Four-dimensional $\mathcal{N} = 2$ superconformal quantum field theories and BPS-quivers

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Alla cara memoria
di Francesco Caracciolo

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Abstract

Supersymmetric $\mathcal{N} = 2$ theories in four dimensions are an interesting laboratory to understand Quantum Field Theory at strong coupling. In these theories many interesting physical quantities are protected by supersymmetry, and hence exactly calculable. One of the most remarkable aspects of extended supersymmetry is the possibility of constructing and studying in detail many four–dimensional SCFTs which do not have any (weakly coupled) Lagrangian formulation and hence are intrinsically strongly coupled. These strongly interacting $\mathcal{N} = 2$ systems can be thought as the basic building blocks of the more general models, providing a generalization of the concept of 'matter' in the non–Lagrangian setting. The prototype of such systems is given by the Argyres–Douglas (AD) $\mathcal{N} = 2$ models, which have an ADE classification; in particular, those of type D_p ($p = 2, 3, \cdots$) have a $SU(2)$ global flavor symmetry that can be gauged. A first generalization of AD models was given by Cecotti–Neitzke–Vafa with the (G, G') SCFT's labeled by pairs out the ADE series. AD models belongs to this class: AD theories are the models of type (G, A_1). The purpose of this thesis is to generalize AD models outside the (G, G') class and to study the properties of the non–lagrangian $\mathcal{N} = 2$ SCFTs that results from such generalization. In particular, we will show that the BPS data are enough to completely characterize a variety of such systems, provided they have the BPS–quiver property. We will propose two different generalizations. The first is that of Arnol'd–models. These models are constructed by geometric engineering of the type IIB superstrings on singular Calabi–Yau hypersurfaces of $\mathbb{C}^4$ obtained as the zero–locus of a quasi–homogenous element out of Arnol'd singularities lists. AD models corresponds to the simple (or zero–modal or minimal) Arnol'd singularities. In between the other Arnol'd models (*i.e.* models associated to singularities with higher modality) there are many elements that are not of (G, G') type, and, in particular, we have examples of models that do not belong to class $\mathcal{S}$. The second and more interesting generalization brought us to the definition of $D_p(G)$ models ($p = 2, 3, \cdots$) that generalize the D_p Argyres–Douglas systems to (typically non–lagrangian) SCFTs with flavor group (at least) G. We construct $D_p(G)$ systems for all simple simply–laced Lie groups G and all $p \geq 2$. Gauging the

G flavor symmetries, these systems contributes to the Yang–Mills beta function as $(p-1)/2p$ adjoint hypermultiplets: only very special elements in each infinite $D_p(G)$ class admit a lagrangian formulation. The construction of $D_p(G)$ models is made rigorous only via BPS–quivers using the categorial methods obtained by Cecotti in his masterpiece about Categorical Tinkertoys. In particular, we provide infinitely many examples of SCFTs with exceptional flavor groups. The $D_p(G)$ classification unveils Lie algebraic number theoretical aspects of many of the properties (flavor numbers, beta functions, superconformal central charges, ...) of the SCFTs in this class. Motivated by the study of these systems, moreover, a new perspective on BPS–quivers emerged, that of `meta`–quivers, that opens the doors to wider generalizations: The `meta`–quiver approach makes the non–perturbative completion possible via an application of homological mirror symmetry.

Contents

Brief introduction and overview

During the last two decades, our perspective about Quantum Field Theory (QFT) has changed quite radically, *e.g.* the simplest nontrivial QFT in four dimensions is no more believed to be the theory of a scalar field with a φ^4 interaction one finds in chapter one of all field theory textbooks, but a $\mathcal{N}=4$ super-Yang-Mills (SYM) model [1]. Wishing to understand a Theory, it is better to start from the simplest examples at hand, and the requirement of space–time supersymmetry (SUSY) is now well–understood to select for us the simplest QFT's: SUSY quantum field theories are the most interesting laboratory to understand Quantum Field Theory at strong coupling. There are various tools of analysis that makes SUSY field theories more accessible than other ordinary QFT's. Schematically these are: the interrelationships between SUSY and the geometry of target manifolds; the (super)string various different engineering, either in terms of branes, or directly from the internal geometry; the relations with different combinatoric objects;[1] the relations with integrable systems; and a quiver description.[2] In this thesis we are going to see manifestations of all of these aspects in the context of four–dimensional $\mathcal{N}=2$ models.

For 4d $\mathcal{N}=2$ supersymmetric theories we have at our disposal many powerful techniques to address nonperturbative questions [2–6]. An especially simple and elegant method is based on BPS–quivers [7, 8], which, for the convenience of the reader, we will review in chapter 1 together with its (combinatorial) consequences. In a sense, BPS–quivers behave like generalized Lagrangians for these systems: one has graphical rûles to gauge a flavor symmetry, to perform a Higgs decoupling limit, to integrate out some massive subsector... and these structures carry over to intrinsically strongly coupled theories, provided they have a BPS–quiver description, giving an interesting handle about them. Notice that, usually, in QFT's we define a model in terms of its UV data, while BPS–quiver theories

[1] *e.g.* 5d $\mathcal{N}=1$ theories and (p,q)–webs; 4d $\mathcal{N}=2$ and pants decompositions of Riemann surfaces; 4d $\mathcal{N}=1$ and bipartite graphs; 3d $\mathcal{N}=2$ and decompositions of 3d manifolds in 3-simplexes, ...

[2] As far as this last point is concerned, there are many possible ways to encode information about a model in a quiver: for example, the spectrum of BPS–solitons of a 2d $\mathcal{N}=(2,2)$ model is encoded in a quiver, or a given representation of the quiver can be used to define the particle and gauge content of a theory (*e.g.*, 4d $\mathcal{N}=1$ quiver theories), or one constructs suitable maps from the set of all possible representations to the set of BPS–states of the model, as we will do in the following chapter for 4d $\mathcal{N}=2$ theories.

are completely characterized by the infrared BPS–data. This is a very interesting change of perspective about the physics of these models. However, a review of the relation in between the BPS–quiver formalism and the ordinary formulation of $\mathcal{N} = 2$ QFT's is out of the scope of the present dissertation: we refer the interested reader to [8, 10–12]. One of the most important consequences of the knowledge of the BPS–quiver of the theory is that it allows to reduce the BPS spectral–problem for models admitting finite chambers of hypermultiplets into the combinatorics of (quantum) cluster algebras. The BPS–spectrum in all chambers is then captured by a very interesting wall–crossing invariant, the quantum monodromy [13], that can be explicitly computed by the mutation–method.

The preliminary step in this approach is to determine the BPS–quiver class associated to the $\mathcal{N} = 2$ quiver theory of interest. For 4d $\mathcal{N} = 2$ theories that have a Type IIB engineering the quiver class is determined by the 2d/4d correspondence: the 4d $\mathcal{N} = 2$ BPS–quiver, is the 2d $\mathcal{N} = (2, 2)$ BPS–quiver of a parent 2d system with $\hat{c} < 2$, that, in principle, is what residues the 3–CY geometry after the decoupling of gravity (*i.e.* in the large volume limit) [7,9,13]. The BPS–quiver class includes (and explains) many interesting 4d $\mathcal{N} = 2$ SUSY QFTs, but a criterion is not known (yet) to understand whether a BPS–quiver exist for a theory of class $\mathcal{S}$:[3] it is believed, however, that a sufficient condition would be that the theory admits *non* exactly–marginal deformations.

In the first chapter of this thesis we introduce the reader to the BPS–quiver theory of [8]. We start by a careful definition of the BPS–spectral problem, with all the subtleties that are related to BPS–particles (wall–crossing phenomenon, quantum Schottky problem...). We then introduce BPS–quivers, define their representations, and reformulate the BPS–spectral problem in terms of representation theory of quivers with (super)potentials, and discuss its solution via the mutation method. In particular, in view of its applications, we will discuss a particular property of finte BPS–spectra, the Coxeter–factorization property: for a Coxeter–factorized BPS–chamber the charge lattice splits naturally in a direct sum of root lattices of simply laced Lie algebras [14].

It has become clear that most of the 4d $\mathcal{N} = 2$ theories do not admit any weakly–coupled lagrangian description. Typically, these systems are interacting quantum systems obtained by weakly gauging the flavor symmetries of strongly interacting superconformal subsystems. The problem of classification of 4d $\mathcal{N} = 2$ models is therefore reformulated as a problem of classification of these superconformal strongly interacting 'building blocks', the tinkertoys [15, 16]. This structure has its own algebraic counterpart: the categorical tinkertoys of [10], that

[3] Recall that a theory is in class $\mathcal{S}[\mathfrak{g}, \Sigma]$ iff it is obtained by the compactification of a 6d $\mathcal{N} = (2, 0)$ theory of type $\mathfrak{g} \in ADE$ on a Riemann surface Σ.

allows for a reformulation of the classification program in a purely categorified representation–theoretical spirit.

The rest of this thesis is devoted to the applications of the BPS–quivers method to the study of strongly interacting 4d $\mathcal{N} = 2$ SCFTs. The prototypical example is that of Argyres–Douglas models [17, 18] obtained by Type IIB geometric engineering on singular Calabi–Yau hypersurfaces of $\mathbb{C}^4$

$$W_G(x, y) + u^2 + v^2 = 0$$

i.e. the zero locus of minimal singularities, which have an ADE classification. This classification can be obtained from the BPS–quiver property, by declaring that a model is of Argyres–Douglas type iff it has only finite BPS–chambers consisting only of hypermultiplets.[4] Argyres–Douglas models of type D_p ($p = 2, 3, \dots$) have an $SU(2)$ flavor symmetry that can be gauged.

A larger family of 4d $\mathcal{N} = 2$ SCFTs was obtained in [13] as part of the family of 'direct sum' SCFTs of type (G, G'). These are obtained by Type IIB engineering on $W_G(x, y) + W_{G'}(u, v) = 0$ where both the polynomials are minimal singularities. Notice that since a minimal singularity has $\hat{c}_G < 1$ the direct sum always has $\hat{c} < 2$, and we obtain well defined four–dimensional models. We propose to search for 4d $\mathcal{N} = 2$ SCFTs that does not belong to the (G, G') class.

The second chapter have appeared in the pubblications [14, 20]. There we introduce Arnol'd models. These are superconformal systems engineered by type IIB superstrings on local Calabi–Yau–hypersurfaces of $\mathbb{C}^4$ obtained out of the quasi–homogenous elements of Arnol'd singularities lists. Singularities undergo different possible classifications. A very interesting one is with respect to increasing *modality*. By 2d/4d correspondence, the modality of a singularity is interpreted as the number of marginal and irrelevant IR chiral operators of the parent 2d $\mathcal{N} = (2, 2)$ $\hat{c} < 2$ system, *i.e.* it is the number of primary operators with dimension $q \geq \hat{c}$. Theories with zero modality correspond to 2d $\mathcal{N} = (2, 2)$ LG minimal models: these are precisely the simple (or minimal) singularities. The associated 4d models are precisely the Argyres–Douglas SCFTs. Increasing in modality we have the unimodal and bi–modal singularities, that are fully classified. In between unimodal and bimodal Arnol'd models there are examples that are not of the (G, G') form. Moreover, for these models, we are able to handle the Quantum Shottky problem with 2d renormalization group. These models are superconformal and non–lagrangian: all the dimensions of the chiral primary operators are of the form $\mathbb{N}/\ell$ for some integer $\ell > 1$. As we are going to show, if this is the

[4] This is a theorem by Gabriel, but we find reference [19] more appropriate.

case, the quantum monodromy operator is expected to be periodic of period ℓ. By the correspondence in between cluster algebras and Y–systems, we predict the existence of *new* periodic Y–systems, and this establishes a correspondence in between periodic TBA integrable Y–systems and non–lagrangian 4d $\mathcal{N} = 2$ SCFTs. We have substantiated this claim with *explicit numerical* checks.

In the third chapter we are going to construct and discuss all the properties of the infinite class of $D_p(G)$ models. These are models labeled by a positive integer ($p = 2, 3, \dots$) that generalize Argyres–Douglas D_p systems to SCFT's with arbitrary simple simply–laced flavor group G (more precisely, these models have flavor symmetry *at least* G: as we will see, there are enhancements). The models of type $D_p(SU(2))$ are precisely the D_p AD theories. In order to construct such families of SCFT's we will have to use the machinery of light subcategories that we will carefully introduce. We are going also to give a long review of the properties of the 4d $\mathcal{N} = 2$ affine models in terms of the representation theory of Euclidean algebras. The properties of the latter will be of fundamental importance for the construction of the $D_p(G)$ systems. Our strategy will be to construct, by 2d/4d correspondence, the quivers with superpotential $\widehat{H} \boxtimes G$. We will then show that these models have a canonical S–duality frame in which are represented by a G SYM sector weakly gauging the G flavor symmetry of some $D_p(G)$ systems. We will study the limit $g_{YM} \to 0$ from the view point of the category of BPS–particles and we will obtain the quiver with superpotential for the corresponding matter system. The construction holds for all G simple and simply–laced.[5] This construction involves the use of the categorification of SUSY QFT. A new perspective emerges, that of `meta`–quivers. As we will discuss, `meta`–quivers make the non–perturbative completion possible by an application of homological mirror symmetry. This is, perhaps, one of the most interesting applications of this formalism. Once we have constructed the models, we move on to the computation of their invariants. By a combination of representation–theoretical methods and 2d/4d correspondence, we are able to compute all invariants one needs to completely characterize the models: beta function of the G SYM sector, flavor charges, superconformal central charges, order of the quantum monodromy... Moreover, we identify the Lagrangian elements in the class, and this gives us very interesting checks of the computations we have made. In particular, we are able to show that all the models of type $\widehat{H} \boxtimes G$ admit Coxeter–factorized BPS–chambers. In the last part of the chapter the $D_2(G)$ models are studied in details. In particular, we obtain the quivers for all Minahan–Nemeshansky theories from their classification. Most of what written in the third chapter of this thesis have appeared in [21, 22].

In the last chapter of this thesis, as an application of the $D_p(G)$ classification,

[5] The author is currently working about the simple non–simply–laced generalization.

we discuss the properties of the rank 1 4d $\mathcal{N} = 2$ SCFTs that are engineered by F–theory on Kodaira singular fibers. This chapter have appeared in [23].

Our results are a manifestation of the incredible power of the representation–theoretical cluster–algebraic approach to 4d $\mathcal{N} = 2$ theories that follows from the BPS–quiver property. Clearly, a lot of work remains to be done in this field of research. We have the feeling that what we have obtained is just the tip of the iceberg that can eventually lead to the classification of 4d $\mathcal{N} = 2$ SCFTs.

Chapter 1

Quivers, BPS–states, and all that

1.1. BPS-states over the Coulomb branch

1.1.1. The Coulomb branch. Consider a 4d $\mathcal{N} = 2$ theory with gauge symmetry group G of rank r. The moduli space of vacua of such theory is divided in branches, that are labeled Coulomb, Higgs or mixed-Higgs, according to the infrared phase they describe: The model is in a Higgs phase if it has a mass gap, it is in a Coulomb phase if it has a $U(1)^r$ abelian gauge interaction mediated by r massless photons, and it is in a mixed-Higgs phase if it has a Coulomb sub–sector with $U(1)^{r-k}$ massless photons with $0 < k < r$. This fact is mirrored by the geometry of the moduli: a Coulomb branch is a rigid special kähler manifold, a Higgs branch is a hyperkähler manifold, while a mixed-Higgs branch is a direct product of a rigid special kähler manifold times a hyperkähler one.

From now on, we will focus on the study of the Coulomb branch of the moduli space. In addition, we will assume that, for generic values of the mass deformations, the system we are describing has a $U(1)^f$ global flavor symmetry. Notice that, for special values of the Coulomb moduli and of the mass deformations, both the gauge and the flavor symmetries can enhance to bigger non-abelian groups (typically in presence of orbifold singularities on the moduli).

1.1.2. The charge lattice and the quantum torus. At a generic point of the Coulomb branch the internal degrees of freedom of the $\mathcal{N} = 2$ multiplets of particles of the system are r electric, r magnetic, and f flavor charges, that are conserved and quantized.[1] Quantization of the charges implies that they are valued in an integer lattice of rank $D = 2r + f$. Such a lattice is called *the charge lattice* of the theory, and denoted $\Gamma \simeq \mathbb{Z}^D$. The elements of the charge lattice are called *charge vectors*. PCT–symmetry acts as an involution on Γ: if a charge $\gamma \in \Gamma$ belongs to the spectrum, so does $-\gamma$. Since we have an abelian $U(1)^r$ theory, we have that electric and magnetic charges obey the Dirac quantization condition, *i.e.*

[1] In our conventions the external degrees of freedom of a multiplet are the ones associated with 4d $\mathcal{N} = 2$ supersymmetry: the spin, the mass and the $U(2)_R$ charges.

there exist an antisymmetric integral pairing

$$\langle -, - \rangle_{\text{Dirac}} \colon \Gamma \longrightarrow \mathbb{Z} \tag{1.1}$$

that makes Γ into a symplectic lattice.[2] From the definition of the Dirac pairing it follows a natural characterization of the flavor charges of the system: A charge $\gamma_f \in \Gamma$ is a flavor charge if and only if it belongs to the radical of the Dirac pairing:

$$\gamma \in \Gamma \text{ is flavor } \Longleftrightarrow \gamma \in \text{rad } \langle -, - \rangle_{\text{Dirac}} \tag{1.2}$$

This statement is at this moment just a tautology, but in the next chapters we will see how to use it to compute the rank of the flavor symmetry group of 4d $\mathcal{N} = 2$ non–lagrangian SCFT's.

To any symplectic lattice $(\Gamma, \langle -, - \rangle_{\text{Dirac}})$ is naturally associated an algebra, $\mathbb{T}_\Gamma(q)$, the *quantum torus of* Γ. This is an infinite dimensional algebra generated as a vector space by elements Y_γ for $\gamma \in \Gamma$ with relations

$$Y_\gamma Y_{\gamma'} \equiv q^{\langle \gamma, \gamma' \rangle_{\text{Dirac}}} Y_{\gamma'} Y_\gamma \qquad \forall\, \gamma, \gamma' \in \Gamma \tag{1.3}$$

On the algebra $\mathbb{T}_\Gamma(q)$ we have an additional group structure defined by the *normal ordered product* $N \colon \mathbb{T}_\Gamma(q) \times \mathbb{T}_\Gamma(q) \to \mathbb{T}_\Gamma(q)$:

$$Y_{\gamma+\gamma'} \equiv N[Y_\gamma, Y_{\gamma'}] \equiv q^{-\frac{1}{2}\langle \gamma, \gamma', \rangle_{\text{Dirac}}} Y_\gamma Y_{\gamma'} \tag{1.4}$$

$N[-,-]$ is associative and commutative and it induces a group homomorphism in between $(\Gamma, +)$ and $(\mathbb{T}_\Gamma, N[-,-])$ that will be very useful in what follows.

1.1.3. BPS–states. The central charge of the $\mathcal{N} = 2$ superalgebra gives a linear map $Z \colon \Gamma \longrightarrow \mathbb{C}$ that depends on all the parameters of the theory (couplings, masses, Coulomb branch parameters, *etc.*), and hence encodes the physical regime in which we study the theory. Let $\{e_i\}_{i=1}^D$ be a set of generators of Γ. By linearity a central charge is specified by its values $Z_i \equiv Z(e_i)$. Let us denote by $M(\gamma)$ the mass of a multiplet of charge $\gamma \in \Gamma$: the representation theory of the 4d $\mathcal{N} = 2$ superalgebra entails that all multiplets undergo the BPS–bound

$$M(\gamma) \;\geq |\; Z(\gamma) \;| \,. \tag{1.5}$$

States that saturates this bound (*i.e.* that have mass *equal* to $|\; Z(\gamma) \;|$) are called *BPS–states* or *BPS–particles*. In particular, BPS–states come in *short* $\mathcal{N} = 2$

[2] Recall the definition of Dirac pairing: let us label $\vec{e}$ the electric charges, $\vec{m}$ the magnetic ones and $\vec{f}$ the flavor ones, then $\langle(\vec{e}_1, \vec{m}_1, \vec{f}_1), (\vec{e}_2, \vec{m}_2, \vec{f}_2)\rangle_{\text{Dirac}} \equiv \vec{e}_2 \cdot \vec{m}_1 - \vec{e}_1 \cdot \vec{m}_2$. The Dirac quantization condition is the statement that the Dirac pairing is always an integer.

multiplets. As $SU(2)_{\text{spin}} \times SU(2)_R$ representations the short multiplets can be always decomposed as

$$[(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})] \otimes \omega \tag{1.6}$$

where ω is the Clifford vacuum of the multiplet. Let us list some examples:

- $\omega = (\mathbf{1},\mathbf{1})$: the half–hypermultiplet $(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$;
- $\omega = (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1})$: the hypermultiplet $2 \times [(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})]$;
- $\omega = (\mathbf{2},\mathbf{1})$: the vectormultiplet $(\mathbf{1},\mathbf{1}) \oplus (\mathbf{3},\mathbf{1}) \oplus (\mathbf{2},\mathbf{2})$;

and so on. In each case consistecy with PCT symmetry constraints the BPS–particles charges.BPS–particles such that ω transforms in a non–trivial representation of $SU(2)_R$ are called exotic. The *absence of exotics* conjecture of [25] is the assertion that at generic point of the Coulomb branch the Clifford vacuum is always a singlet of $SU(2)_R$.[3]

1.1.4. Wall–crossing and quantum monodromy. Consider now a state with charge $\gamma = \gamma_1 + \gamma_2$ where γ_1 and γ_2 are charges of other states of the model. We have that, in general

$$M(\gamma) \geq | Z(\gamma) | = | Z(\gamma_1 + \gamma_2) | \geq | Z(\gamma_1) | + | Z(\gamma_2) | . \tag{1.7}$$

If γ, γ_1, and γ_2 are all BPS the stability of the particle of charge γ is controlled by Pitagora's theorem: if the central charges $Z(\gamma_1)$ and $Z(\gamma_2)$ are aligned, the BPS–particle of charge γ becomes unstable and can decay in its constituents γ_1 and γ_2. This decay may occur if and only if the two charges are non–local, *i.e.* $\langle \gamma_1 , \gamma_2 \rangle_{\text{Dirac}} \neq 0$. The condition that pairs of mutually non–local central charges align defines real–codimension one loci in the space of all possible central charges $\mathbb{C}^D = (\Gamma \otimes \mathbb{C})^\vee$, called *marginal stability walls* (or *walls of the first kind*), that may intersect. Marginal stability walls cuts $\mathbb{C}^D$ into domains of the first kind $\{\mathcal{D}_a\}_a$. In the interior of each such domain, the theory is in a different phase, characterized by a given spectrum of stable BPS–particles. The set of charges of such stable particles is a *BPS–chamber* of the charge lattice $\mathscr{C}_b \subset \Gamma$. Going across a wall of marginal stability a phase transition occurs, the *wall–crossing* phenomenon, and the BPS–chamber changes, typically in a very discontinuous way.

A first formulation of the BPS spectral problem would be the following: Determine the sets of charges $\gamma \in \mathscr{C}_b$ and Clifford vacua ω_γ (for all possible distinct chambers) together with the set of domains of the first kind $\mathcal{D}_a$ such that $\mathscr{C}_b$ is the set of charges of the stable BPS–particles in that infrared phase. Notice that PCT

[3] Very recently, a proof of this conjecture has been sketched in [24].

gives a $\mathbb{Z}_2$ involution on any BPS–chamber $\mathscr{C}_b$. There are some *crucial* subtleties we are neglecting here for simplicity. We will discuss them and give a second formulation of the spectral problem in the following subsection.

Any solution of the BPS spectral problem can be encoded in special inner automorphism of the $\mathbb{T}_\Gamma(q)$ algebra of §.1.1.2. In abstract algebra, an inner automorphisms is described in terms of the adjoint action of an appropriate operator. In our case this operator is called *quantum monodromy* or Kontsevich–Soibelman operator, and denoted by $\mathbb{M}(q)$ [13, 26]. Consider a domain of the first kind $\mathcal{D}_a$. This domain specifies a chamber $\mathscr{C}_b$ in the charge lattice, toghether with an ordering of $\{\arg Z(\gamma) \in S^1\ ,\ \gamma \in \mathscr{C}_b\}$. For each $\gamma \in \mathscr{C}_b$, let j_γ denote the higher spin in the Clifford vacuum ω_γ, then

$$\mathbb{M}(q) \equiv \overrightarrow{\prod_{\gamma\in\mathscr{C}_b}} \prod_{s=-j_\gamma}^{j_\gamma} \Psi(q^s Y_\gamma; q)^{(-)^{2s}} \tag{1.8}$$

where $\Psi(X;q) = \prod_{n\geq 0}(1 - q^{n+1/2}X)$ is the quantum dilogaritm function of Faddeev.[4] The product is taken over all charges in $\mathscr{C}_b$, and is ordered according to $\arg Z(\gamma)$. The KS wall–crossing formula is the statement that the conjugacy class of such operator and of all of its powers does not depend on the particular $\mathcal{D}_a$ nor on the particular $\mathscr{C}_b$ we use to compute it: $\mathbb{M}(q)$ is a wall–crossing *invariant.*

Notice that if a chamber $\mathscr{C}_b$ has a $\mathbb{Z}_m$ involution,[5] by linearity of the central charges, the corresponding domain of the first kind inherits such a symmetry. $\mathscr{C}_b$ splits into m (identical) subchambers $\mathscr{S}_b$: $\mathscr{C}_b = \coprod_{k=0}^{m-1} \mathbf{V}^k \mathscr{S}_b$. Then $\mathbb{M}(q)$ can be factored as a product of $1/m$*–fractional monodromy* $\mathbb{Y}(q)$

$$\mathbb{Y}(q) \equiv \mathbf{V}^{-1} \circ \overrightarrow{\prod_{\gamma\in\mathscr{S}_b}} \prod_{s=-j_\gamma}^{j_\gamma} \Psi(q^s Y_\gamma; q)^{(-)^{2s}} \tag{1.9}$$

and we have

$$\mathbb{M}(q) = \mathbb{Y}(q)^m \tag{1.10}$$

Since physics is PCT symmetric, the half–monodromy operator $\mathbb{K}(q)$ is always defined.

If we can represent $\mathbb{T}_\Gamma(q)$ over the operators of a Hilbert space $\mathcal{H}$, the inner automorphism $\mathbb{M}(q)$ will correspond to a (discrete) symmetry in $\mathcal{O}(\mathcal{H})$ and the

[4] This is true for the examples that we are going to consider, but can be generalized in many ways, for a discussion about this the reader is referred to [13, 27, 28].

[5] *i.e.* a linear transformation $\mathbf{V} : \mathscr{C}_b \to \mathscr{C}_b$ s.t. $\mathbf{V}^m = \mathrm{id}$

statement that the conjugacy class of $\mathbb{M}(q)$ is wall–crossing invariant implies the invariance of the trace $\mathrm{Tr}_{\mathcal{H}}(\mathbb{M}(q))$. A very nice representation of $\mathbb{T}_\Gamma(q)$ was found in [13] that allows to interpret $\mathrm{Tr}_{\mathcal{H}}\mathbb{M}(q)$ as a topological partition function. Here we closely follow [28]. Let the model flow to the UV conformal point where its global $U(1)_R$ symmetry is restored, let R be the corresponding charge. Notice that since we are discussing a topological partition function it won't depend on the moduli. A Melvin cigar is a 3-manifold MC_q defined as a quotient of $\mathbb{C}\times S^1$ with respect to the relation $(z,\theta) \sim (qz, \theta + 2\pi)$. Now, let H_R denote the generator of the Cartan of the $SU(2)_R$ symmetry of the 4d $\mathcal{N} = 2$ model. Consider the topologically twisted theory on the background $MC_q \times_R S^1$, where the R-twist is given by the identification of the $R - H_R$ charge with the holonomy around this second S^1. Notice that by doing this we are breaking 4 of the 8 supercharges of 4d $\mathcal{N} = 2$ SUSY. Let us define

$$Z(t,q) \equiv \mathrm{Tr}_{MC_q\times_R S^1}(-1)^F t^{R-H_R} \equiv \langle(-1)^F t^{R-H_R}\rangle_{MC_q\times_R S^1} \tag{1.11}$$

In this background one obtains a representation of $\mathbb{T}_\Gamma(q)$ from the reduction of the BPS–line operators of the 4d $\mathcal{N} = 2$ theory down to the 1d theory on the S^1 R–circle.[6] Moreover,

$$\mathrm{Tr}_{\mathcal{H}}\,[\mathbb{M}(q)]^k = Z(t = e^{2\pi i k}, q) = \langle(-1)^F \exp(2\pi k R)\rangle_{MC_q\times_R S^1} \tag{1.12}$$

This representation of $\mathbb{M}(q)$ has many interesting facets, let us discuss the two we are going to use. On one hand it is interpreted as the character of a 2d RCFT, as we are going to discuss in chaper 3. On the other hand it corresponds to the vev of the operator $(-1)^F\exp(2\pi R)$ in the topological background we defined.

Now, if the SCFT has R–charges of the form $\mathbb{N}/\ell$, with $\ell > 1$, we *predict* via (1.12), that $\mathrm{Ad}[\big(\mathbb{M}(q)\big)^\ell] = \mathrm{id}_{\mathbb{T}_\Gamma(q)}$. This fact can be used to give a very non–trivial check of all of these ideas, through the prediction of a duality in between non–lagrangian 4d $\mathcal{N} = 2$ SCFT and periodic TBA Y–systems as we will see in the next chapter.

The invariance of the conjugacy class of the quantum monodromy operator with respect to wall–crossing gives an overdetermined set of equations that the BPS spectra have to meet at walls of marginal stability, and this results in the *wall–crossing formulae*. Two distinct domains of the first kind $\mathcal{D}_{a_1}$ and $\mathcal{D}_{a_2}$ can intersect only along their common boundary $\partial\mathcal{D}_{a_1} \cap \partial\mathcal{D}_{a_2}$, where the BPS–spectra undergo the phase transition. Whenever the intersection is along a single marginal stability wall characterized by the requirement

$$Z(\gamma_1)/Z(\gamma_2) \in \mathbb{R}_{\geq 0} \qquad \gamma_1, \gamma_2 \in \Gamma, \tag{1.13}$$

[6] For a more detailed and intriguing description of this story we refer to the original paper [13]

the wall–crossing phase transition is determined by $\langle\gamma_1\,,\gamma_2\rangle_{\text{Dirac}}$: on one side of the wall there is a chamber in which γ_1 and γ_2 do not have any bound–state, while at the other side of the wall there will be all possible bound–states of γ_1 and γ_2. In the following examples the dots on the extreme left and on the extreme right stand for the other charges of BPS particles, shared by the two chambers (for simlpicity we write only the half–monodromy):

- $|\,\langle\gamma_1,\gamma_2\rangle_{\text{Dirac}}\,|= 1$: pentagon wall–crossing ;

$$\cdots\Psi(Y_{\gamma_1})\Psi(Y_{\gamma_2})\cdots\cong\cdots\Psi(Y_{\gamma_2})\Psi(Y_{\gamma_1+\gamma_2})\Psi(Y_{\gamma_1})\cdots$$

- $|\,\langle\gamma_1,\gamma_2\rangle_{\text{Dirac}}\,|= 2$: Kronecker wall–crossing ;

$$\cdots\Psi(Y_{\gamma_1})\Psi(Y_{\gamma_2})\cdots\cong\cdots\left(\overrightarrow{\prod_{n\nearrow}}\Psi(Y_{n\gamma_1+(n+1)\gamma_2})\right)\times$$
$$\times\,\Psi(q^{-1/2}Y_{\gamma_1+\gamma_2})^{-1}\Psi(q^{1/2}Y_{\gamma_1+\gamma_2})^{-1}\left(\overrightarrow{\prod_{m\swarrow}}\Psi(Y_{(m+1)\gamma_1+m\gamma_2})\right)\cdots$$

  where n (resp.m) runs increasing (resp. decreasing) over all positive integers.

- $|\,\langle\gamma_1,\gamma_2\rangle_{\text{Dirac}}\,|= m\geq 3$: m–Kronecker wall–crossing ; nobody has ever worked out the full wall–crossing formula in this case.[7]

In principle, having solved the BPS spectral problem in one given domain of the first kind, by successive wall–crossings one can reach any other domain.

1.1.5. The quantum Shottky problem. In the above discussion we have completely missed one fundamental fact: the central charges are functions of the physical parameters of the theory. Let $\mathscr{P}$ be the subspace of the space of parameters of the theory that corresponds to physically consistent quantum field theories. The central charge Z define a map

$$\Im:\mathscr{P}\to(\Gamma\otimes\mathbb{C})^\vee\simeq\mathbb{C}^D\qquad\lambda_a\mapsto Z_i\equiv Z(e_i)\tag{1.14}$$

Each element of the space $(\Gamma\otimes\mathbb{C})^\vee$ gives a *formal* central charge: the only physically allowed ones are contained in $\Im(\mathscr{P})$. Under which condition $\Im(\mathscr{P})$ has codimension zero? Assume for the moment that the theory admits a lagrangian

[7] The formula is given implicitly: see [29]. This is a very active field of research: the last pubblication about this problem is [30].

formulation. The dimension of the parameter space, in this case, can be computed as

$$\begin{aligned} \dim \mathscr{P} &= \#(\text{gauge couplings}) + \dim\,(\text{Coulomb branch}) + \#(\text{masses}) \\ &= \#(\text{simple factors of G}) + r + f \end{aligned} \tag{1.15}$$

On the contrary the rank of the charge lattice is $D = 2r + f$. Thus

$$\text{codim } \mathscr{P} = r - \#(\text{simple factors of G}). \tag{1.16}$$

This equality holds in a *lagrangian* corner of the parameter space: for a non–lagrangian one we expect that it becames an inequality,

$$r - \#(\text{simple factors of G}) \leq \text{codim } \mathscr{P}, \tag{1.17}$$

since there could be more complicated mechanisms that lead to forbidden directions. From this fact it follows that for a given theory all possible chambers corresponds to physically realized BPS chambers if and only if $G = SU(2)^k$. Such 4d $\mathcal{N} = 2$ systems are called *complete* for this reason. For all other 4d $\mathcal{N} = 2$ theories $\Im(\mathscr{P})$ has non–zero codimension in $(\Gamma \otimes \mathbb{C})^\vee$, and therefore not all possible chambers are physical. To determine if a chamber is physical or not is a Schottky–like problem we will refer to as *the quantum Schottky problem.*

The refined formulation of the BPS spectral problem is the following: Determine the sets of charges $\gamma \in \mathscr{C}_b$ and Clifford vacua ω_γ for the set of domains of the first kind $\mathcal{D}_a \subset (\Gamma \otimes \mathbb{C})^\vee$ such that *1.)* $\Im(\mathscr{P}) \cap \mathcal{D}_a \neq 0$, and *2.)* $\mathscr{C}_b$ is the set of charges of stable BPS–states in that infrared regime.

1.2. The BPS–quiver property.

1.2.1. BPS–quivers. Following [7], we say that a 4d $\mathcal{N} = 2$ theory has the *BPS–quiver property* if it exists a set $\{e_i\}_{i=1}^D$ of generators of its charge lattice Γ such that the charge vectors $\gamma \in \Gamma$ of all BPS–states satisfy

$$\gamma \in \Gamma_+ \qquad \text{or} \qquad \gamma \in -\Gamma_+ \tag{1.18}$$

where $\Gamma_+ \equiv \bigoplus_{i=1}^D \mathbb{Z}_{\geq 0}\, e_i$ is a strict convex cone in $\mathbb{Z}^D$. In a regime specified by the central charge $Z(\cdot)$, without loss of generality by PCT–symmetry, the $\{e_i\}_{i=1}^D$ are required

$$0 < \arg Z(e_i) < \pi \qquad \text{for all } i. \tag{1.19}$$

In a sense, $Z(\cdot)$ selects a preferred *cone of particles* Γ_+, such that $Z(\Gamma_+)$ is strictly contained in a convex domain of the upper half–plane $\text{Im}Z \geq 0$. The requirements in eqns.(1.18)–(1.19) fixes the basis $\{e_i\}_i$ [8] uniquely (up to permutation of the elements), but, physics being PCT invariant, the splitting in between particles and anti–particles of (1.19) is artificial:[8] this ambiguity plays a crucial rôle in the theory that will be discussed in §.1.3. In the rest of this section assume that we have *chosen* the (1.19) splitting (*i.e.* the preferred cone Γ_+).

Eqn.(1.18) is a non-empty requirement on the BPS–spectrum of the theory: let us discuss a *necessary* condition. Let

$$\Theta \equiv \bigcup_{\gamma\in \text{ BPS}} \{\arg Z(\gamma)\} \subset S^1. \tag{1.20}$$

The BPS–quiver property entails that Θ is not dense in S^1. Counterexamples exist even in the class of complete theories. Let $\mathcal{C}_{g\geq 3,0}$ denote a Riemann surface with genus $g \geq 3$ and no boundaries nor punctures. All class $\mathcal{S}[A_1, \mathcal{C}_{g>3,0}]$ do not have BPS–quivers, as follows from classification [7]. These theories are special because all their deformations are in terms of purely marginal operators [6]. The fact that a theory admit deformations that are not purely marginal seems to be the condition that guarantees that there are regions of the moduli in which the BPS–quiver property holds. A well–known example of this feature of 4d $\mathcal{N} = 2$ systems is $\mathcal{N} = 4$ $SU(2)$ SYM. Such theory has a known BPS–spectrum such that Θ is dense in S^1, but, by giving a mass to the adjoint $\mathcal{N} = 2$ hypermultiplet sitting in the $\mathcal{N} = 4$ $SU(2)$ vectormultiplet, one flows to the $\mathcal{N} = 2^*$ $SU(2)$ theory, that, in turn, has the BPS–quiver property.[9]

A *sufficient* (not–necessary) condition for the BPS–quiver property is that the 4d $\mathcal{N} = 2$ system admits a geometric engineering in terms of type IIB superstrings. This idea will be made precise in the next chapters.

A *quiver* is a quadruple $Q \equiv (Q_0, Q_1, s, t)$ where Q_0 and Q_1 are two countable discrete sets that we will call, respectively, the set of nodes and the set of arrows of Q, while s, t are two maps from Q_1 to Q_0 that to each arrow α in Q_1 associate, respectively, the node where α starts, $s(\alpha)$, and the node where α terminates, $t(\alpha)$. Sometimes it is convenient to represent Q with a drawing where the elements of Q_0 (resp.of Q_1) are represented as points of a plane (resp.as arrows in between these points). The *BPS–quiver* of the model is a quiver constructed as follows: to

[8] To get an intuition about this one can think about the representation theory of Lie algebras: Γ is the like a weight lattice, Γ_+ is like a Weyl chamber of the weight lattice, and a BPS–chamber is like the set of weights of an (irreducible) representation. The spectral problem is really a problem in representation theory: each physical regime of the model corresponds to a possible representation of it.

[9] The quiver with superpotential for $SU(2)$ $\mathcal{N} = 2^*$ was determined in [8] and [12].

each generator e_i of Γ_+ we associate a node, and we connect two nodes e_i and e_j with

$$B_{ij} \equiv \langle e_i\,, e_j \rangle_{\text{Dirac}} \qquad \text{(the exchange matrix)} \tag{1.21}$$

arrows that goes from e_i to e_j if $B_{ij} > 0$, and in the opposite direction otherwise.

1.2.2. The path algebra of a quiver. Let a and b be two nodes of Q_0: a *path* from a to b is a sequence of arrows $\{\alpha_n\}_{n=1,\dots,\ell}$ such that $s(\alpha_1) = a$, $t(\alpha_\ell) = b$, and $s(\alpha_n) = t(\alpha_{n-1})$; it is denoted by $(b \mid \alpha_\ell \cdots \alpha_n \alpha_{n-1} \cdots \alpha_1 \mid a)$ (in the rest of this thesis we will shall omit the extrema). The length of a path is the number ℓ of elements of the sequence of arrows (counted with repetition). A path of length ℓ from a node to itself is called an *ℓ-cycle* or simply a *cycle*. The set of all paths in a quiver has the structure of an algebra, the *path algebra* of Q. Over the field $\mathbb{C}$ the path algebra, denoted $\mathbb{C}Q$, is the $\mathbb{C}$-vector space generated by the set of all paths of Q endowed with multiplication

$$(d \mid \beta_{\ell'} \cdots \beta_1 \mid c)(b \mid \alpha_\ell \cdots \alpha_1 \mid a) \equiv \delta_{bc}(d \mid \beta_{\ell'} \cdots \beta_1 \alpha_\ell \cdots \alpha_1 \mid a), \tag{1.22}$$

If a path algebra has a cycle, then it is infinite dimensional as a $\mathbb{C}$–vector space, the typical example being the path algebra of the 1-loop quiver

$$\mathbb{C}\,(\,\bullet \circlearrowright X\,) \equiv \mathbb{C}[X] \tag{1.23}$$

where $\mathbb{C}[X]$ is the algebra of polynomials in one complex variable. A *representation* or a *module* X of $\mathbb{C}Q$ is simply the assignment of a $\mathbb{C}$-vector space X_i to each node $i \in Q_0$ and of a morphism of vector spaces $X_\alpha \colon X_{s(\alpha)} \to X_{t(a)}$ for all $\alpha \in Q_1$.[10] The *dimension vector* of a representation X is

$$\dim X \equiv (\dim(X_i))_{i \in Q_0} \in (\mathbb{Z}_{\geq 0})^{\# Q_0} \tag{1.24}$$

Letting X, Y be two representations of Q, a *morphism* $f : Y \to X$ is the assignment of linear maps $f_a \colon Y_a \to X_a$ for all $a \in Q_0$ such that the following diagram

$$\begin{array}{ccc} Y_{s(\alpha)} & \xrightarrow{f_{s(a)}} & X_{s(\alpha)} \\ {\scriptstyle Y_\alpha}\downarrow & & \downarrow{\scriptstyle X_\alpha} \\ Y_{t(\alpha)} & \xrightarrow{f_{t(a)}} & X_{t(\alpha)} \end{array} \tag{1.25}$$

[10]Notice that $\mathbb{C}Q$ can be interpreted as a small category whose objects are the nodes and whose morphisms are paths. A representation of $\mathbb{C}Q$ is then simply a functor from Q to the category of $\mathbb{C}$-vector spaces $\text{Vect}_{\mathbb{C}}$ that maps the nodes $i \in Q_0$ to vector spaces X_i and arrows $i \xrightarrow{\alpha} j$ to elements in $\text{Hom}(X_i, X_j)$, this point of view will turn out to be very useful when we will get to the `meta`-quiver approach.

commutes. A *subrepresentation* is an injective morphism. Notice that representations form an abelian category: $\mathsf{rep}(Q)$. Path of lenght zero are called lazy paths, and corresponds to the *primitive orthogonal idempotents* of the algebra $\mathbb{C}Q$: these elements are denoted e_i, $i \in Q_0$ and the identity of the algebra can be written uniquely as $1 = \sum e_i$. The modules $S(e_i) \equiv \mathbb{C}e_i$ are the *simple objects* of $\mathsf{rep}(Q)$.

1.2.3. BPS–states and representations. Given the choice of the system of generators of the charge lattice $\{e_i\}_{i=1}^{D}$, let

$$\gamma = \sum_{i=1}^{D} N_i\, e_i \in \Gamma_+ \qquad (N_i \geq 0) \tag{1.26}$$

be the charge vector of a BPS particle. To study the properties of this particle, one reduce the dynamics of the system on its world–line and studies the resulting 1d quantum mechanics. The particle being BPS, one obtains a 1d super–quantum–mechanics (SQM) with 4 supercharges. If the theory has the BPS–quiver property, this 1d theory is a *quiver SQM* [8, 31], defined as follows: one has a gauge group $U(N_i)$ for each node e_i of the quiver, and chiral (super)fields Φ_α in the bifundamental representation $(\overline{N}_i, N_j)$ for each arrow $\alpha\colon e_i \to e_j$. The kinetic term is canonical and in addition one has Fayet–Ilyopoulos (FI) (complex) terms associated with the $U(1)$ factors of the gauge groups.[11] If Q contains a cycle $\alpha_\ell \cdots \alpha_1$, the 1d $\mathcal{N} = 4$ quiver SQM have non–trivial gauge invariant operators

$$\mathrm{tr}[\Phi_{\alpha_\ell} \cdots \Phi_{\alpha_1}]. \tag{1.27}$$

In such a case, in addition to the FI terms, the 1d system can have a superpotential V that is obtained as a linear combination with complex coefficients of terms like (1.27) that correspond to cycles. In particular, for a generic superpotential V, the 2-cycles become mass terms, and the corresponding superfields can be integrated out, being irrelevant for the description of the infrared properties of the model. Notice that for this reason the BPS–quiver we defined are *2-acyclic*, *i.e.* have no 1-cycles nor 2-cycles. We stress that V is not uniquely determined by the quiver.[12] Typically, one has to check if the given superpotential reproduce the correct physics for the 4d model.

The stability of the BPS–particle (1.26) is controlled by the FI terms of the 1d SQM: according to their value the model can have SUSY vacua or not. The degrees of freedom of a BPS–particle of charge $\gamma \in \Gamma_+$ arise quantizing the Kähler

[11] The FI moduli of the 1d system are fixed by the central charge $Z(\gamma)$.

[12] As we are going to discuss in §.1.3.3, for a theory that admits a finite BPS–chamber made only of hypermultiplets the quiver determines a generic V, *i.e.* one such that all 2–cycles can be always integrated out.

moduli space $\mathcal{M}(\gamma)$ of SUSY vauca of the 1d quiver SQM on its worldline: If the 1d system does not have SUSY vacua, we are in a phase of the theory in which the BPS–particle of charge γ is unstable [31]. By standard geometric invariant theory, $\mathcal{M}(\gamma)$ can be presented in two equivalent ways: either it is described as the solution of the F–term and D–term equations of motion modulo the action of the unitary gauge groups $\prod_{i=1}^{D} U(N_i)$, or it is given as the solution of the F–term equations modulo the action of the complexified groups $\prod_{i=1}^{D} GL(N_i, \mathbb{C})$ together with a stability condition. This second point of view is the bridge with representation theory: let us describe it in full details.

Let Q denote the BPS–quiver of the theory. The charge vector of the BPS–state (1.26) is mapped in the dimension vector of a representation $X \in \mathsf{rep}(Q)$ $(N_i \equiv \dim(X_i))$. The complexified gauge groups are, from this point of view, simply the possible rotations of the basis of the vector spaces X_i associated to the nodes of Q:

$$GL(X) \equiv \prod_{i=1}^{D} GL(\dim(X_i), \mathbb{C}). \tag{1.28}$$

The "bosonic" part of the chiral fields Φ_α is the representation of an arrow X_α. Two representations are *isomorphic* iff they are in the same $GL(X)$ orbit. Let V be the superpotential of the 1d SQM. The F–term equations of motion can be interpreted as relations in $\mathbb{C}Q$ that generate an ideal $\partial\mathcal{W} \subset \mathbb{C}Q$. $X \in \mathsf{rep}(Q)$ is a solution of the F–term equations if $X(\partial\mathcal{W}) = 0$, in other words it is a representation of the algebra $\mathbb{C}Q/\partial\mathcal{W}$. The modules of $\mathbb{C}Q/\partial\mathcal{W}$ form an abelian subcategory of $\mathsf{rep}(Q)$

$$\mathsf{rep}(Q, \mathcal{W}) \equiv \{X \in \mathsf{rep}(Q) \text{ such that } X(\partial\mathcal{W}) = 0\}\,. \tag{1.29}$$

Forgetting about dimensions the 1d SQM superpotential V can be thought as a representation of a superpotential $\mathcal{W}$ defined directly on $\mathbb{C}Q$:

$$V(X) \equiv \mathrm{tr} X(\mathcal{W}). \tag{1.30}$$

The $\mathcal{W}$ so defined is named after Derksen–Weyman–Zelevinsky in the math literature [32].[13] We will call the pair $(Q, \mathcal{W})$ a quiver with superpotential. The central

[13] We stress that in full mathematical rigor one should not work with $\mathbb{C}Q/\partial\mathcal{W}$, but with $\mathcal{P}(Q, \mathcal{W})$, the completed path algebra $\widehat{\mathbb{C}Q}$, modulo the closure of the jacobian ideal in the ℓ–adic topology of $\widehat{\mathbb{C}Q}$ (ℓ is the lenght of a path). If there exist a $n > 0$ such that all paths of lenght $\ell \geq n$ of $\mathbb{C}Q$ belongs to $\partial\mathcal{W}$, the two algebras coincide. Since this happens in most of the applications discussed in this thesis, we will avoid to mention $\mathcal{P}(Q, \mathcal{W})$. The reader should just remember that what we really mean with the category $\mathsf{rep}(Q, \mathcal{W})$ might not always coincide with what is written in eqn.(1.29).

charge of the 4d $\mathcal{N}=2$ superalgebra is extended by linearity to all representations of Q:

$$Z(X) \equiv Z(\dim(X)) \equiv \sum_{i=1}^{D} \dim(X_i) \cdot Z(e_i), \tag{1.31}$$

and we say that $X \in \mathsf{rep}(Q, \mathcal{W})$ is *stable* iff

$$0 \leq \arg Z(Y) < \arg Z(X) < \pi \quad \forall\, 0 \neq Y \subset X. \tag{1.32}$$

A stable representation is, in particular, an indecomposable representation.[14] Not all indecomposables are stable: stability entails that an indecomposable representation is, moreover, a *brick*, that is,

$$X \text{ stable} \quad \Rightarrow \quad \operatorname{End} X = \mathbb{C}. \tag{1.33}$$

Being a brick is also a sufficent condition for the given representation to be stable in *some* chamber [33].

In a regime specified by the central charge $Z(\cdot)$, a stable BPS–state with charge vector γ corresponds to the quantization of

$$\mathcal{M}(\gamma) \equiv \Big\{ X \in \mathsf{rep}(Q, \mathcal{W}) \text{ stable and such that } \gamma = \dim(X) \Big\} \Big/ GL(X). \tag{1.34}$$

In particular, the Clifford vacuum ω_γ is given in terms of the Lefshetz and Hodge $SU(2)$ decompositions of $H^p(\mathcal{M}(\gamma), \Omega^q_{\mathcal{M}(\gamma)})$: the absence of exotics is equivalent to the statement that only for $p = q$ the above cohomology is non–trivial,[15] and

$$\omega_\gamma = (\dim(\mathcal{M}(\gamma)) + 1, \mathbf{1}) \quad \text{as a rep. of } SU(2)_{\text{spin}} \times SU(2)_R. \tag{1.35}$$

Representations such that $\mathcal{M}(\gamma)$ is a point are called *rigid*: since $\dim(\mathcal{M}(\gamma)) = 0$, these corresponds to hypers. If $\mathcal{M}(\gamma) \simeq \mathbb{P}^1$, we have a vectormultiplet, and so on for higher spin states. The BPS spectral problem is now formulated as a problem in representation theory.

A last remark is in order: we are counting BPS–states up to PCT. This leads to some subtleties that are better understood via some simple examples. The BPS–hypers comes as a doublet of half–hypers to preserve PCT. We will have $\gamma \in \Gamma_+$ and its anti–particle in $-(\Gamma_+)$. Quantizing $\mathcal{M}(\gamma)$ we will find $\omega_\gamma = (1, 1)$. The other direct summand in the Clifford vacuum is associated with the anti–particle of charge $-\gamma$. If the model admits genuine half–hypers, they come in quaternionic representations of G and we will see only half of their weights in Γ_+, the other half will be in $-(\Gamma_+)$. Of course the same is true for a vectormultiplet in the adjoint: we will find $\mathbb{P}^1$–families of representations only for the positive roots of G.

[14] i.e. it cannot be written as a direct sum. Indeed if $X = X_1 \oplus X_2$, then both X_1 and X_2 are subrepresentations and $Z(X) = Z(X_1) + Z(X_2)$, therefore X will never be stable.

[15] A proof of this fact is non–trivial and it was sketched in [24]

1.3. Mutations

As we have stressed in §.1.2.1 the definition of a BPS–quiver with superpotential $(Q, \mathcal{W})$ for a model depends on an arbitrary choice about the splitting of the spectrum into particles and antiparticles. We are now in position to make that dependence explicit. Suppose we have a 4d $\mathcal{N} = 2$ model which has the BPS quiver property. Let Γ be its rank D charge lattice. Fix a half–plane $H_\theta = \{z \in \mathbb{C}\colon \mathrm{Im}(e^{-i\theta}z) > 0\}$ such that no BPS particle has central charge laying on its boundary ($\theta \in S^1 \setminus \Theta$). We say (conventionally) that the BPS states with central charges in H_θ are *particles*, while those with central charges in $-H_\theta$ are their PCT–conjugate *anti*–particles. Each choice of H_θ leads to:

- a preferred choice of generators $\{e_i^{(\theta)}\}_{i=1}^D$
- a *strict* convex cone $\Gamma_\theta \subset \Gamma$ that has the form $\Gamma_\theta \simeq \bigoplus_{i=1}^D \mathbb{Z}_{\geq 0}\, e_i^{(\theta)}$
- a quiver with superpotential $(Q_\theta, \mathcal{W}_\theta)$ encoded by the exchange matrix

$$B_{ij}^{(\theta)} \equiv \langle e_i^{(\theta)}, e_j^{(\theta)} \rangle_{\text{Dirac}}$$

The BPS particles (as contrasted with antiparticles) correspond to moduli spaces of *stable* representations $X \in \mathsf{rep}(Q_\theta, \mathcal{W}_\theta)$. A representation X is *stable* iff, for all non–zero proper subrepresentation Y, one has $\arg(e^{-i\theta}Z(Y)) < \arg(e^{-i\theta}Z(X))$, where we take $\arg(e^{-i\theta}H_\theta) = [0, \pi]$. The charge $\gamma \in \Gamma_\theta$ of the BPS particle is given by the dimension vector $\sum_i \dim X_i\, e_i^{(\theta)}$ of the corresponding stable representation X. In particular, the representations S_i with dimension vector equal to a generator $e_i^{(\theta)}$ of Γ_θ are *simple*, and hence automatically stable for all choices of the function $Z(\cdot)$ (consistent with the given positive cone $\Gamma_\theta \subset \Gamma$): these BPS states necessarily hypermultiplets, since Q_θ has no loops.

Choosing a different angle θ', we get a convex cone $\Gamma_{\theta'}$ with a different set of generators $e_i^{(\theta')}$, and correspondingly we get another quiver with superpotential $(Q_{\theta'}, \mathcal{W}_{\theta'})$ and a different category of representations $\mathsf{rep}(Q_{\theta'}, \mathcal{W}_{\theta'})$, but the physics does not change: the BPS–spectrum of the chamber *cannot depend on our choice of* θ. Therefore, the physics of the Coulomb phase of a 4d $\mathcal{N} = 2$ model in a regime specified by a central charge Z does not correspond to a unique pair $(Q, \mathcal{W})$, but to an equivalence class of such quivers with superpotentials. This fact is perfectly consistent with physics: it is a Seiberg–like duality [34, 35] in the context of the 1d $\mathcal{N} = 4$ quiver SQM systems.

Let us make more concrete the statements we have made above. The BPS particle of larger (resp.smaller) $\arg(e^{-i\theta}Z(\gamma))$ has a charge γ which has to be a generator $e_{i_1}^{(\theta)}$ of Γ_θ (associated to some node i_1 of Q_θ). We may tilt clockwise (resp.anti–clockwise) the boundary line of H_θ just past the point $Z(e_{i_1}^{(\theta)})$, producing a new half–plane $H_{\theta'}$. In the new frame the state with charge $e_{i_1}^{(\theta)}$ is an *anti–*particle, while its PCT–conjugate of charge $-e_{i_1}^{(\theta)}$ becomes a particle, and in facts a generator of the new positive cone $\Gamma_{\theta'}$. The generators $e_i^{(\theta')}$ of $\Gamma_{\theta'}$ are linear combinations with integral coefficients of the old ones $e_i^{(\theta)}$. For the $1d\ \mathcal{N}=4$ SQM this tilting of H_θ corresponds to a Seiberg–like duality, that in mathematics [32] is called the basic quiver right (resp.left) mutation of Q_θ at the i_1 node, written $\mu_{i_1}^R$ (resp.$\mu_{i_1}^L$). Notice that our conventions differs from those of [8].[16] The explicit expression of the $e_i^{(\theta')}$'s in terms of the $e_i^{(\theta)}$ is

$$e_i^{(\theta')} = \mu_{i_1}^R\big(e_i^{(\theta)}\big) = \begin{cases} -e_{i_1}^{(\theta)} & \text{if } i=i_1 \\ e_i^{(\theta)} + \max\{B_{i_1\,i}^{(\theta)},0\}\, e_{i_1}^{(\theta)} & \text{otherwise.} \end{cases} \tag{1.36}$$

for a right mutation, while for a left mutation one has

$$e_i^{(\theta')} = \mu_{i_1}^L\big(e_i^{(\theta)}\big) = \begin{cases} -e_{i_1}^{(\theta)} & \text{if } i=i_1 \\ e_i^{(\theta)} + \max\{B_{i\,i_1}^{(\theta)},0\}\, e_{i_1}^{(\theta)} & \text{otherwise.} \end{cases} \tag{1.37}$$

Notice that, as expected

$$\mu_i^L \circ \mu_i^R = \mu_i^R \circ \mu_i^L = \mathrm{id}_\Gamma. \tag{1.38}$$

The mutated quiver $Q_{\theta'} = \mu_{i_1}(Q_\theta)$ is specified by the exchange matrix $B_{ij}^{(\theta')} \equiv \langle e_i^{(\theta')}, e_i^{(\theta')}\rangle_{\text{Dirac}}$. One can check by direct computation that *both* for left and right mutations at i_1 we have

$$B_{i\,j}^{(\theta')} = \begin{cases} -B_{ij}^{(\theta)} & \text{if } i=i_1 \text{ or } j=i_1 \\ B_{i\,j}^{(\theta)} + \mathrm{sign}(B_{i\,i_1}^{(\theta)}) \max\{B_{i\,i_1}^{(\theta)}\, B_{i_1\,j}^{(\theta)},0\} & \text{otherwise} \end{cases} \tag{1.39}$$

where there is no summation over i_1. This is expected: if we switch all particles with antiparticles, the left mutation for the particles is the right mutation for the antiparticles and viceversa. The operation (1.39) is called a *basic quiver mutation at the node* i_1, and denoted μ_{i_1}. Notice that, coherently with (1.38), we have

$$\mu_i \circ \mu_i = \mathrm{id}_Q. \tag{1.40}$$

We stress that while μ_i^2 is the identity at the quiver level, the square of a left or a right mutation is a non–trivial transformation of Γ that leads to an element

[16] We call a right mutation one that tilts the plane H_θ clockwise to the right, while in [8] this is a left mutation because the particle rotates away from the left.

of the mutation class that is clearly forbidden physically. For a *generic* quiver superpotential $\mathcal{W}_\theta$, a basic quiver mutation μ_{i_1} coincide with a Seiberg–like duality for the 1d $\mathcal{N}=4$ SQM (which is equivalent to the the DWZ rule [32]).[17] Let us recall it briefly:

- Start with the quiver $(Q_\theta, \mathcal{W}_\theta)$, and reverse the orientation of all arrows through the node i_1

$$\cdots \xrightarrow{\alpha} i_1 \xrightarrow{\beta} \cdots \quad \curvearrowright \quad \cdots \xleftarrow{\alpha^*} i_1 \xleftarrow{\beta^*} \cdots$$

- For all couples of such arrows add a new arrow to the quiver $[\beta\alpha] : s(\alpha) \to t(\beta)$ (this is the meson field of Seiberg–duality) and replace in the superpotential $\mathcal{W}_\theta$ all occurences of the product $\beta\alpha$ with the corresponding meson $[\beta\alpha]$

- Add to the superpotential the term $\alpha^*\beta^*[\beta\alpha]$ for each 3-cycle generated in this way

- Integrate out all possible massive fields. If the superpotential $\mathcal{W}_\theta$ is sufficiently generic, one can integrate out all 2-cycles generated by the above procedure, and the quiver obtained in this way is precisely $Q_{\theta'} = \mu_{i_1}(Q_\theta)$ [32]. The resulting on-shell superpotential is $\mathcal{W}_{\theta'}$ and one writes

$$(Q_{\theta'}, \mathcal{W}_{\theta'}) = \mu_{i_1}(Q_\theta, \mathcal{W}_\theta)$$

As we have discussed, if we move around in the moduli space, the central charges vary. Fix a given H_θ. Walls of the first kind do not affect the quiver description: we have simply a change in the stability conditions. But it may happen that, moving along in the moduli space, the leftmost or the rightmost generator exits H_θ. In such a case the quiver description we are adopting is no longer valid and we have to mutate the quiver accordingly: different quivers with superpotential related by mutation have to be adopted to give a description that covers the whole moduli space. One can think of the various quiver descriptions as if they are like local charts that describe several chambers, but not the totality of them. Two quivers that can be reached through a sequence of mutations are said to belong to the same *mutation class*. From the quantum Shottcky problem we discussed in §.1.1.5, it follows that the moduli space is covered only by a proper mutation–connected subspace of the quiver mutation class. Since, for non–complete theories $\Im(\mathcal{D})$ has nonzero codimension, not all formal BPS–chambers

[17] The rule is similar to the one for 4d $\mathcal{N}=1$ (ordinary) quiver theories, but there are no constraints from anomaly matching.

in $(\Gamma\otimes\mathbb{C})^\vee$ are physical, and therefore not all possible elements of the mutation class can be 'reached' by crossing physical walls of the second kind. On contrast, for complete theories that have the BPS–quiver property all possible elements of the mutation class are physical. Indeed, complete theories have *very special* BPS–quivers: a theory is complete if and only if its quiver has a FINITE mutation class, and this property is strong enough to fully classify them [7].

1.3.1. Quantum mutations. In this section we work at a fixed θ. Let $e_i^{(\theta)}\equiv e_i$ and $B_{ij}\equiv B_{ij}^{(\theta)}$. Using the group homomorphism induced by the normal ordered product (1.4) to embed Γ in $\mathbb{T}_\Gamma(q)$, we see that the BPS–quiver property gives a preferred *minimal* set of generators for Γ_θ in $\mathbb{T}_\Gamma(q)$, $Y_i\equiv Y_{e_i}$, $i=1,...,D$, with relations

$$Y_i\,Y_j = q^{B_{ij}}Y_j\,Y_i \tag{1.41}$$

By a *quantum mutation* of the quantum torus algebra we mean the composition of an (ordered) sequence of elementary mutations at various nodes of Q. The elementary quantum mutation, $\mathcal{Q}_k$, at the k–th node of the 2–acyclic quiver Q is the composition of two transformations [13, 36–39]:

(1) a basic mutation of the quiver at the k–th node, $Q\to\mu_k(Q)$ together with a suitable mutation of the superpotential, $\mathcal{W}\to\mu_k(\mathcal{W})$ [32, 40, 41]. As we have discussed the mutation of the quiver $Q\to\mu_k(Q)$ follows from a change of basis in the charge lattice Γ, which corresponds to choosing a different set of generators of $\mathbb{T}_\Gamma(q)$. Again via the group homomorphism induced by the normal ordered product (1.4) we lift left and right mutations to the generators of the quantum torus:

$$\begin{aligned}\mu_k^R(Y_i)&\equiv Y_{\mu_k^R(e_i)}=q^{-\frac{1}{2}B_{kj}[B_{kj}]_+}Y_iY_k^{[B_{kj}]_+}\\ \mu_k^L(Y_i)&\equiv Y_{\mu_k^L(e_i)}=q^{-\frac{1}{2}B_{kj}[B_{jk}]_+}Y_iY_k^{[B_{jk}]_+}\end{aligned} \tag{1.42}$$

μ_k is not in general an automorphism of the quantum torus algebra; a composition of μ_k's is an algebra automorphism iff it is the identity on the underlying quiver Q since only in this case it leaves invariant the commutation relations.

(2) the adjoint action on $\mathbb{T}_q(\Gamma)$ of the quantum dilogarithm[18] of Y_k

$$Y_\gamma\mapsto\Psi(Y_k;q)^{-1}\,Y_\gamma\,\Psi(Y_k;q). \tag{1.43}$$

Thus, explicitly, the elementary quantum cluster mutation at the k–th node is (we leave as understood the left or right index)

$$\mathcal{Q}_k=\mathrm{Ad}(\Psi(Y_k)^{-1})\circ\mu_k. \tag{1.44}$$

[18] The same specification we did in footnote 4 carries over to this case.

We stress that while μ_k^2 is the identity at the quiver level, $\mu_k^2(Q) \equiv Q$, the square of a left or a right mutation is a non–trivial transformation on the set of generators:

$$(\mu_k^{R,L})^2 \colon Y_i \mapsto q^{-(B_{i\,k})^2/2}\, Y_i\, Y_k^{\pm B_{i\,k}} \equiv Y_{e_i \pm B_{i\,k} e_k} \equiv t_k^{R,L} \tag{1.45}$$

These are called the (right and left) Seidel–Thomas *twists* [39]. The elementary quantum mutations, instead, are involutions of $\mathbb{T}_\Gamma(q)$, *i.e.* one has the identity [13, 36–39]

$$\mathcal{Q}_k^2 = \text{identity on } \mathbb{T}_\Gamma(q). \tag{1.46}$$

Therefore, quantum mutations are the correct lift to the quantum torus of the combinatoric (cluster) structure of basic mutations.

1.3.2. The mutation algorithm. Since the physics does not depend on the conventional choice of the half plane H_θ, the $e_i^{(\theta)}$'s should always be charge vectors of stable BPS hypermultiplets. The idea of the mutation algorithm is to get the full BPS spectrum by collecting all states with charges of the form $e_i^{(\theta)}$'s for all θ. It is easy to see that this gives the full BPS spectrum provided it consists only of hypers (*i.e.* rigid reps) and their number n_h is finite.

The mutation algorithm is the following. Chose a plane H_θ and a corresponding preferred set of generators of Γ, $\{e_i^{(\theta)}\}$. Find the rightmost (resp.leftmost) state in the corresponding H_θ plane. Do the right (resp. left) mutation at $e_{i_1}^{(\theta)}$. The new generators $e_i^{(\theta')}$ are also charge vectors of stable hypers. We can reiterate the procedure by mutating $Q_{\theta'}$ at the node i_2 corresponding to the hypermultiplet with maximal (resp.minimal) $\arg(e^{-i\theta'} Z(\gamma))$. Again we conclude that the BPS spectrum also contains stable hypers with charges $e_i^{(\theta'')}$. Now suppose that after m right (resp. left) mutations we end up with the positive cone $\Gamma_{\theta^{(m)}} \equiv -\Gamma_\theta$; we conclude that $\theta^{(m)} = \theta + \pi$ and hence, with our sequence of m half–plane tiltings, we have scanned the full complex half–plane H_θ, picking up *all* the BPS *particles*, one at each step, according to their decreasing (resp.increasing) phase order in the central charge plane. Thus, whenever this happens, we conclude that we have a BPS chamber in which the BPS spectrum consists of precisely m hypermultiplets. This happens iff there is a sequence of m quiver mutations such that [8]

$$\mu_{i_m} \circ \mu_{i_{m-1}} \circ \cdots \circ \mu_{i_2} \circ \mu_{i_1}\big(e_i^{(\theta)}\big) = -\pi\big(e_i^{(\theta)}\big) \qquad \forall\, i, \tag{1.47}$$

with $i_k \neq i_{k+1}$ $\forall\ k = 1, ..., m-1$, and π a permutation of the D generators $e_i^{(\theta)}$. If, for the given quiver Q_θ, we are able to find a sequence of quiver mutations satisfying equation (1.47) (for some $\pi \in \mathfrak{S}_D$) we may claim to have found a finite

BPS chamber consisting of m hypermultiplets only, and list the quantum numbers $\gamma_\ell \in \Gamma$ of all BPS particles

$$\gamma_\ell = \mu_{i_{\ell-1}} \circ \mu_{i_{\ell-2}} \circ \cdots \circ \mu_{i_1}\big(e^{(\theta)}_{i_\ell}\big) \qquad \ell = 1, 2, \ldots, m. \tag{1.48}$$

In the the rest of this thesis we shall work always at fixed θ, and write the positive cone generators simply as e_i, omitting the angle.

For many purposes, it is convenient to rephrase the algorithm in the language of the previous section. If the sequence of basic quiver mutations μ_{i_a} satisfies eqn.(1.47), from the associated composition of basic quantum cluster mutations one can read off directly the expression of the adjoint action of the half–monodromy operator in that chamber

$$\mathcal{Q}_{i_m} \circ \mathcal{Q}_{i_{m-1}} \circ \cdots \mathcal{Q}_{i_2} \circ \mathcal{Q}_{i_1} = I_\pi \circ \mathrm{Ad}(\mathbb{K}(q)) \tag{1.49}$$

where $\mathbb{K}(q)$ is the quantum half–monodromy and I_π is the unitary operator acting on the generators Y_i of the quantum torus algebra of Q_θ as

$$I_\pi \, Y_i \, I_\pi^{-1} = Y^{-1}_{\pi(i)}. \tag{1.50}$$

The (finite) BPS spectrum may be read directly from the factorization of $\mathbb{K}(q)$ in quantum dilogaritms [13], which is explicit in the LHS of eqn.(1.49).

There are a few strategies to find particular solutions to eqn.(1.47). An elegant one is that of complete families of sink/source factorized subquivers of Q_θ introduced in [14]; this is particularly convenient when the factorized subquivers are Dynkin ones endowed with the standard Coxeter sink/source sequences that we are going to review in §.1.3.4

For general quivers Q, we may perform a systematic search for solutions on a computer; Keller's quiver mutation applet [42] is quite helpful for both procedures. In doing this, it is convenient to rephrase eqn.(1.47) in terms of *tropical* y–seed mutations [43–45]. We recall that the *tropical semifield* $\mathsf{Trop}(u_1, u_2, \ldots, u_r)$ is the free multiplicative Abelian group generated by the indeterminates u_i endowed with the operation $\oplus$ defined by

$$\Big(\prod u_i^{l_i}\Big) \oplus \Big(\prod u_i^{m_i}\Big) = \prod u_i^{\min(l_i, m_i)}. \tag{1.51}$$

To a BPS state of charge $\sum_i n_i e_i$ we associate the tropical y–variable $\prod u_i^{n_i} \in \mathsf{Trop}(u_1, u_2, \ldots, u_r)$. We start with the initial y–seed in which we assign to the i–th node of Q the variable associated to the generator e_i of the positive cone, namely $y_i(0) \equiv u_i$, and we perform the sequence of mutations in eqn.(1.47) on the y–seed using the Fomin–Zelevinski rules

$$y_j(s) = \begin{cases} y_{i_s}(s-1)^{-1} & \text{if } j = i_s \\ y_j(s-1)\, y_{i_s}(s-1)^{[B_{i_s j}(s-1)]_+} \Big(1 \oplus y_{i_s}(s-1)\Big)^{-B_{i_s j}} & \text{otherwise.} \end{cases} \tag{1.52}$$

(here $s = 1, 2, \dots, m$, and $[x]_+ = \max(x, 0)$). Since the tropical variables $y_{i_s}(s-1)$ correspond to BPS particles with charges in the positive cone Γ_θ, one has $1 \oplus y_{i_s}(s-1) \equiv 1$, and eqn.(1.52) reduces to the transformation rule (1.36). In terms of tropical y–variables, then eqn.(1.47) becomes

$$y_j(m) = y_{\pi(j)}(0)^{-1}, \tag{1.53}$$

supplemented by the condition that the tropical quantities $y_{i_s}(s-1)$ are monomials in the u_i's.

In conclusion: given a solution to eqn.(1.47) we have determined a (maybe formal) BPS chamber $\mathscr{C}_{\text{fin}}$ containing finitely many hypers, as well as the quantum numbers $\Gamma|_{\mathscr{C}_{\text{fin}}}$ of all these hypers. In addition, the algorithm specifies the (cyclic) phase order of the central charges $Z(\gamma)$ of the BPS states. From this last information we may read the domain $\mathcal{D}_{\text{fin}} \subset \mathbb{C}^D \equiv (\Gamma \otimes \mathbb{C})^\vee$ of central charges $Z(\cdot) \in (\Gamma \otimes \mathbb{C})^\vee$ for which $\mathscr{C}_{\text{fin}}$ is the *actual* BPS chamber, that is, we may determine the region in the space of the 'physical' parameters of the theory which corresponds to the finite chamber $\mathscr{C}_{\text{fin}}$: in the notation of §.1.1.5, it will be simply $\Im(\mathscr{P}) \cap \mathcal{D}_{\text{fin}}$.

At a generic point in $\mathcal{D}_{\text{fin}}$ the unbroken flavor symmetry is just $U(1)^{\operatorname{rank} F}$. At particular points in parameter space the flavor symmetry may have a non–Abelian enhancement. Let F_{fin} be the flavor symmetry group at a point of maximal enhancement in the domain $\mathcal{D}_{\text{fin}}$. Clearly, the BPS hypers of $\mathscr{C}_{\text{fin}}$ should form representations of F_{fin}. The fact that they do is a non–trivial check of the procedure.

1.3.3. A remark on quiver superpotentials. Given a solution to (1.47) the BPS–spectrum is determined implicitly in all other chambers by wall–crossing. This is rather remarkable: We have seen that most representations of the quiver are unphysical, since they do not solve the F–term equations of motion. In contrast, the BPS–spectrum in a finite chamber made only of hypers has been determined using *only* the combinatorics of quiver mutations that are *independent* of the superpotential, unless the latter is not sufficiently generic. This entails that, for theories that have finite chambers made only of hypermultiplets, the BPS–quiver generic superpotential is in a sense 'predicted' by combinatorics.

1.3.4. Factorized–sequences of mutations. A node of a quiver $i \in Q_0$ is called a *source* (resp.*sink*) if there is no arrow $\alpha \in Q_1$ such that $t(\alpha) = i$ (resp.$s(\alpha) = i$). Let us notice here that with our conventions (1.36)–(1.37) a right (resp.left) mutation on a node i that is a sink (resp.source) have the only effect of reversing the sign of the generator e_i leaving all the other generators unchanged. A sequence of nodes $\Lambda = \{i_1, i_2, \cdots, i_k\}$ of a quiver Q is called a *sink* sequence (resp. a *source* sequence) it the i_s node is a sink (resp. a source) in the mutated quiver $\mu_{i_{s-1}}\mu_{i_{s-2}} \cdots \mu_{i_1}(Q)$ for all $1 \le s \le k$. Let $\boldsymbol{m}_\Lambda = \mu_{i_k}\mu_{i_{k-1}} \cdots \mu_{i_1}$ be the mutation defined by the sequence Λ. A sink (resp. source) sequence Λ is called *full* if contains each node of Q exactly once. If Q is acyclic, and Λ is a full source sequence, the corresponding right mutation sequence acts on the generators of the charge lattice as the Coxeter element $\Phi_Q \colon \Gamma \to \Gamma$, while Λ^{-1} is a full sink sequence and the corresponding right mutation sequence acts as the inversion.

Given a subset S of the set of nodes Q_0, we introduce the notation $Q|_S$ to denote the full subquiver of Q over the nodes S. Consider the node set Q_0 as the disjoint union of a family of sets $\{q_\alpha\}_{\alpha\in A}$:

$$Q_0 = \coprod_{\alpha\in A} q_\alpha \tag{1.54}$$

To each subset of nodes q_α we associate the full subquiver $Q|_{q_\alpha}$ of Q. Given a node $i \in Q_0$, we will denote $q_{\alpha(i)}$ the unique element in the family that contains node i.

Now, consider a finite sequence of nodes $\Lambda = \{i(1), i(2), \ldots, i(m)\}, i(\ell) \in Q_0$ such that

$$\mathbf{m}_\Lambda \equiv \mu_{i(m)} \circ \cdots \circ \mu_{i(1)} \tag{1.55}$$

is a solution to (1.47). Λ is said to be *source-factorized* of type $\{Q|_{q_\alpha}\}_{\alpha\in A}$ if

i) For all $\ell = 1, 2, ..., m$, the ℓ-th node in the sequence $i(\ell)$ is a sink in

$$\mu_{i(\ell-1)} \circ \cdots \circ \mu_{i(1)}(Q)\Big|_{\{i(\ell)\}\cup Q_0\setminus q_{\alpha(i(\ell))}} \tag{1.56}$$

ii) For all $\ell = 1, 2, ..., m$ the ℓ-th node in the sequence $i(\ell)$ is a source in

$$\mu_{i(\ell-1)} \circ \cdots \circ \mu_{i(1)}(Q)\Big|_{q_{\alpha(i(\ell))}} \tag{1.57}$$

In our conventions (1.36)–(1.37) source factorized sequences of mutations are appropriate for right mutations. For the dual left mutations one shall invert sources with sinks in *i*) and *ii*) above.[19]

[19] For saving time and print in view of the applications that we have in mind, here we give just a simplified version: the interested reader is referred to the original paper [14] for the whole beautiful story.

A source-factorized sequence is in particular *Coxeter-factorized* of type $(Q|_{q_\alpha})_\alpha$, provided all $Q|_{q_\alpha}$ are Dynkin ADE quivers with alternating orientation. If this is the case, let us denote by G_α the alternating quiver $Q|_{q_\alpha}$. A sequence of right mutations that is Coxeter–factorized is automatically a solution of (1.47). In particular, if all the alternating ADE subquivers of the family are equal to a given G, one has a $1/h(G)$ fractional monodromy. For a Coxeter–factorized source–sequence, by the right mutation rule (1.36) combined with $i), ii)$, each element of the sequence corresponds to the action of the simple Weyl reflection

$$s_{i(\ell)} \in \mathrm{Weyl}(Q\big|_{q_{\alpha(i(\ell))}}) \tag{1.58}$$

on the charges on nodes $i \in q_{\alpha(i(\ell))}$, and as the identity operation on all other charges! By the standard properties of Weyl reflections of simply–laced root systems,[20] a Coxeter factorized sequence of mutations corresponds to a very peculiar finite BPS–chamber $\mathscr{C}_\Lambda$:

$$\mathscr{C}_\Lambda \simeq \bigoplus_{\alpha\in A} \Delta(G_\alpha) \tag{1.59}$$

where by $\Delta(G)$ is meant the set of roots of G. In other words such a chamber contains one hypermultiplet per *positive* root of G_α. Notice that from this fact it follows that the charge lattice, for a choice of generators compatible with $\mathscr{C}_\Lambda$ is

$$\Gamma \simeq \bigoplus_{\alpha\in A} \Gamma(G_\alpha), \tag{1.60}$$

where we denote with $\Gamma(G_\alpha)$ the root lattice of the Lie algebra of type G_α. It is useful to remark that any quiver that admits in its mutation class a square product form of type $Q\Box G$ admits Coxeter-factorized sequences of type $\{G^{\#Q_0}\}$.[21]

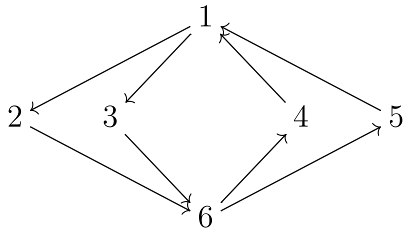
To illustrate the power of the technique, in the next section we shall use it to determine the spectrum in the finite chamber of $SU(2)$ SQCD with four fundamental flavors.

1.3.5. Example: the finite chamber of $SU(2)$ $N_f = 4$ According to ref. [5] $SU(2)$ SQCD with $N_f = 4$ has a BPS chamber with a finite spectrum consisting of 12 hypermultiplets. Let us see how this result follows from the existence of a complete family of Dynkin subquivers.

[20] See, for example, proposition VI.§. 1.33 of [46]

[21] See chapter 2 and §. 3.2 for a defintion.

We write the quiver of $SU(2)$ SQCD with four flavors in the form



(1.61)

which admits the complete family of Dynkin subquivers

$$A_3 \coprod A_3, \tag{1.62}$$

where the two A_3 are the full subquivers over the nodes $\{1,2,3\}$ and, respectively, $\{4,5,6\}$; the sink–factorized sequence of nodes is

$$\Lambda = \{2,3,4,5,6,1\} \tag{1.63}$$

having type $(A_3\,; A_3)$ as it is easy to check using Keller's applet [42]. Under the identification $\Gamma = \Gamma_{A_3} \oplus \Gamma_{A_3}$ we have

$$\boldsymbol{m}_\Lambda(Y_{\alpha\oplus\beta}) = Y_{c(\alpha)\oplus c(\beta)}. \tag{1.64}$$

Since $h(A_3) = 4$, one has $\boldsymbol{m}_\Lambda^4 = 1$ and $\prod_\Lambda \mathcal{Q}_k$ is the 1/4–monodromy. The corresponding monodromy $\mathbb{M}(q)$ satisfies all the physical constraints by comparison with the A_3 AD model.

In conclusion, $SU(2)$ SQCD with $N_f = 4$ has a $\mathbb{Z}_4$–symmetric finite BPS chamber with 12 hypermultiplets whose charge vectors, under the isomorphism $\Gamma = \Gamma_{A_3} \oplus \Gamma_{A_3}$, are

$$\{\alpha \oplus 0 \text{ and } 0 \oplus \alpha \mid \alpha \in \Delta_+(A_3)\}. \tag{1.65}$$

$SU(2)$ SQCD with $N_f = 4$ is a superconformal theory (setting the mass parameters to zero) whose chiral primary operators have integer dimension. Hence the quantum monodromy should have period 1, that is should be the identity on $\mathbb{T}_Q$,

$$\mathrm{Ad}(\mathbb{M}(q)) = \mathbf{1}. \tag{1.66}$$

Using the known spectrum (1.65), we have checked this statement with the method outlined in §.2.5.

Chapter 2

Arnol'd–models

2.1. Introduction

In ref. [13] a large class of $4d$ $\mathcal{N} = 2$ theories were discussed in detail. Those theories are labelled by a pair (G, G') of simply–laced Lie algebras, and are UV superconformal. They belong to the more general class of $4d$ models which may be geometrically engineered by considering the Type IIB superstring on the geometry $\mathbb{R}^{3,1} \times \mathscr{H}$ [18], where $\mathscr{H} \subset \mathbb{C}^4$ is a local 3–CY hypersurface specified by a polynomial equation

$$\mathscr{H}: \ f(x_1, x_2, x_3, x_4) = 0.$$

The resulting four–dimensional theory is $\mathcal{N} = 2$ *superconformal* iff the defining polynomial of $\mathscr{H}$, $f(x_i)$, is *quasi–homogeneous*, which implies that $\mathscr{H}$ is singular at the origin. The four–dimensional theory engineered on a *smooth* hypersurface $f(x_i) = 0$ is then physically interpreted as a massive deformation of the superconformal $\mathcal{N} = 2$ theory associated to the the leading quasi–homogeneous part $f_0(x_i)$ of the polynomial $f(x_i)$, deformed by a set of *relevant* operators corresponding to the lower degree part of the polynomial, *i.e.* to $\Delta f \equiv f(x_i) - f_0(x_i)$. In refs. [9, 18] it was shown that the singularity $f_0(x_i)$ is at finite distance in the complex moduli if it satisfies the condition

$$\sum_{i=1}^{4} q_i > 1, \tag{2.1}$$

where the weights q_i of the quasi–homogeneous polynomial $f_0(x_i)$ are defined trough the identity $\lambda\, f_0(x_i) = f_0(\lambda^{q_i}\, x_i)$, $\lambda \in \mathbb{C}^*$. In the $2d$ language [47–50], the condition (2.1) is equivalent to the statement that the Landau–Ginzburg model with superpotential $W \equiv f_0(x_i)$ has central charge $\hat{c} < 2$. As a consequence, if the homogeneous part of the defining polynomial, $f_0(x_i)$, satisfies eqn.(2.1), the geometry $\mathbb{R}^{3,1} \times \mathscr{H}$ is a valid Type IIB background, and the geometrical engineering produces a consistent $4d$ $\mathcal{N} = 2$ quantum field theory which, typically, has no weakly coupled Lagrangian description.

The (G, G') models studied in ref. [13] correspond to the special case in which

$f_0(x_i)$ is the direct sum of two quasi–homogeneous polynomials

$$f_0(x_i) = W_G(x_1, x_2) + W_{G'}(x_3, x_4), \tag{2.2}$$

where $W_G(x, y)$ stands for the quasi–homogeneous polynomial describing the minimal singularity associated to the ADE algebra G [47,48,51][1]. Of course, the general polynomial $f_0(x_i)$ satisfying eqn.(2.1) has not the 'decoupled' form of eqn.(2.2). Thus one is lead to ask for the extension of the methods and results of [13] to singular hypersurfaces of more general form.

Such an extension is the main purpose of this chapter. There is a particularly important class of non–minimal singularities, namely Arnol'd's 14 exceptional unimodal singularities [51, 52]. They have $\hat{c} < 2$, and hence define superconformal $\mathcal{N} = 2$ theories in four dimensions. These 14 models are, in a sense, the simplest $\mathcal{N} = 2$ superconformal gauge theories which are *not* complete. The associated 14 singularities naturally appear in many different areas of mathematics, and in particular in the representation theory of path algebras of quivers with relations [53,54] (for a review [55]), which is a natural mathematical arena for understanding the BPS spectra of $\mathcal{N} = 2$ theories [7, 31, 56, 57]. Hence this class of $\mathcal{N} = 2$ models appears to be 'exceptional' from the mathematical side as well as from the physical one.

We are going to use these 14 'exceptional' gauge theories as an interesting example to develop our methods, we will then extend our results without efforts to the quasi–homogenous elements out of the bimodal singularities in the last section of this chapter.

When the hypersuface $\mathscr{H}$ has the special form

$$0 = f(x_i) \equiv g(x_1, x_2) + x_3 x_4,$$

the four dimensional $\mathcal{N} = 2$ theory may also be engineered by considering the Abelian $(2, 0)$ six dimensional theory on the curve $\{g(x_1, x_2) = 0\} \subset \mathbb{C}^2$ [13, 18]. From the point of view of singularity and algebra representation theory, the equivalence of the two constructions from $10d$ and $6d$ is just the Knörrer–Solberg periodicity [58] which directly implies the equality of BPS spectra.

The results of [13] and [9, 18] have a peculiar implication from the point of view of the Thermodynamical Bethe Ansatz [59]: They suggest the conjecture[2]

[1] See table 3.1 in the next chapter

[2] From a physical viewpoint (*i.e.* arguing trough string theory), this statement is equivalent to the conjecture that all $4d$ $\mathcal{N} = 2$ models engineered on such a singular hypersurface have at least one chamber with a finite BPS spectrum.

that to each isolated quasi–homogeneous hypersurface singularity, having $\hat{c} < 2$, there is associated a TBA Y–system which is periodic (the two–Dynkin diagrams Y–systems [60] corresponding to direct sums of minimal singularities as in eqn.(2.2), [13]). Here we check this prediction for the 14 Arnol'd exceptional singularities, including the precise value ℓ of the minimal period. It will be highly desirable to have a direct proof of this correspondence, making explicit the underlying connection between singularity theory and cluster categories, in the spirit of ref. [60].

More in general, one expects a Y–system of period ℓ to be associated to any $\mathcal{N} = 2$ superconformal model having a BPS chamber with a finite spectrum and whose chiral primary fields have dimensions of the form $\mathbb{N}/\ell$.

2.2. Arnol'd's 14 exceptional unimodal singularities

We have the identifications[3]

$$E_{12} \equiv A_2 \boxtimes A_6 \qquad E_{14} \equiv A_2 \boxtimes A_7 \tag{2.3}$$

$$W_{12} \equiv A_3 \boxtimes A_4 \qquad U_{12} \equiv D_4 \boxtimes A_3 \tag{2.4}$$

$$Q_{10} \equiv A_2 \boxtimes D_5 \qquad Q_{12} \equiv A_2 \boxtimes D_6 \tag{2.5}$$

of six Arnol'd's models with theories of type (G, G'), $G, G' = ADE$, already studied in [13]. We focus on the remaining 8 Arnol'd exceptional $\mathcal{N} = 2$ theories. These 8 Arnol'd exceptional unimodal singularities (at the quasi–homogeneous value of the modulus) are written in table 2.1 as polynomials $W(x, y, z)$ in the three complex variables x, y, z. The local CY 3–fold $\mathscr{H}$, on which we engineer the corresponding $\mathcal{N} = 2$ model, is then given by the hypersurface in $\mathbb{C}^4$

$$W(x, y, z) + u^2 + \text{lower terms} = 0. \tag{2.6}$$

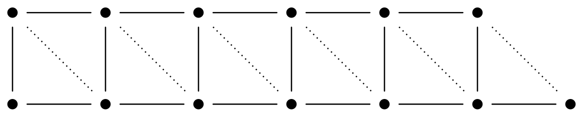
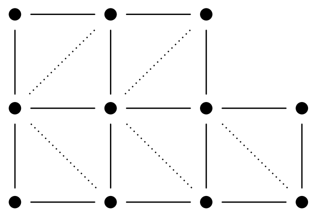
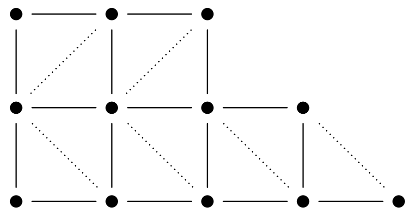
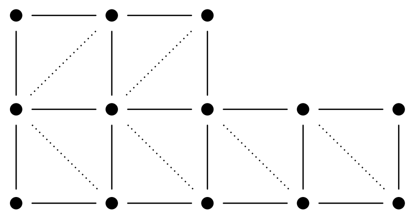
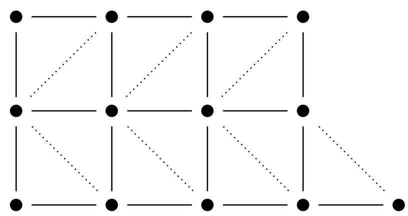
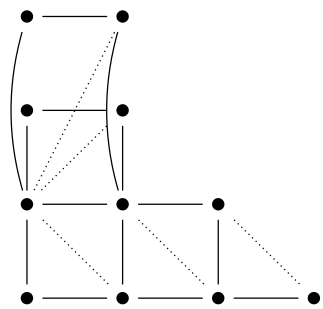
2.2.1. Coxeter–Dynkin graphs, Coxeter transformations The last column of table 2.1 shows the Coxeter–Dynkin diagram of the singularity [52]. We recall its definition: The compact homology of the complex *surface*

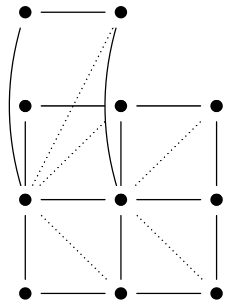
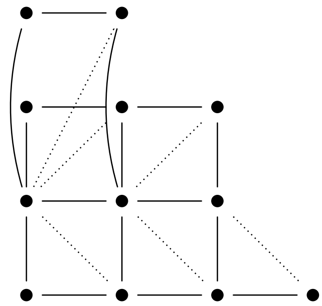
$$\{W(x, y, z) + \cdots = 0\} \subset \mathbb{C}^3$$

is generated by μ 2–spheres [61], where μ is the Milnor number of the singularity (equal to the subfix in the singularity's name). Fixing a strongly distinguished

[3] We postpone the definition of the square tensor product of two acyclic quivers $Q_1 \boxtimes Q_2$ to the next chapter, where we are going to use it extensively.

Table 2.1: Arnol'd's 14 exceptional singularities that are not of the form $W_G + W_{G'}$

name	polynomial $W(x, y, z)$	weights q_i	Coxeter–Dynkin diagram
E_{13}	$x^3 + xy^5 + z^2$	$1/3, 2/15, 1/2$	
Z_{11}	$x^3y + y^5 + z^2$	$4/15, 1/5, 1/2$	
Z_{12}	$x^3y + xy^4 + z^2$	$3/11, 2/11, 1/2$	
Z_{13}	$x^3y + y^6 + z^2$	$5/18, 1/6, 1/2$	
W_{13}	$x^4 + xy^4 + z^2$	$1/4, 3/16, 1/2$	
Q_{11}	$x^2z + y^3 + yz^3$	$7/18, 1/3, 2/9$	

name	polynomial $W(x,y,z)$	weights q_i	Coxeter–Dynkin diagram
S_{11}	$x^2z + yz^2 + y^4$	$5/16, 1/4, 3/8$	
S_{12}	$xy^3 + x^2z + yz^2$	$4/13, 3/13, 5/13$	

basis of (vanishing) 2–cycles δ_j [51, 52], the negative of their intersection form, $-\delta_j \cdot \delta_k$, is an integral symmetric $\mu \times \mu$ matrix, with 2's along the main diagonal, that is naturally interpreted as a 'Cartan matrix'. In fact, for a minimal ADE singularity[4], $-\delta_j \cdot \delta_k$ is the Cartan matrix of the associated simply–laced Lie algebra. However, for a *non*–minimal singularity, it is not true that $-\delta_j \cdot \delta_k \leq 0$ for $j \neq k$, and hence $-\delta_j \cdot \delta_k$ is not a standard Cartan matrix in the Kac sense [62].

Correspondingly, the Coxeter–Dynkin graph becomes a *bi*–graph, *i.e.* a graph with two kinds of edges, solid and dashed. Nodes j, k are connected by $|\delta_j \cdot \delta_k|$ edges; the edges are solid if $\delta_j \cdot \delta_k > 0$, and dashed if $\delta_j \cdot \delta_k < 0$.

It should be stressed that the Coxeter–Dynkin diagram is not unique, since it depends on the particular choice of a (strongly distinguished) homology basis. Two such bases differ by the action of the braid group acting by Picard–Lefshetz transformations [51, 52]. The physical interpretation of this non–uniqueness is well known: In the $2d$ language the Picard–Lefshetz transformations correspond to BPS wall–crossings [50], while from the $4d$ perspective they are understood as SQM Seiberg dualities [34, 35, 63].

One important invariant of the singularities is (the conjugacy class of) its Coxeter transformation, also known as the *strong monodromy* Φ. With respect to a

[4] And a suitable choice of the basis δ_j.

strongly distinguished basis one has

$$\Phi = -(S^{-1})^t S, \tag{2.7}$$

where

$$S_{jk} = \delta_{jk} - \begin{cases} \delta_j \cdot \delta_k & k > j \\ 0 & \text{otherwise}, \end{cases} \tag{2.8}$$

and S encodes the $2d$ BPS spectrum of the Landau–Ginzburg (LG) model with superpotential W [50].

2.3. Arnol'd's $\mathcal{N} = 2$ superconformal theories

2.3.1. $2d/4d$ **correspondence revisited** The $2d/4d$ correspondence of ref. [13] states that the quiver Q of the $4d$ theory engineered on a CY hypersurface $W+u^2 = 0$ is equal to BPS quiver of the $2d$ LG model with $\hat{c} < 2$ having superpotential $W + u^2$. Basically, the nodes of the $4d$ quiver Q are in one–to–one correspondence with the SUSY vacua of the $2d$ model, and two nodes of Q, j and k, are connected by a number of arrows equal to the *signed* number of BPS states interpolating the corresponding $2d$ vacua, $|j\rangle$ and $|k\rangle$. To implement this rule, it is convenient to integrate away the decoupled free $2d$ superfield u, remaining with the LG $2d$ superpotential W. Then, as shown in [50,64], the $2d$ SUSY vacua $|j\rangle$ are in one–to–one correspondence with the elements of a strongly distinguished basis $\{\delta_j\}$ of the vanishing homology of the hypersurface $\mathscr{H}\colon \{W = \text{const.}\}$, and the signed number of BPS particles interpolating between $|j\rangle$ and $|k\rangle$ is given by the corresponding intersection number $\delta_j \cdot \delta_k$. Hence the $2d/4d$ correspondence predicts a quiver with $\delta_j \cdot \delta_k$ arrows between nodes j and k, a negative number again meaning arrows in the opposite direction. In other words, the exchange matrix B_{jk} of Q is given by

$$B = S^t - S, \tag{2.9}$$

where S is as in eqn.(2.8)[5]. Equivalently, the $2d$ quantum monodromy in the sense of ref. [50] is minus the Coxeter transformation Φ of the singular hypersurface (and thus S is identified with the half–plane Stokes matrix of [50]).

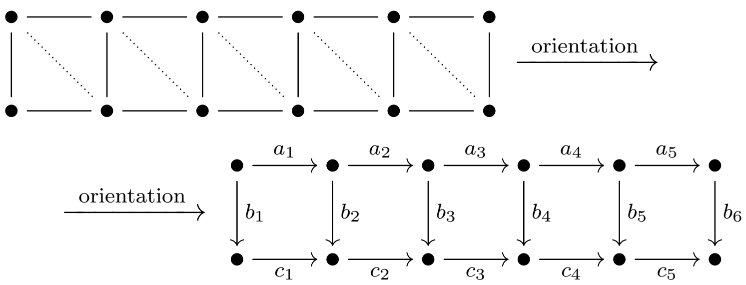
The $2d/4d$ correspondence is rather subtle, since it depends on the correct identification of a *strongly* distinguished basis, and it should be implemented with the necessary care. For this reason, here we present a more intrinsic derivation of

[5] Notice that the notions of a *strongly distinguished homology basis* $\{\delta_j\}$ in the sense of [51,52], and that of a *distinguished set of generators of the charge lattice* $\{e_j\}$ in the sense we discussed in §.1.2.1 agree under the $2d/4d$ correspondence.

the $\mathcal{N} = 2$ Dirac quiver from the Coxeter–Dynkin diagram of the singularity; this method has the additional merit of predicting also the superpotential $\mathcal{W}$ of the quiver (super)quantum mechanics whose SUSY vacua give the $4d$ BPS states. One check that the proposed procedure is equivalent to the proper $2d/4d$ correspondence, is that it reproduces the correct $2d$ quantum monodromy $-\Phi$, which is the mutation–invariant content of the $2d$ BPS quiver.

There is a standard dictionary [65] between Dynkin *bi*–graphs and (classes of) algebras which generalizes Gabriel's relation between representation–finite hereditary algebras and ordinary (simply–laced) Dynkin graphs [66–68]. One picks an orientation of the solid arrows to get a quiver Q; then the dashed arrows are interpreted as a minimal set of relations generating an ideal J in the path algebra $\mathbb{C}Q$ of that quiver. Finally, one considers the basic algebra $\mathbb{C}Q/J$. Of course, the orientation of Q has to be chosen in such a way that the dashed lines make sense as relations in $\mathbb{C}Q$.

Let us illustrate this procedure in the example of the E_{12} Coxeter–Dynkin *bi*–graph



(2.10)

where the quiver in the RHS is supplemented with the relations generating the ideal J determined by the dashed edges in the LHS, namely

$$b_{j+1}\, a_j = c_j\, b_j, \qquad j = 1, 2, \dots, 5. \tag{2.11}$$

These relations just state that the squares in (2.10) are commutative, and hence imply that the resulting algebra $\mathcal{A}'_{E_{12}} \equiv \mathbb{C}Q/J$ is isomorphic to the product $\mathbb{C}\vec{A}_6 \otimes \mathbb{C}\vec{A}_2$, where $\mathbb{C}\vec{A}_n$ stands for the path algebra of the *linear* A_n Dynkin quiver

$$\vec{A}_n: \qquad \overbrace{\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longrightarrow \bullet}^{n \text{ nodes}}\,.$$

The models having this tensor product form were solved in ref. [13] by exploiting the isomorphism $\mathcal{A}'_{E_{12}} \simeq \mathbb{C}\vec{A}_6 \otimes \mathbb{C}\vec{A}_2$ and its generalizations to $\mathbb{C}G \otimes \mathbb{C}G'$ (G, G' being arbitrary ADE Dynkin quivers).

Let[6] C_{ji} be the matrix counting the number of paths between the i–th and j–th node in the quiver Q (identifying paths which differ by an element of J). The Euler form of $\mathcal{A}'_{E_{12}}$ is the non–symmetric bilinear form on the dimension lattice $\Gamma_{E_{12}}$ defined by the matrix C^{-1}, that is

$$\langle X, Y\rangle_E \equiv \sum_{k\geq 0}(-1)^k \dim \mathrm{Ext}^k(X,Y) = (\dim X)^t C^{-1}(\dim Y). \tag{2.12}$$

We stress that $\mathcal{A}'_{E_{12}}$ has global dimension ≤ 2: For all $k \geq 3$, $\mathrm{Ext}^k(X,Y) = 0$. This property is absolutely crucial for the consistency of our manipulations. The Cartan matrix, Dirac pairing, and Coxeter element of the algebra $\mathcal{A}'_{E_{12}}$ are encoded in terms of C^{-1} as follows $\mathcal{A}'_{E_{12}}$ are, respectively,

$$\begin{aligned} &(C^{-1})^t + C^{-1} && \text{(Cartan matrix)}\\ &(C^{-1})^t - C^{-1} && \text{(Dirac pairing)}\\ &-C^t C^{-1} && \text{(Coxeter element)}\end{aligned} \tag{2.13}$$

which agree with the predictions of the $2d/4d$ correspondence since $C^{-1} = S$, as it easy to check going trough the definitions.

However $\mathcal{A}'_{E_{12}} \simeq \mathbb{C}\vec{A}_6 \otimes \mathbb{C}\vec{A}_2$ is not the final story. From the point of view of the quiver supersymmetric quantum mechanics, the relations of J may arise only from the F–term flatness equations $\partial\mathcal{W} = 0$. Hence we have to introduce a SQM superpotential $\mathcal{W}$ and additional Lagrange–multiplier superfields λ_j, one per fundamental relation of J, that is, one λ_j per dashed edge in the *bi*–graph. This is equivalent to replacing the dashed edges of the Coxeter–Dynkin diagram with arrows going in the *opposite* direction. Then, for the E_{12} example, the superpotential is

$$\mathcal{W} = \sum_{j=1}^{5} \lambda_j (b_{j+1}a_j - c_j b_j). \tag{2.14}$$

In this way we get a completed quiver $\widetilde{Q}$, and the algebra $\mathcal{A}'_{E_{12}}$ gets completed to the Jacobian algebra $\mathbb{C}\tilde{Q}/\partial\mathcal{W}$ which is known as the 3–*Calabi–Yau completion of* $\mathcal{A}'_{E_{12}}$, written $\Pi_3(\mathcal{A}'_{E_{12}})$ [69]. The completed algebra is the one relevant for the SQM theory describing the $4d$ BPS states[7].

---

[6] In the math litterature, $C_{ji} \equiv \dim \mathrm{Hom}(P_i, P_j)$. Where P_i denotes the projective cover of the simple representation S_i (S_i is the representation with the one–dimensional space $\mathbb{C}$ at the i–th node, and zero elsewhere). The projective P_j has dimension vector given by $(\dim P_j)_i \equiv$ # of path $i \to j$ (modulo the relations). Since all our algebras are basic, $\mathscr{A} \simeq \oplus_i P_i$ as (right) $\mathscr{A}$–modules.

[7] Let us stress that the algebra $\mathbb{C}\tilde{Q}/\partial\mathcal{W}$ we have defined is not strictly speaking a 3–CY algebra: the 1d SQM we defined is just the degree zero component of it.

This procedure may be repeated word–for–word for all the Coxeter–Dynkin diagrams of the 14 exceptional singularities, and there are no obstacles to generalize it further. This leads to the

> **Algebraic reformulation of the 2d/4d correspondence.** [10] *Let $(Q,\mathcal{W})$ be the BPS–quiver with superpotential of an 4d $\mathcal{N}=2$ theory, and $\mathcal{A}$ be the corresponding jacobian algebra. Then $\mathcal{A}$ is the 3–CY completion of a basic algebra $\mathcal{A}'$ with global dimension ≤ 2 whose Coxeter element $\Phi_{\mathcal{A}'}$ has spectral radius 1 and Jordan blocks of size no more than 2×2.*

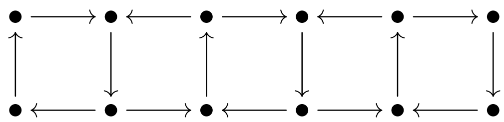
By 2d/4d correspondence, the inverse of the Cartan matrix of $\mathcal{A}'$, is precisely the Stokes matrix of the corresponding 2d $\mathcal{N}=(2,2)$ system S, and therefore the Coxeter element of $\Phi_{\mathcal{A}'}\equiv -C^tC^{-1}$ *coincides* with minus the 2d quantum monodromy $H=(S^{-1})^tS$. The 2d monodromy of a $\mathcal{N}=(2,2)$ system has spectral radius equal to one by definition, since the eigenvalues of its Jordan blocks decomposition are always pure phases. It is well–known [50] that, whenever the 2d theory is not conformal, H can have non–trivial Jordan blocks $J_n(\lambda)$ with $n\leq\hat{c}+1$ (strong monodromy theorem).

The requirement that $\mathcal{A}'$ has global dimension ≤ 2 arises from consistency in between the 3–CY completion and the B matrix one computes out of the Dirac pairing of $\mathcal{A}'$. By Gabriel theorem [68] any basic algebra $\mathcal{A}'$ is Morita equivalent to a *bounded* path algebra. A bounded path algebra is simply a path algebra $\mathbb{C}Q'$ modulo some ideal $\mathcal{I}$ such that *1.)* $\mathcal{I}$ is generated by paths of length ≥ 2 and *2.)* there exists an $m>0$ such that each path of length $\geq m$ in $\mathbb{C}Q'$ is contained in $\mathcal{I}$. Let e_i be the idempotents of the algebra $\mathcal{A}'$. Let S_i denote the corresponding simple $\mathcal{A}'$ modules. The Gabriel quiver Q' of the category of modules of $\mathcal{A}'$ is simply the quiver that has nodes equal to the S_i and in between nodes S_i and S_j dim $\mathrm{Ext}^1(S_i,S_j)$ arrows[8]. The relations in $\mathcal{I}$ have a cohomological interpretation: These are elements of the Ext^2 groups [70, 71]. It is then clear, by going through the various definitions, that the Dirac pairing of C^{-1} is the B matrix of the 3–CY completion quiver iff the global dimension of $\mathcal{A}'$ is ≤ 2. We stress that whenever one can find $\mathcal{A}'$, one obtains also the superpotential for the 4d quiver $(Q,\mathcal{W})$. The inverse problem, however, is not easy. Indeed, given a BPS–quiver in terms of its intersection matrix B, there are many ways of writing B as S^t-S, and many of such possible splitting could, in principle, correspond to a different $\mathcal{A}'$, and therefore to a different superpotential. In the case of Arnol'd models, however, we

[8] $\mathrm{Ext}^1(X,Y)$ can be viewed as the space that parametrizes equivalence classes of short exact sequences $0\to X\to M\to Y\to 0$ (in such a case, indeed, M is said to be the *extension* of X through Y).

know from geometry the precise form of the Coxeter–Dynkin diagrams, and, as we have discussed, this determine a natural candidate $\mathcal{A}'$ that, in turns, gives the right $\mathcal{W}$.

2.3.2. The square and the Coxeter–Dynkin forms of the quiver By repeated mutations (1d Seiberg dualities) we eliminate all diagonal arrows from the completed quiver $\widetilde{Q}$, and we end up with the *square* form of the quiver



(2.15)

where all squares are cyclically oriented[9]. Then the superpotential $\mathcal{W}$ is simply given by the sum of the traces of the products of Higgs fields along each oriented square.

Then

(The square form of the quiver with superpotential) *The quiver of the corresponding $\mathcal{N} = 2$ theory is obtained from the Coxeter–Dynkin diagram in the form of table 2.1 by eliminating the dashed arrows and orienting all the squares. The superpotential $\mathcal{W}$ is the sum of the traces of the cycles corresponding to the oriented squares.*

Of course, the quiver is not unique, and indeed each mutation class contains infinitely many different cluster–equivalent quivers. The one described above is particularly convenient for 'strong coupling' calculations. There is also a 'Coxeter–Dynkin' form of the quiver whose Jacobian algebra corresponds to the 3–CY completion of a *Coxeter–Dynkin algebra of extended canonical type*[10] $\widehat{D}(p,q,r)$ which is a tilting of (and hence derived equivalent to) the one–point extension of the canonical algebra $C(p,q,r)$ at a projective indecomposable. The quiver of $\Pi_3(\widehat{D}(p,q,r))$

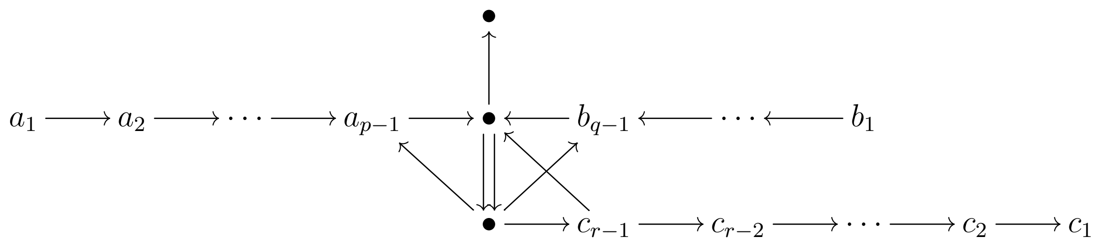
[9] The claim is easily checked with the help of Keller's quiver mutation applet [42].

[10] The the Coxeter–Dynkin algebra of extended canonical type, $\widehat{D}(p,q,r)$ is identified with the path algebra of the quiver in figure 2.1 with the Kronecher subquiver replaced by two dashed lines (*i.e.* by two relations) bounded by the ideal J generated by the two relations. Calling α_i, β_i, $i = 1,2,3$, the single arrows forming the oriented triangles in figure 2.1, the two relations are

$$\alpha_2\beta_2 + \alpha_3\beta_3 = 0 \qquad \text{and} \qquad \alpha_1\beta_1 = \alpha_3\beta_3,$$

from which we deduce the superpotential of the 3–CY completed *canonical* quiver SQM

$$\mathcal{W} = \lambda_1(\alpha_2\beta_2 + \alpha_3\beta_3) + \lambda_2(\alpha_1\beta_1 - \alpha_3\beta_3).$$



values of p, q, r

E_{12}	$2,3,7$	Z_{11}	$2,4,5$	Q_{10}	$3,3,4$	W_{12}	$2,5,5$	S_{11}	$3,4,5$
E_{13}	$2,3,8$	Z_{12}	$2,4,6$	Q_{11}	$3,3,5$	W_{13}	$2,5,6$	S_{12}	$3,4,5$
E_{14}	$2,3,9$	Z_{13}	$2,4,7$	Q_{12}	$3,3,6$			U_{12}	$4,4,4$

Figure 2.1: The quiver corresponding to the 3–Calabi–Yau completion of the Coxeter–Dynkin algebra of extended canonical type $\widehat{D}(p, q, r)$; the table gives the correspondence (singularity type) $\longleftrightarrow$ (p, q, r).

is presented in figure 2.1. For a discussion of the relevant extended canonical algebras and Coxeter–Dynkin algebras, and their relations to Arnol'd's exceptional singularities, see refs. [53, 54].

The *bi*–graph obtained by replacing in figure 2.1 the double arrows by dashed lines and all other arrows by solid edges was show by Ebeling [72] to correspond to the Coxeter–Dynkin diagram of the singularity with respect to a *strongly distinguished* homology basis (related to the previous one by a braid transformation). This is another check of the $2d/4d$ correspondence in the stronger version used here.

2.3.3. Minimal non–complete models The unimodal Arnol'd models are *not* complete theories[11]. For a *non*–complete theory, the computation of the BPS spectrum by any method related to the KS wall–crossing formula — such as cluster–combinatorics, or the stability conditions on quiver representations — is questionable on the grounds of the quantum Schottky problem. In this case, the spectrum we compute does not correspond to any physically realizable regime. Of course, the computation is still mathematically correct, and all the chamber independent quantities, like the conjugacy class of the quantum monodromy and the related

[11] By 2d/4d correspondence all complete theories have $\hat{c} \leq 1$, this is not the case for all Arnol'd models

UV invariants $\mathrm{Tr}[\,\mathbb{M}(q)^k]$, have their physically correct values, and we can always recover the physical spectrum (in principle) by applying the KS wall–crossing formula. However, as physicists, we are interested in knowing whether the spectrum we compute has a direct physical meaning, or if some further mathematical work is required to extract the physically relevant informations.

The purpose of the present subsection is to present some general remark on the question of the *physical realizability* of the special symmetric BPS chambers we use in our computations. The reader may prefer to skip the following qualitative discussion, and jump ahead to the more formal arguments.

The present theories, although non–complete, are *minimally* so, in the sense that the codimension of the image $\Im(\mathscr{P}) \subset \mathbb{C}^D$ is just 1. In other words, there is only one quantum–obstructed variation $(\delta Z_i)_{\mathrm{obs}}$ of the central charge function, normal to the physical submanifold $\Im(\mathcal{D}) \subset \mathbb{C}^D$. In general, modifications $Z_i \to Z_i + \delta Z_i$ correspond to infinitesimal deformations of the periods of the holomorphic 3–form Ω associated to deformations δt_j of the complex structure of the hypersurface $\mathscr{H}$ of the form

$$W(x,y,z) + u^2 + \sum\nolimits_j \delta t_j\, \phi_j = 0, \tag{2.16}$$

where $\{\phi_j\}$ is a basis of chiral primaries for the $2d$ LG model with superpotential $W(x,y,z)$. The offending deformation $(\delta Z_i)_{\mathrm{obs}}$ is the one associated to the unique chiral primary of dimension > 1, namely the Hessian $\mathfrak{H} = \det \partial_\alpha \partial_\beta W$. The problematic deformation is precisely the one defining the 1–parameter family of inequivalent singularities[12], which is the only primary perturbation which changes the behavior at infinity in field–space (and hence may spoil the quantum consistency). Indeed, the 2d renormalization group allows us to identify such directions. The 2d theory has infrared conformal fixed point dictated by the Zamolodchikov's c–theorem [73]. The infrared fixed point is stable under perturbation by irrelevant operators. Correspondingly, variations of the periods of Ω in these directions of the basis of the Milnor fibration [61] are forbidden physically, since these $2d$ deformations renormalize away. The unphysical deformations of the theory $(\delta Z_i)_{\mathrm{obs}}$ are precisely those corresponding to the irrelevant primary perturbations. In computing the BPS spectra, the combinatorics of the quantum clusters is not sensible to this fact: one can compute a mathematically consistent spectrum that corresponds to a BPS chamber that cannot be realized physically due to the above phenomenon — this is an instance of the quantum Schottky problem.

[12] By definition, a *unimodal* singularity has a 1–parameter family of inequivalent singular deformations.

E_{13}	$A_7 \boxtimes A_2\colon\ x^3+y^8+z^2\ \left(\frac{13}{12}\right)$	$xy^5\ \left(\frac{23}{24}\right)$	$\frac{1}{11}$
Z_{11}	$A_4 \boxtimes A_3\colon\ x^4+y^5+z^2\ \left(\frac{11}{10}\right)$	$x^3y\ \left(\frac{19}{20}\right)$	$\frac{1}{9}$
Z_{13}	$A_5 \boxtimes A_3\colon\ x^4+y^6+z^2\ \left(\frac{7}{6}\right)$	$x^3y\ \left(\frac{11}{12}\right)$	$\frac{1}{5}$
W_{13}	$A_5 \boxtimes A_3\colon\ x^4+y^6+z^2\ \left(\frac{7}{6}\right)$	$xy^4\ \left(\frac{11}{12}\right)$	$\frac{1}{5}$
S_{11}	$A_3 \boxtimes D_4\colon\ x^2z+z^3+y^4\ \left(\frac{7}{6}\right)$	$yz^2\ \left(\frac{11}{12}\right)$	$\frac{1}{5}$

Table 2.2: Arnol'd's *superconformal* gauge theories as IR fixed points of superconformal square tensor models [13] perturbed by the less relevant operator.

To address the physical realizability question, we have to make sure that, in the chamber we compute, the Hessian deformation is not switched on. Looking to the Coxeter–Dynkin diagrams in table 2.1, we see that they are all subgraphs of two kinds of (bi)graphs associated to direct sums of minimal singularities of the two forms

$$\begin{array}{c|c} A_n \boxtimes A_m & x^{n+1}+y^{m+1}+z^2 \\ \hline A_n \boxtimes D_4 & x^3+y^3+z^{n+1} \end{array} \tag{2.17}$$

to which the arguments of [13] directly apply. Physically, the 8 Arnol'd superconformal models which are not already of the form $G \boxtimes G'$ may be obtained as follows: one starts with a suitable 'big' $G \boxtimes G'$ theory, and perturbs it by a certain *relevant* operator (that is, relevant at the UV fixed point described by the $G \boxtimes G'$ theory), in such a way that the corresponding $\mathcal{N}=2$ theory will flow in the IR to the Arnol'd superconformal theory we are interested in.

In table 2.2 we list some convenient choices of UV $G \boxtimes G'$ theories and relevant perturbations $\phi_\star$ for five of the 8 non–product Arnol'd theories. The first number in parenthesis is the central charge $\hat{c}$ of the $2d$ UV Landau–Ginzburg; one has $\hat{c} < 2$, and hence the corresponding $4d$ $\mathcal{N}=2$ quantum theories exist by the criterion of refs. [9,13,18]. The second number in parenthesis is the UV dimension (in the $2d$ sense!) of the perturbing chiral primary $\phi_\star$; notice that it is always $2d$ relevant (at the UV fixed point). The last column of table 2.2 is the mass dimension of the $4d$ coupling $t_\star$ corresponding to the deformation $\phi_\star$ (at the UV fixed point) given by [18]

$$[t_\star] = \frac{2(1-q(\phi_\star))}{(2-\hat{c})}. \tag{2.18}$$

The two theories Z_{12} and Q_{11} are better described as the final IR fixed points of RG 'cascades'

$$A_5 \boxtimes A_3 \xrightarrow{x^3y\ (11/12)} Z_{13} \xrightarrow{xy^4\ (17/18)} Z_{12} \tag{2.19}$$

$$A_4 \boxtimes D_4 \xrightarrow{x^2z\ (13/15)} Q_{12} \xrightarrow{yz^3\ (14/15)} Q_{11} \tag{2.20}$$

where the perturbing monomials $\phi_\star$ and their dimensions are written over the corresponding arrow. S_{12} is more tricky; however we may still consider it as the IR fixed point of the model defined by the hypersurface $(y^2z+z^3+x^5)+xz^2+x^3y$ whose UV limit is $A_4 \boxtimes D_4$.

The above RG discussion applies directly to the Arnol'd $\mathcal{N}=2$ theories at their *superconformal point*, that is with all relevant deformations switched off. We are, of course, interested in the massive deformations of the theory which produce interesting chamber–dependent BPS spectra. For the massive case, we may argue as follows: we start with the $A_n \boxtimes G$ deformed hypersurface

$$\lambda\, y^{n+1} + W_G(x,z) + \phi_\star + \sum_i^{*} t_i \phi_i + v^2 = 0, \tag{2.21}$$

where the sum $\sum^*$ is over chiral primaries of dimension q less than $q(\phi_\star)$. By the criterion of [9, 18] , the hypersurface (2.21) corresponds to a physical regime of the (non–complete) $A_n \boxtimes G$ theory for all λ, t_i provided $\lambda \neq 0$. As $\lambda \to 0$, some states become infinitely massive and decouple. The decoupling limit produces a physically realizable regime of the mass–deformed $\mathcal{N}=2$ Arnol'd theory we are interested in.

The physical idea is then to control the realizability of a given BPS chamber for an Arnol'd theory by starting from the $A_n \boxtimes G$ theory (2.21), at large λ, in a BPS chamber which is known to be physical, and then continuously deform λ to zero, while ensuring that no wall of marginal stability is crossed in the process. By construction, we end up into a physical chamber of the (massive) Arnol'd theory, whose BPS spectrum differs from the one of the original $A_n \boxtimes G$ theory only because some particle got an infinite mass in the $\lambda \to 0$ limit and decoupled. In the process, we give volumes to the (special lagrangian) 3–cycles γ_i in the third homology group of the Calabi–Yau 3–fold

$$y^{n+1} + W_G(x,z) + u^2 = 0,$$

by the primary deformation of this singularity. The D3–branes that wrap around these 3–cycles, therefore, get central charges

$$Z(\gamma_i) = \int_{\gamma_i} \Omega\,,$$

becoming the BPS particles of the massive deformation of the corresponding 4 dimensional superconformal theory. If now we deform this theory with $\phi_\star$, along the 2 dimensional RG flow in the infrared some of the above 3–cycles start increasing their volume, that becames bigger and bigger the more close we are to the IR fixed point. Accordingly, the corresponding BPS particle masses increases.

Thus, these particles decouples and are absent from the BPS spectrum of the 4 dimensional theory obtained by engineering type IIB on the corresponding primary deformation of $\mathscr{H}$, the Calabi–Yau 3–fold associated to the IR theory.

As an initial reference chamber of the $A_n \boxtimes G$ theory we take one of those considered in [13]. In general, for a $G' \boxtimes G$ model there is a chamber with a finite spectrum consisting of hypermultiplets with charge vectors

$$\alpha \otimes \beta_a \in \Gamma \simeq \Gamma_G \otimes \Gamma_{G'}, \qquad \alpha \text{ positive root of } G,\ \beta_a \text{ simple roots of } G'. \quad (2.22)$$

There is an obvious duality $G \leftrightarrow G'$ which produces a second finite chamber with the role of G and G' interchanged. It is believed [13] that these two BPS chambers do correspond to physical situations, and hence they may be used as the starting points at large λ for the family of theories (2.21).

Let us sketch the argument of [13] for $\mathcal{N}=2$ models of the form $A_n \boxtimes G$, where G is any ADE Dynkin diagram. Such models are engineered by Type IIB on a hypersurface $\mathscr{H}\colon W_G(x,y,z)+P_{n+1}(v)=0$, where $W_G(x,y,z)$ stands for the usual G minimal singularity and $P_{n+1}(v)$ is a degree $n+1$ polynomial that we take of the Chebyshev form. We can see this geometry as a compactification of IIB down to 6 dimensions on a deformed G–singularity whose deforming parameters depend (adiabatically) on the complex coordinate v. As in ref. [18], the compact 3–cycles on the hypersurface $\mathscr{H}$ are seen as vanishing 2–cycles of the G–type singularity fibered over a curve in the v–plane connecting two zeros of $P_{n+1}(v)$. The G–singularity produces tensionless strings in one–to–one correspondence with the positive roots of G. Let $\delta_\alpha(v)$ be the vanishing cycle over v associated to the positive root α. We define and effective SW differential

$$\lambda_\alpha(v) = \int\limits_{\delta_\alpha(v)} \Omega \sim \big(P_{n+1}(v)\big)^{\Delta_\alpha}\, dv, \quad (2.23)$$

vanishing at the zeros of $P_{n+1}(v)$. Then for each $\alpha \in \Delta_+(G)$ we may repeat the analysis of [18], showing that the spectrum (2.22) corresponds to a physically realizable chamber.

In practice, it may be difficult to check the existence, in the complex λ–plane, of a path from zero to infinity which avoids all wall–crossings while keeping control of the possible mixing between the conserved quantum currents. Therefore, we shall mostly use the above idea in a weak sense, namely, we shall consider a mathematically correct BPS spectrum which is naturally interpreted as the result of the decoupling of some heavy states from the known physical spectrum of the appropriate $A_n \boxtimes G$ model, as a physically sound BPS spectrum which, having a simple physical interpretation, also provides circumstantial evidence for the physical realizability of the corresponding chamber.

2.3.4. Flavor symmetries The number f of flavor charges, or more precisely the dimension of the Cartan subalgebra of the flavor symmetry group F, is an important invariant of the theory, which is independent of the parameters (however, at particular points in the physical domain $\mathscr{P}$ we may have a non–Abelian enhancement of the flavor symmetry, $U(1)^f \to H \times U(1)^{f-\mathrm{rk}(H)}$, which preserves its rank). A general consequence of $2d/4d$ correspondence [7, 13] is that f, which is (by definition) the number of zero eigenvalues of the Dirac pairing matrix B_{ij}, is equal to the number of $2d$ chiral primary operators whose UV dimension q_i is equal to $\hat{c}/2$. Indeed, a charge is flavor only if

$$B\gamma = 0 \Leftrightarrow S\gamma = S^t\gamma \Leftrightarrow \Phi\gamma = -\gamma \tag{2.24}$$

The eigenvalues of Φ are equal to $-\exp[2\pi i(q_i - \hat{c}/2)]$ [50], and so f is equal to the multiplicity of -1 as an eigenvalue of Φ. Then f may be directly read from the factorization of the characteristic polynomial of Φ into cyclotomic polynomials, see table 2 of ref. [53]: f is just the number of Φ_2 factors in the product. Thus

$$f = \begin{cases} 2 & Z_{12}, U_{12} \\ 1 & \text{odd rank} \\ 0 & \text{otherwise.} \end{cases} \tag{2.25}$$

2.3.5. Order of the quantum monodromy The quantum monodromy $\mathbb{M}(q)$ of a $\mathcal{N} = 2$ model engineered by Type IIB on a non–compact CY hypersurface $\mathscr{H} \subset \mathbb{C}^4$, given by the zero locus of a (relevant deformation of a) quasi–homogeneous polynomial $f_0(x_i)$, has a finite order ℓ, that is,

$$\mathbb{M}(q)^\ell = 1, \qquad \text{(identically in } q \in \mathbb{C}^*) \tag{2.26}$$

in the sense of equality of adjoint actions on the quantum torus algebra $\mathbb{T}_\Gamma$. The minimal value of the integer ℓ is easy to predict. Let d, w_1, w_2, w_3, w_4 be integers such that

$$\lambda^d\, f_0(x_i) = f_0(\lambda^{w_i} x_i) \quad \forall\, \lambda \in \mathbb{C}, \tag{2.27}$$

normalized so that $\gcd\{d, w_1, w_2, w_3, w_4\} = 1$. The redefinition $x_i \to \lambda^{w_i}\, x_i$ transforms the CY holomorphic 3–form Ω into $\lambda^{\sum_i w_i - d}\Omega$. Hence the monodromy corresponds to replacing λ with $\exp[2\pi i t/(\sum_i w_i - d)]$ and continuously taking t from 0 to 1. In terms of the original variables, this is

$$x_i \to \exp\left(2\pi i \frac{w_i}{\sum_i w_i - d}\right) x_i \tag{2.28}$$

and the monodromy order ℓ is

$$\ell = \frac{\sum_i w_i - d}{\gcd\{\sum_i w_i - d, w_1, w_2, w_3, w_4\}}. \tag{2.29}$$

E_{13}	7	Z_{11}	7	Z_{12}	5	Z_{13}	8	W_{13}	7
		Q_{11}	8			S_{11}	7	S_{12}	11

Table 2.3: (Reduced) orders of the quantum monodromy for the 8 Arnol'd's exceptional theories which are not of the tensor from $G \boxtimes G'$.

In the case of a singularity of the form $f_0(x, y) + z^2 + v^2$ it is more convenient to consider the reduced order, corresponding to the engineering of the model from the $6d$ $(2, 0)$ theory. It corresponds to setting $w_3 = d, w_4 = 0$ in the above formula.

2.3.6. Arnol'd's exceptional $\mathcal{N} = 2$ models as gauge theories In the title we referred to the Arnol'd's exceptional models as *gauge* theories. Up to now, the gauge aspect of these models has not manifested itself. Although we are mainly interested in 'strong coupled' regimes in which the BPS spectrum contains just finitely many hypermultiplets, these theories do have 'weakly coupled' phases where BPS vector–multiplets are present. At least in the simplest situations, the couplings of these vector–multiplets may be physically interpreted as a super–Yang–Mills sector weakly gauging a subgroup G of the global symmetry group of some 'matter' system (which is non–Lagrangian, in general). Hence the Arnol'd exceptional $\mathcal{N} = 2$ theories behave as gauge theories in some corner of their parameter space, although a full understanding of the phases with stable BPS vector–multiplets requires a more in–depth study. From our general discussion about the quantum Shottky problem, combined with the knowledge that for these model $\mathrm{codim}\Im(\mathscr{P}) = 1$, follows, just counting dimensions, that a minimal non–complete $\mathcal{N} = 2$ model which has, in some limit, the structure of a G SYM weakly coupled to some other sector, the gauge group G must have one of the following forms

$$SU(2)^k, \quad SU(2)^k \times SU(3), \quad SU(2)^k \times SO(5), \quad SU(2)^k \times G_2, \tag{2.30}$$

for some $k \in \mathbb{N}$. For the exceptional Arnol'd models, it is easy to prove the existence of physical limits with $G = SU(2)$, while larger gauge groups are not at all excluded.

To produce a physical regime with a weakly coupled $SU(2)$ SYM sector, it is enough to deform the Arnol'd singularity with suitable lower–order monomials (corresponding to a particular choice of the central charge function Z_i inside the physical region $\Im(\mathscr{P})$) which causes the flow, in the IR, to one of the elliptic–E

complete superconformal gauge theories [7], and specifically

$$\begin{array}{ll} \text{to the } E_8^{(1,1)} \equiv A_2 \boxtimes A_5 \text{ model} & \text{for } E_{12}, E_{13}, E_{14}, \\ \text{to the } E_7^{(1,1)} \equiv A_3 \boxtimes A_3 \text{ model} & \text{for } Z_{11}, Z_{12}, Z_{13}, W_{12}, W_{13}, \\ \text{to the } E_6^{(1,1)} \equiv A_2 \boxtimes D_4 \text{ model} & \text{for } Q_{10}, Q_{11}, Q_{12}, S_{11}, S_{12}, U_{12}. \end{array}$$

The IR effective theory is known to have physical chambers with a stable $SU(2)$ gauge vector coupled to three D–type Argyres–Douglas systems [7]. Since the IR theory is complete, we can tune the coefficients of the defining polynomial of $\mathscr{H}$ to get an arbitrarily weak gauge coupling.

One way to prove the existence of a *formal* BPS chamber with a stable BPS vector–multiplet is to look for a (non–necessarily full) subquiver $\mathcal{S}$ of Q which is mutation equivalent to an acyclic affine quiver. This generalizes the strategy of looking for Kronecker, *i.e.* $\widehat{A}(1,1)$, subquivers used in [7].

If $\mathcal{S}$ is a *full* subquiver of Q, the existence of a mathematical BPS chamber with a stable BPS vector–multiplet is guaranteed: Indeed, the quantization of the $\mathbb{P}^1$ family of brick representations[13] of $\mathcal{S}$ with dimension vector $\sum_i N_i\alpha_i$ equal to the minimal imaginary root δ, extended by zero to a representation of the total quiver Q, produces — for suitable choices of the complex numbers Z_i — a stable BPS vector–multiplet.

If $\mathcal{S}$ is not a full subquiver, the statement remains true, provided the above $\mathbb{P}^1$ family of representations of $\mathcal{S}$, when seen as representations of the total quiver Q, has the following two properties: 1) it satisfies the relations $\partial\mathcal{W} = 0$ induced from the arrows in $Q \setminus \mathcal{S}$, and 2) it does not admit further continuous deformations corresponding to switching on non–trivial maps along the arrows of the full subquiver over the nodes $\mathcal{S}_0$ which are not in $\mathcal{S}$. Indeed, if this *no–extra–deformation* condition is not verified, we have to quantize a moduli space of dimension larger than one, possibly producing higher spin representations of $\mathcal{N} = 2$ supersymmetry, instead than just vector–multiplets.

The quivers of the exceptional Arnol'd models always have affine subquivers (as it is already evident from the Coxeter–Dynkin form of the quiver, see figure 2.1) and we may even find *pairs* of non–overlapping such affine subquivers, leading to the possibilities of chambers with more than one BPS vector–multiplet.

As an (intriguing) example, take the model E_{13} and consider the following pair

[13] The existence of this family of brick representations follows directly from Kac's theorem [74]. For details see *e.g.* [65, 75–77].

of $\mathbb{P}^1$ families of representations with mutually disjoint support

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & 0 & \longleftarrow & 0 & \longrightarrow & 0 & \longleftarrow & \mathbb{C} & \xrightarrow{0} & \mathbb{C} & & \\
\uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow{\scriptstyle [0,1]^t} & & \downarrow{\scriptstyle [1,0]} & & \\
0 & \longleftarrow & 0 & \longrightarrow & 0 & \longleftarrow & \mathbb{C} & \xrightarrow[i]{} & \mathbb{C}^2 & \xleftarrow[1]{} & \mathbb{C}^2 & \xrightarrow[{[1,1]^t}]{} & \mathbb{C}
\end{array} \tag{2.31}$$

$$\begin{array}{ccccccccccccc}
\mathbb{C} & \xrightarrow{i} & \mathbb{C}^2 & \xleftarrow{1} & \mathbb{C}^2 & \xrightarrow{[1,1]^t} & \mathbb{C} & \longleftarrow & 0 & \longrightarrow & 0 & & \\
\uparrow & & \downarrow{\scriptstyle [0,1]^t} & & \uparrow{\scriptstyle [1,0]} & & \downarrow & & \uparrow & & \downarrow & & \\
0 & \longleftarrow & \mathbb{C} & \xrightarrow[0]{} & \mathbb{C} & \longleftarrow & 0 & \longrightarrow & 0 & \longleftarrow & 0 & \longrightarrow & 0
\end{array} \tag{2.32}$$

where the map $\mathbb{C} \xrightarrow{i} \mathbb{C}^2$ defines a line in $\mathbb{C}^2$ and hence a point in $\mathbb{P}^1$. Both representations are pulled back from a representation of a $\widehat{D}_5$ *non*–full subquiver having dimension vector the minimal imaginary root. Note that the representations satisfy the constraints from the F–term flatness conditions $\partial\mathcal{W} = 0$. It remains to check that there are no continuous deformations of these $\mathbb{P}^1$ families obtained by giving non–zero values to the omitted arrow (the arrow with an explicit 0 in eqns.(2.31)(2.32)). Indeed, these arrows are constrained to remain zero by the F–term relations $\partial\mathcal{W} = 0$. Hence, the $\mathbb{P}^1$ family is not further enlarged, and the corresponding BPS vector–multiplet is stable for a suitable choice of the Z_i's. We write δ_1, δ_2 for the charge vectors of the resulting vector–multiplets. Counting arrows, we see that

$$\langle \delta_1, \delta_2 \rangle_{\text{Dirac}} = 1 \tag{2.33}$$

Hence the two vector–multiplets are *not* mutually local. If the mathematical chamber in which both vectors are stable is physically realizable — which is certainly *not* guaranteed, and perhaps unlikely — the physics will not be that of a conventional gauge theory.

2.3.7. A baby example (E_7 AD in diverse chambers) We illustrate the method and its physical meaning in a very baby model: we consider the rank 7 quiver

$$\begin{array}{ccccccc}
1,1 & \xrightarrow{a} & 1,2 & \xleftarrow{e} & 1,3 & & \\
{\scriptstyle d}\uparrow & & \downarrow{\scriptstyle b} & & \uparrow{\scriptstyle g} & & \\
2,1 & \xleftarrow[c]{} & 2,2 & \xrightarrow[f]{} & 2,3 & \longleftarrow & 0
\end{array} \tag{2.34}$$

with superpotential

$$\mathcal{W} = \mathrm{Tr}(dcba) + \mathrm{Tr}(fgeb). \tag{2.35}$$

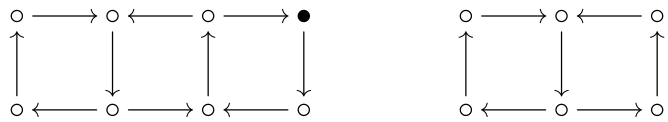


Figure 2.2: The $A_4 \boxtimes A_2$ and the $A_3 \boxtimes A_2$ quivers.

Decoupling limits and all that

Physically, we may realize the $\mathcal{N}=2$ superconformal theory described by (2.34)(2.35) as the IR fixed point of the theory associated to the $A_4 \boxtimes A_2$ quiver (the first quiver in figure 2.2) perturbed by a suitable relevant operator which corresponds to giving a large central charge $|Z_\bullet| \gg \Lambda$ to the black node in figure 2.2, with the effect of decoupling (in the IR) all the degrees of freedom carrying a non–zero $\bullet$ charge. In the same way, the $A_3 \boxtimes A_2$ theory (second quiver in figure 2.2) may be seen as a suitable IR limit of the theory (2.34) where we take $|Z_0| \to \infty$. Since $A_4 \boxtimes A_2$ and $A_3 \boxtimes A_2$ are, respectively, the type–E_8 and type–E_6 Argyres–Douglas models [13], the theory (2.34) should be a rank 7 Argyres–Douglas theory, and hence the type–E_7 one.

Of course, there is an elementary direct proof of this last identification: mutating (2.34) one gets the E_7 quiver in its standard Dynkin form. However, here we are interested in the Dynkin subquiver viewpoint which will turn useful for the more complicated Arnol'd models. The present baby example is conceptually simpler, since the theory is actually complete [7], and all formal manipulations at the quiver level do have a direct physical meaning, and we are allowed to be naive.

From the previous discussion, we see than the theory (2.34) is a decoupling limit of the $4d$ $\mathcal{N}=2$ theory geometrically engineered by the E_8–singular local Calabi–Yau hypersurface

$$x^5 + y^3 + uv = 0, \tag{2.36}$$

deformed by the relevant perturbation

$$x^5 + y^3 + \epsilon\, x^3 y + uv = 0 \tag{2.37}$$

equal to the Hessian $\mathfrak{h}$ of the LHS of (2.36) (*i.e.* the *less relevant* relevant deformation, from both the $2d$ and $4d$ viewpoints). In the IR the theory flows to the fixed point corresponding to the singular hypersurface $y^3 + yx^3 + uv = 0$ (after a rescaling of the x coordinate).

The quiver (2.34) has an obvious decomposition into the complete family of Dynkin subquivers

$$A_2 \coprod A_2 \coprod A_2 \coprod A_1, \tag{2.38}$$

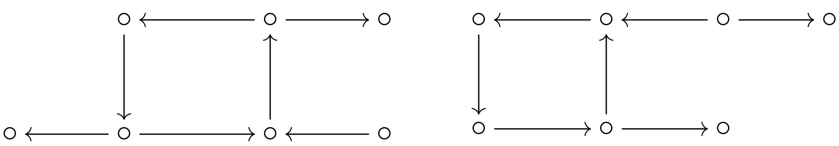


Figure 2.3: Other quivers in the E_7 class with a complete family of Dynkin subquivers

where the three copies of A_2 correspond to the subquivers over the nodes $\{(1,a),(2,a)\}$, $a=1,2,3$. Dually, we have the complete family

$$A_4 \coprod A_3 \tag{2.39}$$

By mutating at 0 we get a quiver which admits the complete families

$$A_3 \coprod A_3 \coprod A_1 \quad \text{and} \quad A_3 \coprod A_2 \coprod A_2. \tag{2.40}$$

Other quivers in the E_7 mutation class may admit a complete family of Dynkin subquivers, see *e.g.* the two quivers in figure 2.3 (as well as, of course, the E_7 Dynkin quiver itself, and the seven A_1's: these two cases being already covered in [13]).

All these decompositions into Dynkin subquivers correspond to BPS chambers of the E_7 AD theory whose BPS spectrum may be easily derived using the combinatorial methods of the present section. Of course, this is not so interesting for Argyres–Douglas theories, but it becomes relevant when applied to more general theories, see the next two sections.

Suppose we start with the E_8 Argyres–Douglas in a BPS chamber where the spectrum is given by a set of hypermultiplets with charge vectors in[14] $\Gamma \simeq \Gamma_{A_2} \otimes \Gamma_{A_4}$ of the form

$$\alpha \otimes \beta_a, \qquad \begin{array}{ll} \alpha \in \Delta_+(A_2), & (\underline{\text{all}} \text{ positive roots of } A_2) \\ \beta_a,\ a=1,2,3,4, & (\underline{\text{simple}} \text{ roots of } A_4) \end{array} \tag{2.41}$$

Such a chamber of the E_8 AD theory is physically realizable since the model is complete. Now switch on the perturbation corresponding to the Hessian $\mathfrak{h}$; by what we saw above, the deformation is expected to give a large mass to the states having a charge vector $\sum_{i,a} d_{i,a}\, \alpha_i \otimes \beta_a$ with $d_{1,4} \neq 0$. Tuning the phase of the

[14] Γ_G denotes the root lattice of the simple simply–laced Lie group G.

parameter ϵ in such a way[15] that the deformation does not trigger spurious wall–crossings, the net effect is just that in the IR two of the states (2.41) decouple, namely those of charge $\alpha_1 \otimes \beta_4$ and $(\alpha_1+\alpha_2)\otimes\beta_4$, the others remaining unaffected.

Stability of quiver representations

Of course, the same spectrum could be obtained directly from the analysis of the stable representations of the quiver (2.34): if $\arg Z_0 < \arg Z_{i,a}$, it is easy to see[16] that the stable representations are the ones with charge vector α_0 plus the stable representations of the subquiver $A_3 \boxtimes A_2$, which (in the corresponding BPS chamber) have charge vectors $\alpha \otimes \beta_a$, $a = 1, 2, 3$. In this case the theory is complete [7] and the two methods give equivalent information. However, in general, the decoupling analysis is more powerful: computing the BPS spectrum mathematically, we get a spectrum which is wall–crossing equivalent to the physical one, but we do not know whether the particular chamber in which we computed it may or may not be physically realized (see discussion in [7]). The decoupling analysis, instead, gives us a physical definition of the BPS chamber we are computing in, and we are guaranteed that our finding have a direct physical meaning as actual particle spectra. Of course, it is also more delicate, since as we move in parameter space we have to control both the wall–crossings and the potential mixing of conserved charges.

Cluster combinatorics

Let us compare the results of the decoupling (or quiver representation theoretical) analysis with those of the cluster–combinatorial method. We consider only the decomposition (2.38) which corresponds to the decoupling limit of the chamber (2.41), leaving the other cases as an amusement for the reader.

Let Λ be the node sequence

$$\begin{aligned}\Lambda = \{&(1,1),(1,3),(2,2),0,(1,2),(2,1),(2,3),(1,1),(1,3),(2,2),\\ &(2,1),(2,3),(1,2),0,(2,2),(1,1),(1,3),(2,1),(2,3),(1,2)\}. \end{aligned} \tag{2.42}$$

[15] This is possible since the model is complete and has a finite spectrum in all chambers, the chamber themselves being finite in number.

[16] Let X be a *stable* representation of the (bound) quiver (2.34) with $(\dim X)_0 = n \neq 0$. Then there is an exact sequence of the form

$$0 \to Y \to X \to S(0)^{\oplus n} \to 0,$$

where $S(0)$ is the simple representation with vector space $\mathbb{C}$ at the 0–th node, and zero elsewhere. If $Y \neq 0$, one has $\arg Z(Y) = \arg(Z(X) - nZ_0) > \arg Z(X)$. Since X is stable, we get a contradiction. Hence $Y = 0$.

It is easy to check (using, say, Keller's applet [42]) that this is a sink–factorized sequence with respect to the complete family of Dynkin subquivers (2.38). $\boldsymbol{m}_\Lambda$ acts as the identity on $\mathbb{T}_Q$, and hence the above sequence corresponds to the full quantum monodromy $\mathbb{M}(q)$. We obtain the charge vectors

$$\gamma_0,\ -\gamma_0,\quad \text{and}\quad \begin{cases} (s_2 s_1)^{k-1}\alpha_2 \otimes \beta_a \\ (s_2 s_1)^{k-1} s_2 \alpha_1 \otimes \beta_a \quad k=1,2,3,\ a=1,2,3 \end{cases} \tag{2.43}$$

where α_i (resp. s_i) are the simple roots (resp. simple reflections) of A_2. Note that $s_2 s_1$ is a Coxeter element of A_2. Then, for a fixed a, we get the spectrum of A_2 AD in the maximal chamber (which, in particular, shows that both the PCT and phase–ordering requirements are automatically satisfied). Hence the spectrum consists of a set of hypermultiplets with charge γ_0 and $\alpha \otimes \beta_a$, $a = 1, 2, 3$ (α any positive root of A_2) which is precisely the spectrum predicted by the other two methods.

This illustrates as the cluster combinatoric captures the BPS spectrum without going trough a detailed analysis.

2.4. BPS spectra

We have several decompositions into complete families of Dynkin quivers which admit full Coxeter–factorized sequences of the standard type, namely,

$$(G_1, G_2, \ldots, G_s) \tag{2.44}$$

Hence the spectrum is always given by equation (1.59), and in particular is consistent with both PCT and the phase–ordering inequalities. This result is confirmed by the stability analysis of the quiver representations, as well as by the physical idea of decoupling states from a parent $A_m \boxtimes G$ theory. The fact that the corresponding monodromies do have the right order ℓ (cfr. table 2.3) guarantees the correctness of the result.

A list of Weyl–factorized sequence types is presented in table 2.4. For the details of these Weyl–factorized sequences, see appendix C.3.

Table 2.4 presents a list of chambers with finite BPS spectra, which have natural physical interpretations, and hence are expected to be physically realized. Of course, as for the E_7 AD model in §.2.3.7, there exist other chambers in which the spectrum has the 'Weyl–factorized' form (1.59). Indeed, the (infinite) mutation classes of the exceptional Arnol'd quivers contain many quivers which admit complete families of Dynkin subquivers: Our combinatoric methods apply to all these chambers in a straightforward way. As an example we show nontrivial chambers

E_{13}	(A_7, A_6)	Q_{10}	(D_4, D_4, A_2)
	(A_6, A_6, A_1)		(A_3, A_3, A_2, A_2)
	$(A_2, A_2, A_2, A_2, A_2, A_2, A_1)$		(D_4, D_4, A_1, A_1)
Z_{11}	(A_4, A_4, A_3)		$(A_2, A_2, A_2, A_2, A_1, A_1)$
	(A_3, A_3, A_3, A_2)	Q_{11}	(D_4, D_4, A_2, A_1)
Z_{12}	(A_4, A_4, A_3, A_1)		$(A_3, A_3, A_2, A_2, A_1)$
	$(A_3, A_3, A_3, A_2, A_1)$		(A_4, A_3, A_2, A_2)
	(A_5, A_4, A_3)	Q_{12}	(D_4, D_4, A_2, A_2)
Z_{13}	(A_5, A_5, A_3)		(A_4, A_4, A_2, A_2)
	$(A_3, A_3, A_3, A_2, A_2)$	S_{11}	(D_4, D_4, A_3)
W_{13}	$(A_3, A_3, A_3, A_3, A_1)$		(A_3, A_3, A_3, A_2)
	(A_4, A_4, A_4, A_1)	S_{12}	(D_4, D_4, A_3, A_1)
	(A_5, A_4, A_4)		$(A_3, A_3, A_3, A_2, A_1)$
			(A_4, A_3, A_3, A_2)

Table 2.4: Types of some Weyl–factorized sequences for the Arnol'd exceptional $\mathcal{N} = 2$ theories. Each of them correspond to a BPS chamber of the corresponding $\mathcal{N} = 2$ theory with a finite spectrum having the direct–sum form (1.59).

for the quivers of type Q_{10} and Q_{12} obtained by elements of the mutation class that are not of the (G, G') type. However, in general it is difficult to establish whether a given chamber is physical or not, even at the heuristic level. This is one reason why here we have not attempted a full classification of all 'factorized' chambers, but limited ourselves to the set of those chambers simply related to the analysis of ref. [13].

2.5. The Y–systems and their periodicity

A general consequence of ref. [13] is that the quantum (fractional) monodromy $\mathbb{M}(q)$ (resp. $\mathbb{Y}(q)$) of a $4d$ $\mathcal{N}=2$ model geometrically engineered by Type IIB on an isolated quasi–homogeneous singularity with $\hat{c}<2$ has finite order. Moreover, in each BPS chamber of such a theory with a *finite* BPS spectrum the (fractional) monodromy is written as a finite product of elementary quantum cluster mutations.

In the classical limit $q \to 1$, the action of $\mathbb{M}(q)$ (resp. $\mathbb{Y}(q)$) on the quantum torus algebra $\mathbb{T}_Q$ reduces to the corresponding KS *rational* symplecto–morphism of the complex torus $T \sim (\mathbb{C}^*)^{\operatorname{rank}\Gamma}$ [4, 26], which is directly related to to the hyperKähler geometry of the $3d$ dimensional version of the theory [4]. As explained in [4, 13], the resulting symplectic rational maps form a Y–system in the sense of the Thermodynamical Bethe Ansatz [59].

The usual TBA periodic Y–systems [59, 60, 78, 79] correspond to 'decoupled' singularities of the form $W_G + W_{G'}$. Ref. [13] predicts the existence of many others such periodic Y–systems associated to non–decoupled singularities. Here we explain how we have checked this prediction for the Arnol'd's exceptional singularities.

We start by reviewing the construction of the Y–system from the quantum monodromy. Recall from chapter 1 that the (fractional) quantum monodromy, as computed from the BPS data in a *finite* chamber, may be seen as the result of a sequence of quantum mutations of the torus algebra $\mathbb{T}_Q$. The action of $\mathbb{M}(q)$ on the quantum torus algebra is specified by the action on the set of generators $\{Y_i\}_{i \in Q_0}$ where, as usual, we write Q_0 for the set of nodes of Q,

$$Y_i \to Y_i' \equiv \operatorname{Ad}(\mathbb{M}^{-1}) Y_i \equiv N[R_i(Y_j)], \tag{2.45}$$

here $N[R_i(Y_j)]$ stands for the normal–order version of the rational function $R_i(Y_j)$ of the operators Y_j [13]. R_i has the log–symplectic property

$$\langle \alpha_i, \alpha_j \rangle_{\text{Dirac}}\, d\log Y_i \wedge d\log Y_j = \langle \alpha_i, \alpha_j \rangle_{\text{Dirac}}\, d\log R_i \wedge d\log R_j. \tag{2.46}$$

The rational map $Y_i \to R_i$ is simply the classical limit of the monodromy action, from which we may recover the full quantum action by taking the normal–order prescription for the operators (this is true [13] for all simply–laced, *i.e.* $|B_{ij}| \leq 1$, quivers). This shows that the quantum monodromy, acting on $\mathbb{T}_Q$ in the adjoint fashion, has finite order ℓ if and only if the rational map $\mathbb{C}^D \to \mathbb{C}^D$

$$R\colon \quad Y_i \mapsto R_i(Y_j) \tag{2.47}$$

has order ℓ.

On the other hand, $\mathrm{Ad}(\mathbb{M}^{-1})$ is an ordered product of basic quantum mutations of the form $\overset{\curvearrowright}{\prod} \mathcal{Q}_k$. The rational map $Y_j \to R_j$ coincides with the composition of the rational functions $R_j^{(k)}$ giving the classical limit of each basic mutation $\mathcal{Q}_k$ in the product. The map $Y_j \to R_j^{(k)}$ is just the elementary mutation at the k–th node of the Y–seed in the sense of Fomin–Zelevinsky[17] [43,80] but for Q replaced by the opposite quiver Q^{op} (see *e.g.* [39]). The Keller applet [42] automatically generates the Y–seed mutations for any quiver Q, and hence, although the actual form of the rational map (2.47) is typically quite cumbersome, it is easily generated by a computer procedure.

By definition, the Y–system associated to a finite chamber of a $\mathcal{N} = 2$ model is simply the recursion relation generated by the iteration of the rational map (2.47), namely

$$Y_j(s+1) = R_j\big(Y_k(s)\big), \qquad s \in \mathbb{Z}. \tag{2.48}$$

Specializing to the $\mathcal{N} = 2$ (G, G') theories studied in [13], eqn.(2.48) reproduces the well–known TBA Y–systems for the integrable $2d$ (G, G') models [59,60,78,79].

Although we have generated at the computer the Y–systems for all the exceptional Arnol'd models, to avoid useless vaste of paper, here we limit ourselves to present the explicit form of just a couple of examples: see the appendices. All the others may be straightforwardly generated, using the explicit Weyl–factorized sequences listed in appendix C.3, by the same computer procedure.

We stress that, although the explicit form of the Y–system depends on the particular finite BPS chamber we use to write the map (2.47), two Y–systems corresponding to different chambers of the *same* $\mathcal{N} = 2$ theory are equivalent, in the sense that they are related by a rational change of variables $Y_j \to Y'_j(Y_k)$. Indeed, the monodromy $\mathbb{M}(q)$ is independent of the chamber up to conjugacy, and so is its classical limit map $Y_j \to R_j$. Hence the rational maps R_j obtained in different chambers are conjugate in the Cremona group.

In conclusion, the (adjoint action of the) quantum monodromy $\mathbb{M}(q)$ has a finite order ℓ if and only if the corresponding Y–system is periodic with (minimal) period ℓ, that is

$$\mathrm{Ad}\big[\mathbb{M}(q)^\ell\big] = \mathrm{Id} \quad \Longleftrightarrow \quad Y_j(s+\ell) = Y_j(s), \ \ \forall\, j \in Q_0, \ s \in \mathbb{Z}. \tag{2.49}$$

[17] For Y variables in the universal semi–field.

For the Arnol'd exceptional models, we know from string theory that $\mathbb{M}(q)$ has the finite orders ℓ listed in table 2.3. This proves that the corresponding Y–systems are periodic of period ℓ. It should be possible to give an interpretation of these new periodic Y–systems in terms of exactly solvable $2d$ theories in analogy with the (G, G') ones [59, 78, 79].

At the mathematical level, we get an unexpected relation between singularity theory and cyclic subgroups of the Cremona groups $\mathrm{Cr}(n)$ of birational automorphisms $\mathbb{P}^n \to \mathbb{P}^n$, both interesting subjects in Algebraic Geometry (the second one being notoriously hard for $n \geq 3$ [81]).

2.5.1. Checking the periodicity Type IIB engineering of the model together with our computation of the BPS spectrum *proves* that the corresponding Y–systems are periodic with the periods listed in table 2.3. However, as a check, we wish to give an independent proof of the periodicity.

In principle, to prove periodicity, one has just to iterate ℓ times the rational map R of eqn.(2.47), and check that the resulting rational map is the identity. Unfortunately, at the intermediate stages of the recursion, one typically gets rational functions so cumbersome that no computer can handle them analytically [82]. Luckily, there is an alternative strategy advocated by Fomin in [82]. The ℓ–fold interation of R, R^ℓ, is a rational map whose fixed–point subvariety $\mathscr{F}$ has some codimension n in $\mathbb{C}^D$. Periodicity is just the statement that $n = 0$.

If we specialize the Y_i's to randomly chosen numbers uniformely distributed in some disk of radius ρ, compute numerically the transformation $R^\ell(Y_i)$, and get back the original point Y_i, we conclude that our randomly chosen point Y_i lays on the fixed–point subvariety $\mathscr{F}$ within the computational numerical accuracy ϵ. The probability that a randomly chosen point appears to be on the fixed locus $\mathscr{F}$ is then of order $(\epsilon/\rho)^{2n}$.

Therefore, the probability that applying R^ℓ to a sequence of k random points we get back the same sequence of points, is of order $(\epsilon/\rho)^{2nk}$. Since $\epsilon/\rho \sim 10^{-11}$, for $n \neq 0$ the probability goes quite rapidly to zero as we increase k. If we do get back the original sequence of points for, say, $k = 5$, we may conclude that $n = 0$ with a confidence level which differs from 100% by a mere 10^{-108} %.

Using this strategy, we have checked all the periodicities listed in table 2.3. The interested reader, may find the details of the check for the S_{11} model in appendix C.2.

We have also checked the order of the 1/4–fractional monodromy for $SU(2)$ SQCD with four flavors, getting 4, namely order 1 for the full monodromy $\mathbb{M}(q)$, in agreement with the physical prediction based on the fact that all chiral primaries have integral dimension.

2.6. Bimodal singularities

In this section we extend, without effort, the results of this chapter to the 4d $\mathcal{N}=2$ SCFT's that correspond to Arnol'd bimodal singularities.

Bimodal singularities are fully classified [51]: they are organized in 8 infinite series and 14 exceptional families. All 14 exceptional families have a quasi–homogeneous point in their moduli and, in between the 8 infinite series, there are 6 families that admit one. With an abuse of language, we will refer to this set as to the set of quasi–homogeneous bimodal singularities. The quasi–homogeneous potentials, $W(x,y,z)$, that corresponds to the elements of this set lead to non–degenerate $2d$ $\mathcal{N}=(2,2)$ Landau–Ginzburg superconformal field theories that have central charge $\hat{c}<2$, therefore, according to [?, 9], these singularities are all at finite distance in Calabi–Yau moduli space: the local CY 3–fold $\mathscr{H}$, given by the hypersurface in $\mathbb{C}^4$

$$\mathscr{H}: W(x,y,z)+u^2+\text{ lower terms}=0. \tag{2.50}$$

is a good candidate for the compactification of type IIB superstring that leads to the engineering of an $\mathcal{N}=2$ superconformal $4d$ theory.

In between the theories obtained engineering Type IIB superstring on bimodal singularities there are the following superconformal square tensor models [13]:

E_{18}	$x^3+y^{10}+z^2$	$A_2\square A_9$	E_{20}	$x^3+y^{11}+z^2$	$A_2\square A_{10}$
W_{18}	$x^4+y^7+z^2$	$A_3\square A_6$	U_{16}	$x^3+xz^2+y^5$	$D_4\square A_4$
$J_{3,0}$	$x^3+y^9+z^2$	$A_2\square A_8$	$W_{1,0}$	$x^4+y^6+z^2$	$A_3\square A_5$

(2.51)

Also the even elements of the Q_{2k} serie belongs to the class of square tensor models, in fact these are precisely the theories of type $A_2\boxtimes D_k$, but, looking at the tables of Coxeter–Dynkin diagrams, one does not recognize the square form: this point will be explained in what follows.

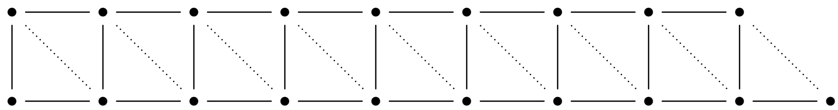
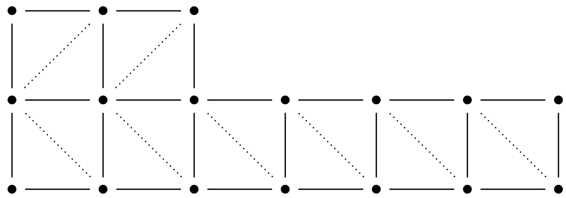
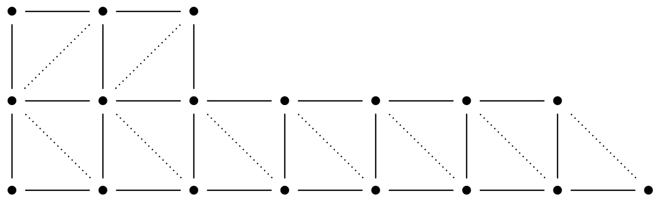
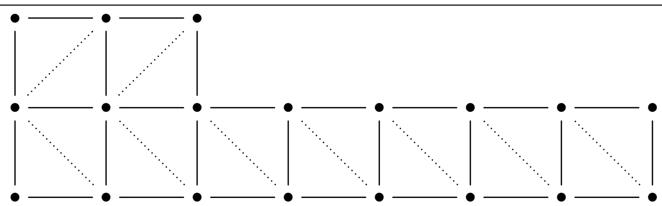
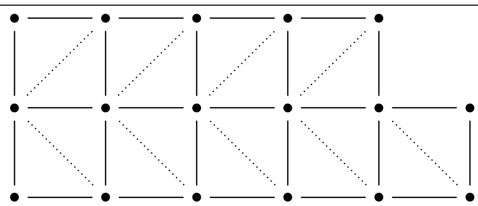
2.6.1. Modality and completeness The Landau–Ginzburg models we are considering here have chiral ring of primary operators [48, 49]

$$\mathcal{R}\simeq\mathbb{C}[x,y,z]/J_W, \tag{2.52}$$

where J_W is the jacobian ideal of W, *i.e.* the ideal of $\mathbb{C}[x,y,z]$ generated by the partials $\partial_i W$. The theories being non–degenerate, the ring is finite–dimensional as a $\mathbb{C}$–algebra and its dimension, μ, is called the Milnor number (or multiplicity) of the singularity $W(x,y,z)$,

$$D=\mu\equiv\dim_{\mathbb{C}}\mathcal{R}. \tag{2.53}$$

Table 2.5: Exceptional bimodal singularities that are not listed in (2.51).

name	$W(x, y, z)$	Coxeter–Dynkin diagram
E_{19}	$x^3 + xy^7 + z^2$	
Z_{17}	$x^3y + y^8 + z^2$	
Z_{18}	$x^3y + xy^6 + z^2$	
Z_{19}	$x^3y + y^9 + z^2$	
W_{17}	$x^4 + xy^5 + z^2$	

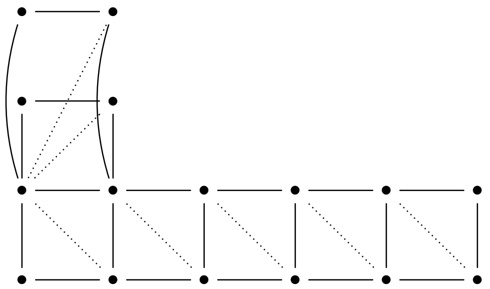
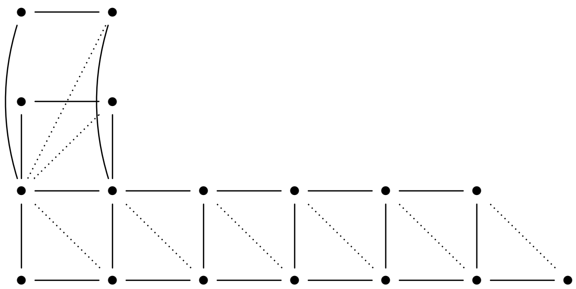
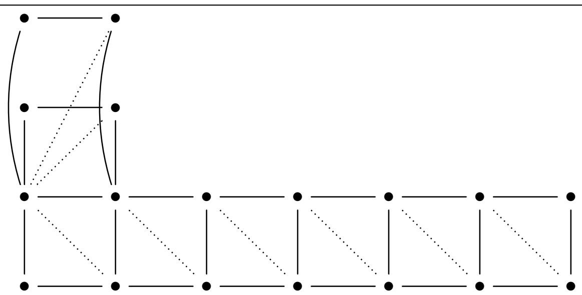
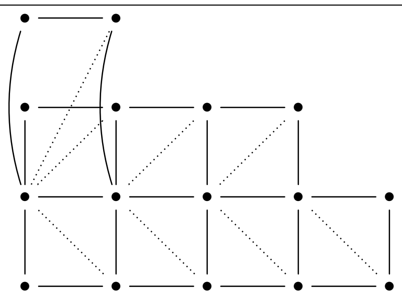
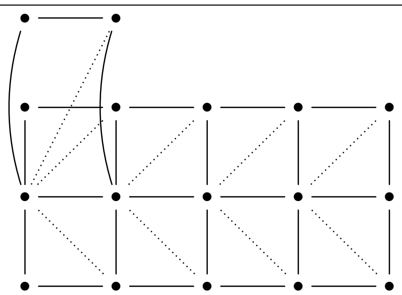
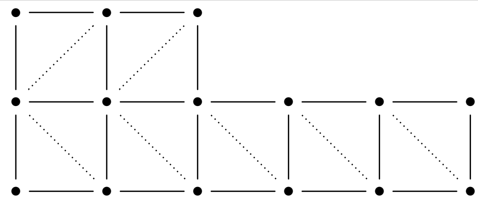
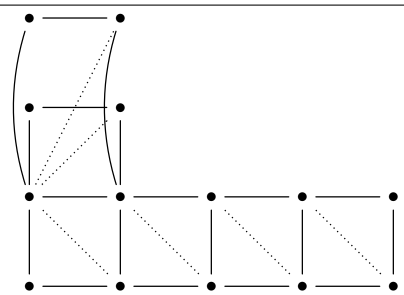
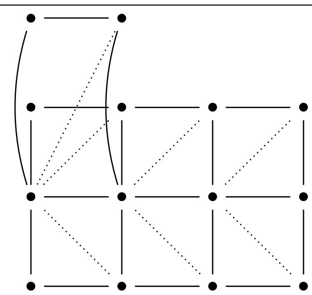
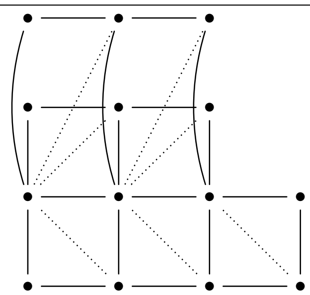
Q_{16}	$x^3 + yz^2 + y^7$	
Q_{17}	$x^3 + yz^2 + xy^5$	
Q_{18}	$x^3 + yz^2 + y^8$	
S_{16}	$x^2z + yz^2 + xy^4$	
S_{17}	$x^2z + yz^2 + y^6$	

Table 2.6: Quasi–homogeneous elements of the 8 infinite series of bimodal singularities that are not in (2.51). We indicate the corresponding Milnor numbers in parenthesis.

name	$W(x,y,z)$	Coxeter–Dynkin diagram
$Z_{1,0}$ (15)	$x^3y + y^7 + z^2$	
$Q_{2,0}$ (14)	$x^3 + yz^2 + xy^4$	
$S_{1,0}$ (14)	$x^2z + yz^2 + y^5$	
$U_{1,0}$ (14)	$x^3 + xz^2 + xy^3$	

	q_x, q_y, q_z	$\hat{c}$	ℓ
E_{19}	1/3 , 1/7 , 1/2	8/7	18
Z_{17}	7/24, 1/8, 1/2	7/6	10
Z_{18}	5/17, 2/17, 1/2	20/17	14
Z_{19}	8/27, 1/9, 1/2	32/27	22
W_{17}	1/4, 3/20, 1/2	6/5	8
Q_{16}	1/3, 1/7, 3/7	25/21	17
Q_{17}	1/3, 2/15, 13/30	6/5	12
Q_{18}	1/3, 1/8, 7/16	29/24	19
S_{16}	5/17, 3/17, 7/17	21/17	13
S_{17}	7/24, 1/6, 5/12	5/4	9
$Z_{1,0}$	2/7, 1/7, 1/2	8/7	6
$Q_{2,0}$	1/3, 1/6, 5/12	7/6	5
$S_{1,0}$	3/10, 1/5, 2/5	6/5	4
$U_{1,0}$	1/3, 2/9, 1/3	11/9	9

Table 2.7: Chiral charges q_i, central charges $\hat{c}$ and orders ℓ of the quantum monodromy $\mathbb{M}(q)$ for the quasi–homogeneous bimodal singularities that are not in (2.51).

$Z_{1,0}$	$x^2y^3(1), xy^6(\frac{8}{7})$	$Q_{2,0}$	$x^2y^2(1), x^2y^3(\frac{7}{6})$
$S_{1,0}$	$zy^3(1), zy^4(\frac{6}{5})$	$U_{1,0}$	$zy^3(1), zy^4(\frac{11}{9})$

E_{19}	$y^{11}(\frac{22}{21}), y^{12}(\frac{8}{7})$	W_{17}	$y^7(\frac{21}{20}), y^8(\frac{6}{5})$		
Z_{17}	$xy^6(\frac{25}{24}), xy^7(\frac{7}{6})$	Z_{18}	$y^9(\frac{18}{17}), y^{10}(\frac{20}{17})$	Z_{19}	$xy^7(\frac{29}{27}), xy^8(\frac{32}{27})$
Q_{16}	$xy^5(\frac{22}{21}), xy^6(\frac{25}{21})$	Q_{17}	$y^8(\frac{16}{15}), y^9(\frac{6}{5})$	Q_{18}	$xy^6(\frac{13}{12}), xy^7(\frac{29}{24})$
S_{16}	$y^6(\frac{18}{17}), y^7(\frac{21}{17})$	S_{17}	$zy^4(\frac{13}{12}), zy^5(\frac{5}{4})$		

Table 2.8: Primary deformations of dimension ≥ 1 of the quasi–homogeneous bimodal singularities that are not square tensor models. The number in parenthesis is the dimension of the deformation.

This number equals the Witten–index $\mathrm{tr}(-)^F$ of the theory, by the spectral flow isomorphism [49, 83]. Moreover, by $2d/4d$ correspondence μ equals the rank of the charge lattice Γ of the 4d theory. $\mathrm{tr}(-)^F$ jumps at the infrared fixed point obtained by perturbing the theory away from criticality with primary relevant perturbations: this corresponds to the fact that taking this infrared limit we are projecting on a proper subalgebra of $\mathcal{R}$. *Modality* is the cardinality of the set of perturbations that generate renormalization flows asymptotically preserving the Witten–index [84]. From this definition, it follows that it can be computed as the number of marginal and irrelevant primary operators in a monomial basis of the chiral ring.

The $4d$ theories obtained by geometric engineering Type IIB on $\mathscr{H}$ are *non–* complete quiver quantum field theories in the sense of [7]. In the case of exceptional bimodal singularities, there are two quantum obstructed deformations, therefore the codimension of $\Im(\mathscr{P}) \subset \mathbb{C}^D$ is 2, while for the non–exceptional bimodals, there is only one such deformation, the other modulus being a marginal deformation — see table 2.8. As an example, consider the $Q_{2,0}$ theory. The marginal deformation of this singularity is x^2y^2. This is an operator equivalent, in the chiral ring, to y^6. By deforming $Q_{2,0}$ with y^6, we obtain the equivalence $Q_{2,0} \sim A_2 \square D_7$.

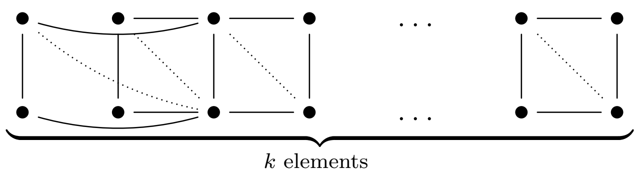
2.6.2. $2d$ **wall–crossings and Coxeter–Dynkin graphs** As already mentioned, Coxeter–Dynkin graphs are *not* unique, depending on the choice of a distinguished homology basis. Equivalent distinguished basis are related by the braid group (Picard–Lefshetz) transformations, *i.e.* by the $2d$ wall–crossing group. Whenever we switch the position of two SUSY vacua, say $|i\rangle$ and $|i+1\rangle$, in the W plane, we cross a $2d$ wall of marginal stability and this has the effect of a phase transition in the $2d$ BPS spectrum. By 2d/4d correspondence, 2d wall–crossing is equivalent to mutations. Assume that the superpotential of the $\mathcal{N} = (2,2)$ $2d$ Landau–Ginzburg superconformal theory has nonzero modality, then there are directions along the $2d$ renormalization group flow along which the Witten–index is conserved also asymptotically. The behaviour of the Coxeter–Dynkin graph under these deformations is encoded in the following proposition:

Proposition(1 of [85]): all the irrelevant and marginal deformations of an $\mathcal{N} = (2,2)$ $2d$ Landau–Ginzburg superconformal theory with $\mu < \infty$ lead to equivalent configurations of vacua and interpolating BPS solitons, where, by equivalent, we mean that they are in the same $2d$ wall–crossing group orbit.

This proposition is the key to understand the phenomenon we encountered with the even elements of the Q series: it is just the $2d$ wall–crossing in action. The diagrams that one can find in the math litterature (and that we reported in table 2.5) are referred to an irrelevant deformation while the ones from which the square tensor form is explicit are obtained directly from the undeformed theory: being the diagrams in a μ=const. stratum of Q_{2k} they are equivalent, *i.e.* the two quivers belongs to the same mutation class.

Indeed, for all k's the result is in perfect agreement with [13]:

Q_{2k}:



We remark that Q_{14} above is just $Q_{2,0}$.

2.6.3. $2d$ **Renormalization group flows** The trivial instances of this RG process are the following theories (we indicate in parenthesis the dimension of $\phi_\star$):

$$\begin{aligned}
A_2\square A_{10}\colon x^3+y^{11}+z^2 &\xrightarrow{xy^7\ (32/33)} E_{19}\\
A_3\square A_7\colon x^4+y^8+z^2 &\xrightarrow{x^3y\ (7/8)} Z_{17}\\
A_3\square A_6\colon x^4+y^7+z^2 &\xrightarrow{xy^5\ (27/28)} W_{17}\\
A_3\square A_6\colon x^4+y^7+z^2 &\xrightarrow{x^3y\ (25/28)} Z_{1,0}\\
A_4\square D_4\colon x^2z+x^3+y^5 &\xrightarrow{xy^3\ (14/15)} U_{1,0}.\\
A_4\square D_4\colon x^2z+z^3+y^5 &\xrightarrow{yz^2\ (13/15)} S_{1,0}
\end{aligned} \tag{2.54}$$

The theories Z_{18}, S_{16} and the ones of the Q series are better described as the final IR points of RG 'cascades'

$$\begin{aligned} A_3\Box A_8\colon x^4+y^9+z^2 &\xrightarrow{x^3y\ (31/36)} Z_{19} \xrightarrow{xy^6\ (26/27)} Z_{18}\\ A_5\Box D_4\colon x^2z+z^3+y^6 &\xrightarrow{yz^2\ (5/6)} S_{17} \xrightarrow{xy^4\ (23/24)} S_{16}\\ A_6\Box D_4\colon x^3+z^3+y^7 &\xrightarrow{yz^2\ (17/21)} Q_{16} \xrightarrow{xy^4\ (19/21)} Q_{2,0}\\ A_7\Box D_4\colon x^3+z^3+y^8 &\xrightarrow{yz^2\ (19/24)} Q_{18} \xrightarrow{xy^5\ (23/24)} Q_{17}. \end{aligned} \tag{2.55}$$

2.6.4. Flavor charges The degeneracies of the chiral ring elements are captured by the Poincaré polynomial:

$$\sum_\alpha t^{q_\alpha} = \prod_i \frac{(1-t^{1-q_i})}{1-t^{q_i}} \tag{2.56}$$

where the q_i are the charges of table 2.7 and the sum is over all the elements of a monomial basis of the chiral ring. Expanding the RHS, f is the (positive or zero) *integer* multiplying the coefficient $t^{\hat{c}/2}$ of the serie. So,

$$f = \begin{cases} 3 \text{ for } Z_{1,0} \\ 2 \text{ for } Q_{2,0}, S_{1,0}, J_{3,0}, Z_{18} \\ 1 \text{ for odd rank exceptionals and } W_{1,0} \\ 0 \text{ otherwise.} \end{cases} \tag{2.57}$$

2.6.5. A remark about weak coupling From table 2.8 we are able to compute the codimension of $\Im(\mathscr{P})$ in $\mathbb{C}^D$

$$\text{codim}(\mathcal{D}) = \begin{cases} 2 \text{ for exceptional bimodals;} \\ 1 \text{ otherwise.} \end{cases}$$

Therefore, if one of the theories we are considering has the structure of a G SYM weakly coupled to some other sector that maybe non–lagrangian, just counting dimensions, we are able to constrain the possible gauge groups G: for non-exceptional bimodals the possibilities are

$$SU(2)^k, \quad SU(2)^k \times SU(3), \quad SU(2)^k \times SO(5), \quad SU(2)^k \times G_2, \tag{2.58}$$

while for exceptional bimodals we have the above cases and the following ones

$$\begin{gathered} SU(2)^k \times SU(3) \times SO(5), \quad SU(2)^k \times SU(3) \times G_2, \\ SU(2)^k \times SO(5) \times G_2, \quad SU(2)^k \times SU(4), \\ SU(2)^k \times SO(6), \quad SU(2)^k \times SO(7) \end{gathered} \tag{2.59}$$

for some $k \in \mathbb{N}$. Since $\mu = \text{rk}(\Gamma)$, the possible number of $SU(2)$ factors appearing here is constrained by the Witten–index of the corresponding 2d theory.

2.6.6. BPS spectra at strong coupling. The quivers with superpotentials of these models are obtained with the same circle of ideas we discussed in §.2.3.1. Again the BPS–spectra of these models are obtained via Weyl–factorized sequences of mutations and determine the existence of 11 new periodic Y–systems that can be straightforwardly generated with the help of the Keller's mutation applet [42]. The sequences of mutations we have used are detailed in appendix C.4. Our result are the following[18]:

- $\mathbf{E_{19}}$: this theory is a one–point extension of $A_2\square A_9$. We have four algebraically trivial finite chambers:

$$\begin{array}{c|c} (A_2\times 8, A_3) & (A_2\times 9, A_1) \\ \hline (A_{10}, A_9) & (A_9\times 2, A_1) \end{array} \tag{2.60}$$

- $\mathbf{Z_{17}}$: we have two algebraically trivial finite chambers:

$$(A_3, A_7, A_7) \;\Big|\; (A_3\times 3, A_2\times 4) \tag{2.61}$$

- $\mathbf{Z_{18}}$: this is the one point extension of the previous one:

$$\begin{array}{c|c} (A_3, A_7, A_8) & (A_3, A_7, A_7, A_1) \\ \hline (A_3\times 3, A_2\times 3, A_3) & (A_3\times 3, A_2\times 4, A_1) \end{array} \tag{2.62}$$

- $\mathbf{Q_{2k}}$: the canonical chambers of $A_2\square D_k$ and the following two algebraically trivial finite chambers:

$$\mathbf{Q_{2,0}}:\ (D_4\times 2, A_2\times 3) \;\Big|\; (A_2\times 2, A_5\times 2) \tag{2.63}$$

$$\mathbf{Q_{16}}:\ (D_4\times 2, A_2\times 4) \;\Big|\; (A_2\times 2, A_6\times 2) \tag{2.64}$$

$$\mathbf{Q_{18}}:\ (D_4\times 2, A_2\times 5) \;\Big|\; (A_2\times 2, A_7\times 2) \tag{2.65}$$

- $\mathbf{Q_{17}}$: this is a one point extension of Q_{16}:

$$\begin{array}{c|c} (D_4\times 2, A_2\times 4, A_1) & (A_2\times 2, A_6\times 2, A_1) \\ \hline (D_4\times 2, A_2\times 3, A_3) & (A_2\times 2, A_6, A_7) \end{array} \tag{2.66}$$

[18]The notation $(..., G\times N, ...)$ means that the Dynkin graph G appears N times in the family.

- For all the others we have two algebraically trivial finite chambers:

$$\mathbf{Z_{19}}:\quad (A_3, A_8, A_8)\ \Big|\ (A_3\times 3, A_2\times 5) \tag{2.67}$$

$$\mathbf{W_{17}}:\quad (A_5, A_6, A_6)\ \Big|\ (A_3\times 5, A_2) \tag{2.68}$$

$$\mathbf{S_{16}}:\quad (D_4\times 2, A_3\times 2, A_2)\ \Big|\ (A_2, A_4, A_5\times 2) \tag{2.69}$$

$$\mathbf{S_{17}}:\quad (D_4\times 2, A_3\times 3)\ \Big|\ (A_2, A_5\times 3) \tag{2.70}$$

$$\mathbf{Z_{1,0}}:\quad (A_3, A_6\times 2)\ \Big|\ (A_3\times 3, A_2\times 3) \tag{2.71}$$

$$\mathbf{S_{1,0}}:\quad (D_4\times 2, A_3\times 2)\ \Big|\ (A_2, A_4\times 3) \tag{2.72}$$

$$\mathbf{U_{1,0}}:\quad (D_4\times 3, A_2)\ \Big|\ (A_3\times 2, A_4\times 2) \tag{2.73}$$

We stress that all these finite BPS chambers have natural physical interpretations as the decoupling of some heavy hypermultiplet from the physical BPS spectrum of a canonical chamber of a square tensor model [13], as we showed in section §2.6.3.

Chapter 3

$\widehat{H} \boxtimes G$ systems and $D_p(G)$ SCFT's

3.1. A heuristic introduction

Consider the type IIB geometric engineering on the local Calabi–Yau hypersurface of equation

$$e^{pZ} + e^{-Z} + W_G(X, Y) + U^2 = 0 \tag{3.1}$$

where $W_G(X, Y)$ stands for (the versal deformation of the) minimal ADE singularity of type G (see table 3.1). The RHS of (3.1), seen as a 2d superpotential, corresponds to a model with central charge $\hat{c}$ at the UV fixed point equal to

$$\hat{c}_{\text{uv}} = 1 + \hat{c}_G < 2 \tag{3.2}$$

where $\hat{c}_G$ is the central charge of the minimal $(2, 2)$ SCFT of type G. Since $\hat{c}_{\text{uv}} < 2$, the criterion of the $2d/4d$ correspondence we reviewed in §.2.3.1 is satisfied, and we get a well–defined QFT in 4D. For $p = 1$ the geometry (3.1) reduces to the known Seiberg–Witten geometry of the pure G SYM, and for all $p \in \mathbb{N}$, we expect the resulting 4D theory is UV asymptotically free; there is, indeed, a scale one cannot get rid off but by changing the underlying geometry discontinuously: the size of the cylinder in the Z coordinate. This claim can be justified also in terms of the standard argument by Tachikawa–Terashima [86]: one can view the geometry (3.1) as a particular fibration of the (deformed) ALE space $W_G(X, Y) + U^2 = 0$. It is well–known that the low energy description of Type II B on $W_G(X, Y) + U^2 = 0$ gives rise to the 6d $\mathcal{N} = (2, 0)$ theory of type G. Then the cylinder in the Z coordinate is naturally interpreted as a Gaiotto plumbing cylinder, and its size can be traded for the scale of the theory. Notice that this is the only scale one can obtain from this geometry: all other cycles are vanishing, therefore it exists a limit in which the theory is conformal up to one single scale. From this we deduce that only one simple factor of the gauge group of these models can be asymptotically free, the case $p = 1$ indicates that the group in question is G. Let g denote the Yang–Mills coupling constant of this gauge group. In the limit $g \to 0$, the SYM sector decouples from the system and one is left with a subsector of the original model with G flavor symmetry (at least). From the above considerations,

	$W_G(X,Y)$	(q_X, q_Y)	$h(G)$
A_{n-1}	$X^n + Y^2$	$(1/n, 1/2)$	n
D_{n+1}	$X^n + XY^2$	$(1/n, (n-1)/2n)$	$2n$
E_6	$X^3 + Y^4$	$(1/3, 1/4)$	12
E_7	$X^3 + XY^3$	$(1/3, 2/9)$	18
E_8	$X^3 + Y^5$	$(1/3, 1/5)$	30

Table 3.1: List of the ADE simple singularities, and some of the corresponding properties.

moreover, we expect that such decoupled system is superconformal. This idea is confirmed by the analysis of the $G = SU(2)$ case: indeed, in this case, the geometry (3.1) gives simply the $\widehat{A}(p,1)$ complete 4d $\mathcal{N} = 2$ theory, that is well–known to have an S–duality frame that represents an $SU(2)$ SYM system weakly gauging the $SU(2)$ flavor symmetry of an Argyres–Douglas system of type D_p. In conclusion, for all other gauge groups, we expect that the geometry (3.1) engineers a G SYM weakly gauging the flavor symmetry of a superconformal sector. Such system generalizes the D_p Argyres–Douglas models to the case of flavor group (at least) G, so we will call these models $D_p(G)$. The purpose of this chapter is to discuss some of the most relevant properties of these models.

3.2. 2d/4d and direct sums of 2d models.

As we have discussed in the previous chapter, the 2d/4d correspondence is the (conjectural) statement that, for each 4d $\mathcal{N} = 2$ supersymmetric quantum field theory with the BPS–quiver property, there is a two–dimensional $\mathcal{N} = (2,2)$ system with $\hat{c} < 2$ such that the exchange matrix B of the 4d quiver Q is

$$B = S^t - S, \tag{3.3}$$

where S is the tt^* Stokes matrix of the 2d system. The inverse process of reconstructing the 2d theory from the 4d one (that is, of finding S given B) involves some subtleties. In the lucky case that Q is acyclic, one has simply

$$S_{ij} = \delta_{ij} - \max\{B_{ij}, 0\}, \tag{3.4}$$

corresponding to the Euler form of the quiver Q. Notice that the path algebra $\mathbb{C}Q$ of *any* quiver without relations Q is always *hereditary*, *i.e.* it has global dimension

≤ 1 [71]. For BPS–quivers the acyclicity condition entails that it cannot have a superpotential and therefore it will not have relations. Formula (3.4), then follows from the algebraic reformulation of 2d/4d correspondence of §.2.3.1. For the constructions of this chapter we need S only for acyclic quivers, and for the slightly more general cases which are derived equivalent to a hereditary *category* [87], the matrix S still being given by the Euler form.

Remark. The 2d/4d correspondence relates 2d superconformal theories to 4d superconformal ones: Indeed the scaling of the 2d theory may be seen as a scaling property of the Seiberg–Witten geometry, which in turn implies a scaling symmetry for the 4d theory. The 2d quantum monodromy is $H = (S^t)^{-1}S$ [50], and the 2d theory is superconformal (in the UV) precisely when H is semi–simple.

Consider now two 2d $(2, 2)$ LG systems with superpotentials $W_1(X_i)$ and $W_2(Y_a)$ and (UV) Virasoro central charges $\hat{c}_1$ and $\hat{c}_2$. Their *direct sum* is defined as the $(2, 2)$ decoupled model with superpotential

$$W(X_i, Y_a) = W_1(X_i) + W_2(Y_a). \tag{3.5}$$

It has $\hat{c} = \hat{c}_1 + \hat{c}_2$. By the 2d/4d correspondence, it defines a 4d $\mathcal{N} = 2$ theory provided $\hat{c} < 2$.

A more general application of the same strategy that lead Cecotti–Neitzke–Vafa to the definition of the (G, G') models is to consider the direct sum of a 2d minimal model ($\hat{c} < 1$) with a 2d model corresponding to a *complete* 4d theory which has $\hat{c} \leq 1$ by definition. Again the direct sum has automatically $\hat{c} < 2$. By construction, the Hilbert spaces of the direct sum models are tensor products $\mathcal{H}_1 \otimes \mathcal{H}_2$ of the Hilbert spaces of the factors, and therefore the tt^* Stokes matrix of the direct sum is the tensor product of the Stokes matrices of the summands [50]

$$S = S_1 \otimes S_2. \tag{3.6}$$

In our case S_2 is the Stokes matrix of an ADE Dynkin quiver (which is a tree, so eqn.(3.4) applies), while S_1 is the Stokes matrix of a $\hat{c} \leq 1$ $(2, 2)$ system. In almost all cases considered in this chapter we will take S_1 to be the Stokes matrix of an *acyclic* affine quiver. The construction may be extended to a general mutation–finite quiver, *provided one knows the right Stokes matrix.*

If the quivers Q_1, Q_2 associated to S_1, S_2 are acyclic, the quiver Q with exchange matrix

$$B = S_1^t \otimes S_2^t - S_1 \otimes S_2 \tag{3.7}$$

is called the triangle tensor product of the two quivers, written $Q_1 \boxtimes Q_2$, and it is equipped with a unique superpotential $\mathcal{W}_\boxtimes$ obtained by 3–CY completion. Notice that (3.6) and (3.7) are perfectly consisten with our discussion in §.2.3.1: the Dirac pairing in (3.7) is precisely the antysimmetrization of the Euler form defined in (3.6). Let $\mathbb{C}Q_1$, $\mathbb{C}Q_2$ be the corresponding path algebras. We can consider the tensor product algebra $\mathbb{C}Q_1 \otimes \mathbb{C}Q_2$ spanned, as a vector space, by the elements $\alpha \otimes \beta$ and endowed with the product

$$\alpha \otimes \beta \cdot \gamma \otimes \delta = \alpha\gamma \otimes \beta\delta. \tag{3.8}$$

Let e_i, (resp. e_a) be the lazy paths ($\equiv$ minimal idempotents) of the algebra $\mathbb{C}Q_1$ (resp. $\mathbb{C}Q_2$). The minimal idempotents of the tensor product algebra are $e_{ia} = e_i \otimes e_a$; for each such idempotent e_{ia} there is a node in the quiver of the algebra $\mathbb{C}Q_1 \otimes \mathbb{C}Q_2$ which we denote by the same symbol. The arrows of the quiver are[1]

$$e_i \otimes \beta \colon e_i \otimes e_{s(\beta)} \to e_i \otimes e_{t(\beta)}, \qquad \alpha \otimes e_a \colon e_{s(\alpha)} \otimes e_a \to e_{t(\alpha)} \otimes e_a. \tag{3.9}$$

However, there are non–trivial relations between the paths; indeed the product (3.8) implies the commutativity relations

$$e_{t(\alpha)} \otimes \beta \cdot \alpha \otimes e_{s(\beta)} = \alpha \otimes e_{t(\beta)} \cdot e_{s(\alpha)} \otimes \beta. \tag{3.10}$$

In the physical context all relations between paths should arise in the Jacobian form $\partial\mathcal{W} = 0$ from a superpotential. In order to set the commutativity relations in the Jacobian form, we have to complete our quiver by adding an extra arrow for each pairs of arrows $\alpha \in Q_1$, $\beta \in Q_2$

$$\psi_{\alpha,\beta} \colon e_{t(\alpha)} \otimes e_{t(\beta)} \to e_{s(\alpha)} \otimes e_{s(\beta)}, \tag{3.11}$$

and introducing a term in the superpotential of the form

$$\mathcal{W}_{\boxtimes} = \sum_{\text{pairs } \alpha,\beta} \psi_{\alpha,\beta} \Big(e_{t(\alpha)} \otimes \beta \cdot \alpha \otimes e_{s(\beta)} - \alpha \otimes e_{t(\beta)} \cdot e_{s(\alpha)} \otimes \beta \Big) \tag{3.12}$$

enforcing the commutativity conditions (3.10). The resulting completed quiver, equipped with this superpotential, is called the *triangle tensor product* of Q_1, Q_2, written $Q_1 \boxtimes Q_2$ [10, 13, 14, 60].

Remark. Notice that the 3–CY completion construction of the quiver with superpotential $Q_1 \boxtimes Q_2$ carries over *formally* for all finite quivers Q_1, Q_2 *without relations*. Indeed, the path algebras $\mathbb{C}Q_1$ and $\mathbb{C}Q_2$, though infinite dimensional, are still hereditary if the quivers Q_1 and Q_2 are finite [71]. Even if this situation

[1] Here $s(\cdot)$ and $t(\cdot)$ are the maps which associate to an arrow its source and target node, respectively.

seems unphysical at first sight, we will see that peculiar decoupling limits (weak coupling regime) can be described using this technology.

Examples. If both Q_1, Q_2 are Dynkin quivers their tensor product corresponds to the (G, G') models constructed and studied by Cecotti–Neitzke–Vafa in [13]. If Q_1 is the Kronecker (affine) quiver $\widehat{A}(1,1)$ and Q_2 is a Dynkin quiver of type G, $\widehat{A}(1,1) \boxtimes G$ is the quiver (with superpotential) of pure SYM with gauge group G [8, 10, 13].

3.3. Light subcategories: a short review.

Consider a BPS–quiver 4d $\mathcal{N} = 2$ gauge theory that behaves, in some duality frame, as SYM with simply–laced[2] gauge group G coupled to some 'matter' system. We fix a quiver Q which 'covers' the region in parameter space corresponding to weak G gauge coupling. We pick a particular pair $(Q, \mathcal{W})$ in the corresponding mutation–class which is appropriate for the G weak coupling regime of its Coulomb branch. $\mathsf{rep}(Q, \mathcal{W})$ should contain, in particular, $\mathbb{P}^1$ families of representations corresponding to the massive W–boson vector–multiplets which are in one–to–one correspondence with the positive roots of G. We write δ_a $(a = 1, 2, \dots, r)$ for the charge vector of the W–boson associated to the *simple–root* α_a of G.

Electric and magnetic weights: Dirac integrality condition

At a generic point in the Coulomb branch we have an unbroken $U(1)^r$ symmetry. The $U(1)^r$ electric charges, properly normalized so that they are integral for all states, are given by the fundamental coroots[3] $\alpha_a^\vee \in \mathfrak{h}$ $(a = 1, 2, \dots, r)$. The a–th electric charge of the W–boson associated to b–th simple root α_b then is

$$q_a(\alpha_b) \equiv \alpha_b(\alpha_a^\vee) \equiv \frac{2\,(\alpha_b, \alpha_a)}{(\alpha_a, \alpha_a)} \equiv C_{ab}, \qquad \text{(the Cartan matrix of } G\text{)}. \tag{3.13}$$

Therefore the vector in $\Gamma \otimes \mathbb{Q}$ corresponding to the a–th unit electric charge is

$$\mathfrak{q}_a = (C)^{-1}_{ab}\, \delta_b. \tag{3.14}$$

Then the magnetic weights (charges) of a representation X are defined through its Dirac electromagnetic pairing with the unit electric vectors

$$m_a(X) \equiv \langle \dim X, \mathfrak{q}_a \rangle_{\text{Dirac}} = C^{-1}_{ab}\ (\dim X)^t\, B\, \delta_b. \tag{3.15}$$

[2] The construction of the light subcategory for G SYM in the non–simply laced case was found in [notsimply], but it is out of the scope of the present work to review also that (very interesting) topic.

[3] Here and below $\mathfrak{h}$ stands for the Cartan subalgebra of the complexified Lie algebra of the gauge group G.

Dirac quantization requires the r linear forms $m_a(\cdot)$ to be *integral* [10]. This integrality condition is quite a strong constraint on the quiver Q, and, if the model has a lagrangian description, it is a criterion strong enough to completely determine it.

The light subcategory $\mathscr{L}(Q,\mathcal{W})$

Assume for simplicity that G is simple and simply laced. At weak Yang–Mills coupling, $g \to 0$, the central charge takes the classical form

$$Z(X) = -\frac{1}{g^2} \sum_i C_a\, m_a(X) + O(1) \qquad C_a > 0. \tag{3.16}$$

States of non–zero magnetic charge have masses of order $O(1/g^2)$ as $g \to 0$, and decouple in the limit. The BPS states which are both stable and light in the decoupling limit must correspond to quiver representations X satisfying the two conditions: 1) $m_a(X) = 0$ for all a (light); 2) if Y is a subrepresentation of X, then $m_a(Y) \leq 0$ for all i (stability in the chamber (3.16)). These two conditions are well known in representation theory, but to understand this link we need to introduce some more abstract nonsense.

Consider an abelian category $\mathscr{A}$. The *Groethendieck group* $K_0(\mathscr{A})$ is the free abelian group on the set of objects of $\mathscr{A}$ modulo all relations $A - B + C$ given by short exact sequences

$$0 \to A \to B \to C \to 0$$

A linear function from $K_0(\mathscr{A})$ to $\mathbb{Z}$ can be viewed as a mapping on the objects of $\mathscr{A}$ that is additive on exact sequences, $\lambda(B) = \lambda(A) + \lambda(C)$. Given a set of linear functions (the controls)

$$\lambda_A \colon K_0(\mathscr{A}) \to \mathbb{Z}, \qquad (A = 1, 2, \ldots, s)$$

the subcategory $\mathscr{B}(\lambda_A) \subset \mathscr{A}$ of objects defined by the requirements that *i)* $\lambda_A(X) = 0$ for all A, and *ii)* for all subobjects Y one has $\lambda_A(Y) \leq 0$ is an exact subcategory[4] of $\mathscr{A}$ that is, moreover, closed under extensions[5]. $\mathscr{B}(\lambda_A)$ is said to be a *controlled subcategory* of $\mathscr{A}$ [10, 88].

Take $\mathscr{A} = \mathsf{rep}(Q,\mathcal{W})$. $\Gamma = K_0(\mathscr{A})$, the isomorphism (of groups) being given by the dimension vector. We stress that the Groethendieck group is the *physically*

[4] *i.e.* it is closed under kernels, cokernels and direct sums.

[5] Recall that given two objects of an abelian category $X, Y \in \mathscr{A}$, $M \in \mathscr{A}$ is an *extension* of X through Y iff $0 \to X \to M \to Y \to 0$ is an exact sequence. The set $\mathrm{Ext}^1(X,Y)$ is simply the set of all extensions of X through Y.

more natural structure: it does not distinguish in between particles and anti–particles[6].

The magnetic charges (3.15) can be extended to linear functions $m_a\colon \Gamma \to \mathbb{Z}$. The two physical conditions that a BPS state is stable and light, then, are precisely the requirements for defining a controlled subcategory $\mathscr{B}(m_a)$ of $\mathsf{rep}(Q,\mathcal{W})$. Such subcategory is said to be a *light subcategory* $\mathscr{L}(Q,\mathcal{W})$ w.r.t. the chosen duality frame. All BPS states with bounded mass in the weak coupling limit $g_{\text{YM}} \to 0$ correspond to representations in $\mathscr{L}(Q,\mathcal{W})$. In facts, for a $\mathcal{N}=2$ theory which has a weakly coupled Lagrangian description the stable objects of $\mathscr{L}(Q,\mathcal{W})$ precisely correspond to the perturbative states that are charged under G: we will have $\mathbb{P}^1$ families that corresponds to gauge bosons, making one copy of the adjoint of G, together with finitely many rigid objects that correspond to hypermultiplets and are organized in definite representations of G.

The light category $\mathscr{L}$ has a quiver (with relations) of its own. However, while typically a full non–perturbative category has a 2–acyclic quiver, the quiver of a light category has, in general, both loops and pairs of opposite arrows $\leftrightarrows$. It depends on the particular superpotential $\mathcal{W}$ whether the pairs of opposite arrows may or may not be integrated away. To motivate this claim, consider a theory with a Lagrangian description: if we take the weak–coupling limit for all simple factors of the gauge group, all states in $\mathscr{L}$ are mutually local so that

$$\langle\cdot,\cdot\rangle_{\text{Dirac}}\big|_{\mathscr{L}} = 0, \tag{3.17}$$

and the net number of arrows between any two nodes of Q' must be zero

$$\#\{\text{arrows }\ i \to j \text{ in } Q'\} - \#\{\text{arrows }\ i \leftarrow j \text{ in } Q'\} = 0. \tag{3.18}$$

In particular, Q', if connected, cannot be 2–acyclic. This is related to the fact that $\mathscr{L}$ is not, typically, the non–perturbative Abelian category representing all BPS states of the theory, but rather it represent only a subsector (the perturbative one) which is not a full QFT. The complete spectrum also contain dyons with masses of order $O(1/g^2)$.

Compatible decoupling limits

The notion of controlled subcategory can be used in many different ways to engineer in representation theory various physical decoupling limits (Higgs decoupling,

[6] Indeed, two mutation equivalent quivers with potential, lead to derived equivalent algebras, that, in particular, have the same Groethendieck group.

acyclic affine quiver $\widehat{H}$	matter content
$\widehat{A}(p,q) \quad p \geq q \geq 1$	$D_p \oplus D_q \ (\oplus D_1)$
$\widehat{D}_r \quad r \geq 4$	$D_2 \oplus D_2 \oplus D_{r-2}$
$\widehat{E}_r \quad r = 6,7,8$	$D_2 \oplus D_3 \oplus D_{r-3}$

Table 3.2: Affine $\mathcal{N} = 2$ complete theories as $SU(2)$ SYM coupled to several D_p systems. D_1 stands for the empty matter and $D_2 \equiv A_1 \oplus A_1$ for a free hypermultiplet doublet.

massive quark,...). An especially simple case is when one has $\lambda_A(e_i) \geq 0$ for all positive generators e_i of Q and all $A = 1, 2, \dots, s$. In this positive case,

$$\mathscr{B}(\lambda_A) = \mathsf{rep}(Q_0, \mathcal{W}\big|_{Q_0}),$$

where Q_0 is the full subquiver of Q over the nodes e_i such that $\lambda_A(e_i) = 0$ for all A. Examples of this sort were the decoupling limits we have considered in §.2.3.3 of the previous chapter.

A very useful property of the light category $\mathscr{L}$, proven in different contexts [10–12], is the following. Assume our theory has, in addition to $g \to 0$, a decoupling limit (*e.g.* large masses, extreme Higgs breaking), which is compatible with parametrically small YM coupling g, and such that the decoupled theory has support in a full subquiver $\widetilde{Q}$ of Q. Then

$$X \in \mathscr{L}(Q) \quad \Rightarrow \quad X\big|_{\widetilde{Q}} \in \mathscr{L}(\widetilde{Q}), \tag{3.19}$$

a relation which just expresses the compatibility of the given decoupling limit with $g \sim 0$. This fact is quite useful since, combined with the Dirac integrality conditions, it allows to construct recursively the category $\mathscr{L}$ for complicate large quivers from the light categories associated to smaller quivers.

Let us focus now on the $G = SU(2)$ case; this will be a very instructive example for the construction of the models discussed in the rest of this chapter.

3.3.1. Affine $\mathcal{N} = 2$ $SU(2)$ gauge theories and Euclidean algebras. The full classification of the $\mathcal{N} = 2$ $SU(2)$ gauge theories whose gauge group is strictly $SU(2)$ and which are both complete and asymptotically–free is presented in reference [7]. Such theories are in one–to–one correspondence with the mutation–classes of quivers obtained by choosing an acyclic orientation of an affine $\widehat{A}\widehat{D}\widehat{E}$ Dynkin graph. For $\widehat{D}_r$ ($r \geq 4$) and $\widehat{E}_r$ ($r = 6, 7, 8$) all orientations are mutation equivalent, while in the $\widehat{A}_r$ case the inequivalent orientations are characterized by the net number p (resp. q) of arrows pointing in the clockwise (anticlockwise) direction along the cycle; we write $\widehat{A}(p, q)$ for the $\widehat{A}_{p+q-1}$ Dynkin graph with such an orientation ($p \geq q \geq 1$). The case $\widehat{A}(p, 0)$ is different because there is a closed *oriented* p–loop. The corresponding path algebra $\mathbb{C}\widehat{A}(p, 0)$ is infinite–dimensional, and it must be bounded by some relations which, in the physical context, must arise from a superpotential, $\partial\mathcal{W} = 0$. For generic $\mathcal{W}$, $\widehat{A}(p, 0)$ is mutation–equivalent to the D_p Argyres–Douglas model which has an $SU(2)$ global symmetry, however, by the triality property of $SO(8)$, the D_4 Argyres–Douglas model is very special: its flavor symmetry gets enhanced to $SU(3)$. One can show in many ways [7, 10] that these $\mathcal{N} = 2$ affine theories correspond to $SU(2)$ SYM gauging the global $SU(2)$ symmetries of a set of Argyres–Douglas models of type D_r as in the table 3.2. To our knowledge the most intuitive way of getting this result is to use the gauging gluing rûle on Gaiotto surfaces that corresponds to affine models. Degenerating the tube that corresponds to the $SU(2)$ gauge group one remains with the surfaces that corresponds to the D_p Argyres–Douglas systems in the table. The same result follows from inspection of the properties of the category $\mathsf{rep}(\widehat{H})$. Since we are going to use heavily these properties in the rest of this chapter (in particular in §.3.4.6), let us review the algebraic proof of the statement in table 3.2 [10, 89].

The path algebras that correspond to the acyclic affine quivers $\mathbb{C}\widehat{H}$ have been widely studied in the math literature under the name of Euclidean algebras. These algebras are, in particular, *hereditary* (*i.e.* of global dimension ≤ 1), and therefore their Euler form is easily determined from the exchange matrix: One has

$$S_{ij} = \delta_{ij} - \max\{B_{ij}, 0\}. \tag{3.20}$$

To the Euler form $\langle -, - \rangle_E$ is associated a quadratic form, called the Tits form[7]:

$$q_{\widehat{H}}(X) \equiv \langle X \,, X \rangle_E \tag{3.21}$$

The Tits form is always positive semi–definite for Euclidean algebras and it is independent on the chosen orientation of $\widehat{H}$. By linearity the Tits form we have defined is extended to the whole charge lattice. The set

$$\Delta_{\widehat{H}} \equiv \{\gamma \in \Gamma_{\widehat{H}} \mid q_{\widehat{H}}(\gamma) \leq 1\} \tag{3.22}$$

is the set of *roots* of Γ. The dimension vectors of the indecomposable representations $\gamma \in \Gamma_{\widehat{H}}$ are in bijection with the *positive roots* of q. A given root $\gamma \in \Delta_{\widehat{H}}$ is

[7] Notice that it also coincides with the Tits form associated to the underlying unoriented Euclidean graph.

$\widehat{H}$	$\dim \delta$	$\mathfrak{d}(\dim X)$
$A(p,q)\colon \begin{array}{ccc} & \bullet\to\dots\to\bullet & \\ \nearrow & & \searrow \\ \spadesuit & & \diamondsuit \\ \searrow & & \nearrow \\ & \bullet\to\dots\to\bullet & \end{array}$	$\begin{array}{ccc} & 1\ \dots\ 1 & \\ 1 & & 1 \\ & 1\ \dots\ 1 & \end{array}$	$n_\spadesuit - n_\diamondsuit$
$\widehat{D}_r\colon \begin{array}{ccccc} \spadesuit_1 & & & & \diamondsuit_1 \\ & \searrow & & \nearrow & \\ & & \bullet\to\dots\to\bullet & & \\ & \nearrow & & \searrow & \\ \spadesuit_2 & & & & \diamondsuit_2 \end{array}$	$\begin{array}{ccc} 1 & & 1 \\ & 2\ \dots\ 2 & \\ 1 & & 1 \end{array}$	$n_{\spadesuit_1} + n_{\spadesuit_2} - n_{\diamondsuit_1} - n_{\diamondsuit_2}$
$\widehat{E}_6\colon \begin{array}{c} a_2 \\ \uparrow \\ a_1 \\ \uparrow \\ b_2 \leftarrow b_1 \leftarrow \spadesuit \to c_1 \to c_2 \end{array}$	$\begin{array}{ccccc} & & 1 & & \\ & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{array}$	$3n_\spadesuit - \sum_{i=1}^{2}(n_{a_i} + n_{b_i} + n_{c_i})$
$\widehat{E}_7\colon \begin{array}{c} c \\ \uparrow \\ a_3 \leftarrow a_2 \leftarrow a_1 \leftarrow \spadesuit \to b_1 \to b_2 \to b_3 \end{array}$	$\begin{array}{ccccccc} & & & 2 & & & \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 \end{array}$	$4n_\spadesuit - 2n_c - \sum_{i=1}^{3}(n_{a_i} + n_{b_i})$
$\widehat{E}_8\colon \begin{array}{c} c \\ \uparrow \\ b_2 \leftarrow b_1 \leftarrow \spadesuit \to a_1 \to a_2 \to a_3 \to a_4 \to a_5 \end{array}$	$\begin{array}{cccccccc} & & 3 & & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$	$6n_\spadesuit - 3n_c - 2\sum_{i=1}^{2} n_{b_i} - \sum_{i=1}^{5} n_{a_i}$

Table 3.3: Dimension vector δ of the minimal imaginary roots and Dlab-Ringel defects $\mathfrak{d}(\dim X)$ of the representations X of $\widehat{H}$. In the last column we adopt the shorthand $n_i \equiv \dim X_i$.

said to be *real* iff $q_{\widehat{H}}(\gamma) = 1$, while it is *imaginary* iff $q_{\widehat{H}}(\gamma) = 0$. There is always a *minimal imaginary root*, δ, such that

$$\begin{aligned} \text{rad}\ \ q_{\widehat{H}}(\cdot) &\equiv \{\gamma \in \Gamma_{\widehat{H}} \mid \langle \gamma\,, - \rangle_E = -\langle -\,, \gamma \rangle_E \} \\ &\simeq \mathbb{Z}\delta. \end{aligned} \tag{3.23}$$

The list of the minimal imaginary roots δ for all $\widehat{H}$ is in table 3.3. The Dlab–Ringel *defect* of a module $X \in \mathsf{rep}(\widehat{H})$ is simply

$$\mathfrak{d}_{\widehat{H}}(X) \equiv \langle \delta\,, \dim X \rangle_E = -\langle \dim X\,, \delta \rangle_E \tag{3.24}$$

Notice that the Coxeter element $\Phi \equiv -(S^{-1})^t S$ of these algebras satisfies a very interesting identity in terms of the defect

$$\Phi^N = \text{Id} + \frac{b\,N}{2}\,\delta \otimes \langle \delta, - \rangle_E \tag{3.25}$$

where b and N are as in table 3.4.

From the definition of the Euler form (2.12), we notice that, for an hereditary algebra

$$q_{\widehat{H}}(X) = \dim\, \text{End}(X) - \dim\, \text{Ext}^1(X, X). \tag{3.26}$$

$\widehat{H}$:	$\widehat{A}(p,q)$	$\widehat{D}_r$	$\widehat{E}_r$
N :	$\text{lcm}\{p,q\}$	$\text{lcm}\{2,2,r-2\}$	$\text{lcm}\{2,3,r-3\}$
b :	$2(1/p+1/q)$	$2(1/(r-2))$	$2((9-r)/6(r-3))$

Table 3.4: Couples (b, N) for eqn.(3.25) and corresponding Euclidean algebras.

Since we know that representations that are stable in some chamber are always *bricks* we have that

$$X \text{ brick } \Rightarrow \dim \text{Ext}^1(X,X) = \begin{cases} 1 & \text{iff } q_{\widehat{H}}(X) = 0 \\ 0 & \text{iff } q_{\widehat{H}}(X) = 1 \end{cases} \tag{3.27}$$

Moreover, our quiver being acyclic, lemma 1 of §.3 of [77] computes the dimension of the moduli space $\mathcal{M}(\gamma)$ for $\gamma = \dim X$ as

$$\dim \mathcal{M}(\dim X) = \dim \text{Ext}^1(X,X). \tag{3.28}$$

The bricks of $\widehat{H}$ are fully classified. Given two dimension vectors γ_1 and γ_2 we say that $\gamma_1 \leq \gamma_2$ iff this inequality holds componentwise.

Characterization of bricks. Let γ be a positive *real* root. The indecomposable representation of $\widehat{H}$ of dimension γ is a *brick* iff one of the following holds: *1.)* $\langle \gamma, \delta \rangle_E \neq 0$; or *2.)* $\gamma \leq \delta$. Moreover, the only other indecomposable representations that are bricks have dimension vector δ.

Let us call W_λ the elements of the $\mathbb{P}^1$ orbit of representations with $\dim W_\lambda \equiv \delta$. This $\mathbb{P}^1$–family is interpreted as the $SU(2)$ W–boson of the model. Notice that the corresponding magnetic charge coincides with the Dlab–Ringel defect:

$$\begin{aligned} m(X) &\equiv \tfrac{1}{2}(\dim X)^t B \delta \\ &= \tfrac{1}{2}(\dim X)^t (S^t - S)\delta \\ &= \tfrac{1}{2}(\langle \delta, \dim X\rangle_E - \langle \dim X, \delta \rangle_E) \\ &= \langle \delta, \dim X \rangle_E \\ &= \mathfrak{d}_{\widehat{H}}(X). \end{aligned} \tag{3.29}$$

This remark leads to the physical interpretation of the characterization of bricks. Indeed, all the positive real roots that have non–vanishing defect correspond to magnetically charged hypermultiplets (the dyons). In addition, there may be other *magnetically neutral* hypers that corresponds to those positive real roots γ such that $m(\gamma) = 0$ *and* $\gamma \leq \delta$. In the limit $g \to 0$, all magnetically charged particles decouple. The light subcategory will contain only the following objects: *(1.)* a $\mathbb{P}^1$ family of indecomposable bricks with charge δ, (*i.e.* the $SU(2)$ W–boson), *(2.)* the rigid bricks with vanishing defect and dimension vectors in the positive roots s.t. $\gamma \leq \delta$. With the exception of the cases $\widehat{A}(1,1), \widehat{A}(2,1), \widehat{A}(2,2)$, and $\widehat{D}_4$, from this fact it follows that the actual light BPS–spectrum will depend on the stability conditions, *i.e.* there are *many* domains of the first kind that are compatible with the limit $g \to 0$, and in each such domain the stable and light BPS–hypers may still not be all mutually local[8]. This is enough to deduce that the theory represents $SU(2)$ weakly gauging an $SU(2)$ flavor symmetry of a strongly coupled non–lagrangian conformal system. The exceptions, indeed, corresponds to the lagrangian asymptotically–free $SU(2)$ SQCD with fundamental hypers in $N_f = 0,1,2,3$. The only SCFT's that have these properties (spectrum made only of hypermultiplets in all chambers[9] + $SU(2)$ flavor symmetry) are the Argyres–Douglas models of type D_p. Our conclusion for the moment is that any $\widehat{H}$ has an S–duality frame in which it can be represented as an $SU(2)$ SYM sector coupled to a certain direct sum $\oplus_i D_{p_i}$. It remains to determine the set of $\{p_i\}_i$ that correspond to the given $\widehat{H}$ model. In order to do that, there are other fundamental properties of the category $\mathsf{rep}(\widehat{H})$ that comes to rescue us.

The category $\mathsf{rep}(\widehat{H})$ has the following structure[10]

$$\mathsf{rep}(\widehat{H}) = \mathcal{P}_{\widehat{H}} \vee \mathcal{T}_{\widehat{H}} \vee \mathcal{Q}_{\widehat{H}} \tag{3.30}$$

where $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{Q}$ are called, respectively, the preprojective, the regular, and the preinjective component. These subcategories of $\mathsf{rep}(\widehat{H})$ are distinguished by the defect: if X is an indecomposable module, then it is, respectively, preprojective, regular or preinjective, if its its defect is negative, zero, or positive. From our definitions, the light subcategory $\mathscr{L}(\widehat{H})$ is precisely $\mathcal{T}_{\widehat{H}}$. The category $\mathsf{rep}(\widehat{H})$ has

8 However, since we know that they can come only in hypermultiplets, their Dirac pairings are constrained to be ≤ 1.

9 This fact alone implies that the underlying quiver is Dynkin of type ADE, by Gabriel's theorem on classification of representation–finite algebras.

10 The notation we use is due to Ringel [65]. Let $\mathscr{A}$ be a Krull–Schmidt category (*i.e.* a $\mathbb{C}$–linear category such that all the endomorphism rings of its indecomposable objects are local — this is always the case for categories of the form $\mathsf{rep}(Q,\mathcal{W})$ that corresponds to 4d $\mathcal{N}=2$ models), an *object class* $\mathscr{B} \subseteq \mathscr{A}$ is a full subcategory of $\mathscr{A}$ that is closed under direct sums, direct summands, and isomorphisms. Any object class of a Krull–Schmidt category is itself a Krull–Schmidt category and it is characterized uniquely by the indecomposable objects that belong to it. Given two object classes $\mathscr{B}_1, \mathscr{B}_2$, we will call

$$\mathscr{B}_1 \vee \mathscr{B}_2 = \mathsf{add}(\mathscr{B}_1, \mathscr{B}_2)$$

the smallest object class containing $\mathscr{B}_1$ and $\mathscr{B}_2$.

Auslander–Reiten translations[11] $\tau, \tau^{-}\colon \mathsf{rep}(\widehat{H}) \to \mathsf{rep}(\widehat{H})$. For indecomposable X,

$$\begin{aligned} &X \in \mathcal{P} \Leftrightarrow \exists\, n > 0 \text{ s.t. } \tau^n X = 0 \\ &X \in \mathcal{Q} \Leftrightarrow \exists\, n > 0 \text{ s.t. } \tau^{-n} X = 0 \\ &X \in \mathcal{T} \Leftrightarrow \tau^n X \neq 0 \ \forall n \in \mathbb{Z} \end{aligned} \tag{3.31}$$

For X indecomposable and not projective

$$\dim \tau X = \Phi \dim X. \tag{3.32}$$

In particular, τ and τ^- are always autoequivalences[12] of the category $\mathcal{T}_{\widehat{H}}$. Moreover, τ induce a *Serre–duality* on $\mathcal{T}_{\widehat{H}}$: again let $X, Y \in \mathcal{T}_\lambda$, we have (Auslander–Reiten formula)

$$\mathrm{Hom}(Y, X) \simeq D\mathrm{Ext}^1(X, \tau Y) \tag{3.33}$$

where D is the usual duality functor of $\mathsf{vec}_{\mathbb{C}}$: $D \equiv \mathrm{Hom}(-, \mathbb{C})$. The categories of regular representations of Euclidean algebras all have the *Ringel property*: in the Ringel notation introduced in footnote 10, this is expressed as

$$\mathcal{T}_{\widehat{H}} = \bigvee_{\lambda \in \mathbb{P}^1} \mathcal{T}_\lambda \tag{3.34}$$

where the elements $\mathcal{T}_\lambda$ are *(standard stable) tubes*. All tubes in (3.34) are Hom–orthogonal, *i.e.* $\mathrm{Hom}(\mathcal{T}_\lambda, \mathcal{T}_{\lambda'}) \neq 0 \Leftrightarrow \lambda = \lambda'$. The Auslander–Reiten functors *respects* this 'fibration' of the additive generators of $\mathcal{T}_{\widehat{H}}$ in tubes[13], *i.e.* given an indecomposable representation $X \in \mathcal{T}_\lambda$, $\tau X \in \mathcal{T}_\lambda$. Notice that these two properties (Hom–orthogonality and τ compatibility) holds also for the object classes[14]

$$\mathcal{T}_{\widehat{H}}(\lambda) \equiv \mathsf{add}(\mathcal{T}_\lambda). \tag{3.35}$$

There are only two kinds of standard stable tubes:

- *homogeneous* tubes, $\mathcal{T}_1$: for all indecomposable $X \in \mathcal{T}_1$ we have $\tau X = X$;
- *non–homogeneous* or *periodic* tubes, $\mathcal{T}_p$: for all indecomposable $X \in \mathcal{T}_p$, $\tau^p X = X$.

[11] For Euclidean algebras $\mathcal{A}$ over $\mathbb{C}$ the AR–translations becomes simply the dual (in $\mathsf{vec}_{\mathbb{C}}$) of the functors $\mathrm{Ext}^1(-, \mathcal{A})$.

[12] Recall that two functors $F\colon \mathscr{A} \to \mathscr{B}$, and $G\colon \mathscr{B} \to \mathscr{A}$ are *adjoint* iff $\mathrm{Hom}_{\mathscr{B}}(FX, Y) = \mathrm{Hom}_{\mathscr{A}}(X, GY)$. Two adjoint functors are an *equivalence* of the categories $\mathscr{A}$ and $\mathscr{B}$ iff they induce isomorphisms in between the corresponding morphism spaces (*i.e.* they are full and faithful).

[13] More rigorously, such 'decomposition' is *defined* via the AR–functors: see footnote 15.

[14] See the definition in footnote 10.

An object $X \in \mathcal{T}_{\widehat{H}}$ is a *regular simple object* iff it has no proper non–zero regular submodule, *i.e.* $\mathfrak{d}_{\widehat{H}}(X) = 0$ and $\mathfrak{d}_{\widehat{H}}(Y) < 0 \quad \forall\, 0 \neq Y \subset X$. If S is a regular simple it is a brick. In particular, $\tau\, S = S$ iff dim S is an imaginary root. If this is the case, clearly, dim $S = \delta$. A τ orbit of regular simples uniquely characterize the tube it belongs to[15]. If the tube is of period one, there is only one regular simple in it, it is a brick, and its dimension vector is δ. On the contrary, if the tube has period p, we have an orbit of regular simples $S_k \equiv \tau^k S$ such that

$$\sum_{k=0}^{p-1} \dim\, \tau^k S = \delta. \tag{3.36}$$

Being bricks, the S_k must correspond to positive real roots and have to be rigid. It is natural to associate to the category $\mathsf{add}(\mathcal{T}_p)$ its Gabriel quiver. It has nodes in correspondence with the S_k and in between two nodes S_i and S_j one draws

$$\dim\, \mathrm{Ext}^1(S_i, S_j) =_{\text{by eqn.(3.33)}} \dim\, \mathrm{Hom}(S_j, \tau S_i) =_{\text{Schur's lemma}} \delta_{j,i+1} \tag{3.37}$$

arrows. The last equality follows from the fact that the S_i are simple objects: by Schur's lemma any morphism in between them is either an isomorphism or zero. The resulting quiver is precisely the $\widehat{A}(p,0)$ quiver that we have discussed at the beginning of this section. Its *nilpotent indecomposable* representations are in one–to–one correspondence with the elements of the stable tube of period $\mathcal{T}_p$. By eqn.(3.36) an indecomposable with dimension vector δ is part of the nilpotent indecomposables of $\widehat{A}(p,1)$: in the terminology of footnote 15 it is the the 'smallest' regular indecomposable of lenght p. From the characterization of bricks, it follows that the information about the BPS–states of the SCFT 'matter system' is contained in the rigid bricks of the tube that have dimensions $\gamma < \delta$. To untangle the rigid bricks from the whole tube we have to impose a relation that selects only the regular indecomposables of length $\leq p-2$. This in turns is a condition on the length of the worlds one can form using the arrows of the quiver $\widehat{A}(p,0)$: imposing that all the worlds of lenght $> p-2$ are zero is equivalent to impose the relations on $\widehat{A}(p,0)$ that would follow from the partials of the superpotential $\mathcal{W} \equiv p$-cycle. As we have said at the beginning of this section, the quiver $(\widehat{A}(p,0), \mathcal{W} = p\text{-cycle})$ is mutation equivalent to the quiver for a D_p system.

[15] Indeed, every indecomposable regular module X is such that it has regular submodules $0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_\ell = X$ and these are the only regular submodules of it. ℓ is called the (regular) length of X. The submodules $X_1, X_2/X_1, X_3/X_2, \ldots, X_\ell/X_{\ell-1}$ are called the *(regular) composition factors* of X, X_1 is its (regular) socle, $X/X_{\ell-1}$ its (regular) top. For each regular simple S there is a unique regular module X with regular top S and composition factors, in decreasing order from the top, $S, \tau S, \tau^2 S, \ldots, \tau^\ell S$. The tube $\mathcal{T}_p$ corresponding to an orbit of regular simples is the set of all indecomposable regulars with composition factors in the orbit.

This shows that each regular tube of period p in $\mathcal{T}_{\widehat{H}}$ corresponds to the gauging of a D_p matter system. The characterization of bricks constraints the possible tubes: at the generic points of the $\mathbb{P}^1$ base of the Ringel–decomposition there are only homogeneous tubes, while at finitely many non–generic points there are non–homogenous tubes. The possible non–homogenous tubes have periods p_i that precisely coincides with the D_{p_i} triples of table 3.2 — see also table 3.6. For example, the systems of type $\widehat{A}(p,1)$ have light subcategories that contain a unique non–homogenous tube of period p. This concludes our summary.

Remark 1. The category $\mathsf{rep}\,\widehat{A}(p,0)$ will play a rôle in what follows. Consider $\mathcal{T}_{\widehat{A}(p,1)}$, the category of regular objects of $\widehat{A}(p,1)$. Let us call $\mathcal{T}_\infty$ the (unique) *homogenous* tube that contains the regular simple

$$\begin{array}{ccccccc} & \mathbb{C} \xrightarrow{\;1\;} \mathbb{C} \xrightarrow{\;1\;} \cdots \xrightarrow{\;1\;} \mathbb{C} & \\ {}^{1}\nearrow & & \searrow^{1} \\ \mathbb{C} & \xrightarrow{\qquad\qquad 0 \qquad\qquad} & \mathbb{C} \end{array} \tag{3.38}$$

It is a theorem by Ringel[16] that

$$\mathcal{T}_{\widehat{A}(p,1)} = \mathsf{rep}\,\widehat{A}(p,0) \vee \mathcal{T}_\infty. \tag{3.39}$$

Remark 2. Notice that via eqn.(3.29), eqn.(3.25) becomes

$$\Phi^N = \mathrm{Id} + b\,N\,m(-)\mathfrak{q}. \tag{3.40}$$

where $\mathfrak{q}$ is the electric weight of $SU(2)$. As we will review in §.3.4.4, this is interpreted as the effect of a $2\pi N$ chiral rotation on the charge lattice: on one hand this is precisely the Witten–effect on Dyon spectra, on the other hand b of table 3.4 can be interpreted as the coefficient of the chiral axial anomaly, and therefore is precisely the coefficient of the beta–function of the $SU(2)$ coupling

$$\mu\frac{\partial}{\partial\mu}\frac{4\pi}{g^2} = \frac{b}{2\pi}.$$

This is a very interesting way to show that each D_p contributes to the beta function of the $SU(2)$ SYM coupling $b(D_p) = 2(p-1)/p$. Using this formula, one check that the models listed in (3.2) precisely correspond to all possible (complete) matter systems which are compatible with the asymptotic freedom of a simple $SU(2)$ gauge group.

16 (6) pag.160 of [65]. We thank Bill Crawley–Boevey for pointing that out.

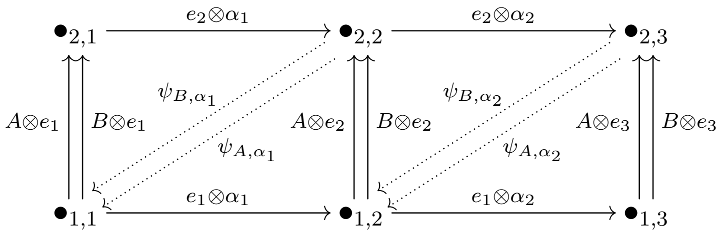


Figure 3.1: The quiver for $SU(4)$ SYM in the notation of (3.12).

3.3.2. The Ringel property of light subcategories. One of the major outcomes of the anlisys of [10] is that the Ringel–decomposition property generalizes to all 4d $\mathcal{N} = 2$ theories with any (simply–laced) gauge group. If we consider the $g \to 0$ limit for one simple simply–laced factor of the gauge group G, the light category has the structure [10]

$$\mathscr{L} = \bigvee_{\lambda\in\mathbb{P}^1} \mathscr{L}_\lambda, \tag{3.41}$$

where the Abelian categories $\mathscr{L}_\lambda$ are called, by analogy with eqn.(3.34), *G–tubes.*

Homogenous G–tubes

Let G be a simple simply–laced lie group. Consider the pure G SYM system. Its quiver is $\widehat{A}(1,1)\boxtimes G$. The charge lattice of this model is $\Gamma_{\widehat{A}(1,1)}\otimes\Gamma_G$, where Γ_G is the root lattice of G. This system has a canonical weak–coupling regime. Let $\widehat{A}(1,1)_a$ denote the full $\widehat{A}(1,1)$ subquiver on the a-th node of the Dynkin subquiver of type G. Let W_a denote the W–boson of the model that corresponds to the simple root e_a in Γ_G^+. The canonical weak–coupling for this system is obtained by declaring that W_a corresponds to the representations that have dimension vector equal to $\delta \otimes e_a$, the minimal imaginary root of the $\hat{A}(1,1)_a$ full subquiver. By (3.6) the Euler form factorizes as the product of the Euler forms of the factors, therefore

$$\begin{aligned}\langle \delta\otimes e_a\,, \delta\otimes e_b\rangle_E &= \langle \delta\,,\delta\rangle_E\cdot\langle e_a\,,e_b\rangle_E\\ &=_{\delta \text{ is an imaginary root}} 0\end{aligned} \tag{3.42}$$

Since the Dirac pairing for this class of models is simply the anti–symmetrization of the Euler form, this ensures that the charges we have chosen are mutually local:

$$\langle \delta\otimes e_a\,, \delta\otimes e_b\rangle_{\text{Dirac}} = 0. \tag{3.43}$$

Let $X \in \mathsf{rep}(\widehat{A}(1,1)\boxtimes G)$. By construction $\dim X = \sum N_{i,b} e_i\otimes e_b$, where $i = 1,2$ and $b = 1,...,r$, the rank of G. Again we use that the Euler form factorizes as the product of the Euler forms of the factors:

$$\begin{aligned}\langle \dim X, \delta\otimes e_c\rangle_E &= \langle \sum_{i,b} N_{i,b} e_i \otimes e_b\,,\delta\otimes e_c\rangle_E \\ &= \sum_{i,b} N_{i,b}\langle e_i,\delta\rangle_E \langle e_b\,,e_c\rangle_E \\ &= -\sum_b \mathfrak{d}(X\big|_{\widehat{A}(1,1)_b})\langle e_b\,,e_c\rangle_E.\end{aligned} \tag{3.44}$$

Where $\mathfrak{d}$ is the defect of the $\widehat{A}(1,1)$ Euclidean algebra (3.24). Analogously, by (3.24), we obtain that

$$\langle \delta\otimes e_c\,,\dim X\rangle_E = \sum_b \mathfrak{d}(X\big|_{\widehat{A}(1,1)_b})\langle e_c\,,e_b\rangle_E \tag{3.45}$$

Therefore (compare with (2.12) and (2.13))

$$\begin{aligned}\langle \dim X\,,\delta\otimes e_c\rangle_{\text{Dirac}} &\equiv \langle \delta\otimes e_c\,,\dim X\rangle_E - \langle \dim X\,,\delta\otimes e_c\rangle_E \\ &= \sum_b \mathfrak{d}(X\big|_{\widehat{A}(1,1)_b})\big(\langle e_b\,,e_c\rangle_E + \langle e_c\,,e_b\rangle_E\big) \\ &\equiv \sum_b (C_G)_{cb}\mathfrak{d}(X|_{\widehat{A}(1,1)_b}).\end{aligned} \tag{3.46}$$

Where $(C_G)_{cb}$ is the Cartan matrix of G. Using the definition (3.15), the a-th magnetic charge for the canonical weak–coupling is

$$\begin{aligned}m_a(X) &= (C_G^{-1})_{ac}\langle \dim X\,,\delta\otimes e_c\rangle_{\text{Dirac}} \\ &= \mathfrak{d}(X|_{\widehat{A}(1,1)_a}) \in \mathbb{Z}\end{aligned} \tag{3.47}$$

Therefore the Dirac integrality condition is satisfied. The (canonical) light subcategory $\mathscr{L}(\widehat{A}(1,1)\boxtimes G)$ of this system is defined as the controlled subcategory with respect to this choice of controls. There are obvious Higgs decoupling limits of the form $G \to SU(2)\times$ 'decoupled photons' that are compatible with the canonical weak–coupling regime: these are given by the full $\widehat{A}(1,1)$ subquivers on the nodes of G. By compatibility with the Higgs decoupling limits, (3.19), we know that

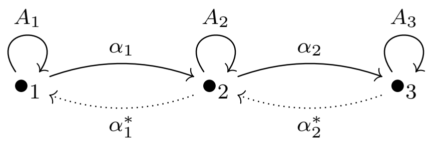
$$X\in\mathscr{L} \Rightarrow X\big|_{\widehat{A}(1,1)_a} \in \mathcal{T}_{\widehat{A}(1,1)_a} \qquad \forall\, a=1,...,r=\text{rank}(G) \tag{3.48}$$

e.g. $X|_{\widehat{A}(1,1)_a}$ is a direct sum of regular indecomposables of the $\widehat{A}(1,1)$. Now, we are going to show that the canonical light subcategory has a Ringel–decomposition of the form in eqn.(3.41) in two ways. The first gives some insight on the quiver of the light subcategory for these models that we will generalize later, the second is just a sketch to show the power of the Auslander–Reiten functor approach to this story; it can be generalized too, but getting the quiver from that might be hard[17].

First proof. From (3.39) in remark 1 of the previous section we know that

$$\mathcal{T}_{\widehat{A}(1,1)} = \mathsf{rep}\,\widehat{A}(1,0) \vee \mathcal{T}_\infty$$

Forgetting about the $\mathcal{T}_\infty$ component, it is natural to associate to the category $\mathscr{L}(A(1,1) \boxtimes G)$ the quiver $A(1,0) \boxtimes G$. For example the quiver for the light subcategory of $SU(4)$ SYM is $\widehat{A}(1,0) \boxtimes A_3$ (the dashed arrows are the relations that implement the commutativity of the tensor product):



(3.49)

$$\mathcal{W}_\boxtimes = \alpha_1^*(\alpha_1 A_1 - A_2\alpha_1) + \alpha_2^*(\alpha_2 A_2 - A_3\alpha_2).$$

Each representation restricted to one node $X|_{\bullet_i}$ can be thought as a representation of the 1–loop Jordan quiver $\widehat{A}(1,0)$. The relations $\partial_{\alpha_i^*}\mathcal{W} = 0$ entails that the arrows α_i induce morphisms

$$\alpha_i \colon X|_{\bullet_{s(\alpha_i)}} \to X|_{\bullet_{t(\alpha_i)}}$$

in between different representations, and the same holds for the relations $\partial_{\alpha_i}\mathcal{W} = 0$: the arrows

$$\alpha_i^* \colon X|_{\bullet_{s(\alpha_i^*)}} \to X|_{\bullet_{t(\alpha_i^*)}}$$

are morphisms too. Embedding back $\mathsf{rep}\,\widehat{A}(1,0)$ in $\mathcal{T}_{\widehat{A}(1,1)}$, we see that whenever the arrows α and α^* are non–zero, they induce morphisms in between objects of $\mathcal{T}_{\widehat{A}(1,1)}$. Since Jordan representations are in particular regular of the Euclidean $\widehat{A}(1,1)$ algebra, they must be direct sums of indecomposables that belongs to homogeneous tubes. Consider an indecomposable representation of $\widehat{A}(1,0)\boxtimes G$ that is supported on more than one node, *i.e.* $X|_{\bullet_i} \neq 0$ and $X|_{\bullet_j} \neq 0$ for $i \neq j$. Since X is indecomposable, the support must be connected, and some of the α, α^* arrows must be non–zero. Thus we have induced homomorphisms in between the various $X|_{\bullet_i}$. This is possible iff $X|_{\bullet_i} \in \mathsf{add}\,(\mathcal{T}_1)_\lambda$ at the same $\lambda\ \forall\ i$. By our discussion around (3.35), these are, indeed, the only possible objects that admits morphisms in between each other. If this is not the case, there is at least an indecomposable summand Y_i of $X|_{\bullet_i}$ for some i that belongs to another tube, at a different λ, but

[17] Even though a possible approach would be to use the Yoneda pairings $\mathrm{Ext}^m_R(X,Y) \otimes \mathrm{Ext}^m_S(P,Q) \to \mathrm{Ext}^{m+n}_{R\otimes S}(X\otimes P, Y\otimes Q)$ where X,Y are R–modules, and P,Q are S–modules.

all the arrows connecting it to other objects that belongs to the λ component must be zero and the representation splits, it is no more indecomposable. This shows that the category of representations of $\widehat{A}(1,0) \boxtimes G$ is Ringel–decomposed over $\mathbb{C}$. Adding the projective closure at $\lambda = \infty$ nothing changes[18]. By construction (all tubes of $\mathcal{T}_{\widehat{A}(1,1)}$ are homogenous), the fibers of indecomposable modules in the Ringel–decomposition of this category are, moreover, all equivalent to one, called the *homogenous G–tube*, and denoted $\mathscr{L}_G^{YM}$.

The relations following from $\partial_{A_k}\mathcal{W}$, are precisely the relations that define a preprojective algebra[19] of type $\mathcal{P}(G)$. We have a forgetful functor from $\mathsf{rep}\widehat{A}(1,0)\boxtimes G$ to the category of modules of the preprojective algebra $\mathsf{mod}\,\mathcal{P}(G)$, given by forgetting the A_i arrows. We know that all indecomposable objects belongs to homogenous G–tubes at fixed λ. This means that focusing on *indecomposable bricks*, all A_i arrows are *fixed*: $A_i = \lambda \cdot \mathrm{id}_{X_i}$, $\forall i$. Forgetting the A_i arrows, we are working directly on $\mathsf{mod}\,\mathcal{P}(G)$. Moreover, by the previous argument, the indecomposable bricks of the latter are precisely in bijection with the bricks of the homogenous G–tube. The category $\mathsf{mod}\,\mathcal{P}(G)$ has only *rigid* bricks with dimension vectors in bijection with the positive roots of G, moreover, precisely one copy of the positive roots of G that we have identified corresponds to bricks of $\mathscr{L}_G^{YM}$ that are stable given a domain of the first kind that is compatible with the canonical weak coupling limit[20]. This concludes the proof that the light subcategory of G SYM contains precisely one adjoint vectormultiplet.

Second proof (skecth). The Auslander–Reiten translations of $\mathsf{rep}\,\widehat{A}(1,1)$ induce the functors

$$\tau \otimes 1, \tau^{-} \otimes 1 : \mathsf{rep}\,A(1,1) \boxtimes G \to \mathsf{rep}\,A(1,1) \boxtimes G. \tag{3.51}$$

[18] Indeed, by S_2 symmetry of the Kronecker quiver $\mathcal{T}_\infty$ is equivalent to $\mathcal{T}_{\lambda=0}$.

[19] Given a (finite, connected) graph L, we define its *double quiver* $\overline{L}$ by replacing each edge $\xrightarrow{\alpha}$ in L with a pair of opposite arrows $\underset{\alpha^*}{\overset{\alpha}{\rightleftarrows}}$. The quotient of the path algebra of $\overline{L}$ by the ideal generated by the relations

$$\sum_{\alpha \in L} (\alpha\alpha^* - \alpha^*\alpha) = 0, \tag{3.50}$$

is called the *preprojective algebra* of the graph L [90,91], which we write as $\mathcal{P}(L)$. A basic result is that $\mathcal{P}(L)$ is finite dimensional if and only if L is an *ADE* Dynkin graph G. In this case, moreover, (3.50) give precisely the set of relations $\partial_{A_i}\mathcal{W} = 0$.

[20] We shall review only the proof of the first part. It is known that for X, Y bricks of $\mathsf{mod}\,\mathcal{P}(G)$ $(X,Y)_C = (\dim X)^t (C_G) \dim Y = \dim \mathrm{Hom}(X,Y) + \dim \mathrm{Hom}(Y,X) - \dim \mathrm{Ext}^1(Y,X)$. Specialize to the case $X = Y$: $0 < (X,X)_C = 2 - \dim \mathrm{Ext}^1(X,X)$. But the minimal value of $(\cdot,\cdot)_{C_G}$ is 2, and this is attained precisely for X with dimensions equal to the positive G root lattice. So $\dim \mathrm{Ext}^1(X,X) = 0$. The dimension of the moduli space of a $\mathcal{P}(G)$–module is proportional to $\dim \mathrm{Ext}^1(X,X)$, so the bricks are rigid.

From (3.48), it is the clear that an indecomposable object $X \in \mathsf{rep}A(1,1) \boxtimes G$ is in the canonical light subcategory $\mathscr{L}$ iff $(\tau \otimes 1)X = X$. The fact that $\mathscr{L}$ inherits the Ringel–decomposition of $\mathcal{T}_{\widehat{A}(1,1)}$ follows from this remark. Indeed, let $S \in (\mathcal{T}_1)_\lambda \subset \mathcal{T}_{\widehat{A}(1,1)}$ be the regular simple that uniquely characterize that tube. All objects $S \otimes R$, where R is a brick of $\mathsf{rep}\, G$ belong to the same $\tau \otimes 1$ orbit, and these are the regular simple objects for the homogeneous G–tube $\mathscr{L}_G^{YM}$, that automatically satisfy the relations of $\partial \mathcal{W}_\boxtimes$. The bricks of $\mathsf{rep}\, G$ have dimension vectors equal to the positive roots of G and are moreover its only indecomposables. The rest of the homogenous G–tube at λ consist of all the (indecomposable) extensions that are compatible with the relations $\partial \mathcal{W}_\boxtimes$ of the objects of the form $X \otimes R$ where $X \in (\mathcal{T}_1)_\lambda$ is an indecomposable regular.

Non–homogenous G–tubes

Whenever we add matter to the system, there will be special points of the $\mathbb{P}^1$ family that correponds to non–homogeneous G–tubes. For almost all $\lambda \in \mathbb{P}^1$, the G–tubes of eqn.(3.41) are homogenous, while at non–generic points, we will find non–homogenous G–tubes.

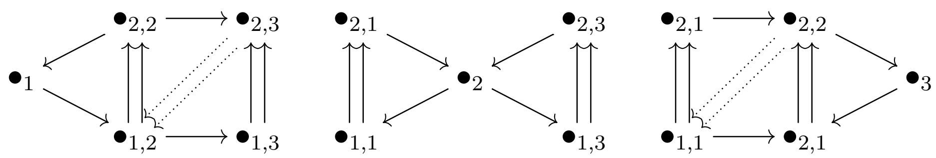
If G is not simple, we have as many coupling constants as simple factors. The above procedure can be carried over for each coupling constant *separately.* At each step we decouple the states that are heavy in the limit in which one of the coupling becomes weak. At the end we will remain with a category that is fibered over a set of $\mathbb{P}^1$'s, one per simple factor of the gauge group, that intersect at non–generic points. At the generic point of each $(\mathbb{P}^1)_k$ we will find the homogeneous G_k–tube of the corresponding simple factor $G_k \subset G$. At the non–generic points of intersection $(\mathbb{P}^1)_{k_1} \cap (\mathbb{P}^1)_{k_2} \cap \cdots (\mathbb{P}^1)_{k_n}$ we are going to find a non–homogeneous $\prod_{i=1}^n G_{k_i}$–tube that represents the gauging of some matter subsystem with $\prod_{i=1}^n G_{k_i}$ flavor group. The more complicated the theory, the more complicated will be the corresponding index variety N such that

$$\mathscr{L} = \bigvee_{\lambda \in N} \mathscr{L}_\lambda. \tag{3.52}$$

The index variety N has a clear geometrical meaning for all theories of class $\mathcal{S}[\mathcal{C}, \mathfrak{g}]$. N is the degeneration limit of $\mathcal{C}$ that corresponds to the S–duality frame captured by $\mathscr{L}$. From this perspective, the classification of $\mathcal{N} = 2$ theories with BPS–quiver property amounts to the classification of possible non–homogenous G–tubes, the Cecotti's *categorical tinkertoy.* The passage from a given light subcategory $\mathscr{L}(Q, \mathcal{W})$ to the full category $\mathsf{rep}(Q, \mathcal{W})$ is said to be a non–perturbative completion or a categorical quantization.

Specialization: a lightning sketch

One can rephrase the specialization procedure of [12] in the language of the previous sections. Consider the quiver of the light subcategory of G SYM, $A(1,0)\boxtimes G$ and set one of the 'loop' arrows, A_i in (3.49), to a special value λ^*. Setting one arrow to λ^*, the possible hom–orthogonal tubes localizes to the λ^*–one. At the tube at $\lambda=\lambda^*$ one finds bricks with the charges of the adjoint representation of G, while at the tubes at $\lambda\neq\lambda^*$ one finds bricks only for the full subquivers obtained deleting the i-th node. Consider the $SU(4)$ SYM example, the quiver is (3.49). Setting A_1 (resp.A_3) to zero one obtains the light subcategory for $SU(3)$ with one hyper in the fundamental $\mathbf{3}$ (resp. antifundamental $\bar{\mathbf{3}}$). Setting $A_2=0$, one obtains $SU(2)$ with one hyper in the bifundamental. Embedding back $\widehat{A}(1,0)\boxtimes G$ in $\widehat{A}(1,1)\boxtimes G$, this procedure determines the superpotential of the specialized quivers



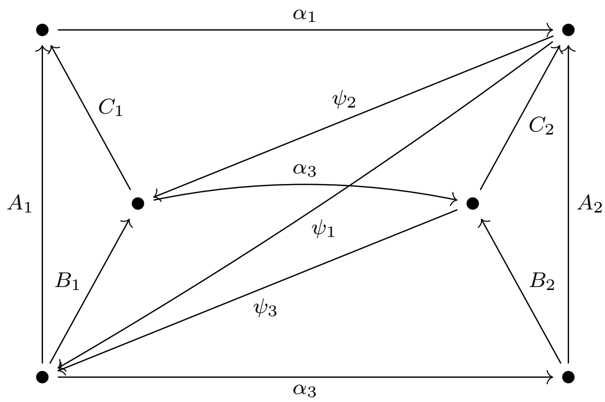
(3.53)

$SU(3)$ w/ $\mathbf{3}$ $\qquad$ $SU(2)^2$ w/ $(\mathbf{2},\bar{\mathbf{2}})$ $\qquad$ $SU(3)$ w/ $\bar{\mathbf{3}}$

The specializations of ADE SYM at *one* Kronecker precisely correspond to the list of cases explicitly discussed in ref. [92], namely

$$\begin{gathered}A_n\to A_{n-k}\times A_{k-1}\\ D_n\to D_{n-1},A_{n-1},D_{n-r}\times A_{r-1}\\ E_6\to D_5,A_5\\ E_7\to D_6,E_6,A_6\\ E_8\to E_7.\end{gathered}\tag{3.54}$$

In all instances the matter representations found in [92] using geometric methods agree with those found in [10] using the Representation Theory of the associated quivers with superpotential. In facts, in both approaches one is effectively reduced to a breaking of the adjoint representation of the Lie groups in the left part of eqn.(3.54) to the Lie groups on the right *times* $U(1)_f$, where — after specialization — $U(1)_f$ is interpreted as the *global* flavor symmetry of the matter sector. (Note that specialization at k vertical Kronecker subquivers reduces the numbers of nodes of the quiver by k; in particular, the exchange matrices of one-Kronecker specializations have odd rank, so that $\det B=0$, consistent with the fact that there is a flavor $U(1)_f$ charge).



$$\mathcal{W} = (\alpha_1 A_1 - A_2\alpha_3)\psi_1 + (\alpha_1 C_1 - C_2\alpha_2)\psi_2 + (\alpha_2 B_1 - B_2\alpha_3)\psi_3$$

Figure 3.2: The quiver and superpotential for $\widehat{A}(2,1) \boxtimes A_2$

3.4. The $\mathcal{N} = 2$ models $\widehat{H} \boxtimes G$

We consider the triangle tensor product $\widehat{H}\boxtimes G$ where $\widehat{H}$ stands for an acyclic affine quiver (listed in the first column of table (3.2)), and G is an ADE Dynkin quiver. Since $\hat{c}(\widehat{H}) = 1$ and $\hat{c}(G) < 1$, the total $\hat{c}$ is always less than 2, and thus all quivers of this form correspond to good $\mathcal{N} = 2$ QFT models by 2d/4d correspondence. With a slight abuse of notation, we shall use the symbol $\widehat{H}\boxtimes G$ to denote both the quiver (with superpotential) and the corresponding 4d $\mathcal{N} = 2$ theory. In addition we shall consider the S_1's corresponding to the four *elliptic* quivers $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$, and $E_8^{(1,1)}$ [7]. However in this last case the quiver (3.7) is not simply a tensor product of the quivers of the 2d direct summands.

Since the 2d minimal models are conformal, the direct sum 4d theory will be a SCFT precisely when the $(2,2)$ theory corresponding to the factor S_1 is conformal. An acyclic affine quiver always corresponds to a 4d theory which is asymptotically–free with $\beta \neq 0$ [50][7], and the 4d models $\widehat{H}\boxtimes G$ are also asymptotically–free with a non–zero β–function. On the contrary, the direct sum of a minimal and an elliptic 2d theories leads automatically to a superconformal 4d model.

If $\widehat{H} = \widehat{A}(1,1)$, the model $\widehat{H} \boxtimes G$ correspond to pure $\mathcal{N} = 2$ SYM with group G. In figure 3.2 we show the quiver (with superpotential) corresponding to the simplest next model *i.e.* $\widehat{A}(2,1) \boxtimes A_2$, the general case being a repetition of this basic structure[21]. We call the full subquiver $\widehat{H}\boxtimes\{\bullet_a\}\subset\widehat{H}\boxtimes G$ 'the affine quiver over the a–th node of the Dynkin graph G', or else 'the affine quiver associated to the the a–th simple root of the group G'; it will be denoted as $\widehat{H}_a$, where $a=1,2\dots,\operatorname{rank}G$.

3.4.1. Weak coupling We claim that the $\mathcal{N}=2$ model $\widehat{H}\boxtimes G$ is SYM with gauge group G coupled to some superconformal $\mathcal{N}=2$ matter (which may contain further SYM sectors). The most convincing proof of this statement consists in computing the BPS mass spectrum as $g_{\rm YM}\to 0$ and showing that the vectors which remain light in the limit form precisely one copy of the adjoint representation of G plus, possibly, G–singlets. This amounts to constructing the light category $\mathscr{L}$, checking that it has the universal structure described in §.3.3.2, and showing that its quiver satisfies all necessary conditions for being considered the BPS–quiver of another 4d $\mathcal{N}=2$ theory.

The charge lattice of these models $\Gamma_{\widehat{H}\boxtimes G}\simeq\Gamma_{\widehat{H}}\otimes\Gamma_G$. We may choose our S–duality frame in such a way that the representations W_a, corresponding to the a–th simple root W–boson, has support in the affine quiver $\widehat{H}_a$ over the a–th simple root. Then, by Kac's theorem [74], its dimension vector must be equal to the minimal imaginary roots of $\widehat{H}$

$$\dim W_a=\delta\otimes e_a. \tag{3.55}$$

This is exactly like the canonical S–duality frame for the pure G SYM models! All set of eqns.(3.42)–(3.47) carries over. In particular, the magnetic charges are given by

$$m_a(X)=\mathfrak{d}\big(\dim X\big|_{\widehat{H}_a}\big) \tag{3.56}$$

where $\mathfrak{d}$ is the Dlab–Ringel defect of the (sub)quiver $\widehat{H}_a$. And $m_a(\delta\otimes e_b)=0$ $\forall\ b$. Mutual locality and Dirac integrality of the magnetic charges are satisfied The magnetic charges $m_a(\cdot)$ define the canonical light category $\mathscr{L}$ as in §. 3.3. The consistency of the light subcategory with the Higgs mechanism expressed in eqn.(3.19), now reads

$$X\in\mathscr{L}\quad\Rightarrow\quad X\big|_{\widehat{H}_a}\in\mathcal{T}_{\widehat{H}}. \tag{3.57}$$

The category $\mathcal{T}_{\widehat{H}}$ is precisely the regular category of $\mathsf{rep}\,\widehat{H}$, see (3.34). Then on $\mathbb{P}^1$ there are ℓ distinct points λ_i such that the associated category $\mathcal{T}_{\lambda_i}$ is a stable tube of period p_i; the $\mathcal{T}_\lambda$'s over all other points of $\mathbb{P}^1$ are homogeneous tubes (period 1). The property (3.57) has an important refinement. For $X\in\mathscr{L}$ one has

$$X\big|_{\widehat{H}_a}\in\mathcal{T}_\lambda(\widehat{H})\qquad\text{the \underline{same} }\lambda\text{ for all }a. \tag{3.58}$$

[21] For $\widehat{H}=\widehat{A}(p,p)$, $\widehat{D}_r$ and $\widehat{E}_r$ we have an equivalent square product quiver without 'diagonal' arrows; for $\widehat{A}(p,q)$ we may reduce to a quiver with just $p-q$ diagonal arrows.

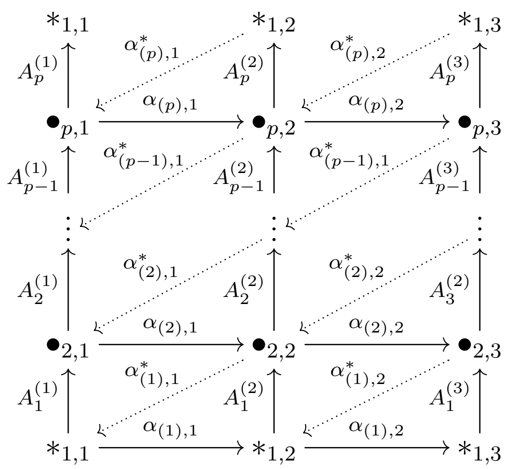
There are many ways to show that this is true. For example the second proof about homogenous G–tubes in §.3.3.2 generalizes to this case[22]. Another way of seeing this is to go to the meta–quiver approach we will discuss later on. However, from (3.58) it follows that the light spectrum consists of vector–multiplets in the adjoint of G — corresponding to the generic point of $\mathbb{P}^1$ — plus the matter which resides at the special values λ_i.

The matter systems associated with two distinct special points decouple from each other as $g_{\text{YM}} \to 0$, so, as long as we are interested in the matter theory itself rather than the full gauged model $\widehat{H} \boxtimes G$, we loose no generality in choosing $\widehat{H}$ to have just *one* special point over which we have a stable tube of period p, $p = 2, 3, \ldots$. This corresponds to the models with $\widehat{H} = \widehat{A}(p,1)$.

3.4.2. $\widehat{A}(p,1) \boxtimes G$ models. From eqn.(3.39) in remark 1 of the previous section we know that

$$\mathcal{T}_{\widehat{A}(p,1)} = \mathsf{rep}\, A(p,0) \vee \mathcal{T}_\infty$$

Forgetting about the $\mathcal{T}_\infty$ component, it is natural to associate to the category $\mathscr{L}(\widehat{A}(p,1) \boxtimes G)$ the quiver with superpotential $\widehat{A}(p,0) \boxtimes G$. For example, the quiver with superpotential $\widehat{A}(p,0) \boxtimes A_3$ is



(3.59)

where nodes $*$ are identified. Here we have introduced the following notation, with respect to the one of §.3.2:

$$A_i^{(a)} \equiv A_i \otimes e_a \qquad \alpha_{(i),a} \equiv e_i \otimes \alpha_a \qquad \alpha^*_{(i),a} \equiv \psi_{A_i,\alpha_a}. \tag{3.60}$$

[22] By (3.57) the canonical light subcategory is the subcategory that has objects $X \in \mathsf{rep}\,\mathcal{W} \boxtimes G$ such that $(\tau^n \otimes 1)X \neq 0$ for all $n \in \mathbb{Z}$

And the $\boxtimes$ superpotential[23] reads

$$\mathcal{W}_\boxtimes=\sum_{i,a}\alpha^*_{(i),a}(\alpha_{(i+1),a}A_i^{(s(\alpha_a))}-A_i^{(t(\alpha_a))}\alpha_{(i),a}). \tag{3.61}$$

For $X\in\mathsf{rep}\,\widehat{A}(p,0)\boxtimes G$, let $X\big|_a$ be its restriction to the a–th affine cyclic subquiver. Again we see that the relations $\partial_{\alpha^*}\mathcal{W}_\boxtimes=0$ entails that the arrows in a family $\widetilde{\alpha}_a\equiv\{\alpha_{(i),a}\}_{i=1}^p$ induce a morphism of representations of $\mathsf{rep}\,\widehat{A}(p,0)$:

$$\widetilde{\alpha}_a\colon X|_{s(\alpha_a)}\to X|_{t(\alpha_a)} \tag{3.62}$$

If we take the partials $\partial_\alpha\mathcal{W}_\boxtimes=0$, instead we obtain relations

$$A_{i-1}^{(s(\alpha))}\alpha^*_{(i-1),s(\alpha)}=\alpha^*_{(i),s(\alpha)}A_i^{(t(\alpha))}. \tag{3.63}$$

Consider the family of arrows $\widetilde{\alpha}^*_a\equiv\{\alpha^*_{(i),a}\}_{i=1}^p$. Again these induce morphisms of representations (as one can easily convince himself with a glance at the quiver (3.59))

$$\widetilde{\alpha}^*_a\colon X|_{s(\alpha^*_a)}\to X|_{t(\alpha^*_a)}. \tag{3.64}$$

because the quiver $\widehat{A}(p,0)$ is symmetric by shift $i\to i-1$. Later in this section we will make this property formal: this shift is the Auslander–Reiten translation τ of $\mathcal{T}_{\widehat{A}(p,1)}$. Moreover, we have the relations $\partial_A\mathcal{W}_\boxtimes=0$

$$\sum_{\alpha:s(\alpha)=a}\alpha^*_{(i),a}\alpha_{(i+1),a}=\sum_{\alpha:t(\alpha)=a}\alpha_{(i),a}\alpha^*_{(i),a}. \tag{3.65}$$

These relations entails that the morphisms $\widetilde{\alpha}_a$ and $\widetilde{\alpha}^*_a$ satisfies the relations of the $\mathcal{P}(G)$ preprojective algebra. The same argument we have discussed for pure G SYM applies here: since the object classes corresponding to the tubes at different values of λ are hom–orthogonal, we obtain a Ringel–decomposition of the category $\mathsf{rep}\,\widehat{A}(p,0)\boxtimes G$ in tubes of indecomposable objects. The tube at $\lambda=\infty$ of the category $\mathcal{T}_{\widehat{A}(p,1)}$ being homogenous nothing changes there, and this entails that we have a Ringel–decomposition of the full light category of $\widehat{A}(p,1)\boxtimes G$

$$\mathscr{L}(\widehat{A}(p,1)\boxtimes G)=\bigvee_{\lambda\in\mathbb{P}^1}\mathscr{L}_\lambda \tag{3.66}$$

For all $\lambda\neq 0$ the indecomposable objects $X\in\mathscr{L}_\lambda$ are such that $X|_a$ belongs to the object class of the homogenous tube at λ, $\mathsf{add}\,(\mathcal{T}_1)_\lambda$: all the arrows $A_i^{(a)}$ are, therefore, isomorphisms. Using these isomorphisms together with the relations (3.65)

[23] i is taken mod p in the summation.

we see that $\mathscr{L}_\lambda$ at $\lambda \neq 0$ is precisely a copy of the homogenous G–tube $\mathscr{L}_G^{YM}$. A way to show this is that, via these isomorphisms, the quiver with superpotential $\widehat{A}(p,0)\boxtimes G$ is mapped into $\widehat{A}(1,0)\boxtimes G$ and the morphisms $\widetilde{\alpha}_a$ and $\widetilde{\alpha}_a^*$ reduce to the α_a and α_a^* arrows. A more rigorous proof can be found in §.3.4.6. On contrast, at $\lambda \neq 0$ the indecomposable $X \in \mathscr{L}_0$ are such that $X|_a$ belong to the object class of a *non–homogenous* tube (of period p). As we have discussed the indecomposable objects of a non–homogenous tube of period p are precisely the indecomposable *nilpotent* representations of $\widehat{A}(p,0)$, therefore not all $A_i^{(a)}$ arrows can be isomorphisms! The BPS particles, stable and light at weak coupling, which correspond to generic λ representations are then vector multiplets forming precisely of one copy of the adjoint representation of G (taking into account also the massless photons in the Cartan subalgebra). The indecomposables of $\mathscr{L}_{\lambda=0}$, instead, contains other representations besides the ones obtained by taking the $\lambda \to 0$ limit of the $\lambda \neq 0$ ones. In facts, one has the inclusion

$$\mathscr{L}^{YM}(\lambda = 0)_G \subseteq \mathscr{L}(\lambda = 0) \tag{3.67}$$

with equality if and only if $p = 1$. For $p > 1$ $\mathscr{L}_{\lambda=0}$ contains in addition representations which (when stable) have the physical interpretation of *matter* BPS–particles charged under the G gauge symmetry. By sending the Yang–Mills coupling to zero, we decouple from the SYM sector the matter system sitting in the category $\mathscr{L}(\lambda = 0)$, which we call $D_p(G)$. Our next task is to characterize the non–perturbative physics of this system.

3.4.3. The quiver with superpotential of the $D_p(G)$ system. The matter $\mathcal{N} = 2$ theory $D_p(G)$ has its own quiver with superpotential $(Q_{\text{mat.}}, \mathcal{W}_{\text{mat.}})$. They are characterized by the property that there is a matter functor $\mathscr{M}$

$$\mathscr{M} : \mathsf{rep}(Q_{\text{mat.}}, \mathcal{W}_{\text{mat.}}) \to \mathsf{rep}\,\widehat{A}(p,0) \boxtimes G \tag{3.68}$$

which preserves indecomposable modules and iso–classes (in the RT jargon, one says that the functor $\mathscr{M}$ *insets* indecomposable modules) as well as the quantum numbers

$$\dim \mathscr{M}(X) = \dim\ X, \tag{3.69}$$

and such that, if X is an indecomposable module of $\mathsf{rep}(Q_{\text{mat.}}, \mathcal{W}_{\text{mat.}})$, then $\mathscr{M}(X)$ is a $\lambda = 0$ indecomposable module of $\mathsf{rep}\,\widehat{A}(p,0) \boxtimes G$ which is *rigid* in the λ direction, that is, $\mathscr{M}(X)$ cannot be continuously deformed to a $\lambda \neq 0$ module.

Let us label $Q_{p,G}$ the quiver of $\widehat{A}(p,0) \boxtimes G$. Since the simple representations with support at a single node of $Q_{p,G}$ are rigid, and there are no other simple repr.'s, the Gabriel quiver [68] of the matter module category has the same nodes as $Q_{p,G}$. It has also the same arrows, since $Q_{p,G}$ is simply–laced and the modules with support on the (full) A_2 subquivers are obviously rigid. Thus the matter quiver is simply

$$Q_{\text{mat.}} \equiv Q_{p,G}. \tag{3.70}$$

The superpotential, however, should be modified to rigidify the parameter λ. In the case $G=A_1$, where the quiver Q_{p,A_1} is just $\widehat{A}(p,0)$, λ–rigidity is achieved by taking $\mathcal{W}_{\text{mat.}}=A_pA_{p-1}\cdots A_1$. Considering the representations of $\mathsf{rep}\,\widehat{A}(p,0)\boxtimes G$ with support in a single affine cyclic subquiver $\widehat{A}(p,0)_a$, and comparing with the A_1 case, we deduce that we have to add to the superpotential $\mathcal{W}_\boxtimes$ at least the extra term

$$\delta\mathcal{W}=\sum_a A_p^{(a)}A_{p-1}^{(a)}\cdots A_1^{(a)}. \tag{3.71}$$

One may wonder whether this modification is enough, or we need to add additional higher order corrections corresponding to cycles not supported in single affine cyclic subquivers. We claim that this is not the case (up to terms which do not modify the universality class of $\mathcal{W}$, and hence may be ignored as far as the BPS spectrum is concerned).

To substantiate the claim, the first thing to check is that the modified superpotential $\mathcal{W}_{\text{mat.}}=\mathcal{W}_\boxtimes+\delta\mathcal{W}$ *does* rigidify λ to zero. For notational simplicity we are going to check this only for the case $G=A_n$, the result can be easily extended to all G in ADE. The only relations that gets modified by $\delta\mathcal{W}$ are the $\partial_A\mathcal{W}$ ones, that becomes (for $G=A_n$)

$$A_{i+1}^{(a)}A_{i+2}^{(a)}\cdots A_p^{(a)}A_1^{(a)}\cdots A_{i-1}^{(a)}=\alpha_{(i),a-1}\alpha^*_{(i),a-1}-\alpha^*_{(i),a}\alpha_{(i+1),a} \tag{3.72}$$

Using $\partial_\alpha\mathcal{W}$ and $\partial_{\alpha^*}\mathcal{W}$ one shows that the maps

$$A_{i+1}^{(a)}A_{i+2}^{(a)}\cdots A_p^{(a)}A_1^{(a)}\cdots A_{i-1}^{(a)}A_i^{(a)}: X_{\bullet_{i,a}}\to X_{\bullet_{i,a}} \tag{3.73}$$

define an endomorphism of a representation of both $\widehat{A}(p,1)\boxtimes G$ and $\mathsf{rep}\,(Q_{p,G},\mathcal{W}_{\text{mat.}})$. Hence, for an *indecomposable* module $X\in\mathsf{rep}(Q_{p,G},\mathcal{W}_\boxtimes+\delta\mathcal{W})$, we have

$$\begin{aligned}
\lambda\cdot\mathrm{Id}_{X_{(j,a)}}+N_{(j,a)} &= A_{j-1}^{(a)}A_{j-2}^{(a)}\cdots A_1^{(a)}A_p^{(a)}\cdots A_j^{(a)}\\
&=_{\text{eqn.(3.73)}}\ \alpha_{(j),a-1}\alpha^*_{(j),a-1}A_j^{(a)}-\alpha^*_{(j),a}\alpha_{(j+1),a}A_j^{(a)}\\
&=_{\text{eqn.(3.63)}}\ \alpha_{(j),a-1}A_{j-1}^{(a-1)}\alpha^*_{(j-1),a-1}-\alpha^*_{(j),a}\alpha_{(j+1),a}A_j^{(a)}\\
&=_{\text{by }\partial_{\alpha^*}\mathcal{W}=0}\ A_{j-1}^{(a)}\alpha_{(j-1),a-1}\alpha^*_{(j-1),a-1}-\alpha^*_{(j),a}\alpha_{(j+1),a}A_j^{(a)}
\end{aligned}$$

where $N_{(j,a)}$ is nilpotent. Taking traces we get

$$\begin{aligned}
\lambda\cdot\dim X_{(j,a)} &= \mathrm{tr}\big(A_{j-1}^{(a)}\alpha_{(j-1),a-1}\alpha^*_{(j-1),a-1}\big)-\mathrm{tr}\big(\alpha^*_{(j),a}\alpha_{(j+1),a}A_j^{(a)}\big)\\
&=_{\text{cyclicity}}\ \mathrm{tr}\big(A_{j-1}^{(a)}\alpha_{(j-1),a-1}\alpha^*_{(j-1),a-1}\big)-\mathrm{tr}\big(\alpha_{(j+1),a}A_j^{(a)}\alpha^*_{(j),a}\big)\\
&=_{\text{by }\partial_{\alpha^*}\mathcal{W}=0}\ \mathrm{tr}\big(A_{j-1}^{(a)}\alpha_{(j-1),a-1}\alpha^*_{(j-1),a-1}\big)-\mathrm{tr}\big(\alpha_{(j+1),a}\alpha^*_{(j+1),a}A_{j+1}^{(a+1)}\big)
\end{aligned}$$

Summing this relation over the nodes of $Q_{p,G}$, we get for all indecomposable module X

$$\lambda \cdot \sum_{j,a} \dim X_{(j,a)} = 0 \tag{3.74}$$

which implies $\lambda = 0$ rigidly. Since all its non–zero indecomposables are rigid in the λ direction, the Abelian category $\mathsf{rep}(Q_{p,G}, \mathcal{W}_\boxtimes + \delta\mathcal{W})$ is rigid in that direction. By this we mean that any object of the category $\mathsf{rep}(Q_{p,G}, \mathcal{W}_\boxtimes + \delta\mathcal{W})$ restricts in each cyclic affine subquiver to a module of the uniserial self–injective Nakayama algebra [68] given by the quotient of the path algebra of $\widehat{A}(p,0)$ by the bilateral ideal generated by all cyclic words $A_i^{(a)} A_{i+1}^{(a)} \cdots A_{i-1}^{(a)}$ for $i = 1, \dots, p$. Note that the module category of this Nakayama algebra is *strictly larger* than the module category of the Nakayama algebra which is the Jacobian algebra of the D_p Argyres–Douglas system, the difference being that the maximal length of the composition series is now p instead of $p-1$. Thus, $D_p(G)$ is not, in any sense, the tensor product of D_p and G. For an explicit counterexample, however, see §.3.4.5.

This rigidity result implies, in particular, that the limit as $\lambda \to 0$ of a $\lambda \neq 0$ brick of $\widehat{A}(p,0) \boxtimes G$ does not satisfy the modified relations, and hence the superpotential $\mathcal{W}_\boxtimes + \delta\mathcal{W}$ has the effect of 'projecting out' the SYM sector.

To get the claim, it remains to construct the functor $\mathscr{M}$. It should be such that, for all representations X of the quiver $Q_{p,G}$ which satisfy the relations $\partial(\mathcal{W}_\boxtimes + \delta\mathcal{W}) = 0$, $\mathscr{M}(X)$ is a representation of the same quiver satisfying $\partial\mathcal{W}_\boxtimes = 0$. We define $\mathscr{M}$ as follows. Let $X|_a$ be a $D_p(G)$ module. Write

$$X\Big|_a = Y_a \oplus Z_a \tag{3.75}$$

where Y_a is a direct sum of indecomposables of lenght p, while the direct summands of Z_a have lengths $\leq p-1$. Then $\mathscr{M}(X)$ is obtained from X by the arrow replacement

$$\alpha_{(*),a} \to \alpha_{(*),a} P_{Z_a}, \qquad \alpha^*_{(*),a-1} \to \alpha^*_{(*),a-1} P_{Z_a}, \tag{3.76}$$

where P_{Z_a} is the projection on the second summand in eqn.(3.75). The functor $\mathscr{M}$, so defined, has the desired properties, and, moreover, $\dim \mathscr{M}(X) \neq \sum_a n_a \dim W_a$ for all bricks X.

3.4.4. Monodromies, beta function, and flavor The 2d quantum monodromy of the $\widehat{A}(p,1) \boxtimes G$ system is the tensor product of the monodromies of the factors

$$H_{p,G} \equiv \Phi_{\widehat{A}(p,1)} \otimes \Phi_G. \tag{3.77}$$

where Φ_Q stands for the Coxeter element of the acyclic quiver Q. Let us denote by $\mathbf{\Phi}_d(X)$ the d-th cyclotomic polynomial. The characteristic polynomial of the

	$\chi_G(X)$
A_n	$\prod_{\substack{d\mid(n+1)\\ d\neq 1}} \mathbf{\Phi}_d(X)$
D_n	$\mathbf{\Phi}_2(X)\prod_{\substack{d\mid 2(n-1)\\ d\nmid(n-1)}} \mathbf{\Phi}_d(X)$
E_6	$\mathbf{\Phi}_3(X)\mathbf{\Phi}_{12}(X)$
E_7	$\mathbf{\Phi}_2(X)\mathbf{\Phi}_{18}(X)$
E_8	$\mathbf{\Phi}_{30}(X)$

Table 3.5: Factorization of the ADE characteristic polynomials $\chi_G(X)$.

Coxeter element of $\widehat{A}(p,1)$ is [87]

$$\det\big[z-\Phi_{\widehat{A}(p,1)}\big]=\mathbf{\Phi}_1(z)\prod_{d\mid p}\mathbf{\Phi}_d(z). \tag{3.78}$$

For each $G=ADE$ we define a function $\delta(d;G):\mathbb{N}\to\mathbb{Z}_{\geq 0}$ by the formula

$$\chi_G(z)\equiv\det\big[z-\Phi_G\big]=\prod_{d\in\mathbb{N}}\mathbf{\Phi}_d(z)^{\delta(d;G)}, \tag{3.79}$$

see table 3.5. By 2d/4d then, f is equal to the the multeplicity of 1 as an eigenvalue of the 2d monodromy — see (2.24). In our case, this is simply the number of solutions to the equations

$$\frac{\ell}{p}+\frac{k_i}{h(G)}\in\mathbb{Z} \qquad \begin{array}{l} i=1,2,3, \quad \ell_i=1,2,\dots,p_i-1,\\ k_i \text{ an exponent of } G. \end{array} \tag{3.80}$$

Let $\varphi(d)$ denote the Euler totient function. For all d that divides p and it is such that $\delta(d;G)\neq 0$, by eqn.(3.78), we will find as many solutions of (3.80) as the roots of $\mathbf{\Phi}_d(X)$: This number is $\varphi(d)$ by definition. Therefore

$$f(p,G)\equiv\sum_{d\mid p}\delta(d;G)\varphi(d) \tag{3.81}$$

A much more elegant number theoretical proof of a general result from which this follows can be found in [22]. Here we prefer not to explain all the details of that proof: we will need only the number of flavors.

Then, after decoupling the G Yang–Mills sector, we remain with a matter system, $D_p(G)$, which has flavor symmetry $G \times F$ whose rank is

$$f(D_p(G)) = r(G) + f(p, G). \tag{3.82}$$

From the 2d monodromy $H_{p,G}$ it is easy to compute the β function of the Yang–Mills coupling g for the model $\widehat{A}(p,1) \boxtimes G$. Indeed, since the matter $D_p(G)$ is superconformal, in the weak YM coupling limit, $g \to 0$, the trace of the energy–momentum tensor is proportional to the YM β–function. Then, by $\mathcal{N} = 2$ supersymmetry, the coefficient b of the β–function

$$\mu \frac{\partial \tau}{\partial \mu} = \frac{i}{2\pi} b,$$

is the same as the coefficient of the chiral $U(1)_R$ anomaly which counts the net chiral number of Fermi zero–modes in the instanton background. A $U(1)_R$ rotation by $2\pi n$ ($n \in \mathbb{Z}$) is equivalent to a shift of the vacuum angle θ by $2\pi b\, n$, which has the effect of changing the electric/magnetic charges of a BPS dyon as [93]

$$(e, m) \to (e + b\, n\, m, m). \tag{3.83}$$

Thus, if we know the action of a chiral $2\pi\, n$ rotation on the charge lattice Γ (that is, on the dimensions of the corresponding modules), we may extract the coefficient b.

The 2d monodromy $H_{p,G}$ acts on the CY 3–form Ω of the geometry (cfr. eqn.(3.1)) as [13]

$$\begin{aligned} \Omega = \frac{dX \wedge dY \wedge dU}{\partial_Z W} &\mapsto \exp\big(2\pi i (q_X + q_Y - 1/2)\big) \frac{dX \wedge dY \wedge dU}{\partial_Z W} \equiv \\ &\equiv \exp(2\pi i / h(G)) \frac{dX \wedge dY \wedge dU}{\partial_Z W}, \end{aligned} \tag{3.84}$$

where $h(G)$ is the Coxeter number of G, and we denote by W the superpotential of the 2d model that corresponds to the geometry (3.1). Hence the action of the chiral $2\pi\, n$ rotation, $\Omega \mapsto e^{2\pi i\, n}\, \Omega$, on the dimension/charge lattice Γ is given by the matrix

$$H_{p,G}^{n\, h(G)} = \Phi_{\widehat{A}(p,1)}^{n\, h(G)} \otimes \Phi_G^{n\, h(G)} \equiv \Phi_{\widehat{A}(p,1)}^{n\, h(G)} \otimes 1. \tag{3.85}$$

It is convenient to take $n = p$; we have (see eqn.(3.25) and combine it with table 3.4)

$$\Phi_{\widehat{A}(p,1)}^{p\, h(G)} = \mathrm{Id} + (p+1)\, h(G)\, \delta \otimes \langle \delta, \cdot \rangle_E \tag{3.86}$$

where δ is the minimal imaginary root of $\widehat{A}_p$, which corresponds to a purely electric charge, while the form $\langle\delta,\cdot\rangle_E$ measures the magnetic charge [10]. Comparing with eqn.(3.83), taking care of the appropriate normalizations, we get

$$p\,b=(p+1)\,h(G), \tag{3.87}$$

that is, the β coefficient b of the $\widehat{A}(p,1)\boxtimes G$ model is

$$b=\frac{p+1}{p}\,h(G), \tag{3.88}$$

whose sign implies asymptotic freedom. b receives a contribution $2h(G)$ from the SYM sector and a negative contribution $-b_{p,G}=-k_{p,G}/2$ from the matter $D_p(G)$ SCFT. We use eqn.(3.88) to extract the central charge of the G–current algebra of the SCFT $D_p(G)$

$$k_{p,G}=2\,\frac{p-1}{p}\,h(G). \tag{3.89}$$

On the other hand, a $U(1)_R$ rotation by 2π defines the 4d quantum monodromy $\mathbb{M}$. For the model $\widehat{A}(p,1)\boxtimes G$, which is just asymptotically–free, $\mathbb{M}$ has not finite order; however, once we decouple the SYM sector, we remain with the SCFT $D_p(G)$ whose 4d quantum monodromy has a finite order $r(p,G)$. As we saw above, the action of a $U(1)_R$ rotation by 2π on the charges of $D_p(G)$ is given by the semi–simple part of the $h(G)$ power of $H_{p,G}$, that is, by

$$\Phi^{h(G)}_{\widehat{A}(p,1)}\Big|_{\text{semi–simple}}\otimes 1. \tag{3.90}$$

The order of the 4d quantum monodromy $\mathbb{M}$ is just the order of this operator. Comparing with eqn.(3.78) we get [21]

$$r(p,G)=\frac{p}{\gcd\{p,h(G)\}}. \tag{3.91}$$

As we have discussed in our heuristic introduction, the underlying 2d $(2,2)$ system of $\widehat{A}(p,1)\boxtimes G$ is encoded in the geometry (3.1). Having identified the coordinate Z there with the plumbing cylinder of the 6d $(2,0)$ systems, is natural to identify, at a heuristic level of rigor, the underlying 2d system for the matter $D_p(G)$ theory with the $Z\to+\infty$ limit of the above. One obtains the 2d superpotential

$$W_{p,G}\equiv e^{pZ}+W_G(X,Y)+U^2 \tag{3.92}$$

An interesting consitency check of our result[24] about the quantum monodromy is given by applying the same method of §.2.3.5 to the geometry $W_{p,G}=0$. We discuss this in appendix §. B.1.

[24] Actually, it is really the same computation, but dressed in another way.

3.4.5. Example: $D_3(SU(3))$ is not $A_2 \boxtimes D_3$. The number of nodes is the same. Let us consider the next quiver invariant: the rank of the B matrix. By 2d/4d the theory $A_2 \boxtimes D_3$ is engineered by the singularity $X^3 + Y^2 + U^2 + UZ^2$. This theory has $q_X = 1/3$, $q_Y = q_U = 1/2$ and $q_Z = 1/3$ by table 3.1. Therefore we have $\hat{c} = 2/3$. The Poincaré polynomial (2.56) for this singularity is very easy:

$$\left(\frac{(1-t^{2/3})}{(1-t^{1/3})}\right)^2 = 1 + 2t^{1/3} + t^{2/3}. \tag{3.93}$$

By 2d/4d correspondence then we see that $A_2 \boxtimes D_3$ has a flavor group of rank 2, so that rank $B = 6 - 2 = 4$. Let us now consider the theory $D_3(SU(3))$. The flavor group has rank at least 2, but $p = 3$, and so, by table 3.5, $\delta(3, A_2) = 1$, and we have enhancement, as discussed in the previous paragraph: $f(p, A_2) = \varphi(3) = 2$. The flavor symmetry group of $D_3(SU(3))$ has rank 4, and its B matrix has rank 2: the two theories are different.

3.4.6. A new viewpoint: META–quivers Usually, by a representation of a quiver Q we mean the assignment of a vector space X_i to each node i of Q and a linear map X_ψ to each arrow ψ, that is, we assign to nodes resp. arrows objects resp. morphisms of the category vec of finite dimensional vector spaces. Of course we may replace vec by any other category $\mathscr{C}$, getting a $\mathscr{C}$–valued representation of Q where the concatenation of arrows along paths in Q is realized as the composition of the corresponding morphisms in $\mathscr{C}$. If $\mathscr{C}$ is $\mathbb{C}$–additive, so that it makes sense to sum morphisms and to multiply them by complex numbers, we may even define $\mathscr{C}$–valued representations of quivers subjected to relations of the standard form. Thus it makes sense to speak of the *category* of $\mathscr{C}$–valued representations of the quiver Q bounded by an ideal I. If, in addition, $\mathscr{C}$ is Abelian, the category of $\mathscr{C}$–valued representations of a quiver with relations is again Abelian, and share most of the properties of the usual module categories.

It is useful to extend the construction to *twisted* $\mathscr{C}$–valued representations. Let $\{\sigma_\ell\}$ be a group of autoequivalences of the category $\mathscr{C}$, where $\sigma_0 = \text{Id}$. To each arrow in Q we assign a valuation in $\{\sigma_\ell\}$. Then a $\mathscr{C}$–valued representation of the *valued* quiver assigns to an arrow a with valuation $\sigma_{\ell(a)}$ a morphism $\psi \in \text{Hom}(\mathcal{O}_{s(a)}, \sigma_{\ell(a)}\mathcal{O}_{t(a)})$. The twisted composition $\star$ of two arrows a and b with $s(b) = t(a)$ has valuation $\sigma_{\ell(a)}\sigma_{\ell(b)}$ and is given by[25]

$$\psi_b \star \psi_a \equiv \sigma_{\ell(a)}(\psi_b) \circ \psi_a \in \text{Hom}\big(\mathcal{O}_{s(a)}, \sigma_{\ell(a)}\sigma_{\ell(b)}\mathcal{O}_{t(b)}\big) \tag{3.94}$$

[25] The fact that the $\{\sigma_a\}$ are taken to be just autoequivalences (and not automorphisms) introduces notorious subtleties; for the particular categories we are interested in, this will not be a problem, since we have an underlying concrete description in terms of elements of objects (the one given by the big BPS–quiver modules). The abstract viewpoint is, however, very useful since it allows to describe the physical phenomena in a unified way for large classes of $\mathcal{N} = 2$ models, abstracting the essential physics from the intricate details of each particular example.

where $\circ$ is the composition in $\mathscr{C}$.

Of course, by this construction we are not introducing any *real* generalization, the resulting Abelian category may always be seen (up to Morita equivalence) as a subcategory of the usual representations of some bigger (possibly infinite) quiver. However, in the case of $\mathcal{N}=2$ QFT's working with twisted $\mathscr{C}$–valued representations turns out to be very convenient both conceptually and technically: we replace a messy BPS quiver with complicate Jacobian relations with a much smaller quiver having few nodes, few arrows and, typically, a higher symmetry which is almost never visible in the messy vec quiver Q_{BPS}. Besides, the messy vec quivers associated to QFT's have no simple universality property useful to characterize and classify them while, for a good choice of $\mathscr{C}$, the smaller $\mathscr{C}$–quivers have rather uniform behaviour.

It is easy to introduce a notion of stability of the (twisted) $\mathscr{C}$–valued representations of Q which is equivalent to the stability for the corresponding vec–valued representation of the messy Q_{BPS}. One introduces a stability function (central charge) $\mathscr{Z}$ for the $\mathscr{C}$–valued representations $\mathcal{X}$

$$\mathscr{Z}\equiv\Big\{Z_i\colon K_0(\mathscr{C})\to\mathbb{C},\ i\in(\text{nodes of }Q)\Big\} \tag{3.95}$$

$$\mathscr{Z}(\mathcal{X})=\sum_i Z_i(\mathcal{X}_i)\in\mathbb{C}, \tag{3.96}$$

where the homomorphism of Abelian groups Z_i coincides with the usual central charge for the subcategory of the representations of the messy quiver Q_{BPS} which map into $\mathscr{C}$–valued representations having support on the i–th node of Q. A $\mathscr{C}$–valued representation $\mathcal{X}$ is $\mathscr{Z}$–stable iff, for all non–zero proper sub–objects $\mathcal{Y}$, $\arg\mathscr{Z}(\mathcal{Y})<\arg\mathscr{Z}(\mathcal{X})$.

Given a $\mathcal{N}=2$ QFT T we say that the quadruple $(Q,I,\mathscr{C},\nu)$ — Q being a finite connected quiver, I a bilateral ideal of relations in $\mathbb{C}Q$, $\mathscr{C}$ a $\mathbb{C}$–additive Abelian category, and ν a valuation of Q in the autoequivalences of $\mathscr{C}$ — is *a* META*–quiver for* T iff the stable, ν–twisted, $\mathbb{C}$–valued representations of Q, subjected to the relations in I, give the BPS spectrum of T (in some chamber).

The META–quiver for the light category of the $\widehat{A}(p,1)\boxtimes G$ model

Let us construct META–quivers for the light category of $\mathcal{N}=2$ models at hand. Let us chose $\mathscr{C}=\mathcal{T}\equiv\mathcal{T}_{\widehat{A}(p,1)}$, *i.e.* the category of regular representations of the affine quiver $\widehat{A}(p,1)$. Let τ be its autoequivalence given by the Auslander–Reiten translation [65, 77]. We consider the quiver of the preprojective algebra of type $\mathcal{P}(G)$ with the following valuation: direct arrows α are valued by Id, while inverse arrows α^* by τ. The LHS of eqn.(3.50) has then valuation τ. From our discussion in §.3.4.2 it should be clear that category of the *twisted* $\mathscr{C}$–valued modules of $\mathcal{P}(G)$ is equivalent to the light category of $\widehat{A}(p,1) \boxtimes G$.

The tube subcategories $\mathcal{T}_{\widehat{A}(p,1)}(\lambda)$ and $\mathcal{T}_{\widehat{A}(p,1)}(\mu)$ are preserved by τ, and Hom–orthogonal for $\lambda \neq \mu$ [65, 77]. Since the arrows α, α^* take value in the Hom groups, they automatically vanish between objects in different tubes. Hence if $\mathcal{X}$ is a $\mathcal{T}_{\widehat{A}(p,1)}$–valued *indecomposable* representation of $\mathcal{P}(G)$

$$\mathcal{X}_i \in \mathcal{T}_{\widehat{A}(p,1)}(\lambda) \quad \text{with the } \underline{\text{same}}\ \lambda \text{ for all nodes } i \text{ of } \overline{G}. \tag{3.97}$$

For $\lambda \neq 0$ we have the equivalence $\mathcal{T}_{\widehat{A}(p,1)}(\lambda) \simeq \mathcal{T}_{\widehat{A}(1,1)}(\lambda)$ (the homogeneous tube), so the category of $\mathcal{T}_{\widehat{A}(p,1)}(\lambda)$–representations of $\mathcal{P}(G)$ coincides in this case with the corresponding light subcategory of SYM. In facts, as we are going to show, the category of twisted $\mathcal{T}$–valued representations of $\mathcal{P}(G)$ contains a canonical subcategory isomorphic to the light Yang–Mills one (for gauge group G).

The SYM sector

There is a more canonical way of looking to the SYM sector. For $\lambda \neq 0$, $\mathcal{T}(\lambda)$ is a homogeneous tube, so $\tau\mathcal{O} = \mathcal{O}$ for all its indecomposable objects. Fix an indecomposable $\mathcal{O}_0 \in \mathcal{T}(\lambda \neq 0)$; we have a functor from the category of modules (in the standard sense) of the preprojective algebra $\mathcal{P}(G)$ to the one of $\mathcal{T}(\lambda \neq 0)$–valued twisted module — that is, to $\mathscr{L}(\lambda \neq 0)$ — given by

$$\mathcal{X}_i = \mathcal{O}_0 \otimes X_i, \quad \mathcal{X}_\alpha = \mathrm{Id} \otimes X_\alpha, \qquad \mathcal{X}_{\alpha^*} = \mathrm{Id} \otimes X_{\alpha^*}. \tag{3.98}$$

In particular, the bricks of $\mathscr{L}(\lambda \neq 0)$ are obtained by taking as $\mathcal{O}_0$ the unique regular brick in $\mathcal{T}(\lambda)$, *i.e.* the $SU(2)$ W–boson representation $W(\lambda)$; one gets

$$\textbf{bricks of } \mathscr{L}(\lambda \neq 0) \quad \equiv \quad W(\lambda) \otimes X, \qquad X \text{ a brick of } \mathsf{mod}\,\mathcal{P}(G). \tag{3.99}$$

The bricks of $\mathsf{mod}\,\mathcal{P}(G)$ have dimension vectors equal to the positive roots of G and are *rigid* (this is an elementary consequence of [94], see [10]), that is, the family of representations (3.99) correspond to BPS vector multiplets with the quantum numbers of the W–bosons of G (the SYM sector).

$\mathcal{T}(\lambda = 0)$ is a tube of period p. Then the

$$\mathcal{X}_i = \bigoplus_{s=1}^{m_i} R_{i,s} \tag{3.100}$$

with $R_{i,s}$ indecomposable regular modules characterized uniquely by their (regular) socle $\tau^{k_{i,s}} S$ and lenght $r_{i,s}$ [77] (S being a reference regular simple). The $\lambda = 0$ $SU(2)$ W–boson representations are the indecomposables of regular lenght p; we write $W(k)$ for the length p regular indecomposable with socle $\tau^k S$ ($0\leq k\leq p-1$). Since

$$\dim \mathrm{Hom}(W(k),W(\ell))=\delta_{k,\ell}, \qquad \dim \mathrm{Hom}(W(k),\tau W(\ell))=\delta_{k,\ell+1}, \quad (3.101)$$

we may promote each ordinary representation X of $\mathcal{P}(G)$ to a representation of the SYM sector of $\mathscr{L}(\lambda=0)$ by replacing the basis vectors $v_{i,s}$ of X_i by $W(k_{i,k})$ in such a way that two basis vectors v, v' related by $v'=\alpha(v)$ (resp. $v'=\alpha^*(v)$) have the $k'=k$ (resp. $k'=k-1$). The assignement of k's may be done consistently since, $\mathcal{P}(G)$ is finite–dimensional (here it is crucial that G is Dynkin) and hence all closed cycles are nilpotent. Again, the bricks of $\mathscr{L}(\lambda=0)$ with $r_{i,k}=p$ for all i,k have the quantum numbers of the W–bosons of G.

Non–perturbative completion

The twisted $\mathcal{T}$–valued representations of $\mathcal{P}(G)$ give just the light category $\mathscr{L}$ of the $\widehat{A}(p,1)\boxtimes G$ model. Physically, one is interested to a META–quiver interpretation of the total non–perturbative category, which includes, besides the light objects, also heavy ones carrying non–zero magnetic charge. In the language of [10], this corresponds to the non–perturbative completion of $\mathscr{L}$. The naive choice $\mathscr{C}=\mathsf{mod}\,\mathbb{C}\widehat{A}(p,1)$ will *not* work, since the Auslander–Reiten translation τ is not an autoequivalence for this module category[26]. This problem may be fixed by recalling the derived equivalence [87]

$$D^b\big(\mathsf{mod}\,\mathbb{C}\widehat{A}(p,1)\big)=D^b\big(\mathsf{Coh}(X_p)\big), \qquad (3.102)$$

where $\mathsf{Coh}(X_p)$ is the Abelian category of coherent sheaves on X_p, the projective line with a marked point (say the origin $\lambda=0$) of weight p (this means that the skyskraper sheaf at $\lambda=0$ has length p in $\mathsf{Coh}(X_p)$). In the category $\mathsf{Coh}(X_p)$ τ is an autoequivalence given by the tensor product with the dualizing sheaf ω_{X_p}. Then the category of the τ–twisted $\mathsf{Coh}(X_p)$–valued $\mathcal{P}(G)$ representations makes sense, and gives the non–perturbative closure (or categorical quantization) of the light category $\mathscr{L}$ in the sense of [10] (the case $G=A_1$ is discussed in that paper).

This construction may be generalized by taking the (τ–twisted) representations of $\mathcal{P}(G)$ valued in the category $\mathsf{Coh}(X_{p_1,p_2,\dots,p_s})$ of the coherent sheaves over a weighted projective line with s marked points of weights $p_1,p_2,\dots,p_s$ (a marked point of weight 1 being equivalent to an unmarked one) [87].

As we shall discuss in section 3.4.7, the (τ–twisted) representations of $\mathcal{P}(G)$ valued in the category $\mathsf{Coh}(X_{p_1,p_2,\dots,p_s})$ will correspond to the (non–perturbative) category of SYM gauging the diagonal symmetry group G of a collection of s decoupled D_{p_i} systems the ranks p_i being equal to the weights of the marked points.

[26] Because of the presence of projective and injective modules.

The product $\circledast$

We saw above that, in order to capture the non–perturbative physics of the 4d $\mathcal{N}=2$ model corresponding to the direct sum of an ADE 2d minimal model and the $W_1 = e^{pZ} + e^{-Z}$ one (or any other affine (2,2) theory [50]), we may consider the τ–twisted representations of $\mathcal{P}(G)$ valued in the coherent sheaves of the geometry associated to W_1. This procedure is a kind of product, which 'morally' is the same as the triangle tensor product $\boxtimes$. We shall denote it by the symbol $\circledast$. So, if A stands for the 4d model whose quiver with potential has the property

$$D^b\big(\mathsf{rep}(Q,\mathcal{W})\big) = D^b\big(\mathsf{Coh}(X_{p_1,p_2,\dots,p_s})\big), \tag{3.103}$$

we write

$$A \circledast G$$

to denote both the META–quiver

$$\big(\mathcal{P}(G), \mathsf{Coh}(X_{p_1,\dots,p_s}), \tau\big), \tag{3.104}$$

as well as the corresponding 4d $\mathcal{N}=2$ QFT associated to the direct sum 2d theory. We shall use this construction in section 4.

Deformed preprojective META–algebras *vs.* $D_p(G)$ SCFT's

In the context of the standard vec–valued representations, the preprojective algebra $\mathcal{P}(L)$ has a generalization, called the *deformed preprojective algebra of weight* λ, written $\mathcal{P}(L)^\lambda$, which is defined by the same double quiver $\overline{L}$ as $\mathcal{P}(L)$ and the deformed relations [95–98]

$$\sum_a (\alpha\,\alpha^* - \alpha^*\,\alpha) = \sum_i \lambda_i\, e_i \equiv \lambda \tag{3.105}$$

where e_i is the lazy path at the i–th node of $\overline{L}$, and the fixed complex numbers λ_i's are the weights (also called Fayet–Illiopoulos terms). In more abstract terms we may say that the RHS of (3.105) is a sum over the nodes i of $\overline{L}$ of fixed *central* elements of $\mathrm{End}(X_i)$ (the endomorphism ring of the object at node i, not to be confused with the End for the representation of the total quiver).

The META counterpart of this construction is to consider the category of the representations of $\overline{L}$ valued in some $\mathbb{C}$–linear Abelian category $\mathscr{C}$ satisfying the relation (3.105), where the RHS is replaced by a sum of prescribed central endomorphisms of the objects at each node. In addition, the representation may be twisted by autoequivalences of $\mathscr{C}$ as in sect.3.4.6.

It is convenient to restrict ourselves to categories $\mathscr{C}$ having a 'trace' map which generalizes the usual trace of vec. That is, for each object $\mathcal{O}\in\mathscr{C}$ we require the existence of a map

$$\mathrm{Tr}\colon \mathrm{End}(\mathcal{O})\to\mathbb{C}, \tag{3.106}$$

which is invariant under the adjoint action of $\mathrm{Aut}(\mathcal{O})$ and has the trace property

$$\mathrm{Tr}(\mathcal{A}\mathcal{B})=\mathrm{Tr}(\mathcal{B}\mathcal{A}),\qquad \forall\,\mathcal{A}\in\mathrm{Hom}(\mathcal{O}_1,\mathcal{O}_2),\quad \mathcal{B}\in\mathrm{Hom}(\mathcal{O}_2,\mathcal{O}_1), \tag{3.107}$$

(the trace in the LHS, resp. RHS, being taken in $\mathrm{End}(\mathcal{O}_2)$, resp. $\mathrm{End}(\mathcal{O}_1)$).

We are particularly interested in the following family of categories. $\mathscr{V}(p)$ is the category whose objects are the pairs $\mathcal{O}\equiv(V,A)$, where V is a $\mathbb{Z}_p$–graded vector space and $A\colon V\to V$ is a *degree* 1 linear map. Its morphisms $\Psi\colon\mathcal{O}_1\to\mathcal{O}_2$ are given by $\mathbb{Z}_p$–graded linear maps $\psi\colon V_1\to V_2$ which satisfy the compatibility condition

$$\psi(A_1 z)=A_2\,\psi(z)\qquad\forall\,z\in V_1, \tag{3.108}$$

with the obvious compositions and identities. Equivalently, $\mathscr{V}(p)$ is the category of the finite–dimensional representations of the *cyclic* affine quiver $\widehat{A}(p,0)$ (no relations). It follows from (3.108) that the degree $k \bmod p$ endomorphisms $E^k\colon(V,A)\to(V,A)$ given by the maps $A^k\colon V\to V$ $(k\in\mathbb{Z}_+)$ belong to the center of $\mathrm{End}(V,A)$. It makes sense, therefore, to consider the $\mathscr{V}(p)$–valued representations of the deformed $\mathcal{P}(L)$ with graded weights of the form $\sum_i\lambda_i\,E_i^s$, where E_i stands for E acting on the object $\mathcal{O}_i\in\mathscr{C}$ sitting at the node i of $\overline{L}$.

$\mathscr{V}(p)$ has a natural autoequivalence σ which acts on objects as $([1],\mathrm{Id})$ where $[1]$ is the operation of shifting the degree by 1. Clearly $\sigma^p=1$. $\mathscr{V}(p)$ has also a natural trace map which on the degree zero endomorphism $\Psi=(\psi)$ is simply

$$\mathrm{Tr}(\Psi)\equiv\mathrm{tr}(\psi), \tag{3.109}$$

which clearly satisfies the trace property (3.107). We generalize the trace to the endomorphisms of degree $-k \bmod p$ by replacing $\mathrm{Tr}(\Psi)$ with

$$\mathbb{T}\mathrm{r}(\Psi)=\mathrm{Tr}(E^k\Psi),\qquad \Psi\in\mathrm{End}(\mathcal{O})\text{ of degree }-k\mod p,\quad 0\le k<p. \tag{3.110}$$

$\mathbb{T}\mathrm{r}(\cdot)$ still satisfies the trace property since E is central in the endomorphism ring.

The Abelian category of (ordinary) representations of the quivers $Q_{p,G}$ with superpotential $\mathcal{W}_{\mathrm{matter}}$, that is, the Abelian category of the $D_p(G)$ theories, is then manifestly equivalent to the category of *twisted* $\mathscr{V}(p)$–valued representations of the deformed preprojective algebra $\mathcal{P}(G)^\lambda$ with the degree -1 weight

$$\lambda=\sum_i E_i^{p-1} \tag{3.111}$$

where the direct arrows of $\overline{G}$ have valuation Id and the inverse ones σ^{-1}.

If t is a non–zero complex number, the weights λ and $t\,\lambda$ produce equivalent representation categories. In the case of vec–valued representations the basic result on the deformed preprojective algebras of a Dynkin graph G [95, 96, 98] says that an indecomposable module X_0 of the undeformed preprojective algebra (at $t = 0$) may be continuously deformed to a module X_t of the $t \neq 0$ one if and only if the trace of the weight λ vanishes on X_0 (the trace of the weight being defined, of course, as the sum of the traces of the endomorphisms at each node). That this is necessary follows from taking the trace of the two sides of eqn.(3.105). If the trace obstruction vanishes, one constructs X_t order by order in t, the procedure stopping since the arrows of X_t have a polynomial dependence on t [98].

After replacing ordinary vec–valued representations by twisted $\mathscr{V}(p)$–valued ones with the weight (3.111), the corresponding statement is that the (twisted) trace of the weight λ, seen as a (sum of) degree -1 endomorphism(s), is an obvious obstruction to the deformation of the representations for $t = 0$ to $t \neq 0$. This is exactly the λ–rigidity result of eqn.(3.74). It is not true, however, that all indecomposable $t = 0$ $\mathscr{V}(p)$–valued representations with $\mathbb{T}\mathrm{r}(\lambda) = 0$ may be deformed to $t \neq 0$ ones. In facts, as we argued in the previous subsection, only the λ–rigid one may be deformed. Let us check that there is an obstruction to the deformation at $t \neq 0$ of the representations[27] corresponding to the W–bosons of G (cfr. eqns.(3.100)(3.101)). The weight at node i is (cfr. eqn.(3.111))

$$0 \neq \lambda_i \equiv E_i^{p-1} = \bigoplus_s A^{p-1}\Big|_{W(k_{i,s})} \in \bigoplus_s \mathrm{End}\big(W(k_{i,s})\big), \tag{3.112}$$

that is, λ_i is block–diagonal and *non–zero*, while

$$(\alpha\alpha^* - \alpha^*\alpha)\Big|_{i \text{ node}} \in \bigoplus_s \mathrm{Hom}\big(W(k_{i,s}), W(k_{i,s}-1)\big), \tag{3.113}$$

i.e. it is block *off*–diagonal. Hence the constraint

$$\sum_\alpha (\alpha\alpha^* - \alpha^*\alpha) = t \sum_i \lambda_i, \tag{3.114}$$

cannot be satisfied for $t \neq 0$. Of course, this just says that the deformation $t\lambda$ projects out the W–bosons from the light category $\mathscr{L}(\lambda = 0)$; what remains is the correct $D_p(G)$ BPS category we were looking for.

3.4.7. Several $D_p(G)$ matter subsectors One could ask what happens if we couple more than one $D_p(G)$ system to a G SYM subsector. Suppose we gauge

[27] We identity the category $\mathcal{T}(\lambda = 0)$ with the category of nilpotent representations of $\widehat{A}(p, 0)$.

ℓ	allowed p_i	UV behavior	$\mathcal{C}$
2	$(p,q)\,:\, p\geq q\geq 1$	AF	$A(p,q)$
3	$(2,2,p)\,:\, p\geq 2$	AF	$\widehat{D}_{p+2}$
	(2,3,3)	AF	$\widehat{E}_6$
	(2,3,4)	AF	$\widehat{E}_7$
	(2,3,5)	AF	$\widehat{E}_8$
	(3,3,3)	CFT	$E_6^{(1,1)}$
	(2,4,4)	CFT	$E_7^{(1,1)}$
	(2,3,6)	CFT	$E_8^{(1,1)}$
4	(2,2,2,2)	CFT	$D_4^{(1,1)}$

Table 3.6: Solutions to eqn.(3.115) and corresponding algebras.

the diagonal group G of ℓ subsystems of type $D_{p_i}(G)$. The requirement of no Landau poles gives

$$b=\Big(2-\sum_{i=1}^{\ell}\frac{p_i-1}{p_i}\Big)h(G)\geq 0 \tag{3.115}$$

The solutions to this condition are listed in table 3.6. The allowed ℓ-uples of p_i are well–known in representation theory: the numbers we obtain for asymptotically–free (resp. for conformal) theories are precisely the tubular types of the $\mathbb{P}^1$ families of regular representations of Euclidean (resp. tubular) algebras [65, 87].

Therefore the asymptotically–free model one gets are precisely the $\widehat{H}\boxtimes G$ models. Such models consist of a G SYM subsector weakly gauging the G-flavor symmetry of several $D_{p_i}(G)$ matter systems according to the following table:

		superconformal system
$A(p,q)\boxtimes G$	$p\geq q\geq 1$	$D_p(G)\oplus D_q(G)\oplus D_1(G)$
$\widehat{D}_r\boxtimes G$	$r\geq 4$	$D_2(G)\oplus D_2(G)\oplus D_{r-2}(G)$
$\widehat{E}_r\boxtimes G$	$r=6,7,8$	$D_2(G)\oplus D_3(G)\oplus D_{r-3}(G)$

(3.116)

The superconformal models correspond to the 2d theories which are the direct sums of the minimal G–models with the ones associated to the four elliptic complete SCFT's [7]

$$D_4^{(1,1)},\quad E_6^{(1,1)},\quad E_7^{(1,1)},\quad E_8^{(1,1)}. \tag{3.117}$$

	superconformal system
$D_4^{(1,1)}\circledast G$	$D_2(G)\oplus D_2(G)\oplus D_2(G)\oplus D_2(G)$
$E_6^{(1,1)}\circledast G$	$D_3(G)\oplus D_3(G)\oplus D_3(G)$
$E_7^{(1,1)}\circledast G$	$D_2(G)\oplus D_4(G)\oplus D_4(G)$
$E_8^{(1,1)}\circledast G$	$D_2(G)\oplus D_3(G)\oplus D_6(G)$

Table 3.7: Matter content of the G–tubular SCFT's.

Equivalently, they may be defined as the models

$$D_4^{(1,1)}\circledast G,\quad E_6^{(1,1)}\circledast G,\quad E_7^{(1,1)}\circledast G,\quad E_8^{(1,1)}\circledast G. \tag{3.118}$$

Their 'messy' vec–quivers Q_{BPS} may also be easily written down since $D_4^{(1,1)}$ is a Lagrangian theory ($SU(2)$ SQCD with $N_f=4$) while [7]

$$E_6^{(1,1)}=D_4\boxtimes A_2,\quad E_7^{(1,1)}=A_3\boxtimes A_3,\quad E_8^{(1,1)}=A_2\boxtimes A_5, \tag{3.119}$$

which allows to write their elliptic Stokes matrices S_{ell} as tensor products of Dynkin ones. The BPS quiver of the SCFT model $H^{(1,1)}\circledast G$ is then given by the exchange matrix

$$B=S_{\text{ell}}^t\otimes S_G^t-S_{\text{ell}}\otimes S_G. \tag{3.120}$$

The periods p_i's of the D_{p_i} matter subsectors may be read from the characteristic polynomial of the Coxeter of the corresponding affine/toroidal Lie algebra [87]

$$\det[z-\Phi]=(z-1)^2\prod_i\frac{z^{p_i}-1}{z-1} \tag{3.121}$$

see table 3.7. This equation also implies that the rank of the flavor group of a $\widehat{H}\boxtimes G$ QFT (resp. $H^{(1,1)}\circledast G$ SCFT) is additive with respect to the matter subsectors

$$\operatorname{rank}F=\sum_i f(p_i,G), \tag{3.122}$$

where $f(p,G)$ is the function defined in (3.81).

From the point of view of section 3.4.6, the module category for G SYM coupled to $\oplus_{i=1}^{\ell}D_{p_i}(G)$ may be more conveniently realized as the (τ–twisted) representations of the preprojective algebra $\mathcal{P}(G)$ valued in the Abelian category $\mathsf{Coh}(X_{p_1,\dots,p_\ell})$.

3.4.8. The BPS spectrum of $\widehat{H}\boxtimes G$ models at strong coupling All the models of type $\widehat{H}\boxtimes G$ admit a finite BPS chamber containing only hypermultiplets with charge vectors

$$e_a\otimes\alpha\in\Gamma_{\widehat{H}}\otimes\Gamma_G,\qquad \alpha\in\Delta_+(G), \tag{3.123}$$

that is, a copy of the positive roots of G per each node of $\widehat{H}$. We get a finite chamber with

$$\#\{\text{ hypermultiplets }\}=\frac{1}{2}\,r(G)\,h(G)\,r(\widehat{H}). \tag{3.124}$$

This result follows from the Weyl–factorized source–sequences approach to the mutation algorithm. Since all the models have chambers of type $(G,...,G)$, these BPS–chambers exhibit, moreover, $1/h(G)$ fractional monodromies. The details of the computation are rather technical: The mutation sequences corresponding to these finite BPS-chambers are constructed in appendix A.3.

3.5. Geometry of $D_p(G)$ SCFT's for G a *classical* group

In the previous section we have defined the four-dimensional $\mathcal{N}=2$ superconformal systems $D_p(G)$ from the study of the light subcategory of the $\widehat{A}(p,1)\boxtimes G$ models. The underlying $(2,2)$ system of the $\widehat{A}(p,1)\boxtimes G$ was given in (3.1) by

$$e^{-Z}+e^{pZ}+W_G(X,Y)+U^2=\text{ lower terms in } X,Y.$$

In particular we have shown that the $\widehat{A}(p,1)\boxtimes G$ system has a G SYM subsector. Geometrically, by the scaling arguments of [86], we would expect that the size of the cylinder in the Z coordinates of (3.1) is related to the size of the G SYM coupling. Thus, formally, the $D_p(G)$ system could be described by the engineering of the Type II B superstring on the limit $Z\to\infty$ of the geometry (3.1):

$$W_{p,G}\equiv e^{pZ}+W_G(X,Y)+U^2+\text{ lower terms }=0, \tag{3.125}$$

with holomorphic top form given as in eqn.(3.84). In this section, we explore the consequences of this geometric picture when G is a (simple, simply–laced) classical Lie group.

The identification of the $D_p(G)$ Lagrangian subclass

The chiral operators of a 4d $\mathcal{N}=2$ SCFT with a weakly coupled Lagrangian formulation have integral dimension. Hence the order of its 4d quantum monodromy is necessarily 1 [13] which — in view of eqns.(3.88)(3.91) — is equivalent to $b \in \mathbb{Z}$. For the $D_p(G)$ models this statement has a partial converse. Indeed, we claim that:

- A model $\widehat{A}(p,1) \boxtimes A_{N-1}$ is Lagrangian iff $b_{p,SU(N)}$ is an integer.
- A model $\widehat{A}(p,1) \boxtimes D_N$ is Lagrangian iff $b_{p,SO(2N)}$ is an *even* integer.
- No model $\widehat{A}(p,1) \boxtimes E_r$ is Lagrangian.

The statement that $\widehat{A}(p,1) \boxtimes E_r$ has no Lagrangian formulation is elementary. The theories of type $D_p(G)$ have $b_{p,G} < h(G)$: If a $D_p(G)$ theory is Lagrangian, its contribution to the YM β–function should be equal to the $U(1)_R$ anomaly coefficient $b(R)$ of a free hypermultiplet in some (generally reducible) representation R of G. Since E_8 has no non–trivial representation with $b(R) < 30$ this is impossible. For E_7 the only representation with $b(R) < 18$ are the $\frac{k}{2}\,\mathbf{56}$, $k = 1, 2$ with $b = 6k$. Then, in order to have a Lagragian model,

$$b_{p,E_7} = \frac{p-1}{p} 18 = 6k \Longleftrightarrow (3-k)p = 3 \Longrightarrow k = 2, p = 3. \tag{3.126}$$

A Lagrangian theory with two half–hypers would have flavor symmetry at least $SO(2)$; but the theory $A(3,1) \boxtimes E_7$ has $f(3, E_7) = 0$ by (3.81), and therefore the model $D_3(E_7)$ cannot be Lagrangian. Finally, the only E_6 representation with $b(R) < 12$ is the $\mathbf{27}$ with $b = 6$; to have a Lagrangian model

$$b_{p,E_6} \equiv \frac{p-1}{p} 12 = 6 \Longleftrightarrow p = 2, \tag{3.127}$$

which would imply F at least $U(1)$, while (3.81) gives $f(2, E_6) = 0$.

Let us now proceed to show the claim for $G = SU(N), SO(2n)$. Some checks of the identifications we have found are in appendix §.A.2 — to be compared with our discussion about specialization.

3.5.1. The case $G = SU(N)$ We have to show that

$$\begin{array}{c} D_p(SU(N)) \text{ admits} \\ \text{a Lagrangian formulation} \end{array} \Longleftrightarrow h(SU(N)) = N = mp. \tag{3.128}$$

Consider the $(2,2)$ superpotentials of type $W_{p,SU(N)}$: Only if $N = mp$ the corresponding $(2,2)$ system admits, at the conformal point, several *exactly marginal* deformations. By 2d/4d correspondence, we known that under such deformations the quiver mutation class is invariant. Since the models of type $D_p(G)$ are defined only by the mutation class of their quivers with superpotential, properly speaking, the 2d/4d correspondence associates to a 4d model the *universal* 2d superpotential $W_{p,G}(t_\alpha)$ over the space of exactly marginal/relevant deformations. The dimensions of the generators of the 2d chiral ring $\mathcal{R}$ are

$$q(X) = \frac{1}{mp} \qquad q(e^Z) = \frac{1}{p} \qquad q(Y) = q(U) = \frac{1}{2}. \tag{3.129}$$

The marginal deformations of $W_{p,SU(mp)}$ correspond to the operators $X^\alpha e^{\beta Z} \in \mathcal{R}$ such that

$$q(X^\alpha e^{\beta Z}) = 1 \Longleftrightarrow \quad \begin{bmatrix} \alpha = m(p-k) \\ \beta = k \end{bmatrix} \quad k = 0, \dots, p. \tag{3.130}$$

The universal superconformal family of superpotentials is then

$$W_{p,SU(mp)} = e^{pZ} + X^{mp} + \sum_{k=1}^{p-1} t_k \, X^{m(p-k)} e^{kZ} + Y^2 + U^2. \tag{3.131}$$

The Seiberg–Witten geometry that corresponds to the $\widehat{A}(p,1) \boxtimes SU(mp)$ model is

$$e^{-Z} + W_{p,SU(mp)} = 0. \tag{3.132}$$

The corresponding Seiberg–Witten curve can be written as

$$e^{-Z} + e^{pZ} + X^{mp} + \sum_{k=1}^{p-1} t_k \, X^{m(p-k)} e^{kZ} = \text{ relevant deformations of } W_{p,SU(mp)} \tag{3.133}$$

with canonical Seiberg–Witten differential $\lambda_{SW} = X\, dZ$. Let us change variables as follows

$$s = e^Z \quad v = X \Longrightarrow \lambda_{SW} = v\, \frac{ds}{s}. \tag{3.134}$$

Multiplying by s the equation (3.133) in the new variables, we obtain the Seiberg–Witten curve

$$1 + p_{mp}(v)s + p_{m(p-1)}(v)s^2 + \cdots + p_{2m}(v)s^{p-1} + p_m(v)s^p + s^{p+1} = 0, \tag{3.135}$$

where the $p_i(v)$ are polynomials of degree i in v. This is a well-known Seiberg–Witten curve (see for example eqn.(2.41) of [99]), and therefore we conclude that all theories $\widehat{A}(p,1) \boxtimes A_{mp-1}$ have a Lagrangian S-duality frame in which they are described as the quiver gauge theory [28]

$$SU(mp) - SU(m(p-1)) - SU(m(p-2)) - \cdots - SU(2m) - SU(m). \tag{3.136}$$

[28] Here, as usual, an edge $-$ denotes a bifundamental hypermultiplet.

Decoupling the first $SU(mp)$ SYM sector, we get that the only Lagrangian theories of type $D_p(SU(N))$ are

$$D_p(SU(mp)) = \left\{ \begin{array}{c} \widehat{A}(p,1) \boxtimes A_{m(p-1)-1} \\ \text{coupled to } mp \text{ fundamental} \\ SU(m(p-1)) \text{ hypers} \end{array} \right\} \tag{3.137}$$

which is indeed a SCFT as expected. Let us perform a couple of consistency checks:

1. The rank of the flavor group F of the theory $\widehat{A}(p,1) \boxtimes A_{mp-1}$ is

$$\delta(d, SU(mp)) = \begin{cases} 1 \text{ if } d \mid mp \text{ and } d \neq 1 \\ 0 \text{ else} \end{cases} \implies f(p, SU(mp)) = \sum_{\substack{d|p \\ d\neq 1}} \varphi(d) = p-1,$$

   which is precisely the number of bifundamentals in the linear quiver (3.136).

2. The rank of the gauge group of the theory (3.136) is

$$r(G) = \sum_{k=1}^{p} (mk-1) = \frac{mp(p+1)}{2} - p \tag{3.138}$$

   In addition we have the $f = p-1$ hypermultiplets. The rank of the charge lattice then matches the number of nodes of the quiver $\widehat{A}(p,1) \boxtimes A_{mp-1}$:

$$2r(G) + f = mp(p+1) - 2p + p - 1 = (p+1)(mp-1). \tag{3.139}$$

3.5.2. Lagrangian subclass for $G = SO(2N)$ The necessary condition that $b_{p,SO(2N)}$ must be an even integer follows from the fact that all representations with $b < h$ of $SO(2N)$ have even Dynkin-index b. So,

$$b_{p,SO(2N)} = \frac{p-1}{p} 2(N-1) \in 2\mathbb{N} \iff N = mp+1. \tag{3.140}$$

Again, this is precisely the case in which the corrisponding $(2,2)$ superpotential at the superconformal point admits marginal deformations. Indeed,

$$W_{p,D_{mp+1}} = e^{pZ} + X^{mp} + XY^2 + U^2, \tag{3.141}$$

while the dimensions of the generators of $\mathcal{R}$ are

$$q(X) = \frac{1}{mp} \qquad q(Y) = \frac{mp-1}{2mp} \qquad q(e^Z) = \frac{1}{p} \qquad q(U) = \frac{1}{2}. \tag{3.142}$$

The marginal deformations of $W_{p,D_{mp+1}}$ are those in eqn.(3.130), and the universal family of superpotentials for $\widehat{A}(p,1)\boxtimes D_{mp+1}$ is

$$e^{-Z}+e^{pZ}+X^{mp}+\sum_{k=1}^{p-1}t_k\,X^{m(k-p)}e^{kZ}+XY^2+U^2=\text{ `lower terms' }. \quad (3.143)$$

The generic 'lower terms' have the form $2\lambda Y$ + 'independent of Y' for some non–zero λ. Integrating out Y we obtain the equivalent geometry

$$e^{-Z}+e^{pZ}+X^{mp}+\sum_{k=1}^{p-1}t_k\,X^{m(k-p)}e^{kZ}-\frac{\lambda^2}{X}+U^2=\text{lower terms}\ , \quad (3.144)$$

which is the Seiberg–Witten (SW) curve of the 4d theory with differential $\lambda_{SW}=XdZ$. Now we change variables $X=v^2$, $s=e^Z$, and we multiply the resulting curve by sv^2. The final form of the SW curve for the model $\widehat{A}(p,1)\boxtimes D_{mp+1}$ is

$$v^2+v^2\,s^{p+1}+\sum_{k=1}^{p}p_{m(p-k+1)+1}(v^2)\,s^k=0, \quad (3.145)$$

where the $p_i(v^2)$ are polynomials of degree $2i$ in v. These Seiberg-Witten curves are well-known (see *e.g.* section 3.6 of [100]): they are part of the family

$$v^2+v^2\,s^{p+1}+\sum_{k=1}^{p}p_{2\ell_k+1+(-1)^k}(v)\,s^k=0, \quad (3.146)$$

that corresponds to linear quiver theories of type

$$SO(2\ell_1)-USp(2\ell_2)-\cdots-SO(2\ell_{k-1})-USp(2\ell_k)-SO(2\ell_{k+1})-\cdots \quad (3.147)$$

where the edges represents *half*-hypermultiplets in the bifundamental repr. In our case

$$2\ell_k+1+(-1)^k=2(m(p-k+1)+1)\Longrightarrow\left[\begin{array}{l}k\text{ even}:\ \ \ell_k=m(p-k+1)\\ k\text{ odd}:\ \ \ell_k=m(p-k+1)+1\end{array}\right. \quad (3.148)$$

for $k=1,\dots,p$. In conclusion: all theories of type $\widehat{A}(p,1)\boxtimes D_{mp+1}$ have a Lagrangian description (in a suitable region of their parameter space) as the linear quiver theory

$$\begin{aligned}&SO(2mp+2)-USp(2m(p-1))-SO(2m(p-2)+2)-USp(2m(p-3))-\cdots\\&\qquad\cdots-SO(2m(p-2\ell)+2)-USp(2m(p-2\ell-1))-\cdots\end{aligned} \quad (3.149)$$

The linear quiver have two possible ends, depending on the parity of p:

$$\begin{aligned} &p \text{ odd}: && \cdots - USp(4m) - SO(2m+2) \\ &p \text{ even}: && \cdots - SO(4m+2) - USp(2m) - \boxed{SO(2)} \end{aligned} \tag{3.150}$$

where the box represents an ungauged flavor group. Consequenctly the only Lagrangian theories of type $D_p(SO(N))$ are the theories:

$$D_p(SO(2mp+2)) = \left\{ \begin{array}{l} \boxed{SO(2(mp+1))} - USp(2m(p-1)) - \cdots \\ \cdots - SO(2m(p-2\ell)+2) - USp(2m(p-2\ell-1)) - \cdots \\ \quad \text{with the same ends as in eqn.(3.150)} \end{array} \right\} \tag{3.151}$$

where again the box represents an ungauged flavor group. A few checks are in order:

1. The rank of the flavor group of $\widehat{A}(p,1) \boxtimes D_{mp+1}$ is

$$f(p, D_{mp+1}) = \begin{cases} 0 & p \text{ odd} \\ 1 & p \text{ even} \end{cases} \tag{3.152}$$

   which is consistent with (3.149): half-hypermultiplets carry no flavor charge.

2. The rank of the gauge group of the $\widehat{A}(p,1) \boxtimes D_{mp+1}$ theory is

$$r(G) = \frac{1}{2}\sum_{k=1}^{p}\Big(2(m(p-k))+1-(-1)^k\Big) = \frac{1}{2}\Big(p(mp+m+1)+\frac{1}{2}\big(1-(-1)^p\big)\Big). \tag{3.153}$$

   Therefore

$$2r(G) + f = p(mp+m+1) + 1 = (p+1)(mp+1) \tag{3.154}$$

   which is the number of nodes for the quiver $\widehat{A}(p,1) \boxtimes D_{mp+1}$.

3. The beta function contribution to the $SO(2mp+2)$ gauge group is

$$4mp - 2m(p-1) = b_{\widehat{A}(p,1)\boxtimes D_{mp+1}}. \tag{3.155}$$

3.6. Computing the 4d a, c, SCFT central charges

3.6.1. 4d quantum monodromy and the SCFT central charge c It is well known that for a $\mathcal{N}=2$ 4d SCFT the corresponding topological theory in curved spacetime develops a superconformal anomaly which is sensitive to the topology of the background manifold [101,102]. If the Euler characteristic χ of the background manifold is zero, such an anomaly is proportional to the central charge c of the $\mathcal{N}=2$ 4d SCFT.

The trace of the 4d quantum monodromy operator $\mathbb{M}(q)$ is a particular instance of topological partition function on the R–twisted Melvin cigar $MC_q \times_g S^1$ [13] which has $\chi=0$. In principle, $\mathrm{Tr}\,\mathbb{M}(q)$ is uniquely fixed once we give the quiver and superpotential of the 4d theory, $(Q,\mathcal{W})$, *via* the BPS spectrum (computed in any chamber) [13].

It turns out that $\mathrm{Tr}\,\mathbb{M}(q)$ is equal to a Virasoro character of a 2d CFT [13]; the effective 2d CFT central charge, $c_{\text{eff}} \equiv c_{\text{eff}}(Q,\mathcal{W})$, then measures an anomaly of the topological partition function which should correspond to the 4d SCFT one. It follows that the 2d effective central charge c_{eff} should be identified with the 4d central charge c, up to normalization. However one has to take into account, in addition, the contribution of the massless sector, which is omitted in the usual definition of $\mathbb{M}(q)$ [13]. For the models of interest in this paper the massless sector consists just of the free photon multiplets, since there are no hypermultiplets which are everywhere light on the Coulomb branch. The number of the free photon multiplets is equal to the dimension of the Coulomb branch, which is $\mathrm{rank}\,B/2$, where B is the exchange matrix of the quiver of the theory. Since one free vector multiplet contributes $+1/6$ to the 4d SCFT central charge c, we get

$$c = \alpha \cdot c_{\text{eff}}(Q,\mathcal{W}) + \frac{\mathrm{rank}\,B}{12}, \tag{3.156}$$

where α is a universal normalization constant still to be determined. To fix α we apply this formula to a free hypermultiplet.

In ref. [13] c_{eff} was computed for the models $G\boxtimes G'$ (G, G' being Dynkin quivers)

$$c_{\text{eff}}(G\boxtimes G') = \frac{r(G)\,r(G')\,h(G)\,h(G')}{h(G)+h(G')}. \tag{3.157}$$

A free hypermultiplet corresponds to $G=G'=A_1$; since it has $c=1/12$, eqns.(3.156)(3.157) give

$$\frac{1}{12} = \alpha\cdot\frac{1\cdot 1\cdot 2\cdot 2}{2+2} \equiv \alpha, \tag{3.158}$$

and the final formula for c is

$$c = \frac{1}{12}\Big(c_{\text{eff}}(Q,\mathcal{W}) + \mathrm{rank}\,B\Big). \tag{3.159}$$

Comparing our analysis with [101] we see that the 2d CFT central charge $c_{\text{eff}}(Q,\mathcal{W})$ captures the scale dimension of the discriminant of the SW curve at the conformal point. Our formula then may be seen as an expression for the discriminant scale in terms of Lie–theoretical invariants of Q.

A few examples are in order.

Example 1: A_{N-1} Argyres–Douglas corresponds to $G = A_{N-1}$, $G' = A_1$ in eqn.(3.157). Then

$$c_{\text{eff}} = \frac{2N(N-1)}{N+2} \tag{3.160}$$

and rank $B = 2[(N-1)/2]$, so

$$c = \frac{\ell(6\ell+5)}{6(2\ell+3)} \quad \text{for} \quad N = 2\ell+1 \tag{3.161}$$

$$c = \frac{(2\ell-1)(3\ell+1)}{12(\ell+1)} \quad \text{for} \quad N = 2\ell \tag{3.162}$$

which are the values reported in [103].

Example 2: D_N Argyres–Douglas corresponds to $G = D_N$, $G' = A_1$. Thus

$$c_{\text{eff}} = \frac{2N \cdot 2(N-1)}{2+2(N-1)} = 2(N-1) \tag{3.163}$$

and rank $B = 2[(N-2)/2]$, so

$$c = \frac{1}{6}(3\ell-2) \quad \text{for} \quad N = 2\ell \tag{3.164}$$

$$c = \frac{1}{2}\,\ell \quad \text{for} \quad N = 2\ell+1, \tag{3.165}$$

in agreement with [103].

Example 3: Nore generally, for all models $A_{k-1} \boxtimes A_{N-1}$ our formula reproduces the value of c conjectured by Xie [103].

3.6.2. Generalization to $\widehat{H} \boxtimes G$ Unfortunately no one has computed c_{eff} for $\mathcal{N} = 2$ models more general than the $G' \boxtimes G$ ones. However in this paper we are interested only in the slightly more general case where the finite–dimensional Lie algebra G' is replaced by the infinite dimensional Kac–Moody Lie algebra $\widehat{H}$, our prime application being to $\widehat{H} = \widehat{A}(p,1)$. The similarity with the case analyzed in [13] suggests that $c_{\text{eff}}(\widehat{H} \boxtimes G)$ is still expressed in terms of Lie–theoretic invariants of the two algebras G and $\widehat{H}$, in facts by eqn.(3.157) where the invariants $r(G')$ and $h(G')$ of G' are replaced by the appropriate invariants of $\widehat{H}$.

In order to make the correct replacements, we have to return to the computation leading to (3.157) was computed, and to track the origin of each Lie–theoretical quantity in the RHS of (3.157). The Coxeter numbers in the numerator arise as (twice) the number of quiver mutations we need to perform to get the complete

BPS spectrum (in each one of the two canonical chambers) by the mutation algorithm of [8]. In other words, it is (twice) the number of mutations after which the mutation algorithm (in those chambers) *stops.* One may see this number also as (twice) the number of mutations we need to perform to collect the contributions to the quantum monodromy $\mathbb{M}(q)$ from *all* the BPS particles. Mathematically, the factor $r(G')\,h(G')$ may be seen as (twice) the size of preprojective component of the AR quiver of $\mathbb{C}G$, generated by repeated application of the inverse AR translation τ^-. On the other hand, the denominator of (3.157) may be understood in terms of the identification of physical observables computed at a chiral phase ϕ with the physical observables at $\phi+2\pi$ twisted by the action of $\mathbb{M}$.

Now, all the these viewpoints about the origin of the $h(G')$'s appearing in the RHS of (3.157) lead to the conclusion that the proper value of h for an affine Lie algebra $\widehat{H}$ is ∞: in one of the two canonical chambers the mutation algorithm will be go on forever visiting particle after particle in the infinite towers of (preprojective) dyons.

Therefore, our educated guess is

$$c_{\text{eff}}(\widehat{H}\boxtimes G)=\lim_{h\to\infty}\frac{r(\widehat{H})\,r(G)\,h(G)\,h}{h+h(G)}\equiv r(\widehat{H})\,r(G)\,h(G), \tag{3.166}$$

For $\widehat{H}\boxtimes G$ one has

$$\operatorname{rank}B=r(G)\left(\sum_{i=1}^{3}p_i-1\right)-\sum_{i=1}^{3}f(p_i,G) \tag{3.167}$$

where $\{p_1,p_2,p_3\}$ are the three periods of $\widehat{H}$ (listed for each acyclic affine quiver $\widehat{H}$ in the second column of the table in eqn.(3.116)) and $f(p,G)$ is the function defined in eqn.(3.81) Then eqn.(3.159) gives the following expression for c of the $\widehat{H}\boxtimes G$ QFT

$$c\big(\widehat{H}\boxtimes G\big)=\frac{1}{12}\left\{\left(\sum_{i=1}^{3}p_i-1\right)r(G)\Big(h(G)+1\Big)-\sum_{i=1}^{3}f(p_i,G)\right\}. \tag{3.168}$$

Let us check that this expression has the right physical properties. First of all, it should be additive in the following sense. Let $\widehat{H}$ be an acyclic affine quiver and $\{p_1,p_2,p_3\}$ its periods (cfr. table in eqn.(3.116)). Since our formula for c refers to the value at the UV fixed point, and the YM coupling is asymptotically free, $c(\widehat{H}\boxtimes G)$ should be the sum of the c's of the four UV decoupled sectors: G SYM, $D_{p_1}(G)$, $D_{p_2}(G)$, and $D_{p_3}(G)$, while $c(D_1(G))\equiv 0$ for all G since it corresponds to the empty matter sector

$$c(\widehat{H}\boxtimes G)=\frac{1}{6}\dim G+\sum_i c(D_{p_i}(G)), \tag{3.169}$$

This gives two conditions which need to be satisfied by the expression (3.168)

$$c(\widehat{A}(1,1) \boxtimes G) = (\, c \text{ of pure SYM with group } G\,) \equiv \frac{1}{6}\dim G \tag{3.170}$$

$$c\big(\widehat{H} \boxtimes G\big) = \sum_{i=1}^{3} c\big(\widehat{A}(p_i,1) \boxtimes G\big) - 2\,c\big(\widehat{A}(1,1) \boxtimes G\big). \tag{3.171}$$

They are both true and corroborate our educated guess (3.166).

From eqn.(3.169) we extract the value of the central charge c for the SCFT $D_p(G)$. It is given by the function

$$c(p,G) = \frac{1}{12}\Big\{(p-1)\,r(G)\Big(h(G)+1\Big) - f(p,G)\Big\}. \tag{3.172}$$

Example 4: the model $\widehat{A}(2,1) \boxtimes A_2$ is $SU(3)$ SYM gauging the $SU(3)_{\text{flavor}}$ of the Argyres–Douglas of type D_4 [21]. From eqn.(3.164) c of D_4 is $2/3$, and then the c of the gauged model should be $c = 8/6 + 2/3 \equiv 2$ in agreement with eqn.(3.168)

$$c(\widehat{A}(2,1) \boxtimes A_2) = \frac{3\cdot 2\cdot 4}{12} = 2. \tag{3.173}$$

In §. 3.6.4 we shall present a number of additional examples of the formula (3.172), which always produces the correct physical results. See also §. 3.7.3 for further applications of the formula.

3.6.3. Computing a The formula for c also determines the central charge a in view of the 2d/4d correspondence. Indeed, consider a $\mathcal{N}=2$ $4d$ SCFT, and let $\mathcal{R}$ be the chiral ring of primary operators [49] of the $(2,2)$ superconformal system associated to it. For an element $\psi \in \mathcal{R}$, let us denote with $q(\psi)$ its 2d R-charge. Deforming the $2d$ superpotential

$$W \longrightarrow W + \sum_{\psi\in\mathcal{R}} u_\psi \psi \qquad u_\psi \in \mathbb{C} \tag{3.174}$$

we induce a massive deformation of the 4d theory with primary operators $\mathcal{O}_\psi$ that have dual parameters u_ψ. Their scaling dimensions are $D[\mathcal{O}_\psi] = 2 - D[u_\psi]$. Let Ω be the Seiberg–Witten form of the theory. At the conformal point the Seiberg–Witten geometry has a holomorphic scaling symmetry (*i.e.* the 2d R-=symmetry) under which the Seiberg-Witten form transforms as

$$\Omega \longrightarrow \lambda^{q(\Omega)}\Omega. \tag{3.175}$$

The 4d scaling dimensions of the mass parameters u_ψ are fixed by requiring that the Seiberg-Witten form have dimension equal to 1

$$D[\Omega]\equiv 1 \Longrightarrow \begin{cases} D[\psi]=q(\psi)/q(\Omega)\\ D[u_\psi]=(1-q(\psi))/q(\Omega)\end{cases} \tag{3.176}$$

It is a known fact [101] that

$$4(2a-c)=\sum_{i=1}^{d}(2D[u_i]-1) \tag{3.177}$$

where d is the dimension of the Coulomb branch of the model and the u_i are the physical deformations that parametrizes it, *i.e.* the deformations with scaling dimensions $D[u_i]>1$. Since $d\equiv \operatorname{rank} B/2$, we have

$$a=\frac{1}{2}\Big(c+\frac{1}{4}\cdot\sum_{i=1}^{d}(2D[u_i]-1)\Big)=\frac{1}{2}\cdot c+\frac{1}{16}\cdot\operatorname{rank} B+\frac{1}{4}\cdot\sum_{i=1}^{d}(D[u_i]-1). \tag{3.178}$$

We define

$$u(\mathcal{R})\equiv\frac{1}{4}\cdot\sum_{\psi\in\mathcal{R}}\left[\frac{1-q(\psi)}{q(\Omega)}-1\right]_+ \tag{3.179}$$

where $[x]_+$ is the function

$$[x]_+=\begin{cases} x & \text{for } x\geq 0\\ 0 & \text{for } x<0.\end{cases} \tag{3.180}$$

This gives the final formula for a

$$a=\frac{1}{2}\cdot c+\frac{1}{16}\cdot\operatorname{rank} B+u(\mathcal{R}). \tag{3.181}$$

For the $D_p(G)$ SCFT this formula may be rewritten in a simpler way. We write $E(G)$ for the set of exponents of G (equal to the degrees of the fundamental Casimirs of G *minus 1*). Then $u(\mathcal{R})$ is given by the function

$$u(p,G)=\frac{1}{4}\sum_{s=1}^{p-1}\sum_{j\in E(G)}\left[j-\frac{h(G)}{p}s\right]_+. \tag{3.182}$$

This formula is very obvious if you recall that $q(\Omega)\equiv 1-\hat{c}/2\equiv 1/h(G)$ (eqn.(3.84)). Hence $4\,u(\mathcal{R})$ is simply $\sum_{RR}[h(G)\,q_{RR}]_+$, where the q_{RR}'s are the $U(1)_R$ charges of the RR vacua whose set is precisely

$$\{q_{RR}\}\equiv\{j/h(G)-s/p\,:\,j\in E(G),\ 1\leq s\leq p-1\}. \tag{3.183}$$

Putting everything together we obtain that the central charge a for the $D_p(G)$ models is

$$a(p;G) = u(p;G) + \frac{1}{48}\Big((p-1)\,r(G)\big(2\,h(G)+5\big) - 5f(p;G)\Big). \qquad (3.184)$$

Correspondingly, at the UV fixed point, the a–central charge for the $\widehat{H}\boxtimes G$ models is

$$a(\widehat{H}\boxtimes G)\big|_{\text{UV}} = \frac{1}{4}\sum_{j\in E(G)} j + \frac{1}{24}r(G)\big(2\,h(G)+5\big) + \sum_{i=1}^{3} a(p_i;G). \qquad (3.185)$$

Example 5: $SU(2)$ SQCD with $N_f \leq 3$ flavors. $N_f = 1,2,3$ correspond, respectively, to the quiver $A(2,1)\boxtimes A_1$, $A(2,2)\boxtimes A_1$, and $\widehat{D}_4 \boxtimes A_1$. One has $c_{\text{eff}}(\widehat{H}\boxtimes A_1) = 2\,r(\widehat{H}) \equiv 2(2+N_f)$, while $\operatorname{rank} B = 2$ for all N_f. Then

$$c = \frac{1}{12}\big(4 + 2N_f + 2\big) = \frac{3}{6} + \frac{2N_f}{12}, \qquad (3.186)$$

$$a = \frac{1}{4} + \frac{9}{24} + N_f\, a(2,A_1) = \frac{5\dim SU(2)}{24} + \frac{2N_f}{24} \qquad (3.187)$$

consistent with $\dim SU(2) = 3$ vector–multiplets and N_f hyper doublets.

3.6.4. Further examples and checks

Example 6: $SU(N)$ linear quivers

For all $m,p\in\mathbb{N}$ we consider the linear quiver theory

$$SU(mp) - SU(m(p-1)) - SU(m(p-2)) - \cdots - SU(2m) - SU(m). \qquad (3.188)$$

As we discussed in §. 3.5.1, such theory has quiver $\widehat{A}(p,1)\boxtimes A_{mp-1}$. The weak coupling computation of c is

$$c = \frac{1}{6}n_v + \frac{1}{12}n_h = \frac{1}{6}\sum_{k=1}^{p}(m^2k^2-1) + \frac{1}{12}\sum_{k=1}^{p-1}(mk)\,m(k+1) = \frac{1}{12}p(m^2p(p+1)-2). \qquad (3.189)$$

The computation from the quiver $\widehat{A}(p,1)\boxtimes A_{mp-1}$ is

$$\frac{1}{12}\Big(\big((p+1)(mp-1)mp\big) + \big((p+1)(mp-1) - p + 1\big)\Big) = \frac{1}{12}p(m^2p(p+1)-2), \qquad (3.190)$$

in perfect agreement with the weak coupling computation. Now consider the central charge a. Let us start computing

$$4\,u(p;A_{mp-1})=\sum_{s=1}^{p-1}\sum_{j=1}^{mp-1}[j-ms]_{+}=\sum_{s=1}^{p-1}\Delta(ms),\tag{3.191}$$

where we have set

$$\Delta(N)=\sum_{j=1}^{N-1}j=\frac{N(N-1)}{2}.\tag{3.192}$$

We already know that

$$f(p;A_{mp-1})=\gcd\{p,mp\}-1=p-1.\tag{3.193}$$

so that (3.185) gives

$$\begin{aligned}a(p;A_{mp-1})&=\frac{1}{4}\sum_{s=1}^{p-1}\Delta(ms)+\frac{1}{48}\Big((p-1)(mp-1)(2mp+5)-5(p-1)\Big)=\\&=\frac{1}{48}(p-1)(4m^2p^2-m^2p-10)\end{aligned}\tag{3.194}$$

On the other hand, let us compute a using weakly coupled QFT in the UV; a for $SU(N)$ SYM is

$$a(SU(N))=\frac{5}{24}\dim SU(N)=\frac{5}{24}(2\Delta(N)+N-1)\tag{3.195}$$

while the hypers in the bifundamental $(N_1,\overline{N}_2)$ contribute with $N_1N_2/24$. Then

$$\begin{aligned}a(\text{linear quiver})\Big|_{\text{QFT}}&=\frac{5}{24}\sum_{s=1}^{p-1}\Big(2\Delta(ms)+ms-1\Big)+\frac{m^2}{24}\sum_{s=1}^{p-1}s(s+1)=\\&=\frac{1}{48}(p-1)(4m^2p^2-m^2p-10),\end{aligned}\tag{3.196}$$

in perfect agreement.

Example 7: SO/USp linear quivers

For all $m,p\in\mathbb{N}$ we consider the linear quiver theory

$$\begin{aligned}&SO(2mp+2)-USp(2m(p-1))-SO(2m(p-2)+2)-USp(2m(p-3))-\cdots\\&\qquad\cdots-SO(2m(p-2\ell)+2)-USp(2m(p-2\ell-1))-\cdots\end{aligned}\tag{3.197}$$

(the two lines are meant to be concatenated). The edges now stand for *bifundamental* HALF hypermultiplets. The linear quiver have two possible ends, depending on the parity of p.

$$\begin{aligned} &p \text{ odd :} \qquad \cdots - USp(4m) - SO(2m+2) \\ &p \text{ even :} \qquad \cdots - SO(4m+2) - USp(2m) - \boxed{SO(2)} \end{aligned} \tag{3.198}$$

where the box means an ungauged flavor group. As we discussed in §. 3.5.2, the BPS quiver of this theory is $\widehat{A}(p,1) \boxtimes D_{mp+1}$.

Let us compute c from weak coupling; we specialize to $p = 2\ell$ even:

$$\begin{aligned} c = \frac{1}{6}\, n_v + \frac{1}{24}\, n_{\frac{1}{2}h} &= \frac{1}{6}\left[\sum_{k=1}^{\ell} \dim SO(4mk+2) + \sum_{k=1}^{\ell} \dim USp(2m(2k-1)\right] + \\ &\quad + \frac{1}{24}\sum_{k=1}^{\ell} \big[2m(2k-1)\big]\big[2m(2k)+2+2m(2k-2)+2\big] = \\ &= \frac{1}{6}\left[\sum_{k=1}^{\ell}(2mk+1)(4mk+1) + \sum_{k=1}^{\ell} m(2k-1)[2m(2k-1)+1]\right] + \\ &\quad + \frac{1}{24}\sum_{k=1}^{\ell} \big[2m(2k-1)\big]\big[2m(2k)+2+2m(2k-2)+2\big] = \\ &= \frac{\ell}{6}\big(8m^2\ell^2 + 4m^2\ell + 6m\ell + 3m + 1\big), \end{aligned} \tag{3.199}$$

while from the quiver

$$c = \frac{1}{12}\big((2\ell+1)(2\ell m+1)4\ell m+(2\ell+1)(2m\ell+1)-1\big) \equiv \frac{\ell}{6}\big(8m^2\ell^2+4m^2\ell+6m\ell+3m+1\big), \tag{3.200}$$

with perfect agreement. Let us consider now p odd $= 2\ell+1$.

$$\begin{aligned} c = \frac{1}{6}n_v + \frac{1}{24}n_{\frac{1}{2}h} &= \frac{1}{6}\left[\sum_{k=0}^{\ell}\dim SO(2m(2k+1)+2) + \sum_{k=1}^{\ell}\dim USp(4mk)\right] + \\ &\quad + \frac{1}{24}\sum_{k=1}^{\ell}4mk\big[2m(2k+1)+2+2m(2k-1)+2\big] = \\ &= \frac{1}{6}\left[\sum_{k=0}^{\ell}(m(2k+1)+1)(2m(2k+1)+1) + \sum_{k=1}^{\ell}2mk[4mk+1]\right] + \\ &\quad + \frac{1}{24}\sum_{k=1}^{\ell}4mk\big[2m(2k+1)+2+2m(2k-1)+2\big] = \\ &= \frac{1}{6}(4m\ell+2m+1)(2m\ell+m+1)(\ell+1) \end{aligned} \tag{3.201}$$

while from the quiver

$$\begin{aligned} c = \frac{1}{12}\Big((2\ell+2)\big((2\ell+1)m+1\big)\big(2m(2\ell+1)\big) + (2\ell+2)\big((2\ell+1)m+1\big)\Big) &= \\ = \frac{1}{6}(4m\ell+2m+1)(2m\ell+m+1)(\ell+1), \end{aligned} \tag{3.202}$$

in complete agreement.

If one considers the $D_p(SO(2mp+2))$ models, one has

$$f(p; D_{mp+1}) = \begin{cases} 1 & p \text{ even} \\ 0 & \text{otherwise} \end{cases} \equiv p+1-2\left[\frac{p+1}{2}\right] \tag{3.203}$$

$$\begin{aligned} u(p; D_{mp+1}) &= \frac{1}{4}\sum_{s=1}^{p-1}\left(\sum_{k=1}^{mp}[2k-1-2ms]_+ + [mp-2ms]_+\right) = \\ &= \frac{1}{4}\sum_{s=1}^{p-1}\sum_{k=1}^{m(p-s)}(2k-1) + \frac{m}{4}\sum_{s=1}^{[p/2]}(p-2s) = \\ &= \frac{m^2}{24}(2p-1)p(p-1) + \frac{m}{4}\,[(p-1)/2]\,[p/2] \end{aligned} \tag{3.204}$$

and then

$$\begin{aligned} a(p; D_{mp+1}) &= \frac{m^2}{24}(2p-1)p(p-1) + \frac{m}{4}\,[(p-1)/2]\,[p/2] + \\ &\quad + \frac{1}{48}\Big\{(p-1)(mp+1)(4mp+5) - 5(p+1) + 10[(p+1)/2]\Big\} \end{aligned} \tag{3.205}$$

for $p = 2q$ even this is

$$\frac{1}{24}\Big(32m^2q^3 - 20m^2q^2 + 24mq^2 + 2m^2q - 15mq + 5q - 5\Big) \tag{3.206}$$

while for $p = 2q + 1$ odd

$$\frac{1}{24}\,q\big(32m^2q + 28m^2q + 24mq + 6m^2 + 9m + 5\big) \tag{3.207}$$

We have already computed the number of vector multiplets and half–hypers in the linear quiver $\widehat{A}(p, 1) \boxtimes D_{mp+1}$: the computation for the theory $D_p(SO(mp + 1))$ changes just n_v by the $SO(mp + 1)$ contribution; for $p = 2q$

$$n_v = \frac{1}{3}(16m^2q^3 - 12m^2q^2 + 12mq^2 + 2m^2q - 9mq + 3q - 3) \tag{3.208}$$

$$n_{\frac{1}{2}h} = \frac{8}{3}mq(4mq^2 + 3q - m) \tag{3.209}$$

and therefore

$$a = \frac{5}{24}n_v + \frac{1}{48}n_{\frac{1}{2}h} = \frac{1}{24}\Big(32m^2q^3 - 20m^2q^2 + 24mq^2 + 2m^2q - 15mq + 5q - 5\Big) \tag{3.210}$$

in agreement with the formula (3.206).

For $p = 2q + 1$, we have

$$n_v = \frac{1}{3}q(16m^2q^2 + 12m^2q + 12mq + 2m^2 + 3m + 3) \tag{3.211}$$

$$n_{\frac{1}{2}h} = \frac{8}{3}mq(4mq^2 + 6mq + 3q + 2m + 3) \tag{3.212}$$

and therefore

$$a = \frac{5}{24}n_v + \frac{1}{48}n_{\frac{1}{2}h} = \frac{1}{24}q\,\big(32m^2q^2 + 28m^2q + 24m + 6m^2 + 9m + 5\big) \tag{3.213}$$

in agreement with formula (3.207).

3.6.5. General properties of the $D_p(G)$ SCFT's We have obtained quite a precise physical picture of the SCFT $D_p(G)$. We know:

- the rank of the flavor group

$$f = r(G) + f(p, G) \tag{3.214}$$

- the dimension of the Coulomb branch

$$d = \frac{1}{2}\Big((p-1)r(G) - f(p,G)\Big) \tag{3.215}$$

- the order of the 4d quantum monodromy

$$r = \frac{p}{\gcd\{p, h(G)\}} \tag{3.216}$$

- the dimension of the discriminant of the SW curve at the UV CFT point[29]:

$$\equiv (p-1)r(G)h(G) \tag{3.217}$$

- the SCF central charge a is $a(p;G)$ defined in eqn.(3.184)
- the SCF central charge c is $c(p;G)$ defined in eqn.(3.172)
- the G–current algebra central charge k_G is

$$k_G \equiv 2\, b_{p,G} = \frac{2(p-1)}{p}\, h(G) \tag{3.218}$$

- the set of the dimensions of the operators parametrizing the Coulomb branch

$$\Big\{\Delta_1, \Delta_2, \dots, \Delta_d\Big\} = \Big\{\, j - \frac{h(G)}{p} s + 1 \,\Big|\, j > \frac{h(G)}{p} s,\ j \in E(G),\ s = 1, \dots, p-1 \,\Big\}. \tag{3.219}$$

From these expressions we may extract some general properties of the $D_p(G)$ SCFT which are *typical* of this class of theories. For instance, as we are going to show, these theories have Coulomb branches of large dimension, $d = O(p)$ (eqn.(3.215)), while, for fixed G, the dimension of their Higgs branches is bounded above by $2\,r(G)\,h(G)^2$ (a sharper inequality holds for $p \geq h(G)$, see below).

We list a few such properties:

The Coulomb branch operator of maximal dimension: Since $h(G) - 1 \in E(G)$ for all G's, we see from (3.219) that the maximal dimension of the Coulomb branch operators is equal to the β–function coefficient $b_{p,G}$,

$$\Delta_d \equiv \frac{p-1}{p}\, h(G) = b_{p,G} \equiv \frac{1}{2}\, k_G, \tag{3.220}$$

and is *always less* that the maximal dimension $\Delta_{\max}(G) \equiv h(G)$ for SYM with gauge groups G.

[29] Just by checking against the formulas by Shapere–Tachikawa [101], one sees that $D(\Delta)_{\text{there}} \equiv c_{eff}(Q, \mathcal{W})_{here}$.

The rank of the flavor group f it is always $\leq 2\,r(G)$ with equality iff $h(G) \mid p$.

The dimensions $\Delta_i \in \mathbb{N} \iff r \in \mathbb{N} \iff b_{p,G} \in \mathbb{N}$.

Asymptotic behavior of a and c: For fixed G and large p the asymptotics of the SCFT central charges a and c are

$$a(p;G) \approx \frac{\dim G}{12}\, p + O(1) \tag{3.221}$$

$$c(p;G) \approx \frac{\dim G}{12}\, p + O(1). \tag{3.222}$$

In particular, $c(p;G) - a(p;G)$ is constant for large p up to a few percent Number–Theoretic modulation (see next item).

Dimension of the Higgs branch: Assume $p \geq h \equiv h(G)$. Then $(r \equiv r(G))$

$$\begin{aligned}\dim_H \text{Higgs branch} \equiv n_h - n_v \equiv 24(c-a) \leq \\ \leq \#(\text{positive roots of } G) + r \equiv \frac{(h+2)r}{2} \in \mathbb{N},\end{aligned} \tag{3.223}$$

with equality *if and only if* $h \mid p$.

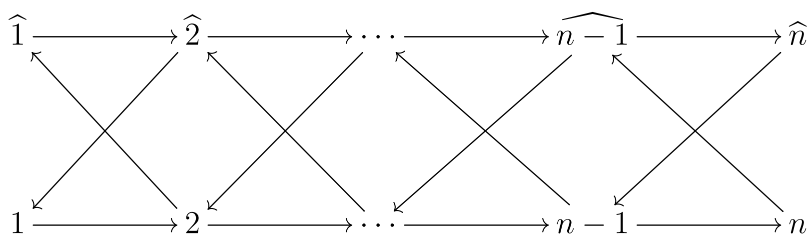
The reader interested in the proofs of eqns.(3.221)–(3.223) is referred to appendix B.2.

3.7. $D_2(G)$ systems and Minahan-Nemeshansky theories

The SCFT's of period $p = 2$ are expected to be particular easy since they generalize to arbitrary $G = ADE$ the D_2 Argyres–Douglas model which is just a free doublet. In this section we study in more detail this simple class of theories.

3.7.1. Quivers and superpotentials The quivers $Q_{2,G}$ have $\widehat{A}(2,0)$ full subquivers over the nodes of G which correspond to quadratical terms in the superpotential $\mathcal{W}_{\text{mat}}$. Therefore the arrows of the $\widehat{A}(2,0)$ vertical subquivers get integrated out from their DWZ–reduced quiver with superpotential, $\mathcal{D}(G)$, which are particularly simple. For $G = A_n$ one gets

$\mathcal{D}(A_n) \equiv$



(3.224)

the $\mathcal{D}(G)$'s for the other simply–laced Lie algebras being represented in figure 3.3. The superpotential for $\mathcal{D}(A_n)$ is

$$\mathcal{W}_{\mathcal{D}} = \sum_a \left(\alpha^*_{(1),a}\alpha_{(2),a} - \alpha_{(1),a}\alpha^*_{(2),a}\right)\left(\alpha^*_{(2),a}\alpha_{(1),a} - \alpha_{(2),a}\alpha^*_{(2),a}\right). \tag{3.225}$$

3.7.2. A finite BPS chamber. The quivers $\mathcal{D}(G)$ contain two full Dynkin G subquivers with alternating orientation and non–overlapping support. *E.g.* the two alternating A_n subquivers of $\mathcal{D}(A_n)$ in (3.224) are the full subquivers over the nodes

$$\left\{1, 2, \widehat{3}, \widehat{4}, 5, 6, \dots\right\} \quad \text{and} \quad \left\{\widehat{1}, \widehat{2}, 3, 4, \widehat{5}, \widehat{6}, \dots\right\}. \tag{3.226}$$

With reference to this example, let us define the following mutation sequence:

$$\mathbf{m}_{2,A_n} \equiv \left(\prod_{a \text{ even}} \mu_a \circ \mu_{\widehat{a}}\right) \circ \left(\prod_{a \text{ odd}} \mu_a \circ \mu_{\widehat{a}}\right) \tag{3.227}$$

The mutation sequence corresponding to the full quantum monodromy associated to the $A_n \oplus A_n$ chamber is symply

$$(\mathbf{m}_{2,A_n})^{n+1} = \underbrace{\mathbf{m}_{2,A_n} \circ \cdots \circ \mathbf{m}_{2,A_n}}_{n+1 \text{ times}}. \tag{3.228}$$

We draw the BPS-quivers of the other $D_2(G)$ models in figure 3.3. From their structure it is clear that they all admit Coxeter–factorized sequences of type (G, G) constructed analogously. We conclude that all the $D_2(G)$ superconformal systems have a (possibly formal) finite–BPS-chamber $\mathscr{C}_{\text{fin}}$ such that

$$\Gamma^+\big|_{\mathscr{C}_{\text{fin}}} \simeq \Delta_+(G) \oplus \Delta_+(G) \tag{3.229}$$

where $\Delta_+(G)$ is the set of positive roots of the Lie algebra G. A few remarks are in order:

1. Applying this result to $D_2(SU(2N))$ we get a refinement of the result of [8] for $SU(N)$ SQCD with $2N$ flavors. The chamber (3.229) has less BPS hypers.
2. Comparing with the previous section we see that
$$c_{\text{eff}} = r(G)h(G) = \#\big\{\text{BPS-particles in the chamber (3.229)}\big\} \tag{3.230}$$
in agreement with a conjecture by Xie and Zhao [104].
3. From the explicit BPS spectrum (3.229) one constructs (new) periodic TBA Y–systems whose periodicity should coincide with that of the 4d quantum monodromy, $r(2, G)$ [13,14]. We have numerically checked this prediction for the corresponding 2d solvable models along the lines of [14], getting perfect agreement.

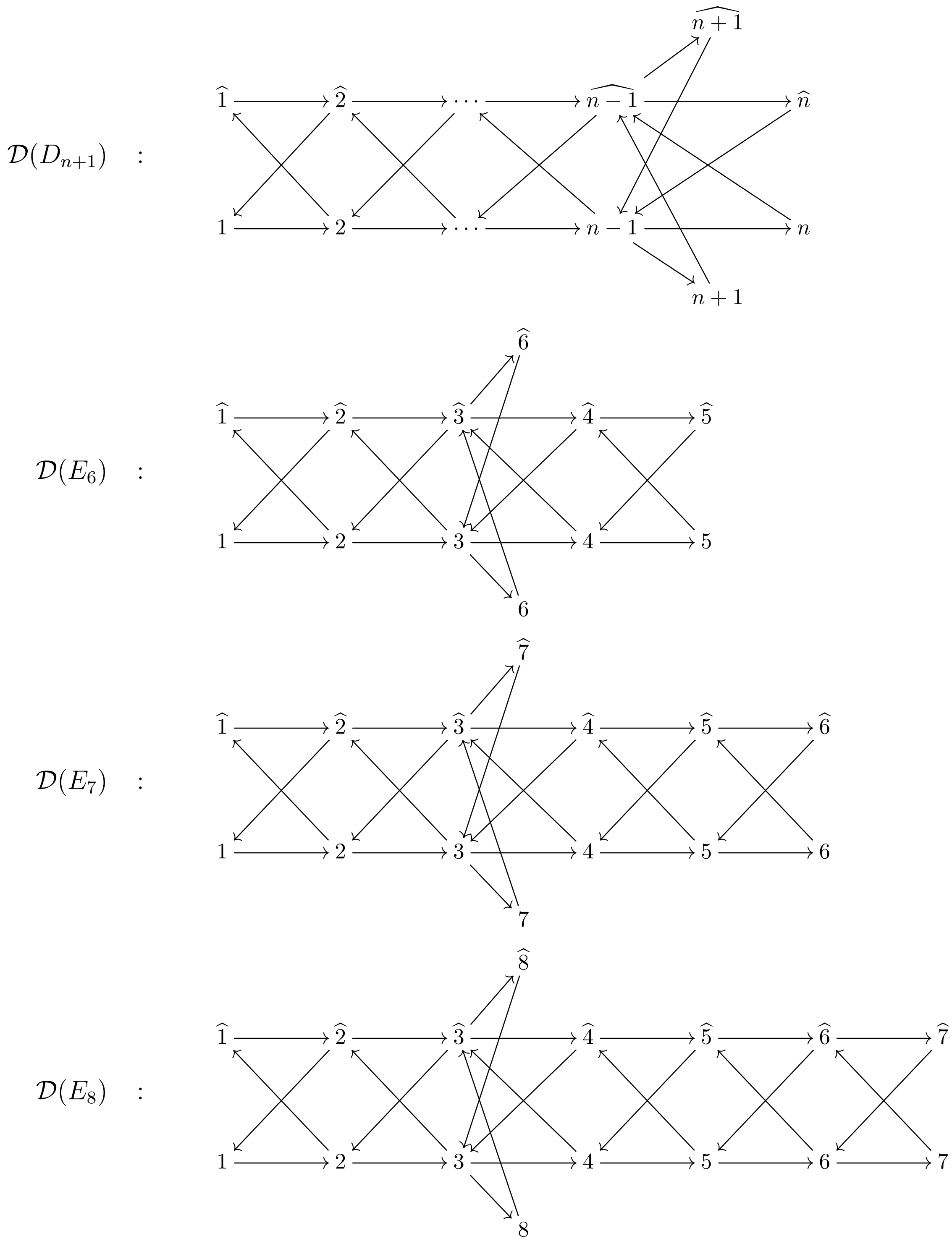
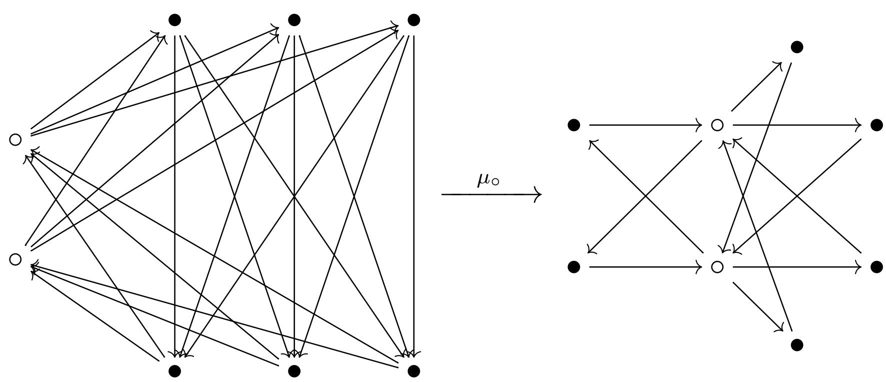


Figure 3.3: The DWZ–reduced quivers $\mathcal{D}(G)$ for the $D_2(G)$ SCFT's.

3.7.3. The exceptional Minahan–Nemeshansky theories.

E_6 **MN** $\equiv D_2(SO(8))$

Notice that $D_2(SO(8))$ theory coincides with the E_6 MN theory: The $D_2(SO(8))$ quiver is mutation equivalent to the quiver in figure (6.11) of [8] we reproduce below on the LHS: It is sufficient to mutate it on one of the white nodes to show this result is true.



(3.231)

As a consistency check, let us now show that all the invariants of the $D_2(SO(8))$ model agrees with the ones of the E_6 MN theory. First of all notice that

$$\delta(2, D_4) = 2 \Longrightarrow f(2, D_4) = \delta(2, D_4) \cdot \varphi(2) = 2 \tag{3.232}$$

Therefore the rank of the flavor group of the $D_2(SO(8))$ model is $4 + 2 = 6$. From this we can recover the rank of B:

$$\operatorname{rank} B = 2 \cdot 4 - 6 = 2 \Longrightarrow \text{dimension of the Coulomb branch } = 1. \tag{3.233}$$

Now,

$$c = \frac{1}{12}\big(4 \cdot 6 + 2\big) = \frac{13}{6} \tag{3.234}$$

and

$$4 \cdot u(2, D_4) = \Big[5 - \frac{6}{2}\Big]_+ + 2\Big[3 - \frac{6}{2}\Big]_+ \Big[1 - \frac{6}{2}\Big]_+ = 2. \tag{3.235}$$

Therefore,

$$a = \frac{13}{12} + \frac{1}{8} + \frac{1}{2} = \frac{41}{24}. \tag{3.236}$$

Moreover, the Coulomb branch coordinate has dimension

$$b_{2,D_4} = \frac{1}{2} \cdot 6 = 3, \tag{3.237}$$

in perfect agreement.

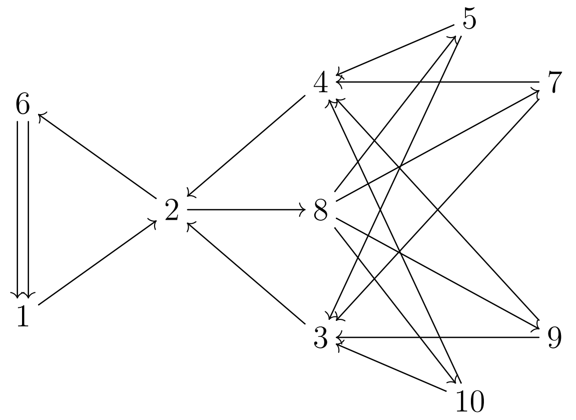


Figure 3.4: Element in the quiver mutation class of $D_2(SO(10))$ describing the S-duality frame of Argyres–Seiberg [105]. The full subquiver of type $A(1,1)$ on the nodes $\{1,6\}$ represents the $SU(2)$ SYM subsector, and the node 2 represents the gauged $SU(2)$ flavor symmetry of E_7 MN.

$D_2(SO(10))$ is E_7 MN coupled to $SU(2)$ SYM

From our results of section 3.5.2 it follows that the theory $D_2(SO(10))$ is Lagrangian. In an S-duality frame, this is just the model $USp(4)$ coupled to 6 hypermultiplets in the fundamental representation, *i.e.*

$$\boxed{SO(10)} - USp(4) - \boxed{SO(2)} \tag{3.238}$$

where the boxes represents ungauged flavor groups ($N_f = 5+1$). One of the most famous examples of S–duality [105] relates precisely this model with an $SU(2)$ SYM sector weakly gauging an $SU(2)$ subgroup of the flavor group of the E_7 MN model. By quiver mutations we are able to give an *explicit proof* of this statement: In the mutation class of the BPS-quiver of $D_2(SO(10))$ there is an element that clearly describes an $SU(2)$ SYM sector weakly gauging the flavor symmetry of a subsystem that we identify with the E_7 MN one. We draw such quiver in figure 3.4. See appendix B.3 for the explicit sequence of mutations.

From this result, it is easy to obtain the quiver for E_7 MN and from such quiver to prove that the E_7 theory has a finte BPS-spectrum. The explicit computation can be found in the next chapter of this thesis.

$D_2(E_6)$ is E_8 MN coupled to $SU(3)$ SYM

We have found how the exceptional MN theories of type E_6 and E_7 appear in between the $D_p(G)$ systems: The question, now, is if we can find also the E_8 MN

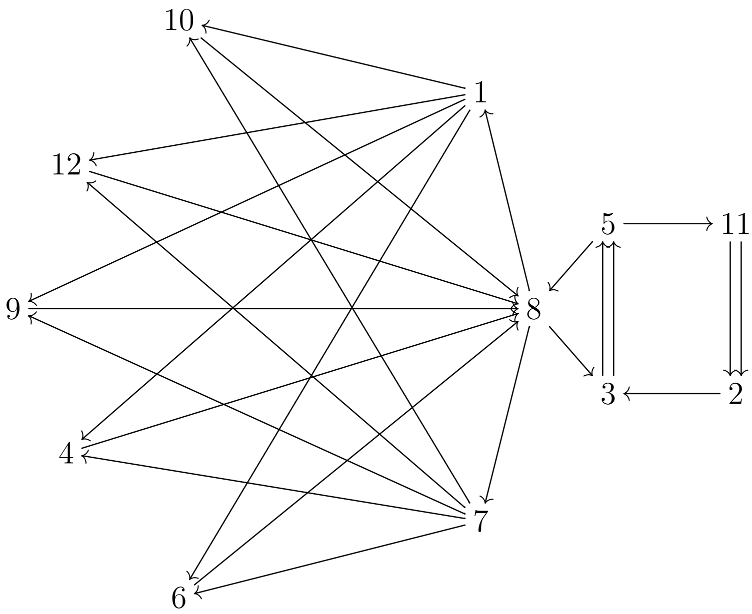


Figure 3.5: An element in the mutation-class of the quiver of the model $D_2(E_6)$. This quiver clearly represents an S-duality frame in which we have an explicit $SU(3)$ SYM sector coupled to the E_8 MN theory. The $SU(3)$ SYM full subquiver is on the nodes $\{2, 3, 5, 11\}$. The node 8 represents the gauged $SU(3)$ flavor symmetry of E_8 MN.

theory. Such theory has rank 1 and it has E_8 flavor symmetry, therefore its charge lattice has dimension 10. Since there are no $D_p(E_8)$ theories with such a small charge lattice, we expect that if the E_8 MN theory appears in between the $D_p(G)$ systems, it will manifest itself with part of its flavor symmetry weakly gauged. The first possibility we have is the $E_7 \otimes SU(2) \subset E_8$ group, but gauging the $SU(2)$ symmetry will leave us with an E_7 flavor symmetry, and there is no $D_p(E_7)$ theories with $10 - 1 + 2 = 11$ nodes. The next possibility is the $E_6 \otimes SU(3) \subset E_8$ group, here we would be gauging the $SU(3)$ flavor symmetry subgroup, remaining with a E_6 flavor group. The theory would have $10 - 2 + 4 = 12$ nodes. And we have a theory with 12 nodes and E_6 flavor symmetry: It is precisely the $D_2(E_6)$ system!!

Let us use our results about the central charges of the $D_2(E_6)$ theory to check if this prediction makes sense. Looking at table 3.5 one obtains that $\delta(2, E_6) = 0$ and therefore $f(2, E_6) = 0$. The rank of the quiver exchange matrix is simply

$$\operatorname{rank} B = 2 \cdot 6 - 6 = 6 \tag{3.239}$$

Moreover $c_{\text{eff}}(D_2(E_6)) = 6 \cdot 12 = 72$, therefore

$$c(2, E_6) = \frac{1}{12}\big(72 + 6\big) = \frac{39}{6}. \tag{3.240}$$

Now, by additivity

$$c(E_8 \text{ MN}) = c(2, E_6) - \frac{1}{6}\dim SU(3) = \frac{39}{6} - \frac{8}{6} = \frac{31}{6} \tag{3.241}$$

which is the correct result!! Let us check that also a is correct. We have

$$\begin{aligned} u(2; E_6) &= \frac{1}{4}\Big([1-6]_+ + [4-6]_+ + [5-6]_+ + [7-6]_+ + [8-6]_+ + [11-6]_+\Big) = \\ &= \frac{1}{4}(1+2+5) = 2 \end{aligned} \tag{3.242}$$

Then

$$a(2; E_6) = 2 + \frac{31}{12} + \frac{1}{16} \cdot 6 = \frac{45}{8} \tag{3.243}$$

so,

$$a(E_8 \text{ MN}) = \frac{45}{8} - \frac{5}{24} \dim SU(3) = \frac{95}{24} \tag{3.244}$$

which is again the correct result !!!

Based on these very strong evidences, we may try to find a representative of the mutation class of the BPS-quiver of $D_2(E_6)$ such that this result is manifest: We draw in figure 3.5 such representative. This concludes our proof of the identification. The explicit mutation sequence is given in appendix B.3.

Chapter 4

H_1, H_2, D_4, E_6, E_7, E_8

4.1. Introduction and interesting numerology

Consider the seven rank 1 4d $\mathcal{N}=2$ SCFT's which may be engineered in F–theory using the Kodaira singular fibers [106–114]

$$H_0,\ H_1,\ H_2,\ D_4,\ E_6,\ E_7,\ E_8. \tag{4.1}$$

H_0 has trivial global symmetry and will be neglected in the following. The other six theories have flavor group F equal, respectively, to

$$SU(2),\ SU(3),\ SO(8),\ E_6,\ E_7 \text{ and } E_8. \tag{4.2}$$

We note that (4.2) is precisely the list of *all* simply–laced simple Lie groups F with the property

$$h(F) = 6\,\frac{r(F)+2}{10-r(F)}, \tag{4.3}$$

where $r(F)$ and $h(G)$ are, respectively, the rank and Coxeter number of F. Physically, the relation (4.3) is needed for consistency with the $2d/4d$ correspondence of [13], and is an example of the restrictions on the flavor group F of a 4d $\mathcal{N}=2$ SCFT following from that principle.

Neglecting H_0, let us list the numbers $2\,h(F)$ for the other six models

$$4,\ 6,\ 12,\ 24,\ 36,\ 60. \tag{4.4}$$

The first four numbers in this list have appeared before in the non–perturbative analysis of the corresponding SCFT's: it is known [8,22] that the (mass–deformed) SCFT's H_1, H_2, D_4 and E_6 have a finite BPS chamber in which the BPS spectrum consists precisely of (respectively) 4, 6, 12 and 24 hypermultiplets. The H_1 SCFT is the $D_3(SU(2))$ model of [21, 22], while the H_2, D_4 and E_6 SCFT's coincide, respectively, with the models $D_2(SU(3))$, $D_2(SU(4))$, and $D_2(SO(8))$ of those papers; then the above statement is a special instance of the general fact that, for all simply–laced Lie groups $G = ADE$, the $D_2(G)$ SCFT has a finite chamber with

$r(G)\,h(G)$ hypermultiplets [22], while, for all $p \in \mathbb{N}$, the model $D_p(SU(2))$ has a special BPS chamber with $2(p-1)$ hypermultiplets[1].

For the four SCFT's H_1, H_2, D_4, E_6, the number of hypermultiplets in the above preferred chamber, n_h, may be written in a number of intriguing ways: we list just a few

$$n_h = 2\,h(F) = \frac{12\,r(F)+24}{10-r(F)} = 12(\Delta-1) = n_7\,\Delta, \tag{4.5}$$

where Δ is the dimension of the field parametrizing the Coulomb branch of the rank 1 SCFT, and n_7 is the number of parallel 7–branes needed to engineer the SCFT in F–theory [106–114]; see Table 4.1.

The special finite BPS chambers with $n_h = 2\,h(F)$ hypers have the particular property of *saturating* the conformal central charge c of the strongly–coupled SCFT. By this we mean that, for these theories, the exact c is equal to the value for n_h *free* hypermultiplets plus the contribution from the massless photon vector multiplet

$$c = \frac{1}{12}\,n_h + \frac{1}{6}, \tag{4.6}$$

that is, c has the same value as the system of free fields with the same particle content as the BPS spectrum in the *special* chamber. In fact, the c–saturating property holds in general for the standard BPS chamber of all $D_2(G)$ SCFT's [22], and also for all $D_p(SU(2))$. It was conjectured by Xie and Zhao [104] that a finite BPS chamber with this property exists for a large class of $\mathcal{N}=2$ models (their examples are close relatives of the present ones). At the level of numerology, for the four SCFT's H_1, H_2, D_4, E_6 we also have a simple relation between the number of hypers in our special chamber, n_h, and the a, k_F conformal central charges: in facts, for all the above SCFT's the central charge a is given by the photon contribution, 5/24, plus *three–halves* the contribution of n_h free hypers

$$a = \frac{1}{24}\,\frac{3\,n_h}{2} + \frac{5}{24} \tag{4.7}$$

$$k_F = \frac{n_h+12}{6}. \tag{4.8}$$

In view of all this impressive numerology involving n_h, it is tempting to **conjecture** that the last two SCFT's in the sequence (4.1), E_7 and E_8, also have

[1] Note that H_1 is the Argyres–Douglas (AD) model of type A_3 [17] which has BPS chambers with any number n_h of BPS hypers in the range $3 \leq n_h \leq 6$; likewise H_2 is the Argyres–Douglas model of type D_4. In both cases it is neither the AD minimal (3 resp. 4 hypers) nor the AD maximal (6 resp. 12 hypers) BPS chamber which is singled out by the property of being c–saturating, but rather their canonical chamber as a $D_p(SU(2))$ resp. a $D_2(G)$ theory [22] (for D_4 AD these two chambers are equivalent).

SCFT	H_1	H_2	D_4	E_6	E_7	E_8
Δ	4/3	3/2	2	3	4	6
n_7	3	4	6	8	9	10
c	1/2	2/3	7/6	13/6	19/6	31/6
a	11/24	7/12	23/24	41/24	59/24	95/24
k_F	8/3	3	4	6	8	12

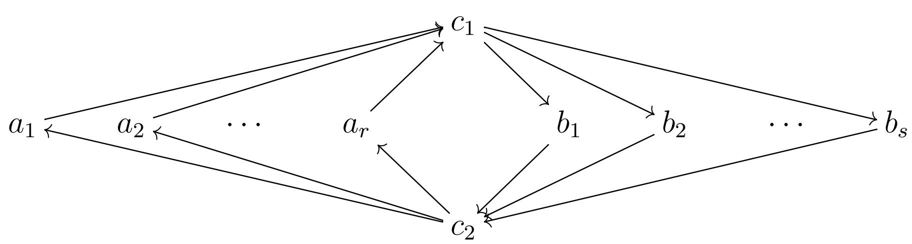
Table 4.1: Numerical invariants for the six SCFT's H_1, H_2, D_4, E_6, E_7 and E_8. The rank of the flavor group, $r(F)$, is equal to the index in the SCFT symbol.

canonical finite BPS chambers with, respectively, 36 and 60 hypermultiplets. This will extend our observations, eqn.(4.5)–(4.8), to the *full* SCFT sequence (4.1), suggesting that the numerology encodes deep physical properties of rank 1 SCFTs.

The purpose of this chapter is to *prove* the above **conjecture**, by constructing explicitly the canonical chambers with $2\,h(F)$ hypers. To get the result we use the BPS quivers for the E_7 and E_8 Minahan–Nemeshanski theories identified in §.3.7.3, together with the mutation algorithm of [8].

4.2. Computing the BPS spectra

4.2.1. The quivers $Q(r,s)$ We begin by fixing uniform and convenient representatives of the quiver mutation–classes for the six $\mathcal{N}=2$ models in eqn.(4.1) with $F \neq 1$. We define $Q(r,s)$ to be the quiver with $(r+s+2)$ nodes



(4.9)

Then the (representative) quivers for our six SCFT's are

SCFT	H_1	H_2	D_4	E_6	E_7	E_8
quiver	$Q(0,1)$	$Q(1,1)$	$Q(2,2)$	$Q(3,3)$	$Q(3,4)$	$Q(3,5)$

(4.10)

(cfr. ref. [7] for H_1, H_2 and D_4, ref. [8] for E_6, and ref. [22] for E_7 and E_8).

The simplest way to get the table (4.10) is by implementing the flavor groups F in eqn.(4.2) directly on the quiver. Indeed, given a $Q(r,s)$ quiver the flavor group F of the corresponding $\mathcal{N}=2$ QFT is canonically identified by the property that its Dynkin graph is the star with three branches of lengths[2] $[r,s,2]$.

4.2.2. The c–saturating chamber for H_1, H_2, D_4 and E_6 The first four quivers in (4.10) may be decomposed into Dynkin subquivers in the sense of [14]

$$\begin{aligned} Q(1,0) &= A_2 \amalg A_1, \qquad & Q(1,1) &= A_2 \amalg A_2,\\ Q(2,2) &= A_3 \amalg A_3, \qquad & Q(3,3) &= D_4 \amalg D_4. \end{aligned} \tag{4.11}$$

For a quiver $G \amalg G'$ the charge lattice is $\Gamma = \Gamma_G \oplus \Gamma_{G'}$, where Γ_G is the root lattice of the Lie algebra G. Since the decomposition has the Coxeter property [14,22], there is a canonical chamber in which the BPS spectrum consists of one hypermultiplet per each of the following charge vectors [14]

$$\Big\{\alpha \oplus 0 \in \Gamma_G \oplus \Gamma_{G'},\ \alpha \in \Delta^+(G)\Big\} \bigcup \Big\{0 \oplus \beta \in \Gamma_G \oplus \Gamma_{G'},\ \beta \in \Delta^+(G')\Big\}, \tag{4.12}$$

where $\Delta^+(G)$ is the set of the positive roots of G. Then the number of hypermultiplets in this canonical finite chamber is

$$n_h = \frac{1}{2}\big(r(G)\,h(G) + r(G')\,h(G')\big), \tag{4.13}$$

which for the four cases in eqn.(4.11) gives (respectively)

$$4,\ 6,\ 12,\ 24, \tag{4.14}$$

i.e. $n_h = 2\,h(F)$ as expected for a c–saturating chamber.

For sake of comparison with the E_7, E_8 cases in the next subsection, we give more details on the computation of the above spectrum for the E_6 Minahan–Nemeshanski theory [112] using the mutation algorithm. The cases H_1, H_2 are similar and simpler, and we have discussed the D_4 theory in §.1.3.5.

The two D_4 subquivers of $Q(3,3)$ are the full subquivers over the nodes $\{a_1,a_2,a_3,c_1\}$ and, respectively, $\{b_1,b_2,b_3,c_1\}$. The quiver $Q(3,3)$ has an automorphism group $\mathbb{Z}_2 \ltimes (\mathfrak{G}_3 \times \mathfrak{G}_3)$, where the two $\mathfrak{G}_3$ are the triality groups of the D_4 subgraphs,

[2] As always, in the length of each branch we count the node at the origin of the star; in particular, a branch of length one is no branch at all, while a branch of length zero means that we delete the origin of the star itself. Note that, for all s, the quiver $Q(2,s)$ is mutation equivalent to the quiver of $SU(2)$ SQCD with $N_f = s+2$ fundamental flavors which has flavor symmetry group $SO(2s+4)$, whose Dynkin graphs is the star with three branches of lengths $[2,s,2]$.

while $\mathbb{Z}_2$ interchanges the two D_4 subquivers (and hence the two $\mathfrak{S}_3$'s). The quiver embedding $D_4 \oplus D_4 \to Q(3,3)$ induces an embedding of flavor groups

$$SU(3) \times SU(3) \to F \equiv E_6, \tag{4.15}$$

where $SU(3)$ is the flavor group of the Argyres–Douglas theory of type D_4 characterized by the fact that $\mathrm{Weyl}(SU(3)) \equiv \mathfrak{S}_3 \equiv$ the triality group of D_4.

The two D_4 subquiver have the 'subspace' orientation; in both bi–partite quivers $Q(3,3)$ and D_4 we call *even* the nodes a_i and b_i and *odd* the c_i ones. Then, by standard properties of the Weyl group, the quiver mutation 'first all even then all odd'

$$\mu_{c_2}\mu_{c_1} \prod_{i=1}^{3} \mu_{b_i} \prod_{i=1}^{3} \mu_{a_i} \tag{4.16}$$

transforms the quiver $Q(3,3)$ into itself while acting on $\Gamma_{D_4} \oplus \Gamma_{D_4}$ as $\mathrm{Cox} \oplus \mathrm{Cox}$, where $\mathrm{Cox} \in \mathrm{Weyl}(D_4)$ is the Coxeter element of D_4. Since $(\mathrm{Cox})^3 = -1$, the quiver mutation

$$\Big(\mu_{c_2}\mu_{c_1} \prod_{i=1}^{3} \mu_{b_i} \prod_{i=1}^{3} \mu_{a_i}\Big)^3 \tag{4.17}$$

is a solution to eqn.(1.47) with $\pi = \mathrm{Id}$. Since there are 24 μ's in eqn.(4.17), we have found a finite BPS chamber with 24 hypers. Eqn.(4.17) is invariant under the automorphism group $\mathbb{Z}_2 \ltimes \big(\mathrm{Weyl}(SU(3)) \times \mathrm{Weyl}(SU(3))\big)$ so that there are points in the parameter domain $\mathcal{D}_{\mathrm{fin}}$ corresponding to the above chamber which preserve a flavor group

$$F_{\mathrm{fin}} \supseteq \mathbb{Z}_2 \ltimes \Big(SU(3) \times SU(3) \times U(1)^2\Big), \tag{4.18}$$

where $\mathbb{Z}_2$ acts by interchanging the two $SU(3)$'s and inverting the sign of the first $U(1)$ charge. The 24 BPS states may be classified in a collection of irrepresentations of the group in the large parenthesis of eqn.(4.18) which form $\mathbb{Z}_2$ orbits. From eqn.(4.17) we read the phase ordering of the *particles* in the 24 BPS hypers (in addition we have, of course, the PCT conjugate *anti*–particles). Ordered in decreasing phase order, we have

$$\begin{gathered} \overbrace{(\mathbf{3},\mathbf{1})_{1,0},\ (\mathbf{1},\mathbf{3})_{-1,0}},\quad \overbrace{(\mathbf{1},\mathbf{1})_{3,1},\ (\mathbf{1},\mathbf{1})_{-3,1}},\quad \overbrace{(\overline{\mathbf{3}},\mathbf{1})_{2,1},\ (\mathbf{1},\overline{\mathbf{3}})_{-2,1}},\\ \overbrace{(\mathbf{1},\mathbf{1})_{3,2},\ (\mathbf{1},\mathbf{1})_{-3,2}},\quad \overbrace{(\mathbf{3},\mathbf{1})_{1,1},\ (\mathbf{1},\mathbf{3})_{-1,1}},\quad \overbrace{(\mathbf{1},\mathbf{1})_{0,1},\ (\mathbf{1},\mathbf{1})_{0,1}}, \end{gathered} \tag{4.19}$$

where overbraces collect representations forming a $\mathbb{Z}_2$–orbit. In terms of dimension vectors of the corresponding quiver representations, the quantum numbers of the

24 BPS particles (in decreasing phase order) is

$$\begin{gathered} a_1,\ a_2,\ a_3;\ \ b_1,\ b_2,\ b_3;\ \ a_1+a_2+a_3+c_1;\ \ b_1+b_2+b_3+c_2; \\ a_2+a_3+c_1,\ a_1+a_3+c_1,\ a_1+a_2+c_1;\ \ b_2+b_3+c_2,\ b_1+b_3+c_2,\ b_1+b_2+c_2; \\ a_1+a_2+a_3+2\,c_1;\ \ b_1+b_2+b_3+2\,c_2; \\ a_1+c_1,\ a_2+c_1,\ a_3+c_1;\ \ b_1+c_2,\ b_2+c_2,\ b_3+c_2;\ \ c_1;\ \ c_2, \end{gathered} \tag{4.20}$$

where, for notational convenience, the positive cone generators $e_{a_i}, e_{b_j}, e_{c_k}$ are written simply as a_i, b_j, c_k, respectively.

4.2.3. The 36–hyper BPS chamber of E_7 MN The quiver $Q(3,4)$ has no obvious *useful* decomposition into Dynkin subquivers. However, with the help of Keller's quiver mutation applet it is easy to check that the composition of the 36 basic quiver mutations at the sequence of nodes

$$\begin{gathered} a_1\ a_2\ a_3\ b_1\ b_2\ b_3\ c_1\ c_2 \\ a_1\ a_2\ b_4\ b_1\ b_2\ b_3\ c_1\ c_2\ a_1\ a_2\ a_3\ b_2\ c_1\ c_2\ b_4\ b_1\ b_2\ b_3\ c_1\ c_2 \\ a_1\ a_2\ a_3\ b_1\ b_2\ b_3\ c_1\ c_2 \end{gathered} \tag{4.21}$$

is a solution to eqn.(1.47) for $Q(3,4)$ with[3]

$$\pi = (a_1\ a_2)(a_3\ b_1\ b_4)(b_2\ b_3)(c_1\ c_2). \tag{4.22}$$

Moreover no proper subsequence of mutations is a solution to eqn.(1.47). Note the similarity with the sequence for E_6 which is a three fold repetition of the first line of (4.21) (the Coxeter sequence of $D_4 \amalg D_4$). Passing from E_6 to E_7 we simply replace the second repetition of the Coxeter sequence for $D_4 \amalg D_4$ with the second line of (4.21) which may also be interpreted as a chain of Coxeter sequences (see remark after eqn.(4.25)).

The solution (4.21) corresponds to the finite BPS chamber for the E_7 Minahan Nemeshanski theory [113] with 36 hypermultiplets we were looking for. The (manifest) automorphism of this finite chamber is given by the centralizer of π in the $Q(3,4)$ automorphism group $\mathfrak{S}_3 \times \mathfrak{S}_4$, which is the subgroup $\mathfrak{G}_2 \times \mathfrak{G}_2$ generated by the involutions $(a_1\ a_2)$ and $(b_2\ b_3)$. Then the BPS hypers in this finite chamber form representations of $F_{\text{fin}} = SU(2) \times SU(2) \times U(1)^5$. From the list of charge vectors of the 36 hypers in Table 4.2 we see that this is indeed true.

We stress that the 36–hyper chamber above *is far from being unique*; a part for the other $m = 36$ solutions to eqn.(1.47) obtained from (4.21) by applying

[3] The fact that π is *not* an involution implies that this mutation cannot arise from Coxeter–factorized subquivers as in the previous examples.

$a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ b_3,\ a_1+a_2+a_3+c_1,\ b_1+b_2+b_3+c_2,\ a_2+a_3+c_1,\ a_1+a_3+c_1,$
$a_1+a_2+a_3+b_4+c_1,\ b_2+b_3+c_2,\ b_1+b_3+c_2,\ b_1+b_2+c_2,\ a_1+a_2+2\,a_3+b_4+2\,c_1,$
$b_1+b_2+b_3+2\,c_2,\ a_1+a_3+b_4+c_1,\ a_2+a_3+b_4+c_1,\ a_1+a_2+b_1+b_2+b_3+c_1+2\,c_2,$
$b_2+c_2,\ b_4,\ a_1+a_2+b_2+c_1+c_2,\ a_1+a_2+a_3+b_2+2\,c_1+c_2,\ b_1+b_4+c_2,\ a_1+a_2+c_1,$
$b_3+b_4+c_2,\ b_1+b_3+b_4+2\,c_2\ ,\ a_1+a_2+a_3+2\,c_1\ ,\ a_2+c_1\ ,\ a_1+c_1,\ b_4+c_2,\ b_3+c_2,$
$a_3+c_1,\ b_1+c_2,\ c_2,\ c_1$

Table 4.2: The charge vectors of the 36 BPS particles in the chamber $\mathcal{C}_{\text{fin}}$ of the E_7 MN theory. To simplify the notation, the positive cone generators $e_{a_i}, e_{b_j}, e_{c_k}$ are written simply as a_i, b_j, c_k, respectively. The particles are listed in decreasing BPS phase order. To get the full BPS spectrum, add the PCT conjugate *anti*–particles.

an automorphism of the quiver $Q(3,4)$, there are other ones; for instance, the sequence of 36 mutations at the nodes

$$\begin{gathered} c_1\ c_2\ a_1\ a_2\ b_1\ b_2\ c_2\ c_1\ a_3\ b_3\ b_4\ b_1\ c_2\ c_1\ b_4\ b_3\ a_1\ a_2 \\ b_1\ b_2\ c_2\ c_1\ a_2\ a_1\ b_3\ b_4\ a_3\ b_1\ c_2\ c_1\ a_3\ b_4\ b_2\ a_1\ a_2\ b_3. \end{gathered} \tag{4.23}$$

is a solution to (1.47) with $\pi = (a_1\ a_2)(a_3\ b_1\ b_2)(b_3\ b_4)(c_1\ c_2)$. The properties of all these chambers look very similar, in particular they are expected to have isomorphic F_{fin}.

4.2.4. The 60–hyper BPS chamber of E_8 MN For the quiver $Q(3,5)$ one checks that the composition of the 60 basic quiver mutations at the sequence of nodes

$$\begin{gathered} a_1\ a_2\ a_3\ b_1\ b_2\ b_3\ c_1\ c_2\ \|\ a_1\ a_2\ b_4\ b_1\ b_2\ b_3\ c_1\ c_2\ \| \\ a_1\ a_2\ a_3\ b_5\ b_1\ b_4\ c_1\ c_2\ \|\ b_1\ b_2\ b_3\ a_3\ c_1\ c_2\ \| \\ a_1\ a_2\ b_2\ b_3\ b_4\ b_5\ c_1\ c_2\ \|\ a_1\ a_2\ a_3\ b_2\ b_3\ b_4\ c_1\ c_2\ \| \\ a_3\ b_1\ b_2\ b_3\ b_4\ b_5\ c_1\ c_2\ \|\ a_1\ a_2\ b_4\ b_5\ c_1\ c_2 \end{gathered} \tag{4.24}$$

is a solution to eqn.(1.47) with

$$\pi = (a_1\ a_2)(a_3\ b_1)(b_2\ b_3)(b_4\ b_5)(c_1)(c_2), \tag{4.25}$$

while no proper subsequence solves it. In eqn.(4.24) the sequence of mutation is divided into pieces by the dividing symbol $\|$; again each piece may be seen as a Coxeter sequence for a suitable $G \amalg G$ (sub)quiver with G either A_3 or D_4.

Thus we have constructed a 60–hyper c–saturating chamber for the E_8 MN theory [113]. The manifest unbroken flavor symmetry in this finite chamber is $SU(2)\times SU(2)\times U(1)^6$ whose Weyl group is realized as permutations of the charge vector sets $\{e_{a_1}, e_{a_2}\}$ and $\{e_{b_2}, e_{b_3}\}$.

$$a_1,\quad a_2,\quad a_3,\quad b_1,\quad b_2,\quad b_3,\quad a_1+a_2+a_3+c_1,\quad b_1+b_2+b_3+c_2,\quad a_2+a_3+c_1,\quad a_1+a_3+c_1,$$
$$a_1+a_2+a_3+b_4+c_1,\quad b_2+b_3+c_2,\quad b_1+b_3+c_2,\quad b_1+b_2+c_2,\quad a_1+a_2+2\,a_3+b_4+2\,c_1,$$
$$b_1+b_2+b_3+2\,c_2,\quad a_1+a_3+b_4+c_1,\quad a_2+a_3+b_4+c_1,\quad a_1+a_2+b_1+b_2+b_3+c_1+2\,c_2,$$
$$a_1+a_2+a_3+b_1+b_2+b_3+b_5+c_1+2\,c_2,\quad b_1+c_2\ ,\ a_3+c_1,\quad a_3+b_4+c_1,$$
$$2\,a_1+2\,a_2+a_3+2\,b_1+b_2+b_3+b_5+2\,c_1+3\,c_2,$$
$$2\,a_1+2\,a_2+a_3+b_1+b_2+b_3+b_5+2\,c_1+2\,c_2,\quad a_3+b_2+b_4+c_1+c_2,$$
$$a_3+b_3+b_4+c_1+c_2,\quad a_1+a_2+a_3+b_1+b_5+c_1+c_2,\quad a_3+b_2+b_3+b_4+c_1+2\,c_2,$$
$$a_1+a_2+a_3+b_5+c_1,\quad a_2+a_3+b_5+c_1,\quad a_1+a_3+b_5+c_1,\quad b_3+c_2,\quad b_2+c_2,$$
$$a_1+a_2+a_3+b_4+b_5+c_1,\quad a_1+a_2+a_3+b_1+b_2+b_3+b_4+2\,c_1+3\,c_2,$$
$$a_1+a_2+b_1+b_2+b_3+c_1+3\,c_2,\quad a_1+a_2+2\,a_3+b_4+2\,b_5+2\,c_1,$$
$$a_1+a_3+b_4+b_5+c_1,\quad a_2+a_3+b_4+b_5+c_1,\quad a_1+a_2+b_2+b_3+c_1+2\,c_2,$$
$$a_1+a_2+b_1+b_2+c_1+2\,c_2,\quad a_1+a_2+b_1+b_3+c_1+2\,c_2,\quad a_3+b_5+c_1,$$
$$2\,a_1+2\,a_2+b_1+b_2+b_3+2\,c_1+3\,c_2,\quad a_3+b_4+b_5+c_1,\quad a_1+a_2+b_1+c_1+c_2,$$
$$a_3+b_4+b_5+c_1+c_2,\quad a_1+a_2+b_3+c_1+c_2,\quad a_1+a_2+b_2+c_1+c_2,$$
$$b_4,\quad b_5,\quad a_1+a_2+c_1,\quad b_4+b_5+c_2,\quad a_2+c_1,\quad a_1+c_1,\quad b_5+c_2,\quad b_4+c_2$$

Table 4.3: Charge vectors of the 60 BPS particles in the chamber $\mathcal{C}_{\text{fin}}$ of the E_8 MN theory (in decreasing phase order). One has to add the PCT conjugate *anti*–particles.

Again the 60–hyper chamber is *not* unique; for instance, another 60–mutation solution is given by the node sequence

$$\begin{gathered} c_2\ c_1\ b_1\ b_4\ b_2\ a_1\ a_3\ a_2\ c_1\ c_2\ b_5\ b_3\ b_2\ b_4\ c_2\ c_1\ a_1\ a_3\ a_2\ b_1\ b_5\ b_3\ c_1\ c_2\ a_1\ a_3\ a_2\ b_2\ b_5\ b_3\\ c_2\ c_1\ b_2\ b_5\ b_3\ b_4\ b_1\ a_1\ c_1\ c_2\ a_3\ a_2\ a_1\ b_1\ c_2\ c_1\ a_3\ a_2\ b_2\ b_5\ b_4\ b_3\ c_1\ c_2\ a_3\ a_2\ b_1\ b_2\ b_5\ b_3. \end{gathered} \tag{4.26}$$

4.2.5. Decoupling and other finite chambers Sending the mass parameter dual to a charge e_{b_j} to infinity, $|Z(e_{b_j})|\to+\infty$, all BPS states with non–zero e_{b_j}–charge get infinitely massive and decouple. The surviving states correspond to representations X of the $Q(r,s)$ quiver with $\dim X_{b_j}=0$, which are stable representations of the quiver $Q(r,s-1)$.

Assuming the decoupling limit may be taken while remaining in the domain $\mathcal{D}_{\text{fin}}$, consistency requires that if we delete from the list of states in Table 4.3 all those which contain the given b_j $(j=1,2,\ldots,5)$ with non–zero coefficient, what remains should be the BPS spectrum of the E_7 Minahan–Nemeshanski model in some (not necessarily canonical) finite chamber. In the same vein, a similar truncation of the list in Table 4.2 should produce a finite BPS spectrum of E_6 MN. The fact the all the BPS spectra so obtained are related by the Wall Crossing Formula to the canonical chamber determined above is a highly non–trivial check on the procedure.

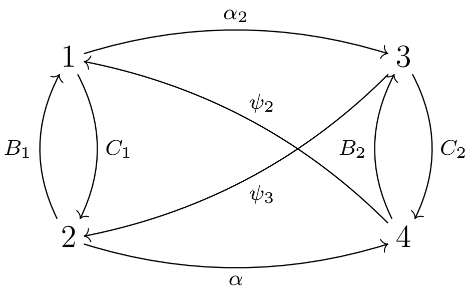
This decoupling procedure applied to E_7 produces BPS chambers with 27 hypermultiplets, which are easily shown to be equivalent to the canonical 24–hypers one. In the E_8 case we get a chamber of E_7 with either 42 or 43 hypers, depending of which e_{b_j} charge we make infinitely heavy.

Appendix A

More details about $\widehat{H} \boxtimes G$ models.

A.1. Detailed study of the light category of $\widehat{A}(2,1) \boxtimes A_2$

The quiver and superpotential for this model are presented in figure **??**. If we are interested in the subcategory $\mathcal{L}$, by eqn.(3.57) we can take A_1, A_2 to be isomorphisms and identify nodes pairwise trough them. Then the fields ψ_1 and $\alpha_1 - \alpha_2$ get massive and may be integrated away. We remain with the quiver and superpotential



(A.1)

$$\mathcal{W}_{\text{eff}} = \psi_2(\alpha C_1 - C_2\alpha_2) + \psi_3(\alpha_2 B_1 - B_2\alpha).$$

The following map is an element of $\operatorname{End} X$

$$(X_1, X_2, X_3, X_4) \mapsto (B_1C_1X_1, C_1B_1X_2, B_2C_2X_3, C_2B_2X_4) \tag{A.2}$$

hence a complex number λ if X is a brick. For $\lambda \neq 0$, B_1, B_2, C_1, C_2 are isomorphisms, which identify the nodes in pairs. The arrows $\alpha - \alpha_2$ and $\psi_2 - \psi_3$ also get massive and may be integrated away, reducing to representations of the preprojective algebra $\mathcal{P}(A_2)$, *i.e.* to the homogeneous $SU(3)$–tube [10]. At $\lambda = 0$ we isolate the non–homogeneous $SU(3)$–tube containing the matter. It corresponds to the representations of the quiver (A.1) bounded by the relations

$$B_1C_1 = C_1B_1 = B_2C_2 = C_2B_2 = \psi_2\alpha = \alpha_2\psi_2 = \psi_3\alpha_2 = \alpha\psi_3 = 0 \tag{A.3}$$
$$C_1\psi_2 - \psi_3B_2 = \psi_2C_2 - B_1\psi_3 = \alpha C_1 - C_2\alpha_2 = \alpha_2B_1 - B_2\alpha = 0. \tag{A.4}$$

Theorem. *The bricks X of the quiver* (A.1) *bounded by the relations* (A.3)(A.4) *are isolated (no moduli). They satisfy*

$$\dim X \leq (1,1,1,1) \tag{A.5}$$

with equality only for modules in the projective closure of the families of representations of the gauge vectors. The dimension vectors of bricks coincide with those for $\mathbb{C}\widehat{A}(4,0)/(\partial[4\text{–cycle}])$.

Proof. By virtue of the relations in the first line, eqn.(A.3), our algebra $\mathscr{A}$ is a *string* algebra. In view of the Butler–Ringel theorem [?], the bricks of $\mathscr{A}$ are isolated iff there is no band which is a brick. In any legitimate string, arrows (direct or inverse) labelled by latin and greek letters alternate. We observe that a sequence of three arrows (direct of inverse) of the form (latin)(greek)(latin) is not legitimate unless the greek arrow points in the opposite direction with respect to the latin ones [same with (latin) $\leftrightarrow$ (greek)]. Indeed by (A.4)

$$\xrightarrow{C_1}\xrightarrow{\alpha}\xrightarrow{B_2} = \xrightarrow{C_1}\xrightarrow{B_1}\xrightarrow{\alpha_2} \qquad \xrightarrow{C_1}\xrightarrow{\alpha}\xleftarrow{C_2} = \xrightarrow{\alpha_2}\xrightarrow{C_2}\xleftarrow{C_2} \qquad \xrightarrow{C_1}\xleftarrow{\psi_3}\xleftarrow{B_2} = \xrightarrow{C_1}\xleftarrow{C_1}\xleftarrow{\psi_2}$$

and the RHS are illegitimate strings. Thus, for all indecomposables of total dimension $\sum_i \dim X_i \geq 4$, the arrows in the string/band should alternate both in alphabets (latin vs. greek) and orientation (direct vs. inverse). Then, given an arrow in the string, the full sequence of its successors is uniquely determined. There are no bands with $\dim X_1 = 0$; if $\dim X_1 \neq 0$ we may cyclically rearrange the band in such a way that the first node is 1 and the first arrow is latin. If it is C_1, the unique continuation of the string is

$$1 \xrightarrow{C_1} 2 \xleftarrow{\psi_3} 3 \xrightarrow{C_2} 4 \xleftarrow{\alpha} 2 \xrightarrow{B_1} 1, \tag{A.6}$$

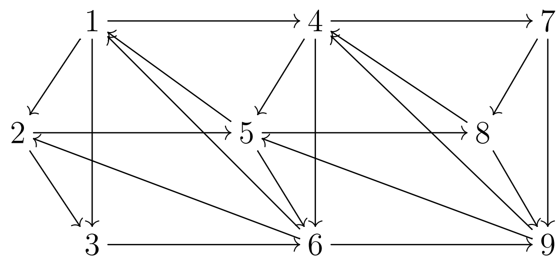
while, if the first arrow is B_1, it is this string segment read from the right. We cannot close (A.6) to make a band since $C_1B_1 = 0$. The string/band may be continued (either ways)

$$\cdots \xleftarrow{\alpha_2} 1 \xrightarrow{C_1} 2 \xleftarrow{\psi_3} 3 \xrightarrow{C_2} 4 \xleftarrow{\alpha} 2 \xrightarrow{B_1} 1 \xleftarrow{\psi_2} 4 \xrightarrow{B_2} 3 \xleftarrow{\alpha_2} 1 \xrightarrow{C_1} \cdots, \tag{A.7}$$

and this structure repeats periodically; all legitimate strings are substrings of a k–fold iteration of the period. Let v_i be the basis elements of X_1 numbered according to their order along the string; from (A.7) we see that $v_1 \mapsto v_1 + v_2$, $v_i \mapsto v_i$ for $i \geq 2$, is a non–trivial endomorphism, so the corresponding string/band module X is not a brick. X may be a brick only if $\dim X_1 \leq 1$; the nodes being all equivalent, $\dim X_i \leq 1$ for all i. Now it is elementary to show that the matter category has a quiver and superpotential equal to those of D_4 [10]. □

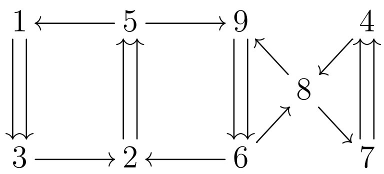
A.2. Mutation checks of the identification §.3.5.

The quiver of $A(2,1) \boxtimes A_3$ is:



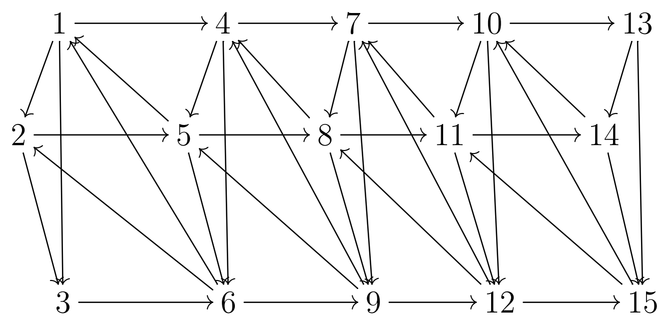
(A.8)

Mutating at the nodes 7 4 8 2 5 9 4 6 9 6 7 6 4 8 we obtain:



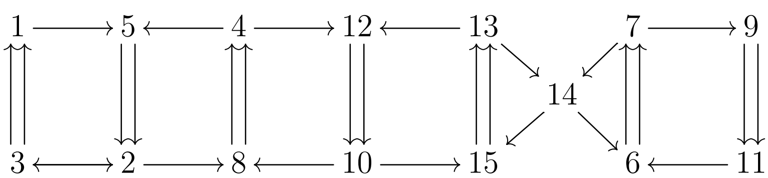
(A.9)

The quiver of $A(2,1) \boxtimes A_5$ is



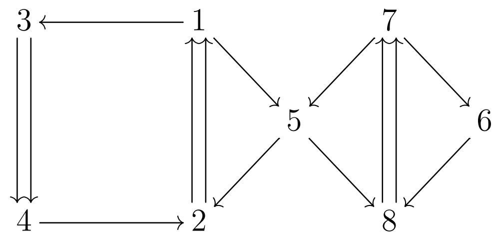
(A.10)

Mutation at 13 10 14 7 11 15 4 8 12 13 2 5 6 4 9 7 9 6 8 7 9 12 7 5 3 10 12 9 10 6 12 10 7 12 7 11 7 12 6 9 13 15 14 11 13 12 8 5 3 15 12 8 5 3 15 11 9 14 6 14 9 6 14 6 14 9 7 14 11 gives:



(A.11)

The quiver of $SU(3) \times SU(2)$ coupled to $(\mathbf{3},\mathbf{2}) \oplus (\mathbf{1},\mathbf{2})$ is



(A.12)

by the sequence of mutations 5 8 3 2 4 8 7 1 6 8 4 5 2 it becomes the quiver $A(3,1) \boxtimes A_2$.

A.3. The BPS spectrum of the $\widehat{H} \boxtimes G$ models

The aim of this appendix is to give a proof of the

Main Claim. All $\widehat{H} \boxtimes G$ models have a finite chamber $\mathscr{C}_{\text{fin}}$ at strong G coupling consisting only of hypers. Moreover, in such a chamber, the cone of particles of the theory has the structure

$$\mathscr{C}_{\text{fin}} \cap \Gamma^+ = \bigoplus_{i=1}^{\text{rk}(\widehat{H})} \Delta^+(G), \tag{A.13}$$

and therefore it consists of $\frac{1}{2}(\text{rk}(\widehat{H}) \times \text{rk}(G) \times h(G))$ hypermultiplets.

First we need to fix the notation. We collect in figure A.1 our conventions about the labelings of the nodes and orientations of the affine quivers.

Remark: Let $a \searrow$ denote the sequence of nodes of $\widehat{H}$

$$a \searrow \equiv \big\{1\ ,\ 2\ ,\ \dots\ ,\ \text{rk}(\hat{H}) - 2\ , \text{rk}(\hat{H}) - 1\ ,\ \text{rk}(\hat{H})\big\}.$$

The corresponding sequence of mutations

$$\mu_{a\searrow}(\hat{H}) \equiv \prod_{a\searrow} \mu_a = \mu_1 \circ \mu_2 \circ \cdots \circ \mu_{\text{rk}(\hat{H})-2} \circ \mu_{\text{rk}(\hat{H})-1} \circ \mu_{\text{rk}(\hat{H})} \tag{A.19}$$

is a sequence of mutations on *sinks* that satisfies properties $i), ii), iii)$. If we interpret them as right mutations, we obtain the minimal BPS chamber of the models associated to the affine quivers.

Label the nodes of the $\widehat{H} \boxtimes G$ quiver, as

$$(i,a), \qquad i = 1,...,\text{rk}(\widehat{H}), \quad a = 1,...,\text{rk}(G). \tag{A.20}$$

$$A(p,q):\quad \begin{array}{ccccc} & \nearrow & 2 \longrightarrow 3 \longrightarrow \dots \longrightarrow p & \searrow & \\ 1 & & & & p+q \\ & \searrow & p+1 \to \dots \to p+q-2 \to p+q-1 & \nearrow & \end{array} \tag{A.14}$$

$$\widehat{D}_p:\quad \begin{array}{ccccc} 1 & & & & p \\ & \searrow & & \nearrow & \\ & & 3 \longrightarrow 4 \longrightarrow \dots \longrightarrow p-2 \longrightarrow p-1 & & \\ & \nearrow & & \searrow & \\ 2 & & & & p+1 \end{array} \tag{A.15}$$

$$\widehat{E}_6:\quad \begin{array}{ccccc} & & 6 & & \\ & & \uparrow & & \\ & & 5 & & \\ & & \uparrow & & \\ 0 \longrightarrow & 1 \longrightarrow & 2 & \longrightarrow 3 & \longrightarrow 4 \end{array} \tag{A.16}$$

$$\widehat{E}_7:\quad \begin{array}{ccccccc} & & & 7 & & & \\ & & & \uparrow & & & \\ 0 \longrightarrow & 1 \longrightarrow & 2 \longrightarrow & 3 & \longrightarrow 4 & \longrightarrow 5 & \longrightarrow 6 \end{array} \tag{A.17}$$

$$\widehat{E}_8:\quad \begin{array}{cccccccc} & & & & & 8 & & \\ & & & & & \uparrow & & \\ 0 \longrightarrow & 1 \longrightarrow & 2 \longrightarrow & 3 \longrightarrow & 4 \longrightarrow & 5 & \longrightarrow 6 & \longrightarrow 7 \end{array} \tag{A.18}$$

Figure A.1: Our conventions on the nodes of the affine quivers.

Let

$$q_k \equiv \{(k,a) \ \mid a = 1, ..., \mathrm{rk}(G)\}, \qquad v_j \equiv \{(i,j) \ \mid i = 1, ..., \mathrm{rk}(\widehat{H})\}. \tag{A.21}$$

We have

$$(\widehat{H} \boxtimes G)\big|_{q_k} = G \quad \text{and} \quad (\widehat{H} \boxtimes G)\big|_{v_j} = \widehat{H}. \tag{A.22}$$

Lemma 1. *Let $\overleftrightarrow{G}$ be the alternating orientation of the Dynkin quiver of type G such that the first node is a source, all odd nodes are sources, and all even nodes are sinks — see figure A.2. All quivers $\widehat{H} \boxtimes G$ are mutation equivalent to $\widehat{H} \boxtimes \overleftrightarrow{G}$.*

Proof. The proof is organized as follows: First we are going to consider the quivers of type A_n, then the quivers of type D_n and finally the exceptionals.

Let $G = A_n$. The orientation for the A_n quivers that we have used elsewhere is simply $(i) \to (i+1)$: When we will speak about the "A_n quiver" or simply "A_n" we will always mean the A_n quiver with this orientation. Let $\mathbf{m}_n$ denote the following sequence of elementary mutations of the A_n quiver:

$$\mathbf{m}_k \equiv \mu_k \circ \mu_{k-1} \circ \cdots \circ \mu_2 \circ \mu_1. \tag{A.24}$$

Notice that $\mathbf{m}_n = \mathrm{id}_{A_n}$. The sequences of mutations

$$\begin{aligned} \mathbf{m}_2 \circ \mathbf{m}_4 \circ \cdots \circ \mathbf{m}_{2(k-2)} \circ \mathbf{m}_{2(k-1)} \circ \mathbf{m}_{2k} \qquad & \text{for } n = 2k+1 \\ \mathbf{m}_2 \circ \mathbf{m}_4 \circ \cdots \circ \mathbf{m}_{2(k-2)} \circ \mathbf{m}_{2(k-1)} \qquad & \text{for } n = 2k \end{aligned} \tag{A.25}$$

are sequences of mutations mapping A_n to $\overleftrightarrow{A}_n$ involving only mutations on sources. By construction of the $\boxtimes$ operation on quivers, the sequence of mutations

$$\mu_{a\searrow}(1) \equiv \prod_{(a,1)\searrow} \mu_{(a,1)} = \mu_{(1,1)} \circ \mu_{(2,1)} \circ \cdots \circ \mu_{(\mathrm{rk}(\hat{H})-2,1)} \circ \mu_{(\mathrm{rk}(\hat{H})-1,1)} \circ \mu_{(\mathrm{rk}(\hat{H}),1)} \tag{A.26}$$

is a mutation on sources for the dynkin subquivers of type A_n, because by definition each $(i,1)$ is a source of the Dynkin on the nodes q_i, and a mutation on sinks for the affine subquiver of type $\widehat{H}$ on the nodes v_1. Analogously, one can define

$$\mu_{a\searrow}(i) \equiv \prod_{(a,i)\searrow} \mu_{(a,i)} = \mu_{(1,i)} \circ \mu_{(2,i)} \circ \cdots \circ \mu_{(\mathrm{rk}(\hat{H})-2,i)} \circ \mu_{(\mathrm{rk}(\hat{H})-1,i)} \circ \mu_{(\mathrm{rk}(\hat{H}),i)}, \tag{A.27}$$

and

$$\mathbf{m}(\hat{H})_k \equiv \mu_{a\searrow}(k) \circ \mu_{a\searrow}(k-1) \circ \cdots \circ \mu_{a\searrow}(2) \circ \mu_{a\searrow}(1). \tag{A.28}$$

Then, by construction of the $\boxtimes$ operation on quivers, $\mathbf{m}(\hat{H})_k$ involves only mutations on sources with respect to the subquivers on the nodes q_i, and on sinks with respect to subquivers on the nodes v_j. Notice that

$$\mathbf{m}(\hat{H})_n = \mathrm{id}_{\widehat{H}\boxtimes A_n} \tag{A.29}$$

$$\overleftrightarrow{A}_{2k+1}: \quad 1 \longrightarrow 2 \longleftarrow 3 \longrightarrow \dots \longleftarrow 2k-1 \longrightarrow 2k \longleftarrow 2k+1$$

$$\overleftrightarrow{A}_{2k}: \quad 1 \longrightarrow 2 \longleftarrow 3 \longrightarrow \dots \longleftarrow 2k-1 \longrightarrow 2k$$

$$\overleftrightarrow{D}_{2k+1}: \quad \begin{array}{ccccccccccc} & & & & & & & & 2k+1 & & \\ & & & & & & & & \uparrow & & \\ 1 & \longrightarrow & 2 & \longleftarrow & 3 & \longrightarrow & \dots & \longleftarrow & 2k-1 & \longrightarrow & 2k \end{array}$$

$$\overleftrightarrow{D}_{2k}: \quad \begin{array}{ccccccccccccc} & & & & & & & & & & 2k & & \\ & & & & & & & & & & \downarrow & & \\ 1 & \longrightarrow & 2 & \longleftarrow & 3 & \longrightarrow & \dots & \longleftarrow & 2k-3 & \longrightarrow & 2k-2 & \longleftarrow & 2k-1 \end{array}$$

$$\overleftrightarrow{E}_{6}: \quad \begin{array}{ccccccccc} & & & & 6 & & & & \\ & & & & \uparrow & & & & \\ 1 & \longrightarrow & 2 & \longleftarrow & 3 & \longrightarrow & 4 & \longleftarrow & 5 \end{array}$$

$$\overleftrightarrow{E}_{7}: \quad \begin{array}{ccccccccccc} & & & & & & 7 & & & & \\ & & & & & & \downarrow & & & & \\ 1 & \longrightarrow & 2 & \longleftarrow & 3 & \longrightarrow & 4 & \longleftarrow & 5 & \longrightarrow & 6 \end{array}$$

$$\overleftrightarrow{E}_{8}: \quad \begin{array}{ccccccccccccc} & & & & & & & & 8 & & & & \\ & & & & & & & & \uparrow & & & & \\ 1 & \longrightarrow & 2 & \longleftarrow & 3 & \longrightarrow & 4 & \longleftarrow & 5 & \longrightarrow & 6 & \longleftarrow & 7 \end{array} \tag{A.23}$$

Figure A.2: Our conventions about the alternating orientations of the Dynkin subquivers.

Our lemma for the case $G = A_n$ is equivalent to the following

Claim. *The sequences of mutations*

$$\begin{aligned} \mathbf{m}(\hat{H})_2 \circ \mathbf{m}(\hat{H})_4 \circ \cdots \circ \mathbf{m}(\hat{H})_{2(k-2)} \circ \mathbf{m}(\hat{H})_{2(k-1)} \circ \mathbf{m}(\hat{H})_{2k} \qquad & \textit{for } n = 2k+1 \\ \mathbf{m}(\hat{H})_2 \circ \mathbf{m}(\hat{H})_4 \circ \cdots \circ \mathbf{m}(\hat{H})_{2(k-2)} \circ \mathbf{m}(\hat{H})_{2(k-1)} \qquad & \textit{for } n = 2k \end{aligned} \tag{A.30}$$

maps the quiver $\widehat{H} \boxtimes A_n$ *into the quiver* $\widehat{H} \boxtimes \overleftrightarrow{A}_n$.

We will proceed by induction on n. For $n = 1, 2$ there is nothing to prove: Let us now show that $(n-1) \Rightarrow (n)$. The case $n = 2k$ is trivial. One considers the subquiver $\widehat{H} \boxtimes A_{2k-1}$, on the nodes

$$\Big\{(i,a) \mid i = 1, \ldots, \mathrm{rk}(\widehat{H}) \ , \ a = 1, \ldots, 2k-1\Big\}.$$

By inductive hypothesis the sequence of mutations

$$\mathbf{m}(\hat{H})_2 \circ \mathbf{m}(\hat{H})_4 \circ \cdots \circ \mathbf{m}(\hat{H})_{2(k-2)} \circ \mathbf{m}(\hat{H})_{2(k-1)} \tag{A.31}$$

maps this subquiver into $\widehat{H} \boxtimes \overleftrightarrow{A}_{2k-1}$. But then each subquiver on the nodes q_i has the form

$$1 \longrightarrow 2 \longleftarrow 3 \longrightarrow \ldots \longleftarrow 2k-1 \longrightarrow 2k \ . \tag{A.32}$$

And we are done. If $n = 2k+1$, analogously, consider the full $\widehat{H} \boxtimes A_{2k}$ subquiver of $\widehat{H} \boxtimes A_{2k+1}$ on the nodes

$$\Big\{(i,a) \mid i = 1, \ldots, \mathrm{rk}(\widehat{H}) \ , \ a = 1, \ldots, 2k\Big\}.$$

By inductive hypothesis the sequence of mutations

$$\mathbf{m}(\hat{H})_2 \circ \mathbf{m}(\hat{H})_4 \circ \cdots \circ \mathbf{m}(\hat{H})_{2(k-2)} \circ \mathbf{m}(\hat{H})_{2(k-1)} \tag{A.33}$$

maps this subquiver into $\widehat{H} \boxtimes \overleftrightarrow{A}_{2k}$. Therefore each subquiver on the nodes q_i, now, looks like:

$$1 \longrightarrow 2 \longleftarrow 3 \longrightarrow \ldots \longleftarrow 2k-1 \longrightarrow 2k \longrightarrow 2k+1 \tag{A.34}$$

Clearly, if we apply to the mutated quiver the sequence of mutations

$$\begin{aligned} \mu_{a\nearrow}(2k+1) &\equiv \prod_{(a,2k+1)\nearrow} \mu_{(a,2k+1)} \\ &= \mu_{(\mathrm{rk}(\hat{H}),2k+1)} \circ \mu_{(\mathrm{rk}(\hat{H})-1,2k+1)} \circ \mu_{(\mathrm{rk}(\hat{H})-2,2k+1)} \circ \cdots \circ \mu_{(2,2k+1)} \circ \mu_{(1,2k+1)}, \end{aligned} \tag{A.35}$$

we obtain the quiver of $\widehat{H} \boxtimes \overleftrightarrow{A}_{2k+1}$. Notice that, by construction, the sequence $\mu_{a\nearrow}(2k+1)$ is on sources with respect to the v_j subquivers and on sinks with respect to the q_i subquivers. Our claim follows if we are able to show that

$$\begin{aligned}&\mu_{a\nearrow}(2k+1) \circ \mathbf{m}(\hat{H})_2 \circ \mathbf{m}(\hat{H})_4 \circ \cdots \circ \mathbf{m}(\hat{H})_{2(k-2)} \circ \mathbf{m}(\hat{H})_{2(k-1)} \\ &= \mathbf{m}(\hat{H})_2 \circ \mathbf{m}(\hat{H})_4 \circ \cdots \circ \mathbf{m}(\hat{H})_{2(k-2)} \circ \mathbf{m}(\hat{H})_{2(k-1)} \circ \mathbf{m}(\hat{H})_{2k}.\end{aligned} \tag{A.36}$$

This equality follows easily from the fact that

$$\begin{aligned}&\mu_{a\searrow}(2k+1) \circ \mathbf{m}(\hat{H})_2 \circ \mathbf{m}(\hat{H})_4 \circ \cdots \circ \mathbf{m}(\hat{H})_{2(k-2)} \circ \mathbf{m}(\hat{H})_{2(k-1)} \circ \mathbf{m}(\hat{H})_{2k} \\ &= \mathbf{m}(\hat{H})_2 \circ \mathbf{m}(\hat{H})_4 \circ \cdots \circ \mathbf{m}(\hat{H})_{2(k-2)} \circ \mathbf{m}(\hat{H})_{2(k-1)} \circ \mu_{a\searrow}(2k+1) \circ \mathbf{m}(\hat{H})_{2k} \\ &= \mathbf{m}(\hat{H})_2 \circ \mathbf{m}(\hat{H})_4 \circ \cdots \circ \mathbf{m}(\hat{H})_{2(k-2)} \circ \mathbf{m}(\hat{H})_{2(k-1)} \circ \mathbf{m}(\hat{H})_{2k+1} \\ &= \mathbf{m}(\hat{H})_2 \circ \mathbf{m}(\hat{H})_4 \circ \cdots \circ \mathbf{m}(\hat{H})_{2(k-2)} \circ \mathbf{m}(\hat{H})_{2(k-1)}.\end{aligned} \tag{A.37}$$

Combined with

$$\mu_{a\nearrow}(2k+1) \circ \mu_{a\searrow}(2k+1) = \mathrm{id}_{\widehat{H}\boxtimes A_{2k+1}} \tag{A.38}$$

that, in turn, follows by our definitions using the fact that $\mu_i \circ \mu_i = \mathrm{id}_Q$ for elementary mutations.

For $G = D_n$, the proof is similar. Consider the $\widehat{H} \boxtimes A_{n-3}$ subquiver on the nodes

$$\big\{(i,a) \mid a = 1, ..., n-3,\ i = 1, .., \mathrm{rk}(\widehat{H})\big\}. \tag{A.39}$$

By our result about $G = A_n$, we know these are mutation equivalent to $\widehat{H} \boxtimes \overleftrightarrow{A}_{n-3}$, and by locality of mutations, the mutations sequences are the same as the one we have obtained previously. If $n = 2\ell + 1$, then one has just to use

$$\mathbf{m}(\hat{H})_2 \circ \cdots \circ \mathbf{m}(\hat{H})_{2(\ell-1)} \tag{A.40}$$

If, instead, $n = 2\ell$, then one needs to apply the mutation sequence

$$\mu_{a\nearrow}(n) \circ \mu_{a\nearrow}(n-1) \circ \mathbf{m}(\hat{H})_2 \circ \cdots \circ \mathbf{m}(\hat{H})_{2(\ell-2)}. \tag{A.41}$$

For E_6, E_7, and E_8 we have, instead, that the sequences are

$$\begin{aligned}E_6\colon &\quad \mu_{a\searrow}(6) \circ \mathbf{m}(\hat{H})_2 \circ \mathbf{m}(\hat{H})_4 \\ E_7\colon &\quad \mu_{a\searrow}(7) \circ \mathbf{m}(\hat{H})_2 \circ \mathbf{m}(\hat{H})_4 \\ E_8\colon &\quad \mu_{a\searrow}(8) \circ \mathbf{m}(\hat{H})_2 \circ \mathbf{m}(\hat{H})_4 \circ \mathbf{m}(\hat{H})_6.\end{aligned} \tag{A.42}$$

□

Lemma 2. All $\widehat{H} \boxtimes G$ models admit Coxeter-factorized sequences of type

$$\big(\underbrace{G, \dots, G}_{\mathrm{rk}(\widehat{H})\ \text{times}}\big). \tag{A.43}$$

Proof. Take the representative $\widehat{H} \boxtimes \overleftrightarrow{G}$ in the mutation class. With the notations of the previous proof, the mutation sequences

$$\mathbf{Cox}_{\widehat{H},G} \equiv \prod_{j \text{ even}} \mu_{a\searrow}(j) \circ \prod_{k \text{ odd}} \mu_{a\searrow}(k) \qquad \text{for } G = A_n, E_6, E_7, E_8 \tag{A.44}$$

$$\mathbf{Cox}_{\widehat{H},D_{2\ell+1}} \equiv \mu_{a\searrow}(2\ell+1) \circ \prod_{j \text{ even}} \mu_{a\searrow}(j) \circ \prod_{k \text{ odd} \leq 2\ell-1} \mu_{a\searrow}(k) \tag{A.45}$$

$$\mathbf{Cox}_{\widehat{H},D_{2\ell}} \equiv \prod_{j \text{ even} \leq 2\ell-2} \mu_{a\searrow}(j) \circ \mu_{a\searrow}(2\ell) \circ \prod_{k \text{ odd}} \mu_{a\searrow}(k) \tag{A.46}$$

acts as the identity on the quiver $\widehat{H} \boxtimes \overleftrightarrow{G}$, and as the Coxeter element of G on the charges of each q_i subquiver for the lattice Γ. Indeed, the sequences are source-sink factorized: Each mutation in the sequences is on a node that is a sink with respect to the full subquivers on the nodes v_j, and a source on the subquivers on the nodes q_i. The full quantum monodromy of the $(\widehat{H}, G)$ model is

$$\mathbb{M}(q) = \mathrm{Ad}(\widehat{\mathbf{Cox}}_{\widehat{H},G})^{h(G)}. \tag{A.47}$$

where the hat means that we are considering the corresponding quantum mutations.

□

Appendix B

More details about $D_p(G)$ systems.

B.1. A further check of (3.91)

Since the theory $D_p(G)$ is $\mathcal{N}=2$ superconformal, its quantum monodromy $M(q)$ has finite order r [13], and all chiral primary operators have dimensions in $\mathbb{N}/\ell$. The order ℓ is a nice invariant which is quite useful to distinguish SCFT models.

As we have discussed in the main body of the text, repeating the scaling arguments at the end of §. 3.3.1, we see that the matter theory $D_p(G)$, at the formal level, is engineered by the local Calabi–Yau geometry

$$W \equiv e^{pZ} + W_G(X_1, X_2) + X_3^2 = 0 \tag{B.1}$$

endowed with the standard holomorphic 3–form

$$\Omega = P.R.\, \frac{dZ \wedge dX_1 \wedge dX_2 \wedge dX_3}{W} \tag{B.2}$$

($P.R.$ stands for 'Poincaré Residue'). At a conformal point $f(X_1, X_2, X_3) \equiv W_G(X_1, X_2) + X_3^2$ is quasi–homogeneous, $f(\lambda^{q_i} X_i) = \lambda\, f(X_i)$ for all $\lambda \in \mathbb{C}$. Thus

$$X_i \to e^{i\alpha q_i}\, X_i, \qquad Z \to Z + \alpha/p, \tag{B.3}$$

is a holomorphic symmetry of the hypersurface (B.1) under which

$$\Omega \to \exp\big(i\alpha(q_1 + q_2 + q_3 - 1)\big)\Omega, \tag{B.4}$$

so that the dimension of X_i is $q_i/(\sum_j q_j - 1) \equiv q_i\, h(G) \in \mathbb{Z}$ while that of e^Z is $h(G)/p$. The order of the quantum monodromy $M(q)$ is then

$$\text{order } M(q) \equiv \ell = \frac{p}{\gcd\{p, h(G)\}}. \tag{B.5}$$

B.2. The proofs of eqns.(3.221)–(3.223)

We present in this appendix a proof which illustrate well the idea that, for the $D_p(G)$ SCFT's, the value of all physical quantities have deep Lie–theoretical meaning. Eqn.(3.221) is elementary:

$$c(p;G) = \frac{1}{12} r(h+1)p + O(1) \tag{B.6}$$

and $\dim G = r(h+1)$ is a well–known identity in Lie theory due to Coxeter.

To get eqn.(3.222) we consider

$$u(p;G) = \frac{1}{4} \sum_{j\in E(G)} \sum_{s=1}^{p-1} \Big[j - h\, s/p \Big]_+ \tag{B.7}$$

For $p \ggg 1$ the sum over s may be evaluated by the Euler–McLaurin summation formula; setting $x = s/p$, the term of order p in $u(p;G)$ is then

$$\frac{p}{4} \sum_{j\in E(G)} \int_0^1 dx\, [j - h\, x]_+ = \frac{p}{8\, h} \sum_{j\in E(G)} j^2 \tag{B.8}$$

where

$$\sum_{j\in E(G)} j^2 = \begin{cases} N(N-1)(2N-1)/6 & SU(N) \\ n(4n-5)(n-1)/3 & SO(2n) \\ 276 & E_6 \\ 735 & E_7 \\ 2360 & E_8 \end{cases} \tag{B.9}$$

Therefore

$$a(p,G) = \frac{1}{48} \left(\frac{6}{h} \sum_{j\in E(G)} j^2 + r(2h+5) \right) p + O(1) \tag{B.10}$$

and eqn.(3.222) is equivalent to the peculiar Lie theoretical identity

$$4 \dim G = \frac{6}{h} \sum_{j\in E(G)} j^2 + r(2h+5). \tag{B.11}$$

However unlikely it looks, this identity is actually true: indeed, plugging in eqn.(B.9), the RHS turns out

$4(N^2-1)$ for $SU(N)$	$4n(2n-1)$ for $SO(2n)$
$312 = 4\times 78$ for E_6	$532 = 4\times 133$ for E_7
$992 = 4\times 248$ for E_8	

(B.12)

The identity (B.11) is more conveniently written as[1]

$$6 \sum_{j\in E(G)} j^2 = 2h^2 r - hr. \tag{B.13}$$

To show eqn.(3.223) we consider first the case $h \mid p$. Let $p = h\ell$; then

$$c - a = \frac{1}{48}\left(2h^2 r - hr - 6\sum_{j\in E(G)} j^2\right)\ell + \frac{1}{48}\left(6\sum_{j\in E(G)} j - 2hr + 2r\right). \tag{B.14}$$

The term linear in ℓ vanishes by the identity (B.13). The equality in Eqn.(3.223) for $h \mid p$ then follows from the identity $\sum_{j\in E(G)} j = rh/2$ which, in view of the discussion after eqn.(3.182), is just 2d PCT.

Returning to the general case, we infer that, as a function of p, $24(c-a)$ is given by a degree–zero polynomial, $\#\Delta^+(G)+r$, plus a small Number–Theoretical modulation depending on the divisibility properties of p and h; the modulation has two sources, from $f(p;G)$ and $u(p;G)$. For $p \geq h$, $u(p;G)$ is simply

$$u(p;G) = \frac{1}{8p}\sum_{j\in E(G)}\left(\frac{j^2p^2}{h} - jp\right) + \frac{h}{8p}\sum_{j\in E(G)}\left\{\frac{jp}{h}\right\}\left(1-\left\{\frac{jp}{h}\right\}\right), \tag{B.15}$$

where $\{x\}$ denotes the fractional part. Therefore (for $p \geq h$)

$$\begin{aligned} 24(c-a) =& \big[\text{polynomial in } p \text{ (of degree zero)}\big] - \\ & - \frac{3h}{p}\sum_{j\in E(G)}\left\{\frac{jp}{h}\right\}\left(1-\left\{\frac{jp}{h}\right\}\right) - \frac{1}{2}\big(r - f(p;G)\big) \end{aligned} \tag{B.16}$$

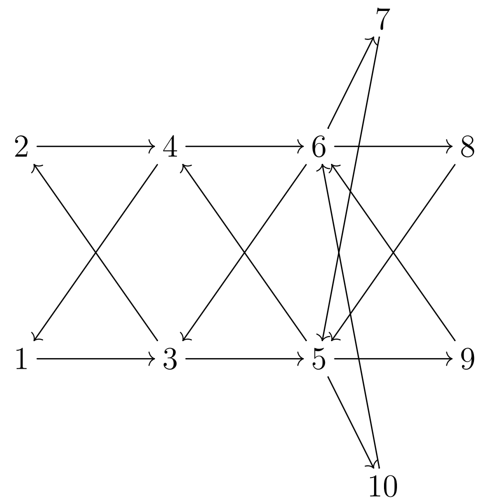
where the second line corresponds to the modulation. Both terms in the modulation are non–positive, and vanish if and only if $h \mid p$. This gives eqn.(3.223), in facts the more precise result ($p \geq h$)

$$24(c-a) = \#\Delta^+(G) + r - \left(\frac{3h}{p}\sum_{j\in E(G)}\left\{\frac{jp}{h}\right\}\left(1-\left\{\frac{jp}{h}\right\}\right) + \frac{1}{2}\big(r - f(p;G)\big)\right). \tag{B.17}$$

[1] We thank the referee for informing us that this identity was proven before, see [?].

B.3. Mutation sequences for the MN theories

The quiver for the model $D_2(SO(10))$ is simply



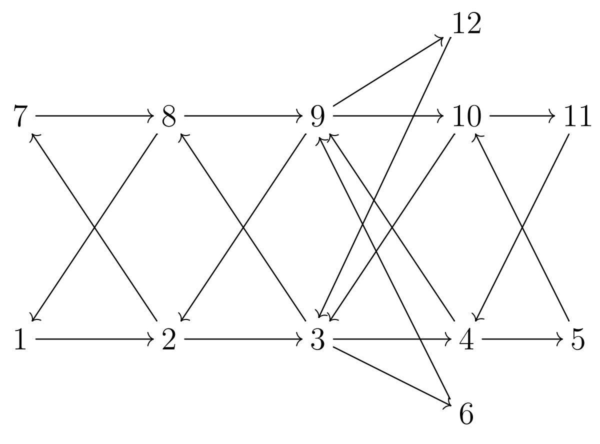
(B.18)

By performing the sequence of mutations

9 4 3 4 10 3 10 8 3 10 5 8 1 2 5

We obtain the element of the mutation class of the quiver of $D_2(SO(10))$ in figure 3.4.

The quiver for $D_2(E_6)$ is



(B.19)

The following sequence of mutations

9 12 4 9 10 12 6 12 3 11 1 9 10 9 12 10 9 12 6 3 12 9 6 5 6 9 10 12 1 7
3 1 7 8 4 8 1 2 1 8 1 4 8 2 8 4 3 5 7 4 7 5 7 1 7 11 7 4 1 8

gives the quiver in figure 3.5.

Appendix C

Technicalities about Arnold models

C.1. The E_7 Y–system from the chamber (2.43)

We illustrate the kind of Y–system one gets from Weyl–factorized sequence using the baby example of §. 2.3.7. There we presented a Weyl–factorized sequence of nodes for the family of subquivers (2.38). Written in terms of the BPS data in the corresponding chamber, the classical monodromy is equal to the Y–seed mutation (for the opposite quiver) associated to this Weyl–factorized sequence. This Y–seed mutation, as generated by the Keller mutation applet [**?**] is (we set $Y_{i,a,s} \equiv Y_{i,a}(s)$)

$$Y_{1,1,s+1} = \frac{1+Y_{2,3,s}}{Y_{2,3,s}Y_{0,s}} \qquad Y_{2,1,s+1} = \frac{Y_{1,2,s}Y_{1,3,s}(1+Y_{2,3,s}+Y_{2,3,s}Y_{0,s})}{1+Y_{1,2,s}}$$

$$Y_{1,2,s+1} = \frac{Y_{2,1,s}(1+Y_{1,2,s})Y_{2,2,s}Y_{2,3,s}Y_{0,s}}{(1+Y_{2,3,s}+Y_{2,1,s}(1+(1+Y_{2,2,s}(1+Y_{1,2,s}))Y_{2,3,s})(1+Y_{2,3,s}(1+Y_{0,s})))}$$

$$Y_{2,2,s+1} = \frac{1+Y_{2,3,s}+Y_{2,2,s}Y_{2,3,s}+Y_{1,2,s}Y_{2,2,s}Y_{2,3,s}}{Y_{1,2,s}+Y_{1,2,s}Y_{2,3,s}}$$

$$Y_{1,3,s+1} = \frac{1+Y_{2,3,s}}{Y_{2,1,s}+Y_{2,1,s}(1+Y_{2,2,s}+Y_{1,2,s}Y_{2,2,s})Y_{2,3,s}}$$

$$Y_{2,3,s+1} = \frac{Y_{1,1,s}Y_{1,2,s}(1+Y_{2,1,s}+Y_{2,3,s}+Y_{2,1,s}(1+Y_{2,2,s}+Y_{1,2,s}Y_{2,2,s})Y_{2,3,s})}{1+Y_{1,2,s}+(1+Y_{1,2,s})(1+(1+Y_{1,2,s}+Y_{1,1,s}Y_{1,2,s})Y_{2,2,s})Y_{2,3,s}}$$

$$Y_{0,s+1} = \frac{1+Y_{2,3,s}+Y_{2,2,s}Y_{2,3,s}+Y_{1,2,s}Y_{2,2,s}Y_{2,3,s}}{Y_{1,1,s}Y_{1,2,s}Y_{2,2,s}Y_{2,3,s}}.$$

This Y–system should be equivalent to the usual E_7 Y–system, differing only by a change of variables $Y_i \to \widetilde{Y}_i$. In particular, it must have the same minimal period ℓ as the usual one, namely 5. We have checked this using the strategy of §. 2.5.1.

C.2. Periodicity of the Y–systems: the S_{11} example

Here is our Mathematica program to test the periodicity for S_{11}.

```
SeedRandom[5]

Clear[i, j, k, Y]
Array[Y, {3, 12}, 0];
Array[X, {1, 12}, 0];
Y[0, 1] = 2 Random[];
Y[0, 2] = 2 Random[];
Y[0, 3] = 2 Random[];
Y[0, 4] = 2 Random[];
Y[0, 5] = 2 Random[];
Y[0, 6] = 2 Random[];
Y[0, 7] = 2 Random[];
Y[0, 8] = 2 Random[];
Y[0, 9] = 2 Random[];
Y[0, 10] = 2 Random[];
Y[0, 11] = 2 Random[];


(*dummy variable to store the initial seed*)


For[k = 1, k < 12, k++,
 X[0, k] = Y[0, k]
 ]
Print["INITIAL SEED="]
For[k = 1, k < 12, k++, Print[{k, Y[0, k]}]]

(*order of the quantum monodromy*)

r = 7;

(*iteration : notice that is based on the 1/2 monodromy so one has to do it twice and at the end implement the nontrivial permutation of the nodes*)

For[i = 1, i < 2 r + 1, i++,

 Y[1, 1] = (((Y[0, 3] + ((1 + (1 + Y[0, 1])*Y[0, 2])*Y[0, 3])*
          Y[0, 4])*Y[0, 5])*
      Y[0, 7] + ((((((1 + Y[0, 1])*Y[0, 2])*Y[0, 3])*Y[0, 4])*
         Y[0, 5])*Y[0, 7])*Y[0, 11])/(1 +
     Y[0, 1] + (1 + Y[0, 1])*
      Y[0, 5] + (1 +
        Y[0, 1] + (1 + Y[0, 1] + Y[0, 1]*Y[0, 3] +
           Y[0, 1]*Y[0, 3]*Y[0, 4])*Y[0, 5])*Y[0, 7]);
 Y[1, 2] = (1 +
     Y[0, 4])/(((1 + Y[0, 1])*Y[0, 2])*
      Y[0, 4] + (((1 + Y[0, 1])*Y[0, 2])*Y[0, 4])*Y[0, 11]);
 Y[1, 3] = (1 +
     Y[0, 1] + (1 + Y[0, 1])*
      Y[0, 5] + (1 + Y[0, 1] + (1 + Y[0, 1])*Y[0, 5])*
      Y[0, 7])/(((Y[0, 1]*Y[0, 3] + Y[0, 1]*Y[0, 3]*Y[0, 4])*Y[0, 5])*
     Y[0, 7]);
 Y[1, 4] = ((((1 + Y[0, 1])*Y[0, 2])*
         Y[0, 4] + (((1 + Y[0, 1])*Y[0, 2])*Y[0, 4])*
         Y[0, 5] + (((1 + Y[0, 1])*Y[0, 2])*
            Y[0, 4] + (((1 + Y[0, 1])*Y[0, 2] +
                 Y[0, 1]*Y[0, 2]*Y[0, 3])*Y[0, 4] +
              Y[0, 1]*Y[0, 2]*Y[0, 3]*Y[0, 4]^2)*Y[0, 5])*Y[0, 7])*
      Y[0, 8] + ((((1 + Y[0, 1])*Y[0, 2])*
           Y[0, 4] + (((1 + Y[0, 1])*Y[0, 2])*Y[0, 4])*
           Y[0, 5] + (((1 + Y[0, 1])*Y[0, 2])*
              Y[0, 4] + (((1 + Y[0, 1])*Y[0, 2] +
                   Y[0, 1]*Y[0, 2]*Y[0, 3])*Y[0, 4] +
                Y[0, 1]*Y[0, 2]*Y[0, 3]*Y[0, 4]^2)*Y[0, 5])*Y[0, 7])*
        Y[0, 8])*
      Y[0, 10] + ((((1 + Y[0, 1])*Y[0, 2])*
            Y[0, 4] + (((1 + Y[0, 1])*Y[0, 2])*Y[0, 4])*
            Y[0, 5] + (((1 + Y[0, 1])*Y[0, 2])*
               Y[0, 4] + (((1 + Y[0, 1])*Y[0, 2] +
                    Y[0, 1]*Y[0, 2]*Y[0, 3])*Y[0, 4] +
                 Y[0, 1]*Y[0, 2]*Y[0, 3]*Y[0, 4]^2)*Y[0, 5])*Y[0, 7])*
         Y[0, 8] + ((((1 + Y[0, 1])*Y[0, 2])*
               Y[0, 4] + (((1 + Y[0, 1])*Y[0, 2])*Y[0, 4])*

               Y[0, 5] + (((1 + Y[0, 1])*Y[0, 2])*
                  Y[0, 4] + (((1 + Y[0, 1])*Y[0, 2] +
                    Y[0, 1]*Y[0, 2]*Y[0, 3])*Y[0, 4] +
                    Y[0, 1]*Y[0, 2]*Y[0, 3]*Y[0, 4]^2)*Y[0, 5])*
```

```
             Y[0, 7])*
          Y[0, 8] + ((((1 + Y[0, 1])*Y[0, 2])*
               Y[0, 4] + ((1 + Y[0, 1])*Y[0, 2])*
               Y[0, 4]^2 + (((1 + Y[0, 1])*Y[0, 2])*
                  Y[0, 4] + ((1 + Y[0, 1])*Y[0, 2])*Y[0, 4]^2)*
               Y[0, 5] + (((1 + Y[0, 1])*Y[0, 2])*
                  Y[0, 4] + ((1 + Y[0, 1])*Y[0, 2])*
                  Y[0, 4]^2 + (((1 + Y[0, 1])*Y[0, 2] +
                  Y[0, 1]*Y[0, 2]*Y[0, 3])*
                  Y[0, 4] + ((1 + Y[0, 1])*Y[0, 2] +
                  2*Y[0, 1]*Y[0, 2]*Y[0, 3])*Y[0, 4]^2 +
                  Y[0, 1]*Y[0, 2]*Y[0, 3]*Y[0, 4]^3)*Y[0, 5])*
               Y[0, 7])*Y[0, 8])*Y[0, 9])*Y[0, 10])*
    Y[0, 11])/(1 + (2 + (1 + Y[0, 1])*Y[0, 2])*
    Y[0, 4] + (1 + (1 + Y[0, 1])*Y[0, 2])*
    Y[0, 4]^2 + (1 + (2 + (1 + Y[0, 1])*Y[0, 2])*
       Y[0, 4] + (1 + (1 + Y[0, 1])*Y[0, 2])*Y[0, 4]^2)*
    Y[0, 5] + (Y[0, 4] + (1 + (1 + Y[0, 1])*Y[0, 2])*
       Y[0, 4]^2 + (Y[0, 4] + (1 + (1 + Y[0, 1])*Y[0, 2])*
          Y[0, 4]^2)*
       Y[0, 5] + (Y[0, 4] + (1 + (1 + Y[0, 1])*Y[0, 2])*

          Y[0, 4]^2 + (Y[0, 4] + (1 + (1 + Y[0, 1])*Y[0, 2])*
             Y[0, 4]^2)*Y[0, 5])*Y[0, 7])*
    Y[0, 8] + ((Y[0, 4] + (1 + (1 + Y[0, 1])*Y[0, 2])*
         Y[0, 4]^2 + (Y[0, 4] + (1 + (1 + Y[0, 1])*Y[0, 2])*
            Y[0, 4]^2)*
         Y[0, 5] + (Y[0, 4] + (1 + (1 + Y[0, 1])*Y[0, 2])*
            Y[0, 4]^2 + (Y[0, 4] + (1 + (1 + Y[0, 1])*Y[0, 2])*
               Y[0, 4]^2)*Y[0, 5])*Y[0, 7])*Y[0, 8])*
    Y[0, 10] + (((1 + Y[0, 1])*Y[0, 2])*
       Y[0, 4] + ((1 + Y[0, 1])*Y[0, 2])*
       Y[0, 4]^2 + (((1 + Y[0, 1])*Y[0, 2])*
          Y[0, 4] + ((1 + Y[0, 1])*Y[0, 2])*Y[0, 4]^2)*
       Y[0, 5] + (((1 + Y[0, 1])*Y[0, 2])*
          Y[0, 4]^2 + (((1 + Y[0, 1])*Y[0, 2])*Y[0, 4]^2)*
          Y[0, 5] + (((1 + Y[0, 1])*Y[0, 2])*
             Y[0, 4]^2 + (((1 + Y[0, 1])*Y[0, 2])*Y[0, 4]^2)*
             Y[0, 5])*Y[0, 7])*
       Y[0, 8] + ((((1 + Y[0, 1])*Y[0, 2])*
            Y[0, 4]^2 + (((1 + Y[0, 1])*Y[0, 2])*Y[0, 4]^2)*
            Y[0, 5] + (((1 + Y[0, 1])*Y[0, 2])*
               Y[0, 4]^2 + (((1 + Y[0, 1])*Y[0, 2])*Y[0, 4]^2)*
               Y[0, 5])*Y[0, 7])*Y[0, 8])*Y[0, 10])*Y[0, 11]);
Y[1, 5] = ((Y[0, 1]*Y[0, 3] + Y[0, 1]*Y[0, 3]*Y[0, 4])*
    Y[0, 7] + ((Y[0, 1]*Y[0, 3] + Y[0, 1]*Y[0, 3]*Y[0, 4])*Y[0, 7])*
    Y[0, 10] + ((Y[0, 1]*Y[0, 3] + Y[0, 1]*Y[0, 3]*Y[0, 4])*
       Y[0, 7] + ((Y[0, 1]*Y[0, 3] + Y[0, 1]*Y[0, 3]*Y[0, 4])*
          Y[0, 7] + ((Y[0, 1]*Y[0, 3] + 2*Y[0, 1]*Y[0, 3]*Y[0, 4] +
              Y[0, 1]*Y[0, 3]*
               Y[0, 4]^2 + (Y[0, 1]*Y[0, 3]*Y[0, 4] +
                 Y[0, 1]*Y[0, 3]*
                  Y[0, 4]^2 + (Y[0, 1]*Y[0, 3]*Y[0, 4] +
                  Y[0, 1]*Y[0, 3]*Y[0, 4]^2)*Y[0, 5])*Y[0, 6])*
            Y[0, 7])*Y[0, 9])*Y[0, 10])*Y[0, 11])/(1 +
   Y[0, 1] + (1 + Y[0, 1])*
    Y[0, 5] + (1 +
      Y[0, 1] + (1 + Y[0, 1] + Y[0, 1]*Y[0, 3] +
         Y[0, 1]*Y[0, 3]*Y[0, 4])*Y[0, 5])*
    Y[0, 7] + (1 +
      Y[0, 1] + (1 + Y[0, 1])*
       Y[0, 5] + (1 +
         Y[0, 1] + (1 + Y[0, 1] + Y[0, 1]*Y[0, 3] +
            Y[0, 1]*Y[0, 3]*Y[0, 4])*Y[0, 5])*Y[0, 7])*
    Y[0, 10] + (1 +
      Y[0, 1] + (1 + Y[0, 1])*
       Y[0, 5] + (1 +
         Y[0, 1] + (1 + Y[0, 1] + Y[0, 1]*Y[0, 3] +
            Y[0, 1]*Y[0, 3]*Y[0, 4])*Y[0, 5])*
       Y[0, 7] + (1 +
         Y[0, 1] + (1 + Y[0, 1])*

          Y[0, 5] + (1 +
             Y[0, 1] + (1 + Y[0, 1] + Y[0, 1]*Y[0, 3] +
                Y[0, 1]*Y[0, 3]*Y[0, 4])*Y[0, 5])*
          Y[0, 7] + (1 +
             Y[0, 1] + (1 + Y[0, 1])*
              Y[0, 4] + (1 + Y[0, 1] + (1 + Y[0, 1])*Y[0, 4])*
              Y[0, 5] + (1 +
                 Y[0, 1] + (1 + Y[0, 1])*
                  Y[0, 4] + (1 + Y[0, 1] +
                    Y[0, 1]*
                    Y[0, 3] + (1 + Y[0, 1] + 2*Y[0, 1]*Y[0, 3])*
```

```
              Y[0, 4] + Y[0, 1]*Y[0, 3]*Y[0, 4]^2)*Y[0, 5])*
         Y[0, 7])*Y[0, 9])*Y[0, 10])*Y[0, 11]);
Y[1, 6] = (1 +
    Y[0, 10] + (1 + (1 + (1 + Y[0, 4])*Y[0, 9])*Y[0, 10])*
     Y[0, 11])/(((((Y[0, 4] + Y[0, 4]*Y[0, 5])*Y[0, 6])*Y[0, 9])*
      Y[0, 10])*Y[0, 11]);
Y[1, 7] = ((Y[0, 1]*Y[0, 3] + Y[0, 1]*Y[0, 3]*Y[0, 4])*
     Y[0, 5] + (Y[0, 1]*Y[0, 3]*Y[0, 4]*Y[0, 5] +
       Y[0, 1]*Y[0, 3]*Y[0, 4]*Y[0, 5]*Y[0, 7])*
     Y[0, 8] + ((Y[0, 1]*Y[0, 3]*Y[0, 4]*Y[0, 5] +
         Y[0, 1]*Y[0, 3]*Y[0, 4]*Y[0, 5]*Y[0, 7])*Y[0, 8])*
     Y[0, 10])/(1 +
    Y[0, 1] + (1 + Y[0, 1])*
     Y[0, 5] + (1 +
       Y[0, 1] + (1 + Y[0, 1] + Y[0, 1]*Y[0, 3] +
          Y[0, 1]*Y[0, 3]*Y[0, 4])*Y[0, 5])*Y[0, 7]);
Y[1, 8] = (1 +
    Y[0, 4])/((Y[0, 4] + Y[0, 4]*Y[0, 7])*
     Y[0, 8] + ((Y[0, 4] + Y[0, 4]*Y[0, 7])*Y[0, 8])*Y[0, 10]);
Y[1, 9] = ((Y[0, 4] + Y[0, 4]*Y[0, 5])*
     Y[0, 6] + ((Y[0, 4] + Y[0, 4]*Y[0, 5])*Y[0, 6])*
     Y[0, 10] + ((Y[0, 4] + Y[0, 4]*Y[0, 5])*
        Y[0, 6] + ((Y[0, 4] + Y[0, 4]*Y[0, 5])*Y[0, 6])*Y[0, 10])*
     Y[0, 11])/(1 +
    Y[0, 4] + (1 + Y[0, 4])*
     Y[0, 10] + (1 +
       Y[0, 4] + (1 +
          Y[0, 4] + (1 + 2*Y[0, 4] +
             Y[0, 4]^2 + (Y[0, 4] +
                Y[0, 4]^2 + (Y[0, 4] + Y[0, 4]^2)*Y[0, 5])*Y[0, 6])*
           Y[0, 9])*Y[0, 10])*Y[0, 11]);
Y[1, 10] = (((1 + Y[0, 4] + (Y[0, 4] + Y[0, 4]*Y[0, 7])*Y[0, 8])*
       Y[0, 9] + (((Y[0, 4] + Y[0, 4]*Y[0, 7])*Y[0, 8])*Y[0, 9])*
       Y[0, 10])*Y[0, 11])/(1 +
    Y[0, 10] + (1 + (1 + (1 + Y[0, 4])*Y[0, 9])*Y[0, 10])*Y[0, 11]);
Y[1, 11] = (((1 + (1 + (1 + Y[0, 1])*Y[0, 2])*Y[0, 4])*Y[0, 9])*
     Y[0, 10] + (((((1 + Y[0, 1])*Y[0, 2])*Y[0, 4])*Y[0, 9])*
       Y[0, 10])*Y[0, 11])/(1 +
    Y[0, 10] + (1 + (1 + (1 + Y[0, 4])*Y[0, 9])*Y[0, 10])*Y[0, 11]);


Y[2, 1] = ((Y[1, 3] + ((1 + (1 + Y[1, 1])*Y[1, 2])*Y[1, 3])*Y[1, 4])*
     Y[1, 7] + (((((1 + Y[1, 1])*Y[1, 2])*Y[1, 3])*Y[1, 4])*Y[1, 7])*
     Y[1, 11])/(1 +
    Y[1, 1] + (1 + Y[1, 1] + Y[1, 1]*Y[1, 3] +
       Y[1, 1]*Y[1, 3]*Y[1, 4])*Y[1, 7]);
Y[2, 2] = (1 +
    Y[1, 4])/(((1 + Y[1, 1])*Y[1, 2])*
     Y[1, 4] + (((1 + Y[1, 1])*Y[1, 2])*Y[1, 4])*Y[1, 11]);
Y[2, 3] = (1 +
    Y[1, 1] + (1 + Y[1, 1])*
     Y[1, 7])/((Y[1, 1]*Y[1, 3] + Y[1, 1]*Y[1, 3]*Y[1, 4])*Y[1, 7]);
Y[2, 4] = (((((1 + Y[1, 1])*Y[1, 2])*Y[1, 4]^2)*
        Y[1, 5] + ((((1 + Y[1, 1])*Y[1, 2] +
               Y[1, 1]*Y[1, 2]*Y[1, 3])*Y[1, 4]^2 +
            Y[1, 1]*Y[1, 2]*Y[1, 3]*Y[1, 4]^3)*Y[1, 5])*Y[1, 7])*
     Y[1, 8] + (((((1 + Y[1, 1])*Y[1, 2])*Y[1, 4]^2)*
          Y[1, 5] + ((((1 + Y[1, 1])*Y[1, 2] +
                Y[1, 1]*Y[1, 2]*Y[1, 3])*Y[1, 4]^2 +
             Y[1, 1]*Y[1, 2]*Y[1, 3]*Y[1, 4]^3)*Y[1, 5])*Y[1, 7])*
       Y[1, 8])*
     Y[1, 10] + (((((1 + Y[1, 1])*Y[1, 2])*Y[1, 4]^2)*
           Y[1, 5] + ((((1 + Y[1, 1])*Y[1, 2] +
                 Y[1, 1]*Y[1, 2]*Y[1, 3])*Y[1, 4]^2 +
              Y[1, 1]*Y[1, 2]*Y[1, 3]*Y[1, 4]^3)*Y[1, 5])*Y[1, 7])*
        Y[1, 8] + (((((1 + Y[1, 1])*Y[1, 2])*Y[1, 4]^2)*
              Y[1, 5] + ((((1 + Y[1, 1])*Y[1, 2] +
                  Y[1, 1]*Y[1, 2]*Y[1, 3])*Y[1, 4]^2 +
                 Y[1, 1]*Y[1, 2]*Y[1, 3]*Y[1, 4]^3)*Y[1, 5])*
              Y[1, 7])*
           Y[1, 8] + (((((1 + Y[1, 1])*Y[1, 2])*
                  Y[1, 4]^2 + ((1 + Y[1, 1])*Y[1, 2])*Y[1, 4]^3)*
               Y[1, 5] + ((((1 + Y[1, 1])*Y[1, 2] +
                  Y[1, 1]*Y[1, 2]*Y[1, 3])*
                  Y[1, 4]^2 + ((1 + Y[1, 1])*Y[1, 2] +
                  2*Y[1, 1]*Y[1, 2]*Y[1, 3])*Y[1, 4]^3 +
                  Y[1, 1]*Y[1, 2]*Y[1, 3]*Y[1, 4]^4)*Y[1, 5])*
               Y[1, 7])*Y[1, 8])*Y[1, 9])*Y[1, 10])*
     Y[1, 11])/(1 + (3 + (1 + Y[1, 1])*Y[1, 2])*
     Y[1, 4] + (3 + (2 + 2*Y[1, 1])*Y[1, 2])*
     Y[1, 4]^2 + (1 + (1 + Y[1, 1])*Y[1, 2])*
     Y[1, 4]^3 + (Y[1, 4] + (2 + (1 + Y[1, 1])*Y[1, 2])*
        Y[1, 4]^2 + (1 + (1 + Y[1, 1])*Y[1, 2])*Y[1, 4]^3)*
```

```
      Y[1, 5] + (Y[1, 4] + (2 + (1 + Y[1, 1])*Y[1, 2])*
         Y[1, 4]^2 + (1 + (1 + Y[1, 1])*Y[1, 2])*
         Y[1, 4]^3 + (Y[1, 4]^2 + (1 + (1 + Y[1, 1])*Y[1, 2])*
            Y[1, 4]^3)*
         Y[1, 5] + (Y[1, 4] + (2 + (1 + Y[1, 1])*Y[1, 2])*
            Y[1, 4]^2 + (1 + (1 + Y[1, 1])*Y[1, 2])*
            Y[1, 4]^3 + (Y[1, 4]^2 + (1 + (1 + Y[1, 1])*Y[1, 2])*
               Y[1, 4]^3)*Y[1, 5])*Y[1, 7])*
      Y[1, 8] + ((Y[1, 4] + (2 + (1 + Y[1, 1])*Y[1, 2])*
           Y[1, 4]^2 + (1 + (1 + Y[1, 1])*Y[1, 2])*
           Y[1, 4]^3 + (Y[1, 4]^2 + (1 + (1 + Y[1, 1])*Y[1, 2])*
              Y[1, 4]^3)*
           Y[1, 5] + (Y[1, 4] + (2 + (1 + Y[1, 1])*Y[1, 2])*
              Y[1, 4]^2 + (1 + (1 + Y[1, 1])*Y[1, 2])*
              Y[1, 4]^3 + (Y[1, 4]^2 + (1 + (1 + Y[1, 1])*Y[1, 2])*
                 Y[1, 4]^3)*Y[1, 5])*Y[1, 7])*Y[1, 8])*
      Y[1, 10] + (((1 + Y[1, 1])*Y[1, 2])*
         Y[1, 4] + ((2 + 2*Y[1, 1])*Y[1, 2])*
         Y[1, 4]^2 + ((1 + Y[1, 1])*Y[1, 2])*
         Y[1, 4]^3 + (((1 + Y[1, 1])*Y[1, 2])*
            Y[1, 4]^2 + ((1 + Y[1, 1])*Y[1, 2])*Y[1, 4]^3)*
         Y[1, 5] + (((1 + Y[1, 1])*Y[1, 2])*
            Y[1, 4]^2 + ((1 + Y[1, 1])*Y[1, 2])*
            Y[1, 4]^3 + (((1 + Y[1, 1])*Y[1, 2])*Y[1, 4]^3)*
            Y[1, 5] + (((1 + Y[1, 1])*Y[1, 2])*
               Y[1, 4]^2 + ((1 + Y[1, 1])*Y[1, 2])*
               Y[1, 4]^3 + (((1 + Y[1, 1])*Y[1, 2])*Y[1, 4]^3)*
               Y[1, 5])*Y[1, 7])*
         Y[1, 8] + ((((1 + Y[1, 1])*Y[1, 2])*
              Y[1, 4]^2 + ((1 + Y[1, 1])*Y[1, 2])*
              Y[1, 4]^3 + (((1 + Y[1, 1])*Y[1, 2])*Y[1, 4]^3)*
              Y[1, 5] + (((1 + Y[1, 1])*Y[1, 2])*
                 Y[1, 4]^2 + ((1 + Y[1, 1])*Y[1, 2])*
                 Y[1, 4]^3 + (((1 + Y[1, 1])*Y[1, 2])*Y[1, 4]^3)*
                 Y[1, 5])*Y[1, 7])*Y[1, 8])*Y[1, 10])*Y[1, 11]);
 Y[2, 5] = (1 + Y[1, 4])/(Y[1, 4]*Y[1, 5]);
 Y[2, 6] = (((((1 + Y[1, 4] + Y[1, 4]*Y[1, 5])*Y[1, 6])*Y[1, 9])*
       Y[1, 10])*Y[1, 11])/(1 +
     Y[1, 10] + (1 + (1 + (1 + Y[1, 4])*Y[1, 9])*Y[1, 10])*Y[1, 11]);
 Y[2, 7] = (Y[1, 1]*Y[1, 3] +
     Y[1, 1]*Y[1, 3]*
      Y[1, 4] + (Y[1, 1]*Y[1, 3]*Y[1, 4] +
        Y[1, 1]*Y[1, 3]*Y[1, 4]*Y[1, 7])*
      Y[1, 8] + ((Y[1, 1]*Y[1, 3]*Y[1, 4] +
          Y[1, 1]*Y[1, 3]*Y[1, 4]*Y[1, 7])*Y[1, 8])*Y[1, 10])/(1 +
     Y[1, 1] + (1 + Y[1, 1] + Y[1, 1]*Y[1, 3] +
        Y[1, 1]*Y[1, 3]*Y[1, 4])*Y[1, 7]);
 Y[2, 8] = (1 +
     Y[1, 4])/((Y[1, 4] + Y[1, 4]*Y[1, 7])*
      Y[1, 8] + ((Y[1, 4] + Y[1, 4]*Y[1, 7])*Y[1, 8])*Y[1, 10]);
 Y[2, 9] = (1 +
     Y[1, 10] + (1 + Y[1, 10])*
      Y[1, 11])/((((1 + Y[1, 4])*Y[1, 9])*Y[1, 10])*Y[1, 11]);
 Y[2, 10] = (((1 + Y[1, 4] + (Y[1, 4] + Y[1, 4]*Y[1, 7])*Y[1, 8])*
        Y[1, 9] + (((Y[1, 4] + Y[1, 4]*Y[1, 7])*Y[1, 8])*Y[1, 9])*
        Y[1, 10])*Y[1, 11])/(1 +
     Y[1, 10] + (1 + (1 + (1 + Y[1, 4])*Y[1, 9])*Y[1, 10])*Y[1, 11]);
 Y[2, 11] = (((1 + (1 + (1 + Y[1, 1])*Y[1, 2])*Y[1, 4])*Y[1, 9])*
      Y[1, 10] + (((((1 + Y[1, 1])*Y[1, 2])*Y[1, 4])*Y[1, 9])*
        Y[1, 10])*Y[1, 11])/(1 +
     Y[1, 10] + (1 + (1 + (1 + Y[1, 4])*Y[1, 9])*Y[1, 10])*Y[1, 11]);

 Y[0, 1] = Y[2, 11];
 Y[0, 2] = Y[2, 2];
 Y[0, 3] = Y[2, 9];
 Y[0, 4] = Y[2, 4];
 Y[0, 5] = Y[2, 6];
 Y[0, 6] = Y[2, 5];
 Y[0, 7] = Y[2, 10];
 Y[0, 8] = Y[2, 8];
 Y[0, 9] = Y[2, 3];
 Y[0, 10] = Y[2, 7];
 Y[0, 11] = Y[2, 1];
 ]

Print["RESULT OF ITERATION="]
For[j = 1, j < 12, j++, Print[{j, Y[0, j]}]]
Print["DIFFERENCE="]
For[j = 1, j < 12, j++, Print[{j, Chop[Y[0, j] - X[0, j]]}]]
```

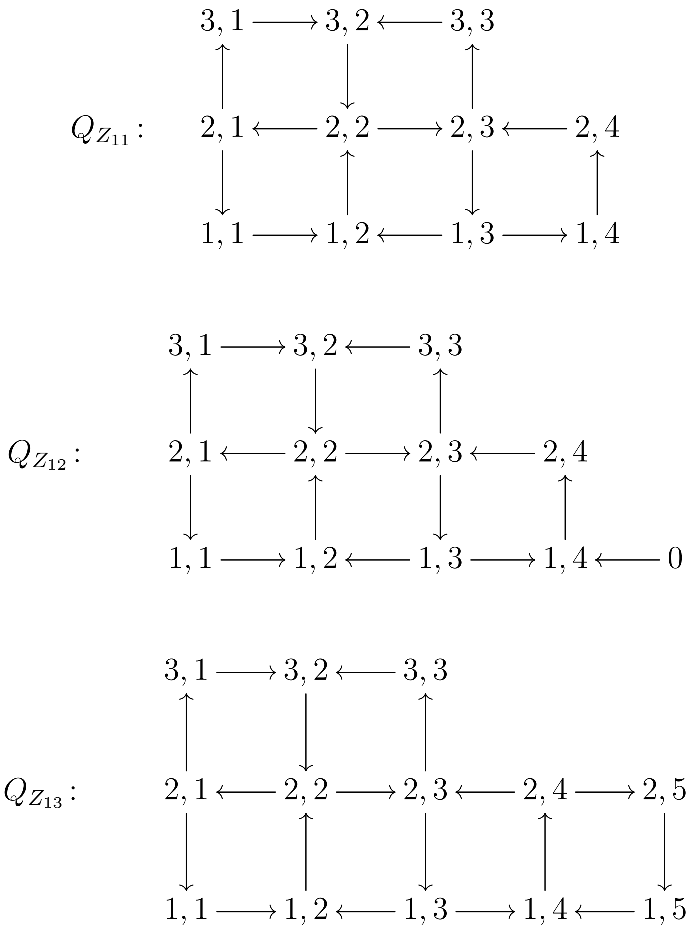


Figure C.1: Quivers for the Z familiy.

C.3. Details on the Weyl–factorized sequences

In this appendix we present the details of the computations summarized in table 2.4. For each model we specify the quiver used and the associated Weyl–factorized sequences, with their types (ie. the element of the Weyl group they generate on the corresp. subquiver) and involutive permutations. For the quivers arising from one–point extensions of known algebras — Z_{12}, Q_{11}, S_{12}, W_{13} — we report only the sequences associated with the quivers in figures C.1, C.2, C.4: obtaining the sequences corresponding to these quivers mutated at 0 is straightforward.

Z_{11}		(3,2),(2,1),(2,3),(1,2),(1,4),(3,1),(2,2),(2,4),(1,1),(1,3),(3,3),(2,1),(2,3), (1,2),(1,4),(3,2),(2,2),(2,4),(1,1),(1,3),(3,1),(3,3),(2,1),(2,3),(1,2),(1,4)
	Type:	$(A_4, s_4 s_2 (c_{A_4})^2 : A_4, s_3 s_1 (c_{A_4})^2 : A_3, (c_{A_3})^2)$
	P_Λ	$< (\gamma_{3,1}, \gamma_{3,3}), (\gamma_{a,4}, \gamma_{a,1}), (\gamma_{a,2}, \gamma_{a,3}), a = 1, 2 >$
		(3,1),(3,3),(1,1),(1,3),(2,2),(2,4),(3,2),(1,2),(1,4),(2,1),(2,3),(3,1),(3,3), (1,1),(1,3),(2,2),(2,4),(3,2),(1,2),(2,1),(2,3)
	Type:	$(A_3, (c_{A_3})^2 : A_3, (c_{A_3})^2 : A_3, (c_{A_3})^2 : A_2, u)$
	P_Λ	$< (\gamma_{1,4}, \gamma_{2,4}), (\gamma_{1,a}, \gamma_{3,a}), a = 1, 2, 3 >$
Z_{12}		(3,2),(2,1),(2,3),(1,2),(1,4),(3,1),(2,2),(1,1),(1,3),0,(2,1), (1,2),(1,4),(1,1),(1,3),(1,2),(2,4),(2,3),(2,2),(3,3),(3,2),0, (1,4),(2,4),(3,1),(3,3),(2,1),(2,3),(1,1),(1,3),0
	Type:	$(A_5, (c_{A_5})^3 : A_4, s_3 s_1 (c_{A_4})^2 : A_3, (c_{A_3})^2)$
	P_Λ	$< (\gamma_{1,1}, \gamma_0), (\gamma_{1,2}, \gamma_{1,4}), (\gamma_{2,1}, \gamma_{2,4}), (\gamma_{2,2}, \gamma_{2,3}), (\gamma_{3,1}, \gamma_{3,3}) >$
		(3,1),(1,1),(2,2),(3,3),(1,3),(2,4),0,(2,1),(3,2),(1,2),(2,3), (1,4),(3,1),(1,1),(2,2),(3,3),(1,3),(2,4),(2,1),(3,2),(1,2),(2,3)
	Type:	$(A_3, (c_{A_3})^2 : A_3, (c_{A_3})^2 : A_3, (c_{A_3})^2 : A_1, s)$
	P_Λ	$< \gamma_{1,a}, \gamma_{3,a} > a = 1, 2, 3, (\gamma_{2,4}, \gamma_{1,4})$
Z_{13}		(3,2),(2,1),(2,3),(2,5),(1,2),(1,4),(2,2),(2,4),(1,1),(1,3),(1,5),(3,1), (2,1),(2,3),(2,5),(1,2),(1,4),(3,3),(2,2),(2,4),(1,1),(1,3),(1,5), (3,2),(2,1),(2,3),(2,5),(1,2),(1,4),(1,1),(3,3),(2,2),(2,4),(1,1),(1,3),(1,5)
	Type:	$(A_5, (c_{A_5})^3 : A_5, (c_{A_5})^3 : A_3, (c_{A_3})^2)$
	P_Λ	$< (\gamma_{3,1}, \gamma_{3,3}), (\gamma_{a,5}, \gamma_{a,1}), (\gamma_{a,4}, \gamma_{a,2}), a = 1, 2 >$
		(3,1),(3,3),(1,1),(1,3),(2,2),(1,5),(2,4),(3,2),(1,2),(2,1),(2,3),(2,5),(1,4), (3,1),(3,3),(1,1),(1,3),(2,2),(3,2),(1,2),(2,1),(2,3),(2,4),(1,5)
	Type:	$(A_3, (c_{A_3})^2 : A_3, (c_{A_3})^2 : A_3, (c_{A_3})^2 : A_2, u : A_2, v)$
	P_Λ	$< (\gamma_{1,b}, \gamma_{2,b}), b = 4, 5, (\gamma_{1,a}, \gamma_{3,a}), a = 1, 2, 3 >$

Table C.1: Weyl–factorized sequences for the quivers of figure C.1.

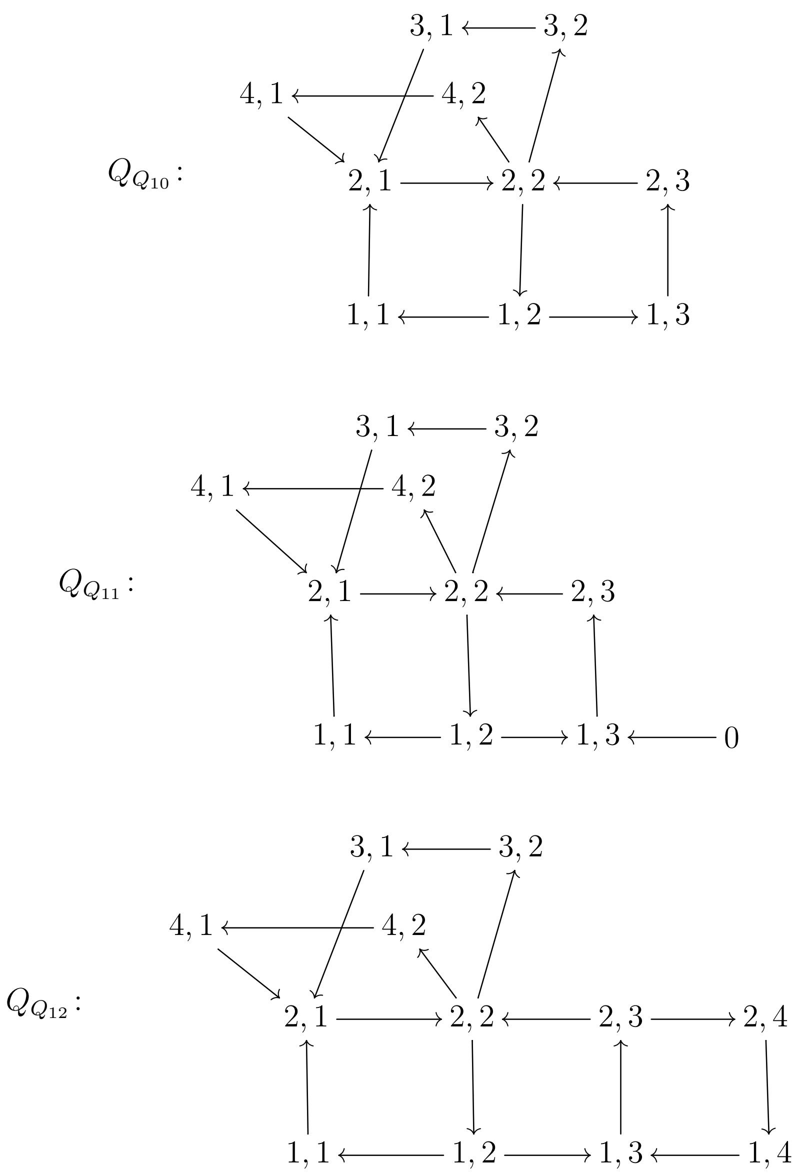


Figure C.2: Quivers for the Q familiy.

Q_{10}		(4,1),(3,1),(2,2),(1,1),(2,3),(2,1),(1,3),(1,2),(4,2),(3,2),(2,2),(1,1), (2,3),(4,1),(3,1),(2,1),(1,3),(1,2)
	Type:	$(A_3, (c_{A_3})^2 : A_3, (c_{A_3})^2 : A_2, v : A_2, v)$
	P_Λ	$< (\gamma_{b,1}, \gamma_{b,2}), b = 3, 4, (\gamma_{a,1}, \gamma_{a,3}), a = 1, 2 >$
		(2,1),(3,2),(4,2),(1,2),(1,3),(4,1),(3,1),(1,1),(2,2),(2,3),(2,1),(4,2), (3,2),(1,2),(4,1),(3,1),(1,1),(2,2),(2,1),(4,2),(3,2),(1,2),(4,1),(3,1), (1,1),(2,2),(1,3)
	Type:	$(D_4, (c_{D_4})^3 : D_4, (c_{D_4})^3 : A_2, v)$
	P_Λ	$< (\gamma_{1,3}, \gamma_{2,3}) >$
Q_{11}		(1,1),(1,3),(2,2),(3,1),(4,1),(1,2),0,(2,1),(2,3),(1,3),(2,2),(3,2),(4,2),0, (1,1),(1,2),(2,1),(1,3),(2,3),(3,1),(4,1),(1,1)
	Type:	$(A_4, s_3 s_1 (c_{A_4})^2 : A_3, (c_{A_3})^2 : A_2, u : A_2, v)$
	P_Λ	$< \gamma_{a,1}, \gamma_{a,2} > a = 3, 4, (\gamma_{2,1}, \gamma_{2,3}), (\gamma_0, \gamma_{1,1}), (\gamma_{1,2}, \gamma_{1,3})$
		0,(2,3),(3,2),(1,2),(4,2),(2,1),(1,3),(2,2),(4,1), (1,1),(3,1),(3,2),(4,2),(1,2),(2,1),(2,2),(3,1),(4,1), (1,1),(3,2),(1,2),(4,2),(2,1),(2,2),(3,1),(4,1), (1,1),(2,3)
	Type:	$(D_4, (c_{D_4})^3 : D_4, (c_{D_4})^3 : A_2, u : A_1, s)$
	P_Λ	$< (\gamma_{1,3}, \gamma_{2,3}) >$
Q_{12}		(1,1),(1,3),(2,4),(2,2),(3,1),(4,1),(2,3),(2,1),(1,2),(1,4),(2,4),(1,3), (2,2),(2,3),(3,2),(4,2),(1,1),(1,2),(2,1),(1,4),(1,1),(2,2),(1,3),(2,4), (4,1),(3,1)
	Type:	$(A_4, s_3 s_1 (c_{A_4})^2 : A_4, s_4 s_2 (c_{A_3})^2 : A_2, v : A_2, v)$
	P_Λ	$< (\gamma_{b,1}, \gamma_{b,2}), b = 3, 4, (\gamma_{a,1}, \gamma_{a,4}), (\gamma_{a,2}, \gamma_{a,3}), a = 1, 2 >$
		(2,1),(4,2),(3,2),(1,2),(2,3),(1,4),(4,1),(3,1),(1,1),(2,2),(1,3),(2,4), (2,1),(4,2),(3,2),(1,2),(4,1),(3,1),(1,1),(2,2),(2,1),(4,2),(3,2),(1,2), (2,3),(1,4),(4,1),(3,1),(1,1),(2,2)
	Type:	$(D_4, (c_{D_4})^3 : D_4, (c_{D_4})^3 : A_2, u : A_2, v)$
	P_Λ	$< (\gamma_{1,a}, \gamma_{2,a}), a = 3, 4 >$

Table C.2: Weyl–factorized sequences for the quivers of figure C.2.

$\mu(Q_{Q_{10}})$:

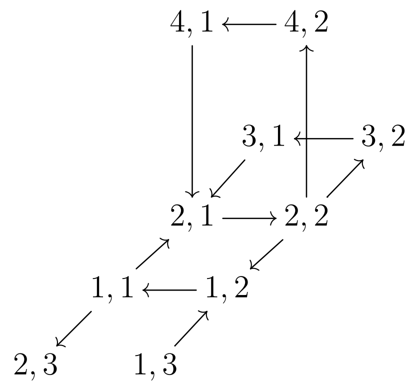


Figure C.3: The quiver in the mutation class of Q_{10} of example (**??**).

Table C.3: Weyl–factorized sequences associated with the quiver of figure C.3. Both of them corresponds to mutations that are the identity on the quiver. The first one, being of order 1 corresponds to the full quantum monodromy, the second to the half–monodromy.

Q_{10}		(4,1),(3,1),(1,1),(2,2),(4,2),(3,2),(1,3),(1,2),(2,1), (4,1),(3,1),(1,1),(2,2),(4,2),(3,2),(2,3),(1,2),(2,1), (4,1),(3,1),(1,3),(1,1),(2,2),(4,2),(3,2),(1,2),(2,1),(2,3)
	Type:	$(A_2, c^3 : A_2, c^3 : A_2, c^3 : A_2, c^3 : A_1, c^2 : A_1, c^2)$
		(2,1),(4,2),(3,2),(1,3),(1,2),(4,1),(3,1),(1,1),(2,2),(2,1), (4,2),(3,2),(1,2),(4,1),(3,1),(1,1),(2,2),(2,1),(4,2),(3,2), (1,2),(4,1),(3,1),(1,1),(2,2),(2,3)
	Type:	$(D_4, (c_{D_4})^3 : D_4, (c_{D_4})^3 : A_1, s : A_1, s)$

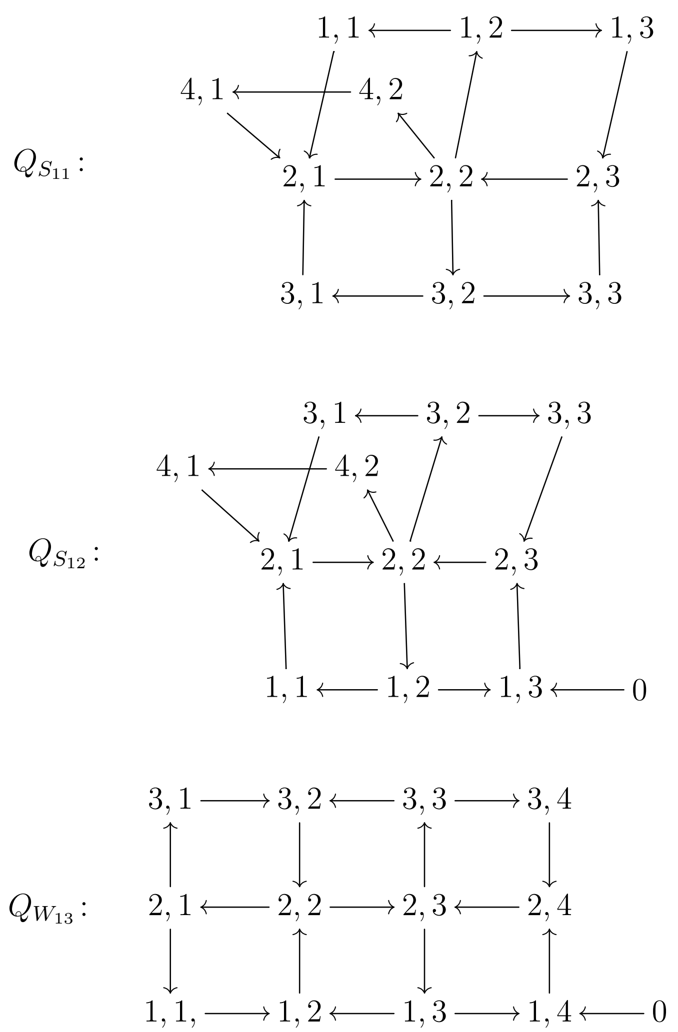


Figure C.4: Quivers for the S familiy and for the W13 theory.

W_{13}		(3,2),(3,4),(2,1),(2,3),(1,2),(1,4),(3,1),(3,3),(2,2),(1,1),(1,3),0, (3,2),(2,1),(2,4),(1,2),(1,4),(3,1),(3,4),(2,3),(1,1),(1,3),0, (3,3),(2,2),(2,4),(1,2),(1,4),(3,2),(3,4),(2,1),(2,3),(1,1),(1,3),0
	Type:	$(A_5, (c_{A_5})^3 : A_4, s_3 s_1 (c_{A_4})^3 : A_4, s_4 s_2 (c_{A_4})^3)$
	P_Λ	$< (\gamma_0, \gamma_{1,1}), (\gamma_{1,2}, \gamma_{1,4}), (\gamma_{a,1}, \gamma_{a,4}), (\gamma_{a,2}, \gamma_{a,3}), a = 2, 3 >$
		(3,1),(1,1),(2,2),(3,3),(1,3),(2,4),0,(2,1),(3,2),(1,2),(2,3),(3,4),(1,4), (3,1),(1,1),(2,2),(3,3),(1,3),(2,4),(2,1),(3,2),(1,2),(2,3),(3,4),(1,4)
	Type:	$(A_3, (c_{A_3})^2 : A_3, (c_{A_3})^2 : A_3, (c_{A_3})^2 : A_3, (c_{A_3})^2 : A_1 : s)$
	P_Λ	$< (\gamma_{1,a}, \gamma_{3,a}), a = 1, 3 >$
S_{11}		(1,1),(1,3),(3,1),(3,3),(2,2),(4,1),(1,2),(3,2),(2,1),(2,3),(4,2), (1,1),(1,3),(3,1),(3,3),(2,2),(1,2),(3,2),(2,1),(2,3),(4,1)
	Type:	$(A_3, (c_{A_3})^2 : A_3, (c_{A_3})^2 : A_3, (c_{A_3})^2 : A_2, v)$
	P_Λ	$< (\gamma_{a,1}, \gamma_{a,3}), a = 1, 2, 3, (\gamma_{4,1}, \gamma_{4,2}) >$
		(2,1),(4,2),(1,2),(3,2),(2,3),(4,1),(1,1),(3,1),(2,2),(1,3),(3,3), (2,1),(4,2),(1,2),(3,2),(2,3),(4,1),(1,1),(3,1),(2,2),(2,1),(4,2), (1,2),(3,2),(4,1),(1,1),(3,1),(2,2),(3,3),(1,3)
	Type:	$(D_4, (c_{D_4})^3 : D_4, (c_{D_4})^3 : A_3, (c_{A_3})^2)$
	P_Λ	$< (\gamma_{1,3}, \gamma_{3,3}) >$
S_{12}		(3,1),(3,3),(1,1),(1,3),(2,2),(4,1),(3,2),(1,2),0,(2,3),(2,1),(4,2), (1,3),(1,1),(2,2),(3,1),(3,3),0,(1,2),(1,1),(1,3),(2,3),(2,1),(3,2),(4,1)
	Type:	$(A_4, s_3 s_1 (c_{A_4})^2 : A_3, (c_{A_3})^2 : A_3, (c_{A_3})^2 : A_2, v)$
	P_Λ	$< (\gamma_{1,1}, \gamma_0), (\gamma_{1,2}, \gamma_{1,3}), (\gamma_{a,1}, \gamma_{a,3}), a = 2, 3, (\gamma_{4,1}, \gamma_{4,2}) >$
		(2,1),(4,2),(3,2),(1,2),(2,3),0,(4,1),(3,1),(1,1),(2,2),(3,3),(1,3),(2,1), (4,2),(3,2),(1,2),(2,3),(4,1),(3,1),(1,1),(2,2),(2,1),(4,2),(3,2),(1,2), (4,1),(3,1),(1,1),(2,2),(1,3),(3,3)
	Type:	$(D_4, (c_{D_4})^3 : D_4, (c_{D_4})^3 : A_3, (c_{A_3})^2 : A_1, s)$
	P_Λ	$< \gamma_{1,3}, \gamma_{3,3} >$

Table C.4: Weyl–factorized sequences for the quivers of figure C.4.

C.4. Mutation sequences for the exceptional bimodals

In this appendix we discuss some selected examples of the Weyl–factorized sequences we have used to generate the BPS spectra of §??. We use the conventions of [?].

- **$\mathbf{E_{19}}$**

  This model is a one point extension of $A_2 \square A_9$. We will discuss all the chambers of it, since it is the simplest one point extension in between the models we are studying in this paper and the other one point extensions behaves similarly.

  The square form of the $Q_{E_{19}}$ quiver is

$$Q_{E_{19}}:\ \begin{array}{ccccccccccccccccc} 2 & \longleftarrow & 4 & \longrightarrow & 6 & \longleftarrow & 8 & \longrightarrow & 10 & \longleftarrow & 12 & \longrightarrow & 14 & \longleftarrow & 16 & \longrightarrow & 18 & & \\ \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \\ 1 & \longrightarrow & 3 & \longleftarrow & 5 & \longrightarrow & 7 & \longleftarrow & 9 & \longrightarrow & 11 & \longleftarrow & 13 & \longrightarrow & 15 & \longleftarrow & 17 & \longrightarrow & 19 \end{array}$$

  The original form of $Q_{E_{19}}$ is related to this one by the following sequence of mutations: (1 3 5 7 9 11 13 15 2 4 6 8 10 12 14 1 3 5 7 9 11 2 4 6 8 10 1 3 5 7 2 4 6 1 3 2)$^{-1}$.

  From $Q_{E_{19}}$ one can easily obtain two of the four chambers (2.60):

  (a) (A_{10}, A_9) chamber:

  - ∗ complete family: A_{10}: odd nodes; A_9: even nodes
  - ∗ Weyl–factorized sequence: 2 6 10 14 18 3 7 11 15 19 4 8 12 16 1 5 9 13 17 2 6 10 14 3 7 11 15 19 4 8 12 18 1 5 9 13 17 2 6 10 16 3 7 11 15 19 4 8 14 18 1 5 9 13 17 2 6 12 16 3 7 11 15 19 4 10 14 18 1 5 9 13 17 2 8 12 16 3 7 11 15 19 6 10 14 18 1 5 9 13 17 4 8 12 16 3 7 11 15 19
  - ∗ permutation:

$$\boldsymbol{m}_\Lambda(Q_{E_{19}}) = \begin{array}{ccccccccccccccccccc} 18 & \longleftarrow & 16 & \longrightarrow & 14 & \longleftarrow & 12 & \longrightarrow & 10 & \longleftarrow & 8 & \longrightarrow & 6 & \longleftarrow & 4 & \longrightarrow & 2 & & \\ \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \\ 19 & \longrightarrow & 17 & \longleftarrow & 15 & \longrightarrow & 13 & \longleftarrow & 11 & \longrightarrow & 9 & \longleftarrow & 7 & \longrightarrow & 5 & \longleftarrow & 3 & \longrightarrow & 1 \end{array}$$

  - ∗ type: $(A_{10}, c^{-5} s_{10} s_8 s_6 s_4 s_2 : A_9, c^5)$

  (b) $(A_2 \times 9, A_1)$ chamber:

* complete family: $A_2 = \{1,2\} + 2k, k = 0, ..., 8,\ A_1 = \{19\}$.
* Weyl–factorized sequence: 1 4 5 8 9 12 13 16 17 2 3 6 7 10 11 14 15 18 1 4 5 8 9 12 13 16 17 19
* permutation:

$$\boldsymbol{m}_\Lambda(Q_{E_{19}}) = \begin{array}{ccccccccccccccccccc} 1 & \longleftarrow & 3 & \longrightarrow & 5 & \longleftarrow & 7 & \longrightarrow & 9 & \longleftarrow & 11 & \longrightarrow & 13 & \longleftarrow & 15 & \longrightarrow & 17 & & \\ \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \\ 2 & \longrightarrow & 4 & \longleftarrow & 6 & \longrightarrow & 8 & \longleftarrow & 10 & \longrightarrow & 12 & \longleftarrow & 14 & \longrightarrow & 16 & \longleftarrow & 18 & \longrightarrow & 19 \end{array}$$

* type: $A_2, s_1 s_2 s_1$ for the even k, $A_2, s_2 s_1 s_2$ for the odd k, A_1, s.

As far as the other two chambers one has to consider the quiver $\mu_{19}(Q_{E_{19}})$:

(a) $(A_9 \times 2, A_1)$ chamber:

* complete family: $A_9 = \{1,3,5,7,9,11,13,15,17\}+k, k = 0,1,\ A_1 = \{19\}$
* Weyl–factorized sequence: 3 7 11 15 2 6 10 14 18 19 1 5 9 13 17 4 8 12 16 3 7 11 15 2 6 10 14 18 1 5 9 13 17 4 8 12 16 3 7 11 15 2 6 10 14 18 1 5 9 13 17 4 8 12 16 3 7 11 15 2 6 10 14 18 1 5 9 13 17 4 8 12 16 3 7 11 15 2 6 10 14 18 1 5 9 13 17 4 8 12 16
* permutation:

$$\boldsymbol{m}_\Lambda(Q_{E_{19}}) = \begin{array}{ccccccccccccccccccc} 18 & \longleftarrow & 16 & \longrightarrow & 14 & \longleftarrow & 12 & \longrightarrow & 10 & \longleftarrow & 8 & \longrightarrow & 6 & \longleftarrow & 4 & \longrightarrow & 2 & & \\ \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \\ 17 & \longrightarrow & 15 & \longleftarrow & 13 & \longrightarrow & 11 & \longleftarrow & 9 & \longrightarrow & 7 & \longleftarrow & 5 & \longrightarrow & 3 & \longleftarrow & 1 & \longleftarrow & 19 \end{array}$$

* type: $(A_9, c^{-5} : A_9, c^5 : A_1, s)$

(b) $(A_2 \times 8, A_3)$ chamber:

* complete family: $A_2 = \{1,2\} + 2k, k = 0, 7,\ A_3 = \{17,18,19\}$
* Weyl–factorized sequence: 1 4 5 8 9 12 13 16 17 2 3 6 7 10 11 14 18 19 15 1 4 5 8 9 12 13 17 16 19 18
* permutation:

$$\boldsymbol{m}_\Lambda(Q_{E_{19}}) = \begin{array}{ccccccccccccccccccc} 1 & \longleftarrow & 3 & \longrightarrow & 5 & \longleftarrow & 7 & \longrightarrow & 9 & \longleftarrow & 11 & \longrightarrow & 13 & \longleftarrow & 15 & \longrightarrow & 19 & & \\ \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \\ 2 & \longrightarrow & 4 & \longleftarrow & 6 & \longrightarrow & 8 & \longleftarrow & 10 & \longrightarrow & 12 & \longleftarrow & 14 & \longrightarrow & 16 & \longleftarrow & 17 & \longleftarrow & 18 \end{array}$$

* type: $A_2, s_1 s_2 s_1$ for the even k, $A_2, s_2 s_1 s_2$ for the odd k, A_3, c^{-2}.

This is the generic situation for all the one–point extensions: mutating at the extra–node leads two different couples of canonical chambers.

- $\mathbf{Z_{17}}$

  This will be our example of a theory in the Z family. The square form of $Q_{Z_{17}}$ is:

$$Q_{Z_{17}}: \begin{array}{ccccccccccccc} 2 & \longrightarrow & 5 & \longleftarrow & 8 & & & & & & & & \\ \uparrow & & \downarrow & & \uparrow & & & & & & & & \\ 3 & \longleftarrow & 6 & \longrightarrow & 9 & \longleftarrow & 11 & \longrightarrow & 13 & \longleftarrow & 15 & \longrightarrow & 17 \\ \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow \\ 1 & \longrightarrow & 4 & \longleftarrow & 7 & \longrightarrow & 10 & \longleftarrow & 12 & \longrightarrow & 14 & \longleftarrow & 16 \end{array}$$

  Related to the original form obtained from its Coxeter–Dynkin diagram of table 2.5 by the sequence of mutations: $(17\ 15\ 13\ 11\ 2\ 1\ 5\ 4\ 3\ 16\ 14\ 12\ 17\ 15\ 16)^{-1}$. We have two (physical) chambers:

  (a) $(A_7 \times 2, A_3)$ chamber:

  - complete family: $A_7 = \{1, 4, 7, 10, 12, 14, 16\}$ $A_7 = \{3, 6, 9, 11, 13, 15, 17\}$ $A_3 = \{2, 5, 8\}$
  - Weyl–factorized sequence: 4 10 14 3 9 13 17 5 1 7 12 16 6 11 15 2 4 10 14 3 9 13 17 1 7 12 16 6 11 15 4 10 14 3 9 13 17 1 7 12 16 6 11 15 4 10 14 3 9 13 8 17 5 1 7 12 16 6 11 15 2 8
  - permutation:

$$\boldsymbol{m}_\Lambda(Q_{Z_{17}}) = \begin{array}{ccccccccccccc} 8 & \longrightarrow & 5 & \longleftarrow & 2 & & & & & & & & \\ \uparrow & & \downarrow & & \uparrow & & & & & & & & \\ 17 & \longleftarrow & 15 & \longrightarrow & 13 & \longleftarrow & 11 & \longrightarrow & 9 & \longleftarrow & 6 & \longrightarrow & 3 \\ \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow \\ 16 & \longrightarrow & 14 & \longleftarrow & 12 & \longrightarrow & 10 & \longleftarrow & 7 & \longrightarrow & 4 & \longleftarrow & 1 \end{array}$$

  - type: $(A_7, c^4 : A_7, c^4 : A_3, c^2)$

  (b) $(A_3 \times 3, A_2 \times 4)$ chamber:

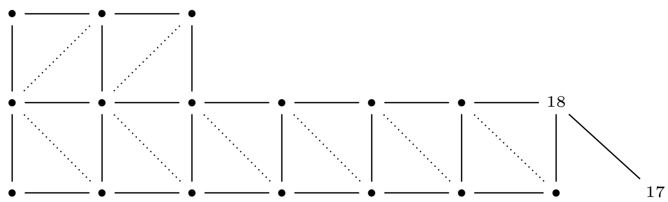
  - complete family: $A_3 = \{1, 3, 2\} + 3k, k = 0, 1, 2$ $A_2 = \{10, 11\} + 2h, h = 0, 1, 2, 3$
  - Weyl–factorized sequence: 1 2 6 8 7 11 12 15 16 3 4 5 9 10 13 14 17 2 1 6 7 8 11 12 15 16 3 5 4 9 10 10
  - permutation:

$$\boldsymbol{m}_\Lambda(Q_{Z_{17}}) = \begin{array}{ccccccccccccc} 1 & \longrightarrow & 4 & \longleftarrow & 7 & & & & & & & & \\ \uparrow & & \downarrow & & \uparrow & & & & & & & & \\ 3 & \longleftarrow & 6 & \longrightarrow & 9 & \longleftarrow & 10 & \longrightarrow & 12 & \longleftarrow & 14 & \longrightarrow & 16 \\ \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow \\ 2 & \longrightarrow & 5 & \longleftarrow & 8 & \longrightarrow & 11 & \longleftarrow & 13 & \longrightarrow & 15 & \longleftarrow & 17 \end{array}$$

∗ type: A_3, c^2, $A_2, s_2 s_1 s_2$ for even k, $A_2, s_1 s_2 s_1$ for odd k.

- $\mathbf{Z_{18}}$

In this case is more convenient to use another representative of Z_{18} under $2d$ wall–crossing. Switching the vacuas $|17\rangle$ and $|18\rangle$, *i.e.* using the α_{17} operation of (**??**), we bring the Coxeter–Dynkin graph of table 2.5 in the form



to this form we apply the method of section §**??** to find a quiver representative of the $4d$ theory. The square form of this quiver is:

$$
Q_{Z_{17}}: \begin{array}{ccccccccccccccc}
2 & \longleftarrow & 5 & \longrightarrow & 8 \\
\downarrow & & \uparrow & & \downarrow \\
3 & \longrightarrow & 6 & \longleftarrow & 9 & \longrightarrow & 11 & \longleftarrow & 13 & \longrightarrow & 15 & \longleftarrow & 18 \\
\uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow \\
1 & \longleftarrow & 4 & \longrightarrow & 7 & \longleftarrow & 10 & \longrightarrow & 12 & \longleftarrow & 14 & \longrightarrow & 16 & \longrightarrow & 17
\end{array}
$$

To obtain the original form use the sequence: (17 18 15 13 11 16 14 18 11 2 1 5 4 3 7 6 8 2 1)$^{-1}$. Now, obtaining the chambers of equation (2.62) is straightforward: one proceeds as for E_{19} using our results about Z_{17}. The more convenient Weyl–factorized sink–sequence to generate the Z_{17} Y–system is the one associated to the chamber $(A_3 \times 3, A_2 \times 4, A_1)$. This is a chamber of $\mu_{17}(Q_{Z_{17}})$, with the obvious complete family of Dynkin's. The sequence is: 3 5 4 9 10 13 14 18 2 1 6 8 7 11 12 15 17 16 3 5 4 9 10 13 14 18 2 1 6 8 7.

- $\mathbf{Z_{19}}$ and $\mathbf{Z_{1,0}}$

These two theories are analogous to Z_{17}. Again we list only the more convenient sink sequence for generating the Y–system.

The square form representative for Z_{19} is obtained with the sequence of mutations 2 1 5 4 3 19 17 18 15 16 19 13 14 17 18 11 12 15 16 19 from the one obtained with §**??**. The result is:

$$Q_{Z_{19}}: \begin{array}{ccccccccccccccc}
2 & \longrightarrow & 5 & \longleftarrow & 8 & & & & & & & & & & \\
\uparrow & & \downarrow & & \uparrow & & & & & & & & & & \\
3 & \longleftarrow & 6 & \longrightarrow & 9 & \longleftarrow & 11 & \longrightarrow & 13 & \longleftarrow & 15 & \longrightarrow & 17 & \longleftarrow & 19 \\
\downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow \\
1 & \longrightarrow & 4 & \longleftarrow & 7 & \longrightarrow & 10 & \longleftarrow & 12 & \longrightarrow & 14 & \longleftarrow & 16 & \longrightarrow & 18
\end{array}$$

The sequence of the chamber $(A_3 \times 3, A_2 \times 5)$ is: 2 1 6 8 7 11 12 15 16 19 3 5 4 9 10 13 14 17 18 2 1 6 7 8 11 12 15 16 19 3 5 4 9.

The $Z_{1,0}$ case is analogous to this one. The square form of the quiver is obtained by removing from the previous one the nodes 16 17 18 19. The transformation that maps it back to the form we deduce from the Coxeter–Dynkin diagram is $(2\ 1\ 5\ 4\ 3\ 15\ 13\ 14\ 11\ 12\ 15)^{-1}$. The sequence of the chamber $(A_3 \times 3, A_2 \times 3)$ is: 2 1 6 8 7 11 12 15 3 5 4 9 10 13 14 2 1 6 8 7 11 12 15 3 5 4 9.

- $\mathbf{W_{17}}$

The study of the stable BPS hypermultiplets of this model is analogous to that of the odd rank elements of the Z family. The square form of the quiver $Q_{W_{17}}$ is

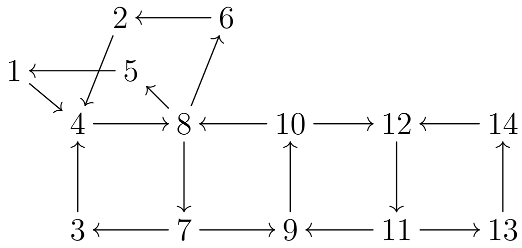
$$Q_{W_{17}}: \begin{array}{ccccccccccc}
2 & \longleftarrow & 5 & \longrightarrow & 8 & \longleftarrow & 11 & \longrightarrow & 14 & & \\
\downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \\
3 & \longrightarrow & 6 & \longleftarrow & 9 & \longrightarrow & 12 & \longleftarrow & 15 & \longrightarrow & 17 \\
\uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow \\
1 & \longleftarrow & 4 & \longrightarrow & 7 & \longleftarrow & 10 & \longrightarrow & 13 & \longleftarrow & 16
\end{array}$$

It is related to the original form by the sequence of mutations: $(\ 1\ 2\ 5\ 4\ 8\ 7\ 11\ 10\ 13\ 3\ 6\ 9\ 12\ 2\ 1\ 4\ 5\ 8\ 7\ 14\ 3\ 6\ 2\ 1)^{-1}$. The two chambers of (2.68) are straightforwardly generated. The more convenient to generate the Y–system is the $(A_3 \times 5, A_2)$ one for which the Weyl–factorized sequence is: 3 5 4 9 11 10 15 16 2 1 6 8 7 12 14 13 17 3 5 4 9 11 10 15 2 1 6 8 7 12 13 14 16.

- $\mathbf{Q_{2,0}}, \mathbf{Q_{16}}$ and $\mathbf{Q_{18}}$

We will discuss $Q_{2,0}$ as an example for all the theories in the Q family of even rank. The square form of $Q_{2,0}$ is

$Q_{Q_{2,0}}:$

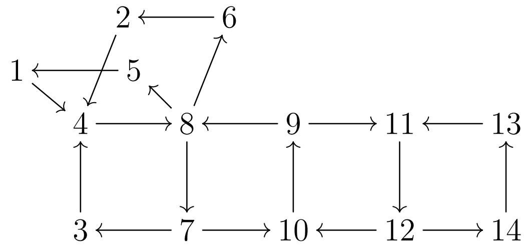


To bring it back to its original form: $(1\ 2\ 3\ 14\ 12\ 10\ 13\ 11\ 14)^{-1}$. The model has two canonical BPS chambers:

(a) $(D_4 \times 2, A_2 \times 3)$ chamber:

  * complete family: $D_4 = \{1,2,3,4\} + 4k, k = 0,1$ $A_2 = \{9,10\} + 2n, n = 0,1,2.$
  * Weyl–factorized sequence: 4 5 6 7 10 11 14 1 2 3 8 9 12 13 4 5 6 7 1 2 3 8 4 5 6 7 10 11 14 1 2 3 8
  * permutation:

$\boldsymbol{m}_\Lambda(Q_{Q_{2,0}}) =$

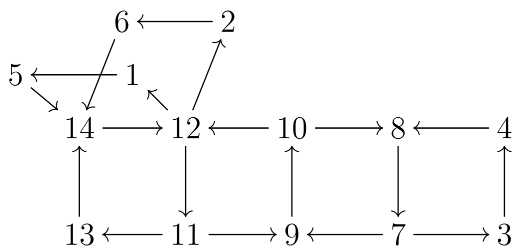


  * type: $D_4, c^3 : D_4, c^3$, while $A_2, s_2 s_1 s_2$ for $n = 0,2$, $A_2, s_1 s_2 s_1$ for $n = 1$.

(b) $(A_2 \times 2, A_5 \times 2)$ chamber:

  * complete family: $A_2 = \{1,5\},\{2,6\}$ $A_5 = \{3,7,10,11,13\} + n, n = 0,1$
  * Weyl–factorized sequence: 2 1 8 12 3 10 13 4 9 14 7 11 8 12 3 10 13 4 9 14 7 11 8 12 3 10 13 5 6 4 9 14 7 11 1 2
  * permutation:

$\boldsymbol{m}_\Lambda(Q_{Q_{2,0}}) =$



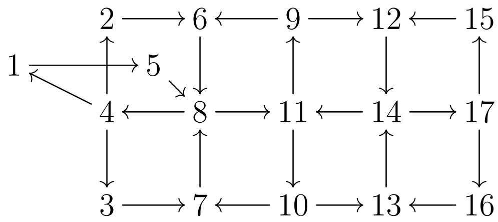
* type: A_5, c^3, $A_2, s_1 s_2 s_1$.

The situation we encountered in chamber (a) is typical for all the even Q's: the D_4 part remains invariant while the A_2 part gets switched. Proceeding analogously one obtains the chambers for Q_{16} and Q_{18} we listed in (2.64) and (2.65).

- $\mathbf{S_{17}}$, $\mathbf{S_{16}}$ and $\mathbf{S_{1,0}}$

The square form of the quiver S_{17} is:

$Q_{S_{17}}$:

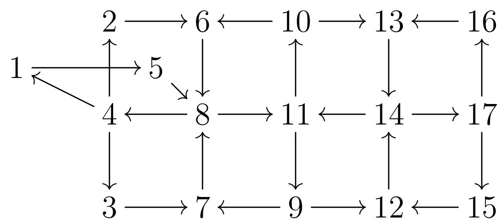


It is related with the Coxeter–Dynkin graph by the sequence of mutations (17 14 15 16 2 1 3 6 7 4 5 $)^{-1}$. The two chambers we listed in (2.70) are easily obtained with the help of the Keller's applet.

(a) $(D_4 \times 2, A_3 \times 3)$ chamber

* complete family: $D_4 = \{1, 2, 3, 4\} + 4k, k = 0, 1$ $A_3 = \{9, 10, 11\} + 3m, m = 0, 1, 2$
* Weyl–factorized sequence: 1 2 3 8 9 10 14 15 16 4 5 6 7 11 12 13 17 1 2 3 8 4 5 6 7 9 10 14 15 16 1 2 3 8 4 5 6 7 11 12 13 17
* permutation:

$\boldsymbol{m}_\Lambda(Q_{S_{17}}) =$



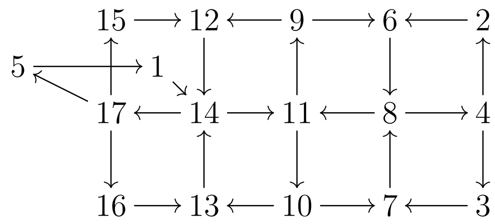
* type: D_4, c^3 and A_3, c^2

(b) $(A_5 \times 3, A_2)$ chamber

* complete family: $A_5 = \{2, 6, 10, 13, 16\}, \{4, 8, 11, 14, 17\}, \{3, 7, 9, 12, 15\}$ $A_2 = \{1, 5\}$
* Weyl–factorized sequence: 5 6 12 4 11 17 7 13 1 2 9 15 8 14 3 10 16 6 12 4 11 17 7 13 2 9 15 8 14 3 10 16 6 12 4 11 17 7 13 2 9 15 8 14 3 10 16 5

∗ permutation:

$\boldsymbol{m}_\Lambda(Q_{S_{17}}) =$

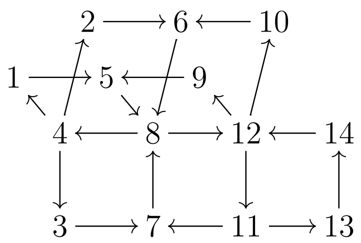


∗ type $A_5 : c^3$, $A_2 : s_2 s_1 s_2$

The cases of S_{16} and $S_{1,0}$ can be treated analogously. The square form of the quiver S_{16} is the same as $Q_{S_{17}}$ with the node 15 removed, and consequent relabeling ($17 \to 16$ and $16 \to 15$), while the quiver of $S_{1,0}$ is obtained by the removal of the nodes 15 16 17. As far as $S_{1,0}$ the shortest sequence to obtain the Y–system is the one leading to the $(D_4 \times 2, A_3 \times 2)$ chamber: 1 2 3 8 9 10 14 4 5 6 7 11 12 13 1 2 3 8 4 5 6 7 1 2 3 8 9 10 14 4 6 7 11 12 13 5. For S_{16} it is the one leading to the $(D_4 \times 2, A_3 \times 2, A_2)$ chamber: 1 2 3 8 9 10 14 15 4 5 6 7 11 12 13 16 1 2 3 8 4 5 6 7 1 2 3 8 9 10 14 4 5 7 6 11 12 13 15.

- $\mathbf{U_{1,0}}$ The square form quiver representative of the $U_{1,0}$ theory is the following:

$Q_{U_{1,0}}$:



Related to the one obtained with the method of §4.1 by the sequence 14 4 9 10 11 13. The analysis of the algebraically trivial chambers of this theory is straightforward. The shortest sink–sequence is the one associated to the $(D_4 \times 3, A_2)$ chamber that reads: 1 2 3 8 9 10 11 14 4 5 6 7 12 13 1 2 3 8 9 10 11 4 5 6 7 12 2 1 3 8 9 10 11 14 4 5 6 7 12.

From the above examples, obtaining the Weyl–factorized sequences for the other chambers we listed in §**??** should now be an easy exercise with the help of Keller's applet [**?**].

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