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"It ain't what you don't know that gets you into trouble.
It's what you know for sure that just ain't so."
— Mark Twain

Abstract

This thesis contains some of the main results obtained during my research activity in these years, in the Statistical Physics sector at SISSA and in the Quantitative Life Sciences sector at ICTP.

Chapter 1 serves as an introduction and is kept brief, because each of the following chapters has a separate introduction containing more details on the different problems that have been considered.

In Chapter 2 several models of wealth dynamics are discussed, with focus on the stationary distributions that they have. In particular, we introduce a stochastic growth model that has a truncated power law distribution as a stationary state, and we give an interpretation for the mechanism generating this cut-off as a manifestation of the shadow banking activity.

Chapter 3 is devoted to the issue of wealth inequality, and in particular to its consequences, when in a system with a power law wealth distribution, economic exchanges are considered. A stylized model of trading dynamics is introduced, in which we show how as inequality increases, the liquid capital concentrates more and more on the wealthiest agents, thereby suppressing the liquidity of the economy.

Finally in Chapter 4, we discuss the issue of complexity and information sensitiveness of financial products. In particular, we introduce a stylized model of binary variables, where the financial transparency can be quantified in bits. We quantify how such information losses create sources of systemic risk, and how they should affect the pricing of financial products.

The results of Chapter 2, Chapter 3 and Chapter 4 are contained in the following publications:

- Davide Fiaschi, Imre Kondor, Matteo Marsili and Valerio Volpati, *The Interrupted Power Law and the Size of Shadow Banking*, PLoS ONE, 9(4): e94237 (2014) .

- João Pedro Jerico, François Landes, Matteo Marsili, Isaac Pérez Castillo and Valerio Volpati, *When does inequality freeze an economy?*, J. Stat. Mech. 073402 (2016) .

- Marco Bardoscia, Daniele d'Arienzo, Matteo Marsili and Valerio Volpati, *Lost in Diversification*, in preparation (2016) .

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Chapter 1

Introduction

Historically, **Statistical Physics** introduced the idea that the complex phenomenology of a macroscopic system can be explained as an emergent process through the interaction of its constituents, when these are modelled as an ensemble of random variables following simple rules [1]. The enormous practical advantage of this approach lies in the fact that a full detailed description of the microscopic dynamics is irrelevant for the understanding of the collective behaviour. Even extremely stylized models can have rich and non predictable macroscopic emergent properties. The conceptual breakthrough of this idea is that what should be considered fundamental - in Science - cannot be reduced to something happening at the scales of elementary particles, but lies as well in the types of interactions taking place at any scale [2].

More recently, these ideas made their way also through non-conventional physical systems, both in natural and social sciences. Apart from bringing into the game very successful and modern quantitative tools (in some cases directly borrowed from Statistical Physics, in other cases from Network Science or from Information Theory, just to name a few), the **Complex Systems Science** has introduced unifying principles in areas where the quantitative understanding used to be limited to few specific sub-fields. One illustrative example of this might be the concept of *complex adaptive system*, initially introduced to explain the robustness of ecological systems [3], and then used for the description of the development of language [4] and societies[5], or to derive principles for the stability of the financial system [6] and for business strategies [7].

In this thesis, I have been mainly interested in the application of these concepts to economic sciences.

In the field of Economics, quite surprisingly, for many years empirically driven analyses have been rather marginal. On the contrary, quantitative models borrowed from classical economic theories have been designed more with the purpose of establishing the internal consistency of some axioms referred to some ideal capitalistic dream, rather than to reproduce empirical evidences (see [8] for a partisan manifesto).

Nevertheless, persistent statistical regularities and patterns in empirical data are frequently observed in many contexts, from stock market returns distributions [9] to corporate growth [10], passing through urban development [11]. This suggests that explaining economic phenomena as emergent statistical properties of a large interacting system should be indeed feasible, as it has been done with encouraging success in these years [12–15].

When these attempts started to become popular, in the middle of the 1990s under the name of *Econophysics*, the attention was primarily focused on the analysis of financial markets [9]. Soon after, another direction, closer to Economics than Finance, has emerged. It studies the distributions of wealth and income in a society and overlaps with the long-standing line of research in Economics studying inequality.

This research line stemmed from one of the most robust empirical stylized facts about economic systems, since the work of Pareto, that is the observation that both the distributions of wealth and income among a group of individuals, as well as the distributions of sizes of cities and firms, approximately follow a power law distribution (called Pareto distribution in Economics) [16, 17]. From the modelling side, a power law distribution does not require sophisticated assumptions, but it can be easily reproduced as the stationary state of a plethora of simple stochastic models [18–21].

In Chapter 2, analyzing data from the Forbes Global 2000, a dataset containing the total assets of the world's largest 2000 firms, we observe that the largest amongst these firms show a deviation from Pareto distribution, because of the presence of a sharp cut-off in the tail of the power law, which is populated exclusively by banks and firms operating in the financial sector. We give an interpretation for the mechanism generating this cut-off as a manifestation of the shadow banking system [22], and we propose a measure of the total asset size involved in this system. Furthermore, we introduce a stochastic growth model that has a truncated power law distribution as a stationary state, which is able to fit the data surprisingly well, and can provide a measure of the shadow banking activity through the years.

The debate on wealth and income distributions, in particular related to the sustainability of economic inequalities, has regained much interest recently, in view of the claim the actual inequality has raised dramatically in recent years, reaching the same levels of the beginning of the 20th century [23]. To address the problem of the consequences of inequality on the efficiency of an economy with Pareto distributed wealth, in Chapter 3 we introduce a simple model of trading dynamics in which a set of zero-intelligent agents randomly trade a set of goods of different prices. In such a model, we show how as inequality in the wealth distribution increases, the liquid capital concentrates more and more on the wealthiest agents, thereby suppressing the probability of successful exchanges, i.e. liquidity.

An interesting role, in the literature of wealth and inequalities, is played by the exponent of the power law, that it has been measured to be close to unity in many systems [24]. For firms size distribution, this exponent is found to be slightly below one in the Forbes Global 2000 dataset, while it is slightly above one in the US household wealth distribution [25], and it is even larger for the US income distribution [26]. As a matter of fact, an economy with a Pareto exponent smaller than one looks very different from an economy for which this exponent is larger than one, the first having the richest agents owning a finite fraction of the total wealth in the system, even in the thermodynamic limit of infinite number of agents. As a result of this *wealth condensation*, the random exchange dynamics that we introduced completely freezes in the condensed phase, giving a measure of when inequality is too large to be sustainable. This is particularly relevant because in the aforementioned datasets on households income and wealth, the exponents of these empirical distribution have been decreasing steadily in recent years.

Another very interesting feature that is contained in the Forbes Global 2000 dataset, is the apparent decoupling between the financial and non-financial sectors. In the last two decades or so, financial firms have grown at a rate which is considerably larger than the growth rate of non-financial firms. This phenomenon can be considered as a sort of "inefficiency" of the financial industry to deliver investments to the real economy. In this respect, a promising direction of research which may provide clues about the role of finance in our global economy, is related to the understanding of the relationship between the faster growth of financial firms (relative to non-financial ones) and the proliferation of financial instruments, as in reference [27].

A different type of financial inefficiency is related to the information processing of complex financial products, considered to be the key factor that lead to the massive devaluation of structured finance type of products in 2007, the main trigger of the global financial crisis [28]. Expectations on the future returns of these type of products can be shown to be sharply dependent on the underlying distribution of the returns of the individual assets they are composed with. As a result, even when they are composed of a large number of such individual components, the diversification principle does not apply and the risk of these financial instruments remains high.

In the last chapter of this thesis, we introduce stylized models of binary variables, in which we can quantify how financial products are sensitive on some side information, affecting the probability distribution of their components. This leads us to give some proposal related to how the financial industry might increase market efficiency and transparency by creating specific *barcodes* for financial products [29].

Chapter 2

Power Law, Interrupted

Power law distributions arise very often, in a large number of surprising empirical regularities in Economics and Finance. In particular, the distributions of wealth and firms sizes, are very often in the literature found to follow power law distributions (called Pareto distributions in Economics)[16, 17].

Analyzing data from the Forbes Global 2000 dataset, we observe that the largest global firms show an anomalous deviation from a Pareto distribution, because of the presence of a sharp cut-off in the tail of the power law, which is populated exclusively by banks and firms operating in the financial sector.

This anomaly in the shape of the top tail of the assets distribution is the starting point of our analysis, and we discuss it in section 2.1.

From a theoretical point of view, the occurrence of power laws (i.e. Pareto distributions) in the size distribution of firms does not require strong assumptions, but it has been related to proportional random growth (PRG) mechanism [18–21] (see section 2.3). Assuming that a PRG dynamics should hold also for financial firms, we can calculate the hypothetical distribution of assets in the absence of any anomaly. Next, we argue that the difference between this hypothetical distribution and the actual one can be taken as a proxy for the size of the so-called *shadow banking* system (see section 2.2), which has been broadly defined as credit intermediation involving entities and activities outside the regular banking system (see [30], p. 3), and it is the subject of much debate in the literature of financial regulation [22, 31, 32].

Finally, in section 2.4 we introduce, as a simple generalization of the model proposed in reference [21], a stochastic growth model that has a truncated power law distribution as a stationary state, which allows a first investigation of the determinants of the observed anomaly.

2.1 A snapshot of the global economy

Financial deepening

If we take the Forbes Global 2000 (FG2000) list [1] as a snapshot of the global economy, we find that financial firms [2] dominate the top tail of the distribution of firms by *asset size*.

In this analysis, we used the asset size, i.e. the total market value of all the investments that are presented on the balance sheet of a firm, as a proxy of the firm size. Firm size can, and often has been in several studies where detailed data on assets were not available, also be measured by other variables such as total sales, number of employees, or market value. However, even though these variables are available in the FG2000, they can be strongly affected by the fluctuations in market prices, and by the conditions of labour and other economic fundamentals. Furthermore, for financial firms in particular, they are not expected to be a good proxy of the actual size of the firm.

Even though financial firms are approximately 30% of the firms that are enlisted in the FG2000 list in terms of numbers, they account for 70% of total assets in the 2004 FG2000 list, a share that rose to 74% in the 2016 list. On the other side, they account for approximately 30% of the total sales, profits and market value, a share that has been roughly constant in the whole period studied.

The predominant role of financial firms in the asset distribution can be described also by noting how they are placed in the list rank. The highest placed firm which is classified as non-financial is General Electric, which ranks only 62nd in the 2016 FG2000 list. This seems to be a recent trend: General Electric was the largest non-financial firm by asset size also in the 2004 FG2000 list, but then it ranked 22nd.

This trend, in which financial firms dominate more and more the tail of the distribution, and form the largest part of the total assets in the economy, is called

[1] The data used are publicly available at http://www.forbes.com/global2000/list/ (FG2000). The FG2000 list refers to the previous year. Thus the 2016 FG2000 list collects firms according to their characteristics in 2015.

[2] we consider as firms belonging to the financial sector all the firms that in the FG2000 list belong to the following industries: Banking, Diversified Financials, Insurance, Consumer Financial Services, Diversified Insurance, Insurance Brokers, Investment Services, Major Banks, Regional Banks, Rental & Leasing, Life & Health Insurance, Thrifts & Mortgage Finance, Property & Casualty Insurance. Their number ranges from 501 in the 2013 list to 597 in the 2008 list.

financial deepening in reference [33], to which we refer for a discussion on the systemic implication of the growth in the size of banks.

Figure 2.1 shows the asset size of all the financial firms, and asset size of all the remaining firms. According to reference [33] (see Chart 2), data from Bank of England shows how the financial deepening has been started to take place most likely in the first half of the 1990s.

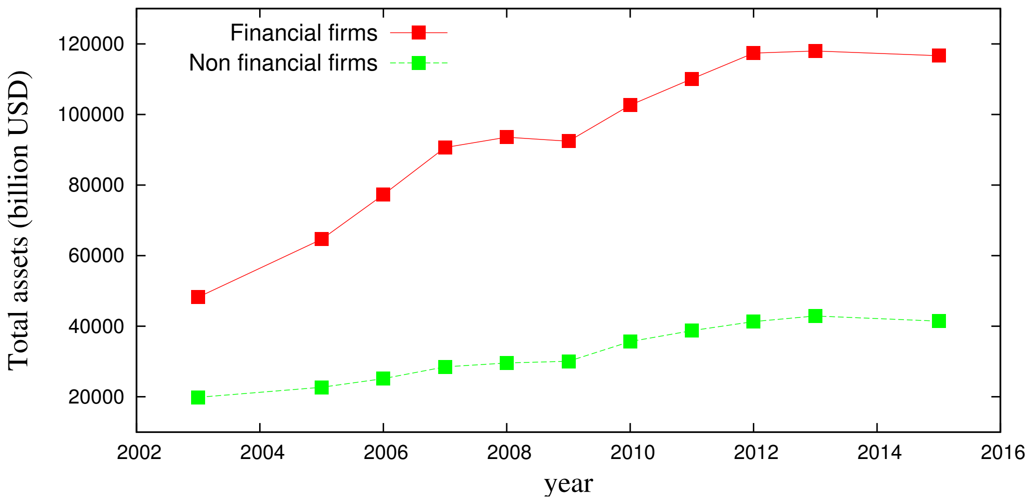

Figure 2.1: Total asset sizes for the financial and the non-financial sectors, in Forbes Global 2000 dataset. In particular, in the years before the 2008-09 global financial crisis, financial firms have been growing at a larger rate than non-financial ones. The FG2000 list refers to the previous year. Thus the 2016 FG2000 list collects firms according to their characteristics in 2015. Data of 2004 (2005 FG2000) and 2014 (2015 FG2000) are missing from the plot.

The interrupted power law

Besides being remarkable in themselves, the sizes of the biggest financial firms also display a peculiar distribution. The 10th largest firm in the 2016 FG2000 list is Bank of America, with 2.18 trillion of U.S. dollars in assets, which is comparable to the Italy's gross domestic product ($2.22 trillion). Yet its size is not much smaller than the largest firm in the list, Industrial and Commerce Bank of China (ICBC), which has assets worth $3.42 trillion. This observation contrasts with the aforementioned empirical fact for the firm sizes S to follow a power law distribution as

$$P(S = x) \sim x^{-\beta-1}, \tag{2.1}$$

with some exponent $\beta > 0$.

Figure 2.2 shows that the rank plot of the firms included in the 2004, 2007 and 2013 lists of FG2000 approximately follows equation (2.1), with an exponent β close to one, corresponding to Zipf's law [24]. However, such a power law distribution in rank seems to apply only from the 20th largest company downward, while the upper tail, which is entirely dominated by financial firms, levels off. If Zipf's law were to hold also for the top 20 companies, we would expect ICBC to be ten times as large as the Bank of America (hence it should have assets worth approximately \$21.8 instead of \$3.42 trillion).

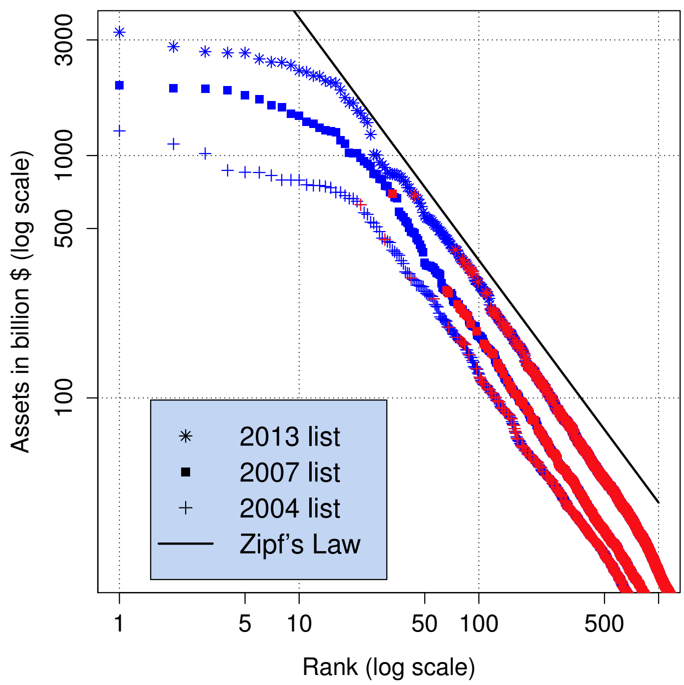

Figure 2.2: Rank plot of the 2004 list (+), 2007 list (□) and 2013 list ($*$) of FG2000 by asset size. Financial firms are shown in blue, while the other firms in red. The straight line corresponds to Zipf's law and is drawn for comparison.

In fact, we recall that, when a random variable S is distributed with a power law distribution, the most probable values for the rank ordered firm sizes $S_{[k]}$ of the k^{th} largest firm, in a sample constituted by N firms, depend on rank with another power law dependence. In fact, these most probable values are given by

$$S_{[k]} \sim \left[\frac{N}{k}\right]^{1/\beta}. \tag{2.2}$$

When a variable is distributed with a power law distribution with exponent β, its *rank ordered statistics* depends on rank with a power law with exponent $1/\beta$.

A simple argument to make sense of this can be given shortly. The cumulative distribution

$$P(S > x) = \int_x^\infty \mathrm{d}x'\, P(S = x'), \tag{2.3}$$

is the probability that a firm has size larger than x. Consequently, the integer value of $NP(S > x)$ is the expected number of firms with sizes larger than x, in a sample of N firms extracted from the distribution. The k^{th} largest observed value $S_{[k]}$ should then be given by

$$NP(S > S_{[k]}) = k. \tag{2.4}$$

Assuming $P(S > S_{[k]}) \sim S_{[k]}^{-\beta}$ yields equation (2.2).

The same truncated power law can be observed also in the more conventional frequency plot (see figure 2.3), but the rank plot emphasizes the behaviour of the distribution in the tail of large firms.

The occurrence of a power law with a cut-off is not entirely peculiar of these firms size datasets. For instance, in cases in which the sample is not very large, or if the exponent β is particularly small, an apparent cut-off could emerge in the rank plot due to incomplete sampling. Following the analysis of reference [34] (see Chapter 6 in the reference), a refinement of equation (2.2) yields for the most probable values

$$S_{[k]} \simeq \left[\frac{(\beta N + 1)C}{\beta k + 1}\right]^{1/\beta}. \tag{2.5}$$

which could be responsible of an apparent cut-off (visible only in the rank plot) looking similar to the one in figure 2.2 .

In other cases, for instance in the distribution of earthquake moments worldwide documented in the Harvard catalogue (see Figure 6.1 in reference [34]), one can find a distribution which has a bulk well described by a power law distribution (with exponent $\beta \simeq 0.7$), while the tail for the largest earthquakes exhibits a significant departure that can be described by an exponential tail [35]. This departure is usually described invoking a change of regime in the mechanism generating the largest earthquake, for which the same mechanism that is responsible for the events in the bulk of the distribution does not hold. The mechanism we are going to invoke to account for this cut-off in the tail of firm size distribution has to do with the shadow banking system, that we are going to introduce in section 2.2.

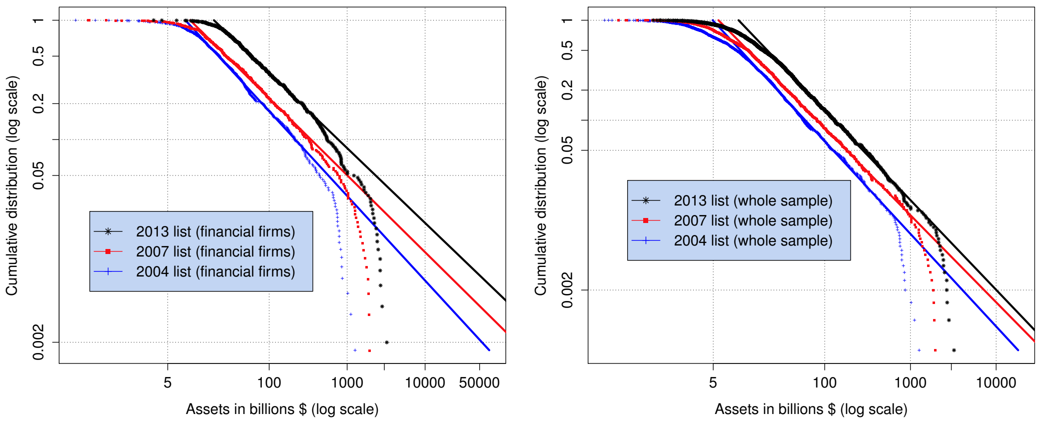

Figure 2.3: Cumulative distribution $P(S > x)$ of asset sizes S for financial (left panel) and all (right panel) firms in 2003, 2006, and 2012 (2004, 2007, and 2013 of FG2000 lists). The straight line is obtained as a linear fit in an intermediate range of $\log P(S > x)$ vs $\log x$ (see table 2.1).

The bulk of the distribution

Table 2.1 reports the ranges considered in the estimate of the power law distribution, and the estimate of the Pareto exponent β of equation (2.1) for all firms in the FG2000 list from 2004 to 2013 (2005 is missing for lack of data, while the analysis has not been extended to the following years). Quite interestingly, Pareto exponent β peaks at the beginning of the period and steadily decreases until it reaches the lowest level in 2007 (2008 list), before the financial crisis. Then it increases suddenly in 2008 and remains relatively stable thereafter. Table 2.1 also reports the estimate of the Pareto exponent of the distribution of financial firms only, β_{fin} ; β_{fin} is smaller than β but it exhibits a behaviour similar to β, with the important exception that it starts to decline again after the crisis.

However, the authors of reference [36] observed that Zipf's law (and the same for power laws in general) holds as a property of a system as a whole, but it may not hold for its parts. As such, it is manifest in samples that preserve a form of *coherence* (with the whole system), but fails to hold in incomplete samples that account for only part of the system (see [36]). Our findings of deviations from a power law behaviour for financial firms - which are more pronounced than for the whole economy, is not entirely surprising and it might indicate that the Pareto distribution of asset sizes should be considered as a property that applies to the whole economy, rather than

List FG2000	S_-	S_+	β	β_{fin}
2004	14.88	665.14	0.926	0.710
			(0.0012)	(0.0019)
2006	11.02	897.85	0.889	0.678
			(0.0005)	(0.0013)
2007	12.18	992.27	0.871	0.645
			(0.0005)	(0.0012)
2008	12.18	1096.63	0.864	0.655
			(0.0006)	(0.0016)
2009	14.88	1339.43	0.899	0.672
			(0.0008)	(0.0012)
2010	14.88	1339.43	0.891	0.674
			(0.0008)	(0.0011)
2011	18.17	1339.43	0.899	0.669
			(0.0006)	(0.0013)
2012	24.53	1635.98	0.905	0.648
			(0.0009)	(0.0012)
2013	24.53	1998.20	0.897	0.627
			(0.0008)	(0.0009)

Table 2.1: The range of assets (in billion \$) $[S_-, S_+]$ where the power law behaviour is estimated (for the whole sample), and the estimated Pareto exponents β both for the whole sample and limited to the financial firms in the FG2000 lists from 2004 to 2013 (data for 2005 are not available). From the standard errors of the estimated Pareto exponents (reported in brackets) we can notice how financial firms size distribution show more pronounced deviations from a power law behaviour.

to a particular sector. This is consistent with empirical findings e.g. in [24], and suggests that, in the absence of anomalies, one should expect a hypothetical assets distribution that would perfectly obey a power law distribution up to the largest firms.

But the most interesting aspect of table 2.1 that we want to point out here is that both estimated exponents β and β_{fin} are less than one for the whole period 2004-2013.

2.2 Shadow banking

Shadow banking (SB) is a relatively new concept; the term itself is attributed to McCulley [37]. SB is a term for the collection of non-bank financial intermediaries that provide services similar to traditional commercial banks but outside normal financial regulations. During the 2007-08 crisis, which is often described as a run on the SB system [38], the private guarantee provided by non-bank institutions proved to be insufficient, and without massive public intervention the collapse of the SB system would have brought down the whole global financial system. The first taxonomy of the different institutions and activities of SB was given by Pozsar [39], who also constructed a map to describe the flow of assets and funding within the system.

The rise of a large part of SB was motivated by regulatory and tax arbitrage, and as such represented the answer of the finance industry to regulation, in particular to capital requirements. In fact, the core activities of investment banks are subject to regulation and monitoring by central banks and other government institutions. As a response, it has been common practice for investment banks to conduct many of their transactions in ways that do not show up on their conventional balance sheet accounting and so are not visible to regulators or unsophisticated investors. Irrespective of the shortcomings or merits of the system, the SB has remained by and large unregulated, its systemic risks implications uncharted, and its connections with the rest of financial system opaque. Indeed, SB is one of the most important issues on the agenda of financial reform [31, 32].

For us, the only property of interest of the SB system is its total volume. Estimates of its size differ in nature: Gravelle and Lavoie [40] distinguish between two broad approaches to measuring the SB sector, one which is based on identifying the entities that contribute to it, and the other based on mapping the activities that constitute it. They also differ quantitatively, because of the difficulty to determine precisely which financial activities should be included in the calculation. For example, the Deloitte Shadow Banking Index [41] shows a rise of the SB system in the US before 2008, but then displays a dramatic drop, suggesting that the phenomenon is now over. The index is built from specific components which are known to have played a major role in the crisis, and its decline after 2008 reflects the deflation of these markets. On the contrary, the Financial Stability Board (FSB) estimates that SB "[...] grew rapidly before the crisis, rising from $26 trillion in 2002 to $62 trillion in 2007. The size of the total system declined slightly in 2008 but increased subsequently to reach $67 trillion in 2011" [30].

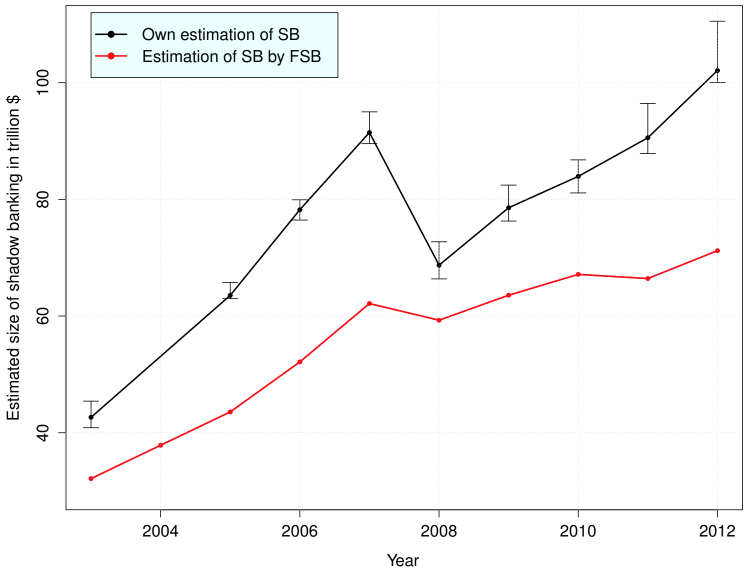

Figure 2.4: Comparison between our index of SB, I_{SB}, with the estimate of the size of SB made by FSB [42] for the period 2003-2012. The reported confidence bands for our estimate of SB are calculated on the basis of ± 2 standard errors in the estimate of the coefficients of the power law distributions.

The shadow banking index

Here we propose an index for the size of the SB system, denoted by I_{SB}, based on the idea that, in an ideal economy where the dynamics of financial firms size should be the same of all other firms, the power law distribution should extend all the way to the largest firms. Since SB is expected to act effectively as a movement of assets outside the largest banks' balance sheets, and the top tail of the distribution is dominated by financial firms, we are led to attribute the mass missing from the distribution of asset sizes to SB. Fitting the middle range of the distribution to a power law (as in the left panel of figure 2.3) leads us to a theoretical estimate $\hat{S}_{[k]}$ of what the size of the k^{th} largest firm should be. Summing the difference between this theoretical estimate and the actual size $S_{[k]}$ of the k^{th} largest firm, over k, i.e.:

$$I_{SB} = \sum_{k=1}^{N} \left(\hat{S}_{[k]} - S_{[k]} \right) \tag{2.6}$$

provides our estimate of the size of the SB system. The sum is limited to the N largest firms. We take $N = 1000$ but the results depend very weakly on the choice of N as long as $S_{[N]}$ is in the range over which the fit is made ($\hat{S}_{[k]} \simeq S_{[k]}$ within this range).

For comparison, figure 2.4 reports also the estimated size of the SB system by FSB [42]. Apart from being of the same order of magnitude, both the FSB estimate and I_{SB} show a strong rise before the crisis in 2007, a drop in 2008 (much more severe for I_{SB}), and a growth after 2008, with I_{SB} increasing at a faster pace, especially in 2011.

In reference [40] it is argued that an entity-based approach to SB, such as that of the FSB, "[...] may omit SB activities undertaken by banks that may contribute to systemic risk." Furthermore, as observed by Adrian *et al.* [22] "[...] the shadow banking system comprises several different entities and activities. In addition, the types of entities and activities which are of particular concern will change in the future, in response to new regulations." Along similar lines, Pozsar *et al* [38] conclude: "[...] the reform effort has done little to address the tendency of large institutional cash pools to form outside the banking system. Thus, we expect shadow banking to be a significant part of the financial system, although almost certainly in a different form, for the foreseeable future." These arguments suggest that the FSB estimate, as well as other estimates which try to map the SB activity, is likely to provide a lower bound to the real size of the SB system. I_{SB} may instead be considered as an upper bound, since it measures the amount of assets that are missing from a hypothetical economy in which a power law distribution holds across all scales of asset sizes, but there might be different mechanisms, other than SB, which could account for deviations from it.

In the next section we are going to describe which type of proportional random growth models (PRG) we are interested in, in order to shed more light on the dynamics that could have shaped the empirical distribution of firms asset.

2.3 Stochastic processes of wealth distribution and the exponent β

The ubiquity and stability of these empirical power law distributions (both in Economics and in other fields) lead several researchers to look for mechanisms that are able to generate such distributions. In particular, power laws can be obtained as stationary distributions of a very general class of simple stochastic processes, the proportional random growth (PRG) models - or processes with multiplicative noise. The simplest PRG model can be defined in terms of the following multiplicative recurrence equation, for the firm size S_i.

$$S_i(t+1) = \eta_i(t+1)S_i(t), \tag{2.7}$$

with any time independent distribution for the multiplicative noise η_i. Rigorously, this process does not have a stationary distribution. Taking the logarithms we have $\log S_i(t) = \log S_i(0) + \sum_{i=1}^{t'} \eta_i(t')$, hence the size probability distribution at large time converges to a log-normal, but there is not a steady state. An argument to show that this is the case can be given by noting that, the variance for the log of the firms size distribution at time t is given by (if it exists!) $\mathbb{V}[\log S_i(t)] = \mathbb{V}[\log S_i(0)] + \mathbb{V}[\eta]t$ and it grows linearly in time without bound.

Nevertheless, at large but finite time, the aforementioned process has a log-normal distribution which in several regimes is indistinguishable from an apparent power law [43]. Additionally, it can be regularized (for instance by adding a friction term to prevent firms to become too small) and in such a way one can show how such a process do have a power law distribution as a stationary distribution, but with a Pareto exponent β *always larger than one* (and very close to one if the friction is small) [17]. As we have seen, the firms size data in the FG2000 list, in the bulk of the distribution, are compatible with a power law but with an exponent that is steadily smaller than one, see table 2.1 .

Bouchaud and Mezard [19] argue that $\beta < 1$ can be obtained within models of PRG with random shocks, by adding trading of assets among firms if this trading is restricted in size and happens within a sparse network. In this model, the firms size evolution is given by

$$S_i(t+1) = \eta_i(t+1)S_i(t) + \sum_{j\neq i} J_{ij}S_j(t) - \sum_{j\neq i} J_{ji}S_i(t)\,, \tag{2.8}$$

where η_i is a Gaussian random variable and J_{ij} the amount of wealth firm j spends buying the product of firm i. In the mean field model, $J_{ij} \equiv J/N$ for all $i \neq j$, with N the total number of firms, the stationary distribution has a power law tail with Pareto exponent β that again is always larger than one, and converge to Zipf ($\beta \to 1$) when $J \to 0$. In the presence of some sparsity in the trading network, both on a regular random network [19] or on some different complex networks [44], this dynamics can be studied numerically and the stationary distribution does still have a power law tail, but now the exponent can be smaller than one.

Malevergne, Saichev and Sornette [21] provide a different mechanism of PRG which can have a condensed phase, by accounting for the entry and exit of firms from the system. In this model, firms evolve independently according to a log-normal stochastic process, like the one of equation (2.7), where η_i is a Gaussian random variable (with drift μ and variance σ). In addition, according to a Poisson point process at rate h, one firm chosen at random disappears from the market. Similarly, new firms enter the market according to a Poisson point process at rate ν, having initial size $S_i(0) = 1$. In reference [21] the possibility of an exogenous growth of economy is considered by having new firms which appears more often ($\nu(t) = \nu e^{d_0 t}$) and with larger initial size ($S_i(0) = e^{c_0 t}$). An exponent smaller than one characterizes an unsustainable economy where the return on investment of the whole economy is larger then the investments in new firms ($\mu - h > d_0 + c_0$), and Zipf's law emerge as an optimal allocation of resources that ensure a maximum sustainable growth ($\mu - h = d_0 + c_0$). In the following we neglect growth (i.e. we take $c_0 = d_0 = 0$). It can be shown that the top tail of the equilibrium size distribution has a power law shape with a Pareto exponent given by

$$\beta = \frac{1}{2}\left[\left(1 - 2\frac{\mu}{\sigma^2}\right) + \sqrt{\left(1 - 2\frac{\mu}{\sigma^2}\right)^2 + 8\frac{h}{\sigma^2}}\,\right]. \tag{2.9}$$

This exponent turn out to be independent of ν, which is a parameter that just specifies how many firms there are on average in the economy, and can be fixed to be $\nu = 1$ without loss of generality.

The simplest way to understand this result is by considering a continuous version of the model, for the density of log firms sizes $\rho = P(\log S = x)$. Such a dynamics can be described by the Fokker-Planck equation

$$\partial_t \rho = \partial_x^2 \rho - \mu' \partial_x \rho - h'\rho + \delta(x), \quad x \in \mathbb{R}. \tag{2.10}$$

The first two terms in this equation are the ones which are usually present in a diffusion process, as they are expected for the evolution of the log of sizes, as in equation (2.7). The last two corresponds to the two Poissonian point processes, the uniform absorption of firms and a source in the origin. The diffusion constant has been put to unity, and μ' and h' are some coarse grained coupling corresponding to the microscopic parameters μ and h.

It can be shown (see appendix 2.A) that the stationary solution of this diffusive equation has a power law tail with exponent given by (2.9), provided the boundary conditions

$$\begin{aligned} \partial_x \rho(0^+) - \partial_x \rho(0^-) &= 1 \\ \rho(0^+) &= \rho(0^-). \end{aligned} \tag{2.11}$$

The first of these boundary condition specifies how new firms are generated in the origin ($S(0) = 1$) while the second guarantees the continuity of the solution.

In the next section we introduce a modification of this last model, in order to reproduce the cut-off displayed by the Forbes data, i.e. in the presence of SB.

2.4 The model: a PRG model with shadow banking

As described in section 2.2, shadow banking is expected to act effectively as a mechanism subtracting assets to the largest banks, who are the most targeted by the financial regulators, hence they have the largest incentive in moving assets away from their balance sheets. A modification of the model of reference [21], which reproduce the observed cut-off, can be obtained by adding a Poisson point process at rate λ, in which the largest firm i^* in the economy (with size $S_{i^*} = \max_i S_i$) moves a fraction ϵ of its assets outside the regular banking system to the SB system, reducing its *observed* size to $(1 - \epsilon)S_{i^*}$. The subtracted wealth is not distributed among the other firms, but it is just removed from the system. Such a modification produces an anomalous extremal dynamics for the firms whose size is among the largest in the economy, which have a diminished growth because of this mechanism.

Like in the model of reference [21], we could consider the exogenous growth of the economy, by having new firms which appears more often and with larger initial size. Since the Forbes dataset contains only information about the largest 2000 firms in the economy, it is not possible to estimate from this data the parameters d_0 and c_0. Hence, since we ultimately want to fit the observed empirical distribution with the present model, we fix d_0 and c_0 to zero, neglecting growth and just focusing our attention on the shape of the normalized stationary distribution.

A continuous equation like equation (2.10) can be written by just adding a reflective boundary condition at some position x_0, to account for the extremal dynamics

$$\partial_t \rho = \partial_x^2 \rho - \mu' \partial_x \rho - h' \rho + \delta(x), \quad x < x_0 \ . \tag{2.12}$$

x_0 can be treated as a parameter, and it can found self consistently, after the solution is found (see appendix 2.A). It can be shown that the stationary distribution has a bulk which is still described by a power law distribution, with the same exponent β of equation (2.9), while the tail is characterized by a sharp cut-off in correspondence of the reflective boundary (see appendix 2.A).

The stationary distribution of such a process reproduces surprisingly well the observed distribution in the data. Table 2.2 reports a calibration of the parameters

Year	μ	σ	β	h	ϵ	λ
2005	0.12	0.20	0.89	0.10	0.1	18
2006	0.10	0.24	0.87	0.09	0.1	15
2007	0.15	0.22	0.86	0.13	0.1	20
2008	0.11	0.28	0.90	0.10	0.1	12
2009	0.04	0.21	0.89	0.04	0.1	6
2010	0.11	0.23	0.90	0.10	0.1	14
2011	0.10	0.17	0.91	0.09	0.1	13
2012	0.09	0.17	0.90	0.08	0.1	12

Table 2.2: Estimates of the parameters of the modified PRG model of the SB system for the period 2005-2012 based on the FG2000 list of firms.

of our modified PRG model based on the FG2000 list of firms for the period 2005-2012 (the analysis has not been extended to the following years). We set $\epsilon = 0.1$ and we located the value of λ that yields the best match between the simulated and the observed firm size distributions. μ and σ are calculated by yearly variations of the firms' asset size in the data between consecutive years (except for 2005 where we use data of 2003, instead of 2004 which is missing). Using these values, h is computed from the estimate of the Pareto index β, inverting equation (2.9). The reported value of λ is the one that minimizes the distance between the observed and the simulated normalized firm size distributions. Specifically, *i)* we compute $Z_k = \langle \log (S_k/S_k^0) \rangle$ with S_k being the k-th largest firm in the simulation, S_k^0 the k-th largest firm in the FG2000 list and $\langle \cdot \rangle$ is the average over 100 simulations. *ii)* We find λ that minimizes the mean square deviation $\sum_k (Z_k - \bar{Z})^2/N$, with $\bar{Z} = \sum_k Z_k/N$.

Figure 2.5 shows the quality of our calibration of the model for 2012. The same fitting procedure was performed for different values of ϵ; for $\epsilon \in (0, 0.1)$ the “flux” $\epsilon\lambda$ of capital flow into the SB system results independent of ϵ. This is reasonable, because when ϵ is very small and λ very large, wealth is repeatedly drawn into the SB system from the same firm (the largest one). The parameter λ, can therefore be interpreted as a proxy for the *intensity* of the activity feeding the SB system.

According to the estimate of λ reported in table 2.2, the *intensity* of SB activity peaked in 2007 before the financial crisis, when the *originate-to-distribute* activities implemented by asset-backed securities and other credit derivatives probably reached their zenith [38]. In 2008 and 2009 the SB activity showed a dramatic fall in agreement with the sharp decline in all economic activities and the supposed breakdown of the

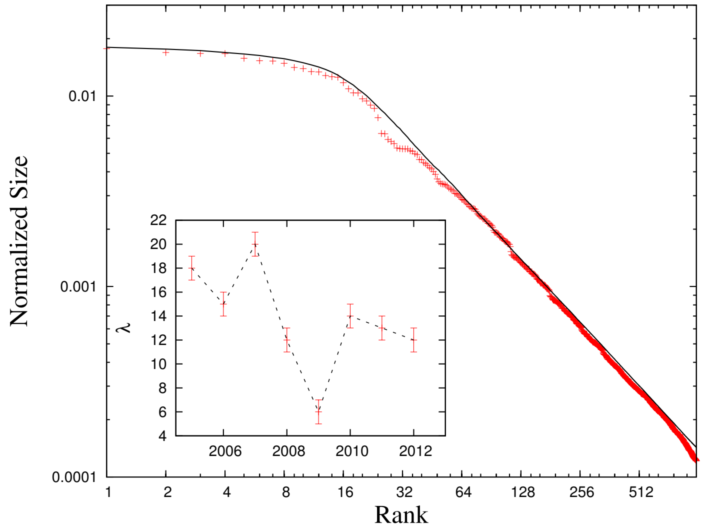

Figure 2.5: Comparison between the empirical FG2000 (red crosses) and the simulated (black bold line) distributions for 2012. Inset: the estimate of λ in the period 2005-2012.

SB system; but from 2010 to 2012 we observe a renewed increase, even though not at pre-crisis rates. This dynamic is fully consistent with the evolution of the size of SB system reported in figure 2.4, being λ a proxy for the *intensity* of the activity feeding the SB system. Considering that the largest firm is of the order of \$ 3 trillion in 2012, our result $\epsilon\lambda \approx 1.2$ suggests a flow of capital into the SB system that is progressing at approximately \$ 3.5 trillions a year.

2.5 Conclusions and outlook

Based on solid evidence in the literature [17], we consider the Pareto distribution for asset sizes as an empirical law of an economy. The observation of power law distributions in Economics is a remarkably solid piece of empirical evidence, dating back to the work of Pareto [16]. This empirical law arises from a generic mechanism – proportional random growth – that is expected to work in particular for financial firms. The actual distribution of firm sizes, at the global scale, closely follows this empirical law in the middle range, but deviates markedly from it in the upper tail, which is populated entirely by financial firms.

We invoke SB as the element that would reconcile observations with the expected law. This allows us to derive an index that identifies the size of SB with the missing mass in the top tail of the asset size distribution. This approach resembles the one leading astrophysicists to invoke *dark matter* and *dark energy* in order to reconcile empirical observations with the law of gravitation (current estimates suggest that dark matter and dark energy account for approximately 95% of the total mass in the universe). Likewise, the observation of a truncated power law in the distribution of asset sizes, points to the existence of *dark assets* that account for the missing mass in the top tail of the distribution.

Our estimate of the SB size is silent about the precise nature of SB activities and entities, as well as about the mechanisms that generate the observed departure from the theoretical power law behaviour. The missing mass from the top tail of the distribution does not necessarily correspond to hidden assets. It may rather refer to assets being redistributed within the system. The creation of Special Investment Vehicles in the securitization process is one example of a mechanism that transfers assets from large banks in the top tail to the bulk of the distribution.

The index is based on a simple and robust statistical feature, depending on a collective property of the economy. It is hard to manipulate and simple to compute, as it requires only data publicly available.

Haldane [45] recently argued that monitoring and regulation based on a detailed classification of financial activities is unlikely to keep pace with the rate of innovations in the financial industry. The increase in complexity of financial markets should rather be tamed by measures based on simple metrics, which are robust to change in regulation and fiscal policy. The index of SB proposed here is a contribution in this direction.

I_{SB} implicitly attributes SB activities to the largest financial firms which populate the top tail of assets distribution. It is well documented that the main financial firms originated most of the SB activities before the crisis [46]. Yet, I_{SB} also crucially depends on the exponent β, whose estimate depends on the shape of the distribution in the intermediate range. In particular, I_{SB} is expected to increase if the exponent β decreases and *vice-versa*. A comparison between table 2.1 and figure 2.4 shows how I_{SB} is (anti)correlated with β and β_{fin}: when the assets distribution gets broader (i.e. β and β_{fin} decrease), I_{SB} increases and *vice-versa*. After the 2007-08 crisis, the correlation of I_{SB} with β_{fin} is much stronger than with β. This is a further indication that the behaviour of financial firms is at the core of the dynamics of I_{SB}.

On the theoretical side, in section 2.4 we discuss a PRG model reproducing the observed behaviour of the largest financial firms based on an anomalous extremal dynamics, by which firms at the top of the distribution of firm sizes shift part of their assets off-balance-sheet. Within this framework, we estimate the intensity of SB activity in 2005-2012, which largely agrees with the observed behaviour of the

SB system. Such an estimate is obtained by comparing the stationary distribution of the model with the empirical distribution in the FG2000 lists. Such an approach could be criticized by arguing that we cannot really assume that each distribution for a given year in the FG2000 lists is a stationary distribution, but instead these snapshots should be considered as *out of equilibrium* snapshots of the economy. A first hint that this could indeed be the case is given by the fact that the FG2000 normalized distributions at different years look very similar, β and β_{fin} changing only very little from year to year, while the estimated parameters μ, σ and h change more pronouncedly in time. Secondly, in the last two decades, financial firms have grown at a rate which is larger than the growth rate of "real economy" firms. In any stochastic model of wealth evolutions, different growth rates among firms hinder the reaching of a stationary state. For these reasons, a full dynamical inference of the model over the data seems to be preferable, with different parameters describing the dynamics in the financial and in the non-financial sectors.

2.A A Fokker-Planck equation for the model: with and without shadow banking

We write down a continuous equation for the density of firms in the model of reference [21] and in our modification with shadow banking (SB). In this model, firms are created with Poissonian rate ν with unit size. Firms disappear from the market at Poissonian rate h, uniformly with no dependance on the size of the firm. While they are alive, these firms experience a log-normal process (drift μ variance σ). As presented, this model predict a power-law distribution for the firm sizes. In order to account for the cut-off present in the data we add a SB mechanism by selecting with Poissonian rate λ the richest firm in the market who decide to move a part of its asset ϵS_{i^*} outside the regular banking system, thus reducing its size to $(1-\epsilon)S_{i^*}$.

The model of reference [21]

The following equation, equation (2.10) in the main text, is expected to give a coarse grained description of the microscopic model, accounting for the evolution of the firms (log of the firms sizes):

$$\partial_t \rho = \partial_x^2 \rho - \mu' \partial_x \rho - h' \rho + \delta(x), \quad x \in \mathbb{R}. \tag{2.13}$$

In this equation, μ' and h' are some coarse grained coupling corresponding to the microscopic parameters μ and h, while the diffusion constant (related to σ has been set to 1 for convenience). Normalizable stationary solutions of the previous equation are

$$\begin{aligned} \rho(x) &= A e^{a_+ x}, \quad x < 0 \\ \rho(x) &= C e^{a_- x}, \quad x > 0 \end{aligned} \tag{2.14}$$

.

By plugging this ansatz into (2.13), we get

$$a_\pm = \frac{\mu' \pm \sqrt{\mu'^2 + 4h'}}{2} = \sqrt{h'} \frac{g \pm \sqrt{g^2+4}}{2}, \quad \text{with } g = \mu'/\sqrt{h'}. \tag{2.15}$$

Without shadow banking, the equation has to be endowed with the two boundary conditions:

$$\begin{aligned} \partial_x \rho(0^+) - \partial_x \rho(0^-) &= 1 \;\; \text{(new firms, rate } \nu = 1) \\ \rho(0^+) &= \rho(0^-) \;\; \text{(continuity)}, \end{aligned} \tag{2.16}$$

which are enough to determine

$$A = C = \frac{1}{a_+ - a_-}. \tag{2.17}$$

In order to compare this result with the one in reference [21], we can perform the change of variable $s = e^x$ so that for large s

$$\rho(s) = \frac{1}{s}\frac{1}{a_+ - a_-}s^{a_-} \sim s^{a_- - 1}. \tag{2.18}$$

Hence a_- should be minus the β exponent of equation (2.9). In fact by fixing

$$\mu' = \frac{\mu - (\sigma^2/2)}{(\sigma^2/2)} \tag{2.19}$$

$$h' = \frac{h}{(\sigma^2/2)} \tag{2.20}$$

we get the expected result

$$\beta = -a_- = 1/2 - \mu/\sigma^2 + 1/2\sqrt{(1 - 2\mu/\sigma^2)^2 + 8h/\sigma^2}. \tag{2.21}$$

We notice, for further convenience, that the presence of the drift term breaks the $x \to -x$ symmetry in $\rho(x)$ so that we have

$$\mathbb{E}[x] = \int x\rho(x) = \frac{g}{h'^3} \tag{2.22}$$

The model with SB; a simplified case, without drift

As suggested by numerical results which show a sharp cut-off in correspondence of a given size, the equation to account for the evolution of the firms with the addition of the SB mechanism can be the same as before, and the extremal process can be inserted here as a reflecting boundary in the Fokker-Planck. Hence we have:

$$\partial_t \rho = \partial_x^2 \rho - h'\rho + \delta(x), \quad x < x_0\,, \tag{2.23}$$

with boundary conditions

$$\begin{aligned} \partial_x \rho(x_0) &= 0 \ \ \text{(SB cut-off)} \\ \partial_x \rho(0^+) - \partial_x \rho(0^-) &= 1 \ \ \text{(new firms, rate } \nu = 1) \\ \rho(0^+) &= \rho(0^-) \ \ \text{(continuity).} \end{aligned} \tag{2.24}$$

The position of the boundary x_0 is now an unknown of the model, and we are going to fix it later.

The normalizable stationary solutions now are

$$\begin{aligned}\rho(x) &= Ae^{\sqrt{h'}x}, \quad x<0 \\ \rho(x) &= Be^{\sqrt{h'}x} + Ce^{-\sqrt{h'}x}, \quad 0<x<x_0\end{aligned} \tag{2.25}$$

Imposing boundary conditions (2.24) we find

$$A = \frac{1+e^{-2\sqrt{h'}x_0}}{2\sqrt{h'}}, \quad B = \frac{e^{-2\sqrt{h'}x_0}}{2\sqrt{h'}}, \quad C = \frac{1}{2\sqrt{h'}}. \tag{2.26}$$

The stationary solution $\rho(x)$ that we just found is not normalized, instead

$$\int_{-\infty}^{x_0} \rho(x) = \frac{1}{h'}. \tag{2.27}$$

In order to normalize it, one should multiply A, B and C by h'

Without any SB mechanism, this distribution is expected to be even for $x \to -x$, because we are in a simplified case with no drift term $\mu' = 0$. SB induce a shift of the distribution to the left proportional to some ϵ', the amount that is subtracted to the top firms, and to h', the probability of subtraction to that given firm. Imposing this condition

$$\mathbb{E}[x] = -\epsilon' h'. \tag{2.28}$$

We want to compute $\int x\rho(x)$ and impose it to be equal to $-\epsilon'$ (since we use the non-normalized $\rho(x)$), in order to determine the unknown x_0. We have

$$\begin{aligned}\int_{-\infty}^{x_0} x\rho(x) &= \int_{-\infty}^{0} \frac{1+e^{-2\sqrt{h'}x_0}}{2\sqrt{h'}} x e^{\sqrt{h'}x} + \int_{0}^{x_0} \frac{e^{-2\sqrt{h'}x_0}}{2\sqrt{h'}} x e^{\sqrt{h'}x} + \\ &+ \int_{0}^{x_0} \frac{1}{2\sqrt{h'}} x e^{-\sqrt{h'}x}.\end{aligned} \tag{2.29}$$

The sum of this 3 integrals can be shown to be

$$\int_{-\infty}^{x_0} x\rho(x) = -\frac{e^{-\sqrt{h'}x_0}}{h'^{3/2}}. \tag{2.30}$$

The model; with drift and SB

We consider here the same equation of the previous section in the presence of drift

$$\partial_t \rho = \partial_x^2 \rho - \mu' \partial_x \rho - h' \rho + \delta(x), \quad x < x_0 \tag{2.31}$$

Again, the normalizable stationary solution is

$$\begin{aligned} \rho(x) &= A e^{a_+ x}, \quad x < 0 \\ \rho(x) &= B e^{a_+ x} + C e^{a_- x}, \quad 0 < x < x_0 \end{aligned} \tag{2.32}$$

with

$$a_\pm = \frac{\mu' \pm \sqrt{\mu'^2 + 4h'}}{2} = \sqrt{h'} \frac{g \pm \sqrt{g^2+4}}{2}, \quad \text{with } g = \mu'/\sqrt{h'}. \tag{2.33}$$

The boundary conditions are:

$$\begin{aligned} \mu' \rho(x) - \partial_x \rho(x_0) &= 0 \;\; \text{(SB cut-off)} \\ \partial_x \rho(0^+) - \partial_x \rho(0^-) &= 1 \;\; \text{(new firms, rate } \nu = 1) \\ \rho(0^+) &= \rho(0^-) \;\; \text{(continuity)}. \end{aligned} \tag{2.34}$$

We find

$$A = B + C, \tag{2.35}$$

$$C = \frac{1}{a_+ - a_-}. \tag{2.36}$$

$$B = -\frac{a_+}{a_-} C \frac{e^{a_- x_0}}{e^{a_+ x_0}}. \tag{2.37}$$

At $g = 0$ these value are equal to the result of the no-drift subsection. The normalization of $\rho(x)$ is still given by noting that

$$\int_{-\infty}^{x_0} \rho(x) = \frac{1}{h'} \tag{2.38}$$

We use the same argument of the previous subsection to find x_0. The integral

$$\begin{aligned} \int_{-\infty}^{x_0} x\rho(x) &= \int_{-\infty}^{0} A x e^{a_+ x} + \int_0^{x_0} B x e^{a_+ x} + \\ &+ \int_0^{x_0} C x e^{a_- x}. \end{aligned} \tag{2.39}$$

can be shown to be

$$\int_{-\infty}^{x_0} x\rho(x) = \frac{g}{h'^{3/2}} - a_+ \frac{e^{a_- x_0}}{h'^2} \tag{2.40}$$

which turn out to be equal to (2.30) for $g = 0$ and to the result without the SB mechanism for $x_0 \to \infty$. The position of the reflective boundary can be fixed by inversion of:

$$a_+ \frac{e^{a_- x_0}}{h'^2} = \epsilon' \ . \tag{2.41}$$

Connection with microscopic parameters

We studied the process with diffusion coefficient equal to 1 (the constant in front of $\partial_x^2 \rho$). In order to find the connection with the microscopic parameters one should consider the diffusion coefficient to be $\sigma^2/2$.

Hence we can use

$$\mu' = \frac{\mu - (\sigma^2/2)}{(\sigma^2/2)} \tag{2.42}$$

$$h' = \frac{h}{(\sigma^2/2)}, \tag{2.43}$$

In addition to these, the parameter ϵ' that we introduced as a mean displacing that is due to SB should be

$$\epsilon' = -\lambda \log(1 - \epsilon). \tag{2.44}$$

Moving to the notation used in the paper, and keeping track also on a factor h' coming from the normalization of $\rho(x)$ we find

$$\frac{a_+ s_0^{a_-}}{4h^2/\sigma^4} = -\lambda \log(1 - \epsilon) \tag{2.45}$$

where $s_0 = e^{x_0}$. The previous equation can be used to fix x_0 when the microscopic parameters of the model are known. As in the previous cases, the exponents read

$$a_- = -1/2 + \mu/\sigma^2 + 1/2\sqrt{(1 - 2\mu/\sigma^2)^2 + 8h/\sigma^2} \tag{2.46}$$

$$a_+ = -1/2 + \mu/\sigma^2 - 1/2\sqrt{(1 - 2\mu/\sigma^2)^2 + 8h/\sigma^2}. \tag{2.47}$$

Inverting (2.45) is easy and one can thus easily compute the full distribution

$$\rho(s) = \begin{cases} As^{a_+} & s \leq 1 \\ Bs^{a_+} + Cs^{a_-} & 1 \leq s \leq s_0 \end{cases} \tag{2.48}$$

Or, for large s the cumulative distribution

$$P(S > s) = -\frac{1}{a_-(a_+ - a_-)}\left(s^{a_-} - \frac{s_0^{a_-}}{s_0^{a_+}} s^{a_+}\right) \tag{2.49}$$

For $s \ll s_0$ this equation shows *the same power law exponent* of the case with no SB, controlled by $\beta = -a_-$. Close to the cut-off the distribution shows a sharp drop to zero.

2.B A comment on power laws and criticality

The ubiquity of power laws distributions in both social and natural sciences asks for a deeper understanding about mechanisms generating them. In section 2.3 we give a short account of the mechanisms based on proportional random growth, or multiplicative noise, which is particularly convincing for wealth distribution models, but it is not appropriate for several different power laws arising in language, biology, earthquakes and so on.

In addition, very often power laws are observed with an exponent β that is very close to unity (Zipf's law), like in the data on firms asset presented in this chapter. We recall that $\beta = 1$ corresponds to a critical point in the space of power law distributions, signalling the condensation transition. In fact, when $\beta < 1$, even in a very large sample, the average properties are controlled by a finite number of events.

In the models described in section 2.3, the specific value $\beta = 1$ corresponds to a special point. While in reference [19], in order to attain distribution with a tail $\beta < 1$ some sparsity has to be introduced, in the model of reference [21] $\beta = 1$ is the point of optimal sustainable growth. In the more general point of view expressed in reference [47], Zipf's law is critical, because in an exponential representation of the probability distribution, it turns out to reduce to a linear relationship between "energy", the logarithm of the p.d.f., and entropy. However, while in the usual context of critical phenomena, criticality requires the fine tuning of some external control parameters, in order to poise the system on the specific critical point, such a mechanism seems to be missing in these distributions in complex systems. In other words, many of these distribution seems to be self poised at critical point [47].

To account for this ubiquity in very diverse systems, a series of less specific mechanisms for the occurring of Zipf's law have been proposed in the literature, which could also be relevant for the present subject of economic distributions. Zipf's law could be attained in a complex system of interacting objects thanks to some learning process [48], to the presence of hidden variables [49] or it can be a trade-off between cooperation and competition [50]. Finally, in reference [51], it is proposed that Zipf's law, as well as power laws in general, can appear even when it is not an inherent property of the real distribution of a given system, but as a result of the effort of extracting more information as possible from an incomplete sample, in a regime where the sampling is poor.

Chapter 3

The Chilling Inequality

The issue of economic inequalities is central in the present economic [52] and political [53] debate. It is beyond the scope of this chapter to discuss why it is so, but it is surely related with the observation that in recent years the levels of inequality have grown dramatically and have reached the same levels as in the beginning of the 20th century [23, 25, 26, 52]. The recent availability of highly detailed datasets on income [26] and wealth [25], that we discuss in section 3.1, shows how empirical household distributions, similar to firm sizes ones, are compatible with a power law description. In this setting, the aforementioned rise in inequality can be described by a decrease in the Pareto exponent, that has been taking place steadily in the last 30 years, both for the income and the wealth distribution.

Without clear yardsticks marking levels of inequality that seriously hamper the functioning of an economy, the debate on inequality very often remains at a qualitative or ideological level. For these reason, from section 3.2 onwards, we introduce a stylized model that addresses the issue of the efficiency of an economy with high degree of inequality. In particular, the main goal of the present work is to isolate the relation between inequality and liquidity in the simplest possible model that allows us to draw sharp and robust conclusions.

Specifically, the model that we introduce in section 3.2 is based on a simplified trading dynamics in which agents with a Pareto distributed wealth randomly trade goods of different prices. Agents receive offers to buy goods and each of these transaction is executed if it is compatible with the budget constraint of the buying agent. This reflects a situation where, at those prices, agents are indifferent between all feasible allocations.

The results, summarized in section 3.3 and section 3.4, show that when inequality in the wealth distribution increases, financial resources (i.e. cash) concentrate more and more in the hands of few agents (the wealthiest), leaving the vast majority without the financial means to trade, resulting in the freezing of the economy.

Our main finding is that, in the simplified setting of our model, there is a sharp threshold beyond which inequality becomes intolerable. More precisely, when the power law exponent of the wealth distribution approaches one from above, liquidity vanishes and the economy halts because all available (liquid) financial resources concentrate in the hands of few agents. This provides a precise, quantitative measure of when inequality becomes too much.

3.1 Inequality in the long run

The debate on the sustainability of economic inequalities has a long history, dating back at least to the work of Kuznets [54] on the u-shaped relationship between inequality and development, who discussed how a society with a too large degree of inequality grows less than a more fair one (at least in developed countries). Following Kuznets, much research in this field has focused on the relation between inequality and growth (see e.g. [55]). Inequality has also been suggested to be positively correlated with a number of indicators of social disfunction, from infant mortality and health to social mobility and crime [56].

Despite these seminal contributions, inequality has been a marginal topic in most economic theories, which relegate it more to an ethical issue, though important, than to one of the main factors concerning growth. The subject has regained considerable interest recently, in view of the claim that levels of inequality have reached the same levels as in the beginning of the 20th century [26, 52]. After the global financial crisis, and the slow growth of the subsequent period (Great Recession), the prevailing view on inequality is changing. In the words of Blanchard, former chief economist at IMF, “ as the effects of the financial crisis slowly diminish, another trend may come to dominate the scene, namely rising inequality. Though inequality has always been perceived to be a central issue, until recently it was not seen as having major implications for macroeconomic developments. This belief is increasingly called into question. How inequality affects both the macroeconomy, and the design of macroeconomic policy, will likely be increasingly important items on our agenda for a long time to come ” [57, 58] .

Piketty and Saez [26], and Saez and Zucman [25] study the evolution of the distributions of income and wealth among the US households over the last century, and they find increasing concentration, of both income and wealth, in the hands of the 0.01% of the richest.

Figure 3.1 shows the Pareto exponents of the wealth and income distributions, obtained assuming that both the data in references [26] and [25] are consistent with a power law distribution

$$P(c_i > x) \sim x^{-\beta}, \tag{3.1}$$

where c_i is the wealth (or income) of the of the $i-$th household. Such an assumption is justified with a good agreement down to the 10% of the richest (see the inset in figure 3.4 in section 3.5).

The Pareto exponents in figure 3.1 are estimated by noting that both references [26] and [25] report the fraction $c_>$ of wealth (or income) in the hands of the $P_> = 10\%, 5\%, 1\%, 0.5\%, 0.1\%$ and 0.01% richest individuals. If the fraction of individuals with wealth (or income) larger than c is proportional to $P_>(c) \sim c^{-\beta}$, the wealth share $c_>$ in the hands of the richest $P_>$ percent of the population satisfies $c_> \sim P_>^{1-1/\beta}$ (for $\beta > 1$). Hence β is estimated from the slope of the relation between $\log P_>$ and $\log c_>$. The error on β is computed as three standard deviations in the least square fit. The quality of the fit is not constant through the years, for instance the data on wealth better fits a power law distribution after the 1960s (see the error bars in figure 3.1).

Wealth inequality is much greater than income inequality. While having approximately similar behaviour along the years, the Pareto exponent for wealth is always clearly smaller than the income one. The exponents β, both for the income and the wealth distributions, have been steadily decreasing in the last 30 years, reaching the same levels it attained at the beginning of the 20th century (for wealth, $\beta = 1.43 \pm 0.01$ in 1917 and $\beta = 1.38 \pm 0.01$ in 2012).

Traditionally, data on wealth are less discussed than data on income, since the latter has always been easier to obtain, due to the fact that taxation in almost every country is mainly based on labour income. Even from the theoretical side, models of wealth are conceptually more complicated, mainly because wealth accumulates gradually over long period of time. However, in the following we concentrate our analysis on wealth. In the next section, we introduce a model that, rather than focusing on the determinants of inequality, focus on a specific consequence of it, i.e. liquidity, which is the ability of the economy to allow agents to exchange goods.

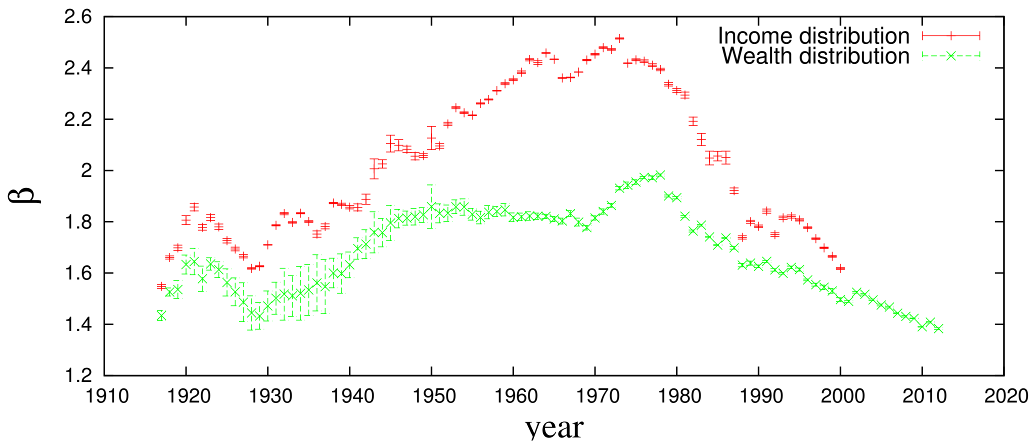

Figure 3.1: Pareto exponent β of the income and wealth distribution as a function of time. Both time series refer to the US. The data on the income distribution is retrieved from [26] and covers the period from 1917 to 2000, the data on the wealth distribution is taken from [25] and covers the period 1917-2012.

3.2 The model: a zero-intelligence agent-based trading dynamics

The model consists of N agents, each with wealth c_i with $i = 1, \dots, N$. Agents are allowed to trade among themselves M objects. Each object $m = 1, \dots, M$ has a price π_m. A given allocation of goods among the agents is described by an $N \times M$ allocation matrix $\mathcal{A}$ with entries $a_{i,m} = 1$ if agent i owns good m and zero otherwise. Agents can only own baskets of goods that they can afford, i.e. whose total value does not exceed their wealth. The wealth not invested in goods

$$c_i - \sum_{m=1}^{M} a_{i,m}\pi_m = \ell_i \geq 0, \qquad i = 1, \dots, N, \tag{3.2}$$

corresponds to the cash (liquid capital) that agent i has available for trading. The inequality $\ell_i \geq 0$ for all i indicate that lending is not allowed. Therefore the set of feasible allocations – those for which $\ell_i \geq 0$ for all i – is only a small fraction of the M^N conceivable allocation matrices $\mathcal{A}$.

Starting from a feasible allocation matrix $\mathcal{A}$, we introduce a random trading dynamics in which a good m is picked uniformly at random among all goods. Its owner then attempts to sell it to another agent i drawn uniformly at random among the other agents. If agent i has enough cash to buy the product m, that is if $\ell_i \geq \pi_m$, the transaction is successful and his/her cash decreases by π_m while the cash of the seller increases by π_m. We do not allow objects to be divided. Notice that the total capital c_i of agents does not change over time, so c_i and the prices π_m are parameters of the model. The entries of the allocation matrix, and consequently the cash, are dynamical variables, which evolve over time according to this dynamics. This model belongs to the class of zero-intelligent agent-based models, in the sense that agents do not try to maximize any utility function.

An interesting property of our dynamics is that the stochastic transition matrix $W(\mathcal{A} \to \mathcal{A}')$ is symmetric between any two feasible configurations $\mathcal{A}$ and $\mathcal{A}'$: $W(\mathcal{A} \to \mathcal{A}') = W(\mathcal{A}' \to \mathcal{A})$. We note that any feasible allocation $\mathcal{A}$ can be reached from any other feasible allocation $\mathcal{A}'$ by a sequence of trades. This implies that the dynamics satisfies the detailed balance condition, with a stationary distribution over the space of feasible configurations that is uniform: $P(\mathcal{A}) = \text{const.}$ Alternative choices of dynamics which also fulfil these conditions are explored in appendix 3.A.

In particular, we focus on realisations where the wealth c_i is drawn from a Pareto distribution $P(c_i > c) = c^{-\beta}$, for c larger than a fixed c_{min}, for each agent i. With c_{min} and β being two parameters describing our wealth distribution, we let β vary to explore different levels of inequality, and compare different economies in which the ratio between the total wealth $C = \sum_i c_i$ and the total value of all objects $\Pi = \sum_m \pi_m$ is kept fixed. We use $C > \Pi$ so as to have feasible allocations. We consider cases where the M objects are divided into a small number K of classes with M_k objects per class ($k = 1, \ldots, K$); objects belonging to class k have the same price $\pi_{(k)}$. If $z_{i,k}$ is the number of object of class k that agent i owns, then (3.2) takes the form $c_i = \sum_{k=1}^{K} z_{i,k}\pi_{(k)} + \ell_i$.

3.3 Main results

The main result of this model is that the flow of goods among agents becomes more and more congested as inequality increases until it halts completely when the Pareto exponent β tends to one from above.

The origin of this behaviour can be understood in the simplest setting where $K = 1$, i.e. all goods have the same price $\pi_m = \pi_{(1)} = \pi$ (we are going to omit the subscript (1) in this case). Figure 3.2 shows the capital composition $\{(\langle z \rangle_i, c_i)\}_{i=1}^{N}$ for all agents in the stationary state, where $\langle z \rangle_i$ is the average number of goods owned by agent i. The population of agents separates into two distinct classes: a *class of cash-poor* agents, who own an average number of goods that is very close to the maximum allowed by their wealth, and a *cash-rich class*, where agents have on average the same number of goods. These two classes are separated by a sharp crossover region. The inset of figure 3.2 shows the cash distribution $P_i(\ell/\pi)$ (where $\ell/\pi = c_i/\pi - z$ represents the number of goods they are able to buy) for some representative agents. While cash-poor agents have a cash distribution peaked at 0, the wealthiest agents have cash in abundance.

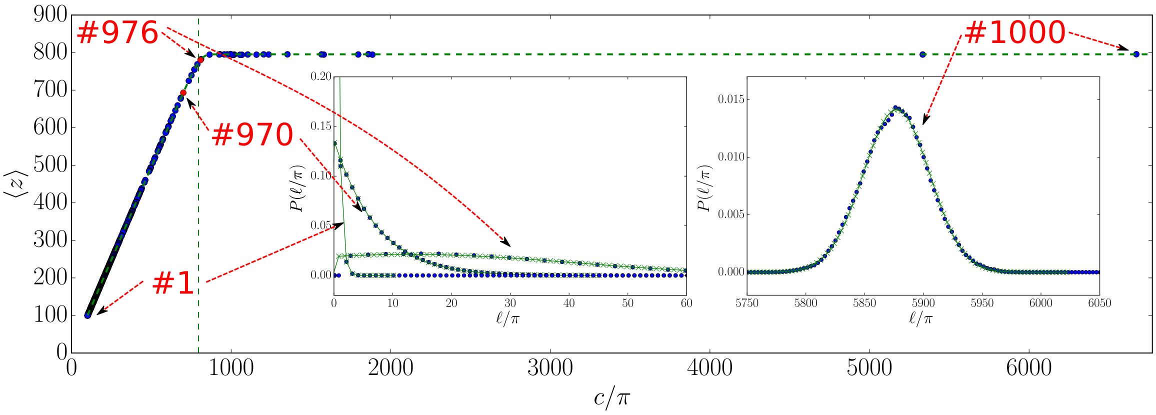

Figure 3.2: Capital composition in an economy with a single type of good, $N = 10^3$ agents, $\beta = 1.8$, $c_{\text{min}} = 1$, $\pi = 0.01$, $M \approx 2.10^5$ and $C/\Pi = 1.1$. Points $\{(\langle z \rangle_i, c_i)\}_{i=1}^N$ denote the average composition of capital for different agents obtained in Monte Carlo simulations. This is compared with the analytical solution obtained from the master equation (green dashed line) given by equation (3.7). The vertical dashed line at $c^{(1)} \simeq 7.98 = M/Np_1^{(\text{suc})}$ indicates the analytically predicted value of the crossover wealth that separates the two classes of agents. Insets: cash distributions $P_i(\ell)$ of the indicated agents.

These two observations allow us to trace the origin of the arrest in the economy back to the shrinkage of the *cash-rich class* to a vanishingly small fraction of the population, as $\beta \to 1^+$. As we'll see in the next section, when β is smaller than 1 the fraction of agents belonging to this class vanishes as $N \to \infty$. In this regime, not only the wealthiest few individuals own a finite fraction of the whole economy's wealth, as observed in reference [19], but they also drain all the financial resources in the economy.

These findings extend to more complex settings. Figure 3.3 illustrates this for an economy with $K = 10$ classes of goods (see figure caption for details) and different values of β. In order to visualise the freezing of the flow of goods we introduce the success rate of transactions for goods belonging to class k, denoted as $p_k^{(\text{suc})}$. Figure 3.3 shows that, as expected, for a fixed value of the Pareto exponent β the success rate increases as the goods become cheaper, as they are easier to trade. Secondly it shows that trades of all classes of goods halt as β tends to unity, that is when wealth

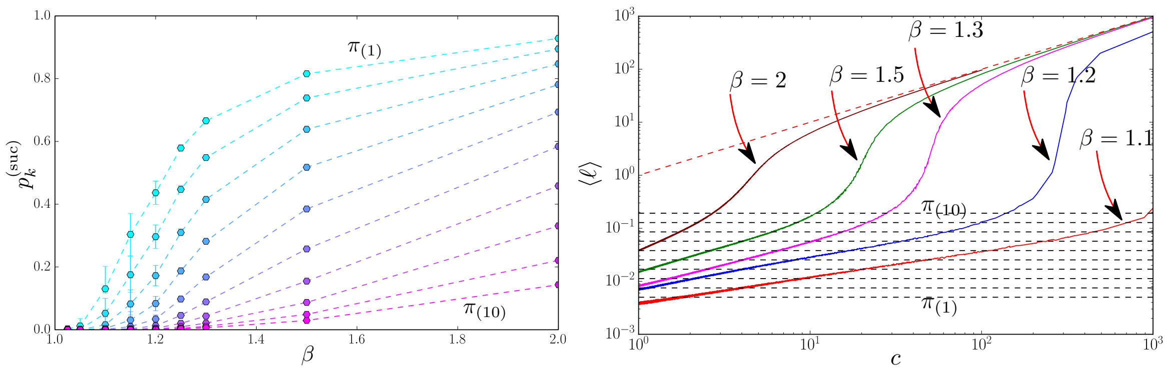

Figure 3.3: Left: Liquidity of goods $\{p_k^{(\mathrm{suc})}\}_{k=1}^K$ as a function of the inequality exponent β for a system of $N = 10^5$ agents exchanging $K = 10$ classes of goods ($\pi_{(k)} = \pi_{(1)} g^{k-1}$ with $g = 1.5$, $\pi_{(1)} = 0.005$, $M_k \pi_{(k)} = \Pi/K$, $c_{\min} = 1$ and $C/\Pi = 1.2$). Note that all success rates $p_k^{(\mathrm{suc})}$ vanish when $\beta \to 1^+$. The curves are ordered from the cheapest (top) to the most expensive (bottom). The markers are the result of numerical simulations, with error bars indicating the minimum and maximum values obtained by averaging over 5 realizations of the wealth allocations (for more details on the simulations see appendix 3.E). Right: for the same simulations with $K = 10$ classes of goods, we plot the time averaged cash $\langle \ell_i \rangle$ as a function of wealth c_i, from $\beta = 1.1$ to $\beta = 2$. The dashed lines indicate the different prices of goods. Agents with $\langle \ell_i \rangle$ below the price of a good typically have not enough cash to buy it. Cash is proportional to wealth for large levels of wealth (see the upper straight red dashed line).

inequality becomes too large, independently of their price.

The decrease of $p_k^{(\mathrm{suc})}$ when inequality increases (i.e. as β decreases) is a consequence of the concentration of cash in the hands of the wealthiest agents. This can be observed in the right panel of figure 3.3, which shows the average cash of agents with a given wealth, for different values of β. The freezing of the economy when β decreases occurs because fewer and fewer agents can dispose of enough cash (i.e. have $\ell > \pi_{(k)}$) to buy the different goods (prices $\pi_{(k)}$ correspond to the dashed lines).

Note finally that $p_k^{(\mathrm{suc})}$ quantifies liquidity in terms of goods. In order to have an equivalent measure in terms of cash that can be compared to the velocity of money, we average $\pi_{(k)} p_k^{(\mathrm{suc})}$ over all goods

$$\bar{p}^{(\mathrm{suc})} = \frac{1}{\Pi} \sum_{k=1}^{K} M_k \pi_{(k)} p_k^{(\mathrm{suc})}. \tag{3.3}$$

This quantifies the frequency with which a unit of cash changes hand in our model economy, as a result of a successful transaction. Its behaviour as a function of β for the same parameters of the economy in figure 3.3 is shown in the right panel of figure 3.4.

3.4 The analytical approach to the stationary state

In order to shed light on the findings described above, in this section we describe how to derive them within an analytic approach. We start by dealing with the simpler case where all the goods in the system have the same price $\pi_m = \pi$, $\forall m$ (i.e. $K = 1$).

A formal approach to this problem consists in writing the complete master equation that describes the evolution of the probability $P(z_1, \ldots, z_N)$ to find the economy in a state where each agent $i = 1, \ldots, N$ has a definite number z_i of goods. Taking the sum over all values of z_j for $j \neq i$, one can derive the master equation for a single agent with wealth c_i (see appendix 3.B for more details). The corresponding marginal distribution $P_i(z)$ in the stationary state can be derived from the detailed balance condition

$$P_i(z+1)\frac{z+1}{M}p^{(\mathrm{suc})} = P_i(z)\frac{1}{N}\left(1-\delta_{z,m_i}\right), \qquad z = 0, 1, \ldots, m_i \tag{3.4}$$

where $m_i = \lfloor c_i/\pi \rfloor$ is the maximum number of goods which agent i can buy with wealth c_i and $p^{(\mathrm{suc})}$ is the probability that a transaction where agent i sells one good (i.e. $z+1 \to z$) is successful. Equation (3.4) says that, in the stationary state, the probability that agent i has z objects and buys a new object is equal to the probability to find agent i with $z+1$ objects, selling successfully one of them. The factor $1 - \delta_{z,m_i}$ enforces the condition that agent i can afford at most m_i goods and it implies that $P_i(z) = 0$ for $z > m_i$. Exchanges are successful if the buyer j does not already have a saturated budget $z_j = m_j$. So the probability $p^{(\mathrm{suc})}$ is also given by

$$\begin{aligned} p^{(\mathrm{suc})} &= 1 - \frac{1}{N-1}\sum_{j\neq i} P(z_j = m_j | z_i = z) & (3.5)\\ &\cong 1 - \frac{1}{N}\sum_{j} P_j(m_j) \qquad (N, M \gg 1)\ . & (3.6)\end{aligned}$$

In equation (3.5), $p^{(\mathrm{suc})}$ is related to agent i only, and depends on the fact that agent i owns z objects. When $N, M \gg 1$, the dependence on z in the conditional probability becomes negligible ($P(z_j = m_j|z_i = z) \simeq P(z_j = m_j)$), and $p^{(\mathrm{suc})}$ can be approximated as in equation (3.6). In such a limit, $p^{(\mathrm{suc})}$ can be approximately considered the same for all agents. This is important, because it implies that for N large the variables z_i can be considered as independent, i.e. $P(z_1, \ldots, z_N) = \prod_i P_i(z_i)$, and the problem can be reduced to that of computing the marginals $P_i(z_i)$ self-consistently.

The solution of equation (3.4) can be written as a truncated Poissonian with parameter $\lambda = M/(Np^{(\mathrm{suc})})$

$$P_i(z) = \frac{1}{Z_i}\left[\frac{\lambda^z}{z!}\right]\Theta\left(m_i - z\right) \tag{3.7}$$

with Z_i is a normalization factor that can be fixed by $\sum_z P_i(z) = 1$. Finally, the value of $p^{(\mathrm{suc})}$ – or equivalently of λ – can be found self-consistently, by solving equation (3.6).

Notice that the most likely value of z for an agent with $m_i = m$ is given by

$$z^{\mathrm{mode}}(m) \equiv \arg\max_z P(z) = \begin{cases} m, & \text{if } m \leq \lambda \\ \lambda, & \text{if } \lambda \leq m \end{cases}. \tag{3.8}$$

This provides a natural distinction between cash-poor agents – those with $m \leq \lambda$ – that often cannot afford to buy further objects, and cash-rich ones – those with $m > \lambda$ – who typically have enough cash to buy further objects.

This separation into two classes of agents was already pointed out in figure 3.2. In terms of wealth, the poor are defined as those with $c_i < c^{(1)}$ whereas the rich ones have $c_i > c^{(1)}$, where the threshold wealth is given by $c^{(1)} = \lambda\pi = M\pi/(Np^{(\mathrm{suc})})$. Notice that when $\lambda \gg 1$, a condition that occurs when the economy is nearly frozen ($p^{(\mathrm{suc})} \ll 1$), the distribution $P_i(z)$ is sharply peaked around $z^{\mathrm{mode}}(m)$ so that its average is $\langle z \rangle \simeq z^{\mathrm{mode}}(m)$. Then the separation between the two classes becomes rather sharp, as in figure 3.2.

In this regime, we can also derive an estimate of $p^{(\mathrm{suc})}$ in the limit $N \to \infty$, for $\beta > 1$. Indeed, we have $P_i(z = m_i) \simeq 1 - \frac{m_i}{\lambda} + O(\lambda^{-2})$ for $\lambda \gg m_i$, so a rough estimate of $P_j(m_j)$ is given by $P_j(m_j) \simeq \max\{0, 1 - m_j/\lambda\}$. Taking the average over agents, as in equation (3.6), and assuming a distribution density of wealth $\rho(c) = \beta c^{-\beta-1}$ for $c \geq 1$ and $\rho(c) = 0$ for $c < 1$, one finds (see appendix 3.C)

$$c^{(1)} \simeq \left[\beta\left(1 - \frac{\Pi}{C}\right)\right]^{1/(1-\beta)}, \tag{3.9}$$

$$p^{(\mathrm{suc})} = \frac{M}{N\lambda} \simeq \frac{\Pi}{C}\frac{\mathbb{E}\left[c\right]}{c^{(1)}}. \tag{3.10}$$

Here $\mathbb{E}\left[c\right] = \beta/(\beta - 1)$ is the expected value of the wealth. Notice that $\mathbb{E}\left[c\right]$ diverges as $\beta \to 1^+$, but also that within this approximation the threshold wealth $c^{(1)}$ diverges much faster, with an essential singularity. More precisely, we note that $\Pi/C < 1$, so that $\beta(1-\Pi/C) \sim (1-\Pi/C)$ is a number smaller than 1 (yet positive). From equation (3.9), we have $c^{(1)} \sim (1-\Pi/C)^{-1/(\beta-1)} \to \infty$. Therefore the liquidity $p^{(\mathrm{suc})}$ vanishes as $\beta \to 1^+$.

For finite N, this approximation breaks down when β gets too close to or smaller than one. Also, $\mathbb{E}[c]$ is ill-defined and in equation (3.10) it should be replaced with $\langle c\rangle \equiv 1/N\sum_i c_i$, which strongly fluctuates between realizations and depends on N. An estimate of $p^{(\mathrm{suc})}$ for finite N and $\beta < 1$ can be obtained by observing that the wealth $c^{(1)}$ marking the separation between the two classes cannot be larger than the wealth $c_{\max}$ of the wealthiest agent. By extreme value theory, the latter is given by $c_{\max} \sim N^{1/\beta}$. Therefore the solution is characterised by $c^{(1)} = \pi\lambda \sim c_{\max} \sim N^{1/\beta}$. Furthermore, for $\beta < 1$ the average wealth is dominated by the wealthiest few, i.e. $\langle c\rangle \sim N^{1/\beta-1}$ and therefore $p^{(\mathrm{suc})} \sim \langle c\rangle/c^{(1)} \sim N^{-1}$. In other words, in this limit the cash-rich class is composed of a finite number of agents, who hold almost all the cash of the economy. Figure 3.5 (left) shows that the rough analytical estimate of equation (3.10) is in good agreement with Monte Carlo simulations.

The analysis carries forward to the general case in which K classes of goods are considered, starting from the full master equation for the joint probability of the ownership vectors $\vec{z}_i = (z_{i,1}\ldots, z_{i,K})$ for all agents $i = 1,\ldots,N$. For the same reasons as before, the problem can be reduced to that of computing the marginal distribution $P_i(\vec{z}_i)$ of a single agent. The main complication is that the maximum number $m_{i,k}$ of goods of class k that agent i can get now depends on how many of the other goods agent i owns, i.e. $m_{i,k}(z_i^{(k)}) = \lfloor (c_i - \sum_{k'(\neq k)} z_{i,k'}\pi_{(k')})/\pi_k\rfloor$, where $z_i^{(k)} = \{z_{i,k'}\}_{k'(\neq k)}$. The detailed balance condition

$$P_i(\vec{z}+\hat{e}_k)\frac{z_k+1}{M}p_k^{(\mathrm{suc})} = P_i(\vec{z})\frac{M_k}{M}\frac{1}{N}\left(1-\delta_{z_k,m_{i,k}(z_{(k)})}\right) \tag{3.11}$$

again yields the stationary state distribution (for $N, M \gg 1$). On the left we have the probability that one of the z_k+1 objects of type k of agent i is picked for a successful sale (here $\hat{e}_k$ is the vector with all zero components and with a k^{th} component equal to one, and $p_k^{(\mathrm{suc})}$ is the probability that a sale of an object of type k is successful). This must balance the probability (on the r.h.s.) that agent i is selected as the buyer of an object of type k, which requires that agent i has less than $m_{i,k}(z_{(k)})$ objects of type k, for the transaction to occur (here M_k/M is the probability that an object of type k is picked at random, and $1/N$ is the probability that agent i is selected as the buyer). It can easily be checked that the solution to this set of equations is given by a product of Poisson laws with parameters $\lambda_k = M_k/(Np_k^{(\mathrm{suc})})$, with the constraint equation (3.2),

$$P_i(z_1, ..., z_K) = \frac{1}{Z_i} \left[\prod_{k=1}^{K} \frac{\lambda_k^{z_k}}{z_k!} \right] \Theta \left(c_i - \sum_k^K z_k \pi_{(k)} \right) , \tag{3.12}$$

with Z_i a normalization factor obeying $\sum_{z_1} ... \sum_{z_K} P_i(z_1, ..., z_K) = 1$. Here the $p_k^{(\mathrm{suc})}$ corresponds to the acceptance rates of transactions of goods of class k and are given by

$$p_k^{(\mathrm{suc})} = 1 - \frac{1}{N} \sum_{i=1}^{N} P \left\{ z_{i,k} = m_{i,k}(z_i^{(k)}) \right\} \tag{3.13}$$

As in the case with $K = 1$, the values of the $p_k^{(\mathrm{suc})}$ need to be found self-consistently, which can be complicated when K and M are large.

When the total number of objects per agent is large for any class k, we expect that $\lambda_1, ..., \lambda_K \gg 1$, and then the values of $z_{i,k}$ are close to their expected values. This implies that the population of agents splits into K classes, where agents with wealth $c_i \in [c^{(k-1)}, c^{(k)}]$ have their budget saturated with goods of class $k' \leq k$ and cannot afford more expensive objects (here $c^{(k)} = \lambda_k \pi_{(k)}$, $k = 1, \ldots, K$ and $c^{(0)} = c_{\min}$). An estimate for the thresholds $c^{(k)}$ can be derived following the same arguments as for $K = 1$, by observing that when analysing the dynamics of goods of type k, all agents in class $k' < k$ are effectively frozen and can be neglected. Combining this with the conservation of the total number of objects of each kind, we obtain a recurrence relation for $c^{(k)}$. We refer the interested reader to the appendix 3.C for details on the derivation, and report here the result in the case of goods with $\pi_{(k)} = \pi_{(1)} g^{k-1}$, $g > 1$ large enough, with $\beta > 1$ and in the limit $N \to \infty$:

$$c^{(k)} \simeq \left[\beta^k - \left(\frac{\beta - \beta^{k+1}}{1 - \beta} \right) \frac{\Pi}{KC} \right]^{\frac{1}{1-\beta}} , \tag{3.14}$$

$$p_k^{(\mathrm{suc})} = \frac{M_k}{N \lambda_k} \simeq \frac{\Pi}{KC} \frac{\mathbb{E}\left[c\right]}{c^{(k)}} . \tag{3.15}$$

In the limit $\beta \to 1^+$ of large inequality, close inspection[1] of equation (3.14) shows that $c^{(k)} \to \infty, \forall k$, which implies that all agents become cash-starved except for the wealthiest few. Since $p_k^{(\mathrm{suc})} \sim \mathbb{E}\left[c\right] / c^{(k)}$, this implies that all markets freeze: $p_k^{(\mathrm{suc})} \to 0, \forall k$. The arrest of the flow of goods appears to be extremely robust against all choices of the parameter $\pi_{(k)}$, as $p_1^{(\mathrm{suc})}$ is an upper bound for the other success rates of transactions $p_k^{(\mathrm{suc})}$. These conclusions are fully consistent with the results of extensive numerical simulations (see figure 3.5 in appendix 3.C).

[1] Note that the term in square brackets is smaller than one, when $\beta \to 1^+$.

3.5 Conclusions and outlook

We have introduced a zero-intelligence trading dynamics in which agents have a Pareto distributed wealth and randomly trade goods with different prices. We have shown that this dynamics leads to a uniform distribution in the space of the allocations that are compatible with the budget constraints.

Unlike traditional models in Economics, in which agents try to maximize an utility function and the properties of the economy are derived from the equilibrium of the economy, the point in configuration space in which all agents' utilities are maximal, the typical properties of our stylized economy are simply a matter of entropy. This is the main difference on how equilibrium is intended in Economics and Statistical Physics. While in the former equilibrium is the solution of a complicated optimization problem, in the latter it is intended as a *statistical ensemble*, in which even sub-optimal configurations are weighted with a non vanishing probability, giving rise to entropy which quantifies the relevance of these configurations. In this respect, the model presented here is a limiting case where only entropy matters.

The main result of this model is that when the inequality in the distribution of wealth increases, the economy converges to an equilibrium where typically (i.e. with probability very close to one) the less wealthy agents have less and less cash available, as their budget becomes saturated by objects of the cheapest type. At the same time this class of cash-starved agents takes up a larger and larger fraction of the economy, thereby leading to a complete halt of the economy when the distribution of wealth becomes so broad that its expected average diverges (i.e. when $\beta \to 1^+$). In these cases, a finite number of the wealthiest agents own almost all the cash of the economy.

The model presented here is intentionally simple, so as to highlight a simple, robust and quantifiable link between inequality and liquidity.

In particular, the model neglects important aspects such as *i)* agents' incentives and preferential trading, *ii)* endogenous price dynamics and *iii)* credit. It is worth discussing each of these issues in order to address whether the inclusion of some of these factors would revert our finding that inequality and liquidity are negatively related.

In particular, our model assumes that all exchanges that are compatible with budget constraints will take place, but in more realistic setting only exchanges that increase each party's utility should take place. Yet if the economy freezes in the case where agents would accept all exchanges that are compatible with their budget, it could be expected to freeze also when only a subset of these exchanges are feasible. Also the model assumes that all agents trade with the same frequency whereas one might expect that rich agents trade more frequently than poorer ones. Could liquidity be restored if trading patterns exhibit some level of homophily, with rich people trading more often and preferentially with rich people?

First we note that both these effects are already present in our simple setting. Agents with higher wealth are selected more frequently as sellers as they own a larger share of the objects. In spite of the fact that buyers are chosen at random, successful trades occur more frequently when the buyer is wealthy. So, in the trades actually observed the wealthier do trade more frequently than the less wealthy, and preferentially with other wealthy agents. Furthermore, if agents are allowed to trade only with agents having a similar wealth (e.g. with the q agents immediately wealthier or less wealthy) it is easy to show that detailed balance still holds with the same uniform distribution on allocations. As long as all the states are accessible, the stationary probability distribution remains the same[2]. Therefore, our conclusions are robust with respect to a wide range of changes in our basic setting that would account for more realistic trading patterns.

Secondly, it is reasonable to expect that prices will adjust – i.e. deflate – as a result of a diminished demand caused by the lack of liquidity. Within our model, the inclusion of price adjustment, occurring on a slower time-scale than trading activity, would reduce the ratio Π/C (between total value of goods and total wealth), but it would also change the wealth distribution. If we think of price adjustment as occurring on a slower time-scale than trading activity, this, within our model, would have the effect of reducing the ratio Π/C between the total value of goods and the total wealth, but it would also change the wealth distribution. Since the freezing phase transition occurs irrespective of the ratio Π/C, the first effect, though it might alleviate the problem, would not change our main conclusion. The second would make it more compelling, because cash would not depreciate as prices do, so deflation would leave wealthy agents – who hold most of the cash – even richer compared to the cash deprived agents, that would suffer the most from deflation. So while price adjustment apparently increases liquidity, this may promote further inequality, that would curtail liquidity further.

Finally, can the liquidity freeze be avoided by allowing agents to borrow? Access to credit, we believe, will hardly improve the situation in line with the results of reference

[2] The dynamics changes and thus $p_k^{(\mathrm{suc})}$ changes, in particular for goods more expensive than $\pi_{(1)}$, the seller is typically cash-rich and thus its neighbours are too. This can induce to have a liquidity of expensive goods higher than that of cheaper ones. However in the limit $\beta \to 1^+$, it is still true that cash concentrates in the hands of a vanishing fraction of agents, and there is still a freeze of the economy.

[20] and for similar reasons. Allowing agents to borrow using goods as collaterals is equivalent to doubling the wealth of cash-starved agents, provided that any good can be used only once as a collateral, and that goods bought with credit cannot themselves be used as collaterals. This would at most blur the crossover between cash-rich agents and cash-starved ones, as intermediate agents would sometimes use credit. This does not change our main conclusion that inequality and liquidity are inversely related and that the economy would halt when $\beta \to 1^+$. Credit may mitigate illiquidity in the short term, but cash deprived agents should borrow from wealthier ones. With positive interest rates, this would make inequality even larger in the long run. So credit is likely to make things worse, in line with the arguments of Piketty [23], who observes that when the rate of return on capital exceeds the growth rate of the economy (which is zero in our setting), wealth concentrates more in the hands of the rich.

Therefore, even though the model presented here can be enriched in many ways, we don't see a way in which the relation between inequality and liquidity could be reversed.

Corroborating the present model with empirical data goes beyond the scope of our work, yet we remark that our findings are consistent with the recent economic trends, as shown in figure 3.4. The direct measure that quantifies the efficiency of an economy, in our simple model ,is the number of possible exchanges that can be realised or equivalently the probability that a random exchange can take place. This probability quantifies the "fluidity" of exchanges and we call it *liquidity* in this chapter.

A quantitative measure of liquidity is provided by the *velocity of money* [60], measured as the ratio between the nominal Gross Domestic Product and the money stock and it quantifies how often a unit of currency changes hand within the economy. In figure 3.4 we report data on the MZM (money with zero maturity), the broadest definition of money stock that includes all money market funds. We refer to [59] for further details. As figure 3.4 shows, the velocity of money has been steadily declining in the last decades.

Our model suggests that this decline and the increasing level of inequality are not a coincidence. Rather the former is a consequence of the latter. In addition, it is worth observing that, alongside with increasing levels of inequality, trade has slowed down after the 2008 crisis. The *U.S. Trade Overview*, published by the International Trade Administration in 2013, observes that "Historically, exports have grown as a share of U.S. GDP. However, in 2013 exports contributed to 13.5% of U.S. GDP, a slight drop from 2012'" [61]. A similar slowing down can be observed at the global level, in the UNCTAD *Trade and Development Report, 2015* (see page 7 in reference

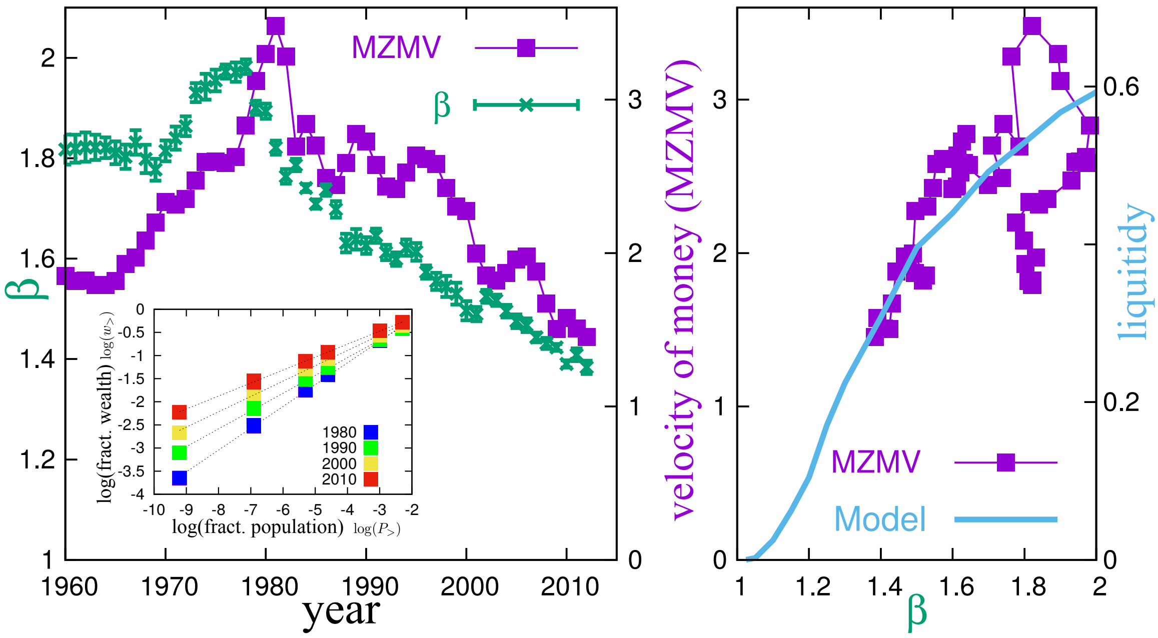

Figure 3.4: Left: Velocity of money of MZM stocks (right y-axis) and Pareto exponent β of the wealth distribution (left y-axis) as a function of time. Both time series refer to the US. The data on the money velocity is retrieved from [59], the data on the wealth distribution is taken from [25]. Inset: relation between the fraction $w_>$ of wealth owned by the $P_>$ percent wealthiest individuals, and $P_>$ for the years 1980, 1990, 2000 and 2010. Right: MZM velocity of money (MZMV, central y-axis) as a function of β, for the same data. Liquidity, defined as the probability that a unit-money random exchange takes place, (right y-axis) as a function of β, in the synthetic economy described by our model (see equation (3.3) and figure 3.3 for details on the numerical simulations).

[62]).

More generally, avoiding deflation -or promoting inflation- has been a major target of monetary policies after 2008, which one could take as an indirect evidence of the slowing down of the economy. In fact liquidity, as intended here, has been the primary concern of monetary polices such as Quantitative Easing aimed at contrasting deflation and the slowing down of the economy, in the aftermath of the 2008 financial crisis.

Furthermore, the fact that inequality hampers liquidity and hence promotes demand for credit suggests that the boom in credit market before 2008 and the increasing levels of inequality might not have been a coincidence.

An interesting side note is that the concentration of capital in the top agents goes hand in hand with a flow of cash to the top. Indeed, in our model an injection of extra capital in the lower part of the wealth pyramid –the so-called *helicopter money* policy– is necessarily followed by a flow of this extra cash to the top, via many intermediate agents, thus generating many transactions on the way. This *trickle up* dynamics should be contrasted with the usual idea of the *trickle down* policy, which advocates injections of money to the top in order to boost investment. In this respect, it is tempting to relate our findings to the recent debate on Quantitative Easing measures, and in particular to the proposal that the (European) central bank should finance households (or small businesses) rather than financial institutions in order to stimulate the economy and raise inflation [63, 64]. Clearly, our results support the helicopter money policy, because injecting cash at the top does not disengages the economy from a liquidity stall.

Extending our minimal model to take into account the endogenous dynamics of the wealth distribution and of prices, accounting for investment and credit, is an interesting avenue of future research, for which the present work sets the stage. In particular, this could shed light on understanding the conditions under which the positive feedback between returns on investment and inequality, that lies at the very core of the dynamics which has produced ever increasing levels of inequality according to [23, 25, 26], sets in.

In fact, a tentative extension of the present model in which wealth is not fixed, but it changes much more slowly with respect to the relaxation of the exchange dynamics can be formulated shortly. In this case in which the two dynamics are decoupled, if the return of investment is assumed to be proportional to the amount of liquid capital at disposal of each agent, the model presented in this chapter can give a purely entropic mechanism for the observed growth in inequality, since wealthier agents are also the only ones who can afford to invest a consistent fraction of their capital.

This model can be thought of the starting point for addressing more complex issues, such as the effect of investment and the interplay between finance and the real economy. In fact, while data on individual wealth (see [25]) usually have an exponent of the power law that is larger than one, in firms sizes data, which are considered in Chapter 2, this exponent is steadily less than one, especially when only financial firms are considered. Finance has been described as an formidable tool to increase the market efficiency, through its ability to perform an optimal inter-temporal resource allocation. In the context of this model, the decoupling discussed in the previous chapter - the financial deepening, might be related to the excessive broadness of the financial firms distribution, resulting in the freezing of the financial market ability to perform this resource allocation task in real economy investments.

3.A About the rules providing detailed balance

The detailed balance condition is a useful criterium to find the stationary state in stochastic processes. Given a dynamics formulated in terms of the transition rates $W(\mathcal{A}_i, \mathcal{A}_j)$ between configurations $\mathcal{A}_i$ and $\mathcal{A}_j$, If one can find a measure $P(\mathcal{A}_i) \geq 0$ over configurations that satisfies the detailed balance condition

$$\forall i, j, \qquad W(\mathcal{A}_i, \mathcal{A}_j) P(\mathcal{A}_i) = W(\mathcal{A}_j, \mathcal{A}_i) P(\mathcal{A}_j), \tag{3.16}$$

and if the system is ergodic[3], then $P(\mathcal{A}_i)$ is the unique stationary distribution. The detailed balance condition is thus a local balance of the probability flux between any pair of configurations.

The simplest way to impose detailed balance is to use symmetrical transfer rates: $W(\mathcal{A}_i, \mathcal{A}_j) = W(\mathcal{A}_j, \mathcal{A}_i)$. In that case, one automatically gets a uniform distribution over the space of configurations: $P(\mathcal{A}) = \text{const}, \forall \mathcal{A}$. The flux $W(\mathcal{A}_1, \mathcal{A}_2)P(\mathcal{A}_1)$ is then also uniform. It is clear that the dynamics defined here has this property, because for any two configurations that differs by the ownership of one object, the rate of the process linking them is equal to $1/(NM)$ in both directions.

What about the rules providing detailed balance, but without symmetry of the rates? In that case, one would need to explicitly find the probability density over the configurations. Since the resulting density would be non-uniform, it would be more difficult to link dynamical observables (rate of money transfer, etc.) to static variables (number of neighbouring configuration to a given configuration). We do not explore these cases.

What are the rules that give symmetrical transfer rates? Here, we consider the simplest case where objects are picked independently of their price[4]. This still leaves us several choices. There are $N-1$ rules which yield symmetric rates (and thus respect detailed balance). The generic case is the following, with $2 \leq n \leq N$:

- rule $\#n$: The integer n is fixed. Select n distinct agents at random. Select one object among the set of all the objects they (collectively) own. This object will be sold (if possible) by the owner to a randomly selected agent among the $n-1$ remaining agents in the set of selected agents.

[3] Meaning that for each pair of configurations $\mathcal{A}_i$ and $\mathcal{A}_j$ there is a path of a finite number of intermediate configurations $\mathcal{A}_{i_k}$ with non-zero rate $W(\mathcal{A}_{i_k}, \mathcal{A}_{i_{k+1}})$,

[4] One could pick an object with a rate proportional to its value. This kind of choice would still give the same phase space and thus the same probability distribution over microstates, but the dynamics could become very different in terms of the speed of transactions, in particular it could fluctuate much more.

This generic rule is a bit cryptic, but has two particular cases that are clearer:

- rule #2: Select two distinct agents at random. Select one object among the set of all the objects they (collectively) own. This object is sold (if possible) by the owner to the other agent.
- rule #N: Pick an object at random. The owner is then the seller. Select a buyer at random among the $N-1$ remaining agents.

Note that the rule $\#n = 1$ does not make sense, so that there are indeed $N-1$ different rules. Here, we always use the rule #N, i.e. simply pick the object at random. As all these rules produce an ergodic dynamics, and since the probability distribution of configurations is the same for all rules (it is $P(\mathcal{A}) = \text{const}$), it does not matter which of these dynamical rules we picked.

3.B The full and the mean field master equations

The proposed dynamics (rule #N in the previous appendix) is very simple: we pick an object at random, pick an agent at random and assign the object to the agent if possible. Let us write down the master equation for this dynamics in the most general way, for a system with N agents, each with capital c_i, for $i = 1, \ldots, N$ and M objects.

The full master equation

We recall that an allocation of goods among the agents is described by an $N \times M$ allocation matrix $\mathcal{A}$ with entries $a_{i,m} = 1$, if agent i owns good m and zero otherwise. A change in this allocation could be described in terms of some operator $\hat{e}_{i,m}$, such that $\hat{e}_{i,m} = 0$ always, except for agent i and good m, in which $\hat{e}_{i,m} = 1$. Then we have the following transition rates

$$\mathcal{R}^m_{i\to j} = W[\mathcal{A} \to \mathcal{A} - \hat{e}_{i,m} + \hat{e}_{j,m}], \tag{3.17}$$

describing the transition of the good m from agent i to agent j. In the case considered in the main text, where the goods are divided into K classes of different prices $\pi_{(k)}$, this transition rates for a good in the class k, can be written explicitly in terms of the $N \times K$ matrix $\mathcal{Z}$ with entries $z_{i,k}$, specifying the number of goods of class k that agent i own

$$\mathcal{R}^k_{i\to j} = W[\mathcal{Z} \to \mathcal{Z} - \hat{e}_{i,k} + \hat{e}_{j,k}] = \frac{M_k}{M}\frac{z_{i,k}}{M_k}\frac{1}{N}\,\theta\left[c_j - \sum_{k'} \left(z_{j,k'} + \hat{e}_{j,k}\right)\pi_{(k')}\right]. \tag{3.18}$$

The first fraction specifies the probability that a good in the class k is picked up, the second the probability that this good belongs to agent i and the third the probability that the agent that is picked up as a buyer is agent j. Finally, the constrained imposed by the θ function guarantees that the budget constraint of agent j is not violated. If we denote $P(\mathcal{Z},t)$ the probability that the system is found in the state $\mathcal{Z} = z_{i,k}$ at time t, its continuous time master equation is

$$\frac{\partial P(\mathcal{Z},t)}{\partial t} = \sum_{\mathcal{Z}'} \{W[\mathcal{Z}' \to \mathcal{Z}]P(\mathcal{Z}',t) - W[\mathcal{Z} \to \mathcal{Z}']P(\mathcal{Z},t)\}. \tag{3.19}$$

where the sum is performed over all the allocation matrix $\mathcal{Z}'$ which are 1 exchange away from $\mathcal{Z}$.

The mean field master equation

The most general master equation (3.19) is of little practical use. In the mean field approximation we write $P(\mathcal{Z},t) = \prod_{i=1}^{N} P(z_{i,k},t)$ inside equation (3.19), and we marginalize over the state of the system of all agents except agent i; we refer to this state using the notation $\mathcal{Z} \setminus i$. This allows us to write K general mean field master equation for a single agent i, each of one reads

$$\frac{\partial P(z_{i,k},t)}{\partial t} = \sum_{z'_{i,k}} \{w[z'_{i,k} \to z_{i,k}]P(z'_{i,k},t) - w[z_{i,k} \to z'_{i,k}]P(z_{i,k},t)\}, \tag{3.20}$$

where we have defined

$$w[z'_{i,k} \to z_{i,k}] = \sum_{\mathcal{Z}\setminus i} \sum_{\mathcal{Z}'\setminus i} W[\mathcal{Z}' \to \mathcal{Z}]P(\mathcal{Z}' \setminus i, t) \tag{3.21}$$

$$w[z_{i,k} \to z'_{i,k}] = \sum_{\mathcal{Z}\setminus i} \sum_{\mathcal{Z}'\setminus i} W[\mathcal{Z} \to \mathcal{Z}']P(\mathcal{Z} \setminus i, t) \tag{3.22}$$

This can be done explicitly, for simplicity, in the case in which all agents have the same capital ($c_i = c$) and all goods have the same price ($\pi_{(k)} = \pi$). In this case, the notation can be further simplified substituting the matrix $\mathcal{Z}$ with a vector $\vec{z} = (z_1 \ldots, z_N)$ specifying the number of goods owned by each agents. The rates of the master equation can be calculated, for instance the transition rate leading to z_i number of goods for the agent i, when the agent is a buyer, is given by

$$
\begin{aligned}
w[z_i - 1 \to z_i] &= \sum_{\vec{z}\setminus i} \sum_{\vec{z}'\setminus i} W[\vec{z}' \to \vec{z}] P(\vec{z}' \setminus i, t) \\
&= \sum_{\vec{z}\setminus i} \sum_{j} W[\vec{z} - e_i + e_j \to \vec{z}] P(\vec{z} \setminus i + e_j, t) \\
&= \sum_{\vec{z}\setminus i} \sum_{j} \frac{z_j}{M} \frac{1}{N} \theta\left[c - z_i \pi\right] P(\vec{z} \setminus i + e_j, t) \\
&\simeq \frac{M - (z_i - 1)}{M} \frac{1}{N} \theta\left[c - z_i \pi\right]
\end{aligned}
$$

In the first passage we used the fact that all the non-zero rates leading to z_i in which agent i is a buyer, involve a single exchange, from an agent j. In the last passage we performed the sum over $\vec{z} \setminus i$ and we put $\sum_{\vec{z}\setminus i} P(\vec{z} \setminus i + e_j, t) = 1$, ignoring the fact that there is a dependence on the configuration of the system in the transition rates. This dependence can be expected to be negligible when the number of agents and the number of goods are both very large, because it depends only on the value of z_i. More precisely, the assumption that is done is that $P(\vec{z} \setminus i) = P(\vec{z} \setminus i | z_i)$.

Analogous calculations can be done for the case in which agent i is a seller, i.e.

$$
\begin{aligned}
w[z_i + 1 \to z_i] &= \sum_{\vec{z}\setminus i} \sum_{\vec{z}'\setminus i} W[\vec{z}' \to \vec{z}] P(\vec{z}' \setminus i, t) \\
&= \sum_{\vec{z}\setminus i} \sum_{j} W[\vec{z} + e_i - e_j \to \vec{z}] P(\vec{z} \setminus i - e_j, t) \\
&= \sum_{\vec{z}\setminus i} \sum_{j} \frac{z_i + 1}{M} \frac{1}{N} \theta\left[c - (z_j + 1)\pi\right] P(\vec{z} \setminus i - e_j, t) \\
&\simeq \frac{z_i + 1}{M} p^{(\mathrm{suc})},
\end{aligned}
$$

where in the last passage $p^{(\mathrm{suc})} = \langle \theta\left[c - (z_j + 1)\pi\right] \rangle$ is introduced as the average number of buyer who can afford the transaction.

Finally, calculating all 4 possible terms giving non vanishing contributions in

(3.20), we get

$$\begin{aligned}\partial_t P(z_i,t) =&P(z_i-1,t)\frac{1}{N}(1-\delta(z_i-1,m))\\ &+P(z_i+1,t)\frac{z_i}{M}\left(p^{(\mathrm{suc})}\right)\\ &-P(z_i,t)\frac{1}{N}(1-\delta(z_i,m))\\ &-P(z_i,t)\frac{z_i}{M}\left(p^{(\mathrm{suc})}\right),\end{aligned} \tag{3.23}$$

where the θ constraints have been turned into δ constraints, by introducing m as the integer part of c/π.

The detailed balance condition implies that, when looking for the stationary solution of this equation $\partial_t P(z_i,t)=0$, the terms have to cancel in pairs, leading to equation (3.4) in the main text.

3.C Computation of $p_k^{(\mathrm{suc})}$ in the large λ limit

Derivation of $p^{(\mathrm{suc})}$ and $c^{(1)}$ in the large λ limit for 1 type of good.

As discussed in the main text, we can compute $p^{(\mathrm{suc})}$ using

$$p^{(\mathrm{suc})} = 1 - \frac{1}{N}\sum_{i=1}^{N} P_i(z=m_i) \tag{3.24}$$

approximating the probability to be on a threshold $P_i(z=m_i)$ by

$$P_i(z=m_i) = \begin{cases}\left(1-\frac{m_i}{\lambda}\right) & \text{for } m_i \ll \lambda \\ 0 & \text{for } m_i > \lambda\end{cases}. \tag{3.25}$$

The first case can be understood by noting that

$$P_i(z=m_i) = \frac{\lambda^{m_i}\frac{1}{m_i!}}{\sum_{x=0}^{m_i}\lambda^x\frac{1}{x!}} = \frac{1}{1+\frac{m_i}{\lambda}+\frac{m_i(m_i-1)}{\lambda^2}+\dots} \simeq \left(1-\frac{m_i}{\lambda}\right), \tag{3.26}$$

where the approximation is valid in the limit $m_i \ll \lambda$. Assuming this approximation to be valid in all the range $m_i < \lambda$ is clearly a bad assumption for all agents with m_i close to λ. However the wealth is power law distributed, and so the weight of agents with $m_i \sim \lambda$ is negligible in the sum over all agents, equation (3.24). The accuracy of this approximation increases when the exponent of the power law β decreases.

Then $p^{(\text{suc})}$ can be computed using

$$p^{(\text{suc})} = 1 - \frac{1}{N}\sum_{i=1}^{N} P_i(z = m_i) \simeq 1 - \int_1^{c^{(1)}=\lambda\pi} dc\, \beta c^{-\beta-1}\left(1 - \frac{c}{\lambda\pi}\right). \tag{3.27}$$

This is an implicit expression for $p^{(\text{suc})}$, since it appears on the l.h.s. of the equation and also on the r.h.s. (because $\lambda = \frac{M}{Np^{(\text{suc})}}$).

When $\beta > 1$ this expression can be expressed to be realization-independent, using

$$p^{(\text{suc})} = \frac{M}{N\lambda} = \frac{\Pi}{C}\frac{\mathbb{E}\left[c\right]}{c^{(1)}}, \tag{3.28}$$

where $\mathbb{E}\left[c\right] = \beta/(\beta - 1)$ is the expected value of the wealth per agent. We also use the fact that we fill in the system a number M of goods in such a way to have a fixed ratio Π/C. Performing the integral on the r.h.s of equation (3.27) gives an equation for $c^{(1)}$:

$$\frac{\Pi}{C}\frac{\mathbb{E}\left[c\right]}{c^{(1)}} = {c^{(1)}}^{-\beta}\left(\frac{1}{1-\beta}\right) - \frac{\beta}{1-\beta}\frac{1}{c^{(1)}}, \tag{3.29}$$

that simplifies into:

$$c^{(1)} = \left[\beta\left(1 - \frac{\Pi}{C}\right)\right]^{1/(1-\beta)}. \tag{3.30}$$

Derivation of $p_k^{(\text{suc})}$ and $c^{(k)}$ in the large λ limit for several types of good.

An analytic derivation for the $p_k^{(\text{suc})}$ and $c^{(k)}$ can be obtained also for the cases of several goods, but only in the limit in which prices are well separated (i.e. $\pi_{(k+1)} \gg \pi_{(k)}$) and the total values of good of any class is approximately constant (we use $M_k\pi_{(k)} = \Pi/K = \text{const}$). In this limit we expect to find a sharp separation of the population of agents into classes. This is because $M_1 \gg M_2 \gg \ldots \gg M_K$ implies that the market is flooded with objects of the class 1, which constantly change hands and essentially follow the laws found in the single type of object case. On top of this dense gas of objects of class 1, we can consider objects of class 2 as a perturbation (they are picked M_2/M_1 times less often!). On the time scale of the dynamics of objects of type 2, the distribution of cash is such that all agents with a wealth less than $c^{(1)} = \pi_{(1)}\lambda_1$ have their budget saturated by objects of type 1 and typically do not have enough cash to buy objects of type 2 nor more expensive ones. Likewise, there is a class of agents with $c^{(1)} < c_i \leq c^{(2)}$ that will manage to afford goods of types 1 and 2, but will hardly ever hold goods more expensive that $\pi_{(2)}$.

In brief, the economy is segmented into K classes, with class k composed of all agents with $c_i \in [c^{(k)}, c^{(k+1)})$ who can afford objects of class up to k, but who are excluded from markets for more expensive goods, because they rarely have enough cash to buy goods more expensive than $\pi_{(k)}$. This structure into classes can be read off from figure 3.3, where we present the average cash of agents, given their cash in a specific case (see caption). The horizontal lines denote the prices $\pi_{(k)}$ of the different objects, and the intersections with the horizontal lines define the thresholds $c^{(k)}$. Agents that have c_i just above $c^{(k)}$ are cash-filled in terms of object of class k, but are cash-starved in terms of objects $\pi_{(k')}, k' > k$.

The liquidities $p^{(\mathrm{suc})}$ can be given by the following expression

$$p_k^{(\mathrm{suc})} = 1 - \frac{1}{N}\sum_{i=1}^{N} P\left\{z_{i,k} = m_{i,k}(z_i^{(k)})\right\} = 1 - \frac{1}{N}\sum_{i=1}^{N} P_i(\text{not accepting good type k}) \tag{3.31}$$

According to the previous discussion of segmentation of the system into K classes, and using the same approximation for this threshold probability discussed in the case of 1 type of good, we assume

$$P_i(\text{not accepting good type k}) = \begin{cases} 1 & \text{for } m_i < \lambda_{k-1} \\ \left(1 - \frac{m_i}{\lambda_k}\right) & \text{for } \lambda_{k-1} < m_i < \lambda_k \ , \\ 0 & \text{for } m_i > \lambda_k \end{cases} \tag{3.32}$$

Then

$$p_k^{(\mathrm{suc})} \simeq 1 - \int_1^{c^{(k-1)}} dc\, \beta c^{-\beta-1} - \int_{c^{(k-1)}}^{c^{(k)}} dc\, \beta c^{-\beta-1}\left(1 - \frac{c}{c^{(k)}}\right) \tag{3.33}$$

In this case now we have

$$p_k^{(\mathrm{suc})} = \frac{M_k}{N\lambda_k} = \frac{\Pi}{KC}\frac{\mathbb{E}\left[c\right]}{c^{(k)}} \tag{3.34}$$

With similar calculations to the ones showed for the previous case, one can easily get to the recurrence relation:

$$c^{(k)} = \left[\beta\left(c^{(k-1)}\right)^{1-\beta} - \beta\frac{\Pi}{KC}\right]^{\frac{1}{1-\beta}}. \tag{3.35}$$

Iterating, we explicit this into:

$$c^{(k)} = \left[\beta^k - \left(\frac{\beta - \beta^{k+1}}{1-\beta}\right)\frac{\Pi}{KC}\right]^{\frac{1}{1-\beta}}, \tag{3.36}$$

A comparison between the analytical estimate and numerical simulations, presented in figure 3.5, shows that this approximation provides an accurate description of the collective behaviour of the model.

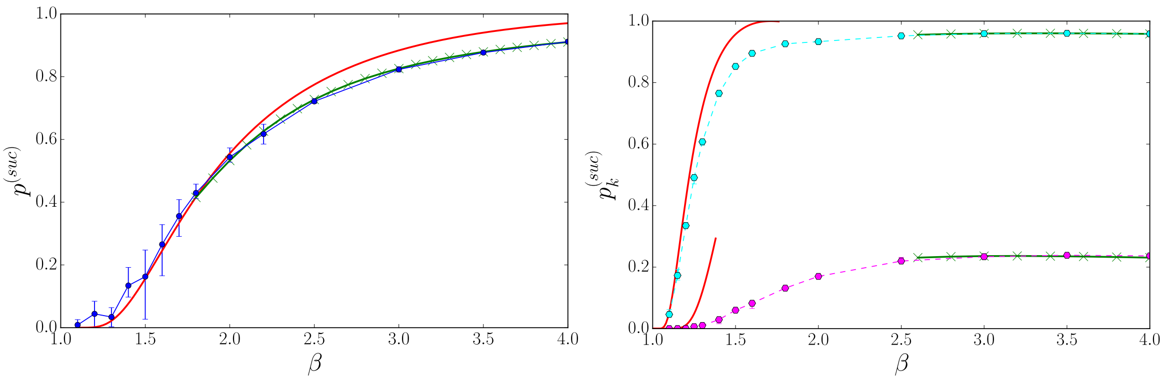

Figure 3.5: Success probability of transaction $p_k^{(\mathrm{suc})}$ as a function of the Pareto exponent β. Comparison between numerical simulations and analytical estimates for one class of goods (left panel) and two classes of goods (right panel). The blue solid circles are the result of Monte Carlo simulations performed for $N = 10^5$ agents and averaged over 5 realizations. Here the error bars indicate the min and max value of $p_k^{(\mathrm{suc})}$ over all realizations (we used the "adjusted Pareto" law for the right panel, see appendix 3.E). The red lines are the analytic estimates according to equation (3.10) and equation (3.14) for left and right panels, respectively. The green crossed lines correspond to numerically (see appendix 3.E) solving the analytical solution (3.12) for a population composed of $N = 64$ (kind of) agents.

See also in figure 3.6 how the liquidity over-concentrates (with respect to capital concentration). There, we compare the liquid and capital concentrations, measured via their Gini coefficients, for various values of β in the system of figure 3.3 ($K = 10, g = 1.5, \pi_{(1)} = 0.001, C/\Pi = 1.2$). In particular, note that the limit $\beta \to 1^+$ is singular, as G_ℓ reaches one around $\beta = 1.1$, with smaller β yielding also $G_\ell \approx 1$. This is an alternative way to see how the concentration of capital generates an over-concentration of liquidities.

3.D An argument for $p^{(\mathrm{suc})} \to 0$ for $\beta < 1$, in the $N \to \infty$ limit

In this appendix we give an different and intuitive argument to justify why we expect the $p^{(\mathrm{suc})}$ to go to zero, when $\beta < 1$, for a very large system, with number of agents $N \to \infty$. We are going to formulate the argument for one type of good, but it can be generalised to the generic case.

Let's rank the N agents from the richest c_1 to the poorest c_N. The argument can be divided into two main steps. First we will show that when $\beta < 1$ there is always a finite n such that the capital of the first n agents, the n richest in the ranking, is

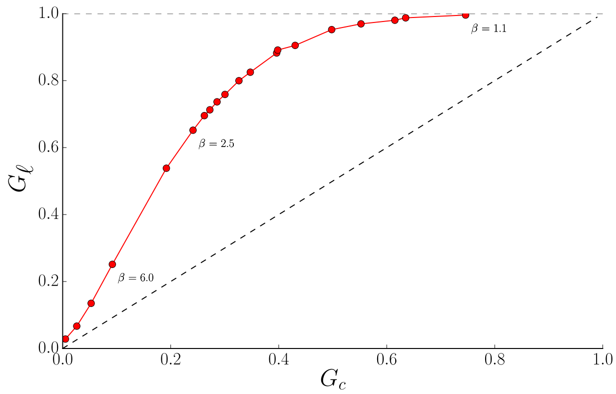

Figure 3.6: Gini coefficient G_ℓ of the cash distribution (liquid capital) in the stationary state of the model as a function of the Gini G_c of the wealth distribution. The dashed line indicates proportionality between cash and wealth, in which case the inequality in both is exactly the same. The wealth follows a Pareto distribution with exponent β that tunes the degree of inequality (the higher is β, the more egalitarian the distribution).

around the same size as total capital of the remaining agents

$$c_{\text{rich}} \equiv \sum_{i=1}^{n} c_i \simeq c_{\text{poor}} \equiv \sum_{i=n+1}^{N} c_i \tag{3.37}$$

and this n stays finite in the $N \to \infty$ limit. More generally we can find a threshold n for which the capital of the richest n agents is equal to any given finite fraction of the total capital of the system.

Secondly we will show that in a typical allocation of goods, the agents belonging to the poorest class of agents is typically with no cash and thus they mostly do not accept a transition when they are selected as buyers. Since the probability of selecting a buyer is uniform and since $\frac{n}{N} \sim 0$ in the large N limit, we will then prove that $p^{(\text{suc})}$ has to be null.

According to *rank ordering statistics* (see [34]), the typical value of the capital of the $i-$th agent in the ranking is

$$c_i^{\text{typ}} = c_1 \frac{1}{i^{1/\beta}} \tag{3.38}$$

The typical value for the total capital of first (second respectively) class of agents is

$$c_{\text{rich}}^{\text{typ}} = c_1 \sum_{i=1}^{n} \frac{1}{i^{1/\beta}} \qquad c_{\text{poor}}^{\text{typ}} = c_1 \sum_{i=n+1}^{N} \frac{1}{i^{1/\beta}} \tag{3.39}$$

When $\beta \geq 1$, the infinite series $\sum_i \frac{1}{i^{1/\beta}}$ is divergent, then in the $N \to \infty$ limit we have

$$\frac{\sum_{i=1}^{n} \frac{1}{i^{1/\beta}}}{\sum_{i=n+1}^{\infty} \frac{1}{i^{1/\beta}}} = 0 \tag{3.40}$$

for any finite n.

When instead $\beta < 1$ the infinite series is convergent, then there is a finite n for which

$$\frac{\sum_{i=1}^{n} \frac{1}{i^{1/\beta}}}{\sum_{i=n+1}^{\infty} \frac{1}{i^{1/\beta}}} \sim 1, \tag{3.41}$$

thus for which $c_{\text{rich}}^{\text{typ}} \simeq c_{\text{poor}}^{\text{typ}}$. More generally, an n can be found in such a way that the ratio of rich-poor capital is equal to any finite value.

In particular, we can also find a finite n for which the partition of the agents in rich and poor is such that, even if we initially distribute the goods among the agents in such a way that the all the poorer are filled with goods, and the only agents with some cash are the first n in the ranking, the goods are equally divided among the rich and the poor classes, i.e.

$$M_{\text{rich}} \simeq M_{\text{poor}}. \tag{3.42}$$

We show now that this configuration is quite close to a typical stationary allocation of goods, under our dynamical process. We can describe the dynamics in this system using a birth-death process using as a variable the "number of holes" in the poor class, i.e. the number of agents in the poor class that are not on their capital threshold. The transition rates for this variable i can be approximately written as

$$p(i \to i+1) \simeq \frac{n}{N} \tag{3.43}$$

$$p(i \to i-1) \simeq \frac{i}{N}. \tag{3.44}$$

In fact, the number of poor agents with some cash increases by one if a rich is selected as a buyer (event with probability n/N), and with probability almost 1 the buyer is going to be a poor agent on the capital threshold (if $N >> n$). This number decreases by one if a poor agent who is able to buy is selected as a buyer. When the goods are equally divided among rich and poor, the selection of the seller gives with the same probability an agent in the poor or in the rich class.

If the dynamics preserves these rates. it is easy to see that under a such birth-death process the typical number of holes in the poor phase are of the same order as n, hence typically agents in the poor class are not able to buy a good, and we can conclude that $p^{(\text{suc})}$ is order n/N in this limit.

3.E Details on the numerical methods

Monte-Carlo simulations

We perform our Monte Carlo simulations of the trading market for $N = 10^5$ agents. Prices generally start from $\pi_{(1)}$ and increase by a factor g between each good class. The minimal wealth is $c_{\min} = 1$. The ratio C/Π is fixed as indicated in captions, and most importantly is kept constant between different realizations. As the total wealth fluctuates, so does the total number of goods.

There are no peculiar difficulties with the numerical method (apart from the large fluctuations in the average wealth, addressed below). The only thing one has to be careful with is to ensure that the stationary state has been reached, i.e. that all observables have a stationary value, an indication that the (peculiar) initial condition has been completely forgotten. The codes for this Monte Carlo simulation are available online [65].

Adjusted Pareto wealth distribution

For $K \geq 2$ we have predictions for the $\beta \sim 1$ regime, in which the average wealth is particularly fluctuating from realization to realization. Because the value of $\mathbb{E}[c]$ controls the number M of objects introduced in the market, this in turns produces large fluctuations in the values of the $p_k^{(\mathrm{suc})}$ which can make it difficult to have robust results.

More importantly, the typical value of the (empirical) average wealth $\langle c \rangle$ is usually quite different from its expectation value $\mathbb{E}[c]$. This effect is well known and well documented for power laws, but we present a concrete example of it in figure 3.7 to emphasize its intensity.

For the sample size that is typically manageable in our simulations, i.e. $N = 10^5$, the typical value for the average value of the wealth (using e.g. $\beta = 1.1$) is of the order of the half of its expected value: $\langle c \rangle \approx 7 \approx \mathbb{E}[c]/2$. This indicates that $N = 10^5$ is (by far) an insufficient size to correctly sample a power-law with exponent $\beta = 1.1$.

To circumvent this problem, we introduce the "adjusted" Pareto distribution. The idea is to draw numbers from a power law distribution as usual, and then to adjust the value of a few of them so that the empirical average matches the expected one. The algorithm is the following: Start from a true random Pareto distribution.

- if $\langle c \rangle < \mathbb{E}[c]$, we select an agent at random and increase its wealth until we have exactly $\langle c \rangle = \mathbb{E}[c]$.

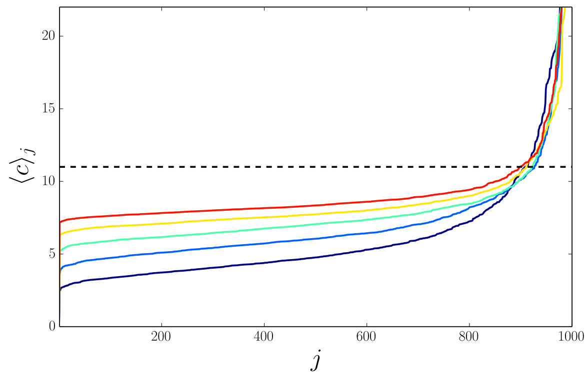

Figure 3.7: Distribution of the average $\langle c \rangle_j$ of power laws depending on their sample size N (from top to bottom, $N = 10^6, 10^5, 10^4, 10^3, 10^2$) for 1000 realizations each ($j = 1, ..., 1000$), using an exponent $\beta = 1.1$. The dashed line indicates the expectation value $\mathbb{E}[c] = c_{\min}\beta/(\beta - 1)$. We see that even for huge samples, the typical $\langle c \rangle_j$'s are significantly smaller than the expected $\mathbb{E}[c]$.

- if $\langle c \rangle > \mathbb{E}[c]$, we select the wealthiest agent and decrease its wealth until we have exactly $\langle c \rangle = \mathbb{E}[c]$, or until its wealth becomes $c_{\min}$. If we reach the latter case (it is quite unlikely), then we perform the same operation on the second-wealthiest agent, and so on until $\langle c \rangle = \mathbb{E}[c]$.

As can be seen in figure 3.7, the most common case is the first one. The corresponding adjustment is equivalent to re-drawing the wealth of a single agent until it is such that $\langle c \rangle = \mathbb{E}[c]$. This is a weak deviation from a true Pareto distribution. The second case is more rare, and mostly consists also in a correction on the wealth of a single agent.

This change in the wealth distribution is very efficient at reducing the variability between different realizations of the same β value. Furthermore it ensures that we can compare our numerical results at finite N with the predictions that implicitly assume $N = \infty$, since we now have $\langle c \rangle = \mathbb{E}[c]$. It is quite crucial to use this "adjusted" Pareto law for the small β's (i.e. for $\beta \leq 1.3$). See figure 3.8 to have an idea of what this modified distribution means: the only changes in the two sample shown would be in the values of the wealthiest agent.

Algorithm computing self-consistent solution $p^{(\mathrm{suc})}, Z(c_i)$

Here we describe the algorithm used to converge to a self-consistent set of values for $\{p_1^{(\mathrm{suc})}, p_2^{(\mathrm{suc})}, Z(c_1), ..., Z(c_N)\}$, i.e. solving equation (3.13) for $K = 2$ (or more simply equation (3.6) in the case of a single type of goods). It can be generalized

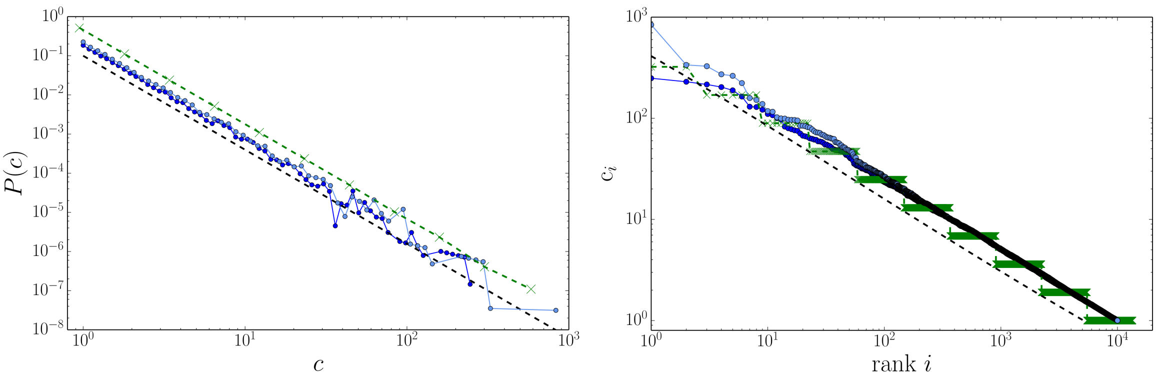

Figure 3.8: Different instances and representation of power-law distributed wealth (or "Pareto distribution"). Blue and pale blue circles are two realisations for $N = 10^4$ agents, green crosses are an example of staircase-like distribution (a useful approximation of a Pareto law that we use elsewhere) and the black dashed line is the law itself (blue dots converge to it in the $N \to \infty$ limit). Left: Probability distribution, with shifts up and down for clarity (i.e. it is not normalized) Right: wealth c_i of each agent, sorted by the rank i. Note that the wealth of the wealthiest agents (low rank) fluctuates a lot from realization to realization.

straightforwardly to $K > 2$, although it may become numerically extremely expensive (see also our code, [65]). The results (green crosses) presented in figure 3.5 were obtained using the method described here.

For each agent there is a constant $Z(c_i)$ to be determined self-consistently. This presents a technical difficulty, as for a true power-law distribution, each agent gets a different wealth and thus the number of constants to compute is N.

A way to tackle this difficulty is to consider a staircase-like distribution of wealth, where agents are distributed in groups with homogeneous wealth c_g and where the number of agents per group is $N_g \sim \int_{c_g}^{c_{g+1}} \rho(c)$ç, so that individual agents approximately follow a power law with exponent β. See figure 3.8 (green crosses) to have an idea of what this modified distribution means concretely. This kind of staircase distribution is not a true power-law, in particular because its maximum is always deterministic and finite. However, as we now have $1 < \mathcal{N} \ll N$, we can numerically solve the $\mathcal{N} + 1$ equations and thus find the exact value of $p^{(\mathrm{suc})}$. Of course, the value of $p^{(\mathrm{suc})}$ found in this way perfectly matches with Monte Carlo results if and only if we use the exact same distribution of wealth and goods in the simulation. This is not surprising at all, and merely validates our iterative scheme.

However, we note that staircase-like wealth distributions turn out to be very good approximations of true power laws, when the wealth levels c_g are sufficiently refined and the number of classes $\mathcal{N}$ sufficiently large. In particular, using $c_g = b^g$ with a

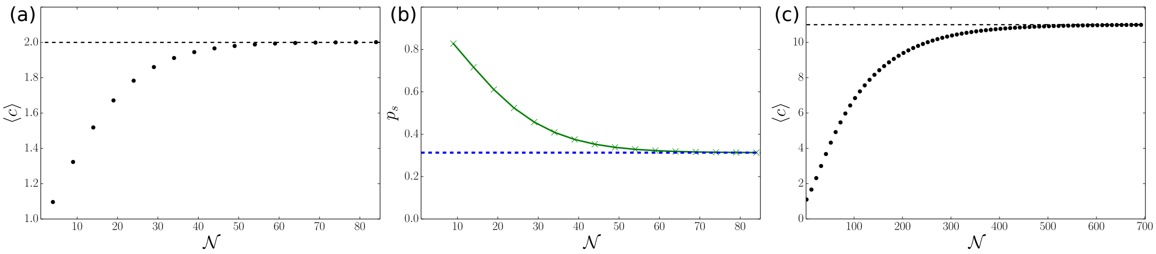

Figure 3.9: (a): Average wealth $\langle c \rangle$ dependence on $\mathcal{N}$ for a staircase distribution, using $\beta = 2$ and $b = 1.1$. Black dots: average computed (exactly) for the staircase distribution. Dashed black: expectation value for the corresponding true power law. Convergence is reached as soon as $\mathcal{N} \approx 50$.
(b): Dependence on $\mathcal{N}$ of the $p_1^{(\mathrm{suc})}$ computed from the iterative method, using a staircase-like distribution of wealth (green crosses). As soon as $\mathcal{N} > 50$, it approaches its "true" value, i.e. the value obtained for a true power-law with exponent $\beta = 2$ (dashed blue line). We used $b = 1.1$.
(c): Average wealth $\langle c \rangle$ dependence on $\mathcal{N}$ for a staircase distribution, using $\beta = 1.1$ and $b = 1.1$. Black dots: average computed (exactly) for the staircase distribution. Dashed black: expectation value for the corresponding true power law. It takes very large $\mathcal{N}$ to converge.

base $b \approx 1^+$, it can be seen that for large enough $\mathcal{N}$, the average wealth $\langle c \rangle$ converges to a value very close to the expected one $\mathbb{E}[c]$ (and no longer depends on $\mathcal{N}$). For large β, typically $\beta \geq 1.5$, convergence is reached rather fast ($\mathcal{N} \geq 50$ is enough), and the iterative method can be used (see for instance figure 3.9a). Under these conditions, the observables (e.g. $p^{(\mathrm{suc})}$) have the same values for a true power law and the corresponding staircase-distribution (see figure 3.9b). However for smaller values of β, convergence is very slow and one needs at least $\mathcal{N} > 200$ to converge (see figure 3.9c). The maximum wealth is then very large, which makes the iterative method useless for practical purposes (overflow errors arise, and the number of terms in the sums to be computed explodes exponentially, along with the computational cost). More details on the algorithm we actually use can be found in reference [65].

Chapter 4

Lost in Diversification

There is a growing consensus around the idea that increasingly complex financial products play an important role in the emergence of new instabilities and systemic risks [66] [67] [27]. Historically, financial innovations have been seen as a formidable tool to increase the efficiency of the market, reducing the risk associated with any investment strategy and ensuring an increasingly optimal resource allocation between investors and the real economy.

After the 2007-2008 global financial crisis, this picture has been showing signs of fraying. The most commonly believed determinant of the crisis is rooted in the financial bubble of the mortgage subprime structured finance market [28]. As it has been discussed widely in the economic literature [68] [69], the formidable complexity of these type of products brought down a curtain of opacity that was able to hide the true risk of the underlying assets (the subprime mortages). In section 4.1 we give a brief account of this history.

While the nature of these instruments as "financial weapons of mass destruction" [70] has been widely recognised, most of the response to the crisis did not address the core issue of the transparency loss implicit in financial transformations. For this reason, from section 4.2 onwards, we introduce a stylized model where both future returns of a pool of assets and some side information related to these assets are treated as random variables. The goal is to try to sort out how relevant the information is for the pricing of a financial product built from the aforementioned pool, and how this information should be transmitted in the most informative way.

In section 4.3, we define the key concepts of financial transformations, relevance of information as well as its price alongside with some general results. Finally, in sections 4.4 and 4.5 we explicitly construct an instance of such class of models for binary variable, derived from the maximum entropy principle.

4.1 The rise and fall of the structured finance market

Classical theories of financial markets are based on a set of very strong assumptions, which often lead to a systematic underestimation of both the non-systemic and the systemic risk. One of those assumption is the efficient market hypothesis [71] [72], which roughly states that the financial markets are able to process all the information coming from the real economy and from the news, and consequently the price of any stock exchanged in the market reflects faithfully this information.

The crisis, and in particular the role played as a trigger by the mortgage backed securities market, exposed a huge inefficiency in the financial market; many of these financial products were commonly perceived by investors as virtually risk-free and certified as such by rating agencies, even though all the information about the status of the underlying assets was available to market participants.

These financial instruments are usually called *Asset Backed Securities* (ABS) and when the individual assets which compose the pool are credits over some residential mortgages, they are called *Residential Mortgages Backed Securities* (RMBS). Although they are fairly complex themselves, they can be used as individual blocks to build even more complex products, like the *Collateralized Debt Obligations* (CDO). See figure 4.1 and its caption for a brief explanation of how these instruments work.

The rise and fall of the structured finance market has been dramatic [28]. In less than a decade, before the crisis, the issuance of these products within the U.S. economy have been growing by a factor larger than 10. About $100 billion in ABSs were issued in the last quarter of 2006 and in the first two quarters of 2007. However, at the beginning of 2008, these quantities dropped to less than $5 billion per quarter. At the same time, a vertical drop in the ratings of these products was observed, 90% of the ABS tranches underwritten by Merrill Lynch in 2007, initially rated AAA, were downgraded to junk (rated below investment grade) in just few weeks [28].

For a variety of reasons, market participants and rating agencies did not accurately measure the risk inherent with financial innovations such as ABSs and CDOs. One of such reasons, in the case of the RMBS market, was the fact that, despite these instruments were accompanied by a large documentation containing full details about the underlying assets, the status of the housing market and the infamous risk profiles of the mortgages typically involved, were effectively hidden to almost all investors [69] .

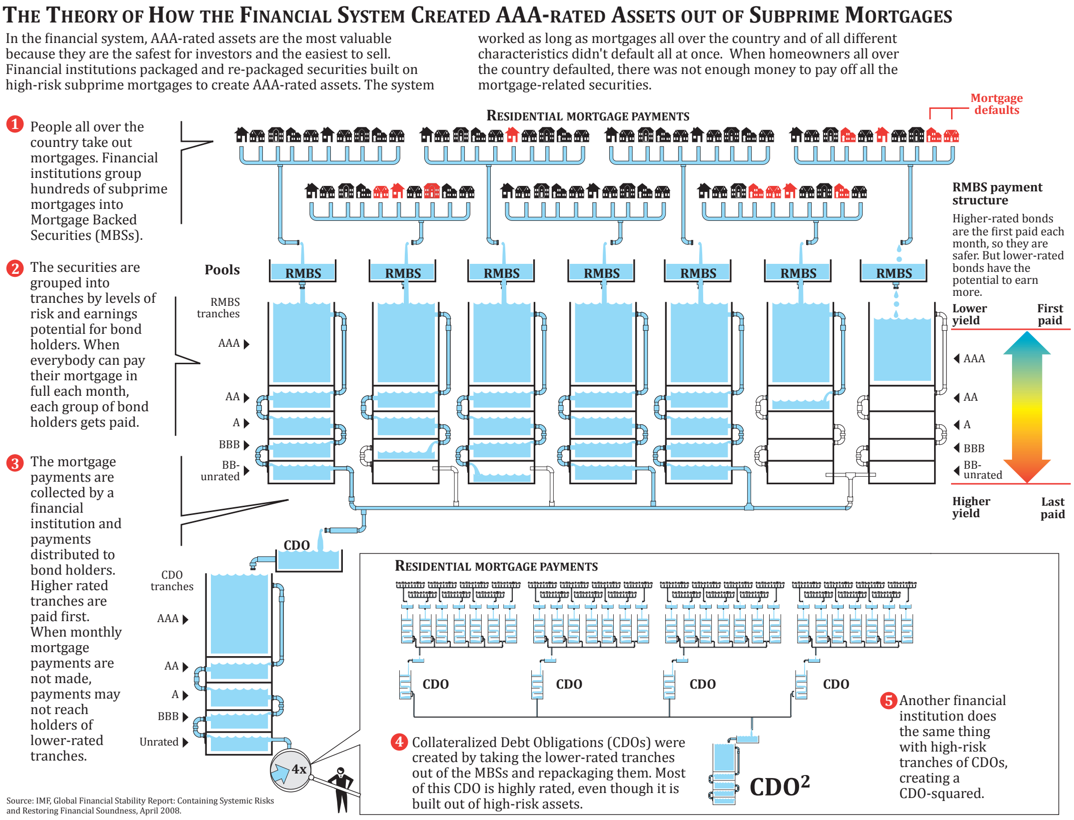

Figure 4.1: Taken from reference [73]. The cash flow coming from individual payments over some credits (residential mortgages in this case) is pictured as a water flow. When these credits are pooled together, the cash flow coming from these payments is collected, and distributed to investors according to a prioritized structure of claims. In practice, an investor who possess a given tranche of a RMBS gets a payoff if more than a given threshold of the payments which constitute the pool are regularly paid. The value of this threshold defines the seniority of the RMBS tranche; while a AAA tranche pays back even when few payments are executed, for a BB- tranche to give a positive payoff, many more (almost all) individual payments have to. The most junior tranches (more risky, rated BB- or unrated in this figure), have been used as the building blocks of different products, CDOs, which work with the same mechanism as the initial RMBSs, just with different underlying assets. This process can be further iterated, junior tranches of CDOs can be used as building blocks of a CDO^2.

After the crisis, the framework and the very existence of these structured finance products has been subject to criticism, and new models for risk assessment have been proposed, mainly focused on the issue of how to take into account the dependencies among assets, when computing future expectations of a set of assets. Nevertheless, the prevailing view is still that securitisation techniques are able to create low risk financial products that are somehow "information insensitive" or "money-like". The subtleness associated with this belief is that these products will consequently be exchanged between investors without the due diligence, in particular without an adequate analysis of the building blocks these products are composed with.

In order to oppose this tendency, the financial industry is pursuing an effort [29] to build an efficient and standardized system, or a common language, through which this information should be easily available to all market participants. Such a *financial barcode*, which should be attached to any financial product, should contain all the information that is relevant in order to make realistic estimates about return and risk of the product, from the risk profiles of the building blocks to the market fundamentals. Yet, it is not clear how such barcodes should be constructed, which information they should contain and whether they should be statically or dynamically updated, when new information is available. In particular, an interesting open question is whether demand for such barcodes may "naturally" arise and how it should be priced, since without a barcode price the seller does not have an incentive for sharing the information.

4.2 The model: how side information affects future returns

Let's suppose we have a pool of stocks $\{X_i\} = \{X_1, \ldots, X_n\}$, where each X_i is the return on an investment, e.g. a loan, a mortgage, an option or a generic financial asset. Together with these stocks, some additional variables $\{Y_i\}$ $=\{Y_1, \ldots Y_n\}$ are available, where each Y_i represents *side information* related to the stock X_i (e.g. the income of the borrower of the loan or information on the fundamentals of the stock). We assume that both the values of $\{X_i\}$ and the values of $\{Y_i\}$ are unknown to the investor, therefore they can be treated as random variables, described by a probability distribution $P(\{X_i\}, \{Y_i\})$.

The investor faces the decision problem of how to evaluate the uncertain future returns of the stocks $\{X_i\}$. He/she can perform this task either using a prior belief, encoded in the marginal distribution $P(\{X_i\})$, or retrieving the information $\{Y_i\}$ and using instead the conditional distribution $P(\{X_i\}|\{Y_i\})$. We are interested to understand in which cases the retrieval of the information is relevant for the risk assessment.

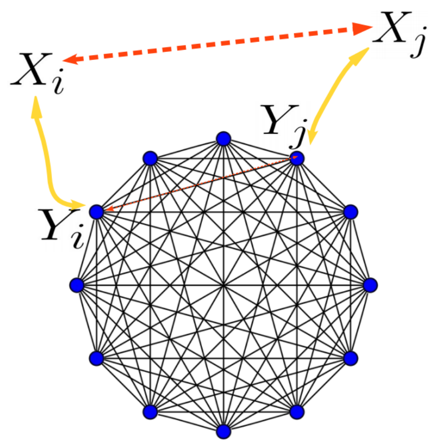

Figure 4.2: The general structure of the models we are going to consider. Each X_i interacts with the system only through its associated Y_i, as expressed in equation (4.1). The dependence between the returns of the assets is a consequence of the dependence among information $\{Y_i\}$.

The general structure of the models we are going to consider is defined by a joint p.d.f of returns and information. We assume the returns to be *conditionally independent* given the information. Namely the joint p.d.f has the form

$$\begin{aligned} P\left(\{X_i\},\, \{Y_i\}\right) &= P\left(\{Y_i\}\right) \times P\left(\{X_i\}\,|\{Y_i\}\right) \\ &= P\left(\{Y_i\}\right) \times \prod_i P\left(X_i\,|Y_i\right). \end{aligned} \tag{4.1}$$

The factorizability of the conditional $P\left(\{X_i\}\,|\{Y_i\}\right)$ implies that any dependence among returns, if it exists, enters in the game only through the $\{Y_i\}$, since each individual X_i becomes independent on the rest of the system when its own Y_i is specified. This assumption encodes the idea that when the most relevant factors affecting future returns are taken into account and quantified in the $\{Y_i\}$ variables, the additional ones act just like noise.

For simplicity, the conditional distributions $P(X_i|Y_i)$ in equation (4.1) are assumed to be identical and we are going to focus on models where the variables $\{Y_i\}$ are identically distributed, i.e. the distribution $P(\{Y_i\})$ is invariant under permutations of the variables $\{Y_i\}$.

In the following, we specialize to the case of binary variables

$$\begin{cases} X_i = \pm 1 \\ Y_i = \pm 1 \ . \end{cases} \tag{4.2}$$

For instance, we have in mind a situation where $X_i = 1$ if the investment yields a positive return or if a loan payment is regularly paid back, while $X_i = -1$ in the opposite case. We assume that the information associated with the stocks can be quantified and then quantized in binary variables as well. For instance, Y_i could be positive when the investment had positive returns at previous times (often financial time series show strong persistence) or when the previous payments have been paid in time by the borrower.

In realistic risk management practices, the probability distributions are never known exactly. Typically only few observables, such as expected returns and some correlations, are measured, often with low precision. These observed moments (e.g. $\mathbb{E}[X_i]$, $\mathbb{E}[X_i X_j]$ or $\mathbb{E}[X_i Y_i]$) can be used to infer the joint probability distribution of returns and information. Within the assumption of our model equation (4.1), and with binary variables, the conditional probability $P(X_i|Y_i)$ (identical for all i) is specified by just two numbers. The marginal distribution of the $\{Y_i\}$ can instead be more complex. Because of the aforementioned ignorance about the distribution and about its moments, we assume $P(\{Y_i\})$ to be the most general compatible with the few observed moments. Such an approach is usually referred to as *maximum entropy principle*, and it has been used in a wide range of applications, as well as in the foundations of Statistical Physics [74]. In the specific example of binary variables, when only first and second moments are known, and when the information variables are identically distributed, the result of the entropy maximization yields a fully connected Ising model [75]. The use of an Ising model for a joint return distribution has already been considered in the literature [76], and it has been show to weight large losses very differently from a standard multivariate Gaussian distribution, widely used by practitioners.

In the cases we are going to consider in more details in sections 4.4 and 4.5, the joint p.d.f. is specified by four parameters, two of them fixing the conditional p.d.f. $P(\{X_i\}|\{Y_i\})$ and two describing $P(\{Y_i\})$. In the next section we highlight some general results that are not specific of any particular choice for the model and we introduce some observables we are interested in.

4.3 General results

Financial products and complexity

A typical practice in finance is the pooling of a large number of assets, obtaining portfolios or more complex investment structures. In our model, such a financial product can be described through a function $F(\{X_i\})$, expressing the return of the product as a function of the returns of the individual assets that compose it. In the following, we are going to consider the *homogeneous portfolio*, or average return

$$\overline{X}(\{X_i\}) = \frac{1}{n}\sum_i X_i. \tag{4.3}$$

The return of such an investment is the average return of the different assets that are pooled together in the portfolio. This corresponds to one of the most basic diversification techniques, where the investor decides that instead of investing his/her total wealth in a single asset X_i, he/she invests a fraction $1/n$ in each of the n assets. The benefit of diversification is that it reduces risk. For example, for n i.i.d. stocks, we have that the variance $\mathbb{V}(\overline{X}) = \mathbb{V}(X_i)/n$ is reduced by a factor of n, with respect to that of the individual stocks. In the next subsection we clarify why the variance can be considered a proxy of the risk of a financial asset.

Another class of products we consider are *Asset Backed Securities* (ABS), the typical products of the structured finance market, whose return function is based over a prioritized structure of claims. In these products, the claims over the cash flow of the underlying asset returns $\{X_i\}$ are prioritized, structured in such a way that the ABS tranche yields a positive return when the total return is larger then a given threshold. The return of these products is

$$F_k(\{X_i\}) = \mathrm{sgn}\left(\sum_i X_i - k\right), \tag{4.4}$$

where $\mathrm{sgn}(x) = 1$ when $x \geq 0$ and $\mathrm{sgn}(x) = -1$ when $x < 0$. Different tranches correspond to different risk profiles that can be obtained with different values of k. The transformation of $\{X_i\}$ into $F_k(\{X_i\})$ is an example of a securitisation and the advantage of it is that it turns a set of risky assets into assets with a controlled risk profile. For example, it is possible to obtain assets that are very safe, i.e. for which $F_k(\{X_i\}) = 1$ with high probability, by taking a sufficiently small value of k.

Given the *complexity* of these products and the large number of assets that are typically used to construct these financial products, the issue of quantifying the risk associated with these investments is a subtle problem [28]. In the following subsection we discuss the lack of transparency that is involved in the processes of diversification and securitization.

Quantifying transparency loss

In order to capture how relevant is the information content of the $\{Y_i\}$ about the $\{X_i\}$ variables and about a generic transformation (i.e. product) $F(\{X_i\})$, the *mutual information* provides a natural way to address this issue in quantitative terms. For a pair of random variables A and B, it is given by

$$I(A;B) = \sum_{a,b} P(a,b) \log \left(\frac{P(a,b)}{P(a)P(b)} \right) = H(A) - H(A|B), \tag{4.5}$$

where the sum is intended over all possible outcomes a and b of the random variables. It can be shown to be equivalent to the reduction in entropy $H(A) = -\sum_a P(a) \log P(a)$ of the random variable A, when the value of the random variable B is given, averaged over the possible outcomes of B, $H(A|B) = \sum_b P(b) H(A|B = b)$.

A general result involving the mutual information is the so called *data processing inequality*, which we can formulate in our setting in the following way. When the $\{X_i\}$ are manipulated and transformed in a product through some $F(\{X_i\})$, some information may or may not be lost, but for sure no information can be gained. This can be formalized using the mutual information as follows

$$I(F(\{X_i\}); \{Y_i\}) \leq I(\{X_i\}; \{Y_i\}). \tag{4.6}$$

In the following, we consider the mutual information as a measure of how much the information contained in the $\{Y_i\}$ variables is informative on the return of the financial product $F(\{X_i\})$.

Another quantity we are going to consider is the *mutual information per bit*, i.e. the ratio between the mutual information and the entropy,

$$\frac{I(F(\{X_i\}); \{Y_i\})}{H(F(\{X_i\}))}, \tag{4.7}$$

which expresses the reduction of the initial ignorance about the return of $F(\{X_i\})$ when the information is given.

The data processing inequality also holds when a manipulation of the information is considered, namely

$$I(\{X_i\}; G(\{Y_i\})) \leq I(\{X_i\}; \{Y_i\}). \tag{4.8}$$

When the pools are composed by a large number of assets, it is difficult for investors to transmit all the original information about the individual assets, from the product originators to the buyers. In practice, this may happen because the information is costly and because sellers have no incentives to share such information and provide details on the final product [69]. Yet, we can imagine that some compressed version of the information, expressed in terms of a function of the original information, $G(\{Y_i\})$, might be used instead of the whole set of information variables $\{Y_i\}$, with some information loss in the process. This is a relevant issue in the financial industry, where the introduction of barcodes for financial products has been discussed [29]. An optimal barcode for a financial product $F(\{X_i\})$ would be one for which the mutual information

$$I(F(\{X_i\}); G(\{Y_i\})) \tag{4.9}$$

is maximal, with respect to all possible compressions G.

A first general result for the case of binary assets and binary information variables can be obtained for the type of products we introduced before, i.e the homogeneous diversification and the ABS. Both these type of products are function of the asset returns $\{X_i\}$ through their aggregate return $X = \sum_i X_i$ (we denote by capital X the sum of the returns X_i in the following). The probability distribution for the aggregate return X have the following property:

$$P\left(X\left|\{Y_i\}\right.\right) = P\left(X\left|Y\right.\right), \tag{4.10}$$

where Y is the aggregate information $Y = \sum_i Y_i$. Such a property holds as a consequence of the permutation symmetry of the sum. This is actually a general result for any function of the stocks that is symmetric under any permutation of the stocks. This implies that the sum (or the average) Y (or Y/n) is an optimal barcode for any financial transformation $F(X)$ which is a function of the average return of the assets, i.e.

$$I(F(X); Y) = I(F(X); \{Y_i\}). \tag{4.11}$$

Pricing information

A task closely related to the relevance of the information $\{Y_i\}$ is its *pricing*, a typical goal of portfolio theory and risk management practices.

It is commonly believed that complex financial products, obtained by pooling together a large number of stocks, have the nice property of reducing the associated risks. For instance, this can be quantified by a reduction in the variance of the corresponding distribution, with respect to the variance of the single asset.

$$\mathbb{V}(F(\{X_i\})) < \mathbb{V}(X_i) \tag{4.12}$$

To understand why variance can be a proxy of the risk of an asset we may think to the following setup of a two times market [77] .

An investor in this market is endowed with an initial wealth $\mathcal{W}_0$ and an utility function $\mathcal{U}(\mathcal{W})$ which quantifies how much the wealth $\mathcal{W}$ satisfies the need of our investor. The utility function $\mathcal{U}$ is usually assumed to be an increasing and concave function of $\mathcal{W}$, to ensure respectively greediness and risk aversion of the investor.

At some time t_0 the investor may decide to invest part of his/her initial wealth $\mathcal{W}_0$ to buy A units of cash (say \$) of an asset with return X_i, resulting in owing $\mathcal{W}_0 - p_{AX_i} + AX_i$ where p_{AX_i} is the *price* of the asset. The return of the asset X_i is known only at some later time t_1, so at time t_0 it can be considered a random variable. The *fair price*, i.e. the price that makes the investor indifferent between buying and not buying the asset is given by the condition

$$\mathcal{U}(\mathcal{W}_0) = \mathbb{E}\left[\mathcal{U}(\mathcal{W}_0 - p_{AX_i} + AX_i)\right], \tag{4.13}$$

where the expected value $\mathbb{E}\left[\cdot\right]$ is intended on the probability distribution of X_i.

If we assume that $A \ll \mathcal{W}_0$ and we Taylor expand expression (4.13), i.e.

$$\mathcal{U}(\mathcal{W}_0) \simeq \mathbb{E}\left[\mathcal{U}(\mathcal{W}_0) + \mathcal{U}'(\mathcal{W}_0)(AX_i - p_{AX_i}) + \frac{1}{2}\mathcal{U}''(\mathcal{W}_0)(AX_i - p_{AX_i})^2\right], \tag{4.14}$$

we get

$$p_{AX} \simeq A\,\mathbb{E}[X_i] + \frac{\mathcal{U}''(\mathcal{W}_0)}{2\mathcal{U}'(\mathcal{W}_0)}\,\mathbb{E}\left[(AX_i - p_{AX_i})^2\right]. \tag{4.15}$$

In the following we are going to discuss in which cases the truncation of this this Taylor expansion is meaningful. By now, if we assume that the second term on the r.h.s is small compared to the first, the price on the r.h.s. can be assumed to be approximately equal to $A\mathbb{E}[X_i]$. In this case, the equation for the price reduces to

$$p_{AX} = A\,\mathbb{E}[X_i] + A^2\frac{\mathcal{U}''(\mathcal{W}_0)}{2\mathcal{U}'(\mathcal{W}_0)}\mathbb{V}[X_i] = A\left(\mathbb{E}[X_i] - \alpha\mathbb{V}[X_i]\right). \tag{4.16}$$

In the last equation $\alpha = -\frac{A\mathcal{U}''(\mathcal{W}_0)}{2\mathcal{U}'(\mathcal{W}_0)}$ is dimensionless, and it is positive (since $\mathcal{U}' > 0$ and $\mathcal{U}'' < 0$ are the general requirement for an utility function). That is, the fair price for A units of an asset with return X_i is given by its expected return, minus a term embedding the uncertainty of the probability distribution, as expressed by its variance; the minus sign precisely highlight the risk aversion.

We further assume investors with constant relative risk aversion (CRRA) utility functions [78] [77]. This means, for instance $\mathcal{U}(\mathcal{W}) = \mathcal{W}^\gamma$, with $\gamma < 1$, or $\mathcal{U}(\mathcal{W}) = \log \mathcal{W}$. In these cases. α is proportional to $A/\mathcal{W}_0$, and it can be assumed to be small from the very beginning. Furthermore, the Taylor expansion leading to equation (4.16) is justified in this case, since the next term in the expansion of (4.13) would be order $A^2/\mathcal{W}_0^2$.

Now we tackle the problem of pricing the *bits* of information: within this formalism, it is natural to price information as the difference in price due to a change in the probability distribution. When the agent acquire some information Y_i, the probability distribution changes from $P(X_i)$ to $P(X_i|Y_i)$. In this case, the calculation of the price he/she is willing to pay for the asset X_i is computed with the new conditional probability distribution. Such a price will depend on the specific value of the information Y_i. We can consider the average of such a price, averaged with respect to the distribution of Y_i, as the price that the investor is willing to pay for *both* the asset *and* the information.

For the cases we are going to consider in our models, the uninformed distribution $P(X_i)$ is the marginal of the joint distribution $P(X_i, Y_i)$. Hence the information on average does not change the expected return of the asset ($\mathbb{E}[X_i] = \mathbb{E}[\mathbb{E}[X_i|Y_i]]$), but produces a reduction in variance ($\mathbb{V}[X_i] \neq \mathbb{E}[\mathbb{V}[X_i|Y_i]]$). $\mathbb{E}[X_i|Y_i]$ and $\mathbb{V}[X_i|Y_i]$ are the conditional cumulants, expected value and variance, at fixed Y_i. As such, they are function of Y_i and they can be averaged with respect to the information probability distribution $P(Y_i)$, as we did in the previous equations in brackets.

In this setup, the fair price that the investor is willing to pay for the information Y_i, is given by the difference in price among the two following cases; the case in which he/she buys the asset and the information, and the one in which he/she buys only the asset. Hence the *price of the information* per unit of asset $A = 1$, can be given by

$$\delta p_i = \alpha \left(\mathbb{V}[X_i] - \mathbb{E}\left[\mathbb{V}[X_i \,|\, Y_i] \right] \right) = \alpha \mathbb{V}\left[\mathbb{E}[X_i \,|\, Y_i] \right], \tag{4.17}$$

where in the last equality we used the *variance decomposition formula* ($\mathbb{V}[X_i] = \mathbb{E}\left[\mathbb{V}[X_i \,|\, Y_i]\right] + \mathbb{V}\left[\mathbb{E}[X_i \,|\, Y_i]\right]$) in order to show that the price of the information is non negative.

Equation (4.17) can be extended beyond the case of the single asset X_i, and we can apply this formalism to estimate the price of the information Y associated with any financial transformation $F(X)$ (that can be a homogeneous diversification X/n or an ABS $F_k(X)$). We get

$$\delta p = \alpha \left(\mathbb{V}\left[F(X)\right] - \mathbb{E}\left[\mathbb{V}\left[F(X) \,|\, Y\right] \right] \right), \tag{4.18}$$

as the definition of the price of the information Y associated with the financial instrument $F(X)$. This price depends on α, which is an investor-dependent quantity, which expresses his/her risk aversion. In the following, when we use such a price we fix $\alpha = 1$, referring just to the reduction in variance given by the information.

A quite general result can be proven (see appendix 4.A) about this expression for the class of the models introduced in (4.1) with binary variables. For those model, as a consequence of the factorizability of the conditional probability $P(X_i \,|\, Y_i)$, we can show that for the average return $\overline{X}$ the following holds

$$\mathbb{E}\left[\mathbb{V}\left[\overline{X} \,|\, Y\right]\right] = \frac{\mathbb{E}[[\mathbb{V}\left[X_i \,|\, Y_i\right]]}{n}. \tag{4.19}$$

We are going to use this result in the following to understand the behaviour of the information price, as a function of the size of the pool, for the homogeneous portfolio.

In the next two sections we are going to specialize to two classes of less general models, symmetric and asymmetric. Symmetric models are intended to model stock price dynamics while asymmetric ones are intended to model those cases where a negative return is a rare event, e.g. credits.

4.4 A model for symmetric assets (stocks)

In this section we consider a symmetric model of binary variables, i.e. for which

$$\begin{cases} X_i = \pm 1 \\ Y_i = \pm 1 \end{cases} \qquad \begin{cases} \mathbb{E}[X_i] = 0 \\ \mathbb{E}[Y_i] = 0 \end{cases} . \tag{4.20}$$

This type of distributions could arise for instance when dealing with binarized high frequency financial data, where observations are very noisy. In this case, volatility is much higher that the expected return, hence the binarization yields at first approximation symmetric variables.

When the asset returns are assumed to be independent, so are the information variables, and the marginals distributions are trivial, i.e $P(\{X_i\}) = P(\{Y_i\}) = (1/2)^N$. In this independent returns case, the unique measurable quantity which is non trivial is $\mathbb{E}[X_i Y_i]$, which measure how likely the return is positive (or negative) when the information is positive (or negative). Since both returns and information are symmetric, the conditional $P(X_i \,|Y_i)$ can be parametrized in terms of a single number. For convenience, we use the following parametrization

$$P(X_i = x_i \,|Y_i = y_i) = \frac{e^{J x_i y_i}}{2\cosh(J)} \ . \tag{4.21}$$

Alternatively to J, we can introduce the probability to be aligned, $p_a = \frac{\mathbb{E}[X_i Y_i]+1}{2} = (1+e^{-2J})^{-1}$, describing how likely each information Y_i is equal to the asset return it is related with.

In the more general case in which the assets returns are not independent and a non zero empirical measure of $\mathbb{E}[X_i X_j]$ is available, we have to make an assumption on which distribution of the information $P(\{Y_i\})$ is compatible with the observed correlation. We recall that in our setting, equation (4.1), a dependence among returns emerges as a consequences of a dependence among information. Equation (4.21) implies $\mathbb{E}[X_i X_j] = \tanh^2(J)\mathbb{E}[Y_i Y_j]$ (we refer to appendix 4.D for computational details), showing explicitly how the asset returns dependence is inherited from the information dependence. When a single measure of correlation is available, and there are no reasons to expect inhomogeneities in the system (i.e. we can assume identical $P(X_i|Y_i)$ and $\{Y_i\}$ identically distributed), the *most general* distribution compatible with a fixed value of $\mathbb{E}[Y_i Y_j]$ is a symmetric fully connected Ising model

$$P(\{Y_i\} = \{y_i\}) = \frac{1}{\mathcal{Z}_y} e^{\frac{C}{2n} y^2}, \tag{4.22}$$

where $y = \sum_i y_i$. By most general distribution, we mean that this is the distribution compatible with the observed moments that has the largest entropy (maximum entropy principle). The independent assets case is recovered when $C = 0$. The parameter J is specified as before by the measurement of $\mathbb{E}[X_i Y_i]$ while C can be fixed by $\mathbb{E}[Y_i Y_j]$ or equivalently by $\mathbb{E}[X_i X_j] = \tanh^2(J)\mathbb{E}[Y_i Y_j]$. In particular, we note that the coupling C changes with the size of the pool n - at fixed returns correlation $\mathbb{E}[X_i X_j]$. In order to find the value of C that corresponds to a given value of the correlation, for a given n, one has to solve the inverse Ising model, the details of which are described in the appendix 4.B.

Here we consider financial transformations that are functions of the average asset performance, as discussed in the previous section. Hence it is useful to compute the joint p.d.f for the sums

$$X = \sum_{i=1}^{n} X_i \qquad \text{and} \qquad Y = \sum_{i=1}^{n} Y_i. \tag{4.23}$$

In general we can write this as

$$P(X = x\,, Y = y) = \sum_{\{X_i\}} \sum_{\{Y_i\}} \left(\prod_{i=1}^{n} P(X_i = x_i\,, Y_i = y_i) \right) \delta\left(\sum_{i=1}^{n} X_i = x \right) \delta\left(\sum_{i=1}^{n} Y_i = y \right), \tag{4.24}$$

but this sum is tricky to compute, because all terms for which the deltas are non zero, *do not* have the same weight. Let's consider instead

$$P(X = x\,, Y = y\,, A = a), \tag{4.25}$$

where the variable A is the number of *aligned* binary variables, i.e. for which the return X_i and its associated Y_i have the same sign. In this case all the terms contributing to this probability have the same weight and we can write

$$P(X = x\,, Y = y\,, A = a) = \frac{e^{Ja}e^{-J(n-a)}e^{C/2ny^2}}{(2\cosh(J))^n \mathcal{Z}_Y} B(x, y, a)\ . \tag{4.26}$$

By $B(x, y, a)$ we denote all possible couples of strings of ± 1 - of length n - such that the first sum to x, the second to y, in such a way they have a aligned variables. To compute this number, we start from all strings summing to x, which is $\binom{n}{\frac{n+x}{2}}$. Starting from each of these string of $\{X_i\}$, we can create a string of $\{Y_i\}$ summing to y and having a aligned variables with the $\{X_i\}$ string in the following way. We initially take $\{Y_i\} = \{X_i\}$ and then we change sign to $\frac{1}{2}(n - a - \frac{y-x}{2})$ variables out of the $\frac{n+x}{2}$ which are positive in the string $\{X_i\}$, and we change sign to $\frac{1}{2}(n - a + \frac{y-x}{2})$ variables out of the $\frac{n-x}{2}$ which are negative.

Hence we get

$$B(x, y, a) = \binom{n}{\frac{n+x}{2}} \binom{\frac{n+x}{2}}{\frac{1}{2}(n - a - \frac{y-x}{2})} \binom{\frac{n-x}{2}}{\frac{1}{2}(n - a + \frac{y-x}{2})}. \tag{4.27}$$

The allowed values of a goes from $a_{\min} = |\frac{x+y}{2}|$ to $a_{\max} = n - |\frac{x-y}{2}|$, so that we can write

$$P(X = x\,, Y = y) = \sum_{a=a_{\min}}^{a_{\max}} P(X = x\,, Y = y, A = a) \tag{4.28}$$

and we finally get

$$P(X = x\,, Y = y) = \sum_{a=a_{\min}}^{a_{\max}} B(x, y, a) \frac{e^{Ja}e^{-J(n-a)}e^{\frac{C}{2n}y^2}}{(2\cosh(J))^n \mathcal{Z}_Y} \tag{4.29}$$

with $B(x, y, a)$ expressed as in equation (4.27). This expression is very useful to compute numerically all desirable quantities like measures of mutual information and variances, because it reduces the sum over an exponential (in n) number of configurations to the sum over a linear set of states.

Independent assets and information

We report in this subsection the results obtained for independent assets, when $C = 0$. Figure 4.3 shows the mutual information per bit and the price of information for an homogeneous portfolio against different values of J - or p_a. With increasing n, the distributions of X and Y converge to two Gaussian distributions, with correlation $\tanh(J)$, because of the central limit theorem. As a result, the mutual information reaches a constant value, which is $I_{n\to\infty}(X;Y) = -1/2\log(1-\tanh(J)^2)$ [79]. Since the entropy is growing logarithmically, as it is known that for a binomial distributed random variable $H(X) \simeq 1/2\log(2\pi e n)$, the mutual information per bit plotted in the left panel of figure 4.3 shows a logarithmic decay to zero. At large n, the knowledge of the information Y is only weakly informative on the random variable X. An alternative way to express this loss of information, is by noting that a constant (in n) mutual information, together with an increasing (in n) entropy $H(X)$, implies

$$\frac{H(X|Y)}{H(X)} \to 1 \qquad \text{when } n \to \infty, \tag{4.30}$$

showing how the knowledge of Y does not reduce the entropy of X.

For independent assets, this loss of information is consistent with what observed on the right panel of figure 4.3, where the information price is going to zero $\sim 1/n$. This can be understood by recalling that the information price, expressed as in (4.18) is a difference between two variances. While the second is decreasing with n in general, as stated in (4.19), the first variance is also having a $\sim 1/n$ behaviour in the case of independent assets. In fact, the variance for the average of a set of random variables can be written as

$$\mathbb{V}\left[\overline{X}\right] = \frac{1}{n^2}\mathbb{V}\left[X\right] = \frac{1}{n}\mathbb{V}\left[X_i\right] + \frac{n(n-1)}{n^2}\mathrm{cov}\left[X_i X_j\right]. \tag{4.31}$$

When n is very large only the covariance term is not going to zero. However, in the case of independent returns, it is exactly zero for all n.

The general picture emerging from figure 4.3 is that the return of a large homogeneous portfolio of independent assets, is only weakly dependent on information. In other words, risks (even unknown ones, since we neglect relevant *bits* of information) can be efficiently diversified.

This indeed is the perfect realization of the diversification task, namely safe financial products can be easily created and they prove to be stable with respect to a change in the market fundamentals. In such a world, it can be expected to see that information has no value, and both buyers and sellers do not have an incentive in building nor buying barcodes for financial products.

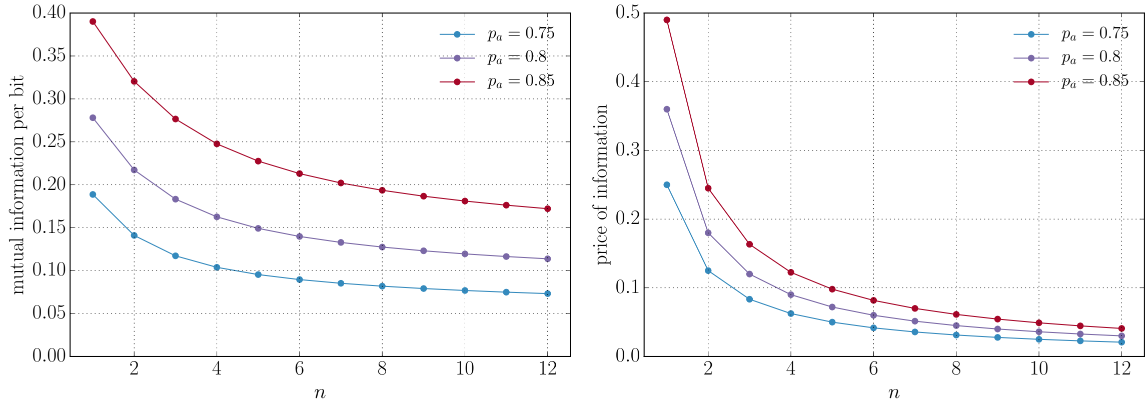

Figure 4.3: Independent assets, homogeneous portfolio $\overline{X}$. Different curves correspond to data for $p_a = 0.75(J = 0.5493), 0.8(J = 0.69315), 0.85(J = 0.8673)$. The mutual information per bit dependence on n shows a logarithmic decay to zero. Correspondingly, the price of the information goes to zero proportional to $1/n$. In the curves with larger value of J, (or with larger p_a), as expected the information is more relevant and has higher price.

Figure 4.4 shows the same plots for an ABS. For the ABS, we fixed k in such a way that, by changing n, the default probability $p_d = P(F_k(X) = -1)$ is constant, namely we studied the size effect on a given tranche. In the case of independent assets, being X normally distributed at large n, this could be achieved by having $k \sim \sqrt{n}$ (since $X = n\mathcal{N}(0,1)$). In particular, in order to have $p_d = 0.01$ we can fix $k = -2.326\sqrt{n}$. In general, when dependencies are included in the model, this inversion is not a trivial task, so in figure 4.4 we fix k numerically, to be the smallest value with a default probability larger than 0.01. Oscillation are observed at small n, due to the fact that the ABS return function distinguishes only even integer values of k, so that it is not possible to fix p_d at the same value for small n.

At variance with the case of the homogeneous portfolio, both the mutual information per bit and the information price go to a constant value. This implies that, for an ABS, the information remains informative about the return on the product and, consistently, the price of such information does not vanish.

For structured finance type of products, like the simple ABS studied here, even at the level of independent and symmetric assets the diversification task does not work: the process of securitisation does not diversify enough to make the information negligible. Therefore, the information stored in the Y variable remains relevant, irrespectively of the size n of the pool of assets involved. Such collective transformations introduce a systemic source of risk that is sensible to the side information Y and that, therefore, cannot be efficiently hedged neglecting Y. In such a case it is meaningful to think that a barcode, containing the value of Y, could be sold together with a tranche $F_k(X)$, and the price of such a barcode could be expressed by the information price (4.18), quantifying the reduction of risk that is on average achieved, when the value of Y is known.

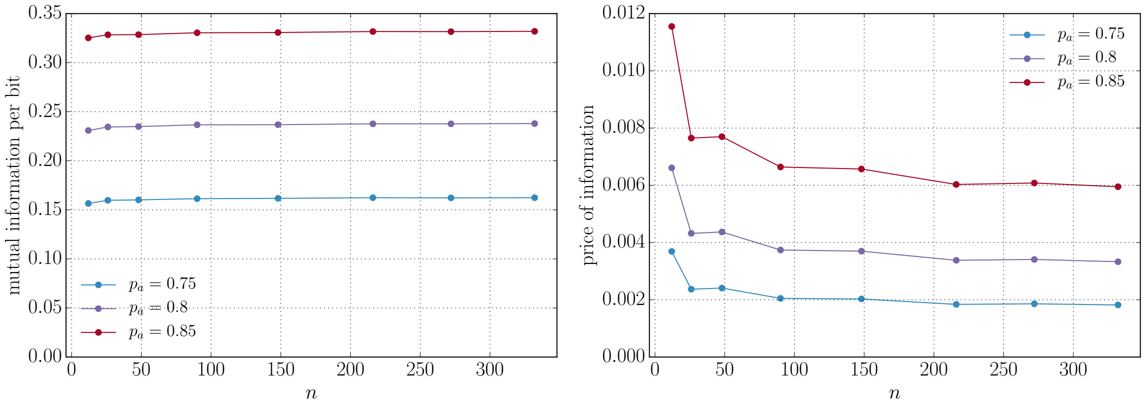

Figure 4.4: Independent assets, ABS $F_k(X)$. Different curves correspond to data for $p_a = 0.75 (J = 0.5493), 0.8 (J = 0.69315), 0.85 (J = 0.8673)$. k is fixed to be the smallest value of with a default probability larger than 0.01. Both the mutual information per bit and the information price go to a constant value, initial oscillation are there because of the impossibility to fix k realizing a precise value of p_d when n is small.

Dependent assets and information

In this subsection, we discuss the results for dependent assets. In figure 4.6 we can see mutual information per bit and the information price for an homogeneous portfolio. Both plots are for a fixed value of $J = 0.5493$ (corresponding to $p_a = 0.75$) and the different curves correspond to different values of $\rho = \mathbb{E}[X_i X_j]$, or equivalently $\mathbb{E}[Y_i Y_j] = \tanh^{-2}(J)\,\rho = 4\,\rho$. This choice is dictated by the observation that both the mutual information per bit and the information price are always increasing in J (or in p_a), while the dependence on the correlation is less trivial. For each value of ρ at a given pool size n, the value of C is computed by the numerical inversion of the Ising model at finite n.

On the left panel we show the behaviour of the mutual information per bit, where all curves eventually reaches a logarithmically decreasing regime, but comparing to the independent assets case, this value is much larger and it is not attained up to very large values of n. In particular, for different values of ρ, the behaviour is not trivial, and for curves corresponding to smaller value of ρ the mutual information per bit is larger, when the pool size n is larger than a given amount. This property can be understood in terms of the behaviour of the mutual information, which is growing logarithmically only when $C = 1$, whereas it reaches a constant value for all values of $C > 1$, as discussed in appendix 4.C. Since weakly correlated assets (hence weakly correlated information) are asymptotically described by a model with C closer to 1 than strongly correlated assets(see appendix 4.B), their mutual information is larger at large n, and so is the mutual information per bit.

In the right panel of figure 4.5, we show the behaviour of the price of the information. The discussion after equation (4.31) is still valid except that here assets are not independent, hence the covariance term is not zero. Each curve is asymptotic to the value of $\rho = \mathbb{E}[X_i X_j]$ which is fixed for all n.

A non trivial observation is that here the mutual information per bit and the information price exhibit a different monotonicity in ρ. In fact, the weaker the correlation among assets returns, the more the information is relevant for large portfolios (when looking at the mutual information per bit), while on the contrary the information price is a decreasing function of this correlation. This is because while the information price asymptotically converges to the value of ρ, the mutual information strongly depends on the assumption on *how* the dependence among information is modelled, in particular here on (4.22). Its behaviour as a function of n and its asymptotic value is dictated by some specific properties of the inverse Ising model.

Figure 4.6 shows the same for an ABS. For ABSs, the introduction of dependences does not change the main qualitative picture of the case of independent asset, meaning that both the mutual information per bit and the information price go to a constant value. This convergence is slower for curves with smaller values of ρ, due to the slow convergence of the inverse Ising. A non-monotonic behaviour, similar to what observed for the homogeneous portfolio is observed also here, and the non-monotonicity extends also to the information price; for products composed with weakly dependent assets the information is more relevant then for products composed with strongly dependent assets, both at the level of entropy reduction and at the level of risk reduction.

4.5 A model for asymmetric assets (credits)

In this section we consider models with binary and homogeneous variables, but we drop the symmetric assumption. In this case, the most general model compatible

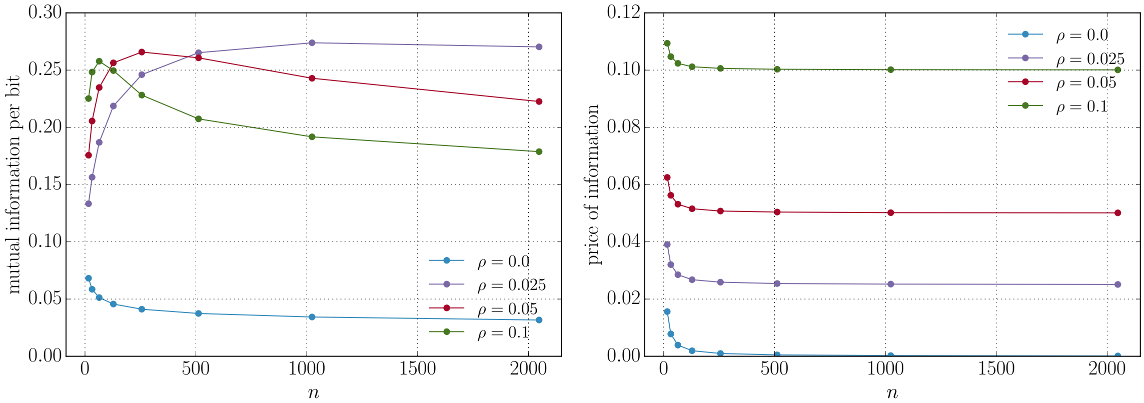

Figure 4.5: Dependent assets, homogeneous portfolio $\overline{X}$. All curves correspond to $p_a = 0.75 (J = 0.5493)$ and different values of $\rho = \mathbb{E}[X_i X_j] = 0.1, 0.05, 0.025, 0.0$. The value of C is computed by the numerical inversion of the Ising model at finite n corresponding to a given value of $\mathbb{E}[Y_i Y_j]$, and $C = 0$ for the curve at $\rho = 0$, which is shown for comparison. On the left the behaviour of the mutual information per bit, which eventually reaches a logarithmically decreasing regime. In the $n-$range showed in the plot only the curves corresponding to $\rho = 0.4$ and 0.2 have reached this regime. On the right the behaviour of the price of the information. Each curve is asymptotic to the value of ρ which is fixed for all n.

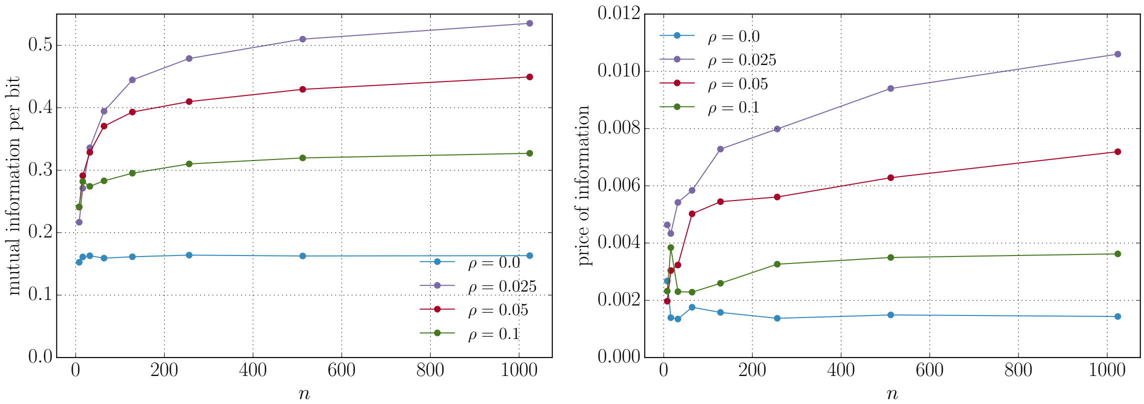

Figure 4.6: Dependent assets, ABS $F_k(X)$. All curves correspond to $p_a = 0.75$ and different values of $\rho = \mathbb{E}[X_i X_j] = 0.1, 0.05, 0.025, 0.0$. The value of C is computed by the numerical inversion of the Ising model at finite n corresponding to a given value of $\mathbb{E}[Y_i Y_j]$, and $C = 0$ for the curve at $\rho = 0$, which is shown for comparison. The tranches correspond to k fixed to be the smallest value of with a default probability larger than 0.01.

with the maximum entropy principle would be specified by:

$$P(X_i = x_i \,|Y_i = y_i) = \frac{e^{Jx_iy_i + H_1x_i}}{2\cosh(Jy_i + H_1)}, \tag{4.32}$$

and by

$$P(\{Y_i\} = \{y_i\}) = \frac{1}{\mathcal{Z}_y} e^{\frac{C}{2n}y^2 + H_2 y}. \tag{4.33}$$

Given the larger number of parameters involved, we might consider two simpler cases; $H_1 = 0$, in which the anisotropy in returns is obtained as a result of anisotropic information or the case in which $H_2 = 0$, in which information are still symmetric and the anisotropy is imposed in the conditional $P(X_i \,|\, Y_i)$. We are going to focus here only the latter case, because asymmetric information variables are not particularly meaningful. In particular, they are not meaningful in all cases in which information is not binary and it is binarized a posteriori. In these cases in fact, the only practical argument to perform the binarization, is to maximize the informative content of information $H(\{Y_i\})$, as symmetric binary variables do.

Analogously to the symmetric case, we compute the joint p.d.f for sums, resulting in

$$P(X = x\,, Y = y) = \sum_{a=a_{\min}}^{a_{\max}} B(x, y, a) \frac{e^{Ja} e^{-J(n-a)} e^{\frac{C}{2n}y^2 + H_1 x + H_2 y}}{2^n \cosh(J + H_1)^{(n+y)/2} \cosh(J - H_1)^{(n-y)/2} \mathcal{Z}_Y} \tag{4.34}$$

with $B(x, y, a)$ expressed as in equation (4.27).

We are going to report here only results about dependent asset, since independent are not showing any new behaviour. In figure 4.8 the behaviour of the mutual information and the information price are shown as a function of n. The parameters J and H_1 are fixed once for all, from measurement of $\mathbb{E}[X_i]$ and $\mathbb{E}[X_iY_i]$. In fact, in the case $H_2 = 0$ the generic form of single returns correlations, as explicitly shown in appendix 4.D, are given by

$$\mathbb{E}[X_i] = \frac{1}{2}\left[\tanh(H_1 + J) + \tanh(H_1 - J)\right] \tag{4.35}$$

$$\mathbb{E}[X_iX_j] = \left(\frac{1 + \mathbb{E}[Y_iY_j]}{4}\right)\big(\tanh^2(H_1 + J) + \tanh^2(H_1 - J)\big) + \tag{4.36}$$

$$+ \left(\frac{1 - \mathbb{E}[Y_iY_j]}{2}\right) \tanh(H_1 + J)\tanh(H_1 - J)$$

$$\mathbb{E}[X_iY_i] = \frac{\exp(J + H_1)}{2\cosh(J + H_1)} + \frac{\exp(J - H_1)}{2\cosh(J - H_1)} - 1. \tag{4.37}$$

Fixing $\mathbb{E}[X_i] = 0.96$ and $\mathbb{E}[X_i Y_i] = 0.02$ (with symmetric Y_is and strongly biased X_is, $\mathbb{E}[X_i Y_i]$ has to be small) and inverting the first and third of the previous equations, we get $J \simeq 0.27976$ and $H_1 \simeq 2.0178$. With these values of J and H_1, the correlation among asset returns $\rho = \frac{\mathbb{E}[X_i X_j] - \mathbb{E}[X_i]^2}{1-\mathbb{E}[X_i]^2}$ results to be still 4 times smaller than the information correlations $\mathbb{E}[Y_i Y_j]$, like in the symmetric case.

Figure 4.7 and figure 4.8 show the usual plots for an homogeneous portfolio and for the ABS. They confirm essentially the picture emerging from the analysis of asset with symmetric returns. In figure 4.7 the regime in which the mutual information per bit decrease with n is not reached up to $n = 2048$ for all curves. The same behaviour for the price of information is also found.

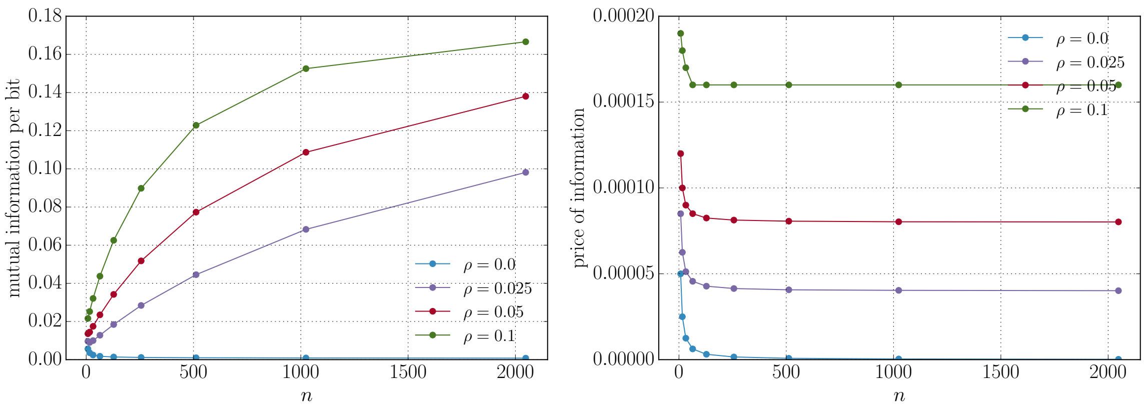

Figure 4.7: Dependent assets, homogeneous portfolio $\overline{X}$. All curves correspond to $\mathbb{E}[X_i] = 0.96$ and $\mathbb{E}[X_i Y_i] = 0.02$ ($J \simeq 0.27976$ and $H_1 \simeq 2.0178$) and different values of $\rho = 0.1, 0.05, 0.025, 0.0$. The value of C is computed by the numerical inversion of the Ising model at finite n corresponding to a given value of $\mathbb{E}[Y_i Y_j] = 4\rho$, while $C = 0.0$ for independent assets. On the left the behaviour of the mutual information per bit, which eventually reaches a logarithmically decreasing regime, but in the $n-$range showed in the plot not a single curve has reached this regime. On the right the behaviour of the price of the information. Each curve is asymptotic to the value of $\mathbb{E}[X_i X_j] = 0.0016\rho$, which is fixed for all n.

4.6 Conclusions and outlook

In this final chapter, we exploited information theoretic concepts and asset pricing theory to investigate the lack of transparency associated with financial transformations, which is widely spread in nowadays financial practices and it had a predominant role in the 2007-2008 global financial crisis.

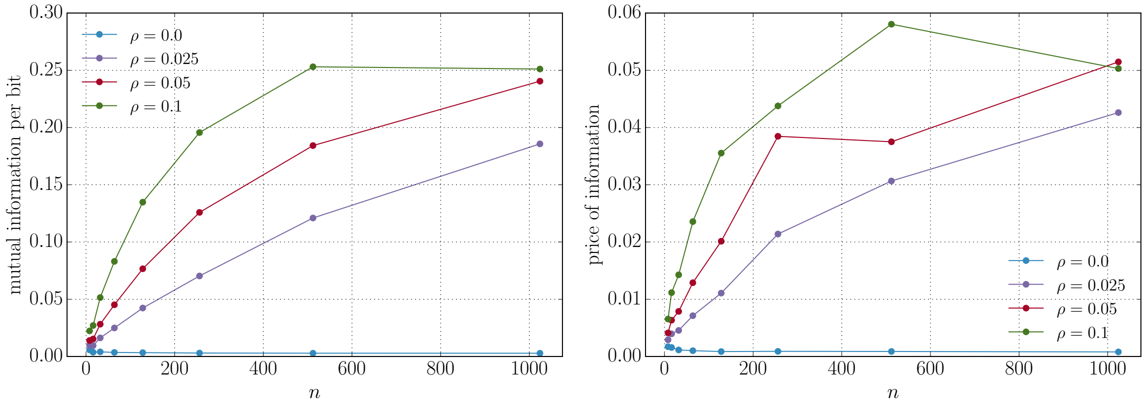

Figure 4.8: Dependent assets, ABS $F_k(X)$. All curves correspond to $\mathbb{E}[X_i] = 0.96$ and $\mathbb{E}[X_i Y_i] = 0.02$ ($J \simeq 0.27976$ and $H_1 \simeq 2.0178$) and different values of $\rho = 0.1, 0.05, 0.025, 0.0$. The value of C is computed by the numerical inversion of the Ising model at finite n corresponding to a given value of $\mathbb{E}[Y_i Y_j] = 4\rho$, while $C = 0.0$ for independent assets. On the left the behaviour of the mutual information per bit, on the right the behaviour of the price of the information. The tranches correspond to k fixed to be the smallest value of with a default probability larger than 0.1.

In an ideal world in which asset returns are independent between each other, standard diversification techniques, like the homogeneous portfolio considered here, allow investors to trade safe financial products, and whose low risk profile is stable with respect to fluctuations in market fundamentals. A very simple principle, which is based on increasing the size of the pool, can be efficiently used to generate instruments with a risk as low as desired.

When instead dependencies among returns are taken into account, or when more involved financial products like structured securitisations are considered, such a *diversification dream* drastically collapses.

In ABSs, the risk can be efficiently hedged by moving the threshold and low risk instruments can be created out of risky assets. However, such an evaluation is extremely unstable. In our model, stability is considered with respect to the acquisition of additionally variables, containing information about the distributions of returns. We explicitly show how in these cases the knowledge of the information remains relevant for the return distribution, both at the level of entropy and at the level of risk, up to very large pools.

The model is motivated by the recent history of the bubble in the subprime mortgage backed securities market, which was nourished by the lack of transparency of structured finance products. [28] [69]. Our results highlight the importance of the proposal [29] to build an efficient and standardized system, or a common language, though which information should be easily available to all market participants. In addition, we proposed a way to price such a *financial barcode*, through the risk reduction due to the information. Such an aspect is particularly relevant, since without a barcode price the seller does not have an incentive for sharing the information with the buyer, and the system is not sustainable.

In the case of homogeneous portfolios, while the actual informative content of the information about the whole distribution of returns, the *mutual information per bit*, depends on the specificities of how dependencies are introduced in the model, the barcode price does not. In particular, when a fully connected Ising model is used to model information dependencies, the information Y remains informative up to very large portfolios and asymptotically is more informative for weakly dependent assets, whose distribution's parameters are closer to the critical point. This implies that a bit of information is more valuable when the correlations are weak.

This suggests that measures of correlations, which are usually rather noisy and unreliable, are not sufficient to understand risk in an environment of complex financial products, but understanding how the returns actually depend on each others is of fundamental need. Eventually, this suggest that that the barcode might include, on top of the actual value of Y, which gives information on the average quality of the assets $\{X_i\}$, also details about the probability distributions of future returns of these $\{X_i\}$.

Our simplified scheme highlights those crucial issues and might be a benchmark for more complex and realistic theoretical models, e.g. with continuous returns as well as a benchmark for the implementation of such barcodes in real financial practices, in particular from the point of view of the regulator. In general, it highlights how systemic lack of transparency in Finance should be considered a key ingredient to compute risk.

4.A The conditional variance

We prove here that the result (4.19), namely

$$\mathbb{E}\left[\mathbb{V}\left[\frac{X}{n} \mid Y\right]\right] = \frac{\mathbb{E}\left[\mathbb{V}\left[X_i \mid Y_i\right]\right]}{n} \tag{4.38}$$

is very general and it can be proven for all the models of binary variables we have introduced.

To compute this variance we use

$$\mathbb{V}\left[\frac{X}{n} \mid Y\right] = \frac{1}{n^2}\mathbb{V}\left[X \mid Y\right] \tag{4.39}$$

and we calculate separately $\mathbb{E}\left[X \mid Y\right]$ and $\mathbb{E}\left[X^2 \mid Y\right]$. For the first we have

$$\begin{aligned} \mathbb{E}\left[X \mid Y\right] &= n\mathbb{E}\left[X_i \mid Y\right] \\ &= n\sum_{Y_i}\mathbb{E}\left[X_i \mid Y_i\right] P\left(Y_i \mid Y\right). \end{aligned} \tag{4.40}$$

We note in that the last equation we can write

$$\mathbb{E}\left[X_i \mid Y_i\right] = a + bY_i \tag{4.41}$$

because this is just a different parametrization (using a and b instead of J and H_1), and

$$P\left(Y_i \mid Y\right) = \frac{n + Y_i Y}{2n}. \tag{4.42}$$

Using those we easily get

$$\mathbb{E}\left[X \mid Y\right] = na + bY. \tag{4.43}$$

Similarly we can write for

$$\begin{aligned} \mathbb{E}\left[X^2 \mid Y\right] &= n + n\left(n-1\right)\mathbb{E}\left[X_i X_j \mid Y\right] \\ &= n + n\left(n-1\right)\sum_{Y_i, Y_j}\mathbb{E}\left[X_i X_j \mid Y_i, Y_j\right] P\left(Y_i, Y_j \mid Y\right) \\ &= n + n\left(n-1\right)\left[a^2 + \frac{2abY}{n} + b^2\sum_{Y_i, Y_j} Y_i Y_j P\left(Y_i, Y_j \mid Y\right)\right] \\ &= n + n\left(n-1\right)\left[a^2 + \frac{2abY}{n} + b^2\left(\frac{Y^2 - n}{n^2 - n}\right)\right]. \end{aligned} \tag{4.44}$$

Putting everything together we get

$$\mathbb{V}\left[\,X \mid Y\,\right] = \frac{(1-a^2-b^2)}{n} - \frac{2ab}{n}Y, \tag{4.45}$$

hence

$$\mathbb{E}\left[\mathbb{V}\left[\frac{X}{n} \mid Y\right]\right] = \frac{(1-a^2-b^2)}{n} - \frac{2ab}{n}\mathbb{E}[Y]. \tag{4.46}$$

We observe that, being $\mathbb{E}\left[\,X_i \mid Y_i\,\right] = a + bY_i$ we have

$$\mathbb{E}\left[\mathbb{V}\left[\,X_i \mid Y_i\,\right]\right] = 1 - a^2 - b^2 - 2ab\mathbb{E}[Y_i], \tag{4.47}$$

hence the equation is proved in general.

4.B Fully connected Ising model, direct and inverse problems

The fully connected Ising model for a set of binary variables $\{Y_i\}$ is defined by the partition function

$$Z(C,H,n) = \sum_{\{Y_i\}} \exp\left\{\frac{C}{2n}Y^2 + HY\right\} = \sum_{Y} B(Y)\exp\left\{\frac{C}{2n}Y^2 + HY\right\}, \tag{4.48}$$

where Y is a shortcut for $Y = \sum Y_i$ and

$$B(Y) \equiv \binom{n}{\frac{n+Y}{2}} \tag{4.49}$$

is the number of configuration $\{Y_i\}$ summing to Y.

From this partition function, different moments of the distributions such as $\mathbb{E}[Y_i]$ or $\mathbb{E}[Y_iY_j]$ can be computed numerically for finite n, and analytically in the large n limit. We refer to this problem as the direct Ising problem. The inverse Ising problem, instead, is the determination of C and H that provide a given value for some observed moments of the distribution.

Both these problems are amenable to an approximated solution in the large n limit. To account for the quadratic term we can use a Hubbard Stratonovich transformation

$$\exp\left\{\frac{C}{2n}Y^2\right\} = \sqrt{\frac{nC}{2\pi}}\int dm\,\exp\left\{-\frac{nC}{2}m^2 + CYm\right\}, \tag{4.50}$$

and use such transformation to rewrite the partition function and get

$$Z(C,H,n) = \sqrt{\frac{nC}{2\pi}}\int dm\, e^{nf(m)} \tag{4.51}$$

with

$$f(m) = \log(2) - \frac{C}{2}m^2 + \log\left[\cosh\left(Cm + H\right)\right]. \tag{4.52}$$

In the large n limit, the previous integral can be computed with the use of the saddle point method. The maximum of the function $f(m)$ is located in the point m^*, determined by the saddle point equation

$$f'(m^*) = 0 \quad \Longrightarrow \quad m^* = \tanh(Cm^* + H), \tag{4.53}$$

and the integral can be approximated to be

$$Z(C, H, n) \simeq \sqrt{\frac{nC}{2\pi}} e^{nf(m^*)} \int dm\, e^{nf''(m^*)} \frac{(m - m^*)^2}{2}. \tag{4.54}$$

The second derivative $f''(m^*)$ can be computed and it is equal to

$$f''(m^*) = -C + C^2\left(1 - m^{*2}\right). \tag{4.55}$$

There are two subtleties associated with the previous approximation when $H = 0$, because the equation (4.53) may have more than one solution and the second derivatives can be null. Specifically, the second derivative is null on the critical point $\{H = 0\,,\, C = 1\}$, separating a phase where equation (4.53) has only one solution ($m^* = 0$ in the region $\{H = 0\,,\, C < 1\}$) to a phase where equation (4.53) has three solutions ($m^* = \{0, \pm m^*\}$ in the region $\{H = 0\,,\, C > 1\}$).

We discuss the case of $H = 0$ separately, also because we have in mind to apply this case to the case of symmetric information.

In this case, if $C \geq 1$, equation (4.53) has three solution, 0 and two $\pm m^*$ corresponding to two separate global maxima of the function $f(m)$. Furthermore, in the special case $C = 1$, the second derivative $f''(m^*)$ is null, and the first derivative which is non-zero is the fourth one. Hence, to have a convergent integral in equation (4.54) additional terms in the power expansion have to be added.

Using the two famous integrals

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi} \qquad\qquad \int_{-\infty}^{\infty} dx\, e^{-x^4} = 2\,\Gamma\left(\frac{5}{4}\right), \tag{4.56}$$

we have that for large n

$$Z(C, H = 0, n) \simeq \begin{cases} \frac{1}{\sqrt{1-C}} 2^n & \text{if } C < 1 \\ 2^n n^{1/4} 12^{1/4} \sqrt{2/\pi}\, \Gamma\left(\frac{5}{4}\right) & \text{if } C = 1\,. \\ \frac{1}{\sqrt{1-C(1-m^{*2})}} e^{nf(m^*)} & \text{if } C > 1 \end{cases} \tag{4.57}$$

From the partition function we can compute the correlation $\rho = \mathbb{E}[Y_i Y_j]$ for large n by using

$$\rho = \mathbb{E}[Y_i Y_j] = \frac{1}{n(n-1)}\mathbb{E}[Y^2] - \frac{1}{n-1} \simeq \frac{1}{n^2}\frac{1}{Z}2n\frac{\mathrm{d}}{\mathrm{d}C}Z - \frac{1}{n}, \tag{4.58}$$

and we get

$$\rho = \mathbb{E}[Y_i Y_j] \sim \begin{cases} \alpha_1 \frac{1}{n} & \text{if } C < 1 \\ \\ \alpha_2 \frac{1}{\sqrt{n}} & \text{if } C = 1 \\ \\ m^{*2} + \frac{\alpha_3}{n} & \text{if } C > 1 \end{cases} \tag{4.59}$$

where α_1, α_1 and α_3 are some constants. The results in equation (4.59) can be used to understand the inverse problem in the large n limit. If we want to find the value of the coupling C that is providing a given value of correlation ρ, the value of C has to be larger than 1, because $C < 1$ is compatible only with $\rho = 0$ in the $n \to \infty$ limit. The specific value of $C(\rho)$ reached in the $n \to \infty$ limit can be obtained by inverting the equation $\rho = m^{*2}$, that in the $H = 0$ case gives

$$C(\rho) \to \frac{\tanh^{-1}\sqrt{\rho}}{\sqrt{\rho}} \qquad \text{for} \quad n \to \infty. \tag{4.60}$$

Such inversion can be obtained also numerically, inverting the equation that provides a given ρ at finite n, and the result is shown in figure (4.9).

4.C Large n behaviour of the mutual information

The definition of the mutual information reads

$$I(X;Y) = H(Y) - H(Y|X). \tag{4.61}$$

An upper bound for the mutual information is given by the entropy $H(Y)$ of the distribution of Y.

In figure 4.10 it is shown the behaviour of the entropy $H(Y)$ when the variables $\{Y_i\}$ are distributed according to a fully connected Ising model (4.33). It is observed to be growing proportionally to $1/2\log(n)$ for all values except for $C = 1$ and $H_2 = 0$ where the slope is steeper and it can be observed to be $3/4\log(n)$. This can be understood in terms of the moments in equation (4.59). For all values except for

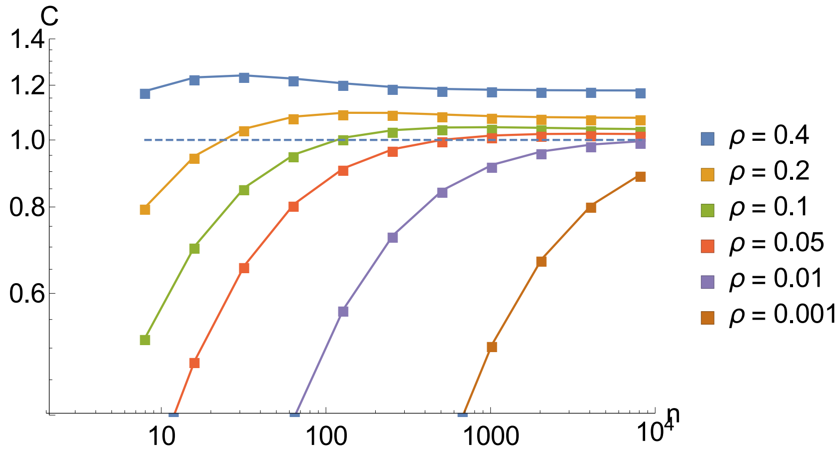

Figure 4.9: Numerical inversion of the fully connected Ising model, as a function of n. The exact expression for $\rho = \mathbb{E}[Y_i Y_j]$ as a function of C and n is numerically inverted (Newton's method), obtaining C for several values of n and ρ. Asymptotically, C converges to the limit $\frac{\tanh^{-1}\sqrt{\rho}}{\sqrt{\rho}}$.

$C = 1$ and $H_2 = 0$ the variance of Y is proportional to n, while on the critical point it is going as $n^{3/4}$. Given that at large n, the Ising spins are essentially independent ($\mathbb{E}[Y_i Y_j] - \mathbb{E}^2[Y_i] \simeq 0$), the entropy is expected to be logarithmic in the variance $\mathbb{V}[Y]$, as for a Normal distribution.

In figure 4.10 it is shown the behaviour of the mutual information $I(X;Y)$, when the $\{X_i\}$ and the $\{Y_i\}$ variables are distributed according to (4.32) and (4.33). It is observed to be constant at large n for all values except for $C = 1$, $H_1 = 0$ and $H_2 = 0$, where it is observed to grow like $(1/4)\log(n)$. This can be understood in terms of the entropy discussed above, since only at the critical point and when $H_1 = 0$ the difference of the two terms in equation (4.61) is not cancelling the logarithmic term.

4.D Empirical averages against model's parameters

Here we derive the relations among empirical averages and model's parameters, in the general case considered in the main text with binary variables distributed according to equations (4.32) and (4.22). The simple trick for the calculation is using the law of iterated expectations conditioning on information and then exploiting the conditional independence of the assets, as we assumed in our model. For the symmetric case we

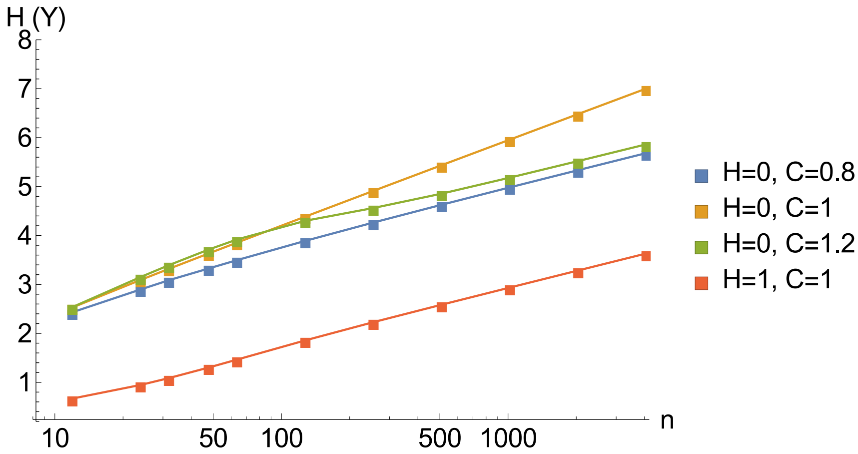

Figure 4.10: $H(Y)$ growth as a function of n in a linear-log plot. The slope is asymptotically $1/2$ for all values except for $C = 1$ and $H_2 = 0$, while it is $3/4$ on the critical point.

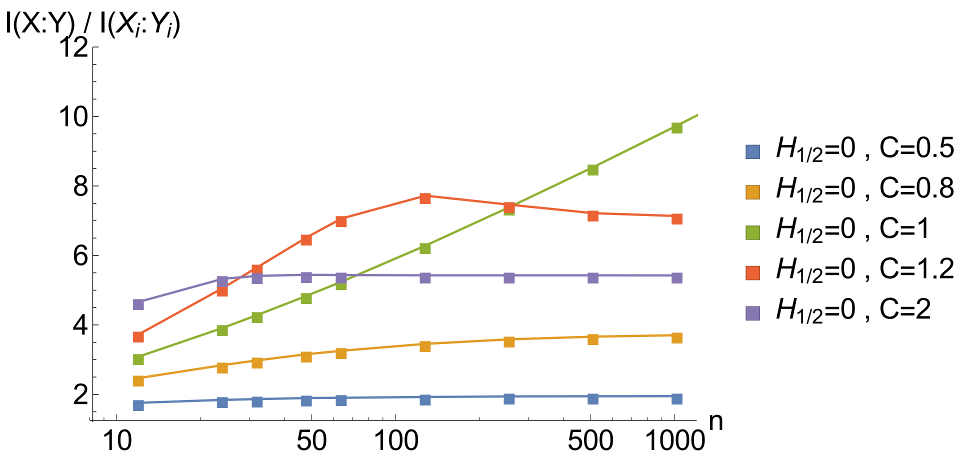

Figure 4.11: $I(X;Y)$ growth as a function of n in a linear-log plot. Asymptotically, for all values except for $C = 1$ and $H_1 = H_2 = 0$ the mutual information converges to a constant, while it grows as $(1/4)\log(n)$ on the critical point.

have:

$$\mathbb{E}[X_iX_j] = \mathbb{E}[\mathbb{E}[X_iX_j|Y_i,Y_j]] = \mathbb{E}\left[\frac{\left(e^{JY_i}-e^{-JY_i}\right)\left(e^{JY_j}-e^{-JY_j}\right)}{4\cosh(J)\cosh(J)}\right] =$$
$$= \frac{1}{4}\mathbb{E}\left[(2\tanh^2(J))P(Y_i=Y_j)+\right.$$
$$\left. + (\tanh(J)^2)(1-P(Y_i=Y_j))\right].$$

This is exactly the expression reported in the main text since $\mathbb{E}[Y_iY_j] = 2P(Y_i = Y_j) - 1$. For the asymmetric model, with $H_2 = 0$ the expected value of an asset reads:

$$\mathbb{E}[X_i] = \mathbb{E}[\mathbb{E}[X_i|Y_i]] = \mathbb{E}\left[\frac{e^{JY_i+H_1}-e^{-JY_i-H_1}}{2\cosh(JY_i+H_1)}\right] =$$
$$= \frac{1}{2}\tanh[J+H_1] + \frac{1}{2}\tanh[H_1-J],$$

while the relation involving the expected value of the product of asset returns reads:

$$\mathbb{E}[X_iX_j] = \mathbb{E}[\mathbb{E}[X_iX_j|Y_i,Y_j]] = \mathbb{E}\left[\frac{\left(e^{JY_i+H_1}-e^{-JY_i-H_1}\right)\left(e^{JY_j+H_1}-e^{-JY_j-H_1}\right)}{4\cosh(JY_i+H_1)\cosh(JY_j+H_1)}\right] =$$
$$= \frac{1}{4}\mathbb{E}\left[(\tanh^2(J+H_1)+\tanh^2(J-H_1))P(Y_i=Y_j)+\right.$$
$$\left. + (\tanh(J+H_1)\tanh(J-H_1))(1-P(Y_i=Y_j))\right].$$

This is exactly the expression (4.35) since $\mathbb{E}[Y_iY_j] = 2P(Y_i = Y_j) - 1$. Finally the correlation among assets and information is related through model's parameters by:

$$\mathbb{E}[X_iY_i] = \mathbb{E}\left[Y_i\mathbb{E}[X_i|Y_i]\right] = \mathbb{E}\left[Y_i\frac{e^{JY_i+H_1}-e^{-JY_i-H_1}}{2\cosh(JY_i+H_1)}\right] =$$
$$= \frac{1}{2}\tanh[J+H_1] - \frac{1}{2}\tanh[H_1-J].$$

Concluding Remarks

Today's global economy is more interconnected and complex than ever, and seems out of any particular institution's control. The diversity of markets and traded products, the complexity of their structure and regulation, make it a daunting challenge to understand behaviours, predict trends or prevent systemic crises.

The standard approach of Economics, that mostly aims at explaining global behaviour in terms of perfectly rational actors and efficient markets, has largely failed [8, 80]. Some alternative approaches, inspired by Statistical Physics and the Complex Systems Science, in which economic phenomena are considered as emergent statistical properties of a large interacting system, and empirical evidences are favoured over mathematical idealizations, can be of great help in dealing with this challenge.

The perspective lying below these approaches describes the economy, likewise an ecological system, as a complex adaptive system. That is, according to the definition in reference [81], a system "[...] composed of individual agents that adjust their behavior or their relative number, with consequences for the system as a whole, and these consequences can in turn affect individual behaviors ". As a result, the economy turns out to be extremely interconnected, a perturbation in a given sector will have consequences on other far related sectors, and these consequences in turn will affect the initial perturbation, possibly amplifying it.

In this thesis, few issues hampering the sustainability of the economic and financial systems have been considered and analyzed with the methods of Statistical Physics: the anomalous size of financial institutions and financial regulation, the trend of rising inequalities and growth, market efficiency and the systemic financial risk. The general idea of this thesis, lying behind the mere models, is that these problems are all deeply interconnected.

Quantitative models, when properly conceived, have the great merit of exposing these interconnections, unravelling the complexity of a large variety of natural and social phenomena. In these respect, economic and social systems are sources of a very large number of interesting problems lacking a satisfactory understanding, a number that evolves together with society itself, because it is intimately connected to the increasing complexity of existing networks and their structure [82].

In the words of Bialek, an ambitious goal for future scientists can be described as "[...] reconcile the physicists' desire for concise, unifying theoretical principles with the obvious complexity and diversity of life" [83]. The present work wishes to be a humble contribution in this direction.

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