Scuola Internazionale Superiore di Studi Avanzati

Doctoral Thesis

Conformal symmetry in String Field Theory and $4D$ Field Theories

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Contents

Chapter 1

Introduction

This thesis is intended as an overview of the two main research topics I have dealt with in the course of my PhD studies: the quest for exact analytic solutions in the context of Witten' s OSFT with the purpose of investigating the moduli space of open strings and the investigation of the structure of trace anomalies in superconformal field theories.
String theory is a widely investigated framework in which it is possible to address the problem of giving a consistent and unified description of our universe. This has long been expected on the basis of the fact that the five known perturbative superstring theories (IIA, IIB, I, heterotic $SO(32)$ and heterotic $E_8 \times E_8$) provide a set of rules to calculate on-shell scattering amplitudes for the modes describing the fluctuations of the strings; such perturbative spectra include, apart from an infinite tower of massive particles, also massless quanta associated with supergravity and super Yang-Mills gauge theories in ten-dimensional space time. These supergravity theories will in general admit black-hole like solutions covering a region of space-time with $9-p$ spatial coordinates. In particular in type IIA/IIB supergravity the black p-brane solutions with even/odd p carry charge under Ramond-Ramond (RR) $(p+1)$-form fields so that they are stable and have a definite tension (mass per unit volume). A fundamental breakthrough came from understanding that they may be viewed as the low-energy supergravity limit of the $(p+1)$-dimensional hypersurfaces where open strings with Dirichlet boundary conditions can end and that therefore these D-branes are actually manifestations of the non perturbative dynamics of String theory. D-branes have actually played a fundamental role in understanding the web of duality symmetries relating the five superstring models to each other and to a yet not fully defined theory, dubbed M-theory, whose low-energy limit is eleven-dimensional supergravity. This picture seems to suggest the attractive possibility all these theories can be formulated as specific limits of a unique non-perturbative string (or M) theory which is explicitly background-independent. The discovery of D-branes is also related to another fundamental achievement of string theory, the AdS/CFT correspondence,

which in its best-known formulation states that quantum type IIB closed string theory on $AdS_5 \times S^5$ with N units of RR 5-form flux is dual to $\mathcal{N} = 4$ $U(N)$ SYM theory on the projective 4- dimensional boundary of AdS_5. The D-brane perspective is that such a gauge theory is realized as the low energy limit of open string theory on N $D3$-branes in flat space-time. So the afore mentioned correspondence is establishing a correspondence between the quantum theory of gravity on a given background space-time and the open string dynamics of the D-brane configuration which is creating it by backreacting on the original flat space-time. The idea that the closed string interactions are already encoded in the open string theory dates back to the early days of string theory when it was soon realized closed strings are to be introduced to unitarize open string amplitudes at one loop. D-branes are a fundamental ingredient in understanding how this open/-closed correspondence is realized at non perturbative level. In fact we find the quantum open string theory on a D-brane configuration has to include closed strings on a given space-time background. This naturally suggests a way to give a completely background independent formulation of quantum gravity could be to consider open string theory on various D-brane configurations. The problem of understanding how different D-brane configurations are related found an impressive explanation by Sen's conjectures about tachyon dynamics. Tachyons are ubiquitous in string theory. In the $(25+1)$ dimensional bosonic string theory, for example, tachyonic states appear in both the closed and open string spectra. That's the reason why superstring theories, which have tachyon free closed string spectrum, are considered to be the best candidates for a unified theory of nature. Nevertheless, according to our current understanding, tachyons are still there for D-branes with odd/even values of p or for Dbrane-anti $\overline{D}$-brane systems in type IIA/IIB superstring theories. Sen has thus conjectured [3, 4], in close analogy with the standard interpretation of tachyonic modes in quantum field theory, that tachyons represent an instability of the D-brane system. An effective tachyon potential $V(T)$ should have a local maximum at $T = 0$, where the D-brane exists, and a local minimum where the tachyon field acquires the vev $T = T_0$. Supposing that the tension T_p of the D-brane is exactly canceled by the negative value $V(T_0)$ this configuration is naturally identified as the vacuum without any D-brane. This in turn implies there are no open string states around the minimum of the potential. Although perturbative states are absent at the closed string tachyon condensation vacuum, the equation of motion for the tachyon must however allow for codimension $25 - p$ lump solution which may be interpreted as lower dimensional Dp-branes. Sen's conjectures not only explain that the presence of tachyon modes in the open string spectrum is not a problem but also relate it to the very dynamical mechanism by which different Dp-brane systems can be generated starting from the perturbative $D25$ vacuum. Such a suggestive picture clearly requires an off-shell formulation of string theory where the study of the moduli space can be naturally addressed. Whereas for superstring theory there is not yet a completely well defined theoretical frame to face this issue, bosonic open string theory has a remarkably simple off-shell formulation. i.e the celebrated Witten's Cubic Open String Field Theory (COSFT), which has the form of a Chern-Simon like theory. Quite extensive numerical studies through level truncation of the string field action have allowed to get a non trivial tachyon potential consistent even quantitatively with Sen's conjecture. In more recent years attention has been focused on the possibility of providing an analytic proof of the conjectures. This was achieved in a quite straightforward way in the so called Boundary String Field Theory (BSFT) [16, 61, 62], where the connection between classical solutions of SFT equations of motion and the Boundary Conformal Field Theory (BCFT) on the world sheet of the propagating string is explicit. In fact, although the interacting SFT seems to be hopelessly complicated due the infinite number of interaction vertices among the infinite tower of fields corresponding to a single string state, world-sheet conformal symmetry is such a strong constraint as to make the problem treatable. In fact string theory vacua simply correspond to two dimensional CFTs with all possible boundary conditions. Marginal deformations which don't break conformal symmetry describe physically equivalent vacua with the same spectrum. Relevant deformations which break conformal invariance describe RG flows taking the world sheet theory to other conformal fixed points representing different vacua or, equivalently, different D-brane configurations. Until Martin Schnabl's groundbreaking analytic tachyon vacuum solution, how to translate such a beautiful picture in the rigorously algebraic language of Witten's COSFT was quite mysterious. We can fairly say a lot of progress has been done since then.

Renormalization group analysis gives a privileged role to scale invariant fixed points by its own very definition. Under such reasonable hypotheses as unitarity, Poincaré invariance(in particular causality), discrete spectrum and the existence of a scale current (rather than just a charge) there is no known example of scale invariant but non conformally invariant field theory. Actually in two dimensions this is a well-established result [77, 78, 89]. In four dimension a perturbative proof has recently been given [91, 92]. This suggests conformal field theories and the exact results that can be obtained in their context, just like e.g. the recently proven and long expected a-theorem [90], may really be of fundamental importance in investigating the landscape of $4d$ quantum field theories. On the other hand, albeit largely studied for a long time, there are plenty of aspects in CFT physics left to be investigated, one of these being the elusive appearance of a CP-violating term in trace anomalies. It has long been known [120] that such a term, that is the Hirzebruch-Pontryagin density, satisfies the Wess-Zumino consistency equation for Weyl transformations . Nevertheless the known examples in which a Pontryagin density actually appears in the trace anomaly are non-unitary theories which therefore cannot be accepted as sensible UV completions. This may actually be a very generic situation with the presence of the Pontryagin density in the trace anomaly always associated to a loss of unitarity. This suggests a lot of physical information may actually be inferred from a precise assessment of trace anomalies. Supersymmetric field theories are known to possess a rich structure in their supercurrent supermultiplet, stemming from some ambiguities in choosing the supersymmetric completion of the multiplets containing the energy-momentum tensor and the supercurrent. Actually it has recently been pointed out there are physically interesting cases in which the very well-known Ferrara-Zumino (FZ) supermultiplet is not well defined and so other choices may be necessary [79–81]. This is quite relevant to the problem of trace anomalies, because different supermultiplets actually correspond to different realizations of the superWeyl group. In fact the solutions of the corresponding Wess-Zumino consistency equation in the formalism of superfields were studied a long time ago for the so called old minimal supergravity, corresponding to the FZ-multplet [93], but such a systematic approach has been missing for other known supergravities. In the following we shall present a first investigation in this direction.

In chapter 1 we give a very fast account of the fundamental steps that have led to understanding how the information about the underlying worldsheet BCFT is encoded in simple subalgebras of the huge $\star$ product algebra. In particular we point out the pivotal role some particular projector elements seem to play in this regard. Then we describe the theoretical framework needed to construct solutions corresponding to relevant deformations, reviewing in particular Erler-Schnabl (ES) tachyon condensation solution and Bonora-Maccaferri-Tolla (BMT) lump solutions.
In chapter 2 we present the computation of the tension for Dp-branes with $p \leq 24$ that was carried out both numerically and analytically leading to a full confirmation of Sen's conjecture about lumps at the tachyon vacuum describing lower dimensional branes. As a technical tool to carry out the computation we also present a different non universal solution representing the tachyon condensation vacuum.
In chapter 3 we present several critical issues have been raised about BMT lumps and address them with both numerical and theoretical analysis. We also propose a mathematical framework, inspired by distribution theory, where the sense in which Witten's EOM should hold is properly formalized.
In chapter 4, after reviewing several formulations of $\mathcal{N} = 1$ $4d$ supergravity, we present a cohomological discussion of Weyl anomalies for different realizations of the super Eeyl group, with particular attention on the appearance of a CP violating Pontryagin density.

Chapter 2

Analytic solutions in COSFT

2.1 Introduction

Dp-branes are defined as $(p+1)$-dimensional hypersurfaces Σ_{p+1} on which open strings with Dirichlet boundary conditions can end. In particular, when p is even/odd in type IIA/IIB string theory, one can see the corresponding spectrum contains a massless set of fields $A_\alpha, \alpha = 0, 1, \ldots, p$ and $X^a, a = p+1, \ldots, 9$, which can be respectively associated with a $U(1)$ gauge field living on the D-brane and the transverse fluctuations of the hypersurface Σ_{p+1} in ten dimensional space-time. Polchinski pointed out stable Dp branes of superstring theory carry charge under Ramond-Ramond $(p+1)$-form fields $A^{(p+1)}_{\mu_1\cdots\mu_{(p+1)}}$ and correspond to BPS states in the low-energy supergravity theory. The Dp-brane low-energy dynamics is determined by the classical Born-Infeld action

$$S = -T_p \int d^{p+1}\xi \; e^{-\varphi} \sqrt{-\det(G_{\alpha\beta} + B_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})} \tag{2.1}$$

where G, B, and φ are the pullbacks of the ten-dimensional metric, antisymmetric tensor, and dilaton to the D-brane world-volume, and F is the field strength of the world-volume $U(1)$ gauge field A_α. The constant T_p is fixed by calculating a string amplitude

$$\tau_p = \frac{T_p}{g_s} = \frac{1}{g_s\sqrt{\alpha'}}\frac{1}{(2\pi\sqrt{\alpha'})^p} \tag{2.2}$$

where τ_p is the mass per unit p-volume, which is called the brane *tension*, and $g_s = e^{\langle\varphi\rangle}$ is the closed string coupling, equal to the exponential of the dilaton expectation value. BPS branes are oriented and so, given a Dp-brane one can consider an anti Dp-brane (a $\overline{D}p$-brane) with opposite orientation and opposite charges. So, although BPS D-branes are stable, a coincident BPS Dpbrane-$\overline{D}p$-brane pair, having null RR charge, can decay into an uncharged vacuum configuration. In fact, one can verify the perturbative spectrum contains two tachyonic modes, corresponding to the open string stretched from the brane to the anti-brane and vice versa. The same happens for wrong-dimension non BPS branes which carry no charges and contain a tachyonic mode in their spectrum. The simplest setup in which, besides an infinite number of other modes with squared mass ≥ 0, a tachyon appears is the Dp-brane in bosonic string theory which has no charge for generic $p \leq 25$. The tension is seen to be (taking $\alpha' = 1$)

$$\tau_p = g_s^{-1} (2\pi)^{-p} , \tag{2.3}$$

The physics of this tachyon field T has been quite a puzzle for a long time because of the non trivial coupling to all the infinite number of other fields in the spectrum. Furthermore, as the modulus of the square mass of T is of the same order of magnitude as that of the other heavy modes, one wouldn't expect there is a low-energy regime where these massive modes decouple. Nevertheless it is convenient to define the effective action $S_{eff}(T, \ldots)$, obtained by formally integrating out all the positive mass fields. This means S_{eff} depends only on the tachyon field and the massless modes. By restricting to space-time independent field configurations and setting the massless fields to zero, one can define the tachyon effective potential such that $S_{eff}(T) = -\int d^{p+1}x\, V(T)$. Than, since the square mass of T is given by $V''(T = 0)$, $V(T)$ must have a maximum at $T = 0$. Sen conjectured this maximum actually describes the unstable Dp-brane and that the following three statements are true

1. The tachyon effective potential $V(T)$ has a local minimum at some value T_0, whose energy density $\mathcal{E} = V(T_0)$, measured with respect to that of the unstable critical point, is equal to minus the tension of the Dp-brane

$$\mathcal{E} = -T_p \, . \tag{2.4}$$

2. The locally stable vacuum is the closed string vacuum, which means no physical open string excitations exist around the minimum of the potential.

3. Although at the closed string vacuum there are no perturbative physical states, the equation of motion derived from S_{eff} has time independent classical lump solutions. If they depend on q spatial coordinates and approach T_0 for one of these coordinates going to infinity, they are expected to represent a $D(p-q)$-brane.

The concept itself of the potential for the zero-momentum tachyon requires an off-shell formulation of string theory, which is therefore needed in order to obtain direct evidence for these conjectures

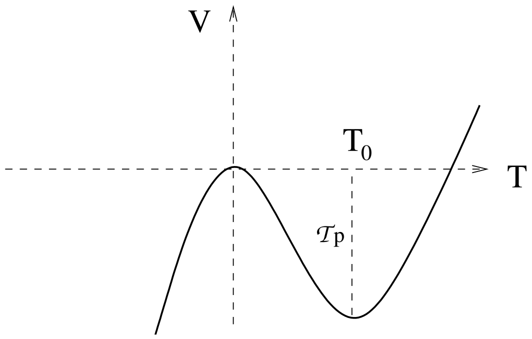

FIGURE 2.1: The tachyon effective potentialon a *Dp*-brane in bosonic string theory.

2.2 Witten's Cubic Open String Field Theory

The basis for the definition of the bosonic string field [5–7, 9, 10] is the first-quantized bosonic open string, which can be worked out in the BRST approach starting from the action

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu, \tag{2.5}$$

where g is the metric on the world-sheet. This action can be gauge-fixed to conformal gauge $g_{\alpha\beta}\sim\delta_{\alpha\beta}$, introducing ghost and antighost fields $c, \widetilde{c}, b, \widetilde{b}$, with ghost numbers 1 and -1 respectively. The gauge-fixed action is

$$S = \frac{1}{2\pi\alpha'}\int d^2z\partial X^\mu\overline{\partial}X_\mu + \frac{1}{2\pi}\int d^2z\ (b\overline{\partial}c + \widetilde{b}\partial\widetilde{c}). \tag{2.6}$$

where $z = -e^{-iw}$, $w = \sigma^1 + i\sigma^2$, is the complex coordinate for the upper half plane. Introducing the standard mode expansion for the matter fields X^μ and the ghosts

$$\begin{aligned}
X^\mu(z,\overline{z}) &= x^\mu - i\alpha' p^\mu \ln|z|^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{\alpha_m^\mu}{m}\left(z^{-m}+\overline{z}^{-m}\right);\\
\text{with}\qquad & p^\mu = \frac{1}{\sqrt{2\alpha'}}\alpha_0^\mu \quad \text{for open string}\\
b(z) &= \sum_n b_n z^{-n-2} && (2.7)\\
c(z) &= \sum_n c_n z^{-n+1} && (2.8)
\end{aligned}$$

and the commutation relations

$$\begin{aligned}
[\alpha_m^\mu, \alpha_n^\nu] &= m\delta_{m+n,0}\eta^{\mu\nu}\ , \quad [x^\mu, p^\nu] = i\eta^{\mu\nu}, && (2.9)\\
\{b_m, c_n\} &= \delta_{m+n,0}. && (2.10)
\end{aligned}$$

we can build up the Hilbert space $\mathcal{H}$ of the first quantized string theory by acting with negative modes $\alpha^\mu_{-n}, b_{-m}, c_{-\ell}$ on the oscillator vacuum $|\Omega\rangle$ which is defined as

$$\begin{aligned} &\alpha^\mu_n|\Omega\rangle = 0 && n > 0 \\ &b_n|\Omega\rangle = 0 && n \geq 0 \\ &c_n|\Omega\rangle = 0 && n > 0 \\ &p^\mu|\Omega\rangle \propto \alpha^\mu_0|\Omega\rangle = 0. && \end{aligned} \tag{2.11}$$

$|\Omega\rangle$ is related to the $SL(2,\mathbb{R})$ invariant vacuum by the relation $|\Omega\rangle = c_1|0\rangle$. The space $\mathcal{H}$ is thus spanned by the states

$$\alpha^{\mu_1}_{-n_1}\cdots\alpha^{\mu_i}_{-n_i}b_{-m_1}\cdots b_{-m_j}c_{-\ell_1}\cdots c_{-\ell_k}|\Omega\rangle,$$

with $n > 0, m > 0, \ell \geq 0$, and i, j, k arbitrary positive integers. In the BRST quantization approach physical states are defined as states belonging to the subspace $\mathcal{H}^1$ of ghost number $+1$ obeying the physical condition of BRST invariance

$$Q_B|\psi\rangle = 0. \tag{2.12}$$

where $Q_B \equiv \oint \frac{dw}{2\pi i}\, j_B(w)$ and

$$j_B \;=\; :cT^{\mathrm{m}}: + \frac{1}{2} :cT^{\mathrm{g}}: + \frac{3}{2}\partial^2 c \tag{2.13}$$

$$\;=\; :cT^{\mathrm{m}}: + :bc\partial c: + \frac{3}{2}\partial^2 c, \tag{2.14}$$

$T_m(z) = \sum_n L^{(m)}_n z^{-n-2}$ stands for the zz component of the world-sheet stress tensor of the matter fields. As $Q_B^2 = 0$, this boils down to a cohomological problem and the real physical Hilbert space is given by

$$\mathcal{H}_{\mathrm{phys}} = \mathcal{H}^1_{\mathrm{closed}}/\mathcal{H}^1_{\mathrm{exact}},$$

i.e. by the cohomology of Q_B with ghost number 1.

A string field is defined as an off-shell generalization of a physical state. In fact in SFT the BRST invariance condition 2.12 is promoted to the linearized equation of motion and the nilpotency of Q_B gives rise to the invariance of solutions under the gauge transformation

$$\delta|\Phi\rangle = Q_B|\chi\rangle \tag{2.15}$$

So, using this gauge freedom to fix the so called Feymann-Siegel gauge

$$b_0|\Phi\rangle = 0 \tag{2.16}$$

we can expand the ghost 1 string field $|\Phi\rangle$ in the Fock space basis

$$\begin{aligned}|\Phi\rangle &= \Big(\phi + A_\mu \alpha^\mu_{-1} + i\alpha b_{-1}c_0 + \frac{i}{\sqrt{2}}B_\mu\alpha^\mu_{-2} + \frac{1}{\sqrt{2}}B_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} \\ &\quad + \beta_0 b_{-2}c_0 + \beta_1 b_{-1}c_{-1} + i\kappa_\mu \alpha^\mu_{-1} b_{-1}c_0 + \cdots\Big)c_1|0\rangle. \end{aligned} \tag{2.17}$$

where every coefficient in the expansion should be thought of as a particle field dependent on the center-of-mass coordinate x of the string. Witten [1] proposed to interpret such a string field as an element of a differential graded algebra (DGA) $\mathcal{A}$. The star product $\star$

$$\star : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}, \tag{2.18}$$

is such that the degree G is additive ($G_{\Psi\star\Phi} = G_\Psi + G_\Phi$). The differentiation $Q : \mathcal{A} \to \mathcal{A}$ has degree one ($G_{Q\Psi} = 1 + G_\Psi$). One has to define an integration operation too

$$\int : \mathcal{A} \to \mathbf{C}\,. \tag{2.19}$$

vanishing for all Ψ with degree $G_\Psi \neq 3$. $Q, \star, \int$ that define the string field theory are assumed to satisfy the following axioms:

(a) Nilpotency of Q: $Q^2\Psi = 0, \quad \forall\Psi \in \mathcal{A}$.

(b) $\int Q\Psi = 0, \quad \forall\Psi \in \mathcal{A}$.

(c) Derivation property of Q:
$Q(\Psi\star\Phi) = (Q\Psi)\star\Phi + (-1)^{G_\Psi}\Psi\star(Q\Phi), \;\; \forall\Psi,\Phi \in \mathcal{A}$.

(d) Cyclicity: $\int \Psi\star\Phi = (-1)^{G_\Psi G_\Phi}\int \Phi\star\Psi, \quad \forall\Psi,\Phi \in \mathcal{A}$.

(e) Associativity: $(\Phi\star\Psi)\star\Xi = \Phi\star(\Psi\star\Xi), \quad \forall\Phi,\Psi,\Xi \in \mathcal{A}$.

This structure is very similar to that of differential forms on a manifold and so it's maybe to be expected the SFT action proposed by Witten

$$S = -\frac{1}{2}\int \Psi\star Q\Psi - \frac{g}{3}\int \Psi\star\Psi\star\Psi\,. \tag{2.20}$$

which is formally similar to the Chern-Simon one on a 3 manifold, is invariant under the gauge transformation

$$\delta\Psi = Q\Lambda + \Psi\star\Lambda - \Lambda\star\Psi\,, \tag{2.21}$$

for any gauge parameter $\Lambda \in \mathcal{A}$ with degree 0. In fact, the form of this gauge transformation is just chosen to generalize that of the nonabelian gauge theory. One can verify this axiomatic approach has concrete realizations, in particular in terms of the space of functionals $\Psi[x(\sigma); c(\sigma), b(\sigma)]$ of the matter, ghost and anti ghost fields describing the open string labeled by the coordinate σ $(0 \leq \sigma \leq \pi.)$. In this case the differential operator Q coincides with the BRST operator Q_B. For the star product to be an associative operation, one has to interpret the product $\Psi \star \Phi$ as the result of gluing the right hand piece $(\pi/2 \leq \sigma \leq \pi)$ of the string Ψ and the left hand piece $(0 \leq \sigma \leq \pi/2.)$ of the string Φ. For the matter sector, this gluing procedure is expressed by the formal functional integral

$$\begin{aligned}
&(\Psi \star \Phi)\,[z(\sigma)] \\
&\equiv \int \prod_{0\leq\widetilde{\tau}\leq\frac{\pi}{2}} dy(\widetilde{\tau})\, dx(\pi-\widetilde{\tau}) \prod_{\frac{\pi}{2}\leq\tau\leq\pi} \delta[x(\tau)-y(\pi-\tau)]\ \Psi[x(\tau)]\Phi[y(\tau)]\,, \\
&\qquad x(\tau)=z(\tau) \quad \text{for} \quad 0\leq\tau\leq\frac{\pi}{2}\,, \\
&\qquad y(\tau)=z(\tau) \quad \text{for} \quad \frac{\pi}{2}\leq\tau\leq\pi\,.
\end{aligned} \tag{2.22}$$

As for the integration operation, a definition consistent with the fact that $\int(a \star b) = \pm\int(b\star a)$ in spite of $a\star b$ and $b\star a$ being two different string states, is the one prescribing to glue the remaining sides of Ψ and Φ, i.e.

$$\int \Psi\star\Phi = \int \Psi[x(\tau)]\Phi[y(\tau)] \prod_{0\leq\tau\leq\pi} \delta[x(\tau)-y(\pi-\tau)]dx(\tau)dy(\tau). \tag{2.23}$$

The identity with respect to the star product is now readily recognized

$$I[x(\sigma)] \equiv \prod_{0\leq\sigma\leq\frac{\pi}{2}} \delta[x(\sigma)-x(\pi-\sigma)] = \prod_n \delta(x_{2n+1})\,. \tag{2.24}$$

and we can also write down the integral of the matter part of a string field as

$$\int \Psi = \int \prod_{0\leq\sigma\leq\pi} dx(\sigma) \prod_{0\leq\tau\leq\frac{\pi}{2}} \delta[x(\tau)-x(\pi-\tau)]\ \Psi[x(\tau)]\,. \tag{2.25}$$

One can proceed for the ghost part analogously, in spite of some further technical difficulties. Even if these definitions may look quite formal, they can be used to carry out computations quite effectively in the oscillator formalism where they are written in terms of creation and annihilation operators acting on the string Fock space. The drawback is that such an approach makes analytic handling of the formulae involved in the computation of physical quantities quite difficult.

2.3 The CFT formulation of SFT

2.3.1 Evaluation of correlation functions

The limit of the operator approach is that, whereas $\int \Psi \star \Phi$ can be easily accommodated into the formalism as the Fock space inner product $\langle\Psi|\Phi\rangle$, $\int \Psi \star \Phi \star \Xi \equiv \langle\Psi|\Phi \star \Xi\rangle$ has a complicated expression in terms of the 3-point vertex $\langle V_3|$. The idea behind the CFT method is instead to use the conformal invariance of the world-sheet theory to make the geometrical meaning of the star product most transparent. In particular, we may consider three strings whose world-sheet has been mapped to the upper half plane(UHP) so that we may think they appeared at $t = -\infty$, i.e. at the origin $z_i = 0$ and then propagated radially till the present ($t = 0$) when they meet at the interaction point $|z_i| = 1$. If we take the world-sheet of the $\star$ product to be the unit disk, it's quite intuitive how to build it by gluing the three half disks corresponding to the strings. For each of them, first we map the local coordinate half-disk in the upper half-plane to the unit half-disk in the right half-plane ($z_i \quad \mapsto \quad w = h(z_i) = \frac{1+iz_i}{1-iz_i}$); then we shrink it to a wedge of angle $\frac{2}{3}\pi$ ($w \quad \mapsto \quad \zeta = \eta(w) = w^{2/3}$). In this way the right and left side of each string can now be sewn together by rotating the wedges in the right position. Overall the mappings to be done are

$$\begin{aligned} g_1(z_1) &= e^{-\frac{2\pi i}{3}} \left(\frac{1+iz_1}{1-iz_1}\right)^{\frac{2}{3}}, \\ \eta \circ h(z_2) = g_2(z_2) &= \left(\frac{1+iz_2}{1-iz_2}\right)^{\frac{2}{3}}, \\ g_3(z_3) &= e^{\frac{2\pi i}{3}} \left(\frac{1+iz_3}{1-iz_3}\right)^{\frac{2}{3}}. \end{aligned} \tag{2.26}$$

Now it's clear we can represent the 3-string vertex $\int \Phi * \Phi * \Phi$ as a 3-point corelation function on the disk, i.e.

$$\int \Phi * \Phi * \Phi = \langle g_1 \circ \Phi(0) g_2 \circ \Phi(0) g_3 \circ \Phi(0)\rangle, \tag{2.27}$$

where $\langle \dots \rangle$ is the correlator on the global disk constructed above, evaluated in the combined matter and ghost CFT and $g_i \circ \Phi(0)$ denotes an active conformal transformation of the operator Φ. If Φ is a primary field of conformal weight h, then $g_i \circ \Phi(0)$ is given by

$$g_i \circ \Phi(0) = \left(g_i'(0)\right)^h \Phi(g_i(0)). \tag{2.28}$$

Of course one may want to represent the world-sheet of the product string as the UHP by the $SL(2,\mathbb{C})$ transformation $z = h^{-1}(\zeta) = -i\frac{\zeta-1}{\zeta+1}$. Then

$$\begin{aligned} \int \Phi * \Phi * \Phi &= \langle f_1 \circ \Phi(0) f_2 \circ \Phi(0) f_3 \circ \Phi(0) \rangle, \\ f_i(z_i) &= h^{-1} \circ g_i(z_i), \end{aligned} \tag{2.29}$$

The above described procedure is easily generalized to arbitrary n-point vertices. Defining

$$\begin{aligned} g_k(z_k) &= e^{\frac{2\pi i}{n}(k-1)} \left(\frac{1+iz_k}{1-iz_k} \right)^{\frac{2}{n}}, \quad 1 \le k \le n \\ f_k(z_k) &= h^{-1} \circ g_k(z_k). \end{aligned} \tag{2.30}$$

one defines

$$\int \Phi * \cdots * \Phi = \langle f_1 \circ \Phi(0) \cdots f_n \circ \Phi(0) \rangle.$$

A particularly remarkable case is the two point vertex $\int \Phi_1 * \Phi_2$, in which case the mappings are

$$f_1(z_1) = h^{-1} \left(\frac{1+iz_1}{1-iz_1} \right) = z_1 = \mathrm{id}(z_1), \tag{2.31}$$

$$f_2(z_2) = h^{-1} \left(-\frac{1+iz_2}{1-iz_2} \right) = -\frac{1}{z_2} \equiv I(z_2). \tag{2.32}$$

The contraction takes the very simple form

$$\int \Phi_1 * \Phi_2 = \langle I \circ \Phi_1(0) \Phi_2(0) \rangle. \tag{2.33}$$

The mapping $I(\xi) = -\frac{1}{\xi}$ is the BPZ mapping in the UHP. In fact BPZ conjugation amounts to transforming the world-sheet $(\tau, \sigma) \to (-\tau, -\sigma)$. Therefore, it can be seen as mapping an incoming string into an outgoing string. One can work out the transformation rule for a primary operator $\mathcal{O}$ of dimension h

$$\mathcal{O}_n^{h\star} \equiv I \circ \mathcal{O}_n^h = \sum_m \mathcal{O}_m^h \oint \frac{d\xi}{2\pi i} (-1)^{h+n} \xi^{-n-m-1} = (-1)^{h+n} \mathcal{O}_{-n}^h \,. \tag{2.34}$$

For the energy momentum tensor its reality implies that the hermitian conjugation of its modes obeys

$$L_n^\dagger = L_{-n} \,. \tag{2.35}$$

From (2.34) and (2.35) we read the relation between BPZ and hermitian conjugations for the Virasoro modes,

$$L_n^\dagger = (-1)^n L_n^\star \,. \tag{2.36}$$

2.3.2 Surface states

The previous discussion is clearly pointing to the fact that in a complete CFT description of SFT we have to account for arbitrary bordered Riemann surfaces (the string world-sheets) having marked points on the boundaries (the open string punctures). According to the usual state-operator correspondence Fock states are defined by insertions of local operators in the far past on the world sheet

$$|\phi\rangle = \phi(0)|0\rangle. \tag{2.37}$$

These states can be used as test states for a special class of string fields $\langle\Sigma|$, called surface states, associated with a a Riemann surface Σ with the topology of a disk, with a marked point P , the puncture, lying on the boundary of the disk, and a local coordinate patch around it. This local coordinate ξ can be thought of as a map from the canonical half-disk $|\xi| \leq 1, \Im(\xi) \geq 0$ into the Riemann surface Σ, where the boundary $\Im(\xi) = 0, |\xi| < 1$ is mapped to the boundary of Σ and $\xi = 0$ is mapped to the puncture P. The open string is the $|\xi| = 1$ arc in the half-disk and the point $\xi = i$ is the string midpoint. The surface Σ minus the image of the canonical ξ half-disk will be called $\mathcal{R}$. So any *global coordinate* u on the disk representing Σ can be seen as a conformal transformation $u = s(\xi)$ with $s(0) = u(P)$ The surface state $\langle\Sigma|$ is thus defined by the relation

$$\langle\Sigma|\,\phi\rangle = \langle s \circ \phi(0)\rangle_\Sigma\ . \tag{2.38}$$

for any state $|\phi\rangle$. $s \circ \phi$ denotes an active conformal transformation of the operator ϕ. If, for example, ϕ is a primary field of dimension h, then $s \circ \phi(x) = (s'(x))^h\, \phi\,(s(x))$. Of course such a description is redundant due to the invariance under conformal transformation and, according to convenience, we can either fix the surface Σ to a canonical one and specify $s(\xi)$ or fix $s(\xi)$ and specify the surface Σ that should be used for calculating the expectation value. For example, in the $\widehat{w}$-presentation, the Riemann surface Σ is mapped such that the image of the canonical ξ half-disk is the canonical half disk $|\widehat{w}| \leq 1, \Re(\widehat{w}) \geq 0$, with $\xi = 0$ mapping to $\widehat{w} = 1$. This is implemented by the map

$$\widehat{w} = \frac{1 + i\xi}{1 - i\xi} \equiv h(\xi)\,. \tag{2.39}$$

The rest $\mathcal{R}$ of the surface will take some definite shape $\widehat{\mathcal{R}}$ in this presentation and will actually carry all the information about the surface state $\langle\Sigma|$.

The star product of two surface states Σ_1 and Σ_2 can be interpreted in terms of purely geometrical operations. We simply need to remove the local coordinate patch from Σ_2 and then glue the right half-string of Σ_1 to the left half-string of Σ_2 while the local coordinate patch is glued in between the other two half-strings.. The Riemann surface $\widehat{\mathcal{S}}$ we get this way identifies the state $|\Sigma_1 * \Sigma_2\rangle$ through the relation

$$\langle \Sigma_1 * \Sigma_2 | \phi \rangle = \langle s \circ \phi(0) \rangle_{\widehat{\mathcal{S}}} . \tag{2.40}$$

Inner products are also associated to the computation of correlation functions on the surface $\check{\mathcal{S}}$ obtained from $\widehat{\mathcal{S}}$ by removing the local coordinate patch and sewing together the remaining half-strings

$$\langle \Sigma_1 | \prod_{i=1}^{n} \mathcal{O}_i(\xi_i) | \Sigma_2 \rangle = \langle \prod_{i=1}^{n} s \circ \mathcal{O}(\xi_i) \rangle_{\check{\mathcal{S}}} . \tag{2.41}$$

where $\prod_{i=1}^{n} \mathcal{O}_i(\xi_i)$, with $\xi_i = e^{\sigma_i}$ are operator insertions on the unit circle.

It is quite remarkable that in this approach we don't need to specify a priori the boundary conditions we are imposing on the world-sheet boundary. Anyhow , if the Boundary Conformal Field Theory (BCFT) under consideration is that of free scalar fields with Neumann boundary conditions describing $D25$-branes in flat space-time, we can represent the surface state i terms of the annihilation and creation operators associated with the scalar fields,a_m, $a_m^\dagger$

$$|\Sigma\rangle = \exp\left(-\frac{1}{2} \sum_{m,n=1}^{\infty} a_m^\dagger V_{mn}^f a_n^\dagger \right) |0\rangle . \tag{2.42}$$

where the matrix V_{mn}^f can be explicitly determined in terms of the mapping $z = f(\xi)$ associated to $\langle \Sigma |$ in the z presentation where the world sheet is mapped to the whole UHP [2]

$$V_{mn}^f = \frac{(-1)^{m+n+1}}{\sqrt{mn}} \oint_0 \frac{dw}{2\pi i} \oint_0 \frac{dz}{2\pi i} \frac{1}{z^m w^n} \frac{f'(z) f'(w)}{(f(z) - f(w))^2} . \tag{2.43}$$

2.3.3 Projectors

Albeit intuitive, this construction does not give an easy operational way to compute the $*$ product of two generic surface states since the world-sheet $\widehat{\mathcal{S}}$ will in general be non-trivial. From this point of view it's quite important to be able to single out subalgebras of states whose $*$product rules take a particularly convenient form. In this regard, of particular interest are the surface states $|\Sigma\rangle$ for which the boundary of $\mathcal{R}$ touches the midpoint $\xi = i$ [13]. In the $\widehat{w}$-presentation this means the boundary of the rest $\widehat{\mathcal{R}}$ touches the point $\widehat{w} = 0$. So in the correlator

$$\langle \Sigma | \prod_{i=1}^{n} \mathcal{O}_i(\xi_i) | \Sigma \rangle = \langle \prod_{i=1}^{n} h \circ \mathcal{O}(\xi_i) \rangle_{\check{\Sigma}} \,. \tag{2.44}$$

the disk $\check{\Sigma}$ is actually pinched at the origin of the $\widehat{w}$-plane so that it is made up of two disks $\widehat{\mathcal{N}}$ and $r \circ \widehat{\mathcal{N}}$, joined at the origin, where r is the conformal map $\widehat{w} \to -\widehat{w}$. So, if we suppose that $\mathcal{O}_i(\xi_i)$ are inserted on the left half-string for $i = 1, \ldots, m$ and on the right half-string for $i = m+1, \ldots, n$, this means they lie on the positive imaginary axis and on the negative one respectively and the correlation function 2.44 factorizes as

$$\langle \prod_{i=1}^{n} h \circ \mathcal{O}(\xi_i) \rangle_{\check{\Sigma}} = \langle \prod_{i=1}^{m} h \circ \mathcal{O}(\xi_i) \rangle_{\widehat{\mathcal{N}}} \, \langle \prod_{i=m+1}^{n} h \circ \mathcal{O}(\xi_i) \rangle_{r \circ \widehat{\mathcal{N}}} \,. \tag{2.45}$$

In the language of the wave functional associated to a state $|\Psi\rangle$, this means that he wave-functional is factorized into a functional Φ_L of the coordinates of the left-half of the string ($X(\sigma)$ for $0 \leq \sigma < \pi/2$) and a functional Φ_R of the coordinates of the right-half of the string ($X(\sigma)$ for $\pi/2 < \sigma \leq \pi$):

$$\Psi[X(\sigma)] = \Phi_L[X(2\sigma)] \Phi_R[X(2(\pi - \sigma))] \,. \tag{2.46}$$

and so the correlation function is factorized too

$$\begin{aligned} \langle \Psi | \prod_{i=1}^{n} \mathcal{O}_i(\xi_i) | \Psi \rangle &= \int [\mathcal{D}X(\sigma)] \prod_{i=1}^{n} \widetilde{\mathcal{O}}_i(X(\sigma_i)) \Psi[X(\pi - \sigma)] \Psi[X(\sigma)] \\ &= \langle \Phi_R^c | \prod_{i=1}^{m} s \circ \mathcal{O}_i(\xi_i) \, | \Phi_L \rangle \langle \Phi_R^c | \prod_{i=m+1}^{n} \widetilde{s} \circ \mathcal{O}_i(\xi_i) \, | \Phi_L \rangle \end{aligned} \tag{2.47}$$

where s and $\widetilde{s}$ denote the conformal transformation $s : \xi \to \xi^2$ and $\xi \to \xi^{-2}$, and the superscript c denotes twist transformation: $\sigma \to (\pi - \sigma)$, needed for the right half-string. Taking the string state is also twist invariant $|\Phi_L\rangle = |\Phi_R\rangle \equiv |\Phi\rangle$, we can rewrite

$$\langle \Phi^c | \prod_{i=1}^{m} s \circ \mathcal{O}_i(\xi_i) \, | \Phi \rangle = \langle \prod_{i=1}^{m} s \circ \mathcal{O}_i(\xi_i) \rangle_{s \circ h^{-1} \circ \widehat{\mathcal{N}}} \,. \tag{2.48}$$

So the half string states are themselves surface states with the same boundary condition as the original split state. Actually these split states turn out to behave like projectors. Infact in this case, the surface $\widehat{\mathcal{S}}$ appearing in the product rule

$$\langle \Sigma * \Sigma | \phi \rangle = \langle h \circ \phi(0) \rangle_{\widehat{\mathcal{S}}} \,. \tag{2.49}$$

is the pinched union of the surface Σ itself and an extra disk with no insertion. This is a situation in which the correlation functions on the splitting surfaces factorizes into the product of correlation functions on the separate surfaces. Normalizing the correlators so that $\langle \mathbf{1} \rangle_\Sigma = 1$ on any disk Σ, we get

$$\langle \Sigma * \Sigma | \phi \rangle = \langle h \circ \phi(0) \rangle_{\widehat{\mathcal{S}}} = \langle h \circ \phi(0) \rangle_\Sigma = \langle \Sigma | \phi \rangle \tag{2.50}$$

This establishes that such surfaces states behave like projectors of rank one

$$|\Sigma * \Sigma\rangle = |\Sigma\rangle \tag{2.51}$$

$$|\Sigma\rangle * |\Upsilon\rangle * |\Sigma\rangle = \langle \Sigma | \Upsilon \rangle |\Sigma\rangle \tag{2.52}$$

Projectors played a fundamental role in VSFT, where they are solutions of the EOM, at least in the matter sector, but seem to be related, although in a less straightforward way, to solutions of the complete SFT too. In particular, if we assume the z-presentation where any surface state is defined by the conformal frame $f(\xi)$, it turns out to be very convenient to consider the one-parameter family of *special projectors*, for which the zero mode $\mathcal{L}_0$ of the energy-momentum tensor in the frame of the projector,

$$\mathcal{L}_0 \equiv \oint \frac{dz}{2\pi i}\, zT(z) = \oint \frac{d\xi}{2\pi i} \frac{f(\xi)}{f'(\xi)} T(\xi)\,, \tag{2.53}$$

and its BPZ conjugate $\mathcal{L}_0^\star = \oint \frac{d\xi}{2\pi i} \left(-\xi^2 \frac{f(-\frac{1}{\xi})}{f'(-\frac{1}{\xi}))} \right) T(\xi)$ obey the crucial algebraic rule

$$[\mathcal{L}_0, \mathcal{L}_0^\star] = s(\mathcal{L}_0 + \mathcal{L}_0^\star) \tag{2.54}$$

where s is a positive real number. Defining $L = \mathcal{L}_0/s$ and $L^+ = \frac{1}{s}(\mathcal{L}_0 + \mathcal{L}_0^\star)$ the algebraic relation 2.54 takes the canonical form $[L, L^\star] = L_0 + L_0^\star$. Sot L^+ has the nice property of increasing the L-level of states, suggesting us the attractive possibility that L^+ may be used as a creation operator for a $\star$-subalgebra. This is for sure the case if the operator algebra 2.54 can be treated as an ordinary matrix algebra when applied to string states. The standard way in which this can be achieved is by splitting operators of the form

$$\mathbf{T}(v) \equiv \oint_C \frac{dt}{2\pi i}\, v(t) T(t)\,, \tag{2.55}$$

with C the unit circle and $t = e^{i\theta}$ the coordinate on it (in our case $\mathcal{L}_0 = \mathbf{T}(v)$ and $v(\xi) = \frac{f(\xi)}{f'(\xi)}$) into

$$\mathbf{T}_L(v) \equiv \oint_{C_L} \frac{dt}{2\pi i}\, v(t) T(t) = \mathbf{T}(v_L), \quad \mathbf{T}_R(v) \equiv \oint_{C_R} \frac{dt}{2\pi i}\, v(t) T(t) = \mathbf{T}(v_R) \tag{2.56}$$

where C_L and C_R are the part of the circle lying in the positive and negative half-plane respectively and v_L and v_R are equal to v on C_L and C_R and zero elsewhere. It is evident that $\mathbf{T}_L(v) + \mathbf{T}_R(v) = \mathbf{T}(v)$ and $v(t) = v_L(t) + v_R(t)$. The BPZ conjugate of such operators is $\mathbf{T}(v^\star)$ where $v^\star(\xi) = -\xi^2 v(-1/\xi)$. For $t = e^{i\theta}$, we have $1/t = t^\star$ so that BPZ even or odd vectors $v^\pm = v \pm v^\star$ satisfy even or odd conditions under the reflection about the imaginary axis $t \to -t^\star$

$$\frac{v^\pm(t)}{t} = \frac{v(t)}{t} \pm \frac{v(-t^\star)}{-t^\star} \tag{2.57}$$

As $t \to -t^\star$ maps C_L and C_R into each other, the *dual* quantities $\widetilde{\mathbf{T}}(v) \equiv \mathbf{T}_L(v) - \mathbf{T}_R(v)$ and $\widetilde{v}(t) \equiv= v_L(t) - v_R(t)$ have opposite parity with respect to $\mathbf{T}(v)$ and $v(t)$. In particular we can split the BPZ-even $L^+ = L^+_L + L^+_R = K_L - K_R$ where K_L and K_R are the left and right part of the BPZ-odd operator $K \equiv \widetilde{L^+} = L^+_L - L^+_R$. A consistent splitting, allowing us to think of L^+_L and L^+_R as operators acting on the left and right parts of a string state Φ respectively, would require of course the further conditions

$$[L^+_L, L^+_R] = 0 \tag{2.58}$$

$$L^+_L (\Phi_1 * \Phi_2) = \left(L^+_L \Phi_1\right) * \Phi_2 \tag{2.59}$$

The first one, in particular, is all but obvious once we realize $[L^+_L, L^+_R] = -\frac{1}{2}[L^+, K]$, with $L^+ = \widetilde{\mathbf{T}}(v^+)$ and $K = \widetilde{\mathbf{T}}(\widetilde{v^+})$, where in the contour integral we have to take

$$\widetilde{v}(t) = v(t)\epsilon(t) = v(t) \cdot \frac{2}{\pi}(\arctan t + \operatorname{arccot} t) \tag{2.60}$$

The discontinuities at the midpoints $t = \pm i$ makes such a splitting anomalous in general unless $v^+(t)$ be strongly vanishing. In particular

$$[L^+_L, L^+_R] = \frac{1}{2}\widetilde{\mathbf{T}}(((v^+)^2 \partial\epsilon) \tag{2.61}$$

constrains the behavior of $v^+(t)$ near the midpoints. Taking the identity state $\mathcal{I}$ as the fundamental state to build the $*$ subalgebra, it is quite natural to require $\mathcal{I}$ is annihilated by the operators L^- and K, which, being BPZ-odd, are naively expected to be derivations of the $*$-algebra. Also these properties can be anomalous and have to be checked case by case. We can now define a family of states parameterized by a real constant $\alpha \in [0, \infty)$

$$P_\alpha = e^{-\frac{\alpha}{2}L^+}\mathcal{I} \tag{2.62}$$

satisfying the product rule $P_\alpha * P_\beta = P_{\alpha+\beta}$. So they interpolate between the identity state $P_0 = \mathcal{I}$ and P_∞, which turns out to be the projector associated with the conformal frame f. Actually all single-split, twist invariant projectors can be related to one another by a reparameterization of the open string coordinate, corresponding to a gauge symmetry of OSFT [20, 21].

2.3.4 Wedge states subalgebra

The standard example of such an abelian identity-based subalgebra identified by a projector is the one corresponding to the case $s = 1$ in 2.54, which is realized choosing the conformal frame $z = \arctan \xi$, which takes the UHP into the semi-infinite cylinder C_π with circumference π. Such a subalgebra can be constructed as a generalization of the standard CFT definition of the three-vertex

$$\langle \phi_1, \phi_2, \phi_3 \rangle = \langle f_1 \circ \phi_1(0)\, f_2 \circ \phi_2(0)\, f_3 \circ \phi_3(0) \rangle_{UHP}, \tag{2.63}$$

where the mappings $f_n(z) = \tan\left(\frac{(2-n)\pi}{3} + \frac{2}{3}\arctan z\right)$ first map the local coordinate half-disk in the upper half-plane to the unit half-disk in the right half-plane, then shrink it to a wedge of angle $\frac{2}{3}\pi$ and finally map them back to the UHP, where the wedges merge into a single world-sheet with three punctures on the boundary.

Generic *wedge states* are usually defined , in the z-presentation where the surface Σ is mapped to the full half z-plane, with the puncture lying at $z = 0$, as surface states associated to the conformal mappings

$$f_r(z) = h^{-1}\left(h(z)^{\frac{2}{r}}\right) = \tan\left(\frac{2}{r}\arctan(z)\right), \tag{2.64}$$

where $h(z) = \frac{1+iz}{1-iz}$. The mappings $f_r(z)$ first map the unit half-disk in the upper half-plane to the unit half-disk in the right half-plane, then shrink it to a wedge of angle $\frac{2}{r}\pi$ and finally map them back to the UHP. The geometrical interpretation of wedge states is very transparent if we map the UHP to the unit disk and then take coordinates where the local coordinate patch is just the unit half-disk with $\Re w > 0$. The wedge state $|r\rangle$ is then identified by a Riemann surface with total opening angle $\pi(r-1)$, i.e. by a surface with branch cuts in general. The star multiplication of two wedge states $|r\rangle$ and $|s\rangle$ is seen to be determined by the simple rule

$$|r\rangle * |s\rangle = |r + s - 1\rangle, \quad r, s \geq 1 \tag{2.65}$$

$|2\rangle$ is the $SL(2, \mathbb{R})$ invariant vacuum $|0\rangle$ and $|1\rangle$ behaves as the identity of the star algebra. The problem of dealing with surfaces with branch cuts can be fixed by working in the cylinder coordinate $\tilde{z} = \arctan(z)$, which maps the UHP to a semi-infinite cylinder C_π of circumference π and the upper half of the unit disk to a strip of this cylinder of width $\pi/2$. The bottom edge of the strip lies on the real axis, and corresponds to the boundary of the open string; the top of the strip at $+i\infty$ corresponds to the open string midpoint, so that the positive and negative vertical edges of the strip correspond to the left and right halves of the open string, respectively.

In this conformal frame the three-vertex correlator can be expressed directly as

$$\langle \widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3 \rangle = \langle \phi_1\Big(\frac{\pi}{2}\Big)\; \phi_2(0)\, \phi_3\Big(-\frac{\pi}{2}\Big)\rangle_{C_{\frac{3\pi}{2}}}, \tag{2.66}$$

where $\widetilde{\phi}_i(\widetilde{x}_i) = \tan \circ \phi_i(\widetilde{x}_i)$ is a local operator ϕ_i expressed in the $\widetilde{z}$ coordinate. Treating one of the strings as a 'test state' $|\chi\rangle$, and then scaling the correlator strip to the canonical cylinder C_π

$$\langle\, \widetilde{\chi}\,,\, \widetilde{\phi}_1\,,\, \widetilde{\phi}_2\,\rangle = \langle\, s \circ \chi\bigg(\pm\frac{3\pi}{4}\bigg)\; s \circ \phi_1\left(\frac{\pi}{4}\right)\; s \circ \phi_2\Big(-\frac{\pi}{4}\Big)\,\rangle_{C_\pi}. \tag{2.67}$$

where the conformal map $s : \widetilde{z} \to \frac{2}{3}\widetilde{z}$ appears. From 2.67 we can read out the rule for the multiplication of two Fock states in the sliver frame

$$\widetilde{\phi}_1(0)|0\rangle * \widetilde{\phi}_2(0)|0\rangle = U_3^\dagger U_3\, \widetilde{\phi}_1\left(\frac{\pi}{4}\right) \widetilde{\phi}_2\left(-\frac{\pi}{4}\right)|0\rangle. \tag{2.68}$$

where the operator $U_r \equiv (2/r)^{\mathcal{L}_0}$ represents the scaling $\widetilde{z} \to \frac{2}{r}\widetilde{z}$, which in the z coordinate is $z \to f_r(z)$, with $f_r(z) = \tan\left(\frac{2}{r}\arctan z\right)$. This can be easily generalized to an arbitrary number of insertions

$$\begin{aligned} &|\widetilde{\phi}_1\rangle * |\widetilde{\phi}_2\rangle * \cdots * |\widetilde{\phi}_n\rangle = \\ &\quad = U_{n+1}^\dagger U_{n+1}\, \widetilde{\phi}_1\left(\frac{(n-1)\pi}{4}\right) \widetilde{\phi}_2\left(\frac{(n-3)\pi}{4}\right) \ldots \widetilde{\phi}_n\left(-\frac{(n-1)\pi}{4}\right)|0\rangle. \end{aligned} \tag{2.69}$$

and on top of that we can consider strips of any length

$$U_r^\dagger U_r\, \widetilde{\phi}_1(\widetilde{x}_1)\widetilde{\phi}_2(\widetilde{x}_2)\ldots\widetilde{\phi}_n(\widetilde{x}_n)|0\rangle, \tag{2.70}$$

for arbitrary real $r \geq 1$ and arbitrary insertion points $\widetilde{x}_i$, $|\operatorname{Re}\widetilde{x}_i| \leq (r-1)\pi/4$. These states don't need to have an explicit construction by gluing Fock states. Their product is given by the cumbersome formula

$$\begin{aligned} &U_r^\dagger U_r\, \widetilde{\phi}_1(\widetilde{x}_1)\ldots\widetilde{\phi}_n(\widetilde{x}_n)|0\rangle * U_s^\dagger U_s\, \widetilde{\psi}_1(\widetilde{y}_1)\ldots\widetilde{\psi}_m(\widetilde{y}_m)|0\rangle = \\ &= U_t^\dagger U_t\, \widetilde{\phi}_1(\widetilde{x}_1 + \frac{\pi}{4}(s-1))\ldots\widetilde{\phi}_n(\widetilde{x}_n + \frac{\pi}{4}(s-1))\, \widetilde{\psi}_1(\widetilde{y}_1 - \frac{\pi}{4}(r-1))\ldots\widetilde{\psi}_m(\widetilde{y}_m - \frac{\pi}{4}(r-1))|0\rangle, \end{aligned} \tag{2.71}$$

where $t = r + s - 1$, which however has a very clear geometric interpretation. We simply glue together the parts of two or more cylinders with strips of length $\pi/2$ cut out and then glue back one such strip to have the cylinder corresponding to the $*$ product. So in the end in the sliver frame all computations boil down to evaluating correlators on strips with operator insertions on their border. In fact one can consider the Virasoro generators

$$\mathcal{L}_n = \oint \frac{d\tilde{z}}{2\pi i} \tilde{z}^{n+1} T_{\tilde{z}\tilde{z}}(\tilde{z}) = \oint \frac{dz}{2\pi i} (1+z^2) \left(\arctan z\right)^{n+1} T_{zz}(z). \tag{2.72}$$

It is possible to recognize a very simple subalgebra made up of the three operators $\mathcal{L}_0$, $\mathcal{L}_0^\dagger = \mathcal{L}_0^\star$ and $\mathcal{L}_{-1}^\dagger = \mathcal{L}_{-1} = K_1$

$$\left[\mathcal{L}_0, \mathcal{L}_0^\dagger\right] = \mathcal{L}_0 + \mathcal{L}_0^\dagger, \tag{2.73}$$

$$\left[\mathcal{L}_0, K_1\right] = K_1, \tag{2.74}$$

$$\left[\mathcal{L}_0^\dagger, K_1\right] = -K_1. \tag{2.75}$$

As discussed in the previous section there is actually the possibility to redefine the algebra in terms of split operators acting on half-strings. In fact

$$\left[\mathcal{L}_0, \mathcal{L}_0^\dagger\right] = \oint \frac{d\xi}{2\pi i}(1+\xi^2)(\arctan\xi + \operatorname{arccot}\xi)T(\xi) = \mathcal{L}_0 + \mathcal{L}_0^\dagger\,. \tag{2.76}$$

One can easily recognize the step function $\epsilon(x)$ in the integrand equal to ± 1 for positive or negative values respectively and rewrite 2.76 as

$$\mathcal{L}_0 + \mathcal{L}_0^\dagger = \frac{\pi}{2}\oint \frac{d\xi}{2\pi i}(1+\xi^2)\epsilon\left(\mathrm{Re}\,\xi\right) T(\xi) = \frac{\pi}{2}\left(K_1^L - K_1^R\right), \tag{2.77}$$

where $K_1 = \mathcal{L}_{-1}^\dagger = \mathcal{L}_{-1}$ and K_1^L and K_1^R are the operators obtained splitting the contour integration over the unit circle into two halves, one in the $\mathrm{Re}\,\xi > 0$ half-plane and the other in $\mathrm{Re}\,\xi < 0$. Since $K_1 = K_1^L + K_1^R$ we also have

$$K_1^L = \frac{1}{2}K_1 + \frac{1}{\pi}\left(\mathcal{L}_0 + \mathcal{L}_0^\dagger\right), \tag{2.78}$$

$$K_1^R = \frac{1}{2}K_1 - \frac{1}{\pi}\left(\mathcal{L}_0 + \mathcal{L}_0^\dagger\right). \tag{2.79}$$

$\left[\mathcal{L}_0 + \mathcal{L}_0^\dagger, K_1\right] = 0$ is responsible for $\left[K_1^L, K_1^R\right] = 0$ giving a completely non anomalous decomposition of the algebra into left and right string operators. The operators K_1^L, K_1^R and K_1 also have rather simple properties with regard to the star product

$$K_1^L\left(\phi_1 * \phi_2\right) = \left(K_1^L \phi_1\right) * \phi_2, \tag{2.80}$$

$$K_1^R\left(\phi_1 * \phi_2\right) = \phi_1 * \left(K_1^R \phi_2\right), \tag{2.81}$$

$$K_1\left(\phi_1 * \phi_2\right) = \left(K_1 \phi_1\right) * \phi_2 + \phi_1 * \left(K_1 \phi_2\right). \tag{2.82}$$

From the previous algebraic rules one can work out that a generic wedge state can be represented as

$$|n\rangle = e^{-(n-1)\frac{\pi}{2}K_1^L}|I\rangle \tag{2.83}$$

This in turn implies

$$\frac{d}{dn}|n\rangle = -\frac{\pi}{2}K_1^L|n\rangle \tag{2.84}$$

that is $e^{tK_1^L}$ creates a semi-infinite strip with a width of t. Similar considerations can be made for the ghost operators

$$\begin{aligned}
\mathcal{B}_0 &= \oint \frac{d\tilde{z}}{2\pi i}\tilde{z}b_{\tilde{z}\tilde{z}}(\tilde{z}) && (2.85)\\
B_1 &= \oint \frac{d\tilde{z}}{2\pi i}b_{\tilde{z}\tilde{z}}(\tilde{z}) = b_1 + b_{-1} && (2.86)\\
B_1^L &= \oint_{C_L} \frac{d\tilde{z}}{2\pi i}b_{\tilde{z}\tilde{z}}(\tilde{z}) = \frac{1}{2}B_1 + \frac{1}{\pi}\left(\mathcal{B}_0 + \mathcal{B}_0^\dagger\right) && (2.87)\\
B_1^R &= \oint_{C_R} \frac{d\tilde{z}}{2\pi i}b_{\tilde{z}\tilde{z}}(\tilde{z}) = \frac{1}{2}B_1 - \frac{1}{\pi}\left(\mathcal{B}_0 + \mathcal{B}_0^\dagger\right), && (2.88)
\end{aligned}$$

A convenient subalgebra is so identified which in the sliver frame can be written as

$$\begin{aligned}
K_1^L &\equiv \int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}T(z),\\
B_1^L &\equiv \int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}b(z),\\
&\qquad\qquad c(z),
\end{aligned} \tag{2.89}$$

where $c(z)$ is inserted exactly on the strip, on the real axis. As suggested by Okawa [18] a very natural language in which to express this algebraic properties is the half-string formalism where any string field can be represented as an operator acting on the Hilbert space of the half string. By converse this means an operator acting on the string field Ψ can be represented as the star multiplication of Ψ with an appropriate state. In general the implementation of this formalism can be quite technical. Nevertheless the existence of the identity string field $|I\rangle$ give an immediate prescription to represent operators as string fields. We can in fact consider the algebra

$$K = \frac{\pi}{2}(K_1)_L|I\rangle, \quad B = \frac{\pi}{2}(B_1)_L|I\rangle, \quad c = \frac{1}{\pi}c(1)|I\rangle. \tag{2.90}$$

2.4 Solutions in split SFT

The above discussion shows that at last the problem of constructing solutions of Witten's cubic OSFT equation of motion can be addressed in a quite manageable algebraic set up, whose fundamental ingredients are the three string fields

$$
\begin{aligned}
K &= \text{Grassmann even, gh\# } = 0 \\
B &= \text{Grassmann odd, gh\# } = -1 \\
c &= \text{Grassmann odd, gh\# } = 1
\end{aligned}
\tag{2.91}
$$

which satisfy the algebraic properties

$$
\begin{aligned}
\{B, c\} = 1,\ [K, B] = 0,\ B^2 = c^2 = 0, \\
[K, c] \equiv \partial c,\ \{B, \partial c\} = 0,
\end{aligned}
\tag{2.92}
$$

as well as the differential relations

$$QB = K \tag{2.93}$$

$$Qc = cKc = c\partial c. \tag{2.94}$$

Q is not the only useful derivation. In fact there is also the reparameterization generator $\mathcal{L}^- \equiv \mathcal{L}_0 - \mathcal{L}_0^\star$, which is the BPZ odd component of the scaling generator in the sliver coordinate frame $\mathcal{L}_0$. It is therefore clear $\mathcal{L}^-$ acts on other fields in the algebra giving twice the scaling dimension of the corresponding operator insertion on the cylinder

$$\frac{1}{2}\mathcal{L}^- K = K, \quad \frac{1}{2}\mathcal{L}^- B = B, \quad \frac{1}{2}\mathcal{L}^- c = -c \tag{2.95}$$

This suggests that the so called $\mathcal{L}^-$ *level expansion*, i.e. the expansion in eigenstates of $\mathcal{L}^-$, is a very natural one for string fields in the KBc algebra, as it can be simply attained expanding in powers of K. Using the simple rules if the KBc algebra Okawa has been able to find a wide class of solutions [18]

$$\Psi = F(K)c\frac{KB}{1 - F(K)^2}cF(K) \tag{2.96}$$

where $F(K)$ can be any function of K. Actually some restriction should be considered in order to have sensible solutions [36, 37, 39, 51]. In fact, although very elegant, these solutions are somewhat formal because generically $F(K)$ has no natural interpretation when one deals with the computation of correlation functions on the cylinder. The classical example is the original Schnabl's solution [17] for tachyon vacuum corresponding to the choice

$$F = e^{-\frac{K}{2}} = \Omega^{\frac{1}{2}} \tag{2.97}$$

where $\Omega^{\frac{1}{2}}$ is the square root of the $SL(2,\mathbb{R})$ vacuum. In fact, to have any hope to check Sen's conjectures, we are forced to give a precise *computational* meaning to the field $\frac{K}{1-e^{-K}}$ appearing 2.96, the obvious choice being the one coming from the geometric series expansion

$$\frac{K}{1-\Omega} = \sum_{n=0}^{N-1} K\Omega^n + \frac{K\Omega^N}{1-\Omega} \tag{2.98}$$

We can rewrite the solution as

$$\Psi = -\sum_{n=0}^{N-1} \psi'_n + \sqrt{\Omega}c\Omega^N \frac{KB}{1-\Omega}c\sqrt{\Omega}, \tag{2.99}$$

where

$$\psi'_n \equiv \frac{d}{dn}\psi_n, \quad \psi_n \equiv \sqrt{\Omega}cB\Omega^n c\sqrt{\Omega}. \tag{2.100}$$

are just the string fields appearing in the original form of Schnabl's solution. If we simply disregard the rest term in 2.99 and take the limit $N \to \infty$ it is possible to verify the EOM hold upon contracting them with any Fock state. Instead, it fails when we compute its contraction with the solution itself. This may be understood as the fact that, truncating 2.100 at N, we are dropping out $\mathcal{O}(N^2)$ terms from the correlation functions, which for large N behave like N^{-2}, thus giving finite contributions over all. In [38] it has been shown that a consistent regularization for Schnabl0 solution can be found introducing the $\mathcal{L}^-$ expansion

$$\frac{K}{1-\Omega} = \sum_{n=0}^{\infty} \frac{B_n}{n!}(-K)^n, \tag{2.101}$$

inside the second term of **??**, where B_n are the Bernoulli numbers. By noticing $\lim_{N\to\infty} K^A\Omega^N \sim N^{-2-A}$ and inspection of the correlators involved in the computation of the energy, one realizes that one can keep only the first term in the Bernoulli series. So one obtains Schnabl's mysterious phantom term ψ_N, which now appears clearly associated to the regularization procedure needed to give a precise meaning to 2.96. This is no accident, but a deep issue related to the fact that, as it was found by Okawa [18], Schnabl's solution can be written as the limit, $\lambda \to 1$, of the pure gauge states

$$\Psi_\lambda = V_\lambda Q U_\lambda = Q\Phi \frac{\lambda}{1-\lambda\Phi}, \tag{2.102}$$

where $U_\lambda = (1-\lambda\Phi)$ and $V_\lambda = \frac{1}{1-\lambda\Phi}$ with $\Phi = Bc(0)|0\rangle$. As in the case before, V_λ is actually defined by the corresponding geometric expansion. One can verify that for $\lambda < 1$ both U_λ and V_λ are well defined in the L_0 and $\mathcal{L}_0$ expansions, so that $V_\lambda \star U_\lambda = U_\lambda \star V_\lambda = \mathcal{I}$ and Ψ_λ is a pure gauge state with zero energy. But for $\lambda = 1$ V_λ apparently diverges in the

$\mathcal{L}_0$ level-expansion [23]. This fact can be considered as the reason why a regularization is needed and a phantom term has to be introduced to get the correct value of the energy. On the other hand one can explicitly prove the cohomology around the solution is empty, as it should be if it correctly describes the tachyon vacuum. In fact let Ψ be the tachyon vacuum solution and Φ a solution of the equaltion of motion *around the tachyon vacuum*

$$\mathcal{Q}\Phi + \Phi * \Phi = 0 \ . \tag{2.103}$$

where $\mathcal{Q}$ is related to the usual BRST operator around the perturbative vacuum by the relation $\mathcal{Q} = Q_B + [\Psi, \cdot]$. Then the existence of a homotopy field A satisfying

$$\mathcal{Q}A = \mathcal{I} \ . \tag{2.104}$$

implies that all *infinitesimal* solutions $\mathcal{Q}\Phi = 0$ can be written as $\Phi = \mathcal{Q}(A * \Phi)$.

2.4.1 Erler-Schanbl Tachyon condensation solution

A particularly important example in the previous class of solutions is the one corresponding to $F(K) = \frac{1}{\sqrt{1+K}}$ [25], i.e.

$$\widehat{\Psi} = \frac{1}{\sqrt{1+K}}\Big[c + cKBc\Big]\frac{1}{\sqrt{1+K}}, \tag{2.105}$$

which is strictly to another simpler, even if non real, solution

$$\Psi = \Big[c + cKBc\Big]\frac{1}{1+K}. \tag{2.106}$$

The inverse of $1+K$ is given a precise meaning through the Schwinger representation

$$\frac{1}{1+K} = \int_0^\infty dt\, e^{-t(1+K)} = \int_0^\infty dt\, e^{-t}\Omega^t, \tag{2.107}$$

This seems to be a well-defined string field and so we find

$$A = B\frac{1}{1+K}. \tag{2.108}$$

is the homotopy field satisfying

$$Q_\Psi A = 1, \tag{2.109}$$

where $Q_\Psi = Q + [\Psi, \cdot]$ is the vacuum kinetic operator.This means the solution 2.106 has no open string degrees of freedom. The energy is easily computed using only the kinetic term

$$E = \frac{1}{6}\langle \Psi, Q_B\Psi\rangle = \frac{1}{6}\Big\langle \Big[c + cKBc\Big]\frac{1}{1+K}cKc\frac{1}{1+K}\Big\rangle, \tag{2.110}$$

where we write

$$\langle \cdot \rangle = \langle I, \cdot \rangle \tag{2.111}$$

One easily finds the expected result for the $D25$ brane tension

$$E = -S(\Psi) = -\frac{1}{2\pi^2}, \tag{2.112}$$

This means Ψ can be considered the tachyon condensation solution appearing in Sen's conjectures.

2.4.2 Solutions from singular gauge transfromations

The fact that Schnabl's solution is in some sense very near to being a pure gauge state, but fails to be so because of some singular behavior which can be traced back to the gauge transformation defining it formally has led to the insight that this can be a quite general feature of solutions in OSFT [24, 55, 56]. This insight is supported by the analogy with Chern-Simons-like theory, where classical solutions are just given by flat connections and even formally pure gauge solution $A = U^{-1}dU$ can be non trivial either because U does not belong to the space of fields used to define the theory or because the inverse $U^{-1}$is not well defined.

Let's assume for definiteness that $\Phi_1 = \Psi$ represents the tachyon vacuum and that a homotopy field A satisfying 2.144 exists. Then any solution Φ of 2.103 can be seen as a left gauge transform

$$\mathcal{Q}U = U\Phi \ , \tag{2.113}$$

so that

$$\Phi = U^{-1}\mathcal{Q}U \ . \tag{2.114}$$

with $U = 1 + A * \Phi$, or a right gauge transform

$$\mathcal{Q}\widetilde{U} = -\Phi * \widetilde{U}, \tag{2.115}$$

so that

$$\Phi = -(\mathcal{Q}U)U^{-1} = \widetilde{U}\mathcal{Q}\widetilde{U}^{-1} \ . \tag{2.116}$$

with $\widetilde{U} = 1 + \Phi * A$. Non trivial solutions must be characterized by a non trivial kernel for U and $\widetilde{U}$

$$U * Z = Z * \widetilde{U} = 0 \ , \tag{2.117}$$

Z turns out to be a solution for

$$Z = P_\Phi * Z = Z * P_\Phi \ . \tag{2.118}$$

where P_Φ is the so called *characteristic* projector defined by

$$P_\Phi = \lim_{N\to\infty} (-\{A, \Phi\})^N \,. \tag{2.119}$$

If such a limit exists and is finite, then there are two cases. If $P_\Phi = 0$, then $Z = 0$, which means U^{-1} and $\widetilde{U}^{-1}$ are probably well defined and Φ is gauge equivalent to the tachyon vacuum. Actually, for every soluten Φ, there is a one parameter family of pure gauge solutions defined by $U_\lambda = 1 + \lambda A * \Phi$, with $\lambda \in [0, 1)$, similarly to the case of Schnabl's solution. If P_Φ is a projector of rank k then the left and right parts of the states in Z each span a k-dimensional subspace so that the kernel Z is a k^2-dimensional subspace. As usually we have to do with rank one projectors corresponding to single pinched surface states, we see that for $k = 1$ $Z = P_\Phi$ up to insertion operators at the midpoint. In critical string theory, at ghostnumber zero, it is generally understood the only such operator is the identity. The characteristic projector P_Φ also satisfies $A * P_\Phi = P_\Phi * A = 0$, which implies $P_\Phi * U = P_\Phi$ and $U * P_\Phi = 0$. The fact that U has a right kernel and not a left one can be traced back to an associativity anomaly $(P_\Phi * U) * P_\Phi = P_\Phi \neq 0 = P_\Phi * (U * P_\Phi)$. Given Φ solving $\mathcal{Q}\Phi + \Phi * \Phi = 0$, its physical meaning is codified in the cohomology of the BRST operator in the Φ-vacuum, defined by $\mathcal{Q}_\Phi \Sigma = \mathcal{Q}\Sigma + \Phi * \Sigma - (-1)^\Sigma \Sigma * \Phi$. One can verify that for a not pure gauge Φ the cohomology is not empty, in fact $\mathcal{Q}_\Phi \Omega_\Phi = 0$ with $\Omega_\Phi = -\{\Phi, A\}$ and $\Omega_\Phi \neq \mathcal{Q}_\Phi \Lambda$. Not surprisingly also the characteristic projector $P_\Phi = \lim_{N\to\infty}(-\{A, \Phi\})^N$ is an element of the ghost number zero cohomology, i.e.

$$\mathcal{Q}_\Phi P_\Phi = 0 \tag{2.120}$$

Recently, in [55], a generalization of the concept of characteristic projector and of eq. 2.120 has been proposed introducing the notion of boundary condition changing projectors as stretched string states connecting the boundary conformal field theories corresponding to two solutions Φ_1 and Φ_2. From this point of view P_Φ is like the characteristic projector of Φ with itself and describes the boundary conditions of a single $BCFT$ with respect to the tachyon vacuum. The simplest example is the case of the pertubative vacuum, corresponding to the sliver projector.

The previous discussion has pointed out that classical solutions of OSFT can be viewed as describing the change of the boundary conditions between different boundary conformal field theories (BCFT). This is a natural point of view in the so called Boundary String Field Theory, but it's quite remarkable that also Witten's covariant cubic string field theory has finally been seen to encode the same kind of information in the classical solutions of its EOM.

2.4.3 Relevant deformations and solutions in OSFT

Whereas marginal boundary deformations, which deform the world sheet while still preserving conformal invariance, simply describe the moduli space of the theory around the perturbative vacuum, relevant deformations are expected to correspond to unstable directions in the string-field potential and to lead to solutions associate with possibly new vacua. Actually, Schanabl's solution for tachyon vacuum, which has the sliver as its characteristic projector, is expected to correspond to such a relevant boundary operator interpolating between the Neumann boundary conditions of the initial $BCFT_0$ describing the perturbative vacuum and the Dirichlet conditions corresponding to a state where no open strings are attached, i.e. tachyon vacuum. In particular, in [25] Erler and Schnabl were able to give an alternative solution for the tachyon vacuum which makes this relationship transparent

$$\psi_0 = \frac{1}{1+K}c(1+K)Bc = c - \frac{1}{1+K}Bc\partial c, \tag{2.121}$$

It's easy to verify this solution describes the tachyon vacuum according to Sen's conjectures and one can also see it can be formally written as a gauge transform of the perturbative vacuum in full agreement of Ellwood's intuition about solutions in OSFT

$$\begin{aligned} \psi_0 &= U_0 Q U_0^{-1} &(2.122)\\ U_0 &= 1 - \frac{1}{1+K}Bc &(2.123)\\ U_0^{-1} &= 1 + \frac{1}{K}Bc. &(2.124) \end{aligned}$$

Of course, in order Ψ to be a different vacuum from the perturbative one, these gauge transformations must be singular because of a non trivial kernel identified by the sliver projector. To enlighten the relation between the sliver projector and the change of boundary conditions associated with the tachyon vacuum, one can consider the midpoint preserving reparameterization $z \to \frac{z}{u}$ mapping the KBc algebra to a isomorphic representation where the gauge transformations and the corresponding solution are given by

$$\begin{aligned} U_u &= 1 - \frac{u}{u+K}Bc &(2.125)\\ U_u^{-1} &= 1 + \frac{u}{K}Bc, &(2.126) \end{aligned}$$

$$\psi_u = \frac{u}{u+K}c(u+K)Bc = uc - \frac{u}{u+K}Bc\partial c. \tag{2.127}$$

It is quite well known such reparameterizations are gauge symmetries of OSFT (see [21] e.g.). Just putting $u = 0$, which of course is not an allowed reparametrization, we get the perturbative vacuum. Instead we expect *any* other value of u to describe the same tachyon vacuum. In fact the mapping $e^{-tK} \to e^{-\frac{t}{u}K}$, while simply changing the length of finite size wedge states, does really deform the boundary near the infinity, i.e. near the midpoint. The sliver is just a wedge state of infinite length whose boundary stretches to the midpoint at infinity and so for such a state every finite value of u induces the same modification on the boundary conditions. In this sense it is quite understandable why the presence of a non trivial kernel in 2.125, i.e. of the sliver, makes the solution 2.127 able to catch the change of boundary conditions which allows to describe the tachyon vacuum with respect to the perturbative one. In fact, by the standard CFT realization of the KBc algebra, we get that the quantity

$$Tr[\frac{u}{u+K}] = \int_0^\infty dT u \left\langle e^{-uT} \right\rangle_{C_T} = \int_0^\infty dT\; u \left\langle e^{-\int_0^T ds\, u} \right\rangle_{C_T}, \tag{2.128}$$

upon rescaling to a canonical cylinder of width 1, can be rewritten

$$\begin{aligned} Tr[\frac{u}{u+K}] &= \int_0^\infty dT\; u \left\langle e^{-\int_0^1 ds\,(Tu)} \right\rangle_{C_1} \\ &= \int_0^\infty dT(-\partial_T) \left\langle e^{-\int_0^1 ds\,(Tu)} \right\rangle_{C_1} \\ &= \left(\lim_{u\to 0^+} - \lim_{u\to\infty} \right) \left\langle e^{-\int_0^{2\pi} \frac{d\theta}{2\pi} u} \right\rangle_{Disk} = Z_{u=0} - Z_{u=\infty}, \end{aligned} \tag{2.129}$$

where

$$Z(u) \equiv \left\langle e^{-\int_0^{2\pi} \frac{d\theta}{2\pi} u} \right\rangle_{Disk} = e^{-u}, \tag{2.130}$$

is the disk partition function obtained by integrating a zero momentum tachyon on the disk boundary. $Z_{u=0} = \langle 1 \rangle_{Disk} = 1$ is the partition function for the $BCFT_0$ with Neumann boundary conditions describing the perturbative open string vacuum. It corresponds to the UV fixed point of the Renormalization Group flow which leads to the IR fixed point $Z_{u=\infty} = 0$, where the boundary conditions switch to pure Dirichlet and no open string degree of freedom is left. It is quite remarkable these values for the partition functions at the two extremes of the RG flow are purely conventional, but their difference is associated unambiguously to the perturbative vacuum as seen from the tachyon vacuum. Furthermore, even if u is a gauge parameter, from the point of view of boundary conformal field theory only $u = 0$ and $u = \infty$ correspond to conformal invariant fixed points whereas conformal invariance is broken for any finite u. From the previous presentation it is quite clear Erler-Schnabl tachyon vacuum solution can be seen as just an example of a more general class of solutions, associated with the world-sheet actions

$$S^{(u)} = S_0 + \int_{\partial Disk} d\theta \phi_u(\theta), \tag{2.131}$$

where S_0 is the action of the $BCFT_0$ corresponding to the definition of OSFT and ϕ_u is a matter relevant vertex operator inducing a RG flow between two fixed conformal points. The parameter u is chosen in such a way that $u = 0$ corresponds to the UV and $u = \infty$ to the IR. In [52] a minimal extension of the KBc algebra was proposed to address such a general problem in the context of OSFT, by considering the identity-based inset ion in the sliver frame $\widetilde{z} = \frac{2}{\pi} \arctan z$

$$\phi = \phi\left(\frac{1}{2}\right)|I\rangle \tag{2.132}$$

with the properties

$$[c, \phi] = 0, \qquad [B, \phi] = 0, \qquad [K, \phi] = \partial\phi, \tag{2.133}$$

such that Q has the following action:

$$Q\phi = c\partial\phi + \partial c\delta\phi. \tag{2.134}$$

For a primary field of weight h we have in particular

$$Q\phi = c\partial\phi + h\partial c\phi. \tag{2.135}$$

A general problem which may affect the consistency of 2.131 is the one of renormalization of UV divergences in the boundary term, but one can avoid this delicate issue requiring $(c\phi)^2 = 0$, which for a boundary primary operator $\phi = \phi^{(h)}$ implies that

$$h_\phi < \frac{1}{2}. \tag{2.136}$$

. This restriction does not however excludes interesting cases, such as tachyon condensation. Including this new string field in the subalgebra KBc it is possible to define deformed wedge states

$$|T, \phi\rangle = e^{-T(K+\phi)}. \tag{2.137}$$

which are defined by contractions with Fock states $\chi = \chi(0)|0\rangle$

$$\langle\chi||T, \phi\rangle = \left\langle e^{-\int_{\frac{1}{2}}^{T+\frac{1}{2}} ds\, \phi(s)} f \circ \chi(0) \right\rangle_{C_{T+1}}, \tag{2.138}$$

where

$$f(z) = \frac{2}{\pi} \arctan z,$$

One can actually prove that, inside the path integral, one can always rewrite the insertion of a wedge state as

$$\langle [...] e^{-T(K+\phi)} [...] \rangle = \langle [...] e^{-TK} e^{-\int_x^{x+T} ds \phi(s)} [...] \rangle. \tag{2.139}$$

This shows the deformed wedge state of length T can be considered as a usual wedge of the same length with the insertion of an operator $e^{-\int_x^{x+T} ds\phi(s)}$ giving modified boundary conditions. This operator is completely well defined without need of any renormalization thanks to the condition 2.136. The minimal extension of the KBc algebra described allows to prove quite easily that

$$\psi_\phi = c\phi - \frac{1}{K+\phi}(\phi - \delta\phi) Bc\partial c \tag{2.140}$$

does indeed satisfy the OSFT equation of motion

$$Q\psi_\phi + \psi_\phi \psi_\phi = 0. \tag{2.141}$$

It is clear that (2.140) is a deformation of the Erler–Schnabl solution, see [25], which can be recovered for $\phi = 1$.

Much like in the Erler-Schnabl (ES) case, we can view this solution as a singular gauge transformation

$$\psi_\phi = U_\phi Q U_\phi^{-1}, \tag{2.142}$$

where

$$U_\phi = 1 - \frac{1}{K+\phi}\phi Bc, \qquad U_\phi^{-1} \;=\; 1 + \frac{1}{K}\phi Bc. \tag{2.143}$$

Of course all of these are quite formal manipulations one should consider with some attention. For example, in 2.143 the collision between the ϕ's in the denominator and numerator of U is potentially dangerous, but the assumption $\lim_{s\to 0} s\phi(s)\phi(0) = 0$ keeps us safe from divergences. In the same formal way, given the kinetic operator $\mathcal{Q}_{\psi_\phi}$ around (2.140), one can prove that

$$\mathcal{Q}_{\psi_\phi} \frac{B}{K+\phi} = Q\frac{B}{K+\phi} + \left\{ \psi_\phi, \frac{B}{K+\phi} \right\} = 1.$$

i.e. that the string $\frac{B}{K+\phi}$ plays the role of a homotopy field for the cohomology of $\mathcal{Q}_{\psi_\phi}$. This means the simple algebraic structure of the $KBc\phi$ should be considered as a valuable instrument in finding candidates for solutions, but hides subtleties which are crucial to the physical interpretation of our formal findings. In particular in [52] it was argued ψ_ϕ is a non trivial acceptable solution if

1. $\frac{1}{K+\phi}$ is singular so that the proposed homotopy field fails to satisfy eq. 2.144 in some restrictive sense: this ensures ψ_ϕ does not describe the tachyon vacuum, but a configuration containing open string dofs.

2. $\frac{1}{K+\phi}(\phi - \delta\phi)$ is regular enough so that the EOM 2.141 can be considered fully satisfied by ψ_ϕ whenever they appear in observable well defined physical quantities.

Of course there is some tension between these two requirements and it's not even clear what their mathematical meaning is. In fact, as noticed before for the KBc algebra, also for its minimal extension such an object like $\frac{1}{K+\phi}$ has a purely formal meaning unless a definite regularization is given. As we have to compute CFT correlators, the most useful *definition* is the one assuming the Schwinger representation as an integral over all deformed wedge states

$$\frac{1}{K+\phi} = \int_0^\infty dt\, \widetilde{\Omega}^t. \tag{2.144}$$

where $\widetilde{\Omega}^t = e^{-t(K+\phi)}$. For the time being we don't even try to justify how such an expression can be treated in a consistent *mathematical* frame, but we take an empirical point of view of testing it at work in the CFT correlators needed to compute physical quantities.

We can see that the conditions 1 and 2 encode very sensible *physical* properties lump solutions associated with BCFTs with boundary relevant deformations ϕ_u are expected to have. ϕ_u should vanish at $u = 0$, which is the UV fixed point described by the the reference conformal field theory $BCFT_0$ and $u \in [0, \infty)$ parameterizes the RG flow leading to another conformal fixed point at $u = \infty$, corresponding to the target conformal field theory $BCFT^*$. If we consider the canonical cylinder of width 1, this means for an arbitrary bulk operator $\mathcal{O}$

$$\lim_{u\to\infty} \left\langle \exp\left[-\int_0^1 ds\, \phi_u(s)\right] \mathcal{O} \right\rangle_{C_1}^{\mathrm{BCFT}_0} = \langle \mathcal{O} \rangle_{C_1}^{\mathrm{BCFT}^*},$$

and

$$\lim_{u\to 0} \left\langle \exp\left[-\int_0^1 ds\, \phi_u(s)\right] \mathcal{O} \right\rangle_{C_1}^{\mathrm{BCFT}_0} = \langle \mathcal{O} \rangle_{C_1}^{\mathrm{BCFT}_0}, \tag{2.145}$$

In particular, in order $BCFT^*$ to keep some open dofs it is necessary to assume its partition function is not vanishing

$$\lim_{u\to\infty} \left\langle e^{-\int_0^1 ds\, \phi_u(s)} \right\rangle_{C_1} \equiv Z^{\mathrm{BCFT}^*} = \texttt{finite}. \tag{2.146}$$

The g-theorem/conjecture tells us the partition function is a monotonically decreasing (and positive) function of u. So what we find is that for non trivial solution it cannot asymptote to zero. On the other hand one can consider the quantity

$$\begin{aligned} Tr[\frac{1}{K+\phi_u}] &= \int_0^\infty dT \left\langle e^{-\int_0^T ds\, \phi_u(s)} \right\rangle_{C_T}, \\ &= \int_0^\infty dT \left\langle e^{-\int_0^1 ds\, \phi_{Tu}(s)} \right\rangle_{C_1} = \int_0^\infty dT\, g(Tu), \end{aligned} \tag{2.147}$$

where we denoted

$$g(v) \equiv Tr[e^{-(K+\phi_v)}] = \left\langle e^{-\int_0^1 ds\, \phi_v(s)} \right\rangle_{C_1}, \tag{2.148}$$

the partition function on the canonical cylinder obtained by integrating ϕ_v on the boundary. So the assumption the RG flow doesn't end up in the tachyon vacuum turns out to be equivalent to the fact that this quantity cannot be finite.

On the other side only $u = 0$ and $u = \infty$ correspond to fixed points where conformal symmetry is restored, meaning that from the point of view of OSFT u is a gauge parameter and cannot appear in physical observables. This is the case if we assume that finite positive values of u can be sent to each other by a linear mapping under rescaling $f_t(z) = \frac{z}{t}$, i.e.

$$f_t \circ \phi_u(z) = \frac{1}{t}\, \phi_{tu}\left(\frac{z}{t}\right). \tag{2.149}$$

A related condition is that

$$Q(c\phi_u) = c\partial c(\phi_u - \delta\phi_u) = c\partial c\, u\partial_u\phi_u. \tag{2.150}$$

under which one can see that

$$\begin{aligned} Tr[\frac{1}{K+\phi_u}(\phi_u - \delta\phi_u)] &= Tr[\frac{1}{K+\phi_u} u\partial_u\phi_u] = \int_0^\infty dT\, Tr[e^{-T(K+\phi_u)} u\partial_u\phi_u], \\ ] &= -\int_0^\infty dt\, \frac{u}{t}\partial_u Tr[e^{-t(K+\phi_u)}] \\ &= -\int_0^\infty dt\, \frac{u}{t}\partial_u \left\langle e^{-\int_0^T \phi_u(s) ds} \right\rangle_{C_T} \\ &= -\int_0^\infty dt\, \frac{u}{t}\partial_u\, g(tu) = -\int_0^\infty dx\, \partial_x\, g(x). \end{aligned} \tag{2.151}$$

This means the string field $\frac{1}{K+\phi}(\phi - \delta\phi)$ is in some sense better behaved that $\frac{1}{K+\phi}$ and its contraction with the identity string field $\mathcal{I}$ is a finite gauge invariant quantity

$$Tr[\frac{1}{K+\phi_u}(\phi_u - \delta\phi_u)] = g(0) - g(\infty) = Z^{UV} - Z^{IR}. \tag{2.152}$$

which, in analogy with 2.129, can be interpreted as the shift in the open string partition function from the UV to the IR, which now, unlike the case of ES solution, describes a non trivial BCFT. So, postponing any attempt to give a general rigorous definition of what a well-behaved string field should be, we focus on the computation of physical well-defined quantities trying to see whether all expected features of lumps can be found for the class of solutions we have just described.

Chapter 3

The energy of the analytic lump solutions

3.1 Introduction

In the previous chapter we showed how the formalism of Witten's OSFT has been developed till the point the authors of [52], following an earlier suggestion of [63], were able to provide a general method to obtain new exact solutions corresponding to boundary relevant deformations, which are expected to play a crucial role in describing tachyon condensation.The great advantage of this strategy is evident in the case of the Erler-Schanbl (ES) tachyon condensation solution. To use it to verify Sen's conjecture is really an elementary task compared both to Schanabl's original solution and to the level truncation computation for the tachyon potential [15]. In the latter case one can simplify a lot the computation by restricting the set of scalar fields which may acquire a vacuum expectation through consideration of the symmetries of the problem. In particular it turns out one can restrict to ghost one scalars in the so-called universal subspace, spanned by the ghost fields and the matter Virasoro generators. Other symmetries to be considered are the twist symmetry and the $SU(1,1)$ symmetry of [12]. Finally gauge fixing removes a lot of fields. One can then enumerate systematically the states by defining the level L of a Fock state with reference to the zero momentum tachyon $c_1|0\rangle$, which is defined to be level zero, in other terms $L = L_0 + 1$. The level truncation approximation (L, N) is obtained by truncating the string field to level L, and keeping interactions terms in the OSFT action up to total level N, with $2L \leq N \leq 3L$. The lowest level truncation, i.e. the zero momentum tachyon , already accounts for 68% of expected energy at the tachyon vacuum. Indeed Gaiotto and Rastelli [14] were able to confirm the result with an impressive accuracy of $3 \cdot 10^{-3}\%$ by considering level truncation up to $l = 18$.

Their computation involves more than 2000 fields and over 10^{10} interaction terms! The construction of lumps in level truncation is by far more complicated because it involves non-universal terms but also possible problems with non-locality as it's necessary to truncate exponentials of momenta. An analytic approach is therefore very desirable.

In this chapter we will analyze a particular solution, generated by an exact RG flow first studied by Witten, [60]. The analysis carried out in the framework of Boundary String Field theory in [61] has pointed out such a solution should describe a D24 brane, with the correct ratio of tension with respect to the starting D25 brane.

3.1.1 Witten's boundary deformation

A particularly important example of boundary relevant deformations is the one first studied by Witten

$$T(X) = a + \sum_{i=p+1}^{25} u_i X_i^2 \tag{3.1}$$

In this case the quadratic nature of the boundary interaction allows to solve the world-sheet theory exactly. The operator $\exp\left(\int_{\partial\Sigma} ds T(X(s))\right)$ doesn't need any regularization because the logarithmic singularities determined by two near $T(X)$'s are integrable. The point $a = u_i = 0$ in the parameter space is the IR conformal point and corresponds to Neumann boundary conditions, $n^\alpha \partial_\alpha X^i = 0$ on $\partial\Sigma$ (n^α is the normal vector to the boundary of Σ). The UV conformal point at $u_i = \infty$ fixes instead Dirichlet conditions $X^i = 0$ for the corresponding fields. The operator $: X^2(z)$ is not primary

$$f\circ : X^2(z) :=: X^2(f(z)) : - \log|f'(z)|^2. \tag{3.2}$$

but by defining

$$\begin{aligned} \phi_u(z) \quad &\equiv \quad u f_u \circ (: X^2 : (uz) + A) = u[: X^2 : (z) + 2(\log u + A)], \qquad (3.3)\\ f_u(z) \quad &= \quad \frac{z}{u}. \qquad (3.4)\end{aligned}$$

one gets a primary-like transformation by letting u free to change along the RG flow

$$f\circ\phi_u(z) = \frac{1}{|\partial_{z'} f^{-1}(z')|}\phi_{u|\partial_{z'}f^{-1}(z')|}(z'). \tag{3.5}$$

In particular, under scaling $f_t(z) = z/t$, we have

$$f_t\circ\phi_u(z) = \frac{1}{t}\phi_{tu}\left(\frac{z}{t}\right) \tag{3.6}$$

and consequently

$$f_t \circ \int_a^b dy\, \phi_u(y) = \int_{\frac{a}{t}}^{\frac{b}{t}} d\widetilde{y}\, \phi_{tu}\,(\widetilde{y})\,. \tag{3.7}$$

This is just the desired behavior under rescaling making u a pure gauge parameter. One can actually verify that u drives the BRST variation of ϕ_u

$$\delta\phi_u(z) = -2u. \tag{3.8}$$

and so the condition (2.150) is satisfied

$$\phi_u - \delta\phi_u = u\partial_u\phi_u. \tag{3.9}$$

So if we consider the operator (defined in the cylinder C_T of width T in the arctan frame)

$$\phi_u(s) = u(X^2(s) + 2\ln u + 2A), \tag{3.10}$$

where A is a constant first introduced in [63], then on the cylinder C_1 of width 1 we have

$$\phi_u(s) = u(X^2(s) + 2\ln Tu + 2A) \tag{3.11}$$

and on the unit disk D,

$$\phi_u(\theta) = u(X^2(\theta) + 2\ln\frac{Tu}{2\pi} + 2A). \tag{3.12}$$

If we set

$$g_A(u) = \langle e^{-\int_0^1 ds\, \phi_u(s)}\rangle_{C_1} \tag{3.13}$$

we have

$$g_A(u) = \langle e^{-\frac{1}{2\pi}\int_0^{2\pi} d\theta\, u\left(X^2(\theta)+2\ln\frac{u}{2\pi}+2A\right)}\rangle_D.$$

According to [60],

$$g_A(u) = Z(2u)e^{-2u(\ln\frac{u}{2\pi}+A)}, \tag{3.14}$$

where

$$Z(u) \equiv \left\langle e^{-\int_0^{2\pi} d\theta \, \frac{u}{4\pi} X^2(e^{i\theta})} \right\rangle_{Disk} = K\,\sqrt{u}\exp(\gamma u)\Gamma(u). \tag{3.15}$$

K is u-independent renormalization constant, which is fixed by the choice of normalization for the zero mode of the coordinate X

$$\lim_{u\to 0} \frac{1}{2\sqrt{\pi u}} = \delta(0) = \int \frac{dx}{2\pi} = \frac{V}{2\pi} =< 0|0 > . \tag{3.16}$$

As $\lim_{u\to 0} Z(2u) \simeq \frac{K}{\sqrt{2u}}$, we choose $K = \frac{1}{\sqrt{2\pi}}$. We should remark this is not the same normalization as the one used in [17, 18, 25], i.e.

$$\langle 0|0\rangle = 2\pi\delta(0) = \mathcal{V}, \tag{3.17}$$

where

$$\delta(k) = \int \frac{dx'}{2\pi} e^{ipx'}, \tag{3.18}$$

with the prime in dx' meaning we can interpret the different normalization as the fact that length is measured in different units, i.e. $V = 2\pi\mathcal{V}$. According to these conventions, as shown in [19], the D25-brane tension (with $\alpha' = 1$) is assumed to be $T_{D25} = \frac{1}{2\pi^2}$ and so the D24 brane tension must be

$$\mathcal{T}_{D24} = \frac{1}{\pi}. \tag{3.19}$$

However, in our conventions the expected result for the D24 brane tension is

$$T_{D24} = \frac{1}{2\pi^2}. \tag{3.20}$$

Requiring finiteness for $u \to \infty$, which means the $BCFT^*$ corresponding to the IR fixed point has a well defined partition function, we get $A = \gamma - 1 + \ln 4\pi$, which implies

$$g_A(u) \equiv g(u) = \frac{1}{\sqrt{2\pi}} \sqrt{2u}\Gamma(2u) e^{2u(1-\ln(2u))} \tag{3.21}$$

and

$$\lim_{u\to\infty} g(u) = 1. \tag{3.22}$$

It should be remarked this value of the limit is completely unambiguous once the foregoing requirements are met and is just the one needed to reproduce the ratio of tensions (using the fact that the total mass, up to some universal constant, is proportional to the partition function, [75])

$$\frac{\tau^{(D24)}}{\tau^{(D25)}} = \frac{g(\infty)}{\frac{g(0)}{V_1}} = \frac{1}{\frac{1}{2\pi}} = 2\pi. \tag{3.23}$$

3.2 The energy functional

The energy of a time independent classical solution is given by

$$E(\psi) = -S[\psi] = \frac{1}{6}Tr[\psi Q\psi] = -\frac{1}{6}Tr[\psi^3] \tag{3.24}$$

In the case of lump solutions the last expression is the most convenient to compute because, contrary to the universal tachyon vacuum case, the deformed wedge states are not BRST closed. Under the assumption $(c\phi_u)^2 = 0$, the identity piece in the solution drops from the evaluation of the energy, and we are left with

$$\begin{aligned} Tr[\psi_u^3] &= -Tr\left[\frac{1}{K+\phi_u}u\partial_u\phi_u Bc\partial c\,\frac{1}{K+\phi_u}u\partial_u\phi_u Bc\partial c\,\frac{1}{K+\phi_u}u\partial_u\phi_u Bc\partial c\right] \\ &= -\int_0^\infty dt_1dt_2dt_3\left\langle e^{-\int_0^T ds\phi_u(s)}Bc\partial cu\partial_u\phi_u(T(x+y))\,\partial cu\partial_u\phi_u(Tx)\,\partial cu\partial_u\phi_u(0)\right\rangle_{C_T} \end{aligned} \tag{3.25}$$

which, in the case of Witten's deformation, can be rewritten in terms of known quantities. In fact

$$\begin{aligned} \langle\psi_u\psi_u\psi_u\rangle &= -\int_0^\infty dt_1dt_2dt_3\mathcal{E}_0(t_1,t_2,t_3)u^3g(uT)\Bigg\{8\Big(-\frac{\partial_{2uT}g(uT)}{g(uT)}\Big)^3 \\ &+ 4\Big(-\frac{\partial_{2uT}g(uT)}{g(uT)}\Big)\Big(G^2_{2uT}(\frac{2\pi t_1}{T})+G^2_{2uT}(\frac{2\pi(t_1+t_2)}{T})+G^2_{2uT}(\frac{2\pi t_2}{T})\Big) \\ &+ 8G_{2uT}(\frac{2\pi t_1}{T})G_{2uT}(\frac{2\pi(t_1+t_2)}{T})G_{2uT}(\frac{2\pi t_2}{T})\Bigg\}. \end{aligned} \tag{3.26}$$

where $T = t_1+t_2+t_3$. Here $g(u)$ is given by

$$g(u) = \frac{1}{\sqrt{2\pi}}\sqrt{2u}\Gamma(2u)e^{2u(1-\ln(2u))} \tag{3.27}$$

and represents the partition function of corresponding BCFT with the relevant deformation $\phi_u(\theta)$ on the boundary disk. $G_u(\theta)$ is the correlator on the boundary, first determined by Witten, [60]:

$$G_u(\theta) = \frac{1}{u} + 2\sum_{k=1}^\infty \frac{\cos(k\theta)}{k+u} \tag{3.28}$$

where we have made the choice $\alpha' = 1$. Finally $\mathcal{E}_0(t_1, t_2, t_3)$ represents the ghost three–point function in C_T.

$$\mathcal{E}_0(t_1,t_2,t_3) = \langle Bc\partial c(t_1+t_2)\partial c(t_1)\partial c(0)\rangle_{C_T} = -\frac{4}{\pi}\sin\frac{\pi t_1}{T}\sin\frac{\pi(t_1+t_2)}{T}\sin\frac{\pi t_2}{T}. \quad (3.29)$$

We change variables $(t_1, t_2, t_3) \to (T, x, y)$, where

$$x = \frac{t_1}{T}, \qquad y = \frac{t_2}{T}.$$

Then the matter part of (3.26) (before integration) can be written as $u^3 F(uT, x, y)$, where

$$\begin{aligned} F(uT,x,y) = {} & g(uT)\Bigg\{8\Big(-\tfrac{\partial_{2uT}g(uT)}{g(uT)}\Big)^3 + 8G_{2uT}(2\pi x)G_{2uT}(2\pi(x+y))G_{2uT}(2\pi y) \\ + {} & 4\Big(-\tfrac{\partial_{2uT}g(uT)}{g(uT)}\Big)\Big(G^2_{2uT}(2\pi x) + G^2_{2uT}(2\pi(x+y)) + G^2_{2uT}(2\pi y)\Big)\Bigg\}. \end{aligned}$$

while the ghost correlator becomes

$$\mathcal{E}_0(t_1,t_2,t_3) \equiv \mathcal{E}(x,y) = -\frac{4}{\pi}\sin\pi x\,\sin\pi y\,\sin\pi(x+y). \quad (3.30)$$

The ghost correlator only depends on x and y, which are scale invariant coordinates.

After the change

$$\int_0^\infty dt_1 \int_0^\infty dt_2 \int_0^\infty dt_3 = \int_0^\infty dT\,T^2 \int_0^1 dx \int_0^{1-x} dy,$$

the energy becomes

$$\begin{aligned} E[\psi_u] &= -S[\psi_u] = -\frac{1}{6}\langle\psi_u\psi_u\psi_u\rangle \\ &= \frac{1}{6}\int_0^\infty dT\,T^2\int_0^1 dx\int_0^{1-x} dy\,\mathcal{E}(x,y)\,u^3F(uT,x,y). \end{aligned} \quad (3.31)$$

It is convenient to change further $x \to y$ and subsequently $y \to 1-y$. The result is

$$E[\psi_u] = \tfrac{1}{6}\int_0^\infty dT\,T^2\int_0^1 dy\int_0^y dx\,\mathcal{E}(1-y,x)\,u^3F(uT,1-y,x), \quad (3.32)$$

where

$$\mathcal{E}(1-y,x) = \frac{4}{\pi}\sin\pi x\,\sin\pi y\,\sin\pi(x-y),$$

and

$$
\begin{aligned}
&F(uT, 1-y, x) \\
&= g(uT)\Bigg\{8\Big(-\frac{\partial_{2uT} g(uT)}{g(uT)}\Big)^3 + 8G_{2uT}(2\pi x)G_{2uT}(2\pi(x-y))G_{2uT}(2\pi y) \\
&+4\Big(-\frac{\partial_{2uT} g(uT)}{g(uT)}\Big)\Big(G_{2uT}^2(2\pi x) + G_{2uT}^2(2\pi(x-y)) + G_{2uT}^2(2\pi y)\Big)\Bigg\}.
\end{aligned}
\tag{3.33}
$$

Summarizing

$$
\begin{aligned}
E[\psi_u] \quad = \quad & \frac{1}{6}\int_0^\infty d(2uT)\ (2uT)^2 \int_0^1 dy \int_0^y dx\, \frac{4}{\pi}\, \sin\pi x \sin\pi y \sin\pi(x-y) \\
& \cdot g(uT)\Bigg\{-\Big(\frac{\partial_{2uT} g(uT)}{g(uT)}\Big)^3 + G_{2uT}(2\pi x)G_{2uT}(2\pi(x-y))G_{2uT}(2\pi y) \\
& -\frac{1}{2}\Big(\frac{\partial_{2uT} g(uT)}{g(uT)}\Big)\Big(G_{2uT}^2(2\pi x) + G_{2uT}^2(2\pi(x-y)) + G_{2uT}^2(2\pi y)\Big)\Bigg\}.
\end{aligned}
\tag{3.34}
$$

Remarkably this expression is explicitly independent of u, which can be absorbed into the integration variable $s = 2uT$ and therefore plays the role of a true gauge parameter of OSFT. On the other hand , as the integration bounds now correspond to the UV ($u = 0$) and IR ($u = \infty$) conformal fixed points describing the perturbative vacuum (D25 brane) and the lower dimensional lump solution (D24 brane in particular), it is expected to produce the D24-brane energy with respect to the D25-brane tension

$$
-\frac{1}{6}Tr[\psi_u^3] = -E^{(UV)} + E^{(IR)}, \tag{3.35}
$$

Notice we should expect $E^{(UV)}$ to be a divergent quantity because, as observed by Witten, the theory at $u = \infty$ has an action, while the theory at $u = 0$ has an action per unit volume.

The expression in (3.34) implies three continuous integrations and, in the most complicated case, three infinite discrete summations. At the best of our ability and knowledge, all these operations cannot be done analytically. Therefore the obvious strategy to evaluate (3.34) is to push as far as possible the analytic computations and bring the integral to a form accessible to numerical evaluation. Actually we first concentrate on studying the behaviors of the integrand in the UV and IR, which are quite crucial for the physical interpretation of ψ_u.

3.3 Behaviour near $s = 0$

Let us consider first the cubic term. We recall that all the summations are convergent at $s = 0$. In (A.14) the expression (A.15) is multiplied by $\frac{1}{6}s^2 g(s)$. Recalling that $g(s) \approx \frac{1}{2\sqrt{\pi s}}$ for $s \approx 0$, we see that the only term that produces a non-integrable singularity in s is the first term on the RHS, which has a cubic pole in s. Altogether the UV singularity due to the cubic term is

$$-\frac{1}{8}\frac{1}{\pi^{\frac{5}{2}}s^{\frac{3}{2}}}. \tag{3.36}$$

As for the quadratic term, we have $\partial_s g(s) \approx -\frac{1}{4\sqrt{\pi}s^{\frac{3}{2}}}$. Once again all the discrete summations are convergent at $s = 0$. Therefore the only UV singular term corresponds to the first term at the RHS of (A.7), i.e. $-\frac{9}{2\pi^2}\frac{1}{s^2}$. According to (A.5) we have to multiply this by $-\frac{1}{12}s^2\partial_s g(s)$. Therefore the contribution of the quadratic term to the UV singularity is

$$-\frac{3}{32}\frac{1}{\pi^{\frac{5}{2}}s^{\frac{3}{2}}}. \tag{3.37}$$

Finally for the last term, the one without G_s, we have

$$s^2 g(s)\left(\frac{\partial_s g(s)}{g(s)}\right)^3 \approx -\frac{1}{16\sqrt{\pi}s^{\frac{3}{2}}}.$$

Therefore altogether this term contributes

$$-\frac{1}{64}\frac{1}{\pi^{\frac{5}{2}}s^{\frac{3}{2}}}. \tag{3.38}$$

So the overall singularity at $s = 0$ is

$$-\frac{15}{64}\int_0 ds\frac{1}{\pi^{\frac{5}{2}}s^{\frac{3}{2}}} = \frac{15}{8}\frac{1}{4\pi^2\sqrt{\pi s}}\bigg|_{s=0} = -\lim_{s\to 0}\frac{15}{8}\frac{1}{2\pi^2}\frac{1}{2\sqrt{\pi s}} = -\frac{15}{8}\frac{1}{2\pi^2}\frac{V}{2\pi}. \tag{3.39}$$

This result shows us the UV divergent contribution to $E[\psi_u]$ doesn't reproduce the expected value for the D25-brane tension

$$E[\psi^{TV}] = -\frac{1}{2\pi^2}\mathrm{g}(0) = -\frac{1}{2\pi^2}\frac{V}{2\pi}$$

In order to subtract this singularity we choose a function $f(s)$ that vanishes fast enough at infinity and such that $f(0) = 1$. For instance $f(s) = e^{-s}$. Then, if we subtract from the energy the expression

$$\frac{15}{8}\frac{1}{4\pi^2\sqrt{\pi}}\int_0^\infty ds\frac{1}{\sqrt{s}}\left(f'(s)-\frac{1}{2s}f(s)\right)=\frac{15}{8}\frac{1}{4\pi^2\sqrt{\pi}}\int_0^\infty ds\frac{\partial}{\partial s}\left(\frac{1}{\sqrt{s}}f(s)\right) \tag{3.40}$$

the energy functional becomes finite, at least in the UV. What remains after the subtraction is the relevant energy.

Notice that the integral in (3.40) does not depend on the regulator f we use, provided it satisfies the boundary condition $f(0) = 1$ and decreases fast enough at infinity. We should notice the lump energy computed in this way is anyhow defined up to a constant. To fix the zero of the energy in a physically meaningful way will require further discussion.

3.4 The behavior near $s = \infty$

The integrand in (A.4) behaves as $1/s^4$ at large s. Therefore the integral (A.4) converges rapidly in the IR.

3.4.1 The quadratic term as $s \to \infty$

With reference to (A.5) we remark first that for large s

$$s^2\partial_s g(s) = -\frac{1}{12\sqrt{2}} + \mathcal{O}\left(\frac{1}{s}\right). \tag{3.41}$$

Therefore this factor does not affect the integrability at large s. The issue will be decided by the other factors. For large s we have

$$E_0^{(2)}(s) = \frac{9}{\pi^2}\frac{1}{s^3} - \frac{6}{\pi^2}\frac{\ln s}{s^4} + \cdots \tag{3.42}$$

and

$$R(s) = -\frac{3}{32\pi}\frac{1}{s} + \frac{1}{16\pi}\frac{1}{s^2} - \frac{3}{32\pi}\frac{\ln s}{s^3} + \cdots . \tag{3.43}$$

Moreover, again for large s,

$$RK(p,s) = \frac{1}{8\pi(p^2-1)}\frac{1}{s} - \frac{1}{8\pi(p^2-1)}\frac{1}{s^2} - \frac{1}{16\pi}\frac{\ln s}{s^3} + \cdots . \tag{3.44}$$

Since

$$\sum_{p=2}^{\infty} \frac{1}{8\pi(p^2-1)} = \frac{3}{32\pi}$$

the coefficient of $1/s$ in (3.43) cancels the corresponding coefficient of (3.44). The coefficient of $1/s^2$ equals $-1/(32\pi)$. This must be multiplied by $\frac{48}{\pi}$ and added to the term $-\frac{9}{2\pi^2}\frac{1}{s^2}$ in (A.7). This is anyhow an integrable term in the IR. This much takes care of the integrability of the $E_0^{(2)}(s)$, $R(s)$ and the first two terms in (3.44) in the IR. Let us now concentrate on the rest of $RK(p,s)$, that is

$$RK'(p,s) = \frac{1}{16\pi}\frac{\ln s}{s^3} + \dots \tag{3.45}$$

In order to estimate the integrability of this term, we can replace the infinite discrete sum with an integral over p, for large p. Now we evaluate the behaviour of $RK'(p,s)$ for any ray, departing from the origin of the (p,s) plane in the positive quadrant, when the rays approach infinity. We can parametrize a ray, for instance, as the line (as, s). It is possible to find an analytic expression for this. We can compute the large s limit for any (positive) value of a. The behaviour is given by the following rule

$$RK'(as,s) \approx \frac{B}{s^3} + \mathcal{O}(s^{-4}). \tag{3.46}$$

In Table 3.1 are some examples (the output is numerical only for economy of space).

a :	45	7	1	$\frac{1}{13}$	$\frac{1}{150}$	$\frac{1}{15000}$
B :	-0.00001	-0.00048	0.00713	-0.04319	-0.09039	-0.18188

Table 3.1: Samples of a and B in eq.(3.46)

It is important to remark that very small values of a are likely not to give a reliable response in the table, because one is bound to come across to the forbidden value $p = 1$, which will give rise to an infinity (see (A.11). Apart from this, on a large range the values of $RK'(as,s)$ are bounded in a.

It is even possible to find an analytic expression of B as a function of a in the large s limit. We have

$$\begin{aligned} B \;=\; & \frac{1}{8a^3\left(-4+a^2\right)^2\pi}\Big(-a\left(16-8a+2a^3+a^4+8a^2(-1+\log 2)\right) \\ & +8a^3\log a + 2(-2+a)^2\left(2+2a+a^2\right)\log(1+a)\Big) \end{aligned} \tag{3.47}$$

which is obviously integrable in the whole range of a. Since $dpds{=}sdads$, this confirms the integral behaviour of $\sum_{p=2}^{\infty} RK'(p,s)$ with respect to the s integration.

To study the integrability for large s and large p in a more systematic way, we divide the positive quadrant of the (p, s) plane in a large finite number N of small angular wedges. We notice that Table 1 means that $RK'(p, s)$ varies slowly in the angular direction – it is actually approximately constant in that direction for large p and s. Therefore it is easy to integrate over such wedges from a large enough value of the radius $r = \sqrt{p^2 + s^2}$ to infinity. The result of any such integration will be a finite number and a good approximation to the actual value (which can be improved at will). Their total summation will also be finite as a consequence of table 1, unless there are pathologies at the extremities. Looking at the asymptotic expansion for large p

$$RK(p, s) = \frac{1}{4\pi}\frac{\log p}{p^3} - \frac{1}{4\pi}(1 + \psi(1 + s))\frac{1}{p^3} + \dots \tag{3.48}$$

and Table 1 we see that also the integration for the very last wedge, a large, will be finite. The contribution of the very first wedge is more problematic for the above explained reason and was computed in Appendix A of [53].

An additional support comes from a numerical analysis of $RK'(p, s)$. It turns out that, for large s, the leading coefficient of $\sum_{p=2}^{\infty} RK'(p, s)$ is

$$\sum_{p=2}^{\infty} RK'(p, s) \approx \frac{-0.0344761...}{s^2} + \cdots . \tag{3.49}$$

This can be rewritten in the (probably exact) analytic way

$$\sum_{p=2}^{\infty} RK'(p, s) = \left(\frac{3}{32\pi} - \frac{1}{4\pi}\left(\gamma + \frac{1}{3}\log 2\right)\right)\frac{1}{s^2} + \cdots . \tag{3.50}$$

Finally, the numerical calculations of the next section further confirm our conclusion.

On the basis of that analysis and the above, we conclude that the quadratic term integrand in s, behaves in the IR in an integrable way, giving rise there to a finite contribution to the energy.

3.4.2 The cubic term as $s \to \infty$

To start with let us recall that for large s

$$s^2 g(s) = \frac{s^2}{\sqrt{2}} + \frac{s}{12\sqrt{2}} + \cdots . \tag{3.51}$$

Looking at (A.14) and (A.15), let us call

$$\begin{aligned}
E_1^{(3)}(s) \quad = \quad & -\frac{3}{2\pi^2}\frac{1}{s^3} + \frac{9}{4\pi^2}\frac{1}{s^2(s+1)} + \frac{3}{\pi^2}\frac{1}{s^2(s^2-1)}\left(-\gamma + \frac{3}{4}(s+1) - \psi(2+s)\right) \\
& + \frac{3}{4\pi^2}\frac{1}{s(s+1)^2} - \frac{7}{2\pi^2}\frac{1}{s(s+1)(s+2)} + \frac{3}{4\pi^2}\frac{1}{s(s^2-1)^2}\cdot \\
& \cdot\left(3(1+s^2) - 8\gamma s + 6s - 8s\psi(2+s) + 4(s^2-1)\psi^{(1)}(2+s)\right) \\
& - \frac{1}{2\pi^2 s(s+1)(s^2-1)}\left(17 + 5s - 12\gamma - 12\psi(3+s)\right) - \frac{3}{2\pi^2 s^2(s^2-1)} \\
& \cdot\left(5 - 4\gamma + s - 2(s+1)\psi(2+s) + 2(s-1)\psi(3+s)\right). \qquad (3.52)
\end{aligned}$$

Then it is easy to prove that

$$\lim_{s\to\infty} s^3 E_1^{(3)}(s) = -\frac{3}{2\pi^2} \qquad (3.53)$$

that is, the nonvanishing, nonintegrable, contribution comes solely from the first term on the RHS of (3.52). Defining $E_0^{(3)}(s) = E_1^{(3)}(s) + \frac{3}{2\pi^2}\frac{1}{s^3}$, one finds

$$E_0^{(3)}(s) \approx \frac{3}{\pi^2}\frac{\ln s}{s^4} + \frac{3\gamma}{\pi^4}\frac{1}{s^4} + \cdots . \qquad (3.54)$$

This corresponds to an integrable singularity at infinity in the s-integration. We expect the nonintegrable contribution coming from (3.53) to be cancelled by the three–cosine pieces. We will see that also the first terms in the RHS of (3.54) gets cancelled.

Let us see the three-cosines pieces in eq.(A.15). The first contribution (A.17), for large s goes as follows

$$S(s) = \frac{5}{256\pi}\frac{1}{s^3} - \frac{2+\ln 8}{32\pi}\frac{1}{s^4} + \cdots . \qquad (3.55)$$

The other contribution is given by $SK(p,s)$. We proceed as for $RK(p,s)$ above.

$$SK(p,s) = \frac{1}{32\pi p(p+2)}\frac{1}{s^3} - \frac{1 - p(p+1)\left(H\left(\frac{p-1}{2}\right) - H\left(\frac{p}{2}\right)\right)}{32\pi p(p+1)}\frac{1}{s^4} + \cdots . \qquad (3.56)$$

Let us consider the first term in the RHS, which, from (A.15), must be multiplied by 3. The sum over p up to ∞ gives the following coefficient of $1/s^3$

$$3\sum_{p=3}^{\infty}\frac{1}{32\pi p(p+2)} = \frac{7}{256\pi}.$$

This must be added to the analogous coefficient in the RHS of (A.17), yielding a total coefficient of $\frac{3}{64\pi}$. In eq.(A.15) this is multiplied by $\frac{32}{\pi}$, which gives $\frac{3}{2\pi^2}$. This cancels exactly the RHS of (3.53). Therefore *in the integral (A.15) there are no contributions of order* $1/s^3$ *for large* s.

As already remarked for the quadratic term, the above takes care of the nonintegrable asymptotic behaviour of (3.53,3.55), but it is not enough as far as the $SK(p,s)$ is concerned. We will proceed in a way analogous to the quadratic term. We will drop the first term in the RHS of (3.56) (since we know how to deal exactly with the latter) and define

$$SK'(p,s) = SK(p,s) - \frac{1}{32\pi p(p+2)}\frac{1}{s^3}. \tag{3.57}$$

In order to estimate the integrability of this expression, we will replace, for large p, the infinite discrete sum with an integral over p. Next we evaluate the behaviour of $SK(p,s)$ for any ray departing from the origin of the (p,s) plane in the positive quadrant when the rays approach infinity, parametrizing a ray as the line (as,s), a being some positive number. The behaviour is given in general by the following rule

$$SK(as,s) \approx \frac{B}{s^5} + \cdots . \tag{3.58}$$

In Table 3.2 are some examples (the output is numerical for economy of space): Also in

a :	45	7	1	1/15	1/85	1/150
B :	-4×10^{-6}	-0.00017	-0.00497	-0.13988	-0.83567	-1.4822

TABLE 3.2: Samples of a and B in eq.(3.46)

this case we warn that it does not make sense to probe extremely small values of a.

On the other extreme, large p and fixed s, we have

$$SK(p,s) = \frac{-1+(1+s)\psi^{(1)}(1+s)}{16\pi(1+s)}\frac{1}{p^3} + \frac{-3-4s+4s(1+s)\psi^{(1)}(1+s)}{32\pi(1+s)} + \cdots . \tag{3.59}$$

This behaviour is of course integrable at $p=\infty$. One can also verify a behavior in p similar to (3.58) and compute a table like Table 2.

Next we study the problem of integrability for large s and large p following the same pattern as for the quadratic term. We divide the positive quadrant of the (p,s) plane in a large finite number N of small angle wedges. We notice that Table 1 means that $s^2SK'(p,s)$ varies slowly in the angular direction – it is actually approximately constant in that direction for large p and s, see (3.60) below. Therefore it is easy to integrate $s^2SK'(p,s)$ over such wedges from a large enough value of the radius $r=\sqrt{p^2+s^2}$ to infinity. The result of any such integration will be a finite number, including the integration for the very last wedge, a very large. To estimate the effectiveness of this approach one should consider the first wedge, which is the most problematic in view of what has been remarked above. But this point is very technical and was dealt with Appendix B of [53].

Additional evidence for convergence can be provided by a numerical analysis. One can see that the behaviour of $s^2SK'(p,s)$ for large p and s may be approximated by by

$$s^2SK'(p,s) \sim \frac{\log r}{r^3} \tag{3.60}$$

which is integrable. We can do better and compute, numerically, the asymptotic behaviour

$$s^2\sum_{p=3}^{\infty} SK'(p,s) \approx -\frac{0.0092Log(s)}{s^2} + \dots \tag{3.61}$$

The numerical calculations of the next section also confirm this. So we conclude that for the cubic term too, the integrand in s behaves in the IR in an integrable way, giving rise there to a finite contribution to the energy.

Finally, on the basis of the heuristic analysis of this section, we conclude that, once the UV singularity is suitably subtracted, the energy integral (3.34) is finite.

3.5 Numerical evaluation

This section is devoted to the numerical evaluation of (3.34) using the results of the previous sections.

The first step is subtracting the UV singularity, which is known in an exact way and can therefore be subtracted as explained in sec.3.3. It remains for us to do it in concrete by choosing a regulator. Since we are interested in enhancing as much as possible the numerical convergence we will choose the following families of f's

$$f(v) = \begin{cases} e^{-\frac{v}{a^2-v^2}} & 0 \le v \le a \\ 0 & v \ge a \end{cases} \tag{3.62}$$

where a is a positive number. It equals 1 at $v = 0$ and 0 at $v = a$. Therefore, for terms in the integrand of (3.34) that are singular in $v = 0$, we will split the integral in two parts: from 0 to a, and from a to ∞. The part from 0 to a will undergo the subtraction explained in sec. 4.

We have checked the regulator for several values of a, $a = 0.01, 0.5, 1, 2, 10, 100, \ldots$. Changing a may affect the fourth digit of the results below, which is within the error bars of our calculations. Therefore in the sequel we will make a favorable choice for the accuracy of the calculations: $a = 1$.

Let us proceed to evaluate the three terms in turn.

3.5.1 The cubic term

In eq.(A.15) we have to pick out the term $-\frac{3}{2\pi^2}\frac{1}{s^3}$ and treat it separately. Let us consider it first in the range $0 \le s \le 1$ and subtract the UV divergence. In the range $1 \le s < \infty$ instead, according to the discussion of the last section, we will combine it with the most divergent of the remaining terms. This will render the corresponding integrals convergent.

1) Let us start with the subtraction for $-\frac{3}{2\pi^2}\frac{1}{s^3}$. Proceeding as explained above the subtracted integrand (after multiplying by $\frac{1}{6}s^2 g(s)$) is

$$-\frac{1}{4\pi^2}\left(\frac{g(s)}{s} - \frac{1}{\sqrt{\pi s}}\frac{e^{\frac{s}{s^2-4}}\left(16+8s-8s^2+2s^3+s^4\right)}{2s\left(s^2-4\right)^2}\right). \tag{3.63}$$

This, integrated from 0 to 1, gives -0.0619767.

2) Now, let us consider the term (3.52). Leaving out the first term we get $E_0^{(3)}(s)$. When multiplied by $\frac{1}{6}s^2 g(s)$ the result has integrable singularity at $s = 0$, therefore it can be directly integrated from 0 to ∞. The result is 0.109048.

3) Next we have the term $S(s)$. When multiplied by $\frac{1}{6}s^2 g(s)$, it is non-integrable at ∞. Thus we split $-\frac{3}{2\pi^2}\frac{1}{s^3}$ as $-\frac{32}{\pi}\frac{5}{256\pi}\frac{1}{s^3} - \frac{32}{\pi}\frac{7}{256\pi}\frac{1}{s^3}$. We add the first addend to $S(s)$ in the range $1 \le s < \infty$, so as to kill the singularity at infinity. Then we multiply the result by $\frac{32}{\pi}\frac{1}{6}s^2 g(s)$. The overall result is integrable both in 0 and at ∞. Finally we integrate from 0 to 1 and from 1 to ∞ the corresponding unsubtracted and subtracted integrands. The result is -0.0190537.

4) Now we are left with the $SK(p,s)$ terms. This must be summed over p from 3 to infinity. After summation this term must be multiplied by $\frac{96}{\pi}\frac{1}{6}s^2 g(s)$. The result is integrable in the UV, but not in the IR. In fact we must subtract the other piece of $-\frac{3}{2\pi^2}\frac{1}{s^3}$, more precisely we should add $-\frac{32}{\pi}\frac{7}{256\pi}\frac{1}{s^3}$ to (A.19) in the range $1 \le s \le \infty$. The best way to do it is to split the integration in the intervals (0,1) and (1,∞), and to subtract from (A.19) the term $\frac{1}{32\pi p(p+2)}\frac{1}{s^3}$. At this point we proceed numerically with Mathematica, both for the summation over p and the integration over s. The result is -0.029204, with possible errors at the fourth digit.

According to the above, the cubic term's overall contribution to the energy is -0.00118596.

3.5.2 The quadratic term

1) Also in this case, looking at (A.6,A.7), we treat separately the term $-\frac{9}{2\pi^2}\frac{1}{s^2}$. This term must be multiplied by $-\frac{1}{12}s^2\partial_s g(s)$. We get as a result

$$s(s) = -\frac{3e^s e^{-(\frac{1}{2}+s)\ln s}\Gamma(s)(-1+2s\ln s - 2s\psi(s))}{32\pi^{\frac{5}{2}}}. \tag{3.64}$$

The resulting term is regular in the IR but singular in the UV. We make the same subtraction as above and obtain

$$\mathfrak{s}(s) = s(s) + \frac{3}{16\pi^2}\frac{1}{\sqrt{\pi s}}\frac{e^{\frac{s}{s^2-4}}\left(16+8s-8s^2+2s^3+s^4\right)}{2s\left(s^2-4\right)^2}. \tag{3.65}$$

It is easy to see that this is now integrable also in the UV. Integrating it between 0 and 1 and $s(s)$ between 1 and infinity one gets 0.0379954.

2) Next comes the integration of the term containing $E_0^{(2)}(s)$, see (A.8) above. This must be multiplied also by $-\frac{1}{3}s^2\partial_s g(s)$. The result is a function regular both at 0 and ∞. One can safely integrate in this range and get 0.0156618.

3) The next term is $R(s)$, (A.12). This behaves like $\frac{1}{s}$ for large s, see (3.43). So we subtract the corresponding divergent term, knowing already that it cancels against the analogous behaviour of the RK piece (see also below). Therefore we define

$$R'(s) = R(s) + \frac{3}{32\pi}\frac{1}{s}. \tag{3.66}$$

This has the right behaviour in the IR, but not the UV. For multiplying by $-\frac{1}{12}s^2\partial_s g(s)$ one gets an ultraviolet singularity. The way out is to limit the subtraction (3.66) to the range $(1,\infty)$. This can be done provided we do the same with the RK term, see below. Finally, in the range (0,1) we will integrate the term containing $R(s)$ without correction, since it can safely be integrated there. In the range $(1,\infty)$ we will integrate the one containing $R'(s)$. The overall result is -0.00392332.

4) There remains the $RK(p,s)$ piece, see (A.13). Again we have to subtract the singularity at ∞ (knowing that it cancels against the previous one). So we define

$$RK'(p,s) = RK(p,s) - 1/(8\pi(p^2-1)s). \tag{3.67}$$

However, when multiplying by $-\frac{4}{\pi}s^2\partial_s g(s)$, this introduces an UV singularity, so in accordance with the previous subtraction, this subtraction has to be limited to the range (1,∞). Consequently we have also to split the integration. Both integrals from 0 to 1 and from 1 to ∞ are well defined. The numerical evaluation gives 0.000235065.

The overall contribution of the quadratic term is therefore 0.049969.

3.5.3 Last contribution

The last one is easy to compute. The integrand is

$$\frac{1}{4\pi^2}s^2 g(s)\left(\frac{\partial_s g(s)}{g(s)}\right)^3. \tag{3.68}$$

This converges very rapidly in the IR. The only problem is with the usual singularity in the UV, where (3.68) behaves like $-\frac{1}{64\pi^2}\frac{1}{\sqrt{\pi}s^{\frac{3}{2}}}$. To this end we will add to (3.68) the function

$$\frac{1}{32\pi^2}\frac{1}{\sqrt{\pi s}}\frac{e^{\frac{s}{s^2-4}}\left(16+8s-8s^2+2s^3+s^4\right)}{2s\left(s^2-4\right)^2}. \tag{3.69}$$

The sum of the two is now well behaved and can be integrated from 0 to ∞. The result is 0.0206096.

3.5.4 Overall contribution

In conclusion the total finite contribution to the energy is 0.0693926.

$$E^{(s)}[\psi_u] \approx 0.0693926, \tag{3.70}$$

where the superscript $^{(s)}$ means that we have subtracted away the UV singularity. This has to be compared with the expected D24 brane tension

$$T_{D24} = \frac{1}{2\pi^2} \approx 0.0506606. \tag{3.71}$$

The two values (3.70) and (3.71) differ by about 27%.

3.5.5 Error estimate

All the numerical calculations of [53] have been carried out with Mathematica. Mathematica can be very precise when performing numerical manipulations. However in our case there are two main sources of error, beside the subtraction of the infinite D25-brane factor and the precision of Mathematica. The first is the summation over p of $RK(p,s)$

and especially $SK(p,s)$. The precision of this summation is probably limited by the computer capacity and seem to affect up to the fourth digit in item 4 of section 6.1 and especially 6.2. Another source of errors is the presence of zeroes in the denominators of the expressions of $RK(p,s)$ and $SK(p,s)$. As we have explained above, they do not correspond to poles, because they are canceled by corresponding zeroes in the numerators; but Mathematica, when operating numerically, is not always able (or we have not been able to use it properly) to smooth out the corresponding functions. This again may affect the fourth digit of item 4 of section 3.5.1 and especially 3.5.2.

It is not easy to evaluate these sources of error. A certain number of trials suggest that a possible error of 1% in the final figure (3.70) does not seem to be unreasonable. We shall see that actually the numerical result (3.70) we have obtained is more precise than that.

However it is clear from now that $E^{(s)}[\psi_u]$ *is not the lump energy* we are looking for. This may be a bit disconcerting at first sight, because, after all, we have subtracted from the energy the UV singularity, which corresponds to tachyon vacuum energy. However one should remember this UV singular term doesn't give the expected value for the D25 brane tension and so the finite contribution after subtraction should not be expected to give the *expected* value for the D24-brane tension. The point is that the subtraction we have carried out to extract this finite value contains an element of arbitrarness. In fact it is a purely *ad hoc* subtraction on the energy functional alone, not a subtraction made in the framework of a consistent scheme. In order to make sure that our result is physical we have to render it independent of the subtraction scheme. ψ_u is a (UV subtracted) solution to the SFT equation of motion *on the perturbative vacuum*; this brings as a consequence the energy 3.34 is affected by a UV singularity whose subtraction is arbitrary. Luckily, what we need to test Sen's conjecture about lower dimensional branes is the solution corresponding to ψ_u *on the tachyon condensation vacuum*, which is expected to be free of UV singularities. As we shall see, the gap between (3.70) and (3.71) is the right gap between the (subtracted) energy of ψ_u and the energy of the lump above the tachyon condensation vacuum.

3.6 The Tachyon Vacuum Solution in $KBc\phi$ algebra

In order to find the lump solution on the tachyon condensation vacuum we need to specify a tachyon condensation solution. Unfortunately the Erler-Schnabl solution is not a convenient choice in our case because it is defined on a perturbative vacuum with no matter degrees of freedom and in particular doesn't have UV singularities. Luckily (even if the way it was found it's quite roundabout, as we were actually looking for a regularized version of ψ_u, [53]) the right solution turns out to be a quite straightforward generalization of ES solution once the change $K \to K + \phi$ is done

$$\psi^{\epsilon}_{\phi} = c(\phi + \epsilon) - \frac{1}{K + \phi + \epsilon}(\phi + \epsilon - \delta\phi)Bc\partial c. \tag{3.72}$$

This is certainly a solution to the equation of motion since it is simply obtained from the lump solution ψ_u by replacing ϕ with $\phi + \epsilon$ and it is certainly regular. Moreover

$$U^{\epsilon}_{\phi} = 1 - \frac{1}{K + \phi + \epsilon}(\phi + \epsilon)Bc, \qquad U^{-1}_{\phi} = 1 + \frac{1}{K}(\phi + \epsilon)Bc. \tag{3.73}$$

For the time being, even if it appears to be an acceptable solution for any ϵ, we assume ϵ small on the basis of the obvious simplification it will take our computations. Like for other solutions associated with boundary relevant deformations, some conditions have to be satisfied in order for ψ^{ϵ}_{ϕ} to be well-defined and not pure gauge. In particular the string field

$$\frac{1}{K + \phi_u + \epsilon}(\phi_u + \epsilon - \delta\phi_u).$$

should be regular. It is certainly well–defined for any ϵ with $\Im\epsilon \neq 0$. Using a Schwinger representation we choose $\Re\epsilon > 0$. The one-point correlator is

$$\begin{aligned}
&\langle\frac{1}{K + \phi_u + \epsilon}(\phi_u + \epsilon - \delta\phi_u)\rangle = \langle\frac{1}{K + \phi_u + \epsilon}(\epsilon + u\partial_u\phi_u)\rangle \\
&= \epsilon\int_0^{\infty} dt\, e^{-\epsilon t}\langle e^{-t(K+\phi_u)}\rangle - \int_0^{\infty} dt\, e^{-\epsilon t}\frac{u}{t}\partial_u\langle e^{-t(K+\phi_u)}\rangle \\
&= \epsilon\int_0^{\infty}\frac{dx}{u}\, g(x)e^{-\epsilon\frac{x}{u}} - \int_0^{\infty} dx\,\partial_x g(x)\, e^{-\epsilon\frac{x}{u}} \\
&= -\int_0^{\infty} dx\,\partial_x\left(g(x)e^{-\epsilon\frac{x}{u}}\right) = g(0) - \lim_{x\to\infty} g(x)e^{-\epsilon\frac{x}{u}},
\end{aligned} \tag{3.74}$$

where $x = tu$. As long as ϵ, u are kept finite, the above limit vanishes and we get

$$\lim_{\epsilon\to 0}\langle\frac{1}{K + \phi_u + \epsilon}(\phi_u + \epsilon - \delta\phi_u)\rangle = g(0). \tag{3.75}$$

If we take the limit $\epsilon \to 0$ first, we get instead

$$\langle\frac{1}{K + \phi_u}(\phi_u - \delta\phi_u)\rangle = g(0) - g(\infty). \tag{3.76}$$

Remark This is a crucial evidence we cannot exchange integration with $\epsilon \to 0$ limit, the reason being that $g(x)$ is not integrable for large x. In fact, just setting $\epsilon = 0$ in the integrand we should find the change in the partition function corresponding to the lumps solution on the perturbative vacuum, instead of the one corresponding to the tachyon vacuum. Such discontinuity of the $\epsilon \to 0$ limit will play a fundamental role in the sequel.

Let us consider next $\langle \frac{1}{K+\phi_u+\epsilon} \rangle$ which is expected to be singular. We have

$$\langle \frac{1}{K+\phi_u+\epsilon} \rangle \;=\; \int_0^\infty dt\, e^{-\epsilon t} \langle e^{-t(K+\phi_u)} \rangle = \int_0^\infty \frac{dx}{u} g(x) e^{-\epsilon \frac{x}{u}}. \tag{3.77}$$

The crucial region is at $x \to \infty$. Since $g(\infty) = finite$ the behaviour of this integral is qualitatively similar to

$$\sim \frac{1}{\epsilon} e^{-\epsilon \frac{x}{u}} \Big|_M^\infty \sim \frac{e^{-\frac{\epsilon M}{u}}}{\epsilon} \tag{3.78}$$

for M a large number. The inverse of ϵ present in this expression makes the integral (3.77) divergent, as it is easy to verify also numerically. This tells us that homotopy operator corresponding to the regularized solution (see below) is well-defined, while if we set $\epsilon = 0$ it becomes singular.

As the above examples show, the $\epsilon \to 0$ limit, in general, is not continuous. This is true in particular for the energy, as we shall see in the next section.

3.7 The energy of the tachyon vacuum

In this section we calculate the energy of the tachyon vacuum solution in the case in which ϕ_u is the Witten deformation discussed previously. The energy is proportional to

$$\begin{aligned} \langle \psi_u^\epsilon \psi_u^\epsilon \psi_u^\epsilon \rangle &= -\lim_{\epsilon\to 0} \Big\langle \Big(\frac{1}{K+\phi_u+\epsilon} (\phi_u + \epsilon + 2u) BcKc \Big)^3 \Big\rangle \\ &= -\lim_{\epsilon\to 0} \int_0^\infty dt_1 dt_2 dt_3 \mathcal{E}_0(t_1,t_2,t_3) e^{-\epsilon T} \Big\langle (\phi_u(t_1+t_2) + \epsilon + 2u) \\ &\qquad \times (\phi_u(t_1) + \epsilon + 2u)(\phi_u(0) + \epsilon + 2u) e^{-\int_0^T ds \phi_u(s)} \Big\rangle_{C_T}, \end{aligned} \tag{3.79}$$

where $T = t_1 + t_2 + t_3$. We map the matter parts to the unit disc:

$$\begin{aligned} \langle \psi_u^\epsilon \psi_u^\epsilon \psi_u^\epsilon \rangle = -\lim_{\epsilon\to 0} \int_0^\infty & dt_1 dt_2 dt_3 \mathcal{E}_0(t_1,t_2,t_3) e^{-2uT\big(\ln(\frac{uT}{2\pi}) + A + \frac{\epsilon}{2u}\big)} \\ & \times u^3 \Big\langle \Big(X^2(\theta_{t_1+t_2}) + 2\big(\ln(\frac{uT}{2\pi}) + A + 1 + \frac{\epsilon}{2u}\big)\Big) \\ & \times \Big(X^2(\theta_{t_1}) + 2\big(\ln(\frac{uT}{2\pi}) + A + 1 + \frac{\epsilon}{2u}\big)\Big) \\ & \times \Big(X^2(0) + 2\big(\ln(\frac{uT}{2\pi}) + A + 1 + \frac{\epsilon}{2u}\big)\Big) e^{-\int_0^{2\pi} d\theta \frac{uT}{2\pi} X^2(\theta)} \Big\rangle_{Disk}. \end{aligned} \tag{3.80}$$

Using Appendix D of [52] and setting $A = \gamma - 1 + \ln 4\pi$, we obtain

$$\langle \psi_u^\epsilon \psi_u^\epsilon \psi_u^\epsilon \rangle = -\lim_{\epsilon\to 0}\int_0^\infty dt_1 dt_2 dt_3 \mathcal{E}_0(t_1,t_2,t_3) u^3 e^{-\epsilon T} g(uT) \tag{3.81}$$
$$\cdot\Big\{8\Big(\frac{h_{2uT}}{2} + \ln(2uT) + \gamma + \frac{\epsilon}{2u}\Big)^3 + 8G_{2uT}(\frac{2\pi t_1}{T})G_{2uT}(\frac{2\pi(t_1+t_2)}{T})G_{2uT}(\frac{2\pi t_2}{T})$$
$$+4\Big(\frac{h_{2uT}}{2} + \ln(2uT) + \gamma + \frac{\epsilon}{2u}\Big)\Big(G^2_{2uT}(\frac{2\pi t_1}{T}) + G^2_{2uT}(\frac{2\pi(t_1+t_2)}{T}) + G^2_{2uT}(\frac{2\pi t_2}{T})\Big)\Big\}.$$

This can also be written as

$$\langle \psi_u^\epsilon \psi_u^\epsilon \psi_u^\epsilon \rangle = -\lim_{\epsilon\to 0}\int_0^\infty dt_1 dt_2 dt_3 \mathcal{E}_0(t_1,t_2,t_3) u^3 e^{-\epsilon T} g(uT)\Bigg\{\Big(\frac{\epsilon}{u} - \frac{\partial_{uT} g(uT)}{g(uT)}\Big)^3 \tag{3.82}$$
$$+2\Big(\frac{\epsilon}{u} - \frac{\partial_{uT} g(uT)}{g(uT)}\Big)\Big(G^2_{2uT}(\frac{2\pi t_1}{T}) + G^2_{2uT}(\frac{2\pi(t_1+t_2)}{T}) + G^2_{2uT}(\frac{2\pi t_2}{T})\Big)$$
$$+8G_{2uT}(\frac{2\pi t_1}{T})G_{2uT}(\frac{2\pi(t_1+t_2)}{T})G_{2uT}(\frac{2\pi t_2}{T})\Bigg\}.$$

Let us make again a change of variables $(t_1,t_2,t_3) \to (T,x,y)$, where

$$x = \tfrac{t_1}{T}, \qquad y = \tfrac{t_2}{T}.$$

Then the matter part of the energy can be written as

$$u^3 e^{-\epsilon T} F_\epsilon(uT,x,y),$$

where

$$F_\epsilon(uT,x,y) = g(uT)\Bigg\{\Big(\frac{\epsilon}{u} - \frac{\partial_{uT} g(uT)}{g(uT)}\Big)^3 \tag{3.83}$$
$$+8G_{2uT}(2\pi x)G_{2uT}(2\pi(x+y))G_{2uT}(2\pi y)$$
$$+2\Big(\frac{\epsilon}{u} - \frac{\partial_{uT} g(uT)}{g(uT)}\Big)\Big(G^2_{2uT}(2\pi x) + G^2_{2uT}(2\pi(x+y)) + G^2_{2uT}(2\pi y)\Big)\Bigg\}.$$

The ghost correlator has been given in the introduction. Making an additional change of coordinate $s = 2uT, x \to y \to 1-y$, yields finally

$$E_0[\psi_u] = \frac{1}{6}\lim_{\epsilon\to 0}\int_0^\infty ds\, s^2 \int_0^1 dy \int_0^y dx\, e^{-\frac{\epsilon s}{2u}} \mathcal{E}(1-y,x)\, F_\epsilon(s/2, 1-y, x) \tag{3.84}$$

with

$$F_\epsilon(s/2,1-y,x) = \mathrm{g}(s)\Big\{\Big(\frac{\epsilon}{2u} - \frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\Big)^3 + G_s(2\pi x)G_s(2\pi(x-y))G_s(2\pi y) + \frac{1}{2}\Big(\frac{\epsilon}{2u} - \frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\Big)\Big(G_s^2(2\pi x) + G_s^2(2\pi(x-y)) + G_s^2(2\pi y)\Big)\Big\}. \tag{3.85}$$

where we have set $\mathrm{g}(s) \equiv g(s/2)$.

3.7.1 The energy in the limit $\epsilon \to 0$

Our purpose here is to study the energy functional $E_\epsilon[\psi_u^\epsilon]$. We notice that (3.84,3.85) for generic ϵ can be obtained directly from (3.34) with the following exchanges: $s \to s$, $g(uT) \to e^{-\frac{\epsilon s}{2u}}\mathrm{g}(s) \equiv \widetilde{\mathrm{g}}_\epsilon(s,u)$. Summarizing, we can write

$$\begin{aligned} E_\epsilon[\psi_u^\epsilon] &= \frac{1}{6}\lim_{\epsilon\to 0}\int_0^\infty ds\, s^2 \int_0^1 dy \int_0^y dx\, \mathcal{E}(x,y)\,\widetilde{\mathrm{g}}_\epsilon(s,u)\Big\{\Big(-\frac{\partial_s\widetilde{\mathrm{g}}_\epsilon(s,u)}{\widetilde{\mathrm{g}}_\epsilon(s,u)}\Big)^3 \\ &\quad + G_s(2\pi x)G_s(2\pi(x-y))G_s(2\pi y) \\ &+ \frac{1}{2}\Big(-\frac{\partial_s\widetilde{\mathrm{g}}_\epsilon(s,u)}{\widetilde{\mathrm{g}}_\epsilon(s,u)}\Big)\Big(G_s^2(2\pi x) + G_s^2(2\pi(x-y)) + G_s^2(2\pi y)\Big)\Big\}. \end{aligned} \tag{3.86}$$

We are of course interested in the limit

$$E_0[\psi_u] = \lim_{\epsilon\to 0} E_\epsilon[\psi_u^\epsilon] \tag{3.87}$$

The dependence on ϵ is continuous in the integrand, therefore a discontinuity in the limit $\epsilon \to 0$ may come only from divergent integrals that multiply ϵ factors. Now, looking at (3.85), we see that we have two types of terms. The first type is nothing but (3.34), with the only difference that the integrand of $d(2uT)$ is multiplied by $e^{-\frac{\epsilon s}{2u}}$. In the previous sections we have shown that, setting formally $\epsilon = 0$ everywhere in (3.84) and (3.85), or in (3.86), and subtracting the UV singularity, we get a finite integral, i.e. in particular the integrand has integrable behaviour for $s \to \infty$. Therefore this first type of term is certainly continuous in the limit $\epsilon \to 0$. However with the second type of terms the story is different. The latter are the terms linear, quadratic or cubic in $\frac{\epsilon}{u}$ in (3.85) (for convenience we will call them ϵ-terms). The factors that multiply such terms in the integrand may be more singular than the ones considered in the previous section. They may give rise to divergent integrals, were it not for the overall factor $e^{-\frac{\epsilon s}{2u}}$. In the $\epsilon \to 0$ limit these terms generate a (finite) discontinuity through a mechanism we shall explain in due course.

To proceed to a detailed proof we will split the s integration into three intervals: $0-m$, $m-M$ and $M-\infty$, where m and M are finite numbers, small (m) and large (M) enough for our purposes. It is obvious that, since possible singularities of the s-integral may arise only at $s=0$ or $s=\infty$, the integral between m and M is well defined and continuously dependent on ϵ, so for this part we can take the limit $\epsilon\to 0$ either before or after integration, obtaining the same result.

In the sequel we will consider the effect of the $\epsilon \to 0$ in the UV and in the IR, the only two regions where a singularity of the mentioned type can arise.

3.7.1.1 The ϵ-terms and the $\epsilon \to 0$ limit in the UV

Here we wish to check that the ϵ-terms do not affect the singularity in the UV, so that the subtraction in section 3.3 remains unaltered. It is enough to limit ourselves to the integral in the interval $(0,m)$, where m is a small enough number. Let us start from the term proportional to $\mathrm{g}(s)\left(\frac{\epsilon}{2u}\right)^3 e^{-\frac{\epsilon s}{2u}}$, coming from the first line of (3.85). To simplify the notation we will denote $\frac{\epsilon}{2u}$ simply by η. Since, near 0, $\mathrm{g}(s)\approx 1/\sqrt{s}$, this first term gives rise to the integral (for $s\approx 0$),

$$\begin{aligned}&\int_0^m ds s^{\frac{3}{2}}\eta^3 e^{-\eta s} \\ &\sim \left(-\eta\, e^{-\eta s}\frac{\sqrt{s}(3+2s\eta)}{2}+\sqrt{\eta}\frac{3\sqrt{\pi}Erf\left(\sqrt{\eta s}\right)}{4}\right)\Bigg|_0^m .\end{aligned}\tag{3.88}$$

Since the error function $Erf(x)\approx x$ for small x, it is evident that this expression vanishes both at $s=0$ and in the limit $\epsilon\to 0$.

The next term to be considered is $\eta^2\partial_s \mathrm{g}(s)$, which leads to the integral

$$\int_0^m ds s^{\frac{1}{2}}\eta^2 e^{-\eta s}\sim\left(-\eta\, e^{-\eta s}\sqrt{s}+\sqrt{\eta}\frac{\sqrt{\pi}Erf\left(\eta s\right)}{2}\right)\Bigg|_0^m \tag{3.89}$$

which again vanishes in the $\epsilon\to 0$ limit.

The following term leads to the integral

$$\int_0^m ds\frac{1}{\sqrt{s}}\,\eta\, e^{-\eta s}\sim\left(\sqrt{\pi\eta}Erf\left(\sqrt{\eta s}\right)\right)\Big|_0^m \tag{3.90}$$

which vanishes as well in the $\epsilon\to 0$ limit.

Finally the term linear in ϵ coming from the last two lines of (3.85) gives rise to an UV behaviour $\sim s^{-\frac{1}{2}}\eta$. Therefore the relevant UV integral is similar to (3.90) and we come to the same conclusion as above.

In conclusion the ϵ-terms do not affect the UV behaviour of the energy integral, and in the $\epsilon \to 0$ limit they yield evanescent contributions.

Finally let us consider what remains after discarding the ϵ-terms. From section 4 the behaviour for $\epsilon \approx 0$ is the following

$$\int_0^m ds\, \frac{e^{-\eta s}}{s^{\frac{3}{2}}} = \left(-2\frac{e^{-\eta s}}{\sqrt{s}} - 2\sqrt{\pi\eta}Erf\left(\sqrt{\eta s}\right)\right)\Big|_0^m. \tag{3.91}$$

In the limit $\epsilon \to 0$ the second term vanishes. The first term gives the expected UV singularity we have subtracted away in section 3.3.

3.7.1.2 The ϵ-terms and the $\epsilon \to 0$ limit in the IR

There is a chance, with ϵ-terms, that the corresponding integrals diverge or produce negative powers of ϵ, leading to finite or divergent contributions in the limit $\epsilon \to 0$.

Let us start again from the term proportional to $\mathrm{g}(s)\eta^3 e^{-\eta s}$, coming from the first line of (3.85). The integration in x, y gives a finite number. $\mathrm{g}(s)$ tends to a constant for $s \to \infty$. To appreciate qualitatively the problem we replace $\mathrm{g}(s)$ by a constant and integrate between M and infinity, M is chosen large enough so that $\mathrm{g}(s) = const$ is a good approximation. The integral is proportional to

$$\begin{aligned}&\int_M^\infty ds\, s^2\eta^3 e^{-\eta s} \sim -e^{-\eta s}\left(2 + 2\eta s + \eta^2 s^2\right)\Big|_M^\infty \\ &= e^{-\eta M}\left(2 + 2\eta M + \eta^2 M^2\right)\end{aligned} \tag{3.92}$$

which does not vanish in the limit $\epsilon \to 0$.

Let us notice that, if we consider an additional term in the asymptotic expression for $\mathrm{g}(s)$, say $\mathrm{g}(s) = a + \frac{b}{s} + \ldots$, the additional $\frac{1}{s}$ term contributes to the RHS of (3.92) an additional term $\sim \eta\, e^{-\eta M}(1 + \eta M)$, which vanishes in the $\epsilon \to 0$ limit. The more so for the next approximants. This is always the case in the following discussion, therefore considering the asymptotically dominant term will be enough for our purposes.

Let us consider next the term proportional to $\eta^2 \partial_s \mathrm{g}(s)$. For large s we have $\partial_s \mathrm{g}(s) \sim \frac{1}{s^2}$. Therefore the integral to be considered is

$$\int_M^\infty ds\eta^2 e^{-\eta s} \sim -e^{-\eta s}\eta\Big|_M^\infty = e^{-\frac{\epsilon M}{u}}\,\frac{\epsilon}{u} \tag{3.93}$$

which vanishes in the limit $\epsilon \to 0$.

The linear term in ϵ coming from the first line of (3.85), that is the term proportional to $\eta\frac{(\partial_s \mathrm{g}(s))^2}{\mathrm{g}(s)}$, leads to a contribution that can be qualitatively represented by the integral

$$\int_M^\infty ds\, \frac{1}{s^2}\eta e^{-\eta s} \sim -\eta\left(\frac{e^{-\eta s}}{s} - \eta Ei\left(-\eta s\right)\right)\Big|_M^\infty \\ = \eta e^{-\eta M} - \eta^2 Ei\left(-\eta M\right) \tag{3.94}$$

which vanishes in the limit $\epsilon \to 0$, because the exponential integral function $Ei(-x)$ behaves like $\log x$ for small x.

Finally let us consider the term linear in ϵ coming from the last two lines of (3.85), i.e.

$$\mathrm{g}(s)\eta\left(G_s^2(2\pi x) + G_s^2(2\pi(x-y)) + G_s^2(2\pi y)\right). \tag{3.95}$$

The integration of x, y of the G_s^2 terms in brackets gives a contribution behaving at infinity as $1/s^2$ (see section 5). Therefore the relevant s contribution for large s is

$$\int_M^\infty ds\eta e^{-\eta s} \sim -e^{-\eta s}\Big|_M^\infty = e^{-\eta M}$$

which is nonvanishing in the limit $\epsilon \to 0$.

Therefore we have found two nontrivial ϵ–terms, the first and the last ones above. Let us call them α and β, respectively. They do not vanish in the limit $\epsilon \to 0$, thus they may survive this limit and represent a finite difference between taking $\epsilon \to 0$ before and after the s-integration. It is therefore of utmost importance to see whether the overall contributions of these two terms survives. This turns out to be the case.

From section 3.4 one can check that the precise form of the first term in question is

$$-\frac{1}{4\pi^2}\lim_{\epsilon\to 0}\int_0^\infty ds\, s^2 \mathrm{g}(s)\eta^3 e^{-\eta s}. \tag{3.96}$$

If one knows the asymptotic expansion of the integrand for large s, it is very easy to extract the exact $\epsilon \to 0$ result of the integral. The asymptotic expansion of $\mathrm{g}(s)$ is $\mathrm{g}(s) \approx 1 + \frac{1}{24s} + \frac{1}{1152s^2} + \ldots$. Integrating term by term from M to ∞, the dominant one gives

$$-\frac{1}{4\pi^2}e^{-\eta M}(2 + 2M\eta + M^2\eta^2) \tag{3.97}$$

which, in the $\epsilon \to 0$ limit, yields $-\frac{1}{2\pi^2}$. The next term gives $\sim e^{-M\eta}(\eta(1 + M\eta)$, which vanishes in the $\epsilon \to 0$ limit, and so on. So the net result of the integral (3.140) in the

$\epsilon \to 0$ limit is $-\alpha$, where

$$\alpha \equiv \frac{1}{2\pi^2}. \tag{3.98}$$

For the β term (the one corresponding to (3.95)) we have

$$-\beta = \frac{1}{12}\lim_{\epsilon\to 0}\int_0^\infty ds\, s^2 \mathrm{g}(s)\eta e^{-\eta s}\left(-\frac{a}{s^2}+\cdots\right), \tag{3.99}$$

where ellipses denote terms that contribute vanishing contributions in the $\epsilon \to 0$ limit and a is the (overall) coefficient of the inverse quadratic term in (A.7). The problem is to compute the latter. With reference to the enumeration in section 3.5.2, the term 1) has the asymptotic expansion

$$\sim e^{-\eta s}\left(-\frac{3\eta}{8\pi^2} - \frac{\eta}{64 s\pi^2} - \cdots\right). \tag{3.100}$$

Integrating from M to ∞ and taking the $\epsilon \to 0$ limit, this gives $-\frac{3}{8\pi^2}$. Proceeding in the same way, term 2) of section 3.5.2 gives $\frac{3}{4\pi^2}$ and term 3) yields $\frac{1}{4\pi^2}$. So altogether we have $\frac{5}{8\pi^2}$ for the three terms contributing to (3.149) considered so far.
It remains term 4) of section 3.5.2. This corresponds to the contribution of $RK(p,s) - \frac{1}{8\pi(p^2-1)}$. One must explicitly sum over p in order to know the asymptotic expansion in s. This has not been possible so far analytically. However Mathematica can compute the coefficient of $1/s^2$ in the asymptotic expansion for large s to a remarkable accuracy. The coefficient turns out to be -0.064317, with an uncertainty only at the fifth digit. Within the same uncertainty this corresponds to the analytic value $-\frac{1}{4\pi}(\gamma + 1/3\log 2)$. We therefore set

$$\beta = -\frac{5}{8\pi^2} + \frac{1}{\pi^2}\left(\gamma + \frac{1}{3}\log 2\right). \tag{3.101}$$

So the overall contribution of the ϵ-terms in the $\epsilon \to 0$ limit is

$$-\alpha - \beta = -\frac{1}{2\pi^2} + \frac{5}{8\pi^2} - \frac{1}{\pi^2}\left(\gamma + \frac{1}{3}\log 2\right) \approx -0.0692292, \tag{3.102}$$

which is accurate up to the fourth digit.

Let us consider now what remains apart from the ϵ-terms. The integrand takes the form

$$\int_0^\infty ds\, F(s)\, e^{-\eta s}, \tag{3.103}$$

where $F(s)$ represents the integrand when $\epsilon = 0$, i.e. the total integrand analyzed in section 3.4. We have already argued that the integration over s and the limit $\epsilon \to 0$ can be safely exchanged, which yields the already found value of 0.0693926.

Concluding we have

$$E_0^{(s)}[\psi_u] = \lim_{\epsilon\to 0} E_\epsilon^{(s)}[\psi_u^\epsilon] \approx 0.000163, \tag{3.104}$$

where again the superscript $^{(s)}$ means that the UV singularity has been subtracted away. The value we have found is consistent with 0 and is what one should expect for the tachyon vacuum obtained from the condensation of all open string dofs Since at this point we can assume the true value of $E_\epsilon^{(s)}[\psi_u]$ to be 0, and since the value (3.102) is much more accurate than (3.70), we can take for the latter the more reliable value

$$E^{(s)}[\psi_u] \approx 0.0692292 \tag{3.105}$$

which we can consider at this point to be exact (even though this will not play any role in the determination of the lump energy). It differs from (3.70) by 2 per mil. Therefore, after all, our numerical evaluation in section 6 was not so bad. Stated differently, the whole procedure of this section is nothing but a more reliable way to compute the energy functional (3.34).

We have already remarked that (3.105) differs from the theoretical value (3.71) of the lump energy by 27%. This is not the expected lump energy, but the mismatch was traced back to the arbitrariness implicit in the subtraction scheme we adopted. Now, thanks to ψ_u^ϵ, we can define a lump solution whose energy is free from such a problem.

3.8 The lump and its energy

In the previous sections we have found various solutions to the equation of motion $Q\psi + \psi\psi = 0$ at the perturbative vacuum. One is ψ_u with UV–subtracted energy (3.105), the others are the ψ_u^ϵ's with vanishing UV–subtracted energy. Using these we can construct a solution to the EOM *at the tachyon condensation vacuum.*

The equation of motion at the tachyon vacuum is

$$\mathcal{Q}\Phi + \Phi\Phi = 0, \quad \text{where} \quad \mathcal{Q}\Phi = Q\Phi + \psi_u^\epsilon\Phi + \Phi\psi_u^\epsilon. \tag{3.106}$$

We can easily show that

$$\Phi_0 = \psi_u - \psi_u^\epsilon \tag{3.107}$$

is a solution to (3.132). The action at the tachyon vacuum is

$$-\frac{1}{2}\langle \mathcal{Q}\Phi, \Phi\rangle - \frac{1}{3}\langle \Phi, \Phi\Phi\rangle. \tag{3.108}$$

Thus the energy is

$$E[\Phi_0] = -\frac{1}{6}\langle \Phi_0, \Phi_0\Phi_0\rangle = -\frac{1}{6}\big[\langle \psi_u, \psi_u\psi_u\rangle - \langle \psi_u^\epsilon, \psi_u^\epsilon\psi_u^\epsilon\rangle - 3\langle \psi_u^\epsilon, \psi_u\psi_u\rangle + 3\langle \psi_u, \psi_u^\epsilon\psi_u^\epsilon\rangle\big]. \tag{3.109}$$

Eq.(3.133) is the lump solution at the tachyon vacuum, therefore, this energy must be the energy of the lump.

We have already shown that $-\frac{1}{6}\langle \psi_u, \psi_u\psi_u\rangle^{(s)} = \alpha + \beta$ and that $\langle \psi_u^\epsilon, \psi_u^\epsilon\psi_u^\epsilon\rangle^{(s)} = 0$, after subtracting the UV singularity. It remains for us to compute the two remaining terms, which we will do in the next subsection. But, before, let us remark one important aspect of (4.10). The UV subtractions are the same in all terms, therefore they neatly cancel out. The two terms $\langle \psi_u^\epsilon, \psi_u\psi_u\rangle$ and $\langle \psi_u, \psi_u^\epsilon\psi_u^\epsilon\rangle$ can be calculated in the same way as the other two, and we limit ourselves to writing down the final result:

$$\begin{aligned}\langle \psi_u^\epsilon, \psi_u\psi_u\rangle \;=\; & -\int_0^\infty ds\, s^2 \int_0^1 dy \int_0^y dx\, e^{-\eta s}\mathcal{E}(1-y,x)\, e^{\eta s y}\, \mathrm{g}(s) \\ & \cdot\Big\{\Big(\eta - \frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\Big)\Big(-\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\Big)^2 + G_s(2\pi x)G_s(2\pi(x-y))G_s(2\pi y) \\ & + \frac{1}{2}\Big(\eta - \frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\Big)G_s^2(2\pi x) + \frac{1}{2}\Big(-\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\Big)\Big(G_s^2(2\pi y) + G_s^2(2\pi(x-y))\Big)\Big\}.\end{aligned} \tag{3.110}$$

and

$$\begin{aligned}\langle \psi_u, \psi_u^\epsilon\psi_u^\epsilon\rangle \;=\; & -\int_0^\infty ds\, s^2 \int_0^1 dy \int_0^y dx\, e^{-\eta s}\mathcal{E}(1-y,x)\, e^{\eta s x}\, \mathrm{g}(s) \\ & \cdot\Big\{\Big(\eta - \frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\Big)^2\Big(-\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\Big) + G_s(2\pi x)G_s(2\pi(x-y))G_s(2\pi y) \\ & + \frac{1}{2}\Big(-\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\Big)G_s^2(2\pi(x-y)) + \frac{1}{2}\Big(\eta - \frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\Big)\Big(G_s^2(2\pi x) + G_s^2(2\pi y)\Big)\Big\}.\end{aligned} \tag{3.111}$$

In the limit $\epsilon \to 0$ the factors $e^{\eta s y}$ and $e^{\eta s x}$, present in (3.110,3.111, respectively), are irrelevant. In fact the integration over y, without this factor, is finite. Therefore we know from above that the integration over y with $e^{\eta s y}$ inserted back at its place is continuous in ϵ for $\epsilon \to 0$ (one can check that the subsequent integration over s does not lead to any complications). Therefore we can ignore these factors in the two integrals above.

The integrals are of the same type as those analyzed in the previous section. Of course they will have both the contribution that comes from setting $\epsilon = 0$, which is proportional to $\alpha+\beta$, like for $\langle\psi_u^\epsilon, \psi_u^\epsilon\psi_u^\epsilon\rangle$. But there are important differences as far as the ϵ terms are concerned. First of all we remark that, in both integrals, the first term in curly brackets does not contain the cubic term in ϵ. Therefore, according to the analysis in the previous section, the α contribution will not be present in either term. On the contrary the β contribution, which comes from the last line in both, will. To evaluate it there is no need of new explicit computations. spon integrating over x, y one can easily realize that the three terms proportional to $G_s^2(2\pi x)$, $G_s^2(2\pi y)$ and $G_s^2(2\pi(x-y))$, give rise to the same contribution. So, when we come to ϵ–terms, each of them will contribute $\frac{1}{3}$ of the β contribution already calculated in the previous section. Summarizing: after subtracting the UV singularity, we will have

$$\frac{1}{6}\langle\psi_u^\epsilon, \psi_u\psi_u\rangle^{(s)} = -\alpha-\beta+\frac{1}{3}\beta, \tag{3.112}$$
$$\frac{1}{6}\langle\psi_u, \psi_u^\epsilon\psi_u^\epsilon\rangle^{(s)} = -\alpha-\beta+\frac{2}{3}\beta. \tag{3.113}$$

Putting together all the results in (4.10), the lump energy above the tachyon vacuum is

$$E[\Phi_0] = \alpha+\beta+0+3(-\alpha-\beta+\frac{1}{3}\beta)-3(-\alpha-\beta+\frac{2}{3}\beta) = \alpha = \frac{1}{2\pi^2}. \tag{3.114}$$

This coincides with the expected theoretical value (3.71). As one can see there is no need to know the value of β, for which, unlike α, a straightforward analytical evaluation is missing, and so this result can be considered an analytical one, as it is only determined by the asymptotics of g(s). Of course the result (3.114) is based on the assumption we made in the last section (before eq.(3.105)) that $E_\epsilon^{(s)}[\psi_u^\epsilon] \equiv 0$. This was proved in part with numerical methods, but its validity is imposed by consistency. In fact, the lump energy comes from the asymptotic region in s and this is the region which is precisely suppressed by the $e^{-\eta s}$ factor produced by the ϵ-regularization. It is therefore not surprising that, modulo the UV subtraction, ψ_u^ϵ represents a tachyon condensation vacuum solution.

In our view, what makes our result a consistent check of Sen's conjecture for lower dimensional lumps is the fact that UV subtractions of the various terms in (4.10) exactly cancel out.

3.9 A D23-brane solution

To prove the consistency of our understanding the lump energy should be computed fro the lump solution on the tachyon condensation vacuum to extract physical predictions from it, we consider the straightforward generalization to lower dimensional lumps.

In the case of a $D23$–brane solution, as suggested in [52], the relevant operator is given by

$$\phi_{(u_1,u_2)} = u_1(: X_1^2 : +2\log u_1 + 2A) + u_2(: X_2^2 : +2\log u_2 + 2A) \tag{3.115}$$

where X_1 and X_2 are two coordinate fields corresponding to two different space directions. There is no interaction term between X_1 and X_2 in the 2D action.

Then we require for ϕ_u the following properties under the coordinate rescaling $f_t(z) = \frac{z}{t}$

$$f_t \circ \phi_{(u_1,u_2)}(z) = \frac{1}{t}\,\phi_{(tu_1,tu_2)}\left(\frac{z}{t}\right). \tag{3.116}$$

The partition function corresponding to the operator (3.115) is factorized, [60, 61]:

$$g(u_1,u_2) = g(u_1)g(u_2) \qquad g(u_i) = \frac{1}{\sqrt{2\pi}}\sqrt{2u_i}\Gamma(2u_i)e^{2u_i(1-\log 2u_i)} \tag{3.117}$$

where in (3.117) we have already made the choice $A = \gamma - 1 + \log 4\pi$. This choice implies

$$\lim_{u_1,u_2\to\infty} g(u_1,u_2) = 1 \tag{3.118}$$

With these properties all the non-triviality requirements of [52, 53] for the solution $\psi_{(u_1,u_2)} \equiv \psi_{\phi_{(u_1,u_2)}}$ are satisfied. One can easily work out the energy

$$E[\psi_{(u_1,u_2)}] = -\frac{1}{6}\langle\psi_{(u_1,u_2)}\psi_{(u_1,u_2)}\psi_{(u_1,u_2)}\rangle \tag{3.119}$$

with obvious modifications with respect to the D24 case. So, for example,

$$\langle X_1^2(\theta)X_2^2(\theta')\rangle_{Disk} = \langle X_1^2(\theta)\rangle_{Disk}\langle X_2^2(\theta')\rangle_{Disk} = Z(u_1)h_{u_1}Z(u_2)h_{u_2} \tag{3.120}$$

and so on. One thus finds

$$\begin{aligned} E[\psi_{(u_1,u_2)}] \;=\; & \frac{1}{6}\int_0^\infty ds\; s^2 \int_0^1 dy \int_0^y dx\, \frac{4}{\pi}\sin\pi x \sin\pi y \sin\pi(x-y)\mathrm{g}(s,vs) \\ & \cdot\Bigg\{ G_s(2\pi x)G_s(2\pi(x-y))G_s(2\pi y) \\ & + v^3\, G_{vs}(2\pi x)G_{vs}(2\pi(x-y))G_{vs}(2\pi y) \\ & -\frac{1}{2}\Big(\frac{\partial_s \mathrm{g}(s,vs)}{\mathrm{g}(s,vs)}\Big)\Big(G_s^2(2\pi x) + G_s^2(2\pi(x-y)) + G_s^2(2\pi y)\Big) \\ & -\frac{1}{2}\Big(v^2\frac{\partial_s \mathrm{g}(s,vs)}{\mathrm{g}(s,vs)}\Big)\Big(G_{vs}^2(2\pi x) + G_{vs}^2(2\pi(x-y)) + G_{vs}^2(2\pi y)\Big) \\ & +\Big(-\frac{\partial_s \mathrm{g}(s,vs)}{\mathrm{g}(s,vs)}\Big)^3\Bigg\} \end{aligned} \tag{3.121}$$

where $s = 2u_1T, v = \frac{u_2}{u_1}$ and, by definition, $\mathrm{g}(s, vs) \equiv g(s/2, vs/2) = g(u_1T, u_2T)$. The derivative ∂_s in $\partial_s \mathrm{g}(s, vs)$ acts on both entries. We see that, contrary to [52], where the u dependence was completely absorbed within the integration variable, in (3.121) there is an explicit dependence on v. This suggests that when the space of boundary perturbations is parameterized by several couplings u_i, their interpretation as gauge parameters of the corresponding solutions in OSFT is much less obvious.

3.9.1 The IR and UV behaviour

First of all we have to find out whether $E[\psi_{(u_1,u_2)}]$ is finite and whether it depends on v.

As the structure of the x, y dependence is the same as for the D24 solution, we can nicely use the results already found , with exactly the same IR ($s \to \infty$) and UV ($s \approx 0$) behaviour, The differences come from the various factors containing g or derivatives thereof. The relevant IR asymptotic behaviour is

$$g(s, vs) \approx 1 + \frac{1+v}{24v}\frac{1}{s} \tag{3.122}$$

for large s (v is kept fixed to some positive value). The asymptotic behaviour does not change with respect to the D24-brane case (except perhaps for the overall dominant asymptotic coefficient, which is immaterial as far as integrability is concerned), so we can conclude that the integral in (3.121) is convergent for large s, where the overall integrand behaves asymptotically as $1/s^2$.

Let us come next to the UV behaviour ($s \approx 0$). To start with let us consider the term not containing G_s. We have

$$\begin{aligned}\frac{1}{4\pi^2} s^2 g(s, vs) \left(\frac{\partial_s g(s, vs)}{g(s, vs)}\right)^3 &= -\frac{1}{16\,(\pi^3\sqrt{v})\,s^2} \\ -\frac{1}{8\pi^3\sqrt{vs}}\big((1+v)(1+2\gamma) &+ 2\log 2 + 2(1+v)\log s + 2v\log(2v)\big) + \mathcal{O}((\log s)^2)\end{aligned} \tag{3.123}$$

The double pole in zero is to be expected. Once we integrate over s we obtain a behaviour $\sim \frac{1}{s}$ near $s = 0$. This singularity corresponds to $\sim \delta(0)^2 \sim V^2$, which can be interpreted as the D25 brane energy density multiplied by the square of the (one-dimensional) volume). In order to extract a finite quantity from the integral (3.121) we have to subtract this singularity. One can find that the function to be subtracted to the LHS of

(3.123) is

$$
\begin{aligned}
h_1(v,s) \;=\;& \Big(-\frac{1}{16\,(\pi^3\sqrt{v})\,s^2} + \frac{1}{16\pi^3\sqrt{v}s} \\
& -\frac{1}{8\pi^3\sqrt{v}s}\left((1+v)(1+2\gamma) + 2\log 2 + 2(1+v)\log s + 2v\log(2v)\right)\Big) \\
& \cdot \frac{e^{\frac{s}{s^2-1}}\left(1+2s-2s^2+2s^3+s^4\right)}{\left(-1+s^2\right)^2} \qquad (3.124)
\end{aligned}
$$

in the interval $0 \le s \le 1$ and 0 elsewhere. It is important to remark that both the singularity and the subtraction are v-dependent.

As for the quadratic terms in G_s and G_{vs} the overall UV singularity is

$$
-\frac{3}{16\,(\pi^3\sqrt{v})\,s^2} - \frac{3(1+v)}{8\,(\pi^3\sqrt{v})\,s} + \mathcal{O}((\log s)^2) \qquad (3.125)
$$

and the corresponding function to be subtracted from the overall integrand is

$$
h_2(v,s) = -\frac{3e^{\frac{s}{s^2-1}}\left(1+2s-2s^2+2s^3+s^4\right)(1+s+2sv)}{16\pi^3 s^2\left(s^2-1\right)^2\sqrt{v}} \qquad (3.126)
$$

in the interval $0 \le s \le 1$ and 0 elsewhere. Also in this case the subtraction is v dependent.

Finally let us come to the cubic term in G_s and G_{vs}. Altogether the UV singularity due to the cubic terms is

$$
-\frac{1}{8\,(\pi^3\sqrt{v})\,s^2} + \frac{(\gamma+\log s)(1+v) + \log 2 + v\log(2v)}{4\pi^3\sqrt{v}s} + \mathcal{O}((\log s)^2) \qquad (3.127)
$$

The overall function we have to subtract from the corresponding integrand is

$$
\begin{aligned}
h_3(v,s) \;=\;& \frac{2}{16\pi^3\sqrt{v}s}\,\frac{e^{\frac{s}{s^2-1}}\left(1+2s-2s^2+2s^3+s^4\right)}{\left(s^2-1\right)^2} \\
& \cdot\left(-1+s+2s(1+v)(\gamma+\log s) + s\log 4 + 2sv\log(2v)\right) \qquad (3.128)
\end{aligned}
$$

for $0 \le s \le 1$ and 0 elsewhere. Also in this case the subtraction is v dependent.

Similarly to the $D25$-brane case, the result of all these subtractions does not depend on the particular functions h_1, h_2, h_3 we have used, provided the latter satisfy a few very general criteria.

After all these subtractions the integral in (3.121) is finite, but presumably v dependent. This is confirmed by a numerical analysis. For instance, for $v = 1$ and 2 we get $E^{(s)}[\psi_{(u_1,u_2)}] = 0.0892632$ and 0.126457, respectively, where the superscript ${}^{(s)}$ means

UV subtracted. It is clear that this cannot represent a physical energy. This confirms the point of view no definite physical meaning can be attached to quantities affected by UV singularities. In fact for the energy to be a gauge invariant quantity, one needs to perform an integration by parts, which is in general not allowed to perform in singular amplitudes. The way out is, like in the D24 lump solution, to compare the (subtracted) energy of $\psi_{(u_1,u_2)}$ with the (subtracted) energy of a solution representing the tachyon condensation vacuum, and show that the result is independent of the subtraction scheme.

3.10 The ϵ-regularization

We can introduce the ϵ-regularized solution corresponding to (3.115) in the usual way as

$$\psi_\phi = c(\phi+\epsilon) - \frac{1}{K+\phi+\epsilon}(\phi+\epsilon-\delta\phi)Bc\partial c \tag{3.129}$$

where ϵ is an arbitrary small number. In the present case

$$\phi \equiv \phi_{(u_1,u_2)} = u_1(:X_1^2: + 2\log u_1 + 2A) + u_2(:X_2^2: + 2\log u_2 + 2A) \tag{3.130}$$

After some manipulations the corresponding energy can be written as

$$\begin{aligned} E[\psi^\epsilon_{(u_1,u_2)}] \;=\; & \frac{1}{6}\lim_{\epsilon\to 0}\int_0^\infty ds\, s^2 \int_0^1 dy \int_0^y dx\, \mathcal{E}(1-y,x)\, \mathrm{g}(s,vs)\, e^{-\eta s} \\ & \cdot\Bigg\{ G_s(2\pi x)G_s(2\pi(x-y))G_s(2\pi y) \\ & + v^3\, G_{vs}(2\pi x)G_{vs}(2\pi(x-y))G_{vs}(2\pi y) \\ & + \frac{1}{2}\Big(\eta - \frac{\partial_s \mathrm{g}(s,vs)}{\mathrm{g}(s,vs)}\Big)\Big(G_s^2(2\pi x) + G_s^2(2\pi(x-y)) + G_s^2(2\pi y)\Big) \\ & + \frac{1}{2}v^2\Big(\eta - \frac{\partial_s \mathrm{g}(s,vs)}{\mathrm{g}(s,vs)}\Big)\Big(G_{vs}^2(2\pi x) + G_{vs}^2(2\pi(x-y)) + G_{vs}^2(2\pi y)\Big) \\ & + \Big(\eta - \frac{\partial_s \mathrm{g}(s,vs)}{\mathrm{g}(s,vs)}\Big)^3 \Bigg\} \end{aligned} \tag{3.131}$$

where $\mathcal{E}(1-y,x) = \frac{4}{\pi}\sin\pi x \sin\pi y \sin\pi(x-y)$ and $\eta = \frac{\epsilon}{2u_1}$.The integrand in (3.131) has the same leading singularity in the UV as the integrand of (3.121). The subleading singularity on the other hand may depend on ϵ. Thus it must undergo an UV subtraction that generically depends on ϵ. We will denote the corresponding subtracted integral by $E^{(s)}[\psi^\epsilon_{(u_1,u_2)}]$. The important remark here is, however, that in the limit $\epsilon \to 0$ both (3.131) and (3.121) undergo the same subtraction. This is an important clue about why the limit $\epsilon \to 0$ is the one to consider in defining lump solutions over the tachyon condensation vacuum. In fact only in this limit we can expect the exact cancellation of UV singularities which is a crucial feature of physical quantities.

The factor of $e^{-\eta s}$ appearing in the integrand of (3.131) changes completely its IR structure. It is in fact responsible for cutting out the contribution at infinity that characterizes (3.121) and (modulo the arbitrariness in the UV subtraction) makes up the energy of the D23 brane.

In keeping with what observed about solution (4.8), we interpret $\psi^{\epsilon}_{(u_1,u_2)}$ as a tachyon condensation vacuum solution and $E^{(s)}[\psi^{\epsilon}_{(u_1,u_2)}]$ the energy of such vacuum. Contrary to our naive expectation, this energy does not vanish and its value is (v,ϵ)-dependent. The reason it does not vanish is that the subtraction itself is (v,ϵ)-dependent and this is due to the arbitrariness of the subtraction scheme. However we can always fix $E^{(s)}[\psi^{\epsilon}_{u_1,u_2}]$ to zero by subtracting a suitable constant. Of course we have to subtract the same constant from $E^{(s)}[\psi_{u_1,u_2}]$.

3.11 The energy of the $D23$–brane

The problem of finding the right energy of the D23 brane consists in constructing a solution over the vacuum represented by $\psi^{\epsilon}_{(u_1,u_2)}$ (the tachyon condensation vacuum). The equation of motion at such vacuum is

$$\mathcal{Q}\Phi + \Phi\Phi = 0, \quad \text{where} \quad \mathcal{Q}\Phi = Q\Phi + \psi^{\epsilon}_{(u_1,u_2)}\Phi + \Phi\psi^{\epsilon}_{(u_1,u_2)} \tag{3.132}$$

One can easily show that

$$\Phi_0 = \psi_{(u_1,u_2)} - \psi^{\epsilon}_{(u_1,u_2)} \tag{3.133}$$

is a solution to (3.132). The action at the tachyon vacuum is

$$-\frac{1}{2}\langle\mathcal{Q}\Phi_0,\Phi_0\rangle - \frac{1}{3}\langle\Phi_0,\Phi_0\Phi_0\rangle. \tag{3.134}$$

Thus the energy is

$$\begin{aligned} E[\Phi_0] \;=\; & -\frac{1}{6}\langle\Phi_0,\Phi_0\Phi_0\rangle = -\frac{1}{6}\big[\langle\psi_{(u_1,u_2)},\psi_{(u_1,u_2)}\psi_{(u_1,u_2)}\rangle - \langle\psi^{\epsilon}_{(u_1,u_2)},\psi^{\epsilon}_{(u_1,u_2)}\psi^{\epsilon}_{(u_1,u_2)}\rangle \\ & -3\langle\psi^{\epsilon}_{(u_1,u_2)},\psi_{(u_1,u_2)}\psi_{(u_1,u_2)}\rangle + 3\langle\psi_{(u_1,u_2)},\psi^{\epsilon}_{(u_1,u_2)}\psi^{\epsilon}_{(u_1,u_2)}\rangle\big]. \end{aligned} \tag{3.135}$$

Eq.(3.133) is the lump solution at the tachyon vacuum, therefore this energy must be the energy of the lump.

The two additional terms $\langle\psi^{\epsilon}_{(u_1,u_2)},\psi_{(u_1,u_2)}\psi_{(u_1,u_2)}\rangle$ and $\langle\psi_{(u_1,u_2)},\psi^{\epsilon}_{(u_1,u_2)}\psi^{\epsilon}_{(u_1,u_2)}\rangle$ are given by

$$\begin{aligned}\langle\psi^{\epsilon}_{(u_1,u_2)},\psi_{(u_1,u_2)}\psi_{(u_1,u_2)}\rangle = &-\lim_{\epsilon\to 0}\int_0^\infty ds\; s^2\int_0^1 dy\int_0^y dx\, e^{-\eta s}\mathcal{E}(1-y,x)\, e^{\eta s y}\,\mathrm{g}(s,vs)\\ &\cdot\Bigg\{\Big(\eta-\frac{\partial_s\mathrm{g}(s,vs)}{\mathrm{g}(s,vs)}\Big)\Big(-\frac{\partial_s\mathrm{g}(s,vs)}{\mathrm{g}(s,vs)}\Big)^2\\ &+G_s(2\pi x)G_s(2\pi(x-y))G_s(2\pi y)+v^3\,G_{vs}(2\pi x)G_{vs}(2\pi(x-y))G_{vs}(2\pi y)\\ &+\frac{1}{2}\Big(\eta-\frac{\partial_s\mathrm{g}(s,vs)}{\mathrm{g}(s,vs)}\Big)\Big(G_s^2(2\pi(x))+v^2G_{vs}^2(2\pi(x))\Big)\\ &+\frac{1}{2}\Big(-\frac{\partial_s\mathrm{g}(s,vs)}{\mathrm{g}(s,vs)}\Big)\Big(G_s^2(2\pi y)+G_s^2(2\pi(x-y))+v^2\Big(G_{vs}^2(2\pi y)+G_{vs}^2(2\pi(x-y))\Big)\Big)\Bigg\}.\end{aligned}\tag{3.136}$$

and

$$\begin{aligned}\langle\psi_{(u_1,u_2)},\psi^{\epsilon}_{(u_1,u_2)}\psi^{\epsilon}_{(u_1,u_2)}\rangle = &-\lim_{\epsilon\to 0}\int_0^\infty ds\; s^2\int_0^1 dy\int_0^y dx\, e^{-\eta s}\mathcal{E}(1-y,x)\, e^{\eta s x}\,\mathrm{g}(s,vs)\\ &\cdot\Bigg\{\Big(\eta-\frac{\partial_s\mathrm{g}(s,vs)}{\mathrm{g}(s,vs)}\Big)^2\Big(-\frac{\partial_s\mathrm{g}(s,vs)}{\mathrm{g}(s,vs)}\Big)\\ &+G_s(2\pi x)G_s(2\pi(x-y))G_s(2\pi y)+v^3\,G_{vs}(2\pi x)G_{vs}(2\pi(x-y))G_{vs}(2\pi y)\\ &+\frac{1}{2}\Big(\eta-\frac{\partial_s\mathrm{g}(s,vs)}{\mathrm{g}(s,vs)}\Big)\Big(G_s^2(2\pi x)+G_s^2(2\pi y)+v^2\Big(G_{vs}^2(2\pi x)+G_{vs}^2(2\pi y)\Big)\Big)\\ &+\frac{1}{2}\Big(-\frac{\partial_s\mathrm{g}(s,vs)}{\mathrm{g}(s,vs)}\Big)\Big(G_s^2(2\pi(x-y))+v^2G_{vs}^2(2\pi(x-y))\Big)\Bigg\}.\end{aligned}\tag{3.137}$$

Now we insert in (3.135) the quantities we have just computed together with (3.121) and (3.131). We have of course to subtract their UV singularities. As we have already remarked above, such subtractions are the same for all terms in (3.135) in the limit $\epsilon\to 0$, therefore they cancel out. So the result we obtain from (3.135) is subtraction-independent and we expect it to be the physical result.

In fact the expression we obtain after the insertion of (3.121,3.131,3.136) and (3.137) in (3.135) looks very complicated. But it simplifies drastically in the limit $\epsilon\to 0$. As was noticed in [53, 57], in this limit we can drop the factors $e^{\eta s x}$ and $e^{\eta s y}$ in (3.136) and (3.137) because of continuity*. What we cannot drop *a priori* is the factor $e^{-\eta s}$.

Next it is convenient to introduce $\widetilde{\mathrm{g}}(s,vs)=e^{-\eta s}\mathrm{g}(s,vs)$ and notice that

$$\eta-\frac{\partial_s\mathrm{g}(s,vs)}{\mathrm{g}(s,vs)}=-\frac{\partial_s\widetilde{\mathrm{g}}(s,vs)}{\widetilde{\mathrm{g}}(s,vs)}\tag{3.138}$$

*It is useful to recall that the limit $\epsilon\to 0$ can be taken safely inside the integration only if the integral without the factor $e^{\eta s x}$ or $e^{\eta s y}$ is convergent. This is true for the x and y integration, but it is not the case for instance for the integral (3.140) below.

Another useful simplification comes from the fact that (without the $e^{\eta sx}$ or $e^{\eta sy}$ factors) upon integrating over x, y the three terms proportional to $G_s^2(2\pi x)$, $G_s^2(2\pi y)$ and $G_s^2(2\pi(x-y))$, respectively, give rise to the same contribution. With this in mind one can easily realize that most of the terms cancel and what remains is

$$\begin{aligned}
E[\Phi_0] &= \frac{1}{6}\lim_{\epsilon\to 0}\int_0^\infty ds\, s^2 \int_0^1 dy \int_0^y dx\, \mathcal{E}(1-y,x)\Bigg\{ \mathrm{g}(s,vs)\,(1-e^{-\eta s}) \\
&\cdot\Bigg[G_s(2\pi x)G_s(2\pi(x-y))G_s(2\pi y) + v^3\, G_{vs}(2\pi x)G_{vs}(2\pi(x-y))G_{vs}(2\pi y) \\
&+\frac{1}{2}\Big(-\frac{\partial_s \mathrm{g}(s,vs)}{\mathrm{g}(s,vs)}\Big)\Big(G_s^2(2\pi x)+G_s^2(2\pi(x-y))+G_s^2(2\pi y)\Big) \\
&+\frac{1}{2}v^2\Big(\eta-\frac{\partial_s \mathrm{g}(s,vs)}{\mathrm{g}(s,vs)}\Big)\Big(G_{vs}^2(2\pi x)+G_{vs}^2(2\pi(x-y))+G_{vs}^2(2\pi y)\Big) \\
&+\Big(-\frac{\partial_s \mathrm{g}(s,vs)}{\mathrm{g}(s,vs)}\Big)^3\Bigg] \\
&+\widetilde{\mathrm{g}}(s,vs)\Big(\frac{\partial_s\widetilde{\mathrm{g}}(s,vs)}{\widetilde{\mathrm{g}}(s,vs)}-\frac{\partial_s \mathrm{g}(s,vs)}{\mathrm{g}(s,vs)}\Big)^3\Bigg\}
\end{aligned} \tag{3.139}$$

The term proportional to $1-e^{-\eta s}$ vanishes in the limit $\epsilon\to 0$ because the integral without this factor is finite (after UV subtraction). Therefore we are left with

$$\begin{aligned}
E[\Phi_0] &= \frac{1}{6}\lim_{\epsilon\to 0}\int_0^\infty ds\, s^2 \int_0^1 dy \int_0^y dx\, \mathcal{E}(1-y,x)\,\mathrm{g}(s,vs)\,e^{-\eta s}\,\eta^3 \\
&= \frac{1}{4\pi^2}\lim_{\epsilon\to 0}\int_0^\infty ds\, s^2 \mathrm{g}(s,vs)\eta^3 e^{-\eta s}.
\end{aligned} \tag{3.140}$$

where $\mathrm{g}(s,vs)=\mathrm{g}(s)\mathrm{g}(vs)$. This quantity is the analog of the coefficient α that determines the energy of the D24 brane solution. This contribution comes from the ϵ^3 term in the last line of (3.139). If one knows the asymptotic expansion of the integrand for large s, it is very easy to extract the exact $\epsilon\to 0$ result of the integral. We recall that the UV singularity has been subtracted away, therefore the only nonvanishing contribution to the integral (3.140) may come from $s\to\infty$. In fact splitting the s integration as $0\le s\le M$ and $M\le s<\infty$, where M is a very large number, it is easy to see the the integration in the first interval vanishes in the limit $\epsilon\to 0$. As for the second integral we have to use the asymptotic expansion of $\mathrm{g}(s,vs)$: $\mathrm{g}(s,vs)\approx 1+\frac{1+v}{12v}\frac{1}{s}+\dots$. Integrating term by term from M to ∞, the dominant one gives

$$\frac{1}{4\pi^2}e^{-\eta M}(2+2M\eta+M^2\eta^2) \tag{3.141}$$

which, in the $\epsilon \to 0$ limit, yields $\frac{1}{2\pi^2}$. The other terms are irrelevant in the $\epsilon \to 0$ limit. Therefore we have

$$E[\Phi_0] = \frac{1}{2\pi^2}. \tag{3.142}$$

We recall that 1 in the numerator on the RHS is to be identified with $\lim_{s\to\infty} g(s, vs)$.

We conclude that

$$T_{23} = \frac{1}{2\pi^2} \tag{3.143}$$

This is the same as T_{24}, so it may at first be surprising. But in fact it is correct because of the normalization discussed in App. C of [52]. Compare with eqs.(C.1) and (C.7) there: when we move from a Dp-brane to a $D(p-1)$-brane, the tension is multiplied by 2π (remember that $\alpha' = 1$), but simultaneously we have to divide by 2π because the volume is measured with units differing by 2π (see after eq. (C.6)).

In more detail the argument goes as follows (using the notation of Appendix C of [52]. The volume in our normalization is $V = 2\pi\mathcal{V}$, where $\mathcal{V}$ is the volume in Polchinski's textbook normalization, [11], see also [19]. The energy functional for the D24 brane is proportional to the 2D zero mode normalization (which determines the normalization of the partition function). The latter is proportional to $\frac{1}{V}$. Since $V = 2\pi\mathcal{V}$, normalizing with respect to $\mathcal{V}$ is equivalent to multiplying the energy by 2π. This implies that

$$T_{D24} = \frac{1}{2\pi}\mathcal{T}_{D24} \tag{3.144}$$

where $\mathcal{T}$ represents the tension in Polchinski's units. The energy functional in (3.34) depends linearly on the normalization of $\mathrm{g}(s, vs)$, which is the square of the normalization of $g(s)$, so is proportional to $\frac{1}{V^2}$. Therefore the ratio between the energy with the two different zero mode normalizations is $(2\pi)^2$. Consequently we have

$$T_{D23} = \frac{1}{(2\pi)^2}\mathcal{T}_{D23} \tag{3.145}$$

Since, from Polchinski, we have

$$\mathcal{T}_{D23} = 2\pi\,\mathcal{T}_{D24} = (2\pi)^2\,\mathcal{T}_{D25} = 2 \tag{3.146}$$

eq.(3.143) follows.

3.12 D(25-p) brane solutions

The previous argument about D-brane tensions can be easily continued and we always find that the value to be expected is

$$T_{25-p} = \frac{1}{2\pi^2}, \qquad \forall p \geq 1 \tag{3.147}$$

An analytic solution with such energy is easily found. We introduce the relevant operator

$$\phi_u = \sum_{i=1}^{p} u_i(: X_i^2 : +2\log u_i + 2A) \tag{3.148}$$

where X_i will represent the transverse direction to the brane and u_i the corresponding 2D couplings. Since the u_i couplings evolve independently and linearly, the partition function will be $g(u_1, \ldots, u_p) = g(u_1)g(u_2)\ldots g(u_p)$.

The derivation of the energy of such solutions is a straightforward generalization of the one above for the D23-brane. The final result for the energy above the tachyon condensation vacuum is

$$\begin{aligned} E[\Phi_0] &= \frac{1}{6}\lim_{\epsilon\to 0}\int_0^\infty ds\, s^2 \int_0^1 dy \int_0^y dx\, \mathcal{E}(1-y,x)\, \mathrm{g}(s, v_1 s, \ldots, v_{p-1}s)\, e^{-\eta s}\, \eta^3 \\ &= -\frac{1}{4\pi^2}\lim_{\epsilon\to 0}\int_0^\infty ds\, s^2 \mathrm{g}(s, v_1 s, \ldots, v_{p-1}s)\eta^3 e^{-\eta s}. \end{aligned} \tag{3.149}$$

where $v_1 = \frac{u_2}{u_1}, v_2 = \frac{u_3}{u_1}, \ldots$. It is understood that the UV singularity has been subtracted away from the integral in the RHS, therefore the only contribution comes from the region of large s. Since, again $\lim_{s\to\infty} \mathrm{g}(s, v_1 s, \ldots, v_{p-1}s) = 1$, we find straightaway that

$$E[\Phi_0] = \frac{1}{2\pi^2}. \tag{3.150}$$

from which (3.147) follows.

Chapter 4

Mathematical issues about solutions

4.1 Introduction

In the foregoing chapter we have shown how it is possible to reproduce the physics of lower dimensional lump solutions in the frame of OSFT analytic solutions corresponding to relevant boundary deformations. We have seen that, in spite of some subtleties and apparent puzzles, it is always possible to address the problem of computing physical well-defined quantities in a consistent setup where all ambiguities disappear. This should be regarded as a remarkable success of BMT lump solutions, even more in the light of the nontrivial way the correct results are obtained. However, the subtleties involved in the computation have also raised some doubts that BMT lump solutions may be considered fully consistent solutions of OSFT [54–56]. In particular Sen's conjecture about lower dimensional $D(25-q)$-branes can be naively stated as the fact that

$$\widetilde{S}(|\Psi\rangle) = S(|\Phi^q\rangle) - S(|\Phi_{TV}\rangle) = -V_{26-q}\mathcal{T}_{25-q}\,. \tag{4.1}$$

where $|\Phi^q\rangle$ is a codimension q lump solution, $|\Phi_{TV}\rangle$ is the tachyon vacuum, $|\Psi\rangle = |\Phi^q\rangle - |\Phi_{TV}\rangle$. S here is understood to be the on-shell action *on the perturbative* $D25$*-brane vacuum*

$$S(|\Psi\rangle) = -\frac{1}{2}\langle\Psi|Q|\Psi\rangle - \frac{1}{3}\langle\Psi|\Psi * \Psi\rangle = \frac{1}{6}\langle\Psi|\Psi * \Psi\rangle\,, \tag{4.2}$$

$\widetilde{S}$ is the action at the tachyon vacuum is $-\frac{1}{2}\langle\mathcal{Q}\Phi,\Phi\rangle - \frac{1}{3}\langle\Phi,\Phi\Phi\rangle$, where $\mathcal{Q}|\Phi^q\rangle = Q|\Phi^q\rangle + |\Phi_{TV}\rangle|\Phi^q\rangle + |\Phi^q\rangle|\Phi_{TV}\rangle$.
The energy of of the lump, $E[\Phi_0] = -\widetilde{S}[\Phi_0]$, we have computed in the previous chapter

is

$$\begin{aligned} E[\Phi_0] &= -\lim_{\epsilon\to 0}\frac{1}{6}\langle\Phi_0^\epsilon,\Phi_0^\epsilon\Phi_0^\epsilon\rangle \\ &= -\frac{1}{6}\lim_{\epsilon\to 0}\left[\langle\psi_u,\psi_u\psi_u\rangle-\langle\psi_u^\epsilon,\psi_u^\epsilon\psi_u^\epsilon\rangle-3\langle\psi_u^\epsilon,\psi_u\psi_u\rangle+3\langle\psi_u,\psi_u^\epsilon\psi_u^\epsilon\rangle\right]. \end{aligned} \tag{4.3}$$

The integrals in the four correlators on the RHS, are IR ($s\to\infty$) convergent. The UV subtractions necessary for each correlator are always the same, therefore they cancel out. In [53], after UV subtraction, we obtained

$$\begin{aligned} &-\frac{1}{6}\langle\psi_u,\psi_u\psi_u\rangle=\alpha+\beta, && \lim_{\epsilon\to 0}\langle\psi_u^\epsilon,\psi_u^\epsilon\psi_u^\epsilon\rangle=0 \\ &\frac{1}{6}\lim_{\epsilon\to 0}\langle\psi_u^\epsilon,\psi_u\psi_u\rangle=\alpha-\frac{2}{3}\beta, && \frac{1}{6}\lim_{\epsilon\to 0}\langle\psi_u,\psi_u^\epsilon\psi_u^\epsilon\rangle=\alpha-\frac{1}{3}\beta \end{aligned} \tag{4.4}$$

where $\alpha+\beta\approx 0.068925$ was evaluated numerically and $\alpha=\frac{1}{2\pi^2}$ was calculated analytically. So $E[\Phi_0]=\alpha$ turns out to be precisely the D24-brane energy, but one could question the non null contributions coming from interference terms contrary to the naive expectation from 4.1. This was interpreted as a clue ψ_u is not really satisfying EOM. Nevertheless one should remark the vanishing of $-3\langle\psi_u^\epsilon,\psi_u\psi_u\rangle+3\langle\psi_u,\psi_u^\epsilon\psi_u^\epsilon\rangle = 3\langle\psi_\epsilon,Q\psi_u\rangle-3\langle\psi_u,Q\psi_\epsilon\rangle$ relies on an integration by parts which is not generally allowed due to the UV singularities affecting this terms. Furthermore Sen's conjecture is explicitly stated under the condition the tachyon vacuum has zero energy. This requirement can be realized in a subtraction independent way only by fixing $|\Phi_{TV}\rangle=0$, i.e. considering the lump solutions *at the tachyon condensation vacuum*. It also raised some criticism the fact that in our approach it seems to be quite crucial to take the limit $\epsilon\to 0$, whereas all other parameters in the solution are let free to flow in keeping with the understanding that they should correspond to gauge symmetries in OSFT and to boundary RG flows in BSFT. If ϵ should be interpreted as a true gauge parameter, than the fact that $E[\Phi_0]$ varies with it could be another hint at some anomalous behavior of EOM, Nevertheless we will argue ϵ cannot consistently be thought of as a gauge parameter, but rather as regulator introduced ad hoc to find a convenient representation for the tachyon vacuum. In this chapter we will show all these issues find a quite natural interpretation when we look at BMT lump solutions in the context of the mathematical formalism in which they are formulated. So the problem in the end is not so much to assess whether BMT lump solutions satisfy or not Witten's Cubic OSFT equation of motion, but to understand in which precise *mathematical* language their peculiar features find a fully consistent decription.

4.2 Nature of the ϵ parameter

It has been pointed out that such string fields as $\frac{1}{K+\phi_u}$ should be regarded as formal objects in need for an operative definition that can be used to compute physical observables associated to them. For instance, up to now we have always used the Schwinger representation

$$\frac{1}{K+\phi_u} = \int_0^\infty dt\, e^{-t(K+\phi_u)} \tag{4.5}$$

Using this definition, we have found that

$$\psi_u = c\phi_u - \frac{1}{K+\phi_u}(\phi_u - \delta\phi_u)Bc\partial c, \tag{4.6}$$

where ϕ_u is Witten's deformation, does consistently describe the expected physical observables for lower dimensional Dp-branes. Anyhow one could also think of different ways of making sense of the formal definition 4.6. In particular, in the limit $\epsilon \to 0$, one could consider

$$\psi_{u,\epsilon} = c\phi_u - \frac{1}{(K+\phi_u+\epsilon)}(\phi_u - \delta\phi_u)Bc\partial c \tag{4.7}$$

$$\psi_u^\epsilon = c(\phi_u+\epsilon) - \frac{1}{K+\phi_u+\epsilon}(\phi_u+\epsilon-\delta\phi_u)Bc\partial c. \tag{4.8}$$

where it is understood

$$\frac{1}{K+\phi_u+\epsilon} = \int_0^\infty dt\, e^{-t(K+\phi_u+\epsilon)} \tag{4.9}$$

It should be noticed that, from a purely formal point of view, all of these three solutions can be considered as different regularizations of the same formal object, but, quite remarkably, whereas 4.6 and 4.7 will be seen to describe the same lump solution, 4.8 is actually a non universal expression for the tachyon condensation vacuum, as it was discussed in the previous chapter. From this point of view, it's quite clear ϵ should be genuinely interpreted as a regularization parameter and actually this regularization procedure is a fundamental ingredient for understanding what kind of physics the solution is describing. On the other hand one could also notice that 4.8 can be obtained from 4.6 by the simple substitution $\phi_u \to \phi_u + \epsilon$ and so ϵ appears on the same footing as other parameters describing boundary RG flows. This led us to numerically investigate the dependence on ϵ of

$$E[\Phi_0] = -\frac{1}{6}\langle\Phi_0, \Phi_0\Phi_0\rangle = -\frac{1}{6}\left[\langle\psi_u, \psi_u\psi_u\rangle - \langle\psi_\epsilon, \psi_\epsilon\psi_\epsilon\rangle - 3\langle\psi_\epsilon, \psi_u\psi_u\rangle + 3\langle\psi_u, \psi_\epsilon\psi_\epsilon\rangle\right]. \tag{4.10}$$

It's convenient to consider separately UV finite quantities for which the numerical evaluation can be carried out in more effective way without having to subtract UV singularities with ad hoc defined functions.

$$\begin{aligned}
\Delta^{(1)}_\epsilon &= \langle\psi_u,\psi_u\psi_u\rangle - \langle\psi^\epsilon_u,\psi^\epsilon_u\psi^\epsilon_u\rangle \qquad (4.11)\\
&= \int_0^\infty ds\, s^2 \int_0^1 dy \int_0^y dx\, e^{-\eta s}\mathcal{E}(1-y,x)\Bigg\{\left(1-e^{-\eta s}\right)\Bigg[\Big(-\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\Big)^3\\
&\quad -\frac{1}{2}\Big(\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\Big)\Big(G_s^2(2\pi(x-y))+G_s^2(2\pi x)+G_s^2(2\pi y)\Big)\\
&\quad +G_s(2\pi x)G_s(2\pi(x-y))G_s(2\pi y)\Bigg]\\
&\quad -e^{-\eta s}\Bigg[\eta^3-3\eta^2\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}+3\eta\Big(\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\Big)^2\\
&\quad +\frac{1}{2}\eta\Big(G_s^2(2\pi(x-y))+G_s^2(2\pi x)+G_s^2(2\pi y)\Big)\Bigg]\Bigg\}
\end{aligned}$$

and

$$\begin{aligned}
&\Delta^{(2)}_\epsilon = \langle\psi^\epsilon_u,\psi_u\psi_u\rangle - \langle\psi_u,\psi^\epsilon_u\psi^\epsilon_u\rangle = \qquad (4.12)\\
= &-\int_0^\infty ds\, s^2\int_0^1 dy\int_0^1 dx\, e^{\eta s y}\,\mathcal{E}(1-y,x)\,H(x,y,\eta,s)\\
&-\eta\int_0^\infty ds\, s^2\int_0^1 dy\int_0^1 dx\, e^{\eta s y}\,\mathcal{E}(1-y,x)\,\mathrm{g}(s)\,e^{-\eta s}\left(-\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\right)^2\\
&-\frac{\eta}{2}\int_0^\infty ds\, s^2\int_0^1 dy\int_0^1 dx\, e^{\eta s y}\,G_s^2(2\pi x)\mathcal{E}(1-y,x)\,\mathrm{g}(s)\,e^{-\eta s}\\
&-\int_0^\infty ds\, s^2\int_0^1 dy\int_0^y dx\,\mathcal{E}(1-y,x)\,\mathrm{g}(s)\,e^{-\eta s}\,e^{\eta s x}\\
&\quad\cdot\left(\eta^2\left(\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\right)-\eta\left(\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\right)^2-\frac{\eta}{2}G_s^2(2\pi x)\right)
\end{aligned}$$

where

$$\begin{aligned}
H(x,y,\eta,s) &= \mathrm{g}(s)e^{-\eta s}\Bigg\{\Big(-\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\Big)^3+G_s(2\pi x)G_s(2\pi(x-y))G_s(2\pi y) \qquad (4.13)\\
&\quad +\frac{1}{2}\Big(-\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\Big)\Big(G_s^2(2\pi(x-y))+G_s^2(2\pi x)+G_s^2(2\pi y)\Big)\Bigg\}
\end{aligned}$$

After integrating over the angular variables x and y, one finds the following numerical results for a sample of values of the parameter $\eta=\frac{\epsilon}{2u}$.

The limit $\lim_{\epsilon\to 0}\Delta^{(1)}_\epsilon$ has been given in the previous chapter: $6(\alpha+\beta)\approx -0.41355$. Since the numbers in Table 1 are accurate up to the third digit (being very conservative the error can be estimated to be ± 0.0005) the dependence on ϵ is evident albeit small. It is also clearly visible that the sequence of numbers tends to the expected value (around

η :	2	1	0.7	0.5	0.1	0.08
$\Delta_\epsilon^{(1)}$:	-0.41968	-0.41958	-0.42028	-0.41860	-0.41868	-0.41853
η :	0.05	0.01	0.005	0.003	0.001	0.0005
$\Delta_\epsilon^{(1)}$:	-0.41831	-0.41660	-0.41625	-0.41587	-0.41483	-0.414009

Table 4.1: Samples of $\Delta_\epsilon^{(1)}$

$\eta = 0.00001$ reliable numerical results becomes hard to retrieve). The smallness of the ϵ dependence (a few percent only) can be seen as a consequence of the fact that the UV singularities cancel out in a very natural way in $\Delta_\epsilon^{(1)}$.

The dependence on ϵ of $\Delta_\epsilon^{(2)}$ is not much easier to detect. In Table 2 we report the numerical results for a sample of the parameter η.

η :	10	2	1	0.7	0.5
$\Delta_\epsilon^{(2)}$:	-0.01431	-0.02704	-0.0308524	-0.0323693	-0.03332
η :	0.4	0.2	0.1	0.08	0.05
$\Delta_\epsilon^{(2)}$:	-0.03398	-0.03525	-0.03567	-0.03550	-0.03613

Table 4.2: Samples of $\Delta_\epsilon^{(2)}$

In chapter 2 the numerical value of $\Delta_\epsilon^{(2)}$ was determined in the $\epsilon \to 0$ limit to be: $\lim_{\epsilon\to 0} \Delta_\epsilon^{(2)} = -2\beta \approx -0.03652$. The results in Table 1 are to be taken with a possible uncertainty of ± 0.0005. We see that they clearly depend on ϵ and that the limit $\epsilon \to 0$ tends to the expected value.

Therefore, even if formally ϵ could be expected to be a gauge parameter for OSFT due to the fact that it can be regarded as one coordinate in the space of relevant boundary deformations $\phi_u(s) + \epsilon$, in our approach it should be treated as a regulator which in the end must be removed by taking the limit to zero. In fact, similarly to what happens for lower dimensional lumps ($p \leq 23$), the dependence of the energy on the ratio $\frac{\epsilon}{u}$ cannot explicitly absorbed into the integration variable unless we fix $\epsilon = \kappa u$ for some positive constant κ; but then, in $\phi_u + \epsilon$, see (3.10), ϵ could be absorbed into a redefinition of A and would disappear from ψ_u^ϵ. As a consequence the latter would actually coincide with ψ_u and $\Delta_\epsilon^{(1)}$ would vanish, which is evidently not the case. So rather than thinking of ϵ as a gauge parameter, we should consider it as a small perturbation we add to break conformal invariance and move from the conformal point corresponding to the lump to the one of tachyon vacuum.
This point of view could look like an ad hoc one to get the right result for the Dp-brane tension. In fact one would be led to think that as ψ_u^ϵ is a solution to the EOM of SFT for any value of ϵ, the term $\langle \psi_u^\epsilon, \psi_u^\epsilon \psi_u^\epsilon \rangle$ is bound to be ϵ-dependent . The point is that ψ_u^ϵ

formally solves the equation of motion but is not necessarily an extreme of the action for $\epsilon \neq 0$ in the presence of UV singularities. The puzzle is explained by the fact that the parameter ϵ is not present in the original action. Therefore one has to prove *a posteriori* that the 'solution' actually corresponds to an extreme of the action*. The variation of the action with ϵ is given by (after replacing the eom) $\delta_\epsilon S \sim \langle \frac{\partial \psi_u^\epsilon}{\partial \epsilon}, Q\psi_u^\epsilon \rangle - \langle Q\frac{\partial \psi_u^\epsilon}{\partial \epsilon}, \psi_u^\epsilon \rangle$. For this to vanish one should be able to 'integrate by parts', which is not possible due to the UV subtractions implicit in the calculation of the correlators, see [53] (and also [47] where similar arguments are developed although not in the same context)†. Now $\delta_\epsilon S$ does not vanish and in order to find an extreme of the action we have to extremize it. This is in keeping with the monotonic dependence on η one can see in Table 1, which tells us that the extreme is met in the limit $\epsilon \to 0$.

So UV singularities are responsible for the fact that a formal gauge parameter doesn't actually behave as such from the point of view of correlation functions. This of course is only related to the fact that in our case the UV subtraction is applied to the three-points correlators. Probably in the frame of a true renormalization scheme for SFT, where the string field ψ_u^ϵ itself would be redefined the formal argument involving the integration by parts of Q should be valid.

4.3 The problem with the Schwinger representation

The previous discussion has pointed out the issue of EOM becomes particularly delicate when UV singularities are involved. In particular one could point out a UV singular action doesn't automatically lead to the expected EOM because of the afore mentioned obstruction in integrating by parts. So in this sense the validity of EOM inside correlators is something that should be tested case by case rather than assumed a priori. In this section we want to argue we don't expect any such anomaly in the EOM to affect the computation for the lump energy we have presented in the foregoing chapter.

Another way to see that EOM are to be considered with some attention in the context of the $KBc\phi$ algebra is to remember $\frac{1}{K+\phi_u}$ is required to be singular so that the corresponding lump solution isn't a pure gauge transform of the tachyon vacuum. There is

*The same consideration applies also to the parameter u, but it was shown in [52] that u actually disappears from the action when we replace ψ_u in it: u is a true gauge parameter.

†Since the UV singularity is linked to the X zero mode, one might expect that with a compactified X this problem should disappear and the integration by parts become possible. However, as long as we consider solution of the type ψ_u, ψ_u^ϵ with a linearly scaling u parameter, this seems to be impossible: the singularity removed from the UV will pop up in the IR, creating analogous problems. The nontrivial boundary contribution in the SFT action, see also section 2, is a new interesting feature which deserves a closer investigation.

an obvious tension between this requirement and the identity

$$\frac{1}{K+\phi_u}(K+\phi_u)=I, \tag{4.14}$$

which is fundamental in the simple algebraic proof that lump solutions satisfy EOM. To illustrate the problem, let us calculate the overlap of both the left and the right hand sides of (4.14) with $Y=\frac{1}{2}\partial^2 c\partial cc$. The right hand side is trivial and, in our normalization, it is

$$\mathrm{Tr}(Y\cdot I)=\lim_{t\to 0}\langle Y(t)\rangle_{C_t}\langle 1\rangle_{C_t}=\frac{V}{2\pi}. \tag{4.15}$$

To calculate the left hand side we need the usual Schwinger representation

$$\mathrm{Tr}\big[Y\cdot\frac{1}{K+\phi_u}(K+\phi_u)\big]=\int_0^\infty dt\mathrm{Tr}\big[Y\cdot e^{-t(K+\phi_u)}(K+\phi_u)\big] \tag{4.16}$$

Making the replacement

$$e^{-t(K+\phi_u)}(K+\phi_u)\to -\frac{d}{dt}e^{-t(K+\phi_u)} \tag{4.17}$$

one obtains

$$\mathrm{Tr}\big[Y\cdot\frac{1}{K+\phi_u}(K+\phi_u)\big]=g(0)-g(\infty)=\frac{V}{2\pi}-g(\infty), \tag{4.18}$$

This seems to be in disagreement with (4.15) unless $g(\infty)$ vanishes. However our previous discussion has shown a non vanishing IR limit of $g(s)$ is the basic requirement ensuring the $BCFT^*$ can contain open string states. In the language of OSFT this fact is associated with the existence of the deformed sliver projector, i.e.

$$\int_0^\infty dt\, e^{-t(K+\phi_u)}(K+\phi_u)=1-\Omega_u^\infty,\qquad \Omega_u^\infty=\lim_{\Lambda\to\infty}e^{-\Lambda(K+\phi_u)} \tag{4.19}$$

From the latter one would be led to say that if $\frac{1}{K+\phi_u}$ is to solve 4.14 it should be written

$$\frac{1}{K+\phi_u}=\int_0^\infty dt\, e^{-t(K+\phi_u)}+\frac{1}{K+\phi_u}\Omega_u^\infty \tag{4.20}$$

which looks different from (4.5). This is quite puzzling. On one side the sliver projector is the object which, according to Ellwood's proposal for lump solutions [24], should encode the information about the boundary conditions associated to lower dimensional Dp-branes. On the other its existence as an element of the $KBc\phi$ seems to give a fatal blow to the very possibility they may solve EOM. This seemingly unsolvable clash has led some authors [54] to exclude the BMT and Ellwood solutions as viable descriptions of the lump. In the upcoming sections we will try to point out the origin of the puzzle and, to some extent, its solution lies in the particular mathematical nature of the projector Ω_u. In particular we will notice every statement about it should always be done only in the context of well defined unambiguous correlators. To start with, let us notice both correlators 4.15 and4.18 are actually UV divergent, so their mismatch should not be considered as a standalone proof that 4.14 is violated in *all* the correlators we have computed.

4.4 A new (formal) representation for $\frac{1}{K+\phi_u}$

In this section we introduce a different operational definition of $\frac{1}{K+\phi_u}$ in which we still use Schwinger representation but we also introduce a small nonnegative parameter ϵ, i.e.

$$\frac{1}{K+\phi_u} = \lim_{\epsilon\to 0}\int_0^\infty dt\, e^{-t(K+\phi_u+\epsilon)} \tag{4.21}$$

For brevity we will also write

$$\frac{1}{K+\phi_u} = \lim_{\epsilon\to 0}\frac{1}{(K+\phi_u+\epsilon)} = e^{-\epsilon\partial_\epsilon}\frac{1}{(K+\phi_u+\epsilon)} \tag{4.22}$$

where $e^{-\epsilon\partial_\epsilon}$ means

$$e^{-\epsilon\partial_\epsilon} = e^{-a\partial_\epsilon}|_{a=\epsilon}$$

It should be noticed that (4.22) may be interpreted as

$$\text{either}\qquad e^{-\epsilon\partial_\epsilon}\int_0^\infty dt\, e^{-t(K+\phi_u+\epsilon)},\qquad\text{or}\qquad \int_0^\infty dt\, e^{-\epsilon\partial_\epsilon}e^{-t(K+\phi_u+\epsilon)} \tag{4.23}$$

In the second case we clearly get back to the original definition used for the lump solution, whereas in the case we take the point of view of first computing correlators and then take the limit $\epsilon\to 0$ (as we did for the "regularized" solution describing the tachyon condensation vacuum) we can in general expect something may change. One can actually verify the solution

$$\psi_{u,\epsilon} = c\phi_u - e^{-\epsilon\partial_\epsilon}\frac{1}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)Bc\partial c \tag{4.24}$$

is expected not to be trivial and to correctly capture the change of boundary entropy related to the physics of lower dimensional Dp branes. The most interesting aspect of this representation is that it explicitly account for the fact that the identity $\frac{1}{K+\phi_u}(K+\phi_u) = I$

is not trivially realized by the Schwinger representation. In fact

$$e^{-\epsilon\partial_\epsilon}\frac{1}{K+\phi_u+\epsilon}(K+\phi_u) = 1 - e^{-\epsilon\partial_\epsilon}\frac{\epsilon}{K+\phi_u+\epsilon} \tag{4.25}$$

The expression $e^{-\epsilon\partial_\epsilon}\frac{\epsilon}{K+\phi_u+\epsilon} = \lim_{\epsilon\to 0}\frac{\epsilon}{K+\phi_u+\epsilon}$ is a very convenient way to take into account in formal manipulations the possible contributions due to Ω_u^∞. It is of course formally vanishing, but we should stick to our understanding only well defined correlators can really probe such an object. It can be noted a correlator containing Ω_u^∞ can be non zero only if without the additional regulator $e^{-\epsilon t}$ it would be divergent or indefinite. In other words such correlators would be scrapped a priori if we took the second possibility described in 4.23. Nevertheless, if we take the other point of view, they may give a finite result. For instance, taking the trace, we are led to evaluate

$$\mathrm{Tr}\left[\frac{\epsilon}{K+\phi_u+\epsilon}\right] = \epsilon\int_0^\infty dt\, e^{-\epsilon t}\, g(ut) \tag{4.26}$$

Since $g(\infty)=1$, the limit $\epsilon\to 0$ is not continuous for the interchange with the integral operation, and this depends on the fact that the integral in the RHS of (4.26) is (linearly) divergent when the factor $e^{-\epsilon t}$ is replaced by 1. On the other hand, if (4.25) is inserted in a correlator (like the energy one) where the integrand without the exponential factor decreases fast enough, then the result of the application of $e^{-\epsilon\partial_\epsilon}$ to $\frac{\epsilon}{K+\phi_u+\epsilon}$ is unambiguously 0. This can be seen by considering for instance the following contraction

$$\begin{aligned}
&\mathrm{Tr}\Big[\partial^2 c\; e^{-(K+\phi)}e^{-\epsilon\partial_\epsilon}\Big(\frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\Big)\Big] \qquad (4.27)\\
&= e^{-\epsilon\partial_\epsilon}\epsilon\int_0^\infty dt\; e^{-\epsilon t}\mathrm{Tr}\big[(\phi_u-\delta\phi_u)e^{-(t+1)(K+\phi)}\big]\langle\partial^2 c(t+1)c\partial c(0)\rangle_{C_{t+1}}\\
&= e^{-\epsilon\partial_\epsilon}\epsilon\int_0^\infty dt\; e^{-\epsilon t}\langle(\phi_u(0)-\delta\phi_u(0))e^{-\int_0^{t+1}ds\phi(s)}\rangle_{C_{t+1}}\langle\partial^2 c(t+1)c\partial c(0)\rangle_{C_{t+1}}\\
&= -e^{-\epsilon\partial_\epsilon}\epsilon\int_0^\infty dt\; e^{-\epsilon t}G(t)\frac{u}{t+1}\partial_u g\big(u(t+1)\big) = 2e^{-\epsilon\partial_\epsilon}\epsilon\int_0^\infty dt\; e^{-\epsilon t}\frac{u}{t+1}\partial_u g\big(u(t+1)\big)
\end{aligned}$$

where the ghost contribution is given by

$$G(t) = \langle\partial^2 c(t+1)(c\partial c)(0)\rangle_{C_{t+1}} = -2.$$

Now we can write eq.(4.27) as

$$\begin{aligned}
&2\big(e^{-\epsilon\partial_\epsilon}\epsilon\big)e^{-\epsilon\partial_\epsilon}\int_0^\infty dt\; e^{-\epsilon t}\frac{u}{t+1}\partial_u g\big(u(t+1)\big)\\
&\qquad = 2\big(e^{-\epsilon\partial_\epsilon}\epsilon\big)\int_0^\infty dt\;\frac{u}{t+1}\partial_u g\big(u(t+1)\big) = 0. \qquad (4.28)
\end{aligned}$$

We note that this last result does not need any UV subtraction.

4.4.1 The energy for the lump solution $\psi_{u,\epsilon}$

The drawback of representing the lump solution as

$$\psi_{u,\epsilon} \;\; = c\phi_u - e^{-\epsilon\partial_\epsilon}\tfrac{1}{(K+\phi_u+\epsilon)}(\phi_u - \delta\phi_u)Bc\partial c \tag{4.29}$$

is that EOM are not *formally* satisfied anymore, as pointed out in [54]. Using in particular

$$Q\left(e^{-\epsilon\partial_\epsilon}\frac{1}{(K+\phi_u+\epsilon)}\right) = -e^{-\epsilon\partial_\epsilon}\frac{1}{(K+\phi_u+\epsilon)}(Q\phi_u)\frac{1}{(K+\phi_u+\epsilon)} \tag{4.30}$$

and proceeding as in section 3.2 of [52], we find

$$\begin{aligned} Q\psi_{u,\epsilon} &= Q\left(c\phi_u - e^{-\epsilon\partial_\epsilon}\frac{1}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)Bc\partial c\right) \\ &= e^{-\epsilon\partial_\epsilon}\left[1+\frac{1}{(K+\phi_u+\epsilon)}(c\partial\phi_u+\partial c\delta\phi_u)\frac{1}{(K+\phi_u+\epsilon)}B-\frac{1}{(K+\phi_u+\epsilon)}K\right](\phi_u-\delta\phi_u)c\partial c \\ &= e^{-\epsilon\partial_\epsilon}\left[\left(c\phi_u-\frac{1}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)\partial c\right)\frac{1}{(K+\phi_u+\epsilon)}+\frac{\epsilon}{(K+\phi_u+\epsilon)}c\right](\phi_u-\delta\phi_u)Bc\partial c \\ &= -\psi_{u,\epsilon}\psi_{u,\epsilon}+e^{-\epsilon\partial_\epsilon}\left(\frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\right) \end{aligned} \tag{4.31}$$

In a setting where correlators divergent or undefined without the introduction of the regulator ϵ are not accepted, we could simply say the second piece in the RHS of the last line vanishes in every correlator of this kind and so in practice 4.31 would boil down to the usual eom $Q\psi_u = -\psi_u\psi_u$. Anyhow we are often led, for the sake of computational easiness, to confront with correlators of more general type,even if in the end we should be careful our physical results do not really depend on that. In fact, if we keep (4.31) in the expression of the energy, we have

$$\begin{aligned} -\langle\psi_u Q\psi_u\rangle \;\; &\to \;\; -\langle\psi_{u,\epsilon}Q\psi_{u,\epsilon}\rangle \\ &= \;\; \langle\psi_{u,\epsilon}\psi_{u,\epsilon}\psi_{u,\epsilon}\rangle + \langle\psi_{u,\epsilon}e^{-\epsilon\partial_\epsilon}\left(\frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\right)\rangle \end{aligned} \tag{4.32}$$

The second term in the RHS equals

$$e^{-\epsilon\partial_\epsilon}\langle\frac{1}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)Bc\partial c\,\frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\rangle \tag{4.33}$$

With the usual procedure we can write this as $(T = t_1 + t_2)$

$$e^{-\epsilon\partial_\epsilon}\left(\epsilon\int_0^\infty dt_1 dt_2\; e^{-\epsilon T}\mathcal{G}(t_1,t_2)u^2 g(uT)\Big\{\Big(-\frac{\partial_{uT}g(uT)}{g(uT)}\Big)^2 + 2G^2_{2uT}(\frac{2\pi t_1}{T})\Big\}\right), \tag{4.34}$$

where the ghost part is given by

$$\mathcal{G}(t_1,t_2) = \langle (Bc\partial c)(t_1)(c\partial c)(0)\rangle_{C_T} = \frac{t_1}{\pi}\sin(\frac{2\pi t_1}{T}) - \frac{2T}{\pi^2}\sin^2(\frac{\pi t_1}{T}). \tag{4.35}$$

Let us show now that (4.34) reduces to the form

$$e^{-\epsilon\partial_\epsilon}\left(\epsilon\int_0^\infty ds\, e^{-\epsilon s}\mathcal{F}(s)\right) \tag{4.36}$$

where $\mathcal{F}(s) \to$ const for large s and the integral is UV finite.

Denoting $x = \frac{t_1}{T}$, Eq.(4.35) can be rewritten as

$$e^{-\widetilde{\eta}\partial_{\widetilde{\eta}}}\widetilde{\eta}\int_0^\infty dss^2\int_0^1 dx\mathcal{E}(x)\; e^{-\widetilde{\eta}s}\mathrm{g}(s)\Big\{\Big(-\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\Big)^2 + \frac{1}{2}G_s^2(2\pi x)\Big\}, \tag{4.37}$$

where $\widetilde{\eta} = \frac{\epsilon}{2u}$ and

$$\mathcal{E}(x) = \langle (Bc\partial c)(x)(c\partial c)(0)\rangle_{C_1} = \frac{-1+\cos(2\pi x)+\pi x\sin(2\pi x)}{\pi^2} \tag{4.38}$$

Since $\int_0^1 dx\mathcal{E}(x) = -\frac{3}{2\pi^2}$, the term with no G_s is given by

$$-\frac{3}{2\pi^2}\widetilde{\eta}\int_0^\infty dss^2\; e^{-\widetilde{\eta}s}\mathrm{g}(s)\Big(-\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\Big)^2 \tag{4.39}$$

As $\mathrm{g}(s) \approx \frac{1}{\sqrt{s}}$ in the UV we are in the case of eq.(8.13) of [53] and so the UV contribution vanishes for $\widetilde{\eta} \to 0$. In the IR we are in the case of eq.(8.17) of [53] and so the IR contribution vanishes too. It can be easily proven that

$$\begin{aligned} 3\int_0^1 dx\mathcal{E}(x)G_s^2(2\pi x) \;&=\; \frac{4}{\pi}\int_0^1 dy\int_0^y dx\sin\pi x\,\sin\pi y\,\sin\pi(x-y) \\ &\cdot\Big(G_s^2(2\pi x)+G_s^2(2\pi(x-y))+G_s^2(2\pi y)\Big) \end{aligned} \tag{4.40}$$

where the expression in the RHS is the same as eq.(3.7) of [53]. Therefore we have

$$\begin{aligned} & e^{-\widetilde{\eta}\partial_{\widetilde{\eta}}}\left(\frac{1}{2}\widetilde{\eta}\int_0^\infty dss^2\; e^{-\widetilde{\eta}s}\mathrm{g}(s)\int_0^1 dx\mathcal{E}(x,0)G_s^2(2\pi x)\right) \\ =\; & e^{-\widetilde{\eta}\partial_{\widetilde{\eta}}}\left(\frac{1}{6}\widetilde{\eta}\int_0^\infty dss^2\; e^{-\widetilde{\eta}s}\mathrm{g}(s)\frac{4}{\pi}\int_0^1 dy\int_0^y dx\sin\pi x\,\sin\pi y\,\sin\pi(x-y)\right. \\ & \left.\cdot\Big(G_s^2(2\pi x)+G_s^2(2\pi(x-y))+G_s^2(2\pi y)\Big)\right) \end{aligned} \tag{4.41}$$

We can now avail ourselves of the results in chapter 2 . The integration over x and y leads to an integrand in s that behaves like a constant for large s, if one abstracts from the factor $e^{-\widetilde{\eta}s}$. Thus we have obtained (4.36). So this is just that kind of correlator which can be given a finite value only with the *ad hoc* prescription of applying $e^{-\widetilde{\eta}\partial_{\widetilde{\eta}}}$ after the integration is done; in this way we obtain -2β, where β is the number introduced in chapter 2. This result is the same as the one obtained by [54]. One however should notice that the term (4.41) appears in the RHS of eq.(4.32) together with $\langle\psi_{u,\epsilon}\psi_{u,\epsilon}\psi_{u,\epsilon}\rangle$. The latter is a UV divergent term (in the case of lower dimensional lumps it was seen to be even gauge-dependent) and needs a UV subtraction, which it was pointed out to be arbitrary. For instance, one could choose the UV subtraction in such a way as to kill the contribution of -2β altogether and there would be no violation of the EOM. So, similarly to what happens for 4.14, 4.32 doesn't seem to lead automatically to the conclusion EOM are violated simply because the amplitudes we are computing are intrinsically ambiguous and in need of a consistent regularization.

This situation looks remarkably similar to what happens in distribution theory when we consider the product of two singular objects. In particular it is well known that, by integrating a vanishing distribution over a non test function, one could obtain a nonvanishing result. The nonvanishing of the second term in the RHS of (4.32) is analogous. The string field $e^{-\epsilon\partial_\epsilon}\frac{\epsilon}{K+\phi_u+\epsilon}(\phi_u-\delta\phi_u)$ plays the role of the vanishing distribution and $\psi_{u,\epsilon}$ the role of the singular test function. In this way, we can get some heuristic understanding of the mechanism by which we get a non vanishing result for 4.33. Apart from the additional ϵ factor in the numerator, the amplitude (4.33) is just the string field $\frac{1}{K+\phi_u+\epsilon}(\phi_u-\delta\phi_u)$ contracted with itself, which can be interpreted as the 'norm' square of this string field, in the limit $\epsilon\to 0$. The above results tell us that this 'norm' is infinite. It is this infinity that multiplied by ϵ allows us to obtain the above finite result. This confirms $\psi_{u,\epsilon}$ should be not be regarded as an acceptable test state in the context of a distribution theory for string fields. We will expand on this mathematical insight into the problem in the next section. Here we can just notice that luckily, and quite beautifully, it's the physics itself of lumps to suggest us we should not attach a particular meaning to 4.32. As was the case in chapter 2 for the lump solution ψ, here also we should consider the lump on the tachyon condensation vacuum, i.e. $\Phi(\varepsilon,\epsilon)=\psi_{u,\varepsilon}-\psi_u^\epsilon$, where ψ_u^ϵ, in the $\epsilon\to 0$ limit, is the tachyon vacuum solution defined in [53]. We get

$$\begin{aligned}
Q\psi_{u,\varepsilon} &= -\psi_{u,\varepsilon}\psi_{u,\varepsilon}+e^{-\varepsilon\partial_\varepsilon}\left(\frac{\varepsilon}{(K+\phi_u+\varepsilon)}(\phi_u-\delta\phi_u)c\partial c\right)\\
Q\psi_u^\epsilon &= -\psi_u^\epsilon\psi_u^\epsilon \\
\mathcal{Q}\Phi(\varepsilon,\epsilon) &= -\Phi(\varepsilon,\epsilon)\Phi(\varepsilon,\epsilon)+e^{-\varepsilon\partial_\varepsilon}\left(\frac{\varepsilon}{(K+\phi_u+\varepsilon)}(\phi_u-\delta\phi_u)c\partial c\right)
\end{aligned}\tag{4.42}$$

where $\mathcal{Q}\Phi=Q\Phi+\psi_u^\epsilon\Phi+\Phi\psi_u^\epsilon$. Moreover

$$\begin{aligned}
-\langle\Phi(\varepsilon,\epsilon)\mathcal{Q}\Phi(\varepsilon,\epsilon)\rangle &= \langle\Phi(\varepsilon,\epsilon)\Phi(\varepsilon,\epsilon)\Phi(\varepsilon,\epsilon)\rangle \\
&\quad+\langle\Phi(\varepsilon,\epsilon)e^{-\varepsilon\partial_\varepsilon}\left(\frac{\varepsilon}{(K+\phi_u+\varepsilon)}(\phi_u-\delta\phi_u)c\partial c\right)\rangle
\end{aligned}\tag{4.43}$$

If we use the just defined representation, the second term in the RHS equals

$$\begin{aligned}
& e^{-\varepsilon\partial_\varepsilon}\langle\psi_{u,\varepsilon}\left(\frac{\varepsilon}{(K+\phi_u+\varepsilon)}(\phi_u-\delta\phi_u)c\partial c\right)\rangle - e^{-\varepsilon\partial_\varepsilon}\langle\psi_u^\epsilon\left(\frac{\varepsilon}{(K+\phi_u+\varepsilon)}(\phi_u-\delta\phi_u)c\partial c\right)\rangle \\
= & -2\beta - e^{-\varepsilon\partial_\varepsilon}\langle\frac{1}{(K+\phi_u+\epsilon)}(\phi_u+\epsilon-\delta\phi_u)Bc\partial c\;\frac{\varepsilon}{(K+\phi_u+\varepsilon)}(\phi_u-\delta\phi_u)c\partial c\rangle \\
= & -2\beta - e^{-\varepsilon\partial_\varepsilon}\langle\frac{1}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)Bc\partial c\;\frac{\varepsilon}{(K+\phi_u+\varepsilon)}(\phi_u-\delta\phi_u)c\partial c\rangle \\
& -e^{-\varepsilon\partial_\varepsilon}\langle\frac{\epsilon}{(K+\phi_u+\epsilon)}Bc\partial c\;\frac{\varepsilon}{(K+\phi_u+\varepsilon)}(\phi_u-\delta\phi_u)c\partial c\rangle
\end{aligned} \tag{4.44}$$

In (4.44) there is no need of UV subtractions. The last two terms in the RHS equal, respectively,

$$\begin{aligned}
& e^{-\varepsilon\partial_\varepsilon}\langle\frac{1}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)Bc\partial c\;\frac{\varepsilon}{(K+\phi_u+\varepsilon)}(\phi_u-\delta\phi_u)c\partial c\rangle \qquad (4.45)\\
= & e^{-\epsilon\partial_\varepsilon}\left(\varepsilon\int_0^\infty dt_1dt_2\; e^{-\epsilon t_2-\varepsilon t_1}\mathcal{G}(t_1,t_2)u^2g(uT)\Big\{\Big(-\frac{\partial_{uT}g(uT)}{g(uT)}\Big)^2+2G^2_{2uT}(\frac{2\pi t_1}{T})\Big\}\right) \\
= & e^{-\tilde\eta\partial_{\tilde\eta}}\left(\tilde\eta\int_0^\infty dss^2\int_0^1 dx\,\mathcal{E}(x)\;e^{-T(\epsilon(1-x)+\varepsilon x)}\mathrm{g}(s)\Big\{\Big(-\frac{\partial_s\mathrm{g}(s)}{\mathrm{g}(s)}\Big)^2+\frac{1}{2}G_s^2(2\pi x)\Big\}\right) \\
= & e^{-\tilde\eta\partial_{\tilde\eta}}\left(\tilde\eta\int_0^\infty dss^2\int_0^1 dx\;\;e^{-\tilde\eta s}\mathcal{E}(1-x)\;e^{s\frac{\epsilon-\varepsilon}{2u}x}\mathrm{g}(s)\Big\{\Big(-\frac{\partial_s\mathrm{g}(s)}{\mathrm{g}(s)}\Big)^2+\frac{1}{2}G_s^2(2\pi x)\Big\}\right)
\end{aligned}$$

and

$$\begin{aligned}
& e^{-\varepsilon\partial_\varepsilon}\langle\frac{\epsilon}{(K+\phi_u+\epsilon)}Bc\partial c\;\frac{\varepsilon}{(K+\phi_u+\varepsilon)}(\phi_u-\delta\phi_u)c\partial c\rangle \qquad (4.46)\\
= & e^{-\varepsilon\partial_\varepsilon}\left(\epsilon\varepsilon\int_0^\infty dt_1dt_2\;e^{-\epsilon t_2-\varepsilon t_1}\mathcal{G}(t_1,t_2)\frac{u}{t_1+t_2}\partial_u g(uT)\right) \\
= & e^{-\varepsilon\partial_\varepsilon}\left(\epsilon\varepsilon\int_0^\infty dT\,T\int_0^1 dx\;e^{-T(\epsilon(1-x)+\varepsilon x)}\mathcal{E}(x)u\partial_u g(uT)\right) \\
= & e^{-\tilde\eta\partial_{\tilde\eta}}\left(\tilde\eta\frac{\epsilon}{2u}\int_0^\infty dss^2\;e^{-\tilde\eta s}\int_0^1 dx\;\mathcal{E}(1-x)\;e^{s\frac{\epsilon-\varepsilon}{2u}x}\partial_s\mathrm{g}(s)\right)
\end{aligned}$$

As we have learnt in section 2 these quantities must be evaluated in the limit $\epsilon\to 0$. We are by now very familiar with this type of integrals and can easily come to the conclusion that both angular integrations are finite even without the $e^{s\frac{\epsilon-\varepsilon}{2u}x}$ factors so that in the limit $\epsilon,\varepsilon\to 0$ the integration is continuous in ϵ,ε and such factors can be dropped. Thus, using always the same representation, the former integral is just -2β. The latter is the same as eq.(4.21) of [54]. It is convergent both in the UV and the IR.

So we find

$$\lim_{\epsilon\to 0}\langle\Phi(\epsilon,\varepsilon)e^{-\varepsilon\partial_\varepsilon}\left(\frac{\varepsilon}{(K+\phi_u+\varepsilon)}(\phi_u-\delta\phi_u)c\partial c\right)\rangle = -2\beta+2\beta-0=0$$

This result is subtraction-independent because all amplitudes on the RHS of 4.43 turn out to be UV finite and so 4.43 can be considered an acceptable test (even if for one particular correlator) that in this context EOM are not anomalous. Furthermore this result confirm what we would have found for ψ_u, i.e. with the prescription of fixing $\epsilon = 0$ before doing integrals, the reason being that the overall s integrand has, in the limit $\epsilon \to 0$, the right convergent behaviour for large s in order to guarantee continuity in ϵ also at $\epsilon = 0$[‡]. This confirms that, as far as physical observables (in particular the energy) are concerned, there is no ambiguity in considering either ψ_u or ψ_u^ϵ as lump solutions of EOM. Another important remark is that the Schwinger representation, when tested in well defined correlators, is actually continuous in the $\epsilon \to 0$ limit, even when EOM are taken into account. 4.47 as opposed to 4.32 is also a very suggestive result in the light of our distribution theory intuition of what is going on: $\psi_u^\epsilon - \psi_{u,\epsilon}$, in particular, appears as the canonical example of what a good test state should behave like, giving correlators with a convergent behavior in the IR and (possibly) no UV singularity.

4.5 A distribution theory inspired approach to the problem

In the previous section we have argued that changing from the usual Schwinger representation

$$\frac{1}{K+\phi} = \int_0^\infty dt\, e^{-t(K+\phi)} \tag{4.47}$$

to the "regularized" one

$$\frac{1}{K+\phi} = \lim_{\epsilon\to 0} \frac{1}{K+\phi+\epsilon} = \int_0^\infty dt\, e^{-t(K+\phi+\epsilon)} \tag{4.48}$$

doesn't change the expected behavior of the lump solution and Sen's conjecture about lower dimensional Dp-branes is fully confirmed once the energy for the lump solution on the tachyon condensation vacuum is considered. In our approach what makes this one the physical quantity to consider in order to properly test the lump solution, rather than the energy on the D25 brane perturbative vacuum used to define the solution, is not only its clear physical meaning, but also at least two nice mathematical characteristics that are for sure consequences of its physicality: it is free of UV singularities and so doesn't need a specific subtraction rule, which at the level of our current knowledge can be done only at the level of correlators making their value completely arbitrary; it is integrable in the IR, even without the regulator factor $e^{-\epsilon t}$, which means the regulator

[‡]What happens here is that we have the difference of two integrals which are divergent (without the $e^{-\tilde{\eta}s}$) but the divergences cancel each other in the limit $\epsilon \to 0$.

can be safely removed inside the integrand. This renders our result unambiguous for the interchange of the two representations. EOM have also been seen to hold if we consider correlators with these characteristics.
Of course the issues we have discussed so far are a manifestation of the general problem of how, in the context of the $KBc\phi$ algebra, it is possible to introduce the inverse of K or $K+\phi$. The standard approach through Schwinger's representation is problematic because such objects like the projectors $\Omega^\infty = \lim_{-\Lambda\to\infty} e^{-\Lambda K}$ or $\Omega_u^\infty = \lim_{\Lambda\to\infty} e^{\Lambda(K+\phi_u)}$ are assumed to be non null and well defined just to be able to reproduce the change of boundary conditions associated with the tachyon vacuum or lump solutions. In the half string formalism we can write $K+\phi = (K_1^L + \phi_u(\frac{1}{2}))|I\rangle$, where K_1^L is the left translation operator, a symmetric operator in the Fock space, and $|I\rangle$ is the star algebra identity. The spectrum of $\mathcal{K}_u \equiv K_1^L + \phi_u(\frac{1}{2})$, which is also a symmetric operator, lies in the real axis and is likely to include also the origin. If it does and the identity string field contains the zero mode of $\mathcal{K}_u$, then a problem of invertibility arises and a prescription about how to treat the singularity is necessary.
As we have seen, the ϵ regularization we have introduced leads to a formal obstruction to invert $K+\phi_u$

$$\frac{1}{K+\phi_u+\epsilon}(K+\phi_u) = 1 - \mathcal{A}_\epsilon \tag{4.49}$$

The quantity

$$\mathcal{A}_0 = \lim_{\epsilon\to 0} \mathcal{A}_\epsilon, \qquad \mathcal{A}_\epsilon = \frac{\epsilon}{K+\phi_u+\epsilon} \tag{4.50}$$

is a representation of the deformed sliver Ω_u^∞. This quantity, whatever it is, is nonvanishing only where $K+\phi_u$ vanishes, i.e. in correspondence with the zero mode of $\mathcal{K}_u$. $\mathcal{A}_0$ has support, if any, only on this zero mode. Identity 4.49 also leads to a modified version of EOM for the regularized solution

$$Q\psi_u + \psi_u\psi_u = \lim_{\epsilon\to 0} \frac{\epsilon}{K+\phi_u+\epsilon}(\phi_u - \delta\phi_u)c\partial c \tag{4.51}$$

The term on the RHS $\Gamma(\epsilon) = \lim_{\epsilon\to 0} \mathcal{A}_\epsilon(\phi_u - \delta\phi_u)c\partial c$ is also proprtional to the formad sliver. All of this is quite reminiscent of what happens in distribution theory, in particular when we try to define the inverse of x. It is well known one can define several distributions

$$(\mathrm{x}^{-1}, \varphi) = \int_0^\infty dx\, x^{-1}(\varphi(x) - \varphi(-x)) \tag{4.52}$$

$$(\mathrm{x}+i0)^{-1} = \mathrm{x}^{-1} - i\pi\delta(x) \tag{4.53}$$

$$(\mathrm{x}-i0)^{-1} = \mathrm{x}^{-1} + i\pi\delta(x) \tag{4.54}$$

which are all consistent inverses although differing by a δ distribution. Thus the problem of defining a regularization of $\frac{1}{x}$ on the full real axis does not have a unique solution. The various distributions that solve this problem differ from one another by a distribution with support on the singularity. In particular (4.52) is well defined for test functions such that $\varphi(x) - \varphi(-x)$ vanishes at the origin and so cannot contain such a δ distribution. On the other hand, it should be stressed that, if one chooses (4.53) or (4.54) as regularizations of $\frac{1}{x}$, the delta function in the RHS is an integral part of the definition of the inverse and so in a genuinely distributional approach no violation of the identity $\frac{1}{x} \cdot x = 1$ can come from such a term. This means all formally violating terms should actually be the zero distribution. This is easily checked in standard distribution theory. In fact one can notice that

$$\lim_{\epsilon\to 0} \frac{1}{|x| + \epsilon} |x| = 1 - \lim_{\epsilon\to 0} \frac{\epsilon}{|x| + \epsilon} \tag{4.55}$$

so that we can empirically interpret $\lim_{\epsilon\to 0} \frac{1}{|x|+\epsilon}$ as the inverse of $|x|$, provided the second term in the RHS of (4.55) vanishes (a similar manipulation leads to (4.49) and (4.51)). One can use rectangular functions vanishing outside $|x| < a$ as test functions. Therefore, up to a multiplicative constant, it is enough to integrate between $-b$ and c ($0 < b, c < a$). The result is

$$\int_{-b}^{c} dx\, \frac{1}{|x| + \epsilon} = \log \frac{(c+\epsilon)(b+\epsilon)}{\epsilon^2} \approx -2\log(\epsilon) + \dots$$

Therefore the second piece on the RHS of (4.55) vanishes, and $\lim_{\epsilon\to 0} \frac{1}{|x|+\epsilon}$ is a formula for the inverse of $|x|$.

Proceeding in the same way for the inverse of $|x|^n + \epsilon$

$$\lim_{\epsilon\to 0} \frac{1}{|x|^n + \epsilon} |x|^n = 1 - \lim_{\epsilon\to 0} \frac{\epsilon}{|x|^n + \epsilon} \tag{4.56}$$

we find that

$$\int_{b}^{c} dx \int_{0}^{\infty} dt\, e^{-t(|x|^n+\epsilon)} \sim \begin{cases} \epsilon^{\frac{1-n}{n}} & n > 1 \\ \epsilon^{\frac{1-n}{n}} \log\epsilon & 0 < n \leq 1 \end{cases} \tag{4.57}$$

and again the second term in the RHS of (4.56) vanishes, so we conclude that $\lim_{\epsilon\to 0} \frac{1}{|x|^n+\epsilon}$ is a good empirical formula for the inverse of $|x|^n$.

The expression $\lim_{\epsilon\to 0} \frac{\epsilon}{|x|^n+\epsilon}$ has support, if any, in $x = 0$, but it is evident from the above that, for instance for $n > 1$, we can have a nonvanishing, and finite, result (a delta-function-like object) only for $\lim_{\epsilon\to 0} \frac{\epsilon^{1-1/n}}{|x|^n+\epsilon}$.

In particular a δ distribution cannot show up because we can never get

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\sqrt{\epsilon}}{x^2 + \epsilon} \tag{4.58}$$

Of course usual distribution theory cannot be applied directly in the framework of SFT in which a position in space (for instance $r = 0$ in ordinary field theory) is replaced by a string configuration (for instance the state representing the zero mode above). Nevertheless it suggests a possible strategy to assess whether such string fields as $\mathcal{A}_\epsilon$ or $\Gamma(\epsilon)$ put any harm to our interpreting ψ_u or $\psi_{u,\epsilon}$ as solutions of EOM. Distributions are objects of the dual of a topological vector space. But this topological vector space is not a generic one, rather it is a space of functions with certain properties, so that its dual can be regarded as a space of 'derivatives' of locally integrable functions. Our aim is to investigate on the possibility to interpret the inverse of $K + \phi_u$ (and $\mathcal{A}_\epsilon$) as a distribution, i.e. a functional in a suitable topological vector space.

The first step consists in constructing this topological vector space. It is probably true that one should start from a very general and abstract point of view like the one envisaged by L.Schwartz, [65]. Here we take a more modest and unsophisticated, but constructive, attitude, by using our knowledge of correlators in open SFT. To be treatable, the vector space should have properties that make it similar to a space of functions, and the duality rule (i.e the rule by which we can evaluate a functional over the test states) should preferably be represented by an integral. This would allow us to use the analogy with ordinary distribution theory as close as possible. Fortunately this is possible in the present case, thanks to the Schwinger representation of the inverse of $K + \phi_u$:

$$\frac{1}{K + \phi_u} = \int_0^\infty dt\, e^{-t(K+\phi_u)} \tag{4.59}$$

This representation makes concrete the abstract properties of the functional in question and 'localizes' the zero mode of $\mathcal{K}_u$ at $t = \infty$ (for the representation (4.59) becomes singular when $K + \phi_u$ vanishes). This 'localization property' makes our life much easier because it allows us to formulate the problem of defining test states, dual functionals and their properties in terms of their t dependence via the Schwinger representation (4.59).

We anticipate that the test states cannot be 'naked' Fock space states because there is no way to interpret such states *as good test states* (see the discussion in the next section).

4.6 The space of test string fields

4.6.1 Good test string fields

Let us now construct a set of string states that have good properties in view of forming the topological vector space of test states we need for our problem[§]. Our final *distribution* or *regularization* will be analogous to the principal value regularization of x^{-1} (4.52), which is characterized by x^{-1} being evaluated on test functions vanishing at the origin.

First of all the states we are looking for must be such that the resulting contractions with $\Gamma(\epsilon) = \mathcal{A}_\epsilon(\phi_u - \delta\phi_u)c\partial c$ be nonsingular (with respect to singularities due to collapsing points). But, especially, they must be characterized by integrable behaviour in the UV and, ignoring the overall $e^{-\epsilon t}$ factor, in the IR. The IR corresponds to $t \to \infty$, where, as was noticed above, the zero mode of $\mathcal{K}_u$ is 'localized'. Therefore the IR behaviour will be crucial in our discussion.

Consider states created by multiple products of the factor $H(\phi_u, \epsilon) = \frac{1}{K+\phi_u+\epsilon}(\phi_u - \delta\phi_u)$ and contract them with

$$\Lambda(\epsilon) = \frac{1}{K + \phi_u + \epsilon}(\phi_u - \delta\phi_u)c\partial c \tag{4.60}$$

From what we said above we are looking for contractions which are finite and whose $\epsilon \to 0$ limit is continuous. More precisely, let us define

$$\mathbf{\Psi}_n(\phi_u, \epsilon) = H(\phi_u, \epsilon)^{n-1} Bc\partial c\, H(\phi_u, \epsilon), \qquad n \geq 2 \tag{4.61}$$

Contracting with $\Lambda(\epsilon)$: $\langle \mathbf{\Psi}_n(\phi_u, \epsilon) Bc\partial cB\, \Lambda(\epsilon)\rangle$, we obtain a correlator whose IR and UV behaviour (before the the $\epsilon \to 0$ limit is taken) is not hard to guess. The correlators take the form

$$\begin{aligned} \langle \mathbf{\Psi}_n(\phi_u, \epsilon) Bc\partial cB\, \Lambda(\epsilon)\rangle = \int_0^\infty ds\, s^n e^{-\widetilde{\eta} s}\, g(s) \,\cdot \\ \cdot \int \prod_{i=1}^n dx_i\, \mathcal{E}\left(\left(-\frac{\partial g(s)}{g(s)}\right)^{n+1} + \ldots + \left(-\frac{\partial g(s)}{g(s)}\right)^{n-k+1} G_s^k + \ldots + G_s^{n+1} \right) \end{aligned} \tag{4.62}$$

where the notation is the same as in the previous section ($s = 2uT$ and $\widetilde{\eta} = \frac{\epsilon}{2u}$), but we have tried to make it as compact as possible. The angular variables x_i have been dropped in $\mathcal{E}$ and G_s. Using the explicit form of G_s, [52], expanding the latter with the

[§]This section is based on the results of [58] binomial formula and integrating over the angular variables, one gets

$$\int \prod_{i=1}^{n} dx_i \, \mathcal{E}\, G_s^k = \sum_{l=0}^{k} \frac{1}{s^{k-l}} \sum_{n_1,\ldots,n_l} \frac{P_l(n_1,\ldots,n_l)}{Q_l(n_1,\ldots,n_l)} \prod_{i=1}^{l} \frac{1}{p_i(n_1,\ldots,n_l)+s} \tag{4.63}$$

The label l counts the number of cosine factors in each term. Here n_i are positive integral labels which come from the discrete summation in G_s; $p_i(n_1,\ldots,n_l)$ are polynomials linear in n_i. Next, P_l and Q_l are polynomials in n_i which come from the integration in the angular variables. Every integration in x_i increases by 1 the difference in the degree of Q_l and P_l, so that generically $\deg Q_l - \deg P_l = n$. But in some subcases the integration over angular variables gives rise to Kronecker deltas among the indices, which may reduce the degree of Q_l. So actually the relation valid in all cases is $\deg Q_l \geq \deg P_l$. But one has to take into account that the number of angular variables to be summed over decreases accordingly.

We are now in the condition to analyze the UV behaviour of (4.62). Let us consider, for instance, the first piece

$$\sim \int_0^\infty ds \, e^{-\tilde{\eta} s} s^n \, g(\frac{s}{2}) \left(\frac{\partial_s g(\frac{s}{2})}{g(\frac{s}{2})}\right)^{n+1} \tag{4.64}$$

Since in the UV $g(\frac{s}{2}) \approx \frac{1}{\sqrt{s}}$, it is easy to see that the UV behaviour of the overall integrand is $\sim s^{-\frac{3}{2}}$, independently of n. As for the other terms, let us consider in the RHS of (4.63) the factor that multiplies $\frac{1}{s^{k-l}}$ (for $l \geq 2$). Setting $s = 0$, the summation over $n_1, \ldots, n_{l-1}$ is always convergent, so that the UV behaviour of each term in the summation is given by the factor $\frac{1}{s^{k-l}}$, with $2 \leq l \leq k$. It follows that the most UV divergent term corresponds to $l = 0$, i.e. $\sim \frac{1}{s^k}$. Since in (4.62) this is multiplied by

$$s^n \, g(\frac{s}{2}) \left(-\frac{\partial g(\frac{s}{2})}{g(\frac{s}{2})}\right)^{n-k+1} \tag{4.65}$$

we see that the UV behaviour of the generic term in (4.62) is at most as singular as $\sim s^{-\frac{3}{2}}$. *In conclusion the states $\mathbf{\Psi}_n$, when contracted with Λ_ϵ, give rise to the same kind of UV singularity $\sim s^{-\frac{3}{2}}$.* Now, for any two such states, say $\mathbf{\Psi}_n$ and $\mathbf{\Psi}_{n'}$, we can form a suitable combination such that the UV singularity cancels. In this way we generate infinite many states, say $\mathbf{\Phi}_n$, which, when contracted with $\Lambda(\epsilon)$, give rise to UV convergent correlators.

Let us consider next the IR properties ($s \gg 1$). All the correlators contain the factor $e^{-\tilde{\eta} s}$ which renders them IR convergent. So as long as this factor is present the states $\mathbf{\Phi}_n$ are both IR and UV convergent when contracted with $\Lambda(\epsilon)$. But the crucial, [58], IR properties (in the limit $\epsilon \to 0$) are obtained by ignoring the exponential factor. This is in order to guarantee the continuity of the $\epsilon \to 0$ limit. So, in analyzing the IR properties we will ignore this factor. The first term (4.64) is very strongly convergent in the IR, because $\partial_s g(\frac{s}{2}) \approx \frac{1}{s^2}$, while $g(\frac{s}{2}) \to 1$. For the remaining terms let us consider in the RHS of (4.63) the factor that multiplies $\frac{1}{s^{k-l}}$ (for $l \geq 2$). To estimate the IR behaviour it is very important to know the degree difference between the polynomials Q_l and P_l. Above we said that this difference is always nonnegative. In principle it could vanish, but from the example with $n = 2$, see [53], we know that there are cancellations and that in fact the difference in degree is at least 2. If this is so in general, we can conclude that the IR behaviour of the summation in the RHS of (4.63) with fixed l is $\sim \frac{1}{s^l}$. However, in order to prove such cancellations, one would have to do detailed calculations, which we wish to avoid here. So we will take the pessimistic point of view and assume that, at least for some of the terms, $\deg Q_l = \deg P_l$ (in which case there remains only one angular integration). In this case the IR behaviour of the corresponding term cannot decrease faster than $\sim \frac{1}{s^{l-1}}$. This has to be multiplied by $\sim \frac{1}{s^{k-l}}$ and by the IR behaviour of (4.65). This means that the least convergent term with fixed k in(4.63) behaves as $\sim \frac{1}{s^{n-k+1}}$. Since $k \leq n+1$, we see that in the worst hypothesis in the integral (4.62) there can be linearly divergent terms. If this is so the UV converging $\mathbf{\Phi}_n$ states are not good test states. However we can repeat for the IR singularities what we have done for the UV ones. Taking suitable differences of the $\mathbf{\Phi}_n$'s (this requires a two steps process, first for the linear and then for the logarithmic IR singularities¶), we can create an infinite set of states, $\mathbf{\Omega}_n$, which, when contracted with $\Lambda(\epsilon)$, yield a finite result and whose $\epsilon \to 0$ limit is continuous. Upon applying $\Gamma(\epsilon)$, instead of $\Lambda(\epsilon)$, such contractions of course vanish. These $\mathbf{\Omega}_n$ are therefore good (and nontrivial) test states. They are annihilated by $\Gamma(\epsilon)$.

We remark that in eq.(4.61) the presence of ϵ in $H(\phi_u, \epsilon)$ is not essential, because in estimating the IR behaviour we have not counted the $e^{-\tilde{\eta}s}$ factor. Using $\frac{1}{K+\phi_u}$ everywhere instead of $\frac{1}{K+\phi_u+\epsilon}$, would lead to the same results. This means that contracting the Ω_n states with Λ_ϵ leads to finite correlators *with or without* ϵ. We stress again that *the* $\epsilon \to 0$ *limit of such correlators is continuous.* This is the real distinctive features of good test states. The property of annihilating $\Gamma(\epsilon)$, is a consequence thereof. This remark will be used later on.

The $\Omega_n(\phi_u, \epsilon)$ are however only a first set of good test states. One can envisage a manifold of other such states. Let us briefly describe them, without going into too many details. For instance, let us start again from (4.61) and replace the first $H(\phi_u, \epsilon)$ factor with $\frac{1}{K+\phi_u+\epsilon} uX^{2k}$ (the term $\delta\phi$ can be dropped). In this way we obtain a new state depending on a new integral label k. However replacing X^2 with X^{2k} is too rough an

¶In the, so far not met, case where a $\log s$ asymptotic contribution appears in the integrand one would need a three step subtraction process.

operation, which renders the calculations unwieldy, because it breaks the covariance with respect to the rescaling $z \to \frac{z}{t}$. It is rather easy to remedy by studying the conformal transformation of X^{2k}. The following corrected replacements will do:

$$\begin{aligned}
uX^2 \quad &\to \quad u\left(X^2 + 2(\log u + \gamma)\right) = \phi_u \equiv \phi_u^{(1)} \\
uX^4 \quad &\to \quad u\left(X^4 + 12(\log u + \gamma)X^2 + 12(\log u + \gamma)^2\right) \equiv \phi_u^{(2)} \\
&\dots \\
uX^{2k} \quad &\to \quad u\left(\sum_{i=0}^{k} \frac{(2k)!}{(2k-2i)!i!} (\log u + \gamma)^i \, X^{2k-2i}\right) \equiv \phi_u^{(k)}
\end{aligned} \tag{4.66}$$

The role of the additional pieces on the RHS is to allow us to reconstruct the derivatives of $g(s)$ in computing the correlators, as was done in [52].

Now let us denote by $\mathbf{\Psi}_n^{(k)}$ the n-th state (4.61) where $\phi_u - \delta\phi_u$ in the first $H(\phi_u, \epsilon)$ factor is replaced by $\phi_u^{(k)}$. Contracting it with $\Lambda(\epsilon)$ it is not hard to see that the term (4.64) will be replaced by

$$\sim \int_0^\infty ds\, e^{-\tilde{\eta}s}\, s^n\, g(\frac{s}{2}) \left(\frac{\partial_s g(\frac{s}{2})}{g(\frac{s}{2})}\right)^{n+k} \tag{4.67}$$

with analogous generalizations for the other terms. It is evident from (4.67) that the UV behaviour becomes more singular with respect to (4.64) while the IR one becomes more convergent. This is a general property of all the terms in the correlator. Thus fixing k we will have a definite UV singularity, the same up to a multiplicative factor for all $\mathbf{\Psi}_n^{(k)}$. Therefore by combining a finite number of them we can eliminate the UV singularity and obtain another infinite set of UV convergent states $\mathbf{\Omega}_n^{(k)}$ for any k ($\mathbf{\Omega}_n^{(1)}$ will coincide with the previously introduced $\mathbf{\Omega}_n$). In general they will be IR convergent (IR subtractions may be necessary for $k = 2$ beside $k = 1$).

It goes without saying that the previous construction can be further generalized by replacing in (4.61) more than one X^2 factors with higher powers X^{2k}.

Qualitatively one can say that the correlators discussed so far have the form of an s integral

$$\int_0^\infty ds\, F(s) \tag{4.68}$$

where the $F(s)$ at the origin behaves as $s^{\frac{k}{2}}$, with integer $k \geq -1$, and F with all possible k's are present. At infinity, excluding the $e^{-\tilde{\eta}s}$, $F(s)$ behaves as $\frac{1}{s^p}$, for any integer $p \geq 2$. In addition, at infinity, we have any possible exponentially decreasing behaviour.

It is evident that *all* the states $\boldsymbol{\Omega}_n^{(k)}$ annihilate $\Gamma(\epsilon)$. In fact the $\boldsymbol{\Omega}_n^{(k)}$ are analogous to the test functions $\varphi(x)$ that vanish at the origin, like the ones considered for the regularization of the distribution x^{-1}. On the other hand, the only possibility of getting a nonzero result while contracting $\Gamma(\epsilon)$ with test states is linked, as usual, to correlators corresponding to IR linearly divergent integrals (without the exponential $e^{-\epsilon t}$). Now, such integrals are characterized by the fact that their $\epsilon \to 0$ limit is discontinuous, therefore the corresponding states can hardly be considered *good test states*. The true question we have to ask, then, is whether the good test states we have constructed are 'enough'.

4.7 The topological vector space of test states

Above we have introduced an infinite set of good test states which will be denoted generically by $\boldsymbol{\Omega}_\alpha$, $\alpha \in \mathbf{A}$ being a multi-index. We recall that in $\boldsymbol{\Omega}_\alpha$ there is also a dependence on the parameter ϵ. Such a dependence improves the IR convergence properties. The linear span of these state will be denoted by $\boldsymbol{\mathfrak{F}}$. It is a vector space. The problem now is to define a topology on it. First of all we define for any two states $\Omega_\alpha, \Omega_\beta$

$$\langle \Omega_\alpha | \Omega_\beta \rangle \equiv \langle \boldsymbol{\Omega}_\alpha B c \partial c B \Lambda_\epsilon \rangle \langle \boldsymbol{\Omega}_\beta B c \partial c B \Lambda_\epsilon \rangle \tag{4.69}$$

where in the RHS feature the previously defined correlators. From the analysis of the previous subsection this is a finite number, generically nonvanishing. Whenever a correlator of this kind depends on ϵ, the limit $\epsilon \to 0$ exists and is finite. The definition (4.69) can be extended by linearity to all finite combinations of the vectors $\boldsymbol{\Omega}$. Thus $\boldsymbol{\mathfrak{F}}$ is an inner product space. This inner product is not a scalar product in general. We will assume that it is nondegenerate (i.e. there are no elements with vanishing inner product with all the elements of the space)[‖] . The existence of an inner product does not mean by itself that $\boldsymbol{\mathfrak{F}}$ is a topological vector space, but it is possible to utilize it to define a topology.

4.7.1 Seminorm topology

There are various ways to introduce a topology in an inner product space $\mathcal{V}$. We will use seminorms. Let us denote by $x, y, \ldots$ the elements of $\mathcal{V}$, and by (x, y) the inner product.

[‖] If the inner product is degenerate the subsequent construction can be equally carried out, but it is more complicated, see for instance [66].

A *seminorm* is a function in $\mathcal{V}$ that satisfies the following axioms

$$\begin{aligned} & p(x) \geq 0 \\ & p(ax) = |a| p(x), \quad a \in \mathbb{C} \\ & p(x+y) \leq p(x) + p(y) \end{aligned} \tag{4.70}$$

Once we have an (infinite) family p_γ (γ is a generic index) of seminorms we can define a topology τ in the following way: a subset V is open if for any $x \in V$ there is a finite subset $p_{\gamma_1}, p_{\gamma_2}, \ldots, p_{\gamma_n}$ of seminorms and a positive number ϵ, such that any other element y satisfying $p_{\gamma_j}(x-y) < \epsilon$, for $j = 1, \ldots n$ belongs to V. A topology τ is *locally convex* if the vector space operations are continuous in τ and if a τ-neighborhood of any point x contains a convex neighborhood of the same point.

What we wish is of course a topology strictly related to the inner product. Therefore we introduce the concept of *partial majorant*. A partial majorant of the inner product $(\cdot, \cdot)$ is a topology τ which is locally convex and such that for any $y \in \mathcal{V}$ the function $\varphi_y(x) = (x, y)$ is τ-continuous. If (x, y) is jointly τ-continuous we say that τ is a *majorant*. Apart from being locally convex (the minimal requirement for the validity of the Hahn-Banach theorem), the main endowment of a majorant topology is the continuity of the inner product simultaneously in both entries.

In addition we say that a topology τ is *admissible* if 1) τ is a partial majorant and 2) for any linear τ-continuous functional $\varphi_0(x)$ there is an element $y_0 \in \mathcal{V}$ such that $\varphi_0(x) = (x, y_0)$. That is, all the continuous linear functionals can be expressed as elements of $\mathcal{V}$ via the inner product.

It is easy to prove that in any inner product space the function p_y defined by

$$p_y(x) = |(x, y)| \tag{4.71}$$

is a seminorm. The corresponding topology is the *weak topology* τ_0. This topology has important properties.

Theorem. The weak topology τ_0 is a partial majorant in $\mathcal{V}$. If the inner product is non-degenerate the space is *separated (Hausdorff)*. Moreover τ_0 is admissible.

When $\mathcal{V}$ is assigned the τ_0 topology, it will be denoted by $\mathcal{V}_w$.

Remark. The topology τ_0 is not a majorant. Indeed a theorem, [66], tells us that the topology τ_0 is a majorant only if the space is finite-dimensional, which is not our case. In view of this, in the applications below we will have to deal only with partial majorants.

Another question one could ask is whether τ_0 is metrizable. Another theorem says, quite predictably, that if τ_0 is metrizable it is also a normed partial majorant.

For later use we have to define the concept of bounded set. A subset B is bounded if for any neighborhood V of 0 there is a positive number λ such that $B \subset \lambda V$. In terms of seminorms we can say that B is bounded if all seminorms are bounded by some finite number in B.

One may wonder why we do not use the seminorm $\|x\| = \sqrt{|(x,x)|}$ to define the topology. This can be done and the corresponding topology is called *intrinsic*, τ_{int}. However such a topology does not guarantee continuity of all the functionals of the type $\varphi_y(x) = (x,y)$, see [66]. So one, in general, has to live with infinitely many seminorms.

Another important question in dealing with topological vector spaces is the existence of a countable base of neighborhoods. A base $\mathcal{B}$ of neighborhoods of the origin is a subset of all the neighborhoods of the origin such that any neighborhood in the given topology contains an element of $\mathcal{B}$. If the space is Hausdorff and the base is countable we say that the space satisfies the second axiom of countability, which is an important property because it permits us to use sequences (instead of filters) to study convergence.

Now, let us return to $\mathfrak{F}$ with the inner product $\langle\cdot|\cdot\rangle$ defined via (4.69). Using it we can define an infinite set of seminorms as above and thereby the weak τ_0 topology. In virtue of the preceding discussion $\mathfrak{F}$ becomes a topological vector space with a separated admissible topology. We can also assume that the second axiom of countability holds for $\mathfrak{F}$. This is due to the fact that, apart from the ϵ dependence, we can numerate the basis of all possible states Ω_α. As for ϵ we can discretize it, i.e replace it with a sequence ϵ_n tending to 0. In this way the index α is replaced by a discrete multi-index ν and we obtain a countable set of seminorms p_ν. The neighborhoods of the origin defined by these seminorms form a countable basis. Finally, $\mathfrak{F}$ with the τ_0 topology is not a normed partial majorant, therefore it is not metrizable.

To stress that $\mathfrak{F}$ is equipped with the τ_0 topology we will use the symbol $\mathfrak{F}_w$.

We could stop at this point, remarking that, since the topology is admissible, any continuous functional can be expressed in terms of $\mathfrak{F}$. The τ_0 topology is so 'coarse' that it accommodates simultaneously test states and distributions. However in $\mathfrak{F}$ we can have a stronger topology. We say that a topology τ_1 is stronger or finer than τ_2 ($\tau_1 \geq \tau_2$) if any open set in τ_2 is an open set also in τ_1. It is a theorem that if τ is locally convex and stronger than τ_0 it is also a partial majorant, which guarantees continuity of the scalar product also wrt τ. We will shortly introduce on $\mathfrak{F}$ the *strong topology*. But to do so we need first to discuss the topology on the dual.

4.7.2 The dual space

Given a topological vector space $\mathcal{V}$ as above, the dual $\mathcal{V}'$ is the space of linear continuous functionals. Let us denote linear continuous functionals by $x', y', ...$ and their evaluation over a point $x \in \mathcal{V}$ by $x'(x), y'(x), ...$

The *weak topology* over $\mathcal{V}'$ can be defined as follows: a sequence of linear continuous functionals x'_n weakly converges to 0, if the numerical sequence $x'_n(x)$ converges to 0 for any $x \in \mathcal{V}$. Alternatively one can define a basis of neighborhoods of zero in $\mathcal{V}'$ as follows:

$$U'_\epsilon(x_1, \dots, x_r) = \left\{x' \in \mathcal{V}' \middle| \, |x'(x_j)| \leq \epsilon, \quad j = 1, \dots r\right\} \tag{4.72}$$

for any subset $\{x_1, \dots x_r\}$ in the family of finite subsets of $\mathcal{V}$. This topology turns $\mathcal{V}'$ into a locally convex topological vector space.

A subset $B' \in \mathcal{V}'$ is (weakly) bounded if for any neighborhood U'_ϵ as in (4.72) there exist a positive number λ such that $\lambda B' \subset U'_\epsilon$.

The space $\mathcal{V}'$ with the weak topology will be denoted $\mathcal{V}'_w$.

We can immediately transfer these concepts to the space $\mathfrak{F}'$ of linear continuous functionals over $\mathfrak{F}$, which is therefore itself a convex topological vector space. The space $\mathfrak{F}'$ with the weak topology will be denoted by ${\mathfrak{F}'}_w$.

4.7.3 The strong topology

Using the weak topology on $\mathcal{V}'$ we can now define the *strong topology* on $\mathcal{V}$. The latter is defined as the uniform convergence topology on all weakly bounded subsets of $\mathcal{V}'$. This means that a sequence x_n converges to 0 in $\mathcal{V}$ if the numerical functions $x'(x_n)$ converge to zero uniformly for x' in any bounded subset B of $\mathcal{V}'$. Alternatively we can define the strong topology by means of a basis of neighborhoods of 0. A neighborhood V_ϵ of 0 is defined by

$$V_\epsilon = \left\{x \in \mathcal{V} \middle| \sup_{x' \in B} |x'(x)| < \epsilon\right\} \tag{4.73}$$

for any ϵ and any bounded set $B \subset \mathcal{V}'$. $\mathcal{V}$ equipped with the strong topology will be denoted by $\mathcal{V}_s$.

We recall that when $\mathcal{V}$ is assigned the weak τ_0 topology, for any continuous functional $x' \in \mathcal{V}'$ we have $x'(x) = (x, y)$ for some $y \in \mathcal{V}$. This is generically not true for the dual of $\mathcal{V}$ when $\mathcal{V}$ is equipped with the strong topology. The dual of $\mathcal{V}_s$ is generally larger than $\mathcal{V}'$. In fact a theorem says that any seminorm which is lower semicontinuous in $\mathcal{V}_w$ is continuous in $\mathcal{V}_s$, in other words there are more continuous seminorms in $\mathcal{V}_s$ than in $\mathcal{V}_w$. Qualitatively speaking, this means that there is in $\mathcal{V}_s$ a smaller number of convergent sequences than in $\mathcal{V}_w$, which implies that there are more continuous functionals.

The dual of $\mathcal{V}_s$ will be denoted by $\mathcal{V}'_s$. For completeness we add that it can itself be equipped with a *strong topology* as follows: a neighborhood V'_ϵ of 0 in $\mathcal{V}'_s$ is defined by

$$V'_\epsilon = \left\{x' \in \mathcal{V}' \middle| \sup_{x\in B}|x'(x)| < \epsilon\right\} \tag{4.74}$$

for any ϵ and any bounded set $B \subset \mathcal{V}_s$. $\mathcal{V}'_s$ equipped with the strong topology will be denoted also as $\mathcal{V}'_{ss}$.

We can immediately transfer these concepts to the space $\mathfrak{F}$ and its duals. The space $\mathfrak{F}'$ with the weak topology will be denoted by $\mathfrak{F}'{}_w$ and $\mathfrak{F}$ with the strong topology will be denoted by $\mathfrak{F}_s$. The dual of the latter will be denoted with the symbol $\mathfrak{F}'_s$.

4.7.4 'Richness' of the space of test states

The space $\mathfrak{F}$ equipped with the weak or strong topology will be our *space of test states*. The dual of the latter, i.e. $\mathfrak{F}'$ or $\mathfrak{F}'_s$ will be our space of *generalized states* or *distributions*. We can equip the latter with the weak or strong topology according to the needs.

As in ordinary distribution theory we have to verify that $\mathfrak{F}$ is a rich enough filter so that no regular behaviour can escape through it. We first remark that the cardinality of the basis $\mathbf{\Omega}_\alpha$ with fixed ϵ is the same as the cardinality of the Fock space states $\mathcal{F}$. If we include the ϵ dependence the cardinality of $\mathfrak{F}$ is larger. Let us also add that in the representation (4.68) of correlators, any kind of inverse integer powers of s appears in the IR, and any kind of half integer power of s appear in the UV. This is what our intuition would suggest to guarantee completeness.

More formally, to be able to claim that $\mathfrak{F}$ is *rich enough*, we must show that a state that annihilates the full $\mathfrak{F}$ can only be 0. To see this let us consider a generic finite linear combination of states $\mathbf{\Omega}_\alpha$, say $\mathbf{\Upsilon}$, and suppose that

$$\langle \mathbf{\Upsilon} | \mathbf{\Omega}_\alpha \rangle = 0, \qquad \forall\, \mathbf{\Omega}_\alpha \in \mathfrak{F} \tag{4.75}$$

If such a state $\mathbf{\Upsilon}$ were to exist it would mean that the inner product (4.69) is degenerate. As far as we can exclude the degeneracy of the inner product we conclude that $\mathfrak{F}$ is a rich enough space of test states.

4.8 Some conclusions and comments

In the light of the construction presented in the previous section, we have got a consistent mathematical framework to assess whether the string fields $\mathcal{A}_0 = \lim_{\epsilon\to 0} \mathcal{A}_\epsilon$ and $\Gamma(\epsilon) = \lim_{\epsilon\to 0} \mathcal{A}_\epsilon(\phi_u - \delta\phi_u)c\partial c$ appearing in the RHS of eqs. (4.49) and (4.51) really lead to a violation of EOM.
The state Λ_ϵ can be accommodated in the dual of $\mathfrak{F}$. This follows from (4.69). Let us keep Ω_β fixed while Ω_α spans $\mathfrak{F}$. A discontinuity of $\langle \Omega_\alpha\, Bc\partial c\, \Lambda_\epsilon\rangle$ would imply a discontinuity of the inner product in the Ω_α entry on the LHS. But this contradicts the fact that in the τ_0 topology the inner product is separately continuous in the two entries. Therefore $\langle \Omega_\alpha\, Bc\partial c\, \Lambda_\epsilon\rangle$ is continuous, i.e. it belongs to $\mathfrak{F}'$ (for any value of ϵ including 0). As a consequence of the construction in sec.4.7, it also belongs to ${\mathfrak{F}_s}'$.

This is probably the simplest way to think of $\Lambda_0 = \lim_{\epsilon\to 0} \Lambda_\epsilon$ as a distribution. In analogy with the example x^{-1} (4.52) we call this the *principal value* regularization of Λ_0, i.e of $\frac{1}{K+\phi_u}$. For the same reason we can also conclude that $\mathcal{A}_0$ is the null distribution in $\mathfrak{F}'$ (see also the discussion below on this point). These conclusions hinge upon the structure of $\mathfrak{F}$, and in particular on the fact that all the test states correlators used to define the inner product are represented, via (4.68), by integrands $F(s)$ that decrease at least as fast as $\frac{1}{s^2}$ in the IR.

With the above principal value regularization it is not possible to capture the contribution (if any) from the 'pointlike' support of $\mathcal{A}_\epsilon$ for $\epsilon \to 0$. This question is important even regardless the invertibility of $K + \phi_u$, for, as we have mentioned in the introduction, it is believed that the limit: $\lim_{t\to\infty} e^{-t(K+\phi_u)}$, represents a sliver-like projector. It would be important to find an adequate mathematical representation of such an object (if it exists). We would now like to explore the possibility to capture such a delta-like object in the functional analytic framework introduced above.

The term

$$\mathcal{A}_0 = \lim_{\epsilon\to 0} \epsilon \int_0^\infty dt\, e^{-t(K+\phi_u+\epsilon)} \tag{4.76}$$

seems to be of the same kind as the one appearing in the RHS of (4.56), which were shown to be vanishing in a distributional sense. Since, what is relevant here is the eigenvalue of $\mathcal{K}_u$ near 0, we can think of replacing $K + \phi_u$ with its eigenvalue and integrating over it to simulate the path integration. The eigenvalue of $\mathcal{K}_u$ is a function of some spectral parameter κ. So we replace $K + \phi_u$ by κ^a, with $a > 0$ (it can only be a power of κ since it must vanish for $\kappa \to 0$). Then we have

$$\mathcal{A}_\epsilon = \epsilon \int_0^\infty dt e^{-t\epsilon} \int_0^m d\kappa\, e^{-t\kappa^a} \approx \epsilon \int_0^\infty dt\, e^{-t\epsilon} t^{-\frac{1}{a}} \sim \begin{cases} \epsilon^{\frac{1}{a}} \log \epsilon & a \leq 1 \\ \epsilon^{\frac{1}{a}} & a > 1 \end{cases} \tag{4.77}$$

where m is an arbitrary small finite number that does not affect the result.

Thus $\mathcal{A}_0 = 0$, at least according to this heuristic treatment. This approach understands a sort of strong operator topology. In order to capture a nonzero contribution in $\mathcal{A}_\epsilon$ one must allow for string states with corresponding integrands in (4.68) that tend to a constant value in the IR, when the factor $e^{-\epsilon t}$ is suppressed. This means that the $\epsilon \to 0$ limit for these states is discontinuous. Therefore they can hardly be considered test states. In conclusion, the empirical formula (4.49) does not seem to be fit to capture the delta-function-like content (if any) of $\frac{1}{K+\phi_u}$. Driven by the analogy with ordinary distributions, such an object could be defined by evaluating $(K + \phi_u)^\lambda$ with complex λ. Unfortunately we are unable to evaluate such an expression using the Schwinger representation.

In general one expects that there are several different ways to represent a regularized inverse of $K + \phi_u$, in analogy with the inverse of x in section 2. But the formalism we can avail ourselves of has at present technical limitations. The only sensible course (at least for the time being) is to use the principal value regulated inverse defined at the beginning of this section. This is what we understand from now on.

We can now go back to the discussion in sec. 2.4.3 about the conditions 1 and 2 required for ψ_ϕ to be a non trivial acceptable solution of EOM. When proving the equation of motion one has to use the regularized inverse of $K+\phi_u$. We have already remarked that in such a way there is no violation to the equation of motion. We have also clarified that the requirement of existence and regularity for $\frac{1}{K+\phi_u}$ means precisely that we can compute it against any test function. As for condition 1, it is of a different nature, it arises from a different requirement: if the homotopy operator $\frac{B}{K+\phi_u}$ applied to a normalized (perturbative) state were to yield a normalized state, the cohomology of $\mathcal{Q}_{\psi_u} = Q + \{\psi_u, \;\cdot\; \}$ would be trivial, and the solution ψ_u would not represent a lump. As shown in [52] there are more than one indication that this is not the case: $\mathrm{tr}\frac{1}{K+\phi_u}$ is infinite and we have just shown that $\frac{1}{K+\phi_u}$ must be understood as a distribution. This is enough to reassure us that $\frac{B}{K+\phi_u}$ is not a good homotopy operator.

4.8.1 Final comments

In the paper [59] we have proposed a framework in which objects such as the inverse of $K + \phi$ can be consistently defined. We have done it by introducing a locally convex topological vector space of string states, with either weak and strong topology, and using the dual space as a distribution space. The inverse of $K + \phi$ turns out to be an object in this space of functionals and to correspond to a regularization we have referred to as 'principal value' regularization. Although we have not done it in detail, also the inverse of K can be treated in a similar way (i.e. using matter as a regulator). Admittedly our approach has been very concrete and case-oriented. For instance, basing the topology on the the inner product (4.69) seems to strongly limit the power of the formalism. A more general approach should be possible along the lines of [65] (which however deals only with finite dimensional vector spaces). It is clear that the basic space is $\boldsymbol{\mathcal{F}}$ introduced in section 4.7 (or rather its generalization including ghosts and zero modes). Maps from the string world-sheet to this space represent string configurations. Therefore the latter space of maps and its topologies is the real thing to be studied. In this framework the Fock space states correspond to constant maps, and it is understandable that they may be of little use as test states. The hard problem is the definition of the topology in the above space of maps and the duality rule. In our construction in sections 4.6 and 4.7 both problems were solved thanks to the knowledge of the exact relevant partition function of [60]. In general one has to make do without it. This seems to be the true challenge.

Chapter 5

Trace anomalies in $\mathcal{N} = 1$ $4D$ supergravities

5.1 Introduction

In this chapter we want to analyze the supersymmetric extension of Weyl transformations in various types of supergravities, the minimal, $20 + 20$ non minimal and $16 + 16$ non minimal N=1 SUGRA in 4D, and study the general structure of trace anomalies [88]. To this end, rather than considering specific cases we carry out a cohomological analysis, whose validity is not limited to one-loop calculations.

The motivation for this research is twofold. On the one hand it has been pointed out recently that 'old' minimal supergravity in 4D (with 12+12 dofs) might be inconsistent due to the presence of an inherent global conserved current, [79]. It has also been suggested that a different type of SUGRA, referred to henceforth as $16 + 16$ *non minimal*, characterized by 16+16 dofs, may be exempt from this risk. This model has been identified with the supergravities studied in [83] and [84]. The study of conformal anomalies in these and other models is interesting not only in itself, but also because it allows us to identify what the 'superWeyl group' is, as will be seen below.

Another motivation arises from the proposal of [85] that a source of CP violation in a 4D theory coupled to gravity could come from the trace anomaly. The trace anomaly may contain, in principle, beside the Weyl density (square of the Weyl tensor)

$$\mathcal{R}_{nmkl}\mathcal{R}^{nmkl} - 2\mathcal{R}_{nm}\mathcal{R}^{nm} + \frac{1}{3}\mathcal{R}^2 \tag{5.1}$$

and the Gauss-Bonnet (or Euler) one

$$\mathcal{R}_{nmkl}\mathcal{R}^{nmkl} - 4\mathcal{R}_{nm}\mathcal{R}^{nm} + \mathcal{R}^2, \tag{5.2}$$

another nontrivial piece, the Pontryagin density

$$\epsilon^{nmlk}\mathcal{R}_{nmpq}\mathcal{R}_{lk}{}^{pq} \tag{5.3}$$

Each of these terms appears in the trace of the e.m. tensor with its own coefficient. The first two are denoted c and a, respectively. They are known at one-loop for any type of (Gaussian) matter [86, 87], and a is the protagonist of recent important developments, [90]. The coefficient of (5.3) is not sufficiently studied. One may wonder whether the appearance of such a term in the trace anomaly is compatible with supersymmetry, i.e. whether (5.3) has a supersymmetric counterpart expressed in terms of covariant superfields. Since it is hard to supersymmetrize these three pieces and relate them to one another in a supersymmetric context, the best course is to proceed in another way, that is to consider a conformal theory in 4D coupled to (external) supergravity formulated in terms of superfields and find all the potential superconformal anomalies. This will allow us to see whether (5.3) can be accommodated in an anomaly supermultiplet as a trace anomaly member.

This type of analysis was carried out long ago for minimal supergravity, see [93] and also [95]. Our purpose here is to extend it to other types of 4D supergravities. in particular to the new minimal SUGRA mentioned above, [83] and [84]. Unfortunately there is no unique choice of the torsion constraints for these theories and no unique superfield formalism, (see [96, 97] and [99, 100] for earlier 'minimal' formulations and [101] for their equivalence; see [102, 103] for earlier non-minimal formulations; see moreover [105–107] and the textbooks [104, 108]). Thus we have chosen to follow the formalism of [109], further expanded in [83, 110, 111]. The analysis of trace anomalies has turned out to be anything but standard, contrary to the case of minimal supergravity. The reason is that in the latter case the cohomological analysis can be done on a differential space formed by polynomials of the superfields. In the other above mentioned versions of supergravities one has to admit in the differential space also nonpolynomial expressions of the superfields, due to the essential role of dimensionless prepotentials in these models. To solve in a satisfactory way the cohomology problem one has to start from minimal supergravity and map its cocycles to the other models with the superfield mappings of ref.[108, 110]. Once this is clarified the possible superconformal anomalies are rather easily identified. Based also on the analysis carried out long ago in [93], one can conclude that there are, not unexpectedly, two independent anomalies corresponding to the square Weyl and Gauss-Bonnet densities, much like in minimal supergravity. The anomaly corresponding to the Gauss-Bonnet density has a particularly complicated form in non-minimal and new minimal supergravities, and could be identified only via the above mentioned mapping method.

The conclusion concerning the Pontryagin density (5.3) is negative: in all types of supergravities the Pontryagin density does not show up in the trace anomaly, but it appears in the chiral (Delbourgo-Salam) anomaly, which, as expected, belongs, together with the trace anomaly, to a unique supermultiplet.

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5.2 Supercurrent Multiplets

Like any other continuous global symmetry, supersymmetry is associated to a conserved current $S_{\mu\alpha}$, $\partial^{\mu} S_{\mu\alpha} = 0$. It is unique up to an improvement term of the form

$$S'_{\mu\alpha} = S_{\mu\alpha} + (\sigma_{\mu\nu})_{\alpha}^{\beta}\, \partial^{\nu} s_{\beta} \ . \tag{5.4}$$

The conserved energy-momentum tensor $T_{\mu\nu}$ can be also modified by a similar improvement.

$$T'_{\mu\nu} = T_{\mu\nu} + \left(\eta_{\mu\nu}\partial^2 - \partial_{\mu}\partial_{\nu}\right) t \tag{5.5}$$

Of course these improvements don't modify the conserved charges. It was originally pointed out by Ferrara and Zumino [82] that in supersymmetric field theory the supercurrent $S_{\mu\alpha}$ and the energy-momentum tensor $T_{\mu\nu}$ belong to the same supermultiplet, i.e. they appear as the components of a real vector superfield $\mathcal{J}_{\alpha\dot{\alpha}}$ which satisfy the conservation law

$$\overline{D}^{\dot{\alpha}} \mathcal{J}_{\alpha\dot{\alpha}} = D_{\alpha} X \ , \tag{5.6}$$

$$\tag{5.7}$$

with X some chiral superfield. These conditions don't completely fix the pair $(\mathcal{J}_{\mu}, X)$ which can be transformed as

$$\begin{aligned} \mathcal{J}'_{\alpha\dot{\alpha}} &= \mathcal{J}_{\alpha\dot{\alpha}} - i\partial_{\alpha\dot{\alpha}}\left(\Xi - \overline{\Xi}\right) = \mathcal{J}_{\alpha\dot{\alpha}} + [D_{\alpha}, \overline{D}_{\dot{\alpha}}]\left(\Xi + \overline{\Xi}\right) \ , && (5.8)\\ X' &= X + \frac{1}{2}\overline{D}^2\Xi \ , && (5.9)\\ \overline{D}_{\dot{\alpha}}\Xi &= 0 \ . && (5.10) \end{aligned}$$

At component level these transformations correspond to the the improvements 5.4 and 5.5. The θ-independent component of $\mathcal{J}_{\alpha\dot{\alpha}}$ is the (R-transformation) axial current $\mathcal{J}_\mu\big| = j_\mu$, corresponding to four bosonic operators, the linear θ-component contains the supersymmetry current $D_\alpha \mathcal{J}_\mu\big| = S_{\mu\alpha} + \frac{1}{3}(\sigma_\mu \overline{\sigma}^\rho S_\rho)$, corresponding to twelve fermionic operators, and at the $\theta\overline{\theta}$ we find the energy-momentum tensor, $2T_{\nu\mu} - \frac{2}{3}\eta_{\nu\mu}T - \frac{1}{4}\epsilon_{\nu\mu\rho\sigma}\partial^{[\rho}j^{\sigma]}$, with six bosonic operators. Furthermore the lowest component of the complex scalar $X\big| = x$ contains two more bosonic operators. That's why this multiplet is called the $12+12$ *minimal supermultiplet*. The axial current is concerned if and only if $X = 0$ up to the redefinition 5.8, i.e. if and only if the theory is superconformal. In fact the θ^2 component of X is $\frac{2}{3}T^\mu_\mu + i\partial^\nu j_\nu$. In this case we find the $8+8$ component supermultiplet of conformal supergravity. Whenever the theory has a continuous R-symmetry , it is possible to define another $\mathcal{R}$-multiplet $\mathcal{R}_{\alpha\dot{\alpha}}$ whose bottom component is the associated conserved $U(1)_R$ current $j^{(R)}_\mu$. $\mathcal{R}_{\alpha\dot{\alpha}}$ is a superfield and satisfies the constraints

$$\overline{D}^{\dot{\alpha}}\mathcal{R}_{\alpha\dot{\alpha}} = \chi_\alpha \ , \tag{5.11}$$

$$\overline{D}_{\dot{\alpha}}\chi_\alpha = \overline{D}_{\dot{\alpha}}\overline{\chi}^{\dot{\alpha}} - D^\alpha \chi_\alpha = 0 \tag{5.12}$$

These constraints, too, don't uniquely fix the supergravity multiplet. In fact usually a theory has several continuous R-symmetries, which differ by a continuous conserved non R-symmetry, which can be written in terms of a real linear superfield J ($D^2 J = 0$) and which acts on the R-multiplet according to the transformations

$$\begin{aligned} &\mathcal{R}'_{\alpha\dot{\alpha}} = \mathcal{R}_{\alpha\dot{\alpha}} + [\mathcal{D}_\alpha, \overline{\mathcal{D}}_{\dot{\alpha}}]\mathcal{J} \ , \\ &\chi'_\alpha = \chi_\alpha + \frac{3}{2}\overline{D}^2 D_\alpha J \ , \\ &D^2 J = 0 \ . \end{aligned} \tag{5.13}$$

A theory endowed with a FZ-multiplet and a $U(1)_R$ symmetry has also a real and well-defined U such that $\overline{D}^2 U = -2X$. One can therefore prove the shift

$$R_{\alpha\dot{\alpha}} = \mathcal{J}_{\alpha\dot{\alpha}} + [D_\alpha, \overline{D}_{\dot{\alpha}}]U \tag{5.14}$$

defines a R supermultiplet with $\chi_\alpha = \frac{3}{2}\overline{D}^2 D_\alpha U$. It was pointed out in [79, 80] that there are case when neither of these choices for the supergravity multiplet is expected to be fully consistent. Whereas of $\mathcal{R}_{\alpha\dot{\alpha}}$ the existence of a continuous R symmetry is a quite obvious necessary condition, for the FZ multiplet the situation is quite more subtle. However one can argue that when the theory has Fayet-Iliopoulos terms $\mathcal{J}_{\alpha\dot{\alpha}}$ is not gauge invariant whereas $\mathcal{R}_{\alpha\dot{\alpha}}$ is a a good gauge invariant operator. Another somehow related situation is the one of sigma models with non exact Kähler form, where $\mathcal{J}_{\alpha\dot{\alpha}}$ turns out to be not globally well defined. These considerations have led the authors of [79, 80] to introduce the so-called $\mathcal{S}$ multiplet

$$\begin{aligned}
&\overline{D}^{\dot\alpha}\mathcal{S}_{\alpha\dot\alpha} = \chi_\alpha + \mathcal{Y}_\alpha \ , \\
&\overline{D}_{\dot\alpha}\chi_\alpha = 0 \ , \qquad D^\alpha\chi_\alpha = \overline{D}_{\dot\alpha}\overline{\chi}^{\dot\alpha} \ , \\
&D_\alpha\mathcal{Y}_\beta + D_\beta\mathcal{Y}_\alpha = 0 \ , \qquad \overline{D}^2\mathcal{Y}_\alpha = 0 \ .
\end{aligned} \tag{5.15}$$

One can solve these constraints straightforwardly, finding the component expression for $\mathcal{S}_{\alpha\dot\alpha}$

$$\begin{aligned}
\mathcal{S}_\mu \ = \ & j_\mu - i\theta\left(S_\mu - \frac{i}{\sqrt{2}}\sigma_\mu\overline{\psi}\right) + i\overline{\theta}\left(\overline{S}_\mu - \frac{i}{\sqrt{2}}\overline{\sigma}_\mu\psi\right) + \frac{i}{2}\theta^2\overline{Y}_\mu - \frac{i}{2}\overline{\theta}^2 Y_\mu \\
& + \left(\theta\sigma^\nu\overline{\theta}\right)\left(2T_{\nu\mu} - \eta_{\nu\mu}A - \frac{1}{8}\epsilon_{\nu\mu\rho\sigma}F^{\rho\sigma} - \frac{1}{2}\epsilon_{\nu\mu\rho\sigma}\partial^\rho j^\sigma\right) \\
& -\frac{1}{2}\theta^2\overline{\theta}\left(\overline{\sigma}^\nu\partial_\nu S_\mu + \frac{i}{\sqrt{2}}\overline{\sigma}_\mu\sigma^\nu\partial_\nu\overline{\psi}\right) + \frac{1}{2}\overline{\theta}^2\theta\left(\sigma^\nu\partial_\nu\overline{S}_\mu + \frac{i}{\sqrt{2}}\sigma_\mu\overline{\sigma}^\nu\partial_\nu\psi\right) \\
& +\frac{1}{2}\theta^2\overline{\theta}^2\left(\partial_\mu\partial^\nu j_\nu - \frac{1}{2}\partial^2 j_\mu\right) \ .
\end{aligned} \tag{5.16}$$

The chiral superfield χ_α is given by

$$\begin{aligned}
\chi_\alpha = -i\lambda_\alpha(y) + \theta_\beta\left(\delta_\alpha{}^\beta D(y) - i(\sigma^{\mu\nu})_\alpha{}^\beta F_{\mu\nu}(y)\right) + \theta^2\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\overline{\lambda}^{\dot\alpha}(y) \ , \\
\lambda_\alpha = 2\sigma^\mu_{\alpha\dot\alpha}\overline{S}^{\dot\alpha}{}_\mu + 3\sqrt{2}i\psi_\alpha \ , \\
D = -4T^\mu{}_\mu + 6A \ , \\
F_{\mu\nu} = -F_{\nu\mu} \ , \qquad \partial_{[\mu}F_{\nu\rho]} = 0 \ ,
\end{aligned} \tag{5.17}$$

and the superfield $\mathcal{Y}_\alpha$ is given by

$$\begin{aligned}
\mathcal{Y}_\alpha = \sqrt{2}\psi_\alpha + 2\theta_\alpha F + 2i\sigma^\mu_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}Y_\mu - 2\sqrt{2}i\left(\theta\sigma^\mu\overline{\theta}\right)(\sigma_{\mu\nu})_\alpha{}^\beta\partial^\nu\psi_\beta \\
+i\theta^2\sigma^\mu_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\partial_\mu F + \overline{\theta}^2\theta_\alpha\partial^\mu Y_\mu - \tfrac{1}{2\sqrt{2}}\theta^2\overline{\theta}^2\partial^2\psi_\alpha \ , \\
\partial_{[\mu}Y_{\nu]} = 0 \ , \\
F = A + i\partial^\mu j_\mu \ .
\end{aligned} \tag{5.18}$$

These are 16 + 16 independent real operators and actually in [81] it was shown this is the most general supergravity multiplet including the conserved energy-momentum tensor $T_{\mu\nu}$ and supersymmetry current $S_\alpha\mu$ as the only operators with spin larger than one. Another fundamental requirement is that the multiplet be indecomposable, i.e. it cannot be separated into two decoupled supersymmetry mutiplet. This does't mean the multiplet is irreducible, that is it may contain non trivial sub-multplets, which are closed under supersymmetry transformation. This is for example the case of the $\mathcal{S}$-multiplet which is intertwined through 5.15 with χ_α and $\mathcal{Y}_\alpha$. In fact there are special cases in which, using the improvement transformations

$$\mathcal{S}_{\alpha\dot\alpha} \to \mathcal{S}_{\alpha\dot\alpha} + [D_\alpha, \overline{D}_{\dot\alpha}]U \ ,$$

$$
\begin{aligned}
&\chi_\alpha \to \chi_\alpha + \frac{3}{2}\overline{D}^2 D_\alpha U \ , \\
&\mathcal{Y}_\alpha \to \mathcal{Y}_\alpha + \frac{1}{2} D_\alpha \overline{D}^2 U \ ,
\end{aligned} \tag{5.19}
$$

we can set either χ_α or $\mathcal{Y}_\alpha$, or both to zero reducing the $\mathcal{S}$-multiplet to the FZ-multiplet or the R multiplet or the superconformal one respectively. If there is a well defined chiral superfield X such that $\mathcal{Y}_\alpha = D_\alpha X$, the $\mathcal{S}$-multplet take a simpler form

$$
\begin{aligned}
&\overline{D}^{\dot\alpha}\mathcal{S}_{\alpha\dot\alpha} = \chi_\alpha + D_\alpha X \ , \\
&\overline{D}_{\dot\alpha}\chi_\alpha = 0 \ , \qquad D^\alpha \chi_\alpha = \overline{D}_{\dot\alpha}\overline{\chi}^{\dot\alpha} \ , \\
&\overline{D}_{\dot\alpha} X = 0 \ .
\end{aligned} \tag{5.20}
$$

which is clearly an interpolation between the FZ-mutiplet and the R-multiplet. All these multiplets contain the energy-momentum tensor and the supersymmetry current, so it appears quite natural to take them as the supergravity current describing the coupling of gravity to matter, namely

$$
\int d^4\theta \mathcal{J}_{\alpha\dot\alpha} H^{\alpha\dot\alpha} \ . \tag{5.21}
$$

The analog is done for $\mathcal{R}_{\alpha\dot\alpha}$ and $\mathcal{S}_{\alpha\dot\alpha}$. $H^{\alpha\dot\alpha}$ is a areal vector field, whose $\theta\overline{\theta}$ component contains the metro field $h_{\mu\nu}$, a two form field $B_{\mu\nu}$ and a real scalar. It can be interpreted as the source of the supergravity current. The coordinate and local supersymmetry transformations are encoded in a complex superfield L_α such that

$$
H_\mu\big| = H_\mu\big|_\theta = H_\mu\big|_{\overline{\theta}} = 0 \ , \tag{5.22}
$$

Imposing gauge invariance of 5.21 imposes some constraints on the parameter L_α which depend on the defining condition of the multiplets. This in turn fixes the field content of $H^{\alpha\dot\alpha}$ after gauge fixing. In this way one can recognize the FZ-multiplet and the $\mathcal{R}$-multiplet correspond to the old minimal supergravity [96–98] and the new minimal supergravity [99, 100] respectively. The $\mathcal{S}$-multiplet has instead been associated with the $16 + 16$ supergravity [83, 84]. An alternative approach would be to let the transformations 5.22 unconstrained and introduce new compensator superfields to have the right content of fields associated with each supergravity. In this way different supergravities don't differ for the definition of their gauge transformations, but for their content of compensator superfields. The advantage of this method is that in this way one can naturally introduce superconformal transformations as transformations with independent parameters directly associated to compensator superfields.

5.3 N=1 minimal supergravity in D=4 and its superfields

For the notation we follow [109]. The superspace of $N = 1$ supergravity is spanned by the supercoordinates $Z^M = (x^m, \theta^\mu, \overline{\theta}_{\dot{\mu}})$. The minimal $N = 1$ supergravity in $D = 4$ can be formulated in terms of the superfields: $R(z), G_a(z)$ and $W_{\alpha\beta\gamma}(z)$. R and $W_{\alpha\beta\gamma}$ are chiral while G_a is real. We will also need the antichiral superfields $R^+(z)$ and $\overline{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}(z)$, conjugate to R and $W_{\alpha\beta\gamma}$, respectively. $W_{\alpha\beta\gamma}$ is completely symmetric in the spinor indices $\alpha, \beta, ...$ These superfields are subject to the constraints:

$$\begin{aligned}
&\nabla^\alpha G_{\alpha\dot{\beta}} = \overline{\nabla}_{\dot{\beta}} R^+, \qquad \overline{\nabla}^{\dot{\beta}} G_{\alpha\dot{\beta}} = \nabla_\alpha R \\
&\nabla^\alpha W_{\alpha\beta\gamma} + \frac{i}{2}(\nabla_{\beta\dot{\beta}} G_\delta{}^{\dot{\beta}} + \nabla_\beta{}^{\dot{\delta}} G_{\dot{\beta}}{}^{\dot{\beta}}) = 0 \\
&\overline{\nabla}^{\dot{\alpha}} W_{\dot{\alpha}\dot{\beta}\dot{\gamma}} + \frac{i}{2}(\nabla_{\beta\dot{\beta}} G^\beta{}_{\dot{\delta}} + \nabla_\beta{}^{\dot{\delta}} G^\beta{}_{\dot{\beta}}) = 0
\end{aligned} \tag{5.23}$$

The latter are found by solving the (super)Bianchi identities for the supertorsion and the supercurvature

$$T^A = dE^A + E^B \phi_B{}^A = \frac{1}{2} E^C E^B T_{BC}{}^A = \frac{1}{2} dz^M dz^N T_{NM}{}^A \tag{5.24}$$

$$R_A{}^B = \frac{1}{2} E^C E^D R_{DCA}{}^B = dz^M dz^N \partial_N \phi_{MA}{}^B + dz^M \phi_{MA}{}^C dz^N \phi_{NC}{}^B$$

where $\phi_{MA}{}^B$ is the superconnection and $E^A = dz^M E_M{}^A$ the supervierbein

$$E_M{}^A E_A{}^N = \delta_M{}^N, \qquad E_A{}^M E_M{}^B = \delta_A{}^B,$$

after imposing by hand the restrictions

$$\begin{aligned}
&T_{\underline{\alpha}\underline{\beta}}{}^{\underline{\gamma}} = 0, \quad T_{\alpha\beta}{}^c = T_{\dot{\alpha}\dot{\beta}}{}^c = 0 \\
&T_{\alpha\dot{\beta}}{}^c = T_{\dot{\beta}\alpha}{}^c = 2i\sigma_{\alpha\dot{\beta}}{}^c \\
&T_{\underline{\alpha}b}{}^c = T_{b\underline{\alpha}}{}^c = 0, \qquad T_{ab}{}^c = 0
\end{aligned} \tag{5.25}$$

where $\underline{\alpha}$ denotes both α and $\dot{\alpha}$. The superdeterminant of the vierbein $E_M{}^A$ will be denoted by E.

The Bianchi identities are

$$\mathcal{D}\mathcal{D}E^A = E^B R_B{}^A, \qquad \nabla T^A = E^B R_B{}^A \tag{5.26}$$

where $\mathcal{D} = dz^M \nabla_M$ and $\nabla_A = E_A{}^M \nabla_M$. Imposing (5.26) one gets all the components of T^A and $R_A{}^B$ in terms of $R, G_a, W^{\alpha\beta\gamma}$ and their conjugates. The other Bianchi identity

$$(\mathcal{D}R)_A{}^B = 0 \tag{5.27}$$

is automatically satisfied.

5.3.1 Superconformal symmetry and (super)anomalies

Superconformal transformations are defined by means of the chiral superfield parameter $\sigma = \sigma(z)$ and its conjugate $\overline{\sigma}$.

$$\begin{aligned}
\delta E_M{}^a &= (\sigma + \overline{\sigma}) E_M{}^a \\
\delta E_M{}^\alpha &= (2\overline{\sigma} - \sigma) E_M{}^\alpha + \frac{i}{2} E_M{}^a \overline{\sigma}_a^{\dot\alpha\alpha} \nabla_{\dot\alpha} \overline{\sigma} \\
\delta E_M{}^{\dot\alpha} &= (2\sigma - \overline{\sigma}) E_M{}^{\dot\alpha} + \frac{i}{2} E_M{}^a \overline{\sigma}_a^{\dot\alpha\alpha} \nabla_\alpha \sigma \\
\delta \phi_{M\alpha\beta} &= E_{M\alpha} \nabla_\beta \sigma + E_{M\beta} \nabla_\alpha \sigma + (\sigma^{ab})_{\alpha\beta} E_{Ma} \nabla_b (\sigma + \overline{\sigma})
\end{aligned} \tag{5.28}$$

where

$$\phi_{M\alpha}{}^\beta = \frac{1}{2} \phi_{Mab} (\sigma^{ab})_\alpha{}^\beta, \qquad \phi_{M\dot\beta}^{\dot\alpha} = \frac{1}{2} \phi_{Mab} (\overline{\sigma}^{ab})^{\dot\alpha}\beta$$

The transformations (5.28) entail

$$\begin{aligned}
\delta E &= 2(\sigma + \overline{\sigma}) E \\
\delta R &= (2\overline{\sigma} - 4\sigma) R - \frac{1}{4} \nabla_{\dot\alpha} \nabla^{\dot\alpha} \overline{\sigma} \\
\delta R^+ &= (2\sigma - 4\overline{\sigma}) R^+ - \frac{1}{4} \nabla^\alpha \nabla_\alpha \sigma \\
\delta G_a &= -(\sigma + \overline{\sigma}) G_a + i \nabla_a (\overline{\sigma} - \sigma) \\
\delta W_{\alpha\beta\gamma} &= -3\sigma W_{\alpha\beta\gamma}
\end{aligned} \tag{5.29}$$

If we promote the superfield σ to a superghost superfield, by inverting the spin-statistics connection, so that it becomes an anticommuting parameter, it is easy to prove that the above transformations are nilpotent.

Let us define the functional operator that implements these transformations, i.e.

$$\Sigma = \int_{x\theta} \delta\chi_i \frac{\delta}{\delta\chi_i}$$

where χ_i represent the various superfields in the game and $_{x\theta}$ denotes integration $d^4 x d^4 \theta$. This operator is nilpotent: $\Sigma^2 = 0$. As a consequence it defines a cohomology problem. The cochains are integrated local expressions of the superfields and their superderivatives, invariant under superdiffeomorphism and local superLorentz transformations. Candidates for superconformal anomalies are nontrivial cocycles of Σ which are not coboundaries, i.e. integrated local functionals Δ_σ, linear in σ, such that

$$\Sigma\,\Delta_\sigma = 0, \qquad \text{and} \qquad \Delta_\sigma \neq \Sigma\,\mathcal{C} \tag{5.30}$$

for any integrated local functional $\mathcal{C}$ (not containing σ).

The complete analysis of all the possible nontrivial cocycles of the operator Σ was carried out long ago in [93]. It was shown there that the latter can be cast into the form

$$\Delta_\sigma = \int_{x\theta} \left[\frac{E(z)}{-8R(z)}\,\sigma(z)\,\mathcal{S}(z) + h.c. \right] \tag{5.31}$$

where $\mathcal{S}(z)$ is a suitable chiral superfield. In [93] all the possibilities for $\mathcal{S}$ were classified. For pure supergravity (without matter) the only nontrivial possibilities turn out to be:

$$\mathcal{S}_1(z) = W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} \qquad \text{and} \qquad \mathcal{S}_2(z) = (\overline{\nabla}_{\dot\alpha}\overline{\nabla}^{\dot\alpha} - 8R)(G_aG^a + 2RR^+) \tag{5.32}$$

(the operator $(\overline{\nabla}_{\dot\alpha}\overline{\nabla}^{\dot\alpha} - 8R)$ maps a real superfield into a chiral one).

It is well-known that the (5.31) cocycles contain not only the trace anomaly, but a full supermultiplet of anomalies. The local expressions of the latter are obtained by stripping off the corresponding parameters from the integrals in (5.31). Let us recall also that the conversion of σ to an anticommuting parameter is not strictly necessary: eq.(5.30) simply corresponds to the Wess-Zumino consistency conditions, i.e. to the invariance under reversing the order of two successive (Abelian) gauge transformations. But an anticommuting σ allows us to use the incomparably simpler formalism of cohomology.

5.3.2 Meaning of superconformal transformations

Eqs. (5.32) are rather implicit and it is opportune to see the corresponding expressions in component fields, at least as far as the dependence on the metric alone is concerned. This reduction has been done in [112]. We repeat it here for pedagogical reasons, but also because the formalism we use is different from the one of [112]. The method below will be used throughout the paper. In general the expressions of the above cocycles in components are extremely complicated and really unmanageable because of the presence of auxiliary fields. We are interested in recognizing the two cocycles (5.32) when only the metric is taken into account while all the other fields are ignored, so that we can compare them with the usual Weyl cocycles (the squared Weyl tensor, the Gauss-Bonnet and the Pontryagin densities). Our task in the sequel is to extract such expressions from (5.32). We will refer to them as the *ordinary* parts of the cocycles.

We first introduce the relevant components fields and clarify the meaning of the components in the parameters superfield $\sigma(z)$. To start with let us define the lowest component fields of the supervierbein as in [109]

$$E_M{}^A(z)\big|_{\theta=\bar\theta=0} = \begin{pmatrix} e_m{}^a(x) & \frac{1}{2}\psi_m{}^\alpha(x) & \frac{1}{2}\overline{\psi}_{m\dot\alpha}(x) \\ 0 & \delta_\mu{}^\alpha & 0 \\ 0 & 0 & \delta^{\dot\mu}{}_{\dot\alpha} \end{pmatrix} \tag{5.33}$$

and

$$E_A{}^M(z)\big|_{\theta=\bar\theta=0} = \begin{pmatrix} e_a{}^m(x) & -\frac{1}{2}\psi_a{}^\mu(x) & -\frac{1}{2}\overline{\psi}_{a\dot\mu}(x) \\ 0 & \delta_\alpha{}^\mu & 0 \\ 0 & 0 & \delta^{\dot\alpha}{}_{\dot\mu} \end{pmatrix} \tag{5.34}$$

where $e_m{}^a$ are the usual 4D vierbein and $\psi_m{}^\alpha(x), \overline{\psi}_{m\dot\alpha}(x)$ the gravitino field components. We have in addition

$$R(z)\big|_{\theta=\bar\theta=0} = -\frac{1}{6}M(x), \qquad G_a(z)\big|_{\theta=\bar\theta=0} = -\frac{1}{3}b_a(x) \tag{5.35}$$

where M is a complex scalar field and b_a is a real vector field. As for the superconnection we have

$$\phi_{mA}{}^B\big|_{\theta=\bar\theta=0} = \omega_{mA}{}^B(x), \qquad \phi_{\mu A}{}^B\big|_{\theta=\bar\theta=0} = 0, \qquad \phi_{\dot\mu A}{}^B\big|_{\theta=\bar\theta=0} = 0, \tag{5.36}$$

and $\omega_{mA}{}^B(x)$ is of course of the Lorentz type. Its independent components turn out to be

$$\begin{aligned} &\omega_{nml} \equiv e_m{}^a e_{lb}\omega_{na}{}^b = \\ &= \frac{1}{2}\Big[e_{na}(\partial_m e_l{}^a - \partial_l e_m{}^a) - e_{la}(\partial_n e_m{}^a - \partial_m e_n{}^a) - e_{ma}(\partial_l e_n{}^a - \partial_n e_l{}^a)\Big] \\ &+\frac{i}{4}\Big[e_{na}\left(\psi_l\sigma^a\overline{\psi}_m - \psi_m\sigma^a\overline{\psi}_l\right) - e_{la}\left(\psi_m\sigma^a\overline{\psi}_n - \psi_n\sigma^a\overline{\psi}_m\right) - e_{ma}\left(\psi_n\sigma^a\overline{\psi}_l - \psi_l\sigma^a\overline{\psi}_n\right)\Big] \end{aligned} \tag{5.37}$$

This has the same symmetry properties in the indices as the usual spin connection and reduces to it when the gravitino field is set to 0.

It is then easy to prove, using (5.25), that

$$R_{nma}{}^b\big|_{\theta=\bar\theta=0} = \partial_n\omega_{ma}{}^b - \partial_m\omega_{na}{}^b + \omega_{ma}{}^c\omega_{nc}{}^b - \omega_{na}{}^c\omega_{mc}{}^b \equiv \mathcal{R}_{nma}{}^b \tag{5.38}$$

This relation will be used later on. In conclusion the independent component fields are the vierbein, the gravitino and the two auxiliary fields M and b_a.

Let us come now to the interpretation of the superconformal transformations (5.28). To this end we expand the chiral superfield $\sigma(z)$ in the following way:

$$\sigma(z) = \omega(x) + i\alpha(x) + \sqrt{2}\Theta^\alpha \chi_\alpha(x) + \Theta^\alpha\Theta_\alpha(F(x) + iG(x)) \tag{5.39}$$

where we have introduced new anticommuting variables Θ^α, which, unlike θ^μ, carry Lorentz indices. This is always possible, see [109]: the first term on the RHS corresponds to $\sigma\big|_{\theta=\bar\theta=0}$, χ_α to $\nabla_\alpha\sigma\big|_{\theta=\bar\theta=0}$, and $F(x)+iG(x)$ to $\nabla^\alpha\nabla_\alpha\sigma\big|_{\theta=\bar\theta=0}$. Comparing now with the first equation in (5.28), and taking into account (5.33,5.34), we see that $\omega(x)$ is the parameter of the ordinary Weyl transformation, while comparing with the second and third equation in (5.28) one can see that ψ_α and $\overline{\psi}^{\dot\alpha}$ transform with opposite signs with respect to the parameter $\alpha(x)$. Thus $\alpha(x)$ is the parameter of an ordinary chiral transformation.

Therefore when (5.32) is inserted in (5.31) the term linear in $\omega(x)$ will represent a conformal anomaly, while the term linear in $\alpha(x)$ will represent a chiral (Delbourgo-Salam) anomaly. Similarly the term linear in χ_α is the supercurrent anomaly. For the meaning of the cocycles linear in $F(x)$ and $G(x)$ see for instance [112]. Not surprisingly all these anomalies form an $N = 1$ supermultiplet.

The next step is to derive the conformal and chiral anomalies in components.

5.3.3 Anomalies in components

To derive the anomalies in components we have to integrate out the anticommuting variables. To this end it is convenient to use, instead of the superdeterminant E, the chiral density $\mathcal{E}$ (see [109]). The latter is defined by

$$\mathcal{E}(z) = a(x) + \sqrt{2}\Theta\rho(x) + \Theta\Theta f(x) \tag{5.40}$$

where $a(x) = \frac{1}{2}e(x) \equiv \frac{1}{2}\det e_m{}^a$. The ρ and f components contain, beside e the gravitino and/or the auxiliary field M, and they vanish when the latter are set to 0. We can rewrite our two integrated cocycles as follows

$$\Delta_\sigma^{(i)} = \int d^4x \left(\int d^2\Theta\, \mathcal{E}(z)\, \sigma(z)\, \mathcal{S}_i(z) \,+\, h.c.\right), \qquad i = 1,2 \tag{5.41}$$

This means that, given the interpretation of the lowest components of $\sigma(z)$ as the parameters of the conformal and chiral transformations, and due to (5.40), the *ordinary* part of the conformal and chiral anomaly terms (i.e. the terms linear in ω and α) will depend on $\nabla\nabla\mathcal{S}_i \equiv \nabla^\alpha\nabla_\alpha\mathcal{S}_i$, because this corresponds to the coefficient of $\Theta\Theta$ in the expansion of $\mathcal{S}_i$. So finally we can write

$$\Delta_\sigma^{(i)} \approx 4 \int d^4x \left(\frac{1}{2} e\,(\omega + i\alpha)\,\nabla\nabla S_i|_{\theta=\bar{\theta}=0} + h.c. \right), \qquad i = 1, 2 \tag{5.42}$$

where $\approx$ means 'up to terms that vanish when all the fields except the metric are set to 0'. The anomalous trace of the energy-momentum tensor and the divergence of the chiral current are obtained from the integral on the RHS of (5.42) by stripping off it the parameters ω and α, respectively.

5.3.3.1 The square Weyl cocycle

Let us start from S_1. The relevant terms to be considered are $\nabla^\alpha \nabla_\alpha W_{\beta\gamma\delta} W^{\beta\gamma\delta}$ and $\nabla^\alpha W^{\beta\gamma\delta} \nabla_\alpha W_{\beta\gamma\delta}$ at $\theta = \bar{\theta} = 0$. The term $W_{\beta\gamma\delta}|_{\theta=\bar{\theta}=0}$ is linear in the gravitino field and in the field b_a. As a consequence this term does not affect the ordinary part of the anomaly. On the contrary the square derivative of W does affect the ordinary part of the anomaly. It is therefore necessary to compute it explicitly. The symmetric part of $\nabla_\alpha W_{\beta\gamma\delta}|_{\theta=\bar{\theta}=0}$ can be computed as follows. Let us consider the identity

$$R_{nma}{}^b = E_n{}^c E_m{}^d R_{cda}{}^b + E_n{}^{\underline{\gamma}} E_m{}^d R_{\underline{\gamma}da}{}^b + E_n{}^c E_m{}^{\underline{\delta}} R_{c\underline{\delta}a}{}^b - E_n{}^{\underline{\gamma}} E_m{}^{\underline{\delta}} R_{\underline{\gamma}\underline{\delta}a}{}^b \tag{5.43}$$

and evaluate it at $\theta = \bar{\theta} = 0$. We know the LHS due to (5.38). The RHS contains various expressions, and in particular the totally symmetrized derivative $\nabla_{(\alpha} W_{\beta\gamma\delta)}$. It is possible to project it out and get

$$\nabla_{(\alpha} W_{\beta\gamma\delta)}|_{\theta=\bar{\theta}=0} = -\frac{1}{16} (\sigma^a \bar{\sigma}^b \epsilon)_{(\alpha\beta} (\sigma^c \bar{\sigma}^d \epsilon)_{\gamma\delta)} \mathcal{R}_{abcd} \tag{5.44}$$

and similarly

$$\nabla_{(\dot{\alpha}} W_{\dot{\beta}\dot{\gamma}\dot{\delta})}|_{\theta=\bar{\theta}=0} = -\frac{1}{16} (\epsilon \bar{\sigma}^a \sigma^b)_{(\dot{\alpha}\dot{\beta}} (\epsilon \bar{\sigma}^c \sigma^d)_{\dot{\gamma}\dot{\delta})} \mathcal{R}_{abcd} \tag{5.45}$$

where $\mathcal{R}_{abcd} = e_a{}^n e_b{}^m \mathcal{R}_{nmcd}$.

Using the second equation in (5.23) one can easily obtain

$$\begin{aligned} \nabla_\alpha W_{\beta\gamma\delta} \;&=\; \nabla_{(\alpha} W_{\beta\gamma\delta)} \\ &\quad + \frac{i}{4} \left(\epsilon_{\alpha\beta} (\sigma^{ab}\epsilon)_{\gamma\delta} + \epsilon_{\alpha\gamma} (\sigma^{ab}\epsilon)_{\beta\delta} + \epsilon_{\alpha\delta} (\sigma^{ab}\epsilon)_{\beta\gamma} \right) (\nabla_a G_b - \nabla_b G_a) \end{aligned} \tag{5.46}$$

and a similar equation for the conjugate derivative. Now let us see, as an example of arguments that will be repeatedly used in the sequel, that $\nabla_a G_b - \nabla_b G_a$ evaluated at $\theta = \bar{\theta} = 0$ does not contribute to the ordinary part of the anomaly. In fact $\nabla_a G_b$ cannot contribute to it, for we have

$$\nabla_a G_b = E_a{}^M \partial_M G_b + E_a{}^M \phi_{Mb}{}^c G_c$$

The last term, when evaluated at $\theta = \overline{\theta} = 0$ is linear in the field b_c. The second term in the RHS can be written

$$E_a{}^M \partial_M G_b\big|_{\theta=\overline{\theta}=0} = -\frac{1}{3} e_a{}^m \partial_m b_a - \frac{1}{2} e_a{}^m \psi_m{}^\alpha \nabla_\alpha G_b\big| - \frac{1}{2} e_a{}^m \overline{\psi}_{m\dot{\alpha}} \nabla^{\dot{\alpha}} G_b\big|$$

where the vertical bar stands for $\big|_{\theta=\overline{\theta}=0}$. Since both $\nabla_\alpha G_b\big|$ and $\nabla^{\dot{\alpha}} G_b\big|$ vanish when the gravitino and the auxiliary fields are set to 0, it follows that also $\nabla_a G_b$ vanishes in the same circumstances. Therefore for our purposes only the completely symmetrized spinor derivative of W matters in (5.46). We will write

$$\nabla_\alpha W_{\beta\gamma\delta} \approx \nabla_{(\alpha} W_{\beta\gamma\delta)}, \qquad \nabla_{\dot{\alpha}} W_{\dot{\beta}\dot{\gamma}\dot{\delta}} \approx \nabla_{(\dot{\alpha}} W_{\dot{\beta}\dot{\gamma}\dot{\delta})} \tag{5.47}$$

to signify that the LHS is equal to the RHS up to terms that vanish when the gravitino and the auxiliary fields are set to 0.

Now it is a lengthy but standard exercise to verify that

$$\nabla^\alpha W^{\beta\gamma\delta} \nabla_\alpha W_{\beta\gamma\delta}\big| \approx \frac{1}{8}\Big(\mathcal{R}_{nmkl}\mathcal{R}^{nmkl} - 2\mathcal{R}_{nm}\mathcal{R}^{nm} + \frac{1}{3}\mathcal{R}^2 + \frac{i}{2}\epsilon^{nmlk}\mathcal{R}_{nmcd}\mathcal{R}_{lk}{}^{cd}\Big) \tag{5.48}$$

where $\mathcal{R}_{nmkl} = e_n{}^a e_m{}^b e_k{}^c e_l{}^d \mathcal{R}_{abcd}$, $\mathcal{R}_{nm} = e_k{}^a e^{kb} \mathcal{R}_{anbm}$ and $\mathcal{R} = e_n{}^a e^{nc} e_m{}^b e^{md} \mathcal{R}_{abcd}$. The first three terms in brackets in the RHS are easily recognized to correspond to the ordinary Weyl density, while the fourth term is the Pontryagin density. We thus have

$$\begin{aligned}
\Delta_\sigma^{(1)} &\approx 4\int d^4x\, e\Big[(\omega + i\alpha)\, \nabla^\alpha W^{\beta\gamma\delta}\, \nabla_\alpha W_{\beta\gamma\delta}\big| + h.c.\Big] \\
&\approx \frac{1}{2}\int d^4x\, e\Big[(\omega + i\alpha)\Big(\mathcal{R}_{nmkl}\mathcal{R}^{nmkl} - 2\mathcal{R}_{nm}\mathcal{R}^{nm} + \frac{1}{3}\mathcal{R}^2 + \frac{i}{2}\epsilon^{nmlk}\mathcal{R}_{nmcd}\mathcal{R}_{lk}{}^{cd}\Big) + h.c.\Big] \\
&= \int d^4x\, e\Big\{\omega\Big(\mathcal{R}_{nmkl}\mathcal{R}^{nmkl} - 2\mathcal{R}_{nm}\mathcal{R}^{nm} + \frac{1}{3}\mathcal{R}^2\Big) - \frac{1}{2}\,\alpha\,\epsilon^{nmlk}\mathcal{R}_{nmpq}\mathcal{R}_{lk}{}^{pq}\Big\}
\end{aligned} \tag{5.49}$$

In the last line one recognizes the conformal Weyl anomaly linear in ω and the Delbourgo-Salam anomaly linear in α.

5.3.3.2 The Gauss-Bonnet cocycle

The second cocycle is determined by $\nabla\nabla\mathcal{S}_2\big|_{\theta=\overline{\theta}=0}$ and its hermitian conjugate. Since G_a, R, R^+ and their first order spinorial derivative evaluated at $\theta = \overline{\theta} = 0$ all vanish when the gravitino and auxiliary fields are set to 0, the *ordinary* part of the cocycle will be determined by

$$\nabla\nabla S_2\big| \approx -4\nabla^\beta\overline{\nabla}_{\dot\alpha}G_a\,\nabla_\beta\overline{\nabla}^{\dot\alpha}G^a\big| + 2\nabla^\alpha\nabla_\alpha R\,\overline{\nabla}_{\dot\alpha}\overline{\nabla}^{\dot\alpha}R^+\big| \tag{5.50}$$

The second term is well known, see [109]. We have $\nabla\nabla R\big| \approx -\frac{1}{3}\mathcal{R}$, so

$$\nabla^\alpha\nabla_\alpha R\,\overline{\nabla}_{\dot\alpha}\overline{\nabla}^{\dot\alpha}R^+\big| \approx \frac{1}{9}\mathcal{R}^2 \tag{5.51}$$

It remains for us to compute $\nabla_\beta\overline{\nabla}_{\dot\alpha}G_a\big|$. From (5.43) we can derive $\overline{\nabla}_{\dot\alpha}\nabla_\beta G_a\big|$. On the other hand we have

$$(\nabla_\beta\overline{\nabla}_{\dot\alpha} + \overline{\nabla}_{\dot\alpha}\nabla_\beta)G_a = R_{\dot\alpha\beta a}{}^bG_b - T_{\dot\alpha\beta}{}^B\nabla_BG_a = -2G_a\sigma^b{}_{\beta\dot\alpha}G_b - 2i\sigma^b{}_{\beta\dot\alpha}\nabla_bG_a \approx 0$$

Therefore

$$\nabla_\beta\overline{\nabla}_{\dot\alpha}G_a \approx -\overline{\nabla}_{\dot\alpha}\nabla_\beta G_a \tag{5.52}$$

Next, using the notation $\overline{\nabla}_{\dot\alpha}\nabla_\alpha G_{\beta\dot\beta} = \sigma^a{}_{\beta\dot\beta}\overline{\nabla}_{\dot\alpha}\nabla_\alpha G_a$, we introduce the following decomposition

$$\overline{\nabla}_{\dot\alpha}\nabla_\alpha G_{\beta\dot\beta} = A_{(\alpha\beta)(\dot\alpha\dot\beta)} + \epsilon_{\alpha\beta}\,B_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta}\,C_{(\alpha\beta)} + \epsilon_{\alpha\beta}\,\epsilon_{\dot\alpha\dot\beta}\,D \tag{5.53}$$

Now we remark that (5.43) contains the part of $\overline{\nabla}_{\dot\alpha}\nabla_\alpha G_{\beta\dot\beta}$ which is symmetric both in the couple α,β and $\dot\alpha,\dot\beta$. After some lengthy but straightforward calculation one can extract it and get

$$A_{(\alpha\beta)(\dot\alpha\dot\beta)} = -\frac{1}{2}R_{abcd}\,(\sigma^{ab}\epsilon)_{\alpha\beta}\,(\epsilon\overline{\sigma}^{cd})_{\dot\gamma\dot\delta} \tag{5.54}$$

Next, contracting the decomposition (5.53) with $\epsilon^{\beta\alpha}$ and using the first equation in (5.23) we get

$$\epsilon^{\beta\alpha}\overline{\nabla}_{\dot\alpha}\nabla_\alpha G_{\beta\dot\beta} = 2\,B_{\dot\alpha\dot\beta} + 2\epsilon_{\dot\alpha\dot\beta}\,D = \nabla_{\dot\alpha}\nabla_{\dot\beta}R^+ \approx -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\,\overline{\nabla\nabla}R^+ \tag{5.55}$$

A similar result one gets by contracting (5.53) with $\epsilon^{\dot\alpha\beta}$. We conclude that

$$\begin{aligned} &B_{(\dot\alpha\dot\beta)}\big| \approx 0, \qquad C_{(\alpha\beta)}\big| \approx 0 \\ &D\big| \approx -\frac{1}{4}\nabla\nabla R\big| \approx -\frac{1}{4}\overline{\nabla\nabla}R^+\big| \approx \frac{1}{12}\mathcal{R} \end{aligned} \tag{5.56}$$

The remaining computation is straightforward. We get

$$\begin{aligned} \nabla\nabla\mathcal{S}_2\big| &\approx -4\nabla^\beta\overline{\nabla}_{\dot\alpha}G_a\,\nabla_\beta\overline{\nabla}^{\dot\alpha}G^a\big| + 2\nabla^\alpha\nabla_\alpha R\,\overline{\nabla}_{\dot\alpha}\overline{\nabla}^{\dot\alpha}R^+\big| \\ &\approx \frac{4}{9}\mathcal{R}^2 - 2\mathcal{R}_{nm}\mathcal{R}^{nm} + \frac{2}{9}\mathcal{R}^2 = \frac{2}{3}\mathcal{R}^2 - 2\mathcal{R}_{nm}\mathcal{R}^{nm} \end{aligned} \tag{5.57}$$

that is

$$\Delta^{(2)}_\sigma = 4\int d^4x\, e\,\omega\big(\frac{3}{\mathcal{R}}^2 - 2\mathcal{R}_{nm}\mathcal{R}^{nm}\big) \tag{5.58}$$

This is not the Gauss-Bonnet density, as one could have expected. But it is easy to recover it by means of a linear combination of $\Delta^{(1)}_\sigma$ and $\Delta^{(2)}_\sigma$:

$$\Delta^{(1)}_\sigma + \frac{1}{2}\Delta^{(2)}_\sigma \approx \int d^4x\, e\Big\{\omega\big(\mathcal{R}_{nmkl}\mathcal{R}^{nmkl} - 4\mathcal{R}_{nm}\mathcal{R}^{nm} + \mathcal{R}^2\big) - \frac{1}{2}\,\alpha\,\epsilon^{nmlk}\mathcal{R}_{nmpq}\mathcal{R}_{lk}{}^{pq}\Big\} \tag{5.59}$$

which contains precisely the Gauss-Bonnet density*.

In conclusion $\Delta^{(1)}_\sigma$ corresponds to a multiplet of anomalies, whose first component is the Weyl density multiplied by ω, accompanied by the Pontryagin density (the Delbourgo-Salam anomaly) multiplied by α. On the other hand $\Delta^{(2)}_\sigma$ does not contain the Pontryagin density and the part linear in ω is a combination of the Weyl and Gauss-Bonnet density.

5.4 Non minimal supergravity

In supergravity there is a freedom in imposing the torsion constraints. A convenient choice is in terms of the so-called 'natural constraints'

$$\begin{aligned} &T_{ab}{}^c = 0, \qquad T_{\alpha\beta}{}^a = T_{\dot\alpha\dot\beta}{}^a = 0, \qquad T_{\alpha\dot\beta}{}^a = 2i\sigma^a_{\alpha\dot\beta}, \\ &T_\gamma{}^{\dot\beta}{}_{\dot\alpha} = (n-1)\delta^{\dot\beta}_{\dot\alpha}T_\gamma, \qquad T^{\dot\gamma}{}_\beta{}^\alpha = (n-1)\delta^\alpha_\beta\overline{T}^{\dot\gamma} \\ &T_{\gamma\beta}{}^\alpha = (n+1)(\delta^\alpha_\gamma T_\beta + \delta^\alpha_\beta T_\gamma), \qquad T^{\dot\gamma\dot\beta}{}_{\dot\alpha} = (n+1)(\delta^{\dot\gamma}_{\dot\alpha}\overline{T}^{\dot\beta} + \delta^{\dot\beta}_{\dot\alpha}\overline{T}^{\dot\gamma}) \\ &T_{\gamma b}{}^a = 2n\,\delta^a_b T_\gamma, \qquad T^{\dot\gamma}{}_b{}^a = 2n\,\delta^a_b\overline{T}^{\dot\gamma} \end{aligned} \tag{5.60}$$

where n is a numerical parameter and $T_\alpha, \overline{T}_{\dot\alpha}$ are new (conjugate) superfields in addition to those of minimal supergravity. The latter is obtained by setting $T_\alpha = 0$. $T_\alpha, \overline{T}_{\dot\alpha}$ are U(1) connections. The $U(1)\times U(1)$ gauge symmetry was added with the purpose of enlarging the minimal supergravity model. The solution for the Bianchi identities can be found in [110]. There are many significant changes with respect to the minimal model.

*For an early appearance of the Gauss-Bonnet and Weyl density anomalies in supergravity see [114, 115].

For instance $W_{\alpha\beta\gamma}$ and R are not chiral anymore, but

$$(\overline{\mathcal{D}}_{\dot{\alpha}} + (3n+1)\overline{T}_{\dot{\alpha}})W_{\alpha\beta\gamma} = 0 \tag{5.61}$$

$$(\overline{\mathcal{D}}_{\dot{\alpha}} + 2(n+1)\overline{T}_{\dot{\alpha}})R = 0 \tag{5.62}$$

where $\mathcal{D}$ replaces ∇ as covariant derivative†.

A distinguished superfield is S (and its conjugate $\overline{S}$), defined by

$$S = \mathcal{D}^{\alpha}T_{\alpha} - (n+1)T^{\alpha}T_{\alpha}, \tag{5.63}$$

which satisfies

$$\mathcal{D}_{\alpha}S = 8T_{\alpha}R^{+} \tag{5.64}$$

The combination

$$Y = 8R + 2(n+1)\overline{S} \tag{5.65}$$

is chiral, $\overline{\mathcal{D}}^{\dot{\alpha}}Y = 0$. The operator

$$\Delta = \mathcal{D}^{\alpha}\mathcal{D}_{\alpha} - 3(n+1)T^{\alpha}\mathcal{D}_{\alpha} - Y \tag{5.66}$$

projects a superfield without Lorentz indices to an antichiral superfield and $\frac{\Delta}{Y}$ is a chiral projector.

The *non minimal model* for supergravity is obtained by further imposing the constraint

$$R = R^{+} = 0 \tag{5.67}$$

with nonvanishing T_{α} and $\overline{T}_{\dot{\alpha}}$.

The non minimal supergravity has 20+20 degrees of freedom. The bosonic dofs are those of the minimal model, excluding R and R^{+}, plus 10 additional ones which can be identified with the lowest components of the superfields S, $\overline{S}, \overline{\mathcal{D}}_{\dot{\alpha}}T_{\alpha} = c_{\alpha\dot{\alpha}} + id_{\alpha\dot{\alpha}}$ and $\overline{\mathcal{D}}_{\alpha}T_{\dot{\alpha}} = -c_{\alpha\dot{\alpha}} + id_{\alpha\dot{\alpha}}$. The additional fermionic dofs can be identified with the lowest components of $T_{\alpha}, \overline{T}_{\dot{\alpha}}$ and $\mathcal{D}_{\alpha}\overline{S}$, $\overline{\mathcal{D}}_{\dot{\alpha}}S$.

†In principle there is no reason to use two different symbols for the covariant derivative, they denote the same covariant derivative in different settings. The use of two different symbols, however, will be instrumental in section 7.

5.4.1 Superconformal transformations in the non minimal model

In the non minimal model there are transformations compatible with the constraints that correspond to local vierbein rescalings. We will refer to them generically as superconformal transformations. They are good candidates for superWeyl transformations (i.e, for supersymmetric extensions of the ordinary Weyl transformations) but, as we shall see, do not automatically correspond to them. They are expressed in terms of an arbitrary (complex) superfield Σ

$$\begin{aligned}
\delta E_\alpha{}^M &= -(2\overline{\Sigma} - \Sigma)\, E_\alpha{}^M \\
\delta E_{\dot\alpha}{}^M &= -(2\Sigma - \overline{\Sigma})\, E_{\dot\alpha}{}^M \\
\delta E_a{}^M &= -(\Sigma + \overline{\Sigma})\, E_a{}^M + \frac{i}{2}\overline{\sigma}_a^{\dot\beta\beta}\,\overline{\mathcal{D}}_{\dot\beta}\left(\overline{\Sigma} - \frac{3n-1}{3n+1}\Sigma\right) E_\beta{}^M \\
&\quad + \frac{i}{2}\overline{\sigma}_a^{\beta\dot\beta}\mathcal{D}_\beta\left(\Sigma - \frac{3n-1}{3n+1}\overline{\Sigma}\right) E_{\dot\beta}{}^M \\
\delta T_\alpha &= -(2\overline{\Sigma} - \Sigma)\, T_\alpha + \frac{3}{3n+1}\mathcal{D}_\alpha\overline{\Sigma} \\
\delta W_{\alpha\beta\gamma} &= -3\Sigma\, W_{\alpha\beta\gamma} \\
\delta G_a &= -(\Sigma + \overline{\Sigma})\, G_a + i\mathcal{D}_a(\overline{\Sigma} - \Sigma) + \frac{1}{3}\overline{\sigma}_a^{\dot\alpha\alpha}\left(T_\alpha\overline{\mathcal{D}}_{\dot\alpha}\overline{\Sigma} - \overline{T}_{\dot\alpha}\mathcal{D}_\alpha\Sigma\right) \\
&\quad - \frac{3n-1}{3(3n+1)}\overline{\sigma}_a^{\dot\alpha\alpha}\left(T_\alpha\overline{\mathcal{D}}_{\dot\alpha}\Sigma - \overline{T}_{\dot\alpha}\mathcal{D}_\alpha\overline{\Sigma}\right) \\
\delta R^+ &= -2(2\overline{\Sigma} - \Sigma)R^+ \\
&\quad + \frac{1}{4(3n+1)}\left(\mathcal{D}^\alpha\mathcal{D}_\alpha + (n+1)T^\alpha\mathcal{D}_\alpha\right)\left[3n(\overline{\Sigma} - \Sigma) - (\overline{\Sigma} + \Sigma)\right]
\end{aligned} \tag{5.68}$$

From (5.67) and (5.68) we see that the superfield Σ is constrained by the linear condition

$$\left(\mathcal{D}^\alpha\mathcal{D}_\alpha + (n+1)T^\alpha\mathcal{D}_\alpha\right)\left[3n(\overline{\Sigma} - \Sigma) - (\overline{\Sigma} + \Sigma)\right] = 0 \tag{5.69}$$

5.4.2 Cocycles in non minimal SUGRA

It the non minimal model it is easy to construct an invariant (0-cocycle)

$$I^{(1)}_{n.m.} = \int_{x,\theta} E W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}\frac{\overline{T}_{\dot\alpha}\overline{T}^{\dot\alpha}}{\overline{S}^2} + h.c. \tag{5.70}$$

and a 1-cocycle

$$\Delta^{(1)}_{n.m.} = \int_{x,\theta} E\,\Sigma\, W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}\frac{\overline{T}_{\dot\alpha}\overline{T}^{\dot\alpha}}{\overline{S}^2} + h.c. \tag{5.71}$$

It is easy to prove that $\delta I^{(1)}_{n.m.} = 0 = \delta\Delta^{(1)}_{n.m.}$ for any n. To this end the condition (5.69) is inessential. If $R \neq 0$ this is not true anymore.

The construction of a second cocycle corresponding to $\Delta_\sigma^{(2)}$ above, is not as straightforward and will be postponed to section 7, after the technique of mapping between different supergravity models has been introduced.

5.5 The 16+16 nonminimal model

One way to define the new minimal model is to introduce a 2-superform B_{AB} and impose natural constraints on its supercurvature. In this way we obtain a 16+16 model. The independent bosonic dofs are the vierbein, the lowest component of $S, \overline{S}$, $c_{\alpha\dot\alpha}$ and $G_{\alpha\dot\alpha}$ (the components of $d_{\alpha\dot\alpha}$ are not independent in this model). The fermionic degrees of freedom are, beside the gravitino field, the lowest components of $T_\alpha, \overline{T}_{\dot\alpha}$ and $\mathcal{D}_\alpha\overline{S}, \mathcal{D}_{\dot\alpha}S$. The new dofs (with respect to the minimal model) are linked to the mode contained in B_{ab}. In new minimal supergravity the range of the parameter n is limited to $n>0$ and $n<-\frac{1}{3}$.

In practice this means that

$$T_\alpha = \mathcal{D}_\alpha\psi, \qquad T_{\dot\alpha} = \mathcal{D}_{\dot\alpha}\psi \tag{5.72}$$

where ψ is a (dimensionless) real superfield. The transformations corresponding to (5.68) on ψ are

$$\delta\psi = \frac{3}{3n+1}(\overline{\Sigma}-\overline{\Lambda}) = \frac{3}{3n+1}(\Sigma-\Lambda) \equiv \frac{3}{3n+1}L \tag{5.73}$$

where $\Lambda(\overline{\Lambda})$ is an arbitrary chiral (antichiral) superfield, and L is a real (vector) superfield. As a consequence the transformations (5.68), compatible with the constraints, for the surviving superfields take the form:

$$\begin{aligned}
&\delta E_\alpha{}^M = -(L+2\overline{\Lambda}-\Lambda)\,E_\alpha{}^M\\
&\delta E_{\dot\alpha}{}^M = -(L+2\Lambda-\overline{\Lambda})\,E_{\dot\alpha}{}^M\\
&\delta T_\alpha = -(L+2\overline{\Lambda}-\Lambda)\,T_\alpha + \frac{3}{3n+1}\mathcal{D}_\alpha L\\
&\delta W_{\alpha\beta\gamma} = -3(L+\Lambda)\,W_{\alpha\beta\gamma}\\
&\delta G_{\alpha\dot\alpha} = -(2L+\Lambda+\overline{\Lambda})\,G_{\alpha\dot\alpha} + i\mathcal{D}_{\alpha\dot\alpha}(\overline{\Lambda}-\Lambda) - \frac{2}{3}\left(T_\alpha\overline{\mathcal{D}}_{\dot\alpha}\overline{\Lambda} - \overline{T}_{\dot\alpha}\mathcal{D}_\alpha\Lambda\right)\\
&+\frac{2(3n-1)}{3(3n+1)}\left(T_\alpha\overline{\mathcal{D}}_{\dot\alpha}L - \overline{T}_{\dot\alpha}\mathcal{D}_\alpha L\right)\\
&\delta S = -2(L+2\overline{\Lambda}-\Lambda)S + 4\mathcal{D}^\alpha(L+\Lambda)T_\alpha - \frac{21n+5}{3n+1}\mathcal{D}^\alpha L\,T_\alpha + \frac{3}{3n+1}\mathcal{D}^\alpha\mathcal{D}_\alpha L
\end{aligned} \tag{5.74}$$

and (5.69) becomes

$$(\mathcal{D}^\alpha \mathcal{D}_\alpha + (n+1)T^\alpha \mathcal{D}_\alpha)\,(L + (3n+1)\Lambda) = 0 \tag{5.75}$$

5.5.1 Cocycles in new minimal 16+16 SUGRA

As in nonminimal SUGRA it is easy to construct an invariant

$$I^{(1)}_{new} = \int_{x,\theta} E W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} \frac{\overline{T}_{\dot\alpha}\overline{T}^{\dot\alpha}}{\overline{S}^2} + h.c. \tag{5.76}$$

and a 1-cocycle

$$\Delta^{(1)}_{new} = \int_{x,\theta} E\,(L+\Lambda) W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} \frac{\overline{T}_{\dot\alpha}\overline{T}^{\dot\alpha}}{\overline{S}^2} + h.c. \tag{5.77}$$

It is easy to prove that $\delta I^{(1)}_{new} = 0 = \delta\Delta^{(1)}_{new}$. Once again we don't need (5.75) to prove this. On the other hand it is not easy to construct a cocycle similar to $\Delta^{(2)}$, i.e. quadratic in the superfield G_a, which, after translation to component form, leads to the Gauss-Bonnet density.

5.6 Reduction to component form

In the following analysis the reduction of the cocycles to ordinary form will play a major role. Thus the purpose of this section is to outline the procedure to derive the component form of the cocycles in nonminimal and new minimal supergravities, as we have done in section 2 for the minimal supergravity anomalies. The operator, [110],

$$\overline{\Delta} = \overline{\mathcal{D}}_{\dot\alpha}\overline{\mathcal{D}}^{\dot\alpha} - 3(n+1)\overline{T}_{\dot\alpha}\overline{\mathcal{D}}^{\dot\alpha} - 2(n+1)\overline{S} \tag{5.78}$$

projects a generic superfield to a chiral superfield. Let U be a superfield without Lorentz indices. It is not hard to see that (see [110])

$$\int_{x,\theta} E\,U = \frac{1}{4n}\int_{x,\theta} E\,\overline{\Delta}\left(e^{-\overline{\Omega}}\,\frac{U}{\overline{\overline{S}}}\right) \tag{5.79}$$

where $\overline{\Omega} = 2(3n+1)\frac{\overline{T}\overline{T}}{\overline{S}}$. Therefore, introducing the appropriate chiral density $\mathcal{E}$, [111], we can write

$$\int_{x,\theta} E\,U = \frac{1}{4n}\int d^4x \int d^2\Theta\,\mathcal{E}\,\overline{\Delta}\left(U\,e^{\overline{\Omega}}\right) \tag{5.80}$$

For instance, when Σ is a chiral superfield the anomaly (5.71) can be written

$$\Delta_\Sigma^{(1)} = \frac{1}{4n}\int d^4x \int d^2\Theta\, \mathcal{E}\, \overline{\Delta}\Big[\Sigma\, W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} \frac{\overline{T}_{\dot\alpha}\overline{T}^{\dot\alpha}}{\overline{S}^2}\Big] + h.c. \tag{5.81}$$

When Σ is not chiral there is in the RHS an additional term which, however, is irrelevant for the following considerations and so will be dropped. In a similar way we can deal with (5.77). After some algebra we have in particular

$$\Delta_\Sigma^{(1)} = -\frac{1}{4n}\int d^4x \int d^2\Theta\, \mathcal{E}\, \left(\Sigma\, W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} + 2\, W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} \frac{\overline{T}_{\dot\alpha}\mathcal{D}^{\dot\alpha}\Sigma}{\overline{S}}\right) + h.c. \tag{5.82}$$

Therefore, proceeding as in section 5, $\Delta_\Sigma^{(1)}$ in components becomes (we disregard a multiplicative factor)

$$\Delta_\Sigma^{(1)} \approx \int d^4x\, e \left[\Sigma\, \mathcal{D}^\alpha W^{\beta\gamma\delta}\, \mathcal{D}_\alpha W_{\beta\gamma\delta}\big| + h.c.\right] \tag{5.83}$$

Now (5.44,5.45) remain valid in the non minimal SUGRA, but (5.46) is replaced by a far more complicated equation, so that (5.47) has to be re-demonstrated. This is not trivial, but can be done (see Appendix A). Thus we can conclude that, up to a multiplicative factor,

$$\Delta_\Sigma^{(1)} \approx \frac{1}{4}\int d^4x\, e\Big\{\omega\Big(\mathcal{R}_{nmkl}\mathcal{R}^{nmkl} - 2\mathcal{R}_{nm}\mathcal{R}^{nm} + \frac{1}{3}\mathcal{R}^2\Big) - \frac{1}{2}\,\alpha\, \epsilon^{nmlk}\mathcal{R}_{nmpq}\mathcal{R}_{lk}{}^{pq}\Big\} \tag{5.84}$$

where $\omega + i\alpha$ is the lowest component of the superfield Σ. In this case too ω corresponds to the ordinary Weyl rescaling, while α is the parameter of a chiral transformation.

The same reduction to ordinary form holds also for (5.77). In this case $\omega + i\alpha$ is the first component of $L + \Lambda$.

At this point it is worth making a comment on the (apparent) singularity of expressions such as (5.70,5.71,5.76,5.77). For instance, the cocycle (5.71), written in terms of superfields has a nonlocal or singular aspect; but one must reflect on the fact that it is nothing but the supersymmetrization of (5.84), which is local. Therefore also (5.71), when expressed in terms of components fields must be local (although it may be non-polynomial if dimensionless prepotentials have to be introduced). The question remains open of whether non-singularity can be be made manifest already at the superfield level. In [111] it was noted that in some cases this is indeed possible by means of opportune field redefinitions.

The scheme outlined in this section is general and will be applied to all the cocycles we will come across.

5.7 Mapping formulas between different supergravity models

A cocycle similar to $\Delta_\sigma^{(2)}$ (i.e. quadratic in G_a) is hard to construct with ordinary means (i.e. with a polynomial cohomological analysis). For this we have to resort to a mapping between different supergravity models. This mapping was outlined in [83, 111] and brought to a more explicit form in [108]. The latter reference is based on different torsion constraints with respect to (5.60). Therefore we have to rederive new appropriate mapping formulas.

Various different models of supergravity are defined by making a definite choice of the torsion constraints and, after such a choice, by identifying the dynamical degrees of freedom. This is the way minimal, nonminimal and new minimal models were introduced. However it is possible to transform the choices of constraints into one another by means of suitable linear transformation of the supervierbein and the superconnection, [108, 110]:

$$E'{}_M{}^A = E_M{}^B X_B{}^A, \qquad E'{}_A{}^M = X^{-1}{}_A{}^B E_B{}^M, \qquad \Phi'{}_{MA}{}^B = \Phi_{MA}{}^B + \chi_{MA}{}^B \tag{5.85}$$

For instance, if we want to pass from a set of unprimed constraints to primed ones the required transformations are as follows

$$E'_\alpha = U\, E_\alpha, \qquad E'^{\dot\alpha} = \overline{U}\, E^{\dot\alpha}, \qquad E' = U^{-2}\overline{U}^{-2} E \tag{5.86}$$

$$E'_{\alpha\dot\alpha} = U\overline{U}\, E_{\alpha\dot\alpha} + i\frac{U\overline{U}}{3n+1}\left(E_{\dot\alpha}{}^M \partial_M \ln\left(\frac{U^{n+1}}{\overline{U}^{n-1}}\right) E_\alpha + E_\alpha{}^M \partial_M \ln\left(\frac{\overline{U}^{n+1}}{U^{n-1}}\right) E_{\dot\alpha}\right) \tag{5.87}$$

where U is a suitable expression of the superfields. Moreover

$$T'_\alpha = UT_\alpha - \frac{1}{6n+2}\,\mathcal{D}'_\alpha \ln\left(\overline{U}^2 U^4\right) \tag{5.88}$$

$$\Phi'_{\alpha\beta\gamma} = U\,\Phi_{\alpha\beta\gamma} - \frac{1}{3n+1}\left(\epsilon_{\alpha\gamma}\mathcal{D}'_\beta + \epsilon_{\alpha\beta}\mathcal{D}'_\gamma\right)\ln\left(\frac{U^{n-1}}{\overline{U}^{n+1}}\right) \tag{5.89}$$

$$W'_{\alpha\beta\gamma} = U\,\overline{U}^2 W_{\alpha\beta\gamma} \tag{5.90}$$

$$8R' + 2(n+1)\overline{S}' = -\left(\overline{\mathcal{D}}'_{\dot\alpha}\overline{\mathcal{D}}'^{\dot\alpha} - 3(n+1)\overline{T}_{\dot\alpha}\overline{\mathcal{D}}'^{\dot\alpha} - 8R - 2(n+1)\overline{S}\right)\overline{U}^2 \tag{5.91}$$

where $\mathcal{D}'$ denotes the covariant derivative in the primed system, together with the conjugate relations. The analogous transformation for the G_a superfield is more complicated:

$$\begin{aligned}
G'_{\alpha\dot\alpha} \;=\; U\overline{U}\Bigg(& G_{\alpha\dot\alpha} - \frac{i}{3}\mathcal{D}'_{\alpha\dot\alpha}\ln\frac{U}{\overline{U}} + \frac{1}{(3n+1)^2}\overline{\mathcal{D}}'_{\dot\alpha}\ln\frac{U^{n+1}}{\overline{U}^{n-1}}\mathcal{D}'_{\alpha}\ln\frac{\overline{U}^{n+1}}{U^{n-1}} \\
& + \frac{1}{3(3n+1)}\overline{\mathcal{D}}'_{\dot\alpha}\ln\frac{U^{n+1}}{\overline{U}^{n-1}}\mathcal{D}'_{\alpha}\frac{U}{\overline{U}} + \frac{1}{3(3n+1)}\mathcal{D}'_{\alpha}\ln\frac{\overline{U}^{n+1}}{U^{n-1}}\overline{\mathcal{D}}'_{\dot\alpha}\frac{U}{\overline{U}} \\
& + \frac{2}{3(3n+1)}\mathcal{D}'_{\alpha}\ln\frac{\overline{U}^{n+1}}{U^{n-1}}\overline{T}_{\dot\alpha} - \frac{2}{3(3n+1)}\overline{\mathcal{D}}'_{\dot\alpha}\ln\frac{U^{n+1}}{\overline{U}^{n-1}}T_{\alpha}\Bigg)
\end{aligned} \tag{5.92}$$

These formulas can be inverted. To this end we have to replace U with U^{-1} everywhere, replace the primed quantities with unprimed ones in the LHS, and the unprimed with the primed ones in RHS; in this case the covariant derivatives on the RHS are the primed ones[‡].

For instance, if we want to pass from the minimal to the nonminimal constraints we have to choose

$$U = \exp\left[2(3n+1)\left(\frac{\overline{\psi}}{6} - \frac{\psi}{3}\right)\right] \tag{5.93}$$

ψ is a 'prepotential' such that $T_\alpha = \mathcal{D}_\alpha\psi$ and $\overline{T}_{\dot\alpha} = \overline{\mathcal{D}}_{\dot\alpha}\overline{\psi}$. Of course if we wish to pass from the nonminimal to the minimal constraints we have simply to use the same formulas with inverted U.

One can verify that

$$(\overline{\nabla}_{\dot\alpha}\overline{\nabla}^{\dot\alpha} - 8R)\overline{U}^2 = -2(n+1)\overline{S} \tag{5.94}$$

We recall that ∇ denotes specifically the covariant derivative in minimal supergravity.

Let us consider next the superconformal transformations. We wish to compare the transformations (5.28,5.29) with (5.68). Given the transformation of T_α and $T_\alpha = \mathcal{D}_\alpha\psi$, we can assume that $\psi, \overline{\psi}$ transform as follows

$$\delta\psi = \frac{3}{3n+1}(\overline{\Sigma} - \overline{\sigma}), \qquad \delta\overline{\psi} = \frac{3}{3n+1}(\Sigma - \sigma), \tag{5.95}$$

where σ is an arbitrary chiral superfield. Taking the variation of both sides of (5.86) and applying (5.95) we can easily see that we can identify the σ superfield in (5.95) with the σ in (5.28). The same is easily done also for (5.90). The transformation of (5.91) is

[‡]For more details on these transformations, see [113].

more complicated. We first derive, using (5.89),

$$\nabla^\alpha\nabla_\alpha\Phi = U^{-2}\left(\mathcal{D}\mathcal{D}\Phi - \frac{4}{3}(3n+1)\mathcal{D}^\alpha\overline{T}\mathcal{D}_\alpha\Phi + \frac{15n-1}{3}T^\alpha\mathcal{D}_\alpha\Phi\right) \tag{5.96}$$

for any scalar superfield Φ. Inverting (5.91) we can write

$$-8R^+ = \left(\mathcal{D}^\alpha\mathcal{D}_\alpha - 3(n+1)T^\alpha\mathcal{D}_\alpha - 2(n+1)S\right)U^{-2} \tag{5.97}$$

The LHS represents R in the minimal model, while the RHS refers to the nonminimal one. Taking the variation of both sides and using (5.95,5.96), one can show that

$$\delta R^+ = -2(2\overline{\sigma}-\sigma)R^+ - \frac{1}{4}\nabla\nabla\sigma \tag{5.98}$$

This is identical to the transformation of R^+ in the minimal model, (5.29).

We can do the same with G_a. Taking the variation of LHS and RHS of the inverted eq.(5.92), and using

$$i\nabla_{\alpha\dot\alpha}(\overline{\sigma}-\sigma) = U^{-1}\overline{U}^{-1}\left(i\mathcal{D}_{\alpha\dot\alpha}(\overline{\sigma}-\sigma) - \frac{1}{3n+1}\overline{\mathcal{D}}_{\dot\alpha}\ln\frac{U^{n+1}}{\overline{U}^{n-1}}\,\mathcal{D}_\alpha\sigma + \frac{1}{3n+1}\mathcal{D}_\alpha\ln\frac{\overline{U}^{n+1}}{U^{n-1}}\,\overline{\mathcal{D}}_{\dot\alpha}\overline{\sigma}\right)$$

one finds

$$\delta G_{\alpha\dot\alpha} = -(\sigma+\overline{\sigma})G_{\alpha\dot\alpha} + i\nabla_{\alpha\dot\alpha}(\overline{\sigma}-\sigma) \tag{5.99}$$

as expected.

Therefore (5.95) connects the superconformal transformations of the minimal and non-minimal models. It is however useful to consider this passage in two steps. Let us split U in (5.93) as follows:

$$U = U_cU_n, \qquad U_c = e^{X-2\overline{X}}, \qquad U_n = e^{\frac{\Omega}{3}-\frac{\overline{\Omega}}{6}} \tag{5.100}$$

where

$$X = \frac{1}{3}(3n+1)\overline{\psi} + \frac{\overline{\Omega}}{6}, \qquad \overline{X} = \frac{1}{3}(3n+1)\psi + \frac{\Omega}{6} \tag{5.101}$$

Recall that $\Omega = 2(3n+1)\frac{T^\alpha T_\alpha}{S}$ and $T_\alpha = \mathcal{D}_\alpha\psi$, etc. It follows that X is a chiral and $\overline{X}$ an antichiral superfield. Moreover $U_c\overline{U}_c^2 = e^{-3X}$ is chiral and $\overline{U}_cU_c^2 = e^{-3\overline{X}}$ is antichiral. Operating on the superfields according to (5.86,5.88,5.90) we see that, for instance $T_\alpha = 0$ is mapped to $T_\alpha = 0$ by the transformation induced by U_c, i.e. after such transformation the model is still minimal supergravity.

For later use we remark that (see also [111])

$$\delta\overline{\Omega} = \Gamma_\Sigma - 6\Sigma, \qquad \Gamma_\Sigma = -\frac{3}{3n+1}\overline{\Delta}\left(\frac{\overline{\Omega}\Sigma}{\overline{S}}\right) \tag{5.102}$$

where $\overline{\Delta}$ is the chiral projector. By repeating the previous verifications one can see that Γ_Σ is an intermediate step between σ and Σ. The important property of Γ_Σ is that it is chiral, but expressed in terms of the nonminimal superfields. Moreover it is consistent with the nonminimal transformation properties and, in particular, $\delta\Gamma_\Sigma = 0$. In parallel with (5.102) we have of course the conjugate formulas.

Analogous things hold if we replace the non minimal with the new minimal model. In this case of course we have to set $\psi = \overline{\psi}$ and the appropriate transformations are (5.73,5.74). It is easy to see that the above superfield redefinitions connect the minimal supergravity transformations with (5.74). Also in this case we have an intermediate step which will turn out instrumental later on. In this case we have

$$\delta\overline{\Omega} = \Gamma_{L+\Lambda} - 6(L+\Lambda), \qquad \Gamma_{L+\Lambda} = -\frac{3}{3n+1}\overline{\Delta}\left(\frac{\overline{\Omega}(L+\Lambda)}{\overline{S}}\right) = \Gamma_L + 6\Lambda \tag{5.103}$$

where Γ_L is chiral.

All this means one important thing: the possibility to construct invariants and cocycles of any supergravity model starting from the invariants and cocycles of a fixed one, for instance the minimal supergravity (such an idea is present in [94]).

5.8 Cocycles from minimal supergravity

We are now ready to construct the cocycles form those of minimal supergravity. The idea is very simple. We start from the cocycles of minimal supergravity and replace the superfields of the latter with the formulas of the previous subsection expressing them in terms of the superfields of other models. Since all the symmetry operations are coherent, the resulting expressions must also be cocycles. The invariants are a subcase of the discussion for 1-cocycles, thus in the sequel we explicitly deal only with the latter. We will consider first the new minimal case.

5.8.1 From minimal to nonminimal cocycles

Let us start from $\Delta_\sigma^{(1)}$. All the superfields therein must be expressed in terms of the new superfields. It is convenient to proceed in two steps, as just outlined. In the first step it is mapped to

$$\int_{x,\theta} \frac{E}{-8R}\sigma\, WW + h.c. = \int_{x,\theta} E'\Gamma_\Sigma \frac{W'W'}{\overline{U}_c^2(\nabla'_{\dot\alpha}\nabla'^{\dot\alpha} - 8R')\overline{U}_c^{-2}} + h.c. \tag{5.104}$$

where WW is a compact notation for $W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}$ and primes denote the superfields in the new representation (which still corresponds to minimal supergravity). We recall that $\nabla'_{\dot\alpha}\nabla'^{\dot\alpha} - 8R'$ projects to a chiral superfield. Therefore we can write

$$\int_{x,\theta} \frac{E\,\sigma}{-8R}\, WW \;=\; \int_{x,\theta} \frac{E'}{-8R'}\Gamma_\Sigma(\nabla'_{\dot\alpha}\nabla'^{\dot\alpha} - 8R')\left(\frac{W'W'\overline{U}_c^{-2}}{(\nabla'_{\dot\alpha}\nabla'^{\dot\alpha} - 8R')\overline{U}_c^{-2}}\right) \tag{5.105}$$

$$= \int_{x,\theta} \frac{E'}{-8R'}\Gamma_\Sigma\, W'W' \tag{5.106}$$

Now we complete the passage to the nonminimal model by performing the U_n transformation. This means

$$\int_{x,\theta} \frac{E'}{-8R'}\Gamma_\Sigma\, W'W' = \int_{x,\theta} E''\Gamma_\Sigma \frac{W''W''}{\overline{U}_n^2\overline{\Delta}''\overline{U}_n^{-2}} \tag{5.107}$$

where $\Delta'' = \mathcal{D}''\mathcal{D}'' - 3(n+1)T''^{\alpha}\mathcal{D}''_\alpha - 2(n+1)S''$ is the antichiral projector in the nonminimal model (endpoint of the overall transformation). For simplicity, from now on, we drop primes, understanding that we are operating in the nonminimal model.

Next we use the identity, demonstrated in [110] by partial integration,

$$4n\int_{x,\theta} E\, e^{\overline{\Omega}}U = \int_{x,\theta} E\,\frac{\Phi}{\overline{S}} \tag{5.108}$$

where U is any superfield expression without Lorentz indices and $\Phi = \overline{\Delta}U$. Applying this identity with $U = \Sigma\frac{e^{-\overline{\Omega}}WW\,\overline{U}^{-2}}{\overline{\Delta U}^{-2}}$ we get

$$\int_{x,\theta} E\,\Gamma_\Sigma \frac{WW}{\overline{U}^2\,\overline{\Delta U}^{-2}} = \frac{1}{4n}\int_{x,\theta} \frac{E}{\overline{S}}\Gamma_\Sigma\, e^{-\overline{\Omega}}WW \tag{5.109}$$

Applying (5.108) again with $U = \Sigma e^{-\overline{\Omega}}\frac{WW}{\overline{S}}$, so that $\Phi = -2(n+1)\Gamma_\Sigma e^{-\overline{\Omega}}WW$, we obtain

$$\frac{1}{4n}\int_{x,\theta} \frac{E}{\overline{S}}\Gamma_\Sigma\, e^{-\overline{\Omega}}WW = -\frac{1}{2(n+1)}\int_{x,\theta} \frac{E}{\overline{S}}\Gamma_\Sigma\, WW \tag{5.110}$$

Replacing now the explicit expression of Γ_Σ, (5.102), and integrating by parts, we find that $\Delta^{(1)}_\sigma$ is mapped to

$$3\frac{5n+1}{n+1}\int_{x,\theta} \frac{E}{\overline{S}^2}\Sigma\,\overline{T}_{\dot\alpha}\overline{T}^{\alpha}\, W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} + h.c. \tag{5.111}$$

which is proportional to the already obtained cocycle $\Delta^{(1)}_{n.m.}$, (5.71). The second cocycle is readily constructed in the same way:

$$\begin{aligned}
\Delta^{(2)}_{\Sigma} &= \int_{x,\theta} E'(\Gamma_\Sigma + \overline{\Gamma}_\Sigma) U^2 \overline{U}^2 \left(G_a(G',T',U) G^a(G',T',U) + 2R(\overline{S}',\overline{T}',U)\, R^+(S',T',U) \right) \\
&= c \int_{x,\theta} E'(\Sigma + \overline{\Sigma})(G'_a G'^a + \ldots) \qquad (5.112)
\end{aligned}$$

after repeated partial integrations. $G_a(G',T',U)$ is given by the inverted (5.92), while $R^+(S,T,U)$ is given by (5.97). c is a suitable number. By construction $\Delta^{(2)}_\Sigma$ satisfies the consistency conditions with generic Σ. Its ordinary form is the same as $\Delta^{(2)}_\sigma$ in section 2.

5.8.2 From minimal to $16+16$ nonminimal cocycles

Let us start again with $\Delta^{(1)}_\sigma$. Proceeding as above with the relevant new formulas outlined at the end of the previous section we get

$$\begin{aligned}
\Delta^{(1)}_{\sigma} &= \int_{x,\theta} \frac{E}{-8R} \sigma W W + h.c. \qquad (5.113) \\
&= \int_{x,\theta} E' \Gamma_{L+\Lambda} \frac{W'W'}{\overline{U}^2 \left(\overline{\mathcal{D}}\overline{\mathcal{D}} - 3(n+1)\overline{T}'_{\dot\alpha}\overline{\mathcal{D}} - 2(n+1)\overline{S}' \right) \overline{U}^{-2}} + h.c. \equiv \widetilde{\Delta}^{(1)}_{L+\Lambda}
\end{aligned}$$

where primed superfields refers to new minimal supergravity. From now on we drop primes, understanding that all the superfields are in the new minimal supergravity. Working out the derivatives in (5.113) we get

$$\widetilde{\Delta}^{(1)}_{L+\Lambda} = -\frac{3}{4} \int_{x,\theta} \frac{E}{\overline{S}} \Gamma_{L+\Lambda} W W \left(1 - \frac{2}{3}(3n+1) \frac{\overline{T}_{\dot\alpha} \overline{T}^{\dot\alpha}}{\overline{S}} \right) + h.c. \qquad (5.114)$$

This is not (5.77) yet, as we would have expected. However, using (5.79) and integrating by parts the spinor derivatives contained in $\Gamma_{L+\Lambda}$, as we have done above for the nonminimal case, one easily finds that $\widetilde{\Delta}^{(1)}_{L+\Lambda}$ is proportional to (5.77).

Let us come now to the second cocycle. As above we start from the minimal cocycle $\Delta^{(2)}_\sigma$ and transform the superfields according to (5.97) and (5.92). We get

$$\begin{aligned}
\Delta^{(2)}_{\sigma} &= \int_{x,\theta} E(\sigma + \overline{\sigma})(G_a G^a + 2RR^+) \\
&= \int_{x,\theta} E'(\Gamma_{L+\Lambda} + h.c.) \left(-\frac{1}{2} \left(G'_{\alpha\dot\alpha} + \frac{4}{9} T_\alpha \overline{T}_{\dot\alpha} \right) \left(G'^{\alpha\dot\alpha} + \frac{4}{9} T^\alpha \overline{T}^{\dot\alpha} \right) \right. \\
&\quad \left. + 2 \left(\frac{1}{6} S - \frac{n}{3}(3n+1) T^\alpha T_\alpha \right) \left(\frac{1}{6} \overline{S} - \frac{n}{3}(3n+1) \overline{T}_{\dot\alpha} \overline{T}^{\dot\alpha} \right) \right) \equiv \Delta^{(2)}_{L+\Lambda} \qquad (5.115)
\end{aligned}$$

where superfields and covariant derivatives in the RHS are new minimal superfields. Of course since nothing has changed concerning the metric, the ordinary form of $\Delta^{(2)}_{\Lambda}$ is the same as the ordinary form of $\Delta^{(2)}_{\sigma}$, computed in section 2.

5.9 Conclusions

In this paper we have determined the possible trace anomalies in the $16+16$ nonminimal supergravity as well as in the non minimal one. There are in all cases two independent nontrivial cocycles whose densities are given by the square Weyl tensor and by the Gauss-Bonnet density, respectively.

Concerning the Pontryagin density, it appears in the anomaly supermultiplets only in the form of chiral anomaly (Delbourgo-Salam anomaly), but never in the form of trace anomaly.

At this point we must clarify the question of whether the cocycles we have found in nonminimal and $16+16$ nonminimal supergravities are the only ones. In this paper we have not done a systematic search of such nontrivial cocycles in the nonminimal and 16+16 nonminimal case, the reason being that when a dimensionless field like ψ and $\overline{\psi}$ are present in a theory a polynomial analysis is not sufficient (and a non-polynomial one is of course very complicated). But we can argue as follows: consider a nontrivial cocycle in nonminimal or $16+16$ nonminimal supergravity; it can be mapped to a minimal cocycle which either vanishes or coincides with the ones classified in [93]. There is no other possibility because in minimal supergravity there are no dimensionless superfields (apart from the vielbein) and the polynomial analysis carried out in [93] is sufficient to identify all cocycles. We conclude that the nonminimal and $16+16$ nonminimal nontrivial cocycles, which reduce in the ordinary form to a nonvanishing expression, correspond to $\Delta^{(1)}_{\sigma}$ and $\Delta^{(2)}_{\sigma}$ in minimal supergravity.

Finally we would like to make a comment on an aspect of our results that could raise at first sight some perplexity. Although one cannot claim the previous results to be a theorem, they nevertheless point in the direction of the non-existence of a supersymmetric anomaly multiplet that has, as its e.m. tensor trace component, the Pontryagin density. On the other hand we know systems with chiral fermions that at first sight can be supersymmetrized and coupled to supergravity. In such system we expect the trace of the e.m. tensor at one loop to contain the Pontryagin density, [117]; thus why couldn't we have an anomaly multiplet that contains as trace component the Pontryagin density? The point is that in such a chiral case there can exist an obstruction to that, as we try to explain next. Suppose that the e.m. tensor of a system like the one just mentioned, has, at one loop, an integrated nonvanishing trace $\Delta_\omega^{(P)}$, containing a term given by ω multiplied by the Pontryagin density. We cannot expect, in general this term to be supersymmetric. On the contrary, denoting by ϵ the supersymmetric local parameter we expect there to exist a partner cocycle $\Delta_\epsilon^{(P)}$ such that

$$\delta_\omega \Delta_\omega^{(P)} = 0, \qquad \delta_\epsilon \Delta_\omega^{(P)} + \delta_\omega \Delta_\epsilon^{(P)} = 0, \qquad \delta_\epsilon \Delta_\epsilon^{(P)} = 0 \tag{5.116}$$

The cocycle $\Delta_\epsilon^{(P)}$ to our best knowledge has not yet been computed in supergravity. So we have to rely on plausibility arguments. There are two possibilities: it might happen that $\Delta_\epsilon^{(P)}$ is trivial, i.e. $\Delta_\epsilon^{(P)} = \delta_\epsilon \mathcal{C}^{(P)}$, so that (5.116) implies that $\delta_\epsilon(\Delta_\omega^{(P)} - \delta_\omega \mathcal{C}^{(P)}) = 0$. The end result would be a supersymmetric Weyl cocycle. This is, for instance, what happens for the chiral ABJ anomaly in rigid supersymmetry, where the supersymmetric partner of the usual chiral anomaly must be trivial, [116], and, precisely as above, the chiral anomaly can be cast in supersymmetric form, see [118].

The second possibility is that no such counterterm $\mathcal{C}^{(P)}$ exists, in which case the cocycle $\Delta_\epsilon^{(P)}$ is nontrivial and there is no possibility to supersymmetrize $\Delta_\omega^{(P)}$. This seems to be the case for the chiral ABJ anomaly in the presence of local supersymmetry, [119]. And this may be the case also for $\Delta_\omega^{(P)}$, which would explain the origin of our inability to find a Weyl cocycle containing the Pontryagin form in the first position (trace anomaly) in terms of superfield[§]. In both cases the origin of the obstruction is the same, i.e. the nontrivial breaking of local supersymmetry. In turn this would explain why a supersymmetry preserving regularization has never been found in such types of systems. Such converging arguments seem to nicely fit together and close the circle.

[§]A plausible explanation for the difference between local and global supersymmetry is that the nontrivial part of $\Delta_\epsilon^{(P)}$ may be an integral of ϵ multiplied by (as it often happens) a total derivative; if ϵ is a (generic) local parameter $\Delta_\epsilon^{(P)}$ is a nonvanishing nontrivial cocycle, but it vanishes it ϵ is constant.

Appendix A

The angular integration

The first step in the evaluation of (3.34) consists in performing the 'angular' x, y integration. This will be done analytically. Let us consider for definiteness the most complicated term, the cubic one in G_s (from now on for economy of notation let us set $s = 2uT$). We represent G_s as the series (3.28) and integrate term by term in x and y. All these integrations involve ordinary integrals which can be evaluated by using standard tables, or, more comfortably, Mathematica. It is a lucky coincidence that most integrals are nonvanishing only for specific values of the integers k. We have, for instance,

$$\begin{aligned}
\int_0^1 dy \int_0^y dx \sin(\pi x)\sin(\pi y)\sin(\pi(x-y))\cos(2\pi k x) &= \frac{1}{8\pi(k^2-1)}, \quad k \neq 1 \\
\int_0^1 dy \int_0^y dx \sin(\pi x)\sin(\pi y)\sin(\pi(x-y))\cos(2\pi x) &= \frac{3}{32\pi}
\end{aligned} \tag{A.1}$$

while the integral $\int_0^1 dy \int_0^y dx \sin(\pi x)\sin(\pi y)\sin(\pi(x-y))\cos(2\pi k x)\cos(2\pi m y)$ vanishes for almost all k, m except $k = m, m \pm 1$ and $k = 1, m$ and $k, m = 1$. For example

$$\begin{aligned}
\int_0^1 dy \int_0^y dx \sin(\pi x)\sin(\pi y)\sin(\pi(x-y))\cos(2\pi k x)\cos(2\pi k y) &= \frac{1}{16\pi(k^2-1)} \qquad \text{(A.2)} \\
\int_0^1 dy \int_0^y dx \sin(\pi x)\sin(\pi y)\sin(\pi(x-y))\cos(2\pi k x)\cos(2\pi(k+1)y) &= -\frac{1}{32\pi k(k+1)}, \quad \text{etc.}
\end{aligned}$$

The integration with three cosines is of course more complicated, but it can nevertheless be done in all cases. The integrals mostly vanish except for specific values of the integers k, m, n inside the cosines. They are non-vanishing for $m = k$ with n generic, and

$m = k, n = k, k \pm 1$:

$$\int_0^1 dy \int_0^y dx \sin(\pi x)\sin(\pi y)\sin(\pi(x-y)) \tag{A.3}$$
$$\cdot\cos(2\pi kx)\cos(2\pi ky)\cos(2\pi n(x-y)) = \frac{n^2+k^2-1}{16\pi((n+k)^2-1)((n-k)^2-1)}$$
$$\int_0^1 dy \int_0^y dx \sin(\pi x)\sin(\pi y)\sin(\pi(x-y))$$
$$\cdot\cos(2\pi kx)\cos(2\pi ky)\cos(2\pi k(x-y)) = -\frac{3(2k^2-1)}{16\pi(4k^2-1)}$$
$$\int_0^1 dy \int_0^y dx \sin(\pi x)\sin(\pi y)\sin(\pi(x-y))$$
$$\cdot\cos(2\pi kx)\cos(2\pi ky)\cos(2\pi(k+1)(x-y)) = \frac{6k^3+9k^2+3k-1}{128\pi(2k+1)(k+1)k}$$
$$\int_0^1 dy \int_0^y dx \sin(\pi x)\sin(\pi y)\sin(\pi(x-y))$$
$$\cdot\cos(2\pi kx)\cos(2\pi ky)\cos(2\pi(k-1)(x-y)) = \frac{6k^3-9k^2+3k+1}{128\pi(2k-1)(k-1)k}$$

and so on. A delicate part of the program consists in finding all nonvanishing terms and identifying the nonoverlapping ranges of summation over k, m and n. Fortunately the triple infinite summation reduces to a finite number of double infinite summations. Mathematica knows how to do the summations over one discrete index, in general not over two.

Let us write down next the result of the angular integration, by considering the three different types of terms in (3.34) in turn.

A.0.1 The term without G_s

This is easy. We get

$$\frac{1}{6}\int_0^\infty ds\, s^2 \int_0^1 dy \int_0^y dx\, \frac{4}{\pi}\sin\pi x \sin\pi y \sin\pi(x-y)\mathrm{g}(s)\left[-\left(\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\right)^3\right] \tag{A.4}$$
$$= \quad -\frac{1}{4\pi^2}\int_0^\infty ds\, s^2 \mathrm{g}(s)\left[-\left(\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\right)^3\right].$$

where we have set $\mathrm{g}(s) \equiv g(s/2)$

A.0.2 The term quadratic in G_s

We have to compute

$$\frac{1}{6}\int_0^\infty d(s)\;(s)^2\int_0^1 dy\int_0^y dx\,\frac{4}{\pi}\,\sin\pi x\,\sin\pi y\,\sin\pi(x-y) \\ \cdot(-\frac{1}{2})\partial_s \mathrm{g}(s)\Big(G_s^2(2\pi x)+G_s^2(2\pi(x-y))+G_s^2(2\pi y)\Big)\Bigg\}. \tag{A.5}$$

Therefore the integrand of the quadratic term in G_s is made of the factor $-\frac{1}{12}s^2\partial_s \mathrm{g}(s)$ multiplied by the factor

$$\frac{4}{\pi}\int_0^1 dy\int_0^y dx\,\sin\pi x\,\sin\pi y\,\sin\pi(x-y) \\ \cdot\Big(G_s^2(2\pi x)+G_s^2(2\pi(x-y))+G_s^2(2\pi y)\Big)\Bigg\}. \tag{A.6}$$

After some work the latter turns out to equal

$$\begin{aligned} & \frac{4}{\pi}\int_0^1 dy\int_0^y dx\,\sin\pi x\,\sin\pi y\,\sin\pi(x-y) \\ & \quad\cdot\Big(G_s^2(2\pi x)+G_s^2(2\pi(x-y))+G_s^2(2\pi y)\Big)\Bigg\} \qquad \text{(A.7)} \\ = \;& -\frac{9}{2\pi^2}\frac{1}{s^2}+\frac{16}{\pi s}\left(\frac{9}{32\pi}\frac{1}{s+1}+\frac{3}{8\pi}\sum_{k=2}^{\infty}\frac{1}{k^2-1}\frac{1}{k+s}\right) \\ & +\frac{48}{\pi}\left(\frac{1}{8\pi}\sum_{\substack{k,n\\ n\neq k,k\pm1}}^{\infty}\frac{n^2+k^2-1}{((n+k)^2-1)((n-k)^2-1)}\frac{1}{(n+s)(k+s)}\right. \\ & -\frac{1}{4\pi}\sum_{k=1}^{\infty}\frac{3k^2-1}{4k^2-1}\frac{1}{(k+s)^2} \qquad \leftarrow R_1(s) \\ & +\frac{1}{64\pi}\sum_{k=1}^{\infty}\frac{3k^2+3k+1}{k(k+1)}\frac{1}{(k+s)(k+s+1)} \qquad \leftarrow R_2(s) \\ & \left.+\frac{1}{64\pi}\sum_{k=2}^{\infty}\frac{3k^2-3k+1}{k(k-1)}\frac{1}{(k+s)(k+s-1)} \qquad \leftarrow R_3(s)\right) \\ \equiv \;& E_1^{(2)}(s)+\frac{48}{\pi}\left(\sum_{p=2}^{\infty}RK(p,s)+R_1(s)+R_2(s)+R_3(s)\right), \end{aligned}$$

where

$$E_1^{(2)}(s)=-\frac{9}{2\pi^2}\frac{1}{s^2}+E_0^{(2)}(s) \tag{A.8}$$

and

$$E_0^{(2)}(s) = \frac{9}{2\pi^2}\frac{1}{s(s+1)} + \frac{3}{2\pi^2}\frac{1}{s(s^2-1)}\Big(3(s+1) - 4\gamma - 4\psi(2+s)\Big), \tag{A.9}$$

where ψ is the digamma function and γ the Euler–Mascheroni constant. To save space, we have introduced in (A.7) in a quite unconventional way the definitions of the quantities $R_i(s)$, $i = 1, 2, 3$. Beside $R_1(s), R_2(s), R_3(s)$, we define

$$RK(k,n,s) = \frac{1}{8\pi}\frac{n^2+k^2-1}{((n+k)^2-1)((n-k)^2-1)}\frac{1}{(n+s)(k+s)} \tag{A.10}$$

and

$$RK(p,s) = \sum_{k=1}^{\infty} RK(k,k+p,s) + \sum_{k=p+1}^{\infty} RK(k,k-p,s) \tag{A.11}$$

with the summation in (A.7) corresponding to: $\sum^{\infty}_{\substack{k,n \\ n\neq k,k\pm 1}} RK(k,n,s) = \sum_{p=2}^{\infty} RK(p,s)$.

A.0.2.1 Performing one discrete summation

As already pointed out it is possible to perform in an analytic way at least one of the two discrete summations above. To start with

$$\begin{aligned} R(s) &= R_1(s) + R_2(s) + R_3(s) = \frac{1}{32\pi}\Big(\frac{s^2(1+3s) - 2(s+1)H(s)}{s(1+s)(s^2-1)} \\ &+\frac{4\left(1+4s(1-\gamma+s-\ln 4) - 4s\psi(1+s) - 2\left(1-7s^2+12s^4\right)\psi^{(1)}(1+s)\right)}{(1-4s^2)^2}\Big). \end{aligned} \tag{A.12}$$

Next

$$\begin{aligned} RK(p,s) = &\frac{1}{4p\left(-1+p^2\right)\pi(-1+p-2s)(1+p-2s)(-1+p+2s)(1+p+2s)} \\ &\cdot\left(4p\left(-1+p^2\right)s\,H\left(\frac{p-1}{2}\right)\right. \\ &-(-1+p+2s)(1+p+2s)\left(-1+p^2-2ps+2s^2\right)H(s) \\ &+(-1+p-2s) \\ &\cdot\left.\Big(-(-1+p)p(1+p+2s) + (1+p-2s)\left(-1+p^2+2ps+2s^2\right)H(p+s)\Big)\right), \end{aligned} \tag{A.13}$$

where $H(s) = \gamma + \psi(s+1)$ is the harmonic number function. It should be remarked that in both (A.12) and (A.13) there are zeros in the denominators, for positive values of s. These however do not correspond to real poles of $R(s)$ and $RK(p,s)$, because they are cancelled by corresponding zeroes in the numerator.

A.0.3 The term cubic in G_s

In (3.34) we have to compute

$$\frac{2}{3\pi}\int_0^\infty ds\, s^2 \int_0^1 dy \int_0^y dx\, \sin\pi x\, \sin\pi y\, \sin\pi(x-y)\, \mathrm{g}(s) G_s(2\pi x) G_s(2\pi(x-y)) G_s(2\pi y). \tag{A.14}$$

The most convenient form of the cubic term in G_s after angular integration is probably the following one (which must be multiplied by $\frac{1}{6}s^2 g(s)$)

$$\frac{4}{\pi}\int_0^1 dy \int_0^y dx \sin(\pi x)\sin(\pi y)\sin(\pi(x-y))\left(\frac{1}{s}+2\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k+s}\right) \tag{A.15}$$

$$\cdot\left(\frac{1}{s}+2\sum_{m=1}^{\infty}\frac{\cos(2\pi my)}{m+s}\right)\left(\frac{1}{s}+2\sum_{n=1}^{\infty}\frac{\cos(2\pi n(x-y))}{k+s}\right)$$

$$= -\frac{3}{2\pi^2}\frac{1}{s^3}+\frac{9}{4\pi^2}\frac{1}{s^2(s+1)}+\frac{3}{\pi^2}\frac{1}{s^2(s^2-1)}\left(-\gamma+\frac{3}{4}(s+1)-\psi(2+s)\right)$$

$$+\frac{3}{4\pi^2}\frac{1}{s(s+1)^2}-\frac{7}{2\pi^2}\frac{1}{s(s+1)(s+2)}+\frac{3}{4\pi^2}\frac{1}{s(s^2-1)^2}\cdot$$

$$\cdot\Big(3(1+s^2)-8\gamma s+6s-8s\psi(2+s)+4(s^2-1)\psi^{(1)}(2+s)\Big)$$

$$-\frac{1}{2\pi^2 s(s+1)(s^2-1)}\Big(17+5s-12\gamma-12\psi(3+s)\Big)-\frac{3}{2\pi^2 s^2(s^2-1)}$$

$$\cdot\Big(5-4\gamma+s-2(s+1)\psi(2+s)+2(s-1)\psi(3+s)\Big)$$

$$+\frac{32}{\pi}\Bigg[3\sum_{\substack{k,n\\ n\neq k,k\pm1}}\frac{n^2+k^2-1}{16\pi((n+k)^2-1)((n-k)^2-1)}\frac{1}{(k+s)^2(n+s)}\leftarrow(S_4,S_5)$$

$$-3\sum_{\substack{k,n\\ n\neq k,k\pm1,k-2}}\frac{n^2+k^2-k}{32\pi(k^2-n^2)((k-1)^2-n^2)}\frac{1}{(k+s)(k+s-1)(n+s)}\leftarrow(S_8)$$

$$-3\sum_{\substack{k,n\\ n\neq k,k\pm1,k+2}}\frac{n^2+k^2+k}{32\pi(k^2-n^2)((k+1)^2-n^2)}\frac{1}{(k+s)(k+s+1)(n+s)}\leftarrow(S_9)$$

$$-3\sum_{k=1}\frac{2k^2-1}{16\pi(4k^2-1)}\frac{1}{(k+s)^3}\leftarrow(S_{10})$$

$$+2\sum_{k=1}\frac{6k^3+9k^2+3k-1}{128\pi k(k+1)(2k+1)}\frac{1}{(k+s)^2(k+s+1)}\leftarrow(S_7)$$

$$+2\sum_{k=2}\frac{6k^3-9k^2+3k+1}{128\pi k(k-1)(2k-1)}\frac{1}{(k+s)^2(k+s-1)}\leftarrow(S_6)$$

$$-2\sum_{k=3}\frac{4k^2-8k+5}{64\pi(2k-1)(2k-3)}\frac{1}{(k+s)(k+s-1)(k+s-2)}\leftarrow(S_2)$$

$$+\sum_{k=2}\frac{6k^3-9k^2+3k-1}{128\pi k(k-1)(2k-1)}\frac{1}{(k+s)(k+s-1)^2}\leftarrow(S_{11})$$

$$-2\sum_{k=2}\frac{4k^2+1}{64\pi(4k^2-1)}\frac{1}{(k+s)(k+s-1)(k+s+1)}\leftarrow(S_1)$$

$$+\sum_{k=1}\frac{6k^3+9k^2+3k+1}{128\pi k(k+1)(2k+1)}\frac{1}{(k+s)(k+s+1)^2}\leftarrow(S_{12})$$

$$-2\sum_{k=1}\frac{4k^2+8k+5}{64\pi(2k+1)(2k+3)}\frac{1}{(k+s)(k+s+1)(k+s+2)}\leftarrow(S_3)\Bigg].$$

The symbols S_i, $i = 1, \ldots, 12$ represents the corresponding terms shown in the formula and correspond to simple summations. As S_4, S_5, S_8, S_9 are shown in correspondence with double summations, they need a more accurate definitions. S_4 is the sum over k from 2 to ∞ of the corresponding term for $n = k + 2$, while S_5 is the sum of the same term from 3 to ∞ for $n = k - 2$; S_8 is the sum over k from 2 to ∞ of the corresponding term for $n = k + 2$. S_9 is the sum over k from 3 to ∞ of the corresponding term for $n = k - 2$.

The first line of the RHS refers to the terms with one cosine, the next four lines to terms with 2 cosines and the remaining ones to terms with three cosines integrated over. In (A.15), $\psi^{(n)}$ is the n-th polygamma function and $\psi^{(0)} = \psi$. There are simple and quadratic poles at $s = 1$, but they are compensated by corresponding zeroes in the numerators. One can also see that all the summations are (absolutely) convergent for any finite s, including $s = 0$.

To proceed further let us define

$$\begin{aligned}
SK0(k,n,s) &= \frac{n^2+k^2-1}{16\pi((n+k)^2-1)((n-k)^2-1)}\,\frac{1}{(k+s)^2(n+s)} \\
SK1(k,n,s) &= \frac{n^2+k^2-k}{32\pi(k^2-n^2)((k-1)^2-n^2)}\,\frac{1}{(k+s)(k+s-1)(n+s)} \\
SK2(k,n,s) &= \frac{n^2+k^2+k}{32\pi(k^2-n^2)((k+1)^2-n^2)}\,\frac{1}{(k+s)(k+s+1)(n+s)}
\end{aligned}$$

and set

$$\begin{aligned}
SK0_+(p,s) &= \sum_{k=1}^{\infty} SK0(k,k+p,s), \qquad SK0_-(p,s) = \sum_{k=p+1}^{\infty} SK0(k,k-p,s), \\
SK1_+(p,s) &= \sum_{k=2}^{\infty} SK1(k,k+p,s), \qquad SK1_-(p,s) = \sum_{k=p+1}^{\infty} SK1(k,k-p,s), \\
SK2_+(p,s) &= \sum_{k=1}^{\infty} SK2(k,k+p,s), \qquad SK2_-(p,s) = \sum_{k=p+1}^{\infty} SK2(k,k-p,s).
\end{aligned}$$

Then the quantity within the square brackets in (A.15) corresponds to

$$3\sum_{p=3}^{\infty}\Big(SK0_+(p,s)+SK0_-(p,s)-SK1_+(p,s)-SK1_-(p,s) \\ -SK2_+(p,s)-SK2_-(p,s)\Big)+\sum_{i=1}^{12} S_i(s). \tag{A.16}$$

A.0.3.1 Performing one discrete summation

Like in the quadratic term we can carry out in an analytic way one discrete summation. We have

$$
\begin{aligned}
& S(s) \qquad \text{(A.17)}\\
= \ & \sum_{i=1}^{12} S_i(s) = \frac{1}{256\pi(-2+s)s^2\left(-1-s+4s^2+4s^3\right)^3\left(18-9s-17s^2+4s^3+4s^4\right)^2}\\
& \cdot\Bigg(\frac{1}{3+s}\Bigg(12\gamma(1+s)(2+s)(3+s)\\
& \cdot\left(-324+13887s^2-48589s^4+72468s^6-44592s^8+11200s^{10}\right)\\
& +s^2\big((1+2s)^3(-845856+1192986s+1878099s^2-2889638s^3-2109474s^4\\
& +3023246s^5+1453619s^6-1668346s^7-622980s^8+493352s^9\\
& +147696s^{10}-82016s^{11}-21440s^{12}+6016s^{13}+1536s^{14})\\
& -192(-2+s)(1+s)^3(3+s)\left(-2+s+s^2\right)^2\\
& \cdot\left(153-132s^2+112s^4+64s^6\right)\ln 4\big)\Bigg)\\
& +12(1+s)(2+s)\\
& \cdot\Big(\left(-324+13887s^2-48589s^4+72468s^6-44592s^8+11200s^{10}\right)\psi(1+s)\\
& +s\Big((-2+s)(-1+s)(1+s)(2+s)(-3+2s)(-1+2s)(1+2s)(3+2s)\\
& \cdot\left(9+138s^2-352s^4+160s^6\right)\psi^{(1)}(1+s)\\
& +2s\left(4-9s^2+2s^4\right)\left(-9+s^2\left(7-4s^2\right)^2\right)^2\psi^{(2)}(1+s)\Big)\Big)\Bigg). \qquad \text{(A.18)}
\end{aligned}
$$

Similarly

$$
\begin{aligned}
& SK(p,s) \qquad \text{(A.19)}\\
\equiv \ & \sum_{i=0}^{2} SKi_+(p,s)+SKi_-(p,s)\\
= \ & \frac{1}{32p^2\pi}\Bigg(\frac{2p}{(1+p)(1+p-2s)^2}+\frac{2p}{(1+p)^2(1+p-2s)}+\frac{p^2}{\left(2+3p+p^2\right)(2+p-2s)}\\
& +\frac{-1+p-p^2-2p^3}{(-1+p)^2\left(2+3p+p^2\right)(1+s)}+\frac{4+p(-3+p(2+(-2+p)p))}{(-2+p)\left(-1+p^2\right)^2(p+s)}+\frac{1+p(3+p)}{(1+p)^2(2+p)(1+p+s)}\\
& +\frac{p}{(-1+p)(2-p+2s)}-\frac{2p}{(1+p)^2(-1+p+2s)}+\frac{2\left(-2+p^2\right)}{\left(-1+p^2\right)(p+2s)}\\
& -\frac{2p}{(-1+p)(1+p+2s)^2}+\frac{4p}{(-1+p)^2(1+p)(1+p+2s)}-\frac{2(-1+p)p}{(-2+p)(1+p)(2+p+2s)}
\end{aligned}
$$

$$
\begin{aligned}
&+2p\Bigg(-\frac{8\left(p-2p^3+p^5-16ps^4\right)\psi(\frac{1+p}{2})}{\left((-1+p^2)^2-8\left(1+p^2\right)s^2+16s^4\right)^2}\\
&+\frac{8p\left(-4+p^2+4s^2\right)\psi(\frac{2+p}{2})}{(-2+p-2s)(p-2s)(2+p-2s)(-2+p+2s)(p+2s)(2+p+2s)}\\
&+\frac{1}{-1+p^2}\\
&\cdot\Big(2\frac{2-4p^4+21p^3s+6s^2-8s^4+p^2\left(2-38s^2\right)+ps\left(-9+28s^2\right)\psi(1+s)}{(-2+p-2s)(-1+p-2s)^2(p-2s)(1+p-2s)^2(2+p-2s)}\\
&-2\frac{\left(-2+4p^4+21p^3s-6s^2+8s^4+ps\left(-9+28s^2\right)+p^2\left(-2+38s^2\right)\right)\psi(p+s)}{(-2+p+2s)(-1+p+2s)^2(p+2s)(1+p+2s)^2(2+p+2s)}\\
&+\frac{\left(-1+p^2-2ps+2s^2\right)\psi^{(1)}(1+s)}{-1+p^2-4ps+4s^2}-\frac{\left(-1+p^2+2ps+2s^2\right)\psi^{(1)}(1+p+s)}{-1+p^2+4ps+4s^2}\Big)\Big)\Bigg).
\end{aligned}
$$

As explained above, in general we cannot proceed further with analytic means in performing the remaining summations and integrations. The strategy from now on consists therefore in making sure that summations and integrals converge (apart from the expected UV singularity, which has to be subtracted).

Appendix B

Reduction formulae

In this appendix we collect from ([110]) the formulas that are needed to reduce superfield expressions to component form. The equations below are not complete, they contain only the terms essential to recover the ordinary parts of the expressions (that is only the parts that survive once all the fields except the metric are disregarded). The complete form can be found in ([110]), or in [109] for the minimal model. The first formula when evaluated at $\theta = \overline{\theta} = 0$, connects the Riemann curvature to specific superfield components and it is basic for reducing cocycles to ordinary form

$$
\begin{aligned}
&\sigma^a_{\alpha\dot\alpha}\sigma^b_{\beta\dot\beta}\sigma^c_{\gamma\dot\gamma}\sigma^d_{\delta\dot\delta}\, R_{cdba} \qquad \text{(B.1)}\\
&\approx 4\epsilon_{\gamma\delta}\epsilon_{\beta\alpha}\Bigg[\frac{1}{4}\Big(\overline{\mathcal{D}}_{\dot\gamma}\overline{W}_{\dot\delta\dot\beta\dot\alpha} + \overline{\mathcal{D}}_{\dot\delta}\overline{W}_{\dot\beta\dot\alpha\dot\gamma} + \overline{\mathcal{D}}_{\dot\beta}\overline{W}_{\dot\alpha\dot\gamma\dot\delta} + \overline{\mathcal{D}}_{\dot\alpha}\overline{W}_{\dot\gamma\dot\delta\dot\beta}\Big)\\
&-\frac{1}{8}\sum_{\dot\gamma\dot\delta}\sum_{\dot\beta\dot\alpha}\epsilon_{\dot\gamma\dot\beta}\sum_{\dot\delta\dot\alpha}\Big[\overline{\mathcal{D}}_{\dot\delta}\mathcal{D}^{\epsilon}G_{\epsilon\dot\alpha} + \frac{1}{4}\overline{\mathcal{D}}_{\dot\delta}\mathcal{D}^{\epsilon}\left(\frac{1}{3}c_{\epsilon\dot\alpha} - i\,n\,d_{\epsilon\dot\alpha}\right) + \frac{i}{2}(n-1)\overline{\mathcal{D}}_{\dot\delta}\mathcal{D}_{\epsilon\dot\alpha}T^{\epsilon}\\
&+n\,\mathcal{D}_{\epsilon\dot\delta}d^{\epsilon}{}_{\dot\alpha} + \frac{1}{6}\mathcal{D}_{\epsilon\dot\delta}c^{\epsilon}{}_{\dot\alpha}\Big] + (\epsilon_{\dot\gamma\dot\alpha}\epsilon_{\dot\beta\dot\delta} + \epsilon_{\dot\delta\dot\alpha}\epsilon_{\dot\beta\dot\gamma})\Lambda\Bigg]\\
&-\frac{1}{2}\epsilon_{\gamma\delta}\,\epsilon_{\dot\beta\dot\alpha}\sum_{\alpha\beta}\sum_{\dot\gamma\dot\delta}\Big[i\,\mathcal{D}_{\beta\dot\gamma}\Big(G_{\alpha\dot\delta} + \frac{1}{3}c_{\alpha\dot\delta} - i\,n\,d_{\alpha\dot\delta}\Big) + \overline{\mathcal{D}}_{\dot\gamma}\mathcal{D}_{\beta}\left(G_{\alpha\dot\delta} + \frac{1}{3}c_{\alpha\dot\delta}\right) + \frac{1}{3}\overline{\mathcal{D}}_{\dot\gamma}\mathcal{D}_{\beta}c_{\alpha\dot\delta}\Big]\\
&-\frac{1}{2}\epsilon_{\dot\gamma\dot\delta}\,\epsilon_{\beta\alpha}\sum_{\dot\alpha\dot\beta}\sum_{\gamma\delta}\Big[i\,\mathcal{D}_{\gamma\dot\beta}\Big(G_{\delta\dot\alpha} + \frac{1}{3}c_{\delta\dot\alpha} - i\,n\,d_{\delta\dot\alpha}\Big) + \overline{\mathcal{D}}_{\dot\beta}\mathcal{D}_{\gamma}\left(G_{\delta\dot\alpha} + \frac{1}{3}c_{\delta\dot\alpha}\right) + \frac{1}{3}\overline{\mathcal{D}}_{\dot\beta}\mathcal{D}_{\gamma}c_{\delta\dot\alpha}\Big]\\
&+4\epsilon_{\dot\gamma\dot\delta}\epsilon_{\dot\beta\dot\alpha}\Bigg[-\frac{1}{4}\Big(\mathcal{D}_{\gamma}W_{\delta\beta\alpha} + \mathcal{D}_{\delta}W_{\beta\alpha\gamma} + \mathcal{D}_{\beta}W_{\alpha\gamma\delta} + \mathcal{D}_{\alpha}W_{\gamma\delta\beta}\Big)\\
&+\frac{1}{8}\sum_{\gamma\delta}\sum_{\beta\alpha}\epsilon_{\gamma\beta}\sum_{\delta\alpha}\Big[\mathcal{D}_{\delta}\overline{\mathcal{D}}^{\dot\epsilon}G_{\alpha\dot\epsilon} + \frac{1}{4}\mathcal{D}_{\delta}\mathcal{D}^{\dot\epsilon}\left(\frac{1}{3}c_{\alpha\dot\epsilon} - i\,n\,d_{\alpha\dot\epsilon}\right) + \frac{i}{2}(n-1)\mathcal{D}_{\delta}\mathcal{D}_{\dot\epsilon\dot\alpha}\overline{T}^{\dot\epsilon}\\
&+n\,\mathcal{D}_{\delta\dot\epsilon}d^{\dot\epsilon}{}_{\alpha} + \frac{1}{6}\mathcal{D}_{\delta\dot\epsilon}c^{\dot\epsilon}_{\alpha}\Big] + (\epsilon_{\gamma\alpha}\,\epsilon_{\beta\delta} + \epsilon_{\delta\alpha}\,\epsilon_{\beta\gamma})\boldsymbol{\Lambda}\Bigg]
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{\Lambda} \;\approx\; & \frac{1}{24}\Big(\mathcal{D}^\alpha\mathcal{D}_\alpha R+\overline{\mathcal{D}}_{\dot\alpha}\overline{\mathcal{D}}^{\dot\alpha}R^+\Big)+\frac{1}{48}\Big(\mathcal{D}^\alpha\overline{\mathcal{D}}^{\dot\alpha}-\overline{\mathcal{D}}^{\dot\alpha}\mathcal{D}^\alpha\Big)G_{\alpha\dot\alpha}\\
&+\frac{1}{24}\Big(-\frac{1}{12}\mathcal{D}^\alpha\overline{\mathcal{D}}^{\dot\alpha}c_{\alpha\dot\alpha}-\frac{i}{4}\,n\,\mathcal{D}^\alpha\overline{\mathcal{D}}^{\dot\alpha}d_{\alpha\dot\alpha}-\frac{i}{2}(n-1)\mathcal{D}^\alpha\mathcal{D}_{\alpha\dot\alpha}\overline{T}^{\dot\alpha}\Big)\\
&+\frac{1}{24}\Big(\frac{1}{12}\overline{\mathcal{D}}^{\dot\alpha}\mathcal{D}^\alpha c_{\alpha\dot\alpha}-\frac{i}{4}\,n\,\overline{\mathcal{D}}^{\dot\alpha}\mathcal{D}^\alpha d_{\alpha\dot\alpha}+\frac{i}{2}(n-1)\overline{\mathcal{D}}^{\dot\alpha}\mathcal{D}_{\alpha\dot\alpha}T^\alpha\Big)-\frac{n}{16}\mathcal{D}^{\alpha\dot\alpha}d_{\alpha\dot\alpha}
\end{aligned}
\qquad (B.2)
$$

Other relations come from constraints among the various superfields

$$
\begin{aligned}
\mathcal{D}^\alpha\mathcal{D}_\alpha R-\overline{\mathcal{D}}_{\dot\alpha}\overline{\mathcal{D}}^{\dot\alpha}R^+ \;\approx\; & -2i\,\mathcal{D}^{\alpha\dot\alpha}G_{\alpha\dot\alpha}\\
&+\Big(\frac{1}{12}\mathcal{D}^\alpha\overline{\mathcal{D}}^{\dot\alpha}c_{\alpha\dot\alpha}+\frac{i}{4}\,n\,\mathcal{D}^\alpha\overline{\mathcal{D}}^{\dot\alpha}d_{\alpha\dot\alpha}+\frac{i}{2}(n-1)\mathcal{D}^\alpha\mathcal{D}_{\alpha\dot\alpha}\overline{T}^{\dot\alpha}\Big)\\
&+\Big(\frac{1}{12}\overline{\mathcal{D}}^{\dot\alpha}\mathcal{D}^\alpha c_{\alpha\dot\alpha}-\frac{i}{4}\,n\,\overline{\mathcal{D}}^{\dot\alpha}\mathcal{D}^\alpha d_{\alpha\dot\alpha}+\frac{i}{2}(n-1)\overline{\mathcal{D}}^{\dot\alpha}\mathcal{D}_{\alpha\dot\alpha}T^\alpha\Big)+i\mathcal{D}^{\alpha\dot\alpha}c_{\alpha\dot\alpha}
\end{aligned}
\qquad (B.3)
$$

together with

$$
\mathcal{D}_\alpha\overline{\mathcal{D}}_{\dot\alpha}R \;\approx\; \frac{i}{8}\,(n+1)\,\overline{\mathcal{D}}_{\dot\alpha}\overline{\mathcal{D}}_{\dot\beta}d_\alpha{}^{\dot\beta}-\frac{i}{4}\,(n-1)\,\overline{\mathcal{D}}_{\dot\alpha}\mathcal{D}_{\alpha\dot\beta}\overline{T}^{\dot\beta} \qquad (B.4)
$$

$$
\overline{\mathcal{D}}_{\dot\alpha}\mathcal{D}_\alpha R \;\approx\; \overline{\mathcal{D}}_{\dot\alpha}\overline{\mathcal{D}}^{\dot\beta}G_{\alpha\dot\beta}-\frac{i}{8}\,n\,\overline{\mathcal{D}}_{\dot\alpha}\overline{\mathcal{D}}^{\dot\beta}d_{\alpha\dot\beta}+\frac{5}{24}\,\overline{\mathcal{D}}_{\dot\alpha}\overline{\mathcal{D}}^{\dot\beta}c_{\alpha\dot\beta}-\frac{i}{4}\,(n-1)\,\overline{\mathcal{D}}_{\dot\alpha}\mathcal{D}_{\alpha\dot\beta}\overline{T}^{\dot\beta} \qquad (B.5)
$$

$$
\begin{aligned}
\mathcal{D}^\gamma W_{\alpha\beta\gamma} \;\approx\; & \frac{1}{16}\Big(\mathcal{D}_\alpha\overline{\mathcal{D}}^{\dot\gamma}G_{\beta\dot\gamma}+\mathcal{D}_\beta\overline{\mathcal{D}}^{\dot\gamma}G_{\alpha\dot\gamma}\Big)-\frac{7}{144}\Big(\mathcal{D}_\alpha\overline{\mathcal{D}}^{\dot\gamma}c_{\beta\dot\gamma}+\mathcal{D}_\beta\overline{\mathcal{D}}^{\dot\gamma}c_{\alpha\dot\gamma}\Big) \qquad (B.6)\\
&+\frac{in}{48}\Big(\mathcal{D}_\alpha\overline{\mathcal{D}}^{\dot\gamma}d_{\beta\dot\gamma}+\mathcal{D}_\beta\overline{\mathcal{D}}^{\dot\gamma}d_{\alpha\dot\gamma}\Big)
\end{aligned}
$$

with the respective conjugate relations.

The last equation above, together with

$$
\mathcal{D}_\alpha W_{\beta\gamma\delta}=\mathcal{D}_{(\alpha}W_{\beta\gamma\delta)}+\frac{1}{4}(\epsilon_{\alpha\beta}\mathcal{D}^\zeta W_{\gamma\delta\zeta}+\epsilon_{\alpha\gamma}\mathcal{D}^\zeta W_{\delta\beta\zeta}+\epsilon_{\alpha\delta}\mathcal{D}^\zeta W_{\beta\gamma\zeta}), \qquad (B.7)
$$

allows us to conclude that

$$
\mathcal{D}_\alpha W_{\beta\gamma\delta}\approx\mathcal{D}_{(\alpha}W_{\beta\gamma\delta)} \qquad (B.8)
$$

Finally we quote in its exact form a constraint equation

$$
\begin{aligned}
\overline{\mathcal{D}}^{\alpha}G_{\alpha\dot\alpha}-\mathcal{D}_\alpha R \;=\; & \frac{1}{4}(n+\frac{1}{3})\overline{\mathcal{D}}\overline{\mathcal{D}}T_\alpha-\frac{1}{4}(n-\frac{1}{3})\overline{\mathcal{D}}^{\dot\alpha}\mathcal{D}_\alpha\overline{T}_{\dot\alpha}-\frac{1}{3}(n+1)(3n-1)\overline{T}^{\dot\alpha}\mathcal{D}_{\dot\alpha}T_\alpha\\
&-\frac{1}{4}(3n-5)\overline{T}^{\dot\alpha}G_{\alpha\dot\alpha}-\frac{1}{6}(6n^2+3n+1)\overline{T}^{\dot\alpha}\mathcal{D}_\alpha\overline{T}_{\dot\alpha}+\frac{1}{6}(n-1)T_\alpha\overline{\mathcal{D}}\overline{T}\\
&-\frac{1}{3}(n-1)T_\alpha\overline{T}\overline{T}-\frac{i}{2}(n+1)\mathcal{D}_{\alpha\dot\alpha}\overline{T}^{\dot\alpha}
\end{aligned}
\qquad (B.9)
$$

which, together with its conjugate, is needed for the cohomological analysis of cocycles.

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