SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
INTERNATIONAL SCHOOL FOR ADVANCED STUDIES
Physics Area
Astroparticle Physics Curriculum

Applications of the Functional Renormalization Group:

from Statistical Models to Quantum Gravity

Thesis submitted for the degree of
Doctor Philosophiæ

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September 2013

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Chapter 1

Introduction

The Renormalization Group underlies most of our modern understanding of Quantum Field Theories.

There are different ways to implement the RG procedure. Whereas standard sliding scale arguments (Gell-mann-Low) are particularly suitable for weakly coupled computations, it is only with Wilson's ideas that nonperturbative insights have been possible.

The Wilsonian RG is based on the idea of performing a coarse-graining by integrating out high energy modes above a certain cutoff scale, and then rescaling the system to recast it in the original form. In field theory, this means that the Fourier modes of the field are integrated above a certain scale; by moving this scale we obtain the RG flow. Even if we start with a theory with a single type of interaction, the coarse-graining procedure in general turns on all couplings compatible with the symmetries of the system.

Define Theory Space as the space of all possible couplings consistent with symmetries. The beta functions for the couplings define a vector field in theory space, and the RG flow can be seen in geometrical terms as a certain trajectory in this space.

Seen in this terms, the infrared physics we get depends upon the differential equation governing the flow as well as on the boundary conditions, that is, the initial point of the flow. If the inital point sits at a finite scale (for example, it is a bare action depending on some UV cutoff Λ), and this scale cannot be removed (for example, the UV cutoff cannot be pushed to infinity), one is in fact considering an Effective Field Theory, whose range of validity is determined by the initial scale of the flow. However, if we want a theory to be called fundamental, we would like to be able to push the initial scale to arbitrarily high values, eventually to infinity. The only known way[1] to perform this limit physically is to hit a fixed point of the RG.

Fixed point theories do not depend on any intrinsic scale, so they can be used to

[1] Of course in general an RG trajectory may end up in more exotic structures, such as limit cycles or even strange attractors. However, there is increasing evidence nowadays that in unitary theories these behaviors are ruled out.

model systems at a critical phase. They are characterized by dimensionless couplings, and physical quantities computed from them exhibit scaling relations which match the observed relations seen in experiments. These relations arise in very different systems sharing the same dimensionality and symmetry groups. This is what is usually referred to as the concept of Universality, the independence of the critical properties of a system from its microscopic details. The RG offers a simple and intuitive explanation of Universality: since the critical properties of a system are determined by fixed points of the RG, microscopic actions defined at different scales that flow to the same fixed point, or equivalently that belong to the same basin of attraction of a fixed point, will describe the same criticality.

We see that in this light the problem of understanding the critical properties realized in nature boils down to that of classifying all the different fixed points compatible with certain universal features. In two dimensions we know that every scale invariant theory is also conformal invariant, so the problem further reduces to the classification of all possible "minimal" Conformal Field Theories, to be seen as the building blocks of more general CFTs. This can be done via algebraic methods, exploiting the properties of the associated Virasoro algebra.

The RG not only explained in simple terms the phenomenon of Universality, but also shed new light on some of the old problems in Quantum Field Theories.

Through the early days of Quantum Field Theory, the concept of Renormalizability has provided guidance in selecting which theories ought to describe Nature. The idea is nowadays very well known. In a general Quantum Field Theory, when the probability for a certain collision process between particles is computed, a certain number of divergent quantities is found. Through the process of renormalization, one can get rid of these infinities. Roughly speaking, for each infinity that is removed, an indetermined quantity is left behind, and this needs to be fixed by experiment. If only a finite number of these quantities is present, then after a finite number of experiments has fixed them, the theory is able to make predictions. Such a theory is called renormalizable. The good thing of renormalizability is that one can classify the possible interactions of a certain model, according if they are renormalizable or not, and decide to keep only the renormalizable ones. This was (partially) the logic that led to the Standard Model of particle physics, which is today showing further remarkable successes.

One other virtue of the renormalization group has been to demistify the concept of renormalizability a little bit. We now look at renormalizable theories as just Effective Field Theories, whose nonrenormalizable interactions become negligible at sufficiently low energies: in this way a theory can look renormalizable even if it's not. This of course leaves open the question of what can be the fundamental theory (assuming it is unique, which is far from being a trivial assumption) underlying all modern Quantum Field Theories.

This becomes particularly relevant when gravity comes into play, since we know that it does not have renormalizable interactions at all (and the only renormalizable[2] one it has, namely the cosmological constant, is also embarassingly problematic for other reasons which we won't discuss in this work). Of course, gravity might be completed at very high energies by a completely new theory, and this might solve the problem. But another intriguing possibility is offered again by an RG analysis, and it is that gravity might be described at high energies by a fixed point theory. Such a scenario was termed Asymptotic Safety by S. Weinberg, who proposed it as a solution for the problem of the nonrenormalizability of Quantum Gravity. It tells us that gravity is really renormalizable, though in a generalized (nonperturbative) way.

All of these breakthroughs would not have been posible were it not for the Renormalization Group. In this thesis we will touch some of these questions and, hopefully, try to give a feeling of why they are so important.

The present dissertation is essentially a collection of three investigations, whose *fil rouge* is the functional Renormalization Group.

We will start from the most natural scenario where to apply Wilsonian ideas, that is, Quantum Field Theory and Statistical Physics. In particular we will study an important class of models, namely scalar models with an $O(N)$ internal symmetry. The main motivation for this study is not only the importance of these models themselves, but also to show how the functional RG is capable of reproducing nontrivial results (even when considering simplified and approximate flows). In particular, we will see how to use it to explore the structure of theory space and how this is deformed when we change the parameters of the theory, like the dimension or the number of fields. To our knowledge, there is no unique framework, perturbative or not, able to reproduce the results we found in all the range of parameters we considered. Approximate schemes (such as the epsilon expansion or large–N expansion) match our results in some limited range, but break down in some other. All this is quite remarkable if we think that our procedure is not ad hoc, but comes from first principles.

Next, we explore the concept of Weyl invariance. In particular we seek for a general prof that a quantization procedure respecting Weyl invariance is always possible, regardless of the field content of the theory. Using the fRG as a tool to quantize a theory, we will indeed prove this statement in a nonperturbative way. The reason for starting this analysis lies in a conjecture, which states that a fixed point for a gravitational system should correspond to a Weyl invariant theory. At present we don't know whether this is true or not, but surely investigating this fact requires more refined techniques, and we hope that our analysis represents the first step in this direction.

Finally, we come to the topic of dynamical gravity. The proper tool to be used in this setting is the Background Field Method, which when combined with the fRG gives important insights on the structure of theory space for quantum gravity. In particular,

[2]More precisely, *superrenormalizable*

a nongaussian fixed point for the flow of the gravitational couplings has been found, in different approximation schemes, in previous studies of the subject. This is an important indication in favor of the Asymptotic Safety scenario for Quantum Gravity. One of the problems of the gravitational beta functions calculated with this technique is that their system is not closed until we specify the anomalous dimensions of the fluctuation fields. Usually this was solved by giving an approximate form of the anomalous dimensions. Here we compute them explicitly, and use this result to close the flow. As we will see a nongaussian fixed point is still found, with real critical exponents.

Plan of the Thesis

The thesis will consist of three main parts:

Capter 1 will deal with the RG in QFT. Original results in this section are based on

- A. Codello and G. D'Odorico, "$O(N)$ Univarsality classes and the Mermin–Wagner theorem", Phys. Rev. Lett. **110** (2013) 141601

Capter 2 will deal with the RG in Weyl invariant systems. Original results in this section are based on

- A. Codello, G. D'Odorico, C. Pagani and R. Percacci, "The Renormalization Group and Weyl–invariance", Class. Quant. Grav. **30** (2013) 115015

Capter 3 will deal with the RG in Quantum Gravity. Original results in this section are based on

- A. Codello, G. D'Odorico and C. Pagani, "Consistent closure of RG flow equations in quantum gravity", [arXiv:1304.4777]
- A. Codello, G. D'Odorico and C. Pagani, "The Background Effective Average Action approach to Quantum Gravity", *in preparation*

Chapter 2

The functional RG in Quantum Field Theory

2.1 Outline

Our modern understanding of quantum or statistical field theory is based on the ideas put forward by K. Wilson and formalized within the framework of the Renormalization Group (RG) [1]. This approach considers all possible theories describing the quantum or statistical fluctuations of a given set of degrees of freedom, the fields, subject only to the constraints imposed by symmetry and dimensionality; this defines what we call theory space. The process of quantization on one side, or averaging on the other, is then seen as a trajectory connecting the bare action or Hamiltonian to the full quantum or statistical effective action. In most cases one needs an ending point for the trajectory: this usually is a fixed-point. RG fixed-points describe scale invariant theories, where fluctuations on all length scales are equally important: these theories, like lighthouses, shed light on the structure of theory space. They attract or repel surrounding theories giving rise to universality, a phenomenon that underlies both non-perturbative renormalization and the understanding of continuous phase transitions [1]. Once all fixed-points are known we can reconstruct the general (topological) properties of the RG flow and acquire a deep understanding of a given class of models. A paradigmatic example of this is the c-theorem [2], which describes the RG flow between two dimensional theories.

Important information about two dimensional theories comes from exact results for particular lattice models; still, our ability to predict the universal features of two dimensional continuous phase transitions resides on our understanding of the structure of theory space. Three dimensional systems are much more difficult to treat exactly; here too, many analytical insights come from the RG study, otherwise one would have to resort to numerical methods. Deep insights, such as the role played by conformal symmetry in constraining statistical fluctuations, are also naturally embedded in the larger framework of RG analysis

[3].

In this chapter, after introducing the functional RG and its various approximations to be used throughout the thesis, we will use it to explore the theory space of a particular class of models, the scalar $O(N)$–models. These are theories with many applications: they can describe long polymer chains ($N = 0$), liquid-vapor ($N = 1$), superfluid helium ($N = 2$), ferromagnetic ($N = 3$) and QCD chiral ($N = 4$) phase transitions [4, 5]. Despite their relevance, there is no complete description of how universality classes of $O(N)$–models depend continuously on both d and N. We will give such a description by studying scaling solutions of the effective average action [4]. As a result we will find many new $N \geq 2$ universality classes describing multi-critical models in fractal dimension $2 \leq d \leq 3$. In the $N = 0$ case we will observe an infinite number of fixed-points in $d = 2$, analogue to the $N = 1$ minimal–models [6]. We will also show how the Mermin-Wagner-Hohenberg-Coleman (or MWHC for short) theorem [7, 8], which states that there cannot be continuous phase transitions in $d = 2$ systems characterized by continuos symmetries, fits in the RG picture.

Structure

This chapter is organized as follows.

In section 2 we review the Wilsonian RG in Quantum Field Theory, and briefly comment about the connection with statistical systems. In section 3 we introduce the functional RG and the Effective Average Action, and describe their main features. In section 4 we sketch the approximation methods that are usually employed to solve the flow of the EAA. A further (diagrammatic) method is described in section 5, and will be relevant when studying quantum gravity. Section 6 finally introduces the $O(N)$ scalar models, and summarizes the classification of their universality classes through the method of scaling solutions. Section 7 is devoted to the conclusions.

2.2 The Wilsonian Renormalization Group

The Effective Average Action method is based on the Wilsonian conceptual framework to treat the RG. It will thus be very instructive to take a look at what this is, and how it is related to Effective Field Theories, to renormalization in QFT, and to the computation of measurable quantities in systems close to criticality via the important concept of Universality.

2.2.1 Quantum Field Theory

The standard Wilsonian RG starts by defining the path integral over field modes $\tilde{\phi}_p$ with momentum p smaller than some UV cutoff Λ, where

$$\phi(x) = \int \frac{d^d p}{(2\pi)^d} \tilde{\phi}_p e^{-ipx} .$$

In this Fourier basis the measure has the natural definition $\mathcal{D}\phi = \left[\prod_p d\tilde{\phi}_p\right]$. The UV cutoff Λ is then imposed in the partition function as follows

$$Z = \int \left[\prod_{|p|\leq\Lambda} d\tilde{\phi}_p\right] e^{-S_\Lambda[\phi]} .$$

The action $S_\Lambda[\phi]$, the bare action at the scale Λ, is called the Wilsonian Effective Action (WEA). Its explicit form will become clear as we go along.

Here and in the following, unless otherwise stated, we work in Euclidean signature, i.e. we assume a Wick rotation has been performed. The reason for this is that in Minkowski space, as we approach a lightlike direction, the individual components of p can be very large while p^2 gets arbitrarily small, and the Wilsonian prescription becomes ineffective. Also, as we shall show later, the Euclidean theory is the most natural one to describe statistical systems. In fact, standard Euclidean QFT can be seen as a zero temperature statistical field theory (see later).

The Wilsonian RG then happens in three steps:

1. Integrate out high energy modes

2. Rescale all momenta

3. Rescale the fields

Step 1 means that we split $\phi_p = \phi_< + \phi_>$, with $\phi_p = \phi_<$ for $|p| \leq b\Lambda$ and $\phi_p = \phi_>$ for $b\Lambda \leq |p| \leq \Lambda$, with $b < 1$, and we write

$$
\begin{aligned}
Z &= \int \left[\prod_{b\Lambda \le |p| \le \Lambda} d\phi_> \right] \left[\prod_{|p| \le b\Lambda} d\phi_< \right] e^{-S_\Lambda[\phi_< + \phi_>]} \\
&= = \int \left[\prod_{|p| \le b\Lambda} d\phi_< \right] e^{-S^{(<)}_\Lambda[\phi_<]} \,.
\end{aligned}
$$

This defines the new action $S^{(<)}_\Lambda[\phi_<]$ as a function of the "slow modes" $\phi_<$, and is what is meant by integrating out high energy modes. $S^{(<)}_\Lambda$ defines a new Effective Field Theory where all momenta are below $b\Lambda$. S_Λ and $S^{(<)}_\Lambda$ therefore belong to different kinematical regions and we cannot directly compare the couplings as they stand. This is the reason for Step 2: after mode elimination, we rescale all momenta as $p \to b\,p$. To recast the action in its original form we will also need to rescale all the fields. However, we will discuss more in detail Step 3, and the "scaling dimension" of fields, later on.

Notice that the three steps described are in fact the continuum version of the Migdal–Kadanoff recipe, which is

1. Perform a coarse–graining
2. Rescale the system to its original size

This general recipe can be easily adapted to very different settings (probability theory, chaos theory, lattice theories, percolation, ...) and is the most general way to extract universal quantities.

If we denote the regularized measure as $\mathcal{D}_\Lambda\phi$ we see that the cutoff is changed leaving the partition function invariant

$$
Z = \int \mathcal{D}_\Lambda\phi\, e^{-S_\Lambda} = \int \mathcal{D}_{\Lambda'}\phi\, e^{-S_{\Lambda'}} \,.
$$

Thus Wilson's physical recipe is to require that the bare constants of the theory depend on Λ in such a way that all observable quantities are Λ–independent.

The RG flow of the Wilsonian EA S_Λ is generated by varying the cutoff scale Λ. Massive modes will be suppressed in the IR by inverse powers of the mass m. This phenomenon is called decoupling. However, in general, even if we start at a certain scale with a local action, when we integrate out modes corresponding to massless excitations, since there is no low energy mass scale to suppress them, we can find that the IR modes resum into nonlocal terms. For the moment let's forget this phenomenon, and consider local actions.

Let us then expand the WEA in a base of local invariants

$$
S_\Lambda[\phi] = \sum_i \mathcal{G}_i(\Lambda)\, I_i[\phi] = \sum_i \Lambda^{\Delta_i} g_i(\Lambda)\, I_i[\phi] \;,
$$

where the g_i are dimensionless, and Δ_i is the dimension of $\mathcal{G}_i = \Lambda^{\Delta_i} g_i(\Lambda)$. Masses are treated as generalized couplings.

This action contains all possible terms allowed by symmetry, of which in general there can be infinitely many. We can imagine a manifold whose coordinates in a local chart are all the possible couplings at a certain scale Λ: this is called Theory Space, it represents the space of all possible theories with a specified field content. Varying the cutoff Λ we generate a flow in theory space, described by the vector field

$$\beta_i(g) = \Lambda \frac{d}{d\Lambda} g_i(\Lambda) = -\Delta_i g_i + (loops) \ .$$

The $\beta_i(g)$ are called beta functions, and encode the loop (quantum) corrections around the classical values. After loop corrections are computed, we would like to remove the cutoff dependence by sending it to infinity. This is also referred to as a continuum limit, since the presence of the UV cutoff means that the physics is defined above a minimum distance scale Λ^{-1}. There is a simple condition under which the continuum limit can be taken [9]. Consider some physical quantity R like a cross–section or a general reaction rate, with mass dimension D, characterizing a physical process at energy scale E. Standard scaling arguments require

$$R = \Lambda^D f\left(\frac{E}{\Lambda}, g_i(\Lambda), X\right) ,$$

where X stands for all other possible dimensionless dependencies of R. We have seen that the RG is constructed in such a way that physical quantities like R do not depend on the renormalization point Λ, so we can simply take $\Lambda = E$ and the previous equation becomes

$$R = E^D f(1, g_i(E), X) \ .$$

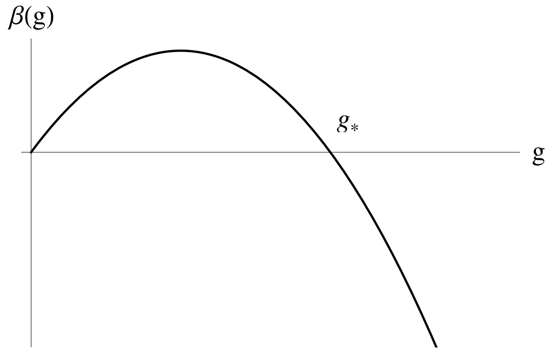
We see that the high energy behaviour of the reaction rate is connected with the behaviour of the couplings $g_i(E)$ as $E \to \infty$. These can have all sort of asymptotic behaviour in general, from limit cycles to chaotic behaviour. However, the simplest types of asymptotics can be understood when theory space is one dimensional, containing a single coupling g. In this case, integrating the flow we find

$$\frac{\Lambda}{\mu} = \exp\left(\int_{g_\mu}^{g_\Lambda} \frac{dg}{\beta(g)}\right) , \tag{2.1}$$

with μ an integration constant. Now suppose that $\beta(g) > 0$ and the integral is convergent, $\int^\infty \frac{dg}{\beta(g)} < \infty$. In this case g moves away from $g = 0$, and becomes infinite at a finite value of E

$$E_\infty = \mu \exp\left(\int_{g_\mu}^{\infty} \frac{dg}{\beta(g)}\right).$$

This is called a "Landau pole". Another possibility is that the beta function starts positive, and then changes sign. To do so it must become zero at a certain point. $\beta(g_\star) = 0$:



The point $g_\star$ is a fixed point of the RG, since when it is reached by an RG trajectory the flow in theory space stops. There are two limiting behaviours for this possibility. The fixed point could be at infinite coupling $g_\star = \infty$, or at zero coupling $g_\star = 0$. In this last case the beta function starts negative and remains negative, so that the coupling is attracted towards the origin. This last situation is called "Asymptotic Freedom", since when it is realized the interactions go to zero at high energy, and the theory becomes free. This is the case for example of Quantum Chromo-Dynamics, where the confining interactions grow with distance, so at sufficiently small length scale the quarks behave as if they were free (a behaviour which was somewhat puzzling before this explanation was found). The theory is thus weakly coupled at high energies, and a perturbative treatment can be used, allowing all the successful computations of QCD.

There is however nothing wrong in supposing that the theory has a nontrivial fixed point $g_\star \neq 0$. In this case it would asymptote to an interacting theory, and perturbative computations may not be possible. Asymptotically we would have in this case

$$R \underset{E\to\infty}{\sim} E^D f\left(1, g_i^\star, X\right)$$

and assuming the function f is well behaved also R will be. For this reason, theories of this type are named "Asymptotically Safe". Trivial fixed points are also called "Gaussian" (since for free theories the path-integral is a gaussian), and nontrivial fixed points are also called "nonGaussian". We will use all these terms interchangeably in the text.

We thus see that the proper condition to take the continuum limit naturally is to reach a fixed point at high energies. It is natural for this to hold in the IR limit as well.

For $g \simeq g_\star$, expanding $\beta(g) \simeq \beta'(g_\star)(g - g_\star)$ and taking $g_{\mu=\bar{m}} \sim g_\star$, we find from eq. (2.1)

$$\bar{m} \sim \Lambda \left| g_\star - g_\Lambda \right|^\nu$$

with $\nu = -\frac{1}{\beta'(g_\star)}$. This is a first example of critical exponent. We will see later that the mass $\bar{m}$ plays the role of an inverse correlation length in statistical field theory. Here, to take the continuum limit we need to send the cutoff $\Lambda \to \infty$ while $\bar{m} \to 0$, which is a reflection of the fact that we are reaching a critical transition and the correlation length diverges.

Having a fixed point theory, we can now perturb it by moving slightly away from the fixed point in theory space, and see what corrections we get. Scaling arguments here can help a lot.

Let's start with the simple case of a gaussian fixed point, which covers all perturbative QFT computations. In this case the term that dominates is the free Lagrangian; take it for definitness to be $\mathcal{L}_0 = \frac{1}{2}(\partial\phi)^2$. Since the action in natural units is dimensionless, if we rescale all energy scales in cutoff units, the dimension of the field is found by requiring that $\left[\int d^d x \, (\partial\phi)^2\right] \sim \Lambda^0$, which gives the usual $[\phi] \sim \Lambda^{\frac{d-2}{2}}$. Then, if we have an interaction $g_i \mathcal{O}_i[\phi]$, knowing the scaling dimension of ϕ we can determine that of $\mathcal{O}_i[\phi]$, and obtain that of g_i. This is called naive or canonical scaling analysis (or canonical power counting).

Before discussing in a little more detailed way the structure of theory space, we can do some heuristic considerations to get the idea [10]. Consider a process at a characteristic energy scale E; then the magnitude of a given term in the action can be estimated as

$$\int d^d x \, \mathcal{O}_i \sim E^{\delta_i - d} \tag{2.2}$$

where δ_i is the (canonical) dimension of the operator $\mathcal{O}_i$. This term then gives an interaction of the order

$$g_i \left(\frac{E}{\Lambda}\right)^{\delta_i - d}. \tag{2.3}$$

If $\delta_i > d$, this term becomes less and less important at low energies, and so is termed *irrelevant*. Similarly, if $\delta_i < d$, the operator is more important at lower energies and is termed *relevant*. An operator with $\delta_i = d$ is equally important at all scales and is *marginal*. This is summarized in the table below, along with the standard terminology from renormalization theory.

δ_i	size as $E \to 0$		
$< d$	grows	relevant	superrenormalizable
$= d$	constant	marginal	strictly renormalizable
$> d$	falls	irrelevant	nonrenormalizable

In most cases there is only a finite number of relevant and marginal terms, so the low energy physics depends only on a finite number of parameters (this statement will be generalized later). For example, this is true of scalar field theory in $d \geq 3$, since an operator $\mathcal{O}_i$ constructed from M ϕ's and N derivatives has dimension

$$\delta_i = M(-1 + d/2) + N. \tag{2.4}$$

It is important to realize that this analysis is based on the assumption that the theory is weakly coupled, so that the free action determines the sizes of typical fluctuations, or matrix elements, of the fields.

In more general situations, the fields will "correct" their scaling dimensions along the flow, so that canonical power counting will have to be corrected as well. For example, in QCD the UV theory is a theory of quarks and gluons with a gaussian FP, while the IR theory is a theory of pions which also has a gaussian FP. However, it is misleading (if not incorrect) to say that the theory flows between two gaussian fixed points, since the UV FP is a free theory for the elementary quark fields q, while the IR FP is a free theory for the bound state $\pi \sim \bar{q}q$. Indeed, in the UV we have $\int d^d x \bar{q}\partial q \sim \Lambda^0$ so $q \sim \Lambda^{\frac{d-1}{2}}$ which is $\Lambda^{\frac{3}{2}}$ in $d = 4$, while since the free pion Lagrangian dominates in the IR, the quark scaling there would be dictated by requiring that $\int d^d x \left(\partial \bar{q} q\right)^2 \sim \Lambda^0$ which gives $q \sim \Lambda^{\frac{d-2}{4}}$ which is $\Lambda^{\frac{1}{2}}$ in $d = 4$. We see in the IR there would be a nontrivial correction to the naive scaling.

To account for this nontrivial scaling we can introduce a wavefunction renormalization constant Z_ϕ for each field ϕ, through the field redefinition $\phi \to Z_\phi^{1/2}\phi$. It is also customary to define the anomalous dimension as

$$\eta_\phi = -\Lambda \frac{d}{d\Lambda} \log Z_\phi \,.$$

We will see later that nontrivial FPs have distinct anomalous dimensions $\eta_\phi^\star \neq 0$. This means, from the previous equation, that Z_ϕ has the asymptotic behaviour $Z_\phi \sim \Lambda^{-\eta_\phi^\star}$. Since the gaussian term now is $\mathcal{L}_0 = \frac{1}{2} Z_\phi \left(\partial\phi\right)^2$, the scaling of the field acquires a correction

$$[\phi] \sim \Lambda^{\frac{d-2}{2} + \frac{\eta_\phi^\star}{2}}$$

which justifies the term "anomalous dimension".

Once we have found a fixed point of the RG flow, critical properties are obtained by linearizing the system around the fixed point, so that the RG equation for small perturbations $\delta g_i = g_i - g_i^\star$ reads

$$\Lambda \frac{d}{d\Lambda} \delta g_i = \left. \frac{\partial \beta_i}{\partial g_j} \right|_{g = g^\star} \delta g_j \equiv M_{ij} \delta g_j \tag{2.5}$$

in which we defined the stability matrix M_{ij} at the fixed point. The general solution is

$$g_i = g_i^\star + \sum_I C_I v_i^I \Lambda^{\lambda^I} \tag{2.6}$$

where v_i^I is the eigenvector of M_{ij} with eigenvalue λ^I

$$\sum_j M_{ij} v_j^I = \lambda^I v_i^I \, . \tag{2.7}$$

The condition for g_i to approach $g_i^\star$ as $\Lambda \to \infty$ is that C_I should vanish for all positive eigenvalues $\lambda^I > 0$. The (linear combinations of) couplings that are attracted to the fixed point are the irrelevant couplings, those that are repelled are the relevant couplings, while the ones with $\lambda^I = 0$ are the marginal ones. This is the generalization of the weak coupling analysis we saw before, valid nonperturbatively. one should keep in mind however that in a general nonperturbative context, the basis of operators that one considers away from the fixed point may differ from the basis of linear perturbations near the fixed point, and in particular what one calls relevant or irrelevant away from the fixed point may change nature once we are near it.

The basin of attraction of the fixed point, that is the set of points in theory space in which every trajectory is attracted to the fixed point, is called the UV *Critical Surface* $\mathcal{S}_{UV}$. The dimensionality of $\mathcal{S}_{UV}$ then is equal to the number of C_I with negative eigenvalues $\lambda^I < 0$, and it sets the number of free parameters to be fixed by experiment to make the theory predictive.

We will call a theory *Asymptotically Safe* if it lies on a finite dimensional UV critical surface (of nonzero dimensionality). Suppose a theory is Asymptotically Safe with a Gaussian fixed point, which means as we have seen that the theory is asymptotically free. Then the beta functions for the dimensionless couplings reduce at the fixed point to $\beta_i(g) = -\Delta_i g_i$ (since loop corrections vanish, and we recover canonical scaling). The stability matrix is then already diagonal

$$M_{ij} = -\Delta_i \delta_{ij} \, . \tag{2.8}$$

Since the theory is Asymptotically Safe, the couplings are constrained to lie on $\mathcal{S}_{UV}$, so all g_i with $\Delta_i < 0$ should vanish. But these are precisely the nonrenormalizable interactions, so this theory must be renormalizable in the usual sense. Thus asymptotic safety can be regarded as a nonperturbative condition for renormalizability (or a condition for nonperturbative renormalizability).

In general, the definition of couplings, and the choice of a basis in theory space, is quite arbitrary. However, the definitions we gave do not suffer from this arbitrariness, since the eigenvalues of the stability matrix are independent of the renormalization scheme. For suppose we change couplings: then the new ones will be functions of

$$\tilde{g}_i\left(\tilde{\Lambda}\right) = \tilde{g}_i\left(\frac{\tilde{\Lambda}}{\Lambda}, g_i(\Lambda)\right) \, . \tag{2.9}$$

Since $\Lambda\partial_\Lambda \tilde{g}_i = 0$ (they are independent of Λ) we find the relation

$$\tilde{\beta}_i\left(\tilde{g}\right) = \sum_j \frac{\partial \tilde{g}_i}{\partial g_j}\beta_j\left(\tilde{g}\right) \tag{2.10}$$

where $\tilde{\beta}_i \equiv \tilde{\Lambda}\partial_{\tilde{\Lambda}}\tilde{g}_i$. This tells us that if a fixed point exists in one renormalization scheme, it will exist also in the other. Differentiating this relation with respect to $\tilde{g}_i$, and using the chain rule, we find that at a fixed point the two stability matrices are related by a similarity transformation

$$\tilde{M}_{ij} = \sum_{kl} U_{ik} M_{kl} U^{-1}_{lj} \tag{2.11}$$

with $U_{ij} = \left(\frac{\partial \tilde{g}_i}{\partial g_j}\right)_{g=g^\star}$, and the eigenvalues of $\tilde{M}$ are the same as those of M. In particular, the critical exponents are invariant (scheme independent).

We have seen that the Wilsonian RG offers a different point of view on renormalizability than it is usually considered in QFT. In fact, a theorem due to Polchinski allows us to be more precise about this [11].

Let us distinguish between renormalizable g_a and nonrenormalizable g_n couplings, with a running over the finite number N of couplings with $\Delta_a \geq 0$, and n running over the infinite number of couplings with $\Delta_n < 0$. If the couplings $g_a\left(\Lambda_0\right)$ and $g_n\left(\Lambda_0\right)$ at some initial cutoff value Λ_0 lie on a generic N–dimensional initial surface[1] $\mathcal{S}_0$, then for $\Lambda \ll \Lambda_0$ they will approach a fixed surface $\mathcal{S}$ that is independent of both Λ_0 and the initial surface. This fixed surface is stable, in the sense that from any point on the surface, the RG trajectory stays on the surface. Such a stable surface defines a finite-parameter set of theories whose physical content is cutoff independent, which is the essential property of renormalizable theories. This also shows that a generic theory defined at cutoff Λ_0 will look for $\Lambda \ll \Lambda_0$ like a perturbatively renormalizable theory. Of course there can be theories whose symmetries and field content do not allow any perturbatively renormalizable interaction, like fermionic theories or gravity; these will in general look like free theories for $\Lambda \ll \Lambda_0$.

The Wilsonian RG, as formulated here, offers many insights into field theoretical questions, but there is an improved version of it which is better suited for quantitative computations, and that can be naturally adapted to study the gravitational case: it is the Effective Average Action (EAA) formalism, and we will describe it in the next section.

We will close this section now by looking at the remarkable phenomenon of universality, which finds an elementary explanation in the context of the RG, and by trying to better clarify the relation between statistical physics and field theory.

[1] This can be for instance the surface with all the nonrenormalizable interactions vanishing

2.2.2 Universality

Equilibrium systems with many degrees of freedom are governed by the laws of statistical mechanics. However, even when these laws give us a rather simple Hamiltonian, the computation of the partition function is usually very difficult. For a realistic Hamiltonian, it can become an hopeless task. Nonetheless, when studying critical phenomena, one finds a remarkable feature, namely that results obtained from simple models match exactly the behaviour of real systems (at least up to so called "corrections to scaling", due to the fact that we can never probe the true asymptotic region close to the critical point).

This phenomenon can be immediately understood in the context of the RG. Critical points of a physical system are associated with fixed points of the RG, the points at which a phase transition happens. Once we tune the parameters of any microscopic Lagrangian or Hamiltonian so that it lies on the critical surface $\mathcal{S}$ of a certain fixed point, the RG flow will drag the theory to the fixed point. So if we start from two different microscopic theories defined on two different points in theory space, but which lie within the same critical surface $\mathcal{S}$, the two theories will flow to the same fixed point, and will thus describe the same phase transition. This is the content of the phenomenon of Universality: the critical phases of a system are determined by the basins of attraction of its fixed points.

From this, one derives the notion of Universality Class of a system, which is given by its symmetry and its dimensionality. Understanding the different universality classes means understanding the different critical phases of a system.

2.2.3 Statistical Field Theory

In this section we want to clarify why the previous construction is relevant to describe statistical systems. We will follow closely Weinberg's treatment [12].

While the aim a QFT is to calculate S–matrix elements, for a statistical system at finite temperature there is no such thing as an S–matrix: since space is filled with debris like black–body radiation, any particle that participate in a collision gets scattered many times before it reaches infinity, so in and out states become ill–defined. Instead, the quantity we are interested in is the partition function

$$Z = Tr \exp\left(-\beta H\right)$$

where H is the Hamiltonian, $\beta = T^{-1}$ and T is the temperature of the system in units in which Boltzmann's constant is one. As usual then one separates $H = H_0 + V$ into free part H_0 and interaction V, goes in interaction representation $V(\tau) = \exp\left(H_0\tau\right) V \exp\left(-H_0\tau\right)$ and uses Dyson's formula to obtain

$$Z = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_1 ... d\tau_n Tr[e^{-\beta H_0} T_\tau\{V(\tau_1)...V(\tau_n)\}] \tag{2.12}$$

where T_τ is the τ–ordering operator. This already renders manifest the fact that (euclidean) QFT is a zero temperature ($\beta \to \infty$) statistical theory. One can define the Green's function for two operators $\chi_{1,2}(\vec{x}_{1,2}, \tau_{1,2})$ as usual

$$G_{12}(\vec{x}_1 - \vec{x}_2, \tau_1 - \tau_2) = Tr \left[e^{-\beta H_0} T_\tau \{\chi_1(\vec{x}_1, \tau_1)\chi_2(\vec{x}_2, \tau_2)\} \right] .$$

Since time here runs over a finite range, as can be seen from (2.12), we can express the propagators as Fourier integrals over momenta $\vec{p}$ but Fourier sums over energies ω. But it is easy to check that they satisfy the following periodicity property:

$$G_{12}(\vec{x}, \tau) = \pm G_{12}(\vec{x}, \tau - \beta)$$

(plus for bosons, minus for fermions), so the Fourier sum runs only over $\omega = n\pi/\beta$, with n even integer for bosons and odd integer for fermions. We see that the only difference with QFT is that every euclidean energy is replaced by an ω satisfying this quantization condition, and so upon making this replacement we can use euclidean QFT to describe a statistical system.

Now, consider for definitness a scalar propagator:

$$\frac{1}{\vec{p}^2 + \omega^2 + m^2} .$$

When a mass goes to zero, a finite value of ω acts as an IR cutoff, and IR divergences arise only from the blowing up of the $\omega = 0$ terms when $\vec{p} \to 0$. Thus, if we want to study long distance behaviour, as for a second order phase transition when the correlation length diverges, we can simply work in the $d-1$ dimensional euclidean field theory defined by $\omega = 0$, and bury all the other terms with $\omega \neq 0$ into effective interactions.

This is the reason why, for instance, we can use a three dimensional euclidean QFT to compute quantities which live in a $3+1$ dimensional world.

The propagator in real space has the asymptotic behaviour

$$G_2\left(r\right) \sim \exp\left(-mr\right) \tag{2.13}$$

and thus we expect correlations on a tipical length scale $mr \sim 1$, or equivalently we find a correlation length $\xi = m^{-1}$. When we approach a phase transition, the mass goes to zero, and the correlation length diverges. This can be seen directly if we recall the definition of the renormalized mass

$$m_R^2 \equiv \lim_{k \to 0} \left({G_2}/{\frac{d^2 G_2}{dk^2}} \right) \tag{2.14}$$

with G_2 the two point function. It is easy to see that for a scalar theory it has the scaling $G_2 \propto k^{-2+\gamma^*}$ at a fixed point, so that indeed $m_R^2 \to 0$.

2.3 Introducing the fRG

2.3.1 Functional regularization

This regularization procedure starts by modifying the path-integral measure introducing a "cutoff action" term ΔS_k designed to cutoff the IR modes. In this way, the modified partition function one considers takes the following form:

$$e^{W_k[J]} \equiv Z_k[J] := \exp\left(-\Delta S_k\left[\frac{\delta}{\delta J}\right]\right) Z[J] = \int_\Lambda \mathcal{D}\varphi\, e^{-S[\varphi]-\Delta S_k[\varphi]+\int J\varphi}\,. \tag{2.15}$$

The simplest way to perform a coarse-graining in this setting is to modify directly the propagation of modes, and this can be achieved with a quadratic regulator:

$$\Delta S_k[\varphi] = \frac{1}{2}\int \frac{d^D q}{(2\pi)^D}\,\varphi(-q)R_k(q)\varphi(q)\,. \tag{2.16}$$

This can be viewed as a momentum-dependent mass term. The regulator function $R_k(q)$, also called "cutoff kernel", should satisfy

$$\lim_{q^2/k^2\to 0} R_k(q) > 0, \tag{2.17}$$

which implements an IR regularization. For instance, if $R_k \sim k^2$ for $q^2 \ll k^2$, the regulator screens the IR modes in a mass-like fashion, $m^2 \sim k^2$. Furthermore,

$$\lim_{k^2/q^2\to 0} R_k(q) = 0, \tag{2.18}$$

which implies that the regulator vanishes for $k \to 0$. As an immediate consequence, we automatically recover the standard generating functional in this limit: $Z_{k\to 0}[J] = Z[J]$. The third condition is

$$\lim_{k^2\to\Lambda\to\infty} R_k(q) \to \infty, \tag{2.19}$$

which induces that the functional integral is dominated by the stationary point of the action in this limit. This justifies the use of a saddle-point approximation which filters out the classical field configuration and the bare action, as we shall see. Incidentally, the regulator is frequently written as

$$R_k(p^2) = k^2\, r(p^2/k^2), \tag{2.20}$$

where $r(y)$ is a dimensionless regulator shape function with a dimensionless momentum argument. The requirements (2.17)-(2.19) translate in an obvious manner into corresponding requirements for $r(y)$.

In general, if our QFT is invariant under some continuous symmetry, like gauge symmetry for nonabelian theories or diffeomorphism invariance for gravity, we want our regulator term to preserve such symmetry. In this case, a slightly more refined construction is needed to perform a covariant coarse–graining. Start by considering the general Hessian

$$\Delta_{AB} = \left.\frac{\delta^2 S\left[\phi\right]}{\delta\phi_A \delta\phi_B}\right|_{\bar{\phi}} .$$

Here, the indices $A, B, ...$ are abstract indices (a notation first introduced by deWitt) encoding all the Lorentz, gauge, spacetime and any other nontrivial structure the field ϕ may have. For instance we can have $\phi_A = \varphi\left(x\right)$ or $\phi_A = H_{\mu\nu abc}\left(x\right)$. The subscript $\bar{\phi}$ indicates that we ought to take the Hessian at a certain reference field configuration, like zero field or on shell or other configurations depending on the problem. The advantage of the abstract index notation is that it allows functionals to be treated as formal (infinite) matrices. The eigenvalues of the "matrix" Δ_{AB} are not invariant under a change of basis: for it to be so we need to "raise an index", that is, construct an endomorphism with a metric g_{AB} defined in field space:

$$\Delta^A_{\ B} = g^{AC}\Delta_{CB} .$$

We will assume that such a metric always exists. We can then diagonalize the endomorphism on a basis of eigenvectors $\mathcal{Y}^A_{(n)}$

$$\Delta^A_{\ B}\mathcal{Y}^B_{(n)} = \lambda_n \mathcal{Y}^A_{(n)} .$$

The eigenvectors $\mathcal{Y}^A_{(n)}$ generalize the flat space Fourier field modes. We will call modes with $\lambda_n \gg k^2$ "rapid" and modes with $\lambda_n \ll k^2$ "slow". The generalized cutoff action will be a function of $\Delta^A_{\ B}$, defined by the coarse–graining requirements:

- $\lim_{k\to 0} R_k\left[\lambda_n\right] \to 0$
- $\lim_{k\to\infty} R_k\left[\lambda_n\right]|_{\lambda\, fixed} \to 0$
- $\lim_{k<<\lambda} R_k\left[\lambda_n\right] \sim 0$

These conditions have the same physical content as before: rapid modes are unaffected by the presence of the regulator, while slow modes perceive it as a mass term forcing their decoupling from the spectrum.

For the applications of this chapter, the cutoff operator can simply be taken as the flat space Laplacian $\Delta = -\partial^2$, and the Fourier analysis done before is sufficient. For nonabelian gauge theories it is chosen to be the gauge Laplacian $\Delta = -D^2$, where D is the gauge covariant derivative, while for gravity it can be taken as the Laplace-Beltrami operator or more general operators (this will be discussed in the last chapter).

2.3.2 Cutoff choice

We have seen in the previous section that only few requirements are imposed on the cutoff ΔS_k. In fact, there is a considerable freedom in choosing the cutoff shape function. Any shape consistent with the requirements would do the job, but of course some are preferable in most situations. Eventually one has to check whether the various quantities depend on the regularization scheme adopted. Let's see the most common shape functions found in the literature.

The simplest one is the "mass" cutoff

$$R_k^{mass}(z) = k^2 \,. \tag{2.21}$$

This strictly speaking does not satisfy all the requirements of the previous section, but can nonetheless be used in certain cases.

Another one, very useful for many analytic calculations, as we shall see, is Litim's, or "optimized" [13], cutoff

$$R_k^{opt}(z) = \left(k^2 - z\right) \theta \left(k^2 - z\right) \,. \tag{2.22}$$

Finally, a third one that is often encountered in the literature is the "exponential" cutoff

$$R_k^{exp}(z) = \frac{z}{\exp\left(z/k^2\right) - 1} \,. \tag{2.23}$$

2.3.3 Effective Average Action

In this section we introduce a generalization of the effective action, called Effective Average Action (EAA), which depends on the infrared cutoff scale k. The main virtue of this definition is that there exists a simple formula for the derivative of the EAA with respect to k, called the Functional RG Equation (FRGE) or Wetterich equation [14], or Exact RG Equation (ERGE).

Definition and general features

The EAA is a functional that smoothly interpolates between the bare action and the standard EA. It is defined via the modified Legendre transform

$$\Gamma_k[\varphi] = \sup_J \left(\int J\varphi - W_k[J] \right) - \Delta S_k[\varphi]. \tag{2.24}$$

The EAA satisfies an integro-differential equation similar to the one satisfied by the standard EA (which is derived in the Appendix). Using the fact that now

$$\frac{\delta\left(\Gamma_k+\Delta S_k\right)}{\delta\varphi}=J_\varphi$$

we find

$$e^{-\Gamma_k[\varphi]-\Delta S_k[\varphi]}=\int\mathcal{D}\chi\exp\left\{-S\left[\varphi+\chi\right]-\Delta S_k\left[\varphi+\chi\right]+\int\chi\left(\frac{\delta\Gamma_k\left[\varphi\right]}{\delta\varphi}+\frac{\delta\Delta S_k\left[\varphi\right]}{\delta\varphi}\right)\right\}.$$

Since the cutoff action is quadratic in the fields we can use the following relation

$$\begin{aligned}-\Delta S_k\left[\varphi+\chi\right]+\Delta S_k\left[\varphi\right]+\int\chi\frac{\delta\Delta S_k\left[\varphi\right]}{\delta\varphi} &= -\frac{1}{2}\int\chi\frac{\delta^2\Delta S_k\left[\varphi\right]}{\delta\varphi\delta\varphi}\chi\\ &= -\Delta S_k\left[\chi\right]\end{aligned}$$

to find the modified integro-differential equation:

$$e^{-\Gamma_k[\varphi]}=\int\mathcal{D}\chi\exp\left\{-S\left[\varphi+\chi\right]-\Delta S_k\left[\chi\right]+\int\chi\frac{\delta\Gamma_k\left[\varphi\right]}{\delta\varphi}\right\}.$$

Let us justify the assertion made in the beginning of this section. The $k\to 0$ limit is trivial: the cutoff action is defined to go to zero in this limit (see previous section), so we immediately have

$$\Gamma_0\left[\varphi\right]=\Gamma\left[\varphi\right].$$

For $k\to\infty$ the cutoff is required to have the asymptotic behaviour $\Delta S_k\left[\varphi\right]\to\frac{1}{2}Ck^2\int\varphi^2$, where the constant C depends on the cutoff shape function. Rescaling the fluctuations as $\chi\to\chi/k$, and assuming that $\frac{\delta\Gamma_k[\varphi]}{\delta\varphi}$ remains finite in the limit $k\to\infty$, we obtain

$$\begin{aligned}e^{-\Gamma_k[\varphi]} &= \int\mathcal{D}\chi\exp\left\{-S\left[\varphi+\frac{\chi}{k}\right]-\frac{1}{k^2}\Delta S_k\left[\chi\right]+\frac{1}{k}\int\chi\frac{\delta\Gamma_k\left[\varphi\right]}{\delta\varphi}\right\}\\ &\underset{k\to\infty}{\to} e^{-S[\varphi]}\int\mathcal{D}\chi\exp\left(-\frac{1}{2}C\int\chi^2\right)\end{aligned}$$

so we conclude

$$\Gamma_\infty\left[\varphi\right]=S\left[\varphi\right]+const\,.$$

We thus see that the bare action to be quantized represents in this framework the initial condition of the flow. In this “fRG quantization” one starts from an intial point in theory space and follows the trajectory defined by the flow down to the endpoint at $k=0$. The initial condition can also be defined at a finite scale Λ; in this case one is doing an Effective Field Theory computation, valid up to the scale Λ. Though Effective Field

Theories may be perfectly reasonable and predictive, if we want a theory to be fundamental we want to be able to push the cutoff scale to infinity. In this case, the initial condition represents a theory with no intrinsic scales, that is, a scale invariant theory. Such theories are obtained when the couplings approach a fixed point of the fRG, namely a point where $\partial_t g_i = 0$. This is the nonperturbative generalization of the concept of renormalization. The trajectory emanating from the fixed point is named "renormalized trajectory". We will have more to say on this point when we will cover the Asymptotic Safety Scenario for Quantum Gravity.

Let's now forget the path-integral quantization, which can be seen here simply as a tool to obtain the fRG equation (much like canonical quantization can be used as a tool to obtain the path-integral). We see that we have obtained a novel quantization procedure; together with the path integral and canonical ones, they offer equivalent but different viewpoints on how to define a QFT starting from a classical theory.

Exact flow equation

The flow of the EAA is described by an exact functional RG equation, which can be derived from the integro-differential equation. Define the "RG time" $t = \ln k$, then a scale derivative gives

$$\begin{aligned}
\partial_t \Gamma_k [\varphi] &= e^{\Gamma_k[\varphi]} \int \mathcal{D}\chi \left(\partial_t \Delta S_k [\chi] - \int \chi \partial_t \frac{\delta \Gamma_k [\varphi]}{\delta \varphi} \right) e^{-S[\varphi+\chi] - \Delta S_k[\chi] + \int \chi \frac{\delta \Gamma_k [\varphi]}{\delta \varphi}} \\
&= \langle \partial_t \Delta S_k [\chi] \rangle - \int \partial_t \frac{\delta \Gamma_k [\varphi]}{\delta \varphi^A} \left\langle \chi^A \right\rangle \\
&= \frac{1}{2} \int \left\langle \chi^A \chi^B \right\rangle \partial_t R_{k,AB}
\end{aligned}$$

where in the last line we used the fact that $\left\langle \chi^A \right\rangle = 0$ by definition. For the same reason, the two point function coincides with the connected one, and we simply have to use the relation between the EAA and the connected Green's functions generator

$$\begin{aligned}
\left\langle \chi^A \chi^B \right\rangle &= \frac{\delta^2 W_k [J]}{\delta J_A \delta J_B} = \left(\frac{\delta^2 \left(\Gamma_k [\varphi] + \Delta S_k [\varphi] \right)}{\delta \varphi^A \delta \varphi^B} \right)^{-1} \\
&= \left(\frac{\delta^2 \Gamma_k [\varphi]}{\delta \varphi^A \delta \varphi^B} + R_{k,AB} \right)^{-1}
\end{aligned}$$

to finally get the Exact Renormalization Group Equation (ERGE):

$$\partial_t \Gamma_k [\varphi] = \frac{1}{2} Tr \left(\frac{\delta^2 \Gamma_k [\varphi]}{\delta \varphi \delta \varphi} + R_k \right)^{-1} \partial_t R_k \, .$$

No approximations were done up to this point and this is the reason why it is called exact. However only in very few cases its exactness has been used with no approximations: its most relevant aspect for what concerns practical applications is that it allows approximations capable of retaining nonperturbative information. We will review the main approximations in the next sections.

This equation has a one-loop structure, and looks like an RG improvement of its one-loop relative

$$\partial_t \Gamma_k [\varphi] = \frac{1}{2} Tr \left(\frac{\delta^2 S [\varphi]}{\delta\varphi\delta\varphi} + R_k \right)^{-1} \partial_t R_k$$

in which the flow is approximated using only the bare hessian. The one-loop structure is maintained in the full flow equation since the cutoff is quadratic in the fields, and we don't pick higher vertices in the rhs of the equation.

The ERGE has many nice features. First, it is functional, which allows to consider the flow of complete functions, containing (in principle) infinitely many couplings. The second feature is that it is a nonlinear PDE, which is a signal of the nontrivial information it encodes. It is also the reason why it is so hard to solve exactly apart fom very special situations. We will see that in certain cases (see the scaling solutions section) it can be converted into an ODE, which can then be solved numerically to obtain nontrivial results. The last crucial feature is the $\partial_t R_k$ term, which acts now as a UV regulator: the equation is thus automatically regulated both in the UV and IR. If we take an initial fixed point condition, the bare scale Λ can be sent to infinity, and then the scale k is no more constrained to be a small scale. By integrating the flow one finds a finite Γ_k at finite k.

2.4 Approximations and computation schemes

The exact equation for the EAA is a functional equation which is very difficult to solve in full generality. Typically one considers a truncation for the EAA hoping this is enough to capture the relevant physical information. Nevertheless it is clear that different approximation techiniques may also be very useful, depending upon the problem at hand. In this section we will describe four such techniques that will be used in different places throughout this thesis.

2.4.1 Vertex expansion

Differentiating the ERGE with respect to φ one obtains an hierarchy of flow equations for the proper vertices of the EAA. For example, the first two equations read:

$$\begin{aligned}\partial_t\Gamma_k^{(1)}[\varphi] &= -\frac{1}{2}\mathrm{Tr}\, G_k[\varphi]\,\Gamma_k^{(3)}[\varphi]G_k[\varphi]\partial_t R_k \\ \partial_t\Gamma_k^{(2)}[\varphi] &= \mathrm{Tr}\, G_k[\varphi]\,\Gamma_k^{(3)}[\varphi]G_k[\varphi]\,\Gamma_k^{(3)}[\varphi]G_k[\varphi]\partial_t R_k \\ &\quad -\frac{1}{2}\mathrm{Tr}\, G_k[\varphi]\,\Gamma_k^{(4)}[\varphi]G_k[\varphi]\partial_t R_k\end{aligned} \tag{2.25}$$

in which we defined the regularized propagator as

$$G_k\,[\varphi] = \left(\Gamma_k^{(2)}\,[\varphi] + R_k\right)^{-1} .$$

Here and in the following, we will often use the abbreviation

$$\Gamma_k^{(n)}[\varphi] \equiv \frac{\delta^n\Gamma_k[\varphi]}{\delta\varphi^n} \tag{2.26}$$

The importance of this expansion lies in the fact that we can truncate the hierarchy at a certain order N, and solve the system of coupled equations to obtain an approximate reconstruction of the EAA. Namely, this is an approximation to the full flow; it can then be combined with a truncation ansatz for the EAA to obtain a powerful technique for nonperturbative computations.

Notice that this is not a loop expansion, since nowhere here we are relying on some perturbative expansion. We will describe the loop expansion in the context of the EAA in the next subsection. The two methods can be seen to agree at one loop.

2.4.2 Loop expansion and perturbation theory

Another working procedure consists in choosing an initial ansatz for the EAA $\Gamma_{0,k}$ and set up an iterative procedure (the subscript indicates that 0 is the loop order of the contribution while k is the RG scale). Indeed one can plug $\Gamma_{0,k}$ into the r.h.s. of the flow equation and integrate the resulting differential equation with the boundary condition $\Gamma_{1,\Lambda} = S_\Lambda$, where S_Λ is the UV (bare) action. The new solution $\Gamma_{k,1}$ can be plugged into the r.h.s. of the flow equation and the procedure can be repeated.

If we set $\Gamma_{0,k} = S_\Lambda$ we recover the standard loop expansion of perturbation theory [15]. However, nothing forbids us to start from a generic fixed point action, satisfying the condition $\partial_t S_* = 0$. We can then perform an iterative solution around this initial condition with little changes.

To see this let us introduce a (generalized) loop counting parameter h; we can expand the EAA around a generic fixed-point action S_*:

$$\Gamma_k = S_* + \sum_{L=1}^{\infty} h^L \Gamma_{L,k}$$

defined by the condition $\partial_t S_* = 0$. The ERGE now takes the form

$$h\partial_t \Gamma_{1,k}[\varphi] + h^2 \partial_t \Gamma_{2,k}[\varphi] + ... = \frac{h}{2} \mathrm{Tr} \left[\frac{\partial_t R_k}{S_*^{(2)}[\varphi] + R_k + h\Gamma_{1,k}^{(2)}[\varphi] + h^2 \Gamma_{2,k}^{(2)}[\varphi] + ...} \right].$$

From this we can read off the L–th loop contribution via

$$\partial_t \Gamma_{L,k}[\varphi] = \frac{1}{(L-1)!} \frac{\partial^{L-1}}{\partial h^{L-1}} \left. \frac{\partial_t \Gamma_k[\varphi]}{h} \right|_{h \to 0}.$$

Let us set up the procedure for the one- and two-loop contributions. The one-loop equation is straightforward:

$$\partial_t \Gamma_{1,k}[\varphi] = \frac{1}{2} \mathrm{Tr}\, G_k[\varphi]\, \partial_t R_k$$

where

$$G_k[\varphi] = \frac{1}{S_*^{(2)}[\varphi] + R_k}.$$

In momentum space we typically have $G_k^{-1} = p^{2-\eta_*} + R_k$, where η_* is the anomalous dimension at the fixed point. Note that in order to start the flow we turned on at least a relevant operator, e.g. a mass. This operator will appear also in G_k but in the following computations we will expand G_k around the fixed-point action so effectively the regularized propagator will have the above form. In some cases a suitable choice of the field configuration can simplify the computation of a particular term, for instance we can choose to work with $\varphi = 0$.

We can integrate the one-loop flow equation and we have

$$\begin{aligned}\Gamma_{1,k} &= -\int_k^\Lambda \frac{dk'}{k'}\partial_{t'}\Gamma_{1,k'} = -\frac{1}{2}\int_k^\Lambda \frac{dk'}{k'}\mathrm{Tr}G_{k'}\partial_{t'}R_{k'} \\ &= \frac{1}{2}\int_k^\Lambda dk'\mathrm{Tr}\partial_{k'}\log G_{k'} = \frac{1}{2}\mathrm{Tr}\log G_k\Big|_k^\Lambda . \qquad (2.27)\end{aligned}$$

Note that we have exchanged the order of the trace and the derivative. This has been possible since we inserted an additional UV regulator Λ. Typically, after having done the trace, one should integrate the beta functional $\partial_t\Gamma_k$ which has no UV or IR singularities. Nevertheless we are interested in recovering the loop expansion and for this reason we also use a UV cutoff which will be eventually removed by the counterterms present in the bare action S_Λ. In the following all manipulations are intended with an implicit UV cutoff Λ.

The one-loop contribution for $k \to 0$ leads to the usual result:

$$\Gamma_{1,k} = \frac{1}{2}\mathrm{Tr}\log S_*^{(2)} - \frac{1}{2}\mathrm{Tr}\log\left(S_*^{(2)} + R_\Lambda\right).$$

If the theory is perturbatively renormalizable we can reabsorbe the divergences into the counterterms of S_Λ.

Now let us consider the two-loop contribution:

$$\partial_t\Gamma_{2,k} = \frac{\partial}{\partial h}\frac{\partial_t\Gamma_k}{h} = -\frac{1}{2}\mathrm{Tr}G_k\left[\Gamma_{1,k}^{(2)}\right]_{\mathrm{ren}} G_k\partial_t R_k = \frac{1}{2}\mathrm{Tr}\left[\Gamma_{1,k}^{(2)}\right]_{\mathrm{ren}}\partial_t G_k.$$

We can plug in the one-loop result previously found. To do that we need to compute $\Gamma_{1,k}^{(2)}$ and we have:

$$\begin{aligned}\delta^2\Gamma_{1,k} &= \delta^2\left(\frac{1}{2}\mathrm{Tr}\log G_k\Big|_k^\Lambda\right) = -\delta\left(\frac{1}{2}G_k^{-1}\delta G_k\right) = \frac{1}{2}G_k\delta G_k^{-1}G_k\delta G_k^{-1} - \frac{1}{2}G_k^{-1}\delta^2 G_k \\ &= \frac{1}{2}G_kS_*^{(3)}G_kS_*^{(3)} - \frac{1}{2}G_k^{-1}\delta\left(-G_kS_*^{(3)}G_k\right) \\ &= \frac{1}{2}G_kS_*^{(3)}G_kS_*^{(3)} + \frac{1}{2}G_k^{-1}\left(-G_kS_*^{(3)}G_kS_*^{(3)}G_k - G_kS_*^{(3)}G_kS_*^{(3)}G_k + G_kS_*^{(4)}G_k\right) \\ &= -\frac{1}{2}G_kS_*^{(3)}G_kS_*^{(3)} + \frac{1}{2}S_*^{(4)}G_k,\end{aligned}$$

where we suppressed all indices. Using the above equation we get

$$\begin{aligned}
\partial_t \Gamma_{2,k} &= -\frac{1}{2}\mathrm{Tr} G_k \left[\Gamma^{(2)}_{1,k}\right]_{\mathrm{ren}} G_k \partial_t R_k = \frac{1}{2}\mathrm{Tr}\left[\Gamma^{(2)}_{1,k}\right]^{ab}_{\mathrm{ren}} \left[\partial_t G_k\right]_{ba} \\
&= \frac{1}{2}\mathrm{Tr}\left[-\frac{1}{2} G_k S^{(3)}_* G_k S^{(3)}_* + \frac{1}{2} S^{(4)}_* G_k\right]^{ab} \left[\partial_t G_k\right]_{ba} \\
&= \frac{1}{2}\partial_t \mathrm{Tr}\left[-\frac{1}{3\cdot 2} G_{k,cd} S^{(3)ade}_* G_{k,ef} S^{(3)bfc}_* G_{k,ab} + \frac{1}{2\cdot 2} S^{(4)ab}_{*cd} G_{k,cd} G_{k,ab}\right]
\end{aligned}$$

So eventually:

$$\Gamma_{2,k} = \mathrm{Tr}\left[-\frac{1}{12} G_{k,cd} S^{(3)ade}_* G_{k,ef} S^{(3)bfc}_* G_{k,ab} + \frac{1}{8} S^{(4)ab}_{*cd} G_{k,cd} G_{k,ab}\right].$$

2.4.3 The derivative expansion

General expansion

One can collect all terms with a fixed number of derivatives in the EAA, thus obtaining an expansion of the form:

$$\begin{aligned}
\Gamma_k[\varphi] &= \int d^d x \left[V_k(\varphi) + \frac{1}{2} Z_k(\varphi)(\partial\varphi)^2 + \frac{1}{2} W_{1,k}(\varphi)\left(\partial^2\varphi\right)^2 \right. \\
&\quad \left. -\frac{1}{2} W_{2,k}(\varphi)\left(\partial^2\varphi\right)\varphi\partial^2\varphi + \frac{1}{4} W_{3,k}(\varphi)(\partial\varphi)^4 + \mathcal{O}\left(\partial^6\right)\right].
\end{aligned}$$

In the following we will focus on the two simplest truncations of this form: the Local Potential Approximation, or LPA, and its slight improvement, the LPA'.

While, as we will see in the following, the LPA already contains a lot of nontrivial physical information, the derivative expansion can be used to have better numerical estimates of the critical quantites computed within this method.

We stress that even if we are projecting the full theory space onto a subspace, so that the resulting flow is not exact, this is a nonperturbative truncation scheme, and thus capable of uncovering highly nontrivial information.

Local Potential Approximation

In the Local Potential Approximation (LPA) one neglects all momentum dependence in the vertices. The EAA will then have the form:

$$\Gamma_k = \begin{pmatrix} kinetic \\ term \end{pmatrix} + \begin{pmatrix} non-derivative \\ interactions \end{pmatrix}.$$

For example, take a scalar field in d dimensions, the LPA reads:

$$\Gamma_k[\varphi] = \int d^d x \left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + V_k(\varphi)\right] .$$

The object $V_k(\varphi)$ is called the Effective Potential.

The ERGE becomes in this case:

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2} Tr \frac{\partial_t R_k\left(-\partial^2\right)}{-\partial^2 + V_k''(\varphi) + R_k\left(-\partial^2\right)} .$$

For constant φ this gives in Fourier space ($z = q^2$):

$$\begin{aligned} \partial_t V_k(\varphi) &= \frac{1}{2}\frac{S_d}{(2\pi)^d}\int_0^\infty dq\, q^{d-1}\frac{\partial_t R_k\left(q^2\right)}{q^2 + V_k''(\varphi) + R_k\left(q^2\right)} \\ &= \frac{1}{2(4\pi)^{d/2}\Gamma(d/2)}\int_0^\infty dz\, z^{\frac{d}{2}-1}\frac{\partial_t R_k(z)}{z + V_k''(\varphi) + R_k(z)} \end{aligned}$$

Choosing the cutoff $R_k(z) = a\left(k^2 - z\right)\theta\left(k^2 - z\right)$, with $a > 0$, the integral can be evaluated analytically in terms of an Hypergeometric function

$$\partial_t V_k(\varphi) = \frac{a c_d k^d}{a + k^{-2}V_k''(\varphi)}\, {}_2F_1\left(\frac{d}{2}, 1, \frac{d}{2} + 1, \frac{a-1}{a + k^{-2}V_k''(\varphi)}\right)$$

where $c_d^{-1} = (4\pi)^{d/2}\Gamma(d/2 + 1)$.

In terms of dimensionless variables $\varphi = k^{d/2-1}\tilde{\varphi}$ and $V_k(\varphi) = k^d \tilde{V}_k(\tilde{\varphi})$ we have

$$\partial_t \tilde{V}_k(\tilde{\varphi}) + d\tilde{V}_k(\tilde{\varphi}) - \left(\frac{d}{2} - 1\right)\tilde{\varphi}\tilde{V}_k'(\tilde{\varphi}) = c_d \frac{a}{a + \tilde{V}_k''(\tilde{\varphi})}\, {}_2F_1\left(\frac{d}{2}, 1, \frac{d}{2} + 1, \frac{a-1}{a + \tilde{V}_k''(\tilde{\varphi})}\right) . \tag{2.28}$$

We can consider the two following cases:

1. For $a = 1$ (optimized cutoff) we find

$$\partial_t \tilde{V}_k(\tilde{\varphi}) = -d\tilde{V}_k(\tilde{\varphi}) + \left(\frac{d}{2} - 1\right)\tilde{\varphi}\tilde{V}_k'(\tilde{\varphi}) + \frac{c_d}{1 + \tilde{V}_k''(\tilde{\varphi})} .$$

2. For $a \to \infty$ (sharp cutoff) we find

$$\partial_t \tilde{V}_k(\tilde{\varphi}) = -d\tilde{V}_k(\tilde{\varphi}) + \left(\frac{d}{2} - 1\right)\tilde{\varphi}\tilde{V}_k'(\tilde{\varphi}) - \frac{d c_d}{2}\log\left(1 + \tilde{V}_k''(\tilde{\varphi})\right) .$$

For small field values the effective potential can be expanded in a Taylor series. Introducing the dimensionless coupling constants $\lambda_{2n} = k^{-d+n(d-2)}\tilde{\lambda}_{2n}$ we have

$$\tilde{V}_k(\tilde{\varphi}) = \sum_{n=0}^{\infty} \frac{\tilde{\lambda}_{2n}}{(2n)!}\tilde{\varphi}^{2n} .$$

A useful fact, which can be immediately verified by acting with ∂_t on the series, is that the scale derivative of the Effective Potential is the generator of the beta functions, in the sense that

$$\tilde{\beta}_{2n} = \frac{(2n)!}{n!} \frac{\partial^n}{\partial\left(\tilde{\varphi}^2\right)^n}\, \partial_t \tilde{V}_k(\tilde{\varphi})\Big|_{\tilde{\varphi}=0} .$$

This equation can be used to systematically generate the beta-functions system to any order. Taking for example the flow equation with Litim's cutoff, the first beta functions read

$$\begin{aligned}
\tilde{\beta}_2 &= -2\tilde{\lambda}_2 - c_d \frac{\tilde{\lambda}_4}{\left(1+\tilde{\lambda}_2\right)^2} \\
\tilde{\beta}_4 &= (d-4)\,\tilde{\lambda}_4 + 6c_d \frac{\tilde{\lambda}_4^2}{\left(1+\tilde{\lambda}_2\right)^3} - c_d \frac{\tilde{\lambda}_6}{\left(1+\tilde{\lambda}_2\right)^2} \\
\tilde{\beta}_6 &= (d-6)\,\tilde{\lambda}_6 - 90c_d \frac{\tilde{\lambda}_4^3}{\left(1+\tilde{\lambda}_2\right)^4} + 30c_d \frac{\tilde{\lambda}_4\tilde{\lambda}_6}{\left(1+\tilde{\lambda}_2\right)^3} - c_d \frac{\tilde{\lambda}_8}{\left(1+\tilde{\lambda}_2\right)^2} \\
\tilde{\beta}_8 &= \ldots
\end{aligned}$$

Implementing this rule in a software like Mathematica or Maple, the system can be pushed to very high orders with little effort.

Statistical quantities can then be also systematically calculated. For example, from the beta function system one can extract the stability matrix at a general fixed point $\tilde{\lambda}^{\star} = \left\{\tilde{\lambda}_2^{\star}, \tilde{\lambda}_4^{\star}, \ldots\right\}$, found as a solution of the system, in the usual way

$$M_{mn} = \frac{\partial \tilde{\beta}_{2n}}{\partial \tilde{\lambda}_{2m}}\bigg|_{\tilde{\lambda}^{\star}} ,$$

diagonalize it on its eigenvector basis, $\sum_n M_{mn} V_n^{(j)} = \ell_j V_m^{(j)}$, and then extract the correlation length critical exponent ν as minus the inverse of the largest IR attractive eigenvalue of M:

$$\nu = -\frac{1}{\ell_{max}} . \tag{2.29}$$

Adding the anomalous dimension

A very useful modification of the LPA, commonly referred to as LPA' in the literature, allows us to introduce the anomalous dimensions in the game without having to deal with all the computational complications of the full derivative expansion. It is obtained by considering the derivative expansion at $\mathcal{O}\left(\partial^2\right)$ and evaluating the flow of Z_k for a constant field configuration, just as in the Effective Potential case.

The starting point is the flow equation for the two-point function, derived in the vertex expansion (we omit for a moment the index k for notational simplicity), in momentum space:

$$\begin{aligned}\dot{\Gamma}^{(2)}_{p,-p} &= \int_q G_q \Gamma^{(3)}_{q,p,-q-p} G_{q+p} \Gamma^{(3)}_{q+p,-q,-p} G_q \dot{R}_q \\ &\quad -\frac{1}{2}\int_q G_q \Gamma^{(3)}_{q,p,-p,-q} G_q \dot{R}_q\end{aligned}$$

where

$$G_q = \frac{1}{\Gamma^{(2)}_{p,-p} + R_q} = \frac{1}{Zq^2 + V^{(2)} + R_q}\,.$$

Performing the variations of the EAA, and choosing a constant field configuration, one arrives at

$$\begin{aligned}\dot{Z}p^2 + \dot{V}^{(2)} &= \int_q G_q^2 G_{q+p}\left[Z^{(1)}\left(q^2 + q\cdot p + p^2\right) + V^{(3)}\right]^2 \dot{R}_q \\ &\quad -\frac{1}{2}\int_q G_q^2\left[Z^{(2)}\left(q^2 + p^2\right) + V^{(4)}\right]\dot{R}_q\,.\end{aligned}$$

The running of Z can be read off from the terms of order p^2. Expanding the propagator and taking care of the angular integrations, we get

$$\begin{aligned}\dot{Z} &= \frac{4v_d}{d}\left(V^{(3)}\right)^2\int_0^\infty dq\, q^{d-1} G_q^2\left[dG'_q + 2q^2\right]\dot{R}_q \\ &\quad +\frac{8v_d}{d}Z^{(1)}V^{(3)}\int_0^\infty dq\, q^{d-1}G_q^2\left[dG_q + (d+2)q^2G'_q + 2q^4G''_q\right]\dot{R}_q \\ &\quad +\frac{4v_d}{d}\left(Z^{(1)}\right)^2\int_0^\infty dq\, q^{d-1}G_q^2\left[(2d+1)G_q + (d+4)q^2G'_q + 2q^4G''_q\right]\dot{R}_q \\ &\quad -2v_dZ^{(2)}\int_0^\infty dq\, q^{d-1}G_q^2\dot{R}_q\end{aligned}$$

in which

$$\begin{aligned}
G'_q &= -\left(Z + R'_q\right) G_q^2 \\
G''_q &= 2\left(Z + R'_q\right)^2 G_q^3 - G_q^2 R''_q \\
v_d &= \left[2\,(4\pi)^{d/2}\,\Gamma\left(d/2\right)\right]^{-1} .
\end{aligned}$$

Using Litim's cutoff $R_q = Z_0\left(k^2 - q^2\right)\theta\left(k^2 - q^2\right)$ and $\partial_t \ln Z_0 = -\eta$, we arrive at

$$\eta_k = c_d \left(\frac{V_k'''}{Z_k}\right)^2 \frac{k^{d+2}}{Z_k\left(k^2 + V_k''/Z_k\right)^4}$$

in which $c_d = 4v_d/d$. In dimensionless variables this becomes

$$\eta = c_d \frac{\left(\tilde{V}_k'''\right)^2}{\left(1 + \tilde{V}_k''\right)^4} . \tag{2.30}$$

2.4.4 Scaling Solutions

Scaling solutions are solutions of $\partial_t \tilde{V}_*(\tilde{\varphi}) = 0$ and correspond to RG fixed-points in the functional space of effective potentials. Every scaling solution, together with its domain of attraction, defines a different universality class.

Let's start for definitness by considering a scalar theory [3] with $\mathbb{Z}_2$-symmetry and let's set the anomalous dimension to zero, $\eta = 0$. Later, when considering $O(N)$ models, the details will change a little bit but the construction will remain basically the same.

By plugging the condition $\partial_t \tilde{V}_*(\tilde{\varphi}) = 0$ into the ERGE eq. (2.28), we find the ODE satisfied by a scaling solution. For example, using the optimized cutoff the equation reads:

$$-d\tilde{V}_*(\tilde{\varphi}) + \frac{d-2}{2}\tilde{\varphi}\,\tilde{V}_*'(\tilde{\varphi}) + c_d \frac{1}{1 + \tilde{V}_*''(\tilde{\varphi})} = 0 . \tag{2.31}$$

The general method can then be described in this terms. One sets up an initial value problem as a function of some initial condition for the effective potential to be specified, call it σ. Then one solves the ODE for different values of σ in a certain range, and the value of σ for which the solution can be extended to the full range of field values determines a scaling solution.

For example, in the model at hand, the $\mathbb{Z}_2$-symmetry of the effective potential requires that its first derivative vanishes at the origin $\tilde{V}_*'(0) = 0$; (2.31) then implies $\tilde{V}_*(0) = \frac{c_d/d}{1+\tilde{V}_*''(0)}$. Since equation (2.31) is a second order non-linear ODE, we need to use numerical methods to solve it. It's easy to set up the initial value problem as a function of the parameter $\sigma = \tilde{V}_*''(0)$ using the two initial conditions just given.

One immediately observes that for most values of the parameter σ the solution ends up in a singularity at a finite value of the dimensionless field. For every d and σ we can

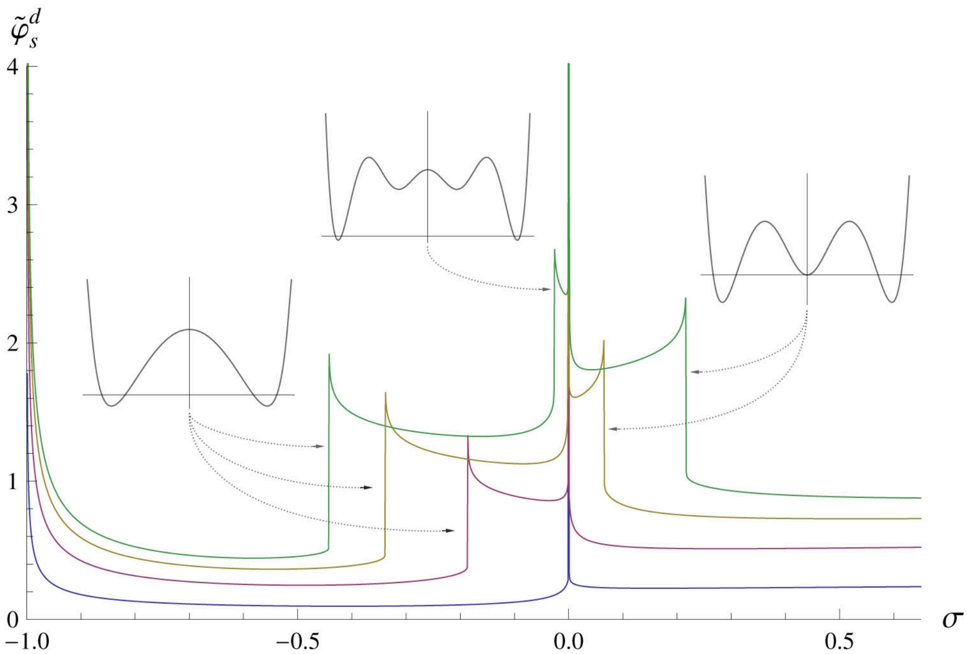


Figure 2.1: The function $\tilde{\varphi}_s^d(\sigma)$ for (starting from below) $d = 4, 3, 2.6, 2.4$. In $d = 4$ one finds only the Gaussian scaling solution, represented by a spike at the origin. As the dimension is lowered new spikes emerge from the Gaussian; these are multi-critical scaling solutions of increasing degree. The three small plots show the general form of the dimensionless effective potentials obtained by integrating (2.31) using, as initial conditions, the positions of the relative spikes, i.e. the values $\sigma_{*,i}$. (Figure adapted from [3]).

call this value $\tilde{\varphi}_s^d(\sigma)$, in this way defining a function [16]. Requiring a scaling solution to be well defined for any $\tilde{\varphi} \in \mathbb{R}$ restricts the admissible initial values of σ to a discrete set $\{\sigma_{*,i}^d\}$ (labeled by i). One can now construct a numerical plot of the function $\tilde{\varphi}_s^d(\sigma)$ to find the $\sigma_{*,i}$ as those values where the function $\tilde{\varphi}_s^d(\sigma)$ has a "spike", since a singularity in $\tilde{\varphi}_s^d(\sigma)$ implies that the relative scaling solution, obtained by integrating the ODE (2.31), is a well defined function for every $\tilde{\varphi} \in \mathbb{R}$. For any d, the function $\tilde{\varphi}_s^d(\sigma)$ gives us a snapshot of theory space, where dimensionless effective potentials are parametrized by σ and where RG fixed-points, i.e. scaling solutions, appear as spikes. By studying $\tilde{\varphi}_s^d(\sigma)$ we will be able to follow the evolution of universality classes as we vary the dimension.

To have a qualitative picture of what happens, start by studying the function $\tilde{\varphi}_s^d(\sigma)$ for $d = 4$. One finds only one spike at $\sigma_{*,1} = 0$ corresponding to the Gaussian scaling solution $\tilde{V}_*(\tilde{\varphi}) = \frac{c_d}{d}$ (the singularity at $\sigma = -1$ is due to the structure of the ODE and does not correspond to any scaling solution). As we decrease the dimension from $d = 4$ we observe a new spike branching to the left of the Gaussian spike: this corresponds to the Ising scaling solution. As we continue to lower d the spike moves to the left and for $d = 3$ the function $\tilde{\varphi}_s^3(\sigma)$ looks as in Figure 2.1. (As one expects, the value of $\sigma_{*,2}$ at which we observe the Ising spike is negative, indicating that the relative scaling solution obtained by integrating the fixed-point equation (2.31) is concave at the origin.)

For non-vanishing anomalous dimension $\eta \neq 0$ the ODE becomes:

$$-d\tilde{V}_*(\tilde{\varphi}) + \frac{d-2+\eta}{2}\tilde{\varphi}\,\tilde{V}_*'(\tilde{\varphi}) + c_d\frac{1-\frac{\eta}{d+2}}{1+\tilde{V}_*''(\tilde{\varphi})} = 0\,, \tag{2.32}$$

and η is given, as we have seen, by eq. (2.30).

Following the $\eta = 0$ case, one can then define the functions $\tilde{\varphi}_s^{d,\eta}(\sigma)$ and identify the discrete values $\{\sigma_{*,i}^{d,\eta}\}$ for which the solutions are well defined for every $\tilde{\varphi} \in \mathbb{R}$. The functions $\tilde{\varphi}_s^{d,\eta}(\sigma)$, for $\eta \neq 0$, turn out to be qualitatively similar to their counter-parts $\tilde{\varphi}_s^d(\sigma)$.

We can then proceed as follows: we fix d and we start with an initial value for η; we compute $\tilde{\varphi}_s^{d,\eta}(\sigma)$ from which we find the values $\{\sigma_{*,i}^{d,\eta}\}$; we use them to solve numerically the ODE (2.32) to obtain the relative scaling solutions; we estimate the anomalous dimension by employing (2.30); we use this value as the ansatz for the next iteration until we converge to a self-consistent solution of (2.32). Using this procedure one is able to find scaling solutions of (2.32) together with the relative anomalous dimensions.

2.5 Vertex Expansion with a background

Before moving to the analysis of $O(N)$ models, we will take a small detour and briefly sketch the vertex expansion in presence of a background, together with the diagrammatic techniques exposed in [17]. We will need this techniques later on when considering gravity.

2.5.1 The vertex expansion

In this section we derive the system of equations governing the RG flow of the proper-vertices of the background EAA, or bEAA, which is the EAA in presence of a background. This generalizes the method already exposed to the case when there are also background fields.

To stress the generality of these rules, we will state them for a general fluctuation φ and a general background $\bar{A}$.

As before, to obtain the equations we take functional derivatives of the flow equation satisfied by the bEAA with respect to the fields φ and $\bar{A}$. When we differentiate with respect to the background field, we have to remember that the cutoff terms present in the flow equation depend explicitly on it: this adds additional terms to the flow equations for the proper-vertices that are not present in the non-background formalism. These terms are in fact crucial to preserve gauge covariance of the gEAA along the flow.

We will however not give all the details of the rules for the background vertices, because we will not use them in this thesis.

Derivation

Taking one functional derivative with respect to the fluctuation or with respect to the background, we obtain the flow equations for the one-vertices of the bEAA:

$$\begin{aligned}
\partial_t \Gamma_k^{(1;0)}[\varphi;\bar{A}] &= -\frac{1}{2} \mathrm{Tr}\, G_k[\varphi;\bar{A}]\, \Gamma_k^{(3;0)}[\varphi;\bar{A}] G_k[\varphi;\bar{A}] \partial_t R_k[\bar{A}] \\
\partial_t \Gamma_k^{(0;1)}[\varphi;\bar{A}] &= -\frac{1}{2} \mathrm{Tr}\, G_k[\varphi;\bar{A}] \left(\Gamma_k^{(2;1)}[\varphi;\bar{A}] + R_k^{(1)}[\bar{A}] \right) G_k[\varphi;\bar{A}] \partial_t R_k[\bar{A}] \\
&\quad + \frac{1}{2} \mathrm{Tr}\, G_k[\varphi;\bar{A}] \partial_t R_k^{(1)}[\bar{A}] \,.
\end{aligned} \tag{2.33}$$

Note that in the second equation of (2.33), where we differentiated with respect to the background field, there are additional terms containing functional derivatives of the cutoff kernel $R_k[\bar{A}]$. Taking further derivatives of equation (2.33), with respect to both the fluctuation field multiplet and background field, gives the following flow equations for the two-vertices[2]:

$$\begin{aligned}
\partial_t \Gamma_k^{(2;0)} &= \mathrm{Tr}\, G_k\, \Gamma_k^{(3;0)} G_k\, \Gamma_k^{(3;0)} G_k \partial_t R_k - \frac{1}{2} \mathrm{Tr}\, G_k\, \Gamma_k^{(4;0)} G_k \partial_t R_k \\
\partial_t \Gamma_k^{(1;1)} &= \mathrm{Tr}\, G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)} \right) G_k\, \Gamma_k^{(3;0)} G_k \partial_t R_k \\
&\quad + \mathrm{Tr}\, G_k\, \Gamma_k^{(3;0)} G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)} \right) G_k \partial_t R_k \\
&\quad - \frac{1}{2} \mathrm{Tr}\, G_k \Gamma_k^{(3;1)} G_k \partial_t R_k - \frac{1}{2} \mathrm{Tr}\, G_k\, \Gamma_k^{(3;0)} G_k \partial_t R_k^{(1)} \\
\partial_t \Gamma_k^{(0;2)} &= \mathrm{Tr}\, G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)} \right) G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)} \right) G_k \partial_t R_k \\
&\quad - \frac{1}{2} \mathrm{Tr}\, G_k \left(\Gamma_k^{(2;2)} + R_k^{(2)} \right) G_k \partial_t R_k \\
&\quad - \mathrm{Tr}\, G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)} \right) G_k \partial_t R_k^{(1)} + \frac{1}{2} \mathrm{Tr}\, G_k \partial_t R_k^{(2)} \,. \qquad (2.34)
\end{aligned}$$

Proceeding in this way we generate a hierarchy of flow equations for the proper-vertices $\Gamma_k^{(n;m)}[\varphi; \bar{A}]$ of the bEAA. In general the flow equation for $\Gamma_k^{(n;m)}[\varphi; \bar{A}]$ involves proper-vertices up to $\Gamma_k^{(n+2;m)}[\varphi; \bar{A}]$ and functional derivatives of the cutoff kernel up to $R_k^{(m)}[\bar{A}]$.

We can define the zero-field proper-vertices as follows:

$$\gamma^{(n;m)}_{k,x_1 \ldots x_n y_1 \ldots y_m} \equiv \Gamma_k^{(n;m)}[0;0]_{x_1 \ldots x_n y_1 \ldots y_m} \,. \qquad (2.35)$$

They can be seen as the coefficients of a Taylor expansion of the functional $\Gamma_k[\varphi; \bar{A}]$ around $\varphi = 0$ and $\bar{A} = 0$:

$$\Gamma_k[\varphi; \bar{A}] = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \int_{x_1 \ldots x_n y_1 \ldots y_m} \gamma^{(n;m)}_{k,x_1 \ldots x_n y_1 \ldots y_m} \varphi_{x_1} \ldots \varphi_{x_n} \bar{A}_{y_1} \ldots \bar{A}_{y_m} \,, \qquad (2.36)$$

The hierarchy of flow equations at $\varphi = 0$ and $\bar{A} = 0$ becomes an infinite system of coupled integro-differential equations for the zero-field proper-vertices $\gamma_k^{(n;m)}$. This system can be used to project the RG flow of all terms of the bEAA which are analytic in the fields φ and $\bar{A}$. In particular these terms can be of non-local character.

Note that in the above considerations is not necessary to expand around the zero-background configuration $\bar{A} = 0$; one can choose to expand around any background, preferably where one is able to perform computations. An example is a constant magnetic field configuration or, in the gravitational case, a sphere or an Einstein space.

[2]Here and in other equations of this section we omit, for clarity, to write explicitly the arguments of the functionals when these are understood.

Compact form

If we introduce the formal operator

$$\tilde{\partial}_t = (\partial_t R_k - \eta_k R_k) \frac{\partial}{\partial R_k}\,, \tag{2.37}$$

where η_k is the multiplet matrix of anomalous dimensions, we can rewrite the flow equation for the bEAA as:

$$\partial_t \Gamma_k[\varphi;\bar{A}] = \frac{1}{2} \mathrm{Tr}\, G_k[\varphi;\bar{A}] \partial_t R_k[\bar{A}] = -\frac{1}{2} \mathrm{Tr}\, \tilde{\partial}_t \log G_k[\varphi;\bar{A}]\,. \tag{2.38}$$

In (2.38) we used the following simple relations:

$$\tilde{\partial}_t G_k = -G_k \partial_t R_k G_k \qquad\qquad \tilde{\partial}_t \log G_k = G_k^{-1} \tilde{\partial}_t G_k = -G_k \partial_t R_k\,.$$

In this way, we can rewrite the flow equation for the one-vertices of the bEAA (2.33) in the following compact form:

$$\begin{aligned} \partial_t \Gamma_k^{(1;0)}[\varphi;\bar{A}] &= \frac{1}{2} \mathrm{Tr}\, \tilde{\partial}_t \left\{ \Gamma_k^{(3;0)}[\varphi;\bar{A}] G_k[\varphi;\bar{A}] \right\} \\ \partial_t \Gamma_k^{(0;1)}[\varphi;\bar{A}] &= \frac{1}{2} \mathrm{Tr}\, \tilde{\partial}_t \left\{ \left(\Gamma_k^{(2;1)}[\varphi;\bar{A}] + R_k^{(1)}[\bar{A}] \right) G_k[\varphi;\bar{A}] \right\}\,, \end{aligned} \tag{2.39}$$

while the flow equations for the two-vertices of the bEAA (2.34) read now:

$$\begin{aligned} \partial_t \Gamma_k^{(2;0)} &= -\frac{1}{2} \mathrm{Tr}\, \tilde{\partial}_t \left\{ \Gamma_k^{(3;0)} G_k \Gamma_k^{(3;0)} G_k \right\} + \frac{1}{2} \mathrm{Tr}\, \tilde{\partial}_t \left\{ \Gamma_k^{(4;0)} G_k \right\} \\ \partial_t \Gamma_k^{(1;1)} &= -\frac{1}{2} \mathrm{Tr}\, \tilde{\partial}_t \left\{ \left(\Gamma_k^{(2;1)} + R_k^{(1)} \right) G_k \Gamma_k^{(3;0)} G_k \right\} + \frac{1}{2} \mathrm{Tr}\, \tilde{\partial}_t \left\{ \Gamma_k^{(3;1)} G_k \right\} \\ \partial_t \Gamma_k^{(0;2)} &= -\frac{1}{2} \mathrm{Tr}\, \tilde{\partial}_t \left\{ \left(\Gamma_k^{(2;1)} + R_k^{(1)} \right) G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)} \right) G_k \right\} \\ &\quad + \frac{1}{2} \mathrm{Tr}\, \tilde{\partial}_t \left\{ \left(\Gamma_k^{(2;2)} + R_k^{(2)} \right) G_k \right\}\,. \end{aligned} \tag{2.40}$$

This compact form can be very useful in actual computations.

2.5.2 Diagrammatic and momentum space techniques

In this section we introduce the diagrammatic representation for the various contributions to the flow equations of the zero-field proper-vertices $\gamma_k^{(n;m)}$, together with the momentum space rules that we will need when considering gravity.

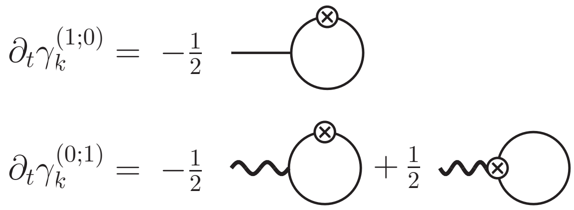


Figure 2.2: Diagrammatic representation of the flow equations for $\partial_t \gamma_k^{(1;0)}$ and $\partial_t \gamma_k^{(0;1)}$ as given in (2.33).

Diagrammatic rules

Diagrammatic techniques have the advantage of making the passage to momentum space straightforward.

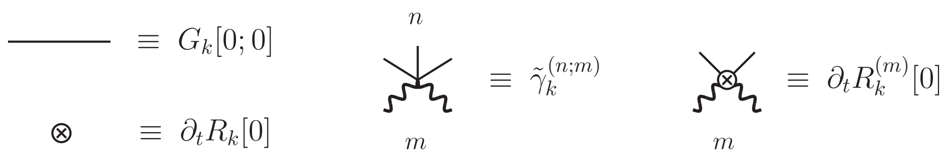
Let us start by introducing the "tilde" bEAA defined by:

$$\tilde{\Gamma}_k[\varphi; \bar{A}] = \Gamma_k[\varphi; \bar{A}] + \Delta S_k[\varphi; \bar{A}]\,, \tag{2.41}$$

and the related "tilde" zero-field proper-vertices

$$\tilde{\gamma}_k^{(n;m)} = \gamma_k^{(n;m)} + \Delta S_k^{(n;m)}[0;0]\,. \tag{2.42}$$

We represent the zero-field regularized propagator $G_k[0;0]$ with an internal continuous line, the cutoff insertions $\partial_t R_k[0]$ are indicated with a crossed circle and the zero-field proper-vertices $\tilde{\gamma}_k^{(n;m)}$ are represented as vertices with n external continuous lines (fluctuation legs) and m external thick wavy lines (background legs). Note that $\Delta S_k^{(n;m)}[0;0] = 0$ if $n > 2$ since the cutoff action is quadratic in the fluctuation fields. This diagrammatic rules are summarized graphically as follows:



Finally, to every closed loop we associate a coordinate or a momentum[3] $\Omega \int_q$ integral (Ω is the space-time volume), together with the factor $\partial_t R_k - \eta R_k$. Here the anomalous dimension η_k pertains to the fields present in the cutoff action. The application of these diagrammatic rules to the flow equations (2.33) for the zero-field one-vertices $\partial_t \gamma_k^{(1;0)}$ and $\partial_t \gamma_k^{(0;1)}$ gives the representation of Figure 2.2, while the flow equations (2.34) for the zero-field two-vertices $\partial_t \gamma_k^{(2;0)}$, $\partial_t \gamma_k^{(1;1)}$ and $\partial_t \gamma_k^{(0;2)}$ can be represented as in Figure 2.3.

[3]We define $\int_x \equiv \int d^d x$ and $\int_q \equiv \int \frac{d^d q}{(2\pi)^d}$.

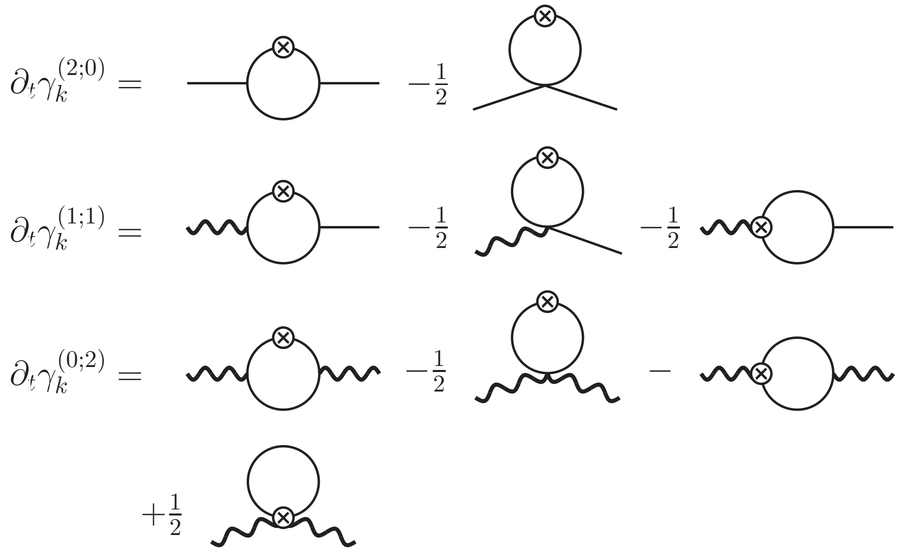
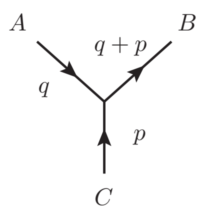


Figure 2.3: Diagrammatic representation of the flow equations for the vertices $\partial_t \gamma_k^{(2;0)}$, $\partial_t \gamma_k^{(1;1)}$ and $\partial_t \gamma_k^{(0;2)}$ as in equation (2.34).

Momentum space representation

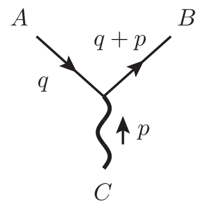
The zero-field proper-vertex $\gamma_k^{(3;0)}$ is represented graphically by the following diagram:



$$= \left[\gamma^{(3;0)}_{q,-q-p,p}\right]^{ABC}$$

Here we use capital letters to indicate general composite indices, in the case of non-abelian gauge theories these have to be interpreted as $A = a\,\alpha, B = b\,\beta, C = c\,\gamma$, while, for example, in the gravitational context they have to interpreted as $A = \alpha\beta, B = \gamma\delta, C = \epsilon\kappa$. Note that each index is associated with a momentum variable, so that A, B, C are the indices of the related momenta $q, p, -q-p$ respectively. Note also that we always define ingoing momenta as being positive.

The two-fluctuations one-background zero-field proper-vertex $\tilde{\gamma}_k^{(2;1)} = \gamma_k^{(2;1)} + \Delta S_k^{(2;1)}[0;0]$ is represented graphically by the diagram:



$$= [\tilde{\gamma}^{(2;1)}_{q,-q-p,p}]^{ABC}$$

The explicit momentum space representation for this vertex involves introducing the "cutoff operator action" $L[\varphi; \bar{A}]$, defined as that action whose Hessian with respect to φ is the cutoff operator, together with its vertices

$$l^{(n;m)}_{x_1 \ldots x_n y_1 \ldots y_m} \equiv L^{(n;m)}[0;0]_{x_1 \ldots x_n y_1 \ldots y_m} \, .$$

Then, the momentum space vertex is

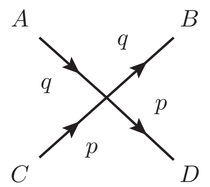
$$[\tilde{\gamma}^{(2;1)}_{q,-q-p,p}]^{ABC} = [\gamma^{(2;1)}_{q,-q-p,p}]^{ABC} + [l^{(2;1)}_{q,-q-p,p}]^{ABC} R^{(1)}_{q+p,q} \, , \tag{2.43}$$

where

$$R^{(1)}_{q+p,q} \equiv \frac{R_{q+p} - R_q}{(q+p)^2 - q^2} \tag{2.44}$$

represents the first finite-difference derivative of the cutoff shape function $R_q \equiv R_k(q^2)$.

And so on. For instance, the four-fluctuation vertex $\gamma^{(4;0)}_k$ (for a particular combination of moments) is represented graphically as:



$$= [\gamma^{(4;0)}_{q,-q,p,-p}]^{ABCD}$$

For the flow of the zero-field fluctuation-fluctuation two-vertex, one finds the following momentum space representation:

$$\begin{aligned} [\partial_t \gamma^{(2;0)}_{p,-p}]^{AB} &= \Omega \int_q (\partial_t R_q - \eta R_q)[G_q]^{12}[\gamma^{(3;0)}_{q,-q-p,p}]^{2A3}[G_{q+p}]^{34}[\gamma^{(3;0)}_{q+p,-q,-p}]^{4B5}[G_q]^{51} \\ &\quad -\frac{1}{2}\Omega \int_q (\partial_t R_q - \eta R_q)[G_q]^{12}[\gamma^{(4;0)}_{q,-q,p,-p}]^{2AB3}[G_q]^{31} \, . \end{aligned} \tag{2.45}$$

In (2.45) η is the multiplet matrix of anomalous dimensions of the fluctuation fields in φ. We are using the generalized notation for the indices introduced before and integers denote dummy indices. With respect to the first equation in Figure 2.3, the first line in (2.45) is the contribution from the first diagram, while the second line is the contribution from the second one.

These are the diagrams we need to evaluate the anomalous dimensions in the gravitational case. But it's clear that in the same way one can construct the momentum space representation for any zero-field vertex flow equation.

For example, the second equation in (2.34), describing the flow of the fluctuation-background zero-field two-vertex, takes the following form:

$$\begin{aligned}
[\partial_t \gamma^{(1;1)}_{p,-p}]^{AB} \;=\;& \Omega \int_q (\partial_t R_q - \eta R_q)[G_q]^{12}[\tilde{\gamma}^{(2;1)}_{q,-q-p,p}]^{2A3}[G_{q+p}]^{34}[\gamma^{(3;0)}_{q+p,-q,-p}]^{4B5}[G_q]^{51} \\
& -\frac{1}{2}\Omega \int_q (\partial_t R_q - \eta R_q)[G_q]^{12}[\tilde{\gamma}^{(3;1)}_{q,-q,p,-p}]^{2AB3}[G_q]^{31} \\
& -\Omega \int_q [l^{(2;1)}_{q,-q-p,p}(\partial_t R^{(1)}_{q+p,q} - \eta R^{(1)}_{q+p,q})]^{4A1} \\
& \times [G_{q+p}]^{12}[\gamma^{(3;0)}_{q+p,-q,-p}]^{2B3}[G_q]^{34}\,. \qquad (2.46)
\end{aligned}$$

2.6 $O(N)$–models and the MWHC theorem

2.6.1 Flow equation and η in the $O(N)$ case

For $O(N)$–symmetric linear sigma models (also called vector models), the LPA' truncation ansatz reads

$$\Gamma_k[\varphi] = \int d^d x \left[U_k(\rho) + \frac{1}{2} Z_k \partial_\mu \varphi_a \partial^\mu \varphi_a \right]$$

where $\rho = \frac{1}{2}\varphi_a\varphi_a$. The Hessian is

$$\Gamma^{(2)}_{k,ab}[\varphi] = -Z_k \partial^2 \delta_{ab} + U'_k(\rho)\,\delta_{ab} + U''_k(\rho)\,\varphi_a\varphi_b\,.$$

A convenient way to invert this formula is by defining the projector $P_{ab} = \varphi_a\varphi_b/\varphi^2$ and using

$$\left[A(1-P)_{ab} + BP_{ab}\right]^{-1} = \frac{1}{A}(1-P)_{ab} + \frac{1}{B}P_{ab}\,.$$

In this way one finds

$$\begin{aligned}\left(\Gamma^{(2)}_{k,ab}[\varphi] + R_{k,ab}[\varphi]\right)^{-1} &= \frac{(1-P)_{ab}}{-Z_k\partial^2 + R_{k,ab} + U'_k(\rho)} \\ &\quad + \frac{P_{ab}}{-Z_k\partial^2 + R_{k,ab} + U'_k(\rho) + 2\rho U''_k(\rho)}\,.\end{aligned}$$

Here the cutoff kernel is taken to have a diagonal structure $R_{k,ab} = R_k\delta_{ab}$. Using the traces $(1-P)_{aa} = N-1$ and $P_{aa} = 1$ we arrive at the flow equation for the effective potential

$$\begin{aligned}\partial_t U_k(\rho) &= \frac{dc_d}{4}(N-1)\int_0^\infty dz\, z^{\frac{d}{2}-1}\frac{\partial_t R_k(z)}{Z_k z + U'_k(\rho) + R_k(z)} \\ &\quad + \frac{dc_d}{4}\int_0^\infty dz\, z^{\frac{d}{2}-1}\frac{\partial_t R_k(z)}{Z_k z + U'_k(\rho) + 2\rho U''_k(\rho) + R_k(z)} \\ &= c_d(N-1)\left(1 - \frac{\eta_k}{d+2}\right)\frac{k^d}{1 + \frac{U'_k(\rho)}{Z_k k^2}} \\ &\quad + c_d\left(1 - \frac{\eta_k}{d+2}\right)\frac{k^d}{1 + \frac{U'_k(\rho) + 2\rho U''_k(\rho)}{Z_k k^2}}\,.\end{aligned}$$

In the last equality we chose the optimized cutoff $R_k = Z_k\left(k^2 - z\right)\theta\left(k^2 - z\right)$ and used the definition $\partial_t Z_k = -\eta_k Z_k$.

In terms of dimensionless variables

$$\rho = Z_k^{-1} k^{d-2}\tilde{\rho}$$

$$U_k(\rho) = k^d \tilde{U}_k(\tilde{\rho})$$

the equation becomes

$$\begin{aligned}\partial_t \tilde{U}_k(\tilde{\rho}) - (d-2+\eta_k)\,\tilde{\rho}\tilde{U}_k'(\tilde{\rho}) + d\tilde{U}_k(\tilde{\rho}) &= c_d\,(N-1)\left(1-\frac{\eta_k}{d+2}\right)\frac{1}{1+\tilde{U}_k'(\tilde{\rho})} \\ &\quad + c_d\left(1-\frac{\eta_k}{d+2}\right)\frac{1}{1+\tilde{U}_k'(\tilde{\rho})+2\tilde{\rho}\tilde{U}_k''(\tilde{\rho})}\,.\end{aligned}$$

A scaling solution is defined by the condition that $\partial_t \tilde{U}_\star(\tilde{\rho}) = 0$, which finally gives us the equation for the scaling potential in the $O(N)$ case

$$\begin{aligned}(d-2+\eta_k)\,\tilde{\rho}\tilde{U}_k'(\tilde{\rho}) - d\tilde{U}_k(\tilde{\rho}) + c_d\,(N-1)\left(1-\frac{\eta_k}{d+2}\right)\frac{1}{1+\tilde{U}_k'(\tilde{\rho})} & \\ + c_d\left(1-\frac{\eta_k}{d+2}\right)\frac{1}{1+\tilde{U}_k'(\tilde{\rho})+2\tilde{\rho}\tilde{U}_k''(\tilde{\rho})} &= 0\,. \qquad (2.47)\end{aligned}$$

All the results of this section will emerge from this apparently simple ODE, using the method of scaling solutions.

The anomalous dimension η will also be needed in what follows. It can be derived using the methods exposed previously. The computation is long and tedious, but straightforward, so we don't need to repeat it here. One needs only to pay some attention to the vector structure in field space, using the projectors. To lowest order its value is related to the running dimensionless effective potential by [4]:

$$\eta = c_d \frac{4\tilde{\rho}_0 \tilde{U}_*''(\tilde{\rho}_0)^2}{\left[1+2\tilde{\rho}_0\tilde{U}_*''(\tilde{\rho}_0)\right]^2}\,, \qquad (2.48)$$

with $\tilde{\rho}_0$ the absolute minimum $\tilde{U}_*'(\tilde{\rho}_0) = 0$.

Every scaling solution, together with its domain of attraction, represents a different universality class; thus by finding the solutions of the system composed of (2.47) and (2.48) one can determine $O(N)$–universality classes. Unlike other implementations of the RG, all the analysis can be made leaving d and N as free parameters, permitting us to study how theory space depends on these.

2.6.2 Scaling solutions in the $O(N)$ case

Mermin-Wagner-Hohenberg-Coleman theorem

We solve the fixed-point equations (2.47) and (2.48) by the method discussed before. For every d and N we find a discrete set of scaling solutions to these equations. These correspond to multi-critical potentials of increasing order with i minima (which we label by i), which are potentials describing multi-critical transitions, in which one needs to tune

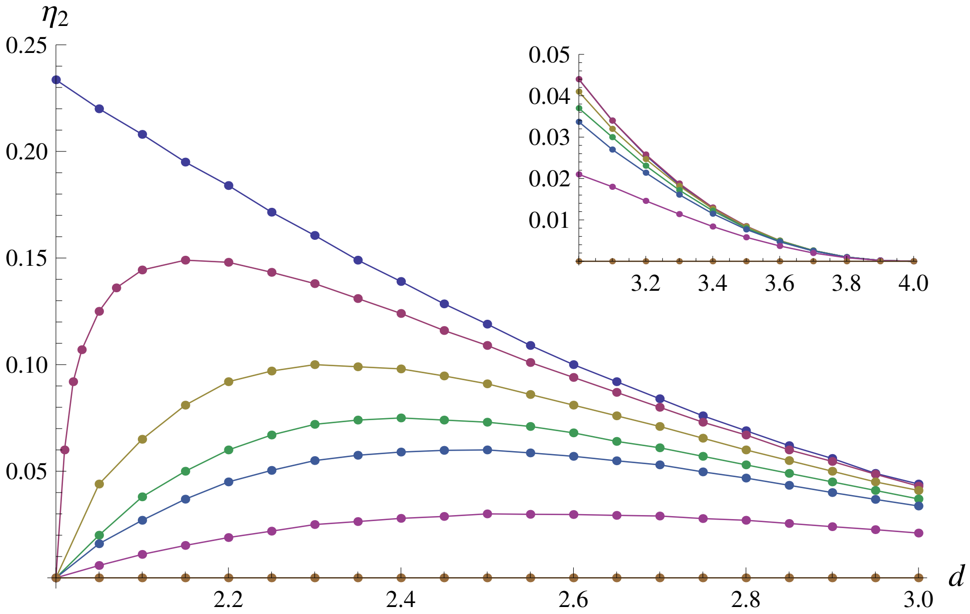


Figure 2.4: η_2 as a function of d for (from above) $N = 1, 2, 3, 4, 5, 10, 100$. In the inset we show the anomalous dimensions in the range $3 \leq d \leq 4$ (note that the $N = 1$ and $N = 2$ curves are almost overlapping).

multiple parameters to reach the critical point. For each of these it is possible to obtain the anomalous dimension η_i as a function of d and N. By studying the function $\eta_i(d, N)$ we can follow the evolution through theory space of the fixed-point representing the i–th multi-critical potential.

For $d > 4$ we find only the Gaussian fixed-point ($i = 1$); at $d = 4 - \epsilon$, just below the upper critical dimension for $O(N)$–models, the Wilson-Fisher fixed-points ($i = 2$) start to branch away from the Gaussian fixed-point. In $d = 3$ these fixed-points describe the known universality classes of the Ising, XY, Heisenberg and other models; our estimates for the anomalous dimensions turn out to be in good agreement with estimates available in the literature [5, 18]. Approaching $d = 2$ one clearly observes that only the $N = 1$ anomalous dimension continues to grow [4]: for all other values of $N \geq 2$ the anomalous dimension bends downward to become zero exactly when $d = 2$. This is a non-trivial fact, not evident from the structure of equation (2.47), telling us that only the $O(N)$–model with discrete symmetry ($N = 1$) can have a second-order phase transition in two dimensions, while all the $O(N)$–models with continuous symmetry ($N \geq 2$) cannot. This result, that here emerges solely from the RG analysis, is commonly known as the Mermin-Wagner-Hohenberg-Coleman (MWHC) theorem [7]. In this respect Figure 2.5 shows the way in

[4] Our result $\eta_2(2, 1) = 0.234$ is in good agreement with the exact result $\eta_{ex} = 0.25$; to provide an error on this estimate one needs to consider higher orders of the derivative expansion [18].

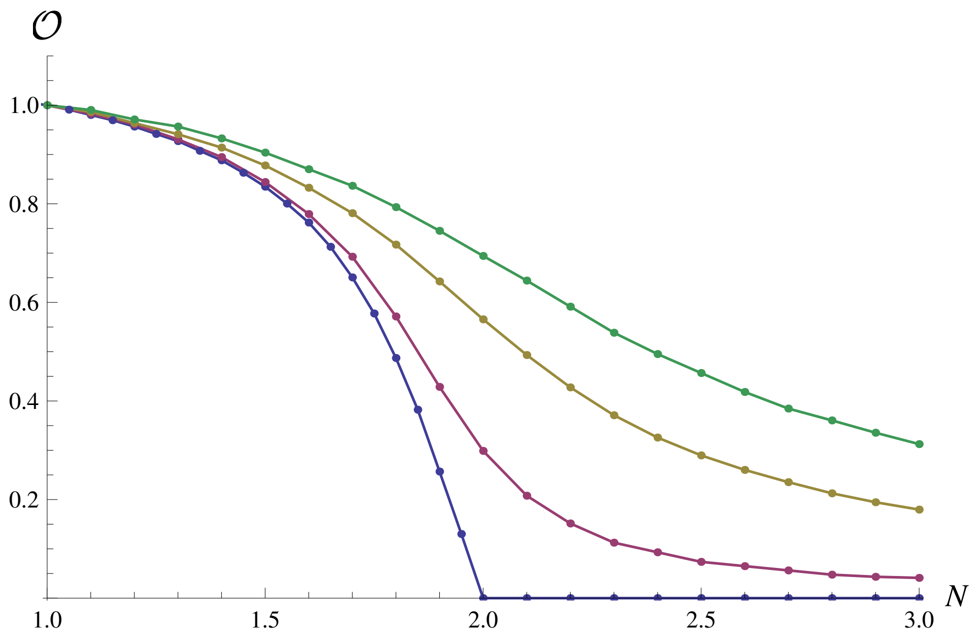


Figure 2.5: $\mathcal{O}(d,N) = \eta_2(d,N)/\eta_2(d,1)$ as a function of N for (from above) $d = 2.1, 2.05, 2.01, 2$. $\mathcal{O}(d,N)$ can be interpreted as the order parameter of a continuos phase transition in which N plays the role of the control parameter.

which the MWHC theorem manifests itself in the RG framework; our analysis can be seen as a RG confirmation of this important theorem and can be the starting point for a new rigorous proof of it. Note also that, as expected from the exact solution [19], the anomalous dimension tends to zero for $N \to \infty$.

That the vanishing of the anomalous dimension implies that there are no continuous phase transitions for the $N \geq 2$ models in $d = 2$ can be confirmed by the analysis of the critical exponent $\nu_2(d,N)$, defined as in (2.29), which indeed blows up for $d \to 2$ and $N \geq 2$ [20]. This allows us to distinguish the Spherical model, related to the $N \to \infty$ limit, from the Gaussian model, both having $\eta = 0$. Only the $N = 1$ model has a finite ν_2 in two dimensions, in all other cases ν_2 diverges upon approaching $d = 2$, as in the $N \to \infty$ limit where one knows exactly that $\nu_2(d,\infty) = \frac{1}{d-2}$.

The critical case $N = 2$ is known to have a distinguished behavior [21]. In this case one can observe all the distinctive properties of the Kosterliz-Thouless phase transition by studying the properties of the RG flow [22].

Our functions $\eta_2(d,N)$ can be compared with large–N expansion analogs [23] which fail to reproduce the small N region, both qualitatively ($N = 1$) and quantitatively ($N < 10$). To our knowledge, our method is the only able to give accurate theoretical estimates valid for every d and N.

To better discriminate between theories which can undergo a continuous phase tran-

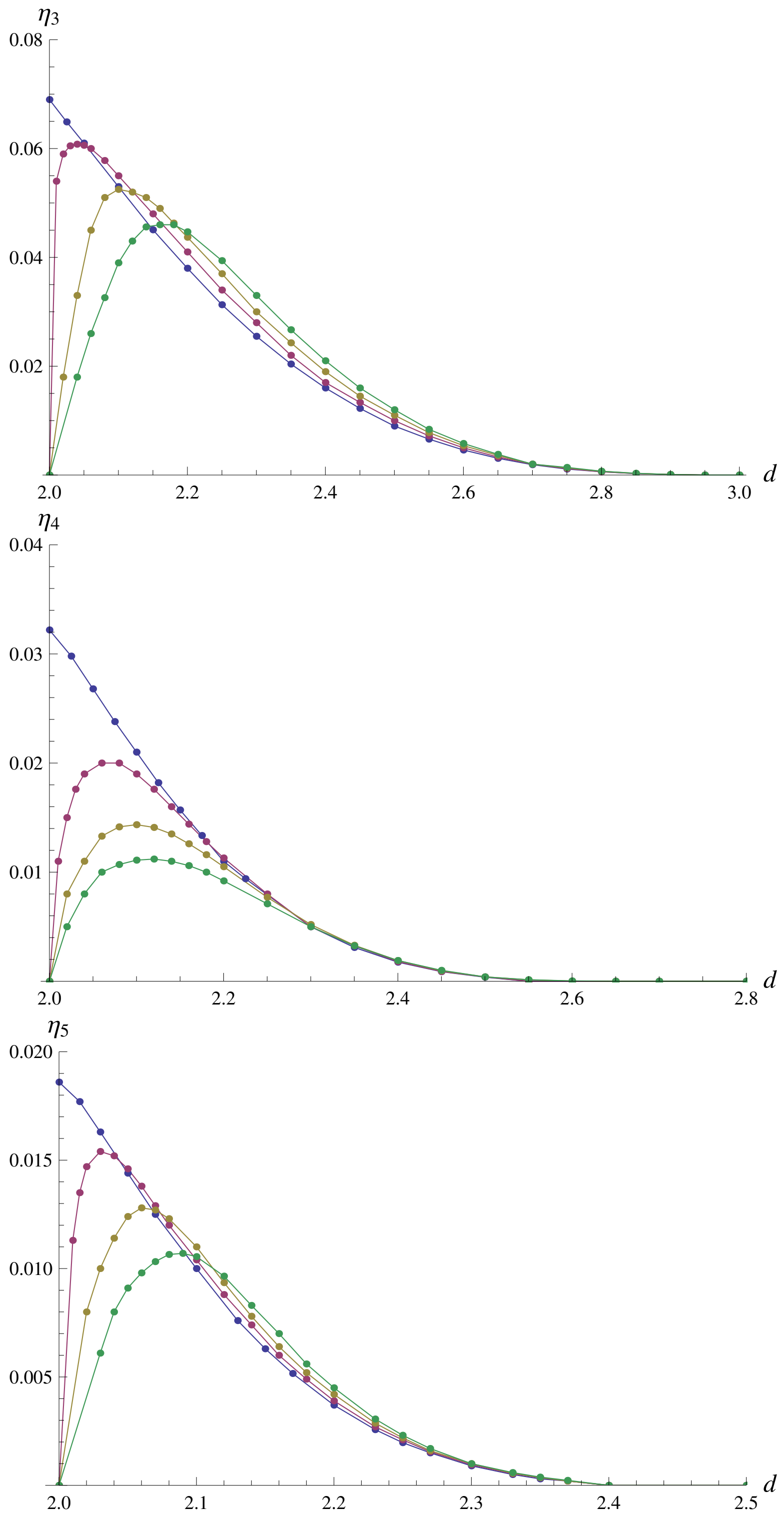


Figure 2.6: η_i as a function of d for (from top to bottom) the tri-critical ($i = 3$), tetra-critical ($i = 4$) and penta-critical ($i = 5$) scaling solutions for (from top of each figure at $d = 2$) $N = 1, 2, 3, 4$.

sition in $d_* = 2$ and those which cannot, we extend the analysis of scaling solutions to non-integer N; in particular we want to see what happens around the critical value $N_* = 2$. The MWHC theorem tells us that at $d = d_*$ the quantity $\mathcal{O}(d, N) = \eta_2(d, N)/\eta_2(d, 1)$ can be seen as a sort of order parameter, meaning it is zero for $N > N_*$ and non-zero for $N < N_*$; but it tells us nothing about its continuity in N. Figure 2.5 shows that the RG analysis can say a lot more about this. First, we see that $\mathcal{O}(d, N)$ evolves continuously with N across N_*; second, we see that $\mathcal{O}(d, N)$ can be written in a scaling form around the transition point $(d_*, N_*) = (2, 2)$; in particular we can write the following scaling relation:

$$\mathcal{O}(d_*, N) \sim \begin{cases} \left(\frac{N_* - N}{N_*}\right)^{\Theta} & N \to N_*^- \\ 0 & N \to N_*^+ \end{cases} , \tag{2.49}$$

where we introduced a new scaling exponent Θ. A fit from the data displayed in Figure 2.5 gives the estimate $\Theta \approx 0.98$ which is quite close to one. Relation (2.49) tells us how theory space deforms as we vary the control parameter N. An interesting question is if relation (2.49) is universal, in the sense that the value of Θ is independent of the details of the implementation of the RG procedure. One can make a similar reasoning by keeping N fixed at N_* and varying d around d_*:

$$\mathcal{O}(d, N_*) \sim \begin{cases} 0 & d \to d_*^- \\ \left(\frac{d - d_*}{d_*}\right)^{\frac{1}{\Delta}} & d \to d_*^+ \end{cases} ; \tag{2.50}$$

where we introduced the new scaling exponent Δ and included the information, taken from [24], that η_2 remains zero for $N \geq N_*$ and $d \leq d_*$. A fit from the data displayed in Figure 2.4 gives the approximate value $\Delta \approx 1.86$. Finally, we found that equation (2.47) has a discrete set of solutions only when the coefficient of the first term on the lhs is negative, thus our analysis applies when $\eta > 2 - d$. This fact prevents us from performing a complete analysis in the range $1 \leq d < 2$, where indeed studies of $O(N)$–models on fractals have shown that the MWHC theorem is still valid [24].

Multi-critical $O(N)$–models in fractal dimension

When new universality classes appear by branching from the Gaussian fixed-point it is easy to determine the relative critical dimensions, since the argument based on canonical dimensions is valid. In particular, the i–th multi-critical scaling solution appears at the upper critical dimension $d_{c,i} = 2 + \frac{2}{i-1}$ [3]. At these dimensions we see non-trivial fixed-points branching from the Gaussian for every $N \geq 2$, corresponding to potentials with i minima when expressed in terms of the variable $2\sqrt{\tilde{\rho}}$.

The critical dimensions $d_{c,i}$ accumulate at $d = 2$ and thus one may naively expect to find, for any N, infinitely many universality classes in two dimension. Our analysis shows instead, see Figure 2.6 for the cases $i = 3, 4, 5$ and $N = 1, 2, 3, 4$, that this happens only in the $N = 1$ case, where the multi-critical fixed-points approach, in the limit $d \to 2$, the fixed-points representing minimal-models [3]. For any other $N \geq 2$ we find that, consistently with the MWHC theorem, the multi-critical scaling solutions, present in the range $2 < d < 3$, are instead absent in $d = 2$. This fact is a strong check of the general validity of the MWHC theorem, which our analysis indicates is also applicable to multi-critical phase transitions. On the other side, we predict the existence of a whole family of $O(N)$–universality classes in fractal dimensions between two and three. To our knowledge these universality classes are new.

The $N \to 0$ limit

We now study the $N \to 0$ limit, describing the universality class of self-avoiding random walks (SAW) [25]. Figure 2.7 (Top) shows η_2 as function of N in the interval between $-2 \leq N \leq 2.5$ for the cases $d = 2$ and $d = 3$. The anomalous dimension is continuous in the whole range; this is an indication that the $N \to 0$ limit is well defined. Figure 2.7 (Top) also shows, interestingly, that both the $d = 2$ and $d = 3$ curves tend to zero as $N \to -2$ where indeed the model is know to have Gaussian critical exponents in both dimensions [26].

We also find multi-critical scaling solutions for $N = 0$. The interesting thing here is that these solutions survive in infinite number when $d \to 2$. A plot of the first four anomalous dimensions is shown in Figure 2.7 (Bottom); these are numerically very similar to those of the $N = 1$ models (see Figure 2.4 and 2.6). This similarity is expected, as one may see by inspection of Figure 2.7 (Top). Even if the anomalous dimension is not a relevant physical parameter in the correspondence with SAW, we can use scaling relations to relate it to the physical critical exponents ν and γ. In $d = 2$ one finds the exact values $\nu_{ex} = \frac{3}{4}$ and $\gamma_{ex} = \frac{43}{32}$ [27], and so $\eta_{ex} = 2 - \frac{\gamma_{ex}}{\nu_{ex}} = \frac{5}{24} \simeq 0.208$; we find $\eta_2(2, 0) = 0.232$. In $d = 3$ one finds from Monte Carlo simulations the values $\nu_{MC} = 0.587$ and $\gamma_{MC} = 1.157$ [5], and so $\eta_{MC} = 2 - \frac{\gamma_{MC}}{\nu_{MC}} \simeq 0.029$; we find $\eta_2(3, 0) = 0.04$. As we said before, we cannot extend our method to $d < 2$ to compare with exact SAW critical exponents found on fractals [28]. In any case, our analysis suggests that there is a countable family of $O(N = 0)$–universality classes in two dimensions. To our knowledge these are novel and may describe multi-critical phase transitions of some polymeric system.

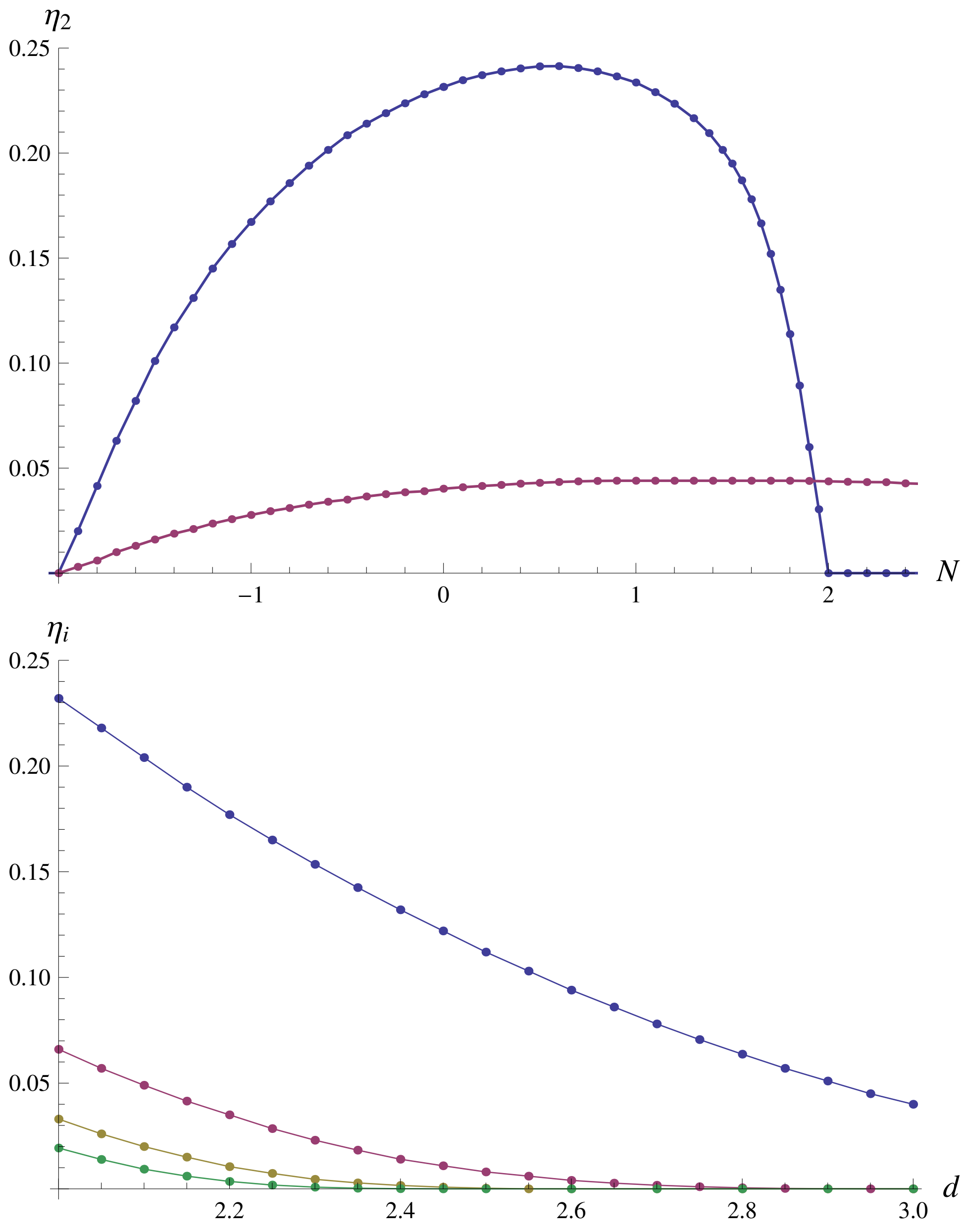


Figure 2.7: (Top) η_2 as a function of N is continuos both in $d = 2$ (upper curve) and in $d = 3$ (lower curve). (Bottom) η_i as a function of d for the first four $N = 0$ multi-critical scaling solutions, i.e. for (from above) $i = 2, 3, 4, 5$

2.7 Discussion and conclusions

In this Chapter, after introducing the EAA and the various approximations to solve its flow, our main result was the study of how universality classes of scalar theories with linearly realized $O(N)$–symmetry vary continuously with the dimension d and with the number of field components N. As we varied these parameters, we followed the evolution of RG fixed-points by studying the scaling solutions of the RG equation (2.47). As in [3], even if all our analysis was based on the study of a simple ODE, we were able to observe a very rich behavior.

Above four dimensions, as expected, we found only the Gaussian universality class; just below $d = 4$ we observed the Wilson-Fisher universality classes appear. In fractal dimension between two and three we found non-trivial fixed-points for all N: these are novel universality classes that can, in principle, be observed in theoretical models on fractal lattices or in real physical systems.

Approaching two dimensions we observed the RG manifestation of the MWHC theorem: only the $N = 1$ universality classes survived down to $d = 2$, while all the $N \geq 2$ ones disappeared. By considering (d, N) as real parameters near $(d_*, N_*) = (2, 2)$ we found that the transition described by the MWHC theorem, between theories that can undergo a continuous phase transition and theories that cannot, is continuous, and that the anomalous dimension, which can be seen as analogous to the order parameter, can be written in scaling form at the critical point $(2, 2)$. Our analysis revealed how different theory spaces parametrized by N are related to each other; this information gives a deep RG understanding of the MWHC theorem and could be used as the starting point for an extension of it.

Finally, we studied the $N \to 0$ limit; we found that it is continuous around $N = 0$ and we observed new $O(N = 0)$–universality classes in $d = 2$. These are analogous to the universality classes of $N = 1$ minimal-models and may describe particular multi-critical transitions of polymeric systems.

Chapter 3

The functional RG and Weyl invariance

3.1 Outline

As we have seen in the previous chapter, the definition of a quantum field theory generally begins with a classical field theory with bare action S, which is then quantized by defining a functional integral. Even if S is scale- or (in curved spacetime) Weyl–invariant, the resulting quantum effective action in general is not, because in the definition of the functional integral one necessarily introduces a mass scale. This is the origin of the celebrated trace anomaly [29].

It has been known since early on that when a dilaton is present, there is a way of perturbatively quantizing the theory which preserves Weyl invariance [30]. This has been rediscovered several times in the literature [31, 32, 33, 34, 35, 36]. Here we will discuss mainly the implications of this type of quantization procedure for the Renormalization Group (RG). Actually, we will adopt a point of view that puts the RG first, and views the quantum effective action as the result of following the RG flow all the way to the IR. By using the functional RG, we will give a nonperturbative proof that this kind of quantization is always possible, also in the case of interacting matter, regardless of its renormalizability properties. We will also show that even though the resulting effective action is Weyl–invariant, the trace anomaly is still present, with all its physical consequences.

Normal physical theories are neither conformal– nor scale–invariant. The renormalization group running describes the dependence of couplings on one dimensionful scale and the theory becomes conformally invariant only at a fixed point. If we now reformulate an arbitrary theory in a Weyl–invariant way, several obvious questions arise: What is the meaning of a cutoff in a Weyl–invariant theory? What distinguishes a fixed point from any other point? Are these Weyl–invariant quantum theories physically equivalent to ordinary non–Weyl–invariant ones? We will address these questions in the course of our derivations and summarize the state of our understanding in the conclusions.

Structure

This capter will be organized as follows.

We will start in section 2 by reviewing the conformal group, the Weyl group and their connection. Section 3 will describe the Weyl invariant quantization method using the fRG. We will first introduce the dilaton by extending th geometry to an integrable Weyl geometry. This is done in subsection 1. We will then use it to obtain a Weyl invariant Effective Action (subsection 2) and Effective Average Action (subsection 3) for free matter. Next (subsection 4) we extend the result to the case of interacting matter. Subsection 5 briefly discusses dynamical gravity, to be discussed in the next chapter. Finally Section 4 is devoted to the conclusions.

3.2 Conformal Field Theories and Weyl–invariant theories

In the previous chapter we examined the fRG in the context of (Euclidean) Quantum Field Theories. In QFT, as applied to particle physics for example, one studies systems which are invariant under the Poincaré group, the group of spacetime translations and Lorentz transformations. However, we have also seen that at phase transitions, the systems are described by a fixed point of the RG, and are thus characterized by a further invariance: scale invariance. In fact, in all known examples of unitary theories, it is found that the systems exhibit a larger symmetry at critical points: they are symmetric under the full conformal group.

We will here briefly review the conformal group, its generalization when gravity is present (Weyl group), and the reasons why the two are worth studying.

3.2.1 Conformal invariance

We have seen in the previous chapter that phase transitions and the critical properties of a system are understood as fixed points of the RG. At a fixed point, by construction the theory becomes invariant under scale transformations, which are transformations that rescale all coordinates by a common constant factor

$$x^\mu \to \lambda x^\mu \,. \tag{3.1}$$

Fields also transform accordingly

$$\phi_i(x) \to \lambda^{-d_i} \phi_i(\lambda x) \,, \tag{3.2}$$

where d_i is their scaling dimension. This means that the theory cannot depend on any scale. Physically, this manifests through the fact that the correlation length diverges at the critical point, and consequently there are fluctuations on all possible scales. In particle physics, instead, it is what we expect in the far UV, at energies much larger than all the scales of the theory, if this is well behaved.

In this way, fixed point theories are invariant under the combined symmetry of Poincaré transformations plus dilatations. The remarkable property that is found is that in fact these theories turn out to be invariant under the extended symmetry group of Poincaré transformations, dilatations, and special conformal transformations, which in infinitesimal form read

$$x^\mu \to x^\mu + (a \cdot x) x^\mu - x^2 a^\mu \,. \tag{3.3}$$

This group is called the Conformal Group, and theories invariant under its transformations are called Conformal Field Theories (CFTs). These transformations have the property of stretching locally the lenghts of vectors while leaving their relative angles invariant. As we will see later, they can also be seen, in a sense, as local scale transformations. Heuristically, if a theory has only short–range interactions, it shouldn't matter too much if the scale transformation depends on position, and this is why we would expect that scale invariance implies conformal invariance in this case. However, a rigorous proof of this fact only exists in two dimensions.

CFTs are quite ubiquituous in physics, they do not appear only in the study of critical phenomena, but also in particle theory model building and in other contexts in high energy physics (most notably the AdS/CFT correspondence). In fact, most of the non–gravitational effective theories that one studies in particle physics can be realized as relevant or marginally relevant deformations of a CFT.

The power of CFTs is that their symmetry constrains so much their structure that they become mathematically better controlled than standard QFTs. Their (lowest) correlators for instance can be solved exactly, and they allow for an exact evaluation of critical exponents via algebraic methods.

These are the main reasons for studying CFTs. We next turn to the case when there is also gravity into play.

3.2.2 Weyl invariance

Definition

Let us begin with some precise definitions. A global scale transformation is a rescaling of all lengths by a fixed, constant factor Ω. In flat space, scale transformations are usually interpreted as the map $x \to \Omega x$. As such, they form a particular subgroup of diffeomorphisms. Alternatively, one can think of rescaling the metric $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$. The two points of view are completely equivalent, since lengths are given by integrating the line element $ds = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$. For our purposes it will be convenient to adopt the second point of view. This is in fact equivalent to assuming that the metric is dimensionful, while the coordinates are mere dimensionless labels.

Let us now define the scaling dimension of a quantity. Consider a theory with fields ψ_a, parameters g_i (which include masses, couplings, wave function renormalizations etc.) and action $S(g_{\mu\nu}, \psi_a, g_i)$. There is a unique choice of numbers w_a (one per field) and w_i (one per parameter) such that S is invariant:

$$S(g_{\mu\nu}, \psi_a, g_i) = S(\Omega^2 g_{\mu\nu}, \Omega^{w_a}\psi_a, \Omega^{w_i} g_i) \ . \tag{3.4}$$

(It does not matter here whether the metric is fixed or dynamical.) The numbers w_a, w_i are called the scaling dimensions, or the weights, of ψ_a and g_i. Here we will assume that the spacetime coordinates are dimensionless and we use natural units where $c = 1$, $\hbar = 1$. Then, the scaling dimensions are equal to the ordinary length dimensions of ψ_a and g_i in the sense of dimensional analysis. Since in particle physics it is customary to use mass dimensions, when we talk of "dimensions" without further specification we will refer to the mass dimensions $d_a = -w_a$ and $d_i = -w_i$. In d spacetime dimensions, the canonical dimensions of scalar, spinor and vector fields are $(d-2)/2$, $(d-1)/2$ and $(d-4)/2$, respectively. One can easily convince oneself that the dimensions of all parameters in the Lagrangian, such as masses and couplings, are the same as in the more familiar case when coordinates have dimension of length.

Changing couplings is usually interpreted as changing theory, so in general the transformations (3.4) are not symmetries of a theory but rather maps from one theory to another. In the case when all the w_i are equal to zero, we have

$$S(g_{\mu\nu}, \psi_a, g_i) = S(\Omega^2 g_{\mu\nu}, \Omega^{w_a}\psi_a, g_i) \ . \tag{3.5}$$

Since these are transformations that map a theory to itself, a theory of this type is said to be globally scale invariant.

Scale transformations with Ω a positive real function of x are called Weyl transformations. They act on the metric and the fields exactly as in (3.4).

A theory invariant under these transformations is said to be Weyl–invariant.

Conformal invariance from Weyl invariance

In flat space, our intuition tells us that local scale transformations should correspond to conformal transformations. In fact we can show that any (diffeomorphism invariant) theory which is Weyl invariant in curved space is also invariant in flat space under the conformal group [37].

A conformal isometry is a coordinate transformation $x^\mu \to y^\mu(x)$ satisfying

$$\frac{\partial y^\alpha(x)}{\partial x^\mu}\frac{\partial y^\beta(x)}{\partial x^\nu} g_{\alpha\beta}(y(x)) = \Omega(x) g_{\mu\nu}(x) \, . \tag{3.6}$$

The infinitesimal form of this transformation, of the form $y^\mu \simeq x^\mu + \epsilon^\mu$, reads:

$$\begin{aligned} \frac{\partial(x^\alpha + \epsilon^\alpha)}{\partial x^\mu}\frac{\partial(x^\beta + \epsilon^\beta)}{\partial x^\nu} g_{\alpha\beta}(x+\epsilon) &= (\delta^\alpha_\mu + \partial_\mu \epsilon^\alpha)(\delta^\beta_\nu + \partial_\nu \epsilon^\beta)(g_{\alpha\beta} + \epsilon\cdot\partial g_{\alpha\beta}) \\ &= g_{\mu\nu} + \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu \, , \end{aligned}$$

We recognize the Lie derivative $\mathcal{L}$ acting on the metric as $g_{\mu\nu} \to g_{\mu\nu} + \mathcal{L}_\epsilon g_{\mu\nu} = g_{\mu\nu} + \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$. This is the infinitesimal form of a diffeomorphism, and we will encounter it again in the next chapter.

Expanding also the factor $\Omega \simeq 1 + \omega$, we arrive at

$$\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = \omega g_{\mu\nu} \, . \tag{3.7}$$

Contracting the indices on both sides of (3.7) we find

$$\omega = \frac{2}{d} \nabla \cdot \epsilon \,, \tag{3.8}$$

which combined with (3.7) gives a condition for ϵ^μ. Any ϵ^μ satisfying (3.7) defines an infinitesimal conformal isometry.

When we go to flat space $g_{\mu\nu} = \delta_{\mu\nu}$ we find:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \omega \delta_{\mu\nu} \,, \tag{3.9}$$

and

$$\omega = \frac{2}{d} \partial \cdot \epsilon \,.$$

Let's derive (3.9) once more and reshuffle the indeces:

$$\begin{aligned} \partial_\lambda \partial_\mu \epsilon_\nu + \partial_\lambda \partial_\nu \epsilon_\mu &= \partial_\lambda \omega \delta_{\mu\nu} \\ \partial_\mu \partial_\nu \epsilon_\lambda + \partial_\mu \partial_\lambda \epsilon_\nu &= \partial_\mu \omega \delta_{\nu\lambda} \\ \partial_\nu \partial_\lambda \epsilon_\mu + \partial_\nu \partial_\mu \epsilon_\lambda &= \partial_\nu \omega \delta_{\lambda\mu} \,. \end{aligned} \tag{3.10}$$

By considering a linear combination of these equations (second plus third minus first row) we easily get:

$$\partial_\mu \partial_\nu \epsilon_\lambda = \frac{1}{2} \left[\partial_\mu \omega \delta_{\nu\lambda} + \partial_\nu \omega \delta_{\lambda\mu} - \partial_\lambda \omega \delta_{\mu\nu} \right] \,, \tag{3.11}$$

and upon contraction this becomes

$$\partial^2 \epsilon_\lambda = \frac{2-d}{2} \partial_\lambda \omega \,. \tag{3.12}$$

Taking a derivative of this equation and acting with ∂^2 on (3.9) we find

$$(2-d) \partial_\mu \partial_\nu \omega = \delta_{\mu\nu} \partial^2 \omega \tag{3.13}$$

whose contraction is

$$(1-d) \partial^2 \omega = 0 \,. \tag{3.14}$$

If $d = 1$, the above equations do not impose any constraint on ω: any coordinate transformation is conformal, which is a somewhat trivial statement since the notion of angle is not defined in this case.

If $d > 1$ the condition is $\partial^2 \omega = 0$, while if $d > 2$, equation (3.13) implies that $\partial_\mu \partial_\nu \omega = 0$. From this it is easy to recover the conformal group.

For example, in $d > 2$ the condition tells us that ω is a linear function of the coordinates:

$$\omega = A + B_\mu x^\mu \,, \tag{3.15}$$

or equivalently that ϵ^μ is at most quadratic in the coordinates:

$$\epsilon^\mu = a^\mu + b^\mu_{\ \nu} x^\nu + c^\mu_{\ \alpha\beta} x^\alpha x^\beta \,. \tag{3.16}$$

The constant term a^μ represents an infinitesimal translation. Substitution of the linear term in (3.9) tells us that it must be of the form $b_{\mu\nu} = \omega_{\mu\nu} + \lambda\delta_{\mu\nu}$ with $\omega_{\mu\nu} = -\omega_{\nu\mu}$. This parametrizes a rotation (Lorentz transformation) and a scale transformation. Finally, substituting the quadratic term into (3.11) one recovers a special conformal transformation.

3.3 A Weyl invariant quantization

3.3.1 Introducing the dilaton

We have seen in the previous section how a Weyl transformation acts on the metric and fields. What about the parameters? They are supposed to be x-independent, so a transformation $g_i \to \Omega(x)^{w_i} g_i$ would not make much sense. One can overcome this difficulty by promoting the dimensionful parameters to fields. One can then meaningfully ask whether (3.4) holds. In general the answer will be negative, but there is a simple procedure that allows one to make a scale invariant theory also Weyl–invariant: it is called Weyl gauging and it was the earliest incarnation of the notion of gauge theory. Here we will restrict ourselves to a special case of Weyl gauging, namely the case when the connection is flat. We pick a mass parameter of the theory, let's call it μ and we promote it to a function that we shall denote χ. We can write

$$\chi(x) = \mu e^{\sigma(x)} \ , \tag{3.17}$$

where μ is constant. The function χ, or sometimes σ, is called the dilaton. Notice that unlike an ordinary scalar field, it has dimension one independently of the spacetime dimensionality. Thus it transforms under Weyl transformation as $\chi \mapsto \Omega^{-1}\chi$. Now we can take any other dimensionful coupling of the theory and write

$$g_i = \chi^{-w_i} \hat{g}_i = \chi^{d_i} \hat{g}_i \ , \tag{3.18}$$

where $\hat{g}_i$ is dimensionless (and therefore Weyl–invariant). In general, a caret over a symbol denotes the same quantity measured in units of the dilaton. In principle one could promote more than one dimensionful parameter, or even all dimensionful parameters, to independent dilatons. This may have interesting applications, but for the sake of simplicity in this chapter we shall restrict ourselves to the case when there is a single dilaton.

With the dilaton we construct a pure-gauge abelian gauge field $b_\mu = -\chi^{-1}\partial_\mu \chi$, transforming under (3.4) as $b_\mu \mapsto b_\mu + \Omega^{-1}\partial_\mu \Omega$. Let ∇_μ be the covariant derivative with respect to the Levi-Civita connection of the metric g. Define a new (non-metric) connection

$$\hat{\Gamma}_\mu{}^\lambda{}_\nu = \Gamma_\mu{}^\lambda{}_\nu - \delta^\lambda_\mu b_\nu - \delta^\lambda_\nu b_\mu + g_{\mu\nu} b^\lambda \ , \tag{3.19}$$

where $\Gamma_\mu{}^\lambda{}_\nu$ are the Christoffel symbols of g. The corresponding covariant derivative is denoted $\hat{\nabla}$. The connection coefficients $\hat{\Gamma}$ are invariant under (3.4). For any tensor t of weight w define the covariant derivative Dt to be

$$D_\mu t = \hat{\nabla}_\mu t - w b_\mu t \ , \tag{3.20}$$

where all indices have been suppressed. We see that the weigth (or the dimension) acts like the Weyl charge of the field. The tensor Dt is covariant under diffeomorphisms and under Weyl transformations. The curvature of D is defined by

$$[D_\mu, D_\nu]v^\rho = \mathcal{R}_{\mu\nu}{}^\rho{}_\sigma v^\sigma \ . \tag{3.21}$$

The tensor $\mathcal{R}_{\mu\nu}{}^\rho{}_\sigma$ is Weyl invariant, and raising and lowering indices one obtains Weyl covariant expressions of different dimensions. A direct calculation gives the explicit expression

$$\begin{aligned}\mathcal{R}_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} + g_{\mu\rho}\left(\nabla_\nu b_\sigma + b_\nu b_\sigma\right) - g_{\mu\sigma}\left(\nabla_\nu b_\rho + b_\nu b_\rho\right) \\ &\quad - g_{\nu\rho}\left(\nabla_\mu b_\sigma + b_\mu b_\sigma\right) + g_{\nu\sigma}\left(\nabla_\mu b_\rho + b_\mu b_\rho\right) - \left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) b^2 \ . \end{aligned} \tag{3.22}$$

From here one finds the analogs of the Ricci tensor and Ricci scalar

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + (d-2)b_\mu b_\nu + (d-2)\nabla_\mu b_\nu - (d-2)b^2 g_{\mu\nu} + \nabla^\rho b_\rho g_{\mu\nu} \ , \tag{3.23}$$

$$\mathcal{R} = R + 2(d-1)\nabla^\mu b_\mu - (d-1)(d-2)b^2 \ . \tag{3.24}$$

It is also possible to define the tensor $\mathcal{C}^\mu{}_{\nu\alpha\beta}$ which is related to $\mathcal{R}^\mu{}_{\nu\alpha\beta}$ by the same formula that relates $C^\mu{}_{\nu\alpha\beta}$ to $R^\mu{}_{\nu\alpha\beta}$, and therefore reduces to the standard Weyl tensor in a gauge where χ is constant.

Now start from a generic action for matter and gravity of the form $S(g_{\mu\nu}, \psi_a, g_i)$. Express every parameter g_i as in (3.18). Replace all covariant derivatives ∇ by Weyl covariant derivatives D and all curvatures R by the Weyl covariant curvatures $\mathcal{R}$. Now all the terms appearing in the action are products of Weyl covariant objects, and local Weyl invariance just follows from the fact that the action is dimensionless. In this way we have defined an action $\hat{S}(g_{\mu\nu}, \chi, \psi_a, \hat{g}_i)$. It contains only dimensionless couplings $\hat{g}_i$, and is Weyl invariant by construction. One can choose a gauge where $\chi = \mu$ is constant (equivalently, $\sigma = 0$), and in this gauge the action $\hat{S}(g_{\mu\nu}, \chi, \psi_a, \hat{g}_i)$ reduces to the original one.

The above construction defines an "integrable Weyl geometry", since the curvature of the Weyl gauge field b_μ is zero. In this integrable case there is also another way of defining a Weyl–invariant action from a non–invariant one, namely to replace all the arguments in S by the corresponding dimensionless quantities $\hat{g}_{\mu\nu} = \chi^2 g_{\mu\nu}$, $\hat{\psi}_a = \chi^{w_a}\psi_a$ and $\hat{g}_i = \chi^{w_i} g_i$ and subsequently reexpress the action in terms of the original fields

$$\hat{S}(g_{\mu\nu}, \chi, \psi_a, \hat{g}_i) = S(\hat{g}_{\mu\nu}, \hat{\psi}_a, \hat{g}_i) \ . \tag{3.25}$$

It is easy to see that this construction gives the same result as the preceding one. This follows from the fact that (3.19) are the Christoffel symbols of $\hat{g}_{\mu\nu}$, that $\hat{\nabla}_\mu \hat{\psi}_a = \chi^{w_a} D_\mu \psi_a$ and that the curvature tensor of $\hat{\Gamma}$ is $\mathcal{R}_{\mu\nu\rho\sigma}$. [1]

[1] If we call $\hat{R}_{\mu\nu\rho\sigma}$ the Riemann tensor of $\hat{g}_{\mu\nu}$, we have $\hat{R}^\mu{}_{\nu\rho\sigma} = \mathcal{R}^\mu{}_{\nu\rho\sigma}$ and $\hat{R}_{\mu\nu\rho\sigma} = \chi^2 \mathcal{R}_{\mu\nu\rho\sigma}$.

The above procedure can be used to rewrite any theory in Weyl–invariant form. Not all Weyl–invariant theories are of this type: there are also theories that are Weyl–invariant without containing a Weyl gauge field b_μ (or a dilaton). In such theories the terms generated by a Weyl transformation that contain the derivatives of the transformation parameter are compensated by terms generated by variations of Ricci tensors. Since Weyl–invariance can be viewed as a gauged version of global scale invariance, this has been called "Ricci gauging" in [38]. It was also shown that such Ricci–gauged theories correspond (under mild additional assumptions) to theories that are conformal–invariant, as opposed to merely scale–invariant, in flat space. The existence of well–behaved theories that are scale– but not conformal–invariant in flat space has been reexamined recently [39, 40, 41, 42].

3.3.2 The effective action of free matter fields coupled to an external gravitational field

The standard measure

In this section we review the evaluation of the effective action for free, massless matter fields conformally coupled to a metric. This will provide the basis for different quantization procedures to be described in the following. Much of the discussion can be carried out in arbitrary even dimension d.

For definiteness let us consider first a single conformally coupled scalar field, with equation of motion $\Delta^{(0)}\phi = 0$, where $\Delta^{(0)} = -\nabla^2 + \frac{d-2}{4(d-1)}R$. Functional integration over ϕ in the presence of a source j leads to a generating functional $W(g_{\mu\nu}, j)$, whose Legendre transform $\Gamma(g_{\mu\nu}, \phi) = W(g_{\mu\nu}, j) - \int j\phi$ is the effective action. For the definition of the functional integral one needs a metric (more precisely an inner product) in the space of the fields. We choose

$$\mathcal{G}(\phi, \phi') = \mu^2 \int dx \sqrt{g}\, \phi\, \phi' \;, \tag{3.26}$$

where μ is an arbitrary mass that has to be introduced for dimensional reasons. The action can be written as

$$S_S(g_{\mu\nu}, \phi) = \frac{1}{2}\int dx\sqrt{g}\, \phi\Delta^{(0)}\phi = \frac{1}{2}\mathcal{G}\left(\phi, \frac{\Delta^{(0)}}{\mu^2}\phi\right) = \frac{1}{2}\sum_n a_n^2 \lambda_n/\mu^2 \;, \tag{3.27}$$

where λ_n are the eigenvalues of $\Delta^{(0)}$, ϕ_n the corresponding eigenfunctions and a_n are the (dimensionless) coefficients of the expansion of ϕ on the basis of the eigenfunctions:

$$\Delta^{(0)}\phi_n = \lambda_n\phi_n \;; \qquad \mathcal{G}(\phi_n, \phi_m) = \delta_{nm} \;; \qquad \phi = \sum_n a_n\phi_n \;; \qquad a_n = \mathcal{G}(\phi, \phi_n) \;. \tag{3.28}$$

(For simplicity we assume that the manifold is compact and without boundary, so that the spectrum of the Laplacian is discrete.) Weyl–covariance means that under a Weyl transformation the operator $\Delta^{(0)}$ transforms as

$$\Delta^{(0)}_{\Omega^2 g} = \Omega^{-1-\frac{d}{2}} \Delta^{(0)}_g \Omega^{\frac{d}{2}-1} \ , \tag{3.29}$$

where we have made the dependence of the metric explicit. For an infinitesimal transformation $\Omega = 1 + \omega$,

$$\delta_\omega \Delta^{(0)} = -2\omega \Delta^{(0)} + \left(\frac{d}{2} - 1\right) [\Delta^{(0)}_g, \omega] \ . \tag{3.30}$$

The functional measure is $(d\phi) = \prod_n da_n$, so the Gaussian integral can be evaluated as

$$e^{-W(g_{\mu\nu}, j)} = \prod_n \left(\int da_n e^{-\frac{1}{2}a_n^2 \lambda_n/\mu^2 - a_n j^n} \right) = \prod_n \sqrt{\frac{\mu}{\lambda_n}} e^{\frac{1}{2}\frac{\mu^2}{\lambda_n}(j^n)^2} = \det \left(\frac{\Delta^{(0)}}{\mu^2} \right)^{-1/2} e^{\frac{1}{2}\int j \Delta^{-1} j} \tag{3.31}$$

up to a field–independent multiplicative constant. From here one gets (using the same notation for the VEV as for the field) $\phi = -\Delta^{(0)-1} j$, so finally the Legendre transform gives

$$\Gamma(\phi, g_{\mu\nu}) = S_S(\phi, g_{\mu\nu}) + \frac{1}{2} \mathrm{Tr} \log \left(\frac{\Delta^{(0)}}{\mu^2} \right) \ . \tag{3.32}$$

An UV regularization is needed to define this trace properly. We see that the scale μ, which has been introduced in the definition of the measure, has made its way into the functional determinant.

Things work much in the same way for the fermion field, which contributes to the effective action a term

$$S_D(\bar\psi, \psi, g_{\mu\nu}) - \frac{1}{2} \mathrm{Tr} \log \left(\frac{\Delta^{(1/2)}}{\mu^2} \right) \ , \tag{3.33}$$

where S_D is the classical action and $\Delta^{(1/2)} = -\nabla^2 + \frac{R}{4}$ is the square of the Dirac operator.

The Maxwell action is Weyl–invariant only in $d = 4$. With our conventions the field A_μ is dimensionless and the Weyl–invariant inner product in field space is:

$$\mathcal{G}(A, A') = \mu^2 \int d^4x \sqrt{g}\, g^{\mu\nu} A_\mu A_\nu \ . \tag{3.34}$$

Using the standard Faddeev-Popov procedure, we add gauge fixing and ghost actions

$$S_{GF} = \frac{1}{2\alpha} \int d^4x \sqrt{g}\, (\nabla_\mu A^\mu)^2 \ ; \qquad S_{gh} = \int d^4x \sqrt{g}\, \bar{C} \Delta^{(gh)} C \ , \tag{3.35}$$

with $\Delta^{(gh)} = -\nabla^2$. Then, in the gauge $\alpha = 1$, the gauge–fixed action becomes

$$S_M + S_{GF} = \frac{1}{2} \int d^4x \sqrt{g}\, A^\mu \Delta^{(1)\nu}_\mu A_\nu = \frac{1}{2} \mathcal{G}\left(A, \frac{\Delta^{(1)}}{\mu^2} A\right) \ , \tag{3.36}$$

where $\Delta^{(1)\nu}_\mu = -\nabla^2\delta^\nu_\mu + R^\nu_\mu$ is the Laplacian on one–forms. Following the same steps as for the scalar field, we obtain a contribution to the effective action equal to

$$S_M(A_\mu, g_{\mu\nu}) + \frac{1}{2}\mathrm{Tr}\log\left(\frac{\Delta^{(1)}}{\mu^2}\right) - \mathrm{Tr}\log\left(\frac{\Delta^{(gh)}}{\mu^2}\right) . \tag{3.37}$$

Note that even though the Maxwell action S_M is Weyl–invariant, the gauge fixing action is not, nor is the ghost action. As a result the operators $\Delta^{(1)}$ and $\Delta^{(gh)}$ are not Weyl–covariant. Instead of an equation like (3.30), they satisfy (in four dimensions)

$$\delta_\omega \Delta^{(gh)} = -2\omega\Delta^{(h)} - 2\nabla^\nu\omega\nabla_\nu ; \tag{3.38}$$

$$\delta_\omega \Delta^{(1)\nu}_\mu = -2\omega\Delta^{(1)\nu}_\mu + 2\nabla_\mu\omega\nabla^\nu - 2\nabla^\nu\omega\nabla_\mu - 2\nabla_\mu\nabla^\nu\omega . \tag{3.39}$$

We shall see in the next section how these non–invariances compensate each other in the effective action, so that the breaking of Weyl–invariance is only due to the presence of the scale μ which was introduced in the inner product.

In general, the need for an inner product in field space can also be seen in a more geometrical way as follows. The classical action, being quadratic in the fields, has the form $\mathcal{H}(\phi,\phi)$, where $\mathcal{H} = \frac{\delta^2 S}{\delta\phi\delta\phi}$ can be viewed as a covariant symmetric tensor in field space: when contracted with a field (a vector in field space) it produces a one–form in field space. Now, the determinant of a covariant symmetric tensor is not a basis-independent quantity. One can only define in a basis-independent way the determinant of an operator mapping a space into itself, i.e. a mixed tensor. One can transform the covariant tensor $\mathcal{H}$ to a mixed tensor $\mathcal{O}$ by "raising an index" with a metric: [2]

$$\mathcal{H}(\phi,\phi') = \mathcal{G}(\phi, \mathcal{O}\phi') . \tag{3.40}$$

It is the determinant of the operator $\mathcal{O}$ that appears in the effective action. Again we see that the scale μ appears through the metric $\mathcal{G}$, which is needed to define the determinant. Notice that since $\mathcal{O}\phi$ is another field of the same type as ϕ, $\mathcal{O}$ must necessarily be dimensionless, and this is guaranteed by the factors of μ contained in $\mathcal{G}$. For example, in the scalar case, $\mathcal{O} = \frac{1}{\mu^2}\Delta^{(0)}$.

Trace anomaly

Under an infinitesimal Weyl transformation the variation of the effective action is

$$\delta_\omega\Gamma = \int dx\, \frac{\delta\Gamma}{\delta g_{\mu\nu}} 2\omega g_{\mu\nu} = -\int dx\sqrt{g}\,\omega\langle T^\mu_\mu\rangle . \tag{3.41}$$

[2] In de Witt's condensed notation, where an index i stands both for a point x in spacetime and whatever tensor or spinor indices the field may be carrying, this equation reads $\mathcal{O}_i{}^j = \mathcal{H}_{ik}\mathcal{G}^{kj}$.

The trace of the energy–momentum tensor vanishes for a Weyl–invariant action, so the appearance of a nonzero trace is the physical manifestation of the anomaly.

For non–interacting fields the one–loop effective action is exact. We use the proper time representation

$$\Gamma = S - \frac{1}{2}\int_{\epsilon/\mu^2}^{\infty} \frac{dt}{t} \mathrm{Tr} e^{-t\Delta} \ , \tag{3.42}$$

where ϵ is a dimensionless UV regulator, and the counterterms needed in this expression are left implicit. We also need the general formula stating the Weyl–covariance of an operator Δ acting on fields of weight w (see (3.30))

$$\delta_\omega \Delta = -2\omega\Delta + w[\Delta, \omega] \ . \tag{3.43}$$

Varying (3.42) and using that the commutator cancels under the trace, one finds

$$\delta_\omega \Gamma = \frac{1}{2}\int_{\epsilon/\mu^2}^{\infty} dt\, \mathrm{Tr}\, \delta_\omega \Delta e^{-t\Delta} = -\int_{\epsilon/\mu^2}^{\infty} dt\, \mathrm{Tr}(\omega\Delta e^{-t\Delta}) = \int_{\epsilon/\mu^2}^{\infty} dt \frac{d}{dt} \mathrm{Tr}(\omega e^{-t\Delta}) = -\mathrm{Tr}\left[\omega e^{-\epsilon\Delta/\mu^2}\right] .$$

For $\epsilon \to 0$ one has from the asymptotic expansion of the heat kernel (see the Appendix):

$$\mathrm{Tr}\left[\omega e^{-\epsilon\Delta/\mu^2}\right] = \frac{1}{(4\pi)^{d/2}} \int d^d x \sqrt{g}\, \omega \left[\frac{\mu^d}{\epsilon^{d/2}} b_0(\Delta) + \frac{\mu^{d-2}}{\epsilon^{d/2-1}} b_2(\Delta) + \ldots + b_d(\Delta) + \ldots\right] , \tag{3.44}$$

where b_i are scalars constructed with i derivatives of the metric. All terms b_i with $i > d$ tend to zero in the limit, so assuming that the power divergences (for $i < d$) are removed by renormalization, there remains a universal, finite limit

$$\delta_\omega \Gamma = -\frac{1}{(4\pi)^{d/2}} \int dx \sqrt{g}\, \omega\, b_d(\Delta) \ . \tag{3.45}$$

which implies that [3]

$$\langle T^\mu_\mu \rangle = \frac{1}{(4\pi)^{d/2}} b_d(\Delta) \ . \tag{3.46}$$

We note that this can also be seen as a direct manifestation of the dependence of the result on the scale μ. In fact one has, formally

$$\mu \frac{d}{d\mu} \frac{1}{2} \mathrm{Tr} \log \frac{\Delta}{\mu^2} = -\mathrm{Tr}\mathbf{1} = -\frac{1}{(4\pi)^{d/2}} \int dx \sqrt{g}\, b_d(\Delta) = -\int dx \sqrt{g}\, \langle T^\mu_\mu \rangle \ , \tag{3.47}$$

where in the second step we have used zeta function regularization [43].

Aside from the different prefactor the calculation follows the same steps in the case of massless spinors. The Maxwell field, however, requires some additional considerations,

[3]Note that in four dimensions the term in b_4 proportional to $\Box R$, which is a total derivative, can be renormalized at will by adding a local term to the effective action.

because the operators $\Delta^{(1)}$ and $\Delta^{(gh)}$ that appear in (3.37) are not covariant. (We restrict ourselves now to $d=4$). The first two steps of the preceding calculation give:

$$\begin{aligned}\delta_\omega\Gamma &= \frac{1}{2}\int_{\epsilon/\mu^2}^{\infty} dt\,\mathrm{Tr}\,\delta_\omega\Delta^{(1)}e^{-t\Delta^{(1)}} - \int_{\epsilon/\mu^2}^{\infty} dt\,\mathrm{Tr}\,\delta_\omega\Delta^{(gh)}e^{-t\Delta^{(gh)}}\\ &= \frac{1}{2}\int_{\epsilon/\mu^2}^{\infty} dt\,\mathrm{Tr}\,(-2\omega\Delta^{(1)}+\rho^{(1)})e^{-t\Delta^{(1)}} - \int_{\epsilon/\mu^2}^{\infty} dt\,\mathrm{Tr}\,(-2\omega\Delta^{(gh)}+\rho^{(gh)})e^{-t\Delta^{(gh)}} \end{aligned} \quad (3.48)$$

where the violation of Weyl covariance is due to

$$\rho^{(gh)} = -2\nabla^\nu\omega\nabla_\nu\ ; \qquad \rho^{(1)\nu}_{\mu} = 2\nabla_\mu\omega\nabla^\nu - 2\nabla^\nu\omega\nabla_\mu - 2\nabla_\mu\nabla^\nu\omega\ . \quad (3.49)$$

Since $\Delta^{(1)}$ maps longitudinal fields to longitudinal fields and transverse fields to transverse fields, $\rho^{(1)}e^{-t\Delta^{(1)}}$ has vanishing matrix elements between transverse gauge fields. Therefore the trace containing $\rho^{(1)}$ can be restricted to the subspace of longitudinal gauge potentials. Let ϕ_n be a basis of eigenfunctions of $\Delta^{(gh)}$ satisfying an orthonormality condition as in (3.28). Then a basis in the space of longitudinal potentials satisfying a similar orthonormality condition with respect to the inner product (3.34) is given by the fields $A^L_{n\mu} = \frac{1}{\sqrt{\lambda_n}}\nabla_\mu\phi_n$. The traces of the terms violating Weyl–covariance are therefore:

$$\frac{1}{2}\mathrm{Tr}\rho^{(1)}e^{-t\Delta^{(1)}} - \mathrm{Tr}\rho^{(gh)}e^{-t\Delta^{(gh)}} = \frac{1}{2}\sum_n \mathcal{G}\left(A^L_n, \rho^{(1)}e^{-t\Delta^{(1)}}A^L_n\right) - \sum_n \mathcal{G}\left(\phi_n, \rho^{(gh)}e^{-t\Delta^{(gh)}}\phi_n\right)\ . \quad (3.50)$$

Noting that

$$\Delta^{(1)}A^L_n = \frac{1}{\sqrt{\lambda_n}}\Delta^{(1)}\nabla_\mu\phi_n = \frac{1}{\sqrt{\lambda_n}}\nabla_\mu\Delta^{(gh)}\phi_n = \lambda_n A^L_n\ ,$$

we can evaluate the matrix elements:

$$\mathcal{G}\left(A^L_n, \rho^{(1)}e^{-t\Delta^{(1)}}A^L_n\right) = -4e^{-t\lambda_n}\mathcal{G}\left(\phi_n, \nabla^\nu\omega\nabla_\nu\phi_n\right)\ ,$$

whereas in the ghost trace we have

$$\mathcal{G}\left(\phi_n, \rho^{(gh)}e^{-t\Delta^{(gh)}}\phi_n\right) = -2e^{-t\lambda_n}\mathcal{G}\left(\phi_n, \nabla^\nu\omega\nabla_\nu\phi_n\right)\ .$$

We see that the sums in (3.50) cancel mode by mode. As a result only the first term remains in each of the traces in (3.48). From this point onwards the calculation proceeds as in the case of the scalar and finally gives

$$\delta_\omega\Gamma = \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left[b_4(\Delta^{(1)}) - 2b_4(\Delta^{(gh)})\right]\ . \quad (3.51)$$

The coefficients of the expansion of the heat kernel for Laplace-type operators are well-known. If there are n_S scalar, n_D spinors, one has in two dimensions

$$\langle T^\mu{}_\mu\rangle = \frac{c}{24\pi}R \tag{3.52}$$

with

$$c = n_S + n_D \tag{3.53}$$

whereas in four dimensions (assuming also the existence of n_M Maxwell fields)

$$\langle T^\mu{}_\mu\rangle = c\,C^2 - aE \tag{3.54}$$

where $E = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the integrand of the Euler invariant, $C^2 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ is the square of the Weyl tensor and the anomaly coefficients are [4]

$$a = \frac{1}{360(4\pi)^2}\left(n_S + 11n_D + 62n_M\right)\ ; \quad c = \frac{1}{120(4\pi)^2}\left(n_S + 6n_D + 12n_M\right)\ . \tag{3.55}$$

The Weyl–invariant measure

Let us now assume that the theory contains also a dilaton χ. For the purposes of this section it will be considered as part of the gravitational sector and treated as an external field. For notational simplicity we will discuss the case $d = 4$ but it is easy to generalize to arbitrary even dimensions.

The crucial observation is that we can now construct Weyl invariant metrics in the spaces of scalar, Dirac and Maxwell fields, replacing the fixed scale μ by the dilaton:

$$\mathcal{G}_S(\phi,\phi') = \int d^4x\sqrt{g}\,\chi^2\phi\phi'\ , \tag{3.56}$$

$$\mathcal{G}_D(\bar\psi,\psi') = \int d^4x\sqrt{g}\,\frac{1}{2}\chi[\bar\psi\psi' + \bar\psi'\psi]\ , \tag{3.57}$$

$$\mathcal{G}_M(A,A') = \int d^4x\sqrt{g}\,\chi^2 A_\mu g^{\mu\nu}A'_\nu\ . \tag{3.58}$$

One can follow step by step the one-loop calculation for the EA, the only change being the replacement of μ by χ. The final result for the one–loop contribution to the effective action can be written as

$$\frac{n_S}{2}\mathrm{Tr}\log\mathcal{O}_S - \frac{n_D}{2}\mathrm{Tr}\log\mathcal{O}_D + \frac{n_M}{2}\mathrm{Tr}\log\mathcal{O}_M - n_M\mathrm{Tr}\log\mathcal{O}_{gh}\ , \tag{3.59}$$

[4] the coefficients c and a were called b and $-b'$ in [29].

where now [5]

$$\mathcal{O}_S = \chi^{-2}\Delta^{(0)} \ , \tag{3.60}$$

$$\mathcal{O}_D = \chi^{-2}\Delta^{(1/2)} \ , \tag{3.61}$$

$$\mathcal{O}_{M\mu}{}^{\nu} = \chi^{-2} g_{\mu\sigma}\left(\Delta^{(1)}\right)^{\sigma\nu} \ , \tag{3.62}$$

$$\mathcal{O}_{gh} = \chi^{-2}\Delta^{(gh)} \ , \tag{3.63}$$

One can then verify that

$$\mathcal{O}^{\Omega}_S(\Omega^{-1}\phi) = \Omega^{-1}\mathcal{O}_S\phi \tag{3.64}$$

$$\mathcal{O}^{\Omega}_D(\Omega^{-3/2}\psi) = \Omega^{-3/2}\mathcal{O}_D\psi \tag{3.65}$$

$$\mathcal{O}^{\Omega}_{M\mu}{}^{\nu}A_\nu = \mathcal{O}_{M\mu}{}^{\nu}A_\nu \tag{3.66}$$

$$\mathcal{O}^{\Omega}_{gh}(\Omega^{-1}c) = \Omega^{-1}\mathcal{O}_{gh}c. \tag{3.67}$$

where the notation $\mathcal{O}^{\Omega}$ stands for the operator $\mathcal{O}$ constructed with the transformed metric $g^{\Omega} = \Omega^2 g$ and dilaton $\chi^{\Omega} = \Omega^{-1}\chi$. These operators map fields into fields transforming in the same way. (As observed earlier, they are dimensionless.) This implies that the eigenvalues of the operators $\mathcal{O}$ are Weyl–invariant and therefore also their determinants are invariant. We conclude that in the presence of a dilaton there exists a quantization procedure for noninteracting matter fields that respects Weyl invariance.

The Wess–Zumino action

We have seen that in the presence of a dilaton one has a choice between different quantization procedures, which can be understood as different functional measures: one of them breaks Weyl–invariance while the other maintains it. Let us denote $\Gamma^{\rm I}$ the effective action obtained with the standard measure and $\Gamma^{\rm II}$ the one obtained with the Weyl–invariant measure. The first is anomalous:

$$\delta_\omega\Gamma^{\rm I} = \int dx\, 2\omega\frac{\delta\Gamma^{\rm I}}{\delta g_{\mu\nu}}g_{\mu\nu} = -\int dx\sqrt{g}\,\omega\langle T^{\mu}{}_{\mu}\rangle^{I} \neq 0 \tag{3.68}$$

whereas the second is Weyl invariant: $\Gamma^{\rm II}(g^{\Omega},\chi^{\Omega}) = \Gamma^{\rm II}(g,\chi)$, or in infinitesimal form

$$0 = \delta_\omega\Gamma^{\rm II} = \int dx\sqrt{g}\,\omega\left(2\frac{\delta\Gamma^{\rm II}}{\delta g_{\mu\nu}}g_{\mu\nu} - \frac{\delta\Gamma^{\rm II}}{\delta\chi}\chi\right) \ . \tag{3.69}$$

[5] To see that this definition is unique, notice that for conformally coupled matter the differential operators keep the same form when written in the metric $\hat{g}$, e.g. for a scalar field $-\nabla^2 + \frac{R}{6} = -D^2 + \frac{\mathcal{R}}{6}$.

The Weyl invariant measure differs from the standard one simply by the replacement of the fixed mass μ by the dilaton χ, therefore we have

$$\Gamma^{\rm II}(g_{\mu\nu},\mu)=\Gamma^{\rm I}(g_{\mu\nu})\ . \tag{3.70}$$

We see that $\Gamma^{\rm II}$ can be obtained from $\Gamma^{\rm I}$ by applying the Stückelberg trick *after* quantization, *i.e.* to the mass parameter μ that has been introduced by the functional measure.

Another useful point of view is the following. Noting that $\Omega=\chi/\mu$ can be interpreted as the parameter of a Weyl transformation, the variation of $\Gamma^{\rm I}$ under a finite Weyl transformation defines a functional $\Gamma_{WZ}(g,\chi)$, the so-called "Wess-Zumino action", by: [6]

$$\Gamma^{\rm I}(g^\Omega)-\Gamma^{\rm I}(g)=\Gamma_{WZ}(g,\mu\Omega)\ . \tag{3.71}$$

It satisfies the so-called Wess-Zumino consistency condition, which can be written in the form

$$\Gamma_{WZ}(g^\Omega,\chi^\Omega)-\Gamma_{WZ}(g,\chi)=-\Gamma_{WZ}(g,\mu\Omega) \tag{3.72}$$

where $g^\Omega=\Omega^2 g$, $\chi^\Omega=\Omega^{-1}\chi$. This shows that variation of the WZ action under a Weyl transformation is exactly opposite to that of the action $\Gamma^{\rm I}$. From the Weyl invariance of $\Gamma^{\rm II}$ and equation (3.70) one finds that $\Gamma^{\rm II}(g_{\mu\nu},\Omega\mu)=\Gamma^{\rm II}(g^\Omega_{\mu\nu},\mu)=\Gamma^{\rm I}(g^\Omega_{\mu\nu})$. Thus, replacing $\mu\Omega$ by χ and using (3.71) we see that the χ-dependence of the Weyl–invariant action is entirely contained in a Wess-Zumino term:

$$\Gamma^{\rm II}(g,\chi)=\Gamma^{\rm I}(g)+\Gamma_{WZ}(g,\chi)\ . \tag{3.73}$$

We will later verify these statements by direct calculation in d=2, where all these functionals can be written explicitly. We can think of the Weyl–invariant effective action as the ordinary effective action to which a Wess-Zumino term has been added, with the effect of canceling the Weyl anomaly. [7]

In the case of non–interacting, massless, conformal matter fields the WZ action can be computed explicitly by integrating the trace anomaly. Let Ω_t be a one-parameter family of Weyl transformations with $\Omega_0=1$ and $\Omega_1=\Omega$, and let $g(t)_{\mu\nu}=g^{\Omega(t)}_{\mu\nu}$.

$$\Gamma_{WZ}(g_{\mu\nu},\Omega)=\int_0^1 dt\int dx\,\frac{\delta\Gamma}{\delta g_{\mu\nu}}\bigg|_{g(t)}\delta g(t)_{\mu\nu}=-\int_0^1 dt\int dx\,\sqrt{g(t)}\langle T_\mu^\mu\rangle_k\Omega(t)^{-1}\frac{d\Omega}{dt}\ . \tag{3.74}$$

In two dimensions, integrating the anomaly (3.52) and using the parametrization (3.17),

[6] Here we view the Wess-Zumino action as a functional of a metric and a dilaton, two dimensionful fields. Sometimes one may prefer to think of it as as a functional of a metric and a Weyl transformation, the latter being a dimensionless function. The two points of view are related by some factors of μ.

[7] This is completely analogous to what happens with gauge invariance in chiral theories [44].

one finds

$$\Gamma_{WZ}(g_{\mu\nu},\mu e^{\sigma}) = -\frac{c}{24\pi}\int d^2x\sqrt{g}\left(R\sigma - \sigma\nabla^2\sigma\right) \ . \tag{3.75}$$

A similar procedure in four dimensions using (3.54) leads to

$$\Gamma_{WZ}(g_{\mu\nu},\mu e^{\sigma}) = -\int dx\,\sqrt{g}\left\{c\,C^2\sigma - a\left[\left(E - \frac{2}{3}\Box R\right)\sigma + 2\sigma\Delta_4\sigma\right]\right\} \ , \tag{3.76}$$

where

$$\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}\nabla^\mu R\nabla_\mu \ . \tag{3.77}$$

At this point the reader will wonder whether the two procedures described above lead to different physical predictions or not. If the metric and dilaton are treated as classical external fields, but we allow them to be transformed, the two quantization procedures yield equivalent physics. In the Weyl–invariant procedure one has the freedom of choosing a gauge where $\chi = \mu$ and in this gauge all the results reduce to those of the standard procedure. In particular we observe that the trace of the energy-momentum tensor derived from the two actions $\Gamma^{\rm I}$ and $\Gamma^{\rm II}$ are the same. This follows from the fact that

$$\int dx\sqrt{g}\,\frac{\delta\Gamma_{WZ}}{\delta g_{\mu\nu}}\bigg|_{(g,\chi=\mu)} 2\omega g_{\mu\nu} = 0 \tag{3.78}$$

which in turn follows from (3.71).

On the other hand if we assume that the metric (and dilaton) are going to be quantized too, the answer hinges on the choice of *their* functional measure.

3.3.3 The Effective Average Action of free matter fields coupled to an external gravitational field

The full power of the FRGE, which is an exact equation, becomes manifest when one considers interacting fields. In this section we shall familiarize ourselves with the FRGE in the context of free matter fields coupled to an external metric and dilaton, where the one–loop approximation is exact. We leave the discussion of interacting matter to the next section.

The EAA and its flow at one loop

As we have seen in the first chapter, the definition of the EAA follows the same steps of the definition of the ordinary effective action, except that one modifies the bare action by adding to it a cutoff term $\Delta S_k(\phi)$ that is quadratic in the fields and therefore modifies the propagator without affecting the interactions. Using the notation of (3.27), the cutoff term is:

$$\Delta S_k(g_{\mu\nu},\phi)=\frac{1}{2}\,\mathcal{G}\left(\phi,\frac{R_k(\Delta)}{\mu^2}\phi\right)=\frac{1}{2}\frac{k^2}{\mu^2}\sum_n a_n^2 r\left(\frac{\lambda_n}{k^2}\right)\ , \tag{3.79}$$

where we have written the cutoff (which has dimension of mass squared) as $R_k(z)=k^2r(z/k^2)$.

The evaluation of the EAA for Gaussian matter fields, conformally coupled to a metric, follows the same steps that led to (3.32). The only differences are the replacement of S by $S+\Delta S_k$ and hence of the "inverse propagator" Δ by the "cutoff inverse propagator" $P_k(\Delta)=\Delta+R_k(\Delta)$, and in the end the subtraction of ΔS_k. The result is

$$\Gamma^{\rm I}_k(g_{\mu\nu},\phi)=S(g_{\mu\nu},\phi)+\frac{1}{2}\mathrm{Tr}\log\left(\frac{P_k(\Delta)}{\mu^2}\right)\ . \tag{3.80}$$

We used here the superscript I to denote that this EAA has been obtained by using the standard measure and reduces to $\Gamma^{\rm I}$ for $k=0$. We would like now to define a Weyl–invariant form of EAA, to be called $\Gamma^{\rm II}_k$ in analogy to the effective action $\Gamma^{\rm II}$ discussed previously.

The first step is to clarify the meaning of the cutoff k in this context. In the usual treatment, *i.e.* in a non–gravitational context, k is a constant with dimension of mass. In the present context these two properties are contradictory. A quantity that has a nonzero dimension cannot generally be a constant: it can only be constant in some special gauge. This means that the cutoff must be allowed to be a generic non-negative function on spacetime.

Now we must give a meaning to the notion that the couplings depend on the cutoff. In a Weyl–invariant theory all couplings are dimensionless, and the only way they can depend on k is via the dimensionless combination $u=k/\chi$. Note that by definition the dilaton cannot vanish anywhere, whereas the cutoff should be allowed to go to zero. So u is a non-negative dimensionless function on spacetime. This raises the question of the meaning of a running coupling whose argument is itself a function on spacetime. In order to avoid such issues we will restrict ourselves to the case when u is a constant, in other words the cutoff and the dilaton are proportional.

With this point understood, the evaluation of the EAA with the Weyl–invariant measure is very simple: as before we just have to replace μ by χ

$$\Gamma^{\rm II}_u(g_{\mu\nu},\phi) = S(g_{\mu\nu},\phi)+\frac{1}{2}\mathrm{Tr}\log\left(\frac{\Delta+R_k(\Delta)}{\chi^2}\right) \tag{3.81}$$

$$= S(g_{\mu\nu},\phi)+\frac{1}{2}\mathrm{Tr}\log\left(\mathcal{O}+u^2r(u^2\mathcal{O})\right)\ . \tag{3.82}$$

In the second line we have reexpressed the EAA as a function of the Weyl–covariant operator $\mathcal{O}=\chi^{-2}\Delta$, the Weyl–invariant cutoff parameter u and the dimensionless function

$r(z/k^2) = R_k(z)/k^2$. It is manifest that all dependence on k is via u and that $\Gamma^{\rm II}_u$ is Weyl–invariant.

Calculating the effective action with the FRGE

In the preceding chapter we have shown that the EAA satisfies the following FRGE:

$$k\frac{d\Gamma_k}{dk} = \frac{1}{2}\mathrm{Tr}\left[\frac{\delta^2(\Gamma_k+\Delta S_k)}{\delta\phi\delta\phi}\right]^{-1} k\frac{d}{dk}\frac{\delta^2\Delta S_k}{\delta\phi\delta\phi}\ , \tag{3.83}$$

which is an exact equation holding for any theory [45]. [8]

The r.h.s. of the FRGE (3.83) can be regarded as the "beta functional" of the theory, giving the k–dependence of all the couplings. To see this let us assume that Γ_k admits a derivative expansion of the form

$$\Gamma_k(\phi, g_i) = \sum_{n=0}^{\infty}\sum_i g_i^{(n)}(k)\mathcal{O}_i^{(n)}(\phi)\ , \tag{3.84}$$

where $g_i^{(n)}(k)$ are coupling constants and $\mathcal{O}_i^{(n)}$ are all possible operators constructed with the field ϕ and n derivatives, which are compatible with the symmetries of the theory. We have

$$k\frac{d\Gamma_k}{dk} = \sum_{n=0}^{\infty}\sum_i \beta_i^{(n)}\mathcal{O}_i^{(n)}\ , \tag{3.85}$$

where $\beta_i^{(n)}(g_j, k) = k\frac{dg_i^{(n)}}{dk} = \frac{dg_i^{(n)}}{dt}$ are the beta functions of the couplings. As before we have introduced $t = \log(k/k_0)$, k_0 being an arbitrary initial value. If we expand the trace on the r.h.s. of (3.83) in operators $\mathcal{O}_i^{(n)}$ and compare with (3.85), we can read off the beta functions of the individual couplings.

The most remarkable property of the FRGE is that the trace on the r.h.s. is free of UV and IR divergences. This is because the derivative of the cutoff kernel goes rapidly to zero for $q^2 > k^2$, and k also acts effectively as a mass. So, even though the EAA defined above is as ill–defined as the usual effective action, its t–derivative is well–defined. Given a "theory space" which consists of a class of functionals of the fields, one can define on it a flow without having to worry about UV regularizations. All the beta functions are finite. This can be done for any theory, whether renormalizable or not.

Then, one can pick an initial point in theory space, which can be identified with the

[8] Note that the structure of (3.83) in field space is the trace of a contravariant two tensor times a covariant two–tensor (in de Witt notation, $((\Gamma_k^{(2)} + \Delta S_k^{(2)})^{-1})^{ij}(\partial_t \Delta S_k^{(2)})_{ji}$, where a superscript (2) denotes second functional derivative and $t = \log k$) and is therefore an invariant expression. In passing from (3.83) to (3.86) one uses the field space metric $\mathcal{G}$ to raise and lower indices and transform the covariant and contravariant tensors into mixed tensors, each of which can be seen as a function of Δ. In practice this amounts to canceling all factors of $\sqrt{g}$ and μ.

bare action at some UV scale Λ, and study the trajectory passing through it in either direction. The EAA can be obtained by solving the first order differential equation (3.83) and taking the limit $k \to 0$.

The issue of the divergences presents itself, in this formulation, when one tries to move Λ to higher energies, which is equivalent to solving the RG equation for growing k. If the trajectory is renormalizable, all dimensionless couplings remain finite in the limit $k \to \infty$. This implies that only the relevant dimensionful coupling diverge, and one expects only a finite number of these. The ambiguities that correspond to these divergences are fixed by the choice of RG trajectory, because the IR limit (i.e. the renormalized couplings) is kept fixed. On the other hand if some dimensionless coupling diverges (e.g. at a Landau pole) the theory ceases to make sense there and the trajectory describes an effective low energy field theory.

Let us now return to the case of free matter in an external gravitational field. In the preceding section we defined two variants of the EAA: the "standard" EAA $\Gamma^{\rm I}_k$ and the Weyl–invariant EAA $\Gamma^{\rm II}_u$, both of which can be written as trace of the logarithm of some function of the kinetic operator. These expressions are formal, because they contain divergences and need to be regularized. In the case of $\Gamma^{\rm II}_u$ this can be done in a Weyl–invariant way by using an UV cutoff that is a multiple of the dilaton, similar to the way we introduced the infrared cutoff k. We do not pursue this here. Instead, we take the derivative of (3.80) with respect to k and using the definition $R_k(\Delta) = k^2 r(\Delta/k^2)$ obtain

$$
\begin{aligned}
k\frac{d\Gamma^{\rm I}_k}{dk} &= \frac{1}{2}\mathrm{Tr}\left(\frac{1}{\Delta + R_k(\Delta)} k\frac{dR_k(\Delta)}{dk}\right) \\
&= \mathrm{Tr}\frac{r(\Delta/k^2) - (\Delta/k^2)r'(\Delta/k^2)}{(\Delta/k^2) + r(\Delta/k^2)} \ .
\end{aligned}
\tag{3.86}
$$

It is easy to see, especially using the form in the first line, that this is a special case of the FRGE (4.2), and the fall–off properties of the function r guarantee that the trace on the r.h.s. is finite.

One can repeat this argument in the case of the Weyl–invariant EAA with little changes, and the flow equation reads

$$
u\frac{d\Gamma^{\rm II}_u}{du} = \mathrm{Tr}\frac{r(\mathcal{O}/u^2) - (\mathcal{O}/u^2)r'(\mathcal{O}/u^2)}{(\mathcal{O}/u^2) + r(\mathcal{O}/u^2)} \ . \tag{3.87}
$$

In this form the r.h.s. of the FRGE is manifestly Weyl–invariant, since u is Weyl–invariant and one has the trace of a function of a Weyl–covariant operator. [9]

The EAA's $\Gamma^{\rm I}_k$ and $\Gamma^{\rm II}_u$ are not well–defined functionals, but their derivatives are well–

[9] Note that $\Delta/k^2 = \mathcal{O}/u^2$ so the r.h.s. of (3.86) and (3.87) are identical. The reason for the lack of invariance of the EAA $\Gamma^{\rm I}$ (and its derivative) is the measure which contains the absolute mass scale μ. If one allowed μ to be transformed, in the same way as we allow the cutoff k to be transformed, the two actions would be seen to be the same.

defined. As explained above, one can integrate the FRGE and obtain, in the IR limit, the ordinary effective action. If one starts from a given Weyl–invariant classical matter action at scale Λ and integrates the flow of $k\frac{d\Gamma^{\rm I}_k}{dk}$, respectively $u\frac{d\Gamma^{\rm II}_u}{du}$, down to $u=0$ one obtains exactly the effective action $\Gamma^{\rm I}$, respectively $\Gamma^{\rm II}$. Furthermore, at each u, $\Gamma^{\rm II}_u$ is obtained from $\Gamma^{\rm I}_k$ by the Stückelberg trick. It is instructive to explicitly illustrate these statements in the case of $d=2$ and, for the c–anomaly, also in the case $d=4$.

$d=2$: the Polyakov action

In this section we consider the effective action of a single scalar field [47]. It has been derived by integrating the FRGE in [48]. The main tool in this derivation is the non–local expansion of the heat kernel in powers of curvature [49, 50]. Keeping terms up to two curvatures one has

$$\mathrm{Tr} e^{-s\Delta} = \frac{1}{4\pi s}\int d^2x\sqrt{g}\left[1+s\frac{R}{6}+s^2 R f_R(s\Delta)R+\ldots\right] , \tag{3.88}$$

where

$$f_R(x)=\frac{1}{32}f(x)+\frac{1}{8x}f(x)-\frac{1}{16x}+\frac{3}{8x^2}f(x)-\frac{3}{8x^2}\ ; \qquad f(x)=\int_0^1 d\xi e^{-x\xi(1-\xi)}\ .$$

The r.h.s. of (3.86) can be written, after some manipulations,

$$k\frac{d\Gamma^{\rm I}_k}{dk}=\int ds\,\tilde h(s)\mathrm{Tr}\,e^{-s\Delta}\ ; \qquad h(z)=\int_0^\infty ds\,\tilde h(s)e^{-s\,z}\ ,$$

where $\tilde h(s)$ is the Laplace anti-transform of $h(z)=\frac{\partial_t R_k(z)}{z+R_k(z)}$. Using the explicit cutoff $R_k(z)=(k^2-z)\theta(k^2-z)$, we have simply $h(z)=2k^2\theta(k^2-z)$ and the integrals give

$$\begin{aligned} k\frac{d\Gamma^{\rm I}_k}{dk} &= \int d^2x\sqrt{g}\Bigg[\frac{k^2}{4\pi}+\frac{1}{24\pi}R \\ &\quad +\frac{1}{64\pi}R\frac{1}{\Delta}\left(\sqrt{\frac{\widetilde\Delta}{\widetilde\Delta-4}}-\frac{\widetilde\Delta+4}{\widetilde\Delta}\sqrt{\frac{\widetilde\Delta-4}{\widetilde\Delta}}\right)\theta(\widetilde\Delta-4)R\Bigg]+O(R^3) \end{aligned} \tag{3.89}$$

with $\widetilde\Delta=\Delta/k^2$. On the other hand, keeping terms at most quadratic in curvature, the EAA can be written in the form

$$\Gamma^{\rm I}_k=\int d^2x\sqrt{g}\,[a_k+b_kR+R\,c_k(\Delta)R]+O\left(R^3\right) \tag{3.90}$$

where $c_k(\Delta)$ is a nonlocal form-factor which, for dimensional reasons, can be written in the form $c_k(\Delta) = \frac{1}{\Delta} c(\widetilde{\Delta})$. The beta functions of a_k, b_k and c_k are then

$$\partial_t a_k = \frac{k^2}{4\pi} \; ; \quad \partial_t b_k = \frac{1}{24\pi} \; ; \quad \partial_t c = \frac{1}{64\pi} \left(\sqrt{\frac{\widetilde{\Delta}}{\widetilde{\Delta}-4}} - \frac{\widetilde{\Delta}+4}{\widetilde{\Delta}} \sqrt{\frac{\widetilde{\Delta}-4}{\widetilde{\Delta}}} \right) \theta(\widetilde{\Delta}-4) \tag{3.91}$$

In order to obtain the effective action, one integrates this flow from some UV scale Λ, that can later be sent to infinity, down to $k=0$. Setting $a_\Lambda = \frac{\Lambda^2}{4\pi}$, one has $a_k = \frac{k^2}{4\pi}$ and therefore the renormalized cosmological term vanishes in the IR limit. The Hilbert term has a logarithmically running coefficient $b_k = b_\Lambda - \frac{1}{24\pi} \log \frac{\Lambda}{k}$. We will not consider this term in the following because it is topological. We assume that c_k vanishes at $k \to \infty$, since the UV action only contains the matter terms. The integral over k is finite even in the limit $\Lambda \to \infty$, and one finds

$$c(\widetilde{\Delta}) = -\frac{1}{96\pi} \frac{\sqrt{\widetilde{\Delta}-4}(\widetilde{\Delta}+2)}{\widetilde{\Delta}^{3/2}} \theta(\widetilde{\Delta}-4) \; . \tag{3.92}$$

The explicit form of c_k can be found also employing the mass cutoff $R_k(z) = k^2$, in which case the computation can also be done analytically, giving

$$c(\tilde{\Delta}) = -\frac{1}{16\pi} \left[\frac{1}{6} - \frac{1}{\tilde{\Delta}} + \frac{\text{Arctanh}\left(\sqrt{\frac{\tilde{\Delta}}{\tilde{\Delta}+4}}\right)}{\tilde{\Delta}^{3/2}\sqrt{\tilde{\Delta}+4}} \right] \tag{3.93}$$

and with the exponential cutoff $R_k(z) = \frac{z}{exp(\frac{z}{k^2})-1}$, in which case it is computed numerically. All three give the same qualitative running, as depicted in Figure 3.2. In the limit $k \to 0$ one obtains, in all cases, the Polyakov action: [10]

$$\Gamma^{\mathrm{I}}(g_{\mu\nu}) = -\frac{1}{96\pi} \int d^2x \sqrt{g} R \frac{1}{\Delta} R \; . \tag{3.94}$$

The function c_k admits a series expansion $c_k(\Delta) = \frac{1}{k^2} \sum_{n=1}^{\infty} c_n \frac{k^{2n}}{\Delta^n}$. Then, one can explicitly perform the variation with respect to the metric and obtain the energy–momentum tensor. In particular, conformal variation of Γ^{I}_k gives the k–dependent trace anomaly:

$$\langle T^\mu_\mu \rangle^I_k = -\frac{2}{\sqrt{g}} g_{\mu\nu} \frac{\delta \Gamma^{\mathrm{I}}_u}{\delta g_{\mu\nu}} = -4c(\tilde{\Delta}) R - \frac{2}{k^2} \sum_{n=0}^{\infty} \sum_{k=1}^{n-1} c_n \left(\frac{1}{\tilde{\Delta}^k} R \right) \left(\frac{1}{\tilde{\Delta}^{n-k}} R \right) \; . \tag{3.95}$$

We observe that the integrated trace anomaly (which is related to the variation of the EAA

[10] Using this action in (3.71) one recovers the WZ action (3.75). Conversely, the Polyakov action can be obtained from the WZ action by using the equation of motion for σ.

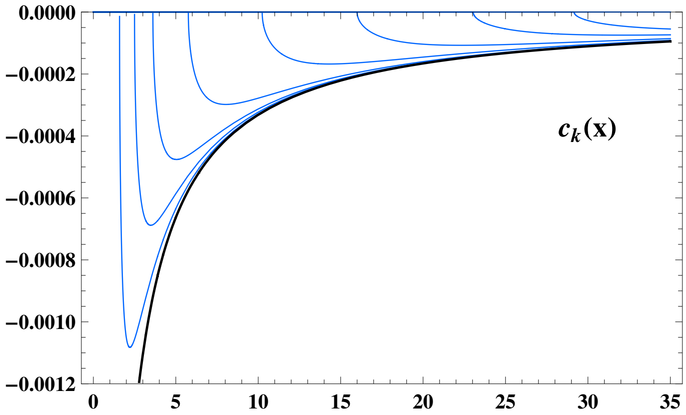


Figure 3.1: Shape of the form factor $c_k(x)$ of eq. (3.92) as a function of $x = \Delta$ for different values of k. The thick line shows the case $k = 0$ (Polyakov action).

under a global scale transformation) can be written more explicitly

$$\int dx\sqrt{g}\,\langle T^\mu_\mu\rangle_k = \int dx\sqrt{g}\left(-4c(\tilde{\Delta})R + \frac{2}{k^2}R\,c'(\tilde{\Delta})R\right)\ . \tag{3.96}$$

For a fixed momentum Δ the linear term of the trace anomaly grows monotonically as k decreases, from zero at infinity to its canonical value at $k = 0$. The second term shows a nontrivial flow for $k \neq 0$, going to zero both in the UV and IR.

Let us now come to the effective action $\Gamma^{\rm II}$. Using the Weyl–invariant measure, the effective action is given by the determinant of the dimensionless operator $\mathcal{O} = \hat{\Delta} = \frac{1}{\chi^2}\Delta$, which can be identified with $\Delta_{\hat{g}}$, the operator constructed with the dimensionless, Weyl–invariant metric $\hat{g}_{\mu\nu} = \chi^2 g_{\mu\nu}$. Therefore, as already discussed, $\Gamma^{\rm II}$ differs from $\Gamma^{\rm I}$ just in the replacement of $\mu^2 g_{\mu\nu}$ by $\chi^2 g_{\mu\nu}$. We have to generalize this for finite $k \neq 0$. As discussed above, we assume that the cutoff is a constant multiple of the dilaton: $k = u\chi$. Neglecting the a– and b–terms, the effective average action can then be written in the manifestly Weyl–invariant form

$$\Gamma^{\rm II}_u(g_{\mu\nu},\chi) = \int d^2x\sqrt{g}\,\mathcal{R}\,\frac{1}{\chi^2\mathcal{O}}c\left(\frac{\mathcal{O}}{u^2}\right)\mathcal{R}\ , \tag{3.97}$$

with the same function c given in (3.92). In particular the Weyl–invariant version of the Polyakov action is obtained in the limit $u \to 0$:

$$\Gamma^{\rm II}(g_{\mu\nu},\chi) = -\frac{1}{96\pi}\int d^2x\sqrt{g}\,\mathcal{R}\,\frac{1}{\chi^2\mathcal{O}}\mathcal{R}\ . \tag{3.98}$$

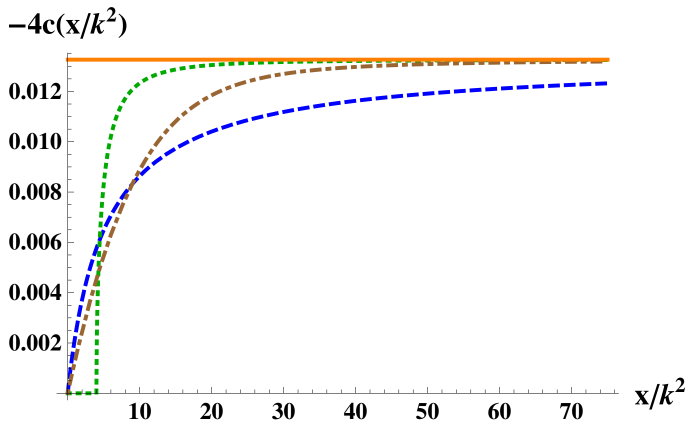


Figure 3.2: Flow of the trace anomaly as a function of x/k for fixed $x=\Delta$, as given by the function $-4c(\frac{x}{k^2})$ of equation (3.95). Note that the origin is the UV limit. The three curves correspond to optimized cutoff (green, dotted), exponential cutoff (brown, dot-dashed) and mass-type cutoff (blue, long -dashed); also plotted is the $k=0$ asymptotic value of the trace anomaly (orange).

It is now easy to check explicitly equation (3.73): for $c=1$, using $\mathcal{R}=R+2\Delta\sigma$, one finds

$$\Gamma^{\rm II}=-\frac{1}{96\pi}\int d^2x\sqrt{g}\,R\,\frac{1}{\Delta}R-\frac{1}{24\pi}\int d^2x\sqrt{g}\,\sigma(\Delta\sigma+R)=\Gamma^{\rm I}+\Gamma_{WZ}\ .$$

We have claimed in the end of section 3.4 that the trace of the energy–momentum tensors computed from $\Gamma^{\rm II}$ and $\Gamma^{\rm I}$ coincide in the gauge $\chi=\mu$. This statement actually holds also for $k\neq 0$. A direct calculation yields

$$\langle T^\mu_\mu\rangle^{\rm II}_u=-\frac{2}{\sqrt{g}}g_{\mu\nu}\frac{\delta\Gamma^{\rm II}_u}{\delta g_{\mu\nu}}=-4c\left(\frac{\mathcal{O}}{u^2}\right)\mathcal{R}-\frac{2}{u^2\chi^2}\sum_{n=1}^{\infty}\sum_{k=1}^{n-1}c_n\left(\frac{u^{2k}}{\mathcal{O}^k}\mathcal{R}\right)\left(\frac{u^{2(n-k)}}{\mathcal{O}^{n-k}}\mathcal{R}\right)\qquad(3.99)$$

One can verify that this is also equal to $\frac{1}{\sqrt{g}}\chi\frac{\delta\Gamma^{\rm II}_u}{\delta\chi}$, thereby obtaining an explicit check of the general statement (3.69). It is also interesting to observe that if we think of $\Gamma^{\rm II}_u$ as a function of k, χ and $g_{\mu\nu}$, and vary each keeping the other two fixed, the metric variation is again given by equation (3.99), the χ variation gives the first term in the r.h.s. of (3.99) and the k variation gives the second term. We also note that the "beta functional" can be written in general as

$$u\frac{d\Gamma^{\rm II}_u}{du}=-\int dx\sqrt{g}\,\frac{2}{u^2\chi^2}\mathcal{R}\,c'\left(\frac{\mathcal{O}}{u^2}\right)\mathcal{R}\ .\qquad(3.100)$$

$d = 4$: the c–anomaly action

One would like to repeat the analysis of the previous section in $d = 4$, to the extent that this is possible. The main difference is that while in $d = 2$ the Polyakov action is the full effective action, in $d = 4$ there are terms with higher powers of curvature. The analysis then has to be limited to the first few terms of the expansion in curvatures:

$$\Gamma^{\rm I}_k = \int d^4x\sqrt{g}\left[a + bR + Rf_R(\Delta)R + C_{\mu\nu\rho\sigma}f_C(\Delta)C^{\mu\nu\rho\sigma} + O(R^3)\right] \tag{3.101}$$

where a, b, f_R and f_C depend on k. Such running structure functions have been already computed in [51]. It was found that the form factor $f_R(\Delta)$ tends to zero in the IR limit, whereas $f_C(\Delta)$ approaches the standard one-loop EA at two curvatures for $k \to 0$:

$$\Gamma^{\rm I} = -\frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,\frac{N_0 + 6N_{1/2} + 12N_1}{120}\,C_{\mu\nu\rho\sigma}\log\left(\frac{\Delta}{k_0^2}\right)C^{\mu\nu\rho\sigma} + \ldots \tag{3.102}$$

where $\Delta = -\nabla^2$ and k_0 is an arbitrary scale. Here we are neglecting the local terms, whose coefficients are arbitrary and can be tuned to zero. To connect this result with standard one loop EAs computed in [49] it is sufficient to change the basis expansion from powers of $(R, C_{\mu\nu\rho\sigma})$ and their derivatives to powers of $(R, R_{\mu\nu})$ and their derivatives. (This requires expressing the Riemann tensor as an infinite nonlocal series in the Ricci tensor.)

An EA of the form (3.102) had been suggested by Deser and Schwimmer [52] as the source of the c–anomaly, namely the terms proportional to $C^2_{\mu\nu\rho\sigma}$ in (3.54). This action (in contrast to the Riegert action discussed below) also produces the correct flat spacetime limit for the correlation functions of the energy momentum tensor $\langle T_{\mu\nu}T_{\rho\sigma}\rangle$ [53].

In the basis of the tensors $(R, R_{\mu\nu})$ the terms cubic in curvature are known explicitly [55]. When the Riemann squared term in the anomaly is expanded in an infinite series in $(R, R_{\mu\nu})$, the action of [55] correctly reproduces the first terms of this expansion [56]. In order to reproduce the full anomaly (both c– and a–terms) one would need also terms in the effective action of order higher than three.

It is possible to write closed form actions that generate the full anomaly. A functional that generates the c–anomaly has been given already in (3.102). Another action that gives both c– and a–anomaly is the Riegert action [57]

$$W(g_{\mu\nu}) = \int dx\,\sqrt{g}\frac{1}{8}\left(E - \frac{2}{3}\Box R\right)\Delta_4^{-1}\left[2c\,C^2 - a\left(E - \frac{2}{3}\Box R\right)\right] + \frac{a}{18}R^2\,. \tag{3.103}$$

It has the drawback that it gives zero for the flat spacetime limit of the correlator of two energy–momentum tensors. This does not mean, however, that one cannot write the full effective action as the sum of the Riegert action and Weyl–invariant terms, because one can write the Deser–Schwimmer action as the Riegert action (with $a = 0$) plus Weyl–invariant terms. In this case the energy–momentum correlator would come from the Weyl–invariant terms, as we shall see below.

The relation between the Wess–Zumino term (3.76) and the Riegert action (3.103) is very similar to the one between the two–dimensional Wess–Zumino action (3.75) and the Polyakov action (3.94): using the Riegert action in (3.71) one recovers the WZ action (3.76). Unlike the two–dimensional case, however, the converse procedure is not unique. The general idea is to replace the dilaton $\chi = \mu e^{\sigma}$, which in the WZ action is treated as an independent variable, by a functional of the metric $g_{\mu\nu}$ having the right transformation properties. One choice, which has been proposed in [31, 59] is

$$\sigma(g_{\mu\nu}) = \log\left(1 - \frac{1}{\Delta + R/6}\frac{R}{6}\right) . \tag{3.104}$$

Another possibility is

$$\sigma(g_{\mu\nu}) = -\frac{1}{4}\frac{1}{\Delta_4}\left(E + \frac{2}{3}\Delta R + b\, C^2\right) , \tag{3.105}$$

where b is an arbitrary constant. In both cases $\sigma(g_{\mu\nu}) \mapsto \sigma(g_{\mu\nu}) - \log\Omega$ under a Weyl transformation. Note that (3.105), for $b = c$ is the equation of motion for the dilaton coming from the WZ action (3.76), while for $b = 0$ it is the equation of motion coming from the a–term of the WZ action. The latter choice exactly reproduces (3.103); other choices of b give the Riegert action plus Weyl–invariant terms, while (3.104) gives another form of the anomaly functional.

In $d = 2$, knowing the explicit form of the effective action, we were able to explicitly check equation (3.73). In $d = 4$ we have only limited knowledge of the effective action. Instead of trying to check equation (3.73) we can use it to obtain some additional information on the effective action $\Gamma^{\rm I}$. We have

$$\Gamma^{\rm I}(g_{\mu\nu}) = \Gamma^{\rm II}(g_{\mu\nu}, \chi) - \Gamma_{WZ}(g_{\mu\nu}, \chi) , \tag{3.106}$$

where the first term in the r.h.s. is Weyl–invariant by construction and the anomaly comes entirely from the second term. For example if we use (3.105) with $b = 0$, the second term exactly reproduces the Riegert action and the correlator of two energy–momentum tensors must come from the first term. We know already that it must contain the term

$$\Gamma^{\rm II}(g_{\mu\nu}, \mu e^{\sigma(g_{\mu\nu})}) = -\frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,\frac{N_0 + 6N_{1/2} + 12N_1}{120}\,\mathcal{C}_{\mu\nu\rho\sigma}\log\left(\frac{\mathcal{O}}{u_0^2}\right)\mathcal{C}^{\mu\nu\rho\sigma} + \ldots \tag{3.107}$$

where $\mathcal{C}_{\mu\nu\rho\sigma}$ is the Weyl tensor constructed with the metric $e^{2\sigma(g)}g_{\mu\nu}$. Expanding this to second order in the curvature of $g_{\mu\nu}$ one reobtains as a leading term the action (3.102). The lack of Weyl–invariance of that action is compensated by higher terms in the expansion. This shows that there is no contradicton between the presence of the Riegert and the Deser–

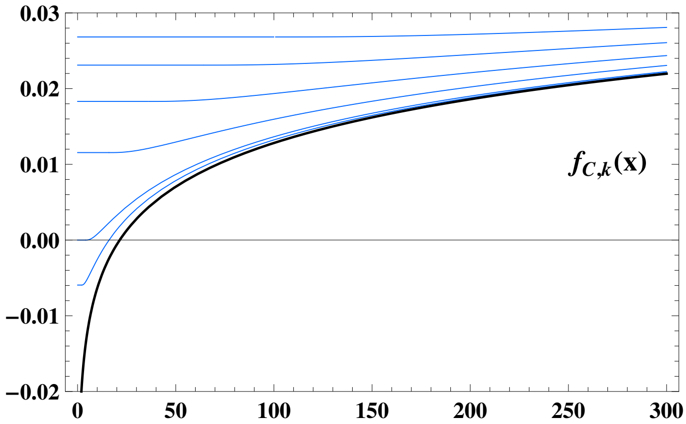


Figure 3.3: Shape of the form factor f_C of eq. (3.108) in the case of a single scalar field, as a function of $x = \Delta$ for different values of k. The thick line shows the case $k = 0$.

Schwimmer terms in the effective action $\Gamma^{\rm I}$, and the flat space limit of energy–momentum tensor correlators. Thus there is also no disagreement with [58] and with [59].

Finally, we can write the explicit form of the interpolating EAA. For a scalar field we have [51]

$$\begin{aligned}
\Gamma^{\rm I}_k(g_{\mu\nu}) &= -\frac{1}{2(4\pi)^2}\int d^4x\sqrt{g}\,C_{\mu\nu\rho\sigma}\left\{\frac{1}{120}\log\left(\frac{k^2}{k_0^2}\right)\right. \\
&+ \theta\left(\tilde\Delta-4\right)\left[-\frac{1}{120}\log\left(\frac{k^2}{k_0^2}\right)-\frac{4\sqrt{\tilde\Delta-4}\sqrt{\tilde\Delta}}{75\tilde\Delta^3}+\frac{11\sqrt{\tilde\Delta-4}\sqrt{\tilde\Delta}}{225\tilde\Delta^2}\right. \\
&- \left.\left.\frac{23}{900}\sqrt{1-\frac{4}{\tilde\Delta}}+\frac{1}{120}\log\left(\frac{\Delta}{2k_0^2}\left(\sqrt{1-\frac{4}{\tilde\Delta}}+1\right)-\frac{k^2}{k_0^2}\right)\right]\right\}C^{\mu\nu\rho\sigma}+\ldots
\end{aligned} \tag{3.108}$$

Notice that the first logarithm in the bracket is both UV and IR divergent, and also note that the Heaviside theta is zero when k is sufficiently large. Thus, the UV divergence is present and must be removed by renormalization, whereas the IR divergence is automatically canceled by the second logarithm. The form factor $f_C(x)$, for fixed x, is plotted in figure 3. Similar formulas, but with different coefficients, hold also for fermions and gauge fields. In the limit $k \to 0$ they all reduce to (3.102)

The calculation of $\Gamma^{\rm II}$ follows the same lines. There are two running structure functions

$f_{\mathcal{R}}$ and $f_{\mathcal{C}}$. The explicit form of $f_{\mathcal{C}}$ for a scalar field is

$$\begin{aligned}\Gamma^{\rm II}_u(g_{\mu\nu},\chi) &= -\frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,\mathcal{C}_{\mu\nu\rho\sigma}\Bigg\{\frac{1}{120}\log u^2 \\ &+ \theta\left(\frac{\mathcal{O}}{u^2}-4\right)\Bigg[-\frac{1}{120}\log u^2-\frac{4u^6\sqrt{\frac{\mathcal{O}}{u^2}-4}\sqrt{\frac{\mathcal{O}}{u^2}}}{75\mathcal{O}^3}+\frac{11u^4\sqrt{\frac{\mathcal{O}}{u^2}-4}\sqrt{\frac{\mathcal{O}}{u^2}}}{225\mathcal{O}^2} \\ &- \frac{23}{900}\sqrt{1-\frac{4u^2}{\mathcal{O}}}+\frac{1}{120}\log\left(\frac{\mathcal{O}}{2}\left(\sqrt{1-\frac{4u^2}{\mathcal{O}}}+1\right)-u^2\right)\Bigg]\Bigg\}\mathcal{C}^{\mu\nu\rho\sigma}\end{aligned} \tag{3.109}$$

The same computation can be repeated in the case of fermions and vectors and a different interpolating function can be found. When $u \to 0$ we get back equation (3.107).

3.3.4 Interacting matter fields

In the preceding sections we have shown that there exists a quantization procedure such that the effective action which is obtained by integrating out free (Gaussian) matter fields remains Weyl invariant. The proof was simple because the integration over matter was Gaussian. Here we generalize the result to the case when there are matter interactions.

As in the preceding section, we begin by considering the case when the initial matter action is Weyl invariant even without invoking a coupling to the dilaton. This is the case for massless, renormalizable quantum field theories such as ϕ^4, Yang-Mills theory and fermions with Yukawa couplings in $d = 4$. The interactions are of the form $S_{int}(g_{\mu\nu}, \Psi_a) = \lambda \int dx \sqrt{g}\, \mathcal{L}_{int}$ where $\mathcal{L}_{int}$ is a dimension d operator and λ is dimensionless. Interactions generate new anomalous terms over and above those that we have already considered for Gaussian matter. The trace anomaly of free matter vanishes in the limit of flat space, but this is not true for interacting fields: the trace is then proportional to the beta function. For the interaction term given above one has in flat space

$$\int dx\, \omega\, \langle T^\mu{}_\mu\rangle = -\delta_\omega S_{int} = \int dx\, \omega\, \beta_\lambda\, \mathcal{L}_{int} \tag{3.110}$$

where $\beta_\lambda = k\frac{d\lambda}{dk}$. (This is somewhat similar to equation (3.47), but there is a sign difference due to the fact that μ does not play the role of a sliding scale there.)

We want to study the effective action of this theory, which is obtained by integrating out the matter fields. In order to be able to make non–perturbative statements we will use the FRGE as a machine for calculating the effective action, as discussed in the introduction and exemplified by the calculations in the preceding sections. The general idea is to begin with some Weyl–invariant bare action at some scale and to integrate the RG flow. If the "beta functional" is itself Weyl–invariant, the action at each scale will be Weyl–invariant. The effective action, which is obtained by letting $k \to 0$, will also be Weyl–invariant.

This statement is seemingly in contrast with (3.110), which implies that Weyl invariance can only be achieved when all beta functions are zero. How can one maintain Weyl–invariance along a flow? The trick is to consider the flow as dependence of λ on the dimensionless parameter $u = k/\chi$. We assume u to be constant to avoid issues related to the interpretation of a coupling depending on a function. [11] Since u is Weyl–invariant, also $\lambda(u)$ is. This is very much in the spirit of Weyl's geometry, where the dilaton is interpreted physically as the unit of mass and u is the cutoff measured in the chosen units.

We now see that with this definition of RG, the running of couplings does not in itself break Weyl invariance. In the spirit of Weyl's theory the dilaton is taken as a reference scale and the couplings are functions of u. Since u is Weyl–invariant,

$$\delta_\omega S_{int} = 0 \ , \tag{3.111}$$

even when the beta function $\beta_\lambda = u\frac{d\lambda}{du}$ is not zero. It is important to stress that this should not be interpreted as vanishing trace of the energy–momentum tensor. We argued in section 3.3.2 that the energy–momentum tensor is the same whether one uses the standard or the Weyl–invariant measure. That argument is not restricted to non–interacting matter and applies here too. So, as in the case of free matter fields discussed at the end of the preceding section, the physical content of the Weyl–invariant theory is exactly the same as in the usual formalism. The recovery of Weyl–invariance is due to additional terms that involve the variation of the action with respect to the dilaton.

Let us return now to the issue of the Weyl–invariance of the flow. As we have seen, in order to have Weyl invariance in the presence of the cutoff k, the latter must be transformed as a field of dimension one. Then, one can construct a Weyl–invariant cutoff action. Since the cutoff action is always quadratic in the quantum fields, one can use exactly the same procedure that we followed in the case of free fields. The r.h.s. of the FRGE is given in (3.83) as a trace of a function involving the Hessian and the k–derivative of the cutoff. If the field ϕ has weight w, the two terms in (3.83) have the transformation properties:

$$\frac{\delta^2(\Gamma_k + \Delta S_k)}{\delta\phi\delta\phi} \quad \mapsto \quad \Omega^{-w}\frac{\delta^2(\Gamma_k + \Delta S_k)}{\delta\phi\delta\phi}\Omega^{-w} \ , \tag{3.112}$$

$$k\frac{d}{dk}\frac{\delta^2\Delta S_k}{\delta\phi\delta\phi} \quad \mapsto \quad \Omega^{-w}k\frac{d}{dk}\frac{\delta^2\Delta S_k}{\delta\phi\delta\phi}\Omega^{-w} \ . \tag{3.113}$$

As a consequence, the trace in the r.h.s. of (3.83) is invariant. Since the beta functional is Weyl–invariant, if we start from some initial condition that is Weyl invariant we will remain within the subspace of theories that are Weyl–invariant. The effective action $\Gamma^{\rm II}$, which is obtained as the limit of the flow for $k \to 0$, will also be Weyl invariant.

The advantage of the calculation based on the FRGE is that it extends easily to ar-

[11] Note that in this way the couplings will remain constant in spacetime. In this sense our approach differs from those in [60, 40, 42], where the couplings are allowed to become functions on spacetime.

bitrary theories. Let us begin by considering the addition of masses, which break Weyl invariance at the classical level but remains within the scope of renormalizable theories. Applying the Stückelberg trick we can convert all mass terms to interactions with the background dilaton, thus reinstating Weyl invariance at the classical level. For example, in the case of a massive scalar field ϕ the mass term is written as $g\chi^2\phi^2$, for some dimensionless coupling g. This becomes a genuine mass term in the gauge where $\chi = \mu$. Then we can repeat the preceding argument. The only difference is that now the dilaton will be present in the action from the beginning, whereas if one had started from a Weyl–invariant theory, the coupling to the dilaton would only arise in the course of the flow as a consequence of its presence in $\mathcal{O}$.

Finally, we can relax all constraints on the functional form of the action $S(g_{\mu\nu}, \psi_a, g_i)$. Let us suppose that we know the form of the action S at some (constant) cutoff $k = u\mu$. It gives rise, via its flow, to an effective action Γ. Using the Stückelberg trick, we can construct an action $\hat{S}(g_{\mu\nu}, \chi, \psi_a, \hat{g}_i)$ and take it as initial point of the flow at cutoff $k = u\chi$ (which could now be some function of position). Flowing towards the IR from this starting point leads to an effective action Γ^{II} that is still Weyl invariant. When Γ^{II} is evaluated at constant $\chi = \mu$ it agrees with the effective action Γ^{I} evaluated in the Weyl–non–invariant flow. In other words, Γ^{II} could be obtained from Γ^{I} using the Stückelberg trick. We thus see that *quantization commutes with the Stückelberg trick.* [12]

As mentioned earlier, renormalizability is not required for these arguments, because the FRGE is UV finite, and divergences manifest themselves when one tries to solve for the flow towards large k. The question whether this theory has a sensible UV limit can be answered by studying the flow for increasing u, but does not spoil our arguments.

3.3.5 Dynamical gravity (Teaser)

Until now we have considered matter fields coupled to an external gravitational field, which is described either by a metric or by a metric and a dilaton. [13] We would like to extend our results also to the case when gravity is dynamical. This means that we have to be able to "quantize gravity". Contrary to what is often stated, there exists a perfectly well defined and workable framework that allows us to compute quantum gravitational effects: it is the framework of effective field theories. By using the background field method to be described in the next chapter, general relativity reduces to a (perturbatively nonrenormalizable) theory of a spin two field propagating on a curved manifold, not unlike the general interacting matter theories discussed up to now. [14] As discussed for example in [63] and in the next

[12] The relation between Γ^{I} and Γ^{II} will always be as in (3.73), but in the general case Γ^{I}, and consequently also the Wess–Zumino action, will contain infinitely many Weyl–non–invariant terms.

[13] For the sake of coupling to spinor fields one should use a frame field rather than a metric. This complication is not relevant for our purposes and will be ignored. We refer to [61, 62] for a recent discussion.

[14] Ordinary perturbation theory is a special case where the background is flat.

chapter, the background field method actually guarantees that the theory is "background independent" in the sense that no background plays a special role. This scheme has the limitation that in trying to calculate the effective action one encounters infinitely many divergences, each requiring a physical measurement to fix the value of the corresponding counterterm. This means that the theory can be adjusted to fit essentially any experimental result and is not predictive. In practice this is not as bad as it seems. As long as one restricts oneself to experiments at energies below the UV cutoff of the theory, a finite number of loops is sufficient to describe the data with a predefined precision. Thus, only finitely many divergences are encountered and one could test the theory by comparing it to a number of experiments that is greater than the number of divergences. This logic has been quite successful in our understanding of strong interactions at low energy.

If this is still regarded as too unsatisfactory, one can entertain the possibility that the theory is on a renormalizable trajectory and therefore, by the arguments we have seen in the previous chapter, can be continued to indefinitely large energies. The advantage of such a situation is that if the attraction basin of the fixed point is finite dimensional, it places infinitely many constraints on experiments at any energy scale, and is therefore highly predictive. [15] This possibility, however, is not essential for our main result. The main fact is that standard quantum field theoretic methods can be used to describe a quantum field theory of gravity which is at least an effective field theory with a limited energy domain of applicability and in the most optimistic scenario may hold up to indefinitely high energy and have a finite number of free parameters.

However, to be able to extend the methods exposed here, we need to know the details of the quantization method to be used for dynamical gravity. We therefore reserve this computation for the next section, where it will be used as a sample application of the background field method.

[15] This is the logic that led to the standard model of particle physics.

3.4 Discussion and conclusions

In discussions of conformal invariance, misunderstandings frequently arise due to the different physical interpretation of the transformations that are used by different authors. In particle physics language, a theory that contains dimensionful parameters is obviously not conformal. Thus conformal invariance is a property of a very restricted class of theories. In particular, in quantum field theory the definition of the path integral generally requires the use of dimensionful parameters (cutoffs, renormalization points) which break conformality even if it was present in the original classical theory. True conformality is only achieved at a fixed point of the renormalization group. Let us call this the point of view I.

On the other hand in Weyl's geometry and its subsequent ramifications, conformal (Weyl) transformations are usually interpreted as relating different local choices of units. Since the choice of units is arbitrary and cannot affect the physics, it follows that essentially any physical theory can be formulated in a Weyl–invariant way. This point of view is more common among relativists. Let us call it the point of view II.

The way in which a generic theory containing dimensionful parameters can be made Weyl–invariant is by allowing those parameters to become functions on spacetime, *i.e.* to become fields. This is the step that the adherents of the interpretation I are generally unwilling to make, since then one would have to ask whether these fields have a dynamics of their own or not, and, in the quantum case, whether they have to be functionally integrated over or not. It can be unnatural to have fields in the theory that do not obey some specific dynamical equation, and it is clear that in general, if one allows all the dimensionful couplings to become dynamical fields, the theory is physically distinct from the original one.

There is however one way in which Weyl–invariance can be introduced in any theory without altering its physical content, and that is to introduce *a single* scalar field, which we called a dilaton (sometimes also called a "Stückelberg" or "Weyl compensator" or "spurion" field) and to assume that all dimensionful parameters are proportional to it. This field carries a nonlinear realization of the Weyl group, since it is not allowed to become zero anywhere. Even though the new field obeys dynamical equations, it does not modify the physical content of the theory because it is exactly neutralized by the enlarged gauge invariance. In practice, it can be eliminated by choosing the Weyl gauge such that it becomes constant.

All this is well–known in the classical case. It had already been observed both in a perturbative and nonperturbative context that the above considerations can be generalized to the context of quantum field theory by treating the cutoff or the renormalization point in the same way as the mass or dimensionful parameters that are present in the action. In the first part of this chapter we have discussed in particular the formulatiom of the renormalization group using the point of view II. It has proven convenient to adopt a non–perturbative definition of the renormalization group, where one considers the dependence of the effective action on an externally prescribed smooth cutoff k. The advantage of this procedure is that the resulting "beta functional" is both UV and IR finite and one can use it to define a first order differential equation whose solution, for $k \to 0$, is the effective action. It can therefore be viewed as a non–perturbative way of defining (and calculating) the effective action. Using this method we have shown in complete generality that one can define a flow of Weyl–invariant actions whose IR endpoint is a Weyl–invariant effective action. This is our main result.

This provides an answer to the following question. Suppose we start from a theory that contains dimensionless parameters, and recast it in a Weyl–invariant form by introducing a dilaton field. If we quantize this Weyl–invariant theory, is the result equivalent to the one we would have obtained by quantizing the original theory? The answer is affirmative, if we use throughout (*i.e.* for all fields) the Weyl–invariant measure. [16] Thus, there is a quantization procedure that commutes with the Stückelberg trick.

It is important to understand that although Weyl invariance is not anomalous, there is still a trace anomaly, in the sense that the trace of the energy–momentum, which is classically zero, is not zero in the quantum theory. This can be easily understood from the fact that in the Weyl–invariant quantization one obtains an effective action that depends not only on the metric but also on the dilaton. Weyl–invariance of the effective action is compatible with a nonvanishing trace, because the latter cancels out against the variation of the dilaton. The physics of the Weyl–invariant quantization procedure is completely equivalent to the standard one. In particular, all the proposed physical applications of the trace anomaly remain valid [64, 66, 65].

Given that in this formalism all theories are conformally invariant, one can also ask what is special about conformal field theories (in the standard sense of quantum field theory), and in particular about fixed points of the renormalization group. The answer is that for generic theories, conformal invariance is only achieved at the price of having a dilaton in the effective action. True conformal field theories are conformal even without the dilaton, so one must expect that as the RG flow approaches a fixed point, the dilaton must decouple.

Weyl–invariance is the statement of conformal invariance in a general relativistic setting, so we expect the formalism developed here to be especially relevant in the discussion of a possible fixed point for gravity. Perhaps another relevant application might be to cosmology, where it is often useful to change conformal frame. Even at a classical level,

[16]By contrast, suppose that after having quantized the matter fields we also quantize the metric and dilaton, using the standard, Weyl–non–invariant measure I. (One does not need to have a full quantum gravity for this argument, it is enough to think of a one loop calculation in the context of an effective field theory). The integration over metric and dilaton will now proceed with total actions $S_G + \Gamma^{\mathrm{I}}$ and $S_G + \Gamma^{\mathrm{II}}$, depending on whether we used for the matter the measures I or II. Clearly the resulting theories are physically inequivalent: In the first case the action is not Weyl invariant, so the dilaton field is physical, in the second case the action is Weyl invariant and the dilaton can be gauged away. So, *all else being equal*, quantizing matter with measures I and II leads to physically different theories.

there has been some controversy on the issue whether such frames should be interpreted as defining physically equivalent situations. Our point of view agrees with that of [68]. The question is much more delicate in the quantum theory, however. For explicit quantum calculations where different conformal frames can be seen to yield equivalent physics, see [69]. The present work provides a general proof that with a suitable quantization procedure, the equivalence between conformal frames can be maintained also in the quantum theory. One could use this to study the relation between $f(R)$ and scalar–tensor theories at the quantum level.

Chapter 4

The Background Effective Average Action approach to Quantum Gravity

4.1 Outline

A promising approach to quantum gravity is the Asymptotic Safety scenario, first proposed by Weinberg [9], which aims to describe quantum gravity within the framework of quantum field theory. As is well known, the quantum field theory of gravity based on the Einstein–Hilbert action is perturbatively non–renormalizable; the Asymptotic Safety scenario suggests instead that the theory is non–perturbatively renormalizable at a non–Gaussian ultraviolet (UV) fixed–point of the renormalization group (RG) flow. To probe if a theory is renormalizable in a non–perturbative way one needs non-perturbative tools. We will use the exact functional RG equation introduced in the first chapter, which, as we have seen, is capable of resolving nonperturbative effects, which the standard perturbative analysis might hide. To accomplish this, we will need to introduce the (background) effective average action (EAA) for quantum gravity, and its flow equation. So far various applications of the EAA formalism to quantum gravity [70, 71] have found a non–Gaussian fixed–point with a finite dimensional UV–critical surface [72], consistent with the Asymptotic Safety scenario. What we will present here can be seen as another piece of evidence in favor of this scenario: in few words our result is a new way to close the flow, which we consider more consistent than the previous ones adopted, by an independent evaluation of the anomalous dimensions for the fluctuations of gravitons and ghosts.

For better clarity, we will present here an extended outline which covers all the central ideas and methodology employed, and a summary of the result. This is intended for readers already familiar with these techniques, or who want to grasp the main physical ideas without having to go through all the fine details. The interested reader will then find all the computational details in the rest of the chapter.

A consistent closure of the fRG flow for gravity

The application of functional renormalization group techniques to theories characterized by local symmetries requires overcoming the problem of performing the coarse–graining procedure in a covariant way. A solution to this problem comes from the combination of the EAA and the background field formalisms [73]. The preservation of gauge invariance along the flow comes at the cost of enlarging theory space to include invariants constructed with both background and fluctuating fields. The EAA becomes a functional of these two fields invariant under diffeomorphisms. In the first part of this chapter we will describe this construction.

One then defines the single–field functional $\bar{\Gamma}_k[\bar{g}] \equiv \Gamma_k[0;\bar{g}]$ and splits the background EAA as:

$$\Gamma_k[\varphi;\bar{g}] = \bar{\Gamma}_k[\bar{g}+h] + \hat{\Gamma}_k[\varphi;\bar{g}]\,, \tag{4.1}$$

where $\varphi = (h_{\mu\nu}, \bar{C}_\mu, C^\nu)$ is the fluctuating multiplet comprising the fluctuating metric and the ghost fields. The background EAA for gravity satisfies the following exact flow equation [70]:

$$\partial_t \Gamma_k[\varphi;\bar{g}] = \frac{1}{2}\mathrm{Tr}\left(\Gamma_k^{(2;0)}[\varphi;\bar{g}] + R_k[\bar{g}]\right)^{-1} \partial_t R_k[\bar{g}]\,. \tag{4.2}$$

It is important to note that the flow (4.2) is driven by the Hessian of the background EAA taken with respect to the fluctuating multiplet φ and thus the flow equation for $\bar{\Gamma}_k[\bar{g}]$, resulting from setting $\varphi = 0$ in (4.2), is *not closed* since its rhs depends also on $\hat{\Gamma}_k[\varphi;\bar{g}]$. This fact forces us to consider the RG flow of the full $\Gamma_k[\varphi,\bar{g}]$ instead of only the flow of $\bar{\Gamma}_k[\bar{g}]$. It is thus of fundamental importance to develop a systematic way to treat truncations of the full background EAA depending on both background and fluctuating fields in order to consistently close the RG flow for the single–field part of the EAA.

Einstein–Hilbert truncation

In actual applications one typically makes an ansatz for the background EAA $\Gamma_k[\varphi,\bar{g}]$; this means that theory space is truncated to some chosen functional and one hopes that this subspace is complete enough to describe the flow in an approximate, but yet physically significant, way.

Our truncation ansatz for the single–field part of the EAA will be the RG improved version of the Einstein–Hilbert action, where both the cosmological constant and Newton's constant become scale dependent quantities:

$$\bar{\Gamma}_k[g] = \frac{1}{16\pi G_k} \int d^d x \sqrt{g}\,(2\Lambda_k - R)\ . \tag{4.3}$$

Quantum fluctuations are responsible for the anomalous scaling of the fields; this fact is accounted for by introducing scale dependent wave–function renormalization constants for all the fluctuating fields present in the theory; in our case we redefine the fluctuating metric and the ghost fields according to:

$$h_{\mu\nu} \to Z_{h,k}^{1/2} h_{\mu\nu} \quad \bar{C}_\mu \to Z_{C,k}^{1/2} \bar{C}_\mu \quad C^\nu \to Z_{C,k}^{1/2} C^\nu \tag{4.4}$$

and we define the fluctuating metric and ghost anomalous dimensions:

$$\eta_{h,k} = -\partial_t \log Z_{h,k} \qquad \eta_{C,k} = -\partial_t \log Z_{C,k}\,. \tag{4.5}$$

Next, we need to make an ansatz for the remainder functional $\hat{\Gamma}_k[\varphi, \bar{g}]$. We will consider the simplest non–trivial case comprised by the classical background gauge–fixing and ghost actions.

There are different possible cutoff choices which can be thought of as the freedom we have in setting up our coarse–graining procedure. In the nomenclature of [71], we will present the results for the type Ia cutoff in order to compare to previous findings [74]. The type Ia cutoff is characterized by having as cutoff operators the covariant Laplacians in both the gravitational and ghost sectors.

Beta functions

To obtain the beta functions of the physical couplings one computes the Hessian of the background EAA with respect to the fluctuating fields φ, inserts it into the rhs of the flow equation and then sets $\varphi = 0$. The trace on the rhs of the flow equation (4.2) can then be expanded in terms of invariants of the background metric using heat kernel techniques in a standard way (see [71] and the Appendix).

After introducing dimensionless cosmological and Newton's constants, $\tilde{\Lambda}_k = k^{-2}\Lambda_k$ and $\tilde{G}_k = k^{d-2} G_k$, one finds the following general form for the beta functions:

$$\begin{aligned} \partial_t \tilde{\Lambda}_k &= -2\tilde{\Lambda}_k + \Big[A_d(\tilde{\Lambda}_k) + C_d(\tilde{\Lambda}_k)\,\eta_{h,k} + E_d(\tilde{\Lambda}_k)\,\eta_{C,k}\Big]\,\tilde{G}_k \\ \partial_t \tilde{G}_k &= (d-2)\tilde{G}_k + \Big[B_d(\tilde{\Lambda}_k) + D_d(\tilde{\Lambda}_k)\,\eta_{h,k} + F_d(\tilde{\Lambda}_k)\,\eta_{C,k}\Big]\,\tilde{G}_k^2\,, \end{aligned} \tag{4.6}$$

where $A_d, B_d, C_d, D_d, E_d, F_d$ are functions of the dimensionless cosmological constant $\tilde{\Lambda}_k$ and their specific functional form depends on both the cutoff type and cutoff shape function.

The beta functions (4.6) for the physical couplings, here represented by Λ_k and G_k, are not closed. This is because of the presence, on the rhs of (4.6), of the anomalous dimensions $\eta_{h,k}$ and $\eta_{C,k}$ of the fluctuating metric and of the ghost fields, reflecting the fact, noticed previously, that the flow of the single–field part of the EAA is not closed. This forces us to consider the flow of the full background EAA in order to find a consistent closure for the beta functions. We will present this in the next section after reviewing the other approximations proposed in the literature.

Closing the flow equations

The first way in which one can close the beta functions (4.6) for the cosmological and Newton's constants is the trivial one where we sets $\eta_{h,k} = \eta_{C,k} = 0$; this amounts to a one–loop approximation. Within this approximation:

$$\begin{aligned} \partial_t \tilde{\Lambda}_k &= -2\tilde{\Lambda}_k + A_d(\tilde{\Lambda}_k)\tilde{G}_k \\ \partial_t \tilde{G}_k &= (d-2)\tilde{G}_k + B_d(\tilde{\Lambda}_k)\tilde{G}_k^2 \,. \end{aligned} \tag{4.7}$$

The second closure of the beta function system (4.6) that one can consider is the RG improvement adopted in many previous studies [70, 75, 71], which is:

$$\eta_{h,k} = \frac{\partial_t G_k}{G_k} = 2 - d + \frac{\partial_t \tilde{G}_k}{\tilde{G}_k} \qquad\qquad \eta_{C,k} = 0\,. \tag{4.8}$$

The identification in (4.8) implies a non–trivial, but difficult to interpret, RG improvement of the beta functions. We will call this procedure the *standard* RG improvement of the beta functions (4.6). Inserting (4.8) in the beta functions (4.6) and solving for $\partial_t \tilde{G}_k$ gives:

$$\begin{aligned} \partial_t \tilde{\Lambda}_k &= -2\tilde{\Lambda}_k + A_d(\tilde{\Lambda}_k)\tilde{G}_k + \frac{B_d(\tilde{\Lambda}_k)C_d(\tilde{\Lambda}_k)}{1 - D_d(\tilde{\Lambda}_k)\tilde{G}_k}\tilde{G}_k^2 \\ \partial_t \tilde{G}_k &= (d-2)\tilde{G}_k + \frac{B_d(\tilde{\Lambda}_k)\tilde{G}_k^2}{1 - D_d(\tilde{\Lambda}_k)\tilde{G}_k}\,. \end{aligned} \tag{4.9}$$

In $d = 4$, the beta functions (4.9) have a non–Gaussian fixed–point which is UV attractive in both directions. Thus, within this truncation, quantum gravity is Asymptotically Safe. [1]

[1] To guarantee predictivity we still need to show that the UV critical surface is finite dimensional; to do this we need to enlarge our truncation and see if we find operators with repulsive UV directions at the non–Gaussian fixed–point. Evidence for the existence of such operators has been found, within truncations closed using (4.8), in [72].

Anomalous dimensions

The third way to close the beta functions system is to separately calculate the anomalous dimensions of the fluctuating metric and ghost fields that enter (4.6). These can be determined as functions of $\tilde{\Lambda}_k$, $\tilde{G}_k$ that can successively be reinserted back in the beta functions (4.6). In this way the latter are closed (within the truncation considered) in a way that takes into account the flow of the wave–function renormalization constants $Z_{h,k}$ and $Z_{C,k}$. In doing so we make a step further in considering the flow in the enlarged theory space where the background EAA lives.

Our calculations of the anomalous dimensions $\eta_{h,k}$ and $\eta_{C,k}$ have been performed using the diagrammatic techniques presented in [17] and reviewed in the first chapter, where one uses the flow equations for the zero–field proper–vertices $\gamma_k^{(n,m)} \equiv \Gamma_k^{(n,m)}[0;0]$ of the background EAA to extract the running of the couplings.

Both $\eta_{h,k}$ and $\eta_{C,k}$ turn out not to depend on the cutoff operator type (i.e. on the cutoff operator used to separate fast from slow field modes) and have the following general form:

$$\begin{aligned} \eta_{h,k} &= \Big[a_d(\tilde{\Lambda}_k) + c_d(\tilde{\Lambda}_k)\,\eta_{h,k} + e_d(\tilde{\Lambda}_k)\,\eta_{C,k}\Big]\,\tilde{G}_k \\ \eta_{C,k} &= \Big[b_d(\tilde{\Lambda}_k) + d_d(\tilde{\Lambda}_k)\,\eta_{h,k} + f_d(\tilde{\Lambda}_k)\,\eta_{C,k}\Big]\,\tilde{G}_k\,, \end{aligned} \tag{4.10}$$

where $a_d, b_d, c_d, d_d, e_d, f_d$ are functions of the dimensionless cosmological constant. Equation (4.10) constitutes a linear system for $\eta_{h,k}$ and $\eta_{C,k}$ that can be solved to yield the anomalous dimensions as functions solely of the physical couplings $\tilde{\Lambda}_k$ and $\tilde{G}_k$.

The linear system (4.10):

$$\bar{\eta}_k = \left(\bar{V} + \mathbf{M}\,\bar{\eta}_k\right)\tilde{G}_k\,, \tag{4.11}$$

with

$$\bar{\eta}_k = \begin{pmatrix} \eta_{h,k} \\ \eta_{C,k} \end{pmatrix} \qquad \bar{V} = \begin{pmatrix} a_d \\ b_d \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} c_d & e_d \\ d_d & f_d \end{pmatrix}$$

is easily solved to give:

$$\bar{\eta}_k = \tilde{G}_k\left(1 - \tilde{G}_k\,\mathbf{M}\right)^{-1}\bar{V}\,; \tag{4.12}$$

or more explicitly:

$$\begin{aligned} \eta_{h,k}(\tilde{\Lambda}_k,\tilde{G}_k) &= \frac{a_d(\tilde{\Lambda}_k)[1-f_d(\tilde{\Lambda}_k)\tilde{G}_k] + b_d(\tilde{\Lambda}_k)e_d(\tilde{\Lambda}_k)\tilde{G}_k}{[1-c_d(\tilde{\Lambda}_k)\tilde{G}_k][1-f_d(\tilde{\Lambda}_k)\tilde{G}_k] - d_d(\tilde{\Lambda}_k)e_d(\tilde{\Lambda}_k)\tilde{G}_k^2}\tilde{G}_k \\ \eta_{C,k}(\tilde{\Lambda}_k,\tilde{G}_k) &= \frac{b_d(\tilde{\Lambda}_k)[1-c_d(\tilde{\Lambda}_k)\tilde{G}_k] + a_d(\tilde{\Lambda}_k)d_d(\tilde{\Lambda}_k)\tilde{G}_k}{[1-c_d(\tilde{\Lambda}_k)\tilde{G}_k][1-f_d(\tilde{\Lambda}_k)\tilde{G}_k] - d_d(\tilde{\Lambda}_k)e_d(\tilde{\Lambda}_k)\tilde{G}_k^2}\tilde{G}_k\,. \end{aligned} \tag{4.13}$$

Inserting back (4.13) in the beta functions (4.6) gives the *new* RG improved form of $\partial_t \tilde{\Lambda}_k$ and $\partial_t \tilde{G}_k$ that accounts for the non–trivial influence that $Z_{h,k}$ and $Z_{C,k}$ have on their flow.

This is the main logic of this chapter. We will now move to the details of the computations.

Structure

This chapter will be organized as follows. Section 2 covers the basics of the Background Field Method quantization of gravity. As an application, it contains also a one–loop computation in the Weyl–invariant formulation. Section 3 reviews the Asymptotic Safety scenario for Quantum Gravity. The background Field Method is then applied to the EAA in Section 4, arriving at the background EAA (bEAA). In Section 5 the beta functions whithin the truncation we use are calculated, and the different ways to close the flow are exposed. Section 6 presents our computation method to find the anomalous dimensions contributions of the gravitons and ghosts; the new flow portrait we find is discussed in section 7. Section 8 is devoted to the conclusions.

4.2 Quantum theory of gravitation

In this section we try to elucidate the construction of a quantum (field) theory of gravity that will underlie the background Effective Average Action approach to be described in the next section. We will do this by briefly reviewing the use of the background field method and the importance of the Background Effective Action approach.

Our approach will be mainly a conservative one: we will look at what can be said about quantum effects in gravity without introducing new physics. It is sometimes pointed out that, since probably the classical gravitational field is not fundamental, but an effective description of some new physics at low energy scales, it is incorrect to quantize it directly. However, we can see this is not the case. Think for example of the vibrations of a crystal lattice. We know that the vibrational excitations should be described eventually in terms of the modes of the microscopic components of the lattice; however, their mean field *can* be quantized, and the resulting particles, the phonons, describe to a very good approximation the physics in the low part of the spectrum. Our starting position will be conceptually the same: even if we don't know what is the proper UV completion of gravity, by studying a quantum field theory of the gravitational field we will reveal interesting properties of it which should hold also as effective properties. In studying these properties, however, we will see that yet another possibility emerges, namely that quantum field theory is all there is to it, and gravity is self complete in the UV because it reaches a nontrivial conformal phase. In this phase gravity is interacting (not free like perturbative computations assume), but has an *antiscreening* behaviour which heals it from possible divergences. We will make these concepts clear in the following sections.

4.2.1 Classical theory

General Relativity is at present the best and most elegant classical theory of gravity we have. Its field equations can be derived from an action principle, starting from the Einstein-Hilbert action:

$$S_{EH}[g] = \frac{1}{16\pi G} \int d^d x \sqrt{g}\,(2\Lambda - R)\ . \tag{4.14}$$

Here $G = 6.67428 \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$ is Newton's gravitational constant and Λ is the cosmological constant. We are disregarding the issue of boundary terms. In units of an energy scale k the two constants have dimensions $[G] = k^{2-d}$ and $[\Lambda] = k^2$. In natural units we can define the fundamental mass scale of gravitational interactions, the Planck mass, as $M_{Planck} = G^{-1/2}$.

The classical equations of motion for the gravitational field are derived minimizing the Einstein-Hilbert action (4.14) with respect to the metric:

$$\frac{\delta S_{EH}[g]}{\delta g_{\mu\nu}} + \frac{\delta S_m[\psi; g]}{\delta g_{\mu\nu}} = 0\,. \tag{4.15}$$

where $S_m[\psi; g]$ is the matter action, with general matter fields ψ. The variation of the gravitational action (4.14) with respect to the metric reads:

$$\delta S_{EH} = \frac{1}{16\pi G}\int d^dx\sqrt{g}\left[\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\right)h^{\mu\nu} + \nabla^2 h - \nabla^\mu\nabla^\nu h_{\mu\nu}\right], \tag{4.16}$$

where $h_{\mu\nu} = \delta g_{\mu\nu}$. We can drop the last two terms in (4.16) since they are total derivatives and contribute only to boundary terms. Since the classical energy momentum tensor $T_{\mu\nu}$ is defined as

$$\delta S_m = \frac{1}{2}\int d^dx\sqrt{g}T_{\mu\nu}h^{\mu\nu}\,, \tag{4.17}$$

putting all together we find Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -8\pi G\,T^{\mu\nu}\,. \tag{4.18}$$

Note that the gravitational coupling constant G does enter in equation (4.18) only if matter is present.

Of course, from the point of view of the action principle, there is nothing wrong in adding to the Einstein–Hilbert action further terms of higher order in the curvature. We will then obtain field equations of higher than second order in the derivatives of the metric. For instance, at second order in the curvatures, other invariants are:

$$\int d^dx\sqrt{g}R^2 \qquad \int d^dx\sqrt{g}R_{\mu\nu}R^{\mu\nu} \qquad \int d^dx\sqrt{g}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}\,.$$

We will see that at low energies these terms can be dropped (see also [76]). However at high energies they will be present, and in particular they will be generated by quantum loop corrections.

4.2.2 Quantum Theory

Background Field Method

As the first chapter should have made clear, the most convenient way to quantize a system from our perspective is functional integral quantization, since it is the most natural way to implement functional RG techniques. However, when a theory possesses local symmetries, as in the case of diffeomorphism invariance for gravity, we immediately run into a problem. Since in field space there exist directions along which the field does not change (the "gauge orbits"), when we integrate along those directions we end up with a divergent integral. Roughly speaking, since the subspace of field space within which the field is constant is given by the action of the invariance group on it, the integral will be proportional to the volume of the invariance group.

The cure to this issue was pointed out by Faddeev and Popov [77] (and DeWitt [78]), namely, that we are parametrizing our theory with more degrees of freedom than we actually need. But we can just “fix the gauge”, by choosing one representative point in field space for every gauge orbit (that is, by taking the quotient space), and integrating only on this subset. This then is like factoring out the volume of the invariance group, and cancels the previous divergence. It can then be shown that the result is pretty much independent on the way in which this gauge fixing is implemented.

However, we also would like not to spoil the symmetry of the theory, so that eventually we will be able to perform the coarse–graining covariantly. Otherwise the RG would switch on all kinds of terms along the flow not invariant under the symmetries of the theory in an uncontrolled way.

The key to obtain both things is the background field method, which we now explain for gravity.

One starts by decomposing the integration variable, the metric, into a background $\bar{g}_{\mu\nu}$ and a fluctuation $h_{\mu\nu}$ part[2]:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

and performs the functional integration only over $h_{\mu\nu}$. A diffeomorphism invariant theory is a theory invariant under

$$\delta g_{\mu\nu} = \mathcal{L}_\varepsilon g_{\mu\nu} = \nabla_\mu \varepsilon_\nu + \nabla_\nu \varepsilon_\mu$$

where ε is any infinitesimal vector field, and $\mathcal{L}_\varepsilon$ is the Lie derivative along this vector field. There are two ways now to realize this transformation. Either as a “quantum” transformation

$$\begin{aligned} \delta_Q h_{\mu\nu} &= \mathcal{L}_\varepsilon \left(\bar{g}_{\mu\nu} + h_{\mu\nu}\right) \\ \delta_Q \bar{g}_{\mu\nu} &= 0 \end{aligned}$$

or a “background” transformation

$$\begin{aligned} \delta_B h_{\mu\nu} &= \mathcal{L}_\varepsilon h_{\mu\nu} \\ \delta_B \bar{g}_{\mu\nu} &= \mathcal{L}_\varepsilon \bar{g}_{\mu\nu} \,. \end{aligned}$$

In the background field method one breaks invariance under the quantum transformation, keeping invariance under the background transformation.

Notice that we can further split the background transformation as one acting on the

[2]The fluctuation is usually multiplied by the gravitational coupling constant but in this section for notational simplicity we will forget this. We are always free to redefine $h_{\mu\nu} = \kappa \tilde{h}_{\mu\nu}$.

fluctuation

$$\begin{aligned}\delta h_{\mu\nu} &= \mathcal{L}_{\varepsilon} h_{\mu\nu} \\ \delta \bar{g}_{\mu\nu} &= 0\end{aligned}$$

plus one acting on the background

$$\begin{aligned}\bar{\delta} h_{\mu\nu} &= 0 \\ \bar{\delta} \bar{g}_{\mu\nu} &= \bar{\mathcal{L}}_{\varepsilon} \bar{g}_{\mu\nu} = \bar{\nabla}_{\mu}\varepsilon_{\nu} + \bar{\nabla}_{\nu}\varepsilon_{\mu} \,.\end{aligned}$$

In this way a background transformation reads $\delta_B = \left(\delta + \bar{\delta}\right)$.

To factor out the volume of the diffeomorphisms (Diff) group from the functional integral, we follow the Faddeev-Popov procedure and insert the identity in the form

$$1 = \int \mathcal{D}f \delta\left[f\right] = \int \mathcal{D}\epsilon \delta\left[f^{\epsilon}\right] \det \left| \frac{\delta f^{\epsilon}}{\delta \epsilon} \right|$$

where $f_{\mu}\left[h, \bar{g}\right]$ is the background gauge–fixing condition (to be specified later), and we normalize the measure dividing by the volume of the Diff group $\mathcal{V}_{diff}$. Then, with obvious notation

$$\begin{aligned}Z &= \int \frac{\mathcal{D}h_{\mu\nu}}{\mathcal{V}_{diff}} \mathcal{D}\epsilon \delta\left[f^{\epsilon}\right] \det \left| \frac{\delta f^{\epsilon}}{\delta \epsilon} \right| e^{-S[\bar{g}+h]} \\ &= \frac{1}{\mathcal{V}_{diff}} \int \mathcal{D}\epsilon \mathcal{D}h^{\epsilon}_{\mu\nu} \delta\left[f^{\epsilon}\right] \det \left| \frac{\delta f^{\epsilon}}{\delta \epsilon} \right| e^{-S[\bar{g}+h^{\epsilon}]} \\ &= \frac{1}{\mathcal{V}_{diff}} \int \mathcal{D}\epsilon \int \mathcal{D}h_{\mu\nu} \delta\left[f\right] \det \mathcal{M} e^{-S[\bar{g}+h]} \\ &= \int \mathcal{D}h_{\mu\nu} \delta\left[f\right] \det \mathcal{M} e^{-S[\bar{g}+h]}\end{aligned}$$

where we used the invariance of the action and measure, we shifted the integration variable and used the fact that $\int \mathcal{D}\epsilon = \mathcal{V}_{diff}$.

We can now use the fact, which we don't prove here [11], that this procedure is independent of the precise form of the gauge fixing functional, to switch to a general functional $B[f]$:

$$Z = \int \mathcal{D}h_{\mu\nu} B\left[f\right] \det \mathcal{M} e^{-S[\bar{g}+h]} \,. \tag{4.19}$$

The gauge fixing that is usually considered has the following form

$$f_{\mu}\left[h, \bar{g}\right] = \left(\delta^{\rho}_{\mu} \bar{\nabla}^{\sigma} - \frac{\beta}{2} \bar{g}^{\rho\sigma} \bar{\nabla}_{\mu}\right) h_{\rho\sigma}$$

and the gauge fixing functional is usually chosen to be of the gaussian type, which means we are adding the gauge fixing action

$$S_{gf}\left[h,\bar{g}\right]=\frac{1}{2\alpha}\int d^dx\sqrt{\bar{g}}\bar{g}^{\mu\nu}f_\mu\left[h,\bar{g}\right]f_\nu\left[h,\bar{g}\right]\,.$$

The matrix $\mathcal{M}$ is defined as

$$\mathcal{M}\left[h,\bar{g}\right]_{\mu\nu}=\left.\frac{\delta f_\mu\left[h^\epsilon,\bar{g}\right]}{\delta\epsilon^\nu}\right|_{\epsilon\to0}$$

Its determinant can be exponentiated using two anticommuting (ghost) fields

$$\det\mathcal{M}=\int\mathcal{D}\bar{C}\mathcal{D}C\exp\left(\int d^dx\sqrt{\bar{g}}\bar{C}_\mu\mathcal{M}^\mu{}_\nu C^\nu\right)\,.$$

The explicit form of the ghost action is found from the gauge fixing action, using

$$\begin{aligned}\frac{\delta f_\mu\left[h^\epsilon,\bar{g}\right]}{\delta\epsilon^\nu}&=\frac{\delta f_\mu\left[h^\epsilon,\bar{g}\right]}{\delta h^\epsilon_{\alpha\beta}}\frac{\delta h^\epsilon_{\alpha\beta}}{\delta\epsilon^\nu}\\&=\left(\delta^\rho_\mu\bar{\nabla}^\sigma-\frac{\beta}{2}\bar{g}^{\rho\sigma}\bar{\nabla}_\mu\right)\left(g_{\nu\rho}\nabla_\sigma+g_{\nu\sigma}\nabla_\rho\right)\end{aligned}$$

and turns out to be

$$S_{gh}\left[h,\bar{C},C,\bar{g}\right]=-\int d^dx\sqrt{\bar{g}}\bar{C}^\mu\left[\bar{\nabla}^\sigma g_{\nu\sigma}\nabla_\mu+\bar{\nabla}^\sigma g_{\mu\nu}\nabla_\sigma-\beta\bar{\nabla}_\mu g_{\nu\sigma}\nabla^\sigma\right]C^\nu\,.$$

Thus eventually our path integral is

$$Z=\int\mathcal{D}h_{\mu\nu}\mathcal{D}\bar{C}\mathcal{D}Ce^{-S_{tot}\left[h,\bar{C},C,\bar{g}\right]}$$

with

$$S_{tot}\left[h,\bar{C},C,\bar{g}\right]=S\left[\bar{g}+h\right]+S_{gf}\left[h,\bar{g}\right]+S_{gh}\left[h,\bar{C},C,\bar{g}\right]\,.$$

As we said the gauge fixing is chosen to respect background gauge invariance

$$\left(\delta+\bar{\delta}\right)S_{gf}=\left(\delta+\bar{\delta}\right)S_{gh}=0 \tag{4.20}$$

so the full action is invariant, $\left(\delta+\bar{\delta}\right)S_{tot}=0$. The background Effective Action (bEA) is then introduced just like the standard EA, and satisfies a similar integro–differential equation:

$$e^{-\Gamma[\varphi;\bar{g}]}=\int\mathcal{D}\chi\exp\left(-S\left[\varphi+\chi;\bar{g}\right]+\int d^dx\sqrt{\bar{g}}\chi\Gamma^{(1;0)}\left[\varphi;\bar{g}\right]\right)$$

where $\varphi=\left\{h_{\mu\nu},\bar{C}_\mu,C^\nu\right\}$. It is also background invariant: $\left(\delta+\bar{\delta}\right)\Gamma=0$. We can also define the single–field EA (gEA) as follows

$$\bar{\Gamma}\left[\bar{g}\right] = \Gamma\left[0; \bar{g}\right] \, .$$

It satisfies $\bar{\delta}\bar{\Gamma}\left[\bar{g}\right] = 0$ and is thus a gauge–invariant functional of the field $\bar{g}$ alone.

Ultraviolet divergences

Before moving on to Asymptotic Safety, it is maybe better to review why quantum effects in gravity are problematic in the first place. It should be stressed that are the perturbative quantum field theoretical effects which are found to be problematic. This remark will be important to understand the next section.

We know from power counting arguments that, given a coupling constant with mass dimension d, a Feynman diagram of order N will contain an integral that goes at large momenta like $\int p^{A-Nd} dp$, where A depends on the process in question but not on N. Interactions with $d < 0$ will have divergent integrals at sufficiently high order: these are the nonrenormalizable interactions. Newton's constant (which can be seen as the coupling constant for gravitons, at least at weak coupling) has dimension $d = -2$, and so even before calculating any loop correction we expect gravity to be nonrenormalizable.

This suspicion is confirmed by actual computations. By adopting the Background Field Method with the Einstein–Hilbert action $\int d^d x \sqrt{g} R$, 't Hooft and Veltman [79] found a divergent one-loop contribution to the EA of the general form

$$\Gamma^{(1)}_{\infty} = \int d^d x \sqrt{g} \left(a_1 R^2 + a_2 R^{\mu\nu} R_{\mu\nu} + a_3 R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \right)$$

with a_i some divergent terms (in dimensional regularization they are proportional to ϵ^{-1}) to be renormalized. The problem is that there are no counterterms in the starting action to absorb these infinities: the theory is nonrenormalizable. It can be shown that on shell the one–loop contribution for pure gravity vanishes (it reduces to an R^2 term, and $R = 0$). If there is matter, the one–loop contribution does not vanish, since for a nonzero stress–energy tensor $R \neq 0$. But one may be tempted to hope that pure gravity instead is renormalizable after all. However, as Goroff and Sagnotti [80] have shown a little later, at two loops there is a term that survives on shell, which is $\Gamma^{(2)}_{GS,\infty} \propto \int d^d x \sqrt{g} Riem^3$. This term does not vanish even if there is no matter. Thus, pure gravity is truly nonrenormalizable, at two loops.

A one–loop computation in the Weyl–invariant formalism

In the previous chapter we anticipated that the Weyl–invariant formulation could be easily adapted to the case of dynamical gravity. To see how the BFM works, let us look at a simple one-loop computation of the beta functions of gravity with matter in such a framework.

That is, we now consider a generic theory of gravity based on an action S which is a diffeomorphism- and Weyl–invariant functional of a metric $g_{\mu\nu}$ and a dilaton χ. By the discussion in the previous chapter, there is a one-to-one correspondence between such functionals and diffeomorphism-invariant functionals of a metric alone. If there were some matter fields that have already been integrated out, the corresponding effective action need not be considered separately and is included here in the gravitational action.

The metric and dilaton now have to be expanded as the sum of a background and a quantum part. Since in the following we will have to refer to the backgrounds much more often than to the full background plus quantum fields, for typographical simplicity we choose to call $\bar{g}$ and $\bar{\chi}$ the full quantum fields, g and χ the background fields, h and η the quantum fields. Thus

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \ ; \qquad \bar{\chi} = \chi + \eta \ . \tag{4.21}$$

The infinitesimal form of diffeomorphism and Weyl transformations is

$$\delta_\xi \bar{g}_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu} \ ; \qquad \delta_\xi \bar{\chi} = \mathcal{L}_\xi \bar{\chi} \ , \tag{4.22}$$

$$\delta_\omega \bar{g}_{\mu\nu} = 2\omega \bar{g}_{\mu\nu} \ ; \qquad \delta_\omega \bar{\chi} = -\omega \bar{\chi} \ , \tag{4.23}$$

where ξ and ω are infinitesimal transformation parameters and $\mathcal{L}_\xi$ is the Lie derivative along ξ. In this case, the "quantum gauge transformation" is

$$\delta_\xi g_{\mu\nu} = 0 \ ; \qquad \delta_\xi \chi = 0 \ ; \tag{4.24}$$

$$\delta_\xi h_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu} \ ; \quad \delta_\xi \eta = \mathcal{L}_\xi \bar{\chi} \ , \tag{4.25}$$

$$\delta_\omega g_{\mu\nu} = 0 \ ; \qquad \delta_\omega \chi = 0 \ , \tag{4.26}$$

$$\delta_\omega h_{\mu\nu} = 2\omega \bar{g}_{\mu\nu} \ ; \quad \delta_\omega \chi = -\omega \bar{\chi} \ , \tag{4.27}$$

while the "background gauge transformation" is

$$\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} \ ; \qquad \delta_\xi \chi = \mathcal{L}_\xi \chi \ ; \tag{4.28}$$

$$\delta_\xi h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu} \ ; \qquad \delta_\xi \eta = \mathcal{L}_\xi \eta \ , \tag{4.29}$$

$$\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu} \ ; \qquad \delta_\omega \chi = -\omega \chi \ , \tag{4.30}$$

$$\delta_\omega h_{\mu\nu} = 2\omega h_{\mu\nu} \ ; \qquad \delta_\omega \eta = -\omega \eta \ , \tag{4.31}$$

We choose as we said background gauge conditions that break the quantum transformations, as required to make the Hessian invertible, but preserve invariance under the background transformations. For diffeomorphisms we choose the gauge fixing action

$$S_{GF} = \frac{1}{2\alpha} \int d^4x \sqrt{g}\, \frac{1}{2} Z \chi^2 F_\mu \bar{g}^{\mu\nu} F_\nu \ , \tag{4.32}$$

where

$$F_\nu = D_\mu h^\mu{}_\nu - \frac{\beta+1}{4} D_\nu h \ , \tag{4.33}$$

and D_μ was defined in the previous chapter. Here α and β are gauge parameters and Z is a wave function renormalization constant to be specified later. The ghost action corresponding to the gauge (4.33) is given by

$$S_{gh} = \int d^4x \sqrt{g}\, \chi^2 \bar{C}_\mu g^{\mu\nu} (\mathcal{O}_{gh})_\nu^\rho C_\rho = \ \mathcal{G}_{gh}\left(\bar{C}, \mathcal{O}_{gh} C\right) \ , \tag{4.34}$$

where $\bar{C}$ and C are dimensionless anticommuting vector fields, $\mathcal{G}_{gh}$ is the Weyl invariant inner product on vector fields defined in (3.58), and

$$(\mathcal{O}_{gh})_\mu^\nu = -\frac{1}{\chi^2}\left(\delta_\mu^\nu D^2 + \frac{1-\beta}{2} D_\mu D^\nu + \mathcal{R}_\mu{}^\nu\right) \tag{4.35}$$

is the Weyl-covariant operator acting on ghosts. To gauge–fix Weyl invariance we impose that $\eta = 0$, a condition that does not lead to ghosts. With this condition we can simply delete from the Hessian the rows and columns that involve the η field and we remain with a Hessian that is a quadratic form in the space of the covariant symmetric tensors $h_{\mu\nu}$.

In this space we choose the Weyl–invariant functional metric

$$\mathcal{G}_G(h, h') = \int d^4x \sqrt{g} \chi^4 h_{\mu\nu} g^{\mu\rho} g^{\nu\sigma} h'_{\rho\sigma} \,, \tag{4.36}$$

which can be used to turn the Hessian into a linear operator $\mathcal{O}_G$ acting on the space of covariant symmetric tensors. Because the original action was Weyl–invariant, this operator is Weyl-covariant, in the sense that

$$(\mathcal{O}_{G(\Omega^2 g_{\mu\nu}, \Omega^{-1}\chi)})_{\mu\nu}{}^{\rho\sigma} (\Omega^2 h_{\rho\sigma}) = \Omega^2 (\mathcal{O}_{G(g_{\mu\nu},\chi)})_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} \ . \tag{4.37}$$

Likewise the operator (4.35) is Weyl-covariant.

Even though we will introduce the background EAA later on, from here it is already clear that the FRGE, which is a sum of traces of functions of these operators, is Weyl–invariant. This shows that there exists a quantization scheme which preserves Weyl–invariance along the flow, so if one starts from a “bare” action which is Weyl invariant, the effective action will also be Weyl–invariant.

Note that if we choose the background gauge such that $\chi = \mu$ (the background gauge being completely independent from the gauge fixing in the functional integral) one obtains an effective action which is a functional of the background metric only. This is exactly the same functional that one would have obtained by integrating with the Weyl-non-invariant measure where χ is replaced by μ and with the action written in the same gauge. Thus, also in the case of dynamical gravity, the choice of the gauge $\chi = \mu$ commutes with quantization.

By a similar argument one sees that the Stückelberg procedure of Weyl-covariantizing an action also commutes with quantization.

The above conclusion is completely general, but in order to illustrate it with a concrete calculation we consider the simple case of gravity with the Einstein-Hilbert action:

$$S(g) = \frac{1}{16\pi G}\int d^4x\sqrt{g}(2\Lambda - R)\,. \tag{4.38}$$

Applying the Weyl covariantization (Stückelberg) procedure described in the previous chapter, this can be rewritten as

$$S(g,\chi) = \int d^4x\sqrt{g}\left[\lambda Z^2\chi^4 - \frac{1}{12}Z\chi^2\mathcal{R}\right] \,, \tag{4.39}$$

where

$$\mathcal{R} = R - 6\chi^{-1}\Box\chi\ ; \qquad Z\chi^2 = \frac{12}{16\pi G}\ ; \qquad \lambda = \frac{2\pi}{9}G\Lambda\ . \tag{4.40}$$

Neglecting interactions, the gravitons contribute to the one loop effective action the terms

$$S(g,\chi) + \frac{1}{2}\mathrm{Tr}\log\mathcal{O}_G - \mathrm{Tr}\log\mathcal{O}_{gh}\ , \tag{4.41}$$

which has to be added to the matter effective action. For the explicit form of the operators we refer to [81]. Using the proper time representation (3.42), and the heat kernel expansion (3.44), the effective action has quartic and quadratic divergences which can be absorbed by redefining the bare couplings Z_B and λ_B that are present in the bare action S. In the gauge $\beta = \alpha = 1$, the renormalization conditions can be written in the form

$$\frac{1}{12}Z\chi^2 = \frac{1}{12}Z_B\chi^2 - \frac{1}{6}\frac{1}{(4\pi)^2}\left(23 + 2n_M - n_D\right)\left(\Lambda_{UV}^2 - k^2\right)\ , \tag{4.42}$$

$$\lambda Z^2\chi^4 = \lambda_B Z_B^2\chi^4 - \frac{1}{8}\frac{1}{(4\pi)^2}\left(2 + n_S + 2n_M - 4n_D\right)\left(\Lambda_{UV}^4 - k^4\right)\ , \tag{4.43}$$

where k can be viewed here as a renormalization scale. Observe that with these renormalization conditions the effective action is the same as one would have obtained by cutting off the t–integration in (3.42) at $t = 1/k^2$. In other words, k behaves exactly as an infrared cutoff. These renormalization conditions may look a bit strange because of the appearance of the field χ. However, we assume here that both Λ_{UV} and k are constant multiples of the dilaton. Defining $\hat{\Lambda}_{UV} = \Lambda_{UV}/\chi$, $u = k/\chi$ the renormalization conditions become

$$\frac{1}{12}Z(u) = \frac{1}{12}Z_B(\hat{\Lambda}_{UV}) - \frac{1}{6}\frac{1}{(4\pi)^2}\left(23 + 2n_M - n_D\right)\left(\hat{\Lambda}_{UV}^2 - u^2\right)\ , \tag{4.44}$$

$$\lambda(u)Z^2(u) = \lambda_B(\hat{\Lambda}_{UV})Z_B^2(\hat{\Lambda}_{UV}) - \frac{1}{8}\frac{1}{(4\pi)^2}\left(2 + n_S + 2n_M - 4n_D\right)\left(\hat{\Lambda}_{UV}^4 - u^4\right)\ . \tag{4.45}$$

Taking a u derivative and adding the matter contributions, the full beta functions are

$$u\frac{dZ}{du} = \frac{1}{4\pi^2}\left(23+2n_M-n_D\right)u^2 \tag{4.46}$$

$$u\frac{d\lambda}{du} = \frac{2+n_S+2n_M-4n_D}{32\pi^2 Z^2}u^2\left[u^2-16\lambda Z\frac{23+2n_M-n_D}{2+n_S+2n_M-4n_D}\right] \tag{4.47}$$

These beta functions agree (for the gravitational part) with those computed in [81] using the FRGE. They depend explicitly on the independent variable u. This is due to the fact that we are measuring all dimensionful quantities in units of the dilaton. If we measured dimensionful couplings in units of k, the beta functions would not contain k explicitly. One can check that rewriting these equations for the variables $\tilde{\Lambda} = \Lambda/k^2 = 6\lambda Z/u^2$ and $\tilde{G} = Gk^2 = 3u^2/4\pi Z$ one recovers the familiar beta functions of the Einstein–Hilbert truncation [46, 82, 83] and in the presence of matter [75, 84, 85].

The equations (4.46), together with the IR boundary conditions $Z(0) = Z_0, \lambda(0) = \lambda_0$, admit the general solution

$$Z(u) = Z_0 + \frac{23+2n_M-n_D}{8\pi^2}u^2 \ , \tag{4.48}$$

$$\lambda(u) = \frac{\pi^2((2+n_S+2n_M-4n_D)u^4+128\pi^2 Z_0^2\lambda_0)}{2(8\pi^2 Z_0+(23+2n_M-n_D)u^2)^2} \ . \tag{4.49}$$

By looking at the behavior for large u, a fixed point for the gravitational couplings in the Einstein–Hilbert truncation is found, with $\lambda(u) \to \lambda_\star = \frac{\pi^2(2+n_S+2n_M-4n_D)}{2(23+2n_M-n_D)^2}$ and[3] $Z(u) \to Z_* u^2$ with $Z_* = \frac{23+2n_M-n_D}{8\pi^2}$. This is an example of Asymptotically Safe behavior in gravity, to which the next section is devoted.

[3]This is consistent with the notion of a fixed point because the wave function renormalization Z is a redundant coupling [86].

4.3 Asymptotically Safe Gravity

4.3.1 The physical idea and its fRG realization

We already introduced the concept of asymptotic safety in the first chapter, but let us now elaborate a bit more on that, and on its relevance for gravitation.

We know from our experience with quantum field theories, and from the general arguments of chapter 1, that upon quantizing a system, all possible terms consistent with the symmetries of our theory will be generated. We also know that at sufficiently low energies, only a finite number of interactions will be needed to explain observed phenomena with some prescribed finite degree of accuracy. This is the reason why we believe that renormalizable interactions like the Standard Model of particle physics are so successful in describing experiments, and is also the logic that lies at the base of Effective Field Theories.

For this reason, in theories like gravity where there is in principle an infinite number of terms consistent with symmetry, we expect an infinite theory space comprising the infinite number of interactions. An EFT can be predictive in this context only once we give a bound on the accuracy we want: then there will be only a finite number N of operators that can give measurable effects with that accuracy, we can fix these operators with N ideal experiments, and all other possible experiments can be predicted with our theory. If the predictions turn out to be wrong, the theory is wrong. But if the predictions turn out to be right, one can still think that after increasing the accuracy, and repeating the same logic, the theory might fail. There is in fact no end to this process, and one will never know what the fundamental theory is in this framework.

For a theory to be fundamental, we woud like it to be predictive also from a theoretical point of view. That is, we would like it to be characterized by a finite number of parameters. Since we already know that, starting from a general surface in theory space, we flow to a finite dimensional surface, we see the issue is related to the initial condition of the flow. If in some way we are able to constrain the theory to lie on some finite dimensional subspace as the initial condition of the flow, then we have gained predictivity at all scales. This is precisely what happens for Asymptotic Safety.

So, let us give a precise definition. A theory[4] will be called Asymptotically Safe, if the following conditions are met:

- It has a fixed point in its RG flow
- The fixed point has a finite number of attractive directions

The second condition tells us that the critical surface of this fixed point is finite dimensional, so the theory is predictive, and the first condition should imply, by the arguments

[4]Note that in our framework, it would be conceptually more correct to identify a theory with a whole RG trajectory. We will not discuss this point here.

of chapter one, that the theory is well behaved at high energy.

If gravity is Asymptotically Safe, its physical properties will change at the UV fixed point, with respect to its IR properties, in a way similar to what happens in nonabelian gauge theories, or asymptotically free theories. In QED, the vacuum becomes a dielectric medium due to virtual electron-positron pairs, which screens the electric charge and causes the effective electric charge to decrease at large distances. This phenomenon is called, for this reason, "screening". If one looks at the nonabelian beta function, one can see that fermions still screen the charge, but now gauge bosons have an opposite effect. This can be understood if one analyzes Gauss's Law in the Coulomb gauge: this has now further terms coming from nonzero structure constants of the gauge algebra. The electric field has a leading $1/r^2$ term, and a correction term, and this correction terms creates an effective dipole pointing towards the original charge, and enhancing its field at larger distances. This phenomenon is called antiscreening, and can be seen as a physical explanation of asymptotic freedom. A similar phenomenon happens in asymptotic safety.

From the fRG point of view, the condition of Asymptotic Safety is in practice the only way in which we can build a continuum limit for gravity. The theory is then quantized by starting from the fixed point, and flowing along a renormalized trajectory. We will see in the next section how we recover standard gravity at ordinary energies in this picture.

As of today, there are different pieces of evidence for the Asymptotic Safety scenario. We can summarize them as follows:

- $2+\epsilon$ expansion [87]

In this case one considers gravity in $d = 2+\epsilon$ dimensions. Since gravity is asymptotically free in two dimensions, one can now trust a one–loop computation, which is found to give a flow for (dimensionless) Newton's constant $\tilde{G}_N$ of the form

$$\mu\frac{d}{d\mu}\tilde{G}_N = \epsilon\tilde{G}_N - \gamma\tilde{G}_N^2 \tag{4.50}$$

with $\gamma > 0$. This has a nontrivial fixed point at $\tilde{G}_N^* = \epsilon/\gamma$. Of course one shouldn't trust a direct continuation to $\epsilon = 2$, because nothing tells us that we are avoiding further poles coming from dimensional continuation in this interval. However, the qualitative flow that is found in four dimensions turns out not to be very different from this one.

- Large N expansion [88, 89]

Here one supplements gravity with a matter action containing a large number $\mathcal{O}(N)$ of matter fields. Then one keeps the product NG_N fixed, and expands in $1/N$. A nontrivial fixed point with a three dimensional critical surface is found.

- Symmetry reduction [90]

In this case, instead of the familiar $3+1$ foliation of geometries one considers a foliation in terms of two–dimensional hypersurfaces Σ and performs the functional integral only over configurations that are constant as one moves from one surface to another. This constancy is formulated in terms of two Killing vector fields, and for this reason this is also called a 2–Killing (or $2+2$) reduction. There are reasons for this choice that we won't discuss here; the main message is that again a nongaussian fixed point is found (with a propagator free of unphysical poles).

- Truncated flows of the EAA

At last we have the general type of computations we are discussing in this thesis. A nontrivial fixed point with a three dimensional critical surface has been found using the EAA under different truncations, for example including all terms up to second order in the curvature, or in a polynomial expansion of $f(R)$ truncations, up to order R^{35} [91]. People is also starting to study general (nonpolynomial) $f(R)$ truncations, with encouraging results [92, 93] (see however also [94, 95]).

The solutions of the truncated flows studied up to now depended on the way the flow was closed. Our new way to close the flow can be seen as a further piece of evidence whithin this class.

Since the fixed point initial condition is the only known condition to take a well defined continuum limit of a quantum field theory, there can be only two situations in which we expect this picture to fail. Either new physics emerges at very high energies, in the form of a theory which is not a QFT (like a String Theory for example), or a continuum limit cannot be taken, either due to a fundamental discreteness of spacetime, or to more exotic structures. At present, we don't think it will be considered too extreme to investigate the possibilities still offered by experimentally tested theories, and see what we can learn.

4.3.2 Gravity at low energy

Consider asymptotically safe pure gravity. It always has a gaussian fixed point. Since every term in the gravitational lagrangian is nonrenormalizable (assuming a zero cosmological constant), every eigenvalue of the stability matrix $\partial\beta_i/\partial g_j$ at $g_i = 0$ is positive. This means that the fixed point at the origin is entirely UV–repulsive, and so it is entirely IR–attractive. Thus, physics at long wavelengths is controlled by this fixed point: whatever direction we move from the origin we always remain in this IR critical surface. This means that for at least a finite region around the origin, every trajectory will flow to this FP. If the UV FP lies within this region, gravity is described by a renormalized trajectory connecting the NGFP in the UV to the GFP in the IR.

Since for $g_i \simeq 0$ the loop contributions are negligible, the beta function is just $\beta_i =$

$-d_i g_i$, which means that for $k \ll M_i$, with M_i some integration constants,

$$g_i(k) \approx \left(\frac{k}{M_i}\right)^{-d_i} .$$

In the natural case where none of the parameters defining the critical surface takes very large or small values, all the M_i will be related to some characteristic energy scale M, and the dimensionful couplings are thus

$$\mathcal{G}_i = k^{d_i} g_i(k) \approx M^{d_i} \tag{4.51}$$

so for Newton's constant M is the Planck mass $M = G^{-1/2}$. Now consider a connected Green's function for a set of gravitational fields at points with typical spacetime separations r. We define running couplings at renormalization points with momenta of order r^{-1}. Eq. (4.51) shows that a graph with $\mathcal{N}_i$ vertices of type i yields a factor proportional to N powers of $G^{1/2}$, where $N = -\sum_i \mathcal{N}_i d_i$. From dimensional analysis this means that the graph will be suppressed by a factor $\left(\sqrt{G}/r\right)^N = (rM)^{-N}$. The leading graphs for r much larger than the Planck length M^{-1} will be those with the smallest value of N. If for the type i interaction we have p_i derivatives and h_i graviton fields, then $d_i = 4 - p_i - g_i$. It is then easy to show using the standard topological relations of Feynman graphs that, for L loops and E external lines,

$$N = \sum_i \mathcal{N}_i(p_i - 2) + 2L + E - 2 \,. \tag{4.52}$$

Thus for a given E the leading graphs in the long wavelength regime $r \gg M^{-1}$ will be the tree graphs ($L = 0$) constructed solely with the Einstein–Hilbert lagrangian ($p_i = 2$). Summing these tree graphs is equivalent to solving the classical field equations: the one–graviton Green's function in the presence of a classical background distribution of energy and momentum satisfies the classical Einstein equations for the gravitational field produced by this energy momentum tensor. In this way GR is recovered.

From eq. (4.52) we also notice that there is a further suppression also for tree graphs of a factor $G^{1/2}$ for each external graviton line. Also, if we compute the metric produced by a mass m, since the coupling $\sqrt{G}T^{\mu\nu}$ in the static case is essentially $\sqrt{G}T^{00} \sim \sqrt{G}m$, the total suppression term for external lines is

$$\left(\frac{Gm}{r}\right)^E .$$

The reason why tree graphs have a detectable effect on planetary motion is that a typical mass like the solar mass $m_\odot$ is so large that the quantity $Gm_\odot/r$ is not a negligible quantity.

The first corrections to GR will arise both from the one loop graphs in pure general relativity ($L = 1$, $p_i = 2$), and from the tree graphs containing one vertex from curvature squared interactions ($L = 0$, $\mathcal{N}_i = 1$, $p_i = 4$). These corrections are suppressed by a factor $(rM)^{-2}$, and thus extremely tiny.

4.4 Background EAA in Quantum Gravity

In this section we present the construction of the background effective average action (bEAA) in quantum gravity.

4.4.1 Cutoff and diffeomorphism invariance

The Background Field Method offers us a path–integral quantization that preserves the local symmetries of a theory, such as, in the case of gravity, diffeomorphism invariance, or invariance under general coordinate transformations. In this way, as we have seen in the first chapter, the coarse–graining can be performed covariantly by choosing a suitable differential operator to be used to separate slow field modes from fast field modes.

Even without the background field method, one may consider small fluctuations around a fixed background, and build the differential operator that enters in the cutoff action by using the fixed background metric. Our theory then would be one of small fluctuations propagating on such a background. However, as soon as we enter in a regime where the fluctuations become strong, this simple approximation breaks down, and in fact the backreaction of the fluctuations on the background can even destroy the very background with which we started.

The background field method resolves this issue by introducing an auxiliary arbitrary background metric (which needs not be on–shell), realizing the concept of "background independence", or the idea that any theory of quantum gravity needs a formulation where no privileged metric is employed.

In the background field method the metric $g_{\mu\nu}$ is split into a background part $\bar{g}_{\mu\nu}$ and a fluctuating one $h_{\mu\nu}$ in the following way:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \sqrt{32\pi G_k}\, h_{\mu\nu}\,, \tag{4.53}$$

where we have re–scaled the fluctuating metric so that the combination $\kappa_k \equiv \sqrt{32\pi G_k}$ acts as the gravitational coupling, G_k being the scale dependent Newton's constant. In this way a gravitational vertex with n–legs is accompanied by a factor $(\sqrt{32\pi G_k})^{n-2}$.

In the construction of the background EAA one introduces in the path integral a source, a gauge–fixing and a cutoff term, in which the background and the fluctuating fields do not appear via their sum $g_{\mu\nu}$. As a consequence the RG flow generates functionals which depend on $\bar{g}_{\mu\nu}$ and $h_{\mu\nu}$ separately.

The cutoff action is taken to be quadratic in the fluctuation metric, while the cutoff operator is constructed with the background metric and can be, for example, the covariant Laplacian. The general form of the cutoff action is:

$$\Delta S_k[\varphi;\bar{g}] = \frac{1}{2}\int d^dx \sqrt{\bar{g}}\varphi\, R_k[\bar{g}]\varphi\,. \tag{4.54}$$

Here $\varphi = (h, \bar{C}, C)$ is the field multiplet combining the fluctuating metric with the fluctuating ghost vector fields $\bar{C}_\mu, C^\mu$. The background effective average action (bEAA) is defined, as in the flat case, by introducing in the integro-differential equation for the background effective action the cutoff action (4.54):

$$e^{-\Gamma_k[\varphi;\bar{g}]} = \int D\chi \exp\left(-S[\chi+\varphi;\bar{g}] - \Delta S_k[\chi;\bar{g}] + \int d^dx\sqrt{\bar{g}}\Gamma_k^{(1;0)}[\varphi;\bar{g}]\chi\right), \tag{4.55}$$

where as usual the field multiplet χ has zero vacuum expectation value $\langle\chi\rangle = 0$.

The bEAA defined in this way is invariant under (background) diffeomorphisms:

$$(\delta + \bar{\delta})\Gamma_k[\varphi;\bar{g}] = 0\,. \tag{4.56}$$

We can now define a single–field functional, that we will call gauge covariant EAA (gEAA), by setting in the bEAA the fluctuation multiplet to zero, $\varphi = 0$, or equivalently $g_{\mu\nu} = \bar{g}_{\mu\nu}$ and $\bar{C}_\mu = C^\mu = 0$:

$$\bar{\Gamma}_k[g] = \Gamma_k[0,0,0;g]\,. \tag{4.57}$$

This is equivalent to a parametrization of the bEAA as the sum of a functional of the full quantum metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, the gEAA, and a "remainder functional" $\hat{\Gamma}_k[\varphi;\bar{g}]$ (rEAA) which remains a functional of both the fluctuation multiplet and the background metric separately:

$$\Gamma_k[\varphi;\bar{g}] = \bar{\Gamma}_k[\bar{g}+h] + \hat{\Gamma}_k[\varphi;\bar{g}]\,. \tag{4.58}$$

The functional $\hat{\Gamma}_k[h,\bar{C},C;\bar{g}]$ plays the role of a generalized gauge-fixing and ghost action as, in the limit $k \to \infty$, it flows to the classical gauge-fixing and ghost actions. It is defined by the property $\hat{\Gamma}_k[0,0,0;\bar{g}] = 0$.

4.4.2 Exact flow equation

The derivation of the ERGE for the bEAA in quantum gravity proceeds along the same lines as that derived in the flat QFT case. We present it here for completeness, and to set up some notation.

Again, we start by differentiating the integro-differential definition of the bEAA (4.55), with respect to the "RG time" $t = \log k/k_0$, obtaining:

$$\begin{aligned} e^{-\Gamma_k[\varphi;\bar{g}]}\partial_t\Gamma_k[\varphi;\bar{g}] &= \int D\chi\left(\partial_t\Delta S_k[\chi;\bar{g}] - \int d^dx\sqrt{\bar{g}}\,\partial_t\Gamma_k^{(1;0)}[\varphi;\bar{g}]\chi\right)\times \\ &\quad \times e^{-S[\varphi+\chi;\bar{g}]-\Delta S_k[\chi;\bar{g}]+\int\sqrt{\bar{g}}\,\Gamma_k^{(1;0)}[\varphi;\bar{g}]\chi}\,. \end{aligned} \tag{4.59}$$

Expressing the terms on the right hand side of (4.59) as expectation values we can rewrite it as follows:

$$\begin{aligned}\partial_t \Gamma_k[\varphi;\bar{g}] &= \langle \partial_t \Delta S_k[\chi;\bar{g}]\rangle - \int d^dx\sqrt{\bar{g}}\,\partial_t \Gamma^{(1;0)}_{k,A}[\varphi;\bar{g}]\,\langle \chi_A\rangle \\ &= \frac{1}{2}\int d^dx\sqrt{\bar{g}}\,\langle \chi_A\chi_B\rangle\,\partial_t R_{k,BA}[\bar{g}]\,. \end{aligned} \tag{4.60}$$

The two-point function of the fluctuation field can be written in terms of the inverse Hessian of the bEAA plus the cutoff action, where the functional derivatives are taken with respect to the fluctuation fields:

$$\langle \chi_A\chi_B\rangle = \left(\Gamma^{(2;0)}_k[\varphi;\bar{g}] + \Delta S^{(2;0)}_k[\varphi;\bar{g}]\right)^{-1} = \left(\Gamma^{(2;0)}_k[\varphi;\bar{g}] + R_k[\bar{g}]\right)^{-1}\,. \tag{4.61}$$

Inserting (4.61) into (4.60), gives:

$$\partial_t \Gamma_k[\varphi;\bar{g}] = \frac{1}{2}\mathrm{Tr}\left(\Gamma^{(2;0)}_k[\varphi;\bar{g}] + R_k[\bar{g}]\right)^{-1}\,\partial_t R_k[\bar{g}]\,. \tag{4.62}$$

The flow equation (4.62) is the exact flow equation for the bEAA for quantum gravity. As we did before, if we define the "regularized propagator" as

$$G_k[\varphi;\bar{g}] = \left(\Gamma^{(2;0)}_k[\varphi;\bar{g}] + R_k[\bar{g}]\right)^{-1}\,, \tag{4.63}$$

then the flow equation for the bEAA (4.62) can be rewritten in the compact form:

$$\partial_t \Gamma_k[\varphi;\bar{g}] = \frac{1}{2}\mathrm{Tr}\,G_k[\varphi;\bar{g}]\partial_t R_k[\bar{g}]\,. \tag{4.64}$$

The flow equation has still a one-loop structure and looks like an RG improvement of the one-loop bEA.

A flow equation for the gEAA can be similarly written down using (4.57) and (4.64):

$$\partial_t \bar{\Gamma}_k[\bar{g}] = \partial_t \Gamma_k[0;\bar{g}] = \frac{1}{2}\mathrm{Tr}\,G_k[0;\bar{g}]\partial_t R_k[\bar{g}]\,. \tag{4.65}$$

Note that $\Gamma^{(2;0)}_k[0,\bar{g}]$ is "super-diagonal" if the ghost action is bilinear in the ghosts and in this case we can immediately do the multiplet trace in (4.65), obtaining[5]:

$$\begin{aligned}\partial_t \bar{\Gamma}_k[\bar{g}] &= \frac{1}{2}\mathrm{Tr}\left(\Gamma^{(2,0,0;0)}_k[0,0,0;\bar{g}] + R_{k,hh}[\bar{g}]\right)^{-1}\partial_t R_{k,hh}[\bar{g}] + \\ &\quad -\mathrm{Tr}\left(\Gamma^{(0,1,1;0)}_k[0,0,0;\bar{g}] + R_{k,\bar{C}C}[\bar{g}]\right)^{-1}\partial_t R_{k,\bar{C}C}[\bar{g}]\,. \end{aligned} \tag{4.66}$$

[5] We introduce here a more precise notation that we will use also in the following: $\Gamma^{(n,m,q;p)}_k$ means we are taking n derivatives with respect to h, m derivatives with respect to $\bar{C}$, q derivatives with respect to C and p derivatives with respect to $\bar{g}$

In (4.66) we defined the cutoff kernels by the relations $R_{k,hh}[\bar{g}] \equiv \Delta S_k^{(2,0,0;0)}[\bar{g}]$ and $R_{k,\bar{C}C}[\bar{g}] \equiv \Delta S_k^{(0,1,1;0)}[\bar{g}]$.

It is important to realize that equation (4.66) is not a closed equation for the gEAA, since it involves the Hessian of the bEAA taken with respect to the fluctuation metric and the ghost fields. This implies that for $k \neq 0$ it is necessary to consider the flow in the extended theory space of all functionals of the fields $h_{\mu\nu}, \bar{C}_\mu, C^\nu$ and $\bar{g}_{\mu\nu}$ invariant under simultaneous physical and background diffeomorphisms, i.e. the flow of $\Gamma_k[h, \bar{C}, C; \bar{g}]$.

The flow equation for rEAA can be deduced directly from (4.58):

$$\partial_t \hat{\Gamma}_k[\varphi; \bar{g}] = \partial_t \bar{\Gamma}_k[\bar{g} + h] - \partial_t \Gamma_k[\varphi; \bar{g}] \,. \tag{4.67}$$

4.5 Beta functions in the Einstein–Hilbert truncation

4.5.1 Local truncations

In this work we will only consider local truncations of the bEAA. Nonlocal truncations can be very useful to match EFT results [96], but the distinctive feature of Asymptotic Safety is that it is predictive in the local part of the Effective Action.

We start by considering as a truncation ansatz for the gEAA the RG improved version of the Einstein-Hilbert action (4.14) where Newton's constant and the cosmological constant become scale dependent quantities:

$$\bar{\Gamma}_k[g] = \frac{\sqrt{2}}{\kappa_k^2} \int d^dx \sqrt{g}\, (2\Lambda_k - R) = \frac{1}{16\pi G_k} \int d^dx \sqrt{g}\, (2\Lambda_k - R) \ . \tag{4.68}$$

Before performing the background splitting, there is still one point that we need to consider. Since we are considering a theory at finite RG scale, we need to take into account the fact that the (fluctuating) fields in general get a nontrivial modification of their scaling along the flow. This is here accounted for by introducing scale dependent wave-function renormalizations for all the fields present in the cutoff action (4.54), i.e. the fields whose modes are cut off in defining the bEAA,

$$h_{\mu\nu} \to Z_{h,k}^{1/2} h_{\mu\nu} \qquad \bar{C}_\mu \to Z_{C,k}^{1/2} \bar{C}_\mu \qquad C^\nu \to Z_{C,k}^{1/2} C^\nu \, . \tag{4.69}$$

In particular, the cutoff term $\partial_t R_k[\bar{g}]$ in the flow equation will contain terms proportional to their anomalous dimensions,

$$\eta_{h,k} = -\partial_t \log Z_{h,k} \qquad \eta_{C,k} = -\partial_t \log Z_{C,k} \, , \tag{4.70}$$

coming from the redefinitions (4.69). The presence of these anomalous dimensions in the beta functions will be the translation of the fact we noticed before, that the flow of the gEAA is not close, unless we combine it with that of the rEAA. In fact, we will use the vertices coming from the rEAA to compute the flow of the anomalous dimensions, and close the gravitational beta functions system.

Expanding the bEAA in powers of the fluctuation metric gives to order $\kappa_k^2 h^2$ the following terms:

$$
\begin{aligned}
\bar{\Gamma}_k[\bar{g}+\kappa_k Z_{h,k}^{1/2}h] &= \frac{\sqrt{2}}{\kappa_k^2}\int d^dx\sqrt{\bar{g}}\left(2\Lambda_k-\bar{R}\right)\\
&+\frac{Z_{h,k}^{1/2}}{\kappa_k}\int d^dx\sqrt{\bar{g}}\left[-\bar{\Delta}h-\bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu}+h_{\mu\nu}\bar{R}^{\mu\nu}+\frac{1}{2}h\left(2\Lambda_k-\bar{R}\right)\right]\\
&+\frac{1}{2}Z_{h,k}\int d^dx\left[\frac{1}{2}h^{\mu\nu}\bar{\Delta}h_{\mu\nu}-\frac{1}{2}h\bar{\Delta}h+h^{\mu\nu}\bar{\nabla}_\nu\bar{\nabla}_\alpha h^\alpha_\mu-h\bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu}\right.\\
&-h^{\mu\nu}h^\alpha_\mu\bar{R}_{\nu\alpha}-h^{\mu\nu}h^{\alpha\beta}\bar{R}_{\alpha\mu\beta\nu}-h\bar{R}^{\mu\nu}h_{\mu\nu}\\
&\left.+\left(\frac{1}{4}h^2-\frac{1}{2}h^{\alpha\beta}h_{\alpha\beta}\right)\left(2\Lambda_k-\bar{R}\right)\right]\\
&+O\left(\kappa_k^{3/2}h^3\right)\,. \qquad (4.71)
\end{aligned}
$$

Note that in (4.71) the kinetic term of the metric fluctuation trace h comes with the wrong sign; this is the signal that the Einstein-Hilbert action (4.14) is unstable in the conformal sector. See [97] for a more detailed discussion of this point.

An important difference between quantum gravity and non-abelian gauge theories is that any ansatz for the gEAA of the first is necessarily non-polynomial in the full quantum metric, because it involves both the inverse metric and the square root of the determinant of the metric. Therefore, the expansion around any background metric does involve an infinite number of terms. Already in the full version of (4.71) all powers of the metric fluctuation are present, giving rise to non-zero contributions to arbitrary high vertices. This is indeed a peculiar property of gravity.

Similarly, we can consider an expansion of the rEAA in powers of the fluctuation metric and of the ghost fields. To second power in $h_{\mu\nu}$ and first in $\bar{C}_\mu, C^\mu$, for instance, we can consider the following truncation ansatz:

$$
\begin{aligned}
\hat{\Gamma}_k[Z_{h,k}^{1/2}h, Z_{C,k}^{1/2}\bar{C}, Z_{C,k}^{1/2}C;\bar{g}] &= \frac{1}{2}Z_{h,k}\int d^dx\sqrt{\bar{g}}\left(h_{\mu\nu}h^{\mu\nu}-h^2\right)m_{h,k}^2\\
&+\frac{1}{2\alpha_k}Z_{h,k}\int d^dx\sqrt{\bar{g}}\bar{g}^{\mu\nu}\left(\bar{\nabla}^\alpha h_{\alpha\mu}-\frac{\beta_k^2}{2}\bar{\nabla}_\mu h\right)^2\\
&-Z_{C,k}\int d^dx\sqrt{\bar{g}}\,\bar{C}^\mu\left[\bar{\nabla}^\alpha g_{\nu\alpha}\nabla_\mu+\bar{\nabla}^\alpha g_{\mu\nu}\nabla_\alpha\right.\\
&\left.-\beta_k\bar{\nabla}_\mu g_{\nu\alpha}\nabla^\alpha\right]C^\nu\,. \qquad (4.72)
\end{aligned}
$$

We will not study the running of the Pauli-Fierz mass $m_{h,k}$ [98], which is here included for completeness. Note that in (4.72) the ghost action involves both covariant derivatives in the full quantum metric ∇_μ and in the background metric $\bar{\nabla}_\mu$. The other pieces in the action (4.72) are the RG improvement of the classical gauge-fixing and ghost actions.

We will limit ourselves to the case where the gauge-fixing parameters are chosen to be $\alpha_k = \beta_k = 1$, where standard heat kernel techniques can be used. The running of Newton's constant and of the cosmological constant have already been studied for general gauge-fixing parameters in [75, 99]. [6]

The Einstein–Hilbert action represents the first term in a derivative expansion, namely to second order in the derivatives of the metric. To fourth order in the derivatives, the next term comprises the following curvature invariants:

$$\bar{\Gamma}_k[g]\big|_{\mathcal{R}^2} = \int d^dx\sqrt{g}\left(\frac{1}{2\lambda_k}C^2 + \frac{1}{\xi_k}R^2 + \frac{1}{\rho_k}E + \frac{1}{\tau_k}\Delta R\right) . \tag{4.73}$$

C^2 is the square of the conformal invariant Weyl tensor given in $d = 4$ by

$$C^2 = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} - \frac{2}{3}R^2 ,$$

while E is the integrand of the Euler topological invariant in four dimension:

$$E = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2 .$$

We are here adopting the basis $\{C^2, R^2, E, \Delta R\}$. By using the relations between the different invariants, one can pass to a different basis, for example with $R_{\mu\nu}R^{\mu\nu}$ in place of C^2.

In this thesis we will limit ourselves to the Einstein–Hilbert truncation.

4.5.2 Variations and functional derivatives

The basic invariants in the Einstein-Hilbert action are the volume and the integral of the Ricci scalar:

$$I_0[g] = \int d^dx\sqrt{g} \qquad\qquad I_1[g] = \int d^dx\sqrt{g}R . \tag{4.74}$$

Note that in $d = 2$ the integrand of the Ricci scalar is proportional to the Euler characteristic for a two dimensional manifold:

$$\chi(\mathcal{M}) = \frac{1}{4\pi}\int_{\mathcal{M}} d^2x\sqrt{g}\,R . \tag{4.75}$$

Up to two curvatures, or four derivatives, the invariants we can construct are:

$$I_{2,1}[g] = \int d^dx\sqrt{g}R^2 \qquad\qquad I_{2,2}[g] = \int d^dx\sqrt{g}R_{\mu\nu}R^{\mu\nu}$$

[6] It is worth noting that the most natural choice for the gauge-fixing parameters should be $\alpha_k = 0$ and $\beta_k = 2/d$, where only traceless transverse gravitons and the conformal factor propagate. It is believed that this values correspond to a fixed point of the RG flow, as is in the case of non-abelian gauge theories.

$$I_{2,3}[g] = \int d^dx\sqrt{g}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \qquad\qquad I_{2,4}[g] = \int d^dx\sqrt{g}\Box R\,. \tag{4.76}$$

The last invariant in (4.76) is a total derivative and is usually dropped. In $d = 4$ the three curvature square invariants are not independent since the linear combination

$$E = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2\,, \tag{4.77}$$

is the integrand of the the Euler characteristic for a four dimensional manifold. There is another interesting combination of the four derivatives invariants, which defines the square of the Weyl conformal tensor:

$$C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - \frac{4}{d-2}R_{\mu\nu}R^{\mu\nu} + \frac{2}{(d-1)(d-2)}R^2\,. \tag{4.78}$$

The Weyl tensor is completely traceless and the action

$$I_C[g] = \int d^dx\sqrt{g}\,C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}\,, \tag{4.79}$$

is invariant under local conformal transformations, i.e. $I_C[e^\sigma g] = I_C[g]$ for any $\sigma(x)$.

We now calculate the variations of the basic invariants just defined. We define $h_{\mu\nu} = \delta g_{\mu\nu}$ to be the first variation of the metric tensor. The first variations of the inverse metric can be deduced from the following relations, valid for any invertible matrix M,

$$M^{-1}M = 1 \qquad \Rightarrow \qquad \delta M^{-1}M + M^{-1}\delta M = 0 \qquad \Rightarrow \qquad \delta M^{-1} = -M^{-1}\delta M\, M^{-1}\,. \tag{4.80}$$

Setting $M_{\mu\nu} = g_{\mu\nu}$ and $\delta M_{\mu\nu} = h_{\mu\nu}$ in (4.80) gives:

$$\delta g^{\alpha\beta} = -g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu} = -h^{\alpha\beta}\,. \tag{4.81}$$

The second variation can be calculated iterating (4.81):

$$\begin{aligned}
\delta^2 g^{\alpha\beta} &= -\delta g^{\alpha\mu}g^{\beta\nu}h_{\mu\nu} - g^{\alpha\mu}\delta g^{\beta\nu}h_{\mu\nu} \\
&= g^{\alpha\lambda}g^{\mu\rho}g^{\beta\nu}h_{\lambda\rho}h_{\mu\nu} + g^{\alpha\mu}g^{\beta\lambda}g^{\nu\rho}h_{\lambda\rho}h_{\mu\nu} \\
&= 2h^{\alpha\lambda}h_\lambda^\beta\,.
\end{aligned} \tag{4.82}$$

The third variation is similarly found to be:

$$\delta^3 g^{\alpha\beta} = -3!h_\rho^\alpha h_\sigma^\rho h^{\sigma\beta}\,. \tag{4.83}$$

Combining (4.81), (4.82) and (4.83) gives the following expansion for the inverse metric around the background metric $\bar{g}_{\mu\nu}$:

$$\begin{aligned} g^{\alpha\beta} &= \bar{g}^{\alpha\beta} + \delta g^{\alpha\beta} + \frac{1}{2}\delta^2 g^{\alpha\beta} + \frac{1}{3!}\delta^3 g^{\alpha\beta} + O\left(h^4\right) \\ &= \bar{g}^{\alpha\beta} - h^{\alpha\beta} + h^{\alpha\lambda}h_\lambda^\beta - h_\rho^\alpha h_\sigma^\rho h^{\sigma\beta} + O\left(h^4\right) \,. \end{aligned} \quad (4.84)$$

It is not difficult to write the general n-th variation of the inverse metric tensor, it can be proven by induction that:

$$\delta^n g^{\alpha\beta} = (-1)^n n! h_{\lambda_1}^\alpha h_{\lambda_2}^{\lambda_1} \cdots h_{\lambda_{n-1}}^{\lambda_{n-2}} h^{\lambda_{n-1}\beta} \,. \quad (4.85)$$

The variations of the determinant of the metric tensor can be easily found using the following relation, valid again for any invertible matrix M,

$$\log \det M = \operatorname{tr} \log M \,. \quad (4.86)$$

A variation of equation (4.86) gives:

$$\delta \det M = \delta e^{\log \det M} = \det M \, \delta \operatorname{tr} \log M = \det M \operatorname{tr}\left(M^{-1}\delta M\right) \,. \quad (4.87)$$

Inserting in (4.87) $M_{\mu\nu} = g_{\mu\nu}$ and $\delta M_{\mu\nu} = h_{\mu\nu}$ brings to

$$\delta\sqrt{g} = \frac{1}{2}\sqrt{g} g^{\alpha\beta}\delta g_{\alpha\beta} = \frac{1}{2}\sqrt{g} h \,. \quad (4.88)$$

The second variation follows easily:

$$\delta^2\sqrt{g} = \frac{1}{4}\sqrt{g}\delta g_\alpha^\alpha \delta g_\beta^\beta - \frac{1}{2}\sqrt{g}\delta g^{\alpha\beta}\delta g_{\alpha\beta} = \sqrt{g}\left(\frac{1}{4}h^2 - \frac{1}{2}h^{\alpha\beta}h_{\alpha\beta}\right) \,. \quad (4.89)$$

For completeness the third variation of the metric determinant is found to be:

$$\delta^3\sqrt{g} = \sqrt{g}\left(\frac{1}{8}h^3 - \frac{3}{4}h h_{\mu\nu}h^{\mu\nu} + h_{\mu\nu}h^{\nu\alpha}h_\alpha^{\ \mu}\right) \,. \quad (4.90)$$

We don't have a closed formula for the n-th variation of the square root of the determinant of the metric, but for any given n these can be easily determined.

We find now the variations of the Christoffel symbols:

$$\Gamma_{\mu\nu}^\alpha = \frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right) \,. \quad (4.91)$$

Using geodesic coordinates, it can be proven that the first variation of the Christoffel symbols is:

$$\delta\Gamma_{\mu\nu}^\alpha = \frac{1}{2}g^{\alpha\beta}\left(\nabla_\mu h_{\nu\beta} + \nabla_\nu h_{\mu\beta} - \nabla_\beta h_{\mu\nu}\right) \,. \quad (4.92)$$

More generally we have the fundamental relation, that can again be proven by induction on n, for the n-th variation of the Christoffel symbols:

$$\delta^n \Gamma^\alpha_{\mu\nu} = \frac{n}{2} \left(\delta^{n-1} g^{\alpha\beta} \right) \left(\nabla_\mu h_{\nu\beta} + \nabla_\nu h_{\mu\beta} - \nabla_\beta h_{\mu\nu} \right) . \tag{4.93}$$

All the non-linearities of the Christoffel symbols are due the inverse metric of which we know exactly the n-variation (4.85). Introducing the tensor:

$$G_{\mu\nu\alpha} = \frac{1}{2} \left(\nabla_\mu h_{\nu\alpha} + \nabla_\nu h_{\mu\alpha} - \nabla_\alpha h_{\mu\nu} \right) , \tag{4.94}$$

we can rewrite the n-th variation of the Christoffel symbols simply as:

$$\delta^n \Gamma^\alpha_{\mu\nu} = n \delta^{n-1} g^{\alpha\beta} \, G_{\mu\nu\beta} . \tag{4.95}$$

Note that the tensor (4.94) is symmetric in the first two indices $G_{\mu\nu\alpha} = G_{\nu\mu\alpha}$. In particular we have the useful contractions:

$$G^{\alpha}_{\ \alpha}{}^{\mu} = \nabla^\alpha h^\mu_\alpha - \frac{1}{2} \nabla^\mu h \qquad\qquad G^\alpha_{\ \mu\alpha} = \frac{1}{2} \nabla_\mu h . \tag{4.96}$$

We turn now the variations of the fundamental building block of all gravitational invariants, the Riemann tensor:

$$R^\alpha_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\lambda\nu} \Gamma^\lambda_{\beta\mu} - \Gamma^\alpha_{\lambda\mu} \Gamma^\lambda_{\beta\nu} . \tag{4.97}$$

The Ricci tensor and the Ricci scalar are defined by the following contractions:

$$R_{\beta\nu} = R^\alpha_{\beta\alpha\nu} \qquad\qquad R = g^{\beta\nu} R_{\beta\nu} . \tag{4.98}$$

The n-th variation of the Riemann tensor is found directly from the definition (4.97) and using the binomial theorem for the variation of a product:

$$\delta^n R^\alpha_{\beta\mu\nu} = \nabla_\mu \delta^n \Gamma^\alpha_{\beta\nu} - \nabla_\nu \delta^n \Gamma^\alpha_{\beta\mu} + \sum_{i=1}^{n-1} \left(\begin{array}{c} n \\ i \end{array} \right) \left(\delta^{n-i} \Gamma^\alpha_{\mu\lambda} \delta^i \Gamma^\lambda_{\beta\nu} - \delta^{n-i} \Gamma^\alpha_{\nu\lambda} \delta^i \Gamma^\lambda_{\beta\mu} \right) . \tag{4.99}$$

This relation together with equation (4.93) or (4.95) and (4.85) gives us, in a closed form, all possible variations of the Riemann tensor.

The n-th variations of the Ricci tensor (4.98) are obtained straightforwardly from (4.99) by contraction:

$$\delta^n R_{\beta\nu} = \delta^n R^\alpha_{\beta\alpha\nu} . \tag{4.100}$$

The n-variation of the Ricci scalar follows from (4.98) and is:

$$\delta^n R = \sum_{i=1}^{n} \begin{pmatrix} n \\ i \end{pmatrix} \delta^{n-i} g^{\beta\nu} \delta^i R_{\beta\nu} \,. \tag{4.101}$$

We can now study some particular examples. From the fundamental relation (4.99), for $i = 1$, we find

$$\delta R^{\alpha}_{\beta\mu\nu} = \nabla_\mu G^{\alpha}_{\beta\nu} - \nabla_\nu G^{\alpha}_{\beta\mu} \,.$$

Using (4.100) and the second relation in (4.96) gives the first variation of the Ricci tensor[7]:

$$\begin{aligned} \delta R_{\mu\nu} &= \nabla_\alpha G^{\alpha}_{\mu\nu} - \nabla_\nu G^{\alpha}_{\mu\alpha} \\ &= \frac{1}{2} \left[\nabla_\alpha \left(\nabla_\mu h^{\alpha}_{\nu} + \nabla_\nu h^{\alpha}_{\mu} - \nabla^\alpha h_{\mu\nu} \right) - \nabla_\nu \nabla_\mu h \right] \\ &= \frac{1}{2} \left(-\nabla^2 h_{\mu\nu} - \nabla_\nu \nabla_\mu h + \nabla_\alpha \nabla_\mu h^{\alpha}_{\nu} + \nabla_\alpha \nabla_\nu h^{\alpha}_{\mu} \right) \,. \end{aligned} \tag{4.102}$$

Combining (4.102) with (4.101) gives the first variation of the Ricci scalar:

$$\begin{aligned} \delta R &= g^{\mu\nu} \delta R_{\mu\nu} + \delta g^{\mu\nu} R_{\mu\nu} \\ &= -\nabla^2 h + \nabla^\mu \nabla^\nu h_{\mu\nu} - h_{\mu\nu} R^{\mu\nu} \,. \end{aligned} \tag{4.103}$$

From (4.99) with $n = 2$ we get the second variation of the Riemann tensor

$$\delta^2 R^{\alpha}_{\beta\mu\nu} = -2\nabla_\mu \left(h^{\alpha\gamma} G_{\gamma\beta\nu} \right) + 2\nabla_\nu \left(h^{\alpha\gamma} G_{\gamma\beta\mu} \right) + 2 \left(G^{\alpha}_{\mu\gamma} G^{\gamma}_{\beta\nu} - G^{\alpha}_{\nu\gamma} G^{\gamma}_{\beta\mu} \right) \,, \tag{4.104}$$

while the second variation of the Ricci tensor is again just the contraction of (4.104):

$$\delta^2 R_{\mu\nu} = -2\nabla_\alpha \left(h^{\alpha\beta} G_{\beta\mu\nu} \right) + 2\nabla_\nu \left(h^{\alpha\beta} G_{\beta\mu\alpha} \right) + 2 \left(G^{\alpha}_{\alpha\beta} G^{\beta}_{\mu\nu} - G^{\alpha}_{\nu\beta} G^{\beta}_{\mu\alpha} \right) \,. \tag{4.105}$$

The second variation of the Ricci scalar is given in terms of (4.81), (4.82), (4.102) and (4.105):

$$\delta^2 R = \delta^2 g^{\mu\nu} R_{\mu\nu} + 2\delta g^{\mu\nu} \delta R_{\mu\nu} + g^{\mu\nu} \delta^2 R_{\mu\nu} \,. \tag{4.106}$$

We can now find the variations of the curvature invariants $I_i[g]$. Using (4.88) and (4.89) we find:

$$\delta I_0[g] = \frac{1}{2} \int d^d x \sqrt{g} h \qquad \delta^2 I_0[g] = \int d^d x \sqrt{g} \left(\frac{1}{4} h^2 - \frac{1}{2} h^{\alpha\beta} h_{\alpha\beta} \right) \,. \tag{4.107}$$

[7] $\nabla_\nu \nabla_\mu h = \nabla_\mu \nabla_\nu h$

Using (4.88) and (4.103) we find:

$$\delta I_1[g] = \int d^dx\,(\delta\sqrt{g}R + \sqrt{g}\delta R) = \int d^dx\sqrt{g}\left(-\nabla^2 h + \nabla^\mu\nabla^\nu h_{\mu\nu} - h_{\mu\nu}R^{\mu\nu} + \frac{1}{2}hR\right). \tag{4.108}$$

For the second variation we have:

$$\delta^2 I_1[g] = \int d^dx\,\left(\delta^2\sqrt{g}R + 2\delta\sqrt{g}\delta R + \sqrt{g}\delta^2 R\right), \tag{4.109}$$

the first two terms in (4.109) are rapidly evaluated using (4.88), (4.89) and (4.103). The last term in (4.109) can be expanded as:

$$\int d^dx\sqrt{g}\delta^2 R = \int d^dx\sqrt{g}\left(\delta^2 g^{\mu\nu}R_{\mu\nu} + 2\delta g^{\mu\nu}\delta R_{\mu\nu} + g^{\mu\nu}\delta^2 R_{\mu\nu}\right). \tag{4.110}$$

Again, the first two terms in (4.110) need just the relations (4.81), (4.82) and (4.102), the last can be written employing (4.105). Modulo a total derivative, we have:

$$\int d^dx\sqrt{g}g^{\mu\nu}\delta^2 R_{\mu\nu} = 2\int d^dx\sqrt{g}\left(G^\alpha_{\alpha\beta}G^{\beta\gamma}{}_\gamma - G^{\alpha\gamma}{}_\beta G^\beta{}_{\gamma\alpha}\right), \tag{4.111}$$

using in (4.111) the relations (4.96) and the product

$$G^{\alpha\gamma}{}_\beta G^\beta{}_{\gamma\alpha} = \frac{1}{4}\left(-\nabla^\gamma h_{\alpha\beta}\nabla_\gamma h^{\alpha\beta} + 2\nabla^\gamma h^{\alpha\beta}\nabla_\alpha h_{\beta\gamma}\right),$$

we find

$$\begin{aligned}\int d^dx\sqrt{g}g^{\mu\nu}\delta^2 R_{\mu\nu} &= 2\int d^dx\sqrt{g}\left(G^\alpha_{\alpha\beta}G^{\beta\gamma}{}_\gamma - G^{\alpha\gamma}{}_\beta G^\beta{}_{\gamma\alpha}\right), \\ &= \int d^dx\sqrt{g}\Big(\nabla_\mu h^{\mu\nu}\nabla_\nu h - \frac{1}{2}\nabla_\mu h\nabla^\mu h \\ &\quad +\frac{1}{2}\nabla^\alpha h_{\mu\nu}\nabla_\alpha h^{\mu\nu} - \nabla^\alpha h^{\mu\nu}\nabla_\mu h_{\nu\alpha}\Big). \end{aligned} \tag{4.112}$$

Inserting in (4.109) the variation (4.110) and (4.112) finally gives:

$$\begin{aligned}\delta^2 I_1[g] &= \int d^dx\sqrt{g}\Big[-\frac{1}{2}h\nabla^2 h + \frac{1}{2}h^{\mu\nu}\nabla^2 h_{\mu\nu} - h^{\mu\nu}\nabla_\alpha\nabla_\mu h^\alpha_\nu + h\nabla^\mu\nabla^\nu h_{\mu\nu} + \\ &\quad +2h^{\mu\nu}h^\alpha_\mu R_{\nu\alpha} - hR^{\mu\nu}h_{\mu\nu} + \left(\frac{1}{4}h^2 - \frac{1}{2}h^{\alpha\beta}h_{\alpha\beta}\right)R\Big]. \end{aligned} \tag{4.113}$$

Commuting covariant derivatives in the third term of (4.113) as

$$\nabla_\alpha\nabla_\mu h^\alpha_\nu = \nabla_\mu\nabla_\alpha h^\alpha_\nu + R_{\mu\alpha}h^\alpha_\nu - R_{\alpha\mu\beta\nu}h^{\alpha\beta},$$

we can recast (4.113) to the form:

$$\begin{aligned}\delta^2 I_1[g] &= \int d^dx\sqrt{g}\left[-\frac{1}{2}h^{\mu\nu}\Delta h_{\mu\nu}+\frac{1}{2}h\Delta h-h^{\mu\nu}\nabla_\nu\nabla_\alpha h^\alpha_\mu+h\nabla^\mu\nabla^\nu h_{\mu\nu}\right.\\ &\left.+h^{\mu\nu}h^\alpha_\mu R_{\nu\alpha}+h^{\mu\nu}h^{\alpha\beta}R_{\alpha\mu\beta\nu}-hR^{\mu\nu}h_{\mu\nu}+\left(\frac{1}{4}h^2-\frac{1}{2}h^{\alpha\beta}h_{\alpha\beta}\right)R\right]\end{aligned}\tag{4.114}$$

which can be later combined with the gauge-fixing action. It is straightforward now to calculate higher order variations of both the actions $I_0[g]$ and $I_1[g]$, since their variations can always be reduced to combinations of variations of the inverse metric, of the metric determinant and of the Christoffel symbols, which are all known exactly. In the same way, we can easily calculate the variations of the higher curvature invariants (4.76). We will not do this here since, in this thesis, we will concentrate to truncations where only variations of $I_0[g]$ and $I_1[g]$ are needed.

The background gauge fixing action is already quadratic in the metric fluctuation, when expanded reads:

$$S_{gf}[h;g]=\frac{1}{2\alpha}\int d^dx\sqrt{g}\left(-h^{\mu\nu}\nabla_\nu\nabla_\alpha h^\alpha_\mu+\beta h\nabla^\mu\nabla^\nu h_{\mu\nu}+\frac{\beta^2}{4}h\Delta h\right)\,. \tag{4.115}$$

Combining (4.115) with (4.114) gives:

$$\begin{aligned}-\frac{1}{2}\delta I_1[g]+S_{gf}[h;g] &= \frac{1}{2}\int d^dx\sqrt{g}\left[\frac{1}{2}h^{\mu\nu}\Delta h_{\mu\nu}-\frac{1}{2}\left(1-\frac{\beta^2}{2\alpha}\right)h\Delta h\right.\\ &+\left(1-\frac{1}{\alpha}\right)h^{\mu\nu}\nabla_\nu\nabla_\alpha h^\alpha_\mu-\left(1-\frac{\beta}{\alpha}\right)h\nabla^\mu\nabla^\nu h_{\mu\nu}\\ &-h^{\mu\nu}h^\alpha_\mu R_{\nu\alpha}-h^{\mu\nu}h^{\rho\alpha}R_{\rho\nu\alpha\mu}+hR^{\mu\nu}h_{\mu\nu}\\ &\left.-\left(\frac{1}{4}h^2-\frac{1}{2}h^{\alpha\beta}h_{\alpha\beta}\right)R\right]\,.\end{aligned}\tag{4.116}$$

We will use (4.116) to construct the Hessian needed in the flow equation for the bEAA. Note that the gauge choice $\alpha=\beta=1$ diagonalizes the Hessian (4.116).

From the variations just obtained we can calculate all the functional derivatives of the previous defined invariants by employing the following relation between variations and functional derivatives[8]:

$$\delta^{(n)}(...)(x)=\frac{1}{n!}\int_{x_1...x_n}\left[(...)^{(n)}(x)\right]^{\mu_1\nu_1...\mu_n\nu_n}(x_1,...,x_n)h_{\mu_1\nu_1}(x_1)...h_{\mu_n\nu_n}(x_n). \tag{4.117}$$

Using (4.117) we can derive all the gravitational vertices needed in the flow equations for the zero-field proper-vertices.

[8] We use the convention $\int_x\equiv\int d^dx\sqrt{g_x}$.

4.5.3 Decomposition and projectors

In this section we decompose the fluctuation metric into its irreducible parts, to identify which degrees of freedom are physical and which are pure gauge, and to construct the projector basis that we will use in the next section to construct the regularized graviton propagator.

We start by decomposing the metric fluctuation into transverse $h^T_{\mu\nu}$ and longitudinal $h^L_{\mu\nu}$ components:

$$h_{\mu\nu} = h^T_{\mu\nu} + h^L_{\mu\nu}\,, \tag{4.118}$$

with the following transversality condition $\nabla^\mu h^T_{\mu\nu} = 0$. The longitudinal part can be written in terms of a vector ξ_μ, which can be further decomposed into a transverse ξ^T_μ vector and the gradient of a scalar σ as $\xi_\mu = \xi^T_\mu + \nabla_\mu\sigma$, with the transversality condition $\nabla^\mu\xi^T_\mu = 0$, so this gives:

$$h^L_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = \nabla_\mu\xi^T_\nu + \nabla_\nu\xi^T_\mu + 2\nabla_\mu\nabla_\nu\sigma\,. \tag{4.119}$$

We can then extract the trace of the fluctuation metric

$$h = g^{\mu\nu}h_{\mu\nu} = g^{\mu\nu}h^T_{\mu\nu} - 2\Delta\sigma\,, \tag{4.120}$$

writing the transverse component of $h_{\mu\nu}$ in the following way:

$$h^T_{\mu\nu} = h^{TT}_{\mu\nu} + \frac{1}{d}g_{\mu\nu}(h + 2\Delta\sigma)\,, \tag{4.121}$$

with $h^{TT}_{\mu\nu}$ the transverse-traceless metric satisfying $g^{\mu\nu}h^{TT}_{\mu\nu} = 0$. Inserting (4.119) and (4.121) in (4.118) gives:

$$h_{\mu\nu} = h^{TT}_{\mu\nu} + \nabla_\mu\xi^T_\nu + \nabla_\nu\xi^T_\mu + 2\nabla_\mu\nabla_\nu\sigma + \frac{1}{d}g_{\mu\nu}(h + 2\Delta\sigma)\,. \tag{4.122}$$

So in the end the metric fluctuation (4.122) contains a transverse-traceless symmetric tensor, a transverse vector and two scalar degrees of freedom, the trace and the longitudinal component of the vector. To see which of these degrees of freedom are physical and which are pure gauge we can insert in (4.122) the gauge transformation of the metric fluctuation parametrized by a vector χ_μ:

$$\delta h_{\mu\nu} = \nabla_\mu\chi_\nu + \nabla_\nu\chi_\mu = \nabla_\mu\chi^T_\nu + \nabla_\nu\chi^T_\mu + 2\nabla_\mu\nabla_\nu\chi\,. \tag{4.123}$$

In (4.123) we used again the decomposition $\chi_\mu = \chi^T_\mu + \nabla_\mu\chi$ with $\nabla^\mu\chi^T_\mu = 0$. Matching (4.123) to

$$\delta h_{\mu\nu} = \delta h^{TT}_{\mu\nu} + \nabla_\mu\delta\xi^T_\nu + \nabla_\nu\delta\xi^T_\mu + 2\nabla_\mu\nabla_\nu\delta\sigma + \frac{1}{d}g_{\mu\nu}(\delta h + 2\Delta\delta\sigma)\,,$$

we find:

$$\delta h^{TT}_{\mu\nu} = 0 \qquad \delta\xi^T_\mu = \chi^T_\mu \qquad \delta\sigma = \chi \qquad \delta h = -2\Delta\chi\,. \tag{4.124}$$

These are the gauge transformation properties of the metric fluctuation components. We see that the transverse-traceless symmetric tensor is a physical degree of freedom, which can be associated with the graviton [9]. Also the following combination of the two scalar degrees of freedom

$$S = h + 2\Delta\sigma\,, \tag{4.125}$$

is gauge invariant $\delta S = 0$. It corresponds to the conformal mode that in the path integral formulation of gravity is dynamical as the graviton. Instead, the transverse vector ξ^T_μ and the scalar field σ are pure gauge fields.

When we will work with the flow equations for the zero-field proper-vertices of the bEAA to compute te running of anomalous dimensions, we will choose a flat space background, where the decomposition (4.122) naturally gives rise to a set of projectors that we will use to express the regularized inverse gravitational propagator $\gamma_k^{(2,0,0;0)} + \mathbf{R}_k[\delta]$ where $\gamma_k^{(2,0,0;0)} = \mathbf{\Gamma}_k^{(2,0,0;0)}[0,0,0;\delta]$. Using the properties of these projectors we can then easily obtain the regularized gravitational propagator $\mathbf{G}_k[0;\delta] = \left(\gamma_k^{(2,0,0;0)} + \mathbf{R}_k[\delta]\right)^{-1}$.

The basic longitudinal projector is defined by $P^{\mu\nu} = \partial^\mu\partial^\nu/\partial^2$ and projects out the longitudinal component of a vector field, $\delta^{\mu\nu} - P^{\mu\nu}$ instead projects out the transverse component of a vector field. The graviton is the transverse part of the traceless component of the metric, in flat space we can define it as follows:

$$\begin{aligned} h^{TT}_{\mu\nu} &= \left[\frac{1}{2}\left(\delta^\alpha_\mu - P^\alpha_\mu\right)\left(\delta^\beta_\nu - P^\beta_\nu\right) + \frac{1}{2}\left(\delta^\alpha_\mu - P^\alpha_\mu\right)\left(\delta^\beta_\nu - P^\beta_\nu\right) + \right. \\ &\quad \left. -\frac{1}{d-1}\left(g_{\mu\nu} - P_{\mu\nu}\right)\left(g^{\alpha\beta} - P^{\alpha\beta}\right)\right] h_{\alpha\beta} \\ &= \left[\tilde{\delta}^{\alpha\beta}_{\mu\nu} - \frac{1}{d-1}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\right] h_{\alpha\beta}\,, \end{aligned} \tag{4.126}$$

where we defined $\tilde{g}^{\mu\nu} = g^{\mu\nu} - P^{\mu\nu}$. We also have the following relations for the scalar degrees of freedom:

$$S = \frac{d}{d-1}\tilde{g}^{kl}h_{kl} \qquad \Box\sigma = \frac{d}{d-1}\left(P^{kl} - \frac{1}{d}g^{kl}\right)h_{kl}\,. \tag{4.127}$$

[9] Up to this point $h^{TT}_{\mu\nu}$ has 5 dofs and so is consistent also with a massive graviton. In order to obtain the usual massless spin 2 particle, we notice that there is still a residual gauge invariance under $h^{TT}_{\mu\nu} \to h^{TT}_{\mu\nu} + \nabla_{(\mu}V_{\nu)}$ with $\nabla_\mu V^\mu = 0$. Fixing this eliminates three further dofs, leaving only the two transverse ones of a massless particle (I'd like to thank G.P. Vacca for pointing this out to me).

Inspired by (4.126) and (4.127) we define the following projectors[10]:

$$\begin{aligned}
P_2^{\mu\nu,\alpha\beta} &= \tilde{\delta}^{\mu\nu,\alpha\beta} - \frac{1}{d-1}\tilde{g}^{\mu\nu}\tilde{g}^{\alpha\beta} \\
P_1^{\mu\nu,\alpha\beta} &= \frac{1}{2}\left(\tilde{g}^{\mu\alpha}P^{\nu\beta} + \tilde{g}^{\mu\beta}P^{\nu\alpha} + \tilde{g}^{\nu\alpha}P^{\mu\beta} + \tilde{g}^{\nu\beta}P^{\mu\alpha}\right) \\
P_S^{\mu\nu,\alpha\beta} &= \frac{1}{d-1}\tilde{g}^{\mu\nu}\tilde{g}^{\alpha\beta} \\
P_\sigma^{\mu\nu,\alpha\beta} &= P^{\mu\nu}P^{\alpha\beta} \\
P_{S\sigma}^{\mu\nu,\alpha\beta} &= \frac{1}{\sqrt{d-1}}\left(\tilde{g}^{\mu\nu}P^{\alpha\beta} + P^{\mu\nu}\tilde{g}^{\alpha\beta}\right) .
\end{aligned} \tag{4.128}$$

The projectors in (4.128) have the following traces (where we use the notation $A = \mu\nu$ and $B = \alpha\beta$ and hats mean contractions):

$$\begin{aligned}
P_2^{\hat{A}B} &= \frac{d^2-d-2}{2} & P_2^{\hat{A}\hat{B}} &= 0 \\
P_1^{\hat{A}B} &= d-1 & P_1^{\hat{A}\hat{B}} &= 0 \\
P_S^{\hat{A}B} &= 1 & P_S^{\hat{A}\hat{B}} &= d-1 \\
P_\sigma^{\hat{A}B} &= 1 & P_\sigma^{\hat{A}\hat{B}} &= 1 \\
P_{S\sigma}^{\hat{A}B} &= 0 & P_{S\sigma}^{\hat{A}\hat{B}} &= 2\sqrt{d-1}
\end{aligned} \tag{4.129}$$

and satisfy the following relations:

$$\begin{aligned}
[P_2 + P_1 + P_S + P_\sigma]^{\mu\nu,\alpha\beta} &= \delta^{\mu\nu,\alpha\beta} \\
\left[(d-1)P_S + P_\sigma + \sqrt{d-1}P_{S\sigma}\right]^{\mu\nu,\alpha\beta} &= g^{\mu\nu}g^{\alpha\beta} \\
\left[2P_\sigma + \sqrt{d-1}P_{S\sigma}\right]^{\mu\nu,\alpha\beta} &= g^{\mu\nu}P^{\alpha\beta} + P^{\mu\nu}g^{\alpha\beta} \\
[P_1 + 2P_\sigma]^{\mu\nu,\alpha\beta} &= \frac{1}{2}\left(g^{\mu\alpha}P^{\nu\beta} + g^{\mu\beta}P^{\nu\alpha} + g^{\nu\alpha}P^{\mu\beta} + g^{\nu\beta}P^{\mu\alpha}\right) \\
P_\sigma^{\mu\nu,\alpha\beta} &= P^{\mu\nu}P^{\alpha\beta} .
\end{aligned} \tag{4.130}$$

We can also introduce the trace projection operator as follows:

$$P^{\mu\nu,\alpha\beta} = \frac{1}{d}g^{\mu\nu}g^{\alpha\beta} , \tag{4.131}$$

[10]Note that the quantity we call $P_{S\sigma}$ is not really a projector, but the sum of two intertwiners. However this does not alter the following discussion.

and from (4.130) this can be expressed in terms of the other projection operators as[11]:

$$\mathbf{P} = \frac{d-1}{d}\mathbf{P}_S + \frac{1}{d}\mathbf{P}_\sigma + \frac{\sqrt{d-1}}{d}\mathbf{P}_{S\sigma}\,, \tag{4.132}$$

so that

$$\mathbf{1} - \mathbf{P} = \mathbf{P}_2 + \mathbf{P}_1 + \frac{1}{d}\mathbf{P}_S + \frac{d-1}{d}\mathbf{P}_\sigma - \frac{\sqrt{d-1}}{d}\mathbf{P}_{S\sigma}\,. \tag{4.133}$$

The non-zero products between these projection operators are:

$$\begin{aligned} &\mathbf{P}_S\mathbf{P}_{S\sigma} + \mathbf{P}_\sigma\mathbf{P}_{S\sigma} = \mathbf{P}_{S\sigma} \qquad && \mathbf{P}_S\mathbf{P}_{S\sigma} = \mathbf{P}_{S\sigma}\mathbf{P}_\sigma \\ &\mathbf{P}_{S\sigma}\mathbf{P}_S = \mathbf{P}_\sigma\mathbf{P}_{S\sigma} \qquad && \mathbf{P}_{S\sigma}\mathbf{P}_{S\sigma} = \mathbf{P}_S + \mathbf{P}_\sigma\,. \end{aligned} \tag{4.134}$$

There is a useful isomorphisms that encodes (4.134) and that can be used to simplify the operations with these projectors. This reads:

$$\mathbf{P}_S \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{P}_\sigma \to \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{P}_{S\sigma} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.135}$$

The general structure of the inverse propagator that we will encounter in the next section is as follows:

$$\mathbf{M} = \lambda_2\mathbf{P}_2 + \lambda_1\mathbf{P}_1 + \lambda_S\mathbf{P}_S + \lambda_\sigma\mathbf{P}_\sigma + \lambda_{S\sigma}\mathbf{P}_{S\sigma}\,. \tag{4.136}$$

We can invert (4.136) to obtain:

$$\mathbf{M}^{-1} = \frac{1}{\lambda_2}\mathbf{P}_2 + \frac{1}{\lambda_1}\mathbf{P}_1 + \frac{\lambda_\sigma}{\lambda_S\lambda_\sigma - \lambda_{S\sigma}^2}\mathbf{P}_S + \frac{\lambda_S}{\lambda_S\lambda_\sigma - \lambda_{S\sigma}^2}\mathbf{P}_\sigma - \frac{\lambda_{S\sigma}}{\lambda_S\lambda_\sigma - \lambda_{S\sigma}^2}\mathbf{P}_{S\sigma}\,. \tag{4.137}$$

The scalar part of (4.137) has been derived by using the isomorphisms (4.135) in the following way:

$$\begin{aligned} (\lambda_S\mathbf{P}_S + \lambda_\sigma\mathbf{P}_\sigma + \lambda_{S\sigma}\mathbf{P}_{S\sigma})^{-1} \quad &\to \quad \begin{pmatrix} \lambda_S & \lambda_{S\sigma} \\ \lambda_{S\sigma} & \lambda_\sigma \end{pmatrix}^{-1} \\ &= \quad \frac{1}{\lambda_S\lambda_\sigma - \lambda_{S\sigma}^2}\begin{pmatrix} \lambda_\sigma & -\lambda_{S\sigma} \\ -\lambda_{S\sigma} & \lambda_S \end{pmatrix} \\ &\to \quad \frac{1}{\lambda_S\lambda_\sigma - \lambda_{S\sigma}^2}\left(\lambda_\sigma\mathbf{P}_S + \lambda_S\mathbf{P}_\sigma - \lambda_{S\sigma}\mathbf{P}_{S\sigma}\right)\,. \end{aligned}$$

Equation (4.137) is the fundamental relation used in the next section to construct the regularized graviton propagator.

[11] We will sometimes suppress indices for notation clarity and we will use boldface symbols to indicate linear operators in the space of symmetric tensors.

4.5.4 Regularized propagator

In this section we construct the regularized graviton propagator that enters the flow equation for the bEAA for quantum gravity. For the truncations we are considering in this chapter, we need the functional derivatives of the basic invariants (4.74) defined before, evaluated at $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$. In momentum space, these are given by the following relations:

$$\begin{aligned} I_0^{(2)}(p,-p)^{\mu\nu,\alpha\beta} &= -\frac{1}{2}\delta^{\mu\nu,\alpha\beta} + \frac{1}{4}g^{\mu\nu}g^{\alpha\beta} \\ -\frac{1}{2}I_1^{(2)}(p,-p)^{\mu\nu,\alpha\beta} + S_{gf}(p,-p)^{\mu\nu,\alpha\beta} &= \frac{1}{2}p^2\delta^{\mu\nu,\alpha\beta} - \frac{1}{2}\left(1-\frac{\beta^2}{2\alpha}\right)p^2 g^{\mu\nu}g^{\alpha\beta} \\ &\quad -\frac{1}{4}\left(1-\frac{1}{\alpha}\right)\Big(g^{\mu\alpha}p^\nu p^\beta + g^{\mu\beta}p^\nu p^\alpha \\ &\quad + g^{\nu\alpha}p^\mu p^\beta + g^{\nu\beta}p^\mu p^\alpha\Big) \\ &\quad + \frac{1}{2}\left(1-\frac{\beta}{\alpha}\right)\Big(g^{\mu\nu}p^\alpha p^\beta + g^{\alpha\beta}p^\mu p^\nu\Big) \end{aligned} \tag{4.138}$$

where in the second line we added the contribution from the gauge-fixing term (4.115). We can now write down, in momentum space, the Hessian of the bEAA we are considering using the projection operators introduced in the previous section. We find the following form:[12]

$$\begin{aligned} \gamma_k^{(2,0,0;0)}(p,-p) &= Z_h\Bigg\{\frac{1}{2}\left(p^2+2m_h^2-2\Lambda\right)\mathbf{P}_2 + \left(\frac{1}{2\alpha}p^2+m_h^2-\Lambda\right)\mathbf{P}_1 \\ &\quad + \left[-\left(\frac{d-2}{2}-\frac{(d-1)\beta^2}{4\alpha}\right)p^2+\frac{\Lambda}{2}\right]\mathbf{P}_S + \left(\frac{(2-\beta)^2}{4\alpha}p^2-\frac{\Lambda}{2}\right)\mathbf{P}_\sigma \\ &\quad + \frac{\sqrt{d-1}}{2}\left[\frac{\beta(\beta-2)}{2\alpha}p^2+2m_h^2+\Lambda\right]\mathbf{P}_{S\sigma}\Bigg\}\,. \end{aligned} \tag{4.139}$$

We need now to choose the tensor structure of the cutoff kernel. We will consider the following form:

$$\mathbf{R}_k[\delta] = \frac{1}{2}Z_h\left[\mathbf{P}_2+\mathbf{P}_1-\frac{d-3}{2}\mathbf{P}_S+\frac{1}{2}\mathbf{P}_\sigma-\frac{\sqrt{d-1}}{2}\mathbf{P}_{S\sigma}\right]R_k(p^2)\,. \tag{4.140}$$

The inverse regularized graviton propagator can thus be written by summing (4.139) and (4.140):

$$\gamma_k^{(2,0,0;0)} + \mathbf{R}_k[\delta] = Z_h\left[\gamma_2\mathbf{P}_2+\gamma_1\mathbf{P}_1+\gamma_S\mathbf{P}_S+\gamma_{S\sigma}\mathbf{P}_{s\sigma}+\gamma_\sigma\mathbf{P}_\sigma\right]\,, \tag{4.141}$$

[12]We retain the Pauli–Fierz mass for generality and we omit to write the k dependence of the running coupling constants where this is nonambiguous.

where the various spin components in (4.141) are:

$$\begin{aligned}
\gamma_2(p^2) &= \frac{1}{2}(p^2 - 2\Lambda) + m_h^2 + \frac{1}{2}R_k(p^2) \\
\gamma_1(p^2) &= \frac{1}{2\alpha}p^2 - \Lambda + m_h^2 + R_k(p^2) \\
\gamma_S(p^2) &= -\left[\frac{d-2}{2} - \frac{(d-1)\beta^2}{4\alpha}\right]p^2 - dm_h^2 + \frac{d-3}{2}\Lambda + \frac{d-1}{4}R_k(p^2) \\
\gamma_\sigma(p^2) &= \frac{(2-\beta)^2}{4\alpha}p^2 - \frac{\Lambda}{2} + \frac{d-1}{4}R_k(p^2) \\
\gamma_{S\sigma}(p^2) &= \sqrt{d-1}\left[\frac{\beta(\beta-2)}{4\alpha}p^2 + m_h^2 + \frac{\Lambda}{2}\right] .
\end{aligned} \tag{4.142}$$

The regularized graviton propagator is defined as the inverse regularized Hessian of the bEAA, i.e the inverse of (4.141):

$$\mathbf{G}_k[0;\delta] = \left(\gamma_k^{(2,0,0;0)} + \mathbf{R}_k[\delta]\right)^{-1} . \tag{4.143}$$

Using the isomorphism described in the previous section to invert (4.141) we find the following general form:

$$\mathbf{G}_k[0;\delta] = G_2\mathbf{P}_2 + G_1\mathbf{P}_1 + G_S\mathbf{P}_S + G_{S\sigma}\mathbf{P}_{S\sigma} + G_\sigma\mathbf{P}_\sigma \,, \tag{4.144}$$

together with the following spin components:

$$\begin{aligned}
G_2(p^2) &= \frac{2}{p^2 + 2m_h^2 - 2\Lambda + R_k(p^2)} \\
G_1(p^2) &= \frac{2\alpha}{p^2 + 2\alpha\left[m_h^2 - \Lambda + R_k(p^2)\right]} \\
G_S(p^2) &= \frac{\gamma_\sigma}{\gamma_S\gamma_\sigma - \gamma_{S\sigma}^2} \\
G_\sigma(p^2) &= \frac{\gamma_S}{\gamma_S\gamma_\sigma - \gamma_{S\sigma}^2} \\
G_{S\sigma}(p^2) &= -\frac{\gamma_{S\sigma}}{\gamma_S\gamma_\sigma - \gamma_{S\sigma}^2} .
\end{aligned} \tag{4.145}$$

Equations (4.144) and (4.145) represent the general form of the regularized graviton propagator on flat space for general values of Λ, m_h, α and β.

In the gauge $\alpha = \beta = 1$ we have:

$$\mathbf{G}_k[0;\delta] = (\mathbf{1} - \mathbf{P})\, G_{TF,k}(p^2) - \frac{2}{d-2}\mathbf{P}\, G_{T,k}(p^2) \,, \tag{4.146}$$

where $\mathbf{P}$ is the trace projector (4.132), and the two graviton propagators are

$$\begin{aligned} G_{TF,k}(p^2) &= \frac{1}{p^2 + R_k(p^2) + m_h^2 - 2\Lambda_k} \\ G_{T,k}(p^2) &= \frac{1}{p^2 + R_k(p^2) + 2\frac{d-1}{d-2}m_h^2 - 2\Lambda_k} \,. \end{aligned}$$

For $m_h = 0$, we have $G_{TF,k}(p^2) = G_{T,k}(p^2)$ and thus

$$\mathbf{G}_k[0;\delta] = (\mathbf{1} - \mathbf{P} - \frac{2}{d-2}\mathbf{P})\, G_{T,k}(p^2)$$

The cutoff kernel (4.140), when written in terms of $\mathbf{P}$ using (4.132) and (4.133) reads as follows:

$$\mathbf{R}_k[\delta] = \frac{1}{2} Z_h \left[\mathbf{1} - \mathbf{P} - \frac{d-2}{2}\mathbf{P}\right] R_k(p^2)\,. \tag{4.147}$$

This shows that the cutoff kernel (4.140) is as the one employed in [83].

4.5.5 Heat Kernel Derivation of $\partial_t G_k$ and $\partial_t \Lambda_k$

General considerations

In this section we calculate the beta function of Newton's constant $\partial_t G$ and the beta function of the cosmological constant $\partial_t \Lambda$ in the Einstein–Hilbert truncation (4.68)

$$\bar{\Gamma}_k[g] = \frac{1}{16\pi G} \int d^d x \sqrt{g}\, (2\Lambda - R)\ . \tag{4.148}$$

We will employ both a type I and a type II cutoff operator (see also the Appendix for these definitions).

Differentiating (4.148) with respect to the RG time gives:

$$\partial_t \bar{\Gamma}_k[\bar{g}] = \partial_t \left(\frac{\Lambda}{8\pi G}\right) \int d^d x \sqrt{g} - \partial_t \left(\frac{1}{16\pi G}\right) \int d^d x \sqrt{g} R\,. \tag{4.149}$$

When we expand the functional traces on the rhs of the flow equation for the gEAA,

$$\begin{aligned} \partial_t \bar{\Gamma}_k[\bar{g}] &= \frac{1}{2} \mathrm{Tr} \left(\Gamma_k^{(2,0,0,0)}[0,0,0;\bar{g}]^{\mu\nu}_{\alpha\beta} + R_k[\bar{g}]^{\mu\nu}_{\alpha\beta}\right)^{-1} \partial_t R_k[\bar{g}]^{\alpha\beta}_{\mu\nu} + \\ & \quad - \mathrm{Tr} \left(\Gamma_k^{(0,1,1,0)}[0,0,0;\bar{g}]^{\mu}_{\nu} + R_k[\bar{g}]^{\mu}_{\nu}\right)^{-1} \partial_t R_k[\bar{g}]^{\mu}_{\nu} \end{aligned} \tag{4.150}$$

using the Heat Kernel expansion, from the terms proportional to the invariants $I_0[g]$ and $I_1[g]$ we can extract the beta functions of the cosmological constant and of Newton's constant. Note that in (4.150) the cutoff kernels in the graviton and ghost sectors are distinguished by the indices. From here on we will consider only the gauge $\alpha = \beta = 1$ that allows us to employ heat kernel methods. We use the general decomposition of the bEAA given in (4.58) to write:

$$\Gamma_k^{(2,0,0;0)}[h,\bar{C},C;\bar{g}]^{\mu\nu}_{\alpha\beta} = \bar{\Gamma}_k^{(2)}[\bar{g}+h]^{\mu\nu}_{\alpha\beta} + \hat{\Gamma}_k^{(2,0,0;0)}[h,\bar{C},C;\bar{g}]^{\mu\nu}_{\alpha\beta} \tag{4.151}$$

and

$$\Gamma_k^{(0,1,1;0)}[h,\bar{C},C;\bar{g}]^{\mu\nu} = Z_C S_{gh}^{(0,1,1;0)}[h,\bar{C},C;\bar{g}]^{\mu\nu} \,. \tag{4.152}$$

In (4.152) we used our ansatz for the rEAA given in equation (4.72). To calculate the gravitational Hessian needed in equation (4.150), we can extract the quadratic part in the fluctuation metric of the action (4.148) using equation (4.116):

$$\begin{aligned}\frac{1}{2}\int d^dx\sqrt{\bar{g}}h_{\mu\nu}\Gamma_k^{(2,0,0;0)}[h,\bar{C},C;\bar{g}]^{\mu\nu}_{\alpha\beta}h^{\alpha\beta} &= \frac{1}{2}Z_h\int d^dx\sqrt{\bar{g}}\left[\frac{1}{2}h^{\mu\nu}\bar{\Delta}h_{\mu\nu} - \frac{1}{4}h\bar{\Delta}h\right.\\ &+m_h^2\left(h^{\alpha\beta}h_{\alpha\beta}-h^2\right) - h^{\mu\nu}h^\alpha_\mu\bar{R}_{\nu\alpha} - h^{\mu\nu}h^{\alpha\beta}\bar{R}_{\alpha\mu\beta\nu}\\ &\left.+h\bar{R}^{\mu\nu}h_{\mu\nu} + \left(\frac{1}{4}h^2 - \frac{1}{2}h^{\alpha\beta}h_{\alpha\beta}\right)(2\Lambda-\bar{R})\right]\,.\end{aligned} \tag{4.153}$$

The gravitational Hessian can now be easily extracted from (4.153) and reads:

$$\Gamma_k^{(2,0,0;0)}[0,0,0;g]^{\mu\nu}_{\rho\sigma} = \frac{Z_h}{2}\left[\delta^{\mu\nu}_{\alpha\beta} - \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\right]\left[\delta^{\alpha\beta}_{\rho\sigma}(\Delta+m_h^2-2\Lambda) + \frac{m_h^2}{d-2}g^{\alpha\beta}g_{\rho\sigma} + U^{\alpha\beta}_{\rho\sigma}\right]\,, \tag{4.154}$$

where the symmetric spin two tensor identity $\delta^{\mu\nu}_{\rho\sigma} = \frac{1}{2}\left(\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\sigma\delta^\nu_\rho\right)$ and the trace projector $P^{\mu\nu}_{\rho\sigma} = \frac{1}{d}g^{\mu\nu}g_{\rho\sigma}$ have been defined in section 4.5.3 and we defined the following tensor:

$$\begin{aligned}U^{\alpha\beta}_{\rho\sigma} &= \left(\delta^{\alpha\beta}_{\rho\sigma} - \frac{1}{2}g^{\alpha\beta}g_{\rho\sigma}\right)R + g^{\alpha\beta}R_{\rho\sigma} + R^{\alpha\beta}g_{\rho\sigma} +\\ &\quad -\frac{1}{2}\left(\delta^\alpha_\rho R^\beta_\sigma + \delta^\alpha_\sigma R^\beta_\rho + R^\alpha_\rho\delta^\beta_\sigma + R^\alpha_\sigma\delta^\beta_\rho\right) - \left(R^{\beta\ \alpha}_{\ \rho\ \sigma} + R^{\beta\ \alpha}_{\ \sigma\ \rho}\right) +\\ &\quad -\frac{d-4}{2(d-2)}\left(-R\,g^{\alpha\beta}g_{\rho\sigma} + g^{\alpha\beta}R_{\rho\sigma} + R^{\alpha\beta}g_{\rho\sigma}\right)\,.\end{aligned} \tag{4.155}$$

With the boldface notation previously introduced we can rewrite (4.154) in the following way:

$$\mathbf{\Gamma}_k^{(2,0,0;0)}[0,0,0;g] = \frac{1}{2}Z_h\left[(\mathbf{1}-\mathbf{P}) - \frac{d-2}{2}\mathbf{P}\right]\left[\mathbf{1}(\Delta+m_h^2-2\Lambda) + m_h^2\frac{d}{d-2}\mathbf{P} + \mathbf{U}\right]\,. \tag{4.156}$$

The ghost action when evaluated at zero fluctuation metric becomes:

$$S_{gh}[0,\bar{C},C;\bar{g}] = \int d^dx\sqrt{\bar{g}}\,\bar{C}^\mu\left[\bar{\Delta}\bar{g}_{\mu\nu} - (1-\beta)\bar{\nabla}_\mu\bar{\nabla}_\nu - \bar{R}_{\mu\nu}\right]C^\nu\,. \tag{4.157}$$

If we then set $\beta = 1$ in (4.157) we find the following ghost Hessian:

$$\Gamma_k^{(0,1,1;0)}[0,0,0;g]^\mu_\nu = Z_C\left(\Delta\delta^\mu_\nu - R^\mu_\nu\right). \tag{4.158}$$

For later use we report here the following traces of the tensors defined in (4.155):

$$\mathrm{tr}\,\mathbf{1} = \frac{d(d+1)}{2} \qquad \mathrm{tr}\,\mathbf{P} = 1 \qquad \mathrm{tr}\,\mathbf{U} = \frac{d(d-1)}{2}R$$

$$\mathrm{tr}\,\mathbf{U}^2 = \frac{d^3-5d^2+8d+4}{2(d-2)}R^2 + \frac{d^2-8d+4}{d-2}R_{\mu\nu}R^{\mu\nu} + 3R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\,. \tag{4.159}$$

We have now to choose the cutoff operator, which can be done in two ways. In the first case, that we call type I, we use in both the graviton and ghost sectors the covariant Laplacian as cutoff operator. In the second case, we employ the differential operator $\mathbf{\Delta}_2 = \Delta\mathbf{1} + \mathbf{U}$ for the graviton modes and $\Delta_1 = \Delta\delta^\mu_\nu - R^\mu_\nu$ for the ghost modes.

Type I

We define the graviton cutoff kernel as:

$$\mathbf{R}_k(\Delta) = \frac{Z_h}{2}\left[(\mathbf{1}-\mathbf{P}) - \frac{d-2}{2}\mathbf{P}\right]R_k(\Delta)\,, \tag{4.160}$$

which corresponds to the flat space expression (4.147) of the previous section. For the ghost cutoff kernel we take:

$$R_k(\Delta)^\mu_\nu = Z_C\delta^\mu_\nu R_k(\Delta)\,. \tag{4.161}$$

Remembering that the anomalous dimension of the fluctuation metric is defined by $\eta_h = -\partial_t \log Z_h$, we see that the flow equation (4.150) for the gEAA becomes:

$$\partial_t\bar\Gamma_k[g] = \frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k(\Delta) - \eta_h R_k(\Delta)}{\mathbf{1}\left(\Delta + m_h^2 - 2\Lambda\right) + m_h^2\frac{d}{d-2}\mathbf{P} + \mathbf{U} + \mathbf{R}_k(\Delta)} - \mathrm{Tr}\frac{\partial_t R_k(\Delta) - \eta_C R_k(\Delta)}{\Delta g^{\mu\nu} - R^{\mu\nu} + R_k(\Delta)^{\mu\nu}}\,. \tag{4.162}$$

Note that the wave-function renormalization factors in (4.162) have deleted each other leaving terms proportional to the anomalous dimension of the fluctuation metric and of the ghost fields. Note also that the possible troublesome conformal instability does not affect (4.162) due to our cutoff choice (4.160).

Deriving the beta functions with a Pauli-Fierz mass requires essentially no additional computational effort than deriving them without it, so we can keep it different from zero. In the end we will put it to zero, but it is nice to see just how it enters in the flow.

We will choose the background as a d-dimensional sphere. On the sphere the Riemann and Ricci tensors are proportional to the Ricci scalar:

$$R_{\mu\nu} = \frac{R}{d} g_{\mu\nu} \qquad\qquad R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right) . \tag{4.163}$$

Considering (4.163), the $\mathbf{U}$ tensor in (4.155) becomes simply:

$$\mathbf{U} = (\mathbf{1} - \mathbf{P}) \frac{d^2 - 3d + 4}{d(d-1)} R + \mathbf{P} \frac{d-4}{d} R . \tag{4.164}$$

Using the fact that now (4.164) is decomposed in the orthogonal basis of the trace and trace-free projectors, we can easily re-express the Hessian (4.156) in the following way:

$$\begin{aligned} \mathbf{\Gamma}_k^{(2,0,0;0)}[0,0,0;g] &= \frac{1}{2} Z_h \left[(\mathbf{1} - \mathbf{P}) \left(\Delta + m_h^2 - 2\Lambda + \frac{d^2 - 3d + 4}{d(d-1)} R \right) \right. \\ & \left. - \frac{d-2}{2} \mathbf{P} \left(\Delta + 2 \frac{d-1}{d-2} m_h^2 - 2\Lambda + \frac{d-4}{d} R \right) \right] . \end{aligned} \tag{4.165}$$

The full regularized graviton propagator then reads:

$$\begin{aligned} & \left[\mathbf{1} \left(\Delta + m_h^2 - 2\Lambda + R_k(\Delta) \right) + m_h^2 \frac{d}{d-2} \mathbf{P} + \mathbf{U} \right]^{-1} = \\ &= (\mathbf{1} - \mathbf{P}) \frac{1}{\Delta + R_k(\Delta) + m_h^2 - 2\Lambda + \frac{d^2-3d+4}{d(d-1)} R} \\ & - \frac{2}{d-2} \mathbf{P} \frac{1}{\Delta + R_k(\Delta) + 2 \frac{d-1}{d-2} m_h^2 - 2\Lambda + \frac{d-4}{d} R} . \end{aligned} \tag{4.166}$$

Notice there is a kinematical singularity in the regularized propagator (4.166) for $d = 2$. We can now define the trace and trace-free parts of the regularized graviton propagator on the d-dimensional sphere as follows:

$$\begin{aligned} G_{TF,k}(z) &= \frac{1}{z + R_k(z) + m_h^2 - 2\Lambda_k + \frac{d^2-3d+4}{d(d-1)} R} \\ G_{T,k}(z) &= \frac{1}{z + R_k(z) + 2 \frac{d-1}{d-2} m_h^2 - 2\Lambda_k + \frac{d-4}{d} R} . \end{aligned} \tag{4.167}$$

Note that when the Pauli-Fierz mass is nonzero, the trace and trace-free regularized propagators in (4.167) are different even at $R = 0$. The ghost regularized propagator on the d-dimensional sphere becomes simply:

$$G_{C,k} = \frac{1}{z + R_k(z) - \frac{R}{d}} . \tag{4.168}$$

To proceed, we insert in the graviton part of the flow equation (4.162) the identity in the space of symmetric rank two tensors in the form $\mathbf{1} = (\mathbf{1} - \mathbf{P}) + \mathbf{P}$. This gives the following relation:

$$\begin{aligned}\partial_t \bar{\Gamma}_k[g] &= \frac{1}{2}\mathrm{Tr}(\mathbf{1} - \mathbf{P})(\partial_t R_k - \eta_h R_k) G_{TF,k} + \frac{1}{2}\mathrm{Tr}\mathbf{P}(\partial_t R_k - \eta_h R_k) G_{T,k} \\ &\quad - \mathrm{Tr}\delta^\mu_\nu (\partial_t R_k - \eta_C R_k) G_{C,k} \\ &= \frac{d^2 + d - 2}{4}\mathrm{Tr}_x(\partial_t R_k - \eta_h R_k) G_{TF,k} + \frac{1}{2}\mathrm{Tr}_x(\partial_t R_k - \eta_h R_k) G_{T,k} \\ &\quad - d\,\mathrm{Tr}_x(\partial_t R_k - \eta_C R_k) G_{C,k}\,. \end{aligned} \tag{4.169}$$

We evaluated the Lorentz traces in (4.169) with the help of (4.159). Note that both the kinematical singularity and the conformal instability are gone due to our choice for the cutoff kernel. Equation (4.169) is the flow equation for the gEAA induced by the Einstein-Hilbert truncation (4.68) with spherical background metric to all orders in the curvature scalar R.

Notice also that, since the spectrum of the covariant Laplacian on the d-dimensional sphere is explicitly known, one could also evaluate the functional traces in (4.169) exactly by summing over the eigenvalues spectrum.

Collecting all terms of zeroth and first order in the scalar curvature that are present on the rhs of (4.169), coming from the expansion of $G_{TF,k}$, $G_{T,k}$ and from the heat kernel expansion, we find the beta functions in terms of the Q–functionals:

$$\begin{aligned}\partial_t \left(\frac{\Lambda_k}{8\pi G_k}\right) &= \frac{1}{(4\pi)^{d/2}}\Bigg\{\frac{d^2+d-2}{4} Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_h R_k) G_{TF,k}\right] + \frac{1}{2} Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_h R_k) G_{T,k}\right] \\ &\quad - d Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_C R_k) G_{C,k}\right]\Bigg\} \\ \partial_t\left(\frac{1}{16\pi G_k}\right) &= \frac{1}{(4\pi)^{d/2}}\Bigg\{\frac{d^2+d-2}{24} Q_{\frac{d}{2}-1}\left[(\partial_t R_k - \eta_h R_k) G_{TF,k}\right] \\ &\quad + \frac{1}{12} Q_{\frac{d}{2}-1}\left[(\partial_t R_k - \eta_h R_k) G_{T,k}\right] - \frac{d}{6} Q_{\frac{d}{2}-1}\left[(\partial_t R_k - \eta_C R_k) G_{C,k}\right] \\ &\quad - \frac{d^2-3d+4}{d(d-1)}\frac{d^2+d-2}{4} Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_h R_k) G^2_{TF,k}\right] \\ &\quad - \frac{d-4}{2d} Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_h R_k) G^2_{T,k}\right] - Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_C R_k) G^2_{C,k}\right]\Bigg\}\end{aligned} \tag{4.170}$$

We can evaluate the beta functions (4.170) using the optimized cutoff shape function. In terms of the dimensionless couplings $\tilde{\Lambda} = k^{-2}\Lambda$, $\tilde{G} = k^{d-2}G$ and $\tilde{m}_h^2 = k^{-2} m_h$ we find the following forms:

$$\begin{aligned}\partial_t\tilde{\Lambda} &= -2\tilde{\Lambda}+\frac{8\pi}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\left\{-4+\frac{d-1}{d}\frac{d+2-\eta_h}{1-2\tilde{\Lambda}+\tilde{m}_h^2}-\right.\\ &\frac{2\left(d^2-3d+4\right)\tilde{\Lambda}}{d^2}\frac{2+d-\eta_h}{\left(1-2\tilde{\Lambda}+\tilde{m}_h^2\right)^2}+\frac{2}{d(d+2)}\frac{2+d-\eta_h}{1-2\tilde{\Lambda}+2\frac{d-1}{d-2}\tilde{m}_h^2}\\ &\left.-\frac{4(d-4)\tilde{\Lambda}}{d^2(d+2)}\frac{2+d-\eta_h}{\left(1-2\tilde{\Lambda}+2\frac{d-1}{d-2}\tilde{m}_h^2\right)^2}-\frac{8\tilde{\Lambda}}{d(d+2)}(d+2-\eta_C)+\frac{4}{2+d}\eta_C\right\}\tilde{G}\\ &+\frac{4\pi}{3(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\left\{-\frac{d^2+d-2}{d}\frac{d-\eta_h}{1-2\tilde{\Lambda}-\tilde{m}_h^2}\right.\\ &\left.+\frac{2}{d}\frac{d-\eta_h}{1-2\tilde{\Lambda}+2\frac{d-1}{d-2}\tilde{m}_h^2}-4(d-\eta_C)\right\}\tilde{\Lambda}\tilde{G} \qquad (4.171)\end{aligned}$$

and

$$\begin{aligned}\partial_t\tilde{G} &= (d-2)\tilde{G}+\frac{16\pi}{(4\pi)^{d/2}d^2\Gamma\left(\frac{d}{2}\right)}\left\{-\frac{4d}{d+2}(d+2-\eta_C)\right.\\ &\left.-\left(d^2-3d+4\right)\frac{d+2-\eta_h}{\left(1-2\Lambda+m_h^2\right)^2}-\frac{2(d-4)}{d+2}\frac{d+2-\eta_h}{\left(1-2\tilde{\Lambda}+2\frac{d-1}{d-2}\tilde{m}_h^2\right)^2}\right\}\tilde{G}^2\\ &+\frac{4\pi}{3(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\left\{\frac{d^2+d-2}{d}\frac{d-\eta_h}{1-2\tilde{\Lambda}+m_h^2}\right.\\ &\left.+\frac{2}{d}\frac{d-\eta_h}{1-2\tilde{\Lambda}+2\frac{d-1}{d-2}\tilde{m}_h^2}-4(d-\eta_C)\right\}\tilde{G}^2\,. \qquad (4.172)\end{aligned}$$

These beta functions represent the generalization of the beta functions for the dimensionless cosmological and Newton's constant in presence of a non-zero Pauli-Fierz mass.

Setting $\tilde{m}_h=0$, we obtain:

$$\begin{aligned}\partial_t\tilde{\Lambda}_k &= -2\tilde{\Lambda}_k+\frac{8\pi}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+2\right)}\left\{\frac{d(d+1)}{4}\frac{d+2-\eta_{h,k}}{1-2\tilde{\Lambda}_k}-d(d+2-\eta_{C,k})\right.\\ &-2\tilde{\Lambda}_k\left[\frac{d(d+1)(d+2)}{48}\frac{d-\eta_{h,k}}{1-2\tilde{\Lambda}_k}-\frac{d(d+2)}{12}(d-\eta_{C,k})\right.\\ &\left.\left.-\frac{d(d-1)}{4}\frac{2+d-\eta_{h,k}}{(1-2\tilde{\Lambda}_k)^2}-(d+2-\eta_{C,k})\right]\right\}\tilde{G}_k\,, \qquad (4.173)\end{aligned}$$

and

$$\begin{aligned}\partial_t\tilde{G}_k &= (d-2)\tilde{G}_k + \frac{16\pi}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+2\right)}\left\{\frac{d(d+1)(d+2)}{48}\frac{d-\eta_{h,k}}{1-2\tilde{\Lambda}_k}\right.\\ &\left.-\frac{d(d+2)}{12}(d-\eta_{C,k})-\frac{d(d-1)}{4}\frac{2+d-\eta_{h,k}}{(1-2\tilde{\Lambda}_k)^2}-(d+2-\eta_{C,k})\right\}\tilde{G}_k^2 \quad (4.174)\end{aligned}$$

These are the beta functions we will study in the following.

Type II

For completeness, let's also report the type II cutoff computation, where we take as cutoff operators $\boldsymbol{\Delta}_2 = \Delta\mathbf{1}+\mathbf{U}$ for the gravitons and $(\Delta_1)^\mu_\nu = \Delta\delta^\mu_\nu - R^\mu_\nu$ for the ghosts. The flow equation for the gEAA, at $m_h = 0$, becomes now simply the following:

$$\partial_t\bar{\Gamma}_k[g] = \frac{1}{2}\mathrm{Tr}_{xL}\mathbf{G}_k(\boldsymbol{\Delta}_2)\partial_t\mathbf{R}_k(\boldsymbol{\Delta}_2) - \mathrm{Tr}_{xL}G_k(\Delta_1)^\mu_\nu\partial_t R_k(\Delta_1)^\nu_\mu\,. \quad (4.175)$$

It is now easy to evaluate the traces in (4.175) using the local heat kernel expansion. Using the following heat kernel coefficients for the cutoff operators we are considering

$$\mathrm{tr}\, b_2(\boldsymbol{\Delta}_2) = \mathrm{tr}\left[\mathbf{1}\frac{R}{6}-\mathbf{U}\right] = -\frac{d(5d-7)}{12}R \qquad \mathrm{tr}\, b_2(\Delta_1) = \mathrm{tr}\left[\delta^\mu_\nu\frac{R}{6}+R^\mu_\nu\right] = \frac{d+6}{d}R\,,$$

we find, to linear order in the curvature, the following expansion:

$$\begin{aligned}\partial_t\bar{\Gamma}_k[g] &= \frac{1}{(4\pi)^{d/2}}\int d^dx\sqrt{g}\left\{\frac{d(d+1)}{4}Q_{\frac{d}{2}}\left[(\partial_t R_k-\eta_h R_k)G_k\right]\right.\\ &-d\,Q_{\frac{d}{2}}\left[(\partial_t R_k-\eta_C R_k)G_{C,k}\right]-\left[\frac{d(5d-7)}{24}Q_{\frac{d}{2}-1}\left[(\partial_t R_k-\eta_h R_k)G_k\right]\right.\\ &\left.\frac{d+6}{6}Q_{\frac{d}{2}-1}\left[(\partial_t R_k-\eta_C R_k)G_{C,k}\right]R\right\}+O\left(\mathcal{R}^2\right)\,. \quad (4.176)\end{aligned}$$

From (4.176) we can extract the following relations that determine the beta functions of Λ and G:

$$\begin{aligned}\partial_t\left(\frac{\Lambda_k}{8\pi G_k}\right) &= \frac{1}{(4\pi)^{d/2}}\left\{\frac{d(d+1)}{4}Q_{\frac{d}{2}}\left[(\partial_t R_k-\eta_h R_k)G_k\right]\right.\\ &\left.-d\,Q_{\frac{d}{2}}\left[(\partial_t R_k-\eta_C R_k)G_{C,k}\right]\right\}\\ \partial_t\left(\frac{1}{16\pi G_k}\right) &= \frac{1}{(4\pi)^{d/2}}\left\{\frac{d(5d-7)}{24}Q_{\frac{d}{2}-1}\left[(\partial_t R_k-\eta_h R_k)G_k\right]\right.\\ &\left.+\frac{d+6}{6}Q_{\frac{d}{2}-1}\left[(\partial_t R_k-\eta_C R_k)G_{C,k}\right]\right\}\,. \quad (4.177)\end{aligned}$$

Inserting in (4.177) the optimized cutoff function we find:

$$\begin{aligned}
\partial_t\left(\frac{\Lambda}{8\pi G}\right) &= \frac{k^d}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+2\right)}\left\{\frac{d(d+1)}{4}\frac{d+2-\eta_h}{1-2\tilde\Lambda}-d(d+2-\eta_C)\right\}\\
\partial_t\left(\frac{1}{16\pi G}\right) &= \frac{k^{d-2}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+1\right)}\left\{\frac{d(5d-7)}{24}\frac{d-\eta_h}{1-2\tilde\Lambda}+\frac{d+6}{6}(d-\eta_C)\right\}. \qquad (4.178)
\end{aligned}$$

So the beta functions for type II cutoff are:

$$\begin{aligned}
\partial_t\tilde\Lambda_k &= -2\tilde\Lambda_k+\frac{8\pi}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+2\right)}\left\{\frac{d(d+1)}{4}\frac{d+2-\eta_{h,k}}{1-2\tilde\Lambda_k}-d(d+2-\eta_{C,k})\right.\\
&\qquad\left.-2\tilde\Lambda_k\left[\frac{d(5d-7)}{24}\frac{d-\eta_{h,k}}{1-2\tilde\Lambda_k}+\frac{d+6}{6}(d-\eta_{C,k})\right]\right\}\tilde G_k \qquad (4.179)
\end{aligned}$$

and

$$\partial_t\tilde G_k=(d-2)\tilde G_k-\frac{16\pi\tilde G_k^2}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+1\right)}\left\{\frac{d(5d-7)}{24}\frac{d-\eta_{h,k}}{1-2\tilde\Lambda_k}+\frac{d+6}{6}(d-\eta_{C,k})\right\}. \qquad (4.180)$$

4.5.6 Closing the flow

To study the flow of the gravitational beta functions of Newton's constant and of the cosmological constant (for $m_{h,k}=0$), we need to know how to deal with the anomalous dimensions. Here we will discuss the two most used ways to do this, the one–loop closure and the "standard improvement" closure.

In $d=4$, from equations (4.173) and (4.174) we find the following set of type I beta functions:

$$\begin{aligned}
\partial_t\tilde\Lambda_k &= -2\tilde\Lambda_k+\frac{\tilde G_k}{12\pi}\left\{\frac{6-8\tilde\Lambda_k-24\tilde\Lambda_k^2-112\tilde\Lambda_k^3}{(1-2\tilde\Lambda_k)^2}\right.\\
&\qquad\left.-\frac{5-11\tilde\Lambda_k-10\tilde\Lambda_k^2}{(1-2\tilde\Lambda_k)^2}\eta_{h,k}+(4+6\tilde\Lambda_k)\eta_{C,k}\right\}\\
\partial_t\tilde G_k &= 2\tilde G_k+\frac{\tilde G_k^2}{12\pi}\left\{-\frac{44-72\tilde\Lambda_k+112\tilde\Lambda_k^2}{(1-2\tilde\Lambda_k)^2}+\frac{1+10\tilde\Lambda_k}{(1-2\tilde\Lambda_k)^2}\eta_{h,k}+6\eta_{C,k}\right\}. \qquad (4.181)
\end{aligned}$$

From equation (4.179) and equation (4.180) we find the following system for the type II beta functions:

$$
\begin{aligned}
\partial_t \tilde{\Lambda}_k &= -2\tilde{\Lambda}_k + \frac{\tilde{G}_k}{12\pi}\left\{\frac{6-44\tilde{\Lambda}_k+80\tilde{\Lambda}_k^2}{1-2\tilde{\Lambda}_k} - \frac{5-13\tilde{\Lambda}_k}{1-2\tilde{\Lambda}_k}\eta_{h,k} + \frac{4+2\tilde{\Lambda}_k-20\tilde{\Lambda}_k^2}{1-2\tilde{\Lambda}_k}\eta_{C,k}\right\} \\
\partial_t \tilde{G}_k &= 2\tilde{G}_k + \frac{\tilde{G}_k^2}{12\pi}\left\{-\frac{92-80\tilde{\Lambda}_k}{1-2\tilde{\Lambda}_k} + \frac{13}{1-2\tilde{\Lambda}_k}\eta_{h,k} + 10\eta_{C,k}\right\}. \qquad (4.182)
\end{aligned}
$$

Though we will apply our method to the type I system, we report here both since the present discussion is general.

The simplest way to close this system is by setting $\eta_{h,k} = \eta_{C,k} = 0$, so that only the first terms inside the parenthesis are retained. In this way one is discarding all the possible non-perturbative information contained in the flow, which reduces to a one–loop approximation. These one-loop beta functions have been analyzed in [83]. A similar one–loop flow is generated by matter interactions and becomes physically significant when the number of matter fields is large [100]. For $\tilde{\Lambda}_k = 0$ one recovers

$$\partial_t \tilde{G}_k = 2\tilde{G}_k - \frac{11}{3\pi}\tilde{G}_k^2 \qquad \text{type I} \qquad (4.183)$$

$$\partial_t \tilde{G}_k = 2\tilde{G}_k - \frac{23}{3\pi}\tilde{G}_k^2 \qquad \text{type II}, \qquad (4.184)$$

which have the same form as the $d = 2 + \epsilon$ computation discussed before. We see that there is a nongaussian fixed point for $\tilde{G}_k$, but this result cannot clearly be trusted in the nonperturbative regime.

So we move to the second way to close the beta function system, which was adopted in all previous studies of these equations. The system (4.181) or (4.182) is here closed by imposing the following relations:

$$Z_{h,k} = \kappa_k^{-1} \qquad Z_{C,k} = 0\,. \qquad (4.185)$$

In this way one retains some of the nonperturbative features contained in the anomalous dimensions. However, this kind of RG improvement of the beta functions is difficult to interpret physically. We will call this procedure the "standard improvement" of the beta functions (4.181) and (4.182). Note also that in this way we are imposing $\eta_{h,*} = 2 - d$ at a fixed–point, independently of the fact if this is Gaussian or not [101]. For type I cutoff, we find the following standard improvement of (4.181):

$$
\begin{aligned}
\partial_t \tilde{\Lambda}_k &= -2\tilde{\Lambda}_k + \frac{1}{6\pi}\frac{(3-4\tilde{\Lambda}_k-12\tilde{\Lambda}_k^2-56\tilde{\Lambda}_k^3)\tilde{G}_k + \frac{1}{12\pi}(107-20\tilde{\Lambda}_k)\tilde{G}_k^2}{(1-2\tilde{\Lambda}_k)^2 - \frac{1}{12\pi}(1+10\tilde{\Lambda}_k)\tilde{G}_k} \\
\partial_t \tilde{G}_k &= 2\tilde{G}_k - \frac{1}{3\pi}\frac{(11-18\tilde{\Lambda}_k+28\tilde{\Lambda}_k^2)\tilde{G}_k^2}{(1-2\tilde{\Lambda}_k)^2 - \frac{1}{12\pi}(1+10\tilde{\Lambda}_k)\tilde{G}_k}\,. \qquad (4.186)
\end{aligned}
$$

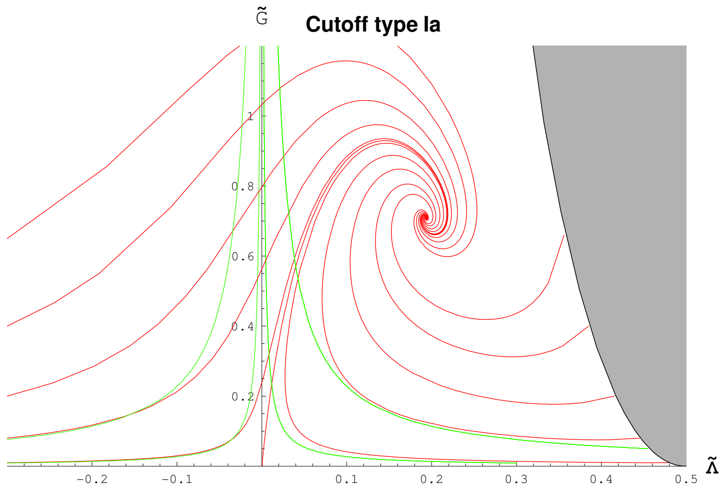


Figure 4.1: The flow for the type I system (4.186). The boundary of the shaded region is a singularity of the beta functions. The curves in light color are "classical" trajectories with constant $\tilde{\Lambda}\tilde{G}$.

The beta functions in (4.186) are exactly those first obtained in [46]. For type II cutoff we find instead the following standard improvement of (4.182):

$$\begin{aligned}
\partial_t \tilde{\Lambda}_k &= -2\tilde{\Lambda}_k + \frac{1}{6\pi}\frac{(3-28\tilde{\Lambda}_k+84\tilde{\Lambda}_k^2-80\tilde{\Lambda}_k^3)\tilde{G}_k+\frac{1}{12\pi}(191-512\tilde{\Lambda}_k)\tilde{G}_k^2}{(1-2\tilde{\Lambda}_k)^2-\frac{13}{12\pi}(1-2\tilde{\Lambda}_k)\tilde{G}_k}\,. \\
\partial_t \tilde{G}_k &= 2\tilde{G}_k - \frac{1}{3\pi}\frac{(23-20\tilde{\Lambda}_k)\tilde{G}_k^2}{1-2\tilde{\Lambda}_k-\frac{13}{12\pi}\tilde{G}_k}\,. \qquad (4.187)
\end{aligned}$$

The system (4.187) has been proposed in [83] together with some variants of it. Note that the beta function (4.186) and (4.187) are rational functions of both $\tilde{G}_k$ and $\tilde{\Lambda}_k$, this can be interpreted as a resummation of an infinite number of perturbative diagrams implemented by the RG improvement (4.185).

The outcome of the numerical integration of (4.186) and (4.187) is shown in Figure 4.1 and Figure 4.2 respectively. The presence of a non-Gaussian fixed point is clearly visible in these pictures. The important point is that the non-Gaussian fixed point is UV attractive in both directions. Notice that the flow near the fixed point is spiraling towards it, i.e. there is a pair of complex conjugated critical exponents with negative real part [13].

This analysis shows that within the truncation we are considering and the standard improvement of the relative beta functions, quantum gravity is asymptotically safe. Actually, we still need to show that the critical surface is finite dimensional. For more details

[13]Here we follow the convention of [83] that a negative value for the critical exponent implies that the relative eigendirection is UV attractive.

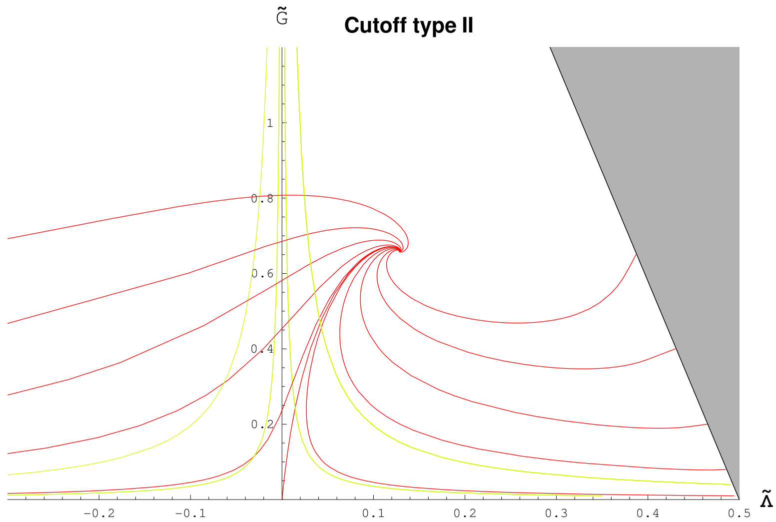


Figure 4.2: The flow for the type II system (4.187). The boundary of the shaded region is a singularity of the beta functions.

see [83].

4.5.7 The third way

The third way to close the beta functions system, that we propose here for the first time, is to separately calculate the anomalous dimensions $\eta_{h,k}$, $\eta_{C,k}$ as functions of $\tilde{\Lambda}_k$, $\tilde{G}_k$ and successively reinsert these back in the beta functions (4.181) or (4.182). In this way we obtain closed beta functions (within the truncation considered) that account for the flow of the wave-function renormalizations $Z_{h,k}$ and $Z_{C,k}$. In doing so we make a step further in considering the flow in the enlarged theory space where the bEAA lives. We are adopting the point of view that Λ_k and G_k are physical couplings while $Z_{h,k}$ and $Z_{C,k}$ are not, but the influence of these last couplings is non-trivial and it is important to account for it.

The calculations of the anomalous dimensions $\eta_{h,k}$ and $\eta_{C,k}$ are done explicitly in the next section using the flow equations for the zero-field proper-vertices of the bEAA, $\gamma_k^{(2,0,0;0)}$ and $\gamma_k^{(0,1,1;0)}$. In doing this computation, we will choose a flat background.

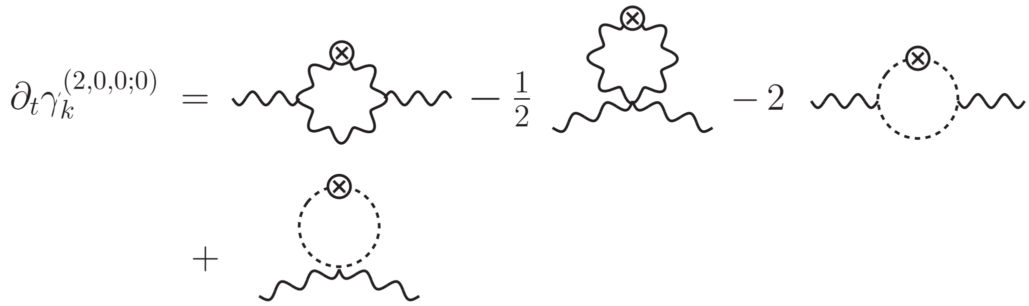


Figure 4.3: Diagrammatic representation of the RG flow equations for the zero–field proper–vertices of the background EAA used to calculate the anomalous dimensions of the fluctuating metric. Wavy lines represent the fluctuating metric, while dotted lines represent the ghosts. The cross–cap stands for a cutoff insertion. Notice the different graphical conventions with respect to Chapter 1

4.6 Anomalous dimensions contributions

4.6.1 Derivation of $\partial_t Z_{h,k}$

In this section we calculate the beta function of the metric fluctuation wave-function renormalization Z_h. We will extract the beta function $\partial_t Z_h$ from the flow equation for the zero-field proper-vertex $\gamma_k^{(2,0,0;0)}$ of the bEAA.

After the multiplet decomposition, and within the truncation we are considering in this chapter, the flow equation becomes as in Figure 4.3. In formulas we have:

$$
\begin{aligned}
[\partial_t \gamma^{(2)}_{p,-p}]^{\mu\nu\,\alpha\beta} \;=\;& \kappa^2 Z_h \int_q (\partial_t R_q - \eta_h R_q)\,[a_{p,q}]^{\mu\nu\,\alpha\beta} - \frac{1}{2}\kappa^2 Z_h \int_q (\partial_t R_q - \eta_h R_q)\,[b_{p,q}]^{\mu\nu\,\alpha\beta} \\
& -2\kappa^2 Z_h \int_q (\partial_t R_q - \eta_C R_q)\,[c_{p,q}]^{\mu\nu\,\alpha\beta} \\
& -2\kappa^2 Z_h \int_q (\partial_t R_q - \eta_C R_q)\,[d_{p,q}]^{\mu\nu\,\alpha\beta}\,. \qquad (4.188)
\end{aligned}
$$

Every diagram in Figure 4.3 is proportional to $\kappa^2 Z_h$ since the metric fluctuation three-vertex comes with a power $\kappa Z_h^{3/2}$, the four-vertex with a power $\kappa^2 Z_h^2$ while the regularized graviton propagators with a factor Z_h^{-1} and graviton cutoff insertion with a factor Z_h. In the ghost diagrams the three-vertex has a power $\kappa Z_h^{1/2} Z_C$, the four-vertex a power $\kappa Z_h Z_C$, the regularized ghost propagator has a power Z_C^{-1} and the ghost cutoff insertion has power Z_C. Also all the volume factors Ω delete each other. The tensor products entering (4.188) are:

$$\begin{aligned}
[a_{p,q}]^{MN} &= [G_q]^{AB}[\gamma^{(3,0,0;0)}_{q,p,-q-p}]^{BMC}[G_{q+p}]^{CD}[\gamma^{(3,0,0;0)}_{q+p,-p,-q}]^{DNE}[G_q]^{EF}[\partial_t R_q]^{FA} &(4.189)\\
[b_{p,q}]^{MN} &= [G_q]^{AB}[\gamma^{(4,0,0;0)}_{q,p,-q-p}]^{BMNC}[G_q]^{CD}[\partial_t R_q]^{DA} &(4.190)\\
[c_{p,q}]^{MN} &= [G^{gh}_q]^{\alpha\beta}[\gamma^{(1,1,1;0)}_{q,p,-q-p}]^{\beta M\gamma}[G^{gh}_q]^{\gamma}[\gamma^{(1,1,1;0)}_{q+p,-p,-q}]^{\gamma N\delta}[G^{gh}_q]^{\delta\alpha}\,, &(4.191)\\
[d_{p,q}]^{MN} &= [G^{gh}_q]^{\alpha\beta}[\gamma^{(2,1,1;0)}_{q,p,-q-p}]^{\beta M\gamma}[G^{gh}_q]^{\gamma\alpha} &(4.192)
\end{aligned}$$

where the vertices entering (4.189) are:

$$\begin{aligned}
\gamma^{(3,0,0;0)} &= 2\Lambda I^{(3)}_0[\delta] - I^{(3)}_1[\delta]\\
\gamma^{(4,0,0;0)} &= 2\Lambda I^{(4)}_0[\delta] - I^{(4)}_1[\delta]\\
\gamma^{(1,1,1;0)} &= \kappa Z_C S^{(1,1,1;0)}_{gh}[0,0,0;\delta]\\
\gamma^{(2,1,1;0)} &= \kappa Z_C S^{(2,1,1;0)}_{gh}[0,0,0;\delta]\,. \qquad (4.193)
\end{aligned}$$

The momentum integrals in (4.188) can be written in spherical coordinates:

$$\int_q \to \frac{S_{d-1}}{(2\pi)^d}\int_0^\infty dq\, q^{d-1}\int_{-1}^1 dx\,\left(1-x^2\right)^{\frac{d-3}{2}}\,, \qquad (4.194)$$

where $S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ is the volume of the d-dimensional sphere and $x = \cos\theta$ with θ the angle between p and q. We can also shift to the variable $z = q^2$ so that

$$\int_0^\infty dq\, q^{d-1} \to \frac{1}{2}\int_0^\infty dz\, z^{\frac{d}{2}-1}\,.$$

We will give the results only in the gauge $\alpha = \beta = 1$.

The different terms are projected with $\mathbf{P}_2$:

$$P_2^{\mu\nu\,\alpha\beta}[a_{p,q}]_{\mu\nu\,\alpha\beta} \qquad (4.195)$$

$$P_2^{\mu\nu\,\alpha\beta}[b_{p,q}]_{\mu\nu\,\alpha\beta} \qquad (4.196)$$

$$P_2^{\mu\nu\,\alpha\beta}[c_{p,q}]_{\mu\nu\,\alpha\beta} \qquad (4.197)$$

$$P_2^{\mu\nu\,\alpha\beta}[d_{p,q}]_{\mu\nu\,\alpha\beta} \qquad (4.198)$$

We will skip all the (lengthy) intermediate contractions and integrations, and just report the results. We just note in passing that, as can be explicitly checked, diagram (d) does not contribute, and so we can drop it.

Once we insert equations (4.195), (4.196) and (4.197) back in (4.188) we obtain within our truncation, the explicit flow of the zero-field proper-vertex $\gamma^{(2,0,0;0)}_{p,-p}$, to all orders in the external momenta p.

From the terms proportional to p^2 in (4.188) we can extract the beta function of the wave-function renormalization of the fluctuation metric. If we define

$$\partial_t Z_h \equiv \beta_{Z_h}\left(\kappa, \Lambda, Z_h, Z_C\right) , \tag{4.199}$$

we can write the anomalous dimension of the fluctuation metric as:

$$\eta_h\left(\kappa, Z_h, Z_C\right) = -\partial_t \log Z_h = -\beta_{Z_h}/Z_h \,. \tag{4.200}$$

When we write everything in terms of Q-functionals we finally find:

$$\begin{aligned}
\eta_h \;=\; & \frac{\kappa^2}{2(4\pi)^{d/2}}\frac{(d+1)}{16\left(d^2-d-2\right)}[8\Lambda Q_{\frac{d}{2}+1}\left[\left(\partial_t R_k-\eta_h R_k\right)G_T^2 G'_{\mathrm{TF}}\right](d-4)^2 \\
& -\frac{8\Lambda^2 Q_{\frac{d}{2}+1}\left[\left(\partial_t R_k-\eta_h R_k\right)G_{\mathrm{TF}}^2 G''_T\right](d-4)^2}{d} \\
& -\frac{8\Lambda^2 Q_{\frac{d}{2}+1}\left[\left(\partial_t R_k-\eta_h R_k\right)G_T^2 G''_{\mathrm{TF}}\right](d-4)^2}{d} \\
& -\frac{8\Lambda^2 Q_{\frac{d}{2}}\left[\left(\partial_t R_k-\eta_h R_k\right)G_{\mathrm{TF}}^2 G'_T\right](d-4)^2}{d} \\
& -\frac{8\Lambda^2 Q_{\frac{d}{2}}\left[\left(\partial_t R_k-\eta_h R_k\right)G_T^2 G'_{\mathrm{TF}}\right](d-4)^2}{d} \\
& +8(d-2)\Lambda Q_{\frac{d}{2}+1}\left[\left(\partial_t R_k-\eta_h R_k\right)G_{\mathrm{TF}}^2 G'_T\right](d-4) \\
& +8(d-3)\Lambda Q_{\frac{d}{2}+2}\left[\left(\partial_t R_k-\eta_h R_k\right)G_{\mathrm{TF}}^2 G''_T\right](d-4) \\
& +8(d-3)\Lambda Q_{\frac{d}{2}+2}\left[\left(\partial_t R_k-\eta_h R_k\right)G_T^2 G''_{\mathrm{TF}}\right](d-4) \\
& +\frac{8(d-6)\Lambda Q_{\frac{d}{2}}\left[\left(\partial_t R_k-\eta_h R_k\right)G_T^2 G_{\mathrm{TF}}\right](d-4)}{d} \\
& +\frac{16(d-3)\Lambda Q_{\frac{d}{2}}\left[\left(\partial_t R_k-\eta_h R_k\right)G_T G_{\mathrm{TF}}^2\right](d-4)}{d} \\
& +\frac{(d((41-4d)d-116)+76)Q_{\frac{d}{2}+1}\left[\left(\partial_t R_k-\eta_h R_k\right)G_T^2 G_{\mathrm{TF}}\right]}{d} \\
& -\frac{(d-2)(d(8d-37)+50)Q_{\frac{d}{2}+1}\left[\left(\partial_t R_k-\eta_h R_k\right)G_T G_{\mathrm{TF}}^2\right]}{d}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{4(d-2)^2(d(d+2)-11)Q_{\frac{d}{2}+1}\left[(\partial_t R_k-\eta_h R_k)\,G_{\rm TF}^3\right]}{d}\\
&-8(d-3)(d-2)(d+4)\Lambda Q_{\frac{d}{2}+1}\left[(\partial_t R_k-\eta_h R_k)\,G_{\rm TF}^2G'_{\rm TF}\right]\\
&+\frac{16(d-2)^2(d+4)\Lambda^2 Q_{\frac{d}{2}+1}\left[(\partial_t R_k-\eta_h R_k)\,G_{\rm TF}^2G''_{\rm TF}\right]}{d}\\
&+\frac{8(d-3)^2(d-2)Q_{\frac{d}{2}+2}\left[(\partial_t R_k-\eta_h R_k)\,G_T^2G'_T\right]}{d^2}\\
&-\frac{2(d-3)(d-2)(d+3)(d(d+2)-4)Q_{\frac{d}{2}+2}\left[(\partial_t R_k-\eta_h R_k)\,G_{\rm TF}^2G'_T\right]}{d^2}\\
&-\frac{2\left(d\left((d-5)(d+1)d^2+96\right)-72\right)Q_{\frac{d}{2}+2}\left[(\partial_t R_k-\eta_h R_k)\,G_T^2G'_{\rm TF}\right]}{d^2}\\
&+\frac{(d-3)(d-2)(d(d(d+3)(d+8)+8)-24)Q_{\frac{d}{2}+2}\left[(\partial_t R_k-\eta_h R_k)\,G_{\rm TF}^2G'_{\rm TF}\right]}{d^2}\\
&-8(d-3)(d-2)(d+4)\Lambda Q_{\frac{d}{2}+2}\left[(\partial_t R_k-\eta_h R_k)\,G_{\rm TF}^2G''_{\rm TF}\right]\\
&-8(d-2)(6Q_{\frac{d}{2}+1}\left[(\partial_t R_k-\eta_C R_k)\,G_C^3\right]\\
&+(d+2)(d+4)\left(Q_{\frac{d}{2}+2}\left[(\partial_t R_k-\eta_C R_k)\,G_C^2G'_C\right]+Q_{\frac{d}{2}+3}\left[(\partial_t R_k-\eta_C R_k)\,G_C^2G''_C\right]\right)\\
&+\frac{8(d-3)^2(d-2)Q_{\frac{d}{2}+3}\left[(\partial_t R_k-\eta_h R_k)\,G_T^2G''_T\right]}{d^2}\\
&-\frac{2(d-3)^2\left(d(d+2)^2-8\right)Q_{\frac{d}{2}+3}\left[(\partial_t R_k-\eta_h R_k)\,G_{\rm TF}^2G''_T\right]}{d^2}\\
&-\frac{2(d-3)^2\left(d(d+2)^2-8\right)Q_{\frac{d}{2}+3}\left[(\partial_t R_k-\eta_h R_k)\,G_T^2G''_{\rm TF}\right]}{d^2}\\
&+\frac{(d-3)(d-2)(d(d(d+3)(d+8)+8)-24)Q_{\frac{d}{2}+3}\left[(\partial_t R_k-\eta_h R_k)\,G_{\rm TF}^2G''_{\rm TF}\right]}{d^2}\\
&+\frac{2(d-8)(d-6)Q_{\frac{d}{2}}\left(G_T^2\right)+(d-2)(d+2)((d-13)d+24)Q_{\frac{d}{2}}\left[(\partial_t R_k-\eta_h R_k)\,G_{\rm TF}^2\right]}{d}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{16(d-3)(d-2)(d+4)\Lambda Q_{\frac{d}{2}}\left[(\partial_t R_k-\eta_h R_k)\,G_{\rm TF}^3\right]}{d}\\
&+\frac{16(d-2)^2(d+4)\Lambda^2 Q_{\frac{d}{2}}\left[(\partial_t R_k-\eta_h R_k)\,G_{\rm TF}^2G'_{\rm TF}\right]}{d}
\end{aligned}
\tag{4.201}
$$

Equation (4.201) gives the anomalous dimension of the fluctuation metric in d-dimensions, for an arbitrary cutoff shape function in the gauge $\alpha=\beta=1$. It is possible to calculate the Q-functionals in (4.201) analytically if we employ the optimized cutoff shape function.

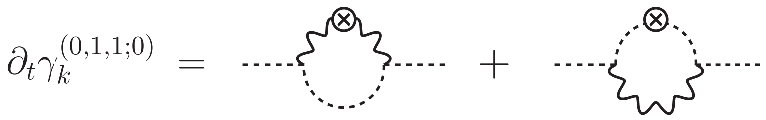


Figure 4.4: Diagrammatic representation of the RG flow equations for the zero–field proper–vertices of the background EAA used to calculate the anomalous dimensions of the ghost fields. Wavy lines represent the fluctuating metric, while dotted lines represent the ghosts. The cross–cap stands for a cutoff insertion.

4.6.2 Derivation of $\partial_t Z_{C,k}$

The flow equation for the zero-field proper-vertex $\gamma_k^{(0,1,1;0)}$ is decomposed as in Figure 4.4. The diagrams in Figure 4.4 involve only one type of vertex: the ghost-ghost-graviton vertex. Note that the cutoff kernel has a different structure in the two sectors and involves different wave-function renormalization. Thus the two diagrams in Figure 4.4 give different contributions.

As before we evaluate the tensor contractions, we Taylor expand in powers of the external momentum, we retain the term of order p^2 and we do the angular integrals. In terms of Q-functionals, setting $m_h = 0$, we find the following expression for the anomalous dimension of the ghost fields:

$$
\begin{aligned}
\eta_C \;=\; & \frac{\kappa^2}{(4\pi)^{d/2}}\Big[\frac{\left(d^2-4\right)}{4(d-2)d^2}\Big(dQ_{\frac{d}{2}+2}\left[\left(\partial_t R_k-\eta_h R_k\right) G_T^2 G_C'\right] \\
& +d(d-1)Q_{\frac{d}{2}+2}\left[\left(\partial_t R_k-\eta_h R_k\right) G_{\mathrm{TF}}^2 G_C'\right] \\
& +(d-1)(4d-5)Q_{\frac{d}{2}+1}\left[\left(\partial_t R_k-\eta_h R_k\right) G_C G_{\mathrm{TF}}^2\right]\Big)+ \\
& \left(d^2-4\right)\Big((d-1)(3d-5)Q_{\frac{d}{2}+1}\left[\left(\partial_t R_k-\eta_C R_k\right) G_C^2 G_{\mathrm{TF}}\right] \\
& -dQ_{\frac{d}{2}+2}\left[\left(\partial_t R_k-\eta_C R_k\right) G_C^2 G_T'\right] \\
& -d(d-1)Q_{\frac{d}{2}+2}\left[\left(\partial_t R_k-\eta_C R_k\right) G_C^2 G_{\mathrm{TF}}'\right]\Big) \\
& +(20-d((d-7)d+28))Q_{\frac{d}{2}+1}\left[\left(\partial_t R_k-\eta_C R_k\right) G_C^2 G_T\right] \\
& +(d(7d-32)+20)Q_{\frac{d}{2}+1}\left[\left(\partial_t R_k-\eta_h R_k\right) G_C G_T^2\right]\Big]
\end{aligned}
\tag{4.202}
$$

A similar form for the anomalous dimension of the ghost fields has been found, using slightly different implementations of the cutoff, in [74, 102].

4.7 New flow portrait

We will here discuss the results we find when we use our consistent closure technique for the gravitational beta functions.

First of all, notice that this line of reasoning has partially been implemented in [74, 102] where the closure (4.8) was extended by separately calculating the ghost anomalous dimension in place of setting it to zero. The flow so obtained is similar to the standard one; in particular, it is still strongly spiraling around the non–Gaussian fixed–point. The authors [74] used a generalized heat kernel technique [103] to determine the anomalous dimension of the ghost fields; since they use a type Ia cutoff we can compare their results directly with ours. Another truncation of the ghost sector has been considered in [104]. The anomalous dimensions (4.10) are dependent on the (possibly scale dependent) gauge–fixing parameters; a first study of this dependence has been made in [102].

Using the optimized cutoff we find as anticipated a linear system for the anomalous dimensions of the graviton and ghost which can be solved to give $\eta_{h,k}$ and $\eta_{C,k}$ as functions of the (dimensionless) Newton's and cosmological constant. Its explicit form is very long and not very enlightning, so we will not write it down in full generality. However the solution in $d = 4$ can be written in few lines:

$$\begin{aligned}\eta_{h,k} \;=\; & -[6\tilde{G}_k(\tilde{G}_k(8\tilde{\Lambda}_k(16\tilde{\Lambda}_k(\tilde{\Lambda}_k(52\tilde{\Lambda}_k-129)+109)-651)+621)\\ & +96\pi(2\tilde{\Lambda}_k-1)(8\tilde{\Lambda}_k(8\tilde{\Lambda}_k(4(\tilde{\Lambda}_k-2)\tilde{\Lambda}_k+5)-11)+1))]/[\\ & [(2\tilde{\Lambda}_k-1)(\tilde{G}_k^2(1200(\tilde{\Lambda}_k-1)\tilde{\Lambda}_k+157)\\ & +48\pi\tilde{G}_k(2\tilde{\Lambda}_k-1)(52(\tilde{\Lambda}_k-1)\tilde{\Lambda}_k-9)\\ & +4608\pi^2(1-2\tilde{\Lambda}_k)^4)]\end{aligned} \tag{4.203}$$

and

$$\begin{aligned}\eta_{C,k} \;=\; & [2\tilde{G}_k(\tilde{G}_k(-16\tilde{\Lambda}_k(\tilde{\Lambda}_k(1200(\tilde{\Lambda}_k-2)\tilde{\Lambda}_k+1643)-693)\\ & -1747)+384\pi(13\tilde{\Lambda}_k-19)(1-2\tilde{\Lambda}_k)^4)]/[\\ & [(1-2\tilde{\Lambda}_k)^2(\tilde{G}_k^2(1200(\tilde{\Lambda}_k-1)\tilde{\Lambda}_k+157)+\\ & 48\pi\tilde{G}_k(2\tilde{\Lambda}_k-1)(52(\tilde{\Lambda}_k-1)\tilde{\Lambda}_k-9)\\ & +4608\pi^2(1-2\tilde{\Lambda}_k)^4)]\end{aligned} \tag{4.204}$$

We can appreciate the fact that these anomalous dimensions have a highly nontrivial structure. In fact, as the study of $O(N)$ models made clear in the first chapter, in the lowest order truncation, all the nonperturbative improvement of the method is essentially contained in the anomalous dimensions contribution. In this case, we see that they are

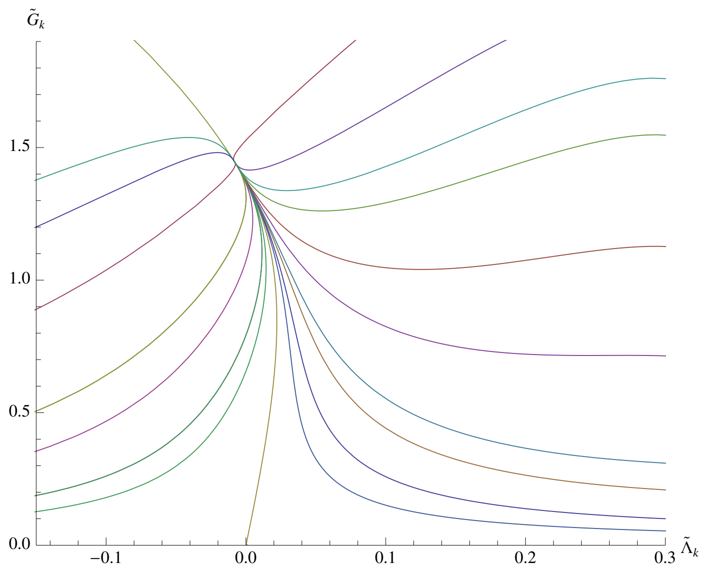


Figure 4.5: RG flow in $d = 4$ in the $(\tilde{\Lambda}_k, \tilde{G}_k)$ plane for the closure of the beta functions obtained by inserting back in (4.6) the independently computed anomalous dimensions $\eta_{h,k}$ and $\eta_{C,k}$.

rational functions of the gravitational couplings, meaning they encode an infinite sum of contributions. They thus carry nonperturbative information, which in previous truncations was discarded (or not fully accounted), and nothing guarantees that they won't in principle spoil the flow portrait previously found. The remarkable thing is that, as we will now show, despite this nontrivial modification, a nonGaussian Fixed Point is still found with the same properties as before. Thus, this can be seen as another piece of evidence in favour of Asymptotic Safety.

If we plug these two functions of the gravitational couplings inside the beta functions found before, we have a closed system for $\tilde{\Lambda}_k$ and $\tilde{G}_k$.

The result of the numerical integration of these beta functions in $d = 4$ is plotted in Figure 4.5 for type Ia cutoff. Note that, despite these new beta functions differ non–trivially from the one–loop (4.7) and standard RG improved (4.9), we still find a UV attractive non–Gaussian fixed–point, but now the critical exponents are real [14]. This is clearly reflected in the fact that the flow next to the non–Gaussian fixed–point is now not spiraling as in the previous truncations. The fixed–point values of the dimensionless couplings and the critical exponents are also given in Table 1. Our fixed–point value of the dimensionless

[14]The critical exponents are calculated as always, from the eigenvalues of the stability matrix

$$\frac{\partial\left(\beta_{\tilde{G}}, \beta_{\tilde{\Lambda}}\right)}{\partial\left(\tilde{G}, \tilde{\Lambda}\right)} \tag{4.205}$$

	$\tilde{\Lambda}_*$	$\tilde{G}_*$	$\theta' \pm i\theta''$	$\tilde{\Lambda}_*\tilde{G}_*$	$\eta_{h,*}$	$\eta_{C,*}$
One–loop	0.121	1.172	$-1.868 \pm 1.398i$	0.142	0	0
[70]	0.193	0.707	$-1.475 \pm 3.043i$	0.137	-2	0
[74]	0.135	0.859	$-1.774 \pm 1.935i$	0.116	-2	-1.8
This work	-0.008	1.446	$-3.323, -1.954$	-0.012	0.07	-1.50

Table 4.1: Fixed–points and critical exponents for the various closures of the beta functions of Λ_k and G_k, in $d = 4$.

cosmological constant is almost zero. The inclusion of matter contributions will change the value $\tilde{\Lambda}_*$ and we need to include these to understand better its UV value. Real critical exponents are also suggested by the analysis of [105].

If we insert the fixed–point values for the cosmological constant and for Newton's constant in (4.13) we determine the fixed–point values for the anomalous dimensions of the fluctuating metric and ghost fields η_{h*} and η_{C*}. The numerical values we find are reported in Table 1, together with previous estimates. The anomalous dimension of $h_{\mu\nu}$ results positive, while the anomalous dimension of the ghost fields is negative, as also found in [74, 102].

4.8 Discussion and Conclusions

In this chapter, after a brief review of the functional RG in Quantum Gravity, we have shown how to account for the non–trivial influence that the anomalous dimensions $\eta_{h,k}$ and $\eta_{C,k}$ of the fluctuating fields have on the RG flow of the cosmological and Newton's constants. We have derived new RG improved beta functions for these couplings which still exhibit a UV attractive non–Gaussian fixed–point, reinforcing the Asymptotic Safety scenario in quantum gravity.

The closure method proposed here is general and can be applied to the beta functions of the higher derivative gravity couplings [109], to the beta functions obtained using the first–order formalism [110] and even to the beta functions present in non–local truncations of the gravitational EAA [96]. It can be extended to applications of the EAA to renormalization of other theories with local symmetries, as non–linear sigma models [111], the theory of membranes [112] or Horava–Liftshitz theories of gravity [113].

A complementary strategy to close the flow of the background EAA has been developed in [106], where bi–metric truncations are constructed using invariants made with both $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$. The problem has also been studied in [107] where the Vilkovisky–DeWitt formalism was used to construct the EAA for quantum gravity. Finally, an analysis similar to ours has been performed in [108] where the flow of the fluctuating metric (zero–field) two–point function in Landau gauge has been used to extract the beta functions of the cosmological and Newton's constants; these exhibiting a non–Gaussian fixed point with real critical exponents.

Chapter 5

Conclusions

In this thesis, we have seen how the functional Renormalization Group can be applied in three main settings.

First, we have seen its uses in flat spacetime Quantum Field Theories. We have put particular emphasis on the scaling solutions, which are in one-to-one correspondence with universality classes. Due to powerful numerical methods able to treat the flow of the effective potential in its full functional form, the universality classes of a given theory can be found and classified. In particular, using the mehod of scaling solutions, we have been able to perform a nonperturbative analysis of the theory space of $O(N)$–scalar models, obtaining a classification of their universality classes. We have found perfect agreement with the MWHC theorem and in fact enlightened some peculiar aspects of it. In the SAW correspondence, we also were able to find multicritical universality classes, which, to the best of our knowledge, are new and deserve further study.

We then moved to the application in the context of Weyl invariant theories, and were able to give a nonperturbative proof that a Weyl invariant quantization is always possible with the aid of a dilaton, regardless of the matter interactions or renormalizability properties of the theory. We have also shown that, despite the Weyl invariance of the resulting Effective Action, the trace anomaly is still present, with all its physical consequences. The analysis of Weyl invariance in the fRG context is particularly relevant given the fact that one hopes, in analogy with critical phenomena in the flat space case, that when gravity is dynamical a fixed point theory should be Weyl invariant. At present there is no proof of this fact, but whether this is true or not, we believe the formalism developed here could be a starting point to address such issues.

In the last part, we explored a new way to consistently close the RG equations in quantum gravity, via the independent evaluation of the contribution of the anomalous dimensions of the ghosts and graviton fluctuations to the flow of the gravitational couplings, namely the dimensionless Newton's and cosmological constant, in the Einstein–Hilbert truncation. As we have already witnessed in the first Chapter, in a somewhat simpler context, the anomalous (scaling) dimensions potentially encode nonperturbative information, which the functional RG is indeed able to handle. In gravity, despite the highly nontrivial correction induced by these anomalous dimensions, a nongaussian fixed point with two attractive directions was found, providing another piece of evidence in favor of the Asymptotic Safety scenario for quantum gravity. Also, the critical exponents turned out to be real, as is expected in a realistic case. We think this is an indication that the present method represents a step forward in capturing a more physical and reliable flow. Future applications to higher derivative gravity and more complicated scenarios will tell.

Appendix A

Functional methods in QFT

In the path–integral approach to QFT one defines the partition function, or vacuum–to–vacuum amplitude, as

$$Z = \int \mathcal{D}\phi e^{-S[\phi]} \,, \tag{1}$$

where ϕ represents a generic field. However, in QFT the physical observable is not the partition function, but rather are the S–matrix elements, which in turn are computed in terms of Green's functions, or n–point functions, or correlators, of the theory

$$G(x_1, ..., x_n) = \langle \phi(x_1)...\phi(x_n) \rangle \ , \tag{2}$$

in which the expectation value is defined using the partition function as follows

$$\langle \mathcal{O}[\phi] \rangle = \frac{1}{Z} \int \mathcal{D}\phi \mathcal{O}[\phi] e^{-S[\phi]} \,. \tag{3}$$

The functional approach then starts with the introduction of an external source J, defining a modified partition function

$$Z[J] = \int \mathcal{D}\phi e^{-S[\phi] + \int J\phi} \,. \tag{4}$$

In this way the correlators can be written as functional derivatives with respect to the source:

$$\langle \phi(x_1)...\phi(x_n) \rangle = \frac{1}{Z[J]} \frac{\delta}{\delta J(x_1)} ... \frac{\delta}{\delta J(x_n)} Z[J] \bigg|_{J=0} \,. \tag{5}$$

The generator of connected correlation functions is defined as

$$W[J] = \log Z[J] \tag{6}$$

and satisfies

$$\frac{\delta W[J]}{\delta J(x)} = \langle\phi\rangle_J \ . \tag{7}$$

Here, $\langle\phi\rangle_J$ is the vacuum expectation value of the field in presence of the source.

We can define the current J_φ as the current with which the vacuum expectation value of the field ϕ is equal to φ:

$$\langle\phi\rangle_{J_\varphi} = \varphi\,. \tag{8}$$

It can be found in principle as a solution to the equation (7).

The Effective Action (EA) is introduced as a Legendre transform of the functional $W[J]$:

$$\Gamma[\varphi] = \int J_\varphi\varphi - W[J_\varphi]\,. \tag{9}$$

One can see that

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = J_\varphi(x) \tag{10}$$

and so setting $J = 0$ we find

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = 0\,. \tag{11}$$

This can be seen as the quantum generalization of the least action principle: the equations of motion derived from it contain the quantum corrections to the classical equations of motion.

Using now (10) together with the definitions of Γ, W and Z, we can find the integro–differential equation satisfied by the EA:

$$e^{-\Gamma[\varphi]} = \int \mathcal{D}\chi \exp\left\{-S\left[\varphi+\chi\right] + \int \chi\frac{\delta\Gamma\left[\varphi\right]}{\delta\varphi}\right\}\,, \tag{12}$$

where we defined the fluctuation χ through $\phi = \varphi + \chi$ and $\langle\chi\rangle = 0$.

One can further show that

$$\frac{\delta^2 W\left[J\right]}{\delta J\left(x\right)\delta J\left(y\right)} = \left(\frac{\delta^2\Gamma\left[\varphi\right]}{\delta\varphi\left(x\right)\delta\varphi\left(y\right)}\right)^{-1}\,. \tag{13}$$

Appendix B

Heat Kernel Techniques

We briefly review the Heat Kernel techniques used to compute functional traces. We refer to [83] for all the references and further details.

The r.h.s. of the ERGE is the trace of a function of a differential operator. To illustrate the methods employed to evaluate such traces, we begin by considering the covariant Laplacian in a metric g, $-\nabla^2$. If the fields carry a representation of a gauge group G and are coupled to gauge fields for G, the covariant derivative ∇ contains also these fields. We will denote $\Delta = -\nabla^2 \mathbf{1} + \mathbf{E}$ a second order differential operator. $\mathbf{E}$ is a linear map acting on the spacetime and internal indices of the fields. In our applications to de Sitter space it will have the form $\mathbf{E} = qR\,\mathbf{1}$ where $\mathbf{1}$ is the identity in the space of the fields and q is a real number.

The trace of a function W of the operator Δ can be written as

$$\mathrm{Tr}W(\Delta) = \sum_i W(\lambda_i) \tag{14}$$

where λ_i are the eigenvalues of Δ. Introducing the Laplace anti-transform $\tilde{W}(s)$

$$W(z) = \int_0^\infty ds\, e^{-zs}\tilde{W}(s) \tag{15}$$

we can rewrite (14) as

$$\mathrm{Tr}W(\Delta) = \int_0^\infty ds \mathrm{Tr}K(s)\tilde{W}(s) \tag{16}$$

where $\mathrm{Tr}K(s) = \sum_i e^{-s\lambda_i}$ is the trace of the heat kernel of Δ. We assume that there are no negative and zero eigenvalues; if present, these will have to be dealt with separately.

The trace of the heat kernel of Δ has the well-known asymptotic expansion for $s \to 0$:

$$\mathrm{Tr}\left(e^{-s\Delta}\right) = \frac{1}{(4\pi)^{\frac{d}{2}}}\left[B_0(\Delta)\, s^{-\frac{d}{2}} + B_2(\Delta)\, s^{-\frac{d}{2}+1} + \ldots + B_d(\Delta) + B_{d+2}(\Delta)\, s + \ldots\right] \quad (17)$$

where $B_n = \int \mathrm{d}^d x\sqrt{g}\mathrm{tr}\mathbf{b}_n$ and $\mathbf{b}_n$ are linear combinations of curvature tensors and their covariant derivatives containing $2n$ derivatives of the metric.

Assuming that $[\Delta, \mathbf{E}] = 0$, the heat kernel coefficients of Δ are related to those of $-\nabla^2$ by

$$\mathrm{Tr}e^{-s(-\nabla^2+\mathbf{E})} = \frac{1}{(4\pi)^{\frac{d}{2}}}\sum_{k,\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\int \mathrm{d}^d x\sqrt{g}\,\mathrm{tr}\,\mathbf{b}_k(\Delta)\mathbf{E}^\ell s^{k+\ell-2}. \quad (18)$$

The first six coefficients have the following form:

$$\mathbf{b}_0 = \mathbf{1} \quad (19)$$

$$\mathbf{b}_2 = \frac{R}{6}\mathbf{1} - \mathbf{E} \quad (20)$$

$$\begin{aligned}\mathbf{b}_4 &= \frac{1}{180}\left(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - R^{\mu\nu}R_{\mu\nu} + \frac{5}{2}R^2 + 6\nabla^2 R\right)\mathbf{1}\\ &\quad + \frac{1}{12}\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}^{\mu\nu} - \frac{1}{6}R\mathbf{E} + \frac{1}{2}\mathbf{E}^2 - \frac{1}{6}\nabla^2\mathbf{E}\end{aligned} \quad (21)$$

$$\begin{aligned}\mathbf{b}_6 &= \frac{1}{180}R\mathbf{1}\left(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - R^{\mu\nu}R_{\mu\nu} + \frac{5}{6}R^2 + \frac{7}{2}\nabla^2 R\right)\\ &\quad + \frac{R}{2}\mathbf{E}^2 + \mathbf{E}^3 + \frac{1}{30}\mathbf{E}\left(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - R^{\mu\nu}R_{\mu\nu} + \frac{5}{2}R^2 + 6\nabla^2 R\right)\\ &\quad + \frac{R}{12}\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}^{\mu\nu} + \frac{1}{2}\mathbf{E}\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}^{\mu\nu} + \frac{1}{2}\mathbf{E}\nabla^2\mathbf{E} - \frac{1}{2}\mathbf{J}_\mu\mathbf{J}^\mu\\ &\quad + \frac{1}{30}\left(2\mathbf{\Omega}^\mu{}_\nu\mathbf{\Omega}^\nu{}_\alpha\mathbf{\Omega}^\alpha{}_\mu - 2R^\mu{}_\nu\mathbf{\Omega}_{\mu\alpha}\mathbf{\Omega}^{\alpha\nu} + R^{\mu\nu\alpha\beta}\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}_{\alpha\beta}\right)\\ &\quad + \mathbf{1}\left[-\frac{1}{630}R\nabla^2 R + \frac{1}{140}R_{\mu\nu}\nabla^2 R^{\mu\nu} + \frac{1}{7560}\left(-64R^\mu{}_\nu R^\nu{}_\alpha R^\alpha{}_\mu + 48R^{\mu\nu}R_{\alpha\beta}R^\alpha{}_\mu{}^\beta{}_\nu\right.\right.\\ &\quad \left.\left. + 6R_{\mu\nu}R^\mu{}_{\rho\alpha\beta}R^{\nu\rho\alpha\beta} + 17R_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta}{}^{\rho\sigma}R_{\rho\sigma}{}^{\mu\nu} - 28R^\mu{}_\alpha{}^\nu{}_\beta R^\alpha{}_\rho{}^\beta{}_\sigma R^\rho{}_\mu{}^\sigma{}_\nu\right)\right]\end{aligned} \quad (22)$$

where $\mathbf{\Omega}^{\mu\nu} = [\nabla^\mu, \nabla^\nu]$ is the curvature of the connection acting on a set of fields in a particular representation of the Lorentz and internal gauge group and $\mathbf{J}_\mu = \nabla_\alpha\mathbf{\Omega}^\alpha{}_\mu$. We neglect total derivative terms.

Let us return to equation (16). If we are interested in the local behavior of the theory (*i.e.* the behavior at length scales much smaller than the typical curvature radius) we can use the asymptotic expansion (17) and then evaluate each integral separately. Then we get

$$\mathrm{Tr}W(\Delta) = \frac{1}{(4\pi)^{\frac{d}{2}}}\big[Q_{\frac{d}{2}}(W)B_0(\Delta) + Q_{\frac{d}{2}-1}(W)B_2(\Delta) + \dots$$
$$+ Q_0(W)B_d(\Delta) + Q_{-1}(W)B_{d+2}(\Delta) + \dots\big] \ , \tag{23}$$

where

$$Q_n(W) = \int_0^\infty ds s^{-n}\tilde{W}(s) \ . \tag{24}$$

In the case of four dimensional field theories, it is enough to consider integer values of n. However, in odd dimensions half-integer values of n are needed and we are also interested in the analytic continuation of results to arbitrary real dimensions. We will therefore need expressions for (24) that hold for any real n.

If we denote $W^{(i)}$ the i-th derivative of W, we have from (15)

$$W^{(i)}(z) = (-1)^i \int_0^\infty ds\, s^i e^{-zs}\tilde{W}(s) \ . \tag{25}$$

This formula can be extended to the case when i is a real number to define a notion of "noninteger derivative". From this it follows that for any real i

$$Q_n(W^{(i)}) = (-1)^i Q_{n-i}(W) \ . \tag{26}$$

For n a positive integer one can use the definition of the Gamma function to rewrite (24) as a Mellin transform:

$$Q_n(W) = \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}W(z) \tag{27}$$

while for m a positive integer or $m = 0$

$$Q_{-m}(W) = (-1)^m W^{(m)}(0) \ . \tag{28}$$

More generally, for n a positive real number we can define $Q_n(W)$ by equation (27), while for n real and negative we can choose a positive integer k such that $n + k > 0$; then we can write the general formula

$$Q_n(W) = \frac{(-1)^k}{\Gamma(n+k)}\int_0^\infty dz\, z^{n+k-1}W^{(k)}(z) \ . \tag{29}$$

This reduces to the two cases mentioned above when n is integer. In the case when n is a negative half integer $n = -\frac{2m+1}{2}$ we will set $k = m + 1$ so that we have

$$Q_{-\frac{2m+1}{2}}(W) = \frac{(-1)^{m+1}}{\sqrt{\pi}}\int_0^\infty dz\, z^{-1/2} f^{(m+1)}(z) \tag{30}$$

In gravitational applications, there are two natural choices of cutoff function: type I cutoff is a function $R_k(-\nabla^2)$ such that the modified inverse propagator is $P_k(-\nabla^2) = -\nabla^2 + R_k(-\nabla^2)$; type II cutoff is the same function but its argument is now the entire inverse propagator: $R_k(\Delta)$, such that the modified inverse propagator is $P_k(\Delta) = \Delta + R_k(\Delta)$.

We now restrict ourselves to the case when $\mathbf{E} = q\mathbf{1}$, so that we can write $\Delta = -\nabla^2 + q\mathbf{1}$. The evaluation of the r.h.s. of the ERGE reduces to knowledge of the heat kernel coefficients and calculation of integrals of the form $Q_n\left(\frac{\partial_t R_k}{(P_k+q)^\ell}\right)$. It is convenient to measure everything in units of k^2. Let us define the dimensionless variable y by $z = k^2 y$; then $R_k(z) = k^2 r(y)$ for some dimensionless function r, $P_k(z) = k^2(y + r(y))$ and $\partial_t R_k(z) = 2k^2(r(y) - yr'(y))$.

In general the coefficients $Q_n(W)$ will depend on the details of the cutoff function. However, if $q = 0$ and $\ell = n + 1$ they turn out to be independent of the shape of the function. Note that they are all dimensionless. For $n > 0$, as long as $r(0) \neq 0$:

$$Q_n\left(\frac{\partial_t R_k}{P_k^{n+1}}\right) = \frac{2}{\Gamma(n)} \int_0^\infty dy \frac{d}{dy}\left[\frac{1}{n}\frac{y^n}{(y+r)^n}\right] = \frac{2}{n!} \ . \tag{31}$$

Similarly, if $r(0) \neq 0$ and $r'(0)$ is finite,

$$Q_0\left(\frac{\partial_t R_k}{P_k}\right) = 2 \ . \tag{32}$$

Finally, for $n = -m < 0$

$$Q_n\left(\frac{\partial_t R_k}{P_k^{1-m}}\right)\Big|_{y=0} = (-1)^m \left(\frac{\partial_t R_k}{P_k^{1-m}}\right)^{(m)}\Big|_{y=0} = \sum_{n=0}^{m}\binom{m}{n}\left(r - y\,r'\right)^{(n)}(y+r)^{(m-1)}\Big|_{y=0} = 0 \tag{33}$$

as $(r - y\,r')^{(n)} = r^{(n)} - y\,r^{(n+1)} - r^{(n)} = -y\,r^{(n+1)}$ which vanishes at $y = 0$.

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Acknowledgements

I wish to start by aknowledging my supervisor, Dr. Roberto Percacci, whose deep knowledge of the subject and advice has proved invaluable during these years. Let me also thank my friends and collaborators Alessandro and Carlo, for their continuous help and support. In particular I'd like to thank Alessandro for freely sharing with me his ideas and thoughts, which are too many, and too good. I'm also grateful to other people I briefly met on the way, G. P. Vacca, A. Bonanno, M. Reuter, O. Zanusso, D. Benedetti, for all their encouragement.

Where would I be now without my course pals, office mates, beer buddys, Dario and Aurora? Probably nowhere near the place I'm standing, where there are so many SISSA people to thank, past and present, present and gone. Thanks to everyone who turned these past SISSA years into something joyful and colorful. Thanks to Federico for always being somewhere else in the world when I'm in Trieste, and to Giorgia for her Theory of Buttons (with Logo). To Eleonora, for the 5 pm caffelatte, and to Giangiacomo, for the Jodorowsky commentary. To Valentina, for that zombie book that everybody bought[1]. To M. d'Anna, for the making of QUAM. To Maurizio, for the daily "Merenda time". Thanks to Adriano, for the great help in the Nijmegen *affaire*. And to Alessio (the red Lupin), Federica, Riccardo, Alessio (the green Lupin), Ambra, Matteo, Rossella, Rita, Elena, Eolo, Goffredo, Enzo, Antonella and everyone else. And if I forgot you, thanks to you too.

To Jamir, the best flatmate I have ever had - and boy, I had many. For the remarkable dinners and *osmizzatas*, for having outlived the Third Flatmate, for the plus and for the minus.

To my present (and almost past!) flatmates, Erica and Laura, for their patience regarding my lack of domestical organization.

To my family, for all their love and support.

To Irene, for being such a wonderful part of my life.

And finally, I'd like to dedicate all these written pages to Francesco. For all those bets we were both losing. For the help you gave me, when I was in need. For all those useless Youtube videos you were always making me watch, that I am now missing. So much.

[1] I'm joking, of course I didn't buy it.