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The Structure and Regularity of Admissible BV Solutions to Hyperbolic Conservation Laws in One Space Dimension

Ph.D. Thesis

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Contents

Introduction

This thesis is devoted to the study of the qualitative properties of admissible BV solutions to the strictly hyperbolic conservation laws in one space dimension by using wave-front tracking approximation. This thesis consists of two parts:

- SBV-like regularity of vanishing viscosity BV solutions to strict hyperbolic systems of conservation laws.
- Global structure of admissible BV solutions to strict hyperbolic conservation laws.

The first problem arises naturally in some problems in the control theory for hyperbolic systems of conservation laws (see for instance [32] for references). Both problems are closely related to each other since they both require deep understanding of the relation between wave-fronts in the approximate solutions and the structure of shocks in the corresponding exact solutions. Up to now, we have got some positive answers to both problems by deriving suitable estimates and structural properties on wave-front tracking approximate solutions and recovering the desired properties for exact solutions as the limits.

0.1 Hyperbolic Conservation Laws.

The study of gas dynamics gave birth to the theory of *hyperbolic conservation laws* (HCLs) about one hundred and fifty years ago. The development of this subject became explosive over the past three decades.

Mathematically, HCLs in one space dimension are described by the quasilinear hyperbolic system

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) + \frac{\partial}{\partial x}f(u(t,x)) = 0, \\ u_{|t=0} = u_0, \end{cases} \tag{0.1.1}$$

where t is time variable, x is one dimension spatial variable, $u(t,x)$ ia a vector of N conserved quantities (e.g. mass, momentum, energy etc.) and f is a smooth vector-valued function with N components, which is called flux function. And (0.1.1) is *strictly hyperbolic* on some domain $\Omega \subset \mathbb{R}^N$ if the eigenvalues $\{\lambda_i(u)\}_{i=1}^N$ of the Jacobian matrix $A(u) := Df(u)$ satisfy

$$\lambda_1(u) < \cdots < \lambda_N(u), \qquad u \in \Omega.$$

A particular feature of nonlinear hyperbolic systems is the appearance of shock waves even with smooth initial data. Therefore, in order to construct solutions globally defined in time, one reasonable option is to consider weak solutions interpreting the equation (0.1.1) in a distributional sense. We recall that for some given $T > 0$, $u \in C([0,T]; L^1_{loc}(\mathbb{R};\mathbb{R}^N))$ is a weak solution to the Cauchy problem (0.1.1) if the initial condition is satisfied and, for any smooth function $\phi \in C^1_c(]0,T[\times\mathbb{R})$, there holds

$$\int_0^T \int_{\mathbb{R}} \phi_t(t,x)u(t,x) + \phi_x(t,x)f(u(t,x))dxdt = 0. \tag{0.1.2}$$

It follows from the weak formulation (0.1.2) and integration by parts that a function with a single jump discontinuity, say

$$u(t,x) = \begin{cases} u^L & \text{if } x < \sigma t, \\ u^R & \text{if } x > \sigma t, \end{cases}$$

for some constants $u^L, u^R \in \mathbb{R}^N, \sigma \in \mathbb{R}$, is a weak solution to (0.1.1) if and only it holds the *Rankine-Hugoniot relations*:

$$f(u^R) - f(u^L) = \sigma(u^R - u^L). \tag{0.1.3}$$

By the strict hyperbolicity of the system, for each fixed $\bar{u} \in \Omega$ and $k \in \{1, \cdots, N\}$, one can construct, in a small neighborhood of $\bar{u}$, the smooth k-th *Hugoniot curve* $S_k[\bar{u}]$ passing though $\bar{u}$, such that each $u \in S_k[\bar{u}]$ satisfies the Rankine-Hugoniot relations:

$$f(u) - f(\bar{u}) = \sigma(u - \bar{u}),$$

for some scalar $\sigma = \sigma_k[\bar{u}, u]$. We say that $[u^{\mathrm{L}}, u^{\mathrm{R}}]$ is a shock discontinuity of the k-th family with speed $\sigma_k[u^{\mathrm{L}}, u^{\mathrm{R}}]$ if $u^{\mathrm{R}} \in S_k[u^{\mathrm{L}}]$.

Since the non-uniqueness of weak solutions to (0.1.1) (see the example in Section 4.4 of [30]), it is necessary to introduce some admissible criteria to select a unique "physical" admissible solution to some initial data.

For the scalar case ($N = 1$), a locally integrable function $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is an *entropy solution* of (0.1.1) (introduced by Volpert in [47]), if

$$\iint \{|u - k|\phi_t - \operatorname{sgn}(u - k)(f(u) - f(k))\phi_x\}dxdt \geq 0 \tag{0.1.4}$$

for each constant k and C^1 function $\phi \geq 0$ with compact support contained in $[0,\infty) \times \mathbb{R}$.

One can construct an entropy solution as a limit of piecewise constant approximations, which is so called front tracking method, as shown in Section 1.3.2, (also see [28]). Other constructions are possible, even for scalar conservation laws in several space dimensions: the vanishing viscosity method by Kruzhkov in [39], nonlinear semigroup theory by Crandall in [27], or finite difference schemes by Smoller in [46].

For the system case ($N \geq 2$), Liu proposed in [41,42] a criterion valid for weak solutions to general hyperbolic systems of conservation laws, which generalizes the classical stability condition introduced by Lax [40]:

Definition 0.1.1. A shock of k-th family joins the state u^{L} on the left to a state $u^{\mathrm{R}} = S_k[u^{\mathrm{L}}]$ on the right with speed σ is *Liu admissible* if

$$\sigma = \sigma_k[u^{\mathrm{L}}, u^{\mathrm{R}}] \leq \sigma_k[u^{\mathrm{L}}, u],$$

for each state u on the Hugoniot curve $S_k[u^{\mathrm{L}}]$ between u^{L} and u^{R}.

It is well known that the space of functions with bounded variation (shortly BV) plays a prominent role in the well-posedness theory for weak solutions to the system (0.1.1). The early construction of admissible BV solutions with small total variations under certain restrictions on the system (0.1.1) were given in [36] by Glimm through Random Choice Method, in [42] by Liu through the Wave Tracing Method and in [18] by Bressan through the Front Tracking Methods. In [38], Iguchi and LeFloch prove, by the front tracking method, the existence of solutions to the systems with general flux functions $f(u)$ that can be approached by a sequence of piecewise genuinely nonlinear flux functions.

For the general strictly hyperbolic systems, the global existence of Liu admissible solutions was established by Bianchini and Bressan in [10], through the vanishing viscosity method, namely, as $\epsilon \downarrow 0$, the limit of solutions u^ϵ of the parabolic system

$$\frac{\partial}{\partial t}u^\epsilon(t,x) + \frac{\partial}{\partial x}f(u^\epsilon(t,x)) = \epsilon\frac{\partial^2}{\partial x^2}u^\epsilon(t,x),$$

with initial condition $u^\epsilon(0,x) = u_0^\epsilon(x),\ -\infty < x < \infty$ of sufficiently small total variation.

We summarize the existence of admissible BV solutions of the Cauchy problem (0.1.1) in the following theorem.

Theorem 0.1.2. *Assume the system* (0.1.1) *is strictly hyperbolic and the initial data u_0 is a L^1 function with sufficiently small total variation. Then there exists a global Liu admissible BV solution. This solution depends continuously in L^1 on its initial values.*

Marson and Ancora have developed in [5] a wave-front tracking algorithm for vanishing viscosity BV solutions to general strictly hyperbolic systems, which opened a door to get additional results on qualitative structure and asymptotic properties of the vanishing viscosity solutions by analysis on the wave-front tracking approximate solutions.

0.2 SBV and SBV-like regularity

Recently there have been interesting advances in the analysis of the measure-theoretic structure of the distributional derivative of BV solutions to genuinely nonlinear hyperbolic conservation laws. The results obtained are that, in addition to the BV bounds, the solution enjoys the strong regularity property that no Cantor part in the space derivative of $u(t)$ appears out of a countable set of times: the fact that the measure $D_x u(t)$ has only absolutely continuous part and jump part yields by definition that $u(t) \in \mathrm{SBV}(\mathbb{R})$. The motivation for studying this SBV-regularizing effect arises from problems in control theory and measure-theoretic questions, see [32] for more references.

The first result has been proved by Ambrosio and De Lellis in [2], for the entropy solution u to the scalar hyperbolic conservation law with strictly convex flux:

$$\partial_t u + \partial_x f(u) = 0 \qquad \text{in } \Sigma \subset [0,T] \times \mathbb{R}. \tag{0.2.1}$$

where $f : \Omega \to \mathbb{R}$ satisfies $f''(u) > 0$ for all u.

More precisely, they proved the following theorem.

Theorem 0.2.1. *Let $u \in L^\infty(\Sigma)$ be an entropy solution of (0.2.1) with locally uniformly convex flux $f \in C^2(\mathbb{R})$. Then there exists $S \subset [0,T]$ at most countable such that for each $t \in [0,T] \setminus S$ the following holds:*

$$u(t,\cdot) \in SBV_{loc}(\Sigma_t) \qquad \text{with } \Sigma_t := \{x \in \mathbb{R} : (t,x) \in \Sigma\}.$$

In particular, $u \in SBV_{loc}(\Sigma)$.

The theorem is optimal since one can construct an entropy solution to Burger's equation with some initial data in L^∞ such that the entropy solution is not in SBV at countable many times.

The intuitive explanation for the SBV regular effect is that if $D_x u(\bar{t},\cdot)$ is not SBV for a certain time $\bar{t}$, then at further times $\bar{t}+\epsilon$ with ϵ arbitrarily small, the Cantor part of $D_x u(t,\cdot)$ transforms into jump singularity. The key of the proof is to construct a bounded monotone $F(t)$ which has a jump with the presence of a Cantor part of $D_x u(t,\cdot)$. Then the monotonicity and boundedness of $F(t)$ imply that the Cantor part of $D_x u(t,\cdot)$ can be nonzero at most for a countable many times.

Following the similar idea, Robyr generalized the SBV regularity to scalar balance laws with the flux having countable many inflection points, see [44].

However, the argument in [2] is hard to be applied for the system case, by the loss of non-crossing properties of characteristics with the appearance of rarefaction waves and the interaction of the waves of different families. Bianchini and Caravenna proved in [11] the SBV regularity for genuinely nonlinear hyperbolic systems of conservation laws in one space dimension. They decompose the derivative of the solution u as

$$D_x u(t) = \sum_i v_i(t)\tilde{r}_i, \tag{0.2.2}$$

with $\tilde{r}_i = r_i$, the i-th right eigenvector of Df, where u is continuous, otherwise the direction of the jump of the i-th family. Each $v_i(t)$ is a bounded measure.

Then, in order to show the SBV regularity of $u(t)$, it suffices to prove that for each $i \in \{1,\cdots,N\}$, $v_i(t)$ has a Cantor part only at countable many times.

Recall that the k-th family is genuinely nonlinear if

$$r_k \cdot \lambda_k(u) \neq 0 \quad \text{for all } u \in \Omega, \tag{0.2.3}$$

and we say that the k-th family is linearly degenerate if

$$r_k \cdot \lambda_k(u) = 0 \quad \text{for all } u \in \Omega.$$

Then the following holds.

Theorem 0.2.2. *Suppose (0.2.3) holds for the k-th characteristic field. Denoting*

$$v_k = \int \big[v_k(t)\big]dt, \qquad v_k(t) = (v_k(t))_{cont} + (v_k(t))_{jump} : \quad (v_k(t))_{jump} \text{ purely atomic,}$$

then there exists a finite, nonnegative Radon measure μ_k^{ICJ} on $\mathbb{R}^+ \times \mathbb{R}$ such that for $s > \tau > 0$

$$|(v_k(s))_{cont}|(B) \leq C\Big\{\mathcal{L}^1(B)/\tau + \mu_k^{ICJ}([s-\tau,s+\tau]\times\mathbb{R})\Big\} \qquad \forall B \text{ Borel subset of } \mathbb{R}. \tag{0.2.4 : k}$$

By a standard argument, one can obtain the SBV regularity for the scalar wave measure v_k.

Corollary 0.2.3. *Let u be a semigroup solution of the Cauchy problem for the strictly hyperbolic system* (0.1.1)*. Consider the k-wave measure $v_k = \tilde{l}_k \cdot u_x$. If* (0.2.3) *holds, then v_k has no Cantor part.*

If all the family are genuinely non-linear, by the wave decomposition (0.2.2) the above estimate yields the $\mathrm{SBV}_{loc}([0,T] \times \mathbb{R}; \mathbb{R}^N)$ regularity of u for all $T > 0$.

Corollary 0.2.4. *Let u be the semigroup solution of the Cauchy problem*

$$\begin{cases} u_t + f(u)_x = 0, \\ u(t=0) = \bar{u}, \end{cases} \qquad u : \mathbb{R}^+ \times \mathbb{R} \to \Omega \subset \mathbb{R}^N, \qquad f \in C^2(\Omega; \mathbb{R}^N),$$

for a strictly hyperbolic system of conservation laws where each characteristic field is genuinely non-linear, with initial datum $\bar{u}$ small in $\mathrm{BV}(\mathbb{R}; \Omega)$. Then $u(t) \in \mathrm{SBV}(\mathbb{R}; \Omega)$ out of at most countably many times.

In Chapter 2, we consider the extension of the results of [11] to the case where the system is only strictly hyperbolic, i.e. no assumption on the nonlinear structure of the eigenvalues λ_i of Df is done. Clearly, by just considering a linearly degenerate eigenvalue, it is fairly easy to see that the solution u itself cannot be in SBV if the initial data u_0 is not a SBV function, so the regularity concerns some nonlinear function of u.

Therefore, for general scalar conservation law, we instead consider the SBV regularity of the characteristic speed $f'(u)$. Write the Cantor part of $D_x g$ as $D_x^c g$ for each BV function g. By Volpert's chain rule, one has

$$D_x^c f'(u) = f''(u) D_x^c u. \tag{0.2.5}$$

Then formally, when $f''(u) \neq 0$, i.e. f is genuinely nonlinear, by Theorem 0.2.1, one has $D_x u$ has no Cantor part which yields $D_x^c f'(u) = 0$. If instead $f''(u) = 0$, by (0.2.5), one still get $D_x^c f'(u) = 0$. In fact, one has the following theorem .

Theorem 0.2.5 ([17])**.** *Suppose that $u \in \mathrm{BV}(\mathbb{R}^+ \times \mathbb{R})$ is an entropy solution of the scalar conservation law* (0.1.1)*. Then there exists a countable set $S \subset \mathbb{R}^+$ such that for every $t \in \mathbb{R}^+ \setminus S$ the following holds:*

$$f'(u(t, \cdot)) \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R}).$$

In particular, if f is uniformly convex or concave, that is, $f'' \neq 0$, then the above theorem yields that u is in SBV.

After generalizing Corollary 0.2.3 to general strictly hyperbolic conservation laws in one space dimension with k-th family being genuinely nonlinear, we proved in [17] the SBV regularity for the i-th component of $D_x \lambda_i(u)$, by using the same strategy for proving Theorem 0.2.5.

In fact, we decompose $D_x u$ as in (0.2.2). Notice that $v_i = \tilde{l}_i \cdot D_x u$ is a scalar valued measure which is called the i*-th wave measure*, where $\tilde{l}_i(t,x)$ are the left eigenvector of

$$A(t,x) = A(u(t,x-), u(t,x+)) := \int_0^1 A(\theta u(t,x-) + (1-\theta) u(t,x+)) d\theta.$$

In the same way we can decompose the a.c. part $D_x^{\mathrm{ac}} u$, the Cantor part $D_x^{\mathrm{c}} u$ and the jump part $D_x^{\mathrm{jump}} u$ of $D_x u$ as

$$D_x^{\mathrm{ac}} u = \sum_{k=1}^{N} v_k^{ac} \tilde{r}_k, \quad D_x^{\mathrm{c}} u = \sum_{k=1}^{N} v_k^{\mathrm{c}} \tilde{r}_k, \quad D_x^{\mathrm{jump}} u = \sum_{k=1}^{N} v_k^{\mathrm{jump}} \tilde{r}_k.$$

We call v_i^{c} the Cantor part of v_i and denote by

$$v_i^{\mathrm{cont}} := v_i^{\mathrm{c}} + v_i^{ac} = \tilde{l}_i \cdot (D_x^{\mathrm{c}} u + D_x^{\mathrm{ac}} u),$$

the continuous part of v_i. According to Volpert's Chain Rule

$$D_x \lambda_i(u) = \nabla \lambda_i(u)(D_x^{\mathrm{ac}} u + D_x^{\mathrm{c}} u) + [\lambda_i(u^+) - \lambda_i(u^-)]\delta_x,$$

and then

$$D_x^{\mathrm{c}} \lambda_i(u) = \nabla \lambda_i \cdot D_x^{\mathrm{c}} u = \sum_k \big(\nabla \lambda_i \cdot \tilde{r}_k\big) v_k^{\mathrm{c}}.$$

We define the *i-th component of $D_x \lambda_i(u)$* as

$$[D_x \lambda_i(u)]_i := \big(\nabla \lambda_i \cdot \tilde{r}_i\big) v_i^{\mathrm{cont}} + [\lambda_i(u^+) - \lambda_i(u^-)] \frac{|v_i^{\mathrm{jump}}(x)|}{\sum_k |v_k^{\mathrm{jump}}(x)|}, \tag{0.2.6}$$

and the *Cantor part of i-th component of $D_x \lambda_i(u)$* to be

$$[D_x^{\mathrm{c}} \lambda_i(u)]_i := \big(\nabla \lambda_i \cdot \tilde{r}_i\big) v_i^{\mathrm{c}}. \tag{0.2.7}$$

As the same intuitive discussion for the scalar case, we have

Theorem 0.2.6. *Let u be a vanishing viscosity solution of the Cauchy problem for the strictly hyperbolic system* (0.1.1) *with small BV norm. Then there exists an at most countable set $S \subset \mathbb{R}^+$ such that i-th component of $D_x \lambda_i(u(t,\cdot))$ has no Cantor part for every $t \in \mathbb{R}^+ \setminus S$ and $i \in \{1,\ 2,\ \ldots,\ N\}$.*

In particular, if we assume that all characteristic fields of the system (0.1.1) are genuinely nonlinear, then Theorem 0.2.6 implies the result of Corollary 0.2.4 since $\nabla \lambda_i \cdot \tilde{r}_i > 0$ in (0.2.7).

0.3 Global structure of BV solutions

From the study of SBV regularity of admissible solution for the strictly hyperbolic system of conservation laws, we know that if one wants to apply the methods developed by the authors of [11], which is based on the decomposition of Radon measure $u_x(t)$ into waves belonging to the characteristic families and the balance of the continuous/jump part of the measures v_i in regions bounded by characteristics, it is necessary to study the corresponding measure and balance equation for the wave-front tracking approximate solutions u^ν. One basic question related to this is how to distinguish the wave-fronts of u^ν converging to the jump set of u and those converging to the continuity set of u.

In fact, in [11], the authors applied the following proposition under the assumption that all characteristic fields of the system (0.1.1) are genuinely nonlinear:

Proposition 0.3.1 ([24]). *Consider a sequence of front tracking approximations* u^ν *converging to* u *in* L^1_{loc}. *For* $\nu > 1$, *let* $\gamma_\nu : [t^-_\nu, t^+_\nu] \mapsto \mathbb{R}$ *be a shock curve of* u^ν *of uniformly large strength:* $|s_\nu(t)| > \epsilon$ *for a.e.* $t \in [t^-_\nu, t^+_\nu]$ *and for some fixed* $\epsilon > 0$. *Assume* $t^-_\nu \to t^-, t^+_\nu \to t^+$, *and* $\gamma_\nu(t) \to \gamma(t)$ *for every* $t \in [t^-, t^+]$. *Then* $\gamma(\cdot)$ *is a shock curve of the limiting solution* u. *That is, for all but countably many times, the derivative* $\dot{\gamma}$ *exists together with distinct right and left limits satisfying the Rankine-Hugoniot relations.*

Furthermore, the authors proved in [24] that the number of the shock curves of u is at most countable which is stated in the regularity theorem

Theorem 0.3.2. *Let* u *be a solution of* (0.1.1) *under the assumption that all the characteristic fields are genuinely nonlinear. Then there exists a countable set* Θ *of irregular points and a countable family of Lipschitz continuous shock curves* $\Gamma := \{x = \gamma_m(t) : t \in [t^-_m, t^+_m], m \geq 1\}$ *such that the following holds:*

(1) For each m *and each* $t \in [t^-_m, t^+_m]$ *such that* $(t, \gamma_m(t)) \notin \Theta$, *the left and right limits of* u *at* $(t, \gamma_m(t))$ *exist and satisfy the Rankine-Hugoniot relations.*

(2) u *is continuous outside the set* $\Theta \cup \Gamma$.

In Chapter 3, we generalize Theorem 0.3.2 to *piecewise genuinely nonlinear* (PGNL) hyperbolic system. For the scalar case, it is to assume that the flux function f has finite number of inflection points. This class of equations has been systematically studied in [42] by Glimm scheme, including the global structure of Liu-admissible solutions. As show in [7], the Liu-admissible solutions constructed by the Glimm scheme in [42] coincide with the vanishing viscosity solution. Thus we can apply the wave-front tracking algorithm in [5] to study the global structure of the Liu-admissible solutions.

The main difficulty to apply the methods in [24] is the appearance of the splitting of the shock waves. Our idea is to introduce the concept of *sud-discontinuities*, artificially decomposing each shock wave into several small waves which are stable under perturbations, see Section 3.4. And we get the similar result of [24] except that there may be several shock curves passing though a discontinuous point of u.

Theorem 0.3.3. *Let* u *be an admissible BV solution of the Cauchy problem* (0.1.1) *with* f *piecewise genuinely nonlinear. Then there exist a countable set* Θ *of interaction points and a countable family* $\mathscr{T}$ *of Lipschitz continuous curves such that* u *is continuous outside* Θ *and Graph*$(\mathscr{T})$.

Moreover, suppose $(t_0, x_0) \in$ *Graph*$(\mathscr{T})\backslash\Theta$. *and there exists exist p Lipschitz continuous curves* $y_1, \cdots, y_p \in \mathscr{T}$ *such that*

- $y_1(t_0) = \cdots = y_p(t_0) = x_0$,
- $y_1(t) \leq \cdots \leq y_p(t)$ *for all* t *in a neighborhood of* t_0.

Then, writting $u^L = u(t_0, x_0-)$, $u^R = u(t_0, x_0+)$, *one has*

$$u^{\mathrm{L}} = \lim_{\substack{x<y_1(t) \\ (x,t)\to(t_0,x_0)}} u(x,t), \qquad u^{\mathrm{R}} = \lim_{\substack{x>y_p(t) \\ (x,t)\to(t_0,x_0)}} u(x,t),$$

and

$$y_1'(t_0) = \cdots = y_p'(t_0) = \sigma[u^{\mathrm{L}}, u^{\mathrm{R}}.]$$

However, the method developed for the PGNL systems does not work for strictly hyperbolic system in general. In fact, in [16], we construct a system which is strictly hyperbolic but with some characteristic field is not PGNL or linearly degenerate and we show that its Liu-admissible solution to some initial data contains shocks which can not be exactly covered by countable many Lipschitz curves.

However, it is still possible to prove that the discontinuity points of the solution to the general system is covered by countably many Lipschitz continuous curves and to give a similar structural theorem for the general systems, except that there these curves may contain continuity points of the solution.

As a first step and a simple model, we consider the general scalar hyperbolic conservation law. In fact, up to now, most of the structural result for the scalar case is concerned about the genuinely nonlinear case, that is, the flux functions are convex or have only countable isolated inflection points. One important work is made by Oleinik in [43] and it is shown that solutions of scalar equation of (0.1.1), with the flux function f strictly convex, are continuous except on the union of an at most countable set of Lipschitz continuous curves. The result is optimal since one can construct a smooth initial data with compact support for which the solution exhibits infinitely many shock waves asymptotically in time, (see [1]). In [29], by introducing an important concept of generalized characteristics, Dafermos generalized this result to scalar balance law with the flux function f has nonlinear degenerate part, that is the scalar equation

$$u_t(t,x) + f_x(u(t,x),t,x) + g(u(t,x)t,x) = 0,$$

where f and g are respectively, C^2 and C^1 smooth functions on $\mathbb{R}^2 \times [0,\infty[$ such that, for fixed (t,x), $f_{uu}(u,t,x)$ is non-negative and does not vanish identically on any u-interval.

Motivated by the recent work of Bianchini and Modena in [13], we construct wave curves functions $X(t,s)$ in the solution u, such that $X(t,\cdot)$ are Lipschitz continuous curves, starting at initial time $t=0$ and moving along characteristics and shocks. Furthermore, along these wave curve, one can obtain a large amount of information concerning u including the structure of the boundary of level sets.

Conversely, from the Coarea formula (see Section 1.2), one can see the relation between the reduced boundary of the level sets and the total variation of the solutions. For the front tracking approximate solutions u^ν, the topological boundary of the level set consists of finite many Lipschitz continuous curves, therefore it coincides with the reduced boundary. Then by the compactness of the uniformly Lipschitz continuous functions, one get up to subsequence, the limits of this curves are Lipschitz curves. Furthermore, we will show that all this curves are also the boundary of level sets of some representative of the solution in L^1 norm sense, which is summarized in the following proposition.

Proposition 0.3.4. *If u is a entropy BV solution to a scalar conservation law, then there is a representation such that up to a $\mathcal{L}^1$-negligible set $S \subset \mathbb{R}$, the reduced boundary of each level set at value $w \notin S$ is made by finite many Lipschitz curves.*

Then, by parameterizing these boundary curves of level set as γ_s with the parameter $s \in [0, \text{Tot.Var.}\{u_0\}]$ and choosing the countable family of these curves to cover the discontinuity of u, one can prove the following theorem:

Theorem 0.3.5. *For the representative of solution $\tilde{u}$, there exist a countable family of graph of Lipschitz curves $\Gamma := \{Graph(\gamma_i)\}$, such that Γ cover the discontinuities of u.*

Notice that in the theorem, the curves in Γ maybe contain continuity points of u, in fact, a simple example is an entropy solution consists of a single rarefaction wave staring from the initial time. Then the collection Γ will contain countably many wave curves which moves along the characteristics of rarefaction waves.

0.4 Main notations

When E is a set, we write the *characteristic function* as

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 2 & \text{if } x \notin E. \end{cases}$$

Throughout the thesis, we write $\mathbb{R}^n$ as Euclidean space with dimension n. We shall denote by $[L^1(\mathbb{R}^n)]^m$ the Lebesgue space of function from $\mathbb{R}^n$ to $\mathbb{R}^m$. When A is a Lebesgue-measurable subset of $\mathbb{R}^n$, its n-dimension Lebesgue measure will be denoted by $|A|$ or $\mathcal{L}^n(A)$.

The Dirac measure of a point $x \in \mathbb{R}^n$ will be denoted by δ_x: $\delta_x(A) = 1$ if $x \in A$, 0 otherwise.

When T is a map from a measure space X, equipped with a measure μ, to a space Y, we write the push-forward measure of μ by T as

$$(T_{\#}\mu)(B) = \mu(T^{-1}(B)), \quad \text{for each measurable set } B \text{ in } Y,$$

where $T^{-1}(B) := \{x \in X : T(x) \in B\}$.

If X is a topological space, we denote by $C(X)$ the space of continuous functions on X, by $C_c(X)$ the space of continuous functions with compact support, by $C^k(X)$ the space of smooth functions with continuous derivative of k-th order and by $C_c^k(X)$ the space of smooth functions with continuous derivative of k-th order and compact support.

When $A \subset X$, we denote by A° the largest open set of contained in A, by $\bar{A}$ the smallest closed set containing A and we set $\partial A = \bar{A} \setminus A^\circ$ as the topological boundary of set A.

Let $\rho > 0$ and $s \in \mathbb{R}^n$, we denote by $B_\rho(x)$ the open ball centered at x with radius ρ, that is,

$$B_\rho(x) := \{x \in \mathbb{R}^n : |x| < \rho\},$$

where $|\cdot|$ is the Euclidean norm of $\mathbb{R}^n$.

For a continuous function $f : [0,T] \to \mathbb{R}^n$, we denote by $\text{Graph}(f)$ the graph of f, that is

$$\text{Graph}(f) := \{(t, f(t)) \in [0,T] \times \mathbb{R}^n : t \in [0,T]\}.$$

If A, B are two sets in $\mathbb{R}^n$, we denote by $\mathrm{dist}(A, B)$ the distance between these two sets and we write the symmetric difference between A and B as

$$A \triangle B := (A \cup B) \setminus (A \cap B).$$

We say a sequence of sets $A_n \subset \mathbb{R}^n$ converge to a set $A \subset \mathbb{R}^n$ in $\mathbf{L}^1$ norm if $|A \triangle A_n| \to 0$ as $n \to \infty$.

We denote by $\mathcal{H}^n$ the Hausdorff n-dimensional measure. In particular, $\mathcal{H}^0$ corresponds to the counting measure.

Let $a, b \in \mathbb{R}$ be two constant, we write $a \wedge b := \min\{a, b\}$.

Chapter 1

Preliminary results

This chapter presents various background materials which will be used in the later chapters. Most of them are statements without details of proof. In the following, we denote by $\hat{\Omega}$ an open set in $\mathbb{R}^n$.

1.1 BV and SBV functions

Consider a interval $J \subset \mathbb{R}$ and a map $u : J \mapsto \mathbb{R}^N$. The *total variation* of u on the subinterval $I \subset J$ is then defined as

$$\text{Tot.Var.}\{u; I\} := \sup \left\{ \sum_{i=1}^{p} |u(x_i) - u(x_{i-1})| \right\}. \tag{1.1.1}$$

where the supremum is taken over all $p \geq 1$ and all points $x_i \in I$ such that $x_0 < x_1 < \cdots < x_p$. If (1.1.1) is bounded, we say that u has bounded total variation on I. If $I = J$, then we simply write $\text{Tot.Var.}\{u\} := \text{Tot.Var.}\{u; J\}$.

Lemma 1.1.1. *Let $u : J \mapsto \mathbb{R}^N$ has bounded total variation. Then, for each point in J, the left and right limits are well defined. Moreover, u has at most countably many points of discontinuity.*

One can approximate the function with bounded total variation by piecewise constant functions, which is described in the following lemma.

Lemma 1.1.2. *Let $u : \mathbb{R} \to \mathbb{R}^N$ be right continuous with bounded variation. Then, for each $\epsilon > 0$, there exists a piecewise constant function u_ϵ such that*

$$\textit{Tot.Var.}\{u^\epsilon\} \leq \textit{Tot.Var.}\{u\}, \qquad \|u^\epsilon - u\|_\infty \leq \epsilon. \tag{1.1.2}$$

Proof. Define

$$U(x) := \text{Tot.Var.}\{u;]-\infty, x]\}.$$

Notice that U is a right continuous, non-decreasing function which satisfies

$$\begin{aligned} U(-\infty) = 0, \qquad U(\infty) = \text{Tot.Var.}\{u\}, \\ |u(y) - u(x)| \leq U(y) - U(x) \quad \text{for all } x < y. \end{aligned} \tag{1.1.3}$$

Given $\epsilon > 0$, let $\bar{N}$ be the larger integer which is smaller than Tot.Var.$\{u\}$ and consider the points

$$x_0 = \infty,\ x_{\bar{N}} = \infty,\ x_j = \min\{x : U(x) \geq j\epsilon\},\ j = q, \cdots, \bar{N} - 1.$$

Defining

$$u^\epsilon = u(x_j) \quad \text{if } x \in [x_j, x_{j+1}[,$$

by (1.1.3) the two estimates in (1.1.2) are both satisfied. □

Lemma 1.1.3. *Suppose $u : R \to \mathbb{R}^N$ is right continuous with bounded total variation and it is continuous at x_0, then we have*

$$\textit{Tot. Var.}\{u;]x_0 - 1/m, x_0 + 1/m[\} \longrightarrow 0,\ \textit{as } m \to \infty.$$

Proof. We denote by I_m the interval $]x_0 - 1/m, x_0 + 1/m[$. It suffices to prove that for each $\epsilon > 0$, there exists a integer $p > 0$ such that for all $m \geq p$,

$$\text{Tot.Var.}\{u;]x_0 - 1/m, x_0 + 1/m[\} < \epsilon.$$

Defining

$$u_m(x) := \begin{cases} u(x_0) & \text{if } x \in I_m, \\ u(x) & \text{if } x \notin I_m, \end{cases}$$

it is easy to see $u_n \to u$ in $\mathbf{L}^1$ as $n \to \infty$. Therefore, one has, by the semicontinuity of total variation,

$$\text{Tot.Var.}\{u\} \leq \liminf_{m\to\infty} \text{Tot.Var.}\{u_m\}.$$

This yields that for each $\epsilon > 0$, there exists $p_1 > 0$ such that for all $m \geq p_1$, one has

$$\text{Tot.Var.}\{u\} \leq \text{Tot.Var.}\{u_m\} + \epsilon/2. \tag{1.1.4}$$

Since u is right continuous as well as u_m, we have

$$\text{Tot.Var.}\{u\} = \text{Tot.Var.}\{u; I_m\} + \text{Tot.Var.}\{u; \mathbb{R} \setminus I_m\},$$
$$\text{Tot.Var.}\{u_m\} = \text{Tot.Var.}\{u_m; I_m\} + \text{Tot.Var.}\{u_m; \mathbb{R} \setminus I_m\}.$$

Since Tot.Var.$\{u; \mathbb{R} \setminus I_m\}$ = Tot.Var.$\{u_m; \mathbb{R} \setminus I_m\}$, the inequality (1.1.4) implies

$$\text{Tot.Var.}\{u; I_m\} \leq \text{Tot.Var.}\{u_m; I_m\} + \epsilon/2. \tag{1.1.5}$$

As u is continuous at x_0, one has there exists the integer $p_2 > 0$ such that for all $m \geq p_2$,

$$\text{Tot.Var.}\{u_m, I_m\} \leq \epsilon/2.$$

Letting $p := \max\{p_1, p_2\}$, we complete the proof. □

Lemma 1.1.4. *If $u : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ is a BV function and is right continuous, then*

$$Du([a, b]) = \textit{Tot. Var.}\{u; [a, b]\}. \tag{1.1.6}$$

Proof. Let
$$v(x) = Du((a, x]) + u(a).$$

Since $Du = Dv$, we have $v = u$ a.e. The right continuity of u yields that $v \equiv u$, that is
$$u(x) = Du((a, x]) + u(a) \quad \text{for } x \in [a, b].$$
This implies the equlity (1.1.6). □

Definition 1.1.5. Let $\hat{\Omega}$ be a open set in $\mathbb{R}^n$, suppose $u \in [L^1(\hat{\Omega})]^N$. The variation $V(u, \hat{\Omega})$ of u is defined by
$$V(u, \hat{\Omega}) := \sup \left\{ \sum_{\alpha=1}^{N} \int_{\hat{\Omega}} u^\alpha \text{div} \phi^\alpha dx : \phi \in [C_c^1(\hat{\Omega})]^{nN}, \|\phi\|_\infty \le 1 \right\}. \tag{1.1.7}$$
We say that u has locally bounded variation if $V(u, \hat{\Omega}) < \infty$, and we write as $u \in [\text{BV}(\hat{\Omega})]^N$.

It is easy to see that for $u : I \subset \mathbb{R} \to \mathbb{R}^N$, Tot.Var.$\{u, I\} < \infty$ implies $V(u, I) < \infty$. On the other hand, for each $u \in [\text{BV}(I)]^N$, one can show that ther exists $\tilde{u}$ in the equivalence class of u such that
$$\text{Tot.Var.}\{\tilde{u}, I\} = V(\tilde{u}, I).$$

In order to get pointwise properties of BV function on $\mathbb{R}^n$ with $n > 1$, we introduce the definition of approximate limit.

Definition 1.1.6 (Approximate limit)**.** Let $u \in [L^1(\hat{\Omega})]^N$, we say that u has an approximate limit at $x \in \hat{\Omega}$ if there exists $z \in \mathbb{R}^N$ such that
$$\lim_{\rho \downarrow 0} ⨍_{B_\rho(x)} |u(y) - z| dy = 0. \tag{1.1.8}$$
The *approximate discontinuity* set S_u consists of points where (1.1.8) does not holds for any $z \in \mathbb{R}$.

At each $x \notin S_u$, the approximate limit of u is uniquely determined by (1.1.8) and denoted by $\tilde{u}(x)$. In the following, we always assume that u is approximate continuous at x if $x \notin S_u$ and $u(x) = \tilde{u}(z)$. By Lebesgue Differentiation Theorem, the the complement of the set of Lebesgue points of u is $\mathcal{L}^n$-negligible, we infer that S_u is $\mathcal{L}^n$-negligible, and $\tilde{u} = u$ $\mathcal{L}^n$-a.e. in $\hat{\Omega} \setminus S_u$.

There are points in S_u which correspond to an approximate jump discontinuity between two different values along a direction. For the definition, we need the convenient notation:
$$\begin{aligned} B_\rho^+(x, \pi) &:= \{y \in B_\rho(x) : \langle y - x, \pi \rangle > 0\}, \\ B_\rho^-(x, \pi) &:= \{y \in B_\rho(x) : \langle y - x, \pi \rangle < 0\}, \end{aligned} \tag{1.1.9}$$
for the two half balls contained in $B_p(x)$ determined by the direction π.

Definition 1.1.7 (Approximate jump points)**.** We say that x is and *approximate jump* point of u if there exist $a, b \in \mathbb{R}^N$ with $a \neq b$ and a unit vector $\pi \in \text{S}^{N-1}$ such that
$$\lim_{\rho \downarrow 0} ⨍_{B_\rho^+(x,\pi)} |u(y) - a| dy = 0, \qquad \lim_{\rho \downarrow 0} ⨍_{B_\rho^-(x,\pi)} |u(y) - b| dy = 0. \tag{1.1.10}$$

The triplet (a, b, π) uniquely determined by (1.1.10) up to a permutation of (a, b) and a change of sing of π. We denote it by $(u^+(x), u^-(x), \pi_u(x))$.

We denote by Du the distributional derivative of u. By Riesz Representation Theorem, (see Section 1.8 of [34]), one knows that $u \in [\mathrm{BV}(\hat{\Omega})]^N$ implies Du is a finite Radon vector measure. Therefore, by Lebesgue Decomposition Theorem, (see Section 1.6 in [34]), measure Du can be decomposed into two mutually singular measures which are the absolutely continuous part $D^a u$ and singular part $D^s u$, with respect to Lebesgue measure.

Furthermore, due to Feder and Vol'pert, $D^s u$ can be written as $D^j u + D^c u$ where

$$D^j u := D^s u \llcorner J_u, \qquad D^c u := D^s u \llcorner (\hat{\Omega} \setminus S_u). \tag{1.1.11}$$

Since Du vanishes on the $\mathcal{H}^{n-1}$-negligible set $S_u \setminus J_u$, (see Lemma 3.7.6 in [3]), one obtains from (1.1.11) that Du can be decomposed into three mutually singular measures:

$$Du = D^a u + D^j u + D^c u.$$

BV functions of one variable has a simple structure for these three measures. More precisely, when $\hat{\Omega} \subset \mathbb{R}$, one has

Proposition 1.1.8. *Let* $u \in [\mathrm{BV}(\hat{\Omega})]^N$. *Then* $S_u = J_u$, $\tilde{u}$ *is continuous on* $\hat{\Omega} \setminus J_u$ *and* $\tilde{u}$ *has classical left and right limits which coincide with* $u^{\pm}(x)$ *at each* $x \in J_u$, *that is*

$$D^j u = \sum_{x \in J_u} (u^+(x) - u^-(x))\delta_x.$$

Definition 1.1.9. we say that $u \in [\mathrm{BV}(\Omega)]^N$ is a *special function with bounded variation*, and we write $u \in [\mathrm{SBV}(\Omega)]^N$, if $D^c u$, the Cantor part of its distributional derivative, is zero.

Suppose $u \in [\mathrm{BV}(\hat{\Omega})]^N$ and $f : \mathbb{R}^N \to \mathbb{R}^p$ is a Lipschitz continuous function. It is not hard to prove that $f(u)$ belongs to $[\mathrm{BV}(\hat{\Omega})]$ and $|Df(u)| \ll |Du|$. Moreover, we have the chain rule for real valued BV functions.

Theorem 1.1.10. *Let* $u \in \mathrm{BV}(\hat{\Omega})$ *and* $f : \mathbb{R} \to \mathbb{R}$ *be a Lipschitz continuous function satisfying* $f(0) = 0$ *if* $|\hat{\Omega}| = 0$. *Then* $f(u)$ *belongs to* $\mathrm{BV}(\hat{\Omega})$ *and*

$$Df(u) = f'(u)\nabla u \mathcal{L}^N + \big(f(u^+) - f(u^-)\big)\,\pi_u \mathcal{H}^{N-1} \llcorner J_u + f'(\tilde{u}) D^c u. \tag{1.1.12}$$

Although there is also a general chain rule formula for the vector valued BV function, here we state a particular theorem in which f is assumed to be smooth, and this is enough for the proofs in the following chapters.

Theorem 1.1.11. *Let* $u \in [\mathrm{BV}(\hat{\Omega})]^N$ *and* $f \in [C^1(\mathbb{R}^N)]^p$ *be a Lipschitz continuous function satisfying* $f(0) = 0$ *if* $|\hat{\Omega}| = 0$. *Then* $f(u)$ *belongs to* $[\mathrm{BV}(\hat{\Omega})]^p$ *and*

$$\begin{cases} \tilde{D} f(u) = f'(u)\nabla u \mathcal{L}^N + f'(\tilde{u}) D^c u, \\ D^j f(u) = (f(u^+) - f(u^-))\,\pi_u \otimes \mathcal{H}^{N-1} \llcorner J_u. \end{cases} \tag{1.1.13}$$

1.2 Coarea formula for BV function

We first introduce a class of sets E whose characteristic function χ_E has bounded variation.

Definition 1.2.1. Let E be an $\mathcal{L}^n$-measurable subset of $\mathbb{R}^n$. For an open set $\hat{\Omega} \subset \mathbb{R}^n$, the perimeter of E in $\hat{\Omega}$ is defined as

$$P(E, \hat{\Omega}) := \sup \left\{ \int_E \mathrm{div}\phi dx : \phi \in [C_c^1(\hat{\Omega})]^n, \quad \|\phi\|_\infty \leq 1 \right\}. \tag{1.2.1}$$

Notice that in the definition, we do not require the set E to be with finite measure which means that the $\mathbf{L}^1$-norm of the characteristic function of set of finite perimeter may not be finite. Therefore, the characteristic function of set of finite perimeter may not be a BV function.

Next, we relate the variation measure of a BV function and the perimeters of its level sets. For $f : \hat{\Omega} \to \mathbb{R}$ and $w \in \mathbb{R}$, define

$$E_w(f) := \{y \in \hat{\Omega} : f(y) \geq w\} \tag{1.2.2}$$

to be the *level set* of E. Then we have the following coarea formula.

Theorem 1.2.2 (Coarea formula in BV)**.** *If $u \in \mathrm{BV}(\hat{\Omega})$, the set $E_w(u)$ has finite perimeter in $\hat{\Omega}$ for $\mathcal{L}^1$-a.e. $w \in \mathbb{R}$ and*

$$|Du|(B) = \int_{-\infty}^{\infty} |D\chi_{E_w(u)}|(B) dw, \qquad Du(B) = \int_{-\infty}^{\infty} D\chi_{E_w(u)}(B) dw, \tag{1.2.3}$$

for each Borel set $B \subset \hat{\Omega}$.

In order to get more precise description of $D\chi_{E_w(u)}$, we need to know the structure of sets of finite perimeter. Let E be a set of locally finite perimeter in $\mathbb{R}^n$. When $n = 1$, the srtucture of sets of finite perimeter is very simple as the following propositon shows.

Proposition 1.2.3. *If E has finite perimeter in (a, b) and $|E \cap (a, b)| > 0$, then there exist an integer $p \geq 1$ and p pairwise disjoint intervals $J_i = [a_{2i-1}, a_{2i}] \subset \mathbb{R}$ such that $E \cap (a, b)$ is equivalent to the union of the J_i and*

$$P(E, (a, b)) = \#\{i \in \{1, \cdots, 2p\} : a_i \in (a, b)\}.$$

The situation is much more complicated for dimension $n > 1$. In fact, there exist open sets of finite perimeter in $\mathbb{R}^n$ whose boundary has strictly positive Lebesuge measure, see Example 3.53 in [3]. This motivates the concept of measure-theoretic boundary for sets of finite perimete.

Definition 1.2.4. Let $y \in \mathbb{R}^n$, we say $y \in \partial^* E$, the *reduced boundary* of E, if

(i) $D\chi_E(B_\rho(y)) > 0, \quad \text{for all } \rho > 0$,

(ii) $\pi_E(y) := \lim_{\rho \downarrow 0} \frac{D\chi_E(B_\rho(y))}{|D\chi_E|(B_\rho(y))}$ exists,

(iii) $|\pi_E(y)| = 1$.

The function $\pi_E : \partial^* E \to S^{n-1}$ is called the *generalized inner normal* to E.

Remark 1.2.5. 1. It holds that (see Section 5.7 in [34])

$$|D\chi_E|(\mathbb{R}^n - \partial^* E) = 0,$$

2. If ∂E ,the topological boundary of E, consists of finite Lipschitz curves, then it coincide with the reduced boundary of E, that is

$$\partial E = \partial^* E$$

and the generalized inner normal to the boundary is equal to the inner normal to the boundary if it exists.

3. $\partial^* E$ is a countably $(n-1)$-rectifiable set and the variation measure $|D\chi_E|$ coincides with $\mathcal{H}^{n-1} \llcorner \partial^* E$, see Section 3.5 in [3].

Then we can rewrite the formulas in (1.2.3) as

$$|Du|(B) = \int_{-\infty}^{\infty} \mathcal{H}^{n-1} \llcorner \partial^* E_w(u)(B) dw \tag{1.2.4a}$$

$$Du(B) = \int_{-\infty}^{\infty} \int_{B \cap \partial^* E_w(u)} \pi_{E_w(u)}(y) d\mathcal{H}^{n-1}(y) dw. \tag{1.2.4b}$$

1.3 The singular conservation law

This section is concerned with the Cauchy problem of a scalar conservation law

$$u_t + f(u)_x = 0 \tag{1.3.1a}$$

$$u(0, x) = u_0(x) \tag{1.3.1b}$$

We assume that $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and $u_0 \in \mathbf{L}^1_{\text{loc}}$ with bounded total viariation. We say a continuous map $u : [0, \infty) \to \mathbf{L}^1_{\text{loc}}(\mathbb{R})$ is an entropy solution of (1.3.1) if it satisfies (1.3.1b) and

$$\iint |u - k|\phi_t + (f(u) - f(k))\text{sgn}(u - k)\phi_x dxdt, \tag{1.3.2}$$

for every $k \in \mathbb{R}$ and every non-negative function $\phi \in C^1_c(\mathbb{R}^2)$ with compact support contained in the half plane $t > 0$.

The existence of entropy solutions to (1.3.1) can be proved by the method of *front tracking approximations.* For a given initial data $u \in \mathbf{L}^1$, we construct in Section 1.3.2 a sequence $\{u^\nu\}_{\nu \geq 1}$ of piecewise constant approximate solutions. As $\nu \to \infty$, a compactness argument yields a subsequence of $\{u^\nu\}_{\nu \geq 1}$ converging to an entropy solution in $\mathbf{L}^1_{\text{loc}}$.

Using the entropy condition (1.3.2), one can show that for any two bounded entropy solutions u, v of (1.3.1),

$$\int_{-\infty}^{\infty} |u(t, x) - v(t, x)| dx \leq \int_{-\infty}^{\infty} |u(0, x) - v(0, x)| dx, \tag{1.3.3}$$

for each $t \geq 0$. This implies the uniqueness and continuous dependence of the entropy solution.

As (1.3.3) shows, the $\mathbf{L}^1$ distance between any two bounded entropy solution does not increasing in time, thus the space of solutions can be extended by continuity to a much larger family of initial condition. In particular, one has the existence of a unique entropy solution to the Cauchy problem (1.3.1) for each initial data $u_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty$

After changing the values of the solution u on a set of measure zero, one get the trajectories of a map

$$(t, u_0) \mapsto S_t u_0 := u(t)$$

defined on $\mathbf{L}^1$ enjoying the properties:

1. *Semigroup property.*

$$S_0 u_0 = u_0, \quad S_t(S_\tau) u_0 = S_{t+\tau} u_0, \quad \forall t, \tau \geq 0, \ \forall u_0 \in \mathbf{L}^1.$$

2. *Lipschitz continuity*

$$\|S_t u_0 - S_t v_0\|_{\mathrm{L}^1} \leq \|u_0 - v_0\|_{\mathrm{L}^1}, \quad \forall t, \tau \geq 0, \ \forall u_0, v_0 \in \mathbf{L}^1.$$

Moreover, for each $u_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty$, the trajectory $(t, u_0) \to S_t u_0$ yields the unique bounded entropy solution of the Cauchy problem (1.3.1).

1.3.1 The Riemann problem

We consider the construction of the entropy solution to (1.3.1) with the initial data of the form

$$u_0(x) = \begin{cases} u^{\mathrm{L}} & \text{if } x > 0, \\ u^{\mathrm{R}} & \text{if } x < 0, \end{cases} \tag{1.3.4}$$

where $u^{\mathrm{L}}, u^{\mathrm{R}} \in \mathbb{R}$. This kind of Cauchy problem is called *Riemann problem.*

If $u^{\mathrm{L}} < u^{\mathrm{R}}$, we define the *convex hull* of f on the interval $[u^{\mathrm{L}}, u^{\mathrm{R}}]$ as

$$\mathrm{conv}_{[u^{\mathrm{L}}, u^{\mathrm{R}}]} f(u) := \sup\{g(u) : g \leq f \text{ on } [u^{\mathrm{L}}, u^{\mathrm{R}}], \ g \text{ is convex}\}.$$

It is a function of class $C^{1,1}$, provided that f is smooth.

Let

$$\lambda(u) := \frac{d}{du} \mathrm{conv}_{[u^{\mathrm{L}}, u^{\mathrm{R}}]} f(u). \tag{1.3.5}$$

It is easy to see that $\lambda(\cdot)$ is a non-decreasing function from $[u^{\mathrm{L}}, u^{\mathrm{R}}]$ to $[\lambda(u^{\mathrm{L}}), \lambda(u^{\mathrm{R}})]$. Thus, its pseudo-inverse function $\lambda \mapsto u(\lambda)$ is a strictly increasing function with a countable number of discontinuity points.

Then, we define a piecewise smooth function

$$u(t,x) = \begin{cases} u^{\mathrm{L}} & \text{if } x/t < \lambda(u^{\mathrm{L}}), \\ u(\lambda) & \text{if } x/t = \lambda \text{ for some } \lambda \in [\lambda(u^{\mathrm{L}}), \lambda(u^{\mathrm{R}})] \\ & \quad \text{which is not a discontinuity point of } u = u(\lambda), \\ u^{\mathrm{R}} & \text{if } x/t > \lambda(u^{\mathrm{R}}), \end{cases} \tag{1.3.6}$$

which is well-defined almost everywhere.

If $u^{\mathrm{L}} > u^{\mathrm{R}}$, all the construction is the same except that we replace $\mathrm{conv}_{[u^{\mathrm{L}}.u^{\mathrm{R}}]}f$ in (1.3.5) with the *concave hull* of f on the interval $[u^{\mathrm{L}}, u^{\mathrm{R}}]$, which is

$$\mathrm{conc}_{[u^{\mathrm{L}},u^{\mathrm{R}}]}f(u) := \inf\{g(u) : g \geq f \text{ on } [u^{\mathrm{L}}, u^{\mathrm{R}}],\ g \text{ is concave}\}.$$

One can show that the function u defined in (1.3.6) is an entropy solution to the Riemann problem with initial data (1.3.4). For the proof, see Section 2.2 of [37].

1.3.2 Front tracking algorithm

In this section, we describe the front tracking algorithm for the Cauchy problem of scalar conservation laws.

Since Tot.Var.$\{u_0\} < \infty$, by Lemma 1.1.2, we can construct a sequence $\{u_0^\nu\}_{\nu\geq 1}$ of piecewise constant functions with finite jump discontinuities, such that

(1) $u_0^\nu(x) \in 2^{-\nu}\mathbb{Z},\ \forall x \in \mathbb{R}$,

(2) $\|u_0^\nu - u_0\|_{L^1} \to 0$,

(3) Tot.Var.$\{u_0^\nu\} \leq$ Tot.Var.$\{u_0\}$,

(4) $\|u_0^\nu\|_\infty \leq \|u_0\|_\infty$.

We approximate f by its piecewise affine interpolation f^ν with grid size of $2^{-\nu}$, i.e.

$$f^\nu(s) = \frac{s - 2^{-\nu}l}{2^{-\nu}} f(2^{-\nu}(l+1)) + \frac{2^{-\nu}(l+1) - s}{2^{-\nu}} f(2^{-\nu}l),$$

for $s \in [2^{-\nu}l, 2^{-\nu}(l+1)]$ with l integer.

For each fixed $\nu \geq 1$, we now try to find a Liu admissible weak solution to the Cauchy problem

$$u_t^\nu + [f^\nu(u^\nu)]_x = 0, \tag{1.3.7a}$$

$$u^\nu(0, x) = u_0^\nu. \tag{1.3.7b}$$

First of all, we consider a Riemann problem for (1.3.7a) with the initial data

$$u^\nu(0,x) = \begin{cases} u^{\mathrm{L}} & \text{if } x < 0, \\ u^{\mathrm{R}} & \text{if } x > 0, \end{cases} \qquad u^{\mathrm{L}}, u^{\mathrm{R}} \in 2^{-\nu}\mathbb{Z}. \tag{1.3.8}$$

Just similar as we discussed in the last section, there are two different ways to construct the solution according to the order between u^{L} and u^{R}.

Case 1. If $u^{\mathrm{L}} < u^{\mathrm{R}}$, let $\breve{f}^\nu = \mathrm{conv}_{[u^{\mathrm{L}},u^{\mathrm{R}}]}f^\nu$. Notice that, the derivative $\frac{d}{du}\breve{f}^\nu$ is a piecewise constant non-decreasing function.

Suppose that $\frac{d}{du}\breve{f}^\nu$ has jumps at points $u^{\mathrm{L}} = w_0 < w_1 < \cdots < w_p = u^{\mathrm{R}}$, we define the increasing sequence of shock speeds as

$$\lambda_l = \frac{f^\nu(w_l) - f^\nu(w_{l-1})}{w_l - w_{l-1}}, \qquad l = 1, \cdots, q. \tag{1.3.9}$$

and a self-similar function

$$w(t,x) = \begin{cases} u^{\mathrm{L}} & \text{if } x < t\lambda_1, \\ w_l & \text{if } t\lambda_l < x < t\lambda_{l+1} \quad (1 \le l \le q-1), \\ u^{\mathrm{L}} & \text{if } x > t\lambda_q. \end{cases} \tag{1.3.10}$$

Case 2. If $u^{\mathrm{L}} > u^{\mathrm{R}}$, let $\hat{f}^\nu = \mathrm{conc}_{[u^{\mathrm{L}},u^{\mathrm{R}}]} f^\nu$. Notice that the derivative $\frac{d}{du}\hat{f}^\nu$ is a piecewise constant non-increasing function.

Suppose that $\frac{d}{du}\hat{f}^\nu$ has jumps at points $u^{\mathrm{R}} = w_0 < w_1 < \cdots < w_p = u^{\mathrm{L}}$, let us define the decreasing sequence of shock speeds as (1.3.9) and a self-similar function

$$w(t,x) = \begin{cases} u^{\mathrm{L}} & \text{if } x < t\lambda_q, \\ w_l & \text{if } t\lambda_{l+1} < x < t\lambda_l \quad (1 \le l \le q-1), \\ u^{\mathrm{L}} & \text{if } x > t\lambda_1. \end{cases} \tag{1.3.11}$$

It can be shown that in both cases, the function w provides an entropy solution of Riemann problem (1.3.7a)-(1.3.8), (see [19] for details).

Now consider the Cauchy problem (1.3.7) for a fixed $\nu \ge 1$. Let $x_1 < \cdots < x_p$ be the points where u_0 has jumps. At each x_i, solving the Riemann problem

$$\begin{cases} (\tilde{u}_i)_t + (f(\tilde{u}_i))_x = 0, \\ \tilde{u}_i(0,x) = \begin{cases} u_0^\nu(x_i-) & \text{if } x < 0, \\ u_0^\nu(x_i+) & \text{if } x > 0, \end{cases} \end{cases} \tag{1.3.12}$$

by the same construction of the entropy solution for (1.3.7a)-(1.3.7b), then $u^\nu(t,x) = \tilde{u}_i(t, x - x_i)$ in a small neighborhood of $(0, x_i)$. The solution can be prolonged up to a first a time $t_1 > 0$ when two or more wave fronts crossing each other, say the wave-fronts emerge form $(0, x_j)$ with the left value u^{L} collide from left with the wave-front from $(0, x_k)$ with the right value u^{R}, at the point (t_1, x_{jk}). Since the value of $u^\nu(t,\cdot)$ always remain with in the set $2^\nu\mathbb{Z}$, we can again solve the new Riemann problem at (t_1, x_{jk}) with initial data of the jump $[u^{\mathrm{L}}, u^{\mathrm{R}}]$. Then the solution can be prolonged up to a time $t_2 > t_1$ where there are wave-front collisions again. Such procedure can be prolonged if the total number of interaction remain finite.

1.3.3 Uniform boundedness estimates on the speed of wave fronts

In this section, we introduce a uniform estimate, obtained by Bianchini and Modena in [13], on the total variation of the speed of fronts of the approximate solutions u^ν which are constructed in the last section.

Recall that for any time $t > 0$ and $\nu > 0$, $u^\nu(t,\cdot)$ is a piecewise constant function which takes value in $2^{-\nu}\mathbb{Z}$. We also assume that $u^\nu(t,\cdot)$ is right continuous except on finite many times of interaction. Letting $J^\nu := [0, \mathrm{Tot.Var.}\{u^\nu\}]$, we need to define three functions:

$$X^\nu : \mathbb{R}^+ \times J^\nu \to \mathbb{R}, \tag{1.3.13a}$$

$$\sigma^\nu : \mathbb{R}^+ \times J^\nu \to [-1,1], \tag{1.3.13b}$$

$$a^\nu : \mathbb{R}^+ \times J^\nu \to \{-1,0,1\}, \tag{1.3.13c}$$

where X^ν describe a path staring from initial time $t = 0$ along fronts of u^ν with $\frac{\partial}{\partial t} X^\nu(t,s) = \sigma^\nu(t,s)$ out of finite interaction and cancellation times, such that, by the notion of push forward,

$$[D_x u^\nu(t)](B) = X^\nu(t,\cdot)_\# \left[a^\nu(t,x)\mathcal{L}^1 \llcorner J\right](B) \qquad \text{for any Borel set } B \subset \mathbb{R}, \tag{1.3.14}$$

that is, for all $\phi \in C_c^1(\mathbb{R})$ and $t \in \mathbb{R}^+$, it holds

$$\int_{\mathbb{R}} u^\nu(t,x) D_x\phi(x)dx = \int_J \phi(X^\nu(t,s))a^\nu(t,s)ds. \tag{1.3.15}$$

Now, we define the wave curve function X^ν by induction. For a fixed $s \in J^\nu$, let

$$X^\nu(0,s) = \bar{x}^\nu(s) := \min\{x \in \mathbb{R} : s \le U^\nu(x)\}, \tag{1.3.16}$$

where $U^\nu(x) := \text{Tot.Var.}\{u_0^\nu;]-\infty, x]\}$.

We define the sign of a wave curve of $s \in J^\nu$ as

$$\mathcal{S}^\nu(s) := \text{sign}\left[u^\nu(\bar{x}^\nu(s)) - u^\nu(\bar{x}^\nu(s)-)\right], \tag{1.3.17}$$

and the right value function

$$w^\nu(s) := u_0^\nu(\bar{x}^\nu(s)-) + \mathcal{S}^\nu(s)\left[s - U^\nu(\bar{x}^\nu(s)-)\right].$$

Then letting $u_1 := u^\nu(0, X_0^\nu(s)-)$ and $u_2 := u^\nu(0, X_0^\nu(s))$, we define

$$\sigma_0 := \begin{cases} \left(\frac{d}{du}\text{conv}_{[u_1,u_2]} f^\nu\right)(w(s)) & \text{if } \mathcal{S}^\nu(s) = +1, \\ \left(\frac{d}{du}\text{conc}_{[u_2,u_1]} f^\nu\right)(w(s)) & \text{if } \mathcal{S}^\nu(s) = -1. \end{cases} \tag{1.3.18}$$

and the wave curve

$$X^\nu(t,s) = X^\nu(0,s) + \sigma_0 t, \tag{1.3.19}$$

until the wave collides with another front at time t_1. And we define the speed function σ^ν on the time interval $[0, t_1[$ as $\sigma^\nu(t,s) = \sigma_0$.

If it is a cancellation and for $t > t_1 + \epsilon$ with sufficiently small $\epsilon > 0$

$$w^\nu(s) < u^\nu(t, X^\nu(s)+) \qquad \text{if } \mathcal{S}^\nu(s) > 0,$$

or

$$w^\nu(s) > u^\nu(t, X^\nu(s)+) \qquad \text{if } \mathcal{S}^\nu(s) < 0,$$

which means that the wave s is cancelled by another front. Then we define

$$T^\nu(s) = t_1 \tag{1.3.20}$$

as the lifespan of the wave curve $X^\nu(s)$.

Otherwise, we can define

$$X^\nu(t,s) = X^\nu(t_1,s) + \sigma_1 t, \tag{1.3.21}$$

where σ_1 is defined as in (1.3.18) by replacing u_1, u_2 with the new $u_1 := u^\nu(t_1, X^\nu(t_1,s)-)$ and $u_2 := u^\nu(t_1, X^\nu(t_1,s))$, up to a time when an interaction or cancellation happens. And we define σ^ν on the time interval $[t_1, t_2[$ as $\sigma^\nu(t,s) = \sigma_1$.

We can continue this procedure until the wave curve X^ν is cancelled.

It is easy to see that for each $s_1 < s_2$ and $t \in [0, T^\nu(s_1) \wedge T^\nu(s_2)]$, it holds the monotonicity

$$X^\nu(t, s_1) \leq X^\nu(t, s_2). \tag{1.3.22}$$

Therefore, we can extend X^ν for all $t \in \mathbb{R}^+$ and preserve its monotonicity.

Next, we define the weight function

$$a^\nu(t,s) = \begin{cases} \mathcal{S}^\nu(s) & \text{if } t \leq T^\nu(s), \\ 0 & \text{if } t > T^\nu(s). \end{cases} \tag{1.3.23}$$

Then one can check that (1.3.14) holds for all $t \in \mathbb{R}^+$. Furthermore, one has the following estimate

$$\int_0^\infty \text{Tot.Var.}\{\sigma^\nu(t,s), [0, T^\nu(s)]\} ds \leq \text{Tot.Var.}\{u_0^\nu\}. \tag{1.3.24}$$

Since one has

$$\text{Tot.Var.}\{u_0^\nu\} \leq \text{Tot.Var.}\{u_0\}, \tag{1.3.25}$$

one get the uniform bound for the left of (1.3.24) for all $\nu \geq 1$.

1.4 The Cauchy problem for systems

In this section is concerned with global existence of solutions to the Cauchy problem

$$u_t + f(u)_x = 0, \qquad u : \mathbb{R}^+ \times \mathbb{R} \to \hat{\Omega} \subset \mathbb{R}^N, \tag{1.4.1}$$

with the initial data $u_{|t=0} = u_0 \in \text{BV}(\mathbb{R}, \hat{\Omega})$, under the assumption that $f \in C^2(\hat{\Omega}, \mathbb{R})$ and the $N \times N$ system is strict hyperbolic in $\hat{\Omega}$, that is, the eigenvalues $\{\lambda_i(u)\}_{i=1}^N$ of the Jacobi matrix $A(u) = Df(u)$ satisfy

$$\lambda_1(u) < \cdots < \lambda_N(u), \qquad u \in \Omega.$$

Furthermore, as we only consider the solutions with small total variation and thus they live in a neighborhood of a point, it is not restrictive to assume that $\hat{\Omega}$ is bounded and there exist constants $\{\check{\lambda}_i\}_{i=0}^N$, such that

$$\check{\lambda}_{i-1} < \lambda_i(u) < \check{\lambda}_i, \qquad \forall u \in \Omega,\ i = 1, \dots, N. \tag{1.4.2}$$

Let $\{r_i(u)\}_{i=1}^N$ and $\{l_j(u)\}_{j=1}^N$ be a basis of right and left eigenvectors, depending smoothly on u, such that

$$l_j(u) \cdot r_i(u) = \delta_{ij} \text{ and } |r_i(u)| \equiv 1, \quad i, j = 1, \dots, N.$$

Given an initial condition u_0 with sufficiently small total variation, we construct a weak, Liu admissible solution u, defined for all $t \geq 0$, by front tracking method developed by Ancona and Marson in [5].

1.4.1 Solution of Riemann problem

First we describe the construction of the solution to the Riemann problem for general hyperbolic systems of conservation laws, which is the Cauchy problem (1.4.1) with piecewise constant initial data of the form

$$u_0 = \begin{cases} u^{\mathrm{L}} & x < 0, \\ u^{\mathrm{R}} & x > 0. \end{cases} \tag{1.4.3}$$

The solution to this problem is the key ingredient for building the front-tracking approximate solution: the basic step is the construction of the admissible *elementary curve* of the k-th family for any given left state u^{L}.

This construction is taken from [6] and the procedure is divided in three steps:

1. find Riemann problems which can be solved using only waves of the i-th family,
2. give the explicit solution of these elementary Riemann problems,
3. show how to piece together these functions in order to obtain the solutions to general Riemann problems.

The first two points actually goes together.

The starting point is that for a fixed point $u^- \in \Omega$ and $i \in \{1, \cdots, N\}$, there are smooth vector valued maps $\tilde{r}_i = \tilde{r}_i(u, v_i, \sigma_i)$ for $(u, v_i, \sigma_i) \in \Omega \times \mathbb{R} \times \mathbb{R}$, v_i and σ_i sufficiently small, with $\tilde{r}_i(u, 0, \sigma) = r_i(u)$ for all u, σ_i. These functions describe the center manifold of traveling profiles. Setting $l_i^0 := l_i(u^0)$, with $u^0 \in \Omega$ fixed, we can normalize $\tilde{r}_i$ such that

$$l_j^0 \cdot \tilde{r}_i(u, v_i, \sigma_i) = 1. \tag{1.4.4}$$

Define the speed function

$$\tilde{\lambda}_i(u, v_i, \sigma_i) := l_i^0 \cdot Df(u)\tilde{r}_i(u, v_i, \sigma_i).$$

For some constants $\delta_0, C_0 > 0$ fixed and $s > 0$, consider the subset of $\mathrm{Lip}([0, s], \mathbb{R}^{N+2})$ given by

$$\begin{aligned} \Gamma_i(s, u^-) := \Big\{ \gamma : \gamma(\xi) = (u(\xi), v_i(\xi), \sigma_i(\xi)) \\ u(0) = u^-, |u(\xi) - u^-| = \xi, v_i(0) = 0, \\ |v_i(\xi)| \leq \delta_1, |\sigma_i(\xi) - \lambda_i(u^0)| \leq 2C_0\delta_1 \leq 1 \Big\}, \end{aligned} \tag{1.4.5}$$

and given a curve $\gamma \in \Gamma_i(s, u^-)$, define the scalar flux function

$$\tilde{f}_i(\tau; \gamma) = \int_0^\tau \tilde{\lambda}_i(u(\xi), v_i(\xi), \sigma_i(\xi)) d\xi. \tag{1.4.6}$$

Recall that the lower convex envelope of $\tilde{f}_i$ in the interval $[a, b] \subset [0, s]$ is given by

$$\begin{aligned} \mathrm{conv}_{[a,b]} \tilde{f}_i(\tau; \gamma) := \inf \Big\{ \theta f_i(\tau'; \gamma) + (1 - \theta) f_i(\tau''; \gamma) : \\ \theta \in [0, 1], \tau', \tau'' \in [a, b], \tau = \theta\tau' + (1 - \theta)\tau'' \Big\}. \end{aligned}$$

Finally define the nonlinear operator $\mathcal{T}_{i,s} : \Gamma_i(s,u^-) \to \mathrm{Lip}([0,s],\mathbb{R}^{N+2})$, $\mathcal{T}_{i,s}(\gamma) = \check{\gamma} = (\check{u},\check{v}_i,\check{\sigma}_i)$, by

$$\begin{cases} \check{u}(\tau) &=& u^- + \int_0^\tau \tilde{r}_i(u(\xi),v_i(\xi),\sigma(\xi))d\xi, \\ \check{v}_i(\tau) &=& \tilde{f}_i(\tau;\gamma) - \mathrm{conv}_{[0,s]}\tilde{f}_i(\tau;\gamma), \\ \check{\sigma}_i(\tau) &=& \frac{d}{d\tau}\mathrm{conv}_{[0,s]}\tilde{f}_i(\tau;\gamma). \end{cases} \tag{1.4.7}$$

In [6] it is shown that, for $0 \le s \le \bar{s} \ll 1$, $\delta_1 \ll 1$ and $C_0 \gg 1$, $\mathcal{T}_{i,s}$ maps $\Gamma_i(s,u^-)$ into itself, and it is a contraction in $\Gamma_i(s,u^-)$ with respect to the distance

$$D(\gamma,\gamma') := \delta_1||u-u'||_{L^\infty} + ||v_i - v_i'||_{L^1} + ||v_i\sigma_i - v_i'\sigma_i'||_{L^1}, \tag{1.4.8}$$

where δ_1 is given in the definition of $\Gamma_i(s,u^-)$, formula (1.4.5), and

$$\gamma = (u,v_i,\sigma_i), \gamma' = (u',v_i',\sigma_i') \in \Gamma_i(s,u^-).$$

Hence, given u^- and $0 \le s \le \bar{s}$, let us denote the fixed point of $\mathcal{T}_{i,s}$ by

$$\bar{\gamma}(\tau;s,u^-) = \big(\bar{u}(\tau;s,u^-),\bar{v}_i(\tau;s,u^-),\bar{\sigma}_i(\tau;s,u^-)\big), \quad \tau \in [0,s].$$

We will give a short sketch of the proof later on.

For $s > 0$ the elementary curve $T_i[u^-] : [0,\bar{s}] \to \mathbb{R}^N$ for i-th family is defined by

$$u^{\mathrm{R}} = T_i[u^-](s) := \bar{u}(s;u^-,s). \tag{1.4.9}$$

This is the set of end points of solutions to (1.4.7).

For the case when $s < 0$, a right state $u^{\mathrm{R}} = T_i[u^{\mathrm{L}}](s)$ can be constructed in the same way as before, except that one replaces $\mathrm{conv}_{[0,s]}\tilde{f}_i(\tau;\gamma)$ in (1.4.7) with the upper concave envelope of $\tilde{f}_i(\tau;\gamma)$ on $[s,0]$, that is

$$\mathrm{conc}_{[a,b]}\tilde{f}_i(\tau;\gamma) := \sup\Big\{\theta f_i(\tau';\gamma) + (1-\theta)f_i(\tau'';\gamma) : \\ \theta \in [0,1], \tau',\tau'' \in [a,b], \tau = \theta\tau' + (1-\theta)\tau''\Big\},$$

and looks at the fixed point of of the integral system (1.4.7) on the interval $[s,0]$, and the elementary curve $T_i[u^-]$ for $s < 0$ is defined accordingly to (1.4.9).

Because of the assumption (1.4.4) and the definition (1.4.9), the elementary curve $T_i[u^{\mathrm{L}}]$ is parameterized by its i-th component relative to the basis $r_1(u^0), \cdots, r_N(u^0)$ i.e.

$$s = l_i^0 \cdot \big(T_i[u^{\mathrm{L}}](s) - u^{\mathrm{L}}\big). \tag{1.4.10}$$

We will also use the notation

$$\sigma_i[u^-](s,\tau) := \bar{\sigma}_i(\tau;u^-,s), \quad \tilde{f}_i[u^-](s,\tau) := \tilde{f}_i(\tau;\bar{\gamma}). \tag{1.4.11}$$

One has thus the following theorem [10].

Theorem 1.4.1. *For every $u \in \Omega$ and $|s| \le \bar{s}$ sufficiently small, there are*

1. *N Lipschitz continuous curves $s \mapsto T_i[u](s) \in \Omega$, $i = 1,\ldots,N$, satisfying $\lim_{s\to 0}\frac{d}{ds}T_i[u](s) = r_i(u)$,*

2. *N Lipschitz continuous functions $(s,\tau)\mapsto\sigma_i[u](s,\tau)$, with $0\le|\tau|\le|s|$, $\operatorname{sgn}\tau=\operatorname{sgn}s$, and $i=1,\dots,N$, satisfying $\tau\mapsto\sigma_i[u](s,\tau)$ increasing,*

with the following properties.
When $u^{\mathrm{L}}\in\Omega$, $u^{\mathrm{R}}=T_i[u^{\mathrm{L}}](s)$, for some s sufficiently small, the admissible solution of the Riemann problem (1.4.1)-(1.4.3) *is given by*

$$u(x,t):=\begin{cases}u^{\mathrm{L}} & x/t\le\sigma_i[u^{\mathrm{L}}](s,0),\\ T_i[u^{\mathrm{L}}](\tau) & x/t=\sigma_i[u^{\mathrm{L}}](s,\tau),|\tau|\in[0,|s|],\operatorname{sgn}\tau=\operatorname{sgn}s,\\ u^{\mathrm{R}} & x/t>\sigma_i[u^{\mathrm{L}}](s,s).\end{cases}\tag{1.4.12}$$

We give a short sketch of the proof of the construction of the curves $T_i[u](s)$ for readers convenience.

Proof. It is clear that

$$u^-+\int_0^\tau\tilde r_i(u(\xi),v_i(\xi),\sigma(\xi))d\xi$$

is $C^{1,1}$ if γ is Lipschitz, as well as $\tilde f_i(\tau;\gamma)$ given by (1.4.6). Moreover, for $|s|\le\bar s\ll 1$, one obtains $|v_i|\le\delta_1$ and $|\sigma_i-\lambda_i(u^-)|\le 2\|f''\|_{L^\infty}\delta_1$ for some constant δ_1, by using the trivial estimates:

$$|\breve v_i(\xi)|\lesssim s\cdot\sup_{\varsigma\in[0,s]}\left|\tilde\lambda_i(u(\varsigma),v_i(\varsigma),\sigma_i(\varsigma))-\lambda_i(u^-)\right|\lesssim s^2,\tag{1.4.13}$$

$$|\breve\sigma(\xi)|\lesssim\sup_{\varsigma\in[0,s]}\left|\tilde\lambda_i(u(\varsigma),v_i(\varsigma),\sigma_i(\varsigma))-\lambda_i(u^-)\right|\lesssim s.\tag{1.4.14}$$

Therefore, the operator $\mathcal{T}_{i,s}$ maps $\Gamma_i(s,u^-)$ into itself.

If $\gamma_1,\gamma_2\in\Gamma_i(s,u^-)$ are two curves, then

$$\big|\tilde r_i(\gamma_1)-\tilde r_i(\gamma_2)\big|,\big|\tilde\lambda_i(\gamma_1)-\tilde\lambda_i(\gamma_2)\big|\le\mathcal{O}(1)\big(|u_1-u_2|+|v_1-v_2|+|v_1\sigma_1-v_2\sigma_2|\big),\tag{1.4.15}$$

where we have used $\tilde r_i(u,0,\sigma)=r_i(u)$. The above estimates imply that

$$\begin{aligned}&\delta_1\left|\int_0^\tau\tilde r_i(\gamma_1)d\xi-\int_0^\tau\tilde r_i(\gamma_2)d\xi\right|\\ &\le\mathcal{O}(1)\delta_1\Big(\bar s\|u_1-u_2\|_{L^\infty}+\|v_1-v_2\|_{L^1}+\|v_1\sigma_1-v_2\sigma_2\|_{L^1}\Big)\\ &\le\frac12 D(\gamma_1,\gamma_2),\end{aligned}$$

for $\bar s\le\delta_1\ll 1$.

By using the elementary estimates

$$\Big\|f-\mathrm{conv}f-(g-\mathrm{conv}g)\Big\|_{L^\infty}\le\frac12\left\|\frac{df}{dx}-\frac{dg}{dx}\right\|_{L^1},$$

$$\left\|\frac{d}{d\tau}\mathrm{conv}f-\frac{d}{d\tau}\mathrm{conv}g\right\|_{L^1}\le\left\|\frac{df}{dx}-\frac{dg}{dx}\right\|_{L^1},$$

we obtain also from (1.4.15)

$$\begin{aligned}
&\Big\|\tilde f(\gamma_1)-\mathrm{conv}\tilde f(\gamma_1)-\big(\tilde f(\gamma_2)-\mathrm{conv}\tilde f(\gamma_2)\big)\Big\|_{L^1}\\
\le&\bar s\Big\|\tilde f(\gamma_1)-\mathrm{conv}\tilde f(\gamma_1)-\big(\tilde f(\gamma_2)-\mathrm{conv}\tilde f(\gamma_2)\big)\Big\|_{L^\infty}\\
\le&\frac{\bar s}{2}\Big\|\frac{d}{dx}\tilde f(\gamma_1)-\frac{d}{dx}\tilde f(\gamma_2)\Big\|_{L^1}\\
\le&\mathcal{O}(1)\frac{\bar s}{2}\big(\|u_1-u_2\|_{L^\infty}+\|v_1-v_2\|_{L^1}+\|v_1\sigma_1-v_2\sigma_2\|_{L^1}\big)\\
\le&\frac{1}{2}D(\gamma_1,\gamma_2),
\end{aligned}$$

and similarly, since $|v_i|\le\delta_1$,

$$\Big\|v_i(\gamma_1)\frac{d}{d\tau}\mathrm{conv}\tilde f(\gamma_1)-v_i(\gamma_2)\frac{d}{d\tau}\mathrm{conv}\tilde f(\gamma_2)\Big\|_{L^1}\le\mathcal{O}(1)\delta_1D(\gamma_1,\gamma_2)\le\frac12D(\gamma_1,\gamma_2).$$

These estimates show that the map $\mathscr{T}_{i,s}$ is a contraction in $\Gamma_i(s,u^-)$, and thus the fixed point $\bar\gamma$ is well defined.

In order to prove that the curve $T_i[u^-](s)$ is Lipschitz, if $s'>s$ and $\bar\gamma'$ is the fixed point to (1.4.7) on the interval $[0,s']$, one just estimate the distance of $\mathcal{T}_{i,s}(\bar\gamma')$ from $\bar\gamma'$. The only components which can vary are v_i and σ_i, simply because we are restricting the convex envelope to the interval $[0,s]\subsetneq[0,s']$. Again by the estimates

$$\begin{aligned}
&\big\|\mathrm{conv}_{[0,s]}f-\mathrm{conv}_{[0,s']}f\big\|_{L^\infty(0,s)},\Big\|\frac{d}{d\tau}\mathrm{conv}_{[0,s]}f-\frac{d}{d\tau}\mathrm{conv}_{[0,s']}f\Big\|_{L^\infty(0,s)}\\
\le&\mathcal{O}(1)\|f''\|_{L^\infty}s'(s'-s),
\end{aligned}\tag{1.4.16}$$

one concludes that

$$D(\bar\gamma,\bar\gamma')\le 2D\big(\mathcal{T}_{i,s}(\bar\gamma'),\bar\gamma'\big)\le\mathcal{O}(1)s'(s'-s),\tag{1.4.17}$$

where we used the contraction factor $1/2$. Being $\bar s\le\delta_1$, the above estimate yields the Lipschitz regularity as well as the existence of the derivative at $s=0$, which can be also shown to be equal to $r_i(u)$ from the definition of $\mathcal{T}_{i,s}(u^-)$.

The final observation is that if γ' is a fixed point to $\mathcal{T}_{i,s'}(u^-)$ such that for some $s\in]0,s'[$ it holds $v_i(\gamma',s)=0$, then $\gamma'\llcorner[0,s]$ is a fixed point to $\mathcal{T}_{i,s}(u^-)$, as well as $\gamma'\llcorner[s,s']$ is a fixed point to $\mathcal{T}_{i,s'-s}[u(\gamma',s)]$: this is consequence of the fact that if $v_i(\gamma',s)=0$, then

$$\mathrm{conv}_{[0,s]}\tilde f_i(\gamma')=\mathrm{conv}_{[0,s']}\tilde f_i(\gamma')\llcorner[0,s].$$

Thus shows that the fixed point given by (1.4.12) can be rewritten as

$$u(x,t):=\begin{cases}u(\gamma,0) & x/t\le\sigma_i(\gamma,0),\\ u(\gamma,\tau) & x/t=\sigma_i(\gamma,\tau),|\tau|\in[0,|s|],\mathrm{sgn}\,\tau=\mathrm{sgn}\,s,\\ u(\gamma,s) & x/t>\sigma_i(\gamma,s),\end{cases}$$

where $\gamma=(u(\gamma),v_i(\gamma),\sigma_i(\gamma))$: this latter formulation is the limit solution to the Riemann problem constructed by vanishing viscosity. □

Finally, using the curves $T_i[u]$ and the solutions to the Riemann problem $[u^{\mathrm{L}}, T_i[u^{\mathrm{L}}](s)]$, one can construct the solution to a general Riemann problem. The idea is that, since the characteristic speeds are well separated, one can piece together the solution to elementary Riemann problems made only of i-th waves.

More precisely, the admissible solution [10] of a Riemann problem for (1.4.1)-(1.4.3), where now u^{R} satisfies only $|u^{\mathrm{L}} - u^{\mathrm{R}}| \ll 1$, is obtained by considering the Lipschitz continuous map

$$\mathbf{s} := (s_1, \dots, s_N) \mapsto T[u^L](\mathbf{s}) := T_N\big[T_{N-1}\big[\cdots\big[T_1[u^{\mathrm{L}}](s_1)\big]\cdots\big](s_{N-1})\big](s_N) = u^{\mathrm{R}}, \tag{1.4.18}$$

which, due to Point (1) of Theorem 1.4.1, is one to one from a neighborhood of the origin onto a neighborhood of u^{L}. Then we can uniquely determine intermediate states $u^{\mathrm{L}} = \omega_0,\ \omega_1,\ \dots,\ \omega_N = u^{\mathrm{R}}$, and the *wave strength* $s_1,\ s_2,\ \dots,\ s_N$ such that

$$\omega_i = T_i[\omega_{i-1}](s_i), \quad i = 1, \dots, N,$$

provided that $|u^{\mathrm{L}} - u^{\mathrm{R}}|$ is sufficiently small.

By Theorem 1.4.1, each Riemann problem with initial data

$$u_0 = \begin{cases} \omega_{i-1} & x < 0, \\ \omega_i & x > 0, \end{cases} \tag{1.4.19}$$

admits a self-similar solution u_i, containing only i-waves. We call u_i the i-th *elementary composite wave* or simply *i-wave*. From the strict hyperbolicity assumption (1.4.2), the speed of each elementary i-th wave in the solution u_i is inside the interval $[\check{\lambda}_{i-1}, \check{\lambda}_i]$ if $s \ll 1$, so we can construct the function

$$u(x,t) = \begin{cases} u^{\mathrm{L}} & x/t \leq \check{\lambda}_0, \\ u_i(x,t) & \check{\lambda}_{i-1} < x/t \leq \check{\lambda}_i, i = 1, \dots, N, \\ u^{\mathrm{R}} & x/t > \check{\lambda}_N. \end{cases} \tag{1.4.20}$$

This function yields the admissible solution to the Riemann problem: it is clearly obtained by piecing together the self-similar solutions of the Riemann problems given by (1.4.1)-(1.4.19).

We end this section with a functional equivalent to the total variation of the solution: assuming for simplicity that $u : \mathbb{R} \to \mathbb{R}$ is piecewise constant with small L^∞-norm with jumps at x_α (as for wavefront approximate solutions), then

$$V(u) := \sum_\alpha \sum_i |s_{i,\alpha}|, \tag{1.4.21}$$

where $s_{i,\alpha}$ are the components of $\mathbf{s}_\alpha$ given by (1.4.18) for the Riemann problem in x_α. It is clear that $V(u)$ is equivalent to Tot.Var.(u), because $T[u^{\mathrm{L}}](\mathbf{s})$ is Lipschitz and invertible.

1.4.2 Construction of solution by wave-front tracking approximation

Now we describe the construction of the wavefront tracking algorithm for general systems of conservation laws, following the approach of [5]. Since the construction is now standard, we only give a short overview about existence, compactness and convergence of the approximation, pointing to the properties needed in our argument: more precisely, we will only consider how one constructs the approximate wave pattern of the k-th genuinely nonlinear family (Section 1.4.2).

In order to construct approximate wave-front tracking solutions, given a fixed $\epsilon > 0$, we first choose a piecewise constant function $u_{0,\epsilon}$ which is a good approximation to initial data u_0 such that

$$\text{Tot.Var.}(u_{0,\epsilon}) \leq \text{Tot.Var.}(u_0), \quad ||u_{0,\epsilon} - u_0||_{L^1} < \epsilon, \tag{1.4.22}$$

and $u_{0,\epsilon}$ only has finitely many jumps. Let $x_1 < \cdots < x_m$ be the jump points of $u_{0,\epsilon}$.

For each $\alpha = 1, \ldots, m$, we approximately solve the Riemann problem with the initial data of the jump $[u_{0,\epsilon}(x_\alpha -), u_{0,\epsilon}(x_\alpha +)]$ by a function $w(x,t) = \phi(\frac{x-x_0}{t})$ where ϕ is a piecewise constant function which will be defined below.

The straight lines where the discontinuities are located are called *wave-fronts* (or just *fronts* for shortness). The wave-fronts travels with constant speed until they meet other wavefronts at a so-called interaction point, and then the corresponding new Riemann problem is approximately solved with a piecewise constant self similar solution. The procedure can be continued up to $t = +\infty$ if the choice of the approximate Riemann solutions produce only finitely many interactions in any compact set of times: for this aim, 3 types of approximate Riemann solutions are considered.

The approximate i-th elementary wave

The key step is to give a procedure to replace the solution to the elementary Riemann problem (1.4.19) with a piecewise constant self-similar function.

Suppose that $u_i(x/t)$ is an i-th elementary composite wave which is obtained by solving Riemann problem with initial data (1.4.19) where $\omega_i = T_i[\omega_{i-1}](s_i)$. For notational convenience, in this section we will write $\sigma_i(\tau) := \sigma_i[\omega_{i-1}](s_i, \tau)$, and for definiteness we consider $s_i > 0$, the other case being completely similar. Let

$$p := \left[\frac{\sigma_i(s_i) - \sigma_i(0)}{\epsilon}\right] + 1,$$

where $[\cdot]$ denotes the integer part, and let

$$\vartheta_{i,\ell} := \sigma_i(0) + \frac{\ell}{p}\left[\sigma_i(s_i) - \sigma_i(0)\right], \qquad \ell = 0, \cdots, p-1.$$

Since $\tau \mapsto \sigma_i(\tau)$ is increasing and continuous, we define the points

$$\tau_{i,\ell} := \min\left\{\tau \in [0, s_i], \sigma_i(s) = \vartheta_\ell\right\},$$

and set

$$\omega_{i-1,\ell} = T_i[\omega_{i-1}](\tau_{i,\ell}). \tag{1.4.23}$$

The i-th elementary composite wave $u_i(x/t)$ will be thus approximated by the function $\tilde{u}_i(x/t)$ given by

$$\tilde{u}_i(x,t) = \begin{cases} \omega_{i-1} & x/t < \vartheta_{i,0}, \\ \omega_{i-1,\ell} & \vartheta_{i-1,\ell-1} < x/t < \vartheta_{i,\ell}, \ \ell = 1, \cdots, p-1, \\ \omega_i & x/t > \vartheta_{i,p-1}. \end{cases} \tag{1.4.24}$$

Notice that $\tilde{u}_i$ consists of p fronts, hence it is piecewise constant. We moreover observe that since at each point $\tau_{i,\ell}$ it holds $\bar{v}_i(\tau_{i,\ell}) = 0$, then one has

$$T_i[\omega_{i-1}](\tau_{i,\ell}) = U(\tau_{i,\ell}; \omega_{i-1}, s_i),$$

where U is the solution of (1.4.7). This shows that an equivalent interpretation of (1.4.23) is $\omega_{i-1,\ell} = U(\tau_{i,\ell}; \omega_{i-1}, s_i)$.

Using the approximate i-th elementary wave we can construct the approximate Riemann solvers.

Approximate Riemann solvers

We present now three types of approximate Riemann solvers, and later we will specify the rule describing in which situation each one is used.

Accurate Riemann solver in this case, one just replaces each i-th elementary composite wave of the exact Riemann solution with the approximate i-th elementary wave defined by (1.4.24) with discretization parameter ϵ: hence the fronts are separated if and only if their difference in speed is $\geq \epsilon$.

Simplified Riemann solver assume that at the interaction point the wave $[u^{\mathrm{L}}, u^M]$ with strength s of the i-th family coming from the left interacts with the wave $[u^M, u^{\mathrm{R}}]$ with strength s' of the i'-th family coming from the right, with $i \leq i'$. The simplified Riemann solver is given by piecing together the elementary approximate solution (1.4.24) to the two Riemann problem

$$[u^L, T_i[u^L](s)] \quad \text{and} \quad [T_i[u^L](s), T_{i'}[T_i[u^L](s)](s')] \quad \text{if } i < i',$$

$$[u^L, T_i[u^L](s+s')] \quad \text{if } i = i',$$

where now the discretization parameter is 2ϵ: hence the fronts are separated if and only if their difference in speed is $\geq 2\epsilon$.

In order to match U^R, one also fix a parameter $\hat{\lambda} > \sup_\Omega \lambda_N(u)$ and consider a non-physical front traveling with speed $\hat{\lambda}$ and of size

$$[T_{i'}[T_i[u^L](s)](s'), u^R] \text{ if } i < i', \quad \text{or} \quad [T_i[u^L](s+s'), u^R] \text{ if } i = i'.$$

Crude Riemann solver this describes the interactions with the nonphysical fronts introduced by the approximate Riemann solver and the i-waves. If $[u^L, u^M]$ is a nonphysical front coming from the left which interacts with an i-th front of strength s coming from the right, then the approximate solution consists of two wave fronts: a single jump $[u^L, T_i[u^L](s)]$ with speed computed by $\tilde{f}_i(s)/s$ and the remaining part of the discontinuity travels as a nonphysical front.

It is customary to think that the nonphysical front corresponds to the $(N+1)$-th characteristic field.

It is not restrictive to assume that at each time $t > 0$ at most one interaction occurs involving only two incoming fronts: in fact, it is enough to change the speed of the front by an arbitrarily small quantity. Since the algorithm provides solutions with uniformly bounded total variation, by letting this error go to 0 the solution still converges to the admissible solution. In this way we can use the Riemann solvers defined above to construct the solution.

Remark 1.4.2. We can divide the wavefronts in an approximate solution into 3 types:

Discontinuity front these are fronts which are also admissible discontinuities;

Rarefaction front these correspond to piecewise constant approximations of rarefactions;

Mixed front these are discontinuities composed of admissible shocks and rarefaction fronts.

The last case, in which the shock is not admissible, can occur because of the definition of the approximate i-th elementary curve, as easily seen even in the scalar case.

Interaction potential and BV estimates

In this section we estimate the growth of total variation due to the nonlinear interaction of waves. We will introduce two quantities, namely the amount of interaction $\mathcal{I}$ and the Glimm interaction potential $\mathcal{Q}$.

Suppose that two wavefronts ς', ς'' interact at $(\bar{t}, \bar{x})$. For definiteness, let ς' be a wavefront of the i'-th family with strength s', and let ς'' be a wavefront of the i''-th family with strength s'', and assume that ς' is located at the left of ς'', so that $i'' \leq i'$. Without loss of generality, we can also assume that $s' > 0$. Denote with $\tilde{f}'_{i'}$, $\tilde{f}''_{i''}$ the corresponding scalar flux functions defined by (1.4.6).

The *amount of interaction* $\mathcal{I}(s', s'')$ between s' and s'' is defined as follows.

If ς', ς'' belong to different characteristic families $i' > i''$, then we define

$$\mathcal{I}(s', s'') := |s's''|. \tag{1.4.25}$$

In the case $i' = i''$, then we have 3 cases to consider, depending on the sign and size of s''. if $g' : [0, a] \to \mathbb{R}$, $g'' : [b, c] \to \mathbb{R}$ are two functions, then define

$$(g' \cup g'')(x) := \begin{cases} g'(x) & x \in [0, a], \\ g''(x - a + b) + g'(a) - g''(b) & x \in]a, a + c - b]. \end{cases}$$

(a) If $s'' > 0$, we set

$$\mathcal{I}(s', s'') := \int_0^{s'+s''} \Big| \big(\mathrm{conv}_{[0,s']} \tilde{f}'_i \cup \mathrm{conv}_{[0,s'']} \tilde{f}''_i\big)(\xi) - \mathrm{conv}_{[0,s'+s'']} (\tilde{f}'_i \cup \tilde{f}''_i)(\xi) \Big| d\xi.$$

(b) If $-s' \leq s'' < 0$, we set

$$\mathcal{I}(s', s'') := \int_0^{s'} \Big| \mathrm{conv}_{[0,s']} \tilde{f}'_i(\xi) - \big(\mathrm{conv}_{[0,s'+s'']} \tilde{f}'_i \cup \mathrm{conc}_{[s'',0]} \tilde{f}''_i\big)(\xi) \Big| d\xi.$$

(c) If $s'' < -s' < 0$, we set

$$\mathcal{I}(s',s'') := \int_{s''}^{0} \Big| \mathrm{conc}_{[s'',0]} \tilde{f}_i''(\xi) - \big(\mathrm{conc}_{[s'',-s']} \tilde{f}_i'' \cup \mathrm{conv}_{[0,s']} \tilde{f}'\big)(\xi) \Big| d\xi.$$

The form of the above amount of interaction $\mathcal{I}(s',s'')$ relies on the analysis of the scalar case, where in that case $\mathcal{I}(s',s'')$ is the area between the curves representing the solutions to the Riemann problems, see [6].

The key estimate proved in [6] (also see Lemma 1 in [5]) is that the quantity $\mathcal{I}(s',s'')$ controls how the wave pattern changes before and after the interaction: if $\mathbf{s}$ is given by solving the Riemann problem at the interaction as in (1.4.18), then

$$\sum_{i=1}^{N+1} |s_i - s_i' - s_i''| \lesssim \mathcal{I}(s',s''), \tag{1.4.26}$$

where $(s_i', s_i'') = (\delta_{i,i'} s', \delta_{i,i''} s'')$. [1]

In particular the functional $V(t)$ given by (1.4.21) increases at most of $\mathcal{O}(1)\mathcal{I}(s',s'')$,

$$V(\bar{t}) - V(\bar{t}-) \lesssim \mathcal{I}(s',s''). \tag{1.4.27}$$

Observe that we also consider the nonphysical waves in the above estimate (1.4.26) as an additional $N+1$-th wave family.

Remark 1.4.3. Note that the form of of the amount of interaction given here is slightly different that the one given in [6], but it is fairly easy to prove that the two forms are equivalent.

In order to bound the increase of the functional $V(t)$, a second functional $\mathcal{Q}$, the *Glimm interaction potential*, is defined as follows: if in $u(t)$ the wavefronts are located at x_α with strength s_α, then

$$\mathcal{Q}(t) := \sum_{\substack{j>i \\ x_\alpha < x_\beta}} \big| s_{j,\alpha} s_{i,\beta} \big| + \frac{1}{4} \sum_{i_\alpha = i_\beta < N+1} \int_0^{|s_\alpha|} \int_0^{|s_\beta|} \big| \sigma_{i_\beta}[\omega_\beta](s_\beta, \tau'') - \sigma_{i_\alpha}[\omega_\alpha](s_\alpha, \tau') \big| d\tau' d\tau''.$$

The last term do not contains the $N+1$-th family because the speed is the constant $\hat{\lambda}$ fixed.

If $\bar{t}$ is the time of interaction of s', s'', then one can prove that (Lemma 5 in [5])

$$\mathcal{Q}(\bar{t}) - \mathcal{Q}(\bar{t}-) \lesssim \mathcal{I}(s',s''). \tag{1.4.28}$$

The above estimate together with (1.4.27) allows to define the *Glimm functional*

$$\Upsilon(t) := V(t) + C_0 \mathcal{Q}(t), \tag{1.4.29}$$

with C_0 suitable constant, so that $\Upsilon(t)$ is monotone decreasing in t.

[1] $\delta_{i,j}$ is Kronecker delta.

Construction of wavefront approximate solutions

The Glimm functional is used to show that one can choose the Riemann solvers defined in Section 1.4.2 in order to have

1. a finite number of interactions points,
2. a finite number of waves,
3. a uniform bound of the total variation of the solution,
4. the total variation of the nonphysical waves converging to 0,
5. an error on the conservation equation converging to 0 weakly in measure.

Hence the limit function will be a solution to (3.0.1) with uniform bounded total variation, and a standard Riemann semigroup comparison technique yields the uniqueness of the limit. We will now sketch the procedure.

The construction starts at initial time $t = 0$ with a given $\epsilon > 0$, by taking u_0^ϵ as a suitable piecewise constant approximation of initial data u_0 satisfying (1.4.22).

At the jump points of u_0^ϵ, we locally solve the Riemann problem by accurate Riemann solver. The approximate solution can be prolonged until a first time t_1 when two wavefronts s, s'' interact. Depending on the amount of interaction at this interaction point, one chooses the appropriate approximate Riemann solver and compute the solution until the next interaction points occurs.

The rule for choosing which Riemann solvers one uses is the following. Fix a parameter $\rho = \rho(\epsilon) > 0$. If s', s'' are physical waves, then one uses the accurate Riemann solver if $\mathcal{I}(s', s'') \geq \rho$, otherwise one applies the simplified Riemann solver. Finally, when one of the waves is nonphysical, then the crude Riemann solver is used.

In [5] it is proved that if $\rho = \rho(\epsilon)$ is chosen sufficiently small, then the construction yields an approximate wavefront solution u_ϵ satisfying the properties (1)-(5) listed above.

For definiteness, for any t we consider $x \mapsto u(t, x)$ right continuous.

Let $\{\epsilon_\nu\}_{\nu=1}^{\infty}$ be a sequence of positive real numbers converging to zero. Consider a corresponding sequence of ϵ_ν-approximate front tracking solutions $u^\nu := u^{\epsilon_\nu}$ of (1.4.1): it is standard to show that the functions $t \mapsto u^\nu(t, \cdot)$ are uniformly Lipschitz continuous in L^1 norm, and the decay of the Glimm Functional yields that the solutions $u^\nu(t, \cdot)$ have uniformly bounded total variation. Then by Helly's theorem, u^ν converges up to a subsequence in $\mathbb{L}^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$ to some function u, which is a weak solution of (1.4.1).

It can be shown that by the choice of the Riemann Solver in Theorem 1.4.1, the solution obtained by the front tracking approximation coincides with the unique vanishing viscosity solution [10]. Furthermore, there exists a closed domain $\mathcal{D} \subset L^1(\mathbb{R}, \Omega)$ and a unique distributional solution u, which is a Lipschitz semigroup $\mathcal{D} \times [0, +\infty[\to \mathcal{D}$ and which for piecewise constant initial data coincides, for a small time, with the solution of the Cauchy problem obtained piecing together the standard entropy solutions of the Riemann problems. Moreover, it lives in the space of BV functions.

For simplicity, the pointwise value of u is its L^1 representative such that the restriction map $t \mapsto u(t)$ is continuous form the right in L^1 and $x \mapsto u(t, x)$ is right continuous from the right.

In the following we will only consider the approximation $u^\nu := u^{\epsilon_\nu}$.

Further estimates

To conclude this section, we consider some natural quantities related to the approximate solution u^ν.

We define the *measure of interaction* μ^{I}_ν and the *measure of interaction and cancellation* μ^{IC}_ν as purely atomic measures concentrated on the interaction points: if $P = (\bar t, \bar x)$ is an interaction point, then the value of $\mu^{\mathrm{I}}_\epsilon(P)$, $\mu^{IC}_\nu(P)$ are given by

$$\mu^{\mathrm{I}}_\nu(\{P\}) := \mathcal{I}(s', s''), \tag{1.4.30a}$$

$$\mu^{IC}_\nu(\{P\}) := \mathcal{I}(s', s'') + \begin{cases} |s'| + |s''| - |s' + s''| & i' = i'', \\ 0 & i' \neq i''. \end{cases} \tag{1.4.30b}$$

Using these measures and the wave strength estimates (1.4.26), one can write balance of waves for approximate solutions, showing that the wave measures s_i of u satisfy a balance equation with source μ^{I}_ν, μ^{IC}_ν. In fact, in each region Γ transversal to the wavefronts, setting

$$W^i_{\nu,\mathrm{in}}(\Gamma) := \sum_{\text{entering }\Gamma} s_i, \quad W^i_{\nu,\mathrm{out}}(\Gamma) := \sum_{\text{exiting }\Gamma} s_i,$$

$$W^{i,\pm}_{\nu,\mathrm{in}}(\Gamma) := \sum_{\text{entering }\Gamma} s^\pm_i, \quad W^{i,\pm}_{\nu,\mathrm{out}}(\Gamma) := \sum_{\text{exiting }\Gamma} s^\pm_i, \qquad s^\pm_i = \max\{\pm s_i, 0\},$$

then one has from the interaction estimates (1.4.26)-(1.4.28) that

$$|W^i_{\nu,\mathrm{out}} - W^i_{\nu,\mathrm{in}}|(\Gamma) \lesssim \mu^{\mathrm{I}}_\nu(\Gamma), \quad |W^{i\pm}_{\nu,\mathrm{out}} - W^{i\pm}_{\nu,\mathrm{in}}|(\Gamma) \lesssim \mu^{IC}_\nu(\Gamma). \tag{1.4.31}$$

We observe that the uniform boundedness of Tot.Var.$(u(t))$ w.r.t. time t and parameter ν together with the Glimm interaction estimates imply that μ^{IC}_ν, μ^{IC}_ν are bounded measures for all ν. Hence, up to subsequences $\nu \to \infty$, there exist bounded measures μ^{I} and μ^{IC} on $\mathbb{R}^+ \times \mathbb{R}$ such that the following weak convergence holds:

$$\mu^{\mathrm{I}}_\nu \rightharpoonup \mu^{\mathrm{I}}, \quad \mu^{\mathrm{IC}}_\nu \rightharpoonup \mu^{\mathrm{IC}}.$$

The key problem in passing to the limit of the balances (1.4.31) is that the map (1.4.18) in nonlinear, so that a stronger convergence of the derivatives of u should be proved. This will be a corollary of our regularity estimates.

Chapter 2

SBV-like regularity for strictly hyperbolic systems of conservation laws

2.1 Overview of the chapter

In this chapter, we study the SBV-like regularity for the vanishing viscosity solutions to the following systems of conservation laws

$$u_t + f(u)_x = 0, \quad u : \mathbb{R}^+ \times \mathbb{R} \to \Omega \subset \mathbb{R}^N, \tag{2.1.1}$$

with initial data

$$u(t = 0) = u_0 \in \mathrm{BV}(\mathbb{R}, \Omega), \tag{2.1.2}$$

under the assumption that

(1) Tot.Var.$\{u_0\}$ is sufficiently small;

(2) The system (2.1.1) is strictly hyperbolic, that is, the Jacobi matrix Df has distinct real eigenvalues in the domain.

Clearly, by just considering a linearly degenerate eigenvalue, it is fairly easy to see that the solution u itself cannot be in SBV if the initial data u_0 is not a SBV function. So instead, we consider SBV regularity of some function related to the solution u, say, the characteristic speed function $f'(u)$ for the scalar case and the i-th -component of $D_x\lambda_i$ for the system case.

We state the main theorems of this chapter: in the following a BV function on $\mathbb{R}$ will be considered defined everywhere by taking the right continuous limit.

In the scalar case, one has

Theorem 2.1.1. *Suppose that $u \in \mathrm{BV}(\mathbb{R}^+ \times \mathbb{R})$ is an entropy solution of the scalar conservation law (2.2.1). Then there exists a countable set $S \subset \mathbb{R}^+$ such that for every $t \in \mathbb{R}^+ \setminus S$ the following holds:*

$$f'(u(t, \cdot)) \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R}).$$

After introducing the definition of i-th component of $D_x\lambda_i(u)$ (see (2.3.11)), we have the SBV-like regularity for the system case.

Theorem 2.1.2. *Let u be a vanishing viscosity solution of the Cauchy problem for the strictly hyperbolic system* (2.3.1) *with small BV norm. Then there exists an at most countable set $S \subset \mathbb{R}^+$ such that i-th component of $D_x\lambda_i(u(t,\cdot))$ has no Cantor part for every $t \in \mathbb{R}^+ \setminus S$ and $i \in \{1,\ 2,\ \ldots,\ N\}$.*

Since in the genuinely nonlinear case $u \mapsto \lambda_i(u)$ is invertible along the i-th admissible curves $T^i_s[u]$ (see Theorem 2.3.2 for the definition), it follows that Theorem 2.4.1 is an extension of the results contained in [11] (and Theorem 2.1.1 is an extension of the results contained in [44] when the source is 0). The example contained in Remark 2.7.2 shows that the results are sharp.

The main point of the proof is the fact that the wave-front tracking approximation for the waves of a genuinely nonlinear family does not essentially differ from the wave-front approximations of genuinely nonlinear systems: in other words, the wave pattern of a genuinely nonlinear characteristic family for a (approximate) solution in a general hyperbolic system has the same structure as if all characteristic families are genuinely nonlinear. Thus the analysis carried out in [11] holds also in this case.

The chapter is organized as follows.

To introduce the argument in the easiest setting, in Section 2.2, we give a proof for the SBV regularity of the characteristic speed for the general scalar conservation laws. The proof is just a slight modification of the proof of Theorem 1.1 in [44].

As one sees in the proof of Theorem 2.1.1, the main tool is to obtain the SBV regularity when only one characteristic field is genuinely nonlinear (Corollary 2.4.2). By inspection, the analysis of [11] relies on the wave-front tracking approximation of [19], which assumes that all characteristic fields are genuinely nonlinear or linearly degenerate. Thus we devote Sections 2.3.2, Section 2.5.1 to introduce the wave-front tracking approximation for general systems [5].

The focus of Section 2.5.2 is the observation that the convergence and regularity estimates of Theorem 10.4 of [19] still holds for the i-th component of u_x, under the only assumption that the i-th characteristic field is genuinely nonlinear: these estimates are needed in order to define the i-th (ϵ_1, ϵ_0)-shocks and to pass to the limit the estimates concerning the interaction, cancellation and jump measures. The latter is responsible for the functional controlling the SBV regularity, Theorem 2.4.1.

After these estimates, for completeness we repeat the proof of the decay of negative waves in Section 2.6.2. Finally we show how to adapt the strategy of the scalar case in Section 2.7.

2.2 The scalar case

In this section, we restrict our attention to the scalar conservation laws and motivate our general strategy with this comparatively simpler situation. Let us consider the entropy solution to the hyperbolic conservation law in one space dimension

$$\begin{cases} u_t + f(u)_x = 0 & u : \mathbb{R}^+ \times \mathbb{R} \to \Omega \subset \mathbb{R},\ f \in C^2(\Omega, \mathbb{R}), \\ u_{|t=0} = u_0 & u_0 \in \mathrm{BV}(\mathbb{R}, \Omega). \end{cases} \tag{2.2.1}$$

In [44], it is proved the SBV regularity result for the convex or concave flux case.

Lemma 2.2.1. *[44] Suppose $f \in C^2(\mathbb{R})$ and $|f''(u)| > 0$. Let $u \in L^\infty(\mathbb{R})$ be an entropy solution of the scalar conservation law* (2.2.1)*. Then there exists a countable set $S \subset \mathbb{R}$ such that for every $\tau \in \mathbb{R}^+ \setminus S$ the following holds:*

$$u(\tau, \cdot) \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R}).$$

Further, by Volpert's Chain Rule (Theorem 3.99 of [3]), it follows that $f'(u(\tau,\cdot)) \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R})$ for $\tau \in \mathbb{R}^+ \setminus S$: actually, since $f'' \neq 0$, the two conditions $f'(u(\tau,\cdot)) \in \mathrm{SBV}_{\mathrm{loc}}$ and $u(\tau,\cdot) \in \mathrm{SBV}_{\mathrm{loc}}$ are equivalent.

Following the same argument together with the analysis in [44], we can get a SBV regularity of the slope of characteristics for the scalar conservation law with general flux as stated in Theorem 2.1.1.

Proof of Theorem 2.1.1. Recall that if $u \in \mathrm{BV}(\mathbb{R}^+ \times \mathbb{R})$ is an entropy solution, then by the theory of entropy solutions, it follows that $u(\tau, \cdot) \in \mathrm{BV}(\mathbb{R})$ is well defined for every $\tau \in \mathbb{R}^+$.

Define the sets

$$\begin{aligned} J_\tau &:= \big\{x \in \mathbb{R} :\ u(\tau, x-) \neq u(\tau, x+)\big\}, \\ F_\tau &:= \big\{x \in \mathbb{R} :\ f''(u(\tau, x)) = 0\big\}, \\ C &:= \big\{(\tau, \xi) \in \mathbb{R}^+ \times \mathbb{R} :\ \xi \in J_\tau \cup F_\tau\big\}. \end{aligned}$$

Set also $C_\tau := J_\tau \cup F_\tau$ as the τ-section of C.

Since the Cantor part $D^{\mathrm{c}}u(\tau,\cdot)$ of $Du(\tau,\cdot)$ and the jump part $D^{\mathrm{j}}u(\tau,\cdot)$ of $Du(\tau,\cdot)$ are mutually singular, then $|D^{\mathrm{c}}u(\tau,\cdot)|(J_\tau) = 0$. Using the fact that $f''(u(\tau,\cdot)) = 0$ on F_τ, by Volpert's Chain Rule one obtains

$$\begin{aligned} |D^{\mathrm{c}} f(u(\tau,\cdot))|(C_\tau) &\leq |D^{\mathrm{c}} f(u(\tau,\cdot))|(J_\tau) + |D^{\mathrm{c}} f(u(\tau,\cdot))|(F_\tau) \\ &= |f''(u(\tau,\cdot)) D^{\mathrm{c}} u(\tau,\cdot)|(J_\tau) + |f''(u(\tau,\cdot)) D^{\mathrm{c}} u(\tau,\cdot)|(F_\tau) = 0. \end{aligned}$$

Let $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R} \setminus C$. Using the finite speed of propagation and the maximum principle for entropy solutions and the fact that $u(t_0, x)$ is continuous at x_0 by the definition of C, it is possible to find a triangle of the form

$$T(t_0, x_0) := \Big\{(t, x) :\ |x - x_0| < b_0 - \bar\lambda(t - t_0),\ 0 < t - t_0 < b_0/\bar\lambda\Big\} \tag{2.2.2}$$

such that $|f''(u(t, x))| \geq c_0 > 0$, for any $(t, x) \in T(t_0, x_0)$. Here c_0 depends on (t_0, x_0) and $\bar\lambda$ is the maximal speed of propagation, which depends only on the L^∞-bound of u_{t_0} (and hence only depends on the L^∞-bound of u by maximal principle).

In particular, in $T(t_0, x_0)$ the solution u of (2.2.1) coincides with the solution of the following problem

$$\begin{cases} w_t + f(w)_x = 0, \\ w(t_0, x) = \begin{cases} u(t_0, x) & |x - x_0| < b_0, \\ \frac{1}{2b_0} \int_{x_0-b_0}^{x_0+b_0} u(t_0, y)dy & |x - x_0| \geq b_0. \end{cases} \end{cases}$$

By Lemma 2.2.1, $w(t, \cdot)$ is SBV regular for any $t > t_0$ out of a countable set of times $S(t_0, x_0)$. Write $T_\tau(t_0, x_0) := T(t_0, x_0) \cap \{t = \tau\}$, thus $u(\tau, \cdot)\llcorner_{T_\tau(t_0,x_0)}$ and $f'(u(\tau, \cdot))\llcorner_{T_\tau(t_0,x_0)}$ are SBV for $\tau \in]t_0,\ t_0 + b/\bar{\lambda}[\setminus S(t_0, x_0)$.

Let B be the set of all points of $\mathbb{R}^+ \times \mathbb{R} \setminus C$ which are contained in at least one of these triangles. (Notice that $T(t_0, x_0)$ is a open set and does not contain the point (t_0, x_0).) Let $\{T(t_i, x_i)\}_{i\in\mathbb{N}}$ be a countable subfamily of the triangles covering B. From the previous observation on the function $u\llcorner_{T(t_i,x_i)}$, the set

$$S_i := \big\{\tau : u(\tau, \cdot)\llcorner_{T_\tau(t_i,x_i)} \notin \mathrm{SBV}(T_\tau(t_i, x_i))\big\}$$

is at most countable.

Let $C' := \mathbb{R}^+ \times \mathbb{R} \setminus (B \cup C)$ and $S_{C'} := \{\tau \in \mathbb{R}^+ : \{t = \tau\} \cap C' \neq \emptyset\}$. It is obvious that for every $t' \in \mathbb{R}^+ \setminus S_{C'}$, $x' \in \mathbb{R}$, either there is a triangle $T \in \{T(t_i, x_i)\}_{i\in\mathbb{N}}$ such that $(t', x') \in T$ and $u(t, \cdot)\llcorner_T$ is SBV function out of countable many times or $(t', x') \in C$.

We claim that the set $S_{C'}$ is at most countable. Indeed, it is enough to prove that the set $S_K := \{\tau \in \mathbb{R}^+ : \{t = \tau\} \cap C' \cap K \neq \emptyset\}$ is at most countable for every compact set $K \subset \mathbb{R}^+ \times \mathbb{R}$ when the triangles $T(t', x')$ have a base of fixed length for every $(t', x') \in C'$: it is fairly simple to see that in this case the set S_K is finite since (t', x') can not be contained in any other $T(t'', x'')$ for $t' \neq t''$ and $(t'', x'') \in C'$.

For any τ not in the countable set

$$S_{C'} \cup \bigcup_{i\in\mathbb{N}} S_i,$$

one obtains the following inequality:

$$|D^c f'(u(\tau, \cdot))(\mathbb{R})| \leq |D^c f'(u(\tau, \cdot))|\Big(\bigcup_{i\in\mathbb{N}} T_\tau(t_i, x_i)\Big) + |D^c f'(u(\tau, \cdot))|(C_\tau) = 0. \qquad (2.2.3)$$

This concludes the proof. □

By a standard argument in the theory of BV functions, we have the following result.

Corollary 2.2.2. *Let $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ be an entropy solution of the scalar conservation law* (2.2.1)*. Then $f'(u) \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$.*

The difference is that now the function $f'(u)$ is considered as a function of two variable.

Proof. The starting point is that up to a countable set of times, $Df'(u(t, \cdot))$ has no Cantor part (Theorem 2.1.1). From the slicing theory of BV function (Theorem 3.107-108 of [3]), we know that the Cantor part of the 2-dimensional measure $D_x f'(u)$ is the integral with respect of t of the Cantor part of $Df'(u(t,\cdot))$. This concludes that $D_x f'(u)$ has no Cantor part, i.e. $D_x^c f'(u) = 0$.

By combining Volpert's Chain Rule and the conservation law (2.2.1), one has

$$D_t^c u = -f'(u) D_x^c u.$$

Using Volpert's rule once again, one obtains

$$D_t^c f'(u) = -f''(u) D_t^c u = -f''(u) f'(u) D_x^c u = -f'(u) D_x^c f'(u) = 0,$$

which concludes that also $D_t f(u)$ has no Cantor part. □

Remark 2.2.3. In [44], it is proved that if f in (2.2.1) has only countable many inflection points. i.e. the set

$$\{u \in \Omega : f''(u) \neq 0\}$$

is at most countable, then the entropy solution of (2.2.1) is SBV regular. It is easy to see that for general hyperbolic scalar conservation laws $f \in C^2$ is not enough to obtain the SBV regularity. In fact, we can consider $f' \equiv$ constant, which means (2.2.1) degenerates into a linear equation. Then the entropy solution u is not SBV regular unless the initial data u_0 is a SBV function.

2.3 Notations and settings for general systems

Throughout the rest of the chapter, the symbol $\mathcal{O}(1)$ always denotes a quantity uniformly bounded by a constant depending only on the system (2.1.1).

2.3.1 Preliminary notation

Consider the Cauchy problem

$$\begin{cases} u_t + f(u)_x = 0 & u : \mathbb{R}^+ \times \mathbb{R} \to \Omega \subset \mathbb{R}^N, \ f \in C^2(\Omega, \mathbb{R}), \\ u_{|t=0} = u_0 & u_0 \in \mathrm{BV}(\mathbb{R}, \Omega). \end{cases} \tag{2.3.1}$$

The only assumption is strict hyperbolicity in Ω: the eigenvalues $\{\lambda_i(u)\}_{i=1}^N$ of the Jacobi matrix $A(u) = Df(u)$ satisfy

$$\lambda_1(u) < \cdots < \lambda_N(u), \qquad u \in \Omega.$$

Furthermore, as we only consider the solutions with small total variation, it is not restrictive to assume that Ω is bounded and there exist constants $\{\check{\lambda}_j\}_{j=0}^N$, such that

$$\check{\lambda}_{k-1} < \lambda_k(u) < \check{\lambda}_k, \qquad \forall u \in \Omega, \ k = 1, \dots, N. \tag{2.3.2}$$

Let $\{r_i(u)\}_{i=1}^N$ and $\{l_j(u)\}_{j=1}^N$ be a basis of right and left eigenvectors, depending smoothly on u, such that

$$l_j(u) \cdot r_i(u) = \delta_{ij} \text{ and } |r_i(u)| \equiv 1, \quad i = 1, \dots, N. \tag{2.3.3}$$

Definition 2.3.1. For $i = 1, \dots, N$, we say that the i-th characteristic field (or i-th family) is *genuinely nonlinear* if

$$\nabla\lambda_i(u) \cdot r_i(u) \neq 0 \quad \text{for all } u \in \Omega,$$

and we say that the i-th characteristic field (or i-th family) is *linearly degenerate* if instead

$$\nabla\lambda_i(u) \cdot r_i(u) = 0 \quad \text{for all } u \in \Omega.$$

In the following, if the i-th characteristic field is genuinely nonlinear, instead of (2.3.3) we normalize $r_i(u)$ such that

$$\nabla\lambda_i(u) \cdot r_i(u) \equiv 1. \tag{2.3.4}$$

2.3.2 Construction of solutions to the Riemann problem

Recall that in Sectionch1:rp, we describe the Riemann problem which is the Cauchy problem (2.3.1) with piecewise constant initial data of the form

$$u_0 = \begin{cases} u^{\mathrm{L}} & x < 0, \\ u^{\mathrm{R}} & x > 0. \end{cases} \tag{2.3.5}$$

The solution to this problem is the key ingredient for building the front-tracking approximate solution: the basic step is the construction of the admissible *elementary curve* of the k-th family for any given left state u^{L}.

A working definition of admissible elementary curves can be given by means of the following theorem.

Theorem 2.3.2. *[6, 10] For every $u \in \Omega$ there exist*

1. *N Lipschitz continuous curves $s \mapsto T^k_s[u] \in \Omega$, $k = 1, \dots, N$, satisfying $\lim_{s\to 0} \frac{d}{ds} T^k_s[u] = r_k(u)$,*
2. *N Lipschitz functions $(s, \tau) \mapsto \sigma^k_s[u](\tau)$, with $0 \le \tau \le s$, $k = 1, \dots, N$, satisfying $\tau \mapsto \sigma^k_s[u](\tau)$ increasing and $\sigma^k_0[u](0) = \lambda_k(u)$*

with the following properties.
When $u^{\mathrm{L}} \in \Omega$, $u^{\mathrm{R}} = T^k_s[u^{\mathrm{L}}]$, for some s sufficiently small, the unique vanishing viscosity solution of the Riemann problem (2.3.1)-(2.3.5) *is defined a.e. by*

$$u(t,x) := \begin{cases} u^{\mathrm{L}} & x/t < \sigma^k_s[u^{\mathrm{L}}](0), \\ T^k_\tau[u^{\mathrm{L}}] & x/t = \sigma^k_s[u^{\mathrm{L}}](\tau), \tau \in \mathcal{I}, \\ u^{\mathrm{R}} & x/t > \sigma^k_s[u^{\mathrm{L}}](s). \end{cases}$$

where $\mathcal{I} := \{\tau \in [0, s] : \sigma^k_s[u^{\mathrm{L}}](\tau) \neq \sigma^k_s[u^{\mathrm{L}}](\tau') \text{ for all } \tau' \neq \tau\}$.

Remark 2.3.3. If i-th family is genuinely nonlinear, then the Lipschitz curve $T^i_s[\bar{u}]$ can be written as

$$T^i_s[\bar{u}] = \begin{cases} R_i[\bar{u}](s) & s \ge 0, \\ S_i[\bar{u}](s) & s < 0, \end{cases}$$

where $R_i[\bar{u}]$, $S_i[\bar{u}]$ are respectively the rarefaction curve and the Rankine-Hugoniot curve of the i-th family with any given point $\bar{u}$ in Ω. Some certain elementary weak solution, called rarefaction waves and shock waves can be defined along the rarefaction curve and Rankine-Hugoniot curve, for example see [19]. The elementary curve $T^i_s[\bar{u}]$ is parametrized by

$$s = l_i(\bar{u}) \cdot (T^i_s[\bar{u}] - \bar{u}). \tag{2.3.6}$$

The vanishing viscosity solution [10] of a Riemann problem for (2.3.1) is obtained by constructing a Lipschitz continuous map

$$(s_1, \dots, s_N) \mapsto T^N_{s_N}\big[T^{N-1}_{s_{N-1}}\big[\cdots\big[T^1_{s_1}[u^{\mathrm{L}}]\big]\big]\big] = u^{\mathrm{R}},$$

which is one to one from a neighborhood of the origin onto a neighborhood of u^{L}. Then we can uniquely determine intermediate states $u^{\mathrm{L}} = \omega_0, \omega_1, \dots, \omega_N = u^{\mathrm{R}}$, and the *wave sizes* $s_1, s_2, \dots, s_N$ such that

$$\omega_k = T^k_{s_k}[\omega_{k-1}], \quad k = 1, \dots, N,$$

provided that $|u^{\mathrm{L}} - u^{\mathrm{R}}|$ is sufficiently small.

By Theorem 2.3.2, each Riemann problem with initial datum

$$u_0 = \begin{cases} \omega_{k-1} & x < 0, \\ \omega_k & x > 0, \end{cases} \tag{2.3.7}$$

admits a vanishing viscosity solution u_k, containing a sequence of rarefactions, shocks and discontinuities of the k-th family: we call u_k the k-th *elementary composite wave*. Therefore, under the strict hyperbolicity assumption, the general solution of the Riemann problem with the initial data (2.3.5) is obtained by piecing together the vanishing viscosity solutions of the elementary Riemann problems given by (2.3.1)-(2.3.7).

Indeed, from the uniform hyperbolicity assumption (2.3.2), the speed of each elementary k-th wave in the solution u_k is inside the interval $[\check{\lambda}_{k-1}, \check{\lambda}_k]$ if $s \ll 1$, so that the solution of the general Riemann problem (2.3.1)-(2.3.5) is then given by

$$u(t,x) = \begin{cases} u^{\mathrm{L}} & x/t < \check{\lambda}_0, \\ u_k(t,x) & \check{\lambda}_{k-1} < x/t < \check{\lambda}_k, k = 1, \dots, N, \\ u^{\mathrm{R}} & x/t > \check{\lambda}_N. \end{cases} \tag{2.3.8}$$

Remark 2.3.4. If the characteristic fields are either genuinely nonlinear or linearly degenerate, the admissible solution of Riemann problem (2.3.1)-(2.3.5) consists of N family of waves. Each family contains either only one shock, one rarefaction wave or one contact discontinuity. However, the general solution of a Riemann problem provided above may contain a countable number of rarefaction waves, shock waves and contact discontinuities.

2.3.3 Cantor part of the derivative of characteristic for i-th waves

Recalling the solution (2.3.8) to the Riemann problem (2.3.1)-(2.3.5), we denote $\tilde{\lambda}_i(u^{\mathrm{L}}, u^{\mathrm{R}})$ as the i-th eigenvalue of the average matrix

$$A(u^{\mathrm{L}}, u^{\mathrm{R}}) = \int_0^1 A(\theta u^{\mathrm{L}} + (1-\theta)u^{\mathrm{R}})d\theta, \tag{2.3.9}$$

and $\tilde{l}_i(u^{\mathrm{L}}, u^{\mathrm{R}})$, $\tilde{r}_i(u^{\mathrm{L}}, u^{\mathrm{R}})$ are the corresponding left and right eigenvector satisfying $\tilde{l}_i \cdot \tilde{r}_i = \delta_{ij}$ and $|\tilde{r}_j| \equiv 1$, for every $i, j \in \{1, \dots, N\}$. Define thus

$$\tilde{\lambda}_i(t,x) = \tilde{\lambda}_i(u(t,x-), u(t,x+)), \tag{2.3.10a}$$

$$\tilde{r}_i(t,x) = \tilde{r}_i(u(t,x-), u(t,x+)), \tag{2.3.10b}$$

$$\tilde{l}_i(t,x) = \tilde{l}_i(u(t,x-), u(t,x+)). \tag{2.3.10c}$$

Since the $\tilde{r}_i$, $\tilde{l}_i$ have directions close to r_i, l_i, one can decompose $D_x u$ into the sum of N measures:

$$D_x u = \sum_{k=1}^{N} v_k \tilde{r}_k,$$

where $v_i = \tilde{l}_i \cdot D_x u$ is a scalar valued measure which we call as *i-th wave measure* [19].

In the same way we can decompose the a.c. part $D_x^{\mathrm{ac}} u$, the Cantor part $D_x^{\mathrm{c}} u$ and the jump part $D_x^{\mathrm{jump}} u$ of $D_x u$ as

$$D_x^{\mathrm{ac}} u = \sum_{k=1}^{N} v_k^{ac} \tilde{r}_k, \quad D_x^{\mathrm{c}} u = \sum_{k=1}^{N} v_k^{\mathrm{c}} \tilde{r}_k, \quad D_x^{\mathrm{jump}} u = \sum_{k=1}^{N} v_k^{\mathrm{jump}} \tilde{r}_k.$$

We call v_i^{c} the Cantor part of v_i and denote by

$$v_i^{\mathrm{cont}} := v_i^{\mathrm{c}} + v_i^{ac} = \tilde{l}_i \cdot (D_x^{\mathrm{c}} u + D_x^{\mathrm{ac}} u)$$

the continuous part of v_i. According to Volpert's Chain Rule

$$D_x \lambda_i(u) = \nabla \lambda_i(u)(D_x^{\mathrm{ac}} u + D_x^{\mathrm{c}} u) + [\lambda_i(u^+) - \lambda_i(u^-)]\delta_x,$$

and then

$$D_x^{\mathrm{c}} \lambda_i(u) = \nabla \lambda_i \cdot D_x^{\mathrm{c}} u = \sum_k \big(\nabla \lambda_i \cdot \tilde{r}_k\big) v_k^{\mathrm{c}}.$$

We define the *i-th component of* $D_x \lambda_i(u)$ as

$$[D_x \lambda_i(u)]_i := \big(\nabla \lambda_i \cdot \tilde{r}_i\big) v_i^{\mathrm{cont}} + [\lambda_i(u^+) - \lambda_i(u^-)] \frac{|v_i^{\mathrm{jump}}(x)|}{\sum_k |v_k^{\mathrm{jump}}(x)|}, \tag{2.3.11}$$

and the *Cantor part of i-th component of* $D_x \lambda_i(u)$ to be

$$[D_x^{\mathrm{c}} \lambda_i(u)]_i := \big(\nabla \lambda_i \cdot \tilde{r}_i\big) v_i^{\mathrm{c}}. \tag{2.3.12}$$

2.4 Main SBV regularity argument

Following [11], the key idea to obtain SBV-like regularity for v_i is to prove a decay estimate for the continuous part of v_i. We state here the main estimate of our chapter.

Theorem 2.4.1. *Consider the general strictly hyperbolic system* (2.3.1)*, and suppose that the i-th characteristic field is genuinely nonlinear. Then there exists a finite, non-negative Radon measure μ_i^{ICJ} on $\mathbb{R}^+ \times \mathbb{R}$ such that for $t > \tau > 0$*

$$\big|v_i^{\mathrm{cont}}(t)\big|(B) \leq \mathcal{O}(1) \left\{ \frac{\mathcal{L}(B)}{\tau} + \mu_i^{\mathrm{ICJ}}([t-\tau, t+\tau] \times \mathbb{R}) \right\} \tag{2.4.1}$$

for all Borel subset B of $\mathbb{R}$.

Different from [11], we assume only one characteristic field to be genuinely nonlinear and no other requirement on the other characteristic fields.

Once Theorem 2.4.1 is proved, then the SBV argument develops as follows [11].

Suppose at time $t = s$, $v_i(s)$ has a Cantor part. Then there exists a $\mathcal{L}^1$-negligible Borel set K with $v_i^{\mathrm{cont}}(s)(K) > 0$ and $D^{\mathrm{jump}} v_i(K) = 0$. Then for all $s > \tau > 0$,

$$0 < |v_i(s)|(K) = |v_i^{\mathrm{cont}}(s)|(K) \leq \mathcal{O}(1)(1)\left\{\frac{\mathcal{L}^1(K)}{\tau} + \mu_i^{\mathrm{ICJ}}([s-\tau, s+\tau] \times \mathbb{R})\right\}.$$

Since $\mathcal{L}^1(K) = 0$, we can let $\tau \to 0$, and deduce that $\mu_i^{\mathrm{ICJ}}(\{s\} \times \mathbb{R}) > 0$. This shows that the Cantor part appears at most countably many times because μ_i^{ICJ} is finite.

Then, we can have the following result which generalizes Corollary 3.2 in [11] to the case when only one characteristic field is genuinely nonlinear and no assumption is made on the others.

Corollary 2.4.2. *Let u be a vanishing viscosity solution of the Cauchy problem for the strictly hyperbolic system* (2.3.1)*, and assume that the i-th characteristic field is genuinely nonlinear. Then $v_i(t)$ has no Cantor part out of a countable set of times.*

As we see in the scalar case, by proving the SBV regularity of the solution under the genuinely nonlinearity assumption of one characteristic field, we can deduce a kind of SBV regularity of the characteristic speed for general systems.

Unlike the scalar case, we do not have the maximum principle to guarantee the small variation of u in the triangle $T(t_0, x_0)$ defined in (2.2.2). However, in the system case, we have the following estimates for the vanishing viscosity solutions.

For $a < b$ and $\tau \geq 0$, we denote by Tot.Var.$\{u(\tau);\ ]a,b[\}$ the total variation of $u(\tau)$ over the open interval $]a,b[$. Moreover, consider the triangle

$$\Delta_{a,b}^{\tau,\eta} := \Big\{(t,x) :\ \tau < t < (b-a)/2\eta,\ a + \eta t < x < b - \eta t\Big\}.$$

The oscillation of u over $\Delta_{a,b}^{\tau,\eta}$ will be denoted by

$$\mathrm{Osc.}\{u;\ \Delta_{a,b}^{\tau,\eta}\} := \sup\Big\{|u(t,x) - u(t',x')| :\quad (t,x),\ (t',x') \in \Delta_{a,b}^{\tau,\eta}\Big\}.$$

We have the following results.

Theorem 2.4.3 (Tame Oscillation, see Section 13 of [10])**.** *There exists $C' > 0$ and $\bar{\eta} > 0$ such that for every $a < b$ and $\tau \geq 0$, one has*

$$\mathrm{Osc.}\{u;\ \Delta_{a,b}^{\tau,\bar{\eta}}\} \leq C' \cdot \mathrm{Tot.Var.}\{u(\tau);\ ]a,b[\}.$$

Adapting the proof of the scalar case, we can prove the main Theorem 2.1.2 of this chapter: the proof of this theorem will be done in Section 2.7.

2.5 Review of wave-front tracking approximation for general system

To prove Theorem 2.4.1, we use the front tracking approximation in described in Section 1.4. Pointing to the properties needed in our argument: more precisely, we will only consider how one constructs the approximate wave pattern of the k-th genuinely nonlinear family (Section 1.4.2).

The main point is that, for general systems, the accurate/simplified/crude Riemann solvers for the k-th wave coincides with the approximate/simplified/crude Riemann solvers when all families are genuinely nonlinear (see below for the definition of accurate/simplified/ crude Riemann solvers). This means that the wave pattern of the k-th genuinely nonlinear family will have the same structure as if all other families are genuinely nonlinear: by this, we mean that shock-shock interaction generates shocks, the jump in characteristic speed across k-th waves is proportional to their size, and one can thus use the k-component of the derivative of λ_k (2.3.11) to measure the total variation of v_k.

2.5.1 Description of the wave-front tracking approximation

The construction of front tracking approximation is described in Section 1.4, the only thing we want to emphasis is that if the k-th characteristic family is genuinely nonlinear, the elementary wave u_k is either a shock wave or a rarefaction wave. The key example of the accurate Riemann solver is thus to consider how these two solutions are approximated.

If k-th elementary wave u_k in (2.3.8) is just a single shock, for example

$$u_k = \begin{cases} u^{\mathrm{L}} & x/t < \sigma, \\ u^{\mathrm{R}} & x/t > \sigma, \end{cases}$$

where σ is the speed of shock wave, then the approximated k-th wave coincides the exact one.

If u_k is a rarefaction wave of the k-th family connecting the left value u^{L} and the right value u^{R}, for example, if $u^{\mathrm{R}} := T^k_s[u^{\mathrm{L}}]$ and

$$u_k = \begin{cases} u^{\mathrm{L}} & x/t < \lambda_k(u^{\mathrm{L}}), \\ T^k_{s^*}[u^{\mathrm{L}}] & x/t \in [\lambda_k(u^{\mathrm{L}}), \lambda_k(u^{\mathrm{R}})],\ x/t = \lambda_k(T^k_{s^*}[u^{\mathrm{L}}]), \\ u^{\mathrm{R}} & x/t > \lambda_k(u^{\mathrm{R}}), \end{cases}$$

where $s^* \in [0, s]$. Then the approximation $\tilde{u}_k$ is a rarefaction fan containing several rarefaction fronts. More precisely, we can choose real numbers $0 = s_0 < s_1 < \cdots < s_n = s$, and define the points $w_i := T^k_{s_i}[u^{\mathrm{L}}]$, $i = 0, \ldots, n$, with the following properties,

$$w_{i+1} = T^k_{(s_{i+1}-s_i)}[w_i],$$

and the wave opening of consecutive wave-fronts are sufficiently small, i.e.

$$\sigma^k_s[u^{\mathrm{L}}](s_{i+1}) - \sigma^k_s[u^{\mathrm{L}}](s_i) \le \epsilon, \quad \forall i = 0, \ldots, n-1.$$

where the function σ^k_s is defined in Theorem 2.3.2. We let the jump $[\omega_i, \omega_{i+1}]$ travel with the speed $\tilde{\sigma}_i := \tilde{\lambda}_k(\omega_i, \omega_{i+1})$ (2.3.10a), so that the rarefaction fan $\tilde{u}_k$ becomes

$$\tilde{u}_k = \begin{cases} u^{\mathrm{L}} & x/t < \tilde{\sigma}_1, \\ \omega_i & \tilde{\sigma}_i \le x/t < \tilde{\sigma}_{i+1},\ i = 1, \ldots, n-1, \\ u^{\mathrm{R}} & x/t \ge \tilde{\sigma}_n. \end{cases}$$

2.5.2 Jump part of i-th waves

The derivative of u^ν is clearly concentrated on polygonal lines, being a piecewise constant function with discontinuities along lines. Suppose the i-th family is genuinely nonlinear. To select the wave fronts belonging to i-th family of u^ν converging to the jump part of u, we use the following definition.

Definition 2.5.1 (Maximal (ϵ^0, ϵ^1)-shock front)**.** [19] A maximal (ϵ^0, ϵ^1)-shock front for the i-th family of an ϵ_ν-approximate front-tracking solution u^ν is any maximal (w.r.t. inclusion) polygonal line $(t, \gamma^\nu(t))$ in the (t,x)-plane, $t_0 \leq t \leq t_1$, satisfying:

(i) the segments of γ^ν are i-shocks of u^ν with size $|s^\nu| \geq \epsilon^0$, and at least once $|s^\nu| \geq \epsilon^1$;

(ii) the nodes are interaction points of u^ν;

(iii) it is on the left of any other polygonal line which it intersects and which have the above two properties.

Let $M^{\nu,i}_{(\epsilon^0,\epsilon^1)}$ be the number of maximal (ϵ^0, ϵ^1)-shock front for the i-th family. Denote

$$\gamma^{\nu,i}_{(\epsilon^0,\epsilon^1),m} : [t^{\nu,i,-}_{(\epsilon^0,\epsilon^1),m}, t^{\nu,i,+}_{(\epsilon^0,\epsilon^1),m}] \to \mathbb{R}, \quad m = 1, \ldots, M^{\nu,i}_{(\epsilon^0,\epsilon^1)},$$

as the maximal (ϵ^0, ϵ^1)-shock fronts for the i-th family in u^ν. Up to a subsequence, we can assume that $M^{\nu,i}_{(\epsilon^0,\epsilon^1)} = \bar{M}^i_{(\epsilon^0,\epsilon^1)}$ is a constant independent of ν because the total variations of u^ν are bounded.

Consider the collection of all maximal (ϵ^0, ϵ^1)-shocks for the i-th family and define

$$\mathscr{T}^{\nu,i}_{(\epsilon^0,\epsilon^1)} = \bigcup_{m=1}^{\bar{M}^i_{(\epsilon^0,\epsilon^1)}} \mathrm{Graph}(\gamma^{\nu,i}_{(\epsilon^0,\epsilon^1),m}),$$

and let $\{\epsilon^0_k\}_{k\in\mathbb{N}}$, $\{\epsilon^1_k\}_{k\in\mathbb{N}}$ be two sequences satisfying $0 < 2^k \epsilon^0_k \leq \epsilon^1_k \searrow 0$.

Up to a diagonal argument and by a suitable labeling of the curves, one can assume that for each fixed k, m the Lipschitz curves $\gamma^{\nu,i}_{(\epsilon^0_k,\epsilon^1_k),m}$ converge uniformly to a Lipschitz curve $\gamma^i_{(\epsilon^0_k,\epsilon^1_k),m}$. Let

$$\mathscr{T}^i := \bigcup_{m,k} \mathrm{Graph}(\gamma^i_{(\epsilon^0_k,\epsilon^1_k),m}).$$

denote the collection of all these limiting curves in u.

For fixed (ϵ^0, ϵ^1), we write for shortness

$$\tilde{l}^\nu_i(t,x) := \tilde{l}_i(u^\nu(t,x-), u^\nu(t,x+)) \tag{2.5.1}$$

and define

$$v^{\nu,\mathrm{jump}}_{i,(\epsilon^0,\epsilon^1)} := \tilde{l}^\nu_i \cdot u^\nu_x \llcorner_{\mathscr{T}^{\nu,i}_{(\epsilon^0,\epsilon^1)}}. \tag{2.5.2}$$

Following the same idea of the proof of Theorem 10.4 in [19], the next lemma holds if only the i-th characteristic field is genuinely nonlinear.

Lemma 2.5.2. *The jump part of v_i is concentrated on $\mathscr{T}^i$.*

Moreover there exists a countable set $\Theta \subset \mathbb{R}^+ \times \mathbb{R}$, such that for each point

$$P = (\tau, \xi) = (\tau, \gamma^i_m(\tau)) \notin \Theta$$

where i-th shock curve γ^i_m is approximated by the sequence of (ϵ^0, ϵ^1)-shock fronts $\gamma^{\nu,i}_{(\epsilon^0,\epsilon^1),m}$ of the approximate solutions u^ν, the following holds

$$\lim_{r\to 0+} \limsup_{\nu\to\infty} \left(\sup_{\substack{x<\gamma^{\nu,i}_{(\epsilon^0,\epsilon^1),m}(t) \\ (t,x)\in B(P,r)}} \left| u^\nu(t,x) - u^- \right| \right) = 0, \tag{2.5.3a}$$

$$\lim_{r\to 0+} \limsup_{\nu\to\infty} \left(\sup_{\substack{x>\gamma^{\nu,i}_{(\epsilon^0,\epsilon^1),m}(t) \\ (t,x)\in B(P,r)}} \left| u^\nu(t,x) - u^+ \right| \right) = 0. \tag{2.5.3b}$$

Moreover, we can choose a sequence $\{\nu_k\}_{k=1}^\infty$ such that

$$v_i^{\text{jump}} = \text{weak}^* - \lim_k \sum_{i=1}^N v^{\nu_k,\text{jump}}_{i,(\epsilon^0_k,\epsilon^1_k)}. \tag{2.5.4}$$

The key argument of the proof is that we can use the tools of the proof of Theorem 10.4 in [19] because the wave structure of the i-th genuinely nonlinear family has the following properties:

1. the interaction among two shocks of the i-th family generates only one shock of the i-th family,
2. the strength of i-th waves can be measured by the jump of the i-th characteristic speed λ_i,
3. the speed of i-th waves is very close to the average of the jump of λ_i across the discontinuity.

These properties are a direct consequence of the behavior of the approximate Riemann solvers on the i-th waves if the i-th family is genuinely nonlinear (Section 1.4.2).

Before proving the lemma, we recall some definitions which will be used in the proof.

Definition 2.5.3 ([19], Definition 7.2)**.** Let $\hat{\lambda}$ be a constant larger than the absolute value of all characteristic speed. We say a curve $x = y(t),\ t \in [a,b]$ is *space-like* if

$$|y(t_2) - y(t_1)| > \hat{\lambda}(t_2 - t_1) \quad \text{for all } a < t_1 < t_2 < b.$$

We recall that a *minimal generalized i-characteristic* is an absolutely continuous curve starting from (t_0, x_0) satisfying the differential inclusion

$$x^\nu(t; t_0, x_0) := \min\left\{ x^\nu(t) : x^\nu(t_0) = x_0,\ \dot{x}^\nu(t) \in \left[\lambda_i\big(u^\nu(t, x(t)+)\big), \lambda_i\big(u^\nu(t, x(t)-)\big)\right] \right\}$$

for a.e. $t \geq t_0$.

For any given $(T,\bar{x})\in\mathbb{R}$, we consider the minimal (maximal) generalized i-characteristic through $(T,\bar{x})$, defined as

$$\chi^{-(+)}(t)=\min(\max)\{\chi(t):\chi \text{ is a generalized } i\text{-characteristic}, \chi(T)=\bar{x}\}.$$

From the properties of approximate solutions, we conclude that there is no wave-front of i-th family crossing χ^{+} from the left or crossing χ^{-} from the right.

Sketch of the proof. Let Θ be the set defined by all jump points of the initial datum, the atoms of μ^{IC} (see (1.4.2)). For any point $P\in\mathscr{T}^i\setminus\Theta$, if (2.5.3a) or (2.5.3b) does not hold, then this means that the approximate solutions u^ν have some uniform oscillation. Indeed, if (2.5.3a) not true , there exist $P_\nu,\ Q_\nu\to P$ and $P_\nu,\ Q_\nu$ on the left of $\gamma^{\nu,i}_{(\epsilon^0,\epsilon^1),m}$, $\overline{P_\nu Q_\nu}$ is space-like such that

$$u(P_\nu)\to u^-$$

and

$$|u^\nu(P_\nu)-u^\nu(Q_\nu)|\geq\epsilon_0,$$

for some constant $\epsilon_0>0$. It is not restrictive to assume that the direction $\overrightarrow{P_\nu Q_\nu}$ towards $\gamma^{\nu,i}_{(\epsilon^0,\epsilon^1),m}$. Let $\Lambda_k(\overline{P_\nu Q_\nu})$ be the total wave strength of fronts of k-th family which across the segment $\overline{P_\nu Q_\nu}$. Then, one has $\Lambda_j(\overline{P_\nu Q_\nu})\geq c\epsilon_0$ for some $j\in\{1,\cdots,d\}$ and some constant $c>0$. We consider three cases.

1 $j>i$, we take the maximal forward generalized j-characteristic χ^+ through P_ν and minimal generalized j-characteristic χ^- through Q_ν.

If χ^+ and χ^- interact each other at O_ν before hitting $\gamma^{\nu,i}_{(\epsilon^0,\epsilon^1),m}$. We consider the region Γ_ν bounded by $\overline{P_\nu Q_\nu}$, χ^+ and χ^-. Since no fronts can leave Γ_ν through χ^+ and χ^-. By (1.4.25) and (1.4.30b), one obtains that there exists a constant $c_1>0$ such that $\mu^{IC}_\nu(\Gamma_\nu)\geq c_1\epsilon_0$.

If χ^+ interact $\gamma^{\nu,i}_{(\epsilon^0,\epsilon^1),m}$ at A_ν and χ^- interact $\gamma^{\nu,i}_{(\epsilon^0,\epsilon^1),m}$ at B_ν, we consider the region Γ_ν bounded by $\overline{P_\nu Q_\nu}$, χ^+, χ^- and $\gamma^{\nu,i}_{(\epsilon^0,\epsilon^1),m}$. Then either there exists constant $0<c_0'<1$ scuh that $\mu^{IC}_\nu(\Gamma_\nu)>c_0'\epsilon_0$ or there exists constant $0<c_0''<1$ such that fronts with total strength lager than $c_0''\epsilon_0$ hitting $\overline{A_\nu B_\nu}$. By (1.4.25) and (1.4.30b), we it determine that $\mu^{IC}_\nu(\bar{\Gamma}_\nu)\geq c_0\epsilon^2$ on the closure of Γ_ν.

Thus, let $B(P,r_\nu)$ be the ball center at P containing Γ_ν with radius $r_\nu\to 0$ as $\nu\to 0$. This implies that $\mu^{IC}(\{P\})>0$ against the assumption $P\notin\Theta$.

2 $j<i$, we consider the minimal backward generalized j-characteristic through the point P_ν and the maximal backward generalized j-characteristic through the point Q_ν. Then by the similar argument for the case $j>i$, we get $\mu^{IC}(\{P\})>0$ against the assumptions.

3 $j=i$ and for any $j'\neq i$, $1\leq j'\leq d$, $\Lambda_{j'}(\overline{P_\nu Q_\nu})\to 0$ as $\nu\to\infty$. In this case, suppose that $\overline{P_\nu Q_\nu}$ intersects the curve $\gamma^{\nu,i}_{(\epsilon^0,\epsilon^1),m}$ at B_ν. Because of genuine nonlinearity, the minimal generalized i-characteristic χ through P_ν will hit $\gamma^{\nu,i}_{(\epsilon^0,\epsilon^1),m}$ if no previous

large interactions or cancellations occur on $\gamma^{\nu,i}_{(\epsilon^0,\epsilon^1),m}$. We consider the triangle region Γ_ν bounded by the segment $P_\nu B_\nu$, the curve $\gamma^{\nu,i}_{(\epsilon^0,\epsilon^1),m}$ and χ. Since no fronts of i-family can exit from Γ_ν through χ, one obtains $\mu^{IC}_\nu(\Gamma_\nu)$ uniformly positive which contradicts the assumption $\mu^{IC}(P) = 0$.

Therefore, we conclude that (2.5.3) is true. And (2.5.3b) is similar to prove.

For $P \notin \mathscr{T}^i \cup \Theta$, if $v_i^{\text{jump}}(P) > 0$, i.e. P is a jump point of u, by the similar argument of Step 8 in the proof of Theorem 10.4 in [19] this shows that the waves present in the approximate solutions are canceled, and thus $\mu^{\text{IC}}(P) > 0$. It is impossible since $P \notin \Theta$. This concludes that v_i^{jump} is concentrated on $\mathscr{T}^i$, because by (2.5.3) the jumps in the approximate solutions are vanishing in a neighborhood of every $P \notin \mathscr{T}^i \cup \Theta$.

We are left with the proof of (2.5.4). At jump point $(t, \gamma^i_{(\epsilon^0,\epsilon^1),m}(t)) \in \mathscr{T}^i \setminus \Theta$, according to (2.5.3a), (2.5.3b), there exists a sequence $(t^\nu, \gamma^{\nu,i}_{(\epsilon^0,\epsilon^1),m}(t)(t^\nu))$ such that

$$\big(t, \gamma^i_{(\epsilon^0,\epsilon^1),m}(t)\big) = \lim_{\nu\to\infty} \big(t^\nu, \gamma^{\nu,i}_{(\epsilon^0,\epsilon^1),m}(t)(t^\nu)\big) \tag{2.5.5}$$

and its left and right values converges to the left and right values of the jump in $(t, \gamma^i_{(\epsilon^0,\epsilon^1),m}(t))$.

Since $f \in C^2$, by the definition (2.3.9) the matrix $A(u^{\text{L}}, u^{\text{R}})$ depends continuously on the value $(u^{\text{L}}, u^{\text{R}})$, and since its eigenvalues are uniformly separated the same continuity holds for its eigenvalues $\tilde\lambda_k(u^{\text{L}}, u^{\text{R}})$, left eigenvectors $\tilde l_k(u^{\text{L}}, u^{\text{R}})$ and right eigenvectors $\tilde r_k(u^{\text{L}}, u^{\text{R}})$. Using the notation (2.3.10a) and (2.5.1), one obtains

$$\tilde l_i\big(t, \gamma^i_{(\epsilon^0,\epsilon^1),m}(t)\big) = \lim_{\nu} \tilde l_i^\nu\big(t^\nu, \gamma^{\nu,i}_{(\epsilon^0,\epsilon^1),m}(t^\nu)\big), \tag{2.5.6}$$

and similar limits holds for $\tilde r_i$, $\tilde\lambda_i$.

Up to a subsequence $\{\nu_k\}$, from the convergence of the graphs of $\mathscr{T}^{\nu_k,i}_{(\epsilon^0_k,\epsilon^1_k)}$ to $\mathscr{T}^i$ and (2.5.3a), (2.5.3b), it is fairly easy to prove that

$$Du\llcorner_{\mathscr{T}^i} = \text{weak}^*-\lim_{k\to\infty} Du^{\nu_k}\llcorner_{\mathscr{T}^{\nu_k,i}_{(\epsilon^0_k,\epsilon^1_k)}}. \tag{2.5.7}$$

According to (2.5.2), (2.5.6) and (2.5.7), one concludes the weak convergence of $v^{\nu_k,\text{jump}}_{i,(\epsilon^0_k,\epsilon^1_k)}$ to v_i^{jump}. □

2.6 Proof of Theorem 2.4.1

2.6.1 Decay estimate for positive waves

The Glimm Functional for BV functions to general systems has been obtained in [6], and when u is piecewise constant, it reduced to (1.4.2): and we will write it as $\mathcal{Q}$ also the formulation of the functional given in [6]. Moreover, for the same constant $C_0 > 0$ of the Glimm Functional $\Upsilon(t)$ (1.4.29), the sum $\text{Tot.Var.}(u) + C_0\mathcal{Q}(u)$ is lower semi-continuous w.r.t the L^1 norm (see Theorem 10.1 of [19]).

For any Radon measure μ, we denote $[\mu]^+$ and $[\mu]^-$ as the positive and negative part of μ according to Hahn-Jordan decomposition. The same proof of the decay of the Glimm Functional $\Upsilon(t)$ yields that for every finite union of the open intervals $J = I_1 \cup \cdots \cup I_m$

$$[v_i]^\pm(J) + C_0\mathcal{Q}(u) \le \liminf_{\nu\to\infty} \big\{[v_i^\nu]^\pm(J) + C_0\mathcal{Q}(u^\nu)\big\}, \quad i = 1, \dots, n, \tag{2.6.1}$$

as $u^\nu \to u$ in L^1.

In [19] the authors prove a decay estimate for positive part of the i-th wave measure under the assumption that i-th characteristic field is genuinely nonlinear and the other characteristic fields are either genuinely nonlinear or linearly degenerate. By inspection, one can verify that the proof also works (with a little modification) under no assumptions on the nonlinearity on the other characteristic fields, since the essential requirements of strict hyperbolicity and of the controllability of interaction amounts by Glimm Potential still hold: the main variation is that one should replace the original Glimm Potential in [19] with the generalized one given in [6].

We thus state the following theorem, which is the analog of Theorem 10.3 in [19].

Theorem 2.6.1. *Let the system* (2.1.1) *be strictly hyperbolic and the i-th characteristic field be genuinely non-linear. Then there exists a constant C'' such that, for every $0 \leq s < t$ and every solution u with small total variation obtained as the limit of wave-front tracking approximation, the measure $[v_i(t)]^+$ satisfies*

$$[v_i(t)]^+(B) \leq C'' \left\{ \frac{\mathcal{L}^1}{t-s}(B) + [\mathcal{Q}(s) - \mathcal{Q}(t)] \right\} \tag{2.6.2}$$

for every B Borel set in $\mathbb{R}$.

The estimate (2.6.2) gives half of the bound (2.4.1). In this section, we always assume that the i-th family is genuinely nonlinear.

2.6.2 Decay estimate for negative waves

To simplify the notation, we omit the index (ϵ^0, ϵ^1) in $v^{\nu,\text{jump}}_{i,(\epsilon^0,\epsilon^1)}$ in the rest of the proof. In order to get the uniform estimate for the *continuous part* $v_i^{\nu,\text{cont}} := v_i^\nu - v_i^{\nu,\text{jump}}$, we need to consider the distributions

$$\mu_i^\nu := \partial_t v_i^\nu + \partial_x(\tilde{\lambda}_i^\nu v_i^\nu), \qquad \mu_i^{\nu,\text{jump}} := \partial_t v_i^{\nu,\text{jump}} + \partial_x(\tilde{\lambda}_i^\nu v_i^{\nu,\text{jump}}).$$

Estimate for μ_i^ν

Let $y_m : [\tau_m^-, \tau_m^+] \to \mathbb{R}$, $m = 1, \dots, L^\nu$, be time-parameterized segments whose graphs are the i-th wave-fronts of u^ν and define

$$u_m^{\text{L}} := u(t, y_m(t)-), \quad u_m^{\text{R}} = u(t, y_m(t)+), \qquad t \in]\tau_m^-, \tau_m^+[.$$

For any test function $\phi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ one obtains

$$-\int_{\mathbb{R}^+ \times \mathbb{R}} \phi \, d\mu_i^\nu = \sum_{m=1}^{L^\nu} \left[\phi(\tau_m^+, y_m(\tau_m^+)) - \phi(\tau_m^-, y_m(\tau_m^-) \right] \tilde{l}_i \cdot (u_m^{\text{R}} - u_m^{\text{L}}). \tag{2.6.3}$$

For any m, since the i-th characteristic field is genuinely nonlinear, one has

$$|\tilde{l}_i(u^{\text{L}}, u^{\text{R}}) - l_i(u^{\text{L}})| = \mathcal{O}(1)(1)|u_m^{\text{R}} - u_m^{\text{L}}|,$$

where $u_m^{\text{R}} = T_{s_i}^i[u_m^{\text{L}}]$ for some size s_i. Then it follows from (2.3.6) that

$$s_i \cong \tilde{l}_i \cdot (u_m^{\text{R}} - u_m^{\text{L}}). \tag{2.6.4}$$

Let $\{(t_k, x_k)\}_k$ be the collection of points where the i-th fronts interact. The computation (2.6.3) yields that μ_i^ν concentrates on the interaction points, i.e.

$$\mu_i^\nu = \sum_k p_k \delta_{(t_k,x_k)},$$

where p_k is the difference between the strength of the i-th waves leaving at (t_k, x_k) and the i-th waves arriving at (t_k, x_k). We estimate the quantity p_k depending on the type of interaction:

Since in [11], it is proved that the total size of nonphysical wave-fronts are of the same order of ϵ_ν, when decomposing u_x^ν, we only consider the physical fronts. If at (t_k, x_k), two physical fronts with i-th component size s_i', s_i'' interact and generate an i-th wave or a rarefaction fan with total size $s_i = \sum_m s_i^m$, from (2.6.3) and (2.6.4), one has

$$p_k \cong s_i - s_i' - s''. \tag{2.6.5}$$

Notice that s' or s'' or both may vanish in (2.6.5) if one of incoming physical fronts does not belong to the i-th family.

According to the estimate in [5] (Lemma 1), the difference of sizes between the incoming and outgoing waves of the same family is controlled by the Amount of Interaction (see Section 1.4.2), so that one concludes

$$|\mu_i^\nu|(\{(t_k, x_k)\}) \leq \mathcal{O}(1)(1)\mathcal{I}(s_i, s_i')$$

and thus

$$|\mu_i^\nu|(\{t_k\} \times \mathbb{R}) \leq \mathcal{O}(1)(1)\{\Upsilon^\nu(t_k^-) - \Upsilon^\nu(t_k^+)\}.$$

This yields

$$|\mu_i^\nu|(\mathbb{R}^+ \times \mathbb{R}) \leq \mathcal{O}(1)(1)\Upsilon^\nu(0),$$

i.e. $|\mu_i^\nu|$ is a finite Radon measure.

Estimate for $\mu_i^{\nu,\text{jump}}$

Let $\gamma_m^i : [\tau_m^-, \tau_m^+] \to \mathbb{R}$, $m = 1, \ldots, \bar{M}^i_{(\epsilon^0,\epsilon^1)}$, be the curves whose graphs are the segments supporting the fronts of u^ν belonging to $\mathscr{T}^{\nu,i}_{(\epsilon^0,\epsilon^1)}$, and write

$$u_m^{\mathrm{L}} := u\big(t, \gamma_m^i(t)-\big), \quad u_m^{\mathrm{R}} := u\big(t, \gamma_m^i(t)+\big), \qquad t \in]\tau_m^-, \tau_m^+[.$$

For any test function $\phi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ by direct computation one has as in (2.6.3)

$$-\int_{\mathbb{R}^+\times\mathbb{R}} \phi \, d\mu_i^{\nu,\text{jump}} = \sum_{m=1}^{\bar{M}^i_{(\epsilon^0,\epsilon^1)}} \big[\phi(\tau_m^+, y_m(\tau_m^+)) - \phi(\tau_m^-, y_m(\tau_m^-))\big]\tilde{l}_i \cdot (u^{\mathrm{R}} - u^{\mathrm{L}}),$$

which yields

$$\mu_i^{\nu,\text{jump}} = \sum_k q_k \delta_{(\tau_k,x_k)},$$

where (τ_k, x_k) are the nodes of the jumps in $\mathscr{T}^{\nu,i}_{(\epsilon^0,\epsilon^1)}$ and the quantities q_k can be computed as follows: if the i-th incoming waves have sizes s' and s'', and the outgoing i-th shock has size s, then (see [11])

$$q_k \cong \begin{cases} -s' & (t_k, x_k) \text{ terminal point of a front not merging into another front,} \\ s & (t_k, x_k) \text{ initial point of a maximal front,} \\ s - s' - s'' & (t_k, x_k) \text{ is a triple point of } \mathscr{T}^{\nu,i}_{(\epsilon^0,\epsilon^1)}, \\ s - s' & (t_k, x_k) \text{ interaction point of a front with waves not belonging to } \mathscr{T}^{\nu,i}_{(\epsilon^0,\epsilon^1)}. \end{cases} \tag{2.6.6}$$

one has by the interaction estimates

In fact, since $s \leq 0$ on shocks the second case of (2.6.6) implies $q_k \leq 0$. For the triple point, one has that

$$q_k \leq \mu^{\mathrm{IC}}_\nu(\tau_k, x_k).$$

When a shock front in $\mathscr{T}^{\nu,i}_{(\epsilon^0,\epsilon^1)}$ interacts with a front not belonging to $\mathscr{T}^{\nu,i}_{(\epsilon^0,\epsilon^1)}$, there are three situations:

- It interacts with a rarefaction front of i-th family, then one has by the interaction estimates
$$q_k \leq \mu^{\mathrm{IC}}_\nu(\tau_k, x_k).$$
- It interacts with a front of different family, then also one gets
$$q_k \leq \mu^{\mathrm{I}}_\nu(\tau_k, x_k).$$
- It interacts with a shock of i-th family which does not belong to $\mathscr{T}^{\nu,i}_{(\epsilon^0,\epsilon^1)}$, then
$$q_k \leq 0.$$

Suppose now that (τ_k, x_k) is a terminal point of an (ϵ^0, ϵ^1)-shock front γ_m. By the definition of (ϵ^0, ϵ^1)-shock, for some $t \leq \tau_k$ the shock front γ_m has size $s_0 \leq -\epsilon^1$, and at (τ_k, x_k) the size s_1 of the outgoing i-th front must be not less than $-\epsilon^0$ as a result of interaction between two wave-fronts belonging to different family or cancellation between two wave-fronts belonging to the same family along γ_k. Hence we obtain

$$\epsilon^1 - \epsilon^0 \leq |s_0| - |s_1| \leq \mathcal{O}(1)(1)\mu^{\mathrm{IC}}_\nu(\gamma_k).$$

This yields

$$\begin{aligned} q_k \cong -s_1 + (s_1 + q_k) \leq & \frac{\epsilon^0}{\epsilon^1 - \epsilon^0}(\epsilon^1 - \epsilon^0) + \mathcal{O}(1)(1)\mu^{\mathrm{I}}_\nu(t_k, x_k) \\ \leq & \frac{\mathcal{O}(1)(1)\epsilon^0}{\epsilon^1 - \epsilon^0}\mu^{\mathrm{IC}}_\nu(\gamma_k) + \mathcal{O}(1)(1)\mu^{\mathrm{I}}_\nu(t_k, x_k). \end{aligned}$$

Since the end points correspond to disjoint maximal i-th fronts, due to genuinely nonlinearity, it follows that

$$\sum_{(t_k,x_k)\text{ end point}} q_k \leq \mathcal{O}(1)(1)\mu^{\mathrm{IC}}_\nu(\mathbb{R}^+ \times \mathbb{R}),$$

so that it is a uniformly bounded measure. We thus conclude that the distribution

$$\bar{\mu}^\nu := -\mu_i^{\nu,\mathrm{jump}} + \mathcal{O}(1)(1)\mu_\nu^{\mathrm{IC}} + \sum_{(t_k,x_k)\text{ end point}} q_k \delta_{(t_k,x_k)}$$

is non-negative, so it is a Radon measure and thus also $\mu_i^{\nu,\mathrm{jump}}$ is a Radon measure.

In order to obtain a lower bound, one considers the Lipschitz continuous test function

$$\phi_\alpha(t) := \chi_{[0,T+\alpha]}(t) - \frac{t-T}{\alpha}\chi_{[T,T+\alpha]}(t), \quad \alpha > 0,$$

which is allowed because v_i^ν is a bounded measure. Since $\bar{\mu}$ is non-negative, one obtains

$$\begin{aligned}
\bar{\mu}^\nu\big([0,T]\times\mathbb{R}\big) \leq & \int_{\mathbb{R}^+\times\mathbb{R}} \phi_\alpha d\bar{\mu} \\
= & -\int_{\mathbb{R}^+\times\mathbb{R}} \phi_\alpha d\mu_i^{\nu,\mathrm{jump}} + \mathcal{O}(1)(1)\int_{\mathbb{R}^+\times\mathbb{R}} \phi_\alpha d\mu_\nu^{\mathrm{IC}} + \sum_{(t_k,x_k)\text{ end point}} q_k\phi_\alpha(t_k) \\
\leq & \int_{\mathbb{R}^+\times\mathbb{R}} \big[(\phi_\alpha)_t + \tilde{\lambda}_i^\nu(\phi_\alpha)_x\big] d\big[v_i^{\nu,\mathrm{jump}}(t)\big]dt + \big[v_i^{\nu,\mathrm{jump}}(0)\big](\mathbb{R}) + \mathcal{O}(1)(1)\mu_\nu^{\mathrm{IC}}\big([0,T+\alpha]\times\mathbb{R}\big) \\
\leq & -\frac{1}{\alpha}\int_T^{T+\alpha} \big[v_i^{\nu,\mathrm{jump}}(t)\big](\mathbb{R})dt + \big[v_i^{\nu,\mathrm{jump}}(0)\big](\mathbb{R}) + \mathcal{O}(1)(1)\mu_\nu^{\mathrm{IC}}\big([0,T+\alpha]\times\mathbb{R}\big).
\end{aligned}$$

Letting $\alpha \searrow 0$ and since $[v_i^{\nu,\mathrm{jump}}(\mathbb{R})](0)$ is negative, one concludes

$$\bar{\mu}^\nu\big([0,T]\times\mathbb{R}\big) \leq -\big[v_i^{\nu,\mathrm{jump}}(T)\big](\mathbb{R}) + \mathcal{O}(1)(1)\mu_\nu^{\mathrm{IC}}\big([0,T+\alpha]\times\mathbb{R}\big) \leq \mathcal{O}(1)(1)\Upsilon^\nu(0).$$

We conclude this section by writing the uniform estimate

$$-\mathcal{O}(1)(1)\Upsilon^\nu(0) \leq \mu_i^{\nu,\mathrm{jump}} \leq \mathcal{O}(1)(1)\mu_\nu^{\mathrm{IC}}.$$

In particular, the definitions of the measures μ_i^ν, $\mu_i^{\nu,\mathrm{jump}}$ give the following balances for the i-th waves across the horizontal lines:

$$\big[v_i^\nu(t+)\big](\mathbb{R}) - \big[v_i^\nu(t-)\big](\mathbb{R}) = \mu_i^\nu\big(\{t\}\times\mathbb{R}\big), \tag{2.6.7a}$$

$$\big[v_i^{\nu,\mathrm{jump}}(t+)\big](\mathbb{R}) - \big[v_i^{\nu,\mathrm{jump}}(t-)\big](\mathbb{R}) = \mu_i^{\nu,\mathrm{jump}}\big(\{t\}\times\mathbb{R}\big). \tag{2.6.7b}$$

The limits are taken in the weak topology. Notice that we can always take that $t \mapsto v_i^\nu(t), v_i^{\nu,\mathrm{jump}}(t)$ is right continuous in the weak topology.

Balances of i-th waves in the region bounded by generalized characteristics

Given an interval $I = [a,b]$, we define the region $A_{[a,b]}^{\nu,(t_0,\tau)}$ bounded by the minimal i-th characteristics $a(t)$, $b(t)$ of u^ν starting at (t_0,a) and (t_0,b) by

$$A_{[a,b]}^{\nu,(t_0,\tau)} := \Big\{(t,x) : t_0 < t \leq t_0+\tau,\ a(t) \leq x \leq b(t)\Big\},$$

and its time-section by $I(t) := [a(t),b(t)]$. Let $J := I_1 \cup I_2 \cup \cdots \cup I_M$ be the union of the disjoint closed intervals $\{I_i\}_{i=1}^M$, and set

$$J(t) := I_1(t) \cup \cdots \cup I_M(t), \quad A_J^{\nu,(t_0,\tau)} := \bigcup_{m=1}^M A_{I_m}^{\nu,(t_0,\tau)}.$$

We will now obtain wave balances in regions of the form $A_J^{\nu,(t_0,\tau)}$. Due to the genuinely non-linearity of the i-th family, the corresponding proof in [11] works, we will repeat it for completeness.

The balance on the region $A_J^{\nu,(t_0,\tau)}$ has to take into account also the contribution of the flux Φ_i^ν across boundaries of the segments $I_m(t)$: due to the definition of generalized characteristic and the wave-front approximation, it follows that Φ_i^ν is an atomic measure on the characteristics forming the border of $A_J^{\nu,(t_0,\tau)}$, and moreover a positive wave may enter the domain $A_J^{\nu,(t_0,\tau)}$ only if an interaction occurs at the boundary point $(\hat{t},\hat{x})$, which gives the estimate

$$\Phi_i^\nu\big(\{(\hat{t},\hat{x})\}\big) \leq \mathcal{O}(1)(1)\mu_i^{\mathrm{IC}}\big(\{(\hat{t},\hat{x})\}\big). \tag{2.6.8}$$

One thus obtains that

$$\big[v_i^\nu(\tau)\big](J(\tau)) - \big[v_i^\nu(t_0)\big](J) = \mu_i^\nu\big(A_J^{\nu,(t_0,\tau)}\big) + \Phi_i^\nu\big(A_J^{\nu,(t_0,\tau)}\big) + \mathcal{O}(1)(1)\epsilon_\nu, \tag{2.6.9}$$

where the last term depends on the errors due to the wave-front approximation (a single rarefaction front may exit the interval I_m at t_0).

The same computation can be done for the jump part $v_i^{\nu,\mathrm{jump}}$, obtaining

$$\big[v_i^{\nu,\mathrm{jump}}(\tau)\big](J(t)) - \big[v_i^{\nu,\mathrm{jump}}(t_0)\big](J) = \mu_i^{\nu,\mathrm{jump}}\big(A_J^{\nu,(t_0,\tau)}\big) + \Phi_i^{\nu,\mathrm{jump}}\big(A_J^{\nu,(t_0,\tau)}\big). \tag{2.6.10}$$

Since the flux $\Phi_i^{\nu,\mathrm{jump}}$ only involves the contribution of (ϵ^0,ϵ^1)-shocks, it is clearly non-positive.

Subtracting (2.6.10) to (2.6.9), one finds the following equation for $v_i^{\nu,\mathrm{cont}}$:

$$\begin{aligned}&\big[v_i^{\nu,\mathrm{cont}}(\tau)\big](J(\tau)) - \big[v_i^{\nu,\mathrm{cont}}(t_0)\big](J)\\ =&\big(\mu_i^\nu - \mu_i^{\nu,\mathrm{jump}}\big)\big(A_J^{\nu,\tau}\big) + \big(\Phi_i^\nu - \Phi_i^{\nu,\mathrm{jump}}\big)\big(A_J^{\nu,(t_0,\tau)}\big) + \mathcal{O}(1)(1)\epsilon_\nu.\end{aligned}$$

Denote the difference between the two fluxes by

$$\Phi_i^{\nu,\mathrm{cont}} := \Phi_i^\nu - \Phi_i^{\nu,\mathrm{jump}}.$$

Since $\Phi_i^{\nu,\mathrm{jump}}$ removes only some terms in the negative part of Φ_i^ν, one concludes that

$$\Phi_i^\nu - \Phi_i^{\nu,\mathrm{jump}} \leq \big[\Phi_i^\nu\big]^+ \leq \mu_\nu^{\mathrm{IC}}. \tag{2.6.11}$$

Setting

$$\mu_{i,\nu}^{\mathrm{ICJ}} := \mu_\nu^{\mathrm{IC}} + \big|\mu_i^{\nu,\mathrm{jump}}\big|,$$

and using the estimate $|\mu_i^\nu| \leq \mathcal{O}(1)(1)\mu_\nu^{\mathrm{IC}}$, one has

$$\mu_i^\nu - \mu_i^{\nu,\mathrm{jump}} \leq \mathcal{O}(1)(1)\mu_{i,\nu}^{\mathrm{ICJ}}. \tag{2.6.12}$$

Decay estimate

Due to the semigroup property of solutions, it is sufficient to prove the estimate for the measure $[v_i^{\nu,\mathrm{cont}}(t=0)]^-$. Consider thus a closed interval $I=[a,b]$ and let $z(t) := b(t)-a(t)$ where

$$a(t) := x^\nu(t;0,a), \quad b(t) := x^\nu(t;0,b)$$

and the minimal forward characteristics stating at $t = 0$ from a and b. For $\mathcal{L}^1$-a.e. t one has

$$\dot{z}(t) = \tilde{\lambda}_i(t, b(t)) - \tilde{\lambda}_i(t, a(t)).$$

By introducing a piecewise Lipschitz continuous non-decreasing potential Φ to control the waves on the other families [19], with $\Phi(0) = 1$, one obtains

$$\Big|\dot{z}(t) + \xi(t) - \big[v_i^\nu(t)\big](I(t))\Big| \leq \mathcal{O}(1)(1)\epsilon_\nu + \dot{\Phi}(t)z(t), \tag{2.6.13}$$

where

$$\xi(t) := \big(\tilde{\lambda}_i(t, a(t)+) - \tilde{\lambda}_i(t, a(t)-)\big) + \big(\tilde{\lambda}_i(t, b(t)+) - \tilde{\lambda}_i(t, b(t)-)\big).$$

We consider two cases.

Case 1. If

$$\dot{z}(t) - \dot{\Phi}(t)z(t) < \frac{1}{4}\big[v_i^{\nu,cont}(0)\big](I)$$

for all $t > 0$, then

$$\frac{d}{dt}\Big[e^{-\int_0^t \dot{\Phi}(s)ds} z(t)\Big] = e^{-\int_0^t \dot{\Phi}(s)ds}\big\{\dot{z}(t) - \dot{\Phi}(t)z(t)\big\} < \frac{e^{-\int_0^t \dot{\Phi}(s)ds}}{4}\big[v_i^{\nu,cont}(0)\big](I).$$

Integrating the above inequality from 0 to τ and remembering that $\Phi(0) = 1$ and $v_i^{\nu,\text{jump}}(0)$ is non-positive, one has

$$\begin{aligned} -\mathcal{L}^1(I) = -z(0) \leq e^{-\int_0^\tau \dot{\Phi}(s)ds} z(\tau) - z(0) &\leq \frac{1}{4}\int_0^\tau e^{-\int_0^t \dot{\Phi}(s)ds} d\tau \big[v_i^{\nu,cont}(0)\big](I) \\ &\leq \frac{1}{4}\tau\big[v_i^{\nu,\text{cont}}(0)\big](I), \end{aligned}$$

which reads as

$$-\big[v_i^{\nu,\text{cont}}(0)\big](I) \leq 4\frac{\mathcal{L}^1(I)}{\tau}.$$

Case 2. Assume instead that

$$\dot{z}(\bar{t}) - \dot{\Phi}(\bar{t})z(\bar{t}) \geq \frac{1}{4}\big[v_i^{\nu,cont}(0)\big](I) \tag{2.6.14}$$

at some time $\bar{t} > 0$. From (2.5.2) and the fact that the i-th family is genuinely nonlinear and the fronts in $\mathscr{T}^{\nu,i}_{(\epsilon^0,\epsilon^1)}$ satisfy Rankine-Hugoniot conditions (up to a negligible error), we have

$$v_i^{\nu,\text{jump}}(t, a(t)) = \lambda_i(t, a(t)+) - \lambda_i(t, a(t)-).$$

Then by the balance (2.6.10), we conclude that

$$\begin{aligned} \xi(t) &\geq \frac{3}{4}\Big[\big[v_i^{\nu,\text{jump}}(t)\big](a(t)) + \big[v_i^{\nu,\text{jump}}(t)\big](b(t)) - 2\epsilon^1\Big] \\ &\geq \frac{3}{4}\Big[\big[v_i^{\nu,\text{jump}}(t)\big](I(t)) - 2\epsilon^1\Big]. \end{aligned} \tag{2.6.15}$$

As $v^{\nu,\text{jump}}$ is non-positive, (2.6.13) and (2.6.15) yield that

$$\begin{aligned} \dot{z}(t) - \dot{\Phi}z(t) &\leq \big[v_i^{\nu,\text{cont}}(t)\big](I(t)) + \big[v_i^{\nu,\text{jump}}(t)\big](I(t)) - \xi(t) + \mathcal{O}(1)(1)\epsilon_\nu \\ &\leq \big[v_i^{\nu,\text{cont}}(t)\big](I(t)) + \mathcal{O}(1)(1)\epsilon_\nu + 2\epsilon^1. \end{aligned}$$

Recall the assumption (2.6.14), at time $\bar{t}$, we get

$$\big[v_i^{\nu,\mathrm{cont}}(0)\big](I)/4 \le \big[v_i^{\nu,\mathrm{cont}}(\bar{t})\big](I(\bar{t})) + \mathcal{O}(1)(1)\epsilon_\nu + 2\epsilon^1.$$

By the balance for $v^{\nu,\mathrm{cont}}$ we get in Section 2.6.2, one obtains

$$\big[v_i^{\nu,\mathrm{cont}}(0)\big](I)/4 \le \big[v_i^{\nu,\mathrm{cont}}(0)\big](I) + \mu_\nu^{\mathrm{ICJ}}\big(A_I^{\nu,(0,\bar{t})}\big) + \mathcal{O}(1)(1)\epsilon_\nu + 2\epsilon^1.$$

Combining the conclusion for the two cases one gets the uniform bound r.w.t ν

$$-\big[v_i^{\nu,\mathrm{cont}}(0)\big](I) \le \mathcal{O}(1)(1)\bigg\{\frac{\mathcal{L}^1(I)}{t} + \mu_\nu^{\mathrm{ICJ}}\big(A_I^{\nu,(0,t)}\big) + \epsilon^1 + \epsilon_\nu\bigg\}.$$

This gives the estimate (2.4.1) for the case of a single interval for the approximate solution.

By analogous computation for the region which is a finite union of intervals, as we have done in Section 2.6.2, one obtains the same bound as above, and since $v_i^{\nu,\mathrm{cont}}$ is a Radon measure, the same result holds for any Borel sets, i.e.

$$-\big[v_i^{\nu,\mathrm{cont}}(0)\big](B) \le \mathcal{O}(1)(1)\bigg\{\frac{\mathcal{L}^1(B)}{t} + \mu_\nu^{\mathrm{ICJ}}\big(\overline{A_B^{\nu,(0,t)}}\big) + \epsilon^1 + \epsilon_\nu\bigg\},$$

where B is any Borel set in $\mathbb{R}$ and

$$A_B^{\nu,(0,t)} := \Big\{\big(\tau, x^\nu(\tau;0,x_0)\big):\ x\in B,\ 0<\tau\le t\Big\}.$$

As the solution is independent on the choice of the approximation, we can consider a particular converging sequence $\{u^\nu\}_{\nu\ge 1}$ of ϵ_ν-approximate solutions with the following additional properties:

$$\mathcal{Q}(u^\nu(0,\cdot)) \to \mathcal{Q}(u_0).$$

By lower semi-continuity of $[v_i(0)]^- + C_0\mathcal{Q}(u(0))$ (2.6.1), one gets

$$[v_i(0)]^- + C_0\mathcal{Q}(u(0)) \le \mathrm{weak}^* - \liminf_{\nu\to\infty}\left\{[v_i^\nu(0)]^- + C_0\mathcal{Q}(u^\nu(0))\right\}. \tag{2.6.16}$$

Since $v_i^{\mathrm{jump}}(0)$ has only negative part, from (2.6.16) and (2.5.4), up to a subsequence, one obtains for any open set $U\subset\mathbb{R}$,

$$\begin{aligned}
\big[v_i^{\mathrm{cont}}(0)\big]^-(U) &= [v_i(0)]^-(U) + \big[v_i^{\mathrm{jump}}(0)\big](U)\\
&\le \liminf_{\nu\to\infty}\left\{\big[v_i^\nu(0)\big]^-(U) + C_0\mathcal{Q}(u^\nu(0))\right\} - C_0\mathcal{Q}(u(0)) + \lim_{\nu\to\infty}\big[v_i^{\nu,\mathrm{jump}}(0)\big](U)\\
&= \liminf_{\nu\to\infty}\left\{\big[v_i^{\nu,\mathrm{cont}}(0)\big]^-(U) + C_0\mathcal{Q}(u^\nu(0))\right\} - C_0\mathcal{Q}(u(0))\\
&\le \liminf_{\nu\to\infty}\mathcal{O}(1)\bigg\{\frac{\mathcal{L}^1(U)}{t} + \mu_i^{\nu,\mathrm{ICJ}}\big(\overline{A_U^{\nu,(0,t)}}\big) + \epsilon^1 + \epsilon_\nu + \mathcal{Q}(u^\nu(0)) - \mathcal{Q}(u(0))\bigg\}\\
&\le \mathcal{O}(1)\bigg\{\frac{\mathcal{L}^1(U)}{t} + \mu_i^{\mathrm{ICJ}}\big([0,t]\times\mathbb{R}\big)\bigg\},
\end{aligned}$$

where μ_i^{ICJ} is defined as weak*-limit of measure $\mu_i^{\nu,\mathrm{ICJ}}$ (up to a subsequence). Then the outer regularity of Radon measure yields the inequality for any Borel set.

The above estimate together with Theorem 2.6.1 gives (2.4.1).

2.7 SBV regularity for the i-th component of the i-th eigenvalue

This last section concerns the proof of Theorem 2.1.2, adapting the strategy of Section 2.2.

Proof of Theorem 2.1.2. As in the scalar case, we define the sets

$$\begin{aligned} J_\tau &:= \big\{x \in \mathbb{R} :\ u^{\mathrm{L}}(\tau,x) \neq u^{\mathrm{R}}(\tau,x)\big\}, \\ F_\tau &:= \big\{x \in \mathbb{R} :\ \nabla\lambda_i(u(\tau,x)) \cdot r_i(u(\tau,x)) = 0\big\}, \\ C &:= \big\{(\tau,\xi) \in \mathbb{R}^+ \times \mathbb{R} :\ \xi \in J_\tau \cup F_\tau\big\}, \quad C_\tau := J_\tau \cup F_\tau. \end{aligned}$$

By definition of continuous part

$$\big|v_i^{\mathrm{cont}}(\tau)\big|(J_\tau) = 0,$$

and since

$$\nabla\lambda_i\big(u(\tau, F_\tau \setminus J_\tau)\big) \cdot r_i\big(u(\tau, F_\tau \setminus J_\tau)\big) = 0,$$

we conclude that

$$\begin{aligned} &\big|\nabla\lambda_i(u) \cdot r_i(u) v_i^{\mathrm{cont}}(\tau)\big|(C_\tau) \\ =&\big|\nabla\lambda_i(u) \cdot r_i(u) v_i^{\mathrm{cont}}(\tau)\big|(J_\tau) + \big|\nabla\lambda_i(u) \cdot r_i(u) v_i^{\mathrm{cont}}(\tau)\big|(F_\tau \setminus J_\tau) = 0. \end{aligned}$$

For any $(t_0,x_0) \in \mathbb{R}^+ \times \mathbb{R} \setminus C$, there exist strictly positive $b_0 = b_0(x_0,t_0)$, $c_0 = c_0(x_0,t_0)$ such that

$$\big|\nabla\lambda_i \cdot r_i(u(t_0,x))\big| \geq c_0 > 0,$$

for every x in the open interval $I_0 :=]-b_0+x_0, x_0+b_0[$, because $u(t_0,x)$ is continuous at $x_0 \notin C_{t_0}$. Hence by Theorem 2.4.3, we know that there is a triangle

$$T_0 := \Big\{(t,x) :\ |x-x_0| < b_0' - \bar\eta(t-t_0),\ 0 < t-t_0 < b_0'/\bar\eta\Big\},$$

with the basis $I_0' :=]-b_0'+x_0, x_0+b_0'[\subset I_0$, such that

$$\big|\nabla\lambda_i \cdot r_i(u(t_0,x))\big| \geq \frac{c_0}{2} > 0, \tag{2.7.1}$$

by taking $b_0' \ll 1$ in order to have that the total variation remains sufficiently small.

Since $u \llcorner T_0$ coincides with the solution to

$$\begin{cases} \partial_t w + f(w)_x = 0, \\ w(x,t_0) = \begin{cases} u_{t_0}(x) & |x-x_0| < b_0', \\ \frac{1}{2b_0'}\int_{x_0-b_0'}^{x_0+b_0'} u_{t_0}(y)dy & |x-x_0| \geq b_0, \end{cases} \end{cases} \tag{2.7.2}$$

and by taking b_0' sufficiently small, we still have that (2.7.1) holds for the range of w. In particular w is SBV outside a countable number of times, and the same happens for u in T_0.

As in the scalar case, one thus verifies that there is a countable family of triangles $\{T_i\}_{i=1}^{\infty}$ covering the complement of C outside a set whose projection on the t-axis is countable. The same computation of the scalar case concludes the proof: for any τ chosen as in (2.2.3)

$$\left|(\nabla\lambda_i \cdot r_i)v_i^{\mathrm{c}}\right|(\mathbb{R}) \leq \left|(\nabla\lambda_i \cdot r_i)v_i^{\mathrm{c}}\right|(C_\tau) + \left|(\nabla\lambda_i \cdot r_i)v_i^{\mathrm{c}}\right|\left(\bigcup_i T_i \cap \{t=\tau\}\right) = 0.$$

Recall the definition (2.3.12), we conclude that the i-th component of $D_x\lambda_i(u(t,\cdot))$ has no Cantor part for every $t \in \mathbb{R}^+ \setminus S$ and $i \in \{1,\ 2,\ \ldots,\ N\}$. □

Similar to the scalar case, it is easy to get the following corollary from the Theorem 2.1.2 and (2.3.11).

Corollary 2.7.1. *Suppose u be a vanishing viscosity solution of the Cauchy problem for the strictly hyperbolic system* (2.1.1)-(2.1.2). *Let u be the vanishing viscosity solution of the problem* (2.1.1), (2.1.2). *Then the scalar measure $[D_x\lambda_i(u)]_i$ has no Cantor part in $\mathbb{R}^+ \times \mathbb{R}$.*

Remark 2.7.2. As we mentioned in the introduction, it no longer holds the SBV regularity of admissible solution to the general strictly hyperbolic system of conservation laws.

Consider the following equations

$$\begin{cases} u_t = 0, \\ v_t + ((1+v+u)v)_x = 0. \end{cases}$$

Since $D_x\lambda_2((u,v)) = D_xu + 2D_xv$, then it is clear that $D_x\lambda_2$ can have a Cantor part since the first equation is just trivial which means that the component u is not SBV regular if the initial data is not.

While from Theorem 2.4.1 the Cantor part of the second component of $D_x\lambda_2(u)$,

$$[D_x\lambda_2(u)]_2 = \big(D_u\lambda_2 \cdot r_2\big)\big(l_2 \cdot D_x^c(u,v)\big) = \frac{2}{1+u+2v}\big(vD_x^c u_x + (1+u+2v)D_x^c v\big)$$

vanishes. (Notice that since the Cantor part of (D_xu, D_xv) concentrates on the set of continuous points of (u,v), we do not need to specify the coefficients at the jump points of (u,v).)

Chapter 3

Global structure of admissible BV solutions to the piecewise genuinely nonlinear system

This chapter is concerned with the qualitative structure of admissible solutions to the strictly hyperbolic $N \times N$ system of conservation laws in one space dimension of the form

$$\begin{cases} u_t + f(u)_x = 0, & u : \mathbb{R}^+ \times \mathbb{R} \to \Omega \subset \mathbb{R}^N, \ f \in C^2(\Omega, \mathbb{R}), \\ u_{|t=0} = u_0, & u_0 \in \mathrm{BV}(\mathbb{R}; \Omega). \end{cases} \tag{3.0.1}$$

Assume that the system is strictly hyperbolic in $\Omega \subset \mathbb{R}^N$.

Furthermore, as we only consider the solutions with small total variation and thus they live in a neighborhood of a point, it is not restrictive to assume that Ω is bounded and there exist constants $\{\check{\lambda}_i\}_{i=0}^N$, such that

$$\check{\lambda}_{i-1} < \lambda_i(u) < \check{\lambda}_i, \qquad \forall u \in \Omega, \ i = 1, \dots, N. \tag{3.0.2}$$

Let $\{r_i(u)\}_{i=1}^N$ and $\{l_j(u)\}_{j=1}^N$ be a basis of right and left eigenvectors, depending smoothly on u, such that

$$l_j(u) \cdot r_i(u) = \delta_{ij} \text{ and } |r_i(u)| \equiv 1, \quad i, j = 1, \dots, N.$$

The integral curve of the vector field $r_i(u)$ with initial datum u_0

$$\frac{du}{d\omega} = r_i(u(\omega)), \quad u(0) = u_0,$$

will be denoted by $R_i[u_0](\omega)$, and it is called the i*-th rarefaction curve* through u_0,

Recall the Rankine-Hugoniot condition

$$f(u_1) - f(u_0) = \sigma(u_1 - u_0) \quad \text{if} \quad u(t,x) = u_0 + (u_1 - u_0)\chi_{x \geq \sigma t} \quad \text{is a weak solution,}$$

generates N distinct smooth curves $S_i[u_0]$ starting from any $u_0 \in \Omega$ and N smooth functions $\sigma_i[u_0]$ such that

$$\sigma_i[u_0](s)[S_i[u_0](s) - u_0] = f(S_i[u_0](s)) - f(u_0),$$

and moreover

$$S_i[u_0](0) = u_0, \qquad \sigma_i[u_0](0) = \lambda_i(u_0), \qquad \frac{d}{ds} S_i[u_0](0) = r_i(u_0).$$

The curve $S_i[u_0]$ is called *the i-th Hugoniot curve* issuing from u_0; we will also say that $[u_0, u_1]$ is an *i-th discontinuity* with speed $\sigma_i(u_0, u_1)$.

We are now ready to introduce the definition of piecewise genuinely nonlinear systems.

Definition 3.0.3. We say that i-th characteristic field of the system (3.0.1) is *piecewise genuinely nonlinear* if the set $Z_i := \{u : \nabla\lambda_i \cdot r_i(u) = 0\}$ is the union of $(N-1)$-dimensional distinct manifolds Z_i^j, $j = 1, \dots, J_i$ transversal to the vector field $r_i(u)$ and such that each rarefaction curve $R_i[u_0]$ crosses all the Z_i^j.

This implies that along R_i, the function λ_i has J_i critical points (see Figure 3). Without loss of generality we can also assume that the points $\omega^j[u_0]$ given by

$$R_i[u_0](\omega^j[u_0]) \in Z_i^j,$$

are monotone increasing w.r.t. $j = 1, \dots, J_i$.

We will denote by $\Delta_i^j \subset \mathbb{R}^N$ the region between Z_i^j and Z_i^{j+1}:

$$\begin{aligned} \Delta_i^j &:= \big\{u \in \Omega : \ \omega^j[u] < 0 < \omega^{j+1}[u]\big\}, \\ \Delta_i^0 := \big\{u \in \Omega : \omega^1[u] > 0\big\}, \ \Delta_i^{J_i} &:= \big\{u \in \Omega : \omega^{J_i}[u] < 0\big\}. \end{aligned} \tag{3.0.3}$$

Without any loss of generality (the analysis of the other case being completely similar), we set

$$\nabla\lambda_i \cdot r_i(u) < 0 \ \text{ if } j \text{ is even}, \ u \in \Delta_i^j,$$

$$\nabla\lambda_i \cdot r_i(u) > 0 \ \text{ if } j \text{ is odd}, \ u \in \Delta_i^j.$$

In what follow, we assume that *each characteristic field of* (3.0.1) *is piecewise genuinely nonlinear.* We will thus call the hyperbolic system *piecewise genuinely nonlinear.*

Remark 3.0.4. From the above definitions it follows that we do not allow characteristic families to be linearly degenerate. Thus our assumptions are slightly stricter than the natural extension of the setting of [19], where the families are either genuinely nonlinear or linearly degenerate.

It is however immediate to verify that the proof of regularity for linearly degenerate characteristic families does not depends on the properties of the remaining families, so that the results which we state in this chapter are valid also if some family is linearly degenerate. In fact, the regularity results we state are valid for a piecewise genuinely nonlinear family i, even if the system is not piecewise genuinely nonlinear.

Let $[u^-, u^+]$, $u^+ = S_i[u^-](s)$, be an admissible i-discontinuity. For us, this means that it is the limit of the vanishing viscosity approximation, and it can be shown to be equivalent to the following stability condition (used in [41]):

$$\forall\, 0 \le |\tau| \le s \ \Big(\sigma_i[u^-](\tau) \ge \sigma_i(u^-, u^+)\Big).$$

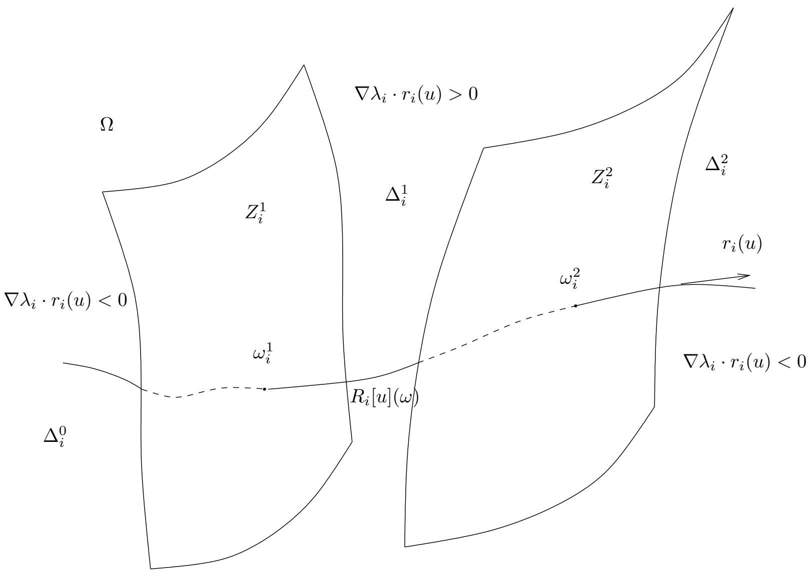

Figure 3.1:

Following the notation of [42], we call the jump $[u^-, u^+]$ *simple* if

$$\forall\, |\tau| \in]0, |s|[\ \Big(\sigma_i[u^-](\tau) > \sigma_i(u^-, u^+)\Big).$$

If $[u^-, u^+]$ is not simple, then we call it a *composition of the waves* $[u_0, u_1]$, $[u_1, u_2], \cdots$, $[u_\ell, u_{\ell+1}]$ with $u_0 = u^1$ and $u_{\ell+1} = u^+$, if

$$u_k = S_i[u^-](s_k) \quad \text{and} \quad \sigma_i(u_{k-1}, u_k) = \sigma_i(u^-, u^+),\ k = 1, \dots, \ell+1, \tag{3.0.5}$$

where

$$0 = s_0 < s_1 < s_2 < \cdots < s_\ell < s \quad (\text{or } s < s_\ell < \cdots < s_1 < s_0 = 0),$$

and there are no other points τ such that (3.0.5) holds. (For general f, it may happens that the set where $\sigma(u, u^-) = \sigma(u^+, u^-)$ is not finite, but this does not happen for piecewise genuinely nonlinear systems, as it will be shown as a consequence of Lemma 3.2.3).

In [42], under assumption of piecewise genuinely nonlinearity, by using Glimm scheme it is proved that if the initial data has small total variation, there exists a weak admissible BV solution of (3.0.1). Therefore, this solutions enjoy the usual regularity properties of BV function: u either is approximately continuous or has an approximate jump at each point $(x,t) \in \mathbb{R}^+ \times \mathbb{R} \setminus \mathcal{N}$, where $\mathcal{N}$ is a subset whose one-dimensional Hausdorff measure $\mathcal{H}^1$ is zero. In the same paper, the author shows much stronger regularity that u holds. The set $\mathcal{N}$ contains at most countably many points, and u is continuous (not just approximate continuous) outside $\mathcal{N}$ and countably many Lipschitz continuous curves.

In [24], the authors adopt wave-front tracking approximation to prove the similar result for (3.0.1) with the assumption that each characteristic field is genuinely nonlinear. Moreover, the authors were able to prove that outside the countable set Θ there exist right and left limits u^-, u^+ on the jump curves, and these limits are stable w.r.t. wavefront approximate solutions: more precisely, for each jump point $(\bar{t}, \bar{x})$ not belonging to the countable set Θ (the points where a strong interaction occurs, see the definition at page 66), there exists a shock curve $x = y_\nu(t)$ for the approximate solution u_ν converging to it and such that its left and right limit converge to u^-, and u^+ uniformly. This means that

$$\lim_{r\to 0+}\left(\limsup_{\nu\to\infty}\sup_{\substack{x<y_\nu(t)\\(x,t)\in B(P,r)}}\left|u_\nu(x,t)-u^-\right|\right)=0,$$

$$\lim_{r\to 0+}\left(\limsup_{\nu\to\infty}\sup_{\substack{x>y_\nu(t)\\(x,t)\in B(P,r)}}\left|u_\nu(x,t)-u^+\right|\right)=0.$$

In [19] (Theorem 10.4), the author generalizes this result to the case when some characteristic field may be linearly degenerate.

In our setting, in order to prove this additional regularity estimates on shocks, some additional difficulties arise: in fact the proof in [19] is based on the wave structure of the solution to genuinely nonlinear or linearly degenerate systems, where only one shock curve passes through the discontinuous point (which is not a point where a strong interaction occurs, i.e. not in Θ). In our case, instead, it may happen that the shock is composed by several waves as in (3.0.5), and these waves separate even if the point does not belong to the countable Θ.

For example, consider a scalar equation where f has two inflection points. It is thus clearly piecewise genuinely nonlinear (see Figure 3.2). Let u_0 be the initial data

$$u_0=\begin{cases}u_1 & \text{if } x<x_1,\\ u_2 & \text{if } x_1<x<x_2,\\ u_3 & \text{if } x_2<x<x_3,\\ u_4 & \text{if } x>x_3.\end{cases}$$

By carefully choosing f and the points x_1, x_2, x_3 and the value $u_1, \dots, u_4$, one can obtain the wave pattern shown in Figure 3.3: the point where the two jumps meet is not a strong interaction point, however the waves join together.

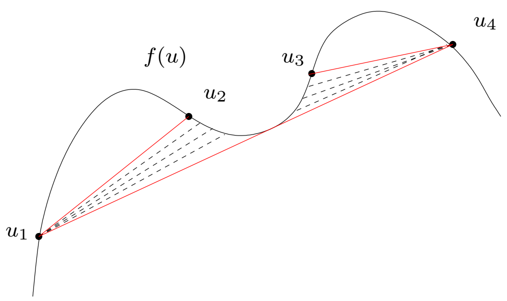

Figure 3.2:

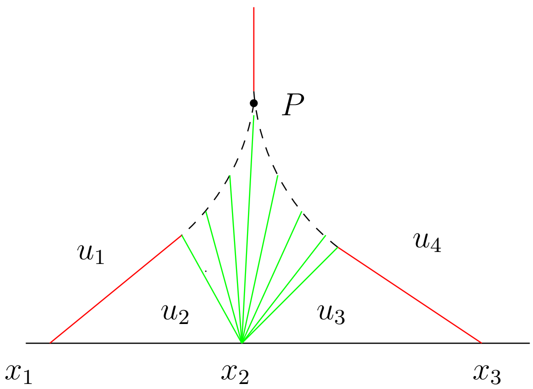

Figure 3.3:

In a similar way, one can construct examples where the shock splits, even without a strong interaction. Clearly such wave pattern can not be reproduced if f is convex or concave.

In this chapter we prove the following theorem:

Theorem 3.0.5. *Let u be an admissible BV solution of the Cauchy problem* (3.0.1) *with f piecewise genuinely nonlinear. Then there exist a countable set Θ of interaction points and a countable family $\mathscr{T}$ of Lipschitz continuous curves such that u is continuous outside Θ and Graph$(\mathscr{T})$.*

Moreover, suppose $u(t_0, x)$ is discontinuous at $x = x_0$ as a function of x, and $(t_0, x_0) \notin \Theta$. Write $u^L = u(t_0, x_0-)$, $u^R = u(t_0, x_0+)$ and suppose that $u^{\mathrm{R}} = S_i[u^{\mathrm{L}}](s)$ with $s > 0$ $(s < 0)$.

- *If $[u^L, u^R]$ is simple, there exists a Lipschitz curve $y(t) \in \mathscr{T}$, s.t. $y(t_0) = x_0$ and*

$$u^{\mathrm{L}} = \lim_{\substack{x<y(t)\\(x,t)\to(t_0,x_0)}} u(x,t), \qquad u^{\mathrm{R}} = \lim_{\substack{x>y(t)\\(x,t)\to(t_0,x_0)}} u(x,t) \quad \text{and} \quad \dot{y}(t_0) = \sigma(u^{\mathrm{L}}, u^{\mathrm{R}}).$$

- *If $[u^{\mathrm{L}}, u^{\mathrm{R}}]$ is a composition of the waves $[u^{\mathrm{L}}, u_1]$, $[u_1, u_2], \cdots, [u_\ell, u^{\mathrm{R}}]$, then there exist p Lipschitz continuous curves $y_1, \cdots, y_p \in \mathscr{T}$, $p \le \ell + 1$ satisfying*

  - $y_1(t_0) = \cdots = y_p(t_0) = x_0$,
  - $y_1'(t_0) = \cdots = y_p'(t_0) = \sigma(u^{\mathrm{L}}, u^{\mathrm{R}})$,
  - $y_1(t) \le \cdots \le y_p(t)$ *for all t in a neighborhood of t_0.*

  Moreover,

$$u^{\mathrm{L}} = \lim_{\substack{x<y_1(t)\\(x,t)\to(t_0,x_0)}} u(x,t), \qquad u^{\mathrm{R}} = \lim_{\substack{x>y_p(t)\\(x,t)\to(t_0,x_0)}} u(x,t), \tag{3.0.6}$$

  and if in a small neighborhood of (t_0, x_0), y_j and y_{j+1} are not identical, one has

$$u_j = \lim_{\substack{y_j(t)<x<y_{j+1}(t)\\(x,t)\to(t_0,x_0)}} u(x,t). \tag{3.0.7}$$

As in [19], the above result is based on the following convergence result for approximate wave-front solutions, which implies the stability of the wave pattern w.r.t. L^1_{loc}-convergence of solutions (see Remark 3.3.1).

Theorem 3.0.6. *Consider a sequence of wave-front tracking approximate solutions u_ν converging to u in $L^1_{loc}(\mathbb{R}^+ \times \mathbb{R})$. Suppose $P = (\tau, \xi)$ is a discontinuity point of u and write $u^L = u(\tau, \xi-)$, $u^R = u(\tau, \xi+)$. Assume $[u^{\mathrm{L}}, u^{\mathrm{R}}]$ is the composition of ℓ waves, and let $\mathscr{T} \ni y_j : [t_j^-, t_j^+] \to \mathbb{R}$, $j = 1, \cdots, \ell$, be ℓ Lipschitz continuous curves (not necessarily distinct) passing through the point P, such that u is discontinuous across y_j and*

$$y_1(t) \le \cdots \le y_\ell(t) \qquad \text{in a small neighborhood of } \tau.$$

Then up to a subsequence, there exist $y_{j,\nu} : [t_{j,\nu}^-, t_{j,\nu}^+] \to \mathbb{R}$, $j = 1, \cdots, \ell$, which are discontinuity curves of u_ν not necessarily distinct, such that $t_{j,\nu}^- \to t_j^-$, $t_{j,\nu}^+ \to t_j^+$ and

$$y_{j,\nu}(t) \to y_j(t) \qquad \text{for every } t \in [t_j^-, t_j^+].$$

Moreover, one has

$$\lim_{r\to 0+}\left(\limsup_{\nu\to\infty}\sup_{\substack{x<y_{1,\nu}(t)\\(x,t)\in B(P,r)}}\left|u_\nu(x,t)-u^L\right|\right)=0,$$

$$\lim_{r\to 0+}\left(\limsup_{\nu\to\infty}\sup_{\substack{x>y_{\ell,\nu}(t)\\(x,t)\in B(P,r)}}\left|u_\nu(x,t)-u^R\right|\right)=0.$$

Note that it is possible that the curve y_j coincide for all j, while the curves $y_{\nu,j}$ do not have any common point $\nu > 0$.

A brief outline of this chapter follows.

In Section 3.1, we briefly describe the wave-front tracking approximate scheme for general strictly hyperbolic system, as presented in [5]. In particular, we introduce the definition of interaction and cancellation measures. Even if this part is well known in the literature, we reproduce the essential ideas for reader's convenience.

In Section 3.2, since we have already described the construction of front tracking approximations for general hyperbolic systems of conservation laws, here we only give some remark on the speciality of piecewise genuinely nonlinear case.

In Section 3.3, we give the proofs of Theorem 3.0.5 and Theorem 3.0.6, by proving that the approximate subdiscontinuity curves converge to the curves in the family $\mathcal{T}$ defined in the statements. For the interesting case of shocks, the proof works as follows: if the statements of the theorems were false, then waves not supported by the curves y_i would exist in the approximating solution in a neighborhood of the point $(t_0, x_0) \notin \Theta$. These waves cannot be shocks (otherwise they will converge to some of the limiting curves y_i) Thus by the structure of the system they must interact in the vicinity of the shock. In the limit $\nu \searrow 0$, this will imply that the point under consideration is in Θ.

In Section 3.4, we construct a strictly hyperbolic 2×2 system of conservation laws, which is not piecewise genuinely nonlinear and whose admissible solution to a particular initial datum does not have the structural properties described in Theorem 3.0.5. In fact, it is not possible to find finitely many curves supporting a shock of the second characteristic family in a small neighborhood of any point, even if the set of times t where the discontinuities of the second characteristic family are present has positive Lebesgue measure. In particular, it is not possible to even state (3.0.6). This shows that the assumption of piecewise genuinely nonlinearity cannot be removed.

Notation. Throughout the chapter, we write $A \lesssim B$ ($A \gtrsim B$) if there exists a constant $C > 0$ which only depends on the system (3.0.1) such that $A \leq CB$ ($A \geq CB$).

3.1 Description of wave-front tracking approximation

We have describe the necessary notation and construction for front-tracking approximation and shown that the limit is vanishing viscosity solutions. In [10] it is proved that if $u^{\mathrm{L}}, u^{\mathrm{R}} \in \Delta_i^j$ with some j odd (even) and $u^{\mathrm{R}} = T_i[u^{\mathrm{L}}](s)$, $s > 0$ ($s < 0$), the solution u of the Riemann problem with the initial date (1.4.3) is a *centered rarefaction wave*, that is for

$t > 0$,

$$u(x,t) = \begin{cases} u^{\mathrm{L}} & \text{if } x/t < \lambda_i(u^{\mathrm{L}}), \\ R_i[u^{\mathrm{L}}](\tau) & \text{if } x/t \in [\lambda_i(u^{\mathrm{L}}), \lambda_i(u^{\mathrm{R}})],\ x/t = \lambda_i(R_i[u^{\mathrm{L}}](\tau)), \\ u^{\mathrm{R}} & \text{if } x/t > \lambda_i(u^{\mathrm{R}}), \end{cases}$$

where $\tau \in [0,s]$ ($\tau \in [s,0]$) such that $s = l_i^0 \cdot (R_i[u^{\mathrm{L}}](s) - u^{\mathrm{L}})$. This is a consequence of the fact that $\nabla\lambda_i \cdot r_i(u) > 0$ in Δ_i^j. Notice that u is Lipschitz continuous for $t > 0$.

As shown in [10] (see also Remark 4 in [5] and Section 4 of [41]), under the assumption of piecewise genuine nonlinearity, the solution of the Riemann problem provided by (1.4.12) is a composed wave of the i-th family containing a finite number of rarefaction waves and admissible discontinuities. Recalling Theorem 2.3.2, one knows that the open intervals where the v_i-component of the solution to (1.4.7) vanishes correspond to rarefaction waves, while the closed intervals where the v_i-component of the solution to (1.4.7) is different from zero correspond to admissible discontinuities.

3.2 Construction of subdiscontinuity curves

In this section we define the family of approximate subdiscontinuity curves. The key point is that due to the piecewise genuine nonlinearity assumption, one can select finitely many subdiscontinuities of a given jumps where the flux f_i is convex (or concave, see below). These components behave very similarly to the genuinely nonlinear case: the main property is that they cannot be split by interactions, but only completely removed by cancellation. Thus for these components one can adapt the procedure used to define the discontinuity curves for genuinely nonlinear systems.

We now define the (i,j)-subdiscontinuities s_i^j of an i-th shock s_i. The index j refers to the regions Δ_i^j defined in (3.0.3). Let $[u^{\mathrm{L}}, u^{\mathrm{R}}]$, $u^{\mathrm{R}} = T_i[u^{\mathrm{L}}](s_i)$, be a wavefront of i-th family in the approximate solution u_ν. For definiteness, we assume $s_i > 0$. Since the derivative of the curve T_i is very close the i-eigenvector r_i, it follows that the curve $\bar{u}(\cdot; u^L, s_i)$ [1] intersects transversally the surfaces Z_i^j. Let thus $0 \le \tau^{j_1} \le \tau^{j_1+1} \le \dots \le \tau^{j_2} \le s_i$ be the values such that

$$u^{j_1+k} = \bar{u}(\tau^{j_1+k}; s_i, u^L) \in Z_i^{j_1+k}, \quad k = 1, \dots, j_2 - j_1.$$

If $\tau^{j_1} > 0$, set $\tau^{j_1-1} = 0$ and if $\tau^{j_2} < s_i$, set $\tau^{j_2+1} = s_i$.

Definition 3.2.1. We say that the wavefront $[u^{\mathrm{L}}, u^{\mathrm{R}}]$ has a (i,j)*-subdiscontinuity* $[u^j, u^{j+1}]$ *of strength* $s_i^j = \tau^{j+1} - \tau^j$ when the latter is different from 0, with j odd for $s_i < 0$ and j is even for $s_i > 0$.

Notice that obviously only mixed fronts and discontinuity fronts can have (i,j)-subdiscontinuities s_i^j, because rarefaction fronts are contained in regions where the i-th eigenvalue is *increasing* across the discontinuity, while by the above definition the subdiscontinuities belong to the part of the wavefront in which the i-th eigenvalue is *decreasing*.

[1] the fixed point of the operator $\mathcal{T}_{i,s}$, see (1.4.7).

Observe moreover that the wave decomposition given by (3.0.5) are such that in each component there is at least a subdiscontinuity.

The above observation implies that the subdiscontinuities are quite stable, in the sense that they do not split when involved in an interaction: this is a direct consequence of the construction of the (approximate) Riemann solution, and it will be proved in Lemma 3.2.3.

The second step is to define for the subdiscontinuities s_i^k of a wavefront s_i the components which has a uniform strength in some time interval. The following definition is an adaptation of the definition of (δ, i)-approximate discontinuities [22].

Definition 3.2.2. For $\delta \neq 0$ fixed, a (δ, i, j)*-approximate subdiscontinuity curve* is a polygonal line in (x,t)-plane with nodes $(t_0, x_0), (t_1, x_1), \cdots, (t_n, x_n)$ such that

1. (t_k, x_k) are interaction points with $0 \leq t_0 < t_1 < \cdots < t_n$,
2. j is odd for $s_i < 0$ or j is even for $s_i > 0$,
3. for $1 \leq k \leq n$, the segment $[(t_{k-1}, x_{k-1}), (t_k, x_k))$ is the support of an (i,j)-subdiscontinuity front with strength $|s_i^j| \geq \delta/2$, and there is at least one time $t \in [t_0, t_n]$ such that $|s_i^j| \geq \delta$.

In order to count them, the following property of piecewise genuinely nonlinear system comes in handy.

Lemma 3.2.3. *The solution of a Riemann problem given by any approximate Riemann solver contains at most one subdiscontinuity* s_i^j *for all* $i = 1, \dots, N$, $j = 1, \dots, J_i$.

While the proof can be obtained directly from the analysis of [6], we repeat it.

Proof. For approximate solutions to a Riemann problem, the proof reduces in proving that the speed $\bar\sigma_i(\tau; u^-, s)$ obtained by solving the system (1.4.7) is constant in each subdiscontinuity component.

Assume that this is not the case, and for definiteness let $s_i > 0$, so that in the subdiscontinuities we have $D\lambda_i \cdot r_i < 0$. Then by inspection of system (1.4.7) one has that if $\bar\tau \in (\tau^j, \tau^{j+1})$, j even, is the point where $d\bar\sigma_i/d\tau > 0$, then $\bar v_i(\tau_\ell) = 0$ for a sequence $\tau_\ell \to \bar\tau$. Hence $\tilde\lambda_i(\bar\gamma(\tau_\ell)) = \lambda_i(\bar u(\tau_\ell))$, which implies

$$\frac{d\bar\sigma}{d\tau}(\bar\tau) = D\lambda_i \cdot \frac{d\bar u}{d\tau}(\bar\tau).$$

By using $d\bar u/d\tau = \tilde r_i(\bar\gamma)$, since $v_i(\tau) = 0$ one obtains

$$\frac{d\bar u}{d\tau}(\bar\tau) = r_i(\bar u(\bar\tau)),$$

so that

$$0 < \frac{d\bar\sigma}{d\tau}(\bar\tau) = D\lambda_i(\bar u(\bar\tau)) \cdot r_i(\bar u(\bar\tau)) \leq 0,$$

which is a contradiction. □

Remark 3.2.4. The same proof shows that a composite wave with strength s can have at most $[J_i/2]+1$ components[2]. In fact, the extremal values of a component have $\bar{v}_i = 0$, and thus only one can be present in the regions Δ_i^j for j even if $s < 0$ or j odd for $s > 0$. Moreover it is clear that the points u_k of (3.0.5) are uniquely determined by the condition of being the unique point in some $\Delta_i^{j_k}$, j_k even for $s < 0$ or j_k odd for $s > 0$, such that $\lambda_i(u_k) = \sigma_i[u^{\mathrm{L}}](s,s)$.

Define the family of curves $\mathscr{T}^j_{\delta,i}(\nu)$ as follows: if $\{y_\ell : I_\ell \to \mathbb{R}\}_{\ell=1}^L$ have been chosen, for a jump point $(t,x) \notin \cup_\ell \mathrm{graph}\, y_\ell$ such that the subdiscontinuity s_i^j has strength $\geq \delta$, let y_{L+1} be the unique curve supporting an approximate (δ,i,j)-subdiscontinuity passing through (t,x) such that

1. it is the leftmost among all approximate (δ,i,j)-subdiscontinuities passing trough (t,x),
2. it is maximal w.r.t. set inclusion.
3. it is disjoint from all the curves y_ℓ, $\ell = 1,\dots,L$.

The uniqueness follows from the fact that the above lemma implies uniqueness of the curve y_{L+1} in the future. In particular, in the past the curve y_{L+1} never meets another wave y_ℓ, $\ell \leq L$.

The next proposition implies that the number $M^j_{\delta,i}(\nu)$ of curves in $\mathscr{T}^j_{\delta,i}(\nu)$ is finite, independently of ν.

Proposition 3.2.5. *For fixed j and δ, $M^j_{\delta,i}(\nu)$ is uniformly bounded w.r.t. ν.*

Proof. First of all, for all fixed times t the number of (δ,i,j)-subdiscontinuities is clearly bounded by $2\mathrm{Tot.Var.}(u(t))/\delta$. Suppose that there is a sequence of times $\{t_\ell\}_{\ell=1}^{L_\nu}$ such that at each t_ℓ there exists an approximate (δ,i,j)-subdiscontinuity curve γ_ℓ whose interval of definition does not contain $t_{\ell'}$, $\ell' < \ell$. Since we can take t_ℓ increasing and at a fixed time the number of subdiscontinuity curves is finite, we thus conclude that L_ν many of them are created and canceled.

Since the number of curves $M^j_{\delta,i}(\nu)$ is increasing with δ decreasing, we can assume that

$$\delta \leq \mathrm{dist}(Z_i^j, Z_i^{j+1}).$$

It follows that a subdiscontinuity s_i^j of $[u^{\mathrm{L}}, u^{\mathrm{R}}]$ can have size $< \delta/2$ only if $u^{\mathrm{L}} \in s_i^j$ or $u^{\mathrm{R}} \in s_i^j$.

As a consequence of Lemma 3.2.3, the only way to decrease the strength of an approximate (i,j)-subdiscontinuity γ_i^j from $s_i^j(t_1) \geq \delta$ to a $s_i^j(t_2) < \delta/2$ at a later time $t_2 > t_1$ is only by interaction and cancellation: this is a direct consequence of the fact that we cannot split the subdiscontinuities. Hence we can reduce a subcomponent s_i^j of $[u^{\mathrm{L}}, u^{\mathrm{R}}]$ only by varying the end points of the curve $T_i[u^{\mathrm{L}}](s)$.

Due to the Lipschitz dependence of the curve $T_i[u^{\mathrm{L}}](s)$ from u^{L} and s, and the transversality of the surfaces Z_i^j, it follows that to reduce the size of a subdiscontinuity from δ to

[2] $[\cdot]$ is the integer part of a real number.

$\delta/2$ we have to vary s or u^{L} of at least $\delta/2$. In both cases, from Glimm interaction estimate, it follows that the amount of interaction along the curve supporting s_i^j is of at least $\delta^2/4$: indeed, by direct inspection, interactions of the same family just increase s_i^j (remember that it belong to the end portions of the jump $[u^{\mathrm{L}}, u^{\mathrm{R}}]$).

Hence, by Glimm interaction estimates, it follows that the amount of interaction/ cancellation along $\gamma_i^j([t_1, t_2])$ is $\geq \mathcal{O}(1)\delta/2$.

From the uniform boundedness of Glimm functional and of the disjointness of the subdiscontinuity curves, we conclude that L_ν is uniformly bounded, which implies the uniformly boundedness of the number $M_{\delta,i}^j(\nu)$ w.r.t. ν. □

Remark 3.2.6. In Section 3.4, we show why the piecewise genuine nonlinearity is essential for the validity of the above proposition. In fact, an explicit example in a 2×2 hyperbolic system shows that the statement is false in the general case.

We will denote the curves of $\mathscr{T}_{\delta,i}^j(\nu)$ as $y_{\delta,i}^{j,\ell}(\nu)$, $\ell = 1, \dots, M_{\delta,i}^j$, with $M_{\delta,i}^j$ independent of ν. By standard compactness estimates, completely similar to the genuinely nonlinear case, one can fairly easily prove that up to subsequences we can assume that $y_{\delta,i}^{j,\ell}(\nu) \to y_{\delta,i}^{j,\ell}$ in the uniform topology, with $y_{\delta,\ell}^{j,\ell}$ non necessarily distinct curves in $\mathbb{R}^+ \times \mathbb{R}$.

Let us denote by

$$\mathscr{T}_{\delta,i}^j := \big\{ y_{\delta,i}^{j,\ell} : I_{\delta,i}^{j,\ell} \to \mathbb{R}, \ell = 1, \dots, M_{i,\delta}^j \big\}, \quad i \in 1, \dots, N, j = 1, \dots, J_i,$$

the collection of all these limiting curves for fixed δ, i, j, and set moreover

$$\mathscr{T}_{\delta,i} := \bigcup_j \mathscr{T}_{\delta,i}^j, \qquad \mathscr{T}_i := \bigcup_\delta \mathscr{T}_{\delta,i}.$$

With an abuse of notation, we will also write $\mathscr{T}_i$ for the graph in $\mathbb{R}^+ \times \mathbb{R}$ of the curves in $\mathscr{T}_i$.

Definition 3.2.7. Let Θ consist of all jump points of initial data and the atoms of interaction and cancellation measure μ^{IC}.

Lemma 3.2.8. *Let $y_{\delta,i}^{j,\ell} : I_\ell \to \mathbb{R}$ be an (δ, i, j)-subdiscontinuity curve in $\mathscr{T}_{\delta,i}^j$. If $(t, y_{\delta,i}^{j,\ell}(t)) \notin \Theta$, then the derivative $\dot{y}_{\delta,i}^{j,\ell}(t)$ exists.*

Proof. By the definition of $\mathscr{T}_{\delta,i}^j$, there exists a curve $y_{\delta,i}^{j,\ell}(\nu) \in \mathscr{T}_{\delta,i}^j(\nu)$ converging uniformly to $y_{\delta,i}^{j,\ell}$. Since $(t, y_{\delta,i}^{j,\ell}(t))$ is not an atom for μ^{IC}, then for all $\eta > 0$ there exists $r > 0$ such that

$$\mu_\nu^{IC}\big(B((t, y_{\delta,i}^{j,\ell}(t)), r)\big) \leq \eta.$$

For a discontinuity of size $\delta > 0$, it follows from the Glimm estimate that its change in speed is proportional to μ^{IC}/δ, and thus the approximating curve $y_{\delta,i}^{j,\ell}(\nu)$ have a speed whose total variation is $\leq \eta/\delta$. The conclusion follows from the l.s.c. of the Lipschitz constant w.r.t. uniform convergence. □

To conclude this section, we give a definition of a partial order relation among subdiscontinuities s_i^j of the same family but with different j. For definiteness, we assume that $s_i^j > 0$ so that the index j is even.

Consider the calligraphic ordering $\prec$ in $\mathbb{R}^2$:

$$(x,y) \prec (x',y') \quad \Longleftrightarrow \quad \Big(x < x' \ \vee \ \big(x = x' \wedge y < y'\big)\Big).$$

Let $P_i(u) = u_i$ be the projection of the vector u on its i-th component and let $y^{j,\ell}_{\delta,i}$, $y^{j',\ell'}_{\delta',i}$ be two subdiscontinuity curves, corresponding to the subdiscontinuities

$$U := \big[u^{j,\ell}_{\delta,i}-, u^{j,\ell}_{\delta,i}+\big], \quad U' := \big[u^{j,\ell}_{\delta',i}-, u^{j',\ell'}_{\delta',i}+\big],$$

and with j even. Then we define

$$y^{j,\ell}_{\delta,i} \prec y^{j',\ell'}_{\delta',i} \quad \Longleftrightarrow \quad \exists t, u \in U, u' \in U' \ \big((y^{j,\ell}_{\delta,i}(t), P_i(u)) \prec (y^{j',\ell'}_{\delta',i}(t), P_i(u'))\big). \tag{3.2.1}$$

It is fairly easy to see that the above definition does not depend on the points u, u', but maybe it is not clear if it is independent of t. However a direct inspection on the Riemann solver formula implies that this monotonicity is preserved, so that $\prec$ is a partial ordering on $\cup_{j \text{ even}} \mathscr{T}^j_i$. The fact that it is not a linear order is due to the possibility that the interval of existence of the curves $y^{j,\ell}_{\delta,i}$ are disjoint.

A completely similar partial ordering can be introduced on $\cup_{j \text{ odd}} \mathscr{T}^j_i$, by taking

$$y^{j,\ell}_{\delta,i} \prec y^{j',\ell'}_{\delta',i} \quad \Longleftrightarrow \quad \exists t, u \in U, u' \in U' \ \big((y^{j,\ell}_{\delta,i}(t), -P_i(u)) \prec (y^{j',\ell'}_{\delta',i}, -P_i(u'))\big).$$

3.3 Proof of the main theorems

In this section we give a proof of Theorems 3.0.5 and 3.0.6. The theorems contain 3 different statements:

1. outside the interaction points Θ and the discontinuity curves $\cup_i \mathscr{T}_i$, the solution is continuous and the limit of the wavefront approximations converge pointwise,

2. on the discontinuity points in $\cup_i \mathscr{T}_i$ which are not interaction point in Θ, the solution is right and left continuous, and there are curves converging to the discontinuity curve such that the wavefront approximations converges pointwise on both sides of these curves;

3. if the discontinuity is a composed shock and the components split in a neighborhood of the point, a similar continuity and convergence result holds in the region between the two curves.

A consequence of the proof is that the stability of the wave structure is preserved under L^1-convergence of solutions: this result is contained in the remark ending this section.

First we prove that u is continuous outside the points of interactions Θ and the discontinuity curves $\cup_i \mathscr{T}_i$. Consider a point

$$P = (\tau,\xi) \notin \Theta \cup \bigcup_i \mathscr{T}_i,$$

and assume by contradiction that it is not a continuity point of u. Then, by the L^1-convergence of approximate solutions u_ν, there exists $\eta > 0$ and a sequence of points $P_\nu = (x_{P_\nu}, t_{P_\nu})$, $Q_\nu = (x_{Q_\nu}, t_{Q_\nu})$ converging to P such that

$$\big|u_\nu(Q_\nu) - u_\nu(P_\nu)\big| \geq \eta, \tag{3.3.1}$$

up to subsequences. Due to the finite finite speed of propagation, we can assume that the segment $[P_\nu, Q_\nu]$ is space-like, i.e. its slope $\check{\lambda}$ is higher that all the characteristic speeds (see Figure 3.4), otherwise by the estimate

$$\sup_{a+\check{\lambda}t<x<b-\check{\lambda}t} \big|u(t,x) - c\big| \leq \mathcal{O}(1) \sup_{a<x<b} \big|u(0,x) - c\big| \tag{3.3.2}$$

the inequality (3.3.1) is impossible in an arbitrarily small neighborhood of P.

Three cases have to be considered.

Case 1.1: If there exists $i < i'$ such that the total wave strength of the i-th and i'-th families crossing the segments $[P_\nu, Q_\nu]$ are uniformly larger than $\eta/4$, then it follows that in Γ_ν, a small neighborhood of P, these waves are either created, canceled or have interacted. In all these cases, the amount of interaction on the region Γ_ν is uniformly large, that is $\mu_\nu^{IC}(\Gamma_\nu) \geq \eta^2/16$, which implies the point $P \in \Theta$ against the assumption.

Case 1.2: If instead only one family i has total variation of order $\eta/2$ and there a large discontinuity, since $P \notin \Theta$ this discontinuity contains some subdiscontinuity which is not canceled in a neighborhood of P, contradicting $P \notin \cup_i \mathscr{T}_i$.

Case 1.3: If finally the discontinuities are arbitrarily small as $\nu \to 0$, then they must belong to one of the regions Δ_i^j. Since in these regions the characteristic speed is genuinely nonlinear, then these waves must interact either in the future or in the past (depending from the sign of j, and they cannot be canceled or created because $P \notin \Theta$). In all cases, one concludes the $P \in \Theta$, yielding a contradiction.

Note that we have proved that at these points the convergence is pointwise, not in L^1.

Next, consider a point $P = (\tau, \xi) \in \cup_i \mathscr{T}_i \setminus \Theta$. It is clear that P belongs to $\mathscr{T}_i$ for only one family i, otherwise $P \in \Theta$. Since $x \mapsto u(\tau, x)$ has bounded variation in $\mathbb{R}$, the limits

$$u^L := \lim_{x \to \xi-} u(x,\tau), \qquad u^R := \lim_{x \to \xi+} u(x,\tau) \tag{3.3.3}$$

exist, and moreover $u^{\mathrm{R}} = T_i[u^{\mathrm{L}}](s)$. Without loss of generality we can assume $s > 0$, and let $\{y_i^j, j = j_1, \dots, j_P, j \text{ even}\}$ be the subdiscontinuity curves passing through P. Since $P \notin \Theta$, these curves are defined in a neighborhood of τ, and by the ordering we have that $j \mapsto y_i^j$ is increasing in the sense of (3.2.1). Let $y_i^j(\nu)$, $j = j_1, \dots, j_P$, j even, be the corresponding curves (for the approximate solutions u_ν) converging to y_i^j: their existence follows from the definition of $\mathscr{T}_i$ and the fact that $P \notin \Theta$.

The same analysis performed in the continuity points implies that on the left of $y_i^{j_1}$ the solution converges pointwise to a value $u^- \in \mathbb{R}^N$:

$$\lim_{r \to 0+} \left(\limsup_{\nu \to \infty} \sup_{\substack{x < y_i^{j_1}(\nu,t) \\ (x,t) \in B(P,r)}} \big|u_\nu(x,t) - u^-\big| \right) = 0, \tag{3.3.4a}$$

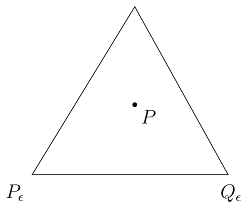

Figure 3.4:

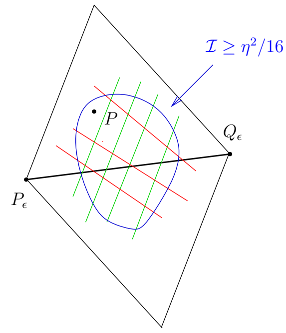

Figure 3.5:

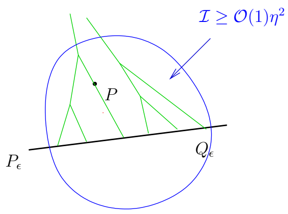

Figure 3.6:

and the same on the right side of $y_1^{J_p}(\nu)$:

$$\lim_{r\to 0+}\left(\limsup_{\nu\to\infty}\sup_{\substack{x>y_i^{jP}(\nu,t)\\(x,t)\in B(P,r)}}\left|u_\nu(x,t)-u^+\right|\right)=0, \tag{3.3.4b}$$

for some $u^+ \in \mathbb{R}^N$.

In fact, if the equality (3.3.4a) is not true, there a sequence of points $P_\nu = (x_{P_\nu}, t_{P_\nu})$, $Q_\nu = (x_{Q_\nu}, t_{Q_\nu})$ converging to P such that

$$\left|u_\nu(Q_\nu) - u_\nu(P_\nu)\right| \geq \eta, \tag{3.3.5}$$

and each segment $[P_\nu, Q_\nu]$ is space-like. Let us consider three corresponding cases as for continuous points.

Case 2.1: If there exists $i \neq i'$ such that the total wave strength of the i-th and i'-th families crossing the segments $[P_\nu, Q_\nu]$ are uniformly larger than $\eta/4$, then it follows that either a uniformly large amount of cancellation occurs in a small neighborhood of P, or waves with a uniformly large total strength interact with the curve $y_\nu^{j_1}$. Both contradicts the assumption $P \in \Theta$.

Case 2.2: If instead only one family i has total variation of order $\eta/2$ and there a large discontinuity, since $P \notin \Theta$ this discontinuity contains some subdiscontinuity which is not canceled in a neighborhood of P. This implies that there exist a subdiscontinuity curve $y^{j_0} \in \mathscr{T}_i$ such that $y^{j_0} \prec y^{j_1}$, which contradicts the assumption that y^{j_1} is the most left (in the sense of order (3.2.1)) subdiscontinuity curve passing though the point P.

Case 2.3: As the same as the situation in Case 1.3.

The equality (3.3.4b) is analogous to prove.

Similarly, if two curves y_i^j, y_i^{j+1} split for $t < t_P$ (or $t > t_P$), then also the subdiscontinuity curves $y_i^j(\nu)$, $y_i^{j+1}(\nu)$ converging y_i^j, y_i^{j+1} to have different locations for $t < t_P - \delta$ (or $t > t_P + \delta$), and the same analysis done before implies that

$$\lim_{r\to 0+}\left(\limsup_{\nu\to\infty}\sup_{\substack{y_i^j(\nu,t)<x<y_i^{j+1}(\nu,t)\\(x,t)\in B(P,r),t\le t_P-\delta}}\left|u_\nu(x,t)-u^{j-}\right|\right)=0,$$

or

$$\lim_{r\to 0+}\left(\limsup_{\nu\to\infty}\sup_{\substack{y_i^j(\nu,t)<x<y_i^{j+1}(\nu,t)\\(x,t)\in B(P,r),t>t_P+\delta}}\left|u_\nu(x,t)-u^{j+}\right|\right)=0,$$

for some $u^{j\pm}$.

The fact that $u^- = u^{\mathrm{L}}$, $u^+ = u^{\mathrm{R}}$ is a consequence of this convergence and equation (3.3.3) together with (3.3.2), while the fact that $u^{j-} = u^{j+}$ is given by the decomposition of shocks (3.0.5) follows from Lemma 3.2.8 and Remark 3.2.4.

Finally, the R-H conditions for the curves in $\cup_i \mathscr{T}_i$, that is

$$\dot y(t) = \sigma(u^L, u^R),$$

follows from the left and right limits (3.3.4) and the construction of wave-front tracking approximation.

Remark 3.3.1. If u_ν is a sequence of exact solutions to (3.0.1) with uniformly bounded total variation such that $u_\nu \to u$ in L^1_{loc}, then by a standard diagonal argument on the approximating wavefront solutions $u_{\nu,\nu}$ one obtains the following.

1. If P is a continuity point of u but not an interaction point in Θ, then u_ν converges strongly to u, i.e. for all η there exists r such that

$$\left|u_\nu(B(P,r)) - u(P)\right| \le \eta.$$

2. If P is a discontinuity point but not an interaction point, then there exists discontinuities in u_ν converging to the discontinuity of u in P and such that the values of u_ν converges to the values of u on the left and on the right of the discontinuity in the sense of Theorem 3.0.5.

Remark 3.3.2. By the same corresponding argument in proof for Theorem 10.4 in [19], we can prove that Theorem 3.0.5 still holds if some characteristic fields of the system are linearly degenerate.

Remark 3.3.3. A fairly easy consequence of the convergence of the wave structure is that the wave strength s_i converges weakly. In fact, the convergence of u^{L}, u^{R} on each shock apart the point in Θ yields that the decomposition of the measures

$$u_x^\pm = \sum_i v_i^\pm \hat r_i(t,x),$$

$$\text{where } \hat r_i(t,x) = \begin{cases} r_i(u(t,x)) & (t,x) \text{ continuity point of } u,\\ u^{\mathrm{R}} - u^{\mathrm{L}}/|u^{\mathrm{R}} - u^{\mathrm{L}}| & (t,x) \text{ discontinuity point in } \cup \mathscr{T}_i \setminus \Theta, \end{cases}$$

converges weakly: indeed, even if the decomposition is nonlinear, the convergence of u given by Theorem 3.0.6 yields that the vectors $\hat{r}_i$ converges in L^1 w.r.t. the measure $|u_x|(t)$ outside the countable number of times $P_t\Theta$.

Thus it is possible to pass to the limit to the wave balances (1.4.31) as in [22], obtaining as in [22] that

$$\big|\partial_t v^i + \partial_x(\hat{\lambda}_i v^i)\big| \leq \mu^I, \quad \big|\partial_t |v|^i + \partial_x(\hat{\lambda}_i |v|^i)\big| \leq \mu^{IC}.$$

3.4 A counterexample on general strict hyperbolic systems

In this last section we prove that the assumption of piecewise genuinely nonlinearity cannot be omitted: by an explicit example of 2×2 system, we show that the set of $2th$-discontinuities of its admissible solution does not contain any segment, even if it is of positive $\mathcal{H}^1$-measure. Hence the pointwise convergence of the limits in the left and right of a discontinuity cannot the proved, since there is not a clear boundary.

Consider a 2×2 system of the following form

$$\begin{cases} u_t + f(u,v)_x = 0, \\ v_t - v_x = 0. \end{cases} \tag{3.4.1}$$

where f is a smooth function. The Jacobian matrix of flux function is

$$DF(u,v) = \begin{pmatrix} f_u & f_v \\ 0 & -1 \end{pmatrix},$$

and the eigenvalues, eigenvectors are

$$\lambda_1 = -1, \quad \lambda_2 = f_u, \qquad r_1(u,v) = (f_v, -f_u - 1)^T, \quad r_2 = (1,0)^T.$$

The system is strict hyperbolic if $f_u > -1$.

We will choose f in order to have

$$Z_2 = \{(u,v) : \nabla\lambda_2 \cdot r_2\} = \{(u,v) : f_{uu}(u,v) = 0\} = \{v = 0\}. \tag{3.4.2}$$

This yields that the vector field r_2 is tangent to the manifold Z_2, therefore the second characteristic family is not piecewise genuinely nonlinear or linearly degenerate.

Define $f(u,v) = e^{-1/v}u^2/2$ when $v > 0$ and $f(u,0) \equiv 0$. The value of f for $v < 0$ will be computed below, in order to have the wave pattern we desire.

Let the initial data be

$$u_0(x) = \begin{cases} u_\ell & x < 0, \\ u_r & x > 0, \end{cases} \qquad v_0(x) = \begin{cases} -a & x < h, \\ a & x > h, \end{cases} \tag{3.4.3}$$

for some small constants $u_\ell > u_r$ and $a, h > 0$.

Since the second equation in (3.4.1) is a linear transport equation, one has

$$v(x,t) = \begin{cases} -a & x + t < h, \\ a & x + t > h. \end{cases} \tag{3.4.4}$$

Then one can solve the system (3.4.1) by regarding it as a scalar conservation laws of u

$$u_t + f(u,v)_x = 0$$

with discontinuous coefficient v. The definition of f for $v<0$ is chosen in order to have a solution whose wave pattern is given by Figure 3.8: a centered rarefaction waves at $t=0$ for u which after crossing the shock of v becomes a centered compressive waves, generating a shock.

If u^- is the value of u before crossing the jump of v of size $2a$, then by Rankine-Hugoniot conditions

$$-(u^+ - u^-) = f(u^+, a) - f(u^-, -a).$$

This yields

$$u^+ = e^{1/a}\Big(\sqrt{1+2e^{-1/a}(f(u^-,-a)+u^-)} - 1\Big). \tag{3.4.5}$$

The equation for the wave with value u^+ and converging to the point $(2h, 0)$ is

$$x = e^{-1/a}u^+(t-2h),$$

while the equation for the wave u^- starting at 0 is

$$x = f_u(u^-, -a)t.$$

Since they have to meet at the same point along the line $x = h - t$, one obtains

$$e^{-1/a}u^+\big(-2f_u(u^-,-a)-1\big) = f_u(u^-,-a). \tag{3.4.6}$$

Hence, substituting (3.4.5) into the expression (3.4.6), we obtain the ODE defining $f(u,-a)$

$$f_u(u,-a) = -\frac{e^{-1/a}(e^{-1/a}u^2/2 - f(u,-a))}{1+2e^{-1/a}(e^{-1/a}u^2/2 - f(u,-a))} = \frac{1-g(u,-a)}{2g(u,-a)-1}, \tag{3.4.7}$$

where $g(u,-a) = \sqrt{1+2e^{-1/a}(f(u,-a)+u)}$. By setting $f(0,-a)=0$, we can solve this ODE obtaining a function $f(u,-a)$, in a neighborhood of $u=0$, smoothly depending on the parameter a: the explicit solution is given by

$$f(u,-a) = \frac{1}{4}e^{1/a}\Big(\sqrt{1+4e^{-1/4}u} - 1 - 2e^{-1/a}u\Big).$$

It is easy to see that $f(\cdot, a)$ is concave for $a<0$, because

$$f_{uu} = \frac{-g_u}{2g+1} < 0.$$

Finally, since $g(u,a)$ tends to 0 exponentially fast as $a \to 0$, one can also see that f is smooth across the line $v=0$.

Notice that there is shock of 2-th family starting from the point $(h,0)$. However, we can modify the initial data a little to get rid of this shock. In fact, recalling the formula (3.4.5) and letting

$$u_1 = \frac{-1+\sqrt{1+2e^{-1/a}(f(u_r,-a)+u_r)}}{e^{-1/a}}.$$

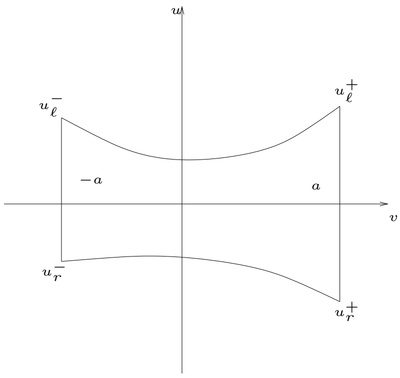

Figure 3.7:

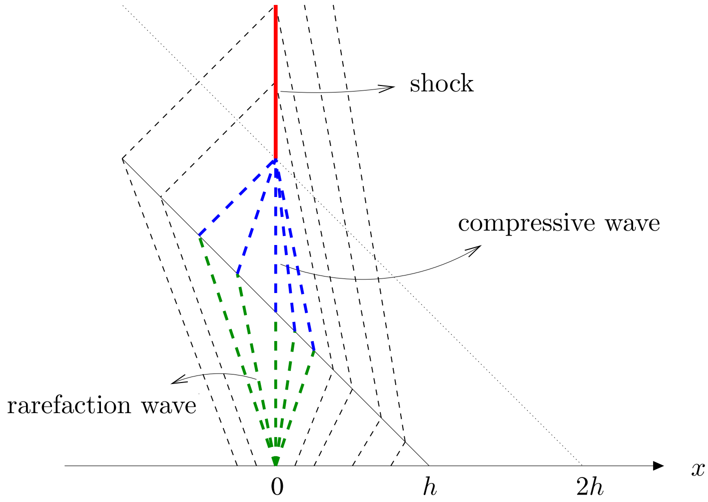

Figure 3.8:

We can replace u_0 in the initial data by

$$\tilde{u}_0 = \begin{cases} u_l & x < 0, \\ u_r & 0 < x < h, \\ u_1 & x > h. \end{cases}$$

By the total variation estimates for the general system

$$\text{Tot.Var.}\{u(t, \cdot)\} \lesssim \text{Tot.Var.}\{u_0(\cdot)\},$$

it is not restrictive to assume that the total variation of $\tilde{u}_0$ is sufficiently small.

Using this function f it is now easy to construct the example. In fact, if $\{(a_\ell, b_\ell)\}_\ell$ is a sequence of open sets in $[0, 1]$ whose complement is a Cantor set K of positive Lebesgue measure, take in fact initial data for u as

$$u(0, x) = u^- \chi_{x<0} + u^+ \chi_{x>0} + \sum_\ell u_{0,\ell} \Big(\chi_{(a_\ell, (a_\ell + b_\ell)/2)} - \chi_{(a_\ell+b_\ell)/2, b_\ell)} \Big).$$

where the sequence $\{u_{0,\ell}\}$ is chosen to get rid of extra shocks of 2-th family starting at points $(a_\ell, 0), (b_\ell, 0)$, and define

$$v(0, x) = \sum_\ell v_{0,\ell} \Big(\chi_{(a_\ell, (a_\ell + b_\ell)/2)} - \chi_{(a_\ell+b_\ell)/2, b_\ell)} \Big).$$

Then one can verify that the waves pattern is as in Figure 3.9.

Thus the times where $u(t)$ has a discontinuities are given exactly by K: the solution oscillates on the Riemann invariants of Figure 3.7.

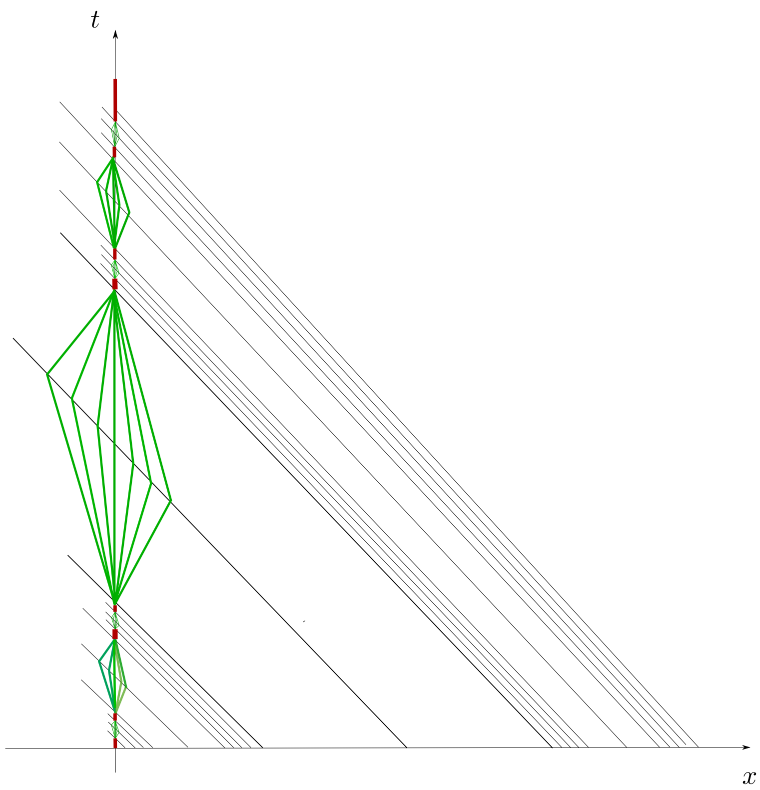

Figure 3.9:

Chapter 4

Global structure of entropy solutions to general scalar conservation law

4.1 Overview

As we discuss in the last chapter, the method developed to study the pointwise global structure of solution to piecewise genuinely nonlinear systems can not be applied to the general hyperbolic systems. Indeed, up to now, there is no corresponding result even for the general scalar conservation law. In this chapter, we prove the structural properties for the equation

$$u_t + f(u)_x = 0, \qquad u : \mathbb{R} \to \Omega \subset \mathbb{R}, \tag{4.1.1}$$

with the initial data

$$u(0, \cdot) = u_0(\cdot) \in \mathrm{BV}(\mathbb{R}). \tag{4.1.2}$$

We only assume that f is a locally Lipschitz continuous function. It is not restrictive to assume that $\|f'\|_\infty \leq 1$, i.e. the absolute value of characteristic speed is bounded by 1. It is well known that the Cauchy problem (4.1.1)-(4.1.2) admits a unique bounded entropy solution $u = u(t, x)$ defined for all $t \geq 0$, with

$$\text{Tot.Var.}\{u(t, \cdot)\} \leq \text{Tot.Var.}\{u_0\}, \qquad \|u(t, \cdot)\|_\infty \leq \|u_0\|_\infty \qquad \forall t \geq 0. \tag{4.1.3}$$

Since the entropy solution u can be constructed as the limit of front tracking approximate solutions u^ν, (see Section 1.3.2), it is also possible to study the pointwise structure by analyzing the correspond properties of the approximations. In fact, as shown in Section 1.3.3, for front tracking approximations, it is possible to characterize the approximations u^ν by the wave curves $X^\nu(\cdot, s)$. Since they are uniform Lipschitz curves, up to subsequence, for each $s \in [0, \text{Tot.Var.}\{u_0\}]$, the wave curves $X^\nu(\cdot, s)$ converge to a Lipschitz curve $X(\cdot, s)$. However, the bounded function a as the weak limit of a^ν may be no longer a monotone function take value in the set $\{1, -1, 0\}$ by the effect of cancellation of the weak convergence. Therefore, it is not a good Lagrangian formula for the solution.

Recalling the coarea formula for BV function (as shown in Section 1.2), the measure of distributional derivative $D_x u^\nu$ can be computed by the boundary curves of their level sets. Since the limit of these level sets of the approximations u^ν converge to the level set of the solution u as $\nu \to \infty$, one thus concludes the convergence of the boundary curves for the approximations. After parameterizing these curves by the parameter $s \in [0, \text{Tot.Var.}\{u_0\}]$, one obtain the Lagrangian formula for the solution such that the same formula (1.3.14) for the entropy solution of (4.1.1).

Next, we will select countable family of these curves to cover the discontinuity points of the solution and the left and right continuity of the solution along these curves.

This chapter is organized as follows. In Section 4.2, we study the properties of the level sets for the approximations. Then in Section 4.3, we show the corresponding properties of level set for the solution by passing to the limits. In Section 4.4, we parameterize the boundary curves of level sets of the solution by the parameter $s \in [0, \text{Tot.Var.}\{u_0\}]$, such that the same formula (1.3.14) holds for the entropy solution u as well. In Section 4.5, we prove the main theorem on the structural regularity of the entropy solution.

4.2 Estimates on the level sets of the front tracking approximations

First, we introduce two notations for the level set and the half closed set bounded by two Lipschitz curves.

For a fixed time $T^* \leq T$, let $\Omega = [0, T] \times \mathbb{R}$, and $M = \|u_0\|_\infty$, and we write $I := [-M, M]$. We define the *level set* at value $w \in \mathbb{R}$ of a real valued function f defined on Ω as

$$E_w(f) := \{(t, x) \in \Omega, f(t, x) \geq w\}.$$

Let $\gamma_1, \gamma_2 : [0, T^*] \to \mathbb{R}$ be two Lipschitz continuous function, we denote by $[\gamma_1, \gamma_2[$ the half closed set bounded by γ_1 and γ_2, that is

$$[\gamma_1, \gamma_2[:= \{(t, x) \in [0, T^*] \times \mathbb{R} : \gamma_1 \leq x < \gamma_2\}.$$

It is easy to see that each level sets of front tracking approximations are bounded by finitely many 1-Lipschitz curves whose initial points is at time $t = 0$. In fact, for each $w \in I$, $E_w(u^\nu)$ consists of finite connect component set, say $E_w(u^\nu) = \bigcup_i^{N_w^\nu} A_{i,w}^\nu$, where A_i is the connect component. $A_{i,w}^\nu$ could be bounded or unbounded.

Since we construct the front tracking approximations u^ν to be right continuous out of finite interaction points, one concludes that there are three cases for the connect component of level sets:

1. $A_{i,w}^\nu$ is bounded, then there are two 1-Lipschitz curves $\gamma_{2i,w}^\nu, \gamma_{2i+1,w}^\nu : [0, T^*] \to \mathbb{R}$ with some $T^* \leq T$, such that $A_{i,w}^\nu = [\gamma_{2i,w}^\nu, \gamma_{2i+1,w}^\nu[$. What is more, if $T^* < T$, then $\gamma_{2i,w}^\nu(T^*) = \gamma_{2i+1,w}^\nu(T^*)$.

2. $A^{\nu}_{i,w}$ is half bounded, then there exists an 1-Lipschitz curve $\gamma^{\nu}_{2i,w}$ or $\gamma^{\nu}_{2i+1,w} : [0,T] \to \mathbb{R}$, such that $A^{\nu}_{i,w} = [\gamma^{\nu}_{2i,w}, +\infty[:= \{(t,x) \in [0,T] \times \mathbb{R} : \gamma^{\nu}_{2i,w}(t) \leq x < \infty\}$ or $A^{\nu}_{i,w} =]-\infty, \gamma^{\nu}_{2i+1,w}[:= \{(t,x) \in [0,T^*] \times \mathbb{R} : -\infty < x < \gamma^{\nu}_{2i+1,w}(t)\}$.

3. $A^{\nu}_{i,w}$ is unbounded, that is $A^{\nu}_{i,w} = \Omega$.

4.2.1 Bounds on the initial points of the boundary curves of level sets

The approximate initial data being piecewise constant with finite jumps yields that for each value $w \in I$ and $\nu \geq 1$, $\mathcal{H}^0(\partial E_w(u^{\nu}))$ is finite. Consequently, for each $u^{\nu} \geq 1$, the topological boundary $E_w(u^{\nu})$ is made by finite many Lipschitz curves, starting from at initial time $t = 0$ where u^{ν} has jumps. By Remark.. one has that the topological boundary coincides with the reduced boundary of $E_w(u^{\nu})$, that is

$$\partial^* E_w(u^{\nu}) = \partial E_w(u^{\nu}).$$

This yields, by coarea formula (1.2.4a),

$$\int_{-\infty}^{\infty} \mathcal{H}^0(\partial E_w(u^{\nu}_0))dw = \text{Tot.Var.}\{u^{\nu}_0\}, \tag{4.2.1}$$

As Lemma 1.1.2 shows, we can choose the approximate initial data u^{ν}_0 such that

$$\text{Tot.Var.}\{u^{\nu}_0\} \to \text{Tot.Var.}\{u_0\}, \quad \text{as } \nu \to \infty,$$

and $\mathcal{H}^0(\partial E_w(u^{\nu}_0))$ is increasing with respect to ν. Then by the coarea formula

$$\int_{-\infty}^{\infty} \mathcal{H}^0(\partial^* E_w(u_0))dw = \text{Tot.Var.}\{u_0\},$$

one get, up to a subsequence $\{\nu'\}$ for a.e. $w \in I$,

$$\mathcal{H}^0(\partial E_w(u^{\nu}_0)) \to \mathcal{H}^0(\partial^* E_w(u_0)) \quad \text{as } \nu' \to \infty.$$

In particular, the number of the boundary point of the level set $E_w(u^{\nu'})$)for the approximations is uniformly bounded for a.e w by $\mathcal{H}^0(\partial^* E_w(u_0))$. From now on, we denote by u^{ν} the subsequence $u^{\nu'}$.

4.2.2 Bound estimates on the derivative of the boundary curves of level sets

We write

$$\partial E_w(u^{\nu}) = \bigcup_{i}^{N^{\nu}_w} \text{Graph}(\gamma^{\nu}_{i,w}),$$

where $\gamma^{\nu}_{i,w} : [0, T_{i,w}] \to \mathbb{R}$ are 1-Lipschitz curves and $N^{\nu}_w = \mathcal{H}^0(\partial E_w(u^{\nu}_0))$ are uniformly bounded with respect to ν for a.e. w, then by coarea formula, one obtain

$$|D_x u^{\nu}|(\Omega) = \int_I \sum_{i}^{N^{\nu}_w} \left[\int_0^{T^{\nu}_{i,w}} |\dot{\gamma}^{\nu}_{i,w}(t)|dt\right] dw, \tag{4.2.2}$$

which immediately yields the formula for the total variation of u at time $t \in]0,T[$:

$$\text{Tot.Var.}\{u(t,\cdot)\} = \int_{-\infty}^{\infty} \sum_{i}^{N_w^\nu} |\dot\gamma_{i,w}^\nu(t)| \cdot l_{i,w}^\nu(t) dw \tag{4.2.3}$$

where

$$l_{i,w}^\nu(t) = \begin{cases} 1 & \text{if } t \le T_{i,w}^\nu, \\ 0 & \text{if } t > T_{i,w}^\nu. \end{cases}$$

This implies that the decrease of the total variation of $u(t)$ as t increase is caused by the fact that the curves end at some time $T^* > 0$. This corresponds to the disappearance of a connected component of the level set.

Recall the notions of $X^\nu(t,s)$, $T^\nu(s)$ in Section 1.3.3, and the estimate

$$\int_{J^\nu} \text{Tot.Var.}\left\{\frac{\partial}{\partial t}X^\nu(\cdot,s);[0,T^\nu(s)]\right\}ds \le \mathcal{O}(1)\text{Tot.Var.}\{u_0\}^2. \tag{4.2.4}$$

Now we show the relation between the wave curves X^ν and the boundary curves $\gamma_{i,w}^\nu$ of the level sets. Define

$$w^\nu(s) := u(x^\nu(s)-) + \mathcal{S}^\nu(s)[s - U(x^\nu(s)-)], \tag{4.2.5}$$

then it is easy to see that $s \mapsto w^\nu(s)$ is well defined and for each $s \in J^\nu = [0, \text{Tot.Var.}\{u_0^\nu\}]$, there exist a unique boundary curves $\gamma_{i,w}^\nu$ of the level set $E_w(u^\nu)$ with $w = w^\nu(s)$, starting at the point $(0, x^\nu(s))$, such that

$$\gamma_{i,w}^\nu(t) = X^\nu(t,s), \qquad \text{for all } t \in [0, T^\nu(s)],$$

and $T^\nu(s) = T_{i,w}^\nu$.

On the other hand, if for some $w \in I$, there is a boundary curve $\gamma_{i,w}^\nu$ of $E_w(u^\nu)$ starting at the point $(0,\bar x)$, then letting

$$s = U(\bar x-) + \text{sgn}[u_0^\nu(\bar x) - u_0^\nu(\bar x-)](w - u_0^\nu(\bar x-)),$$

one can show that

$$X^\nu(t,s) = \gamma_{i,w}^\nu(t), \qquad \text{for all } t \in [0, T^\nu(s)].$$

Therefore, this gives an one to one map from $\{X^\nu\}$ to $\{\gamma_{i,w}^\nu\}$ and concludes, from (4.2.4), the bounded estimate of the second derivatives of the boundary curves of level sets.

$$\int_I \sum_{i}^{N_w^\nu} \left[\text{Tot.Var.}\{\dot\gamma_{i,w}^\nu : [0, T_{i,w}^\nu]\}\right] dw = \mathcal{O}(1)\text{Tot.Var.}\{u_0\}^2. \tag{4.2.6}$$

4.3 Level sets in the exact solutions

First, we give an easy lemma about the convergence of the level sets for approximations.

Lemma 4.3.1. *Suppose that a sequence of functions f_n convergence to f as $n \to \infty$ in the norm $L^1(\Omega)$, then up to a subsequence $\{f_{n'}\}$, for a.e. $w \in I$, one has*

$$E_w(f_{n'}) \to E_w(f),$$

i.e. $|E_w(f) \triangle E_w(f_{n'})| \to 0$ as $n' \to \infty$.

Proof. Write $E_n(w) := \big\{(t,x) \in \Omega : f_n(t,x) > w \geq f(t,x) \text{ or } f(t,x) > w \geq f_n(t,x)\big\}$, one can easily check that

$$|f_n - f|(t,x) = \int_{-\infty}^{\infty} \chi_{E_n(w)}(t,x)dw.$$

Then by Fubini Theorem, one has

$$\begin{aligned} & \int_\Omega |f_n - f|(t,x)dtdx \\ = & \int_\Omega \left[\int_{-\infty}^{\infty} \chi_{E_n(w)}(t,x)dw\right] dtdx \\ = & \int_{-\infty}^{\infty} \left[\int_\Omega \chi_{E_n(w)}(t,x)dtdx\right] dw. \end{aligned}$$

This yields that, up to a subsequence $\{f_{n'}\} \subset \{f_n\}$, for a.e. $w \in \mathbb{R}$,

$$\int_\Omega \chi_{E_{n'}(w)}(t,x)dtdx \longrightarrow 0 \text{ as } n' \to \infty,$$

which gives the convergence of level sets in L^1 norms. □

In particular, one has for a.e. w, $E_w(u^\nu) \to E_w(u)$ in $L^1(\Omega)$ up to a subsequence as $u^\nu \to u$ in $L^1(\Omega)$. We take this subsequence of the approximation as the sequence u^ν we consider in the following.

We also has the stability of level sets as the following lemma shows.

Lemma 4.3.2. *Suppose $f \in L^1(\Omega)$ and there a sequence $\{w_h\} \subset \mathbb{R}$ such that w_ℓ monotonely converge to $w \in \mathbb{R}$, then*

$$E_{w_h}(f) \longrightarrow E_w(f) \quad \textit{in } L^1.$$

Proof. We assume that $\{w_\ell\}$ is an increasing sequence, (the decreasing case is similar to prove). Then it is equivalent to show

$$\mathcal{L}^2\{x \in \Omega : w_h \leq f < w\} \longrightarrow 0, \quad \text{as } h \to \infty.$$

Letting $A_\ell := \{w_\ell \leq f < w_{\ell+1}\}$ and $A := \bigcup_{\ell=1}^{\infty} A_\ell = \{w_1 \leq f < w\}$, one has that $\mathcal{L}^2(A) < \infty$. Since $\{A_\ell\}$ are pair-wisely disjoint, one obtains

$$\sum_{\ell=h}^{\infty} |A_\ell| \to 0 \quad \text{as } h \to \infty.$$

So

$$\mathcal{L}^2\{x \in \Omega : w_h \leq f < w\} = \mathcal{L}^2\left\{\bigcup_{\ell=h}^{\infty} A_\ell\right\} = \sum_{\ell=h}^{\infty} |A_\ell| \to 0 \quad \text{as } h \to \infty.$$

□

Write $\mathcal{N}_0 := \{w \in \mathbb{R} : E_w(u^\nu) \nrightarrow E_w(u) \text{ as } \nu \to \infty\}$ and $\mathcal{N}_1 := \{w \in \mathbb{R} : \mathcal{H}^0(\partial^* E_w(u_0)) = \infty\}$ (which means $E_w(u_0)$ does not have finite perimeter). Let $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1$. From Lemma 4.3.1 and (4.2.1), we know that $|\mathcal{N}| = 0$.

Letting $S_N := \{w \in \mathbb{R} : \mathcal{H}^0(\partial^* E_w(u_0)) \leq N\} \setminus \mathcal{N}$, from

$$N|S_N^c| \leq \int_{S_N^c} \mathcal{H}^0(\partial^* E_w(u_0))dw \leq \text{Tot.Var.}\{u_0\},$$

one gets $|S_N^c| \leq \frac{1}{N}\text{Tot.Var.}\{u_0\}$.

Then we can choose a compact subset $K_N \subset S_N$, such that

$$|S_N \setminus K_N| \leq \frac{1}{N}\text{Tot.Var.}\{u_0\}.$$

Therefore,

$$|K_N^c| = |S_N^c \cup (S_N \setminus K_N)| \leq \frac{2}{N}\text{Tot.Var.}\{u_0\}. \tag{4.3.1}$$

Furthermore, we can assume that $\{K_N\}$ such that $K_N \subset K_{N+1}$. Let $K = \bigcup_{N=1}^{\infty} K_N$, one has $|\mathbb{R} \setminus K| = 0$. Notice that we can always let $-M \in K_N$, and $\mathbb{R} \subset \overline{K}$.

For each $w \in K$, one has

- the number of the boundary curves of the level set $E_w(u^\nu)$, which we denote by N_w^ν, is uniformly bounded by $N_w := \mathcal{H}^0(E_w(u_0))$,
- $E_w(u^\nu) \nrightarrow E_w(u)$ as $\nu \to \infty$.

It is not restrictive to assume that $N_w^\nu \equiv N_w$ is a constant in the following.

By the compactness of uniform Lipschitz continuous functions and Lemma 4.3.1, one can get a good representative in L^1-norm for the limit of the level sets of approximations ,and it holds the convergence of the boundary curves of almost of all level sets of approximations.

Proposition 4.3.3. *For a fixed $w \in K_N$, suppose $\partial E_w(u^\nu) = \bigcup_{i=1}^{N_w} Graph(\gamma_{i,w}^\nu)$, where $\gamma_{i,w}^\nu : [0, T_{i,w}^\nu] \to \mathbb{R}$ are 1-Lipschitz curves. Then $E_w(u^\nu)$ converge in L^1-norm to some set $\tilde{E}_w(u)$ which is equivalent to the level set $E_w(u)$ in L^1-norm and bounded by N_w 1-Lipschitz curves i.e.*

$$\partial \tilde{E}_w(u) = \bigcup_{i=1}^{N_w} Graph(\gamma_{i,w}),$$

where $\gamma_{i,w} : [0, T_{i,w}] \to \mathbb{R},\ i \in \{1, \cdots, N_w\}$ are 1-Lipschitz curves with $T_{i,w} \leq \lim_{\nu\to\infty} T_{i,w}^\nu$.

Furthermore, one has

$$\gamma_{i,w}^\nu \longrightarrow \gamma_{i,w} \quad \text{as } \nu \to \infty \quad \text{for } 1 \leq i \leq N_w,$$

uniformly on $[0, T_{i,w}]$.

Proof. For a fixed $w \in K_N$, suppose

$$E_w(u^\nu) = \bigcup_{i=1}^{\bar{N}_w} A_{i,w}^\nu$$

where $\bar{N}_{i,w}$ is fixed integer for all $\nu \geq 1$ and $A_{i,w}^\nu$ are the connect components of $E_w(u^\nu)$. We first consider the case when $A_{i,w}^\nu$ is bounded, say there exist two 1-Lipschitz curves $\gamma_{2i,w}^\nu, \gamma_{2i+1,w}^\nu : [0, T_{2i,w}^\nu] \to \mathbb{R}$ such that

$$A_{i,w}^\nu = [\gamma_{2i,w}^\nu, \gamma_{2i+1,w}^\nu[.$$

By the compactness of family of uniform Lipschitz continuous functions, there exist two 1-Lipschitz curves $\gamma_{2i,w}, \gamma_{2i+1,w} : [0, T^*_{2i,w}]$ such that up to a subsequence $\{\nu'\} \subset \{\nu\}$,

$$\gamma^{\nu'}_{2i,w} \to \gamma_{2i,w}, \quad \gamma^{\nu'}_{2i+1,w} \to \gamma_{2i+1,w}, \quad \text{uniformly on } [0, T^*_{2i,w}] \quad \text{as } \nu' \to \infty.$$

and

$$T^{\nu'}_{2i,w} \longrightarrow T^*_{2i,w} \quad \text{as } \nu' \to \infty. \tag{4.3.2}$$

This implies that

$$A^{\nu'}_{i,w} \longrightarrow A_{i,w} := [\gamma_{2i,w}, \gamma_{2i+1,w}[\quad \text{in } L^1\text{-norm} \quad \text{as } \nu' \to \infty. \tag{4.3.3}$$

Similarly, if $A^{\nu}_{i,w}$ is unbounded, say $A^{\nu}_{i,w} =]-\infty, \gamma^{\nu}_{2i+1,w}[$ or $A^{\nu}_{i,w} = [\gamma^{\nu}_{2i,w}, \infty[$, then there exist 1-Lipschitz curves $\gamma_{2i,w}$ or $\gamma_{2i+1,w}$ such that up to a subsequence $\nu' \subset \{\nu\}$, it holds

$$\gamma^{\nu'}_{2i,w} \to \gamma_{2i,w}, \quad \text{or } \gamma^{\nu'}_{2i+1,w} \to \gamma_{2i+1,w} \quad \text{uniformly on } [0, T^*_{2i,w}] \quad \text{as } \nu' \to \infty.$$

Then we define

$$A_{i,w} :=]-\infty, \gamma_{2i+1,w}[\quad \text{or } A_{i,w} := [\gamma_{2i,w}, \infty[,$$

and one has

$$A^{\nu'}_{i,w} \longrightarrow A_{i,w} \quad \text{in } L^1 \text{ norm} \quad \text{as } \nu' \to \infty.$$

Let $\tilde{E}_w(u) = \bigcup_{i=1}^{\bar{N}_w} A_{i,w}$, it holds

$$E_w(u^{\nu'}) \longrightarrow \tilde{E}_w(u) \quad \text{in } L^1 \text{ norm as } \nu' \to \infty, \tag{4.3.4}$$

that is

$$\int_{\mathbb{R}^2} \chi_{E_w(u^{\nu'})}(y)dy \longrightarrow \int_{\mathbb{R}^2} \chi_{\tilde{E}_w(u)}(y)dy.$$

Since by Lemma 4.3.1, we have $E_w(u^\nu) \to E_w(u)$ in $L^1(\Omega)$ for each $w \in K_N$, one has

$$E_w(u) = \tilde{E}_w(u) \quad \mathcal{L}^2\text{-a.e}, \tag{4.3.5}$$

and the convegence (4.3.4) hold for the whole sequence u^ν. Furthermore, since $E_w(u^\nu)$ and $\tilde{E}_w(u)$ are all bounded by finite 1-Lipschitz curves, one also obtain that

$$\gamma^{\nu}_{i,w} \longrightarrow \gamma_{i,w} \quad \text{as } \nu \to \infty \quad \text{for } 1 \le i \le N_w,$$

for the whole sequence, not just for a subsequence as shown before. In fact, by contradiction, if it is not true, namely, there is a subsequence $\{\nu'\}$ and $\epsilon_0 > 0$ such that, for some time t_0, one has $|\gamma^{\nu}_{i,w}(t_0) - \gamma_{i,w}(t_0)| \ge \epsilon_0$, which yields

$$\mathcal{L}^2\{E_w(u^\nu) \triangle \tilde{E}_w(u)\} \ge \mathcal{O}(1)(1)\epsilon_0^2,$$

since they are all 1-Lipschitz curves. This contradicts with L^1 convergence of $E_w(u^\nu)$ to $\tilde{E}_w(u)$.

As it may happen that in (4.3.3), $\gamma^{\nu}_{2i,w}$ and $\gamma^{\nu}_{2i+1,w}$ approach to each other more and more closed as $\nu \to \infty$, which means that there exist a time $T_{2i,w} \le T^*_{2i,w}$ such that

$$\gamma_{2i,w}(T_{2i,w}) = \gamma_{2i+1,w}(T^*_{2i,w}).$$

Then, by maximal principle of the entropy solution, one has

$$\gamma_{2i,w}(t) = \gamma_{2i+1,w}(t), \quad \text{for all } T_{2i,w} \leq t \leq T^*_{2i,w}.$$

Now we restrict the curves $\gamma_{2i,w}$ and $\gamma_{2i+1,w}$ to be defined on the time interval $[0.T_{2i,w}]$. And one still has that

$$A_{i,w} = [\gamma_{2i,w}, \gamma_{2i+1,w}[.$$

□

Remark 4.3.4. From the proof, we see that for each $w \in K$, $\tilde{E}_w(u)$ has an analogous structure with $E_w(u^\nu)$ as we discussed in Section 4.2.

As $\mathbb{R} \subset \bar{K}$, one can choose a countable set $W \subset K$ which is dense in $\mathbb{R}$. Also, we assume that $-M \in W$.

We reconstruct the solution u by defining

$$\tilde{u}(t,x) := \sup\{w \in W : (t,x) \in \tilde{E}_w(u)\}. \tag{4.3.6}$$

Since $\tilde{E}_{-M}(u) = \Omega$, one has for every $(t,x) \in \Omega$, the set $\{w \in W : (t,x) \in \tilde{E}_w(u)\} \neq \emptyset$. Thus $\tilde{u}$ is well-defined for all $(t,x) \in \Omega$.

Letting $D_w := E_w(u) \triangle \tilde{E}_w(u)$, by (4.3.5), one has $D := \bigcup_{w \in W} D_w$ is $\mathcal{L}^2$-negligible. For each $(t,x) \notin D$, one has $\tilde{u}(t,x) \leq u(t,x)$ since $\tilde{E}_w(u) \setminus D = E_w(u) \setminus D$ for each $w \in W$. If $\tilde{u}(t,x) < u(t,x)$, then there exists $\bar{w} \in W$ such that $\tilde{u}(t,x) < \bar{w} \leq u(t,x)$, which contradicts the definition of $\tilde{u}$. Thus one has

$$\tilde{u} = u \quad \text{a.e. in } \Omega.$$

For each $w \in K$, we claim that

$$\tilde{E}_w(u) = E_w(\tilde{u}).$$

In fact, fix a $\bar{w} \in K$. For each $(t,x) \in \tilde{E}_{\bar{w}}(u)$, since

$$\forall w_1, w_2 \in K, w_1 < w_2, \text{ one has } \tilde{E}_{w_2}(u) \supseteq \tilde{E}_{w_1}(u), \tag{4.3.7}$$

there exist a sequence $\{w_n\}$ increasingly converge to $\bar{w}$ such that $(t,x) \in \tilde{E}_{w_n}(u)$.

Then by the definition (4.3.6), one has $\tilde{u}(t,x) \geq \bar{w}$, this yields

$$\tilde{E}_{\bar{w}}(u) \subseteq E_{\bar{w}}(\tilde{u}).$$

For each $(t,x) \in E_{\bar{w}}(u)$, then by the definition (4.3.6), there exists a sequence $\{w_m\}$ such that

$$w_m \geq \tilde{u}(t,x) - \frac{1}{m} \geq \bar{w} - \frac{1}{m} \quad \text{and} \quad (t,x) \in \tilde{E}_{w_m}(u).$$

Then, according to (4.3.7) and Remark 4.3.4, one obtains $(t,x) \in \tilde{E}_{\bar{w}}(u)$, which implies

$$E_{\bar{w}}(\tilde{u}) \subseteq \tilde{E}_{\bar{w}}(u).$$

As there is the formula

$$\tilde{u}(t,x) = \int_0^\infty \chi_{E_w(\tilde{u})}(t,x)dw - \int_{-\infty}^0 1 - \chi_{E_w(\tilde{u})}(t,x)dw,$$

and the right continuity of $\chi_{E_w(\tilde{u})}(t,\cdot) = \chi_{\tilde{E}_w(u)}(t,\cdot)$ for all $w \in K$, one has that $\tilde{u}(t,\cdot)$ is also right continuous.

These conclude the existence of the representative of the solution with fine structure of level sets.

Theorem 4.3.5. *If u is an entropy BV solution to a scalar conservation law, then there is a L^1-representative $\tilde{u}$ of u such that, $\tilde{u}(t,\cdot)$ is right continuous for each $t \in [0,T]$ and up to a $\mathcal{L}^1$-negligible set $S \subset I$, the (reduced) boundary of each level set $E_w(\tilde{u})$ at value $w \notin S$ is made by finite many 1-Lipschitz curves.*

Recall Remark 1.2.5. Since the topological boundaries of these level set are Lipschitz continuous, they coincide with the reduced boundaries, and the inner normals coincide with the generalized inner normals $\pi_{E_w(\tilde{u})}$.

Denoting by $\pi^x_{E_w(\tilde{u})}$ the x-component of the normal vector $\pi_{E_w(\tilde{u})}$, one has

$$\pi^x_{E_w(\tilde{u})}(t,\gamma_{i,w}(t)) d\mathcal{H}^1 \llcorner \gamma_{i,w}(t) = \operatorname{sgn}\left(\pi^x_{E_w(\tilde{u})}(t,\gamma_{i,w}(t))\right) dt.$$

Then according to the coarea formula (1.2.4b), one has

$$D_x u(B) = \int_{-\infty}^{\infty} \sum_i^{N_w} \left[\int_0^{T_{i,w}} \chi_B(t,\gamma_{i,w}(t)) \operatorname{sgn}\left(\pi^x_{E_w(\tilde{u})}(t,\gamma_{i,w}(t))\right) dt \right] dw, \tag{4.3.8}$$

for each Borel subset B of Ω.

Similarly, one has, by the formula (1.2.4a),

$$|D_x u|(B) = \int_{-\infty}^{\infty} \sum_i^{N_w} \left[\int_0^{T_{i,w}} \chi_B(t,\gamma_{i,w}(t)) dt \right] dw, \tag{4.3.9}$$

for each Borel subset B of Ω.

Now, we can prove the estimate of second derivative of boundary curves by passing (4.2.6) to the limit. For any $\phi \in C_c^\infty((0,T)\times\mathbb{R})$ with $\|\phi\|_\infty \le 1$, by (4.2.6) and the definition of total variation, one has

$$\begin{aligned}
&\int_K \sum_i^{N_w} \left| \int_0^{T_{i,w}} \gamma_{i,w}(t) \frac{d^2}{dt^2}\phi(t,w) dt \right| dw \\
&= \lim_{\nu\to\infty} \int_K \sum_i^{N_w} \left| \int_0^{T^\nu_{i,w}} \gamma^\nu_{i,w}(t) \frac{d^2}{dt^2}\phi(t) dt \right| dw \\
&\le \liminf_{\nu\to\infty} \int_K \left[\sum_i^{N_w} \left| \frac{D^2}{Dt^2}\gamma^\nu_{i,w} \right| ([0,T^\nu_{i,w}]) \right] dw = \mathcal{O}(1) \text{Tot.Var.}\{u_0\}^2.
\end{aligned}$$

Therefore, by Riesz Representation Theorem, one obtain that, for a.e. $w \in \mathbb{R}$, the second distributional derivative of $\gamma_{i,w}$ is a finite Radon measure and

$$\int_K \left[\sum_i^{N_w} \left| \frac{D^2}{Dt^2}\gamma_{i,w} \right| ([0,T_{i,w}]) \right] dw = \mathcal{O}(1)(1) \text{Tot.Var.}\{u_0\}^2. \tag{4.3.10}$$

4.4 Lagrangian representative for the entropy solution

For notational simplicity, we just denote by u the representative $\tilde{u}$ in this section. We now parameterize the boundary curves $\{\gamma_{i,w}\}$ of level sets with the parameter $s \in J := [0, \text{Tot.Var.}\{u_0\}]$, in order to give a Lagrangian representative for the entropy solution u.

Recall the notation

$$U(x) := \text{Tot.Var.}\{u_0;]-\infty, x]\}.$$

For each $s \in J$, letting

$$\bar{x}(s) := \min\{x : U(x) \geq s\},$$

we choose a boundary curve with value $w(s)$ as following:

(1) If $u_0(\bar{x}(s)-) = u_0(\bar{x}(s)+)$, and $\bar{x}(s)$ is not a local strictly maximal or minimal point of u_0, let $w = u_0(\bar{x}(s))$.

(2) $u_0(\bar{x}(s)-) \neq u_0(\bar{x}(s)+)$, let

$$w(s) := s - U(\bar{x}(s)) + \ \text{sgn}(u_0(\bar{x}(s)+) - u_0(\bar{x}(s)-))u_0(\bar{x}(s)-).$$

If there is a boundary curve with value w starting from the point $(0, \bar{x}(s))$, we define it as the curve γ_s.

On the other hand, for each fixed $w \in \mathbb{R}$ and $i \in \{1, \cdots, N_w\}$, let $(0, x_{i,w})$ be the the initial point of the boundary curve $\gamma_{i,w}$, we set

$$s = U(x_{i,w}-) + |w - u_0(x_{i,w}-)|.$$

It is easy to check that $\gamma_{i,w}$ is γ_s with respect to the parameterization.

Therefore, we get almost one to one maps from $\{\gamma_s, s \in J\}$ to $\{\gamma_{i,w}, w \in \mathbb{R}, 1 \leq i \leq N_w\}$. We can also define $T(s) := T_{i,w}$ and $S(s) := \text{sgn}\left(\pi^x_{E_w(\tilde{u})}(t, \gamma_{i,w}(t))\right)$ according this map. Moreover, one has the monotonicity as shown in the following lemma:

Lemma 4.4.1. *$\forall s_1, s_2 \in J$ and $s_2 < s_2$, one has $\gamma_{s_1}(t) < \gamma_{s_2}(t)$ for $t \in [0, T_{s_1} \wedge T_{s_2}]$.*

Thus, we can define γ_s for all $s \in J$ by taking the right limit. Then we can extend γ_s for all $t \in [0, T]$.

We can find an extension function $X(t, s) : [0, T] \times J \to \mathbb{R}$ satisfying

1. for each $t \in [0, T]$, $X(t, \cdot)$ is an non-decreasing function;
2. if $X(\bar{t}, s_1) = X(\bar{t}, s_2)$ for some $\bar{t} < T$ and $s_1 \neq s_2$, then $X(t, s) = X(t, s')$ for all $t \geq \bar{t}$ and $s_1 \leq s < s' \leq s_2$;
3. $X(t, s) = \gamma_s$ for each $t \in [0, T(s)]$ and $s \in J$.

We call $X : [0, T] \to \mathbb{R}$ the wave curve function and $X(\cdot, s)$ the *wave curves* with parameter s or simply the wave s.

Moreover, since all the boundary curves of level sets are 1-Lipschitz curves, then immediately, one has for each fixed $s \in J$,

$$|X(t_1, s) - X(t_2, s)| \leq |t_1 - t_2|, \quad \forall t_1, t_2 \in [0, T(s)], \tag{4.4.1}$$

which means the absolute values of the speeds of the wave curves are bounded by 1.

In order to use X to represent the derivative of u, we need to define a cancellation time function a as following:

$$a(t,s) := \begin{cases} S(s) & t \in [0, T(s)], \\ 0 & t \in (T(s), T]. \end{cases} . \tag{4.4.2}$$

By Coarea formula, Fubini Theorem and the parameterization of the boundary curves, one has

$$\begin{aligned} &\int_\Omega \phi(t,x)|D_x u|(dt,dx), \\ =&\int_I \left(\sum_i \int_0^{T_{i,w}} \phi(\gamma_{i,w}(t,x))dt \right) dw, \\ =&\int_J \left(\int_0^T \phi(t, X(t,s))|a(t,s)|dt \right) ds, \end{aligned}$$

and

$$\begin{aligned} &\int_\Omega \phi(t,x) D_x u(dt,dx), \\ =&\int_J \left(\int_0^T \phi(t, X(t,s))a(t,s)dt \right) ds, \end{aligned}$$

In particular, one has

$$\int_{\mathbb{R}} \phi(x)|D_x u(t)|(dx) = \int_J \phi(X(t,s))|a(t,s)|ds. \tag{4.4.3}$$

and

$$\int_{\mathbb{R}} \phi(x) D_x u(t)(dx) = \int_J \phi(X(t,s))a(t,s)ds. \tag{4.4.4}$$

Before proving the main theorem, we apply the formula (4.4.3) and (4.4.4) to prove a useful oscillation estimate. First, we need an equivalent formula for (4.4.4).

Lemma 4.4.2. *For any fixed* $\bar{t} \in [0,T]$, *one has for all* $\phi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned} &\int_{\mathbb{R}} \phi(x) Du(\bar{t}, dx) \\ =&\int_J \phi(X(\bar{t},s))a(\bar{t},s)ds = \int_J \phi(X(\bar{t},s))\mathcal{S}(s)ds. \end{aligned} \tag{4.4.5}$$

Proof. Let

$$\begin{aligned} S^+ &:= \{s \in J : T(s) < \bar{t},\ \mathcal{S}(s) = 1\}, \\ S^- &:= \{s \in J : T(s) < \bar{t},\ \mathcal{S}(s) = -1\}. \end{aligned}$$

Since

$$a(t,s) = \begin{cases} 0 & \text{if } t > T(s), \\ \mathcal{S}(s) & \text{if } t \le T(s), \end{cases}$$

it suffices to prove

$$\int_{S^+} \phi(X(s))ds = \int_{S^-} \phi(X(\bar{t}, s))ds. \tag{4.4.6}$$

Recall that for each $s_1 \in S^+$, the curve $X(\cdot, s_1)$ on $[0, T(s_1)]$ is a boundary curve of a level set. From the proof of Proposition 4.3.3, one knows that the bounded component is bounded by two Lipschitz curves. Since the X is parametrized of the boundary curves of level sets, there exists $s_2 \in S^-$ such that

$$X(T(s_1), s_1) = X(T(s_2), s_2). \tag{4.4.7}$$

Vise versa, for each $s_2 \in J_2$, there exists $s_1 \in J_1$ such that (4.4.7) holds.

Therefore, one gets $|S^+| = |S^-|$ and the equality (4.4.6). This concludes the lemma. □

Definition 4.4.3. We say a curve $\Upsilon : [a, b] \to \mathbb{R}$ is *space-like* if it is of the form $\{t = \Upsilon(x) : x \in [a, b]\}$ with

$$|\Upsilon(x_2) - \Upsilon(x_1)| < x_2 - x_1 \quad \text{for all } a < x_1 < x_2 < b.$$

We denote by Tot.Var.$\{u; \Upsilon\}$ the total variation of the solution along the curve $\Upsilon : [a, b] \to \mathbb{R}$, that is

$$\sup \left\{ \sum_{i=1}^{m} |u(\Upsilon(x_i), x_i) - u(\Upsilon(x_{i-1}), x_{i-1})| : a < x_0 < x_1 < \cdot < x_m < b \right\}.$$

Lemma 4.4.4. *Suppose that* $\Upsilon : [a, b] \to \mathbb{R}$ *is a space-like curve such that* $\Upsilon(a) = \Upsilon(b) = \bar{t}$ *and* $\Upsilon(x) > \bar{t}$, $\forall x \in]a, b[$. *Then, one has*

$$\text{Tot.Var.}\{u; \Upsilon\} \leq \text{Tot.Var.}\{u(\bar{t}, \cdot);]a, b[\}. \tag{4.4.8}$$

Proof. Suppose that $a < x_0 < x_1 < \cdots < x_m < b$ are given and $t_i = \Upsilon(x_i)$.

We define the sets

$$S(t_i, x_i) = \{s \in J : X(t_i, s) \leq x_i\} \text{ for } i \in \{0, \cdots, m\}.$$

as the collection of waves that arrive at the line $\{t = t_i\} \times \{x \leq x_i\}$.

Then, for each $j \in \{1, \cdots, m\}$, by Lemma 1.1.4 and Lemma 4.4.2, one has

$$\begin{aligned} &|u(t_j, x_j) - u(t_{j-1}, x_{j-1})| \\ =&|Du(t_j,] - \infty, x_j]) - Du(t_{j-1},] - \infty, x_{j-1}]) \\ =&\left| \int_{S(t_j, x_j)} \mathcal{S}(s)ds - \int_{S(t_{j-1}, x_{j-1})} \mathcal{S}(s)ds \right|. \end{aligned} \tag{4.4.9}$$

Let $\hat{\Upsilon}$ be the piecewise affine curve connecting the points $\{(t_i, x_i)\}$ i.e.

$$\hat{\Upsilon}(x) = \frac{x_i - x}{x_i - x_{i-1}} \Upsilon(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} \Upsilon(x_i), \quad \text{for } x \in [x_{i-1}, x_i].$$

Define the set

$$S_i(\Upsilon) := \{s \in J : X(\Upsilon(x), s) = x, \ x_{i-1} < x \leq x_i\} \text{ for } i \in \{1, \cdots, m\}.$$

as the collections of all waves that passing through the segments that connecting the points (t_{i-1}, x_{i-1}) and (t_i, x_i).

Since Υ is a space-like curve, all the wave curves passing the line $\{t = t_j\} \times \{x \le x_j\}$ must come from the wave curves passing the line $\{t = t_{j-1}\} \times \{x \le x_{j-1}\}$ and the segment connecting the points (t_{j-1}, x_{j-1}) and (t_i, x_i) as shown in the Figure 4.1. This implies that $S(t_j, x_j) = S(t_{j-1}, x_{j-1}) \cup S_j(\Upsilon)$. Then, the last term of (4.4.9) equals

$$\left| \int_{S_j(\Upsilon)} \mathcal{S}(s)ds \right|$$

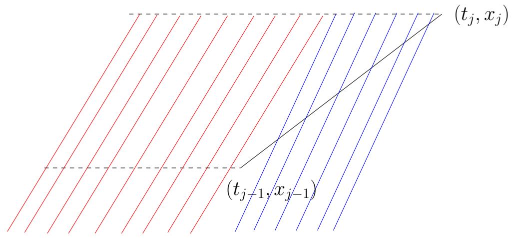

Figure 4.1:

Since $\hat{\Upsilon}$ is also a space-like curve, all wave curves passing through it must also pass through the segment $\{t = \bar{t}\} \times]a, b[$. Then, according to Lemma 4.4.2, one has

$$\begin{aligned}
\sum_{i=1}^{m} |u(t_i, x_i) - u(t_{i-1}, x_{i-1})| &= \left| \int_{\bigcup_i S_i(\Upsilon)} \mathcal{S}(s)ds \right| \\
&= \left| \int_J \chi_{]a,b[}(X(\bar{t}, s))S(s)ds \right| \\
&= \left| \int_J \chi_{]a,b[}(X(\bar{t}, s))a(\bar{t}, s)ds \right| \\
&\le Du(\bar{t},]a, b[) = \text{Tot.Var.}\{u(\bar{t}, \cdot);]a, b[\}.
\end{aligned}$$

This concludes the bounded estimates (4.4.8) by taking the points $\{(t_i, x_i)\}$ arbitrarily. □

Define the triangle $\triangle^{\eta}_{t_0,x_0}$ as

$$\triangle^{\eta}_{t_0,x_0} := \{(t, x) : t_0 < t < t_0 + \eta, x_0 - \eta + < x < x_0 + \eta - t\}.$$

Then, applying Lemma 4.4.4, one can prove the *tame oscillation property* as the following

Proposition 4.4.5. *For each $(t_0, x_0) \in]0, t[\times\mathbb{R}$ and $\eta > 0$ such that $\triangle^{\eta}_{t_0,x_0} \subset \Omega$, one has*

$$\sup \left\{ |u(t, x) - u(t', x')| : \ (t, x), \ (t', x') \in \triangle^{\eta}_{t_0,x_0} \right\} \le 2\,Tot.\,Var.\{u(t_0, \cdot);]x_0 - \eta, x_0 + \eta[\}.$$

Proof. For each couple of points (t_1, x_1) and (t_2, x_2) in $\triangle^{\eta}_{t_0,x_0}$, we construct the space-like curves $\Upsilon_1 : [x_1, x_0 + \eta] \to \mathbb{R}$, $\Upsilon_2 : [x_2, x_0 + \eta] \to \mathbb{R}$ by

$$\Upsilon_1(x) := \max\{t_1 - x + x_1, t_0\}, \qquad \Upsilon_2(x) := \max\{t_2 - x + x_2, t_0\}.$$

Since $\Upsilon_1(x) = \Upsilon_2(x)$ for all x sufficiently close to $x_0 + \eta$, we have, by Lemma 4.4.4,

$$\begin{aligned}|u(t_1, x_1) - u(t_2, x_2)| &\leq \text{Tot.Var.}\{u : \Upsilon_1\} + \text{Tot.Var.}\{u; \Upsilon_2\} \\ &\leq 2\text{Tot.Var.}\{u(t_0, \cdot);]x_0 - \eta, x_0 + \eta[\}.\end{aligned}$$

□

Remark 4.4.6. A similar tame oscillation property can be prove for the system case, here we use the small idea of proof for Theorem 9.3 in [19], which works for the admissible solution obtained as the limit of front tracking approximations. The main difference is that in [19], the author first proves the corresponding result of Lemma 4.4.4 for the approximates, then the bounded estimate for the admissible solutions is obtained by passing to the limit.

For any subset $A \subset \Omega$, we define the collection of waves that cancelled in the set A as

$$\text{Cancel}(A) = \{s \in J : \ (T(s), X(T(s), s)) \in A\}.$$

We give a total variation estimates for the space-like curve below some fixed time, where the amount of cancellation should be taken into account.

Lemma 4.4.7. *Let $\Upsilon : [a, b] \to \mathbb{R}$ be a space-like curves such that $\Upsilon(a) = \Upsilon(v) = \bar{t}$ and $\Upsilon(x) < x, \ \forall x \in]a, b[$ and denote by $\hat{\Gamma}$ be the region bounded by the curve Υ and $\{t = \bar{t}\}$, namely,*

$$\{(t, x) : \ a < x < b, \ \Upsilon(x) < t \leq \bar{t}\}.$$

Then, we have the estimate

$$\textit{Tot.Var.}\{u; \Upsilon\} - |\textit{Cancel}(\hat{\Gamma})| \leq \textit{Tot.Var.}\{u(\bar{t}, \cdot);]a, b[\}. \tag{4.4.10}$$

Proof. Notice that all the wave curves passing through Υ must arrive at the segment $\{t = \bar{t}\} \times]a, b[$. Since the waves cancelled in the region $\hat{\Gamma}$ should not taken into account of the total variation of $u(t, \cdot)$ on $]a, b[$. By using the similar argument in the proof for Lemma 4.4.4, we conclude the estimate (4.4.10). □

4.5 Pointwise structure

Recalling the estimate (4.3.10), we have that for a.e. $s \in J$, $\frac{D^2}{Dt^2} X(\cdot, s)$ is a finite Radon measure on $[0, T(s)]$. Define the measure μ by

$$\mu(B) = \int_J \left(\int_0^T \chi_B(t.s) |a(t, s)| \left| \frac{D^2}{Dt^2} X(dt, s) \right| \right) ds, \quad \text{for each Borel set } B \subset [0, T] \times J,$$

and the measure of interaction as the push forward measure

$$\hat{\mu} = X_\sharp \mu.$$

We define the set of *interaction points* Θ_1 as the collection of all atom point of $\hat{\mu}$. Since $\hat{\mu}$ is finite measure, one has Θ_1 is at most countable.

We say a point $(t,x)\in\Omega$ is a *cancellation point* if the set

$$\mathcal{T}(s) := \{s \in J : X(t,s) = t \text{ and } T(s) = t\} \tag{4.5.1}$$

has positive $\mathcal{L}^1$ measure. We denote by Θ_2 the set of all cancellation points in Ω. Since the entropy solution u is a BV function on Ω, by the formula (4.4.4), one concludes that Θ_2 is at most countable.

Letting $\Theta := \Theta_1 \cup \Theta_2$, we now prove our main theorem about the pointwise structure of entropy solutions.

Theorem 4.5.1. *For the representative of solution $\tilde{u}$, there exist a countable family of graph of Lipschitz curves $\Gamma := \{Graph(\gamma_i)\}$, such that Γ cover the discontinuities of $\tilde{u}$. The speeds of the wave curves equal to the characteristic speed at the continuity points and equal to the speed of the jump at the jump points, that is*

$$\frac{\partial}{\partial t}X(t,s) = \begin{cases} f'(u(t,X(t,s)) & \text{if } (t,X(t,s)) \text{ is a continuity point of } u, \\ \frac{f(u(t,X(t,s)+))-f(u(t,X(t,s)-))}{u(t,X(t,s)+)-u(t,X(t,s)-)} & \text{if } (t,X(t,s)) \text{ is a jump point of } u. \end{cases}$$

Proof. For notational simplicity, we just denote by u the representative $\tilde{u}$ in the following. Choose a countable dense set $D \subset J$, define $\Gamma := \bigcup_{s\in D} \mathrm{Graph}(X(\cdot,s))$.

Step 1. We claim that outside Γ, u is continuous.

By contradiction, if it is not true, we suppose that there is a point $P=(\tau,\xi)\notin\Gamma$ such that u is discontinuous at P. Letting

$$u^{\mathrm{L}} := u(\tau,\xi-), \quad u^{\mathrm{R}} := u(\tau,\xi+),$$

we claim that $|u^{\mathrm{L}} - u^{\mathrm{R}}| > 0$.

By contradiction, suppose that $u^{\mathrm{L}} = u^{\mathrm{R}}$, then by Lemma 1.1.3, one has

$$\text{Tot.Var.}\{u,(\xi-\epsilon,\xi+\epsilon)\} \to 0 \text{ as } \epsilon \to 0. \tag{4.5.2}$$

Define the oscillation of u over $\triangle^{\epsilon}_{\tau,\xi}$ as

$$\text{Osc.}\{u;\triangle^{\epsilon}_{\tau,\xi}\} := \sup\left\{|u(t,x)-u(t',x')| : \ (t,x),\ (t',x') \in \triangle^{\epsilon}_{\tau,\xi}\right\}.$$

By Proposition 4.4.5 we have

$$\text{Osc.}\{u;\triangle^{\epsilon}_{\tau,\xi}\} = \mathcal{O}(1)\text{Tot.Var.}\{u,]\xi-\epsilon,\xi+\epsilon[\}. \tag{4.5.3}$$

On the other hand, consider the triangle

$$\nabla^{\epsilon}_{t_0,x_0} := \{(t,x) : \tau-\epsilon < t < \tau, \xi-\epsilon-(t-\tau) < x < \xi+\epsilon+(t-\tau)\}.$$

By Lemma 4.4.7 and the argument in the proof for Proposition 4.4.5, one has

$$\text{Osc.}\{u;\nabla^{\epsilon}_{t_0,x_0}\} - |\text{Cancel}(\nabla^{\epsilon}_{t_0,x_0})| \le \text{Tot.Var.}\{u(\bar{t},\cdot);]\xi-\epsilon,\xi+\epsilon[\}. \tag{4.5.4}$$

If $\text{Osc.}\{u;\nabla^{\epsilon}_{t_0,x_0}\}$ do not tend to zero, the cancellation will be uniformly large in any small neigbourhood of the point (τ,ξ) with contradiction to $P\notin\Theta$.

Therefore, combining (4.5.3) and (4.5.4), one obtain that P is a continuous point, which contradicts the assumption that u is discontinuous at P. This concludes our claim that $u^{\mathrm{L}} \neq u^{\mathrm{R}}$.

We define the set

$$A_{\tau,\xi} := \{s \in J : X(\tau, s) = \xi\}.$$

Then, from the formula (4.4.4), one has

$$\mathcal{L}^1(A_{\tau,\xi}) > 0,$$

which implies that there exists $s_1 < s_2$ such that

$$X(\tau, s_1) = X(\tau, s_2) = \xi.$$

Therefore, $\forall s \in (s_1, s_2)$, one has $X(\tau, s) = \xi$. In particular there is a curve in Γ passing though P, which contradicts the fact that $P \notin \Gamma$.

Step 2. Now, we compute the speed of the wave curves. First we define the generalized characteristic speed function as

$$\tilde{\lambda}(t,x) = \begin{cases} f'(u(t,x)) & \text{if } (t,x) \text{ is a continuity point of } u, \\ \frac{f(u(t,x+))-f(u(t,x-))}{u(t,x+)-u(t,x-)} & \text{if } (t,x) \text{ is a jump point of } u. \end{cases} \tag{4.5.5}$$

As shown in Step 1, for each $(t,x) \notin \Theta$, (t,x) is either a continuity point or a jump point of u. Therefore, $\tilde{\lambda}$ is well defined for all $(t,x) \notin \Theta$.

By conservation law, chain rule and Rankine-Hugniot Relations, the measure $D_x u$ satisfies the continuity equation

$$(D_x u)_t + (\tilde{\lambda} D_x u)_x = 0, \tag{4.5.6}$$

in the sense of distribution, that is for all $\phi \in C_c(\Omega^\circ)$,

$$\int_\Omega (\phi_t + \tilde{\lambda}\phi_x)(t,x) D_x u(dtdx) = 0. \tag{4.5.7}$$

According to the formula (4.4.4), we rewrite (4.5.7) as

$$\int_J \int_0^T \phi_t(t, X(t,s)) + \tilde{\lambda}(t, X(t,s))\phi_x(t, X(t,s))a(t,s)dtds = 0. \tag{4.5.8}$$

Notice that for a.e. $s \in J$, $X(t,s)$ is a Lipschitz continuous function with respect to $t \in [0, T(s)]$, $\frac{\partial}{\partial t}X(t,s)$ exists for a.e. $t \in [0, T(s)]$, one obtains that (4.5.8) is equivalent to

$$\int_J \int_0^T \left[\frac{d}{dt}\phi(t, X(t,s)) + \left(\tilde{\lambda}(t, X(t,s)) - \frac{\partial}{\partial t}X(t,s)\right)\phi_x(t, X(t,s))\right] a(t,s)dtds = 0. \tag{4.5.9}$$

We claim that

$$\int_J \int_0^T \frac{d}{dt}\phi(t, X(t,s))a(t,s)dtds = 0. \tag{4.5.10}$$

In fact, by the definition of the function a in (4.4.2), the left of (4.5.10) equals

$$\begin{aligned} &\int_J \int_0^{T(s)} \frac{d}{dt}\phi(t, X(t,s))a(t,s)dtds \\ =& \int_J [\phi(T(s), X(T(s),s)) - \phi(0, X(0,s))]a(T(s),s)ds. \end{aligned} \tag{4.5.11}$$

Let

$$J_1 = \{s \in J : \mathcal{S}(s) = 1, T(s) < T\},$$
$$J_2 = \{s \in J : \mathcal{S}(s) = -1, T(s) < T\}.$$

Since $\phi(0, X(0,s)) = \phi(T, X(T,s)) = 0$, the integral (4.5.11) turns out to be

$$\int_{J_1} \phi(T(s), X(T(s),s))ds - \int_{J_2} \phi(T(s), X(T(s),s))ds. \tag{4.5.12}$$

By the same argument in the proof for Lemma 4.4.2, one get that (4.5.11) is zero. Thus, (4.5.9) implies that for all $\phi \in C_c(\Omega^\circ)$, one has

$$\int_J \int_0^T (\tilde{\lambda}(t, X(t,s)) - \frac{\partial}{\partial t}X(t,s))\phi_x(t, X(t,s))a(t,s)dtds = 0. \tag{4.5.13}$$

For a fixed point $(\tau, \eta) \notin \Theta$, define the set

$$\mathcal{S}_{\tau,\eta} := \{s \in J : X(\tau, s) = \eta\}. \tag{4.5.14}$$

By monotonicity of $X(t,\cdot)$, $\mathcal{S}_{\tau,\eta}$ must be a closed interval (or a single point), namely,

$$\mathcal{S}_{\tau,\eta} = [s_1, s_2] \quad \text{(res. } s_1 = s_2).$$

We define a family of rectangles

$$H_\epsilon(\tau,\eta) := [\tau - \epsilon, \tau + \epsilon] \times [s_\epsilon^-, s_\epsilon^+], \tag{4.5.15}$$

where

$$s_\epsilon^- := \min\{s \in J : X(\tau, s) = X(\tau, s - \epsilon)\},$$
$$s_\epsilon^+ := \max\{s \in J : X(\tau, s) = X(\tau, s + \epsilon)\}.$$

By density, there exists a sequence of smooth function $\{\phi_\ell\} \subset C_c^\infty(\Omega^\circ)$, such that

$$\phi_\ell(t, X(t,s)) \longrightarrow \chi_{H_\epsilon(\tau,\eta)}(t,s) \text{ pointwisely as } \ell \to \infty.$$

Then we get from (4.5.13) that

$$\begin{aligned} 0 =& \int_J \int_0^T \left[\tilde{\lambda}(t, X(t,s)) - \frac{\partial}{\partial t}X(t,s)\right] a(t,s)\chi_{H_\epsilon(\tau,\eta)}(t-\tau, s-\eta)dtds \\ =& \frac{1}{H_\epsilon(t,s)} \int_{H_\epsilon} \left[\tilde{\lambda}(t, X(t,s)) - \frac{\partial}{\partial t}X(t,s)\right] a(t,s)dtds \end{aligned} \tag{4.5.16}$$

Let $\epsilon \to 0$, there are two cases.

1. If $s_1 = s_2$, then for a.e. $(t,x) \in A(t,x) \setminus \Theta$, one has

$$\tilde{\lambda}(t, X(t,s)) = \frac{\partial}{\partial t}X(t,s) \text{ for } t \in [0, T(s)].$$

2. If $s_1 = s_2$, since (τ, η) is not a cancellation point, one has

- there exist a positive measure set $A \subset [s_1, s_2]$ such that

$$|A| = |u(\tau, \eta+) - u(\tau, \eta-)| \text{ and } a(t,s) = \mathrm{sgn}(u(\tau, \eta+) - u(\tau, \eta-)) \ \forall s \in A.$$

- the $B := \{s \in J : a(t,s) = -\mathrm{sgn}(u(\tau, \eta+) - u(\tau, \eta-))\}$ is $\mathcal{L}^1$-negligible.

We claim that for a.e. $s \in [s_1, s_2]$, $\tilde{\lambda}(t, X(\tau, s)) = \frac{\partial}{\partial t} X(\tau, s)$.

Otherwise, as $\tilde{\lambda}(\tau, X(\tau, s)) \equiv$ constant for all $s \in [s_1, s_2]$, if there exist two set S^-, S^+ with $|S^-| = |S^+|$, such that

$$\int_{S^-} \frac{\partial}{\partial t} X(t,s) ds = - \int_{S^+} \frac{\partial}{\partial t} X(t,s) ds.$$

This contradicts with the assumption that (τ, η) is not an interaction point.

□

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