *Suppose that $R_t : \Delta\times\Delta \longrightarrow$ Mat is a quasi-normal mapping and $R_t(a,b)\neq 0$. Suppose that superboxes a and b lie in the jth and ith level of Δ respectively ($j < i$). Then the pair (a,b) is the $\big(c(b) - c(\bar{a})\big)$th pair in the tuple (1.1.5).*

Indeed, by Lemma 1.6 the nonzero matrix $R_t(a,b)$ must correspond to a pair from the appropriate tuple of the form (1.1.5). The second sentence of the lemma is obvious.

Proof of Statement 3. Fix some admissible path $\eta = \{b_0, \dots, b_{s+2}\}$ from $\Omega(a_j, \bar{a}_i, s+2)$ (by definition, $b_0 = a_j$ and $b_{s+2} = \bar{a}_i$). Let us denote by k the number of jumping vertices in η. Also, let $b_{m_1}, \dots, b_{m_k}$ be all jumping vertices of η, where $m_1 < m_2 < \dots < m_k$. Set also $m_0 = -1$, $m_{k+1} = s+2$. It is evident that for any $1 \le u \le k+1$ the number of superboxes on η between $b_{m_{u-1}+1}$ and b_{m_u} (including $b_{m_{u-1}+1}$ but not b_{m_u}) is equal to $c(b_{m_{u-1}+1}) - c(b_{m_u})$. Therefore the fact that all superboxes b_u with $0 \le u < s+1$ are either walking or sleeping or jumping can be expressed as follows

$$\sum_{u=1}^{k+1} \big(c(b_{m_{u-1}+1}) - c(b_{m_u})\big) + \#\{\text{sleeping vertices of } \eta\} + k = s+2. \tag{1.2.50}$$

Lemma 1.8. *Under the condition* (1.2.29) *with* $k \ge 3$ *if* $P_\eta \neq 0$ *for a path* $\eta \in \Omega(a_j, \bar{a}_i, s+2)$ $(j < i)$ *with* $p_j - p_i \ge s$, *then there is only one jumping vertex and there are no sleeping vertices in* η.

Proof. Since any path $\eta \in \Omega(a_j, \bar{a}_i, s+2)$ has to contain at least one jumping vertex (in order to jump somehow from jth to ith level) the lemma is actually equivalent to the fact that

$$\#\{\text{sleeping vertices of } \eta\} + k = 1 \tag{1.2.51}$$

Assume the converse, i.e.

$$\#\{\text{sleeping vertices of } \eta\} + k \ge 2. \tag{1.2.52}$$

Given a superbox $x \in \Delta$, denote by $p(x)$ the number of superboxes in the level of x. Assume that the superboxes b_{m_u} and b_{m_u+1} lie in different levels. By Remark 1.4, $R_t(\bar{b}_{m_u}, b_{m_u+1}) \neq 0$. Therefore, according to Lemma 1.7 either $(\bar{b}_{m_u}, b_{m_u+1})$ or $(b_{m_u+1}, \bar{b}_{m_u})$ is the $\big(c(b_{m_u+1}) - c(b_{m_u})\big)$th pair in the tuple (1.1.5). Combining this with Remark 1.4 and assumptions of Statement 3,

$$c(b_{m_u+1}) - c(b_{m_u}) > \min\{s-1, |p(b_{m_u+1}) - p(\bar{b}_{m_u})| - 1\}. \tag{1.2.53}$$

Further, since $c(b_0) = 1$ and $c(b_{s+2}) = 0$ (recall that $b_0 = a_j$, $b_{s+2} = a_i$, and $m_{k+1} = s+2$), we have

$$\sum_{u=1}^{k+1} \big(c(b_{m_{u-1}+1}) - c(b_{m_u})\big) = \sum_{u=1}^{k} \big(c(b_{m_u+1}) - c(b_{m_u})\big) + 1. \tag{1.2.54}$$

Substituting it into (1.2.50) and using assumption (1.2.52) we obtain

$$\sum_{u=1}^{k} \big(c(b_{m_u+1}) - c(b_{m_u})\big) \le s-1. \tag{1.2.55}$$

Since all terms in the sum in the left-hand side of the previous inequality are positive, we have $c(b_{m_u+1}) - c(b_{m_u}) \le s-1$ for any $1 \le u \le k$. Combining it with (1.2.53) we obtain that

$$c(b_{m_u+1}) - c(b_{m_u}) \ge |p(b_{m_u+1}) - p(\bar{b}_{m_u})|. \tag{1.2.56}$$

Besides, if the superboxes b_{m_u} and b_{m_u+1} lie in the same level, then the inequality (1.2.56) holds automatically.

On the other hand, by our constructions the superboxes b_{m_u+1} and $\bar{b}_{m_{u+1}}$ lie in the same level of Δ. This fact together with inequalities (1.2.56) and (1.2.55) implies that

$$p_j - p_i \le \sum_{i=1}^{k} |p(b_{m_u+1}) - p(\bar{b}_{m_u})| \le \sum_{u=1}^{k} c(b_{m_u+1}) - c(b_{m_u}) \le s-1,$$

which contradicts the assumption $p_j - p_i \geq s$ of the lemma. The proof of the lemma is completed. $\square$

Now, if η has only one jumping vertex and no sleeping vertices, then from (1.2.50) and (1.2.54) it follows that $c(b_{m_1+1}) - c(b_{m_1}) = s$. Besides, in this case the superbox b_{m_1} lies in the jth level and the superbox b_{m_1+1} lies in the ith level. But then from Remark 1.4 and Lemma 1.7 it follows that if $P_\eta \neq 0$ then the pair $(\bar{b}_{m_1}, b_{m_1+1})$ is exactly the sth pair of the tuple (1.1.5), which together with (1.2.48) and (1.2.49) implies (1.2.47). The proof of Statement 3 is completed. $\square$

As we have already mentioned, Proposition 1.3 follows immediately from Statement 3 by induction w.r.t. s, starting with $s = 1$ (for which the assumptions of Statement 3 hold automatically).

1.2.7 Final steps of the proof of Theorem 1.1

The "if" part of Proposition 1.3 implies that the tuple $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$ constructed in Subsection 1.2.5 is a normal moving frame of the curve $\Lambda(\cdot)$. Moreover, by the constructions of Subsection 1.2.3 the space $V_i(t) = \text{span}\{E_{\sigma_i}(t)\}$ is the canonical complement of $\left(\Lambda_{(p_i)}\right)^{(1)}(t)$ in $\Lambda_{(p_i-1)}(t)$ for any $1 \leq i \leq d$, where σ_i is the special superbox of the ith level, and by constructions of Subsection 1.2.4 the curves $E_{\sigma_i}(t)$ are horizontal sections of the bundle $\mathfrak{B}_i$, defined in Subsection 1.2.3 .

Now suppose that $(\{\widetilde{E}_a(t)\}_{a\in\Delta}, \{\widetilde{F}_a(t)\}_{a\in\Delta})$ is another normal moving frame of the curve $\Lambda(\cdot)$. From the second line of the structural equation (1.1.7) (where all $E_a(t)$ and $F_a(t)$ are replaced by $\widetilde{E}_a(t)$ and $\widetilde{F}_a(t)$) and the definition of Darboux frame it follows that conditions (1.2.29) (again with all $E_a(t)$ replaced by $\widetilde{E}_a(t)$) hold for any $1 \leq j < i \leq d$ and $k = 1, 2$. Indeed, $\omega(\widetilde{E}_{a_i}(t), \widetilde{E}'_{a_j}(t)) = \omega(\widetilde{E}_{a_i}(t), \widetilde{F}_{a_j}(t)) = 0$ and $\omega(\widetilde{E}_{a_i}(t), \widetilde{E}''_{a_j}(t)) = -\omega(\widetilde{E}'_{a_i}(t), \widetilde{E}'_{a_j}(t)) = -\omega(\widetilde{F}_{a_i}(t), \widetilde{F}_{a_j}(t)) = 0$. Further, by Proposition 1.3, from the normality of the frame $(\{\widetilde{E}_a(t)\}_{a\in\Delta}, \{\widetilde{F}_a(t)\}_{a\in\Delta})$ it follows that conditions (1.2.29) (again with all $E_a(t)$ replaced by $\widetilde{E}_a(t)$) hold for any $1 \leq j < i \leq d$ and $3 \leq k \leq p_j - p_i + 1$. Therefore, Lemma 1.5 implies that $\text{span}\{\widetilde{E}_{\sigma_i}(t)\} = \text{span}\{E_{\sigma_i}(t)\} = V_i(t)$. Besides, from the second line of the structural equation (1.1.7) (where again all $E_a(t)$ and $F_a(t)$ are replaced by $\widetilde{E}_a(t)$ and $\widetilde{F}_a(t)$) and Proposition 1.1 it follows that the curves $\widetilde{E}_{\sigma_i}$ are horizontal sections of the bundle $\mathfrak{B}_i$, which together with (1.2.33) implies relations (1.1.8). This completes the proof of Theorem 1.1.

1.3 Nonmonotonic curves satisfying condition (G)

Now consider possibly non-monotonic curves with fixed Young diagram D and reduced Young diagram Δ, satisfying condition (G) (see Subsection 1.2.3). For such curves the canonical complements $V_i(t)$ to $\left(\Lambda_{(p_i)}\right)^{(1)}(t)$ in $\Lambda_{(p_i-1)}(t)$ are defined as well. Denote by Γ_i^+ and Γ_i^- the positive and the negative index of the quadratic form $\dot{\Lambda}(t)|_{(\Lambda_{(p_i-1)})^{(p_i-1)}(t)}$ and let $r_i^+ = \Gamma_i^+ - \Gamma_{i-1}^+$ and $r_i^- = \Gamma_i^- - \Gamma_{i-1}^-$. Actually the numbers r_i^+ and r_i^- are equal to the positive and negative inertia index of the canonical quadratic forms $Q_{i,t}$ on $V_i(t)$. These numbers do not depend on t and they will be called *the ith positive inertia index and the ith negative inertia index of the curve* $\Lambda(t)$ respectively. Similarly to Definition 1.3 one can define the normal (quasi-normal) moving frame for a curve in a Lagrange Grassmannian, satisfying condition (G). The only modification comparing to this definition is that one should replace the second line in the structural equation (1.1.7) by $E'_a = F_a(t) I_{r_i^+, r_i^-}$, $a \in \mathcal{F}_1 \cap \Upsilon_i$, where r_i^+ and r_i^- are the ith positive and negative inertia indices of the curve $\Lambda(t)$, and the matrix $I_{r_i^+,r_i^-}$ is the diagonal $(r_i^+ + r_i^-)\times(r_i^+ + r_i^-)$–matrix such that its first r_i^+ diagonal entries are equal to 1 and others are equal to -1. Also, set $O(r_i^+, r_i^-) = \{A \in \mathbb{R}^{r_i^+ \times r_i^-} : A^T I_{r_i^+,r_i^-} A I_{r_i^+,r_i^-} = \mathrm{Id}\}$. Continuing the normalization procedure by complete analogy with obvious modifications, one gets the following generalization of Theorem 1.1 to nonmonotonic curves satisfying condition (G):

Theorem 1.3. *For any curve $\Lambda(t)$ with the Young diagram D in the Lagrange Grassmannian, satisfying condition (G), there exists a normal moving frame $\big(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta}\big)$. A moving frame*

$$\big(\{\tilde{e}_\alpha(t)\}_{\alpha\in D}, \{\tilde{f}_\alpha(t)\}_{\alpha\in D}\big)$$

is a normal moving frame of the curve $\Lambda(\cdot)$ if and only if for any $1 \le i \le d$ there exists a constant matrix $U_i \in O(r_i^+, r_i^-)$ such that for all t

$$\widetilde{E}_a(t) = E_a(t)U_i, \quad \widetilde{F}_a(t) = F_a(t) I_{r_i^+,r_i^-} U_i I_{r_i^+,r_i^-}, \qquad \forall a \in \Upsilon_i, \tag{1.3.1}$$

where r_i^+ and r_i^- are the ith positive and the negative inertia indices of the curve $\Lambda(t)$.

Further, take a Young diagram D, as before, and fix a tuple of nonnegative integers $\{r_i^-\}_{i=1}^d$ such that $0 \le r_i^- \le r_i$ for any $1 \le i \le d$. Let $\mathfrak{Q}_D$ be the quiver, defined in Subsection 1.1.3 . A representation of the quiver $\mathfrak{Q}_D$ will be called *compatible with the Young diagram D and the tuple* $\{r_i^-\}_{i=1}^d$, if for any $1 \le i \le d$ the space of the representation corresponding to the vertex Υ_i is a r_i-dimensional pseudo-Euclidean space with negative inertia index r_i^- and the linear mappings $\mathcal{R}(a,b)$ of the representation corresponding to arrows (a,b) satisfy the following relations: $\mathcal{R}(a,b)^* = \mathcal{R}(b,a)$ and $\mathcal{R}\big(a, r(a)\big)$ is antisymmetric w.r.t. the corresponding pseudo-Euclidean structure. Then by complete analogy with Theorem 1.2 we have

Theorem 1.4. *For the given one-parametric family $\Xi(t)$ of representations of the quiver $\mathfrak{Q}_D$ compatible with the Young diagram D with $|D|$ boxes and the tuple of nonnegative integers $\{r_i^-\}_{i=1}^d$ there exists the unique, up to a symplectic transformation, curve $\Lambda(t)$, satisfying condition (G), in the Lagrange Grassmannian of $2|D|$-dimensional symplectic space with the Young diagram D such that the quiver of curvatures of $\Lambda(t)$ is isomorphic to $\Xi(t)$ and its ith negative inertia index is equal to r_i^- for any $1 \le i \le d$. If, in addition, all rows of D have different length, then given a tuple of smooth functions $\{\rho_{a,b}(t) : (a,b) \in \Delta\times\Delta, (a,b)$ is an essential pair$\}$ there exists the unique, up to a symplectic transformation, curve $\Lambda(t)$, satisfying condition (G), in the Lagrange Grassmannian of $2|D|$-dimensional symplectic space with the Young diagram D such that for any essential pair $(a,b) \in \Delta\times\Delta$ and any t its (a,b)-curvature map at t coincides with $\rho_{a,b}(t)$ and its ith negative inertia index is equal to r_i^- for any $1 \le i \le d$.*

1.4 Consequences for geometric structures on manifolds

Let $\mathfrak{A}$ be a geometric structure on a manifold M, as in the Introduction, and h be the maximized or a critical Hamiltonian associated with the geometric structure $\mathfrak{A}$. Assume that the point $\lambda \in T^*M$ satisfies: $h(\lambda) > 0$, $dh(\lambda) \neq 0$, and the germ of the Jacobi curve $\mathfrak{J}_\lambda(t)$ at $t = 0$ has Young diagram D with the reduced diagram Δ and with p_1 boxes in the first row. Let, as before, $W_\lambda = T_\lambda\mathcal{H}_{h(\lambda)}/\{\mathbb{R}\vec{h}(\lambda)\}$ be the symplectic space, where the Jacobi curve $\mathfrak{J}_\lambda(t)$ lives. The point λ will be called *D-regular* if, in addition to above,

$$\mathfrak{J}_\lambda^{(p_1)}(0) = W_\lambda \tag{1.4.1}$$

and the germ of the Jacobi curve $\mathfrak{J}_\lambda(t)$ at $t = 0$ satisfies condition (G). The latter holds automatically in the case of the maximized Hamiltonian by Lemma 1.3. Here for simplicity we will work mainly with D-regular points for some Young diagram D. Let

$$\mathfrak{J}_\lambda(0) = \bigoplus_{a\in\Delta} \widetilde{\mathcal{V}}_a(\lambda) \tag{1.4.2}$$

be the canonical splitting of the subspace $\mathfrak{J}_\lambda(0)$ (w.r.t. the canonically parametrized curve $\mathfrak{J}_\lambda(0)$) and $\mathrm{proj}_\lambda : T_\lambda\mathcal{H}_{h(\lambda)} \to W_\lambda$ be the canonical projection on the factor space. Set

$$\mathcal{V}_a(\lambda) = (\mathrm{proj}_\lambda)^{-1}\big(\widetilde{\mathcal{V}}_a(\lambda)\big) \cap \Pi_\lambda, \tag{1.4.3}$$

where Π_λ is the vertical subspace of $T_\lambda\mathcal{H}_{h(\lambda)}$, defined by (0.1.4). Taking into account that proj_λ establishes an isomorphism between Π_λ and $\mathfrak{J}_\lambda(0)$, we get from (1.4.2) and (1.4.3) the following *canonical splitting of the tangent space $T_\lambda\big(T^*_{\pi(\lambda)}M\big)$ to the fiber of T^*M at λ*:

$$T_\lambda T^*_{\pi(\lambda)}M = \bigoplus_{a\in\Delta} \mathcal{V}_a(\lambda) \oplus \mathrm{span}\,\{\epsilon(\lambda)\}, \tag{1.4.4}$$

where ϵ is the Euler field of T^*M, i.e. the infinitesimal generator of the homotheties of the fibers of T^*M. Besides, each subspace $\mathcal{V}_a(\lambda)$ is endowed with the canonical pseudo-Euclidean structure and the corresponding curvature maps between the subspaces of the splitting are intrinsically related to the geometric structure $\mathfrak{A}$.

Moreover, let $\mathfrak{R}_\lambda(a,b) : \mathcal{V}_a(\lambda) \to \mathcal{V}_b(\lambda)$ and $\mathfrak{R}_\lambda : \Pi_\lambda \to \Pi_\lambda$ be the (a,b)-curvature map and the big curvature map of the Jacobi curve $\mathfrak{J}_\lambda(\cdot)$ at $t = 0$. These maps are intrinsically related to the geometric structure $\mathfrak{A}$. They are called the *(a,b)-curvature map* and the *big curvature map* of the geometric structure $\mathfrak{A}$ at the point λ. Also, the canonical complement $\mathfrak{J}_\lambda^{\mathrm{trans}}(t)$ at $t = 0$ gives rise a canonical complement of Π_λ in W_λ. For any $a \in \Delta$, denote

$$\mathcal{V}_a^{\mathrm{trans}}(\lambda) = V_a^{\mathrm{trans}}(0). \tag{1.4.5}$$

Then the space

$$\mathrm{Hor}(\lambda) = \bigoplus_{a\in\Delta} \mathcal{V}_a^{\mathrm{trans}}(\lambda) \oplus \mathbb{R}\vec{h}(\lambda)$$

is transversal to the tangent space $T_\lambda\big(T^*_{\pi(\lambda)}M\big)$ to the fiber of T^*M at λ. Thus, if for some diagram D the set U of its regular D-points is open in $T^*M\backslash h^0$, then for any $q \in \pi(U)$ the subsets $T_q^*M \cap U$ of the linear space T_q^*M is endowed with very rich additional structures: at each point $\lambda \in T_q^*M \cap U$ there is the canonical splitting of tangent spaces (smoothly depending on λ) such that the subspaces of the splitting are parametrized by the superboxes of the reduced diagram Δ, the dimension of each subspace is equal to the size of the corresponding superbox, these subspaces are endowed with the canonical pseudo-Euclidean structures, and the canonical linear mappings between these subspaces (i.e. the (a,b)-curvature map) are defined. Besides, the distribution of "horizontal" subspaces $\mathrm{Hor}(\lambda)$ defines the *connection on $U \subset T^*M$, canonically associated with geometric structure $\mathfrak{A}$.*

Finally, let $\lambda \in T^*M$ and let $\lambda(t) = e^{t\vec{h}}\lambda$. Assume that $(E_a^\lambda(t), F_a^\lambda(t))_{a\in\Delta}$ is a normal moving frame of the Jacobi curve $\mathfrak{J}_\lambda(t)$ attached at point λ. Let ϵ be the Euler field on T^*M, as in the Introduction. Clearly $T_\lambda(T^*M) = T_\lambda\mathcal{H}_{h(\lambda)} \oplus \mathbb{R}\epsilon(\lambda)$. The flow $e^{t\vec{h}}$ on T^*M induces the pushforward maps $\big(e^{t\vec{h}}\big)_*$ between the corresponding tangent spaces $T_\lambda T^*M$ and $T_{e^{t\vec{h}}\lambda}T^*M$, which in turn induce naturally the maps between the spaces $T_\lambda(T^*M)/\mathbb{R}\vec{h}(\lambda)$ and $T_{e^{t\vec{h}}\lambda}T^*M/\mathbb{R}\vec{h}(e^{t\vec{h}}\lambda)$. The map $\mathcal{K}^t$ between $T_\lambda(T^*M)/\mathbb{R}\vec{h}(\lambda)$ and $T_{e^{t\vec{h}}\lambda}T^*M/\mathbb{R}\vec{h}(e^{t\vec{h}}\lambda)$, sending

$E_a^\lambda(0)$ to $\left(e^{t\vec{h}}\right)_* E_a^\lambda(t)$, $F_a^\lambda(0)$ to $\left(e^{t\vec{h}}\right)_* F_a^\lambda(t)$ for any $a \in \Delta$, and the equivalence class of $\epsilon(\lambda)$ to the equivalence class of $\epsilon(e^{t\vec{h}}\lambda)$, is independent of the choice of normal moving frames. The map $\mathcal{K}^t$ is called *the parallel transport* along the extremal $e^{t\vec{h}}\lambda$ at time t. For any $v \in T_\lambda(T^*M)/\mathbb{R}\vec{h}(\lambda)$, its image $v(t) = \mathcal{K}^t(v)$ is called *the parallel transport of v at time t*. Note that from the definition of the Jacobi curves and the construction of normal moving frame it follows that the restriction of the parallel transport $\mathcal{K}_t$ to the vertical subspace $T_\lambda(T^*_{\pi(\lambda)}M)$ of $T_\lambda(T^*M)$ can be considered as a map onto the vertical subspace $T_{e^{t\vec{h}}\lambda}(T^*_{\pi(e^{t\vec{h}}\lambda)}M)$ of $T_{e^{t\vec{h}}\lambda}(T^*M)$. A vertical vector field V is called *parallel* if $V(e^{t\vec{h}}\lambda) = \mathcal{K}^t\big(V(\lambda)\big)$.

Chapter 2

Jacobi Equations and Comparison Theorems for Corank 1 sub-Riemannian Structures with Symmetries

We will obtain explicit expressions for the curvature map of the corank 1 sub-Riemannian structures with symmetries in terms of the curvature tensor of the reduced Riemannian manifold and the magnetic field. We will also estimate the number of conjugate points along the sub-Riemannian extremals in terms of the bounds for the curvature tensor of this Riemannian manifold and the magnetic field in the case of a uniform magnetic field. The results of this chapter can be found in [18].

2.1 Sub-Riemannian structures

As was already mentioned in the Introduction, if $\mathfrak{A}_q$ is an intersection of an ellipsoid centered at the origin with a linear subspace $\mathcal{D}_q$ in T_qM (where both the ellipsoids and the subspaces $\mathcal{D}_q$ depend smoothly on q), then $\mathfrak{A}$ is called a *sub-Riemannian structure on M with underlying distribution $\mathcal{D}$*. In this case $\mathfrak{A}_q$ is the unit sphere w.r.t. the unique Euclidean norm $||\cdot||_q$ on $\mathcal{D}_q$, i.e. fixing an ellipsoid in $\mathcal{D}_q$ is equivalent to fixing an Euclidean norm on $\mathcal{D}_q$ for any $q \in M$. This reformulation justifies the term "sub-Riemannian". In the sequel, we will assume that the distribution D is nonholonomic. Then from Rashevskii-Chow theorem (see e.g. [11]) it follows that any given two points on M can be connected by an admissible curve. The maximized Hamiltonian, defined by (0.1.1) is equal to $||p|_{\mathcal{D}_q}||_q$, i.e. it is equal to the norm of the restriction of the functional $p \in T_q^*M$ on $\mathcal{D}_q$ w.r.t. the Euclidean norm $||\cdot||_q$ on $\mathcal{D}_q$. Actually it is more convenient to work with the half of the square of this maximized Hamiltonian. Thus, in the sequel the sub-Riemannian Hamiltonian h is the following one:

$$h(\lambda) \triangleq \frac{1}{2}||p|_{\mathcal{D}_q}||^2, \quad \lambda = (p, q) \in T^*M, \; q \in M, \; p \in T_q^*M, \tag{2.1.1}$$

The Hamiltonian h is nonnegative quadratic form on the fibers. First it implies the monotonicity of the corresponding Jacobi curves. Further assume that in this case relation (1.4.1) holds for some λ and p_1. Then there is a neighborhood U of $\pi(\lambda)$ in M and an open and dense subset $\mathcal{O}$ of U that satisfies the following property: for any $\tilde{q} \in \mathcal{O}$ there exists a neighborhood $\widetilde{U} \in \mathcal{O}$ and a Young diagram D such that for each $\hat{q} \in \widetilde{U}$ the intersection of the set of its

D-regular points with $T^*_{\tilde{q}}M$ is an nonempty Zariski open subset of $T^*_{\tilde{q}}M$. Besides, the canonical splitting, the canonical Euclidean structures on the subspaces of the splitting, the curvature maps, and the canonical connection above depend algebraically on points of the fibers of T^*M. Thus, to any sub-Riemannian metric satisfying assumptions above one can assign very rigid additional structures on T^*M.

Condition (1.4.1) has the following equivalent description in terms of the extremal $e^{t\vec{h}}\lambda$. Projections of the Pontryagin extremals to the base manifold M are called extremal trajectories. Conversely, an extremal projected to the given extremal trajectory is called its *lift*. From the Pontryagin Maximum Principle it follows that the set of all lifts of given extremal trajectory can be provided with the structure of linear space. The dimension of this space is called *corank of the extremal trajectory*. It turns out that if condition (1.4.1) holds, then corank of the extremal trajectory $\pi(e^{t\vec{h}}\lambda)$ is equal to 1. Conversely, if corank of the extremal trajectory $\pi(e^{t\vec{h}}\lambda)$ is equal to 1, then $J^{(p_1(t))}_{e^{t\vec{h}}\lambda}(0) = W_{e^{t\vec{h}}\lambda}$ for t from generic set. Note also that if corank of the extremal trajectory is greater than 1, then this extremal trajectory is the projection of a so-called abnormal extremal (a Pontryagin extremal living on zero level set of the corresponding Hamiltonian).

Recently, A. Agrachev proved ([2]) that *any sub-Riemannian metric on a completely nonholonomic vector distribution has at least one corank 1 extremal trajectory or, equivalently, not all extremal trajectories of it are projections of abnormal extremals.* Therefore the constructions above can be implemented for any sub-Riemannian metric on any completely nonholonomic vector distribution.

In the case of a Riemannian metric the canonical connection above coincides with the Levi-Civita connection ([6]) and the splitting of the tangent spaces to the fibers is trivial. Moreover, there is only one curvature map and it is naturally related to the Riemannian sectional curvature tensor. Denote this box by a. The structure equation for a normal moving frame is of the form:

$$\begin{cases} E_a'(t) = F_a(t) \\ F_a'(t) = -E_a(t)\mathcal{R}_t(a,a). \end{cases} \tag{2.1.2}$$

Remark 2.1. *Note that from* (2.1.2) *it follows that if* $(\widetilde{E}_a(t), \widetilde{F}_a(t))$ *is a Darboux moving frame such that* $\widetilde{E}_a(t)$ *is an orthonormal frame of* $\Lambda(t)$ *and* $\operatorname{span}\{\widetilde{F}_a(t)\} = \Lambda^{\text{trans}}(t)$. *Then there exists a curve of antisymmetric matrices* $B(t)$ *such that*

$$\begin{cases} \widetilde{E}_a'(t) = \widetilde{E}_a(t)B(t) + \widetilde{F}_a(t) \\ \widetilde{F}_a'(t) = -\widetilde{E}_a(t)\widetilde{\mathcal{R}}_t(a,a) + \widetilde{F}_a(t)B(t), \end{cases} \tag{2.1.3}$$

where $\widetilde{\mathcal{R}}_t(a,a)$ *is the matrix of the curvature map* $\mathfrak{R}_t(a,a)$ *on* $\Lambda(t)$ *w.r.t. the basis* $\widetilde{E}_a(t)$.

In [6] and [5] it was shown that in the considered case the canonical connection coincides with the Levi-Civita connection and the unique curvature map $\mathfrak{R}_\lambda(a,a) : \mathcal{V}_a(\lambda) \longrightarrow \mathcal{V}_a(\lambda)$ (where $\mathcal{V}_a(\lambda) = \Pi_\lambda$) was expressed by the Riemannian curvature tensor. In order to give this expression let R^∇ be the Riemannian curvature tensor. Below we will use the identification between the tangent vectors and the cotangent vectors of the Riemannian manifold M given by the Riemannian metric. More precisely, given $p \in T_q^*M$ let $p^h \in T_qM$ such that $p \cdot v = \langle p^h, v\rangle$ for any $v \in T_qM$. Since tangent spaces to a linear space at any point are naturally identified with the linear space itself we can also identify in the same way the space $T_\lambda(T^*_{\pi(\lambda)}M)$ with $T_{\pi(\lambda)}M$.

$$\mathfrak{R}_\lambda(a,a)v = R^\nabla(p^h, v^h)p^h, \quad \forall \lambda = (q,p) \in \mathcal{H}_{h^{-1}(\lambda)}, q \in M, p \in T_q^*M, \quad v \in \Pi_\lambda. \tag{2.1.4}$$

Given a vector $X \in T_qM$ denote by ∇_X its lift to the Levi-Civita connection, considered as an Ehresmann connection on T^*M. Then by constructions the Hamiltonian vector field $\vec{h}$ is horizontal and satisfies $\vec{h} = \nabla_p$. Take any $v, w \in \Pi_\lambda$ and let V be a vertical vector field such that $V(\lambda) = v$. ¿From (2.1.4) , structure equation (2.1.2), and the fact that the Levi-Civita connection (as an Ehresmann connection on T^*M) is a Lagrangian distribution it follows that the Riemannian curvature tensor satisfies the following identity:

$$\langle R^\nabla(p^h, v^h)p^h, w^h\rangle = -\sigma\left([\nabla_{p^h}, \nabla_{V^h}](\lambda), \nabla_{w^h}\right). \tag{2.1.5}$$

For the nontrivial case of sub-Riemannian structures, i.e. when $\mathcal{D} \subsetneq TM$, let us consider the simplest case: the sub-Riemannian structure on a nonholonomic corank 1 distribution. Fix dim $M = n(n \geq 3)$. Recall that our considerations are local, thus we can select a nonzero 1-form ω_0 satisfying $\omega_0|_\mathcal{D} = 0$. Then $d\omega_0|_\mathcal{D}$ is well-defined nonzero 2-form up to a multiplication of nonzero function. Therefore, for any $q \in M$, the skew-symmetric linear map $J_q : \mathcal{D}_q \longrightarrow \mathcal{D}_q$ satisfying $d\omega_0(q)(X,Y) = \langle J_qX, Y\rangle_q, \forall X, Y \in \mathcal{D}_q$ is well-defined up a nonzero constant. Let $\mathcal{D}_q^\perp$ be as in (0.1.3). Then one has the following series of natural identifications:

$$T_q^*M/\mathcal{D}_q^\perp \sim \mathcal{D}_q^* \overset{\langle\cdot,\cdot\rangle}{\sim} \mathcal{D}_q, \tag{2.1.6}$$

where $\mathcal{D}_q^* \subseteq T_q^*M$ is the dual space of $\mathcal{D}_q$. According to this identification, J_q can be taken as the linear map from the fiber T_q^*M of T^*M to $T_q^*M/\mathcal{D}_q^\perp$ (in this case, $J_q|_{\mathcal{D}_q^\perp} = 0$).

Let D be the Young diagram consisting of two columns, with $(n-2)$ boxes in the first column and 1 box in the second column. Then the set of D-regular points coincides with $\{(p,q) \in \mathcal{H}_{\frac{1}{2}} : J_qp \neq 0\}$(see Proposition 2.1 below for the proof in the particular case with symmetries) . In the case of $n > 3$, the reduced Young diagram consists of three boxes: two in the first column and one in the second. The box in the second column will be denoted by a, the upper box in the first column will be denoted by b and the lower box in the first column will be denoted by c. Note that size(a) = size$(b) = 1$ and size$(c) = n-3$. When $n = 3$, the reduced Young diagram consists of two boxes, a and b as above and the box c does not appear. All formulae for $n > 3$ will be true for $n = 3$ if one avoids the formulae containing the box c. In this case, the symmetric (Darboux) compatible mapping (with Young diagram D) is normal if and only if $R_t(a,b) = 0$ and the canonical splitting of Π_λ has the form: $\Pi_\lambda = \mathcal{V}_a(\lambda) \oplus \mathcal{V}_b(\lambda) \oplus \mathcal{V}_c(\lambda)$, where $\mathcal{V}_a(\lambda), \mathcal{V}_b(\lambda)$ are of dimension 1 and $\mathcal{V}_c(\lambda)$ is of dimension $n-3$. These subspaces can be described as follows. As the tangent space of the fibers of T^*M can be naturally identified with the fibers themselves (the fibers are linear spaces), one can show that

$$\mathcal{V}_a(\lambda) = \mathcal{D}_{\pi(\lambda)}^\perp.$$

Using the fact that $\mathcal{V}_b(\lambda) \oplus \mathcal{V}_c(\lambda) \oplus \mathbb{R}p$ is transversal to $\mathcal{D}_q^\perp$, one can get the following identification

$$\mathcal{V}_b(\lambda) \oplus \mathcal{V}_c(\lambda) \oplus \mathbb{R}p \sim T_q^*M/\mathcal{D}_q^\perp, \tag{2.1.7}$$

Finally, combining (2.1.6) and (2.1.7), we have that

$$\mathcal{V}_b(\lambda) \oplus \mathcal{V}_c(\lambda) \oplus \mathbb{R}p \sim \mathcal{D}_q^* \sim \mathcal{D}_q, \tag{2.1.8}$$

Under the identifications, one can show that (see step 1 in Subsection 2.2.3 below):

$$\mathcal{V}_b(\lambda) = \mathbb{R}J_qp, \quad \mathcal{V}_c(\lambda) = (\text{span}\{p, Jp\})^\perp. \tag{2.1.9}$$

Regarding the $(a,b)-$curvature map, even in the considered case it is difficult to get the explicit expression in terms of sub-Riemannian structures without additional assumptions. Here we calculate them in the special case of sub-Riemannian structures on corank 1 distribution, having additional infinitesimal symmetries. After an appropriate factorization, such structure can be reduced to a Riemannian manifold equipped with a symplectic form (a magnetic field) and the curvature maps can be expressed in terms of the Riemannian curvature tensor and the magnetic field. The main results of this chapter are the explicit expressions of the curvature maps (Theorems 2.1-2.3 below) and the estimation of the number of conjugate points along sub-Riemannian extremals (Theorem 2.4 below) in terms of the Riemannian curvature tensor of the reduced manifold and the magnetic field (the latter is done in the case of the uniform magnetic field).

2.2 Algorithm for calculation of canonical splitting and (a, b)-curvature map

We begin with the discussion of sub-Riemannian structures with additional symmetries and show that they can be reduced to a Riemannian manifold with a symplectic form. Then we describe the algorithm of finding of normal moving frames for the Jacobi curves of the extremals of such structures. As a result, we write down the canonical complement $\mathcal{V}^{\text{trans}}(\lambda)$ using the symplectic form σ, Lie derivatives w.r.t. $\vec{h}$ and the tensor J. Further, we establish certain calculus relating Lie derivatives and the covariant derivative of the reduced Riemannian structure. As a result, we can characterized sub-Riemannian connection in terms of Levi-Civita connection and the tensor J.

2.2.1 Corank 1 sub-Riemannian structures with symmetries

As before, assume that $\mathcal{D}$ is a nonholonomic corank 1 distribution. Assume that the sub-Riemannian structure $(M, \mathcal{D}, \langle\cdot,\cdot\rangle)$ has an additional infinitesimal symmetry, i.e. a vector field X_0 such that

$$e_*^{tX_0}\mathcal{D} = \mathcal{D}\ ,\ (e^{tX_0})^* \langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle\ .$$

Assume also that X_0 is transversal to the distribution $\mathcal{D}$, $\mathbb{R}X_0 \oplus \mathcal{D}_q = T_qM, \forall q \in M$. In this case, the 1−form ω_0, defined by $\omega_0|_{\mathcal{D}} = 0$, as before, can be determined uniquely by imposing the condition $\omega_0(X_0) = 1$. Therefore $d\omega_0|_{\mathcal{D}}$ and the operator J_q are also determined uniquely. Let ξ be the 1-foliation generated by X_0. Denote by $\widetilde{M}$ the quotient of M by the leaves of ξ and denote the factorization map by $\mathrm{pr} : M \longrightarrow \widetilde{M}$. Since our construction is local, we can assume that $\widetilde{M}$ is a manifold. The sub-Riemannian metric $\langle\cdot,\cdot\rangle$ induces a Riemannian metric g on $\widetilde{M}$. Also $d\omega_0$ and J_q induce a symplectic form Ω and a type $(1, 1)$ tensor on $\widetilde{M}$, respectively. We denote the $(1, 1)$ type tensor by J as well. Actually, Ω can be seen as a magnetic field and J can be seen as a Lorenzian force on Riemannian manifold $\widetilde{M}$. The projection by pr of all sub-Riemannian geodesics describes all possible motion of a charged particle (with any possible charge) given by the magnetic field Ω on the Riemannian manifold $\widetilde{M}$(see e.g. [20, Chapter 12] and the references therein).

Define $u_0 : T^*M \longrightarrow \mathbb{R}$ by $u_0(p, q) \triangleq p \cdot X_0(q),\ (p, q) \in T^*M, q \in M, p \in T_q^*M$. Since X_0 is a symmetry of the sub-Riemannian structure, the function u_0 is the first integral of the extremal flow, i.e., $\{h, u_0\} = 0$, where $\{\cdot,\cdot\}$ is the Poisson bracket.

2.2.2 Algorithm of normalization

In the considered case the structural equation for the normal moving frame is of the form:

$$\begin{cases} E_a'(t) = E_b(t) \\ E_b'(t) = F_b(t) \\ E_c'(t) = F_c(t) \\ F_a'(t) = -E_c(t)R_t(a,c) - E_a(t)R_t(a,a) \\ F_b'(t) = -E_c(t)R_t(b,c) - E_b(t)R_t(b,b) - F_a(t) \\ F_c'(t) = -E_c(t)R_t(c,c) - E_b(t)R_t(c,b) - E_a(t)\mathcal{R}_t(c,a). \end{cases} \tag{2.2.1}$$

Assume that each element of the set $\{\mathcal{E}_a(\lambda), \mathcal{E}_b(\lambda), \mathcal{E}_c(\lambda), \mathcal{F}_a(\lambda), \mathcal{F}_b(\lambda), \mathcal{F}_c(\lambda)\}$ is either a vector field or a tuple of vector fields, depending on the size of the corresponding box in the Young diagram such that

$$\begin{aligned} & (\mathcal{E}_a(e^{t\vec{h}}\lambda), \mathcal{E}_b(e^{t\vec{h}}\lambda), \mathcal{E}_c(e^{t\vec{h}}\lambda), \mathcal{F}_a(e^{t\vec{h}}\lambda), \mathcal{F}_b(e^{t\vec{h}}\lambda), \mathcal{F}_c(e^{t\vec{h}}\lambda)) \\ = \; & \mathcal{K}^t(\mathcal{E}_a(\lambda), \mathcal{E}_b(\lambda), \mathcal{E}_c(\lambda), \mathcal{F}_a(\lambda), \mathcal{F}_b(\lambda), \mathcal{F}_c(\lambda)), \end{aligned}$$

where $\mathcal{K}^t$ is the parallel transport, defined in Section 1.4. Recall that for any vector fields X, Y one has the following formula: $\frac{d}{dt}\big|_{t=0} e_*^{-tX} Y = \mathrm{ad}_X Y$. So, the derivative w.r.t. t on the level of curves can be substituted by taking the Lie bracket with $\vec{h}$ on the level of sub-Riemannian structure. The normalization procedure of chapter 1 can be described in the following steps:

Step 1 The vector field $\mathcal{E}_a(\lambda)$ can be characterized , uniquely up to a sign, by the following conditions: $\mathcal{E}_a(\lambda) \in \Pi_\lambda$, $\mathrm{ad}\vec{h}\,\mathcal{E}_a(\lambda) \in \Pi_\lambda$, and

$$\sigma(\mathrm{ad}\,\vec{h}\,\mathcal{E}_a(\lambda), (\mathrm{ad}\vec{h})^2\mathcal{E}_a(\lambda)) = 1.$$

Then by the first two lines of (2.2.1) $\mathcal{E}_b(\lambda) = \mathrm{ad}\vec{h}\,\mathcal{E}_a(\lambda)$ and $\mathcal{F}_b(\lambda) = (\mathrm{ad}\vec{h})^2\mathcal{E}_a(\lambda)$.

Step 2 The subspace $\mathcal{V}_c(\lambda)$ is uniquely characterized by the following two conditions:

1. $\mathcal{V}_c(\lambda)$ is the complement of $\mathcal{V}_a(\lambda) \oplus \mathcal{V}_b(\lambda)$ in Π_λ;
2. $\mathcal{V}_c(\lambda)$ lies in the skew symmetric complement of

$$\mathcal{V}_a(\lambda) \oplus \mathcal{V}_b(\lambda) \oplus \mathbb{R}(\mathrm{ad}\vec{h})^2\mathcal{E}_a(\lambda) \oplus \mathbb{R}(\mathrm{ad}\vec{h})^3\mathcal{E}_a(\lambda).$$

It is endowed with the canonical Euclidean structure, which is the restriction of $\dot{\mathfrak{J}}_\lambda(0)$ on it.

Step 3 The restriction of the parallel transport $\mathcal{K}^t$ to $\mathcal{V}_c(\lambda)$ is characterized by the following two properties:

1. $\mathcal{K}^t$ is an orthogonal transformation of spaces $\mathcal{V}_c(\lambda)$ and $\mathcal{V}_c(e^{t\vec{h}}\lambda)$;
2. The space $\mathrm{span}\{\frac{d}{dt}((e^{-t\vec{h}})_*(\mathcal{K}^t v))|_{t=0} : v \in \mathcal{V}_c(\lambda)\}$ is isotropic.

Then $\mathcal{V}_c^{trans}(\lambda) = \mathrm{span}\{\frac{d}{dt}((e^{-t\vec{h}})_*(\mathcal{K}^t v))|_{t=0} : v \in \mathcal{V}_c(\lambda)\}$.

Step 4 To complete the construction of normal moving frames it remains to fix $\mathcal{F}_a(\lambda)$. The field $\mathcal{F}_a(\lambda)$ is uniquely characterized by the following two conditions (see line 4 of (2.2.1)):

1. The tuple $\{\mathcal{E}_a(\lambda), \mathcal{E}_b(\lambda), \mathcal{E}_c(\lambda), \mathcal{F}_a(\lambda), \mathcal{F}_b(\lambda), \mathcal{F}_c(\lambda)\}$ constitutes a Darboux frame;
2. $\sigma(\mathrm{ad}\,\vec{h}\mathcal{F}_a(\lambda), \mathcal{F}_b(\lambda)) = 0$.

To find $\mathcal{F}_a(\lambda)$, one can choose any $\widetilde{\mathcal{F}}_a(\lambda)$ such that $\{\mathcal{E}_a(\lambda), \mathcal{E}_b(\lambda), \mathcal{E}_c(\lambda), \widetilde{\mathcal{F}}_a(\lambda), \mathcal{F}_b(\lambda), \mathcal{F}_c(\lambda)\}$ constitutes a Darboux frame. Then

$$\mathcal{F}_a(\lambda) = \widetilde{\mathcal{F}}_a(\lambda) - \sigma(\mathrm{ad}\vec{h}\,\widetilde{\mathcal{F}}_a(\lambda), \mathcal{F}_b(\lambda))\mathcal{E}_a(\lambda). \tag{2.2.2}$$

2.2.3 Preliminary implementation of the algorithm

In order to implement the algorithm for the corank 1 sub-Riemannian structure with symmetries, let us analyze the relation between T^*M and $T^*\widetilde{M}$ in more detail. Let Ξ be the 1-foliation such that its leaves are integral curves of $\vec{u}_0$. Let $\mathrm{PR} : T^*M \to T^*M/\Xi$ be the canonical projection to the quotient manifold.

Fix a constant c. The quotient manifold $\{u_0 = c\}/\Xi$ can be naturally identified with $T^*\widetilde{M}$. Indeed, a point $\tilde{\lambda}$ in $\{u_0 = c\}/\Xi$ can be identified with a leaf $\mathrm{PR}^{-1}(\tilde{\lambda})$ of Ξ which has a form $((e^{-tX_0})^*p, e^{tX_0}q)$, where $\lambda = (p, q) \in \mathrm{PR}^{-1}(\tilde{\lambda})$, $q \in M$ and $p \in T_q^*M$. On the other hand, any element in $T^*\widetilde{M}$ can be identified with a one-parametric family of pairs $(e^{tX_0}q, (e^{-tX_0})^*(p|_{\mathcal{D}}))$. The mapping $I : \{u_0 = c\}/\Xi \to T^*\widetilde{M}$ sending $(e^{tX_0}q, (e^{-tX_0})^*p)$ to $(e^{tX_0}q, (e^{-tX_0})^*(p|_{\mathcal{D}}))$ is one-to-one (because $p(X_0) = u_0$ is already prescribed and equal to c) and it defines the required identification. Therefore, for any vector field X on $T^*\widetilde{M}$, we can assign the vector field $\underline{X}$ on T^*M s.t. $PR_*\underline{X} = (I^{-1})_*X$ and $\pi_*\underline{X} \in \mathcal{D}$.

Let $\tilde{\sigma}$ be the standard symplectic form on $T^*\widetilde{M}$. Note that $(I \circ \mathrm{PR})^*\tilde{\sigma}$ is a 2-from on $\{u_0 = c\}$. Let, as before, σ be the standard symplectic form on T^*M. Let ω_0 be the 1-form as in Subsection 2.2.1. Then σ and $\pi^*d\omega_0$ induce two 2-forms on $\{u_0 = c\}$ by restriction. The following lemma describes the relation between these 2-forms.

Lemma 2.1. *The following formula holds on* $\{u_0 = c\}$.

$$\sigma = (I \circ PR)^*\tilde{\sigma} - u_0\pi^*d\omega_0. \tag{2.2.3}$$

Proof. First define a 1-form ς_0 on T^*M by

$$\varsigma_0(v) = u_0\omega_0(\pi_*v), \ \ v \in T_\lambda^*M, \ \ \lambda = (p, q) \in T^*M, q \in M, p \in T_q^*M.$$

Let ς and $\tilde{\varsigma}$ be the tautological (Liouville) 1–forms on T^*M and $T^*\widetilde{M}$ respectively. Then on the set $\{u_0 = c\}$ one has $\varsigma = (I \circ \mathrm{PR})^*\tilde{\varsigma} + \varsigma_0$. Therefore, by definition of standard symplectic form on a cotangent bundle, we have

$$\sigma = (I \circ \mathrm{PR})^*\tilde{\sigma} - d\varsigma_0 = (I \circ \mathrm{PR})^*\tilde{\sigma} - du_0 \wedge \pi^*\omega_0 - u_0\pi^*d\omega_0. \tag{2.2.4}$$

We complete the proof of the lemma by noticing that $d\varsigma_0 = u_0\pi^*d\omega_0$ on $\{u_0 = c\}$. □

Before going further, let us introduce some notations. Given $v \in T_\lambda T_q^*M$ ($\sim T_q^*M$), where $q = \pi(\lambda)$, we can assign a unique vector $v^h \in T_{\mathrm{pr}(q)}\widetilde{M}$ to its equivalence class in $T_q^*M/\mathcal{V}_a(\lambda)$ by using the identifications (2.1.7) and (2.1.8). Conversely, to any $X \in T_{\mathrm{pr}(q)}\widetilde{M}$ one can assign an equivalence class of $T_\lambda(T_q^*M)/\mathcal{V}_a(\lambda)$. Denote by $X^v \in T_\lambda T_q^*M$ the unique representative of this equivalence class such that $du_0(X^v) = 0$.

Lemma 2.2. *For any vectors* $X, V \in T_\lambda T^*M$ *with* $\pi_*V = 0$ *we have* $\sigma(X, v) = g(\pi_*X, V^h)$.

Proof. Let $\lambda = (p, q) \in T^*M$, $p \in T_q^*M, q \in M$ and ς be the tautological (Liouville) 1-form on T^*M as before. Extend the vector X to a vector field and V to a vertical vector field in a neighbourhood of λ. It follows from the definition of the canonical symplectic form and the verticality of V that

$$\begin{aligned}\sigma(X, V) = -d\varsigma(X, V) = V(\varsigma(X)) + \varsigma([X, V]) = \\ V(p \cdot \pi_*X) - p \cdot \pi_*[V, X] = V \cdot \pi_*X.\end{aligned}$$

In the last equality here we use again the identification between $T_\lambda T_q^*M$ and T_q^*M. Finally, $V \cdot \pi_*X = g(V^h, \pi_*X)$ by the definition of V^h. □

Lemma 2.1 implies that the sub-Riemannian Hamiltonian vector field can be decomposed into the Riemannian Hamiltonian vector field and another part depending on the tensor J.

Lemma 2.3. *The following formula holds.*

$$\vec{h}(\lambda) = \underline{\nabla_{p^h}} - u_0(Jp^h)^v, \tag{2.2.5}$$

where $\lambda = (p, q) \in T^*M, q \in M, p \in T_q^*M$ *and* ∇_{p^h} *is the lift of* p^h *to* $T^*\widetilde{M}$ *w.r.t. the Levi-Civita connection.*

Proof. Denote by $\tilde{h}$ the Riemannian Hamiltonian function on $T^*\widetilde{M}$. Since the Hamiltonian vector field $\vec{\tilde{h}}$ is horizontal w.r.t. the Levi-Civita connection and its projection to $\widetilde{M}$ is equal to p^h, we have $\vec{\tilde{h}} = \nabla_{p^h}$. Further, it follows from the definition of I that $(I \circ \mathrm{PR})^*\tilde{h} = h$ and $(I \circ \mathrm{PR})_*(\underline{\nabla_{p^h}}) = \nabla_{p^h}$. Thus, for any vector X tangent to $\{u_0 = c\}$, we have

$$\begin{aligned}
\sigma(\underline{\nabla_{p^h}}, X) &= ((I \circ \mathrm{PR})^*\tilde{\sigma} - u_0\pi^*d\omega_0)(\underline{\nabla_{p^h}}, X) \\
&= \tilde{\sigma}(\nabla_{p^h}, (I \circ \mathrm{PR})_*X) - u_0 d\omega_0(p^h, \pi_*X) \\
&= d\tilde{h}((I \circ \mathrm{PR})_*X) - u_0 d\omega_0(p^h, \pi_*X) \\
&= (I \circ \mathrm{PR})^*d\tilde{h}(X) - u_0 d\omega_0(p^h, \pi_*X) \\
&= d((I \circ \mathrm{PR})^*\tilde{h})(X) - u_0 g(Jp^h, \pi_*X) \\
&= dh(X) + u_0\sigma((Jp^h)^v, X)
\end{aligned}$$

It follows that $\vec{h}(\lambda)$ and $\underline{\nabla_{p^h}} - u_0(Jp^h)^v$ are equal modulo $\mathbb{R}\vec{u}_0$, which is the symplectic complement of the tangent space to $\{u_0 = c\}$. But $\pi_*\vec{h}(\lambda), \pi_*(\underline{\nabla_{p^h}}) \in D_q$ and $\pi_*\vec{u}_0 = X_0 \notin D_q$, which implies (2.2.5). □

Now we give more precise description of normal moving frames following the steps as in Subsection 2.2.2. Assume that $\mathcal{V}_a^{trans}(\lambda), \mathcal{V}_b^{trans}(\lambda), \mathcal{V}_c^{trans}(\lambda)$ are defined by (1.4.5).

Step 1 First define the vector field $\widetilde{\mathcal{E}_a}$ on T^*M by

$$\widetilde{\mathcal{E}_a}(\lambda) \in \Pi_\lambda,\ \widetilde{\mathcal{E}_a}(\lambda) \in \mathcal{D}^\perp,\ du_0(\widetilde{\mathcal{E}_a}(\lambda)) = 1. \tag{2.2.6}$$

For further calculations it is convenient to denote $\widetilde{\mathcal{E}_a}$ by ∂_{u_0}, because to take the Lie brackets of $\widetilde{\mathcal{E}_a}$ with $\vec{h}$ is the same as to make "the partial derivatives w.r.t. u_0" in the left-hand side of (2.2.5). Indeed, by (2.2.5) $\mathrm{ad}\vec{h}\ \partial_{u_0} = (Jp^h)^v \in \Pi_\lambda$ and then $\pi_*((\mathrm{ad}\vec{h})^2\ \partial_{u_0}) = -Jp^h$. Besides, by direct computations,

$$\pi_*[\vec{h}, (Jp^h)^v] = -Jp^h. \tag{2.2.7}$$

Then from Lemma 2.2 it follows immediately that

$$\sigma(\mathrm{ad}\vec{h}\ \partial_{u_0}, (\mathrm{ad}\vec{h})^2\ \partial_{u_0}) = \|Jp^h\|^2.$$

As a direct consequence of the last identity we get

Proposition 2.1. *A point* $\lambda = (p, q) \in T^*M$ *is a* $D-$*regular point if and only if* $J_q p \neq 0$.

Remark 2.2. *Note that if* $\mathcal{D}$ *is a contact distribution the operators* J_q *are non-singular, and all points of* T^*M *out of the zero section are* D*-regular.*

Further from step 1 of Subsection 2.2.2, we have that

$$\mathcal{E}_a(\lambda) = \frac{\partial_{u_0}}{\|Jp^h\|}, \tag{2.2.8}$$

$$\mathcal{E}_b(\lambda) = \frac{(Jp^h)^v}{\|Jp^h\|} + \vec{h}\left(\frac{1}{\|Jp^h\|}\right)\partial_{u_0}, \tag{2.2.9}$$

$$\mathcal{F}_b(\lambda) = \frac{1}{\|Jp^h\|}[\vec{h},(Jp^h)^v] + 2\vec{h}\left(\frac{1}{\|Jp^h\|}\right)(Jp^h)^v + (\vec{h})^2\left(\frac{1}{\|Jp^h\|}\right)\partial_{u_0}. \tag{2.2.10}$$

Step 2 Let us characterize the space $\mathcal{V}_c(\lambda)$. For this let $\widetilde{\Pi}_\lambda = \{v \in \Pi_\lambda : du_0(v) = 0\}$ and let $\pi_0 : \Pi_\lambda \to \widetilde{\Pi}_\lambda$ be the projection from Π_λ to $\widetilde{\Pi}_\lambda$ parallel to $\mathcal{E}_a(\lambda)$. Note that $\pi_0(v) = (v^h)^v$. Since $\mathcal{V}_c(\lambda) \in \Pi_\lambda$ and $\mathcal{V}_c(\lambda)$ lies in the skew symmetric complement of $(\mathrm{ad}\vec{h})^2\mathcal{E}_a(\lambda)$, we have, using (2.2.7) and Lemma 2.2, that

$$\mathcal{V}_c(\lambda) \equiv (\mathrm{span}\{p^h, Jp^h\}^\perp)^v \mod \mathbb{R}\mathcal{E}_a(\lambda). \tag{2.2.11}$$

Further, let $\widetilde{\mathcal{V}}_c(\lambda) = \pi_0\big(\mathcal{V}_c(\lambda)\big)$. Using the condition that $\mathcal{V}_c(\lambda)$ is in the skew symmetric complement of $(\mathrm{ad}\vec{h})^3\mathcal{E}_a(\lambda)$, we have

$$\mathcal{V}_c(\lambda) = \{v + \mathcal{A}(\lambda, v)\mathcal{E}_a(\lambda) : v \in \widetilde{\mathcal{V}}_c(\lambda)\}. \tag{2.2.12}$$

where $\mathcal{A}(\lambda, v)$ is the linear functional on the Whitney sum $T^*M \oplus T^*M$ over M, given by

$$\mathcal{A}(\lambda, v) = \sigma\left(v, \frac{(\mathrm{ad}\vec{h})^2\,(Jp^h)^v}{\|Jp^h\|}\right). \tag{2.2.13}$$

Step 3 Since the normal moving frame is a Darboux frame, the space $\mathcal{V}_c^{\mathrm{trans}}(\lambda)$ lies in the skew symmetric complement of $\mathcal{V}_b(\lambda)$. Besides, its image under π_* belongs to $\mathcal{D}\big(\pi(\lambda)\big)$. Then, using Lemma 2.2 we obtain that

$$(\mathrm{pr}\circ\pi)_*\big(\mathcal{V}_c^{\mathrm{trans}}(\lambda)\big) \equiv \mathrm{span}\{p^h, Jp^h\}^\perp \mod \mathbb{R}\mathrm{p}^\mathrm{h}, \tag{2.2.14}$$

where, as before, $\mathrm{pr} : M \to \widetilde{M}$ is the canonical projection. For $\mathcal{V}_c^{\mathrm{trans}}(\lambda) \in T_\lambda(T^*M)/\mathbb{R}\vec{h}(\lambda)$ one can take a canonical representative in $T_\lambda(T^*M)$ which projects exactly to $\mathrm{span}\{p^h, Jp^h\}^\perp$ by $(\mathrm{pr}\circ\pi)_*$. In the sequel, this canonical representative will be denoted by $\mathcal{V}_c^{\mathrm{trans}}(\lambda)$ as well.

Further, given any $X \in \mathrm{span}\{p^h, Jp^h\}^\perp$ denote by ∇_X^c the lift of X to $\mathcal{V}_c^{\mathrm{trans}}(\lambda)$, i.e. the unique vector $\nabla_X^c \in V_c^{\mathrm{trans}}(\lambda)$ such that $(\mathrm{pr}\circ\pi)_*\nabla_X^c = X$. Then there exist the unique $B \in \mathrm{End}\big(\widetilde{\mathcal{V}}_c(\lambda)\big)$ and $\alpha, \beta \in \mathcal{V}_c(\lambda)^*$ such that

$$\nabla_{v^h}^c = \underline{\nabla_{v^h}} + B\big(\pi_0(v)\big) + \alpha(v)\frac{(Jp^h)^v}{\|Jp^h\|^2} + \beta(v)\partial_{u_0}, \quad \forall v \in \mathcal{V}_c(\lambda) \tag{2.2.15}$$

where, as before, ∇ stands for the lifts to the Levi-Civita connection on $T^*\widetilde{M}$. Let us describe the operator B and the functionals α and β more precisely. First we prove the following lemma, using the property (1) of the parallel transport $\mathcal{K}^t$ listed in Subsection 2.2.2:

Lemma 2.4. *The linear operator B is antisymmetric w.r.t. the canonical Euclidean structure in $\mathcal{V}_c(\lambda)$.*

Proof. Fix a point $\bar{\lambda} \in T^*M$ and consider a small neighborhood U of $\bar{\lambda}$. Let $\mathcal{E}_c(\lambda) = \{\mathcal{E}_c^i(\lambda)\}_{i=1}^{n-3}$ be a frame of $\mathcal{V}_c(\lambda)$) (i.e. $\mathcal{V}_c(\lambda) = \mathrm{span}\{\mathcal{E}_c(\lambda)\}$) for any $\lambda \in U$ such that the following four conditions hold

1. $\mathcal{E}_c(\lambda)$ is orthogonal w.r.t. the canonical Euclidean structure on $\mathcal{V}_c(\lambda)$;

2. Each vector field $\mathcal{E}_c^i(\lambda)$ is parallel w.r.t the canonical parallel transport $\mathcal{K}_t$, i.e. $\mathcal{E}_c^i(e^t\vec{h}\lambda) = \mathcal{K}^t\mathcal{E}_c^i(\lambda)$ for any λ and t such that $\lambda, e^{t\vec{h}}\lambda \in U$;

3. The vector fields $(Jp^h)^v$ and $\mathcal{E}_c^i(\lambda)$ commute on $U \cap T^*_{\pi(\bar{\lambda})}M$;

4. The vector fields $\vec{u}_0$ and $\mathcal{E}_c^i(\lambda)$ commute on $U \cap T^*_{\pi(\bar{\lambda})}M$.

Note that the frame $\mathcal{E}_c(\lambda)$ with properties above exists, because the Hamiltonian vector field $\vec{h}$ is transversal to the fibers of T^*M and it commutes with $\vec{u}_0$.

¿From the property (2) of the parallel transport $\mathcal{K}^t$ (see property (2) in step 3 of Subsection 2.2.2) it follows that

$$\nabla^c_{(\mathcal{E}_c^i(\lambda))^h} = -\mathrm{ad}\vec{h}\,\mathcal{E}_c^i(\lambda) \tag{2.2.16}$$

Let $\widetilde{\mathcal{E}}^i(\lambda) = \pi_0(\mathcal{E}_c^i(\lambda))$ for $1 \leq i \leq n-3$ and $\widetilde{\mathcal{E}}^{n-2}(\lambda) = \frac{(Jp^h)^v}{\|Jp^h\|}$. Also let $\widetilde{\mathcal{E}}(\lambda) = \{\widetilde{\mathcal{E}}^i(\lambda)\}_{i=1}^{n-2}$. Using the above defined identification $I : \{u_0 = c\}/\Xi \to T^*\widetilde{M}$, where $c = u_0(\bar{\lambda})$, one can look on the restriction of the tuple of vector fields $\widetilde{\mathcal{E}}(\lambda)$ to the submanifold $\{u_0 = c\}$ as on the tuple of the vertical vector fields of $T^*\widetilde{M}$ (which actually span the tangent to the intersection of the fiber of $T^*\widetilde{M}$ with the level to the corresponding Riemannian Hamiltonian). Then first the tuple $\widetilde{\mathcal{E}}(\lambda)$ is the tuple of orthonormal vector fields (w.r.t. the canonical Euclidean structure on the fibers of $T^*\widetilde{M}$, induced by the Riemannian metric g). Further, by Remark 2.1 the Levi-Civita connection of g is characterized by the fact that there exists a field of antisymmetric operators $\widetilde{B} \in \mathrm{End}(\mathrm{span}\,\widetilde{\mathcal{E}}(\lambda))$ such that

$$[\nabla_{p^h}, \widetilde{\mathcal{E}}^i(\lambda)] = -\nabla_{\left(\widetilde{\mathcal{E}}^i(\lambda)\right)^h} - \widetilde{B}\widetilde{\mathcal{E}}^i(\lambda) \tag{2.2.17}$$

¿From (2.2.16) and (2.2.17), using (2.2.5),(2.2.12), and the property (3) of $\mathcal{E}_c^i(\lambda)$, one has

$$\begin{aligned}\nabla^c_{\left(\mathcal{E}_c^i(\lambda)\right)^h} &= -\mathrm{ad}\vec{h}\,\mathcal{E}_c^i(\lambda) = -[\underline{\nabla_{p^h}} - u_0(Jp^h)^v, \widetilde{\mathcal{E}}^i(\lambda) + \mathcal{A}(\lambda, \mathcal{E}^i)\frac{\partial_{u_0}}{\|Jp^h\|}] \\ &= \underline{\nabla_{\left(\widetilde{\mathcal{E}}^i(\lambda)\right)^h}} + \widetilde{B}\,\widetilde{\mathcal{E}}^i(\lambda) - \mathcal{A}(\lambda, \mathcal{E}^i(\lambda))\frac{(Jp^h)^v}{\|Jp^h\|} \quad \mathrm{mod}\ \mathbb{R}\partial_{u_0}.\end{aligned} \tag{2.2.18}$$

Note that one has the following orthogonal splitting of the space span$\{\widetilde{\mathcal{E}}(\lambda)\}$:

$$\mathrm{span}\{\widetilde{\mathcal{E}}(\lambda)\} = \widetilde{\mathcal{V}}_c(\lambda) \oplus \mathbb{R}(Jp^h)^v. \tag{2.2.19}$$

The operator B is exactly the endomorphism of $\widetilde{\mathcal{V}}_c(\lambda)$ such that $B\tilde{v}$ is the projection of $\widetilde{B}\tilde{v}$ to $\widetilde{\mathcal{V}}_c(\lambda)$ w.r.t. the splitting (2.2.19) for any $\tilde{v} \in \widetilde{\mathcal{V}}_c(\lambda)$. Obviously, the antisymmetricity of $\widetilde{B}$ implies the antisymmetricity of B. The proof of the lemma is completed. □

Now we are ready to find B explicitly using the fact that $\mathcal{V}_c^{\mathrm{trans}}(\lambda)$ is isotropic. For this let φ be the projection from $(\mathbb{R}p^h)^\perp$ to span$\{p^h, Jp^h\}^\perp$ parallel to Jp^h. Obviously,

$$\varphi(\tilde{v}) = \tilde{v} - g(\tilde{v}, Jp^h)\frac{Jp^h}{\|Jp^h\|^2}, \quad \forall \tilde{v} \in \widetilde{\mathcal{V}}_c(\lambda). \tag{2.2.20}$$

Lemma 2.5. *The operator B satisfies*

$$(B\tilde{v})^h = -\frac{u_0}{2}\varphi \circ J\tilde{v}^h, \quad \forall \tilde{v} \in \widetilde{\mathcal{V}}_c(\lambda) \tag{2.2.21}$$

or, equivalently,

$$B\tilde{v} = \frac{u_0}{2}\left(-(J\tilde{v}^h)^v + g(J\tilde{v}^h, Jp^h)\frac{(Jp^h)^v}{\|Jp^h\|^2}\right), \quad \forall \tilde{v} \in \widetilde{\mathcal{V}}_c(\lambda). \tag{2.2.22}$$

Proof. Since $\mathcal{V}_c^{\mathrm{trans}}(\lambda)$ is an isotropic subspace, we have

$$\sigma(\nabla^c_{v_1^h}, \nabla^c_{v_2^h}) = 0, \quad \forall v_1, v_2 \in \mathcal{V}_c$$

On the other hand, from (2.2.15) and the fact that $V_c^{\mathrm{trans}}(\lambda)$ lies in the skew symmetric complement of $\mathcal{V}_a(\lambda) \oplus \mathcal{V}_b(\lambda)$ it follows that

$$\sigma(\nabla^c_{v_1^h}, \nabla^c_{v_2^h}) = \sigma\Big(\underline{\nabla_{v_1^h}} + B\tilde{v}_1, \underline{\nabla_{v_2^h}} + B\tilde{v}_2\Big), \tag{2.2.23}$$

where $\tilde{v}_i = \pi_0(v_i)$, $i = 1, 2$. Then, using (2.2.3), the fact that the Levi-Civita connection (as an Ehresmann connection) is a Lagrangian distribution in $T^*\widetilde{M}$ and Lemma 2.2, we get

$$\begin{aligned}0 = \sigma(\nabla^c_{v_1^h}, \nabla^c_{v_2^h}) = \Big((I \circ \mathrm{PR})^*\tilde{\sigma} - u_0\pi^* d\omega_0\Big)\Big(\underline{\nabla_{v_1^h}} + B\tilde{v}_1, \underline{\nabla_{v_2^h}} + B\tilde{v}_2\Big) = \\ - u_0 d\omega_0(v_1^h, v_2^h) - g\big((B\tilde{v}_1)^h, v_2^h\big) + g\big((B\tilde{v}_2)^h, v_1^h\big) = \\ - u_0 g(Jv_1^h, v_2^h) - g\big((B\tilde{v}_1)^h, v_2^h\big) + g\big((B^*\tilde{v}_1)^h, v_2^h\big).\end{aligned}$$

Taking into account that B is antisymmetric, we get identity (2.2.21). Then, using relation (2.2.20) and Lemma 2.2, one easily gets identity (2.2.22). □

Further we need the following notation. Given a map $S : T^*M \oplus W_\lambda \longrightarrow \mathbb{R}$, define a map $S^{(1)} : T^*M \oplus T^*M \longrightarrow \mathbb{R}$ by

$$S^{(1)}(\lambda, v) = \frac{d}{dt}S(e^{t\vec{h}}\lambda, \mathcal{K}^t v)\Big|_{t=0}, \quad \lambda, v \in T^*M, \tag{2.2.24}$$

where in the second argument we use again the natural identification of $T^*_{\pi(\lambda)}M$ with $T_\lambda(T^*_{\pi(\lambda)}M)$.

Lemma 2.6. *The functionals α and β from (2.2.15) satisfy the following identities*

1. $\alpha(v) = -\sigma(\underline{\nabla_{v^h}}, ad\vec{h}\ (Jp^h)^v)$;

2. $\beta(v) = -\left(\frac{1}{\|Jp^h\|}\mathcal{A}\right)^{(1)}(\lambda, (v^h)^v) = -\frac{1}{\|Jp^h\|}\mathcal{A}^{(1)}(\lambda, (v^h)^v) - \vec{h}\left(\frac{1}{\|Jp^h\|}\right)\mathcal{A}(\lambda, (v^h)^v)$.

Proof. First, from step 2 in Subsection 2.2.2 it follows that for any $v \in \mathcal{V}_c(\lambda)$, we have

$$\begin{aligned}0 = \sigma(\nabla^c_{v^h}, \mathrm{ad}\vec{h}\ (Jp^h)^v) &= \sigma\Big(\underline{\nabla_{v^h}} + B\big(\pi_0(v)\big) + \alpha(v)\frac{(Jp^h)^v}{\|Jp^h\|^2} + \beta(v)\partial_{u_0}, \mathrm{ad}\vec{h}\ (Jp^h)^v\Big) = \\ &\quad \sigma(\underline{\nabla_{v^h}}, \mathrm{ad}\vec{h}\ (Jp^h)^v) + \alpha(v).\end{aligned}$$

Therefore, $\alpha(v) = -\sigma(\underline{\nabla_{v^h}}, \mathrm{ad}\vec{h}\ (Jp^h)^v)$.

Further, take the tuple of vertical vector fields $\mathcal{E}_c(\lambda) = \{\mathcal{E}_c^i(\lambda)\}_{i=1}^{n-3}$ as in the proof of Lemma 2.4. Then from (2.2.15),(2.2.16), and the fact that the vector fields $\vec{h}$ and $\vec{u}_o$ commute it follows that

$$\begin{aligned} \beta\big(\mathcal{E}_c^i(\lambda)\big) &= \sigma\big(\vec{u}_0, \nabla^c_{(\mathcal{E}_c^i(\lambda))^h}\big) = -\sigma(\vec{u}_0, \operatorname{ad}\vec{h}\mathcal{E}_c^i(\lambda)) \\ &= -[\vec{h}, \mathcal{E}_c^i(\lambda)](u_0) = -\big(\vec{h} \circ \mathcal{E}_c^i(\lambda)\big)(u_0) = -\vec{h}\big(\sigma(\vec{u}_0, \mathcal{E}_c^i(\lambda))\big). \end{aligned} \tag{2.2.25}$$

Then from by (2.2.12) it follows

$$\sigma(\vec{u}_0, \mathcal{E}_c^i(\lambda)) = \frac{1}{\|Jp^h\|}\mathcal{A}(\lambda, \widetilde{\mathcal{E}}_c^i(\lambda)). \tag{2.2.26}$$

The item (2) of the lemma follows immediately from (2.2.25) and (2.2.26). □

Step 4 According to the algorithm, described in Subsection 2.2.2, first find some vector field $\widetilde{\mathcal{F}}_a(\lambda)$ such that the tuple $\{\mathcal{E}_a(\lambda), \mathcal{E}_b(\lambda), \mathcal{E}_c(\lambda), \widetilde{\mathcal{F}}_a(\lambda), \mathcal{F}_b(\lambda), \mathcal{F}_c(\lambda)\}$ constitutes a Darboux frame. Let $\mathfrak{V}_0$ be a vector in $\mathcal{V}_c(\lambda)$ such that

$$\sigma(\mathfrak{V}_0, \nabla^c_{v^h}) = \beta(v), \quad \forall v \in \mathcal{V}_c(\lambda). \tag{2.2.27}$$

Also, let $\mathfrak{W}_0$ be a vector in $\mathcal{V}_c^{\text{trans}}(\lambda)$ such that

$$\sigma(v, \mathfrak{W}_0) = \mathcal{A}(\lambda, v), \quad \forall v \in \mathcal{V}_c(\lambda). \tag{2.2.28}$$

Note that by constructions the map $v \mapsto \nabla^c_{v^h}$ is an isomorphism between $\mathcal{V}_c(\lambda)$ and $\mathcal{V}_c^{\text{trans}}(\lambda)$. Let $\mathfrak{V}_1$ be a vector in $\mathcal{V}_c(\lambda)$ such that $\mathfrak{W}_0 = \nabla^c_{\mathfrak{V}_1^h}$. Then from (2.2.27) and (2.2.28) it follows that

$$\mathcal{A}(\lambda, \mathfrak{V}_0) = \beta(\mathfrak{V}_1). \tag{2.2.29}$$

Lemma 2.7. *A vector field $\widetilde{\mathcal{F}}_a(\lambda)$ can be taken in the following form*

$$\widetilde{\mathcal{F}}_a(\lambda) = -\|Jp^h\|\vec{u}_0 + \|Jp^h\|\mathfrak{V}_0 - \mathfrak{W}_0 + \|Jp^h\|(\vec{h})^2\left(\frac{1}{\|Jp^h\|}\right)\mathcal{E}_b(\lambda) - \|Jp^h\|\vec{h}\left(\frac{1}{\|Jp^h\|}\right)\mathcal{F}_b(\lambda) \tag{2.2.30}$$

Proof. Note that such vector field $\widetilde{\mathcal{F}}_a(\lambda)$ is defined modulo $\mathbb{R}\mathcal{E}_a(\lambda) = \mathbb{R}\partial_{u_0}$. Therefore we can look for $\widetilde{\mathcal{F}}_a(\lambda)$ in the form

$$\widetilde{\mathcal{F}}_a(\lambda) = \gamma_1\vec{u}_0 + \gamma_2\mathcal{E}_b(\lambda) + \gamma_3\mathcal{F}_b(\lambda) + v_c + \bar{v}_c, \tag{2.2.31}$$

where $v_c \in \mathcal{V}_c(\lambda)$ and $\bar{v}_c \in \mathcal{V}_c^{\text{trans}}(\lambda)$. Then

1. From relations $\sigma\big(\mathcal{E}_a(\lambda), \widetilde{\mathcal{F}}_a(\lambda)\big) = 1$ and (2.2.8) it follows that $\gamma_1 = -\|Jp^h\|$;
2. From relations $\sigma\big(\mathcal{E}_b(\lambda), \widetilde{\mathcal{F}}_a(\lambda)\big) = 0$ and (2.2.9) it follows that $\gamma_3 = -\|Jp^h\|\vec{h}\left(\frac{1}{\|Jp^h\|}\right)$;
3. From relations $\sigma\big(\mathcal{F}_b(\lambda), \widetilde{\mathcal{F}}_a(\lambda)\big) = 0$ and (2.2.10) it follows that $\gamma_2 = \|Jp^h\|(\vec{h})^2\left(\frac{1}{\|Jp^h\|}\right)$;
4. From relations $\sigma(\widetilde{\mathcal{F}}_a(\lambda), \nabla^c_v) = 0$ for any $v \in \mathcal{V}_c(\lambda)$ and the decomposition (2.2.15) it follows that $\sigma(v_c, \nabla^c_{v^h}) = \|Jp^h\|\beta(v)$ for any $v \in \mathcal{V}_c(\lambda)$. Hence $v_c = \|Jp^h\|\mathfrak{V}_0$;
5. From relations $\sigma(\widetilde{\mathcal{F}}_a(\lambda), v) = 0$ for any $v \in \mathcal{V}_c(\lambda)$ and relation (2.2.12) it follows that $\sigma(\bar{v}_c, v) = \mathcal{A}(\lambda, v)$ for any $v \in \mathcal{V}_c(\lambda)$. Hence $\bar{v}_c = -\mathfrak{W}_0$.

Combining items (1)-(5) above we get (2.2.31). □

The canonical $\mathcal{F}_a(\lambda)$ is obtained from $\widetilde{\mathcal{F}}_a(\lambda)$ by formula (2.2.2).

Now as a direct consequence of structure equation (2.2.1), we get the following preliminary descriptions of $(a, b)-$ curvature maps.

Proposition 2.2. *Let V be a parallel vector field such that $V(\lambda) = v$. Then the curvature maps satisfy the following identities:*

$$g((\mathfrak{R}_\lambda(c,c)v)^h, w^h) = -\sigma(ad\vec{h}\ \nabla^c_{V^h}, \nabla^c_{w^h}), \quad \forall w \in \mathcal{V}_c(\lambda) \tag{2.2.32}$$

$$\mathfrak{R}_\lambda(c,b)v = \sigma(ad\vec{h}\ \nabla^c_{V^h}, \mathcal{F}_b(\lambda))\frac{(Jp^h)^v}{\|Jp^h\|} = \sigma(ad\vec{h}\ \mathcal{F}_b(\lambda), \nabla^c_{v^h})\frac{(Jp^h)^v}{\|Jp^h\|} \tag{2.2.33}$$

$$\mathfrak{R}_\lambda(c,a)v = \sigma(ad\vec{h}\ \nabla^c_{V^h}, \mathcal{F}_a(\lambda))\partial_{u_0} \tag{2.2.34}$$

$$\mathfrak{R}_\lambda(b,b)(\frac{(Jp^h)^v}{\|Jp^h\|}) = -\sigma(ad\vec{h}\ \mathcal{F}_b(\lambda), \mathcal{F}_b(\lambda))(\frac{(Jp^h)^v}{\|Jp^h\|}) \tag{2.2.35}$$

$$\mathfrak{R}_\lambda(a,a)\partial_{u_0} = -\sigma(ad\vec{h}\ \mathcal{F}_a(\lambda), \mathcal{F}_a(\lambda))\partial_{u_0} \tag{2.2.36}$$

2.3 Calculus and the canonical splitting

2.3.1 Some useful formulas

Constructions of the previous section show that in order to calculate the $(a, b)-$ curvature maps it is sufficient to know how to express the Lie bracket of vector fields on the cotangent bundle T^*M via the covariant derivatives of the Levi-Civita connection on $T^*\widetilde{M}$. For this, we need special calculus which will be given in Proposition 2.3 below.

Let A be a tensor of type $(1, K)$ and B be a tensor of type $(1, N)$ on $\widetilde{M}$, $K, N \geq 0$. Define a new tensor $A \bullet B$ of type $(1, K + N - 1)$ by

$$A \bullet B(X_1, ..., X_{K+N-1}) = \sum_{i=0}^{K-1} A(X_1, ..., X_i, B(X_{i+1}, \dots X_{i+N}), X_{i+N+1}, ..., X_{K+N-1}).$$

This definition needs a clarification in the cases when either $K = 0$ or $N = 0$. If $K = 0$, then we set $A \bullet B = 0$, and if $N = 0$, i.e. B is a vector field on $\widetilde{M}$, then we set $A \bullet B(X_1, ..., X_{K-1}) = \sum_{i=0}^{K-1} A(X_1, ..., X_i, B, X_{i+1}, ..., X_{K-1})$. Also define by induction $A^{i+1} = A \bullet A^i$. For simplicity, in this subsection, we denote

$$Ap^h = A(\underbrace{p^h, p^h, ..., p^h}_{K}), \ Ap = (Ap^h)^v. \tag{2.3.1}$$

Besides, we denote by ∇A the covariant derivative (w.r.t. the Levi-Civita connection) of the tensor A, i.e. ∇A is a tensor of type $(1, K + 1)$ defined by

$$\nabla A(X_1, ..., X_K, X_{K+1}) = (\nabla_{X_{K+1}} A)(X_1, ..., X_K). \tag{2.3.2}$$

Also define by induction $\nabla^{i+1} A = \nabla(\nabla^i A)$.

Now we are ready to give several formulae, relating Lie derivatives w.r.t. the $\vec{h}$ and classical covariant derivatives, which will be the base for our further calculations:

Proposition 2.3. *The following identities hold:*

(1) $[Ap, Bp] = (B \bullet A)p - (A \bullet B)p$;

(2) $[\underline{\nabla_{Ap^h}}, Bp] = -\underline{\nabla_{(A\bullet B)p^h}} + ((\nabla_{Ap^h}B)p^h)^v$;

(3) $[\underline{\nabla_{Ap^h}}, \underline{\nabla_{Bp^h}}] = \underline{\nabla_{(\nabla_{Ap^h}B)p^h - (\nabla_{Bp^h}A)p^h}} + (R^\nabla(Ap^h, Bp^h)p^h)^v - \Omega(Ap^h, Bp^h)\vec{u}_0$,
where the 2*-form* Ω *is as in subsection 2.2.1 (recall that* $\Omega(X,Y) = g(JX,Y)$*).*

(4) $\underline{\nabla_p}(g(Ap^h, Bp^h)) = g((\nabla A)p^h, Bp^h) + g(Ap^h, (\nabla B)p^h)$.

Proof. Obviously, it is sufficient to prove all items of the proposition in the case, when the tensors A and B have the form $A = SX$ and $B = TY$, where S and T are tensors of the type $(0,K)$ and $(0,L)$ respectively and X and Y are vector fields. By analogy with (2.3.1), let

$$Sp^h = S(\underbrace{p^h, p^h, ..., p^h}_{K}) \text{ and } Tp^h = T(\underbrace{p^h, p^h, ..., p^h}_{L}).$$

Then directly from definitions we have

$$(A \bullet B)p^h = Bp(Sp^h)X, \tag{2.3.3}$$

where by $Bp(Sp^h)$ we mean the derivative of the function Sp^h in the direction Bp. Therefore

$$[Ap, Bp] = [Sp^hX^v, Tp^hY^v] = Ap(Tp^h)Y^v - Bp(Sp^h)X^v = (B \bullet A)p - (A \bullet B)p,$$

which completes the proof of item (1).

For the proof of the remaining items one can use the following scheme: First one shows that it is sufficient to prove them in the case $K = L = 0$, i.e. when A and B are vector fields in $\widetilde{M}$. Then one checks them in the latter case. As a matter of fact, the required identities in the latter case follow directly from the definitions of the Levi-Civita connection for items 2 and 4 and from the definition of the Riemannian curvature tensor for item (3), where the nonholonomicity of the distribution $\mathcal{D}$ causes the appearance of the additional term.

Let us prove item (2). The left-hand side of the required identity for $A = SX$ and $B = TY$ has the form

$$[\underline{\nabla_{Ap^h}}, Bp] = [\underline{\nabla_{Sp^hX}}, Tp^hY^v] = Sp^hX(Tp^h)Y^v - Bp(Sp^h)\underline{\nabla_X} + Sp^hTp^h[\underline{\nabla_X}, Y^v] \tag{2.3.4}$$

Using (2.3.3), the first term in the right-hand side of the required identity can be written as follows:

$$\underline{\nabla_{(A\bullet B)p^h}} = Bp(Sp^h)\underline{\nabla_X}. \tag{2.3.5}$$

Further, let us analyze the second term of the right-hand side of the required identity:

$$(\nabla_{Ap^h}B)p^h = (\nabla_{Sp^hX}Tp^hY)p^h = Sp^hX(Tp^h)Y + Sp^hTp^h\nabla_XY. \tag{2.3.6}$$

Comparing (2.3.4) with (2.3.5) and (2.3.6) we conclude that in order to prove the item (2) it is sufficient to show that $[\underline{\nabla_X}, (Y)^v] = (\nabla_XY)^v$. The last identity directly follows from the definition of the covariant derivative.

Let us prove item (3). The required identity is equivalent to the following one

$$[\underline{\nabla_{Ap^h}}, \underline{\nabla_{Bp^h}}] - \underline{\nabla_{(\nabla_{Ap^h}B)p^h - (\nabla_{Bp^h}A)p^h}} = (R^\nabla(Ap^h, Bp^h)p^h)^v - \Omega(Ap^h, Bp^h)\vec{u}_0. \tag{2.3.7}$$

Note that both sides of the last identity are tensorial: the result of the substitution $A = SX$ to both of them is equal to S multiplied by the result of the substitution of $A = X$ (and the same for the corresponding substitutions of B). Therefore it is sufficient to prove this identity in the case when $A = X$ and $B = Y$, where X and Y are vector fields on $\widetilde{M}$. Since the Levi-Civita connection is torsion-free, i.e. $\nabla_X Y - \nabla_Y X = [X, Y]$, the required identity in this case has the form

$$\left([\underline{\nabla_X}, \underline{\nabla_Y}] - \underline{\nabla_{[X,Y]}}\right)(\lambda) = (R^\nabla(X,Y)p^h)^v - \Omega(X,Y)\vec{u}_0(\lambda). \tag{2.3.8}$$

Let us prove identity (2.3.8). For this let $\mathcal{D}^L = \{v \in T_\lambda T^*M : \pi_* v \in \mathcal{D}_q\}$ be the pullback of the distribution D w.r.t. the canonical projection π. Then we have the following splitting of the tangent space $T_\lambda T^*M$ to the cotangent bundle at any point λ:

$$T_\lambda T^*M = \mathcal{D}^L(\lambda) \oplus \mathbb{R}\vec{u}_0. \tag{2.3.9}$$

Denote by π_1^L and π_2^L the projection onto $\mathcal{D}^L$ and the projection onto $\mathbb{R}\vec{u}_0$ w.r.t. the splitting (2.3.9), respectively. By definition, for any vector field Z on $\widetilde{M}$, one has $\underline{\nabla_Z} \in \mathcal{D}^L$. Thus by definition of the Riemannian curvature tensor,

$$(R^\nabla(X,Y)p^h)^v = \pi_1^L([\underline{\nabla_X}, \underline{\nabla_Y}](\lambda)) - \underline{\nabla_{[X,Y]}}(\lambda) \tag{2.3.10}$$

It remains only to prove that

$$\pi_2^L([\underline{\nabla_X}, \underline{\nabla_Y}]) = -\Omega(X,Y)\vec{u}_0. \tag{2.3.11}$$

Note that from (2.2.4) it follows that $\mathcal{D}^L$ is the symplectic complement of the vector field ∂_{u_0}. Besides, by definition, $\sigma(\vec{u}_0, \partial u_0) = 1$. Therefore,

$$\pi_2^L([\underline{\nabla_X}, \underline{\nabla_Y}]) = \sigma([\underline{\nabla_X}, \underline{\nabla_Y}], \partial_{u_0})\vec{u}_0 \tag{2.3.12}$$

Using again (2.2.4) and the definition of the form Ω we get

$$\sigma([\underline{\nabla_X}, \underline{\nabla_Y}], \partial_{u_0}) = \omega_0(\pi_*[\underline{\nabla_X}, \underline{\nabla_Y}]) = -d\omega_0(\pi_*\underline{\nabla_X}, \pi_*\underline{\nabla_Y}) = -\Omega(X,Y),$$

where ω_0 is the 1-form on M defined in Subsection 2.2.1. This completes the proof of the formula (2.3.11) and of the item (3).

Finally, let us prove item (4). As in the proof of item (2), we can substitute into the left-hand side and right-hand side of the required identity $A = SX$ and $B = TX$ to conclude that it is sufficient to show that

$$p^h(g(X,Y)) = g(\nabla_{p^h}Y, Y) + g(X, \nabla_{p^h}Y),$$

but the latter is actually the compatibility of the Levi-Civita connection with the Riemannian metric. □

Remark 2.3. *Note that if $K = 0$ then item (2) has the form*

$$[\underline{\nabla_A}, Bp] = ((\nabla_{A^h}B)p^h)^v \tag{2.3.13}$$

and if $N = 0$ then item (2) has the form

$$[\underline{\nabla_{Ap^h}}, B] = -\underline{\nabla_{(A\bullet B)p^h}}; \tag{2.3.14}$$

2.3.2 Calculation of the canonical splitting

Using formulae given by Proposition 2.3, we are ready to express the canonical splitting of W_λ $(= T_\lambda\mathcal{H}_{\frac{1}{2}}/\mathbb{R}\vec{h})$ in terms of the Riemannian structure and the tensor J on $\widetilde{M}$. Note that by (2.2.8) the subspace $\mathcal{V}_a(\lambda)$ is already expressed in this way. To express the subspace $\mathcal{V}_b(\lambda)$ and $\mathcal{V}_b^{\text{trans}}(\lambda)$ we need the following

Lemma 2.8. *The following identities hold:*

(1) $\vec{h}\left(\frac{1}{\|Jp^h\|}\right) = -\frac{1}{\|Jp\|^3}g(Jp^h, \nabla J(p^h, p^h))$;

(2) $(\vec{h})^2\left(\frac{1}{\|Jp^h\|}\right) = \frac{3}{\|Jp^h\|^5}g^2(Jp^h, \nabla J(p^h, p^h)) - \frac{1}{\|Jp^h\|^3}g(\nabla J(p^h, p^h), \nabla J(p^h, p^h))$
$- \frac{1}{\|Jp^h\|^3}g(Jp^h, \nabla^2 J(p^h, p^h, p^h)) + \frac{u_0}{\|Jp^h\|^3}\big(g(J^2p^h, \nabla J(p^h, p^h)) + g(Jp^h, \nabla J(Jp^h, p^h))$
$+ g(Jp^h, \nabla J(p^h, Jp^h))\big)$.

Proof. (1) Using item (4) of Proposition 2.3 we have

$$\underline{\nabla_{p^h}}\left(g(Jp^h, Jp^h)\right) = 2g(\nabla J(p^h, p^h), Jp^h); \tag{2.3.15}$$

Besides,

$$(Jp^h)^v\left(g(Jp^h, Jp^h)\right) = 2g(J^2p^h, Jp^h) = 0. \tag{2.3.16}$$

Combining the last two identities with (2.2.5) we immediately get the first item of the lemma.

(2) Using item (4) of Proposition 2.3, we get from (2.3.15) that

$$\left(\underline{\nabla_{p^h}}\right)^2\left(g(Jp^h, Jp^h)\right) = 2\underline{\nabla_{p^h}}\left(g(\nabla J(p^h, p^h), Jp^h)\right) = 2g(\nabla^2 J(p^h, p^h, p^h), Jp^h)$$
$$+2g(\nabla J(p^h, p^h), \nabla J(p^h, p^h));$$

Further,

$$(Jp^h)^v\left(g(\nabla J(p^h, p^h), Jp^h)\right) =$$
$$\left(g(\nabla J(Jp^h, p^h), Jp^h)\right) + \left(g(\nabla J(p^h, Jp^h), Jp^h)\right) + \left(g(\nabla J(p^h, p^h), J^2p^h)\right)$$

Using the last two identities together with (2.3.16), one can get the second item of the lemma by straightforward computations. □

Now substituting item (1) of Lemma 2.8 into (2.2.9) we get the expression for the subspace $\mathcal{V}_b(\lambda)$. Now let us find the expression for $\mathcal{V}_b^{\text{trans}}(\lambda)$. First by (2.2.5) and item (2) of Proposition 2.3 we have

$$[\vec{h}, (Jp^h)^v] = [\underline{\nabla_{p^h}} - u_0(Jp^h)^v, (Jp^h)^v] = -\underline{\nabla_{Jp^h}} + (\nabla J(p^h, p^h))^v \tag{2.3.17}$$

Substituting the last formula and the items (1) and (2) of Lemma 2.8 into (2.2.10) we will get the required expression for $\mathcal{V}_b^{\text{trans}}(\lambda)$.

Further, according to (2.2.12) in order to find the expression for $\mathcal{V}_c(\lambda)$ we have to express $\mathcal{A}(\lambda, v)$.

Lemma 2.9. *Let* $v \in \Pi_\lambda$. *Then*

$$\mathcal{A}(\lambda, v) = \frac{2}{\|Jp^h\|}g(v^h, \nabla J(p^h, p^h)) - \frac{u_0}{\|Jp^h\|}g(v^h, J^2p^h). \tag{2.3.18}$$

Proof. Using relation (2.3.17) and items (2) and (3) of Proposition 2.3, we get

$$(\mathrm{pr}\circ\pi)_*\big((\mathrm{ad}\vec{h})^2(Jp^h)^v\big) = -2\nabla J(p^h,p^h) + u_0 J^2 p^h.$$

Then

$$\begin{aligned}\sigma(v, \frac{1}{\|Jp^h\|}\mathrm{ad}^2\vec{h}(Jp^h)^v) &= \frac{1}{\|Jp^h\|}\sigma(v, -2\nabla J(p^h,p^h) + u_0J^2p^h + \|Jp^h\|^2p^h)\\ &= \frac{2}{\|Jp^h\|}g(v^h, \nabla J(p^h,p^h)) - \frac{u_0}{\|Jp^h\|}g(v^h, J^2p^h),\end{aligned}$$

which completes the proof of the lemma. □

In order to express $\mathcal{V}_c^{\mathrm{trans}}(\lambda)$ it is sufficient to express the operator B and functionals α and β, defined by (2.2.15). The operator B is already expressed by (2.2.22). Further, from decomposition (2.2.3), Lemma 2.2, and the fact that the Levi-Civita connection is a Lagrangian distribution it follows that

$$\begin{aligned}\alpha(v) &= -\sigma(\underline{\nabla_{v^h}}, \underline{-\nabla_{Jp^h}} + (\nabla J(p^h,p^h))^v) &(2.3.19)\\ &= -u_0 d\omega_0(v^h, Jp^h) - g(v^h, \nabla J(p^h,p^h))\\ &= u_0 g(v^h, J^2p^h) - g(v^h, \nabla J(p^h,p^h))\end{aligned}$$

Note that from (2.2.22), (2.3.18), and (2.3.19) it follows by straightforward computations that

$$B\big(\pi_0(v)\big) + \alpha(v)\frac{(Jp^h)^v}{\|Jp^h\|^2} = -\frac{u_0}{2}(Jv^h)^v - \frac{1}{2}\mathcal{A}(\lambda, v)\frac{(Jp^h)^v}{\|Jp^h\|}. \qquad (2.3.20)$$

To derive the formula for β we need to study the operator $\mathcal{A}^{(1)}$. For later use we will work in more general setting. Let $\mathfrak{S}$ be a tensor of type $(1,K)$ on $\widetilde{M}$. This tensor induces a map $S: T^*M \oplus T^*M \longrightarrow \mathbb{R}$ by

$$S(\lambda, v) = g(\mathfrak{S}p^h, v^h),\ \lambda = (p,q) \in T^*M, p \in M, p \in T_q^*M. \qquad (2.3.21)$$

where $\mathfrak{S}p^h$ is as in (2.3.1).

Proposition 2.4. *Let* $v \in \mathcal{V}_c(\lambda)$.

$$S^{(1)}(\lambda, v) = -\frac{1}{2}S\left(\lambda, \frac{(Jp^h)^v}{\|Jp^h\|}\right)\mathcal{A}(\lambda, v) + g(v^h, (\nabla\mathfrak{S})p^h - u_0(\mathfrak{S}\bullet J)p^h + \frac{1}{2}u_0(J\bullet\mathfrak{S})p^h)$$

Proof. Take $v \in \mathcal{V}_c(\lambda)$ and let $\tilde{v} = \pi_0(v)$. Let V and $\widetilde{V}$ be parallel vector fields such that $V(\lambda) = v$ and $\widetilde{V}(\lambda) = \tilde{v}$. We first show that the following identity holds.

$$[\vec{h}, \widetilde{V}](\lambda) = -\nabla_{\tilde{v}^h} - \frac{1}{2}\mathcal{A}(\lambda, \tilde{v})\frac{(Jp^h)^v}{\|Jp^h\|} + \frac{u_0}{2}(J\tilde{v}^h)^v. \qquad (2.3.22)$$

For this first by (3.4.2) and (2.2.16) we have

$$[\vec{h}, V](\lambda) = -\nabla_{\tilde{v}^h} - B(\tilde{v}) - \alpha(v)\frac{(Jp^h)^v}{\|Jp^h\|^2} - \beta(v)\partial_{u_0}. \qquad (2.3.23)$$

On the other hand from (2.2.12) it follows that $v = \tilde{v} + \mathcal{A}(\lambda, \tilde{v})\mathcal{E}_a(\lambda)$. Hence from (2.2.8), (2.2.9), and the second relation of Lemma 2.6 one gets

$$\begin{aligned}&[\vec{h}, V](\lambda) - [\vec{h}, \widetilde{V}](\lambda) = [\vec{h}, \mathcal{A}(\lambda, \tilde{v})\mathcal{E}_a(\lambda)]\\ &= \mathcal{A}(\lambda, \tilde{v})\frac{(Jp^h)^v}{\|Jp^h\|} + \Big(\frac{1}{\|Jp^h\|}\mathcal{A}\Big)^{(1)}(\lambda, \tilde{v})\partial_{u_0} = \mathcal{A}(\lambda, \tilde{v})\frac{(Jp^h)^v}{\|Jp^h\|} - \beta(v)\partial_{u_0}.\end{aligned}$$

Therefore, by (2.3.23) and (2.3.20) we have

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} e^{-t\vec{h}}\tilde{v}(t) &= -\nabla_{\tilde{v}^h} - B(\tilde{v}) - \alpha(v)\frac{(Jp^h)^v}{\|Jp^h\|^2} - \mathcal{A}(\lambda,\tilde{v})\frac{(Jp^h)^v}{\|Jp^h\|} \\ &= -\nabla_{\tilde{v}^h} - \frac{1}{2}\mathcal{A}(\lambda,\tilde{v})\frac{(Jp^h)^v}{\|Jp^h\|} + \frac{u_0}{2}(J\tilde{v}^h)^v \end{aligned}$$

The proof of (2.3.22) is completed.

Further, from Lemma 2.2 and definition of S given by (2.3.21) it follows that

$$S(\lambda,v) = \sigma(v, \underline{\nabla_{\mathfrak{S}p^h}})$$

$$S^{(1)}(\lambda,v) = \sigma\left([\vec{h},\tilde{V}(\lambda)], \underline{\nabla_{\mathfrak{S}p^h}}\right) + \sigma(\tilde{v}, [\vec{h}, \nabla_{\mathfrak{S}p^h}]) \tag{2.3.24}$$

The first term in identity (2.3.24) can be calculated using the relation (2.3.22) and Lemmas 2.2 and 2.3. Then we apply Proposition 2.3 and relation (2.3) to get $(\mathrm{pr}\circ\pi)_*\left(\mathrm{ad}\vec{h}(\nabla_{\mathfrak{S}p^h})\right) = (\nabla\mathfrak{S})p^h - u_0(\mathfrak{S}\bullet J)p^h$ and we can calculate the second term using again Lemma 2.2. Putting all the calculations together, we completed the proof of the proposition. □

As a straightforward consequence of the previous Proposition and lemma 2.8 we get

Corollary 2.1. *Let $v \in \mathcal{V}_c(\lambda)$.*

$$\begin{aligned} \mathcal{A}^{(1)}(\lambda,v) = \frac{1}{\|Jp^h\|} g\Big(v^h, 2\nabla^2 J(p^h,p^h,p^h) - 3u_0\nabla J(Jp^h,p^h) \\ -2u_0\nabla J(p^h,Jp^h) + \frac{1}{2}u_0^2 J^3 p^h\Big) - \mathcal{A}(\lambda,v)\mathcal{A}\left(\lambda, \frac{(Jp^h)^v}{\|Jp^h\|}\right). \end{aligned} \tag{2.3.25}$$

The function β can be expressed by substituting (2.3.25) and item (1) of Lemma 2.8 into item (2) of Lemma 2.6. In this way one gets the required expression for the subspace $\mathcal{V}_c^{\mathrm{trans}}(\lambda)$. To summarize, we have

$$\nabla^c_{v^h} = \underline{\nabla_{v^h}} - \frac{1}{2}\mathcal{A}(\lambda,v)\frac{(Jp^h)^v}{\|Jp^h\|} - \frac{u_0}{2}(Jv^h)^v + \beta(v)\partial_{u_0}. \tag{2.3.26}$$

To finish the representation of the canonical splitting, we find more detailed expression for $\mathcal{V}_a^{\mathrm{trans}}(\lambda) = \mathbb{R}\mathcal{F}_a(\lambda)$ on the base of equations (2.2.2) and (2.2.30). For this we will describe the properties of vectors $\mathfrak{V}_0$, $\mathfrak{V}_1$, and $\mathfrak{W}_0$ from Step 4 of Subsection 2.2.3 which will be used in the calculations of the curvature maps (section 2.4).

Lemma 2.10. *Let $v \in \mathcal{V}_c(\lambda)$ and V be a parallel vector field such that $V(\lambda) = v$. Then the following identities hold:*

(1) $\mathfrak{V}_1^h = (\mathrm{pr}\circ\pi)_*\mathfrak{W}_0 = -\frac{2}{\|Jp^h\|}\nabla J(p^h,p^h) + \frac{u_0}{\|Jp^h\|}J^2p^h + u_0\|Jp^h\|p^h + \frac{2}{\|Jp^h\|^3}g(\nabla J(p^h,p^h),Jp^h)Jp^h.$

(2) $\sigma\Big(\mathfrak{W}_0, ad\vec{h}(\nabla^c_{V^h})\Big) = g\Big((\mathfrak{R}_\lambda(c,c)v)^h, \mathfrak{V}_1^h\Big),$

$\sigma(\mathfrak{W}_0, ad\vec{h}\mathcal{F}_b(\lambda)) = -g\Big((\mathfrak{R}_\lambda(c,b)\mathfrak{V}_1)^h, \frac{Jp^h}{\|Jp^h\|}\Big);$

Proof.

(1) ¿From (2.2.28) and Lemma 2.9 it follows that

$$(\mathrm{pr}\circ\pi)_*\mathfrak{W}_0 = -\frac{2}{\|Jp^h\|}\nabla J(p^h,p^h) + \frac{u_0}{\|Jp^h\|}J^2p^h, \ \mathrm{mod}\,\mathrm{span}\{p^h, Jp^h\}.$$

Note that by constructions $(\mathrm{pr}\circ\pi)_*\mathfrak{W}_0 \in \mathrm{span}\{p^h, Jp^h\}^\perp$. Let us work with the orthogonal splitting $T_q\widetilde{M} = \mathrm{span}\{p^h, Jp^h\}^\perp \oplus \mathbb{R}p^h \oplus \mathbb{R}Jp^h$. Assume that the vector $\frac{2}{\|Jp^h\|}\nabla J(p^h,p^h) - \frac{u_0}{\|Jp^h\|}J^2p^h$ has the following decomposition w.r.t. this splitting:

$$\frac{2}{\|Jp^h\|}\nabla J(p^h,p^h) - \frac{u_0}{\|Jp^h\|}J^2p^h = -(\mathrm{pr}\circ\pi)_*\mathfrak{W}_0 + \gamma_1 p^h + \gamma_2 Jp^h.$$

Then

$$\gamma_1 = g\left(\frac{2}{\|Jp^h\|}\nabla J(p^h,p^h) - \frac{u_0}{\|Jp^h\|}J^2p^h, p^h\right).$$

Note that $g(\nabla J(p^h,p^h),p^h) = \nabla^h_p g(Jp^h,p^h) = 0$. So, $\gamma_1 = u_0\|Jp^h\|$.

Finally,

$$\gamma_2 = \frac{1}{\|Jp^h\|^2}g\left(\frac{2}{\|Jp^h\|}\nabla J(p^h,p^h) - \frac{u_0}{\|Jp^h\|}J^2p^h, Jp^h\right)$$

Note that since J is antisymmetric, we have $g(J^2p^h, Jp^h) = 0$. Therefore,

$$\gamma_2 = \frac{2}{\|Jp^h\|^3}g(\nabla J(p^h,p^h), Jp^h),$$

which completes the proof of item (1).

(2) Relations in this item are direct consequences of relations (2.2.32) and (2.2.33) respectively.

□

2.4 Curvature maps via the Riemannian curvature tensor and the tensor J on $\widetilde{M}$

Let $\lambda = (p,q)$, $q \in M$, $p \in T_q^*M$ be the given D-regular point, as before. Fix $v \in \mathcal{V}_c(\lambda)$. As before, denote by R^∇ the Riemannian curvature tensor.

Theorem 2.1. *The curvature map $\mathfrak{R}_\lambda(c,c)$ can be represented as follows*

$$g\Big(\big(\mathfrak{R}_\lambda(c,c)(v)\big)^h, v^h\Big) = g(R^\nabla(p^h,v^h)p^h,v^h) + u_0 g(v^h,\nabla J(p^h,v^h)) + \frac{u_0^2}{4}\|Jv^h\|^2 - \frac{1}{4}\mathcal{A}^2(\lambda,v),$$

where $\mathcal{A}$ is as in (2.3.18)

Proof. Take $v \in \mathcal{V}_c(\lambda)$ and parallel vector fields V such that $V(\lambda) = v$. As in the proof of Lemma 2.4 we can take V such that

$$[(Jp^h)^v, V](\bar\lambda) = 0, \quad \bar\lambda \in U \cap T_q^*M, \tag{2.4.1}$$

where U is a neighborhood of λ. For simplicity denote $\bar\sigma = (I\circ\mathrm{PR})^*\tilde\sigma$.

Recall that by Proposition 2.2, (relation (2.2.32) there)

$$g((\mathfrak{R}_\lambda(c,c)v)^h, w^h) = -\sigma(\mathrm{ad}\vec{h}\ \nabla^c_{V^h}, \nabla^c_{v^h}).$$

Let us simplify the right-hand side of the last identity. First, from the last line of the structural equations (2.2.1) it follows that

$$(\mathrm{pr}\circ\pi)_*(\mathrm{ad}\vec{h}(\nabla^c_{V^h}))\in\mathbb{R}p^h. \tag{2.4.2}$$

Then from (2.3.26) it follows that

$$\sigma(\mathrm{ad}\vec{h}(\nabla^c_{V^h}),\nabla^c_{v^h})=\sigma(\mathrm{ad}\vec{h}(\nabla^c_{V^h}),\underline{\nabla_{v^h}}) \tag{2.4.3}$$

Further, from the decomposition (2.2.3) it follows that the form $u_0\pi^*d\omega_0=\sigma-\bar{\sigma}$ is semi-basic (i.e. its interior product with any vertical vector field is zero). Besides, since $v\in\mathcal{V}_c(\lambda)$, from (2.2.11) it follows that $\pi^*d\omega_0(\vec{h},\underline{\nabla_{v^h}})=g(Jp^h,v^h)=0$. Therefore,

$$g\big((\mathfrak{R}_\lambda(c,c)v)^h,v^h\big)=-\bar{\sigma}(\mathrm{ad}\vec{h}\ \nabla^c_{V^h},\nabla_{v^h}). \tag{2.4.4}$$

Also, from relation (2.2.32) it follows that it is enough to consider $\mathrm{ad}\vec{h}\ \nabla^c_{V^h}$ modulo $\mathcal{V}_a(\lambda)\oplus\mathcal{V}_b(\lambda)$.

We also need the following

Lemma 2.11. *Let V,W be vector fields of T^*M such that $\pi_*V=\pi_*W=0$. Then*

(1) $([(Jp^h)^v,(JV^h)^v])^h=J([(Jp^h)^v,(V^h)^v])^h$.

(2) $\sigma([(Jp^h)^v,\underline{\nabla_{V^h}}],\underline{\nabla_{W^h}})=-g(W^h,\nabla J(p^h,V^h))$.

Proof. (1) It is clear that if item (1) holds for vector field V then also holds for vector field aV. Thus in order to prove item (1) it is sufficient to prove it when V is constant on the fibers of T^*M, i.e., when V^h is a vector field on $\widetilde{M}$. But in this case from item 1 of Proposition 2.3 for $K=1,N=0$ it follows that both sides of the formula of our item 1 are equal to $-J^2v^h$.

(2) Both sides are linear on vector field V, thus it is sufficient to prove it when V is constant on the fibers of T^*M, which is a direct consequence of identity (2.3.13) and Lemma 2.2. □

Now we are ready to start our calculations:

$$\begin{aligned}\mathrm{ad}\vec{h}(\nabla^c_{V^h}) &= [\underline{\nabla_{p^h}},\underline{\nabla_{V^h}}]-u_0[(Jp^h)^v,\underline{\nabla_{V^h}}]-\frac{\mathcal{A}(\lambda,v)}{2\|Jp^h\|}[\underline{\nabla_{p^h}},(Jp^h)^v]\\ &-\frac{u_0}{2}[\underline{\nabla_{p^h}},(JV^h)^v]+\frac{u_0^2}{2}[(Jp^h)^v,(JV^h)^v],\quad \mathrm{mod}\,\mathcal{V}_a(\lambda)\oplus\mathcal{V}_b(\lambda)\end{aligned} \tag{2.4.5}$$

Note that the last term of (2.4.5) vanishes by item (1) of Lemma 2.11 and relation (2.4.1). Therefore, by (2.4.4),

$$\begin{aligned}&g\big((\mathfrak{R}_\lambda(c,c)v)^h,v^h\big)=-\bar{\sigma}([\underline{\nabla_{p^h}},\underline{\nabla_{V^h}}],\underline{\nabla_{v^h}})+u_0\bar{\sigma}([(Jp^h)^v,\underline{\nabla_{V^h}}],\underline{\nabla_{v^h}})+\\ &\frac{\mathcal{A}(\lambda,v)}{2\|Jp^h\|}\bar{\sigma}([\underline{\nabla_{p^h}},(Jp^h)^v],\underline{\nabla_{v^h}}))+\frac{u_0}{2}\bar{\sigma}([\underline{\nabla_{p^h}},(JV^h)^v],\underline{\nabla_{v^h}})\end{aligned} \tag{2.4.6}$$

Now we analyze the right-hand side of the last equation term by term. First, it follows from identity (2.1.5) that

$$\bar{\sigma}([\underline{\nabla_{p^h}},\underline{\nabla_{V^h}}],\underline{\nabla_{v^h}})=-g(R^\nabla(p^h,v^h)p^h,v^h). \tag{2.4.7}$$

Also it follows from item (2) of Lemma 2.11 that

$$\bar{\sigma}([(Jp^h)^v,\underline{\nabla_{V^h}}],\underline{\nabla_{v^h}})=g(\nabla J(p^h,v^h),v^h). \tag{2.4.8}$$

Also it follows from identity (2.3.17) that

$$\bar{\sigma}([\underline{\nabla_{p^h}},(Jp^h)^v],\underline{\nabla_{v^h}}))=g(v^h,\nabla J(p^h,p^h)). \tag{2.4.9}$$

To analyze the fourth term of (2.4.6) we need the following

Lemma 2.12. *The following identity holds:*

$$(\mathrm{pr}\circ\pi)_*([\underline{\nabla_{p^h}},\underline{\nabla_{v^h}})]) = \frac{u_0}{2}(Jv^h)^v - \frac{1}{2}\mathcal{A}(\lambda, v)\frac{(Jp^h)^v}{\|Jp^h\|} \quad \mathrm{mod}\ \mathbb{R}p^h. \tag{2.4.10}$$

Proof. First, it follows from the equations (2.1.3) and the identity (2.2.17) that

$$(\mathrm{pr}\circ\pi)_*([\underline{\nabla_{p^h}},\underline{\nabla_{v^h}})]) = -\widetilde{B}v^h, \ \mathrm{mod}\ \mathbb{R}p^h$$

, where $\widetilde{B}$ is as in (2.2.17). Further, comparing identities (2.2.18) and (2.3.26), we get $\widetilde{B}(v^h)^v = -\frac{u_0}{2}(Jv^h)^v + \frac{1}{2}\mathcal{A}(\lambda, v)\frac{(Jp^h)^v}{\|Jp^h\|}$. The proof of the proposition is completed. □

Finally, it follows from identity (2.4.10) that

$$\bar{\sigma}([\underline{\nabla_{p^h}}, (JV^h)^v], \underline{\nabla_{v^h}}) = \bar{\sigma}(\pi_*([\underline{\nabla_{p^h}}, \underline{\nabla_{v^h}}]), (Jv^h)^v) = g(\frac{u_0}{2}(Jv^h)^v - \frac{1}{2}\mathcal{A}(\lambda, v)\frac{(Jp^h)^v}{\|Jp^h\|}, Jv^h). \tag{2.4.11}$$

Substituting identities (2.4.7), (2.4.8), (2.4.9), and (2.4.11) into (2.4.6), we get the required expression for $\mathfrak{R}_\lambda(c, c)$. □

Theorem 2.2. *The curvature maps $\mathfrak{R}_\lambda(c, b)$ and $\mathfrak{R}_\lambda(c, b)$ can be represented as follows*

1) $\mathfrak{R}_\lambda(c, b)v = \rho_\lambda(c, b)(v)\mathcal{E}_b(\lambda)$, *where* $\rho_\lambda(c, b) \in \mathcal{V}_c(\lambda)^*$ *and it satisfies*

$$\begin{aligned}\rho_\lambda(c, b)(v) &= \frac{1}{\|Jp^h\|}g(R^\nabla(p^h, Jp^h)p^h, v^h) - \frac{3}{\|Jp^h\|}g(v^h, \nabla^2 J(p^h, p^h, p^h)) \\ &+ \frac{4u_0}{\|Jp^h\|}g(v^h, \nabla J(Jp^h, p^h) + \nabla J(p^h, Jp^h)) + \frac{u_0^2}{\|Jp^h\|}g(Jv^h, J^2p^h) \\ &+ \frac{8}{\|Jp^h\|^3}g(Jp^h, \nabla J(p^h, p^h))g(v^h, \nabla J(p^h, p^h)) - \frac{4u_0}{\|Jp^h\|^3}g(Jp^h, \nabla J(p^h, p^h))g(v^h, J^2p^h);\end{aligned}$$

2) $\mathfrak{R}_\lambda(b, b)\mathcal{E}_b(\lambda) = \rho_\lambda(b, b)\mathcal{E}_b(\lambda)$, *where*

$$\begin{aligned}\rho_\lambda(b, b) &= \frac{1}{\|Jp^h\|^2}g(R^\nabla(Jp^h, p^h)Jp^h, p^h) - \frac{10}{\|Jp^h\|^4}g^2(\nabla J(p^h, p^h), Jp^h) \\ &+ \frac{6}{\|Jp^h\|^2}\|\nabla J(p^h, p^h)\|^2 + \frac{3}{\|Jp^h\|^2}g(Jp^h, \nabla^2 J(p^h, p^h, p^h)) - \frac{2u_0}{\|Jp^h\|^2}g(Jp^h, \nabla J(p^h, Jp^h)) \\ &- \frac{3u_0}{\|Jp^h\|^2}g(Jp^h, \nabla J(Jp^h, p^h)) - \frac{6u_0}{\|Jp^h\|^2}g(J^2p^h, \nabla J(p^h, p^h)) + \frac{u_0^2}{\|Jp^h\|^2}\|J^2p^h\|^2\end{aligned}$$

Sketch of the proof. Recall that by Proposition 2.2 (relations (2.2.33) and (2.2.35) there)

$$\begin{aligned}\rho_\lambda(c, b)v &= \sigma(\mathrm{ad}\vec{h}\ \mathcal{F}_b(\lambda), \nabla^c_{v^h}) \\ \rho_\lambda(b, b) &= -\sigma(\mathrm{ad}\vec{h}\ \mathcal{F}_b(\lambda), \mathcal{F}_b(\lambda)).\end{aligned} \tag{2.4.12}$$

First it follows from (2.2.10) that

$$\begin{aligned}\mathrm{ad}\vec{h}\ \mathcal{F}_b(\lambda) &= \frac{1}{\|Jp^h\|}(\mathrm{ad}\vec{h})^2(Jp^h)^v + 3\vec{h}\left(\frac{1}{\|Jp^h\|}\right)(\mathrm{ad}\vec{h})(Jp^h)^v \\ &+ 3(\vec{h})^2\left(\frac{1}{\|Jp^h\|}\right)(Jp^h)^v + (\vec{h})^3\left(\frac{1}{\|Jp^h\|}\right)\partial_{u_0}\end{aligned} \tag{2.4.13}$$

Note that the last two terms of (2.4.13) belong to the space $\mathcal{V}_a(\lambda) \oplus \mathcal{V}_b(\lambda)$, which lies in the skew-symmetric complement of $\nabla^c_{vh} \in \mathcal{V}^{\text{trans}}_c(\lambda)$ w.r.t. σ. Therefore

$$\rho_\lambda(c,b)v = \sigma\left(\frac{1}{\|Jp^h\|}(\text{ad}\vec{h})^2(Jp^h)^v + 3\vec{h}\left(\frac{1}{\|Jp^h\|}\right)(\text{ad}\vec{h})(Jp^h)^v, \nabla^c_{vh}\right) \tag{2.4.14}$$

In a similar way, since $\mathcal{V}_a(\lambda) = \mathbb{R}\partial_{u_o}$, we have $\sigma(\partial_{u_0}, \mathcal{F}_b(\lambda)) = 0$. Therefore

$$\rho_\lambda(b,b) = -\sigma(\frac{1}{\|Jp^h\|}(\text{ad}\vec{h})^2(Jp^h)^v + 3\vec{h}\left(\frac{1}{\|Jp^h\|}\right)(\text{ad}\vec{h})(Jp^h)^v + 3(\vec{h})^2\left(\frac{1}{\|Jp^h\|}\right)(Jp^h)^v, \mathcal{F}_b(\lambda)). \tag{2.4.15}$$

Note that $(\text{ad}\vec{h})(Jp^h)^v$ is computed in (2.3.17) and $(\vec{h})^2\left(\frac{1}{\|Jp^h\|}\right)$ is computed in item (2) of Lemma 2.8. Furthermore, from relations (2.3.17) and (2.2.5), using items (1), (2), and (3) of Proposition 2.3, it follows that

$$\begin{aligned}(\text{ad}\vec{h})^2(Jp^h)^v &= [\underline{\nabla_{p^h}} - u_0(Jp^h)^v, -\underline{\nabla_{(Jp^h)^v}} + (\nabla J(p^h,p^h))^v] \\ &= -2\underline{\nabla_{\nabla J(p^h,p^h)}} + u_0\underline{\nabla_{J^2p^h}} + \|Jp^h\|^2\vec{u}_0 - (R^\nabla(p^h,Jp^h)p^h)^v + \nabla^2 J(p^h,p^h,p^h) \\ &\quad - u_0(\nabla J(Jp^h,p^h))^v - 2u_0(\nabla J(p^h,Jp^h))^v + u_0(J\nabla J(p^h,p^h))^v\end{aligned} \tag{2.4.16}$$

Substituting all this into (2.4.14) and (2.4.15) and using identity (2.2.3) and Proposition 2.3 one can get both items of the theorem by long but straightforward computations. □

Further, let $\mathfrak{V}_1$ be as in Step 4 of Subsection 2.2.3. Note that the expression for $\mathfrak{V}^h_1$ can be found in item (2) of Lemma 2.10.

Theorem 2.3. *The curvature maps $\mathfrak{R}_\lambda(c,a)$ and $\mathfrak{R}_\lambda(a,a)$ can be represented as follows*

$$\begin{aligned}\mathbf{1)}\ \mathfrak{R}_\lambda(c,a)v &= \rho_\lambda(c,a)(v)\frac{\partial_{u_0}}{\|Jp^h\|}, \text{ where } \rho_\lambda(c,a) \in \mathcal{V}_c(\lambda)^* \text{ and it satisfies} \\ \rho_\lambda(c,a)v &= \|Jp^h\|\left(\frac{1}{\|Jp^h\|}\mathcal{A}\right)^{(2)}(\lambda,v) - g\Big((\mathfrak{R}_\lambda(c,c)v)^h, \mathfrak{V}^h_1\Big) + \|Jp^h\|\vec{h}\left(\frac{1}{\|Jp^h\|}\right)\rho_\lambda(c,b)v \\ \mathbf{2)}\ \mathfrak{R}_\lambda(a,a)\partial_{u_0} &= \rho_\lambda(a,a)\partial_{u_0}, \text{ where } \rho_\lambda(c,a) \in \mathcal{V}_c(\lambda)^* \text{ and it satisfies} \\ \rho_\lambda(a,a) &= \vec{h}\left(\rho_\lambda(c,b)(\mathfrak{V}^h_1)\right) + \|Jp^h\|\vec{h}\left(\frac{1}{\|Jp^h\|}\right)\vec{h}(\rho_\lambda(b,b)) + \rho_\lambda(c,a)(\mathfrak{V}_1) \\ &- \|Jp^h\|\vec{h}\left(\frac{1}{\|Jp^h\|}\right)\rho_\lambda(c,b)(\mathfrak{V}_1) + \|Jp^h\|\vec{h}^2\left(\frac{1}{\|Jp^h\|}\right)\rho_\lambda(b,b) + \|Jp^h\|\vec{h}^4\left(\frac{1}{\|Jp^h\|}\right)\end{aligned}$$

where $\rho_\lambda(c,b)$ and $\rho_\lambda(b,b)$ are as in Theorem 2.2, $\mathcal{A}$ is expressed in (2.3.18) *and $\mathfrak{W}^h_1$ is expressed by item (1) of Lemma 2.10.*

Proof: **1)** Recall that by Proposition 3.2, (relation (2.2.34) there)

$$\rho_\lambda(c,a)v = \sigma(\text{ad}\vec{h}\nabla^c_{Vh}, \mathcal{F}_a(\lambda)) \tag{2.4.17}$$

Since $\mathcal{E}_a(\lambda)$ lies in the skew-symmetric complement of $\mathcal{F}^{\text{trans}}_c(\lambda)$ w.r.t. σ, then it follows from relations (2.2.2) and (2.4.17) that

$$\rho_\lambda(c,a)v = \sigma(\text{ad}\vec{h}\nabla^c_{Vh}, \widetilde{\mathcal{F}}_a(\lambda)) \tag{2.4.18}$$

Further it follows from relations (2.2.30) and (2.4.2) that

$$\rho_\lambda(c,a)v = \sigma(\text{ad}\vec{h}\nabla^c_{Vh}, -\|Jp^h\|\vec{u}_0 - \mathfrak{W}_0 - \|Jp^h\|\vec{h}\left(\frac{1}{\|Jp^h\|}\right)\mathcal{F}_b(\lambda)) \tag{2.4.19}$$

Now let us analyze the right-hand side of identity (2.4.18) term by term. First from identity (2.3.26) it follows that

$$\sigma(\mathrm{ad}\vec{h}\nabla^c_{V^h},\vec{u}_0) = -\vec{h}(\beta(V)) \tag{2.4.20}$$

Substituting relation (2.4.20) into identity (2.4.18) and using item (2) of Lemma 2.10, we have

$$\rho_\lambda(c,a)v = -\|Jp^h\|\vec{h}(\beta(V)) - g\Big((\mathfrak{R}_\lambda(c,c)v)^h, \mathfrak{W}_1^h\Big) + \|Jp^h\|\vec{h}\left(\frac{1}{\|Jp^h\|}\right)\rho_\lambda(c,b)v. \tag{2.4.21}$$

Taking into account item (2) of Lemma 2.6, we get the item 1) of the theorem.

2) Recall that by Proposition 3.2, (relation (2.2.36) there)

$$\rho_\lambda(a,a) = -\sigma(\mathrm{ad}\vec{h}\mathcal{F}_a(\lambda),\mathcal{F}_a(\lambda)) \tag{2.4.22}$$

Further, from the fourth line of structural equations (2.2.1) it follows that

$$(\mathrm{pr}\circ\pi)_*\mathrm{ad}\vec{h}\mathcal{F}_a(\lambda) = 0, \quad \mathrm{mod}\ \mathbb{R}p^h, \quad \sigma(\mathrm{ad}\vec{h}\mathcal{F}_a(\lambda),\mathcal{F}_b(\lambda)) = 0 \tag{2.4.23}$$

Then it follows from relations (2.2.30) and (2.2.2) that

$$\rho_\lambda(a,a) = -\sigma(\mathrm{ad}\vec{h}\mathcal{F}_a(\lambda), -\|Jp^h\|\vec{u}_0 - \mathfrak{W}_0) \tag{2.4.24}$$

Now let us analyze the right-hand side of identity (2.4.24). First since $[\vec{h},\vec{u}_0]=0$, we get

$$\sigma(\mathrm{ad}\vec{h}\mathcal{F}_a(\lambda),\vec{u}_0) = -\vec{h}(\sigma(\vec{u}_0,\mathcal{F}_a(\lambda))) \tag{2.4.25}$$

Let us calculate $\sigma(\vec{u}_0,\mathcal{F}_a(\lambda))$. Since

$$\widetilde{\mathcal{F}}_a(\lambda) = \vec{u}_0, \quad \mathrm{mod}\ \mathcal{V}_b(\lambda)\oplus\mathcal{V}_c(\lambda)\oplus\mathcal{V}_b^{\mathrm{trans}}(\lambda)\oplus\mathcal{V}_c^{\mathrm{trans}}(\lambda),$$

we get

$$\sigma(\vec{u}_0,\widetilde{\mathcal{F}}_a(\lambda)) = 0 \tag{2.4.26}$$

Further, it follows from relation (2.2.2) that

$$\sigma(\vec{u}_0,\mathcal{F}_a(\lambda)) = -\frac{1}{\|Jp^h\|}\sigma(\mathrm{ad}\vec{h}\,\widetilde{\mathcal{F}}_a(\lambda),\mathcal{F}_b(\lambda)) = -\frac{1}{\|Jp^h\|}\sigma(\mathrm{ad}\vec{h}\,\mathcal{F}_b(\lambda),\widetilde{\mathcal{F}}_a(\lambda)) \tag{2.4.27}$$

Furthermore, it follows from the line before last of structural equations (2.2.1) and relation (2.2.30) that

$$\sigma\big(\mathrm{ad}\vec{h}\,\mathcal{F}_b(\lambda),\widetilde{\mathcal{F}}_a(\lambda)\big) = \sigma\left(\mathrm{ad}\vec{h}\,\mathcal{F}_b(\lambda), -\|Jp^h\|\vec{u}_0 - \mathfrak{W}_0 - \|Jp^h\|\vec{h}\left(\frac{1}{\|Jp^h\|}\right)\mathcal{F}_b(\lambda)\right) \tag{2.4.28}$$

Substituting it into (2.4.27) and using relation (2.4.13), item (2) of Lemma 2.10 and the second identity of (2.4.12), we get

$$\sigma(\vec{u}_0,\mathcal{F}_a(\lambda)) = -(\vec{h})^3\left(\frac{1}{\|Jp^h\|}\right) - \frac{1}{\|Jp^h\|}\rho_\lambda(c,b)(\mathfrak{W}_1) + \vec{h}\left(\frac{1}{\|Jp^h\|}\right)\rho_\lambda(b,b). \tag{2.4.29}$$

Finally, we have

$$\sigma(\mathrm{ad}\vec{h}\,\widetilde{\mathcal{F}}_a(\lambda),\mathfrak{W}_0) = \sigma\big(\mathrm{ad}\vec{h}\,\mathfrak{W}_0,\widetilde{\mathcal{F}}_a(\lambda)\big) = -\rho_\lambda(c,a)\mathfrak{W}_1. \tag{2.4.30}$$

Substituting identities (2.4.25),(2.4.29) and (2.4.30) into (2.4.24), we obtain the required expression for $\rho_\lambda(a,a)$. □

Note that using the calculus developed in the previous section and the previous theorem, one can express the curvature maps $\mathfrak{R}_\lambda(c,a)$ and $\mathfrak{R}_\lambda(a,a)$ explicitly in terms of the Riemannian metric on $\widetilde{M}$ and the tensor J, but the expressions are too long to be presented here. Instead we analyze in more detail the expressions for curvature maps in the case of a uniform magnetic field, i.e. when $\nabla J = 0$. Remarkably, the curvature maps $\mathfrak{R}_\lambda(c,a)$ and $\mathfrak{R}_\lambda(a,a)$ vanish in this case.

Corollary 2.2. *Assume that J defines a uniform magnetic field , i.e., $\nabla J = 0$. Then the curvature maps have the following form*

(1) $g\Big(\big(\mathfrak{R}_\lambda(c,c)(v)\big)^h, v^h\Big) = g(R^\nabla(p^h, v^h)p^h, v^h) + \frac{u_0^2}{4}\Big(\|Jv^h\|^2 - \frac{1}{\|Jp^h\|^2} g^2(v^h, J^2p^h)\Big);$

(2) $\mathfrak{R}_\lambda(c,b)v = \Big(\frac{1}{\|Jp^h\|} g(R^\nabla(p^h, Jp^h)p^h, v^h) + \frac{u_0^2}{\|Jp^h\|} g(Jv^h, J^2p^h)\Big)\,\mathcal{E}_b(\lambda);$

(3) $\rho_\lambda(b,b) = \frac{1}{\|Jp^h\|^2} g(R^\nabla(Jp^h, p^h)Jp^h, p^h) + \frac{u_0^2}{\|Jp^h\|^2}\|J^2p^h\|^2;$

(4) $\mathfrak{R}_\lambda(c,a) = 0;$

(5) $\mathfrak{R}_\lambda(a,a) = 0,$

where $\rho_\lambda(b,b)$ is as in Theorem 2.2.

Proof Items (1), (2) and (3) are direct consequences of Theorems 2.1 and 2.2. Now we will show the proofs for items (4) and (5). We will denote by X, Y, Z, W, V the vector fields on $\widetilde{M}$. Assume that $v \in \mathcal{V}_c(\lambda)$ and V is a parallel vector field such that $V(\lambda) = v$. The following two propositions will be needed.

Lemma 2.13. *If $\nabla J = 0$, then*

(1) For any positive integer $k \in \mathbb{N}$, $\nabla(J^k) = 0$, $\nabla^k J = 0$;

(2) $J(R^\nabla(X,Y)Z) = R^\nabla(X,Y)JZ$;

(3) $g(R^\nabla(X,Y)JW, Z) = -g(R^\nabla(X,Y)W, JZ)$;

Proof. The item (1) is proved by definition; The item (2) is an analogy of [17, Chapter IX, Proposition 3.6 (2)]; The item (3) follows from item (2) immediately. □

Lemma 2.14. *For $\forall v \in \mathcal{V}_c(\lambda)$, the following identities hold:*

(1) $\mathcal{A}(\lambda, v) = -\frac{u_0}{\|Jp^h\|} g(v^h, J^2p^h),$

(2) $\mathcal{A}^{(1)}(\lambda, v) = \frac{u_0^2}{2\|Jp^h\|} g(v^h, J^3p^h),$

(3) $\mathcal{A}^{(2)}(\lambda, v) = -\frac{u_0^3\|J^2p^h\|^2}{4\|Jp^h\|^3} g(v^h, J^2p^h) - \frac{u_0^3}{4\|Jp^h\|} g(v^h, J^4p^h).$

Proof. The items (1) (2) are direct consequences of Lemma 2.9 and Corollary 2.1, respectively; The item (3) can be proved by applying Proposition 2.4 to $\mathcal{A}^{(1)}$. □

Let us prove $\mathfrak{R}_\lambda(c,a) = 0$. It follows from item (1) of Lemma 2.8 that

$$\vec{h}\left(\frac{1}{\|Jp^h\|}\right) = 0. \tag{2.4.31}$$

Then it follows from item 1) of Theorem 2.3 that

$$\rho_\lambda(c,a)v = \mathcal{A}^{(2)}(\lambda, v) - g\Big(\big(\mathfrak{R}_\lambda(c,c)v\big)^h, \mathfrak{V}_1^h\Big) \tag{2.4.32}$$

Further it follows from item (1) of Lemma 2.10 that

$$\mathfrak{V}_1^h = \frac{u_0}{\|Jp^h\|} J^2p^h + u_0\|Jp^h\|p^h. \tag{2.4.33}$$

Substituting identity (2.4.33) into the expression of $\mathfrak{R}_\lambda(c,c)$, we get

$$g\Big(\left(\mathfrak{R}_\lambda(c,c)v\right)^h, \mathfrak{V}_1^h\Big) = g(R^\nabla(p^h,v^h)p^h, \frac{u_0}{\|Jp^h\|}J^2p^h + u_0\|Jp^h\|p^h)$$
$$+\frac{u_0^2}{4}g\left(Jv^h, \frac{u_0}{\|Jp^h\|}J^3p^h + u_0\|Jp^h\|Jp^h\right) \tag{2.4.34}$$
$$-\frac{1}{4\|Jp^h\|^2}g(Jv^h,J^2p^h)g\left(\frac{u_0}{\|Jp^h\|}J^2p^h + u_0\|Jp^h\|p^h, J^2p^h)\right),$$

¿From item (3) of Lemma 2.14 it is easy to see that the sum of the last two items of (2.4.34) is equal to $-\mathcal{A}^{(2)}(\lambda,v)$. Thus

$$\rho_\lambda(c,a)v = -g(R^\nabla(p^h,v^h)p^h, \frac{u_0}{\|Jp^h\|}J^2p^h + u_0\|Jp^h\|p^h) \tag{2.4.35}$$

Finally by items (2), (3) of Lemma 2.13 and algebraic properties of the Riemannian curvature tensor we conclude that $\rho_\lambda(c,a)v = 0$.

Now let us prove that $\mathfrak{R}_\lambda(a,a) = 0$. First using that $R_\lambda(c,a) = 0$ and relation (2.4.31) we get from item 2) of Theorem 2.3 that

$$\rho_\lambda(a,a) = \vec{h}\left(\rho_\lambda(c,b)(\mathfrak{V}_1)\right) \tag{2.4.36}$$

Let us show that $\rho_\lambda(c,b)(\mathfrak{V}_1) = 0$. Indeed, from item (2) of the present corollary it follows

$$\rho_\lambda(c,b)(\mathfrak{V}_1) = \frac{1}{\|Jp^h\|}g(R^\nabla(p^h,Jp^h)p^h,\mathfrak{V}_1^h) + \frac{u_0^2}{\|Jp^h\|}g(J\mathfrak{W}_1^h, J^2p^h) \tag{2.4.37}$$

Note that the first term of the right-hand side of last identity coincides with the right-hand side of (2.4.35), taken with the opposite sign. Hence, it vanishes. The second term also vanishes due to relation (2.4.33) and the antisymmetricity of J. By this we complete the proof of the corollary. □

Finally consider even more particular but important case when $\nabla J = 0$ and $J^2 = -\mathrm{Id}$, i.e. when the tensor J defines a complex structure on $\widetilde{M}$ and the pair (g,J) defines a Kählerian structure on $\widetilde{M}$. As a direct consequence of the previous theorem, one has

Corollary 2.3. *Assume that J defines a complex structure on $\widetilde{M}$, i.e. $\nabla J = 0$ and $J^2 = -\mathrm{Id}$. Then*

$$\begin{aligned}
g((\mathfrak{R}_\lambda(c,c)(v))^h, v^h) &= g(R^\nabla(p^h,v^h)p^h,v^h) + \frac{u_0^2}{4}\|v\|^2,\\
\mathfrak{R}_\lambda(b,c)(v) &= g(R^\nabla(p^h,Jp^h)p^h,v^h)\mathcal{E}_b(\lambda),\\
\rho_\lambda(b,b) &= g(R^\nabla(p^h,Jp^h)p^h,Jp^h) + u_0^2,\\
\mathfrak{R}_\lambda(c,a) &= 0 \quad \textit{and} \quad \mathfrak{R}_\lambda(a,a) = 0,
\end{aligned}$$

2.5 Comparison Theorems

In the present section we restrict ourselves to sub-Riemannian structures with a transversal symmetry on a *contact* distribution such that the corresponding tensor J satisfies $\nabla J = 0$. We give estimation of the number of conjugate points (the Comparison Theorem) along the normal sub-Riemannian extremals (Theorem 2.4 below) in terms of the bounds for the curvature of the Riemannian structure on $\widetilde{M}$ and the tensor J. The main tool here is the Generalized Sturm

Theorem for curves in Lagrangian Grassmannians ([10] and [15]), applied to our structure equation (2.2.1).

Let, as before, $\lambda = (p, q) \in T^*M, q \in M, p \in T_q^*M$. Define the following two quadratic forms on the space $\mathcal{V}_b(\lambda) \oplus \mathcal{V}_c(\lambda)$

$$\widetilde{Q}_\lambda(v) = \|Jv^h\|^2 - \frac{1}{\|Jp^h\|^2} g(Jv^h, Jp^h)^2 \qquad (2.5.1)$$

$$Q_\lambda(v) = \widetilde{Q}_\lambda(v) - \frac{3}{4}\widetilde{Q}_\lambda(v_c), \qquad (2.5.2)$$

where the vector $v_c \in \mathcal{V}_c(\lambda)$ comes from the decomposition $v = v_b + v_c$ with $v_b \in \mathcal{V}_b(\lambda)$. The quadratic form $\widetilde{Q}_\lambda$ has the natural geometric meaning: the number $\widetilde{Q}_\lambda(v)$ is equal to the square of the area of the parallelogram spanned by the vectors Jv^h and Jp^h in $T_{\mathrm{pr}(q)}\widetilde{M}$ divided by $\|Jp^h\|^2$. In particular, the quadratic forms $\widetilde{Q}_\lambda$ are positive definite. The reason for introducing the form Q_p is that the identities in the Corollary 2.2 can be rewritten as follows, using the big curvature map $\mathfrak{R}_\lambda$ of the sub-Riemannian structure:

$$g\Big(\big(\mathfrak{R}_\lambda(v)\big)^h, v^h\Big) = g(R^\nabla(p^h, v^h)p^h, v^h) + u_0^2 Q_\lambda(v_{bc}), \qquad (2.5.3)$$

where the vector $v_{bc} \in \mathcal{V}_b(\lambda) \oplus \mathcal{V}_c(\lambda)$ comes from the decomposition $v = v_a + v_{bc}$ with $v_a \in \mathcal{V}_a(\lambda)$.

Now fix $T > 0$. In the sequel given a real analytic function $\varphi : [0, T] \to \mathbb{R}$ denote by $\sharp_T\{\varphi(x) = 0\}$ the number of zeros of φ on the interval $[0, T]$ counted with multiplicities. Given a normal sub-Riemannian extremal $\lambda : [0, T] \to \mathcal{H}_{\frac{1}{2}}$ denote by $\sharp_T(\lambda(\cdot))$ the number of conjugate point to 0 on $(0, T]$. Let

$$\phi_\omega(t) = \begin{cases} \sin\frac{\sqrt{\omega}t}{2}\left(\sqrt{\omega}t\cos\frac{\sqrt{\omega}t}{2} - 2\sin\frac{\sqrt{\omega}t}{2}\right), & \text{if } \omega \neq 0, \\ t^4 & \text{if } \omega = 0 \end{cases}; \qquad (2.5.4)$$

$$\psi_\omega(t) = \begin{cases} \sin\sqrt{\omega}t, & \text{if } \omega \neq 0, \\ t & \text{if } \omega = 0 \end{cases}. \qquad (2.5.5)$$

Further, define the following integer valued function on $\mathbb{R}^2$:

$$Z_T(\omega_b, \omega_c) \stackrel{def}{=} \sharp_T\Big\{\phi_{\omega_b}(t)\psi_{\omega_c}^{n-3}(t) = 0\Big\} \qquad (2.5.6)$$

An elementary analysis shows that

$$Z_T(\omega_b, \omega_c) = \begin{cases} (n-3)[\frac{T\sqrt{\omega_c}}{\pi}] + [\frac{T\sqrt{\omega_b)}}{2\pi}] + \sharp_T\{\tan(\frac{\sqrt{\omega_b}}{2}x) - \frac{\sqrt{\omega_b}}{2}x = 0\}, \text{if } \omega_b > 0,\ \omega_c > 0; \\ [\frac{T\sqrt{\omega_b}}{2\pi}] + \sharp_T\{\tan(\frac{\sqrt{\omega_b}}{2}x) - \frac{\sqrt{\omega_b}}{2}x = 0\}, \quad \text{if } \omega_b > 0,\ \omega_c \leq 0; \\ (n-3)[\frac{T\sqrt{\omega_c}}{\pi}], \quad \text{if } \omega_b \leq 0,\ \omega_c > 0. \\ 0, \quad \text{if } \omega_b \leq 0,\ \omega_c \leq 0. \end{cases}$$

(2.5.7)

Theorem 2.4. *Let $\mathfrak{c}_b$,$\mathfrak{c}_c$,$\mathfrak{C}_b$, and $\mathfrak{C}_c$ are constants such that the curvature tensor R^∇ of the Riemannian metric g on $\widetilde{M}$ satisfies*

$$\mathfrak{c}_b\|v_b^h\|^2 + \mathfrak{c}_c\|v_c^h\|^2 \leq g(R^\nabla(p^h, v_b^h + v_c^h)p^h, v_b^h + v_c^h) \leq \mathfrak{C}_b\|v_b^h\|^2 + \mathfrak{C}_c\|v_c^h\|^2, \qquad (2.5.8)$$
$$\forall \lambda \in \mathcal{H}_{\frac{1}{2}}, v_b \in \mathcal{V}_b(\lambda), v_c \in \mathcal{V}_c(\lambda).$$

Also let k_b, k_c, K_b, K_c be constants such that

$$k_b\|v_b^h\|^2 + k_c\|v_c^h\|^2 \le Q_\lambda(v_b + v_c) \le K_b\|v_b^h\|^2 + K_c\|v_c^h\|^2, \quad \forall \lambda \in \mathcal{H}_{\frac{1}{2}}, v_b \in \mathcal{V}_b(\lambda), v_c \in \mathcal{V}_c(\lambda). \tag{2.5.9}$$

Let $\lambda(\cdot)$ be a normal sub-Riemannian extremal on $\mathcal{H}_{\frac{1}{2}} \cap \{u_0 = \bar{u}_0\}$ Then the number of conjugate points $\sharp_T(\lambda(\cdot))$ to 0 on $(0, T]$ along $\lambda(\cdot)$ satisfies the following inequality

$$Z_T(\mathfrak{c}_b + k_b\bar{u}_0^2, \mathfrak{c}_c + k_c\bar{u}_0^2) \le \sharp_T(\lambda(\cdot)) \le Z_T(\mathfrak{C}_b + K_b\bar{u}_0^2, \mathfrak{C}_c + K_c\bar{u}_0^2). \tag{2.5.10}$$

Remark 2.4. *If the sectional curvature of the Riemannian metric g on $\widetilde{M}$ is bounded from below by a constant $\mathfrak{c}$ and bounded from above by a constant $\mathfrak{C}$, then in* (2.5.8) *one can take $\mathfrak{c}_b = \mathfrak{c}_c = \mathfrak{c}$ and $\mathfrak{C}_b = \mathfrak{C}_c = \mathfrak{C}$.*

Proof. We start with some general statements. Let, as before, W be a linear symplectic space and $\Lambda : [0, T] \to L(W)$ be a monotonically nondecreasing curve in the Lagrange Grassmannians $L(W)$ with the constant Young Diagram D. In this case the set of all conjugate points to 0 is obviously discrete. Denote by $\sharp_T(\Lambda(\cdot))$ the number of conjugate points (counted the multiplicities) of $\Lambda(\cdot)$ on $(0, T]$. Then $\sharp(\Lambda(\cdot)) = \sum_{0<\tau\le T} \dim(\Lambda(\tau) \cap \Lambda(0))$. We will use the following corollary of the generalized Sturm theorems from [15] and [10]:

Theorem 2.5. *Let h_τ, H_τ be two quadratic non-stationary Hamiltonians on W such that for any $0 \le \tau \le T$, the quadratic form $h_\tau - H_\tau$ is non-positive definite. Let $P_\tau, \widetilde{P}_\tau$ be linear Hamiltonian flows generated by h_τ, H_τ, respectively:*

$$\frac{\partial}{\partial \tau} P_\tau = \vec{h}_\tau P_\tau, \quad \frac{\partial}{\partial \tau} \widetilde{P}_\tau = \vec{H}_\tau \widetilde{P}_\tau, \quad P_0 = \widetilde{P}_0 = id.$$

Further, let $\Lambda(\cdot), \widetilde{\Lambda}(\cdot)$ be nondecreasing trajectories of the corresponding flows on $L(W)$, both having constant Young diagram D:

$$\Lambda(\tau) = P_\tau \Lambda(0), \quad \widetilde{\Lambda}(\tau) = \widetilde{P}_\tau \Lambda(0), \quad 0 \le \tau \le T.$$

Then $\sharp_T(\Lambda(\cdot)) \le \sharp_T(\widetilde{\Lambda}(\cdot))$.

The detailed proof of this statement (even a in slightly general setting) can be found in [21] (see also [4]). As the direct consequence of this theorem and the structural equations (1.1.7) we get the following

Corollary 2.4. *Let $\Lambda, \widetilde{\Lambda} : [0, T] \to L(W)$ be two monotonically nondecreasing curves in the Lagrangian Grassmannian $L(W)$ with the same Young diagram D. Assume that $\Lambda(\cdot)$ and $\widetilde{\Lambda}(\cdot)$ have normal moving frames $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$ and $(\{\widetilde{E}_a(t)\}_{a\in\Delta}, \{\widetilde{F}_a(t)\}_{a\in\Delta})$ respectively such that if R_t is the matrix of the big curvature map of $\Lambda(\cdot)$ w.r.t. the basis $(\{E_a(t)\}_{a\in\Delta}$ and $\widetilde{R}_t$ is the matrix of the big curvature map of $\widetilde{\Lambda}(\cdot)$ w.r.t. the basis $(\{\widetilde{E}_a(t)\}_{a\in\Delta}$, then the symmetric matrix $R_t - \widetilde{R}_t$ is non-positive definite. Then $\sharp_T(\Lambda(\cdot)) \le \sharp_T(\widetilde{\Lambda}(\cdot))$.*

Now let the diagram D be as for the case of sub-Riemannian structures on corank 1 distributions. Let, as before, $\mathfrak{J}_\lambda(\cdot)$ is the Jacobi curve attached at the point λ. Given constants ω_b and ω_c let $\Gamma_{\omega_b,\omega_c}(\cdot)$ be the curve in $L(W)$ with the Young diagram D such that its curvature maps satisfy:

$$\mathfrak{R}_t(a, a) = 0, \mathfrak{R}_t(c, a) = 0, \mathfrak{R}_t(c, b) \equiv 0, \mathfrak{R}_t(b, b)E_b = \omega_b E_b, \mathfrak{R}_t(c, c) = \omega_c Id \quad \forall t \tag{2.5.11}$$

Then from the identity (2.5.3), conditions (2.5.8) and (2.5.9), and Corollary 2.4 it follows immediately that

$$\sharp_T(\Gamma_{\mathfrak{c}_b + k_b\bar{u}_0^2, \mathfrak{c}_c + k_c\bar{u}_0^2}(\cdot)) \le \sharp_T(\mathfrak{J}_\lambda(\cdot)) \le \sharp_T(\Gamma_{\mathfrak{C}_b + K_b\bar{u}_0^2, \mathfrak{C}_c + K_c\bar{u}_0^2}(\cdot)) \tag{2.5.12}$$

In order to prove Theorem 2.4 it remains to show that

$$\sharp_T\big(\Gamma_{\omega_b,\omega_c}(\cdot)\big) = Z_T(\omega_b,\omega_c). \tag{2.5.13}$$

Let us prove identity (2.5.13). Let $(E_a(t), E_b(t), E_c(t), F_a(t), F_b(t), F_c(t)$ be a normal moving frame of the curve $\Gamma_{\omega_b,\omega_c}(\cdot)$. Substituting (2.5.11) into the structural equation (2.2.1) we get

$$\begin{cases} E_a'(t) = E_b(t) \\ E_b'(t) = F_b(t) \\ E_c'(t) = F_c(t) \\ F_a'(t) = 0 \\ F_b'(t) = -\omega_b E_b(t)\mathcal{R}_t(b,b) - F_a(t) \\ F_c'(t) = -\omega_c E_c(t). \end{cases} \tag{2.5.14}$$

¿From this we obtained the following two separated equations for E_a and for E_c, respectively:

$$\begin{cases} E_a^{(4)} + \omega_b E_a'' = 0 \\ E_c'' + \omega_c E_c = 0 \end{cases} \tag{2.5.15}$$

Assume first that $\omega_b \neq 0$ and $\omega_c \neq 0$. Then there exist vectors $\alpha_1, \dots, \alpha_4$ and β_1^k, β_2^k, $k = 1, \dots n-3$ in W such that

$$\begin{aligned} &E_a(t) = e^{i\sqrt{\omega_b t}}\alpha_1 + e^{-i\sqrt{\omega_b t}}\alpha_2 + \alpha_3 + t\alpha_4, \\ &E_b(t) = i\sqrt{\omega_b}e^{i\sqrt{\omega_b}}\alpha_1 - i\sqrt{\omega_b}e^{-it\sqrt{\omega_b}}\alpha_2 + \alpha_4, \\ &E_c(t) = \big(e^{i\sqrt{\omega_c t}}\beta_1^1 + e^{-i\sqrt{\omega_c t}}\beta_2^1, \dots, e^{i\sqrt{\omega_c t}}\beta_1^{n-3} + e^{-i\sqrt{\omega_c t}}\beta_2^{n-3}\big). \end{aligned} \tag{2.5.16}$$

Besides, by constructions vectors $\alpha_1, \dots, \alpha_4, \beta_1^1, \beta_2^1, \dots, \beta_1^{n-3}, \beta_2^{n-3}$ have to be linearly independent.

Introducing some coordinate in W we can look on the tuple

$$\big(E_a(t), E_b(t), E_c(t), E_a(0), E_b(0), E_c(0)\big)$$

as on $2(n-1) \times 2(n-1)$–matrix, representing each involved vector as a column. Let $d(t)$ be the determinant of this matrix. Obviously, $\bar{t}$ is conjugate point to 0 of multiplicity l if and only if $\bar{t}$ is zero of multiplicity l of function $d(t)$. On the other hand, using expressions (2.5.16) it is easy to show that the function $d(t)$ is equal, up to a nonzero constant factor, to

$$\begin{vmatrix} e^{i\sqrt{\omega_b t}} & i\sqrt{\omega_b}e^{i\sqrt{\omega_b t}} & 1 & i\sqrt{\omega_b} \\ e^{-i\sqrt{\omega_b t}} & -i\sqrt{\omega_b}e^{-i\sqrt{\omega_b t}} & 1 & -i\sqrt{\omega_b} \\ 1 & 0 & 1 & 0 \\ t & 1 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} e^{i\sqrt{\omega_c t}} & 1 \\ e^{-i\sqrt{\omega_c t}} & 1 \end{vmatrix}^{n-3},$$

which in turn is equal, up to a nonzero constant factor, to the function $\phi_{\omega_b}(t)\psi_{\omega_c}^{n-3}(t)$ appearing in the definition (2.5.6) of the function $Z_T(\omega_b, \omega_c)$. The case when one or both ω_b and ω_c are equal to zero can be treated analogously. This completes the proof of (2.5.13) and Theorem 2.4 itself. □

Now let us state separately what Theorem 2.4 says about the intervals along normal extremals of the considered sub-Riemannian structure which do not contain conjugate points or contain at least one conjugate point:

Corollary 2.5. *Under the same estimates on the curvature of the Riemannian metric g on $\widetilde{M}$ and on the quadratic forms Q_λ as in Theorem 2.4 the following statement hold for a normal sub-Riemannian extremal on $\mathcal{H}_{\frac{1}{2}} \cap \{u_0 = \bar{u}_0\}$:*

(1) If $\mathfrak{C}_b + K_b\bar{u}_0^2 > 0$ and $\mathfrak{C}_c + K_c\bar{u}_0^2 > 0$, then there are no conjugate points to 0 in the interval $(0, \min\{\frac{2\pi}{\sqrt{\mathfrak{C}_b+K_b\bar{u}_0^2}}, \frac{\pi}{\sqrt{\mathfrak{C}_c+K_c\bar{u}_0^2}}\})$;

(2) If $\mathfrak{C}_b+K_b\bar{u}_0^2 > 0$ and $\mathfrak{C}_c+K_c\bar{u}_0^2 \leq 0$, then there are no conjugate points to 0 in $(0, \frac{2\pi}{\sqrt{\mathfrak{C}_b+K_b\bar{u}_0^2}})$;

(3) If $\mathfrak{C}_b+K_b\bar{u}_0^2 \leq 0$ and $\mathfrak{C}_c+K_c\bar{u}_0^2 > 0$, then there are no conjugate points to 0 in $(0, \frac{\pi}{\sqrt{\mathfrak{C}_c+K_c\bar{u}_0^2}})$;

(4) If $\mathfrak{C}_b + K_b\bar{u}_0^2 \leq 0$ and $\mathfrak{C}_c + K_c\bar{u}_0^2 \leq 0$, then there are no conjugate points to 0 in $(0, \infty)$;

(5) If $\mathfrak{c}_b+k_b\bar{u}_0^2 > 4(\mathfrak{c}_c+k_c\bar{u}_0^2) > 0$, then there is at least one conjugate point to 0 in $(0, \frac{2\pi}{\sqrt{\mathfrak{c}_b+k_b\bar{u}_0^2}}]$;

(6) If $\mathfrak{c}_c + k_c\bar{u}_0^2 \geq \frac{1}{4}(\mathfrak{c}_b + k_b\bar{u}_0^2) > 0$, then there is at least $n-3$ conjugate points to 0 in $(0, \frac{\pi}{\sqrt{\mathfrak{c}_c+k_c\bar{u}_0^2}}]$ (at least $n-2$ conjugate points in the case $\mathfrak{c}_b + k_b\bar{u}_0^2 = 4(\mathfrak{c}_c + k_c\bar{u}_0^2) > 0$);

(7) If $\mathfrak{c}_b + k_b\bar{u}_0^2 > 0$ and $\mathfrak{c}_c + k_c\bar{u}_0^2 \leq 0$, then there is at least one conjugate point to 0 in $(0, \frac{2\pi}{\sqrt{\mathfrak{c}_b+k_b\bar{u}_0^2}}]$

(8) If $\mathfrak{c}_b + k_b\bar{u}_0 \leq 0$ and $\mathfrak{c}_c + k_c\bar{u}_0^2 > 0$, then there is at least $n-3$ conjugate points to 0 in $(0, \frac{\pi}{\sqrt{\mathfrak{c}_c+k_c\bar{u}_0^2}}]$

Finally note that if in addition $J^2 = -\mathrm{Id}$ then the quadratic forms Q_λ have the following simple form:

$$Q_\lambda(v_c + v_b) = \|v_b^h\|^2 + \frac{1}{4}\|v_c^h\|^2 \quad \forall v_b \in \mathcal{V}_b(\lambda), v_c \in \mathcal{V}_c(\lambda).$$

Therefore in this case one can take $k_b = K_b = 1$ and $k_c = K_c = \frac{1}{4}$.

Chapter 3

Hyperbolic flows in sub-Riemannian structures with symmetries

For the sub-Riemannian structures with multidimensional transversal commutative infinitesimal symmetries, we proceed with the *Poission (symplectic) reduction* to obtain the reduced flows of the sub-Riemannian geodesic flows (on the common level set of all integrals in the cotangent bundle). We give conditions for this flow to be hyperbolic by applying the criteria of [8] for the hyperbolicity of Hamiltonian flows. The main results of this chapter can be found in [19].

3.1 Reduction

In this chapter we consider sub-Riemannian metrics $\langle\cdot,\cdot\rangle$ on distribution $\mathcal{D}$ of corank s, having s transversal infinitesimal symmetries, i.e. s vector fields $X_1,\ldots,X_s$ on M such that

$$e_*^{tX_i}\mathcal{D}=\mathcal{D}\ ,\ (e^{tX_i})^*\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle,\quad 1\le i\le s,$$

and $TM=\mathcal{D}\oplus\operatorname{span}\{X_1,\ldots,X_s\}$. Suppose further that the symmetries $\{X_i:\ 1\le i\le s\}$ are commutative (see Remark 1 for noncommutative case), i.e.

$$[X_i,X_j]=0,\quad \forall 1\le i,j\le s. \tag{3.1.1}$$

Denote by $\widetilde{M}$ the quotient of M by the leaves of the integral manifold of the involutive distribution spanned by $X_1,\ldots X_s$ and denote the factorization map by $\mathrm{pr}:\mathrm{M}\to\widetilde{\mathrm{M}}$. Then $\widetilde{M}$ is a Riemannian manifold equipped with the Riemannian metric g induced from the sub-Riemannian metric.

For any vector field X_i define the "quasiimpluses" $u_i:T^*M\to\mathbb{R}$ by

$$u_i(p,q)=p\cdot X_i(q),\ q\in T_q^*M, q\in M,\ \forall 1\le i\le s.$$

Let h be the sub-Riemannian Hamiltonian as in (2.1.1). Since X_i is a symmetry,

$$\{h,u_i\}=0,\forall 1\le i\le s. \tag{3.1.2}$$

and from (3.1.1) it follows that

$$\{u_i,u_j\}=0,\ \forall 1\le i,j\le s, \tag{3.1.3}$$

where $\{\ ,\ \}$ is the Poisson bracket. In other words, u_i, $1\le i\le s$ are first integrals in involution of the Hamiltonian system $e^{t\vec{h}}$.

Now we can apply the Poisson reduction (see e.g. [1], [9]) to our Hamiltonian flow $e^{t\vec{h}}$. For simplicity, denote by $\mathfrak{u} = (u_1, ..., u_s)$ and let $\vec{\mathfrak{u}} = (\vec{u}_1, ..., \vec{u}_s)$. Now take a common level set

$$\mathcal{H}_{\frac{1}{2},\bar{\mathfrak{u}}} \triangleq \{h = \frac{1}{2}\} \cap \{\mathfrak{u} = \bar{\mathfrak{u}}\}$$

for some $\bar{\mathfrak{u}} \in \mathbb{R}^s$.

Note that $\operatorname{span}\{\vec{h}, \vec{u}_i,\ 1 \le i \le s\} = \ker\, \sigma|_{T_\lambda \mathcal{H}_{\frac{1}{2},\bar{\mathfrak{u}}}}$. Therefore,

$$W_\lambda^{\bar{\mathfrak{u}}} = T_\lambda \mathcal{H}_{\frac{1}{2},\bar{\mathfrak{u}}}/\operatorname{span}\{\vec{h}(\lambda), \vec{u}_i(\lambda),\ 1 \le i \le s\}$$

is a symplectic space with the symplectic form $\sigma^{\bar{\mathfrak{u}}}$ naturally inherited from the symplectic form σ on M. Moreover, let Π_λ be as in (0.1.4) and

$$\Pi_\lambda^{\bar{\mathfrak{u}}} = T_\lambda(\mathcal{H}_{\frac{1}{2},\bar{\mathfrak{u}}}) \cap \Pi_\lambda$$

is a Lagrangian subspace in $W_\lambda^{\bar{\mathfrak{u}}}$.

Further, it follows from relations (3.1.2) and (3.1.3) that $\mathcal{H}_{\frac{1}{2},\bar{\mathfrak{u}}}$ is an invariant set of the flow $e^{t\vec{h}}$ and $e_*^{t\vec{h}}\vec{u}_i = \vec{u}_i$, $\forall 1 \le i \le s$. Hence, $e^{t\vec{h}}$ induces a symplectic transformation $e_*^{t\vec{h}} : W_\lambda^{\bar{\mathfrak{u}}} \longrightarrow W_{e^{t\vec{h}}\lambda}^{\bar{\mathfrak{u}}}$. Set

$$\mathfrak{J}_\lambda^{\bar{\mathfrak{u}}}(t) \triangleq e_*^{-t\vec{h}}\Pi_{e^{t\vec{h}}\lambda}^{\bar{\mathfrak{u}}}/\operatorname{span}\{\vec{h}, \vec{u}_i, 1 \le i \le s\}. \tag{3.1.4}$$

The curve $t \mapsto \mathfrak{J}_\lambda^{\bar{\mathfrak{u}}}(t)$ is a curve in the Lagrange Grassmannian of the symplectic space $W_\lambda^{\bar{\mathfrak{u}}}$. It is called *the reduced Jacobi curve of the extremal $e^{t\vec{h}}$ attached at $\lambda \in T^*M$ obtained by the reduction by the first integrals $\mathfrak{u}$ to the level set $\{\mathfrak{u} = \bar{\mathfrak{u}}\}$, or shortly, the reduced Jacobi curve of the extremal $e^{t\vec{h}}$* .

Remark 3.1. *If the symmetries $\{X_i :\ 1 \le i \le s\}$ are not commutative but*

$$g = \operatorname{span}_{\mathbb{R}}\{X_1, \dots X_s\}$$

is a Lie algebra, then the reduced Jacobi curve can be defined for extremals lying on certain levels of the corresponding first integrals. Indeed, assume that the derived Lie algebra $g^2 = [g, g]$ has dimension k. We can always assume that the first k of the symmetries $X_1, \dots, X_s$ span the algebra g^2. Then on the level sets $\mathfrak{u} = (\underbrace{0, \dots, 0}_{k\ times}, c_{k+1}, ..., c_s)$ the integrals $u_1, \dots, u_s$ commute and all constructions above work. However, to consider extremals of the original sub-Riemannian structure on the level sets $\mathfrak{u} = (\underbrace{0, \dots, 0}_{k\ times}, c_{k+1}, ..., c_s)$ is the same as to consider extremals of the sub-Riemannian structure obtained from the original one by the reduction by first k-symmetries $X_1, \dots, X_k$. After such a reduction we will get the sub-Riemannian structures on the distribution of corank $s - k$ having $s - k$ commutative symmetries (induced by $X_{k+1}, \dots, X_s$). Hence the case of non-commutative symmetries can be reduced in essence to the case of commutative symmetries.

Proposition 3.1. *The reduced Jacobi curve $\mathfrak{J}_\lambda^{\bar{\mathfrak{u}}}(t)$ is a regular monotonically nondecreasing curve in $L(W_\lambda^{\bar{\mathfrak{u}}})$.*

Proof. The intersection of the set $\mathcal{H}_{\frac{1}{2}}$ with a fiber of T^*M is a cylinder with an elliptic base, while the intersection of $\{\mathfrak{u} = \bar{\mathfrak{u}}\}$ with this fiber is a linear subspace transversal to the generator of this cylinder. So, the intersection of the set $\mathcal{H}_{\frac{1}{2},\bar{\mathfrak{u}}}$ with a fiber of T^*M is an ellipsoid. The proposition follows from the fact that the velocity of the reduced Jacobi curve is equivalent (under linear substitutions of variables in the corresponding quadratic forms) to the second fundamental form of this ellipsoid, following from [3, Proposition 1] □

Let $\Xi^{\bar{u}}$ be the s-foliation on T^*M such that its leaves are integral curves of $\vec{\bar{u}}$. Similar to Subsection 2.2.3 (the corank 1 case there), one can show that $\{u = \bar{u}\}/\Xi^{\bar{u}}$ is identified with the cotangent bundle $T^*\widetilde{M}$. Denote the identification by $I^{\bar{u}}$. As in Subsection 2.2.3, for any vector field X on $T^*\widetilde{M}$, we can assign the vector field $\underline{X}$ on T^*M s.t. $PR_*\underline{X} = (I^{-1})_*X$ and $\pi_*\underline{X} \in \mathcal{D}$, where $\mathrm{PR} : T^*M \to T^*M/\Xi^{\bar{u}}$ is the canonical projection to the quotient manifold.

Further, if denote $N_{\bar{u}} = \mathcal{H}_{\frac{1}{2},\bar{u}}/\Xi^{\bar{u}}$, then it follows from relations (3.1.2) and (3.1.3) that $e^{t\vec{h}}$ induces a Hamiltonian flow $(e^{t\vec{h}})_{\mathrm{red}}$ on $N_{\bar{u}}$. It is called *the reduced flow by the first integrals* u. Then the reduced Jacobi curve $\mathfrak{J}^{\bar{u}}_\lambda(\cdot)$ is actually the Jacobi curve associated with the Hamiltonian flow $e^{t\vec{h}} : N_{\bar{u}} \to N_{\bar{u}}$, defined as (0.1.5).

Now let $\mathcal{D}_q^\perp$ be as in (0.1.3). Then similar to (2.1.6) one has the following series of natural identifications:

$$T_q^*M/\mathcal{D}_q^\perp \sim \mathcal{D}_q^* \overset{\langle\cdot,\cdot\rangle}{\sim} \mathcal{D}_q \sim T_{\mathrm{pr(q)}}\widetilde{M}. \tag{3.1.5}$$

where $\mathcal{D}_q^* \subseteq T_q^*M$ is the dual space of $\mathcal{D}_q$. Given $v \in T_\lambda T_q^*M$ ($\sim T_q^*M$), where $q = \pi(\lambda)$, we can assign a unique vector $v^h \in T_{\mathrm{pr}(q)}\widetilde{M}$ to its equivalence class in $T_q^*M/\mathcal{D}_q^\perp$ by using the identifications (3.1.5). Conversely, to any $X \in T_{\mathrm{pr}(q)}\widetilde{M}$ one can assign an equivalence class of $T_\lambda(T_q^*M)/\mathcal{D}_q^\perp$. Denote by $X^v \in T_\lambda T_q^*M$ the unique representative of this equivalence class such that $du_i(X^v) = 0$, $\forall 1 \le i \le s$.

3.2 Reduced curvature map

Since the curve $\mathfrak{J}^{\bar{u}}_\lambda(\cdot)$ is regular in the corresponding Lagrangian Grassmannian, its reduced Young diagram consist of one box, which will be denoted by c. Since $\mathfrak{J}^{\bar{u}}_\lambda(0)$ and $\Pi^{\bar{u}}_\lambda$ can be naturally identified, there is a canonical splitting of $W^{\bar{u}}_\lambda$:

$$W^{\bar{u}}_\lambda = \Pi^{\bar{u}}_\lambda \oplus \widetilde{\mathfrak{J}}^{\bar{u}}(\lambda), \tag{3.2.1}$$

where $\widetilde{\mathfrak{J}}^{\bar{u}}(\lambda) = (\mathfrak{J}^{\bar{u}}_\lambda)^{\mathrm{trans}}(0)$ is the canonical complement of the reduced Jacobi curve $\mathfrak{J}^{\bar{u}}_\lambda(\cdot)$ at 0. Furthermore, one can define the curvature map $\mathfrak{R}^{\bar{u}}_\lambda : \Pi^{\bar{u}}_\lambda \to \Pi^{\bar{u}}_\lambda$ such that $\mathfrak{R}^{\bar{u}}_\lambda = \mathfrak{R}_{cc}(0)$, where $\mathfrak{R}_{cc}(t)$ is the curvature maps of the regular Jacobi curve $\mathfrak{J}^{\bar{u}}_\lambda(\cdot)$ at $t = 0$. Recall that $\widetilde{\mathfrak{J}}^{\bar{u}}(\lambda) \in T_\lambda T^*M/\mathrm{span}\{\vec{h}(\lambda), \vec{\bar{u}}\}$. As a canonical representative of $\widetilde{\mathfrak{J}}^{\bar{u}}(\lambda)$ one can take the representative, which projects to $\mathcal{D}$ by π_* and projects to $(\mathbb{R}p^h)^\perp$ by $(\mathrm{pr} \circ \pi)_*$. In the sequel, this canonical representative will be denoted by $\widetilde{\mathfrak{J}}^{\bar{u}}(\lambda)$ as well.

Besides, let $\omega = (\omega_i)_{1 \le i \le s}$ be the $\mathbb{R}^s$-valued 1-form defined by $\omega_i|_{\mathcal{D}} = 0$ and $\omega_i(X_j) = \delta_{ij}$, $\forall 1 \le i,j \le s$. Then $d\omega = (d\omega_i)_{1 \le i \le s}$ induces a $\mathbb{R}^s$-valued 2-form on $\widetilde{M}$. We denote the 2-form by $\Omega = (\Omega_i)_{1 \le i \le s}$. Now we define a tuple $J = \{J_i(\tilde{q})\}_{i=1}^s$ of s tensors of type $(1,1)$ on $\widetilde{M}$ as follows

$$g_{\tilde{q}}(J_i(\tilde{q})v, w) = \Omega_i(\tilde{q})(v, w), \quad v, w \in T_{\tilde{q}}\widetilde{M}, \tilde{q} \in \widetilde{M}, \ \forall 1 \le i \le s.$$

Besides we can define the following s-dimensional pencils of 2-forms Ω^{u} and $(1,1)$-tensors J^{u}:

$$\Omega^{\mathrm{u}} = \sum_{i=1}^{s} u_i \Omega_i, \quad J^{\mathrm{u}} = \sum_{i=1}^{s} u_i J_i, \quad u = (u_1, \dots u_s) \in \mathbb{R}^s. \tag{3.2.2}$$

Now we are ready to give the expression for the reduced curvature map using the curvature tensor R^∇ of the Riemannian metric on $\widetilde{M}$ and the tensors J^{u}. Let r^{u}_λ be the following quadratic form induced by the reduced curvature map, i.e.

$$r^{\mathrm{u}}_\lambda(v) = g((\mathfrak{R}^{\mathrm{u}}_\lambda v)^h, v^h), \quad \forall v \in \Pi^{\mathrm{u}}_\lambda.$$

The form r^{u}_{λ} is called the *(reduced) curvature form.*

Theorem 3.1. *Let $v \in \Pi^{\mathrm{u}}_{\lambda}$. Then*

$$r^{\mathrm{u}}_{\lambda}(v) = g(R^{\nabla}(p^h, v^h)p^h, v^h) + g(\nabla J^{\mathrm{u}}(p^h, v^h), v^h) + \frac{1}{4}g(J^{\mathrm{u}}v^h, J^{\mathrm{u}}v^h) + \frac{3}{4}(g(p^h, J^{\mathrm{u}}v^h))^2.$$

The proof of the theorem above is given in section 3.4.

3.3 Conditions for hyperbolicity of extremal flows

First let us recall some elements of Hyperbolic Dynamics. More details can be found in [16].

Definition 3.1. *Let e^{tX}, $t \in \mathbb{R}$ be the flow generated by the vector field X on a manifold $\mathcal{M}$. A compact invariant set $W \in \mathcal{M}$ of the flow e^{tX} is called a hyperbolic set if there exists a Riemannian structure in a neighborhood of W, a positive constant δ, and a splitting: $T_z\mathcal{M} = E^+_z \oplus E^-_z \oplus \mathbb{R}X(z)$, $z \in W$ such that $X(z) \neq 0$ and*

(1) $e^{tX}_* E^+_z = E^+_{e^{tX}z}$, $e^{tX}_* E^-_z = E^-_{e^{tX}z}$,

(2) $\|e^{tX}_* \zeta^+\| \geq e^{\delta t}\|\zeta^+\|$, $\forall t > 0, \forall \zeta^+ \in E^+_z$,

(3) $\|e^{tX}_* \zeta^-\| \leq e^{-\delta t}\|\zeta^-\|$, $\forall t > 0, \forall \zeta^- \in E^-_z$.

If the entire manifold $\mathcal{M}$ is a hyperbolic set, then the flow e^{tX} is called an Anosov flow

The following theorem is a direct consequence Theorem 2 in [8]:

Theorem 3.2. *Let $\bar{\mathrm{u}} \in \mathbb{R}^s$. Assume that $K_{\bar{\mathrm{u}}} \subset N_{\bar{\mathrm{u}}}$ is a compact invariant set of the flow $(e^{t\vec{h}})_{red}$ on $N_{\bar{\mathrm{u}}}$. If the curvature form $r^{\bar{\mathrm{u}}}_{\lambda}$ is negative at every point of $K_{\bar{\mathrm{u}}}$, then $K_{\bar{\mathrm{u}}}$ is a hyperbolic set of the flow $(e^{t\vec{h}})_{red}$ on $N_{\bar{\mathrm{u}}}$.*

Now denote by $S_1\widetilde{M}$ the unit tangent bundle. Combining the previous theorem with Theorem 3.1 and using that $g(w, Jv)^2 \leq g(Jv, Jv)$ for $v, w \in S^1_q\widetilde{M}$, we get the following

Theorem 3.3. *Assume that the reduced Riemannian manifold $(\widetilde{M}, g)$ is compact and has sectional curvature bounded from above by $k_{\max}$. If a vector $\bar{\mathrm{u}} \in \mathbb{R}^s$ satisfies*

$$\max_{v,w \in S_1\widetilde{M}, v \perp w} g(v, \nabla J^{\bar{\mathrm{u}}}(w; v)) + g(J^{\bar{\mathrm{u}}}v, J^{\bar{\mathrm{u}}}v)) < -k_{\max}, \tag{3.3.1}$$

where $J^{\bar{\mathrm{u}}}$ is as in (3.2.2)*, then the flow $(e^{t\vec{h}})_{red}$ is an Anosov flow on $N_{\bar{\mathrm{u}}}$.*

Note that the left-hand side of the inequality (3.3.1) is always positive, because the second term inside the max is positive and the first term can be made nonnegative, if necessary, by changing the sign of w. Hence, Theorem 3.3 makes sense only if $k_{\max} < 0$. Note also that if $\bar{u} = 0$, then the flow $(e^{t\vec{h}})_{\text{red}}$ on N_0 is exactly the Riemannian geodesic flow of the Riemannian structure $(\widetilde{M}, g)$ (by the identification between $T\widetilde{M}$ and $T^*\widetilde{M}$ via the Riemannian metric g). So, Theorem 3.3 gives the classical result of Hyperbolic Dynamics ([16], [12]): *Geodesic Flows of compact Riemannian manifold with negative sectional curvatures are Anosov flows.* The flow $(e^{t\vec{h}})_{\text{red}}$ on $N_{\bar{\mathrm{u}}}$ can be considered as a perturbation of the Riemannian geodesic flow: the flow $(e^{t\vec{h}})_{\text{red}}$ on $N_{\bar{\mathrm{u}}}$ remains to be an Anosov flow for $\bar{\mathrm{u}} \in \mathbb{R}^s$ sufficiently close to the origin. Our Theorem 3.3 gives more explicit estimation of the domain of $\bar{\mathrm{u}}$ around the origin for which the flow $(e^{t\vec{h}})_{\text{red}}$ on $N_{\bar{u}}$ remains to be an Anosov flow. Finally note that in the case of $s = 1$ our Theorem 3.3 actually coincides with the main result of [14]. As was mentioned before, in this case the projections to $\widetilde{M}$ of all sub-Riemannian extremals from $\mathcal{H}_{\frac{1}{2},\bar{u}}$ describe all possible motion of a charged particle given by the magnetic field Ω on the Riemannian manifold $\widetilde{M}$ and Theorem 3.3 for $s = 1$ gives the sufficient condition for a magnetic flow to be an Anosov flow. Thus, this theorem for arbitrary s can be seen as a generalization of this situation.

3.4 Proof of Theorem 3.1

We first express the canonical complement in terms of the Levi-Civita connection of the Riemannian metric and the tensor J^u and then we can give the proof of Theorem 3.1 using the calculus formulae from Subsection 2.3.1. Let $\tilde{\sigma}$ be the standard symplectic form on T^*M, as before.

3.4.1 The canonical complement $\widetilde{\mathfrak{J}}^u(\lambda)$

To express the canonical complement $\widetilde{\mathfrak{J}}^u(\lambda)$ in more detail, we need the decomposition of the symplectic form σ and the Hamiltonian field $\vec{h}$, which are analogs of Lemmas 2.1 and 2.3.

Lemma 3.1. *On the level set* $\{u = \bar{u}\}$, $\quad \sigma = (I^{\bar{u}} \circ \mathrm{PR})^*\tilde{\sigma} - (\mathrm{pr} \circ \pi)^*\Omega^{\bar{u}}$.

Lemma 3.2. *Let* $p \in T_q^*M, q \in M$. *Denote by* ∇_{p^h} *the lift of* p^h *to* $T^*\widetilde{M}$ *with respect to the Levi-Civita connection. Then*

$$\vec{h}(p,q) = \underline{\nabla_{p^h}} - (J^u p^h)^v. \tag{3.4.1}$$

Given any $X \in \Pi_\lambda^u$ denote by $\widetilde{\nabla}_{X^h}$ the lift of X to $\widetilde{\mathfrak{J}}^u(\lambda)$: the unique vector $\widetilde{\nabla}_{X^h} \in \widetilde{\mathfrak{J}}^u(\lambda)$ such that $(\mathrm{pr} \circ \pi)_*\widetilde{\nabla}_{X^h} = X^h$. Then there exist the unique $B \in \mathrm{End}(\Pi_\lambda^u)$ such that

$$\widetilde{\nabla}_{v^h} = \underline{\nabla_{v^h}} + Bv, \quad \forall v \in \Pi_\lambda^u. \tag{3.4.2}$$

Similar to Lemma 2.4, one can show that B is antisymmetric w.r.t. the canonical Euclidean structure in Π_λ^u (with complete similar proof). Moreover, we have

Lemma 3.3. *The operator* B *satisfies*

$$(B\tilde{v})^h = -\frac{1}{2}J^u\tilde{v}^h + \frac{1}{2}g(J^u\tilde{v}^h, p^h)p^h, \quad \forall \tilde{v} \in \Pi_\lambda^u \tag{3.4.3}$$

Proof. Since $\widetilde{\mathfrak{J}}^u(\lambda)$ is an isotropic subspace, we have

$$\sigma(\widetilde{\nabla}_{v_1^h}, \widetilde{\nabla}_{v_2^h}) = 0, \quad \forall v_1, v_2 \in \Pi_\lambda^u$$

On the other hand, using Lemma 3.1, the fact that the Levi-Civita connection (as an Ehresmann connection) is a Lagrangian distribution in $T^*\widetilde{M}$ and Lemma 2.2, we get

$$\begin{aligned}
0 = \sigma(\widetilde{\nabla}_{v_1^h}, \widetilde{\nabla}^c_{v_2^h}) &= \Big((I^u \circ \mathrm{PR})^*\tilde{\sigma} - (\mathrm{pr} \circ \pi)^*\Omega^u\Big)\Big(\underline{\nabla_{v_1^h}} + B\tilde{v}_1, \underline{\nabla_{v_2^h}} + B\tilde{v}_2\Big) = \\
&- \Omega^u(v_1^h, v_2^h) - g\big((B\tilde{v}_1)^h, v_2^h\big) + g\big((B\tilde{v}_2)^h, v_1^h\big) = \\
&- g(J^u v_1^h, v_2^h) - g\big((B\tilde{v}_1)^h, v_2^h\big) + g\big((B^*\tilde{v}_1)^h, v_2^h\big).
\end{aligned}$$

Taking into account that B is antisymmetric, we get

$$(B\tilde{v})^h = -\frac{1}{2}J^u\tilde{v}^h + \alpha(\tilde{v})p^h, \tag{3.4.4}$$

for some $\alpha \in (\Pi^{\mathrm{u}}_\lambda)^*$. Since $B\tilde{v} \in \Pi^{\mathrm{u}}_\lambda$, then $B\tilde{v}$ is tangent to the level of h. Thus it follows from (3.4.1) that

$$0 \;=\; \sigma(\vec{h}, -\frac{1}{2}(J^{\mathrm{u}}\tilde{v}^h)^v + \alpha(\tilde{v})(p^h)^v) = -\frac{1}{2}g(J^{\mathrm{u}}p^h, J\tilde{v}^h) + \alpha(\tilde{v}).$$

From the last identity and the identity (3.4.4), we get the relation (3.4.3). The proof of the lemma is completed. □

Corollary 3.1. *The canonical complement $\widetilde{\mathfrak{J}}^{\mathrm{u}}(\lambda)$ can be expressed as follows:*

$$\widetilde{\mathfrak{J}}^{\mathrm{u}}(\lambda) = \{\underline{\nabla_{v^h}} - \frac{1}{2}(J^{\mathrm{u}}v^h)^v + \frac{1}{2}g(J^{\mathrm{u}}v^h, p^h)(p^h)^v,\ v \in \Pi^{\mathrm{u}}_\lambda\}.$$

3.4.2 Proof of the formula for the reduced curvature map

As a direct consequence of structure equation (1.1.7), we get the following preliminary descriptions of the reduced curvature maps.:

Proposition 3.2. *Let $v \in \Pi^{\mathrm{u}}_\lambda$. Let V be a parallel vector field such that $V(\lambda) = v$. Then the reduced curvature map satisfies the following identity:*

$$g((\mathfrak{R}^{\mathrm{u}}_\lambda v)^h, v^h) = -\sigma(ad\vec{h}\ (\widetilde{\nabla}_{V^h}), \widetilde{\nabla}_{v^h}). \tag{3.4.5}$$

Also, the following lemma will be needed in the calculation of the reduced curvature map, which is an analog of Lemma 2.11.

Lemma 3.4. *The following identities holds:*

(1) $([(J^{\mathrm{u}}p^h)^v, (J^{\mathrm{u}}V^h)^v])^h = J([(J^{\mathrm{u}}p^h)^v, (V^h)^v])^h$,

(2) $\sigma([(J^{\mathrm{u}}p^h)^v, \underline{\nabla_{V^h}}], \underline{\nabla_{v^h}}) = g(\nabla J^{\mathrm{u}}(p^h, v^h), v^h)$,

Now we can show the proof of Theorem 3.1.

Proof of Theorem 3.1 In the following calculations, we adapt the Einstein summation convention and all indices range from 1 to s. As in the proof of Lemma 2.4, we can take a parallel vector field V such that $V(\lambda) = v$ and

$$[(J^{\mathrm{u}}p^h)^v, V](\bar{\lambda}) = 0, \quad \bar{\lambda} \in U \cap T^*_qM, \tag{3.4.6}$$

where U is a neighborhood of λ. For simplicity denote $\bar{\sigma} = (I^{\mathrm{u}} \circ \mathrm{PR})^*\sigma$.

Let us simplify the right-hand side of the identity (3.4.5). First, from the last line of the structural equations (1.1.7) it follows that

$$(\mathrm{pr} \circ \pi)_*(\mathrm{ad}\vec{h}(\widetilde{\nabla}_{V^h})) \in \mathbb{R}p^h. \tag{3.4.7}$$

Then from Corollary 3.1 it follows that

$$\sigma(\mathrm{ad}\vec{h}(\widetilde{\nabla}_{V^h}), \widetilde{\nabla}_{v^h}) = \sigma(\mathrm{ad}\vec{h}(\widetilde{\nabla}_{V^h}), \underline{\nabla_{v^h}} + \frac{1}{2}g(J^{\mathrm{u}}v^h, p^h)(p^h)^v) \tag{3.4.8}$$

Besides, it follows from 3.1 and Proposition 2.3 that

$$\sigma(\mathrm{ad}\vec{h}(\widetilde{\nabla}_{V^h}), (p^h)^v) = -\sigma(\widetilde{\nabla}_{V^h}, \mathrm{ad}\vec{h}(p^h)^v) = \sigma(\widetilde{\nabla}_{V^h}, \underline{\nabla_{p^h}}) = 0. \tag{3.4.9}$$

Hence it follows from (3.4.8) and (3.4.9) that

$$\sigma(\mathrm{ad}\vec{h}(\widetilde{\nabla}_{V^h}), \widetilde{\nabla}_{v^h}) = \sigma(\mathrm{ad}\vec{h}(\widetilde{\nabla}_{V^h}), \underline{\nabla_{v^h}}) \tag{3.4.10}$$

Now we are ready to start our calculations:

$$\begin{aligned} \mathrm{ad}\vec{h}(\widetilde{\nabla}_{V^h}) &= [\underline{\nabla_{p^h}}, \underline{\nabla_{V^h}}] - [(J^{\mathrm{u}}v^h)^v, \underline{\nabla_{V^h}}] - \frac{1}{2}[\underline{\nabla_{p^h}}, (J^{\mathrm{u}}V^h)^v] + \frac{1}{2}[(J^{\mathrm{u}}p^h)^v, (J^{\mathrm{u}}V^h)^v] \\ &+ \frac{1}{2}g(J^{\mathrm{u}}v^h, p^h))[\underline{\nabla_{p^h}}, (p^h)^v] - \frac{1}{2}g(J^{\mathrm{u}}v^h, p^h))[(J^{\mathrm{u}}p^h)^v, (p^h)^v], \quad \mathrm{mod}\ \mathbb{R}(p^h)^v \end{aligned} \tag{3.4.11}$$

Note that the fourth term of the right-hand side of (3.4.11) vanishes by item (1) of Proposition 3.4 and relation (3.4.6). Also the last term vanishes by item (1) of Proposition 2.3. Therefore, by relation (3.4.5), we get

$$\begin{aligned} g((\mathfrak{R}^{\mathrm{u}}_{\lambda}v)^h, v^h) &= -\bar{\sigma}([\underline{\nabla_{p^h}}, \underline{\nabla_{V^h}}], \underline{\nabla_{v^h}}) + \Omega^{\mathrm{u}}((\mathrm{pr}\circ\pi)_*([\underline{\nabla_{p^h}}, \underline{\nabla_{V^h}}]), v^h) \\ &+ \bar{\sigma}([(J^{\mathrm{u}}p^h)^v, \underline{\nabla_{V^h}}], \underline{\nabla_{v^h}}) - \Omega^{\bar{\mathrm{u}}}((\mathrm{pr}\circ\pi)_*([(J^{\mathrm{u}}p^h)^v, \underline{\nabla_{V^h}}]), v^h) \\ &+ \frac{1}{2}\bar{\sigma}([\underline{\nabla_{p^h}}, (J^{\mathrm{u}}v^h)^v], \underline{\nabla_{v^h}}) - \frac{1}{2}\Omega^{\mathrm{u}}((\mathrm{pr}\circ\pi)_*([\underline{\nabla_{p^h}}, (J^{\mathrm{u}}v^h)^v]), v^h) \\ &- \frac{1}{2}g(J^{\mathrm{u}}v^h, p^h))\bar{\sigma}([\underline{\nabla_{p^h}}, (p^h)^v], \underline{\nabla_{v^h}}) + \frac{1}{2}\Omega^{\mathrm{u}}((\mathrm{pr}\circ\pi)_*([\underline{\nabla_{p^h}}, (p^h)^v]), v^h)g(J^{\mathrm{u}}v^h, p^h) \end{aligned} \tag{3.4.12}$$

Now we analyze the right-hand side of the last identity term by term. First, it follows identity (2.1.5) that

$$\bar{\sigma}([\underline{\nabla_{p^h}}, \underline{\nabla_{V^h}}], \underline{\nabla_{v^h}}) = -g(R^{\nabla}(p^h, v^h)p^h, v^h). \tag{3.4.13}$$

Also it follows from item (2) of Lemma 3.4 that

$$\bar{\sigma}([(J^{\mathrm{u}}p^h)^v, \underline{\nabla_{V^h}}], \underline{\nabla_{v^h}}) = g(\nabla J^{\mathrm{u}}(p^h, v^h), v^h). \tag{3.4.14}$$

Also it follows from straightforward computations that

$$\Omega^{\mathrm{u}}((\mathrm{pr}\circ\pi)_*([\underline{\nabla_{p^h}}, (J^{\mathrm{u}}v^h)^v]), v^h) = -\Omega^{\mathrm{u}}(J^{\mathrm{u}}v^h, v^h) = \|J^{\mathrm{u}}v^h\|^2 \tag{3.4.15}$$

and

$$\Omega^{\mathrm{u}}((\mathrm{pr}\circ\pi)_*([\underline{\nabla_{p^h}}, (p^h)^v]), v^h) = -\Omega^{\bar{\mathrm{u}}}(p^h, v^h) = g(p^h, J^{\mathrm{u}}v^h). \tag{3.4.16}$$

Also we have

$$\begin{aligned} &-\Omega^{\mathrm{u}}((\mathrm{pr}\circ\pi)_*([\underline{\nabla_{p^h}}, \underline{\nabla_{V^h}}]), v^h) - \frac{1}{2}\bar{\sigma}([\underline{\nabla_{p^h}}, (J^{\mathrm{u}}v^h)^v], \underline{\nabla_{v^h}}) \\ &= g((\mathrm{pr}\circ\pi)_*([\underline{\nabla_{p^h}}, \underline{\nabla_{V^h}}]), J^{\mathrm{u}}v^h) - \frac{1}{2}\bar{\sigma}([\underline{\nabla_{p^h}}, \underline{\nabla_{v^h}}], (J^{\mathrm{u}}v^h)^v) \\ &= \frac{1}{2}g((\mathrm{pr}\circ\pi)_*([\underline{\nabla_{p^h}}, \underline{\nabla_{V^h}}]), J^{\mathrm{u}}v^h) \end{aligned} \tag{3.4.17}$$

Also it follows from item (2) of Proposition 2.3 that

$$\bar{\sigma}([\underline{\nabla_{p^h}}, (p^h)^v], \underline{\nabla_{v^h}}) = \bar{\sigma}(-\underline{\nabla_{p^h}}, \underline{\nabla_{v^h}}) = 0. \tag{3.4.18}$$

In order to calculate the other terms in the right-hand side of identity (3.4.12), we need the following lemma.

Lemma 3.5. *The following relations hold:*

(1) $(\mathrm{pr}\circ\pi)_*([(J^{\mathrm{u}}p^h)^v, \underline{\nabla_{V^h}}]) = J^{\mathrm{u}}v^h$,

(2) $(\mathrm{pr}\circ\pi)_*([\underline{\nabla_{p^h}}, \underline{\nabla_{V^h}}]) = \frac{1}{2}J^{\mathrm{u}}v^h - \frac{1}{2}g(J^{\mathrm{u}}v^h, p^h)p^h$.

Proof. (1) Applying the Jacobi identity and the relation (3.4.6), we get

$$0 = [\underline{\nabla_{p^h}}, [(J^{\mathrm{u}}p^h)^v, (V^h)^v]] = [[\underline{\nabla_{p^h}}, (J^{\mathrm{u}}p^h)^v], (V^h)^v] + [(J^{\mathrm{u}}p^h)^v, [\underline{\nabla_{p^h}}, (V^h)^v]].$$

Then it follows immediately that

$$\begin{aligned} & (\mathrm{pr} \circ \pi)_*([(J^{\mathrm{u}}p^h)^v, \underline{\nabla_{V^h}}]) = -(\mathrm{pr} \circ \pi)_*([(J^{\mathrm{u}}p^h)^v, [\underline{\nabla_{p^h}}, (V^h)^v]]) \\ = \;& (\mathrm{pr} \circ \pi)_*([[\underline{\nabla_{p^h}}, (J^{\mathrm{u}}p^h)^v], (V^h)^v]) = -(\mathrm{pr} \circ \pi)_*([\underline{\nabla_{J^{\mathrm{u}}p^h}}, (V^h)^v]) = J^{\mathrm{u}}v^h. \end{aligned}$$

Note that the last identity follows from direct computations.

(2) As an analog of Lemma 2.12, we have

$$(\mathrm{pr} \circ \pi)_*([\underline{\nabla_{p^h}}, \underline{\nabla_{V^h}}]) = \frac{1}{2}J^{\mathrm{u}}v^h - \frac{1}{2}g(J^{\mathrm{u}}v^h, p^h)p^h, \;\; \mathrm{mod}\; p^h. \tag{3.4.19}$$

Moreover, it follows from identity (3.4.18) of Proposition 2.3 that

$$\bar{\sigma}([\underline{\nabla_{p^h}}, \underline{\nabla_{V^h}}], (p^h)^v) = \bar{\sigma}([\underline{\nabla_{p^h}}, (p^h)^v], \underline{\nabla_{v^h}}) = 0 \tag{3.4.20}$$

Comparing identities (3.4.19) and (3.4.20), we get the item (2) of the lemma. □

Now applying item (1) of Lemma 3.5, we get

$$\Omega^{\mathrm{u}}((\mathrm{pr} \circ \pi)_*([(J^{\mathrm{u}}p^h)^v, \underline{\nabla_{V^h}}]), v^h) = \Omega^{\mathrm{u}}(J^{\mathrm{u}}v^h, v^h) = -\|J^{\mathrm{u}}v^h\|^2. \tag{3.4.21}$$

Applying item (2) of Lemma 3.5, we get from (3.4.17) that

$$-\Omega^{\mathrm{u}}(\pi_*([\underline{\nabla_{p^h}}, \underline{\nabla_{V^h}}]), v^h) - \frac{1}{2}\bar{\sigma}([\underline{\nabla_{p^h}}, (J^{\mathrm{u}}v^h)^v], \underline{\nabla_{v^h}}) = \frac{1}{4}\|J^{\mathrm{u}}v^h\|^2 - \frac{1}{4}\left(g(p^h, J^{\mathrm{u}}v^h)\right)^2 \tag{3.4.22}$$

Substituting identities (3.4.13)-(3.4.16), (3.4.18), (3.4.21), (3.4.22) into identity (3.4.12), we get the required expression for the reduced curvature map. □

Bibliography

[1] Ralph Abraham and Jerrold E. Marsden. *Foundations of Mechanics.* Westview Press, 2nd edition, 1994.

[2] A. Agrachev. Any sub-Riemannian metric has points of smoothness. *Russian Math.Dokl.*, 79:1–3, 2009.

[3] A. Agrachev and I. Zelenko. Geometry of Jacobi curves. I. *J. Dynamical and Control Systems*, 8(1):93–140, 2002.

[4] A. Agrachev and I. Zelenko. Geometry of jacobi curves. II. *J. Dynamical and Control Systems*, 8(2):167–215, 2002.

[5] A. A. Agrachev. Geometry of optimal control problems and Hamiltonian systems. In *Nonlinear and Optimal Control Theory*, Lecture Notes in Mathematics, Volumn 1932, pages 1–59. Springer Berlin / Heidelberg, 2008.

[6] A. A. Agrachev and R. V. Gamkrelidze. Feedback-invariant optimal control theory - i. regular extremals. *J. Dynamical and Control Systems*, 3:343–389, 1997.

[7] A. A. Agrachev and P. Lee. Generalized Ricci curvature bounds for three dimensional contact subriemannian manifolds. *arXiv:0903.2550.*

[8] Andrei A. Agrachev and N. Chtcherbakova. Hamiltonian systems of negative curvature are hyperbolic. *Russian Math. Dokl.*, 400:295–298, 2005.

[9] Andrei A. Agrachev, N. Chtcherbakova, and I. Zelenko. On curvatures and focal points of dynamical Lagrangian distributions and their reductions by first integrals. *J. Dynamical and Control Systems*, 11:297–327, 2005.

[10] Andrei A. Agrachev and R. Gamkerlidze. Symplectic methods in optimization and control. In B. Jakubczyk and W. Respondek, editors, *Geometry of Feedback and Optimal Control*, pages 19–77. Marcel Dekker, 1998.

[11] Andrei A. Agrachev and Yuri L. Sachkov. *Control Theory from the Geometric Viewpoint.* Encyclopaedia of Mathematical Sciences, Control Theory and Optimization(87). Springer-Verlag, 2004.

[12] D. V. Anosov. Geodesic flows on the closed Riemannian manifold of negative curvature. *Proceedings of the Steklov Institute of Mathematics*, 90:3–209. AMS, Providence, RI, 1967.

[13] H. Derksen and J. Weyman. Quiver representations. *Notices of AMS*, 52(2):200–206, 2005.

[14] N. Gouda. Magnetic flows of Anosov type. *Tôhoku Math. J.*, 49:165–183, 1997.

[15] Arnold V. I. The Sturm's theorem and symplectic geometry. *Function Anal. and its Appl.*, 19(1):1–10, 1985.

[16] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems.* Encyclopedia of Mathematics and its Applications(54). Cambridge Univ. Press, 1995.

[17] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry*, volume 2. Wiley-Interscience, 1 edition, 1996.

[18] C. Li and I. Zelenko. Jacobi equations and comparison theorems for corank 1 sub-Riemannian structures with symmetries. *arXiv: 0908.4397v1 [math. DG], SISSA preprint 53/2009/M.*

[19] Chengbo Li. Hyperbolic flows in sub-Riemannian structures with multidimensional symmetres. *in preparation.*

[20] R. Montgomery. *A Tour of Subriemannian Geometries, Their Geodesics, and Applications.* Mathematical Surveys and Monographs, Volume 91. American Mathematical Society, 2002.

[21] E. A. K. Nyandjou. Generalized Sturm theorem for self-adjoint differential operators of higher order. *Diploma Dissertation (under supervision of I. Zelenko), ICTP*, 2007.

[22] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mischenko. *The Mathematical Theory of Optimal Processes.* Wiley, New York, 1962.

[23] I. Zelenko. Complete systems of invariants for rank 1 curves in Lagrange Grassmannians. *Differential Geometry and its Applications, Proc. Conf. Prague, August 30-September 3,2004, Charles University, Prague*, pages 365–379, 2005.

[24] I. Zelenko and C. Li. Differential geometry of curves in Lagrange Grassmannians with given Young diagram. *Differential Geometry and its Applications, doi: 10.1016/j.difgeo.2009.07.002.*

[25] I. Zelenko and C. Li. Parametrized curves in lagrange grassmannians. *C. R. Acad. Sci. Paris, Ser. I*, 345(11):647–652, 2007.