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CFTs and the Bootstrap

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Abstract

This thesis deals with investigations in the field of higher dimensional CFTs. The first part is focused on the technology neccessary for the calculation of general conformal blocks in 4D CFTs. These special functions are neccessary for general boostrap analysis in 4D CFTs. We show how to reduce the calculation of arbitrary conformal blocks to the calculation of a minimal set of "seed" conformal blocks through the use of differential operators. We explicitly write the set of operators necessary and show a general basis for the case of external traceless symmetric operators. We then compute in closed analytical form this set of seeds. We write in a compact form the set of quadratic Casimir equations and proceed to solve them in closed form with the use of an educated Ansatz. Various details on the form of the ansatz are deduced with the use of the so called shadow formalism. The second part of this thesis deals with numerical investigations of the bootstrap equation for external scalar operators. We compute bounds on the OPE coefficients in 4D CFTs for theories with and without global symmetries, and write the bootstrap equations for theories with $SO(N) \times SO(M)$ and $SU(N) \times SO(M)$ symmetries. The last part of the thesis presents the Multipoint bootstrap, a conformal-bootstrap method advocated in ref. [25]. In contrast to the most used method based on derivatives evaluated at the symmetric point $z = \bar{z} = 1/2$, we can consistently "integrate out" higher-dimensional operators and get a reduced, simpler, and faster to solve, set of bootstrap equations. We test this "effective" bootstrap by studying the 3D Ising and $O(n)$ vector models and bounds on generic 4D CFTs, for which extensive results are already available in the literature. We also determine the scaling dimensions of certain scalar operators in the $O(n)$ vector models, with $n = 2, 3, 4$, which have not yet been computed using bootstrap techniques.

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Foreword

The work presented in this thesis is based on the work done by the author and collaborators.

Chapter 3 is based on:

A. C. Echeverri, E. Elkhidir, D. Karateev and M. Serone, "Deconstructing Conformal Blocks in 4D CFT," JHEP **1508** (2015) 101 [arXiv:1505.03750 [hep-th]].

Chapter 4 is based on :

A. Castedo Echeverri, E. Elkhidir, D. Karateev and M. Serone, "Seed Conformal Blocks in 4D CFT," JHEP **1602**, 183 (2016) doi:10.1007/JHEP02(2016)183 [arXiv:1601.05325 [hep-th]].

Chapter 5 is based on:

F. Caracciolo, A. C. Echeverri, B. von Harling and M. Serone, "Bounds on OPE Coefficients in 4D Conformal Field Theories," JHEP **1410** (2014) 20 [arXiv:1406.7845 [hep-th]].

Chapter 6 is based on:

A. Castedo Echeverri, B. von Harling and M. Serone, "The Effective Bootstrap," arXiv:1606.02771 [hep-th].

Contents

Introduction

The field of theoretical physics encompasses many and different realms of reality. From the microscopic study of fundamental interactions to the understanding of massive objects in the vastness of the universe, passing through statistical physics and the study of condensed matter systems. The energy scales that these different fields of study probe are highly disparate but a common denominator seems to govern the behaviour of all of them: Field Theories. The history of field theories as tools to describe nature is one of success. Ubiquitous nowadays, we have become accustomed to their use for describing all types of phenomena, their power undeniable. Quantum field theories (QFTs) in particular seem to provide some of the most accurate and detailed descriptions of fundamental forces in the universe, providing a window into the high energy behaviour of nature, one of the most relevant examples being the Standard Model of particle physics.

One of the main guiding principles behind the study of QFTs, and field theories in general, is the use of symmetries. The presence of symmetries proves to be one of the most powerful tools available, as it dictates sets of constrains the theory must satisfy, simplifying the task of finding solutions and restricting the allowed types of interactions within the system. For example, the gauge symmetries imposed in the Standard Model ($SU(3) \times SU(2) \times U(1)$) along with the lack of anomalies in the system basically dictates the form of the theory. Thus the quest for finding new symmetry principles has driven most of the research within the physics community. One of such principles is that of scale invariance. The study of scale invariant theories has always played a special role in theoretical physics. The absence of characteristic scales within a physical system is a very studied phenomena. The study of renormalization group (flow) within QFTs shows that in a generic quantum system interactions vary with energy. The beta functions found this way describe the specific change of the couplings of a theory at different energy scales, and in the specific case where these beta functions are vanishing they describe critical points in the flow. One example being the deep UV limit of Quantum Chromodynamics which is absent of divergences and thus a finite and complete theory in its own right. It turns out that when a system is invariant under scale transformations the symmetries of the system are generally enhanced to conformal invariance. This surprising fact has been proven in 2D [1, 2], and the general case is

still being studied [3–5]. With this in mind we might start studying Conformal Field Theories (CFTs) as a starting point instead of trying to flow theories by tuning their parameters. The idea of studying CFTs as a starting point found its perfect realization in the case of String theory, where it represents a basic pillar the theory is built upon. Strings are described by 2D CFTs on the world-sheet, and most of the nice properties that arise in String Theory find their origin in the CFT principles that determine the consistency of the theory.

The relevance of CFTs does not end here. They play a very important role in describing condensed matter systems such as that of the Ising phase transition in three dimensions. Another relevant field where CFTs appear is that of AdS/CFT. The discovery of a duality between CFTs in D-dimensions and quantum gravity in D+1 dimensions [6, 7] provides a new setting where CFTs prove essential. Following this duality then we can find certain CFTs as holographic definitions of Quantum gravity theories. It seems then an interesting challenge to try to understand and classify all CFTs with the hope of obtaining new insights in a multitude of physical systems.

The study of CFTs in D>2 has experienced a huge boost in recent years with the advent of the "Conformal Bootstrap". This field of study has its roots in the work of Ferrara, Gatto and Grillo [10] and Polyakov [11] in the 1970s. The idea behind it was to "bootstrap" or constrain the space of consistent CFTs by means of symmetry considerations alone. Conformal symmetry allows one to determine completely two point correlators of local operators. The three-point functions of the system are determined as well kinematically up to constants. It seems that Conformal symmetry is powerful enough to allow this trend to continue. When considering higher n-point functions the combined use of Conformal symmetry and the Operator Product Expansion (OPE) allows one to determine any correlation function in terms of the CFT data alone, which consists of the spectrum of operators in the theory as well as the three point function coefficients.

The basis of the bootstrap method is to write a four point function of local operators [1] (we will refer to them as external) in several distinct ways using the OPE. This leads to a "bootstrap equation", written in terms of OPE coefficients and conformal partial waves, which is a statement about crossing symmetry. In the case where we have only bosonic operators:

$$\Big\langle \Big(\phi(x_1) \underset{OPE}{\times} \phi(x_2)\Big)\Big(\phi(x_3) \underset{OPE}{\times} \phi(x_4)\Big)\Big\rangle = \Big\langle \Big(\phi(x_1) \underset{OPE}{\times} \phi(x_3)\Big)\Big(\phi(x_2) \underset{OPE}{\times} \phi(x_4)\Big)\Big\rangle$$

[1] Recent studies have shown that more powerful constraints appear when considering several four point correlation functions at the same time [61].

Where the OPE is taken in two distinct channels. Performing the OPE in both sides the equality becomes:

$$\sum_{\mathcal{O}}(\lambda_{\phi\phi\mathcal{O}})^2\mathcal{W}_{\mathcal{O}}(x_1,x_2,x_3,x_4)=\sum_{\mathcal{O}}(\lambda_{\phi\phi\mathcal{O}})^2\mathcal{W}_{\mathcal{O}}(x_1,x_3,x_2,x_4)$$

And $\mathcal{W}_{\mathcal{O}}$ are called Conformal Partial Waves (CPWs) which are functions fixed by conformal invariance and contain the contribution of a tower of operators to the four point function in consideration. They are labeled by $\mathcal{O}$, the primary operators present in the OPE. These CPWs are written in terms of (several for the general case) conformal blocks, which are universal functions, in the sense that they do not depend on the specific theory being considered. The fulfillment of this crossing symmetry relation is highly non-trivial, for it depends on the possible operators entering the OPE and the OPE coefficients. These ideas were applied in D=2 [12], together with the use of the infinite dimensional algebra present in D=2, the upside being the fact that the allowed primary operators in the so called "minimal models" render a finite set. The revival of the program took place in the seminal paper by Rattazzi, Rychkov, Tonni and Vicchi in 2008 [9] where they studied 4D CFTs by means of scalar correlators. One of the main ingredients used there were the closed expressions for the conformal blocks found by Dolan and Osborn [13, 14], resuming an infinite tower of operators for each conformal block [2]. The knowledge of the conformal blocks seems essential in order to carry on the bootstrap analysis. Closed expressions for the conformal blocks appearing in four scalar correlators are known in even dimensions [15]. The difficulty of finding closed expressions for odd dimensions has not hindered the progress, however the clever use of recursion relations for general D allows one to write approximate expressions for the conformal blocks. As a matter of fact some of the most impressive results in the bootstrap field come from studies in D=3 [101, 103, 105].

Most of the results arising from the Conformal Bootstrap up to now are based on the study of four point functions of scalars. Although undoubtedly interesting, it opens the question as to how to bootstrap tensor or even fermionic operators. For the case of four external scalars the only conformal blocks contributing are those of traceless symmetric operators (TSO) since those are the only ones present, due to symmetry, in the OPE. This type of exchanged operators will be present as well when the fields entering the four point function are tensors or fermionic fields, thus these new conformal blocks have to be computed. There are also the conformal blocks corresponding to the exchange

[2] The conformal blocks $g_{\mathcal{O}}$ contain the contribution of a primary operator $\mathcal{O}$ and its infinite number of descendants. An extra difficulty in $D \geq 3$ to carry the bootstrap is the existence of infinitely many primaries.

of all other possible types of exchanges (not TSO), these again have to be calculated. A great deal of progress has been made, however [19] has shown how to relate the conformal blocks of any four point function of external traceless symmetric tensors to the known case of external scalars. This methodology takes the known scalar blocks of Dolan and Osborn and treats them as a seed to generate higher spin conformal blocks. The key concept behind this idea is the fact that we can relate conformal blocks where an operator $\mathcal{O}$ is exchanged, with conformal blocks of higher spin external operators where the same operator $\mathcal{O}$ is exchanged.This idea goes a long way. The existence of these "seeds" simplifies the calculation of general blocks enormously. As an example, the case of D=3 is especially relevant. Integer spin (bosonic) representations in D=3 are exhausted by TSO and as such only a single seed (the scalar blocks) is needed for any correlator of external TSO. A practical application of this principle has been used in [38] allowing one to bootstrap a four point function of identical fermions, whose conformal blocks are again related to the scalar seed blocks. Furthermore the case of D=3 requires only an extra seed in order to account for half spin operators exchange in the OPE. The possible seeds in this case are the conformal blocks appearing in a four point function correlator of 2 fermions and 2 scalars. The calculation of this last seed has been performed in [37] allowing for the calculation of any possible blocks in 3D.

Given all this progress one may be interested in finding the set of all possible seeds in general dimension D. However in $D \geq 4$ the number of seed correlators to consider are infinite[3]. With this idea in mind we will try to explore the specific case of $D = 4$. Chapters 3 and 4 deal with the formalism necessary to obtain all possible conformal blocks in $D = 4$. In Chapter 3 we show how CPWs of spinor/tensor correlators can be related to each other by means of differential operators. We will show as well the necessary "seeds" that we need in order to calculate any other CPW. We will proceed in Chapter 4 where we will compute in closed analytical form the (infinite) set of "seed" conformal blocks by solving the set of Casimir equations that the blocks satisfy.

Chapters 5 and 6 instead contain numerical investigations of the bootstrap equation with four external scalars. In Chapter 5 we obtain bounds on OPE coefficients in 4D CFTs by means of semidefinite programming, whereas in Chapter 6 we explore the Multipoint Bootstrap method as an alternative to the more standard derivative method that has been used so far. We show the validity of the method with specific examples where we reproduce known results in the bootstrap in D=3 and D=4.

[3]Further progress for the case of bosonic operators in general D dimensions has been shown in [39, 42, 43] where methods to calculate the possible seeds are presented.

Chapter 1

1 CFT Primer

In this introductory chapter we will set the basic language that we will use throughout the rest of the thesis. We will make an effort to explain the main tools and concepts behind CFTs in higher dimensions, hopefully giving an overview that will render this document self contained.

We will start in section 1.1 introducing the conformal group in higher dimensions, analyzing the generators of the group and their commutation relations. We will analyze field representations of the Conformal Group in section 1.2. The radial quantization of CFTs will be presented in section 1.3, the understanding of which will be essential in the following sections. Section 1.4 will show the constraints set by unitarity in the allowed spectrum of operators in a CFT, an important result that will be used throughout the whole thesis. We will introduce the embedding formalism in section 1.5 and quickly move onto the twistor formalism, very adequate for 4D CFTs, which we will use heavily in the following chapters. We fill finish with section 1.7 which will focus on the Operator Product Expansion (OPE) of great importance for CFTs and for the bootstrap program in particular.

1.1 The Conformal Group

Conformal transformations in flat space can be defined in several equivalent ways. We will adopt here an active point of view for coordinate transformations when analyzing space-time symmetries. Consider a coordinate transformation in $x \to x'$ such that the line element changes accordingly as $ds^2 \to ds'^2$ where:

$$ds'^2 = \eta_{\mu\nu} dx'^\mu dx'^\nu = \Lambda(x) ds^2 \tag{1.1}$$

The last equality is imposed as the constraint that defines conformal transformations. We can study now this equation written as follows:

$$\frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} \eta_{\mu\nu} = \Lambda(x) \eta_{\alpha\beta} \tag{1.2}$$

Let's write the coordinate transformation at the infinitesimal level as $x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$ and plug it in the previous equation to obtain the following general constraint:

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = f(x)\eta_{\mu\nu} \tag{1.3}$$

Where $f(x)$ is some function related to $\Lambda(x)$. We can trace the above equation with $\eta^{\mu\nu}$ which allows us to write the expression for $f(x)$:

$$f(x) = \frac{2}{D}(\partial \cdot \epsilon) \tag{1.4}$$

Where D is the dimensionality of our space, and finally we can write $\Lambda(x) = 1 + \frac{2}{D}(\partial\cdot\epsilon) + O(\epsilon^2)$. There are several useful relations that can be derived from here on. By applying ∂^{ν} on equation (1.3):

$$2\partial^2\epsilon_{\mu} = (2 - D)\partial_{\mu}f(x) \tag{1.5}$$

Applying ∂^{μ} and using the expression for $f(x)$ we arrive at:

$$(D - 1)\partial^2 f(x) = 0 \tag{1.6}$$

We can see that for the specific case of $D = 1$ we obtain no constraints.

Let's obtain another useful relation by applying ∂_{ν} to the previous equation in combination with equation (1.3):

$$(2 - D)\partial_{\mu}\partial_{\nu}f(x) = \delta_{\mu\nu}\partial^2 f(x) \tag{1.7}$$

There is a final expression that will be useful in the following, to reach it we have to take derivatives ∂_{η} of eq (1.3) and permute indices to obtain:

$$\begin{aligned} \partial_{\rho}\partial_{\mu}\epsilon_{\nu} + \partial_{\rho}\partial_{\nu}\epsilon_{\mu} &= \tfrac{2}{D}\eta_{\mu\nu}\partial_{\rho}(\partial \cdot \epsilon) \\ \partial_{\nu}\partial_{\rho}\epsilon_{\mu} + \partial_{\mu}\partial_{\rho}\epsilon_{\nu} &= \tfrac{2}{D}\eta_{\rho\mu}\partial_{\nu}(\partial \cdot \epsilon) \\ \partial_{\mu}\partial_{\nu}\epsilon_{\rho} + \partial_{\nu}\partial_{\mu}\epsilon_{\rho} &= \tfrac{2}{D}\eta_{\nu\rho}\partial_{\mu}(\partial \cdot \epsilon) \end{aligned} \tag{1.8}$$

Subtracting the first line from the sum of the last two leads to:

$$2\partial_{\mu}\partial_{\nu}\epsilon_{\rho} = \frac{2}{D}\left(-\eta_{\mu\nu}\partial_{\rho} + \eta_{\rho\mu}\partial_{\nu} + \eta_{\nu\rho}\partial_{\mu}\right)(\partial \cdot \epsilon) \tag{1.9}$$

The case of $D = 2$ is of specific interest, and using the previous formulas (in Minkowsky signature) we reach:

$$\partial_0\epsilon_0 = \partial_1\epsilon_1, \quad \partial_0\epsilon_1 = -\partial_1\epsilon_0, \tag{1.10}$$

Which we recognise as the familiar Cauchy-Riemann equations appearing in complex analysis. This then allows one to show the holomorphicity of $\epsilon(z)$ and carry on the analysis for the 2D conformal algebra, steaming into a infinite dimensional algebra. We are focused however in higher dimensions, so for $D \geq 3$ the constraints reduce to $\partial^2 f(x) = 0$ which means this function is at most linear in x, which translated to $\epsilon(x)$ means it is at most a quadratic function of x:

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \tag{1.11}$$

We should study now the various terms in this equation, which we can do separately as the constraints of conformal invariance have to be independent of the position x^μ. The constant term a_μ is not constrained from the previous equations and it describes infinitesimal translations. The term $b_{\mu\nu}$ can be studied by plugging the general solution into eq. (1.3) for which we obtain:

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{D}\left(\eta^{\rho\sigma}b_{\sigma\rho}\right)\eta_{\mu\nu} \tag{1.12}$$

We see we can split $b_{\mu\nu}$ into a symmetric part proportional to $\eta_{\mu\nu}$ corresponding to scale transformations $(x'^\mu = (1+\alpha)x^\mu)$ and an antisymmetric part which will correspond to rotations. Finally the quadratic term $c_{\mu\nu\rho}$ can be studied by plugging it into eq. (1.9). We find out that it can be written in terms of a single vector as:

$$c_{\mu\nu\rho} = \eta_{\mu\rho}b_\nu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu, \quad b_\mu = \frac{1}{D}c^\rho_{\rho\mu} \tag{1.13}$$

This corresponds to Special Conformal Transformations (SCT) which have the infinitesimal form $x'^\mu = x^\mu + 2(x\cdot b)x^\mu - (x\cdot x)b^\mu$. These transformations are slightly less intuitive than the previous one. Their finite transformations are as follows:

$$x'^\mu = \frac{x^\mu - (x\cdot x)b^\mu}{1 - 2(b\cdot x) + (b\cdot b)(c\cdot x)} \tag{1.14}$$

Expanding the denominator for small b^μ one can recover the infinitesimal version. To gain further understanding of Special Conformal Transformations we can rewrite the finite form in the following way:

$$\frac{x'^\mu}{x'\cdot x'} = \frac{x^\mu}{x\cdot x} - b^\mu \tag{1.15}$$

From here we observe that SCTs can be understood as an inversion of x^μ ($x^\mu \to x^\mu/x^2$), followed by a translation b^μ, and followed again by an inversion.

Let's find the generators of the group corresponding to all the transformations we found. Let's consider the action of the group upon pure functions, or scalar fields. The coordinate transformations can be written as $x' = g(x)$ (or $x = g^{-1}(x')$), so we have $\phi'(x) = \phi(g^{-1}(x'))$. Our generators act upon our functions as $\phi'(x) = e^{-iT}\phi(x)$. By matching both of these formulas and expanding both sides for transformations close to the identity, we obtain the generators of the group:

Translations	$P_\mu = i\partial_\mu$
Dilations	$D = ix^\mu\partial_\mu$
Rotations	$M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$
SCTs	$K_\mu = i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)$

And the non-vanishing commutation relations are the following:

$$\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho})\\
&[M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu)\\
&[M_{\mu\nu}, K_\rho] = -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu)\\
&[K_\mu, P_\nu] = -2i(\eta_{\mu\nu}D + M_{\mu\nu})\\
&[D, P_\mu] = -iP_\mu\\
&[D, K_\mu] = iK_\mu
\end{aligned} \tag{1.16}$$

This corresponds to the SO(D,2) algebra, a fact that will be more clearly stated in section 1.5.

1.2 Field representations of the Conformal Group

In the previous sections we found the space-time part of the generators corresponding to conformal transformations. We are left with the task of finding how quantum fields are affected by conformal transformations by finding the extra piece of the generator that does the job. Consider a generic field $\phi^a(x)$ where a denotes possible tensor indices and we demand our field is in an irreducible representation of the Lorentz group. We start by studying the little group that leaves the point $x = 0$ invariant. In the case of Poincare, we remind the reader, works in the following way. The little group corresponds to the Lorentz group, and we introduce a matrix representation to define the action of Lorentz transformations on our field $\phi^a(0)$:

$$U(\Lambda)\phi^a(0)U^{-1}(\Lambda) = S^a{}_b(\Lambda^{-1})\phi^b(0) \tag{1.17}$$

where the U are the unitary transformations corresponding to the specific transformation, and $S^a{}_b$ is the matrix representation we define. Writing $U(\Lambda) = e^{-\frac{i}{2}w_{\mu\nu}M^{\mu\nu}}$, and expanding both sides for transformations close to the identity, for the $x = 0$ little group we get:

$$[M_{\mu\nu}, \phi^a(0)] = i(\mathcal{S}_{\mu\nu})^a{}_b\phi^b(0) \tag{1.18}$$

where the $(\mathcal{S}_{\mu\nu})^a{}_b$ is the spin operator corresponding to our field ϕ^a. We could have expanded for general x in equation (1.17), however for the following this proves more direct. In order to get the transformation of our fields at a generic point x we simply translate it:

$$\left[M_{\mu\nu}, e^{-ix\cdot P}\phi^a(0)e^{ix\cdot P}\right] = e^{-ix\cdot P}\left[e^{ix\cdot P}M_{\mu\nu}e^{-ix\cdot P}, \phi^a(0)\right]e^{ix\cdot P} \tag{1.19}$$

And we evaluate $e^{ix\cdot P}M_{\mu\nu}e^{-ix\cdot P}$ by using the Baker-Campbell-Hausdorff expansion:

$$e^{-A}Be^{A} = B + [B, A] + \frac{1}{2}[[B, A], A] + \frac{1}{3!}[[[B, A], A], A] + ... \tag{1.20}$$

Which in our case stops after the second term in the expansion. Our final expression is:

$$[M_{\mu\nu}, \phi^a(x)] = -i(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi^a(x) + i(\mathcal{S}_{\mu\nu})^a{}_b\phi^b(0) \tag{1.21}$$

Now the case of the conformal group is almost identical. The transformations that make up the little group leaving invariant $x = 0$ are the Lorentz transformations as before, plus dilations and special conformal transformations. We will denote the operators corresponding to these transformations at $x = 0$, $\mathcal{S}_{\mu\nu}$ (as before), Δ and k_μ respectively.

$$[D, \phi^a(0)] = -i\Delta\phi^a(0), \quad [M_{\mu\nu}, \phi^a(0)] = i(\mathcal{S}_{\mu\nu})^a{}_b\phi^b(0), \quad [K_\mu, \phi^a(0)] = ik_\mu\phi^a(0), \tag{1.22}$$

Now doing the same exercise, with the only difference that we must expand to three terms equation (1.20) for the case of K_μ, we obtain:

$$
\begin{aligned}
&[P_\mu, \phi^a(x)] = -i\partial_\mu \phi^a(x) \\
&[D, \phi^a(x)] = -i(\Delta + x^\mu \partial_\mu)\phi^a(x) \\
&[M_{\mu\nu}, \phi^a(x)] = -i(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi^a(x) + i(\mathcal{S}_{\mu\nu})^a{}_b \phi^b(0) \\
&[K_\mu, \phi^a(x)] = -i(2x_\mu \Delta - 2x^\lambda \mathcal{S}_{\lambda\mu} + 2x_\mu(x^\rho \partial_\rho) - x^2 \partial_\mu - k_\mu)^a{}_b\, \phi^b(x)
\end{aligned}
\tag{1.23}
$$

Notice that for the representations of the little group, since we demanded that the field $\phi^a(x)$ belonged to an irreducible representation of the Lorentz group, D being a Lorentz scalar, by Schur's Lemma, must be proportional to the identity since it commutes with all the generators of said representation. Thus the commutation relation $[D, K_\mu]_R = -iK_\mu$ requires $K_\mu(k_\mu)$ to vanish.

1.3 Radial Quantization

A method of quantization more convenient for CFTs than the usual, where one foliates space-time in equal time hypersurfaces, is that of Radial Quantization ([85], see also [22]). More familiar in the context of 2D CFTs and String Theory, the concept of Radial Quantization translates to any dimension and allows us to have a clear picture of the Hilbert Space of our theory. In this type of quantization we foliate our space-time by using S^{D-1} spheres of various radii centered at $x = 0$. It is convenient to work in the Euclidean. Mapping our space time (by a conformal transformation) to the cylinder we can recover a more usual picture and our spheres slices become horizontal slices in the cylinder with time translation invariance along it. The explicit mapping of coordinates is $\tau = \log r$. We see that the τ parameter shifts under dilations $r \to e^{\lambda} r$ as $\tau \to \tau + \lambda$. We can use the usual Hamiltonian quantization with τ being our time coordinate.

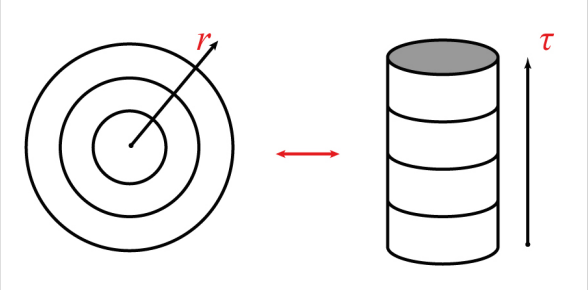

Figure 1.1: Left: Flat space geometry $\mathbb{R}^D$ foliated by S^{D-1} spheres. Right:The spheres of constant r are mapped to horizontal slices in the cylinder, which corresponds to $S^{D-1} \times \mathbb{R}$.

In this picture the dilation operator (in the Euclidean) becomes the Hamiltonian that evolves our theory. States living on the spheres are classified according to their scaling dimension and their $SO(D)$ spin:

$$\begin{aligned} D\,|\Delta, \ell\rangle_a &= i\Delta\,|\Delta, \ell\rangle_a \\ M_{\mu\nu}\,|\Delta, \ell\rangle_a &= (\mathcal{S}_{\mu\nu})^b_a\,|\Delta, \ell\rangle_b \end{aligned} \tag{1.24}$$

Being D and $M_{\mu\nu}$ the only commuting generators in the algebra.

Let's consider for a moment a state in the Hilbert space generated by the action of a generic field at the origin acting on the vacuum of our theory:

$$|\phi\rangle \equiv \phi(0)\,|0\rangle \tag{1.25}$$

The eigenvalue of this operator under the dilation operator, assuming our vaccum is invariant under the action of the generators of the group, is simply $i\Delta$. Acting repeatedly with the K_μ operator we notice we lower the dimension of our state:

$$D(K_\mu\,|\phi\rangle) = i(\Delta - 1)(K_\mu\,|\phi\rangle) \tag{1.26}$$

Assuming the dimensions are bounded from below we must eventually hit zero (we will see in the next section that indeed we can obtain lower bounds on the dimension of states). These states, annihilated by K_μ are called $\mathbf{primary\ operators}$ and they mark a "lowest weight" state in the representation. We can act with P_μ on our state and we will raise the value of the dilation eigenvalue by one each time. These states are called $\mathbf{descendants}$. We see that we can classify representations of the whole conformal group according to their Lorentz quantum number, and the lowest scaling dimension Δ, corresponding to a state annihilated by K_μ, all other states being descendants of the primary and forming the whole (infinite) conformal multiplet. Notice as well that operators away from the origin are not eigenstates of the dilation operator, but rather a superposition of states with definite scaling dimension:

$$\phi(x)\,|0\rangle = e^{ix\cdot P}\phi(0)e^{-ix\cdot P} = e^{ix\cdot P}\,|\phi\rangle = \sum_n \frac{1}{n!}(ix\cdot P)^n\,|\Delta\rangle \tag{1.27}$$

This mapping from operators to states is the so called operator-state correspondence. We can define the Hilbert space of the theory by the insertion of primary states at the origin. Any other state can be mapped back to the origin as shown and any state at the origin can be put in correspondence with a local operator.

Another interesting fact that can be easily recovered from the radial quantization picture is the fact that conjugation in radial quantization is related to the inversion operation ([22]). On the cylinder a reflection transformation $\theta : \tau \to -\tau$ can be used to understand conjugation. In the Hamiltonian formulation $\phi(\tau, \mathbf{n}) \equiv e^{\tau H_{cyl}}\phi(0, \mathbf{n})e^{-\tau H_{cyl}}$, so for an hermitian Minkowskian field $\phi(0, \mathbf{n})$:

$$\phi(\tau, \mathbf{n})^\dagger = (e^{\tau H_{cyl}}\phi(0, \mathbf{n})e^{-\tau H_{cyl}})^\dagger = e^{-\tau H_{cyl}}\phi(0, \mathbf{n})e^{\tau H_{cyl}} = \phi(-\tau, \mathbf{n}) \tag{1.28}$$

So $\langle\phi(-\tau, \mathbf{n})\phi(\tau, \mathbf{n})\rangle = \left\langle\phi(\tau, \mathbf{n})^\dagger\phi(\tau, \mathbf{n})\right\rangle$, corresponds to the (positive) norm of the

state $|\phi(\tau, \mathbf{n})\rangle$. Mapping back to Euclidean space the reflection operator maps to an inversion $x \to x/x^2$. As such the conjugate of $\phi(x)\,|0\rangle$ will be:

$$\langle 0|\,[\phi(x)]^\dagger = \langle 0|\,\mathcal{R}[\phi(x)] \tag{1.29}$$

where $\mathcal{R}$ is the inversion operator. Operators inserted in an inversion invariant way compute the norm of a state in radial quantization, this is called reflection positivity and it is the Euclidean equivalent of unitarity. This fact translates into the conjugation properties of algebra generators. The following is true in radial quantization:

$$P_\mu^\dagger = K_\mu = \mathcal{R} P_\mu \mathcal{R} \tag{1.30}$$

As we showed in section 1.1 special conformal transformations can be recovered by an inversion, followed by translation and a final inversion. In radial quantization it relates both generators by conjugation.

1.4 Unitarity Bounds

With the use of the radial quantization formalism [78] we can obtain one of the most interesting results in general CFTs, the unitarity bounds. This famous result, originally presented in [90], tells us that the conformal dimensions of primary fields must be bounded from below by a minimal value, which depends on the spin. Let's see how we can obtain it. Consider the following matrix:

$$A_{\nu\, a,\mu\, b} = {}_a\langle \Delta|\, K_\nu P_\mu\, |\Delta\rangle_b \tag{1.31}$$

In a reflection positive (unitary) theory we should have only positive eigenvalues, otherwise we would obtain negative norm states (we remind the reader that the operator K is the conjugate of P in radial quantization). Using the commutators of our algebra we obtain:

$${}_a\langle \Delta|\, K_\nu P_\mu\, |\Delta\rangle_b = {}_a\langle \Delta|\, [K_\nu, P_\mu]\, |\Delta\rangle_b + {}_a\langle \Delta|\, P_\mu K_\nu\, |\Delta\rangle_b \tag{1.32}$$

The only contribution comes from the first term since we are dealing with primary states. The commutator will give us a term proportional to the eigenvalue Δ, the other term will be:

$$B_{\nu\, a,\mu\, b} = {}_a\langle \Delta|\, iM_{\mu\nu}\, |\Delta\rangle_b \tag{1.33}$$

The condition that the eigenvalues $\lambda_A \geq 0$ puts a limit to the dimension of the states that depends on the maximum eigenvalue of this matrix B: $\Delta \geq \lambda_{max}(B)$.

To compute the eigenvalues of B we can use a trick. Notice the following:

$$-iM_{\mu\nu} = \frac{-i}{2}(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu) M_{\alpha\beta} \tag{1.34}$$

Now the term in parenthesis is nothing but the vector representation of the Lorentz generator $V^{\alpha\beta}_{\mu\nu}$. Thus we can write the previous equation as $(V \cdot M)_{\mu\nu}$ where the product is defined in the vector space as $A \cdot B = \frac{1}{2} A_{\alpha\beta} B_{\alpha\beta}$. We can compare this to a standard problem in quantum mechanics where we have to calculate the eigenvalues of:

$$L^i \cdot S^i \tag{1.35}$$

In quantum mechanics the diagonalization is easily performed using the identity:

$$L^i \cdot S^i = \frac{1}{2}[(L+S)^2 - L^2 - S^2] \tag{1.36}$$

And the operators S^2 and L^2 are Casimirs, so the eigenvalues are known ,$s(s+1)/2$ and $\ell(\ell+1)/2$ respectively. The operator $(L+S)^2$ is the Casimir of the tensor product representation $\ell \otimes s$ and its eigenvalues are $j(j+1)$ where $j = |\ell - s|, ..., \ell + s$

In our case we have a representation R , the vector representation $V_{\ell=1}$, and the tensor product representation $R \otimes V$. Thus the maximal eigenvalue is given by:

$$\lambda_{max}(B) = \frac{1}{2}[Cas(V_{\ell=1}) + Cas(R) - minCas(R \otimes V)] \tag{1.37}$$

As an example consider the traceless symmetric representations ℓ, the tensor representation $\ell \otimes V$ for $\ell \geq 1$ is the $\ell - 1$ representation. The $SO(D)$ Casimir value in this case is $\ell(\ell + D - 2)$ and the bound becomes $\Delta \geq \ell + D - 2$

Let's focus on arbitrary representations in $D = 4$, where we have $SO(4) = SU(2) \times SU(2)$, so representations are labeled by two integers ℓ and $\bar{\ell}$. In this case [78]

$$\Delta \geq \frac{\ell + \bar{\ell}}{2} + 2, \quad \ell \neq 0 \ \ and \ \ \bar{\ell} \neq 0 \tag{1.38}$$

We can easily check we recover the bound for traceless symmetric operators when $\ell = \bar{\ell}$.

1.5 Embedding Formalism

We will explore in this section an embedding of D dimensional space-time into a D+2 dimensional spacetime (this idea goes back to Dirac [74], and among the first that used it were Mack and Salam [73], for recent papers on the subject see [18] and [66]). It turns out that our conformal algebra is in fact isomorphic to $SO(D,2)$. For the case when we stay in the euclidean this isomorphism maps the Conformal group directly to the Lorentz group in D+2 dimensions $SO(D+1,1)$. Thus conformal transformations acquire the familiar form we are more used to of Lorentz transformations. The algebra we wrote at the end of the previous section does not make this equivalence obvious at first sight, but with a little rewriting we will see it very clearly. Let's choose a set of coordinates in 6D $X^A = X^1, ..., X^5, X^6$, for concreteness, which means we will be studying the Conformal group in 4D. The metric of this spacetime is $\eta_{AB} = diag(-1,1,1,1,1,-1)$ and we will do a change of coordinates such that:

$$X^+ = X^6 + X^5, \quad X^- = X^5 - X^6 \tag{1.39}$$

The conformal algebra generators can be assigned in the following way to make the connection with $SO(4,2)$ more obvious:

$$\begin{aligned} L_{\mu\nu} &= M_{\mu\nu}, \quad L_{\mu+} = P_\mu \\ L_{\mu-} &= K_\mu, \quad L_{+-} = D \end{aligned} \tag{1.40}$$

And we can check that they satisfy the algebra of SO(4,2):

$$[L_{MN}, M_{RS}] = i(\eta_{NR}M_{MS} + \eta_{MS}M_{NR} - \eta_{MR}M_{NS} - \eta_{NS}M_{MR}) \tag{1.41}$$

Now that we have rediscovered the algebra in a more clear way, let's push this trick and try to embed our 4D spacetime in a consistent way into this 6D space. First of all the action of the conformal group on coordinates will act linearly in 6D space, $X^M \to \Lambda^M_N X^N$, with Λ^M_N being a matrix of SO(4,2). We need however a correct mapping that respects all the degrees of freedom of our 4D space and reproduces correctly the conformal transformations we found. Let's restrict to a null cone in our extended space:

$$X^2 = 0 \tag{1.42}$$

This constraint already reduces one coordinate degree of freedom, and we can solve it for example by $X^- = -\frac{X_\mu X^\mu}{X^+}$. Finally we impose the following equivalence property:

$$X \sim \lambda X, \ \lambda \in \mathbb{R} \tag{1.43}$$

This type of embedding is named "projective null cone" for obvious reasons. Given the freedom of rescaling of our coordinates, we can "gauge" this freedom by choosing a specific X^+, this would give us a one to one correspondence between our embedding and 4D. We will not do this but it simplifies the general analysis, for example we could have chosen $X^+ = 1$, this gauge "slice" is called the Poincare section. A transformation $\Lambda \in SO(4,2)$ takes X to ΛX by matrix multiplication. To get back to the Poincare section, we must further rescale $\Lambda X \to \Lambda X/(\Lambda X)^+$. This combined transformation is precisely the nonlinear action of the conformal group on 4D. The standard 4D coordinates are recovered with the following map:

$$x^\mu = \frac{X^\mu}{X^+} \tag{1.44}$$

We will not do the exersise in detail, but given these constraints and mapping one can show that conformal transformations acting on x^μ are mapped to Lorentz transformations acting on the "projective light cone" and viceversa.

What should we do at the level of fields? We can assign appropriate mappings to any type of field as long as we impose sets of constraints such that we respect their transformation properties and degrees of freedom. Let's clarify this point by taking scalar fields as an example. Take $\phi(x)$ a 4D primary scalar operator with scaling dimension Δ and $\Phi(X)$ the corresponding, to be, 6D field, which in the projective cone should be an homogeneous function $\Phi(\lambda X) = \lambda^{-n}\Phi(X)$ with $\lambda \in \mathbb{R}$. It should transform as a scalar under an $SO(4,2)$ transformation $\Phi(X) \to \Phi'(\Lambda X) = \Phi(X)$ with $\Lambda \in SO(4,2)$. We can assign the relation:

$$\phi(x) = (X^+)^n \Phi(X) \tag{1.45}$$

Given this relation we can check that if $n = \Delta$ then 6D $SO(4,2)$ transformations of the $\Phi(X)$ field correspond to conformal transformations of our 4D field. This makes the task of calculating correlations functions much easier, since we can work in 6D and identify at a glance if a correlation function is Lorentz invariant (in 6D), the only extra thing to do is to verify that correlation functions satisfy the homogeneity of our fields $\Phi(\lambda X) = \lambda^{-\Delta}\Phi(X)$. Let's see this with an example. Take the two point function $\langle \Phi_1(X_1)\Phi_2(X_2)\rangle$, it is fixed by conformal invariance (SO(4,2) in 6D) and the null condition $X_i^2 = 0$ to have the form:

$$\langle \Phi_1(X_1)\Phi_2(X_2)\rangle \propto \frac{1}{X_{12}^m}, \qquad X_{ij} \equiv X_i \cdot X_j \tag{1.46}$$

If we further impose the homogeneity of our fields we obtain:

$$\langle \Phi_1(X_1)\Phi_2(X_2)\rangle \propto \frac{\delta_{12}}{X_{12}^\Delta} \tag{1.47}$$

And if we want to get back to the 4D form we just have to use eq. (1.45).

The formalism works similarly with other types of fields, such as vectors, only this time we have to impose additional constraints in order to reduce the number of components of our fields as it was done in [66]. A further step was taken in [18] where they developed a way to get rid of indices and simplify the calculation of correlators even further. This "free index" formalism is quite powerful, however it is tailored for traceless symmetric fields and as such it does not help with any other types of fields[1]. We will not go through this formalism in this thesis, instead we will go with another formalism more apt for the 4D case, such that it allows us to include all types of fields, such as spinors and antisymmetric tensors and treat them in the same footing.

1.5.1 Twistor Formalism

Arbitrary 4D Lorentz representations can be built from products of spinors, meaning that if we can uplift spinor operators to the embedding space, then we can uplift any representation in 4D. Using the local isomorphism between $SO(4,2)$ and $SU(2,2)$ we will achieve this in a unified manner. The spinorial representations $\mathbf{4}_\pm$ of $SO(4,2)$ are mapped to the fundamental and anti-fundamental representations of $SU(2,2)$. Twistor space consists of four-component objects:

$$Z_A = \begin{pmatrix} \lambda_\alpha \\ \mu^{\dot\alpha} \end{pmatrix} \tag{1.48}$$

Spinor indices $\alpha, \beta...$ and $\dot\alpha, \dot\beta, ...$ are mapped to twistor indices $A, B, ...$ We also have the duals $\bar W^A$, and an invariant pairing $\bar W^A Z_A$ under "twistor" ($SU(2,2)$) transformations of the form $Z \to UZ$, $\bar W \to \bar W \bar U$, where U is a transformation matrix satisfying $\bar U U = U\bar U = 1$, and $\bar U \equiv \rho U^\dagger \rho$, ρ being the $SU(2,2)$ metric. In this language we complexify our 6D coordinates as $\mathbf{X}_{AB} \equiv X_M \Gamma^M_{AB}$, with Γ^M_{AB} a chiral gamma matrix, satisfying:

[1] It does however apply to any dimensions.

$$\left\{\Gamma^M, \Gamma^N\right\} = 2\eta^{MN} \tag{1.49}$$

where η^{MN} is our previously defined 6D metric, and the six dimensional Gamma matrices Γ^M are constructed by means of the 6D matrices Σ^M and $\bar{\Sigma}^M$, analogues of σ^μ and $\bar{\sigma}^\mu$ in 4D:

$$\Gamma^M = \begin{pmatrix} 0 & \Sigma^M \\ \bar{\Sigma}^M & 0 \end{pmatrix} \tag{1.50}$$

In our basis of choice the Σ^M are antisymmetric, and its explicit form is given by:

$$\begin{aligned} \Sigma^M_{ab} &= \left\{ \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot{\gamma}}\epsilon^{\dot{\beta}\dot{\gamma}} \\ -\bar{\sigma}^{\mu\dot{\alpha}\gamma}\epsilon_{\beta\gamma} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2\epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \begin{pmatrix} -2\epsilon_{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix} \right\}, \\ \bar{\Sigma}^{Mab} &= \left\{ \begin{pmatrix} 0 & -\epsilon^{\alpha\gamma}\sigma^\mu_{\gamma\dot{\beta}} \\ \epsilon_{\dot{\alpha}\dot{\gamma}}\bar{\sigma}^{\mu\dot{\gamma}\beta} & 0 \end{pmatrix}, \begin{pmatrix} -2\epsilon^{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2\epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix} \right\}, \end{aligned} \tag{1.51}$$

where, in order, $M = \{\mu, +, -\}$. The null condition $X^2 = 0$ in this language implies $\mathbf{X}\bar{\mathbf{X}} = \bar{\mathbf{X}}\mathbf{X} = 0$, where $\bar{\mathbf{X}}^{AB} \equiv X_M \bar{\Gamma}^{MAB} = \frac{1}{2}\epsilon^{ABCD}X_{CD}$. Given a spinor primary $\psi_\alpha(x)$ with dimension Δ, it can be shown that the combination:

$$\begin{aligned} \Psi_A(\mathbf{X}) &\equiv (\mathbf{X}^+)^{1/2-\Delta} \begin{pmatrix} \psi_\alpha(x) \\ -(x\cdot\bar{\sigma})^{\dot{\alpha}\beta}\psi_\beta(x) \end{pmatrix} \\ \Phi^A(\mathbf{X}) &\equiv (\mathbf{X}^+)^{1/2-\Delta} \begin{pmatrix} \bar{\phi}_{\dot{\beta}}(x)(x\cdot\bar{\sigma})^{\dot{\beta}\alpha} \\ \bar{\phi}_{\dot{\alpha}}(x) \end{pmatrix} \end{aligned} \tag{1.52}$$

Transforms as a twistor under the conformal group [66], with the previously defined map between 6D and 4D $x^\mu = X^\mu/X^+$. This choice satisfies a transversality condition $\bar{\mathbf{X}}^{AB}\Psi_B(\mathbf{X}) = 0$, which is essentially a constraint to match degrees of freedom, and has degree $-1/2 + \Delta$ in X. We can do a slightly better job of uplifting our spinors by choosing a different (but equivalent) lift of $\psi_\alpha(x)$. One can always solve the transversality condition $\bar{\mathbf{X}}\Psi = \bar{\Phi}\mathbf{X} = 0$ as $\Psi = \mathbf{X}\bar{\Psi}$ and $\bar{\Phi} = \Phi\bar{\mathbf{X}}$ since on the cone $\mathbf{X}\bar{\mathbf{X}} = \bar{\mathbf{X}}\mathbf{X} = 0$. In this way we associate our spinors in the follwing way:

$$\begin{aligned} \psi_\alpha(x) &= (X^+)^{\Delta-1/2}\mathbf{X}_{\alpha A}\bar{\Psi}^A(X), \\ \bar{\phi}^{\dot{\alpha}}(x) &= (X^+)^{\Delta-1/2}\bar{\mathbf{X}}^{\dot{\alpha}A}\Phi_A(X), \end{aligned} \tag{1.53}$$

where $\bar{\mathbf{X}}^{\dot{\beta}B} = \epsilon^{\dot{\beta}\dot{\gamma}}\bar{\mathbf{X}}^B_{\dot{\gamma}}$. The twistors $\bar{\Psi}(X)$ and $\Phi(X)$ are subject to an equivalence relation,

$$\begin{aligned}\bar{\Psi}(X) &\sim \bar{\Psi}(X) + \bar{\mathbf{X}}V,\\ \Phi(X) &\sim \Phi(X) + \mathbf{X}\bar{W},\end{aligned} \tag{1.54}$$

where V and $\bar{W}$ are generic twistors. In this way the degree of our twistors is $1/2+\Delta$ and we are essentially trading the transversality condition for a sort of gauge redundancy. As an example, a two-point function of twistor fields is fixed by conformal invariance and homogeneity to have the form:

$$\left\langle \bar{\Psi}^A(X)\Phi_B(Y)\right\rangle = \frac{\delta^A_B}{(X\cdot Y)^{\Delta+1/2}} \tag{1.55}$$

And thanks to the gauge redundancy we can discard terms of the form $\bar{\mathbf{X}}^{AC}Y_{CB}$. We can project using eq. (1.53) to obtain the correct form in 4D. This step however will hardly be necessary and we will keep our discussion in 6D.

Now that we have a correct way to uplift spinors we can consider fields in arbitrary representations of the Lorentz group. Consider a general primary in the $(\ell,\bar{\ell})$ representation of the Lorentz group and with dimension Δ

$$f^{\dot\beta_1\ldots\dot\beta_{\bar\ell}}_{\alpha_1\ldots\alpha_\ell}(x) \tag{1.56}$$

where dotted and undotted indices are symmetrized. We can generalize the uplift from eq. (1.53) and encode it in a multi-twistor field $F^{A_1\ldots A_\ell}_{B_1\ldots B_{\bar\ell}}$ of degree $\Delta+(\ell+\bar\ell)/2$ as follows:

$$f^{\dot\beta_1\ldots\dot\beta_{\bar\ell}}_{\alpha_1\ldots\alpha_\ell}(x) = (X^+)^{\Delta-(\ell+\bar\ell)/2}\mathbf{X}_{\alpha_1 A_1}\ldots\mathbf{X}_{\alpha_\ell A_\ell}\mathbf{X}^{\dot\beta_1 b_1}\ldots\mathbf{X}^{\dot\beta_{\bar\ell}B_{\bar\ell}}F^{A_1\ldots A_\ell}_{B_1\ldots B_{\bar\ell}}(X) \tag{1.57}$$

And given the gauge redundancy any two fields F and $\hat{F}=F+\bar{\mathbf{X}}V$ or $\hat{F}=F+\mathbf{X}\bar{W}$ are equivalent uplifts of f. We will adopt an index free-notation by contracting with auxiliary (commuting and independent) spinors S_A and $\bar{S}^B$:

$$F(X,S,\bar{S}) \equiv F^{A_1\ldots A_\ell}_{B_1\ldots B_{\bar\ell}}(X)S_{A_1}\ldots S_{A_\ell}\bar{S}^{B_1}\ldots\bar{S}^{B_{\bar\ell}} \tag{1.58}$$

In this language the gauge redundancy means that we can restrict S and $\bar{S}$ to be transverse:

$$\bar{\mathbf{X}}S = \mathbf{X}\bar{S} = 0 \tag{1.59}$$

And this can be solved as $S=\mathbf{X}\bar{T}$, $\bar{S}=\bar{\mathbf{X}}T$ for some T,$\bar{T}$. Consequently the product $\bar{S}S$ vanishes as well. Coming back to indices, this means that $F^{A_1\ldots A_\ell}_{B_1\ldots B_{\bar\ell}}$ must also have

a gauge redundancy under shifts proportional to $\delta^{A_i}_{B_i}$. The general projection to 4D is as follows:

$$f^{\dot\beta_1\dots\dot\beta_{\bar\ell}}_{\alpha_1\dots\alpha_\ell}(x)=\frac{(X^+)^{\Delta-(\ell+\bar\ell)/2}}{\ell!\bar\ell!}\left(\mathbf{X}\frac{\partial}{\partial S}\right)_{\alpha_1}\dots\left(\mathbf{X}\frac{\partial}{\partial S}\right)_{\alpha_\ell}\left(\bar{\mathbf{X}}\frac{\partial}{\partial\bar S}\right)^{\dot\beta_1}\dots\left(\bar{\mathbf{X}}\frac{\partial}{\partial\bar S}\right)^{\dot\beta_{\bar\ell}}F(X,S,\bar S) \tag{1.60}$$

As we have mentioned before it will be hardly necessary to project back to 4D so we will not stretch the discussion more on this point. From now on we will use the "scalar" (under SU(2,2)) form of the fields (1.58) and forget about indices altogether. This will be extremely useful as we will be able to construct "scalar" quantities under SU(2,2) and work with them maintaining all symmetry properties at every step of our calculations.

1.6 Three-Point Function Classification

The calculation of n-point correlation functions in a CFT in the basis of pure symmetry considerations is a key pillar behind most of our results and one of the most important features of CFTs. Two point functions are entirely determined by symmetry as we saw in an example in the previous section, focusing on the 4D case from now on. As for three point functions, symmetry alone is enough to determine kinematically their form up to a constant (one for each of the independent tensor structures satisfying the symmetry constraints). These results will prove of great importance when we talk about the Operator Product Expansion in the next section. General three-point functions in 4D CFTs involving bosonic or fermionic operators in irreducible representations of the Lorentz group have recently been classified and computed in ref. [31] using the twistor formalism described in the previous sections. We will here briefly review the main results of ref. [31].

The 4D three-point functions are conveniently encoded in their scalar 6D counterpart $\langle O_1O_2O_3\rangle$ which must be a sum of $SU(2,2)$ invariant quantities constructed out of the X_i, S_i and $\bar S_i$, with the correct homogeneity properties under rescaling. Notice that quantities proportional to $\bar S_i\mathbf{X}_i$, $\overline{\mathbf{X}}_iS_i$ or $\bar S_iS_i$ $(i=1,2,3)$ are projected to zero in 4D.

The non-trivial $SU(2,2)$ possible invariants are ($i \neq j \neq k$, indices not summed) [21]:

$$I_{ij} \equiv \bar{S}_i S_j \,, \tag{1.61}$$

$$K_{i,jk} \equiv N_{i,jk} S_j \overline{\mathbf{X}}_i S_k \,, \tag{1.62}$$

$$\overline{K}_{i,jk} \equiv N_{i,jk} \bar{S}_j \mathbf{X}_i \bar{S}_k \,, \tag{1.63}$$

$$J_{i,jk} \equiv N_{jk} \bar{S}_i \mathbf{X}_j \overline{\mathbf{X}}_k S_i \,, \tag{1.64}$$

where

$$N_{jk} \equiv \frac{1}{X_{jk}} \,, \quad N_{i,jk} \equiv \sqrt{\frac{X_{jk}}{X_{ij} X_{ik}}} \,. \tag{1.65}$$

Two-point functions are easily determined again through the use of SU(2,2) "scalar" quantities and the homogeneity properties of our fields. One has

$$\langle O_1(X_1, S_1, \bar{S}_1) O_2(X_2, S_2, \bar{S}_2) \rangle = X_{12}^{-\tau_1} I_{21}^{\ell_1} I_{12}^{\bar{\ell}_1} \delta_{\ell_1, \bar{\ell}_2} \delta_{\ell_2, \bar{\ell}_1} \delta_{\Delta_1, \Delta_2} \,, \tag{1.66}$$

where $X_{ij} \equiv X_i \cdot X_j$ and $\tau_i \equiv \Delta_i + (\ell_i + \bar{\ell}_i)/2$. As can be seen from eq.(1.66), any operator $O^{\ell,\bar{\ell}}$ has a non-vanishing two-point function with a conjugate operator $O^{\bar{\ell},\ell}$ only. The proportionality constant in front of this expression can be fixed by rescaling our fields appropriately, and it is a freedom we use to fix it to one for every two point function.

The main result of ref. [31] can be recast in the following way. The most general three-point function $\langle O_1 O_2 O_3 \rangle$ can be written as[2]

$$\langle O_1 O_2 O_3 \rangle = \sum_{s=1}^{N_3} \lambda_s \langle O_1 O_2 O_3 \rangle_s \,, \tag{1.67}$$

where

$$\langle O_1 O_2 O_3 \rangle_s = \mathcal{K}_3 \Big(\prod_{i \neq j = 1}^{3} I_{ij}^{m_{ij}} \Big) C_{1,23}^{n_1} C_{2,31}^{n_2} C_{3,12}^{n_3} \,. \tag{1.68}$$

In eq.(1.68), $\mathcal{K}_3$ is a kinematic factor that depends on the scaling dimension and spin of the external fields,

$$\mathcal{K}_3 = \frac{1}{X_{12}^{a_{12}} X_{13}^{a_{13}} X_{23}^{a_{23}}} , \tag{1.69}$$

with $a_{ij} = (\tau_i + \tau_j - \tau_k)/2$, $i \neq j \neq k$. The index s runs over all the independent tensor structures parametrized by the integers m_{ij} and n_i, each multiplied by a constant coefficient λ_s which is undetermined by symmetry considerations alone. The invariants $C_{i,jk}$ equal to one of the three-index invariants (4.30)-(1.64), depending on the value of

$$\Delta\ell \equiv \ell_1 + \ell_2 + \ell_3 - (\bar{\ell}_1 + \bar{\ell}_2 + \bar{\ell}_3) \,, \tag{1.70}$$

[2] The points X_1, X_2 and X_3 are assumed to be distinct.

of the external fields. Three-point functions are non-vanishing only when $\Delta\ell$ is an even integer [31, 84]. We have

- $\Delta\ell = 0$: $C_{i,jk} = J_{i,jk}$.
- $\Delta\ell > 0$: $C_{i,jk} = J_{i,jk}, K_{i,jk}$.
- $\Delta\ell < 0$: $C_{i,jk} = J_{i,jk}, \overline{K}_{i,jk}$.

A redundance is present for $\Delta\ell = 0$. It can be fixed by demanding, for instance, that one of the three integers n_i in eq.(1.68) vanishes. The total number of $K_{i,jk}$'s ($\overline{K}_{i,jk}$'s) present in the correlator for $\Delta\ell > 0$ ($\Delta\ell < 0$) equal $\Delta\ell/2$ ($-\Delta\ell/2$). The number of tensor structures is given by all the possible allowed choices of nonnegative integers m_{ij} and n_i in eq.(1.67) subject to the above constraints and the ones coming from matching the correct powers of S_i and $\bar{S}_i$ for each field. The latter requirement gives in total six constraints.

Let's look at an example. Consider the following three point function:

$$\left\langle \Psi(X_1, \bar{S}_1)\bar{\Psi}(X_2, S_2)O^{\ell,\ell}(X_0, S_0, \bar{S}_0)\right\rangle \tag{1.71}$$

Where we have made explicit the dependence on the auxiliary twistor variables. The first operator would correspond to an anti-fermion in 4D and the second would be its conjugate, these operators are contracted with a single $\bar{S}_1$ and a S_2 respectively. The third operator corresponds to a traceless symmetric operator in the (ℓ, ℓ) representation of the Lorentz group, thus contracted with S_0 and $\bar{S}_0$ (a total of ℓ times each). The value of $\Delta\ell$ in this case is 0. Correspondingly the $C_{i,jk}$ are simply the $J_{i,jk}$ structures and we can write the final result for $\ell \geq 1$:

$$\left\langle \Psi(X_1, \bar{S}_1)\bar{\Psi}(X_2, S_2)O^{\ell,\ell}(X_0, S_0, \bar{S}_0)\right\rangle = \mathcal{K}_3(\lambda_1 I_{10}I_{02}J_{0,12}^{\ell-1} + \lambda_2 I_{12}J_{0,12}^{\ell}) \tag{1.72}$$

By considering all the constraints previously discussed one can obtain this result. We can also easily see at a glance these are the only structures allowed. We must build tensor structures made out of: one $\bar{S}_1$, one S_2 and ℓ number of S_0 and $\bar{S}_0$. By looking at the invariants at our disposal (1.61)-(1.64) we realize that in order to put the S_0 and $\bar{S}_0$ into SU(2,2) invariants we can either pair one of each of them with the $\bar{S}_1$ and S_2 obtaining a I_{10} and I_{02} respectively, or we can pair them all between themselves. The first case leaves us with $\ell - 1$ number of S_0 and $\bar{S}_0$ that we can package into $J_{0,12}$ structures. In the second case we pair all the S_0 and $\bar{S}_0$ into $J_{0,12}$ leaving $\bar{S}_1$ and S_2 by themselves, which we can only pair them into a I_{12} structure. We have an undetermined coefficient λ_i for each of the independent structures.

Conserved 4D operators are encoded in multitwistors O that satisfy the current conservation condition

$$D \cdot O(X, S, \bar{S}) = 0\,, \qquad D = \left(X_M \Sigma^{MN} \frac{\partial}{\partial X^N}\right)_a^b \frac{\partial}{\partial S_a} \frac{\partial}{\partial \bar{S}^b}\,. \tag{1.73}$$

When eq.(1.73) is imposed on eq.(1.67), we generally get a set of linear relations between the OPE coefficients λ_s's, which restrict the possible allowed tensor structures in the three point function. Under a 4D parity transformation, the invariants (1.61)-(1.64) transform as follows:

$$\begin{aligned} I_{ij} &\to -I_{ji}\,, \\ K_{i,jk} &\to +\overline{K}_{i,jk}\,, \\ \overline{K}_{i,jk} &\to +K_{i,jk}\,, \\ J_{i,jk} &\to +J_{i,jk}\,. \end{aligned} \tag{1.74}$$

1.7 OPE

We have shown in previous sections that radial quantization allows us to understand the Hilbert space of CFTs by inserting primary operators at the origin, the hamiltonian evolution is done by means of the dilation operator and we have complete Hilbert spaces in each of the spheres around the origin by evolving the states within each sphere. Let's take the product of two (scalar) operators $O_i(x)O_j(0)$ at distinct points. The action of this product onto the vacuum of our theory $|0\rangle$ generates a state on any sphere surrounding both operators (by "dilation" evolution), and can be decomposed as follows:

$$O_i(x)O_j(0)\,|0\rangle = \sum_k C_{ijk}(x, P)O_k(0)\,|0\rangle \tag{1.75}$$

Since any state is a linear combination of primaries and descendants. The $C_{ijk}(x, P)$ is an operator that produces primaries and descendants and k runs over primary operators. This is simply the statement that we have a complete basis in each Hilbert space at each sphere surface. This expression is true as long as there are no other operators inserted below $|x|$. By means of the state-operator correspondence we can simply write:

$$O_i(x_1)O_j(x_2) = \sum_k C_{ijk}(x_{12}, \partial_2)O_k(x_2) \tag{1.76}$$

which we can use inside any correlation function where the other operators are inserted outside a sphere surrounding x_1 and x_2. In a CFT this expression becomes an exact

formula, whereas in regular QFT it is understood in the limit where the two operators are very close together, and as such it is only used in the asymptotic short-distance limit. The understanding of this formula thanks to radial quantization in CFTs makes it a very powerful tool. Conformal symmetry alone determines the functions C_{ijk} up to a constant (in the case of operators with spin we will see that we need several constants), that from here on we will call OPE coefficients. Once we determined the OPE structure (the explicit form of the C_{ijk} functions) we can use it to express any n-point functions as a sum of $(n-1)$-functions:

$$\langle O_1(x_1)O_2(x_2)\Pi_i O_i(x_i)\rangle = \sum_k C_{12k}(x_{12},\partial_2)\,\langle O_k(x_2)\Pi_i O_i(x_i)\rangle \tag{1.77}$$

This fact is of great value, correlation functions of arbitrarily high order can be computed by applying the OPE recursively and written as sum of two point functions. For this we need to know all operator dimensions Δ_i (and thus the content of the theory) and all the OPE coefficients, which together form what is know as the CFT data. We will see in the next chapter how the OPE can be used as well to constraint the CFT data, in what it is know as the Conformal Bootstrap. Let's briefly indicate a way in which one could determine the precise form of the C_{ijk} functions. We remind the reader that three point functions are determined by symmetry alone, up to coefficients that multiply each of the independent tensor structures allowed by symmetry, these constants are the OPE coefficients. Schematically:

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle = \sum_k C_{12k}(x_{12},\partial_2)\,\langle O_k(x_2)O_3(x_3)\rangle \tag{1.78}$$

where we have used the OPE between the first two fields. We can see now how to determine the function C_{12k}. First of all notice that the two point functions are orthogonal, that is O_k (for real fields) must correspond to O_3 in order for the two point function to be non-vanishing. At this point the sum collapses to a single term. By using the know expression of the three point function we can match terms in order to determine C_{123}, which will be multiplied by an overall coefficient λ_{123}, the OPE coefficient. For the scalar case this function has the following form:

$$C_{123}(x,\partial) = \frac{\lambda_{123}}{|x|^{\Delta_1+\Delta_2-\Delta_3}}\left(1+\frac{\Delta_3+\Delta_1-\Delta_2}{2\Delta_3}x\cdot\partial+...\right) \tag{1.79}$$

where the other terms correspond to higher order terms in x and derivatives. In actuality the precise form of the OPE functions will not be needed during the rest of the thesis.

This connection we have seen, between three point function and the OPE turns out to be extremely useful. As a byproduct of this connection we will mention the fact that

the three point function does indeed tell us about the structure of the OPE as well as of the allowed operators in the OPE (which are in reality equivalent statements). Thus if we find vanishing three point functions, as a consequence of symmetries, then we can conclude that any of the operators in the three point function is not allowed in the OPE of the remaining operators, or equivalently the corresponding OPE coefficient is zero.

Let's finish this section by pointing out the fact that the OPE is a convergent expansion, even at finite distance, we made the intuitive argument being a consequence of the completeness of the Hilbert space, for more details we refer the reader to [22].

Chapter 2

2 Conformal Blocks

At this point we have seen many interesting facts about CFTs. We learned about conformal kinematics that allow us to determine two and three point functions (up to constants) completely. We explored the radial quantization of CFTs and understood an important fact that operators in the spectrum of any CFT must satisfy, the unitarity bounds. We saw the Operator Product Expansion and its powerful application for determining general n-point functions given that we have knowledge of the CFT data. So far we have only mentioned this fact, it is now time to put it to practice. Let's consider the correlation function of 4 identical scalar operators (we will work in 4D throughout this section) of dimension $[\phi] = d$:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \tag{2.1}$$

We can invoke conformal symmetry and try to determine kinematically the form of this correlator. Let's introduce two parameters at this point, the conformal cross ratios:

$$u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v \equiv \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \tag{2.2}$$

where again $x_{ij} \equiv |x_i - x_j|$. These two combinations of coordinates happen to be invariant under any type of conformal transformations. These types of parameters appear first at the level of four point functions. We can see that under the Poincare group $|x_i - x_j|$ is invariant already so we only need two different points. Dilation however, requires ratios of differences, so we need at least four different points to have non-trivial ratios. We can prove that special conformal transformations also only require four different points (had we worked in 6D the necessity of four different points would have come as a consequence of homogeneity). At the level of four point functions, the conformal ratios, previously defined, are the only invariants possible. We understand the fact that these quantities exist, as the impediment to determine the four point function kinematically. We can indeed find a structure that transforms accordingly respecting the scaling of the operators and their tensor structures (none for the case of scalars), however we will always have the possibility of having some arbitrary function of both u and v.

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{G(u,v)}{x_{12}^{2d}x_{34}^{2d}} \tag{2.3}$$

where at this point $G(u,v)$ is an undetermined function of the cross ratios. As we mentioned already we can use the OPE to determine higher n-point functions in terms of sums of two point functions. Let's see how this works at the level of four point functions. Let's write again the form of the OPE for convenience:

$$\phi(x_1)\phi(x_2) = \sum_O \lambda_{\phi\phi o}\tilde{C}_{\phi\phi O}(x_{12},\partial_2)O(x_2) \tag{2.4}$$

$$\phi(x_3)\phi(x_4) = \sum_{O'} \lambda_{\phi\phi O'}\tilde{C}_{\phi\phi O'}(x_{34},\partial_4)O'(x_4) \tag{2.5}$$

where we have extracted the OPE coefficient from the C_{ijk}, thus the tilde on top of the new (completely kinematic) functions $\tilde{C}_{ijk}$. The sums go through O and O', all the primaries allowed by conformal invariance in these specific OPE channels. In the case of scalars it can be proven that the only operators present are traceless symmetric operators. Introducing this into the four point function we obtain the s-channel expansion:

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_O\sum_{O'} \lambda_{\phi\phi O}\lambda_{\phi\phi O'}\tilde{C}_{\phi\phi O}(x_{12},\partial_2)\tilde{C}_{\phi\phi O'}(x_{34},\partial_4)\,\langle O(x_2)O'(x_4)\rangle \tag{2.6}$$

Again, the two point functions have been chosen orthonormal, such that the double sums become one:

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_O (\lambda_{\phi\phi O})^2\mathcal{W}_O(x_1,x_2,x_3,x_4) \tag{2.7}$$

The function $\mathcal{W}_O$ resums the contribution of the primary O and its descendants to the four scalar correlator. These functions are known as Conformal Partial Waves (CPWs), and for the case of scalar correlators only a single set of them contributes to the four point function. The number of them depends on the number of tensor structures appearing in the OPE, we will discuss this point in more detail in subsequent chapters. The CPW can be written in a form that resembles the four point function (2.1):

$$\mathcal{W}_O(x_1,x_2,x_3,x_4) = \frac{g_O(u,v)}{x_{12}^{2d}x_{34}^{2d}} \tag{2.8}$$

The function $g_O(u,v)$ is known as conformal block, thus we can write the function $G(u,v)$ appearing in (2.3) as:

$$G(u,v) = \sum_O (\lambda_{\phi\phi O})^2 g_O(u,v) \tag{2.9}$$

We see that once we determine $g_O(u,v)$, which could be done by applying the operators in (2.6), we have determined completely the four point function up to the OPE coefficients, thus as said, the knowledge of the CFT data (OPE coefficients, and spectrum of primary operators) allows us to determine any n-point function. With the knowledge of the conformal blocks we can try to impose constraints on the CFT data, with a procedure known as Conformal Bootstrap. We will outline the basic equations necessary in section (2.3). We will focus in the next two sections on methods to obtain closed expressions for the conformal blocks.

Let's finish this section by indicating a change of variables that will prove extremely useful in the following. The map is as follows:

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}) \tag{2.10}$$

The new variables z and $\bar{z}$ will prove essential when solving the conformal blocks in closed form in 4D.

2.1 Casimir equation

Let's review a method proposed by Dolan and Osborn [14] to obtain closed expression for the conformal blocks in even dimensions. We will focus on the correlator of four scalars. The heart of the method consists in finding and solving a differential equation that each conformal block, corresponding to the exchange of a primary operator and its tower of descendants, satisfies. We will be using the 6D embedding formalism to simplify the calculations. Let's first recall some facts about the action of conformal generators, $\hat{L}_{MN}$ upon fields, which act as differential operators:

$$\left[\hat{L}_{MN}, \phi(x)\right] \equiv L_{MN}(x,\partial)\phi(x) \tag{2.11}$$

In 6D the quadratic Casimir of the conformal group has the simple expression:

$$\hat{C} = \frac{1}{2}\hat{L}_{MN}\hat{L}^{MN}\,. \tag{2.12}$$

This operator acting on primary operators gives the Casimir eigenvalue (and by definition of the Casimir operator, the same eigenvalue applies to each of the descendants of the primary):

$$[\hat{C}, \mathcal{O}^{(\ell,\ell)}(x)] = E_{\Delta,\ell}\mathcal{O}^{(\ell,\ell)}(x) \tag{2.13}$$

where

$$E_{\Delta,\ell} = \Delta(\Delta - d) + \ell(\ell + d - 2) \tag{2.14}$$

Let's apply a conformal generator to the product of two fields:

$$\begin{aligned}&\left[\hat{L}_{MN}, \phi_1(x), \phi_2(y)\right] = \left[\hat{L}_{MN}, \phi_1(x)\right]\phi_2(y) + \phi_1(x)\left[\hat{L}_{MN}, \phi_2(y)\right] = \\ &\left(L^{(1)}_{MN}(x, \partial_x) + L^{(2)}_{MN}(y, \partial_y)\right)(\phi_1(x)\phi_2(y))\end{aligned} \tag{2.15}$$

Applying the second order differential operator

$$\frac{1}{2}\left(L^{(1)}_{MN} + L^{(2)}_{MN}\right)\left(L^{MN}_{(1)} + L^{MN}_{(2)}\right) \tag{2.16}$$

is then equivalent to applying the Casimir operator to the first two operators

$$\langle[\hat{C}, \phi_1(x_1)\phi_2(x_2)]\phi_3(x_3)\phi_4(x_4)\rangle \tag{2.17}$$

By using the OPE and singling out a conformal partial wave corresponding to the contribution of a primary, we can then achieve our final equation. We simply have to apply the OPE to the first two operators, thus obtaining the action of the casimir into the primary (and its descendants) as in eq. (2.13). Even though the OPE provides us an infinite sum, all terms are proportional to the same eigenvalue. Now using the differential operator form of (2.16) we obtain an equation for the specific conformal partial wave $W_{\mathcal{O}}$. When acting on scalar operators at x_1 and x_2, the Lorentz generator in 6D can be written as $L_{MN} = L_{1,MN} + L_{2,MN}$, where

$$L_{i\,MN} = i\left(X_{i\,M}\frac{\partial}{\partial X_i^N} - X_{i\,N}\frac{\partial}{\partial X_i^M}\right). \tag{2.18}$$

After a bit of algebra to pass from conformal partial wave to conformal block we can finally write the differential equation:

$$\Delta_2^{a,b,0} g_{\mathcal{O}}(z, \bar{z}) = \frac{1}{2}E_{\Delta,\ell}g_{\mathcal{O}}(z, \bar{z}) \tag{2.19}$$

with $a = \frac{\Delta_2 - \Delta 1}{2}$, $b = \frac{\Delta_3 - \Delta 4}{2}$ and

$$\Delta_\epsilon^{(a,b;c)} = D_z^{(a,b;c)} + D_{\bar{z}}^{(a,b;c)} + \epsilon \frac{z\bar{z}}{z-\bar{z}}\Big((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\Big)\,, \tag{2.20}$$

in terms of the second-order holomorphic operator

$$D_z^{(a,b;c)} \equiv z^2(1-z)\partial_z^2 - \Big((a+b+1)z^2 - cz\Big)\partial_z - abz\,. \tag{2.21}$$

It is possible to solve eq. (2.19) in a closed form, as it was done in [14]. This method will be useful in what follows since it provides a differential equation for any conformal partial wave once the OPE structure of the fields involved in the 4 point function is understood. For a general 4 point function the equations that emerge from this procedure are too cumbersome and highly complex. However we will see in Chapter 4 a direct application of this method.

2.2 Shadow formalism

Another method to obtain conformal blocks (CBs) in closed analytical form uses the so called shadow formalism. It was first introduced by Ferrara, Gatto, Grillo, and Parisi [86–89] and used in ref. [13] to get closed form expressions for the scalar CBs. In this section we will go through the formalism using the recent formulation given in ref. [21]. We will adopt a notation that will be used in the following chapters. We will construct the conformal partial waves by seeking an object that satisfies all the requirements to be a CPW: It satisfies the casimir equation. It satisfies the homogeneity properties in all the X_i accordingly, and finally it is invariant under conformal transformations. The heart of the method is to find a sort of "projector" that once inserted into the four-point function, does the job of extracting the contribution of an operator $\mathcal{O}$ to the four point function, what we have called the conformal block $g_{\mathcal{O}}(z,\bar{z})$. This projector takes the following form [86–89]:

$$\int d^dx \mathcal{O}(x)\,|0\rangle\,\langle 0|\,\tilde{\mathcal{O}}(x) \tag{2.22}$$

The operator $\tilde{\mathcal{O}}(x)$ is called the "shadow operator", and it is a non-local operator of dimension $\tilde{\Delta} = 4 - \Delta$. The generalization to operators with spin requires the definition of an analogous projector, this time for the specific spinning operators we might want to extract the CPW of. Once the projector is inserted into the four point function we have to perform a "conformal integral" of two three point functions. We will be working in 4D so it is convenient to use the embedding formalism in the twistor approach, in order to deal with general representations of operators in a simpler way. The projectors

we will use respect the gauge invariance imposed by the twistor embedding. The CPW associated to the exchange of a given operator O_r with spin $(\ell, \bar{\ell})$ in a correlator of four operators $O_n(X_n)$, $n = 1, 2, 3, 4$ (in embedding space and twistor language) is given by

$$W^{(i,j)}_{O^{(\ell,\bar{\ell})}}(X_i) = \tilde{\nu} \int d^4X_0 \langle O_1(X_1)O_2(X_2)O_r(X_0, S, \bar{S})\rangle_i \overleftrightarrow{\Pi}_{\ell,\bar{\ell}} \langle \widetilde{O}_r(X_0, T, \bar{T})O_3(X_3)O_4(X_4)\rangle_j \Big|_M , \tag{2.23}$$

where $\tilde{\nu}$ is a normalization factor, the projector gluing two 3-point functions is given by

$$\overleftrightarrow{\Pi}_{\ell,\bar{\ell}} = (\overleftarrow{\partial}_S X_0 \overrightarrow{\partial}_T)^{\ell} (\overleftarrow{\partial}_{\bar{S}} \overline{X}_0 \overrightarrow{\partial}_{\bar{T}})^{\bar{\ell}} , \tag{2.24}$$

and $\widetilde{O}_r$ is the shadow operator

$$\widetilde{O}_r(X, S, \bar{S}) \equiv \int d^4Y \frac{1}{(X \cdot Y)^{4-\Delta+\ell+\bar{\ell}}} O_{\bar{r}}(Y, Y\bar{S}, \bar{Y}S) . \tag{2.25}$$

We can see clearly that this object does satisfy the Casimir equation, this fact is easily seen by applying the Casimir operator, which goes through the integral and "hits" the three point function. Being built out of scalar quantities in $SU(2,2)$ we have ensured the invariance under conformal transformations and the homogeneity properties of each X_i are inherited from the three point functions. The gluing projector in the middle is the only choice respecting all the properties previously mentioned when acting on three point functions, namely they respect the gauge invariance of our objects. Furthermore the shadow operator is a non-local operator and as such it does not violate unitarity by formally having conformal dimension $\Delta - 4$. The factors of X_0 appearing outside of the three point functions (as well as the Ys in the shadow operator integral) are there to ensure the correct homogenous behaviour for the conformal integrals.

In eq.(2.23) we have omitted for simplicity the dependence of O_n on their auxiliary twistors S_n, $\bar{S}_n$. The subscripts i and j in $\langle O_1 O_2 O_r\rangle$ and $\langle \widetilde{O}_r O_3 O_4\rangle$ denote the three point functions stripped off their OPE coefficients:

$$\langle O_1 O_2 O_3\rangle \equiv \sum_i \lambda^i_{O_1 O_2 O_3} \langle O_1 O_2 O_3\rangle_i . \tag{2.26}$$

The integral in eq.(2.23) would actually determine the CPW associated to the operator $O_r(X, S, \bar{S})$ plus its unwanted shadow counterpart, that corresponds to the exchange of a similar operator but with the scaling dimension $\Delta \to 4 - \Delta$, which can be traced back to the symmetry of the projector we used under $\mathcal{O} \leftrightarrow \tilde{\mathcal{O}}$. This fact is also present in the form of the Casimir equation, which has a symmetry under the exchange $\Delta \leftrightarrow \Delta - 4$ (see e.g eq(2.14)). The two contributions can be distinguished by their different behaviour under the monodromy transformation $X_{12} \to e^{4\pi i} X_{12}$. In particular,

the physical CPW should transform with the phase $e^{2i\pi(\Delta-\Delta_1-\Delta_2)}$, independently of the Lorentz quantum numbers of the external and exchanged operators. This projection on the correct monodromy component explains the subscript M in the bar at the end of eq.(2.23).

The basic objects we have to deal with are integrals in the projective null-cone since we are working in the embedding formalism in 6D. These integrals are conveniently "gauge-fixed" to match their projection to 4D, details on this procedure can be found in section 2.3 of [21]. We will be using several common tricks for the calculation of these integrals (which after gauge fixing become simply integrals in flat space). For example a three point integral can be written as follows:

$$\int D^d X_0 \frac{1}{X_{10}^a X_{20}^b X_{30}^c} = \alpha(a,b,c) \frac{1}{X_{12}^{h-c} X_{13}^{h-b} X_{23}^{h-a}} \tag{2.27}$$

Where $h \equiv d/2$ and $a+b+c=d$ so that the projective measure of the integral is well-defined. $X_{ij} \equiv X_i \cdot X_j$. And:

$$\alpha(a,b,c) = \frac{\pi^d \Gamma(h-a)\Gamma(h-b)\Gamma(h-c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \tag{2.28}$$

This comes from a couple of basic ingredients. First, the basic building block for all our computations will be (see eq 2.17 in [21]), :

$$I(Y) = \int D^d X_0 \frac{1}{X_{0Y}^d} = \frac{\pi^{d/2}\Gamma[d/2]}{\Gamma[d]\,(Y^2)^{d/2}} \tag{2.29}$$

Which if used with Feynman/Schwinger parametrization to rewrite the denominator, allows us to use eq. (2.29) to obtain eq.(2.27). It is also useful to have an expression with open indices:

$$\int D^d X_0 \frac{X_0^{m_1}...X_0^{m_n}}{X_{0Y}^{d+n}} = \frac{\Gamma[d]}{2^n \Gamma[d+n]} \left(\prod_i (\partial_Y)_{m_i} I(Y) \right) = \frac{\pi^{d/2}}{\Gamma[d+n]} \frac{\Gamma[d/2+n] Y^{m_1}...Y^{m_n}}{(-Y^2)^{d/2+n}} - traces \tag{2.30}$$

We simply used eq. (2.29) and took partial derivatives to obtain the necessary factors. The analogous expression involving three distinct points is:

$$\int D^d Y \frac{Y_{M_1} Y_{M_2}...Y_{M_n}}{X_{0Y}^{a+n} X_{Y3}^b X_{Y4}^c} = \frac{\pi^{d/2}\Gamma[d/2+n]}{\Gamma[a]\Gamma[b]\Gamma[c]} \left(\int_0^\infty dx dy x^{b-1} y^{c-1} \frac{X_{x,y}^{M_1}...X_{x,y}^{M_n}}{[2xX_{03}+2yX_{04}+2xyX_{34}]^{d/2+n}} \right) - traces \tag{2.31}$$

Which we wrote with the help of equations (2.30) and (2.29), and where we have defined:

$$X^M_{x,y} \equiv X^M_0 + xX^M_3 + yX^M_4 \tag{2.32}$$

Although this seems quite involved, in practical computations only a handful of terms contribute. We have to subtract the traces of this object since it is clear that it is traceless from the left hand side of (2.31). These formulas are only necessary for the computation of the shadow operator/three point function, which in turn can be determined up to a constant by conformal symmetry. We will make use of these formulas to compute our blocks.

Let's reproduce the conformal blocks for traceless symmetric operators in a four point function of scalars in this formalism. We need the expressions of the three point functions (stripped off the OPE coefficients), these are given by:

$$\langle \Phi_1(X_1)\Phi_2(X_2)O^{(\ell,\ell)}(X_0)\rangle \;\; = \;\; \mathcal{K}_3(\tau_1,\tau_2,\tau)J^\ell_{0,12}\,,$$

where

$$\mathcal{K}_3(\tau_1,\tau_2,\tau_3) = X_{12}^{\frac{\tau_3-\tau_1-\tau_2}{2}} X_{13}^{\frac{\tau_2-\tau_1-\tau_3}{2}} X_{23}^{\frac{\tau_1-\tau_2-\tau_3}{2}}, \tag{2.33}$$

is a kinematic factor and

$$J_{i,jk} \equiv \frac{1}{X_{jk}} \bar{S}_i \mathbf{X}_j \overline{\mathbf{X}}_k S_i \tag{2.34}$$

is one of the $SU(2,2)$ invariants for three-point functions. The shadow counterpart is simply:

$$\langle \widetilde{\overline{O}}^{(\ell,\ell)}(X_0)\Phi_3(X_3)\bar{\Phi}_4(X_4)\rangle \propto \langle \overline{O}^{(\ell,\ell)}(X_0)\Phi_3(X_3)\bar{\Phi}_4(X_4)\rangle\Big|_{\Delta\to 4-\Delta} \;\; = \;\; \mathcal{K}_3\Big|_{\Delta\to 4-\Delta} J^\ell_{0,34}.$$

This can be shown by using eq. (2.25) and performing the three point integral with the use of eq. (2.31). After putting all together the numerator of the integrand inside the expression corresponding to eq.(2.23) becomes:

$$\mathcal{N}_\ell \;\; \equiv (\bar{S}X_2\bar{X}_1S)^\ell \overleftrightarrow{\Pi}_{\ell,\ell} (\bar{T}X_4\bar{X}_3T)^\ell, \tag{2.35}$$

Our Conformal partial wave is determined by:

$$W^{O_{\ell,\ell}} = \frac{\nu}{X_{12}^{\frac{2d-\Delta+\ell}{2}} X_{34}^{\frac{2d-\tilde{\Delta}+\ell}{2}}} \int D^4 X_0 \frac{\mathcal{N}_\ell}{X_{01}^{\frac{\Delta+\ell}{2}} X_{02}^{\frac{\Delta+\ell}{2}} X_{03}^{\frac{\tilde{\Delta}+\ell}{2}} X_{04}^{\frac{\tilde{\Delta}+\ell}{2}}}\Big|_{M=1}, \tag{2.36}$$

And ν is a constant depending on the dimension of our field Φ and the dimension of the exchanged operator Δ and its spin ℓ. The next step is to evaluate the numerator (2.35). One can show that it satisfies a recursion relation that identifies it with a gegenbauer polynomial:

$$\mathcal{N}_\ell = (l!)^4 (-1)^\ell s^{\ell/2} C_\ell^1(t) \tag{2.37}$$

where we have:

$$s \equiv X_{12} X_{34} \prod_{n=1}^{4} X_{0n},\ t \equiv \frac{1}{2\sqrt{s}} \Big(X_{02} X_{03} X_{14} - X_{01} X_{03} X_{24} - (3 \leftrightarrow 4) \Big), \tag{2.38}$$

Expanding the polynomial $C_\ell^1(t)$, the integral (2.36) becomes a sum of basic conformal four-point integrals. These integrals can be evaluated with methods similar to the ones used for the three point integrals. It will not be necessary however since the integral in eq. (2.36) has been shown to resum in a very compact form by Dolan and Osborn in reference [13]. The result is as follows:

$$W^{O_{\ell,\ell}} = \frac{G_{\Delta,\ell}(U,V)}{X_{12}^{2d} + X_{34}^{2d}} \tag{2.39}$$

where U and V are the counterparts of u and v in 6D, and:

$$G_{\Delta,\ell}(z,\bar{z}) = (-1)^\ell \frac{z\bar{z}}{z-\bar{z}} \left(k_{\frac{\Delta+\ell}{2}}^{(0,0;0)}(z) k_{\frac{\Delta-\ell-2}{2}}^{(0,0;0)}(\bar{z}) - (z \leftrightarrow \bar{z}) \right), \tag{2.40}$$

expressed in terms of the function[1]

$$k_\rho^{(a,b;c)}(z) \equiv z^\rho \, {}_2F_1(a+\rho,\, b+\rho;\, c+2\rho;\, z)\,. \tag{2.41}$$

This result will be of great importance in Chapter 4, where we will use equation (2.36) and its closed form (2.40) in order to rewrite other CPWs in terms of this known four point conformal integral.

[1] We adopt here the notation first used in ref. [9] for this function, but notice the slight difference in the definition: $k_\rho^{there} = k_{\rho/2}^{here}$.

2.3 Conformal Bootstrap

Once we have obtained expressions for the conformal blocks, we can start asking if we can use these expressions to put some sort of constraints on our CFT. The answer is indeed positive and it requires one more ingredient, crossing symmetry. In the case of bosonic operators this crossing symmetry is nothing but the Bose symmetry of our correlator. Sticking to the example of four identical scalars, one basic functional constrain comes from the change $x_2 \leftrightarrow x_4$. Bose symmetry dictates that nothing is changed, but at the level of the cross ratios u and v, this change induces $u \leftrightarrow v$. The whole change in our four point function (2.3) becomes:

$$\frac{G(u,v)}{x_{12}^{2d}x_{34}^{2d}} = \frac{G(v,u)}{x_{14}^{2d}x_{23}^{2d}} \tag{2.42}$$

and after a small rewriting the final constrain reads:

$$\left(\frac{v}{u}\right)^d G(u,v) = G(v,u) \tag{2.43}$$

Another condition comes from the change $x_1 \leftrightarrow x_2$, the constrain is:

$$G(u,v) = G(u/v, 1/v) \tag{2.44}$$

Let's focus on 4D for the rest of the discussion, and we will use the explicit closed expressions for the conformal blocks (2.40). Writing our functions as:

$$G(u,v) = \sum_O (\lambda_{\phi\phi O})^2 g_O(u,v) \tag{2.45}$$

we can ask under which conditions our four point function is consistent with crossing symmetry. One important point is that the coefficients $\lambda_{\phi\phi O}$ are real, given that we consider unitary (reflection positive in the Euclidean) theories [9], so their squares are simply positive numbers. The condition arising from $x_1 \leftrightarrow x_2$ happens to be satisfied block by block [13].:

$$g_O(u,v) = (-1)^l g_O(u/v, 1/v) \tag{2.46}$$

The spin dependence in this case is irrelevant since for the case of identical scalars the only operators exchanged are traceless symmetric operators with even spin. The demonstration of this fact is easily done by showing that these operators are the only ones that have a non-vanishing three point function:

$$\left\langle \phi(x_1)\phi(x_2)O^{\ell,\bar{\ell}}(x_3)\right\rangle \neq 0, \quad \Leftrightarrow \quad \ell = \bar{\ell} \ \ and \ \ \ell \ \ even \tag{2.47}$$

The first condition coming from $x_2 \leftrightarrow x_4$ gives however a nontrivial condition, which can be written as follows:

$$1 = \sum_O \lambda^2_{\phi\phi O} F_O(z,\bar{z}), \quad \lambda^2_{\phi\phi O} \geq 0 \tag{2.48}$$

with

$$F_O(z,\bar{z}) \equiv \frac{v^d g_O(u,v) - u^d g_O(v,u)}{u^d - v^d} \tag{2.49}$$

The left hand side term corresponds to the contribution of the unit operator to the four point function. This term is fixed to be one, and it is a fact that can be traced back to the three point function of:

$$\langle \phi(x_1)\phi(x_2)\mathbb{1}\rangle \tag{2.50}$$

This is nothing but the two point function, which by convention we normalize such that the contribution of the unit operator to the four point function turns out to be simply 1. Equation (2.48) is known as the bootstrap equation and it represents a condition that our CFT must satisfy to be consistent, according to unitarity, and crossing symmetry. The bootstrap equation imposes severe constraints on the allowed spectrum and interaction strengths (OPE coefficients) of the theory, a fact that is intuitively seen by understanding that the precise combinations of these in the right hand side of eq. (2.48) must sum precisely to one. In Chapters 5 and 6 we will see numerical applications of the bootstrap program, where we will put constraints to the allowed spectrum of operators as well as to several OPE coefficients, by carefully studying equation (2.48).

It is also worth noticing that the closed expressions presented in this chapter for the conformal blocks have been found only for even dimensions (we have shown expressions in 4D) [2]. For odd dimensions the task of finding closed expressions for the conformal blocks seems to be much harder. The most used formulas, as of today, have been found by solving the Casimir equations recursively ([54], [25]), with very powerful recursion relations that allow the fast evaluation of general blocks. These expressions, even though not closed, are still sufficient to carry on the bootstrap program and offer a window into known CFTs in the realm of 3D and 5D, such as the 3D Ising

[2] For closed expressions in other even dimensions, such as 6D, see [14, 15]

Model ([101], [103]) which has become one of the most successful benchmarks of the Bootstrap program in higher dimensions.

Chapter 3

3 Deconstructing Conformal Blocks in 4D CFTs

We have shown in previous chapters how to decompose 4-point functions in terms of conformal blocks of primary operators appearing in the OPE in different kinematical channels. Our examples have been restricted to the case of 4 scalars however. In order to go further we need to address the case of "spinning" operators, that is, operators other than scalars. In the 4D case, external operators can be in any representation of the Lorentz group, and this manifests itself, first, as tensor structures allowed in the 4-point function. Tensor/spinor correlators can be decomposed similarly to the scalar case, but this time there will be a function of U and V corresponding to each of the independent tensor structures allowed. These tensor structures are kinematically determined and their number, N_4, grows very rapidly with the spin of the external operators. For each primary operator exchanged we will talk of Conformal Partial Waves (CPWs) which encompass the entire contribution of a primary operator to the whole correlator, given by several conformal blocks, one for each tensor structure.

We should start studying the OPE structure of different operators, for it is essential when trying to understand the primary operators contributing to any 4-point function. The first step is to find all possible 3-point functions and their classification, for they are directly correlated, as we mentioned, with the OPE. This has been done recently in [31] and reviewed in Chapter 1. Once we have understood the classes of operators that contribute we can try to solve for the different CPWs by means of the methods introduced in Chapter 2. However we will try a slightly different route. If it is possible to relate a 3-point function to a simpler one by means of some operators, then a relation between CPWs of different 4-point functions exists. By mean of this observation, building on previous work [18], in ref. [19] the CPWs associated to a correlator of traceless symmetric operators (in arbitrary space-time dimensions), which exchange a traceless symmetric operator, have been related to the scalar conformal block of refs. [13, 14]. In 4D we will have however, CPWs associated to the exchange of non-traceless symmetric operators. And even for the case of traceless symmetric exchange, the work of refs. [18, 19] does not allow the study of correlators with external non-traceless symmetric fields. Thus in this chapter we will generalize the relations

between CPWs for the specific case of 4D CFTs which we will describe using the 6D embedding formalism in twistor space. We will show in full generality how to use differential operators to simplify the calculation of CPWs for external traceless symmetric operators. We will see that for the case of non-traceless symmetric exchange this method deconstructs the CPWs to a series of "seed" CPWs which can then be solved with the methods introduced in Chapter 2.

In section 3.1 we show in more detail how we can relate different CPWs to one another, using the shadow formalism to clarify this point. We introduce a set of general differential operators in section 3.2, and we show the basis of operators necessary for our study case in section 3.3. Section 3.4 is then dedicated to introducing 4-point function tensor structures and the simplest set of seed CPWs. We end the chapter with some examples.

3.1 Relation between CPW

In 4D CFTs, for a given four-point function, CBs and CPWs are labelled by the quantum numbers of the exchanged primary operator and thus they depend on its scaling dimension Δ and representation $(\ell, \bar{\ell})$ of the 4D Lorentz group. The simplest case of four-point functions involving scalar fields only is the best known. In any channel, the exchanged operators have $\bar{\ell} = \ell$, i.e. they are all and only traceless symmetric tensors. In this case CPW and CB are equivalent up to a kinematic factor and their analytic form has been derived in a remarkable compact form in refs. [13, 14] for any Δ and ℓ. Four-point functions involving tensor (or fermion) operators are considerably more complicated because different tensor structures arise and more operators can be exchanged. A generic fermion-tensor four-point function can be parametrized as

$$\langle \mathcal{O}_1^{I_1}(x_1)\mathcal{O}_2^{I_2}(x_2)\mathcal{O}_3^{I_3}(x_3)\mathcal{O}_4^{I_4}(x_4)\rangle = \mathcal{K}_4 \sum_{n=1}^{N_4} g_n(u,v)\mathcal{T}_n^{I_1I_2I_3I_4}(x_i)\,, \tag{3.1}$$

where I_i are schematic Lorentz indices of the operators $\mathcal{O}_i(x_i)$,

$$\mathcal{K}_4 = \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\tau_1-\tau_2}{2}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\tau_3-\tau_4}{2}} (x_{12}^2)^{-\frac{\tau_1+\tau_2}{2}} (x_{34}^2)^{-\frac{\tau_3+\tau_4}{2}} \tag{3.2}$$

is a kinematic factor, $x_{ij}^2 = (x_i - x_j)_\mu (x_i - x_j)^\mu$, $\tau_i = \Delta_i + (\ell_i + \bar{\ell}_i)/2$, u and v are the usual conformally invariant cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,, \tag{3.3}$$

and $\mathcal{T}_n^{I_1I_2I_3I_4}(x_i)$ are kinematically determined tensor structures. Their total number N_4 depends on the Lorentz properties of the external primaries. For correlators involving scalars only, one has $N_4 = 1$, but in general $N_4 > 1$ and rapidly grows with the spin of the external fields. For instance, for four traceless symmetric operators with identical spin ℓ, one has $N_4(\ell) \sim \ell^7$ for large ℓ [31]. One can infer that the number of allowed tensor structures in three and four-point functions is related:[1]

$$N_4 = \sum_r N_{3r}^{12} N_{3\bar{r}}^{34} \,. \tag{3.4}$$

All the non-trivial dynamical information of the 4-point function is encoded in the N_4 functions $g_n(u, v)$. As we mentioned, a bootstrap analysis requires to rewrite the 4-point function (3.1) in terms of the operators exchanged in that channel. In the s-channel (12-34), for instance, we have

$$\langle \mathcal{O}_1^{I_1}(x_1)\mathcal{O}_2^{I_2}(x_2)\mathcal{O}_3^{I_3}(x_3)\mathcal{O}_4^{I_4}(x_4)\rangle = \sum_{i,j}\sum_{\mathcal{O}_r} \lambda^i_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_r}\lambda^j_{\bar{\mathcal{O}}_{\bar{r}}\mathcal{O}_3\mathcal{O}_4} W^{(i,j)I_1I_2I_3I_4}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4,\mathcal{O}_r}(x_i)\,, \tag{3.5}$$

where i and j run over the possible independent tensor structures associated to the three point functions $\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_r\rangle$ and $\langle \bar{\mathcal{O}}_{\bar{r}}\mathcal{O}_3\mathcal{O}_4\rangle$, λ's being their corresponding structure constants and $W^{(p,q)I_1I_2I_3I_4}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4}(u, v)$ are the associated CPWs. The sum over the exchanged primary operators $\mathcal{O}_r$ includes a sum over all possible representations $(\ell, \bar{\ell})$ that can appear in the 4-point function and, for each representation, a sum over all the possible primaries, i.e. a sum over all possible scaling dimensions $\Delta_{\mathcal{O}_r}$. It is useful to define $\delta = |\bar{\ell} - \ell|$ and rearrange the sum over $(\ell, \bar{\ell})$ in a sum over, say, ℓ and δ. There is an important difference between these two sums. For any 4-point function, the sum over ℓ extends up to infinity, while the sum over δ is always finite. More precisely, we have

$$\begin{aligned} &\delta = 0,\, 2\,,\, \ldots\,,\, p-2,\, p, \qquad \mathcal{O}_r \;\; \text{bosonic} \\ &\delta = 1,\, 3\,,\, \ldots\,,\, p-2,\, p, \qquad \mathcal{O}_r \;\; \text{fermionic.} \end{aligned} \tag{3.6}$$

In both cases, the integer p is defined to be

$$p = \min(\ell_1 + \bar{\ell}_1 + \ell_2 + \bar{\ell}_2, \ell_3 + \bar{\ell}_3 + \ell_4 + \bar{\ell}_4)\,, \tag{3.7}$$

and is automatically an even or odd integer when $\mathcal{O}_r$ is a boson or a fermion operator. There are several CPW for each exchanged primary operator $\mathcal{O}_r$, depending on the number of allowed 3-point function structures. They admit a parametrization like the

[1] We do not have a formal proof of eq.(3.4), although the agreement found in ref. [31] using eq.(3.4) in different channels is a strong indication that it should be correct.

4-point function itself,

$$W^{(i,j)I_1I_2I_3I_4}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4,\mathcal{O}_r}(x_i) = \mathcal{K}_4 \sum_{n=1}^{N_4} g^{(i,j)}_{\mathcal{O}_r,n}(u,v)\mathcal{T}^{I_1I_2I_3I_4}_n(x_i)\,, \tag{3.8}$$

where $g^{(i,j)}_{\mathcal{O}_r,n}(u,v)$ are the CBs, scalar functions of u and v that depend on the dimensions and spins of the external and exchanged operators. Imposing crossing symmetry by requiring the equality of different channels is the essence of the bootstrap approach. In order to successfully bootstrap the correlator (3.1), it is necessary to know the explicit form of the CPWs (3.8), in particular the CBs $g^{(i,j)}_{\mathcal{O}_r,n}(u,v)$.

The computation of CPW of tensor correlators is possible, but technically is not easy. In particular it is desirable to have a relation between different CPW, so that it is enough to compute a small subset of them, which determines all the others. In order to understand how this reduction process works, it is very useful to embed the CPW in the 6D embedding space with an index-free notation. We use here the formalism in terms of twistors as reviewed in section 1.6. It is useful to consider the parametrization of CPW in the shadow formalism [86–89]. It has been shown in ref. [21] that a generic CPW can be written in 6D as

$$W^{(i,j)}_{O_1O_2O_3O_4,O_r}(X_k) \propto \int d^4Xd^4Y \langle O_1(X_1)O_2(X_2)O_r(X,S,\bar{S})\rangle_i G \langle \bar{O}_{\bar{r}}(Y,T,\bar{T})O_3(X_3)O_4(X_4)\rangle_j\,. \tag{3.9}$$

In eq.(3.9), $O_k(X_k) = O_k(X_k,S_k,\bar{S}_k)$ are the index-free 6D fields associated to the 4D fields $\mathcal{O}_k(x_k)$, $O_r(X,S,\bar{S})$ and $\bar{O}_{\bar{r}}(Y,T,\bar{T})$ are the exchanged operator and its conjugate, G is a sort of "propagator", function of X,Y and of the twistor derivatives $\partial/\partial S$, $\partial/\partial T$, $\partial/\partial\bar{S}$ and $\partial/\partial\bar{T}$, and the subscripts p and q label the three-point function tensor structures. Finally, in order to remove unwanted contributions, the transformation $X_{12} \to e^{4\pi i}X_{12}$ should be performed and the integral should be projected to the suitable eigenvector under the above monodromy. We do not provide additional details, which can be found in ref. [21], since they are irrelevant for our considerations. Suppose one is able to find a relation between three-point functions of this form:

$$\langle O_1(X_1)O_2(X_2)O_r(X,S,\bar{S})\rangle_p = D_{pp'}(X_{12},S_{1,2},\bar{S}_{1,2})\langle O'_1(X_1)O'_2(X_2)O_r(X,S,\bar{S})\rangle_{p'}\,, \tag{3.10}$$

where $D_{pp'}$ is some operator that depends on $X_{12},S_{1,2},\bar{S}_{1,2}$ and their derivatives, but is crucially independent of X, S, and $\bar{S}$, and $O'_k(X_k)$ are some other, possibly simpler, tensor operators. As long as the operator $D_{pp'}(X_{12},S_{1,2},\bar{S}_{1,2})$ does not change the monodromy properties of the integral, one can use eq.(3.10) in both three-point functions entering eq.(3.9) and move the operator $D_{pp'}$ outside the integral. In this

way we get, with obvious notation,

$$W^{(p,q)}_{O_1O_2O_3O_4,O_r}(X_i) = D^{12}_{pp'}D^{34}_{qq'}W^{(p',q')}_{O'_1O'_2O'_3O'_4,O_r}(X_i)\,. \tag{3.11}$$

Using the embedding formalism in vector notation, ref. [19] has shown how to reduce, in any space-time dimension, CPW associated to a correlator of traceless symmetric operators which exchange a traceless symmetric operator to the known CPW of scalar correlators [13, 14].

Focusing on 4D CFTs and using the embedding formalism in twistor space, we will see how the reduction of CPW can be generalized for arbitrary external and exchanged operators.

3.2 Differential Representation of Three-Point Functions

We look for an explicit expression of the operator $D_{pp'}$ defined in eq.(3.10) as a linear combination of products of simpler operators. They must raise (or more generically change) the degree in $S_{1,2}$ and have to respect the gauge redundancy we have in the choice of O. As we recalled in subsection 1.6, multitwistors of the form

$$O \sim O + \mathcal{O}(\bar{S}X)G + \mathcal{O}(\overline{X}S)G'\,, \qquad O \sim O + \mathcal{O}(X^2)G\,, \tag{3.12}$$

where G and G' are some other multi-twistors fields, are equivalent uplifts of the same 4D tensor field. Eq.(3.10) is gauge invariant with respect to the equivalence classes (3.12) only if we demand

$$D_{pp'}\mathcal{O}(\overline{\mathbf{X}}_i\mathbf{X}_i, \overline{\mathbf{X}}_iS_i, \overline{S}_i\mathbf{X}_i, X_i^2, \overline{S}_iS_i) = \mathcal{O}(\overline{\mathbf{X}}_i\mathbf{X}_i, \overline{\mathbf{X}}_iS_i, \overline{S}_i\mathbf{X}_i, X_i^2, \overline{S}_iS_i)\,, \quad i = 1, 2\,. \tag{3.13}$$

It is useful to classify the building block operators according to their value of Δl, as defined in eq.(1.70).

At zero order in derivatives, we have three possible operators, with $\Delta l = 0$:

$$\sqrt{X_{12}}, I_{12}\,, I_{21}\,. \tag{3.14}$$

At first order in derivatives (in X and S), four operators are possible with $\Delta l = 0$:

$$\begin{aligned}
D_1 &\equiv \frac{1}{2}\overline{S}_1\Sigma^M\overline{\Sigma}^N S_1\Big(X_{2M}\frac{\partial}{\partial X_1^N} - X_{2N}\frac{\partial}{\partial X_1^M}\Big)\,,\\
D_2 &\equiv \frac{1}{2}\overline{S}_2\Sigma^M\overline{\Sigma}^N S_2\Big(X_{1M}\frac{\partial}{\partial X_2^N} - X_{1N}\frac{\partial}{\partial X_2^M}\Big)\,,\\
\widetilde{D}_1 &\equiv \overline{S}_1\mathbf{X}_2\overline{\Sigma}^N S_1\frac{\partial}{\partial X_2^N} + 2I_{12}\,S_{1a}\frac{\partial}{\partial S_{2a}} - 2I_{21}\,\overline{S}_1^a\frac{\partial}{\partial\overline{S}_2^a}\,,\\
\widetilde{D}_2 &\equiv \overline{S}_2\mathbf{X}_1\overline{\Sigma}^N S_2\frac{\partial}{\partial X_1^N} + 2I_{21}\,S_{2a}\frac{\partial}{\partial S_{1a}} - 2I_{12}\,\overline{S}_2^a\frac{\partial}{\partial\overline{S}_1^a}\,.
\end{aligned} \tag{3.15}$$

The extra two terms in the last two lines of eq.(3.15) are needed to satisfy the condition (3.13). The $SU(2,2)$ symmetry forbids any operator at first order in derivatives with $\Delta\ell = \pm 1$.

When $\Delta\ell = 2$, we have the two operators

$$d_1 \equiv S_2\overline{X}_1\frac{\partial}{\partial\overline{S}_1}\,,\qquad d_2 \equiv S_1\overline{X}_2\frac{\partial}{\partial\overline{S}_2}\,, \tag{3.16}$$

and their conjugates with $\Delta\ell = -2$:

$$\overline{d}_1 \equiv \overline{S}_2 X_1\frac{\partial}{\partial S_1}\,,\qquad \overline{d}_2 \equiv \overline{S}_1 X_2\frac{\partial}{\partial S_2}\,. \tag{3.17}$$

The operator $\sqrt{X_{12}}$ just decreases the dimensions at both points 1 and 2 by one half. The operator I_{12} increases by one the spin $\bar{\ell}_1$ and by one ℓ_2. The operator D_1 increases by one the spin ℓ_1 and by one $\bar{\ell}_1$, increases by one the dimension at point 1 and decreases by one the dimension at point 2. The operator $\widetilde{D}_1$ increases by one the spin ℓ_1 and by one the spin $\bar{\ell}_1$ and it does not change the dimension of both points 1 and 2. The operator d_1 increases by one the spin ℓ_2 and decreases by one $\bar{\ell}_1$, decreases by one the dimension at point 1 and does not change the dimension at point 2. The action of the remaining operators is trivially obtained by $1 \leftrightarrow 2$ exchange or by conjugation.

Two more operators with $\Delta\ell = 2$ are possible:

$$\begin{aligned}
\widetilde{d}_1 &\equiv X_{12}S_1\overline{\Sigma}^M S_2\frac{\partial}{\partial X_1^N} - I_{12}S_{1a}\overline{\mathbf{X}}_2^{ab}\frac{\partial}{\partial\overline{S}_1^b}\,,\\
\widetilde{d}_2 &\equiv X_{12}S_2\overline{\Sigma}^M S_1\frac{\partial}{\partial X_2^N} - I_{21}S_{2a}\overline{\mathbf{X}}_1^{ab}\frac{\partial}{\partial\overline{S}_2^b}\,,
\end{aligned} \tag{3.18}$$

together with their conjugates with $\Delta\ell = -2$. We will shortly see that the operators (3.18) are redundant and can be neglected.

The above operators satisfy the commutation relations

$$
\begin{aligned}
&[D_i,\widetilde{D}_j]=[d_i,d_j]=[\bar{d}_i,\bar{d}_j]=[d_i,\widetilde{d}_j]=[\bar{d}_i,\overline{\widetilde{d}}_j]=[\widetilde{d}_i,\widetilde{d}_j]=[\overline{\widetilde{d}}_i,\overline{\widetilde{d}}_j]=0\,,\qquad i,j=1,2\,,\\
&[D_1,D_2]=4I_{12}I_{21}\Big(-X_1^M\frac{\partial}{\partial X_1^M}+X_2^M\frac{\partial}{\partial X_2^M}\Big)\,,\\
&[\widetilde{D}_1,\widetilde{D}_2]=4I_{12}I_{21}\Big(X_1^M\frac{\partial}{\partial X_1^M}-X_2^M\frac{\partial}{\partial X_2^M}+S_1\frac{\partial}{\partial S_1}+\bar{S}_1\frac{\partial}{\partial \bar{S}_1}-S_2\frac{\partial}{\partial S_2}-\bar{S}_2\frac{\partial}{\partial \bar{S}_2}\Big)\,,\\
&[\widetilde{d}_1,\overline{\widetilde{d}}_2]=2X_{12}I_{12}I_{21}\Big(-X_1^M\frac{\partial}{\partial X_1^M}+X_2^M\frac{\partial}{\partial X_2^M}-\bar{S}_1\frac{\partial}{\partial \bar{S}_1}+S_2\frac{\partial}{\partial S_2}\Big)\,,\\
&[d_i,\bar{d}_j]=2X_{12}\Big(S_j\frac{\partial}{\partial S_j}-\bar{S}_i\frac{\partial}{\partial \bar{S}_i}\Big)(1-\delta_{i,j})\,,\qquad i,j=1,2\,,\\
&[d_i,D_j]=-2\delta_{i,j}\widetilde{d}_i\,,\qquad i,j=1,2\,,\\
&[d_1,\widetilde{D}_1]=2\widetilde{d}_2\,,\qquad\qquad [d_2,\widetilde{D}_1]=0\,,\\
&[\widetilde{d}_1,D_1]=0\,,\qquad\qquad [\widetilde{d}_2,D_1]=-2I_{12}I_{21}d_2\,,\\
&[\widetilde{d}_1,\widetilde{D}_1]=2I_{12}I_{21}d_2\,,\qquad [\widetilde{d}_2,\widetilde{D}_1]=0\,,\\
&[d_1,\overline{\widetilde{d}}_1]=-X_{12}\widetilde{D}_2\,,\qquad [d_1,\overline{\widetilde{d}}_2]=X_{12}D_2\,.
\end{aligned}
\tag{3.19}
$$

Some other commutators are trivially obtained by exchanging 1 and 2 and by the parity transformation (3.25). The operators $\sqrt{X}_{12}$, I_{12} and I_{21} commute with all the differential operators. Acting on the whole correlator, we have

$$
S_i\frac{\partial}{\partial S_i}\to\ell_i\,,\quad \bar{S}_i\frac{\partial}{\partial \bar{S}_i}\to\bar{\ell}_i\,,\quad X_i^M\frac{\partial}{\partial X_i^M}\to-\tau_i\,,
\tag{3.20}
$$

and hence the above differential operators, together with X_{12} and $I_{12}I_{21}$, form a closed algebra when acting on three-point correlators. Useful information on conformal blocks can already be obtained by considering the rather trivial operator $\sqrt{X_{12}}$. For any three point function tensor structure, we have

$$
\langle O_1O_2O_3\rangle_s=(\sqrt{X_{12}})^a\langle O_1^{\frac{a}{2}}O_2^{\frac{a}{2}}O_3\rangle_s\,,
\tag{3.21}
$$

where a is an integer and the superscript indicates a shift in dimension. If $\Delta(\mathcal{O})=\Delta_{\mathcal{O}}$, then $\Delta(\mathcal{O}^a)=\Delta_{\mathcal{O}}+a$. Using eqs.(3.21) and (4.27), we get for any 4D CPW and pair of integers a and b:

$$
W^{(p,q)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4,\mathcal{O}_r}=x_{12}^a x_{34}^b W^{(p,q)}_{\mathcal{O}_1^a\mathcal{O}_2^a\mathcal{O}_3^b\mathcal{O}_4^b,\mathcal{O}_r}\,.
\tag{3.22}
$$

In terms of the conformal blocks defined in eq.(3.8) one has

$$
\mathcal{G}^{(p,q)}_{\mathcal{O}_r,n}(u,v)=\mathcal{G}^{(p,q)a,a,b,b}_{\mathcal{O}_r,n}(u,v)\,,
\tag{3.23}
$$

where the superscripts indicate the shifts in dimension in the four external operators. Equation (3.23) significantly constrains the dependence of $\mathcal{G}^{(p,q)}_{\mathcal{O}_r,n}$ on the external operator dimensions Δ_i. The conformal blocks can be periodic functions of Δ_1, Δ_2 and Δ_3, Δ_4, but can arbitrarily depend on $\Delta_1 - \Delta_2$, $\Delta_3 - \Delta_4$. This is in agreement with the known form of scalar conformal blocks. Since in this paper we are mostly concerned in deconstructing tensor structures, we will neglect in the following the operator $\sqrt{X_{12}}$.

The set of differential operators is redundant, namely there is generally more than 1 combination of products of operators that lead from one three-point function structure to another one. In particular, without any loss of generality we can forget about the operators (3.18), since their action is equivalent to commutators of d_i and D_j. On the other hand, it is not difficult to argue that the above operators do not allow to connect any three-point function structure to any other one. For instance, it is straightforward to verify that there is no way to connect a three-point correlator with one $(\ell, \bar{\ell})$ field to another correlator with a $(\ell \pm 1, \bar{\ell} \mp 1)$ field, with the other fields left unchanged. This is not an academic observation because, as we will see, connections of this kind will turn out to be useful in order to simplify the structure of the CPW seeds. The problem is solved by adding to the above list of operators the following second-order operator with $\Delta\ell = 0$:

$$\nabla_{12} \equiv \frac{(\overline{\mathbf{X}}_1 \mathbf{X}_2)^a_b}{X_{12}} \frac{\partial^2}{\partial \overline{S}^a_1 \partial S_{2,b}} \tag{3.24}$$

and its conjugate ∇_{21}. The above operators transform as follows under 4D parity:

$$D_i \to D_i\,, \quad \widetilde{D}_i \to \widetilde{D}_i\,, \quad d_i \leftrightarrow -\bar{d}_i\,, \quad \tilde{d}_i \leftrightarrow \widetilde{\bar{d}}_i\,, \quad (i = 1, 2)\,, \quad \nabla_{12} \leftrightarrow -\nabla_{21}\,. \tag{3.25}$$

It is clear that all the operators above are invariant under the monodromy $X_{12} \to e^{4\pi i} X_{12}$. The addition of ∇_{12} and ∇_{21} makes the operator basis even more redundant. It is clear that the paths connecting two different three-point correlators that make use of the least number of these operators are preferred, in particular those that also avoid (if possible) the action of the second order operators ∇_{12} and ∇_{21}. We will not attempt here to explicitly construct a minimal differential basis connecting two arbitrary three-point correlators. Such an analysis is in general complicated and perhaps not really necessary, since in most applications we are interested in CPW involving external fields with spin up to two. Given their particular relevance, we will instead focus in the next section on three-point correlators of two traceless symmetric operators with an arbitrary field $O^{(\ell,\bar{\ell})}$.

3.3 Differential Basis for Traceless Symmetric Operators

In this section we show how three-point correlators of two traceless symmetric operators with an arbitrary field $O^{(\ell_3,\bar{\ell}_3)}$ can be reduced to seed correlators, with one tensor structure only. We first consider the case $\ell_3 = \bar{\ell}_3$, and then go on with $\ell_3 \neq \bar{\ell}_3$.

3.3.1 Traceless Symmetric Exchanged Operators

The reduction of traceless symmetric correlators to lower spin traceless symmetric correlators has been successfully addressed in ref. [19]. In this subsection we essentially reformulate the results of ref. [19] in our formalism. This will turn out to be crucial to address the more complicated case of antisymmetric operator exchange. Whenever possible, we will use a notation as close as possible to that of ref. [19], in order to make any comparison more transparent to the reader.

Three-point correlators of traceless symmetric operators can be expressed only in terms of the $SU(2,2)$ invariants I_{ij} and $J_{i,jk}$ defined in eqs.(1.61)-(1.64), since $\Delta\ell$ defined in eq.(1.70) vanishes. It is useful to consider separately parity even and parity odd tensor structures. Given the action of parity, eq.(1.74), the most general parity even tensor structure is given by products of the following invariants:

$$(I_{21}I_{13}I_{32} - I_{12}I_{31}I_{23}), (I_{12}I_{21}), (I_{13}I_{31}), (I_{23}I_{32}), J_{1,23}, J_{2,31}, J_{3,12}\,. \tag{3.26}$$

These structures are not all independent, because of the identity

$$J_{1,23}J_{2,31}J_{3,12} = 8(I_{12}I_{31}I_{23} - I_{21}I_{13}I_{32}) - 4(I_{23}I_{32}J_{1,23} + I_{13}I_{31}J_{2,31} + I_{12}I_{21}J_{3,12})\,. \tag{3.27}$$

In ref. [31], eq.(3.27) has been used to define an independent basis where no tensor structure contains the three $SU(2,2)$ invariants $J_{1,23}$, $J_{2,31}$ and $J_{3,12}$ at the same time. A more symmetric and convenient basis is obtained by using eq.(3.27) to get rid of the first factor in eq.(3.26). We define the most general parity even tensor structure of traceless symmetric tensor correlator as

$$\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1 & \ell_2 & \ell_3 \\ m_{23} & m_{13} & m_{12} \end{bmatrix} \equiv \mathcal{K}_3 (I_{12}I_{21})^{m_{12}} (I_{13}I_{31})^{m_{13}} (I_{23}I_{32})^{m_{23}} J_{1,23}^{j_1} J_{2,31}^{j_2} J_{3,12}^{j_3}\,, \tag{3.28}$$

where ℓ_i and Δ_i are the spins and scaling dimensions of the fields, the kinematical

factor $\mathcal{K}_3$ is defined in eq.(4.29) and

$$\begin{aligned} j_1 &= \ell_1 - m_{12} - m_{13} \geq 0\,, \\ j_2 &= \ell_2 - m_{12} - m_{23} \geq 0\,, \\ j_3 &= \ell_3 - m_{13} - m_{23} \geq 0\,. \end{aligned} \tag{3.29}$$

Notice the similarity of eq.(3.28) with eq.(3.15) of ref. [19], with $(I_{ij}I_{ji}) \to H_{ij}$ and $J_{i,jk} \to V_{i,jk}$. The structures (3.28) can be related to a seed scalar-scalar-tensor correlator. Schematically

$$\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1 & \ell_2 & \ell_3 \\ m_{23} & m_{13} & m_{12} \end{bmatrix} = \mathcal{D} \begin{bmatrix} \Delta_1' & \Delta_2' & \Delta_3 \\ 0 & 0 & \ell_3 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.30}$$

where $\mathcal{D}$ is a sum of products of the operators introduced in section 3.2. Since symmetric traceless correlators have $\Delta\ell = 0$, it is natural to expect that only the operators with $\Delta\ell = 0$ defined in eqs.(3.14) and (3.15) will enter in $\mathcal{D}$. Starting from the seed, we now show how one can iteratively construct all tensor structures by means of recursion relations. The analysis will be very similar to the one presented in ref. [19] in vector notation. We first construct tensor structures with $m_{13} = m_{32} = 0$ for any ℓ_1 and ℓ_2 by iteratively using the relation (analogue of eq.(3.27) in ref. [19], with $D_1 \to D_{12}$ and $\widetilde{D}_1 \to D_{11}$)

$$\begin{aligned} &D_1 \begin{bmatrix} \Delta_1 & \Delta_2+1 & \Delta_3 \\ \ell_1-1 & \ell_2 & \ell_3 \\ 0 & 0 & m_{12} \end{bmatrix} + \tilde{D}_1 \begin{bmatrix} \Delta_1+1 & \Delta_2 & \Delta_3 \\ \ell_1-1 & \ell_2 & \ell_3 \\ 0 & 0 & m_{12} \end{bmatrix} = \\ &(2+2m_{12}-\ell_1-\ell_2-\Delta_3) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & m_{12} \end{bmatrix} - 8(\ell_2 - m_{12}) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & m_{12}+1 \end{bmatrix}. \end{aligned} \tag{3.31}$$

The analogous equation with D_2 and $\widetilde{D}_2$ is obtained from eq.(3.31) by exchanging $1 \leftrightarrow 2$ and changing sign of the coefficients in the right hand side of the equation. The sign change arises from the fact that $J_{1,23} \to -J_{2,31}$, $J_{2,31} \to -J_{1,23}$ and $J_{3,12} \to -J_{3,12}$ under $1 \leftrightarrow 2$. Hence structures that differ by one spin get a sign change. This observation applies also to eq.(3.33) below. Structures with $m_{12} > 0$ are deduced using (analogue of eq(3.28) in ref. [19])

$$\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1 & \ell_2 & \ell_3 \\ m_{23} & m_{13} & m_{12} \end{bmatrix} = (I_{12}I_{21}) \begin{bmatrix} \Delta_1+1 & \Delta_2+1 & \Delta_3 \\ \ell_1-1 & \ell_2-1 & \ell_3 \\ m_{23} & m_{13} & m_{12}-1 \end{bmatrix}. \tag{3.32}$$

Structures with non-vanishing m_{13} (m_{23}) are obtained by acting with the operator D_1 (D_2):

$$4(\ell_3-m_{13}-m_{23})\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ \ell_1 & \ell_2 & \ell_3\\ m_{23} & m_{13}+1 & m_{12}\end{bmatrix}=D_1\begin{bmatrix}\Delta_1 & \Delta_2+1 & \Delta_3\\ \ell_1-1 & \ell_2 & \ell_3\\ m_{23} & m_{13} & m_{12}\end{bmatrix}$$
$$+4(\ell_2-m_{12}-m_{23})\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ \ell_1 & \ell_2 & \ell_3\\ m_{23} & m_{13} & m_{12}+1\end{bmatrix}-$$
$$\tfrac{1}{2}(2+2m_{12}-2m_{13}+\Delta_2-\Delta_1-\Delta_3-\ell_1-\ell_2+\ell_3)\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ \ell_1 & \ell_2 & \ell_3\\ m_{23} & m_{13} & m_{12}\end{bmatrix}, \tag{3.33}$$

and is the analogue of eq (3.29) in ref. [19]. In this way all parity even tensor structures can be constructed starting from the seed correlator.

Let us now turn to parity odd structures. The most general parity odd structure is given by

$$\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ \ell_1 & \ell_2 & \ell_3\\ m_{23} & m_{13} & m_{12}\end{bmatrix}_{odd}\equiv(I_{12}I_{23}I_{31}+I_{21}I_{32}I_{13})\begin{bmatrix}\Delta_1+1 & \Delta_2+1 & \Delta_3+1\\ \ell_1-1 & \ell_2-1 & \ell_3-1\\ m_{23} & m_{13} & m_{12}\end{bmatrix}. \tag{3.34}$$

Since the parity odd combination $(I_{12}I_{23}I_{31}+I_{21}I_{32}I_{13})$ commutes with $D_{1,2}$ and $\widetilde{D}_{1,2}$, the recursion relations found for parity even structures straightforwardly apply to the parity odd ones. One could define a "parity odd seed"

$$16\ell_3(\Delta_3-1)\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ 1 & 1 & \ell_3\\ 0 & 0 & 0\end{bmatrix}_{odd}=(d_2\bar{d}_1-\bar{d}_2d_1)D_1D_2\begin{bmatrix}\Delta_1+2 & \Delta_2+2 & \Delta_3\\ 0 & 0 & \ell_3\\ 0 & 0 & 0\end{bmatrix} \tag{3.35}$$

and from here construct all the parity odd structures. Notice that the parity odd seed cannot be obtained by applying only combinations of $D_{1,2}$, $\widetilde{D}_{1,2}$ and $(I_{12}I_{21})$, because these operators are all invariant under parity, see eq.(3.25). This explains the appearance of the operators d_i and $\bar{d}_i$ in eq.(3.35). The counting of parity even and odd structures manifestly agrees with that performed in ref. [18].

Once proved that all tensor structures can be reached by acting with operators on the seed correlator, one might define a differential basis which is essentially identical to

that defined in eq.(3.31) of ref. [19]:

$$\left\{\begin{array}{ccc}\Delta_1 & \Delta_2 & \Delta_3\\ \ell_1 & \ell_2 & \ell_3\\ m_{23} & m_{13} & m_{12}\end{array}\right\}_0 = (I_{12}I_{21})^{m_{12}} D_1^{m_{13}} D_2^{m_{23}} \widetilde{D}_1^{j_1} \widetilde{D}_2^{j_2} \left[\begin{array}{ccc}\Delta_1' & \Delta_2' & \Delta_3\\ 0 & 0 & \ell_3\\ 0 & 0 & 0\end{array}\right], \qquad (3.36)$$

where $\Delta_1' = \Delta_1 + \ell_1 + m_{23} - m_{13}$, $\Delta_2' = \Delta_2 + \ell_2 + m_{13} - m_{23}$. The recursion relations found above have shown that the differential basis (3.36) is complete: all parity even tensor structures can be written as linear combinations of eq.(3.36). The dimensionality of the differential basis matches the one of the ordinary basis for any spin ℓ_1, ℓ_2 and ℓ_3. Since both bases are complete, the transformation matrix relating them is ensured to have maximal rank. Its determinant, however, is a function of the scaling dimensions Δ_i and the spins ℓ_i of the fields and one should check that it does not vanish for some specific values of Δ_i and ℓ_i. We have explicitly checked up to $\ell_1 = \ell_2 = 2$ that for $\ell_3 \geq \ell_1 + \ell_2$ the rank of the transformation matrix depends only on Δ_3 and ℓ_3 and never vanishes, for any value of Δ_3 allowed by the unitarity bound [90]. On the other hand, a problem can arise when $\ell_3 < \ell_1 + \ell_2$, because in this case a dependence on the values of Δ_1 and Δ_2 arises and the determinant vanishes for specific values (depending on the ℓ_i's) of $\Delta_1 - \Delta_2$ and Δ_3, even when they are within the unitarity bounds.[2] This issue is easily solved by replacing $\widetilde{D}_{1,2} \to (\widetilde{D}_{1,2} + D_{1,2})$ in eq.(3.36), as suggested by the recursion relation (3.31), and by defining an improved differential basis

$$\left\{\begin{array}{ccc}\Delta_1 & \Delta_2 & \Delta_3\\ \ell_1 & \ell_2 & \ell_3\\ m_{23} & m_{13} & m_{12}\end{array}\right\} = (I_{12}I_{21})^{m_{12}} D_1^{m_{13}} D_2^{m_{23}} \sum_{n_1=0}^{j_1} \binom{j_1}{n_1} D_1^{n_1} \widetilde{D}_1^{j_1-n_1} \sum_{n_2=0}^{j_2} \binom{j_2}{n_2} D_2^{n_2} \widetilde{D}_2^{j_2-n_2} \left[\begin{array}{ccc}\Delta_1' & \Delta_2' & \Delta_3\\ 0 & 0 & \ell_3\\ 0 & 0 & 0\end{array}\right] \qquad (3.37)$$

where $\Delta_1' = \Delta_1 + \ell_1 + m_{23} - m_{13} + n_2 - n_1$, $\Delta_2' = \Delta_2 + \ell_2 + m_{13} - m_{23} + n_1 - n_2$. A similar basis for parity odd structures is given by

$$\left\{\begin{array}{ccc}\Delta_1 & \Delta_2 & \Delta_3\\ \ell_1 & \ell_2 & \ell_3\\ m_{23} & m_{13} & m_{12}\end{array}\right\}_{odd} = (d_2\bar{d}_1 - \bar{d}_2 d_1) D_1 D_2 \left\{\begin{array}{ccc}\Delta_1+2 & \Delta_2+2 & \Delta_3\\ \ell_1-1 & \ell_2-1 & \ell_3\\ m_{23} & m_{13} & m_{12}\end{array}\right\}. \qquad (3.38)$$

In practical computations it is more convenient to use the differential basis rather than the recursion relations and, if necessary, use the transformation matrix to rotate the results back to the ordinary basis. We have explicitly constructed the improved differential basis (3.37) and (3.38) up to $\ell_1 = \ell_2 = 2$. The rank of the transformation matrix depends on Δ_3 and ℓ_3 for any value of ℓ_3, and never vanishes, for any value of Δ_3 allowed by the unitary bound.[3]

[2] A similar problem seems also to occur for the basis (3.31) of ref. [19] in vector notation.

[3] The transformation matrix is actually not of maximal rank when $\ell_3 = 0$ and $\Delta_3 = 1$. However,

3.3.2 Antisymmetric Exchanged Operators

In this subsection we consider correlators with two traceless symmetric and one antisymmetric operator $O^{(\ell_3,\bar{\ell}_3)}$, with $\ell_3 - \bar{\ell}_3 = 2\delta$, with δ an integer. A correlator of this form has $\Delta\ell = 2\delta$ and according to the analysis of section 1.6, any of its tensor structures can be expressed in a form containing an overall number δ of $K_{i,jk}$'s if $\delta > 0$, or $\overline{K}_{i,jk}$'s if $\delta < 0$. We consider in the following $\delta > 0$, the case $\delta < 0$ being easily deduced from $\delta > 0$ by means of a parity transformation. The analysis will proceed along the same lines of subsection 3.3.1. We first show a convenient parametrization for the tensor structures of the correlator, then we prove by deriving recursion relations how all tensor structures can be reached starting from a single seed, to be determined, and finally present a differential basis.

We first consider the situation where $\ell_3 \geq \ell_1 + \ell_2 - \delta$ and then the slightly more involved case with unconstrained ℓ_3.

Recursion Relations for $\ell_3 \geq \ell_1 + \ell_2 - \delta$

It is convenient to look for a parametrization of the tensor structures which is as close as possible to the one (3.28) valid for $\delta = 0$. When $\ell_3 \geq \ell_1 + \ell_2 - \delta$, any tensor structure of the correlator contains enough $J_{3,12}$'s invariants to remove all possible $K_{3,12}$'s invariants using the identity

$$J_{3,12}K_{3,12} = 2I_{31}K_{1,23} - 2I_{32}K_{2,31}\,. \tag{3.39}$$

There are four possible combinations in which the remaining $K_{1,23}$ and $K_{2,31}$ invariants can enter in the correlator: $K_{1,23}I_{23}$, $K_{1,23}I_{21}I_{13}$ and $K_{2,31}I_{13}$, $K_{2,31}I_{12}I_{23}$. These structures are not all independent. In addition to eq.(3.39), using the two identities

$$\begin{aligned} 2I_{12}K_{2,31} &= J_{1,23}K_{1,23} + 2I_{13}K_{3,12}\,, \\ 2I_{21}K_{1,23} &= -J_{2,31}K_{2,31} + 2I_{23}K_{3,12}\,, \end{aligned} \tag{3.40}$$

this case is quite trivial. The exchanged scalar is free and hence the CFT is the direct sum of at least two CFTs, the interacting one and the free theory associated to this scalar. So, either the two external ℓ_1 and ℓ_2 tensors are part of the free CFT, in which case the whole correlator is determined, or the OPE coefficients entering the correlation function must vanish.

we can remove half of them and keep only, say, $K_{1,23}I_{23}$ and $K_{2,31}I_{13}$. The most general tensor structure can be written as

$$\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1 & \ell_2 & \ell_3 \\ m_{23} & m_{13} & m_{12} \end{bmatrix}_p \equiv \Big(\frac{K_{1,23}I_{23}}{X_{23}}\Big)^{\delta-p}\Big(\frac{K_{2,31}I_{13}}{X_{13}}\Big)^{p} \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1-p & \ell_2-\delta+p & \ell_3 \\ m_{23} & m_{13} & \widetilde{m}_{12} \end{bmatrix}, \quad p=0,\dots,\delta\,, \tag{3.41}$$

expressed in terms of the parity even structures (3.28) of traceless symmetric correlators, where

$$\begin{aligned} &j_1=\ell_1-p-\widetilde{m}_{12}-m_{13}\geq 0\,,\\ &j_2=\ell_2-\delta+p-\widetilde{m}_{12}-m_{23}\geq 0\,,\\ &j_3=\ell_3-m_{13}-m_{23}\geq 0 \end{aligned} \qquad \widetilde{m}_{12}=\begin{cases} m_{12} & if \quad p=0 \ or \ p=\delta \\ 0 & otherwise \end{cases}\,. \tag{3.42}$$

The condition in m_{12} derives from the fact that, using eqs.(3.40), one can set m_{12} to zero in the tensor structures with $p\neq 0,\delta$, see below. Attention should be paid to the subscript p. Structures with no subscript refer to purely traceless symmetric correlators, while those with the subscript p refer to three-point functions with two traceless symmetric and one antisymmetric field. All tensor structures are classified in terms of $\delta+1$ classes, parametrized by the index p in eq.(3.41). The parity odd structures of traceless symmetric correlators do not enter, since they can be reduced in the form (3.41) by means of the identities (3.40). The class p exists only when $\ell_1\geq p$ and $\ell_2\geq\delta-p$. If $\ell_1+\ell_2<\delta$, the entire correlator vanishes.

Contrary to the symmetric traceless exchange, there is no obvious choice of seed that stands out. The allowed correlator with the lowest possible spins in each class, $\ell_1=p$, $\ell_2=\delta-p$, $m_{ij}=0$, can all be seen as possible seeds with a unique tensor structure. Let us see how all the structures (3.41) can be iteratively constructed using the operators defined in section 3.2 in terms of the $\delta+1$ seeds. It is convenient to first construct a redundant basis where $m_{12}\neq 0$ for any p and then impose the relation that leads to the independent basis (3.41). The procedure is similar to that followed for the traceless symmetric exchange. We first construct all the tensor structures with $m_{13}=m_{32}=0$ for any spin ℓ_1 and ℓ_2, and any class p, using the following relations:

$$\begin{aligned} &D_1\begin{bmatrix} \Delta_1 & \Delta_2+1 & \Delta_3 \\ \ell_1-1 & \ell_2 & \ell_3 \\ 0 & 0 & m_{12} \end{bmatrix}_p + \widetilde{D}_1\begin{bmatrix} \Delta_1+1 & \Delta_2 & \Delta_3 \\ \ell_1-1 & \ell_2 & \ell_3 \\ 0 & 0 & m_{12} \end{bmatrix}_p = (\delta-p)\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & m_{12} \end{bmatrix}_{p+1} \\ &-8(\ell_2-\delta+p-m_{12})\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & m_{12}+1 \end{bmatrix}_p + (2m_{12}-\ell_1-\ell_2-\Delta_3+2+\delta-p)\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & m_{12} \end{bmatrix}_p, \end{aligned} \tag{3.43}$$

together with the relation

$$\begin{bmatrix} \Delta_1-1 & \Delta_2-1 & \Delta_3 \\ \ell_1+1 & \ell_2+1 & \ell_3 \\ 0 & 0 & m_{12}+1 \end{bmatrix}_p = (I_{12}I_{21}) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & m_{12} \end{bmatrix}_p . \tag{3.44}$$

Notice that the operators $D_{1,2}$ and $\widetilde{D}_{1,2}$ relate nearest neighbour classes and the iteration eventually involves all classes at the same time. The action of the D_2 and $\widetilde{D}_2$ derivatives can be obtained by replacing $1 \leftrightarrow 2$, $p \leftrightarrow (\delta - p)$ in the coefficients multiplying the structures and $p+1 \to p-1$ in the subscripts, and by changing sign on one side of the equation. Structures with non-vanishing m_{13} and m_{23} are obtained using

$$\begin{aligned} 4(\ell_3 - m_{13} - m_{23} + \delta - p) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1 & \ell_2 & \ell_3 \\ m_{23} & m_{13}+1 & m_{12} \end{bmatrix}_p - 4(\delta - p) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1 & \ell_2 & \ell_3 \\ m_{23}+1 & m_{13} & m_{12} \end{bmatrix}_{p+1} = \\ 4(\ell_2 - \delta + p - m_{23} - m_{12}) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1 & \ell_2 & \ell_3 \\ m_{23} & m_{13} & m_{12}+1 \end{bmatrix}_p + D_1 \begin{bmatrix} \Delta_1 & \Delta_2+1 & \Delta_3 \\ \ell_1-1 & \ell_2 & \ell_3 \\ m_{23} & m_{13} & m_{12} \end{bmatrix}_p \\ -\frac{1}{2}(2m_{12} - 2m_{13} + \Delta_2 - \Delta_1 - \Delta_3 - \ell_1 - \ell_2 + \ell_3 + 2\delta - 2p + 2) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1 & \ell_2 & \ell_3 \\ m_{23} & m_{13} & m_{12} \end{bmatrix}_p \end{aligned} \tag{3.45}$$

together with the corresponding relation with $1 \leftrightarrow 2$ and $p \to p+1$. All the structures (3.41) are hence derivable from $\delta+1$ seeds by acting with the operators $D_{1,2}$, $\widetilde{D}_{1,2}$ and $(I_{12}I_{21})$. The seeds, on the other hand, are all related by means of the following relation:

$$(\delta-p)^2 \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ p+1 & \delta-p-1 & \ell_3 \\ 0 & 0 & 0 \end{bmatrix}_{p+1} = R \begin{bmatrix} \Delta_1+1 & \Delta_2+1 & \Delta_3 \\ p & \delta-p & \ell_3 \\ 0 & 0 & 0 \end{bmatrix}_p , \tag{3.46}$$

where

$$R \equiv -\frac{1}{2}\bar{d}_2 d_2 . \tag{3.47}$$

We conclude that, starting from the single seed correlator with $p=0$,

$$\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ 0 & \delta & \ell_3 \\ 0 & 0 & 0 \end{bmatrix}_0 \equiv \left(\frac{K_{1,23}I_{23}}{X_{23}}\right)^{\delta} \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ 0 & 0 & \ell_3 \\ 0 & 0 & 0 \end{bmatrix} , \tag{3.48}$$

namely the three-point function of a scalar, a spin δ traceless symmetric operator and the antisymmetric operator with spin $(\ell_3+2\delta, \ell_3)$, we can obtain all tensor structures of higher spin correlators.

Let us now see how the constraint on m_{12} in eq.(3.42) arises. When $p \neq 0, \delta$, namely when both K_1 and K_2 structures appear at the same time, combining eqs.(3.40), the

following relation is shown to hold:

$$\begin{aligned}\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ \ell_1 & \ell_2 & \ell_3\\ m_{23} & m_{13} & m_{12}+1\end{bmatrix}_p &= -\frac{1}{4}\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ \ell_1 & \ell_2 & \ell_3\\ m_{23} & m_{13} & m_{12}\end{bmatrix}_p - \begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ \ell_1 & \ell_2 & \ell_3\\ m_{23} & m_{13}+1 & m_{12}\end{bmatrix}_p - \begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ \ell_1 & \ell_2 & \ell_3\\ m_{23}+1 & m_{13} & m_{12}\end{bmatrix}_p\\ &-8\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ \ell_1 & \ell_2 & \ell_3\\ m_{23}+1 & m_{13}+1 & m_{12}\end{bmatrix}_p + \begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ \ell_1 & \ell_2 & \ell_3\\ m_{23} & m_{13}+1 & m_{12}\end{bmatrix}_{p-1} + 4\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ \ell_1 & \ell_2 & \ell_3\\ m_{23} & m_{13}+2 & m_{12}\end{bmatrix}_{p-1}\\ &+\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ \ell_1 & \ell_2 & \ell_3\\ m_{23}+1 & m_{13} & m_{12}\end{bmatrix}_{p+1} + 4\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ \ell_1 & \ell_2 & \ell_3\\ m_{23}+2 & m_{13} & m_{12}\end{bmatrix}_{p+1}.\end{aligned} \tag{3.49}$$

Using it iteratively, we can reduce all structures with $p \neq 0, \delta$ to those with $m_{12} = 0$ and with $p = 0, \delta$, any m_{12}.[4] This proves the validity of eq.(3.41). As a further check, we have verified that the number of tensor structures obtained from eq.(3.41) agrees with those found from eq.(3.38) of ref. [31].

Recursion Relations for general ℓ_3

The tensor structures of correlators with $\ell_3 < \ell_1 + \ell_2 - \delta$ cannot all be reduced in the form (3.41), because we are no longer ensured to have enough $J_{3,12}$ invariants to remove all the $K_{3,12}$'s by means of eq.(3.39). In this case the most general tensor structure reads

$$\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ \ell_1 & \ell_2 & \ell_3\\ m_{23} & m_{13} & m_{12}\end{bmatrix}_{p,q} \equiv \eta\Big(\frac{K_{1,23}I_{23}}{X_{23}}\Big)^{\delta-p}\Big(\frac{K_{2,31}I_{13}}{X_{13}}\Big)^{q}\Big(\frac{K_{3,12}I_{13}I_{23}}{\sqrt{X_{12}X_{13}X_{23}}}\Big)^{p-q}\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ \ell_1-p & \ell_2-\delta+q & \ell_3\\ m_{23} & m_{13} & \widetilde{m}_{12}\end{bmatrix}, \tag{3.50}$$

with $p = 0, \ldots, \delta$, $q = 0, \ldots, \delta$, $p - q \geq 0$ and

$$\begin{aligned} j_1 &= \ell_1 - p - \widetilde{m}_{12} - m_{13} \geq 0\,,\\ j_2 &= \ell_2 - \delta + q - \widetilde{m}_{12} - m_{23} \geq 0\,,\\ j_3 &= \ell_3 - m_{13} - m_{23} \geq 0\,,\end{aligned} \qquad \widetilde{m}_{12} = \begin{cases} m_{12} & if \quad q=0 \ \ or \ \ p=\delta\\ 0 & otherwise\end{cases}$$
$$\eta = \begin{cases} 0 & if \quad j_3 > 0 \ \ and \ \ p \neq q\\ 1 & otherwise\end{cases}. \tag{3.51}$$

The parameter η in eq.(3.51) is necessary because the tensor structures involving $K_{3,12}$ (i.e. those with $p \neq q$) are independent only when $j_3 = 0$, namely when the traceless symmetric structure does not contain any $J_{3,12}$ invariant. All the tensor structures (3.50) can be reached starting from the single seed with $p = 0$, $q = 0$, $\ell_1 = 0$, $\ell_2 = \delta$ and $m_{ij} = 0$. The analysis follows quite closely the one made for $\ell_3 \geq \ell_1 + \ell_2 - \delta$, although it is slightly more involved. As before, it is convenient to first construct a redundant basis where $m_{12} \neq 0$ for any p, q and we neglect the factor η above, and impose only later the relations that leads to the independent basis (3.50). We start from the structures with $p = q$, which are the same as those in eq.(3.41): first construct

[4] One has to recall the range of the parameters (3.42), otherwise it might seem that non-existant structures can be obtained from eq.(3.49).

the structures with $m_{13} = m_{23} = 0$ by applying iteratively the operators $D_{1,2} + \widetilde{D}_{1,2}$, and then apply D_1 and D_2 to get the structures with non-vanishing m_{13} and m_{23}. Structures with $p \neq q$ appear when acting with D_1 and D_2. We have:

$$
\begin{aligned}
D_1 \left[\begin{array}{ccc} \Delta_1 & \Delta_2+1 & \Delta_3 \\ \ell_1-1 & \ell_2 & \ell_3 \\ m_{23} & m_{13} & m_{12} \end{array} \right]_{p,p} &= 2(\delta-p) \left[\begin{array}{ccc} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1 & \ell_2 & \ell_3 \\ m_{23} & m_{13} & m_{12} \end{array} \right]_{p+1,p} \\
-4(\ell_2+p-\delta-m_{12}-m_{23}) \left[\begin{array}{ccc} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1 & \ell_2 & \ell_3 \\ m_{23} & m_{13} & m_{12}+1 \end{array} \right]_{p,p} &+ 4(\ell_3-m_{13}-m_{23}) \left[\begin{array}{ccc} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1 & \ell_2 & \ell_3 \\ m_{23} & m_{13}+1 & m_{12} \end{array} \right]_{p,p} \\
&+\frac{1}{2}\Big(2m_{12}-2m_{13}+\Delta_2-\Delta_1-\Delta_3-\ell_1-\ell_2+\ell_3+2(\delta-p+1)\Big) \left[\begin{array}{ccc} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1 & \ell_2 & \ell_3 \\ m_{23} & m_{13} & m_{12} \end{array} \right]_{p,p} .
\end{aligned}
\tag{3.52}
$$

The action of D_2 is obtained by exchanging $1 \leftrightarrow 2$ and $\delta - p \leftrightarrow q$ in the coefficients multiplying the structures and replacing the subscript $(p+1, p)$ with $(p, p-1)$. For $m_{13} + m_{23} < \ell_3$ the first term in eq.(3.52) is redundant and can be expressed in terms of the known structures with $p = q$. An irreducible structure is produced only when we reach the maximum allowed value $m_{13} + m_{23} = \ell_3$, in which case the third term in eq.(3.52) vanishes and we can use the equation to get the irreducible structures with $p \neq q$. Summarizing, all tensor structures can be obtained starting from a single seed upon the action of the operators $D_{1,2}$, $(D_{1,2} + \widetilde{D}_{1,2})$, $I_{12}I_{21}$ and R.

Differential Basis

A differential basis that is well defined for any value of ℓ_1, ℓ_2, ℓ_3 and δ is

$$
\begin{aligned}
\left\{ \begin{array}{ccc} \Delta_1 & \Delta_2 & \Delta_3 \\ \ell_1 & \ell_2 & \ell_3 \\ m_{23} & m_{13} & m_{12} \end{array} \right\}_{p,q} &= \eta \, (I_{12}I_{21})^{\widetilde{m}_{12}} D_1^{m_{13}+p-q} D_2^{m_{23}} \sum_{n_1=0}^{j_1} \left(\begin{array}{c} j_1 \\ n_1 \end{array} \right) D_1^{n_1} \widetilde{D}_1^{j_1-n_1} \sum_{n_2=0}^{j_2} \left(\begin{array}{c} j_2 \\ n_2 \end{array} \right) \\
&\quad D_2^{n_2} \widetilde{D}_2^{j_2-n_2} R^q \left[\begin{array}{ccc} \Delta_1' & \Delta_2' & \Delta_3 \\ 0 & \delta & \ell_3 \\ 0 & 0 & 0 \end{array} \right]_0 ,
\end{aligned}
\tag{3.53}
$$

where $\Delta_1' = \Delta_1 + \ell_1 + m_{23} - m_{13} + n_2 - n_1 - p + q$, $\Delta_2' = \Delta_2 + l_2 + m_{13} - m_{23} + n_1 - n_2 + 2q - \delta$, and all parameters are defined as in eq.(3.51). The recursion relations found above have shown that the differential basis (3.53) is complete. One can also check that its dimensionality matches the one of the ordinary basis for any ℓ_1, ℓ_2, ℓ_3 and δ. Like in the purely traceless symmetric case, the specific choice of operators made in eq.(3.53) seems to be enough to ensure that the determinant of the transformation matrix is non-vanishing regardless of the choice of Δ_1 and Δ_2. We have explicitly checked this result up to $\ell_1 = \ell_2 = 2$, for any ℓ_3. The transformation matrix is always of maximal rank, except for the case $\ell_3 = 0$ and $\Delta_3 = 2$, which saturates the unitarity bound for $\delta = 1$. Luckily enough, this case is quite trivial, being associated to the

exchange of a free $(2,0)$ self-dual tensor [91] (see footnote 3). The specific ordering of the differential operators is a choice motivated by the form of the recursion relations, as before, and different orderings can be trivially related by using the commutators defined in eq.(3.19).

3.4 Computation of Four-Point Functions

We have shown in section 3.1 how relations between three-point functions lead to relations between CPW. The latter are parametrized by 4-point, rather than 3-point, function tensor structures, so in order to make further progress it is important to classify four-point functions. It should be clear that even when acting on scalar quantities, tensor structures belonging to the class of 4-point functions are generated. For example $\widetilde{D}_1 U = -U J_{1,24}$. We lack a general classification of 4-point functions structures in 4D CFTs at the moment, however we will give an overview of the formalism and the tensor structures that can arise using twistor formalism.

3.4.1 Tensor Structures of Four-Point Functions

In 6D, the index-free uplift of the four-point function (3.1) reads

$$\langle O_1 O_2 O_3 O_4 \rangle = \mathcal{K}_4 \sum_{n=1}^{N_4} g_n(U,V)\, \mathcal{T}^n(S_1, \bar{S}_1, .., S_4, \bar{S}_4), \tag{3.54}$$

where $\mathcal{T}^n$ are the 6D uplifts of the tensor structures appearing in eq.(3.1). The 6D kinematic factor $\mathcal{K}_4$ and the conformally invariant cross ratios (U,V) are obtained from their 4D counterparts by the replacement $x_{ij}^2 \to X_{ij}$ in eqs.(3.2) and (3.3).

The tensor structures $\mathcal{T}^n$ are formed from the three-point invariants (1.61)-(1.64) (where i,j,k now run from 1 to 4) and the following new ones:

$$J_{ij,kl} \equiv N_{kl}\, \bar{S}_i \mathbf{X}_k \overline{\mathbf{X}}_l S_j\,, \tag{3.55}$$

$$K_{i,jkl} \equiv N_{jkl}\, S_i \overline{\mathbf{X}}_j \mathbf{X}_k \overline{\mathbf{X}}_l S_i\,, \tag{3.56}$$

$$\overline{K}_{i,jkl} \equiv N_{jkl}\, \bar{S}_i \mathbf{X}_j \overline{\mathbf{X}}_k \mathbf{X}_l \bar{S}_i\,, \tag{3.57}$$

where $i \neq j \neq k \neq l = 1,2,3,4$; $K_{i,jkl}$ and $\overline{K}_{i,jkl}$ are totally anti-symmetric in the last three indices and the normalization factor is given by

$$N_{jkl} \equiv \frac{1}{\sqrt{X_{jk} X_{kl} X_{lj}}}. \tag{3.58}$$

The invariants $J_{ij,kl}$ satisfy the relations $J_{ij,kl} = -J_{ij,lk} + 2I_{ij}$. Given that, and the 4D parity transformations $K_{i,jkl} \leftrightarrow \overline{K}_{i,jkl}$ and $J_{ij,kl} \leftrightarrow -J_{ji,lk}$, a convenient choice of index ordering in $J_{ij,kl}$ is $(i < j,\, k < l)$ and $(i > j,\, k > l)$. Two other invariants $H \equiv S_1 S_2 S_3 S_4$ and $\bar{H} \equiv \bar{S}_1 \bar{S}_2 \bar{S}_3 \bar{S}_4$ formed by using the epsilon $SU(2,2)$ symbols, are redundant. For instance, one has $X_{12} H = K_{2,14} K_{1,23} - K_{1,24} K_{2,13}$.

Any four-point function can be expressed as a sum of products of the invariants (1.61)-(1.64) and (3.55)-(3.57). However, not every product is independent, due to several relations between them. We report in Appendix A a small subset of them. Having a general classification of 4-point tensor structures is crucial to bootstrap a four-point function with non-zero external spins. When we equate correlators in different channels, we have to identify all the factors in front of the same tensor structure, thus it is important to have a common basis of independent tensor structures. For specific cases the task of finding an independent basis of structures seems possible and in the few cases we will present that are revelevant for the discussion we have indeed found such basis.

II

3.4.2 Relation between "Seed" Conformal Partial Waves

Using the results of the last section, we can compute the CPW associated to the exchange of arbitrary operators with external traceless symmetric fields, in terms of a set of seed CPW, schematically denoted by $W^{(p,q)}_{\mathcal{O}^{\ell+2\delta,l}}(\ell_1, \ell_2, \ell_3, \ell_4)$. We have

$$W^{(p,q)}_{O^{\ell+2\delta,\ell}}(\ell_1, \ell_2, \ell_3, \ell_4) = D^{(p)}_{(12)} D^{(q)}_{(34)} W_{O^{\ell+2\delta,\ell}}(0, \delta, 0, \delta)\,, \tag{3.59}$$

where $D^{(p)}_{12}$ schematically denotes the action of the differential operators reported in the last section, and $D^{(q)}_{34}$ are the same operators for the fields at X_3 and X_4, obtained by replacing $1 \to 3$, $2 \to 4$ everywhere in eqs.(3.15)-(3.18) and (3.24). For simplicity we do not report the dependence of W on U, V, and on the scaling dimensions of the external and exchanged operators. The seed CPW are the simplest among the ones appearing in correlators of traceless symmetric tensors, but they are *not* the simplest in general. These will be the CPW arising from the four-point functions with the *lowest* number of tensor structures with a non-vanishing contribution of the field $O^{\ell+2\delta,\ell}$ in

some of the OPE channels. Such minimal four-point functions are[5]

$$\langle O^{(0,0)}(X_1)O^{(2\delta,0)}(X_2)O^{(0,0)}(X_3)O^{(0,2\delta)}(X_4)\rangle = \mathcal{K}_4 \sum_{n=0}^{2\delta} g_n(U,V) I_{42}^n J_{42,31}^{2\delta-n}\,, \quad (3.60)$$

with just

$$N_4^{seed}(\delta) = 2\delta + 1 \quad (3.61)$$

tensor structures. In the s-channel (12-34) operators $O^{\ell+n,\ell}$, with $-2\delta \leq n \leq 2\delta$, are exchanged. We denote by $W_{seed}(\delta)$ and $\overline{W}_{seed}(\delta)$ the single CPW associated to the exchange of the fields $O^{\ell+2\delta,\ell}$ and $O^{\ell,\ell+2\delta}$ in the four-point function (3.60). They are parametrized in terms of $2\delta+1$ conformal blocks as follows ($\mathcal{G}_0^{(0)} = \overline{\mathcal{G}}_0^{(0)}$):

$$\begin{aligned} W_{seed}(\delta) &= \mathcal{K}_4 \sum_{n=0}^{2\delta} \mathcal{G}_n^{(\delta)}(U,V) I_{42}^n J_{42,31}^{2\delta-n}\,, \\ \overline{W}_{seed}(\delta) &= \mathcal{K}_4 \sum_{n=0}^{2\delta} \overline{\mathcal{G}}_n^{(\delta)}(U,V) I_{42}^n J_{42,31}^{2\delta-n}\,. \end{aligned} \quad (3.62)$$

In contrast, the number of tensor structures in $\langle O^{(0,0)}(X_1)O^{(\delta,\delta)}(X_2)O^{(0,0)}(X_3)O^{(\delta,\delta)}(X_4)\rangle$ grows rapidly with δ. Denoting it by $\widetilde{N}_4(\delta)$ we have, using eq.(6.6) of ref. [31]:

$$\widetilde{N}_4(\delta) = \frac{1}{3}\Big(2\delta^3 + 6\delta^2 + 7\delta + 3\Big)\,. \quad (3.63)$$

It is important to stress that a significant simplification occurs in using seed CPW even when there is no need to reduce their number, i.e. $p = q = 1$. For instance, consider the correlator of four traceless symmetric spin 2 tensors. The CPW $W_{O^{\ell+8,\ell}}(2,2,2,2)$ is unique, yet it contains 1107 conformal blocks (one for each tensor structure allowed in this correlator), to be contrasted to the 85 present in $W_{O^{\ell+8,\ell}}(0,4,0,4)$ and the 9 in $W_{seed}(4)$! We need to relate $\langle O^{(0,0)}(X_1)O^{(2\delta,0)}(X_2)O^{(\ell+2\delta,\ell)}(X_3)\rangle$ and $\langle O^{(0,0)}(X_1)O^{(\delta,\delta)}(X_2)O^{(\ell+2\delta,\ell)}(X_3)\rangle$ in order to be able to use the results of section 3.3 together with $W_{seed}(\delta)$. As explained at the end of Section 3.2, there is no combination of first-order operators which can do this job and one is forced to use the operator (3.24):

$$\langle O_{\Delta_1}^{(0,0)}(X_1)O_{\Delta_2}^{(\delta,\delta)}(X_2)O_{\Delta}^{(\ell,\,\ell+2\delta)}(X)\rangle_1 = \Big(\prod_{n=1}^{\delta} c_n\Big)(\bar{d}_1\nabla_{12}\widetilde{D}_1)^\delta \langle O_{\Delta_1+\delta}^{(0,0)}(X_1)O_{\Delta_2}^{(2\delta,0)}(X_2)O_{\Delta}^{(\ell,\,\ell+2\delta)}(X)\rangle_1\,, \quad (3.64)$$

[5] Instead of eq.(3.60) one could also use 4-point functions with two scalars and two $O^{(0,2\delta)}$ fields or two scalars and two $O^{(2\delta,0)}$ fields. Both have the same number $2\delta+1$ of tensor structures as the correlator (3.60).

where[6]

$$c_n^{-1} = 2(1-n+2\delta)\Big(2(n+1)+\delta+\ell+\Delta_1-\Delta_2+\Delta\Big)\,. \tag{3.65}$$

Equation (3.64) implies the following relation between the two CPW:

$$W_{O^{\ell+2\delta,\ell}}(0,\delta,0,\delta) = \Big(\prod_{n=1}^{\delta} c_n^{12}c_n^{34}\Big)\Big(\nabla_{43}d_3\widetilde{D}_3\Big)^{\delta}\Big(\nabla_{12}\bar{d}_1\widetilde{D}_1\Big)^{\delta} W_{seed}(\delta)\,, \tag{3.66}$$

where $c_n^{12} = c_n$ in eq.(3.65), c_n^{34} is obtained from c_n by exchanging $1 \to 3, 2 \to 4$ and the scaling dimensions of the corresponding external operators are related as indicated in eq.(3.64).

Summarizing, the whole highly non-trivial problem of computing $W^{(p,q)}_{O^{\ell+2\delta,\ell}}(\ell_1,\ell_2,\ell_3,\ell_4)$ has been reduced to the computation of the $2\times(2\delta+1)$ conformal blocks $\mathcal{G}_n^{(\delta)}(U,V)$ and $\overline{\mathcal{G}}_n^{(\delta)}(U,V)$ entering eq.(3.62). Once they are known, one can use eqs.(3.66) and (3.59) to finally reconstruct $W^{(p,q)}_{O^{\ell+2\delta,\ell}}(\ell_1,\ell_2,\ell_3,\ell_4)$.

As a brief example of the power of this method we want to mention the simplification that occurs when calculating conformal blocks for the correlator of four energy-momentum tensors (for more details see section 6.2 of [36]). The exchanged operators can be in the representations $(\ell+2\delta,\ell)$ and $(\ell,\ell+2\delta)$ where $\delta = 0,1,...,4$. In the most general case of four distinct non conserved operators, no parity imposed, one should compute $1107^2 \sim 10^6$ conformal blocks, that are reduced to 49 using the differential basis, $W_{seed}(\delta)$ and $\overline{W}_{seed}(\delta)$. For four identical spin 2 tensors, namely for four energy momentum tensors, one gets $N_4 = 22_+ + 3_-$ tensor structures allowed in the four point function ([36])). The number of parity even structures agrees with what found in ref. [69], while to the best of our knowledge the 3 parity odd structures found is a new result.

Notice that even if the number of tensor structures is significantly reduced when conservation is imposed, they are still given by a linear combination of all the tensor structures. It might be interesting to see if there exists a formalism that automatically gives a basis of independent tensor structures for conserved operators and the use of the much larger basis of allowed structures.

[6] Notice that the scalings dimension Δ_1 and Δ_2 in eq.(3.65) do not exactly correspond in general to those of the external operators, but should be identified with Δ_1' and Δ_2' in eq.(3.53). It might happen that the coefficient c_n vanishes for some values of Δ_1 and Δ_2. As we already pointed out, there is some redundancy that allows us to choose a different set of operators. Whenever this coefficient vanishes, we can choose a different operator, e.g. $\widetilde{D}_1 \to D_1$.

3.5 Example: Four Fermions Correlator

In this section we would like to elucidate various aspects of our construction. We will give an example in which we deconstruct a correlation function of four fermions. We leave the domain of traceless symmetric external operators to show the generality of our formalism. It might also have some relevance in phenomenological applications beyond the Standard Model [58]. Our goal here is to deconstruct the CPW in the s-channel associated to the four fermion correlator

$$\langle \bar{\psi}^{\dot{\alpha}}(x_1)\psi_{\beta}(x_2)\chi_{\gamma}(x_3)\bar{\chi}^{\dot{\delta}}(x_4)\rangle\,. \tag{3.67}$$

For simplicity, we take $\bar{\psi}$ and $\bar{\chi}$ to be conjugate fields of ψ and χ, respectively, so that we have only two different scaling dimensions, Δ_ψ and Δ_χ. Parity invariance is however not imposed in the underlying CFT. The correlator (3.67) admits six different tensor structures. An independent basis of tensor structures for the 6D uplift of eq. (3.67) can be found using the relation (A.10). A possible choice is

$$\langle \Psi(X_1,\,\bar{S}_1)\,\bar{\Psi}(X_2,\,S_2)\,\bar{\mathcal{X}}(X_3,\,S_3)\,\mathcal{X}(X_4,\,\bar{S}_4)\rangle = \frac{1}{X_{12}^{\Delta_\psi+\frac{1}{2}}X_{34}^{\Delta_\chi+\frac{1}{2}}}\Big(g_1(U,V)I_{12}I_{43}+ \tag{3.68}$$

$$g_2(U,V)I_{42}I_{13}+g_3(U,V)I_{12}J_{43,21}+g_4(U,V)I_{42}J_{13,24}+g_5(U,V)I_{43}J_{12,34}+g_6(U,V)I_{13}J_{42,31}\Big).$$

For $l\geq 1$, four CPW $W^{(p,q)}_{O^{\ell,\ell}}$ $(p,q=1,2)$ are associated to the exchange of traceless symmetric fields, and one for each antisymmetric field, $W_{O^{\ell+2,\ell}}$ and $W_{O^{\ell,\ell+2}}$. Let us start with $W^{(p,q)}_{O^{\ell,\ell}}$. The traceless symmetric CPW are obtained as usual by relating the three point function of two fermions and one $O^{\ell,\ell}$ to that of two scalars and one $O^{\ell,\ell}$. This relation requires to use the operator (3.24). There are two tensor structures for $l\geq 1$:

$$\begin{aligned}
\langle \Psi(\bar{S}_1)\bar{\Psi}(S_2)O^{\ell,\ell}\rangle_1 &= \mathcal{K}I_{12}J^{\ell}_{0,12}=I_{12}\langle \Phi^{\frac{1}{2}}\Phi^{\frac{1}{2}}O^{l,l}\rangle_1, \\
\langle \Psi(\bar{S}_1)\bar{\Psi}(S_2)O^{\ell,\ell}\rangle_2 &= \mathcal{K}I_{10}I_{02}J^{\ell-1}_{0,12}=\frac{1}{16l(\Delta-1)}\nabla_{21}\Big(\widetilde{D}_2\widetilde{D}_1+\kappa I_{12}\Big)\langle \Phi^{\frac{1}{2}}\Phi^{\frac{1}{2}}O^{\ell,\ell}\rangle_1,
\end{aligned} \tag{3.69}$$

where $\kappa=2\Big(4\Delta-(\Delta+\ell)^2\Big)$, the superscript n in Φ indicates the shift in the scaling dimensions of the field and the operator $O^{\ell,\ell}$ is taken at X_0. Plugging eq.(3.69) (and the analogous one for $\mathcal{X}$ and $\bar{\mathcal{X}}$) in eq.(4.27) gives the relation between CPW. In order to simplify the equations, we report below the CPW in the differential basis, the relation

with the ordinary basis being easily determined from eq.(3.69):

$$\begin{aligned}
W^{(1,1)}_{O^{\ell,\ell}} =& I_{12}I_{43}W^{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}_{seed}(0)\,,\\
W^{(1,2)}_{O^{\ell,\ell}} =& I_{12}\nabla_{34}\widetilde{D}_4\widetilde{D}_3W^{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}_{seed}(0)\,,\\
W^{(2,1)}_{O^{\ell,\ell}} =& I_{43}\nabla_{21}\widetilde{D}_2\widetilde{D}_1W^{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}_{seed}(0)\,,\\
W^{(2,2)}_{O^{\ell,\ell}} =& \nabla_{21}\widetilde{D}_2\widetilde{D}_1\nabla_{34}\widetilde{D}_4\widetilde{D}_3W^{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}_{seed}(0)\,,
\end{aligned} \tag{3.70}$$

where $\widetilde{D}_3$ and $\widetilde{D}_4$ are obtained from $\widetilde{D}_1$ and $\widetilde{D}_2$ in eq.(3.15) by replacing $1\to 3$ and $2\to 4$ respectively. The superscripts indicate again the shift in the scaling dimensions of the external operators. As in ref. [19] the CPW associated to the exchange of traceless symmetric fields is entirely determined in terms of the single known CPW of four scalars $W_{seed}(0)$. For illustrative purposes, we report here the explicit expressions of $W^{(1,2)}_{O^{l,l}}$:

$$\begin{aligned}
\mathcal{K}_4^{-1}W^{(1,2)}_{O^{\ell,\ell}} = 8I_{12}I_{43}\Big(&U\Big(V-U-2\Big)\partial_U+U^2\Big(V-U\Big)\partial_U^2+\Big(V^2-(2+U)V+1\Big)\partial_V+\\
&V\Big(V^2-(2+U)V+1\Big)\partial_V^2+2UV\Big(V-U-1\Big)\partial_U\partial_V\Big)\mathcal{G}_0^{(0)}\\
+4UI_{12}J_{43,21}\Big(&U\partial_U+U^2\partial_U^2+\Big(V-1\Big)\partial_V+V\Big(V-1\Big)\partial_V^2+2UV\partial_U\partial_V\Big)\mathcal{G}_0^{(0)},
\end{aligned} \tag{3.71}$$

where $\mathcal{G}_0^{(0)}$ are the known scalar conformal blocks [13, 14]. It is worth noting that the relations (A.3)-(A.10) have to be used to remove redundant structures and write the above result (3.71) in the chosen basis (3.68).

The analysis for the antisymmetric CPW $W_{O^{\ell+2,\ell}}$ and $W_{O^{\ell,\ell+2}}$ is simpler. The three point function of two fermions and one $O^{\ell,\ell+2}$ field has a unique tensor structure, like the one of a scalar and a $(2,0)$ field F. One has

$$\begin{aligned}
\langle\Psi(\bar{S}_1)\bar{\Psi}(S_2)O^{\ell+2,\ell}\rangle_1 &= \mathcal{K}I_{10}K_{1,20}J^{\ell}_{0,12} = \frac{1}{4}\bar{d}_2\langle\Phi^{\frac{1}{2}}F^{\frac{1}{2}}O^{\ell+2,\ell}\rangle_1\,,\\
\langle\Psi(\bar{S}_1)\bar{\Psi}(S_2)O^{\ell,\ell+2}\rangle_1 &= \mathcal{K}I_{02}\overline{K}_{2,10}J^{\ell}_{0,12} = \frac{1}{2}\bar{d}_2\langle\Phi^{\frac{1}{2}}F^{\frac{1}{2}}O^{\ell,\ell+2}\rangle_1
\end{aligned} \tag{3.72}$$

and similarly for the conjugate $(0,2)$ field $\bar{F}$. Using the above relation, modulo an

irrelevant constant factor, we get

$$\begin{aligned} W_{O^{\ell+2,\ell}} &= \bar{d}_2 d_4 W_{seed}^{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}(1)\,, \\ W_{O^{\ell,\ell+2}} &= \bar{d}_2 d_4 \overline{W}_{seed}^{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}(1)\,, \end{aligned} \tag{3.73}$$

where $W_{seed}(1)$ and $\overline{W}_{seed}(1)$ are defined in eq.(3.62). Explicitly, one gets

$$\begin{aligned} \frac{\sqrt{U}}{4}\mathcal{K}_4^{-1} W_{O^{\ell+2,\ell}} =& I_{12}I_{43}\Big(\mathcal{G}_2^{(1)} + (V-U-1)\mathcal{G}_1^{(1)} + 4U\mathcal{G}_0^{(1)}\Big) - 4UI_{42}I_{13}\mathcal{G}_1^{(1)} + UI_{12}J_{43,21}\mathcal{G}_1^{(1)} \\ & - UI_{42}J_{13,24}\mathcal{G}_2^{(1)} + UI_{43}J_{12,34}\mathcal{G}_1^{(1)} - 4UI_{13}J_{42,31}\mathcal{G}_0^{(1)}\,. \end{aligned} \tag{3.74}$$

The same applies for $W_{O^{\ell,\ell+2}}$ with $\mathcal{G}_n^{(1)} \to \overline{\mathcal{G}}_n^{(1)}$. The expression (3.74) shows clearly how the six conformal blocks entering $W_{O^{\ell,\ell+2}}$ are completely determined in terms of the three $\mathcal{G}_n^{(1)}$.

Chapter 4

4 Seed Conformal Blocks

We have come a long way in the computation of general Conformal Partial Waves in 4D. The procedure introduced in the previous chapter has shown how to relate, by means of differential operators, mixed tensor CBs appearing in an arbitrary spinor/tensor 4-point correlator to a basis of minimal mixed tensor CBs. These "seed" blocks arise from 4-point functions involving two scalars and two tensor fields in the $(0,p)$ and $(p,0)$ representations of the Lorentz group, with p an arbitrary integer ($(1,0)$ is a fermion). Such 4-point functions are the simplest ones (i.e. with the least number of tensor structures) where $(\ell+p,\ell)$ or $(\ell,\ell+p)$ mixed symmetry (bosonic or fermionic) tensors can be exchanged in some OPE limit, for any ℓ.

We are left with the task of calculating these "seed" CBs. Fortunately, we will show, the use of the methods presented in the introduction, namely the shadow formalism and the Casimir equation, will allow us to find closed expressions for the blocks.

We will start in section 4.1 with a summary of the last chapter, trivially generalizing to the case when p is not only even, but odd as well. In section 4.2 we will write the explicit form of the Casimir equation for the seed CPW. We will see that the Casimir equation becomes a system of $p+1$ coupled differential equations for the $p+1$ CBs $G_e^{(p)}$. We will use the shadow formalism in section 4.3 to obtain some knowledge on the analytic structure of the blocks, and in some cases obtain complete solutions. With the insight gained from this exercise we will proceed in section 4.4 to obtain a final solution to the system, using generalizations of the methods introduced in ref. [14] (and further refined in ref. [15]) to compute 6D symmetric CBs for scalar correlators. We write a proper ansatz for the solution that allows us to reduce a very complicated system of $p+1$ quadratic coupled differential equations in two variables to a simple algebraic linear system for some unknown coefficients $c^e_{m,n}$. These coefficients are then determined by solving the linear system. The set of non-trivial coefficients $c^e_{m,n}$ admits a very useful geometric interpretation. They span a two-dimensional lattice in the (m,n) plane. For each CB labelled by e, the shape of the lattice is an octagon, with p and e dependent edges. For large p, the total number of coefficients $c^e_{m,n}$ grows like p^3 and their explicit form becomes more and more complicated as p increases. We point out that a similar geometric interpretation applies also to the set of non-trivial

coefficients $x_{m,n}$ entering the solution for the symmetric scalar blocks in even number of dimensions.

4.1 Deconstructing Conformal Partial Waves

We have shown in the previous chapter that CPWs associated to an operator $\mathcal{O}^{(\ell,\ell+p)}$ (and similarly for its conjugate $\overline{\mathcal{O}}^{(\ell+p,\ell)}$) exchanged in the OPE channel $(12)(34)$ of a 4-point function $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle$, can be obtained from a single CPW $W^{seed}_{\mathcal{O}^{(\ell,\ell+p)}}$ as follows:

$$W^{(i,j)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4,\mathcal{O}^{(\ell,\ell+p)}} = \mathcal{D}^i_{12}\mathcal{D}^j_{34}W^{seed}_{\mathcal{O}^{(\ell,\ell+p)}}\,, \tag{4.1}$$

where $\mathcal{D}^i_{12}$ and $\mathcal{D}^i_{34}$ are differential operators that depend on $\mathcal{O}_{1,2}$ and $\mathcal{O}_{3,4}$, respectively. For even integer $p=2n$, the seed CPWs are those associated to 4-point functions of two scalar fields with one $(2n,0)$ and one $(0,2n)$ mixed symmetry tensor, while for odd integer $p=2n+1$, they consist of 4-point functions of two scalar fields with one $(2n+1,0)$ and one $(0,2n+1)$ mixed symmetry fermion-tensor:[1]

$$\langle\phi_1(x_1)F_{2,\alpha_1\alpha_2\ldots\alpha_{2n}}(x_2)\phi_3(x_3)\overline{F}_4^{\dot\beta_1\dot\beta_2\ldots\dot\beta_{2n}}(x_4)\rangle\,, \qquad p=2n\,, \tag{4.2}$$

$$\langle\phi_1(x_1)\psi_{2,\alpha_1\alpha_2\ldots\alpha_{2n+1}}(x_2)\phi_3(x_3)\overline{\psi}_4^{\dot\beta_1\dot\beta_2\ldots\dot\beta_{2n+1}}(x_4)\rangle\,, \quad p=2n+1\,. \tag{4.3}$$

In the above correlators, in the OPE channel $\langle(12)(34)\rangle$ primary operators $\mathcal{O}^{(\ell,\ell+\delta)}$ and their conjugates $\overline{\mathcal{O}}^{(\ell+\delta,\ell)}$ can be exchanged only with the values of δ indicated in eq. (3.6) and any ℓ. There are several 4-point functions in which the operators $\mathcal{O}^{(\ell,\ell+p)}$ and $\overline{\mathcal{O}}^{(\ell+p,\ell)}$ are exchanged and in which the corresponding CPWs have a unique structure. Among these, the correlators (4.2) and (4.3) are the ones with the minimum number of tensor structures and hence the simplest. This is understood by noticing that for *any* value of δ (and not only for $\delta=p$) the operators $\mathcal{O}^{(\ell,\ell+\delta)}$ and their conjugates $\overline{\mathcal{O}}^{(\ell+\delta,\ell)}$ appear in both the OPE's with one tensor structure only, since there is only one tensor structure in the corresponding three-point functions:

$$\langle\phi(x_1)F_{\alpha_1\ldots\alpha_{2n}}(x_2)\mathcal{O}^{\dot\beta_1\ldots\dot\beta_{\ell+\delta}}_{\alpha_1\ldots\alpha_\ell}(x_0)\rangle\,, \qquad \langle\overline{\mathcal{O}}^{\dot\beta_1\ldots\dot\beta_\ell}_{\alpha_1\ldots\alpha_{\ell+\delta}}(x_0)\phi(x_3)\overline{F}^{\dot\beta_1\ldots\dot\beta_{2n}}(x_4)\rangle\,, \tag{4.4}$$

$$\langle\phi(x_1)\psi_{\alpha_1\ldots\alpha_{2n+1}}(x_2)\mathcal{O}^{\dot\beta_1\ldots\dot\beta_{\ell+\delta}}_{\alpha_1\ldots\alpha_\ell}(x_0)\rangle\,, \quad \langle\overline{\mathcal{O}}^{\dot\beta_1\ldots\dot\beta_\ell}_{\alpha_1\ldots\alpha_{\ell+\delta}}(x_0)\phi(x_3)\overline{\psi}^{\dot\beta_1\ldots\dot\beta_{2n+1}}(x_4)\rangle \tag{4.5}$$

This implies then that the number of 4-point tensor structures appearing in eqs.(4.2) and (4.3) is the minimum possible and equals to $N_4=p+1$.

Summarizing, the problem of computing CPWs and CBs associated to the exchange

[1] The previous chapter focused strictly on the even p case, but it is obvious that the same result applies to odd p.

of mixed symmetry operators $\mathcal{O}^{(\ell,\ell+p)}$ and $\overline{\mathcal{O}}^{(\ell+p,\ell)}$ in any 4-point function is reduced to the computation of the $p+1$ CBs appearing in the decomposition of $W^{seed}_{\mathcal{O}^{(\ell,\ell+p)}}$ and $\overline{W}^{seed}_{\mathcal{O}^{(\ell+p,\ell)}}$.

Despite this simplification, the above computation is still technically challenging. We still have to deal with a complicate calculation, given we have to deal with several tensor strucutres. A possible independent basis for the $p+1$ tensor structures appearing in our seed correlators (4.2) and (4.3) can be obtained from the above 6D $SU(2,2)$ invariant scalar quantitites:

$$\langle \Phi_1(X_1) F_2^{(p,0)}(X_2,S_2) \Phi_3(X_3) \overline{F}_4^{(0,p)}(X_4,\overline{S}_4)\rangle = \mathcal{K}_4 \sum_{n=0}^{p} g_n(U,V) I_{42}^n J_{42,31}^{p-n}\,, \tag{4.6}$$

where $\mathcal{K}_4$, U and V are the 6D analogues of eqs.(3.2)-(3.3), obtained by replacing $x_{ij}^2 \to X_{ij}$. We denote the 6D seed CPW associated to the exchange of the fields $O^{(\ell,\ell+p)}$ and $\overline{O}^{(\ell+p,\ell)}$ in the 4-point function (4.6) by $W^{seed}(p)$ and $\overline{W}^{seed}(p)$, respectively. They are parametrized in terms of $p+1$ CBs as follows:

$$\begin{aligned} W^{seed}(p) &= \mathcal{K}_4 \sum_{e=0}^{p} G_e^{(p)}(U,V) I_{42}^e J_{42,31}^{p-e}, \\ \overline{W}^{seed}(p) &= \mathcal{K}_4 \sum_{e=0}^{p} \overline{G}_e^{(p)}(U,V) I_{42}^e J_{42,31}^{p-e}. \end{aligned} \tag{4.7}$$

For simplicity, we have dropped in eq.(4.7) the dependence of $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ on Δ and ℓ. The CBs depend also on the external operator dimensions, more precisely on a and b, defined as

$$a \equiv \frac{\tau_2-\tau_1}{2} = \frac{\Delta_2-\Delta_1}{2} + \frac{p}{4}\,, \qquad b \equiv \frac{\tau_3-\tau_4}{2} = \frac{\Delta_3-\Delta_4}{2} - \frac{p}{4}\,. \tag{4.8}$$

For simplicity of notation, we no longer distinguish between even and odd values of p, since we can consider both cases simultaneously. It is then understood that in the corrrelator (4.6) $F_2^{(p,0)}$ and $\overline{F}_4^{(0,p)}$ are 6D uplifts of 4D fermion fields for p odd.

It is possible to get $W^{seed}(p)$ from $\overline{W}^{seed}(p)$, or vice versa, by using differential operators introduced in chapter 3 and a parity transformation $\mathcal{P}$. We have

$$\overline{W}^{seed}(p) = \mathcal{P}\; W_{\Phi_1 \overline{F}_2 \Phi_3 F_4, O^{(\ell,\ell+p)}}\,, \tag{4.9}$$

where

$$W_{\Phi_1 \overline{F}_2 \Phi_3 F_4, O^{(\ell,\ell+p)}} = \frac{1}{2^{2p}\,(p!)^2} \Big(\prod_{n=1}^{p} c_n \Big)\, (\nabla_{12} \bar{d}_1 \widetilde{D}_1)^p (\nabla_{43} d_3 \widetilde{D}_3)^p W^{seed}(p) \Big|_{a\to a-\frac{p}{2},\, b\to b+\frac{p}{2}} \tag{4.10}$$

is the CPW associated to the parity dual 4-point function $\langle \Phi_1 \overline{F}_2^{(0,p)} \Phi_3 F_4^{(p,0)} \rangle$, and

$$(c_n)^{-1} = (4+3p-2a-\tau-2n)(4+3p+2b-\tau-2n)\,, \quad \tau = \Delta+\ell+\frac{p}{2}\,. \tag{4.11}$$

However, we will not use eq.(4.9) to compute $\overline{W}^{seed}(p)$, because we will find an easier way to directly compute both $W^{seed}(p)$ and $\overline{W}^{seed}(p)$.

Instead of eq.(4.6), we could have considered the alternative 4-point function

$$\langle \Phi_1(X_1) F_2^{(p,0)}(X_2) \overline{F}_3^{(0,p)}(X_3) \Phi_4(X_4) \rangle \tag{4.12}$$

to calculate an analogue seed CPW $\widetilde{W}^{seed}(p)$. Since eq.(4.12) is equal to eq.(4.6) under the permutation $3 \leftrightarrow 4$, the CBs appearing in the decomposition of $W^{seed}(p)$ and $\widetilde{W}^{seed}(p)$ are related as follows:

$$\widetilde{G}_e^{(p)}(U,V;a,b) = V^a G_e^{(p)}\Big(\frac{U}{V},\frac{1}{V};a,-b\Big)\,, \quad e=0,\dots,p\,. \tag{4.13}$$

The 4D CBs are directly identified with their 6D counterparts. One has simply

$$G_e^{(p)}(U,V) = G_e^{(p)}(u,v)\,, \quad \overline{G}_e^{(p)}(U,V) = \overline{G}_e^{(p)}(u,v) \tag{4.14}$$

where $G_e^{(p)}(u,v)$ and $\overline{G}_e^{(p)}(u,v)$ are the 4D CBs entering the r.h.s. of eq.(3.8) when expanding the 4D CPWs $W^{seed}_{\mathcal{O}^{(\ell,\ell+p)}}$ and $\overline{W}^{seed}_{\mathcal{O}^{(\ell+p,\ell)}}$.

4.2 The System of Casimir Equations

In this section we derive the system of second order Casimir equations for the seed conformal blocks defined in eq. (4.7). We will follow the procedure outlined in chapter 2 with the obvious generalizations. First of all the generalization of the casimir eigenvalue:

$$[\hat{C}, \mathcal{O}^{(\ell,\ell)}(x)] = E_\ell^0 \mathcal{O}^{(\ell,\ell)}(x) \tag{4.15}$$

Now we are considering general representations of the Lorentz group (no longer only traceless symmetric operators $\mathcal{O}^{(\ell,\ell)}$), which we are parametrizing by ℓ and p. In this case the eigenvalue takes the following form.

$$E_\ell^p = \Delta\,(\Delta-4) + \ell^2 + (2+p)(\ell+\frac{p}{2}) \tag{4.16}$$

This is the Casimir eigenvalue associated to an operator in the $(\ell+p,\ell)$ or $(\ell,\ell+p)$

Lorentz representations. Using then eq.(3.5) one derives a differential equation for each CPW, for any fixed Δ and ℓ.

The derivation exposed in chapter 2 can be generalized for CPWs entering 4-point correlators of tensor fields. As we have seen in section 4.1, in the most general case the exchange of a given field $\mathcal{O}^{(\ell,\bar{\ell})}$ is not parametrized by a single CPW, but by a set of CPWs $W^{(i,j)}$, whose number depends on the number of tensor structures defining the three-point functions $(12\mathcal{O})$ and $(34\overline{\mathcal{O}})$. In order to derive the second order differential equation satisfied by $W^{(i,j)}$ one has to properly identify the OPE coefficients λ^i appearing in the generalization of the OPE with those in eq.(3.5). This is not needed for the seed correlators (4.6) since the CPW is unique, like in the scalar correlator. For each p, we have

$$C\, W^{seed}(p) = E_\ell^p\, W^{seed}(p), \tag{4.17}$$

where C is the explicit differential form of the Casimir operator to be determined and E_ℓ^p is as in eq.(4.16). An identical equation is satisfied by $\overline{W}^{seed}(p)$. Contrary to the scalar case, even in this simple set-up, the single differential equation (4.17) for $W^{seed}(p)$ turns into a system of equations for the $p+1$ CBs $G_e^{(p)}$. Let us see how this system of equations can be derived for any p.

The action of the Lorentz generators $L_{i,MN}$ on tensor fields should include, in addition to the orbital contribution (2.18), the spin part. Recall that $SO(2,4) \simeq SU(2,2)$ and at the level of representations $\mathbf{8_{spin}} \simeq \mathbf{4} + \overline{\mathbf{4}}$, where $\mathbf{4}$ and $\overline{\mathbf{4}}$ represent twistor indices. Denoting by $[\Sigma_{MN}]_a{}^b$ and $[\overline{\Sigma}_{MN}]^a{}_b$ the generators of $SU(2,2)$ fundamental/anti-fundamental (twistor) representations (see Appendix A of ref. [31] for details and our conventions), one can label the 6D spin representations by two integers $(s,\,\bar{s})$ which count the number of twistor indices in the $\mathbf{4}$ and $\overline{\mathbf{4}}$ representations respectively. The Lorentz generators acting on generic 6D fields in the $(s,\,\bar{s})$ representation are then given by

$$\begin{aligned}
[L_{i\,MN}]^{b_1..\,b_s;\,c_1..\,c_{\bar{s}}}_{a_1..\,a_{\bar{s}};\,d_1..\,d_s} &= i(X_{i\,M}\partial_{i\,N} - X_{i\,N}\partial_{i\,M})(\delta^{c_1}_{a_1}..\,\delta^{c_{\bar{s}}}_{a_{\bar{s}}})(\delta^{b_1}_{d_1}..\,\delta^{b_s}_{d_s}) \\
&+ i\Big([\Sigma_{MN}]^{c_1}_{a_1}\delta^{c_2}_{a_2}..\delta^{c_{\bar{s}}}_{a_{\bar{s}}} + [\Sigma_{MN}]^{c_2}_{a_2}\delta^{c_1}_{a_1}..\delta^{c_{\bar{s}}}_{a_{\bar{s}}} + ..\Big)\delta^{b_1}_{d_1}..\delta^{b_s}_{d_s} \\
&+ i\Big([\overline{\Sigma}_{MN}]^{b_1}_{d_1}\delta^{b_2}_{d_2}..\delta^{b_s}_{d_s} + [\overline{\Sigma}_{MN}]^{b_2}_{d_2}\delta^{b_1}_{d_1}..\delta^{b_s}_{d_s} + ..\Big)\delta^{c_1}_{a_1}..\delta^{c_{\bar{s}}}_{a_{\bar{s}}}\,.
\end{aligned} \tag{4.18}$$

We can get rid of all the twistor indices by defining the index-free Lorentz generators

$$L_{iMN} = i(X_{iM}\partial_{iN} - X_{iN}\partial_{iM}) + i(S_i\overline{\Sigma}_{MN}\partial_{S_i}) + i(\bar{S}_i\Sigma_{MN}\partial_{\bar{S}_i}). \tag{4.19}$$

Given any 6D tensor $O(X,S,\bar{S})$, we have

$$[\hat{L}_{MN}, O_i(X_i,S_i,\bar{S}_i)] = L_{iMN} O_i(X_i,S_i,\bar{S}_i)\,, \tag{4.20}$$

where $\hat{L}_{MN}$ satisfy the Lorentz algebra

$$[\hat{L}_{MN},\hat{L}_{RS}] = i\Big(\eta_{MS}\hat{L}_{NR} + \eta_{NR}\hat{L}_{MS} - \eta_{MR}\hat{L}_{NS} - \eta_{NS}\hat{L}_{MR}\Big). \tag{4.21}$$

The explicit form of the Casimir differential operator entering eq.(4.17) is obtained by plugging eq.(4.19) in eq.(2.12). The single equation (4.17) for the CPW turns into a system of second-order coupled differential equations for the $p+1$ conformal blocks $G_e^{(p)}$, $e=0,\dots,p$, since the coefficients multiplying the $p+1$ tensor structures in eq.(4.7) should vanish independently. Schematically

$$(C-E_\ell^p)\Big(\mathcal{K}_4\sum_{e=0}^{p} G_e^{(p)}(U,V) I_{42}^e J_{42,31}^{p-e}\Big) = \mathcal{K}_4\sum_{e=0}^{p} Cas_e^{(p)}(G) I_{42}^e J_{42,31}^{p-e} = 0 \;\; \Rightarrow Cas_e^{(p)}(G)=0\,, \tag{4.22}$$

where $Cas_e^{(p)}(G)$ are the $p+1$ Casimir equations, in general each one involving all conformal blocks $G_e^{(p)}$. Determining the Casimir system $Cas_e^{(p)}(G)$ is conceptually straightforward but technically involved. The main complication arises from the spin part of the Lorentz generator (4.19) that generates products of $SU(2,2)$ invariants not present in eq.(4.7). The new invariants are linearly dependent and must be eliminated using relations among them. See Appendix A of ref. [36] for a list of such relations. This is a lengthy step, that however can be automatized in a computer. When redundant structures have been eliminated, one is finally able to read from eq.(4.22) the Casimir system $Cas_e^{(p)}(G)$. Despite the complicacy of the computation, the final linear system of $p+1$ coupled partial differential equations can be written into the following remarkably compact form:

$$Cas_e^{(p)}(G) = \Big(\Delta_{2+p}^{(a_e,b_e;c_e)} - \frac{1}{2}\left(E_\ell^p - \varepsilon_e^p\right)\Big) G_e^{(p)} + A_e^p\, z\bar{z}\, L(a_{e-1})\, G_{e-1}^{(p)} + B_e\, L(b_{e+1})\, G_{e+1}^{(p)} = 0\,, \tag{4.23}$$

where $e=0,\dots,p$,

$$\varepsilon_e^p \equiv \tfrac{3}{4}p^2 - (1+2e)\,p + 2e\,(2+e), \quad A_e^p \equiv 2(p-e+1), \qquad B_e \equiv \frac{e+1}{2}\,, \tag{4.24}$$

and the coefficients E_ℓ^p are given in eq.(4.16). In eq.(4.23) it is understood that $G_{-1}^{(p)} = G_{p+1}^{(p)} = 0$. An identical system of equations is satisfied by the conjugate CBs $\overline{G}_e^{(p)}$. Interestingly enough, only two differential operators enter into the Casimir system: the second-order operator already featured in the scalar case $p=0$, with coefficients

a_e, b_e and c_e given by

$$a_e \equiv a, \quad b_e \equiv b + (p - e), \quad c_e \equiv p - e\,, \tag{4.25}$$

and the new linear operator $L(\mu)$ given by

$$L(\mu) \equiv \frac{1}{z - \bar{z}} \left(z(z-1)\partial_z - \bar{z}(\bar{z}-1)\partial_{\bar{z}} \right) + \mu. \tag{4.26}$$

Another remarkable property of the Casimir system (4.23) is that, for each given e and p, at most three conformal blocks mix with each other in a sort of "nearest-neighbour interaction": G_e mixes only with G_{e+1} and G_{e-1}. The Casimir equations at the "boundaries" $Cas_0^{(p)}$ and $Cas_p^{(p)}$ involve just two blocks. For $p = 0$, the second and third terms in eq.(4.23) vanish and the system trivially reduces to the single equation we discovered in the scalar case.

Finding the solution of the system (4.23) is a complicated task, that we address in the next sections.

4.3 Shadow Formalism Approach

In this section we apply the shadow formalism, using the recent formulation given in ref. [21] and outlined in chapter 2, to get compact expressions for $W^{seed}(p)$ and $\overline{W}^{seed}(p)$ in an integral form for any p and ℓ. Using these expressions, we compute the CBs $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ for $\ell = 0$ and generic p. We then provide a practical way to obtain $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ for any ℓ in a compact form. We finally use this method to compute $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ for $p = 1$ and $G_e^{(p)}$ for $p = 2$ explicitly.

Despite the power of the above technique, it is computationally challenging to go beyond the $p = 2$ case. Moreover, as we will see, we do not have any control on the final analytic form of CBs. In light of this, we will provide the full analytic solution for $G_e^{(p)}$ and $\overline{G}_e^{(p)}$, for any p, only in section 4.4, where we solve directly the set of Casimir differential equations by using an educated ansatz for the solution. The results obtained in this section are however of essential help to argue the proper ansatz. They will also allow us to get the correct physical asymptotic behaviour of $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ that will be used as boundary conditions to solve the Casimir system of equations (4.23). Finally, the explicit computation of $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ for $p = 1$ and $G_e^{(p)}$ for $p = 2$ using the shadow formalism provides an important consistency check for the validity of the full general solution (4.101) to be found in section 4.4.

4.3.1 CPW in Shadow Formalism

Let's write again the generic formula to obtain the CPW associated to the exchange of a given operator O_r with spin $(\ell,\bar{\ell})$ in a correlator of four operators $O_n(X_n)$, $n=1,2,3,4$:

$$W^{(i,j)}_{O^{(\ell,\bar{\ell})}}(X_i)=\nu\int d^4X_0\langle O_1(X_1)O_2(X_2)O_r(X_0,S,\bar{S})\rangle_i\overleftrightarrow{\Pi}_{\ell,\bar{\ell}}\langle\widetilde{O}_r(X_0,T,\bar{T})O_3(X_3)O_4(X_4)\rangle_j\Big|_M\,, \tag{4.27}$$

We use eq.(4.27) to get an integral expression of $W^{seed}(p)$ and $\overline{W}^{seed}(p)$ in eq.(4.7). The explicit expressions of the needed 3-point functions in this case (again stripped off their OPE coefficients) are given by

$$\begin{aligned}\langle\Phi_1(X_1)F_2(X_2)O^{(\ell,\ell+p)}(X_0)\rangle &= \mathcal{K}_3(\tau_1,\tau_2,\tau)I^p_{02}J^\ell_{0,12}\,,\\ \langle\Phi_1(X_1)F_2(X_2)\overline{O}^{(\ell+p,\ell)}(X_0)\rangle &= \mathcal{K}_3(\tau_1,\tau_2,\tau)K^p_{0,12}J^\ell_{0,12}\,,\end{aligned} \tag{4.28}$$

where

$$\mathcal{K}_3(\tau_1,\tau_2,\tau_3)=X_{12}^{\frac{\tau_3-\tau_1-\tau_2}{2}}X_{13}^{\frac{\tau_2-\tau_1-\tau_3}{2}}X_{23}^{\frac{\tau_1-\tau_2-\tau_3}{2}}, \tag{4.29}$$

is a kinematic factor and

$$K_{i,jk}\equiv\sqrt{\frac{X_{jk}}{X_{ij}X_{ik}}}S_j\overline{\mathbf{X}}_iS_k\,,\qquad \overline{K}_{i,jk}\equiv\sqrt{\frac{X_{jk}}{X_{ij}X_{ik}}}\bar{S}_j\mathbf{X}_i\bar{S}_k\,,\quad J_{i,jk}\equiv\frac{1}{X_{jk}}\bar{S}_i\mathbf{X}_j\overline{\mathbf{X}}_kS_i \tag{4.30}$$

are $SU(2,2)$ invariants for three-point functions. The "shadow" 3-point function counterparts are given by

$$\begin{aligned}\langle\widetilde{O}^{(\ell,\ell+p)}(X_0)\Phi_3(X_3)\bar{F}_4(X_4)\rangle\propto\langle O^{(\ell,\ell+p)}(X_0)\Phi_3(X_3)\bar{F}_4(X_4)\rangle\Big|_{\Delta\to4-\Delta} &= \mathcal{K}_3\Big|_{\Delta\to4-\Delta}\overline{K}^p_{3,04}J^\ell_{0,34},\\ \langle\widetilde{\overline{O}}^{(\ell+p,\ell)}(X_0)\Phi_3(X_3)\bar{F}_4(X_4)\rangle\propto\langle\overline{O}^{(\ell+p,\ell)}(X_0)\Phi_3(X_3)\bar{F}_4(X_4)\rangle\Big|_{\Delta\to4-\Delta} &= \mathcal{K}_3\Big|_{\Delta\to4-\Delta}I^p_{40}J^\ell_{0,34}.\end{aligned}$$

Notice again the simplification we have due to the fact the three point functions have a unique tensor structure. The calculation of our CPW is much simpler in this case, having gotten ridden of the indices (i,j) in 4.27. Using the above relations, after a bit of algebra, one can write

$$W^{seed}(p) = \frac{\nu}{X_{12}^{a_{12}+\frac{\ell}{2}}X_{34}^{a_{34}+\frac{\ell+p}{2}}}\int D^4X_0\frac{\mathcal{N}_\ell(p)}{X_{01}^{a_{01}+\frac{\ell}{2}}X_{02}^{a_{02}+\frac{\ell+p}{2}}X_{03}^{a_{03}+\frac{\ell+p}{2}}X_{04}^{a_{04}+\frac{\ell}{2}}}\Big|_{M=1} \tag{4.31}$$

$$\overline{W}^{seed}(p) = \frac{\overline{\nu}}{X_{12}^{a_{12}+\frac{\ell+p}{2}}X_{34}^{a_{34}+\frac{\ell}{2}}}\int D^4X_0\frac{\overline{\mathcal{N}}_\ell(p)}{X_{01}^{a_{01}+\frac{\ell+p}{2}}X_{02}^{a_{02}+\frac{\ell}{2}}X_{03}^{a_{03}+\frac{\ell}{2}}X_{04}^{a_{04}+\frac{\ell+p}{2}}}\Big|_{M=1} \tag{4.32}$$

where

$$a_{01}=\frac{\Delta}{2}+\frac{p}{4}-a,\qquad a_{02}=\frac{\Delta}{2}-\frac{p}{4}+a,\qquad a_{12}=\frac{\Delta_1+\Delta_2}{2}-\frac{\Delta}{2},$$
$$a_{03}=\frac{4-\Delta}{2}+\frac{p}{4}+b,\quad a_{04}=\frac{4-\Delta}{2}-\frac{p}{4}-b,\quad a_{34}=\frac{\Delta_3+\Delta_4}{2}-\frac{4-\Delta}{2}\,, \tag{4.33}$$

and

$$\mathcal{N}_\ell(p) \equiv (\bar{S}S_2)^p(\bar{S}X_2\bar{X}_1S)^\ell \overleftrightarrow{\Pi}_{\ell,\ell+p}(\bar{S}_4X_3\bar{T})^p(\bar{T}X_4\bar{X}_3T)^\ell, \tag{4.34}$$
$$\overline{\mathcal{N}}_\ell(p) \equiv (\bar{S}_4S)^p(\bar{S}X_3\bar{X}_4S)^\ell \overleftrightarrow{\Pi}_{\ell+p,\ell}(S_2\overline{X}_1T)^p(\bar{T}X_1\bar{X}_2T)^\ell. \tag{4.35}$$

We will not need to determine the normalization factors ν and $\bar{\nu}$ in eqs.(4.31) and (4.32). Notice that the correct behaviour of the seed CPWs under $X_{12}\to e^{4\pi i}X_{12}$ is saturated by the factor X_{12} multiplying the integrals in eqs.(4.31) and (4.32). Hence the latter should be projected to their trivial monodromy components $M=1$, as before. Notice that eqs.(4.34) and (4.35) are related by a simple transformation:

$$\overline{\mathcal{N}}_\ell(p)=\mathcal{P}\mathcal{N}_\ell(p)\Big|_{1\leftrightarrow 3,\ 2\leftrightarrow 4}, \tag{4.36}$$

where $\mathcal{P}$ is the parity operator.

We can recast the expression (4.34) in a compact and convenient form using some manipulations. We define a couple new variables and remind the reader of the variable s:

$$s\equiv X_{12}X_{34}\prod_{n=1}^{4}X_{0n},\ t\equiv\frac{1}{2\sqrt{s}}\Big(X_{02}X_{03}X_{14}-X_{01}X_{03}X_{24}-(3\leftrightarrow 4)\Big),\ u\equiv\frac{X_{02}X_{03}X_{34}}{\sqrt{s}}. \tag{4.37}$$

Then we look for a relation expressing the generic $\mathcal{N}_\ell(p)$ with the known $\mathcal{N}_\ell(0)$:

$$\mathcal{N}_\ell(0)=(-1)^\ell(\ell!)^4\ s^{\ell/2}C_\ell^1(t)\,, \tag{4.38}$$

where C_ℓ^p are Gegenbauer polynomials of rank p. Starting from eq.(4.34), after acting with the S and T derivatives, one gets

$$\mathcal{N}_\ell(p)=(\ell!)^2(\overrightarrow{\partial}_{\bar{S}}\overline{X}_0\overrightarrow{\partial}_{\bar{T}})^{\ell+p}\left((\bar{S}S_2)^p(\bar{S}_4X_3\bar{T})^p(\bar{S}\Omega\bar{T})^\ell\right), \tag{4.39}$$

where we have defined $\Omega_{ab}=(X_2\bar{X}_1X_0\bar{X}_3X_4)_{ab}$. In order to relate $\mathcal{N}_\ell(p)$ above to $\mathcal{N}_{\ell+p}(0)$ in eq.(4.38), we look for an operator $\mathcal{D}$ satisfying

$$\mathcal{D}^p\,(\overrightarrow{\partial}_{\bar{S}}\overline{X}_0\overrightarrow{\partial}_{\bar{T}})^{\ell+p}(\bar{S}\Omega\bar{T})^{\ell+p}=(\overrightarrow{\partial}_{\bar{S}}\overline{X}_0\overrightarrow{\partial}_{\bar{T}})^{\ell+p}\left((\bar{S}S_2)^p(\bar{S}_4X_3\bar{T})^p(\bar{S}\Omega\bar{T})^\ell\right). \tag{4.40}$$

By looking at eq.(4.40) we deduce that $\mathcal{D}$ should be bilinear in $\bar{S}_4$ and S_2 and should commute with $(\overrightarrow{\partial}_{\bar{S}}\overline{X}_0\overrightarrow{\partial}_{\bar{T}})$. In addition to that, it should have the correct scaling in X's and should be gauge invariant, namely it should be well defined on the light-cone $X^2=0$ and preserve the conditions (1.59). It is not difficult to see that the choice

$$\mathcal{D}=(\bar{S}_4X_0\bar{\Sigma}^N S_2)\frac{\partial}{\partial X_2^N} \tag{4.41}$$

fulfills all the requirements. One has $\mathcal{D}(\bar{S}\Omega\bar{T})=8X_{01}X_{04}(\bar{S}S_2)(\bar{S}_4X_3\bar{T})$. Iterating p times gives the desired relation:

$$\mathcal{N}_\ell(p)\propto\frac{1}{(X_{01}X_{04})^p}\mathcal{D}^p\mathcal{N}_{\ell+p}(0)\,. \tag{4.42}$$

The operator $\mathcal{D}$ annihilates the invariants $J_{42,01}$ and $J_{42,30}$ and all the scalar products with the exception of X_{12}, in which case we have $\mathcal{D}X_{12}=X_{01}\,J_{42,01}$. The action on the s, t, and u variables is

$$\mathcal{D}\,s=X_{12}^{-1}\,s\,J_{42,01},\;\;\mathcal{D}\,t=-\frac{1}{2}\,X_{12}^{-1}\,(u^{-1}\,J_{42,30}+t\,J_{42,01}),\;\;\mathcal{D}\,u^{-1}=\frac{1}{2}\,X_{12}^{-1}\,u^{-1}\,J_{42,01}\,, \tag{4.43}$$

and on Gegenbauer polynomials is

$$\mathcal{D}\,C_n^\lambda(t)=2\lambda\,C_{n-1}^{\lambda+1}(t)\,\mathcal{D}\,t. \tag{4.44}$$

Using recursively the identity for Gegenbauer polynomials

$$\frac{n}{2\lambda}\,C_n^\lambda(t)-t\,C_{n-1}^{\lambda+1}(t)=-C_{n-2}^{\lambda+1}(t)\,, \tag{4.45}$$

we can write the following expression for $\mathcal{N}_\ell(p)$:

$$\mathcal{N}_\ell(p)\propto s^{\frac{\ell}{2}}\;\sum_{w=0}^{p}\binom{p}{w}u^w\,C_{\ell-w}^{p+1}(t)\;I_1^{p-w}I_2^w, \tag{4.46}$$

where $\binom{p}{w}$ is the binomial coefficient and for compactness we have defined the dimensionful tensor structures

$$I_1\equiv X_{03}\,J_{42,30}\,,\qquad I_2\equiv X_{01}\,J_{42,01}. \tag{4.47}$$

Combining together eqs.(4.31), (4.32), (4.36), (4.37) and (4.46) we can finally write

$$\begin{aligned} W^{seed}(p) &= \nu' \sum_{w=0}^{p} \binom{p}{w} \frac{1}{X_{12}^{a_{12}+\frac{w}{2}} X_{34}^{a_{34}+\frac{p-w}{2}}} \int D^4 X_0 \frac{C_{\ell-w}^{p+1}(t)\ I_1^{p-w} I_2^{w}}{X_{01}^{a_{01}+\frac{w}{2}} X_{02}^{a_{02}+\frac{p-w}{2}} X_{03}^{a_{03}+\frac{p-w}{2}} X_{04}^{a_{04}+\frac{w}{2}}}\Big|_{M=1}, \\ \overline{W}^{seed}(p) &= \bar{\nu}' \sum_{w=0}^{p} \binom{p}{w} \frac{1}{X_{12}^{a_{12}+\frac{p-w}{2}} X_{34}^{a_{34}+\frac{w}{2}}} \int D^4 X_0 \frac{C_{\ell-w}^{p+1}(t)\ I_1^{w} I_2^{p-w}}{X_{01}^{a_{01}+\frac{p-w}{2}} X_{02}^{a_{02}+\frac{w}{2}} X_{03}^{a_{03}+\frac{w}{2}} X_{04}^{a_{04}+\frac{p-w}{2}}}\Big|_{M=1} \end{aligned} \tag{4.48}$$

where ν' and $\bar{\nu}'$ are undetermined normalization factors.

4.3.2 Seed Conformal Blocks and Their Explicit Form for $\ell = 0$

The computation of the CBs $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ starting form eq.(4.48) is a non-trivial task for generic ℓ and p, since we are not aware of a general formula for an integral that involves $C_{\ell-w}^{p+1}(t)$ for $p \neq 0$. For any given ℓ, one can however expand the Gegenbauer polynomial, in which case the CBs $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ can be computed. After a discussion on the structure of the CBs for generic ℓ, we compute in this subsection the CBs $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ for $\ell = 0$ and generic p.

Looking at eq.(4.48) and recalling the definition of t in eq.(4.37), one realizes that the Gegenbauer polynomials, when expanded, do not give rise to intrinsically new integrals but just amounts to shifting the exponents in the denominator. The tensor structures in the numerators bring p open indices in the form $X_0^{N_1} \dots X_0^{N_p}$, which can be removed by using eq.(3.21) in ref. [21], reducing the problem to scalar integrals in $2h = 2(2+p)$ effective dimensions, of the form:

$$I^{(h)}_{A_{02},\,A_{03},\,A_{04}} \equiv \int D^{2h} X_0 \frac{1}{X_{01}^{A_{01}} X_{02}^{A_{02}} X_{03}^{A_{03}} X_{04}^{A_{04}}}\Big|_{M=1}, \tag{4.49}$$

where $A_{01} + A_{02} + A_{03} + A_{04} = 2h$. The capital A_{0i} are used for the exponents in the denomentaor with all possible shifts introduced by the Gegenbaur polynomial. This integral is given by the following expression

$$I^{(h)}_{A_{02},\,A_{03},\,A_{04}} \propto X_{13}^{A_{04}-h} X_{14}^{A_{02}+A_{03}-h} X_{24}^{-A_{02}} X_{34}^{h-A_{03}-A_{04}} \times R^{(h)}(z,\bar{z};\, A_{02}, A_{03}, A_{04}), \tag{4.50}$$

where $R^{(h)}$ is given by

$$R^{(h)}(z,\bar{z};\, A_{02}, A_{03}, A_{04}) \equiv \Big(-\frac{\partial}{\partial v}\Big)^{h-1} f(z;\, A_{02}, A_{03}, A_{04}) f(\bar{z};\, A_{02}, A_{03}, A_{04}), \tag{4.51}$$

$$f(z;\, A_{02}, A_{03}, A_{04}) \equiv {}_2F_1(A_{02} - h + 1,\, -A_{04} + 1;\, -A_{03} - A_{04} + h + 1; \tag{4.52}$$

The derivative $-\partial/\partial v$ in $(z,\bar{z})$ coordinates is given by

$$-\frac{\partial}{\partial v}=\frac{1}{z-\bar{z}}\Big(z\frac{\partial}{\partial z}-\bar{z}\frac{\partial}{\partial\bar{z}}\Big). \tag{4.53}$$

In the case of $\ell=0$, all the above manipulations simplify drastically. The Gegenbauer polynomial $C^{p+1}_{\ell-w}(t)$ vanishes for all the values w except for $w=0$, leaving only one type of tensor structure I^p_1 for $W^{seed}(p)$ and I^p_2 for $\overline{W}^{seed}(p)$. This leads to a one-to-one correspondence between CBs and integrals:

$$\begin{aligned}
G^{(p)}_e &\propto X_{13}^{p-e}X_{34}^{e}\,\mathcal{K}_4^{-1}\,I^{(2+p)}_{a_{02}+\frac{p}{2},a_{03}+\frac{p}{2},a_{04}+e}\propto(z\bar{z})^{\frac{\Delta+\frac{p}{2}}{2}}R^{(2+p)}(z,\bar{z};\,a_{02}+\frac{p}{2},a_{03}+\frac{p}{2},a_{04}+e), \qquad (4.54)\\
\overline{G}^{(p)}_e &\propto X_{12}^{e}X_{13}^{p-e}\,\mathcal{K}_4^{-1}\,I^{(2+p)}_{a_{02}+e,\,a_{03}+p-e,\,a_{04}+\frac{p}{2}}\propto(z\bar{z})^{\frac{\Delta-\frac{p}{2}}{2}+e}R^{(2+p)}(z,\bar{z};\,a_{02}+e,a_{03}+p-e,a_{04}+\frac{p}{2}).
\end{aligned}$$

We have omitted here the relative factors between different CBs. They must be restored if one wants to check that $G^{(p)}_e$ and $\overline{G}^{(p)}_e$ in eq.(4.54) satisfy the Casimir system (4.23). For generic ℓ the CBs are a sum of expressions like eq.(4.54) with different shifts of the parameters A_{0i}, weighted by the relative constants and powers of v (coming from the Gegenbauer polynomial). Since all these terms have $p+1$ derivatives with respect to v, the highest power in $1/(z-\bar{z})$ appearing in $G^{(p)}_e$ and $\overline{G}^{(p)}_e$ is

$$\Big(\frac{1}{z-\bar{z}}\Big)^{1+2\,p}. \tag{4.55}$$

The asymptotic behaviour of the CBs when $z,\bar{z}\to 0$ ($u\to 0$, $v\to 1$) for $\ell=0$ is easily obtained from eq.(4.54) by noticing that $R^{(h)}(z,\bar{z};\,A_{02},A_{03},A_{04})$ is constant in this limit. We then have

$$\lim_{z\to 0,\,\bar{z}\to 0}G^{(p)}_e\propto(z\bar{z})^{\frac{\Delta}{2}+\frac{p}{4}}\,,\qquad\qquad \lim_{z\to 0,\,\bar{z}\to 0}\overline{G}^{(p)}_e\propto(z\bar{z})^{\frac{\Delta}{2}-\frac{p}{4}+e}\,. \tag{4.56}$$

By knowing that the CBs should be proportional to the factor in eq.(4.55), we can refine eq.(4.56) and write

$$\lim_{z\to 0,\,\bar{z}\to 0}G^{(p)}_e\propto\frac{(z\bar{z})^{\frac{\Delta}{2}+\frac{p}{4}}}{(z-\bar{z})^{1+2p}}(z^{1+2p}-\bar{z}^{1+2p})\,, \tag{4.57}$$

$$\lim_{z\to 0,\,\bar{z}\to 0}\overline{G}^{(p)}_e\propto\frac{(z\bar{z})^{\frac{\Delta}{2}-\frac{p}{4}+e}}{(z-\bar{z})^{1+2p}}(z^{1+2p}-\bar{z}^{1+2p})\,. \tag{4.58}$$

Notice that the behavior (4.57) and (4.58) of the CBs for $z,\bar{z}\to 0$ when $\ell=0$ is not guaranteed to be straightforwardly extended for any $\ell\neq 0$. Indeed, we see from eq.(4.48) that for a given p, the generic CPW is obtained when $\ell\geq p$, in which case all terms in the sum over w are present. All the values of $\ell<p$ should be treated

separately.

4.3.3 Computing the Conformal Blocks for $\ell \neq 0$

A useful expression of the CBs for generic values of ℓ can be obtained using eq.(4.42) and the known closed form of $W^{seed}(0)$. Recall that

$$W^{seed}(0) = \left(\frac{X_{14}}{X_{13}}\right)^b \left(\frac{X_{24}}{X_{14}}\right)^{-a} \frac{G^{(0)}(z,\bar{z})}{X_{12}^{\frac{\Delta_1+\Delta_2}{2}} X_{34}^{\frac{\Delta_3+\Delta_4}{2}}}\,, \tag{4.59}$$

where a and b are as in eq.(4.8) for $p=0$ and $G^{(0)}(z,\bar{z})$ are the known scalar CBs [13,14]

$$G^{(0)}(z,\bar{z}) = G^{(0)}(z,\bar{z};\Delta,l,a,b) = (-1)^\ell \frac{z\bar{z}}{z-\bar{z}} \left(k^{(a,b;0)}_{\frac{\Delta+\ell}{2}}(z) k^{(a,b;0)}_{\frac{\Delta-\ell-2}{2}}(\bar{z}) - (z \leftrightarrow \bar{z})\right)\,, \tag{4.60}$$

expressed in terms of the function $k^{(a,b;c)}_\rho(z)$ defined in eq (2.41).

Comparing eq.(4.59) with eq.(4.48) for $p=0$, one can extract the value of the shadow integral in closed form for generic spin ℓ [21]:

$$I_\ell \equiv \int D^4 X_0 \frac{C^1_\ell(t)}{X_{01}^{a_{01}} X_{02}^{a_{02}} X_{03}^{a_{03}} X_{04}^{a_{04}}}\Big|_{M=1} \propto \left(\frac{X_{14}}{X_{13}}\right)^b \left(\frac{X_{24}}{X_{14}}\right)^{-a} \frac{G^{(0)}(z,\bar{z};\Delta,\ell,a,b)}{X_{12}^{\frac{\Delta}{2}} X_{34}^{\frac{4-\Delta}{2}}}\,. \tag{4.61}$$

Using the relations (4.38) and (4.42) one can recast $W^{seed}(p)$ and $\overline{W}^{seed}(p)$ in the form

$$\begin{aligned} W^{seed}(p) &\propto \frac{\mathcal{D}_{N_1}\ldots\mathcal{D}_{N_p}}{X_{12}^{a_{12}+\frac{\ell}{2}} X_{34}^{a_{34}}} X_{12}^{\frac{\ell+p}{2}} \int D^4 X_0 \frac{C^1_{\ell+p}(t) X_0^{N_1}\ldots X_0^{N_p}}{X_{01}^{a_{01}+\frac{p}{2}} X_{02}^{a_{02}} X_{03}^{a_{03}} X_{04}^{a_{04}+\frac{p}{2}}}\Big|_{M=1}\,, \\ \overline{W}^{seed}(p) &\propto \frac{\overline{\mathcal{D}}_{N_1}\ldots\overline{\mathcal{D}}_{N_p}}{X_{12}^{a_{12}} X_{34}^{a_{34}+\frac{\ell}{2}}} X_{34}^{\frac{\ell+p}{2}} \int D^4 X_0 \frac{C^1_{\ell+p}(t) X_0^{N_1}\ldots X_0^{N_p}}{X_{01}^{a_{01}} X_{02}^{a_{02}+\frac{p}{2}} X_{03}^{a_{03}+\frac{p}{2}} X_{04}^{a_{04}}}\Big|_{M=1}\,, \end{aligned} \tag{4.62}$$

where $\overline{\mathcal{D}} = \mathcal{P}\mathcal{D}|_{1\leftrightarrow 3,2\leftrightarrow 4}$, as follows from eq.(4.36), $\mathcal{D} = \mathcal{D}_M X_0^M$, $\overline{\mathcal{D}} = \overline{\mathcal{D}}_M X_0^M$. The tensor integral is evaluated using $SO(4,2)$ Lorentz symmetry. One writes

$$\int D^4 X_0 \frac{C^1_{\ell+p}(t) X_0^{M_1}\ldots X_0^{M_p}}{X_{01}^{a_{01}+\frac{p}{2}} X_{02}^{a_{02}} X_{03}^{a_{03}} X_{04}^{a_{04}+\frac{p}{2}}} = \sum_n A_n(X_i)\ \tau_n^{M_1\ldots M_p}(X_i)\,, \tag{4.63}$$

where n runs over all possible rank p traceless symmetric tensors τ_n which can be constructed from X_1, X_2, X_3, X_4 and η_{MN}'s, with arbitrary scalar coefficients A_n to be determined. Performing all possible contractions, which do not change the monodromy of the integrals, the A_n coefficients can be solved as linear combinations of the scalar block integrals I_ℓ defined in eq.(4.61), with shifted external dimensions.

In this way, we have computed the CBs $G_e^{(p)}$ with $p = 1, 2$ and $\overline{G}_e^{(p)}$ with $p = 1$ for general Δ, ℓ, a, b. We have also verified that the CBs $\overline{G}_e^{(1)}$ obtained from $G_e^{(1)}$ using eqs.(4.10) and (4.9) agree with those arising from the direct shadow computation. There is a close connection among the CBs $G_e^{(p)}$ and $\overline{G}_{p-e}^{(p)}$, for any p. More on this point in section 4.4. In all cases the CBs satisfy the Casimir system (4.23).

As mentioned at the end of subsection 4.3.2, the asymptotic behaviour of the CBs for $z, \bar{z} \to 0$ depends on whether $\ell \geq p$ or not. For $p = 1$ we can expand the obtained solutions, which for $\ell \geq 1$ read as

$$\lim_{z\to 0,\,\bar{z}\to 0} G_e^{(1)} \propto \frac{(z\bar{z})^{\frac{\Delta-\ell}{2}+\frac{1}{4}}}{(z-\bar{z})^3}\Big(\bar{z}^{\ell+e+2} - (z \leftrightarrow \bar{z})\Big)\,, \qquad \ell \geq 1 \tag{4.64}$$

$$\lim_{z\to 0,\,\bar{z}\to 0} \overline{G}_e^{(1)} \propto \frac{(z\bar{z})^{\frac{\Delta-\ell}{2}-\frac{1}{4}}}{(z-\bar{z})^3}\Big(z^e\bar{z}^{\ell+3} - (z \leftrightarrow \bar{z})\Big)\,, \qquad \ell \geq 1\,, \tag{4.65}$$

while for $\ell = 0$ they obviously match eqs.(4.57) and (4.58). The above relations, together with eqs.(4.55), (4.57) and (4.58), will allow us to settle the problem of the boundary values of the CBs for any value of p and ℓ, that will be reported in eqs.(4.74) and (4.78). The explicit form of $G_e^{(p)}$ found for $p = 2$ using the shadow formalism provides a further check of the whole derivation.

4.4 Solving the System of Casimir Equations

The goal of this section is to find the explicit form of the conformal blocks $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ appearing in eq.(4.7) by solving the Casimir system (4.23). In doing it we adopt and expand the methods introduced by Dolan and Osborn in refs. [14, 15] to obtain 6D scalar conformal blocks. We will mostly focus on the blocks $G_e^{(p)}$, since the same analysis will apply to $\overline{G}_e^{(p)}$ with a few modifications that we will point out.

Before jumping into details let us outline the main logical steps of our derivation. We first find, with the guidance of the results obtained in section 4.3, the behaviour of $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ in the limit $z, \bar{z} \to 0$ in which the Casimir system (4.7) can be easily solved. Using this information and eq.(4.55), we then write an educated ansatz for the form of the CBs. Using this ansatz, we reduce the problem of solving a system of linear partial differential equations of second order in two variables to a system of linear algebraic equations for the unknown coefficients entering the ansatz. Then we show that the non-zero coefficients in the ansatz admit a geometric interpretation. They form a two-dimensional lattice with an octagon shape structure. This interpretation allows us to precisely predict which coefficients enter in our ansatz for any value of p. Finally, we show that the linear algebraic system admits a recursive solution and we discuss the complexity of deriving full solutions for higher values of p.

4.4.1 Asymptotic Behaviour

Not all solutions of the Casimir system (4.23) give rise to sensible CBs. The physical CBs are obtained by demanding the correct boundary values for $G_e^{(p)}$ and $\overline{G}_e^{(p)}$. Possible boundary values are given by considering the OPE limit $z, \bar{z} \to 0$ of $W^{seed}(p)$ and $\overline{W}^{seed}(p)$. The limits of $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ for $z, \bar{z} \to 0$ could be computed by a careful analysis of tensor structures. This analysis has been partly done in section 4.3, where we have obtained the boundary values of $G_e^{(p)}$ and $\overline{G}_e^{(p)}$ for $z, \bar{z} \to 0$ for special values of p and/or ℓ. Luckily enough, there will be no need to extend such analysis because the form of the system (4.23) in the OPE limit, together with eqs.(4.57), (4.64) and (4.65), will clearly indicate the general form of the boundary values of $G_e^{(p)}$ and $\overline{G}_e^{(p)}$.

Let us then consider the form of the conformal blocks $G_e^{(p)}$ in the limit $z, \bar{z} \to 0$, with $z \to 0$ *taken first*. In this limit

$$G_e^{(p)} \to N_e z^{\lambda^{(e)}} \bar{z}^{\bar{\lambda}^{(e)}}\,, \tag{4.66}$$

where N_e, $\lambda^{(e)}$ and $\bar{\lambda}^{(e)}$ are parameters to be determined. For simplicity of notation we have omitted their p-dependence. The differential operators (2.20) and (4.26), when acting on eq.(4.66) give, at leading order in z and $\bar{z}$,

$$\Delta_\epsilon^{(a_e,b_e;c_e)} \to \lambda^{(e)}(\lambda^{(e)}-1) - c_e(\lambda^{(e)}+\bar{\lambda}^{(e)}) + \bar{\lambda}^{(e)}(\bar{\lambda}^{(e)}-1) - \epsilon\lambda^{(e)}\,, \tag{4.67}$$

$$L(\mu) \to \frac{1}{\bar{z}}(\lambda^{(e)} - \bar{\lambda}^{(e)})\,. \tag{4.68}$$

Let us now focus on the specific "boundary" equation $Cas_e^{(p)}$ with $e = p$. In the limit $z, \bar{z} \to 0$ it reads

$$\begin{aligned} Cas_p^{(p)}(G) \to \quad & N_p\left(\lambda^{(p)}(\lambda^{(p)}-1) + \bar{\lambda}^{(p)}(\bar{\lambda}^{(p)}-1) - (p+2)\lambda^{(p)} - \frac{1}{2}(E_{\ell,p} - \epsilon_p^p)\right) z^{\lambda^{(p)}} \bar{z}^{\bar{\lambda}^{(p)}} \\ & + 2N_{p-1}(\lambda^{(p-1)} - \bar{\lambda}^{(p-1)}) z^{\lambda^{(p-1)}+1} \bar{z}^{\bar{\lambda}^{(p-1)}} = 0\,. \end{aligned} \tag{4.69}$$

For generic values of ℓ, we have $\lambda^{(e)} \neq \bar{\lambda}^{(e)}$. Hence we cannot have $\lambda^{(p-1)} + 1 < \lambda^{(p)}$ in eq.(4.69), since this would imply that the last term dominates in the limit and N_{p-1} vanishes, in contradiction with the initial hypothesis (4.66).

Let us first consider the case in which $\lambda^{(p-1)} + 1 > \lambda^{(p)}$, so that the terms in the second row of eq.(4.69), coming from $G_{p-1}^{(p)}$, vanish. It is immediate to see that the only sensible solution for $\lambda^{(p)}$ and $\bar{\lambda}^{(p)}$ that reproduce the known OPE limit for the $p = 0$ case is

$$\lambda^{(p)} = \frac{\Delta - \ell}{2} + \frac{p}{4}, \qquad \bar{\lambda}^{(p)} = \frac{\Delta + \ell}{2} + \frac{p}{4}\,. \tag{4.70}$$

Notice that eq.(4.70) agrees with the asymptotic behaviour for the CBs $G_e^{(p)}$ found in eq.(4.64) for $e = p = 1$ and $\ell \geq 1$. Consider now the equation $Cas_{p-1}^{(p)}$. For $z, \bar{z} \to 0$ we have

$$\begin{aligned} Cas_{p-1}^{(p)}(G) \to \quad & N_{p-1}\Big(\lambda^{(p-1)}(\lambda^{(p-1)}-1)+\bar{\lambda}^{(p-1)}(\bar{\lambda}^{(p-1)}-1)+(\lambda^{(p-1)}+\bar{\lambda}^{(p-1)})-(p+2)\lambda^{(p-1)} \\ & -\frac{1}{2}(E_{\ell,p}-\epsilon_{p-1}^{p})\Big)z^{\lambda^{(p-1)}}\bar{z}^{\bar{\lambda}^{(p-1)}}+\frac{p}{2}N_p(\lambda^{(p)}-\bar{\lambda}^{(p)})z^{\lambda^{(p)}}\bar{z}^{\bar{\lambda}^{(p)}-1} \\ & +4N_{p-2}(\lambda^{(p-2)}-\bar{\lambda}^{(p-2)})z^{\lambda^{(p-2)}+1}\bar{z}^{\bar{\lambda}^{(p-2)}}=0\,. \end{aligned} \tag{4.71}$$

According to eq.(4.64), we expect $\lambda^{(p-2)} = \lambda^{(p-1)} = \lambda^{(p)}$, $\bar{\lambda}^{p-1} = \bar{\lambda}^{(p)} - 1$, $\bar{\lambda}^{p-2} = \bar{\lambda}^{(p)} - 2$ in eq.(4.71). In this case the last term is higher order in z and eq.(4.71) is satisfied by simply taking

$$\frac{N_{p-1}}{N_p} = -\frac{\ell p}{2(\ell+p)}\,. \tag{4.72}$$

Notice that we have tacitly assumed above that $\lambda^{(p)} - \bar{\lambda}^{(p)} = -\ell$ does not vanish, i.e. $\ell \neq 0$. For $\ell = 0$, more care is required and one should consider the first subleading term in $\bar{z}$ in the expansion (4.66).

The above analysis can be iteratively repeated until the last equation $Cas_0^{(p)}$ is reached and all the coefficients N_e, $\lambda^{(e)}$ and $\bar{\lambda}^{(e)}$ are determined. Analogously to the $\ell = 0$ case in eq.(4.71), all the low spin cases up to $\ell = p$ should be treated separately at some step in the iteration, as already pointed out in subsection 4.3.2. Skipping the detailed derivation, the final values of $\lambda^{(e)}$ and $\bar{\lambda}^{(e)}$ are given by

$$\begin{aligned} &\lambda^{(e)} = \lambda^{(p)}\,, && \forall \ell = 0,1,2,\dots \\ &\bar{\lambda}^{(e)} = \bar{\lambda}^{(p)} - (p-e)\,, && \forall \ell = p-e, p-e+1,\dots \\ &\bar{\lambda}^{(e)} = \bar{\lambda}^{(p)}\,, && \forall \ell = 0,1,\dots,p-e-1 \end{aligned} \tag{4.73}$$

where $\lambda^{(p)}$ and $\bar{\lambda}^{(p)}$ are as in eq.(4.70) and $e = 0, \dots, p-1$. The asymptotic behaviour of the CBs in the OPE limit is given for any ℓ and p by

$$\lim_{z\to 0,\,\bar{z}\to 0} G_e^{(p)} \propto \frac{(z\bar{z})^{\lambda^{(p)}}}{(z-\bar{z})^{1+2p}}\Big(\bar{z}^{\bar{\lambda}^{(e)}-\lambda^{(p)}+1+2p} - (z \leftrightarrow \bar{z})\Big)\,. \tag{4.74}$$

We do not report the explicit form of the normalization factors N_e, since they will be of no use in what follows.

We still have to consider the case in which $\lambda^{(p-1)} + 1 = \lambda^{(p)}$ in eq.(4.69). By looking at eq.(4.65), it is clear that this case corresponds to the asymptotic behaviour of the conjugate CBs $\overline{G}_e^{(p)}$. We do not report here the similar derivation of the Casimir equations for $\overline{G}_e^{(p)}$ in the OPE limit. It suffices to say that the analysis closely follows

the ones made for $G_e^{(p)}$ starting now from the equation with $e = 0$. If we denote by

$$\overline{G}_e^{(p)} \to \bar{N}_e z^{\omega^{(e)}} \bar{z}^{\bar{\omega}^{(e)}} \tag{4.75}$$

the boundary behaviour of $\overline{G}_e^{(p)}$ when $z, \bar{z} \to 0$ ($z \to 0$ taken first), one finds

$$\begin{aligned} \omega^{(e)} &= \omega^{(0)} + e\,, & \forall \ell &= 0, 1, 2, \dots \\ \bar{\omega}^{(e)} &= \bar{\omega}^{(0)}\,, & \forall \ell &= p - e, p - e + 1, \dots \\ \bar{\omega}^{(e)} &= \bar{\omega}^{(0)} + e\,, & \forall \ell &= 0, 1, \dots, p - e - 1 \end{aligned} \tag{4.76}$$

where

$$\omega^{(0)} = \frac{\Delta - \ell}{2} - \frac{p}{4}, \qquad \bar{\omega}^{(0)} = \frac{\Delta + \ell}{2} - \frac{p}{4}\,. \tag{4.77}$$

The asymptotic behaviour of the conjugate CBs are given for any ℓ and p by

$$\lim_{z\to 0,\, \bar{z}\to 0} \overline{G}_e^{(p)} \propto \frac{(z\bar{z})^{\omega^{(e)}}}{(z - \bar{z})^{1+2p}} \left(\bar{z}^{\bar{\omega}^{(e)} - \omega^{(e)} + 1 + 2p} - (z \leftrightarrow \bar{z}) \right). \tag{4.78}$$

4.4.2 The Ansatz

The key ingredient of the ansatz is the function $k_\rho^{(a,b;c)}(z)$ defined in eq.(2.41), which is an eigenfunction of the hyper-geometric like operator $D_z^{(a,b;c)}$:

$$D_z^{(a,b;c)}\; k_\rho^{(a,b;c)}(z) = \rho\,(\rho + c - 1)\; k_\rho^{(a,b;c)}(z). \tag{4.79}$$

Using eq.(4.79) one can define an eigenfunction of the operator $\Delta_0^{(a,b;c)}$ as the product of two k's:

$$\mathcal{F}_{\rho_1,\,\rho_2}^{(a,b;c)}(z, \bar{z}) \;\equiv\; k_{\rho_1}^{(a,b;c)}(z) k_{\rho_2}^{(a,b;c)}(\bar{z}), \tag{4.80}$$

$$\mathcal{F}_{\rho_1,\,\rho_2}^{\pm\,(a,b;c)}(z, \bar{z}) \;\equiv\; \mathcal{F}_{\rho_1,\,\rho_2}^{(a,b;c)}(z, \bar{z}) \pm \mathcal{F}_{\rho_1,\,\rho_2}^{(a,b;c)}(\bar{z}, z). \tag{4.81}$$

These functions played an important role in ref. [14] for the derivation of an analytic closed expression of the scalar CBs in even space-time dimensions. In our case, the situation is much more complicated, because we have different blocks appearing in the Casimir equations. We notice, however, that the second order operator Δ in each equation $Cas_e^{(p)}$ acts only on the block $G_e^{(p)}$, while the blocks $G_{e-1}^{(p)}$ and $G_{e+1}^{(p)}$ are multiplied by first order operators only. Since, as we will shortly see, first order derivatives and factors of z and $\bar{z}$ acting on the functions $\mathcal{F}$ can always be expressed in terms of functions $\mathcal{F}$ with shifted parameters, a reasonable ansatz for the CBs is to take each G_e proportional to a sum of functions of the kind $\mathcal{F}_{\rho_1,\,\rho_2}^{(a_e,b_e;c_e)}(z, \bar{z})$ for some ρ_1 and ρ_2. Taking also into account eq.(4.55), found using the shadow formalism, the

form of the ansatz for the blocks $G_e^{(p)}$ should be[2]

$$G_e^{(p)}(z,\bar{z}) = \left(\frac{z\bar{z}}{z-\bar{z}}\right)^{2p+1} g_e^{(p)}(z,\bar{z}), \quad g_e^{(p)}(z,\bar{z}) \equiv \sum_{m,n} c_{m,n}^e \mathcal{F}^{-\,(a_e,b_e;c_e)}_{\rho_1+m,\,\rho_2+n}(z,\bar{z}), \tag{4.82}$$

where $c_{m,n}^e$ are coefficients to be determined and the sum over the two integers m and n in eq.(4.82) is so far unspecified. Notice that all the functions $\mathcal{F}$ entering the sum over m and n have the same values of a_e, b_e and c_e. Matching eq.(4.82) in the limit $z, \bar{z} \to 0$ with eq.(4.74) allows us to determine ρ_1 and ρ_2, modulo a shift by an integer. We take

$$\rho_1 = \bar{\lambda}^{(p)}\,, \qquad \rho_2 = \lambda^{(p)} - p - 1\,, \tag{4.83}$$

in which case the sum over n is bounded from below by $n_{min} = -p$. At this value of n, we have $m(n_{min}) = e - p$. There is no need to discuss separately the behaviour of the blocks with $\ell \leq p$. Their form is still included in the ansatz (4.82) with the additional requirement that some coefficients $c^e_{m,n_{min}}$ should vanish. This condition will automatically be satisfied in the final solution. In the next subsections we will discuss the precise range of the sum over m and n and explain how the coefficients $c^e_{m,n}$ can be determined.

4.4.3 Reduction to a Linear System

The eigenfunctions $\mathcal{F}^{\pm\,(a,b;c)}_{\rho_1,\,\rho_2}(z,\bar{z})$ have several properties that would allow us to find a solution to the system (4.23). In order to exploit such properties, we first have to express the system (4.23) for $G_e^{(p)}$ in terms of the functions $g_e^{(p)}(z,\bar{z})$ defined in eq.(4.82). We plug the ansatz (4.82) in eq.(4.23) and use the following relations

$$\begin{aligned}
\Delta_\epsilon^{(a,b;c)}\left(\frac{z\bar{z}}{z-\bar{z}}\right)^k &= \left(\frac{z\bar{z}}{z-\bar{z}}\right)^k\left(\Delta^{(a,b;c)}_{\epsilon-2k} + k\,(k-\epsilon+c-1) - k\,(k-\epsilon+1)\,\frac{z\bar{z}(z+\bar{z})-2z\bar{z}}{(z-\bar{z})^2}\right),\\
L(\mu)\left(\frac{z\bar{z}}{z-\bar{z}}\right)^k &= \left(\frac{z\bar{z}}{z-\bar{z}}\right)^k\left(L(\mu) + k\,\frac{z+\bar{z}-2z\bar{z}}{(z-\bar{z})^2}\right),
\end{aligned} \tag{4.84}$$

to obtain the system of Casimir equations for $g_e^{(p)}$:

$$\widetilde{Cas}_e^{(p)}(g) \equiv Cas^0\, g_e^{(p)} + Cas^+\, g_{e+1}^{(p)} + Cas^-\, g_{e-1}^{(p)} = 0\,. \tag{4.85}$$

We split each Casimir equation in terms of three differential operators Cas^0, Cas^+, Cas^-, that act on $g_e^{(p)}$, $g_{e+1}^{(p)}$ and $g_{e-1}^{(p)}$, respectively. In order to avoid cluttering, we have omitted the obvious e and p dependences of such operators. Their explicit form is

[2]Recall that the conformal blocks are even under $z \leftrightarrow \bar{z}$ exchange, that leaves u and v unchanged.

as follows:

$$
\begin{aligned}
Cas^0 &= \left(\frac{z-\bar z}{z\bar z}\right)^2\left(\Delta_0^{(a_e,b_e;c_e)}+(1+2p)(2p-2-e)-\frac{1}{2}\left(E^p_\ell-\varepsilon^p_e\right)\right)\\
&\quad -3p\,\frac{z-\bar z}{z\bar z}\times\Big((1-z)\partial_z-(1-\bar z)\partial_{\bar z}\Big)-p\,(1+2p)\,\frac{z+\bar z-2}{z\bar z} \qquad (4.86)\\
Cas^+ &= B_e\,\frac{z-\bar z}{z\bar z}\times\frac{z-\bar z}{z\bar z}L(b_{e+1})+(1+2p)\,B_e\,\frac{z+\bar z-2z\bar z}{z\bar z}\frac{1}{z\bar z}, \qquad (4.87)\\
Cas^- &= A^p_e\,\frac{z-\bar z}{z\bar z}\times(z-\bar z)L(a_{e-1})+(1+2p)\,A^p_e\,\frac{z+\bar z-2z\bar z}{z\bar z}. \qquad (4.88)
\end{aligned}
$$

Notice that the action of $\Delta_0^{(a_e,b_e;c_e)}$ in eq.(4.86) on $g_e^{(p)}$ is trivial and gives just the sum of the eigenvalues of the $\mathcal{F}^{\pm\,(a,b;c)}_{\rho_1,\,\rho_2}(z,\bar z)$ entering $g_e^{(p)}$. It is clear from the form of the ansatz (4.82) that the system (4.85) involves three different kinds of functions $\mathcal{F}$, with different values of a, b and c (actually only b and c differ, recall eq.(4.25)).

Using properties of hypergeometric functions, however, we can bring the Casimir system (4.85) into an algebraic system involving functions $\mathcal{F}^{-\,(a_e,b_e;c_e)}_{\rho_1+r,\,\rho_2+t}(z,\bar z)$ only, with different values of r and t, but crucially with the same values of a_e, b_e and c_e. In order to do that, it is useful to interpret each of the terms entering the definitions of Cas^0, Cas^+ and Cas^- as an operator acting on the functions $\mathcal{F}^-$. Their action can be reconstructed from the more fundamental operators provided in the appendix B. For each function $\mathcal{F}^-$ appearing in the ansatz (4.82), we have

$$
\begin{aligned}
Cas^0\,\mathcal{F}^{-(a,b;c)}_{\rho_1+m,\,\rho_2+n}(z,\bar z) &= \sum_{(r,t)\in\mathcal{R}_0} A^0_{r,t}(m,n)\,\mathcal{F}^{-\,(a,b;c)}_{\rho_1+m+r,\,\rho_2+n+t}(z,\bar z)\,, \qquad (4.89)\\
Cas^+\,\mathcal{F}^{-\,(a,b;c)}_{\rho_1+m,\,\rho_2+n}(z,\bar z) &= \sum_{(r,t)\in\mathcal{R}_+} A^+_{r,t}(m,n)\,\mathcal{F}^{-\,(a,b+1;c+1)}_{\rho_1+m+r,\,\rho_2+n+t}(z,\bar z)\,, \qquad (4.90)\\
Cas^-\,\mathcal{F}^{-\,(a,b;c)}_{\rho_1+m,\,\rho_2+n}(z,\bar z) &= \sum_{(r,t)\in\mathcal{R}_-} A^-_{r,t}(m,n)\,\mathcal{F}^{-\,(a,b-1;c-1)}_{\rho_1+m+r,\,\rho_2+n+t}(z,\bar z)\,, \qquad (4.91)
\end{aligned}
$$

where A^0, A^- and A^+ are coefficients that in general depend on all the parameters involved: a, b, Δ, ℓ, e and p but not on z and $\bar z$, namely they are just constants. For future purposes, in eqs.(4.89)-(4.91) we have only made explicit the dependence of A^0, A^- and A^+ on the integers m and n. The sum over (r,t) in each of the above terms runs over a given set of pairs of integers. We report in fig. 4.1 the values of (r,t) spanned in each of the three regions $\mathcal{R}_0$, $\mathcal{R}_+$ and $\mathcal{R}_-$. We do not report the explicit and quite lengthy expression of the coefficients $A^0_{r,t}$, $A^+_{r,t}$ and $A^-_{r,t}$, but we refer the reader again to appendix B where we provide all the necessary relations needed to derive them. Using eqs.(4.82) and (4.89)-(4.91), the Casimir system (4.85) can be rewritten in terms of only the functions $\mathcal{F}^-$ with the same set of coefficients: a_e, b_e

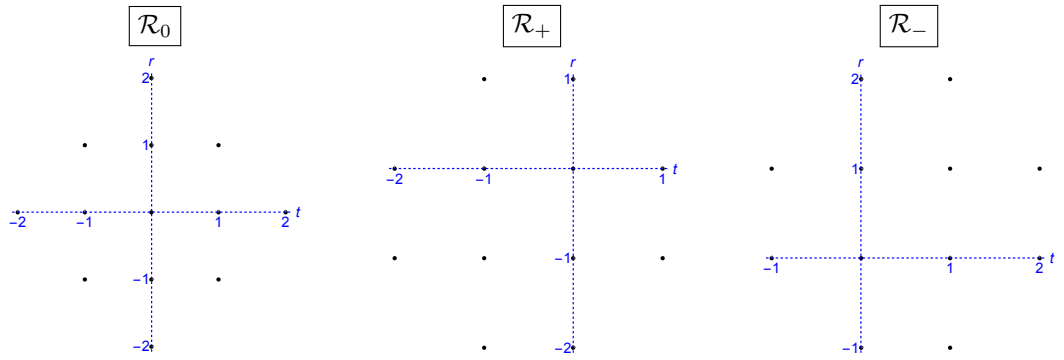

Figure 4.1: Set of points in the (r,t) plane forming the regions $\mathcal{R}_0$ (13 points), $\mathcal{R}_+$ (12 points) and $\mathcal{R}_-$ (12 points) defined in eqs.(4.89)-(4.91).

and c_e:[3]

$$\sum_{m,n}\Big(\sum_{(r,t)\in\mathcal{R}_0}A^0_{r,t}(m,n)\,c^e_{m,n}+\sum_{(r,t)\in\mathcal{R}_+}A^+_{r,t}(m,n)\,c^{e+1}_{m,n}+\sum_{(r,t)\in\mathcal{R}_-}A^-_{r,t}(m,n)\,c^{e-1}_{m,n}\Big)\mathcal{F}^{-\,(a_e,b_e;c_e)}_{\rho_1+m+r,\,\rho_2+n+t}=0\,. \tag{4.92}$$

The functions $\mathcal{F}^-$ appearing in eq.(4.92) are linearly independent among each other, since they all have a different asymptotic behaviour as $z,\bar{z}\to 0$. Hence the only way to satisfy eq.(4.92) is to demand that terms multiplying different $\mathcal{F}^-$ vanish on their own:

$$\begin{aligned}&\sum_{(r,t)\in\mathcal{R}_0}A^0_{r,t}(m'-r,n'-t)c^e_{m'-r,n'-t}+\sum_{(r,t)\in\mathcal{R}_+}A^+_{r,t}(m'-r,n'-t)c^{e+1}_{m'-r,n'-t}\\&+\sum_{(r,t)\in\mathcal{R}_-}A^-_{r,t}(m'-r,n'-t)c^{e-1}_{m'-r,n'-t}=0\,,\qquad\forall m',n',\quad e=0,\dots p\,,\end{aligned} \tag{4.93}$$

where $m'=m+r$, $n'=n+t$. The Casimir system is then reduced to the over-determined *linear algebraic system of equations* (4.93).

4.4.4 Solution of the System

In order to solve the system (4.93), we have to determine the range of values of (m,n) entering the ansatz (4.82), that also determines the size of the linear system. By rewriting the known $p=1$ and $p=2$ CBs found using the shadow formalism in the form of eq.(4.82), after some work we managed to deduce the range in (m,n) of the coefficients $c^e_{m,n}$ for any p (a posteriori extensively tested). For each value of e, the non-trivial coefficients $c^e_{m,n}$ span a two-dimensional lattice in the (m,n) plane. For each e, the shape of the lattice is an octagon, with p and e dependent edges. The generic octagon is depicted in fig. 4.2 with all the dimensions. The position of the

[3] It is understood that $c^{-1}_{m,n}=c^{p+1}_{m,n}=0$ in eq.(4.92).

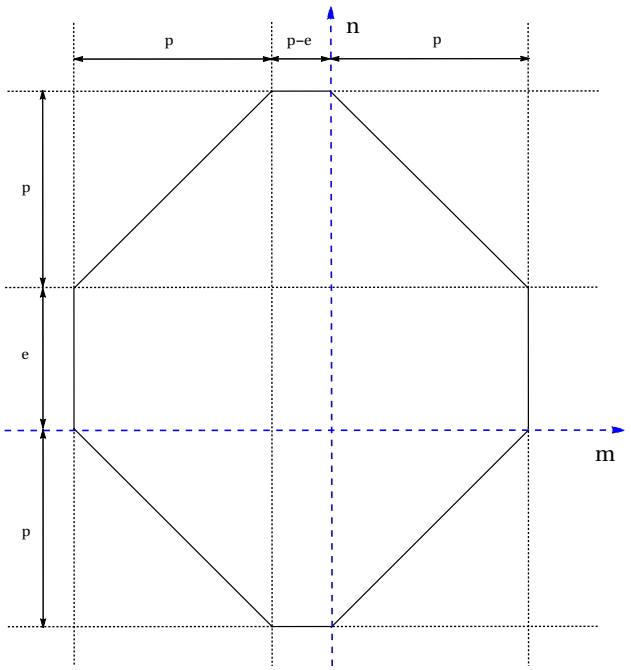

Figure 4.2: The dimensions of the generic octagon enclosing the lattice of non-vanishing coefficients $c^e_{m,n}$ entering the ansatz for mixed tensor CBs in eq.(4.101).

octagon in the (m, n) plane is given by

$$n_{min} = -\,p, \;\; n_{max} = e + p, \;\; m_{min} = e - 2\,p, \;\; m_{max} = p\,. \tag{4.94}$$

For $e = 0$ and $e = p$, the octagons collapse to hexagons. The number N^e_p of points inside a generic octagon is

$$N^e_p = 2p\,(2p - e) + (1 + e)\,(3p + 1 - e) \tag{4.95}$$

and correspond to the number of non-trivial coefficients $c^e_{m,n}$ entering the ansatz (4.82). The total number N_p of coefficients to be determined at level p is then

$$N_p \equiv \sum_{e=0}^{p} N^e_p = (1 + p)\left(1 + \frac{17}{6}\,p + \frac{25}{6}\,p^2\right). \tag{4.96}$$

The size of the linear system grows as p^3. The first values are $N_1 = 16$, $N_2 = 70$, $N_3 = 188$, $N_4 = 395$. For illustration, we report in fig. 4.3 the explicit lattice of non-trivial coefficients $c^e_{m,n}$ for $p = 3$.

The system (4.93) is always over-determined, since we have more equations than coefficients. It can be easily seen since the system (4.93) is spanned by the values (m', n') whose range is bigger than the range of $(m, n) \in Oct^{(p)}_e$ (spanning all the

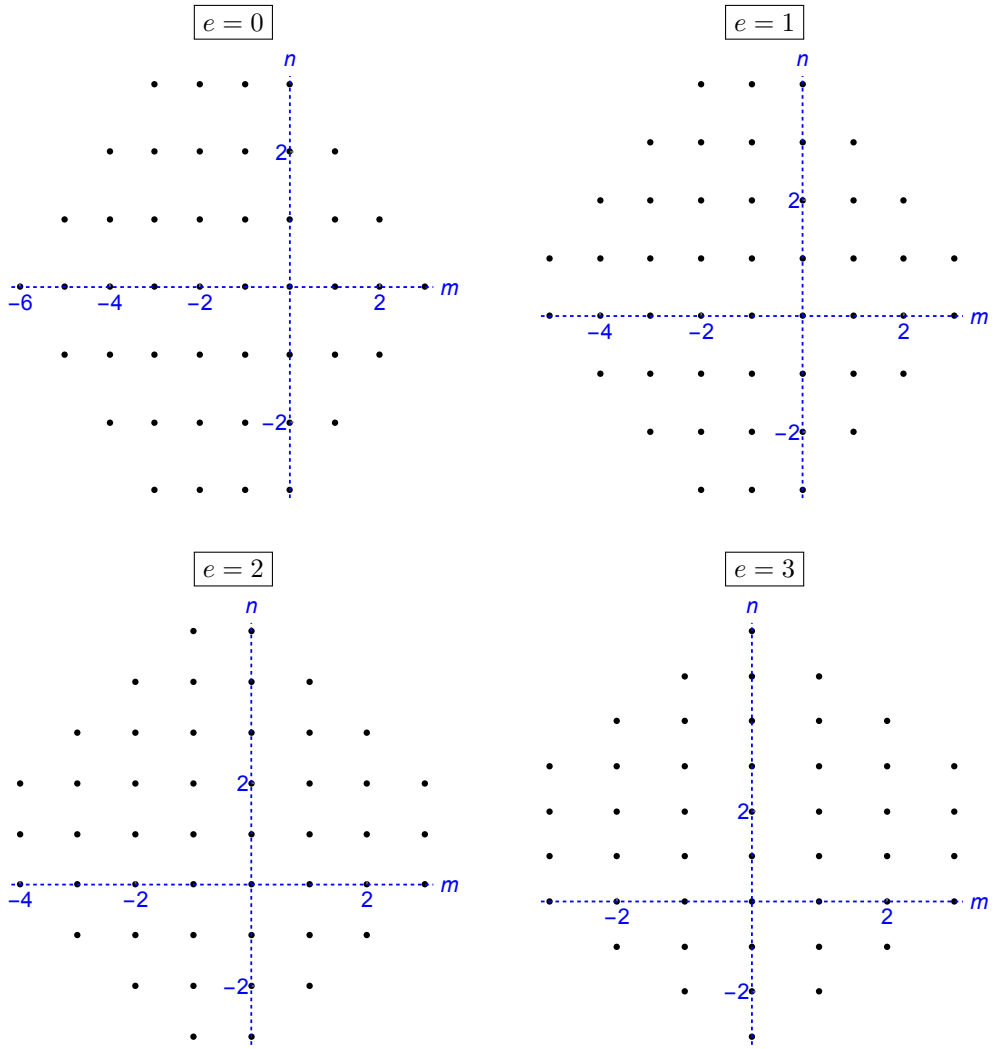

Figure 4.3: Set of non-vanishing coefficients $c^e_{m,n}$ (represented as black dots) entering the ansatz for mixed tensor CBs in eq.(4.101) for $p = 3$ and $e = 0, 1, 2, 3$. For $e = 0$ and $e = p$ the octagons collapse to hexagons.

coefficients to be determined) due to the presence of $(r, t) \in [-2, 2]$. There are only $N_p - 1$ linearly independent equations, since the system of Casimir equations can only determine conformal blocks up to an overall factor. The most important property of the system (4.93) is the following: while the number of equations, as we have seen, grows with p, the total number of coefficients $c^e_{m,n}$ entering any given equation in the system (4.93) does not. This is due to the "local nearest-neighbour" nature of the interaction between the blocks, for which at most three conformal blocks can enter the Casimir system (4.23), independently of the value of p. More precisely, all the equations (4.93) involve from a minimum of one coefficient $c^e_{m,n}$ up to a maximum of 37 ones. Thirty seven corresponds to the total number of coefficients A^0, A^+ and A^- entering eqs.(4.89)-(4.91) (see fig.4.1). The only coefficients that enter alone in some

equations[4] are the ones corresponding to the furthermost vertices of the hexagons, namely

$$c^p_{0,-p}, c^p_{0,2p}, c^0_{p,0}, c^0_{-2p,0}\,. \tag{4.97}$$

It is not difficult to understand that. For instance, let us take $n' = -2-p$ and $e = p$ in eq.(4.93), with m' generic. Since $n_{min} = -p$, a non-vanishing term can be obtained only by taking $t = -2$. Considering that $c^{p+1} = 0$ and $\mathcal{R}_-$ does not include $t = -2$ (see fig.4.1), this equation reduces to

$$A^0_{0,-2}(m,-p)|_{e=p}\, c^p_{m,-p} = 0\,, \quad \forall m\,, \tag{4.98}$$

where $m' = m$, since the point in $\mathcal{R}_0$ with $t = -2$ has $r = 0$. This equation forces all the coefficients $c^p_{m,-p}$ to vanish, unless the factor $A^0_{0,-2}(m,-p)$ vanishes on its own. One has

$$A^0_{0,-2}(m,n)|_{e=p} \propto (m+n+p)\Delta+(m-n-p)\ell+m^2+\frac{1}{2}m(p-2)+(n+p)(n+\frac{3}{2}p-2)\,.$$

This factor is generally non-vanishing, unless $m = 0$ and $n = -p$, in which case it vanishes for any Δ, ℓ and p. In this way eq.(4.98) selects $c^p_{0,-p}$ as the only non-vanishing coefficient at level $n = -p$ for $e = p$. Notice that it is crucial that $A^0_{0,-2}(m,n)|_{e=p}$ vanishes automatically for a given pair (m,n), otherwise either the whole set of equations would only admit the trivial solution $c^e_{m,n} = 0$, or the system would be infinite dimensional. A similar reasoning applies for the other three coefficients. One has in particular

$$\begin{aligned} A^0_{0,2}(0,2p)|_{e=p} \qquad & c^p_{0,2p} = 0\,, \\ A^0_{2,0}(p,0)|_{e=0} \qquad & c^0_{p,0} = 0\,, \\ A^0_{-2,0}(-2p,0)|_{e=0} \qquad & c^0_{-2p,0} = 0\,, \end{aligned} \tag{4.99}$$

that are automatically satisfied because the three coefficients $A^0_{0,2}$, $A^0_{2,0}$ and $A^0_{-2,0}$ vanish when evaluated for the specific values reported in eq.(4.99) for any Δ, ℓ and p.

The system (4.93) is efficiently solved by extracting a subset of $N_p - 1$ linearly independent equations. This can be done by fixing the values $(r,t) = (r^*,t^*)$ entering the definitions of (m',n'). There are 4 very special subsets of the $N_p - 1$ equations (corresponding to very specific values (r^*,t^*)) which allows us to determine the solution iteratively starting from eq.(4.93). They correspond to a solution where one of the four coefficients (4.97) is left undetermined, in other words (r^*,t^*) can be set to be $(0,-2)$, $(0,2)$, $(2,0)$ or $(-2,0)$. For instance, if we choose $c_0 \equiv c^p_{0,-p}$ as the undetermined

[4] Of course, this is clear only a posteriori after one has understood the pattern of which coefficients $c^e_{m,n}$ are non-vanishing, as explained above.

coefficient, a recursion relation is found from eq.(4.93) by just singling out the term with $t=-2$ in A^0 and setting $(r^*,t^*)=(0,-2)$. Such a choice leads to $m'=m$, $n'=n-2$, and one finally gets[5]

$$\begin{aligned} -A^0_{0,-2}(m,n)c^e_{m,n} &= \sum_{\substack{(r,t)\in\mathcal{R}_0\\(r,t)\neq(0,-2)}} A^0_{r,t}(m-r,n-2-t)c^e_{m-r,n-2-t} \\ &+ \sum_{(r,t)\in\mathcal{R}_+} A^+_{r,t}(m-r,n-2-t)c^{e+1}_{m-r,n-2-t} \\ &+ \sum_{(r,t)\in\mathcal{R}_-} A^-_{r,t}(m-r,n-2-t)c^{e-1}_{m-r,n-2-t}\,. \end{aligned} \tag{4.100}$$

It is understood in eq.(4.100) that $c^e_{m,n}=0$ if the set (m,n) lies outside the e-octagon of coefficients. The recursion (4.100) allows us to determine all the coefficients $c^e_{m,n}$ at a given $e=e_0$ and $n=n_0$ in terms of the ones $c^e_{m,n}$ with $n<n_0$ and c^e_{m,n_0} with $e>e_0$. Hence, starting from c_0, one can determine all $c^e_{m,n}$ as a function of c_0 for any p. The overall normalization of the CBs is clearly irrelevant and can be reabsorbed in a redefinition of the OPE coefficients. However, some care should be paid in the choice of c_0 if one wants to avoid the appearance of spurious divergencies in the CBs for specific values of ℓ and Δ. These divergencies are removed by a proper Δ and ℓ dependent rescaling of c_0. From eq.(4.93) one can easily write the three other relations similar to eq.(4.100) to determine recursively $c^e_{m,n}$ starting from $c^p_{0,2p}$, $c^0_{p,0}$ or $c^0_{-2p,0}$.

We can finally write down the full analytic solution for the CBs $G^{(p)}_e$:

$$G^{(p)}_e(z,\bar z)=\left(\frac{z\bar z}{z-\bar z}\right)^{2p+1}\sum_{(m,n)\in Oct^{(p)}_e} c^e_{m,n}\mathcal{F}^{-\,(a_e,b_e;c_e)}_{\frac{\Delta+\ell+\frac{p}{2}}{2}+m,\,\frac{\Delta-\ell+\frac{p}{2}}{2}-(p+1)+n}(z,\bar z), \tag{4.101}$$

where $c^e_{m,n}$ satisfy the recursion relation (4.100) (or any other among the four possible ones) and (m,n) runs over the points within the e-octagon depicted in fig.4.2.

A similar analysis can be performed for the conjugate blocks $\overline{G}^{(p)}_e$. We do not report here the detailed derivation that is logically identical to the one above, but just the final solution. We notice that there is a relation among the CBs $G^{(p)}_{p-e}$ and $\overline{G}^{(p)}_e$. More specifically, one has

$$\overline{G}^{(p)}_e(z,\bar z)=\left(\frac{z\bar z}{z-\bar z}\right)^{2p+1}\sum_{(m,n)\in Oct^{(p)}_{p-e}} \bar c^e_{m,n}\mathcal{F}^{-\,(a_e,b_e;c_e)}_{\frac{\Delta+\ell-\frac{p}{2}}{2}+e+m,\,\frac{\Delta-\ell-\frac{p}{2}}{2}+e-(p+1)+n}(z,\bar z). \tag{4.102}$$

[5] Now that the range of the coefficients in (m,n) has been established, there is no more reason to distinguish (m,n) from (m',n').

where

$$\bar{c}^{e}_{m,n}(a,b,\Delta,\ell,p) = 4^{e} c^{p-e}_{m,n}\Big(-a+\frac{p}{2},-b-\frac{p}{2},\Delta,\ell,p\Big)\,. \tag{4.103}$$

Generating the full explicit solution from eq.(4.100) can be computationally quite demanding for large values of p. While at the beginning of the recursion only a few terms in eq.(4.100) are non-vanishing, for $p \geq 2$ one reaches a point where all 36 terms in eq.(4.100) contribute. We have obtained fully explicit solutions for any a, b, Δ and ℓ for $p = 1, 2, 3, 4$. As we mentioned, we have alternatively computed the blocks $G^{(p)}_{e}$ for $p = 1, 2$ and $\overline{G}^{(p)}_{e}$ for $p = 1$ using the shadow formalism and found complete agreement with the solutions (4.101) and (4.102) obtained via the Casimir system. By choosing specific values for the parameters a and b, we also have determined the coefficients $c^{e}_{m,n}$ for $p = 8$, i.e. the value of p that is obtained in the 4-point function of four energy momentum tensors, see eq.(3.7). For concreteness, we report in appendix C the explicit form of the 16 coefficients $c^{e}_{m,n}$ for $p = 1$ and $a = -b = 1/2$. The general form of $c^{e}_{m,n}$ for $p = 1, 2, 3, 4$ can be downloaded from https://sites.google.com/site/dskarateev/downloads.

It is important to remind the reader that the CBs $G^{(p)}_{e}$ computed here are supposed to be the seed blocks for possibly other 4-point correlation functions, whose CBs are determined by acting with given operators on $G^{(p)}_{e}$ [36]. The complexity of the form of the blocks $G^{(p)}_{e}$ at high p is somehow compensated by the fact that the operators one has to act with become simpler and simpler, the higher is p. An example should clarify the point. Let us consider a 4-point function of spin two operators. In this case, one has to determine conformal blocks associated to the exchange of operators $\mathcal{O}^{(\ell,\ell+p)}$ (and $\overline{\mathcal{O}}^{(\ell+p,\ell)}$) for $p = 0, 2, 4, 6, 8$ (and any ℓ). The conformal blocks associated to the traceless symmetric operators are obtained by applying up to 8 derivative operators in several different combinations to the scalar CB $G^{(0)}_{0}$. Despite the seed block is very simple, the final blocks are given by (many) complicated sum of derivatives of $G^{(0)}_{0}$. The $p = 8$ CBs, instead, are essentially determined by the very complicated $G^{(8)}_{e}$ blocks, but no significant extra complications come from the external operators. An example of such phenomenon is the four fermion correlator shown (though in a less significative way) in section 3.5. For any given 4-point function, after the use of the differential operators introduced in Chapter 3, there is no need to compute the coefficients $c^{e}_{m,n}$ for any a and b but only for the values of interest. This considerably simplifies the expression of $c^{e}_{m,n}$. In bootstrap applications, one might also consider to compute the $c^{e}_{m,n}$'s numerically by fixing a, b, ℓ and Δ. Needless to say, when the problem becomes purely numerical, the coefficients $c^{e}_{m,n}$ are found immediately for any sensible value of p.

4.4.5 Analogy with Scalar Conformal Blocks in Even Dimensions

It is worth to point out in more detail some similarities between the CBs $G_e^{(p)}$ for mixed symmetry tensors computed above and the scalar conformal blocks G_d in $d > 2$ even space-time dimensions. The quadratic Casimir equation for scalar CBs in any number of dimensions is

$$\Delta_{d-2}^{(a,b;0)} G_d(z,\bar{z}) = \frac{1}{2} E_\ell(d) G_d(z,\bar{z})\,, \tag{4.104}$$

where

$$E_\ell(d) = \Delta\,(\Delta - d) + \ell(\ell + d - 2) \tag{4.105}$$

is the quadratic Casimir eigenvalue for traceless symmetric tensors. The explicit analytical form of scalar blocks in $d = 2, 4, 6$ dimensions has been found in refs. [13, 14]. Subsequently, the same authors found a relation between scalar blocks in any even space-time dimensionality, eq.(4.36) of ref. [15], that allows us to iteratively determine G_d for any d, starting from G_2. In particular, the $d = 4$ and $d = 6$ solutions found in ref. [14] have the form

$$G_d(z,\bar{z}) = \left(\frac{z\bar{z}}{z-\bar{z}}\right)^{d-3} g_d(z,\bar{z})\,, \qquad g_d(z,\bar{z}) = \sum_{m,n} x_{m,n} \mathcal{F}^{-\,(a,b;0)}_{\frac{\Delta+\ell}{2}+m,\,\frac{\Delta-\ell+2-d}{2}+n}(z,\bar{z}), \tag{4.106}$$

where a and b are as in eq.(4.8) with $p = 0$ and $x_{m,n}$ are coefficients that in general depend on Δ, l, a and b. In $d = 4$ there is only one non-vanishing coefficient centered at $(m,n) = (0,0)$, while in $d = 6$ there are five of them. They are at $(m,n) = (0,-1)$, $(m,n) = (-1,0)$, $(m,n) = (0,0)$, $(m,n) = (1,0)$ and $(m,n) = (0,1)$. These five points form a quadrilateral-shaped (a slanted square) in the (m,n) plane, centered at the origin. The explicit form of the coefficients $x_{m,n}$ is known, but it will not be needed in what follows.[6] It is natural to expect that eq.(4.106) should apply for any even $d \geq 4$, with a number of non-vanishing coefficients that increases with d.[7] This is not difficult to prove. From the first relation in eq.(4.84) we can get the form of the Casimir equation for the function $g_d(z,\bar{z})$ defined in eq.(4.106), that can be written as

$$\left(\frac{1}{\bar{z}} - \frac{1}{z}\right)\left(\Delta_0^{(a,b;0)} + 6 - 2d - \frac{1}{2}E_\ell(d)\right) g_d = (d-4)\Big((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\Big) g_d\,. \tag{4.107}$$

Using the techniques explained in subsection 4.4.3 and the results of appendix B, it is now straightforward to identify which is the range of (m,n) of the non-vanishing

[6] En passant, notice that there is a typo in eq.(2.20) of ref. [14] where the block G_6 is reported. In the denominator appearing in the last row of that equation, one should replace $(\Delta+\ell-4)(\Delta+\ell-6) \to (\Delta-\ell-4)(\Delta-\ell-6)$.

[7] See also ref. [27], where similar considerations were conjectured.

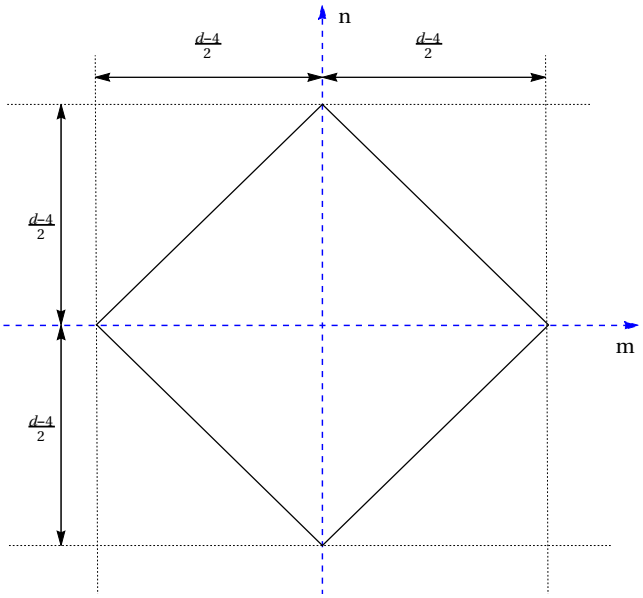

Figure 4.4: The dimensions of the generic slanted square enclosing the lattice of non-vanishing coefficients $x_{m,n}$ entering the ansatz for scalar symmetric CBs in eq.(4.106).

coefficients $x_{m,n}$ for any d (see fig.4.4).[8] In d dimensions, the minimum and maximum values of m and n are given by

$$n_{min} = \frac{4-d}{2}\,, \quad n_{max} = \frac{d-4}{2}\,, \quad m_{min} = \frac{4-d}{2}\,, \quad m_{max} = \frac{d-4}{2}\,. \tag{4.108}$$

The number $\widetilde{N}_d$ of coefficients $x_{m,n}$ entering the ansatz (4.106) for scalar blocks in d even space-time dimensions is easily computed by counting the number of lattice points enclosed in the slanted square. We have

$$\widetilde{N}_d = \frac{d^2}{2} - 3d + 5\,. \tag{4.109}$$

For large d, $\widetilde{N}_d \propto d^2$ and matches the behavior of $N_e^p \propto p^2$ for large p in eq.(4.95).

In light of the above analogy between scalar CBs G_d in even d dimensions and mixed tensor CBs $G_e^{(p)}$ in four dimensions, it would be interesting to investigate whether there exist a set of differential operators that link the blocks $G_e^{(p+1)}$ (or $G_e^{(p+2)}$) to the blocks $G_e^{(p)}$, in analogy to the operator (4.35) of ref. [15] relating G_{d+2} to G_d. It would be very useful to find, in this or some other way, a more compact expression for the blocks $G_e^{(p)}$.

[8] Alternatively, one might use eq.(4.36) of ref. [15] to compute G_d and then recast it in the form (4.106).

Let us finally emphasize a technical, but relevant, point where the analogy between G_d in d dimensions and $G_e^{(p)}$ in 4 dimensions does not hold. A careful reader might have noticed that in the Casimir equation for g_d the term proportional to $(z+\bar{z})-2$, namely the third term in the r.h.s. of the first equation in eq.(4.84), automatically vanishes. Indeed, if we did not know the power $d-3$ in the ansatz (4.106), we could have guessed it by demanding that term to vanish. On the contrary, no such simple guess seems to be possible for the power $2p+1$ entering $G_e^{(p)}$, given also the appearance of the operator L defined in eq.(4.84). As discussed, we have fixed the power $2p+1$ by means of the shadow formalism.

Chapter 5

5 Numerical Bounds on OPE coefficients

In this chapter and the next we will proceed with some numerical studies of the bootstrap equation. In the present chapter we will numerically study bounds on OPE coefficients in 4D. We will only study scalar correlators, with the motivation of gaining some insight into the behaviour of CFTs with large amounts of matter. We will focus our attention on studying bounds on the coefficient κ of the two-point function between two conserved currents associated with a global symmetry of a CFT, and as we will see we can relate this to a specific OPE coefficient within the theory. Our main motivation comes from theoretical considerations in the context of composite Higgs models, in which the CFT is the hidden sector which gives rise to the Higgs, and a subgroup of the global symmetry of the CFT is weakly gauged in order to get the Standard Model gauge interactions. These composite Higgs models are related, through the AdS/CFT correspondence, to Randall-Sundrum theories [96] with matter in the bulk, which are promising solutions to the gauge hierarchy problem. Particularly interesting are the models where the Higgs is a pseudo Nambu-Goldstone Boson (pNGB) of an approximate spontaneously broken global symmetry of the CFT, which corresponds to gauge-Higgs unification models in 5D warped theories. Neither the UV completion of the 5D models nor the explicit form of the 4D CFT is known so far. Calculability of the dual 5D models would require that the CFT is in some large N limit, but this is not a necessary requirement. On the contrary, various phenomenological bounds tend to favour models at small N, so we will not assume the existence of a large N limit in the CFT. Constructing such a CFT is not a trivial task, so we look for possible consistency relations. When the global symmetry of the CFT is gauged, the coefficient κ of the current-current two-point function governs the leading contribution of the CFT to the one-loop evolution of the corresponding gauge coupling. This contribution should not lead to Landau poles for the SM gauge couplings. We also require that the CFT has no relevant deformations, in order not to reintroduce the hierarchy problem. This leads to the constraint that the dimension of the lowest-lying scalar singlet operator should be $\Delta_S \geq 4$. All our considerations apply independently of the pNGB nature or not of the Higgs.

Motivated by the above considerations, we extend the analysis of ref. [50], where lower bounds on κ have been set starting from crossing constraints imposed on a four-point function of scalar operators in the fundamental representation of SO(N) and SU(N), in two ways. First, we see how the bounds found in ref. [50] are modified when the lowest-lying singlet scalar operator is assumed to have a scaling dimension $\Delta_S \geq \Delta_{\min}$, where we choose $\Delta_{\min} = 2, 3, 4$ for concreteness. Second, we extend the analysis to non-simple groups of the form SO(N)×SO(M). We study non-simple groups because they easily allow us to generalize the bounds for the groups SO(N) and SU(N), which are obtained by considering a single field in the fundamental representation of the group, to multiple fields. Analogous to what was found in ref. [50] for singlet operators, the lower bounds on vector currents for SU(N) groups that we find are, within the numerical precision, identical to those obtained for SO($2N$). Hence we only report lower bounds for SO(N) and SO(N)×SO(M) global symmetries. We have derived the bootstrap equations also for groups of the form SO(N)×SU(M), but no bounds are reported for this case. Unfortunately, the numerical implementation of these equations turned out to be too demanding for our computing resources. Given the above equality between SO($2N$) and SU(N) bounds, we believe that the lower bounds for SO(N)×SU(M) should be very similar (if not identical) to those that we have obtained for SO(N)×SO($2M$). In addition to that, we study the constraints on the OPE coefficients of spin $\ell = 2$ and $\ell = 4$ tensors coming from two identical scalar operators ϕ, as a function of the scaling dimension of the tensors, in the general case in which no global symmetry is assumed. In analogy to the vector-current case, we analyze how these bounds change when one assumes a lower bound on the scale dimension of the scalar operators appearing in the $\phi\phi$ OPE.

All our numerical results are based on semi-definite programming methods, as introduced in ref. [50] in the context of the bootstrap approach, with a few technical modifications which are discussed in subsection 5.2.

We will start this chapter by giving a phenomenological motivation for this study. In section 5.2 we will briefly review how the bounds on OPE coefficients are obtained, for more details on the specific techniques used we refer the reader to [50]. In section 5.3 we report our results for the OPE coefficients of tensor $\ell = 2$ and $\ell = 4$ operators. Section 5.4 contains the most important results of the paper. We report here the lower bounds on κ associated with SO($2N$) (or SU(N)) vector currents, when the global symmetry of the CFT is SO($2N$) (or SU(N)) and $SO(2N) \times SO(M)$. [1] In appendix D we report the crossing equations for SO(N)×SO(M) and SO(N)×SU(M).

[1] Results for SO(N) groups with odd N are analogous to those for SO($2N$) and do not need any special treatment.

5.1 Motivation for a Gap in the Scalar Operator Dimension

The motivation to consider CFTs with a gap in the scaling dimension of scalar gauge-singlet operators comes from applications in the context of physics beyond the Standard Model (SM) that addresses the gauge hierarchy problem. The latter can be formulated from a CFT point of view, see e.g. ref. [9]. Neglecting the cosmological constant, the SM can be seen as an approximate CFT with one relevant deformation of classical mass dimension $\Delta_{H^\dagger H} = 2$, corresponding to the Higgs mass term $H^\dagger H$. Relevant deformations grow in going from the UV towards the IR. If we assume that the Higgs mass term is generated at some high scale Λ_{UV}, we would expect from naturalness that the Higgs mass-squared term is of order $\Lambda_{UV}^{4-\Delta_{H^\dagger H}} = \Lambda_{UV}^2$ in the IR. There are essentially two ways to solve this hierarchy problem: i) invoke additional symmetries that keep the relevant deformation small in the IR (e.g. supersymmetry); ii) assume that the Higgs is a composite field of a strongly interacting sector, in which case the operator $H^\dagger H$ can have a large anomalous dimension that makes it effectively marginal or irrelevant.

A model along the lines of ii), conformal technicolor [97], where the strongly coupled sector is assumed to be a CFT in the UV, was in fact the motivation for the pioneering work [9]. Conformal technicolor is an interesting attempt to solve one of the long-standing problems of standard technicolor theories: how to reconcile the top mass with Flavour Changing Neutral Current (FCNC) bounds. In order to get a sizable top mass and at the same time avoid dangerous FCNCs, one has to demand that the scale dimension Δ_H of the Higgs field H is as close to one as possible. In order not to reintroduce the hierarchy problem, however, one has to keep $\Delta_{H^\dagger H} \gtrsim 4$ at the same time. The analyses in refs. [9, 50] have shown that generally these two conditions are in tension and that one needs $\Delta_H \gtrsim 1.52$ in order to have $\Delta_{H^\dagger H} \gtrsim 4$.

An alternative, phenomenologically more promising, solution is to rely on a different mechanism to generate SM fermion masses: partial compositeness [98]. To this end, one assumes that the SM fermions mix with fermion resonances of the strongly coupled sector. Due to this mixing, SM vectors and fermions become partially composite. In particular, the lighter the SM fermions are, the weaker is the mixing. This simple, yet remarkable, observation allows one to significantly alleviate most flavor bounds. The Yukawa couplings are effective couplings that arise from the mixing terms once the strongly coupled states are integrated out.

This idea is particularly appealing when one assumes that the strongly coupled sector is an approximate CFT spontaneously broken at some scale μ. In this case, the hierarchy

of the SM Yukawa couplings is naturally obtained by assigning different scale dimensions Δ_ψ^i to the fermion operators mixing with the different SM fermions [99]. In particular, there is no longer the need to keep Δ_H close to one since the effective size of the SM Yukawa couplings is governed by Δ_ψ^i. One assumes that $\Delta_\psi^i > 5/2$ for all SM fermions except the top, so that the mixing terms are irrelevant deformations of the CFT and naturally give rise to suppressed Yukawa couplings in the IR. For the top, on the other hand, one assumes that $\Delta_\psi^t \simeq 5/2$, corresponding to a nearly marginal deformation of the CFT.

One might wonder whether CFTs with all the necessary requirements to give rise to theoretically and phenomenologically viable composite Higgs models exist at all. A possible issue might arise in weakly gauging the SM subgroup of the global symmetry of the CFT. Since partial compositeness requires a fermion operator in the CFT for each SM fermion, dangerous Landau poles can potentially appear in the theory. Indeed, it has recently been shown that Landau poles represent the main obstruction in obtaining UV completions of composite Higgs models with a pNGB Higgs, based on supersymmetry [100]. It is then of primary importance to try to understand if and at what scale Landau poles will arise. In theories with a pNGB Higgs, the relevant deformation $H^\dagger H$ can naturally be small, since it is protected by a shift symmetry. Moreover, it is not defined in the UV, where the global symmetry is restored. Nevertheless, in order not to introduce other possible fine-tunings, one should demand that any scalar operator which is not protected by any symmetry, namely which is neutral under all possible global symmetries of the CFT, should be marginal or irrelevant.

Summarizing, we can identify four properties that a CFT needs to have for a theoretically and phenomenologically viable composite Higgs model with partial compositeness:

1. A global symmetry $G \supseteq G_{\rm SM} = \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$.
2. No scalar operator with dimension $\Delta < 4$ which is neutral under G.
3. No Landau poles for the SM gauge couplings below the scale $\Lambda_{\rm UV}$ when we gauge $G_{\rm SM}$.[2]
4. The presence of fermion operators with $\Delta_\psi^i \geq 5/2$ in some representation of G, such that some of its components can mix with each of the SM fermion fields. At least one fermion operator should have dimension $\simeq 5/2$.[3]

Of course, these are only necessary but not sufficient conditions to get a viable CFT. In particular, one might want to address the mechanism which gives rise to the spontaneous breaking of the conformal symmetry as well as of the global symmetry in CFTs with a pNGB Higgs.

[2] Ideally, we might want to have $\Lambda_{\rm UV} \sim M_{\rm Planck}$.
[3] The right-handed top, in principle, might be directly identified with a field of the CFT.

The consistency of a CFT which fulfils the above four conditions can be checked using crossing symmetry of four-point functions of the CFT. The first and second condition can be imposed by hand, assuming the existence of the global symmetry and that the lowest-dimensional scalar operator in the singlet channel has dimension $\Delta_S \geq 4$. One can extract information on the third condition by analyzing the bounds on the coefficients of current-current two-point functions. Finally, the fourth condition can again be implemented by assumption. The ideal configuration would be to analyze four-point functions involving fermion operators, which by assumption should appear in the CFT, and to extract any possible information from these correlators. Although this is in principle possible to do, correlation functions involving fermions in a non-supersymmetric setting have not been worked out so far. Postponing to a future project the analysis of such correlation functions, in this paper we start to address these issues by replacing fermions with scalars with dimension $1 \leq d < 2$ in the third requirement.

Let us estimate how severe the Landau pole problem can be in the simplest composite Higgs model where the Higgs is the pNGB associated with the SO(5)$\to$SO(4) symmetry breaking pattern. Let us consider the SU$(3)_c$ coupling g_c, because it runs fastest and possibly leads to the lowest-lying Landau pole, and let us denote by

$$\beta_{\rm CFT} = g_c^3 \frac{\kappa}{16\pi^2} \tag{5.1}$$

the CFT contribution to its one-loop β-function. Assuming that the only non-SM fields which are charged under SU$(3)_c$ arise from the CFT, a Landau pole develops at around

$$\Lambda_L \simeq \mu \exp\left(\frac{2\pi}{(\kappa-7)\alpha_c(\mu)}\right) \tag{5.2}$$

for $\kappa > 7$, where $\alpha_c = g_c^2/(4\pi)$ and $\mu \sim \mathcal{O}(\text{TeV})$ is the scale where the CFT breaks spontaneously. Composite fermions coming from the CFT and mixing with SM fermions must be color triplets and in representations of SO(5) that give rise to electroweak SU(2) doublets and singlets. If we assume them to be in the fundamental representation $\mathbf{5}$ of SO(5), the fermion components in a given $\mathbf{5}$ can mix with both the left-handed and right-handed components of a quark field. We then need $n_f = 6$ $\mathbf{5}$s, one for each quark field, for a total of $6 \times 5 = 30$ SU$(3)_c$ triplet Dirac fermions. In order to have an idea of the scales which are involved, it is useful to consider the (unrealistic) limit of a free CFT. In this case, we get

$$\kappa_{\rm free} = \frac{2}{3} \times 30 = 20\,, \tag{5.3}$$

corresponding to

$$\Lambda_L \sim 200 \ \ \text{TeV}\,, \tag{5.4}$$

for $\mu \simeq 1$ TeV. It is clearly very important to set lower bounds on κ in a generic CFT, given the exponential sensitivity of Λ_L on this quantity.

In the following, we will analyze bounds on the coefficients for SO$(2N)$ (or, equivalently, SU(N)) currents obtained from four-point functions of scalar operators in the fundamental representation of the group in presence of a gap in the operator dimension in the scalar gauge-singlet channel. In order to mimic the presence of more than one field multiplet, we will also consider fields in the bi-fundamental representation of the product group SO$(2N)\times$SO(M).

5.2 Bounds on OPE Coefficients and Numerical Implementation

The bootstrap equation (2.48) has originally been used to set bounds on the scalar operator dimensions that can appear in a CFT. Shortly after that, ref. [45] has shown how to obtain bounds on the OPE coefficient $\lambda_{\mathcal{O}_0}$ of an operator $\mathcal{O}_0$ appearing in the $\phi\phi$ OPE. Let us assume that a linear functional α can be found, such that

$$\alpha(F_{d,\Delta_0,\ell_0}) = 1\,, \qquad \alpha(F_{d,\Delta,\ell}) \geq 0 \quad \forall(\Delta,\ell) \neq (\Delta_0,\ell_0)\,. \tag{5.5}$$

And are explicitely writing the dependance on the dimension of the external operators $[\phi] = d$ as well as the dimension Δ and spin ℓ of the exchanged operator. Applying such a functional to eq. (2.48) gives

$$|\lambda_{\mathcal{O}_0}|^2 = \alpha(1) - \sum_{(\Delta,\ell)\neq(\Delta_0,\ell_0)} |\lambda_{\mathcal{O}}|^2 \alpha(F_{d,\Delta,\ell}) \;\leq\; \alpha(1)\,. \tag{5.6}$$

The optimal bound is obtained by minimizing $\alpha(1)$ among all the functionals α which satisfy eq. (5.5). One can use the functional α also to rule out the existence of certain CFTs. For instance, if under a certain assumption on the CFT data one finds a functional α and an operator $\mathcal{O}_0$ for which $|\lambda_{\mathcal{O}_0}|^2 < 0$, then that CFT is ruled out.

The above procedure is easily generalized in presence of global symmetries. Let's briefly recap the form of the bootstrap equations in the case of global symemtries for completeness. When the CFT has a global symmetry G, the bootstrap analysis can be generalized using scalar fields ϕ_a (real or complex) in some representation r of G [48]. The symmetry implies that all the field components of the multiplet must have the same dimension d. Moreover, it allows us to easily classify the operators appearing in the $\phi_a\phi_b$ OPE in terms of the irreducible representations appearing in the product $r \otimes r$. A similar analysis applies for complex fields in the $\phi_a\phi_b^\dagger$ OPE. It is useful to

introduce another function, similar to the F of eq. (2.49):

$$H_{d,\Delta,l}(z,\bar{z}) \equiv \frac{v^d g_{\Delta,l}(u,v) + u^d g_{\Delta,l}(v,u)}{u^d + v^d}\,. \tag{5.7}$$

In presence of a global symmetry G, eq. (2.48) generalizes to a system of $P+Q$ equations of the form

$$\begin{aligned}
&\sum_i \eta^p_{F,i} \sum_{\mathcal{O}\in r_i} |\lambda_{\mathcal{O}_i}|^2 F_{d,\Delta,\ell}(z,\bar{z}) = \omega^p_F\,, \quad p = 1,\dots,P\,,\\
&\sum_i \eta^q_{H,i} \sum_{\mathcal{O}\in r_i} |\lambda_{\mathcal{O}_i}|^2 H_{d,\Delta,\ell}(z,\bar{z}) = \omega^q_H\,, \quad q = P+1,\dots,P+Q\,.
\end{aligned} \tag{5.8}$$

Here, i runs over all possible irreducible representations that can appear in the s- and t-channel decomposition, $\eta^p_{F,i}$ and $\eta^q_{H,i}$ are numerical factors that depend on G and $\lambda_{\mathcal{O}_i}$ is a short-hand notation for the $\langle\phi_a\phi_b\mathcal{O}\rangle$ three-point function coefficient. Furthermore, $\omega^p_F = 1$ and $\omega^q_H = -1$ if the singlet representation appears in the left-hand side of eq. (5.8), and $\omega^p_F = \omega^q_H = 0$ otherwise. The explicit form of eq. (5.8) for the cases of interest will be given in section 5.4 and appendix D.

Let us assume that we want to bound the OPE coefficient of an operator $\mathcal{O}_0$ with dimension Δ_0 and spin ℓ_0 in the representation r_1. We look for a set of linear functionals α_m $(m = 1,\dots,P+Q)$ such that

$$\begin{aligned}
&\sum_{p=1}^{P} \alpha_p\Big(\eta^p_{F,1} F_{d,\Delta_0,\ell_0}\Big) + \sum_{q=P+1}^{P+Q} \alpha_q\Big(\eta^q_{H,1} H_{d,\Delta_0,\ell_0}\Big) = 1\,,\\
&\sum_{p=1}^{P} \alpha_p\Big(\eta^p_{F,1} F_{d,\Delta,\ell}\Big) + \sum_{q=P+1}^{P+Q} \alpha_q\Big(\eta^q_{H,1} H_{d,\Delta,\ell}\Big) \geq 0\,, \qquad \forall(\Delta,\ell) \neq (\Delta_0,\ell_0)\,,\\
&\sum_{p=1}^{P} \alpha_p\Big(\eta^p_{F,i} F_{d,\Delta,\ell}\Big) + \sum_{q=P+1}^{P+Q} \alpha_q\Big(\eta^q_{H,i} H_{d,\Delta,\ell}\Big) \geq 0\,, \qquad \forall(\Delta,\ell)\,, i \neq 1\,.
\end{aligned} \tag{5.9}$$

Applying such a functional to eq. (5.8) gives

$$|\lambda_{\mathcal{O}_0}|^2 \leq \sum_{p=1}^{P} \alpha_p(\omega^F_p) + \sum_{q=P+1}^{P+Q} \alpha_q(\omega^H_q)\,. \tag{5.10}$$

In our paper, we will mainly be interested in the OPE coefficient associated with a conserved vector current J_μ of a global symmetry, which has $\Delta_0 = 3$ and $\ell_0 = 1$. We shall denote this coefficient by λ_J. As we will discuss in section 5.4 (see eq. (5.28)), upper bounds on $|\lambda_J|^2$ turn into lower bounds on the coefficient κ introduced in eq. (5.1).

Following ref. [9], we consider functionals that act as linear combinations of derivatives

on a generic function $f(z,\bar{z})$,

$$\alpha(f(z,\bar{z})) = \sum_{m+n\leq 2k} a_{mn}\partial_z^m\partial_{\bar{z}}^n f(z,\bar{z})|_{z=\bar{z}=1/2}\,, \tag{5.11}$$

where a_{mn} are real coefficients. Due to the symmetries of the conformal blocks F and H, the sum can be restricted to $m<n$ and even values of $m+n$ when α acts on F, and $m<n$ and odd values of $m+n$ when it acts on H.

We numerically search for functionals α which satisfy eqs. (5.5) and (5.9) by following the method developed in refs. [46,50]. We refer the reader to these references for further details. For this method, one approximates the derivatives of the conformal blocks $F_{d,\Delta,\ell}$ and $H_{d,\Delta,\ell}$ in eq. (5.11) with polynomials $P_l^{mn}(\Delta_\ell(1+x))$, where $x\in[0,\infty)$ and $\Delta_\ell=\ell+2$ is the unitarity bound on the scaling dimension for an operator of spin l ($\Delta_0=1$ for $l=0$). The requirements in eq. (5.5) or eq. (5.9) imply that the linear combination of polynomials of the form $a_{mn}P_\ell^{mn}$ must be positive-semidefinite on the positive real x-axis, for any value of ℓ. There are two great virtues in setting up the problem in this way. Firstly, there is no need to discretize the dimension Δ and to put a cut-off value $\Delta_{\rm max}$, like in the linear programming methods used in ref. [9]. In particular, we can probe all Δ continuously up to infinity. Secondly, one can exploit numerical packages that allow us to handle very large systems of equations quite efficiently. A key variable in the numerical algorithm is the coefficient k entering in eq. (5.11). The larger k, the larger is the space of possible viable functionals, and hence the stronger are the bounds. Of course, the larger k, the more time-consuming is the numerical evaluation. For our computations, we have chosen $k=9,10,11$, depending on the complication of the problem.

Let's discuss a bit more in detail how this procedure works. Consider the simplest case of an external singlet operator. The constraints that the functional α needs to fulfill in order to get bounds on OPE coefficients are given in eq. (5.5). It is convenient to first rescale the bootstrap equation by a (Δ,ℓ)-independent function $g(z,\bar{z})$ (see ref. [50] for more details). In particular, the positivity constraints on the rescaled conformal blocks $E^+_{d,\Delta,\ell}\equiv g(z,\bar{z})F_{d,\Delta,\ell}$ then read

$$a_{mn}\,\partial_z^m\partial_{\bar{z}}^n\,E^+_{d,\Delta,\ell}\,\geq\,0\qquad\forall(\Delta,\ell)\neq(\Delta_0,\ell_0)\,, \tag{5.12}$$

where summation over m and n is understood and the derivatives are evaluated at $z=\bar{z}=1/2$. The crucial insight is that the derivatives of $E^+_{d,\Delta,\ell}$ allow for an approximation

$$\partial_z^m\partial_{\bar{z}}^n\,E^+_{d,\Delta,\ell}\,\simeq\,\chi_l(\Delta)U^{mn}_{\ell,d,+}(\Delta)\,, \tag{5.13}$$

where $\chi_l(\Delta)$ is a positive definite function of Δ and $U^{mn}_{\ell,d,+}(\Delta)$ is a polynomial in Δ.

We use 5 roots for this approximation (see ref. [50] for more details). An analogous approximation can be found for the rescaled conformal blocks $U^{mn}_{\ell,d,-}(\Delta) \equiv \tilde{g}(z,\bar{z})H_{d,\Delta,\ell}$ that appear when dealing with global symmetries. Making use of a theorem by Hilbert, the positivity constraints in eq. (5.12) can equivalently be formulated as the requirement that there exist positive semidefinite matrices A_ℓ and B_ℓ such that

$$a_{mn}\, U^{mn}_{\ell,d,+}(\Delta_l(1+x)) \;=\; X_p\, A_\ell\, X_p^T \;+\; x\, X_q\, B_\ell\, X_q^T \qquad \forall l \neq l_0\,. \tag{5.14}$$

Here $X_p \equiv (1, x, ..., x^p)$ is a vector and p and q are determined by the degree of the polynomial $U^{mn}_{\ell,d,+}$. Furthermore, Δ_l is the unitarity bound on the operator dimension. The task now consists of finding coefficients a_{mn} and a set of matrices A_ℓ and B_ℓ such that eq. (5.14) is fulfilled (additional constraints arise from e.g. the normalization condition $\alpha(F_{d,\Delta_0,\ell_0}) = 1$ in eq. (5.5) and the minimization of $\alpha(1)$ in eq. (5.6)). This can be formulated as a positive semidefinite program for which there exist powerful numerical codes. The existence of such coefficients and matrices guarantees the positivity of the functional for all $\Delta \geq \Delta_\ell$ (corresponding to $x \geq 0$).

The above algorithm, however, still requires to truncate the system at a given maximal spin L. This is in principle a serious problem, because one might have

$$\alpha(F_{d,\Delta,\ell}) < 0 \qquad \text{for} \quad \ell > L\,. \tag{5.15}$$

If L is chosen sufficiently large, $\mathcal{O}(10)$ or more, we do not expect possible violations in the semidefinite positiveness of α of the form (5.15) to be important for the numerical value of the bound. Indeed, large spin ℓ implies large dimensions Δ according to the unitarity bound, and the contribution to the four-point function of operators with large Δ is exponentially suppressed in Δ [22]. Nevertheless, it would be more reassuring to have more control on such effects. For parametrically large ℓ, the conformal blocks $F_{d,\Delta,\ell}$ and $H_{d,\Delta,\ell}$ and their derivatives allow for simple analytic expressions. For large ℓ, the terms involving the highest derivatives dominate. Using these analytic expressions, we can find the value $\ell_{\max}$, which *depends on* k, for which the contribution of the large-ℓ conformal blocks is largest.

We use Mathematica 9.0 to calculate the coefficients of the polynomials $U^{mn}_{\ell,d,\pm}$ and to set up the positive semidefinite program. The data is written to file and handed to the numerical code SDPA-GMP 7.1.2 [129] (using their sparse data format) which solves the positive semidefinite program. We use the same parameter set for the SDPA-GMP as ref. [50] (see the table in their Appendix B). For the calculations, we have used the Zefiro cluster of the INFN which is located in Pisa (Italy). This cluster consists of 25 computers, each of which has 512 GB RAM and 4 processors with 16 cores. For the plots, we have calculated points with a spacing of $\delta d = 3 \cdot 10^{-2}$ or $\delta\Delta = 3 \cdot 10^{-2}$. In

order to obtain smooth plots, we interpolate between these points.

The Multiple Precision Arithmetic Library (GMP) allows us to carry out calculations up to high precision. This is necessary because the numerical values of the coefficients of the polynomials $U^{mn}_{\ell,d,\pm}$ span several orders of magnitudes. An important source for this spread are conformal blocks with large spins $\ell \gtrsim 10$. For these values of l, an asymptotic expression for the conformal blocks and its derivatives is a good approximation. Taking $z = 1/2 + a + b$ and $\bar{z} = 1/2 + a - b$, for $l^2 \gg \Delta - \ell - 2$ one finds [9]

$$\partial_a^{2m}\partial_b^{2n} F_{d,\Delta,\ell}|_{a=b=0} \simeq \frac{\text{const.}}{(2m+1)(2n+1)}(2\sqrt{2}\ell)^{2m+2n+2}e^{-c\,\ell}\,, \tag{5.16}$$

where $c = -\log(12 - 8\sqrt{2}) \simeq 0.377$ and const. is a positive constant of $\mathcal{O}(1)$ that only depends on d. A straightforward generalization of this result allows us to also find an asymptotic analytic expression for the conformal block H defined in eq. (5.7):

$$\partial_a^{2m}\partial_b^{2n} H_{d,\Delta,\ell}|_{a=b=0} \simeq \frac{\text{const.}}{(2n+1)}(2\sqrt{2}\ell)^{2m+2n+1}e^{-c\,\ell}\,. \tag{5.17}$$

From the above two results, we find that the spread among the coefficients of the polynomials for a given spin $l \gtrsim 10$ is at least of order $\mathcal{O}(\ell^{2k+2})$ for conformal blocks F and $\mathcal{O}(\ell^{2k+1})$ for H. In addition, these results allow us to estimate the value ℓ_{max} for which derivatives of the conformal blocks have a maximum (in which case potential violations of the positivity constraint in eq. (5.5) could give a large correction in eq. (5.6)). To this end, notice that for a given ℓ, the largest coefficients arise from the highest derivatives with $m + n < 2k$. Maximizing these coefficients with respect to ℓ then , for $2k \gg 1$:

$$\ell_{\text{max}} \sim \frac{2k}{c}\,, \tag{5.18}$$

where $c = -\log(12 - 8\sqrt{2}) \simeq 0.377$. For $k \sim 10$, eq. (5.18) gives $\ell_{\text{max}} \sim 50 \div 60$. Ideally, one would include all spins from $\ell = 0$ up to $L = \ell_{\text{max}}$. This is computationally quite demanding. Fortunately, we have found that it is sufficient to take $L = 20$ to get numerically stable bounds. Changing L to $L = 22$ or $L = 24$ does not significantly alter the bounds. Nevertheless, in order to have more control on the higher-l states, we have included two other states in the constraints, at $\ell = \ell_{\text{max}}$ and at an intermediate value $\ell \approx (L + \ell_{\text{max}})/2$.[4] We have numerically tested that this implementation works better than including states at very large values of ℓ, such as $\ell = 1000, 1001$ as done in e.g. ref. [50]. We can always check the positivity of α a posteriori. We have found that by imposing constraints at $\ell = 0, ..., 20, 1000, 1001$ the functional often becomes

[4] More precisely, we include spins $\ell = 35, 52$ for calculations with $k = 9$, $\ell = 37, 56$ for $k = 10$ and $\ell = 40, 60$ for $k = 11$.

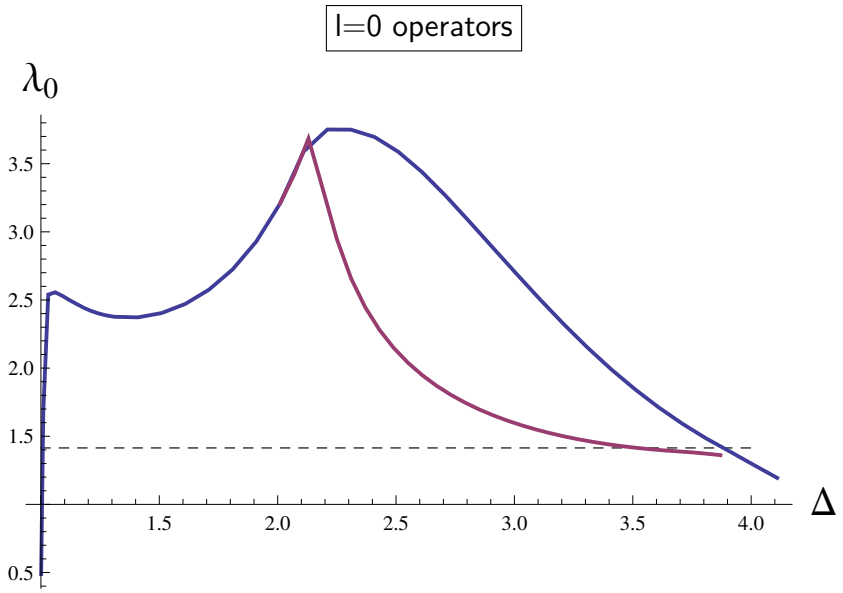

Figure 5.1: Upper bounds on the three-point function coefficient λ_0 between two scalar operators of dimension $d = 1.6$ and a scalar operator $\mathcal{O}$ of dimension Δ calculated at $k = 11$ with no assumptions on the spectrum (blue line) and assuming that no scalar operator in the OPE is present below $\Delta_0 = 2$ (red line). For illustrative purposes, we show the free-theory value for $d = 1$ (in which case $\Delta = 2$), $\lambda_0^{\text{free}} = \sqrt{2}$, as a black dashed line.

negative for values $\ell \neq 0, ..., 20, 1000, 1001$ whereas for our implementation α remains positive for most of the ℓ that we have checked. In practice, however, we have not detected deviations in the results among the two different implementations, confirming that values of $\ell > L$ are numerically negligible.

An additional source for the spread arises from the approximation in eq. (5.13). The functions $\chi_\ell(\Delta)$ are numerically small for large spins ℓ and therefore increase the spread among the various coefficients of the polynoms $U^{mn}_{\ell,d,\pm}(\Delta) \simeq \partial_z^m \partial_{\bar{z}}^n E^{\pm}_{d,\Delta,\ell} \,/\, \chi_\ell(\Delta)$ that determine the positive semidefinite program.

In order to reduce the numerical spread among the polynom coefficients (which allows one to reduce the required precision and thereby speeds up the calculation), we rescale them by both an (m, n)-dependent factor and an l-dependent factor before handing them to the SDPA-GMP. Both of these rescalings transform the positivity constraint eq. (5.14) into an equivalent constraint. Indeed, the (m, n)-dependent factor amounts to a redefinition of the coefficients a_{mn}, whereas the ℓ-dependent rescaling can be absorbed into the matrices A_ℓ and B_ℓ. Note, however, that e.g. the effect of the rescaling on the normalization condition $\alpha(F_{d,\Delta_0,\ell_0}) = 1$ in eq. (5.5) needs to be taken into account when calculating the bound from eq. (5.6).

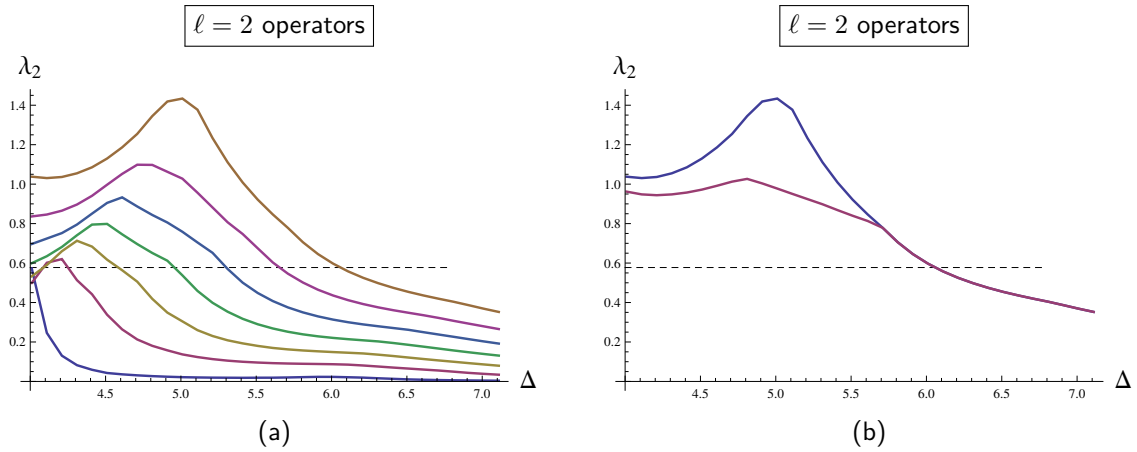

Figure 5.2: Upper bounds on the three-point function coefficient λ_2 between two scalar operators of dimension d and a tensor operator $\mathcal{O}$ with spin $\ell = 2$ and dimension Δ calculated at $k = 11$. (a) Starting from below, the lines correspond to the values $d = 1.01, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6$. No assumption on the spectrum is made. (b) For $d = 1.6$ with no assumption on the spectrum (blue line, as in (a) for $d = 1.6$) and assuming that no scalar operator in the OPE is present below $\Delta_0 = 2$ (red line). For illustrative purposes, we show the free-theory value for $d = 1$ (in which case $\Delta = 4$), $\lambda_2^{\text{free}} = 1/\sqrt{3}$, as a black dashed line in both panels.

5.3 Bounds on OPE Coefficients for Tensor Operators

In this section, we report our results for the upper bounds on the three-point function coefficient $\lambda_{\mathcal{O}}$ appearing in the OPE of two identical scalar operators ϕ of scaling dimension d. The operator $\mathcal{O}$ is a traceless symmetric tensor of even spin l. The coefficient λ is normalized such that its free-theory value is

$$\lambda_{\mathcal{O}_\ell}^{\text{free}} \equiv \lambda_\ell^{\text{free}} = \sqrt{2}\frac{\ell!}{\sqrt{(2\ell)!}}\,. \tag{5.19}$$

We do not report the results for the $\ell = 0$ case, which were first derived in ref. [45] and subsequently improved in ref. [50]. Our results agree with fig. 10 of ref. [50]. These bounds change if we assume that the first scalar operator which appears in the $\phi\phi$ OPE has a dimension $\Delta_0 > \Delta_\ell$, where Δ_l is the unitarity bound on Δ. As expected, the upper bounds do not significantly change when d is close to 1, since by continuity the theory is close to the free theory, where the only scalar operator arises exactly at $\Delta_0 = 2$. For values of d not too close to 1, on the other hand, the bound is significantly improved and becomes more stringent as Δ_0 increases. In fig. 5.1, we report the bounds for $d = 1.6$ and $\Delta_0 = 2$.

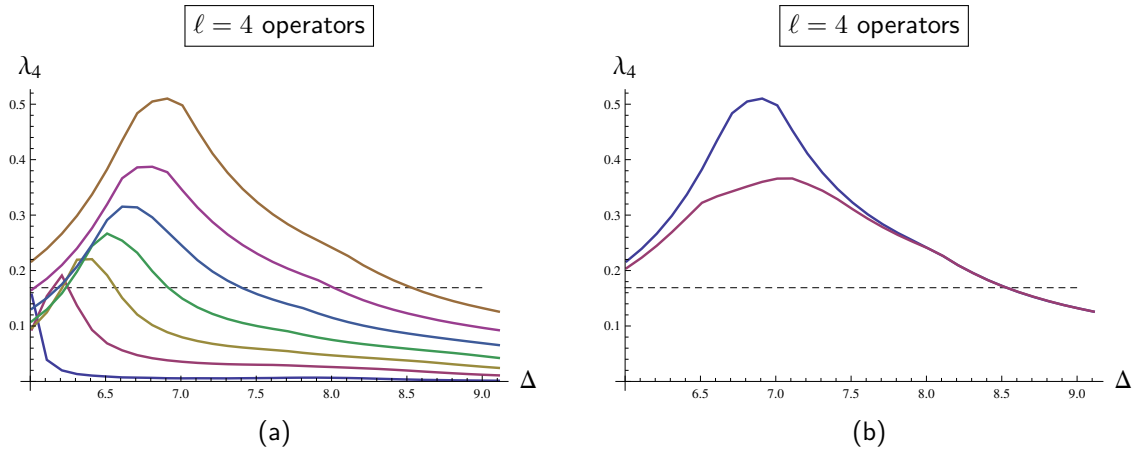

Figure 5.3: Upper bounds on the three-point function coefficient λ_4 between two scalar operators of dimension d and a tensor operator $\mathcal{O}$ with spin $\ell = 4$ and dimension Δ calculated at $k = 11$. (a) Starting from below, the lines correspond to the values $d = 1.01, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6$. No assumption on the spectrum is made. (b) For $d = 1.6$ with no assumption on the spectrum (blue line, as in (a) for $d = 1.6$) and assuming that no scalar operator in the OPE is present below $\Delta_0 = 2$ (red line). For illustrative purposes, we show the free-theory value for $d = 1$ (in which case $\Delta = 6$), $\lambda_4^{\text{free}} = 1/\sqrt{35}$, as a black dashed line in both panels.

Analogously, one can study the upper bounds on λ_2 for generic tensor operators $\mathcal{O}$ with spin $\ell = 2$ and dimension $\Delta \geq 4$. Upper bounds on the central charge $c \propto 1/\lambda_2^2$ associated with the energy-momentum tensor (the lowest-dimensional operator in the $\ell = 2$ sector, with $\Delta = 4$), have been extensively analyzed in refs. [46, 47, 50], with and without the assumption of a lower bound on the dimension of the lowest-lying scalar operator appearing in the $\phi\phi$ OPE. In fig. 5.2 (a), we report the upper bounds on the coupling λ_2 between two scalar operators of dimension d and a tensor operator $\mathcal{O}$ with spin $\ell = 2$ and dimension Δ for different d and as a function of Δ. As can be seen, the larger d is, the less stringent is the upper bound, in agreement with the naive expectation for which $d - 1$ can be seen as a measure (for d not too far from 1) of how strongly coupled the CFT is. Like for scalar operators, the bounds change if we make some assumptions on the CFT spectrum. As for the scalar case, the upper bounds do not significantly change when d is very close to 1, but for values of d not too close to 1, they become more stringent as Δ_0 increases. For illustration, in fig. 5.2 (b), we report the upper bounds on λ_2 as a function of Δ for $d = 1.6$, assuming that the lowest *scalar* operator appearing in the $\phi\phi$ OPE has a dimension $\Delta_0 \geq 2$.

Similarly, one can analyze tensor operators at higher ℓ. In figs. 5.3 (a) and (b), we report the same as above for $\ell = 4$ operators. As expected, the absolute scale of λ_ℓ becomes lower and lower as ℓ increases, with the allowed values of λ_ℓ quickly decreasing as l becomes larger. Notice that the maximal allowed value of both λ_2 and λ_4 is

centered at values of Δ that increase as d is increased.

5.4 Bounds on Current-Current Two-Point Functions

At leading order, the CFT contribution to the one-loop beta function of a gauge field A_μ, external to the CFT, is governed by the coefficient of the two point-function of the corresponding current. Denoting by

$$\mathcal{L}_{\text{gauged}} = \mathcal{L}_{\text{CFT}} + g J^\mu_A A^A_\mu - \frac{1}{4} F^A_{\mu\nu} F^{\mu\nu}_A \tag{5.20}$$

the total Lagrangian after the gauging, we can consider the effective action $\Gamma(A)$ defined as (in euclidean signature)

$$e^{-\Gamma(A)} = \int \mathcal{D}\Phi_{\text{CFT}}\; e^{-\int d^4x\, \mathcal{L}_{\text{gauged}}}\,, \tag{5.21}$$

where the functional integration is over all the CFT states and we have omitted color indices. In general

$$\Gamma(A) \supset -\frac{1}{4}\int d^4x\, Z F^A_{\mu\nu} F^{\mu\nu}_A\,, \tag{5.22}$$

where $Z = (1 + \delta Z_{\text{CFT}})$ and δZ_{CFT} is the CFT contribution to the wave function renormalization of the gauge field, which in turns gives us the one-loop contribution of the CFT to the RG running of g:

$$\beta_{\text{CFT}} = g\mu \frac{d}{d\mu}\sqrt{Z} = \frac{1}{2} g\mu \frac{d}{d\mu}\delta Z_{\text{CFT}}\,. \tag{5.23}$$

By taking two functional derivatives with respect to $A^A_\mu(p)$ and $A^B_\nu(-p)$ in eq. (5.21), we readily get

$$\delta_{AB}\delta Z_{\text{CFT}}(\delta_{\mu\nu}p^2 - p_\mu p_\nu) = -g^2 \langle J^A_\mu(-p) J^B_\nu(p)\rangle_{g=0}\,, \tag{5.24}$$

where the subscript in the correlator specifies that the two-point function is computed in the *unperturbed* CFT setting $g = 0$. The normalization of the current is uniquely fixed by Ward identities. Following the notation of ref. [50], we parametrize the two-point function in configuration space as follows:[5]

$$\langle J^A_\mu(x) J^B_\nu(0)\rangle_{g=0} = \frac{3\kappa\delta^{AB}}{4\pi^4}\left(\delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}\right)\frac{1}{x^6}\,. \tag{5.25}$$

[5] Notice that the definition of κ here is not identical to that of ref. [50] which tacitly applies to CFTs with one charged multiplet only. In general, $\kappa_{\text{here}} \propto \sum_i \kappa^i_{\text{there}} T(r_i)$ where i runs over all the charged fields of the CFT in the representations r_i and $\delta^{AB} T(r_i) = \text{Tr}(t^A_{r_i} t^B_{r_i})$.

The "vector central charge" κ is roughly a measure of how many charged degrees of freedom are present in the CFT, similar to the standard central charge c being a measure of the total number of degrees of freedom of the CFT. Modulo irrelevant contact terms, the momentum space correlation function reads

$$\langle J_\mu^A(-p) J_\nu^B(p)\rangle_{g=0} = (\delta_{\mu\nu}p^2 - p_\mu p_\nu)\frac{\kappa}{16\pi^2}\delta^{AB}\log\left(\frac{p^2}{\mu^2}\right) \tag{5.26}$$

and hence

$$\beta_{\text{CFT}} = g^3\frac{\kappa}{16\pi^2}\,. \tag{5.27}$$

We extract κ by rescaling the vector current so that it appears as the coefficient of the three-point function $\langle\phi_i\phi_j J_\mu^A\rangle$:

$$\lambda_J^2 = \frac{\rho}{\kappa}\,. \tag{5.28}$$

Upper bounds on λ_J^2 turn into lower bounds on κ. The constant factor ρ is easily found by matching the result with the free-theory case, in which both λ_J^2 and κ are calculable. In what follows, we will analyze the lower bounds on κ for different vector currents that come from the crossing symmetry constraints applied to four-point functions of scalars.

5.4.1 SO(N) Global Symmetry

We consider a four-point function of real scalars that are taken to be the components of a single field in the fundamental representation of SO(N) with dimension d. The crossing symmetry relations have been derived in ref. [48]. We report them here for completeness:

$$\sum_{S^+}|\lambda_{\mathcal{O}}^S|^2\begin{pmatrix}0\\F\\H\end{pmatrix}+\sum_{T^+}|\lambda_{\mathcal{O}}^T|^2\begin{pmatrix}F\\(1-\frac{2}{N})F\\-(1+\frac{2}{N})H\end{pmatrix}+\sum_{A^-}|\lambda_{\mathcal{O}}^A|^2\begin{pmatrix}-F\\F\\-H\end{pmatrix}=\begin{pmatrix}0\\1\\-1\end{pmatrix}. \tag{5.29}$$

Here S, T and A refer respectively to the singlet, rank-2 symmetric and antisymmetric (adjoint) representations of the operators $\mathcal{O}$ which define the different conformal blocks. For the superscript $+$, only even spins are included in the sum whereas for $-$ only odd spins are summed over. For simplicity, we have omitted the labels d, Δ, ℓ and the arguments $z, \bar{z}$ of the conformal blocks F and H. Bounds on κ (as defined in eq. (5.25)) in this set-up have already been found in ref. [50]. In this subsection we will see how these bounds change if assumptions on the dimensionality of the lowest-lying scalar operator in the singlet channel are made.

First of all, let us consider the free theory of a real scalar in the fundamental representation of SO(N) in order to fix the constant ρ in eq. (5.28). The free-theory values of

the OPE coefficients in the three different channels read

$$\begin{aligned}
\lambda^{\text{free}}_{A,\ell} &= \frac{1}{\sqrt{2}}\lambda^{\text{free}}_{\ell} \qquad (\ell \text{ odd})\,, \\
\lambda^{\text{free}}_{T,\ell} &= \frac{1}{\sqrt{2}}\lambda^{\text{free}}_{\ell} \qquad (\ell \text{ even})\,, \\
\lambda^{\text{free}}_{S,\ell} &= \frac{1}{\sqrt{N}}\lambda^{\text{free}}_{\ell} \quad (\ell \text{ even})\,,
\end{aligned} \tag{5.30}$$

where $\lambda^{\text{free}}_{\ell}$ is given in eq. (5.19). We in particular get $\lambda^{\text{free}}_{J} = \lambda^{\text{free}}_{A,1} = 1/\sqrt{2}$. Matching eq. (5.27) with the one-loop contribution to the β-function of a scalar in an SO(N) gauge theory gives

$$\kappa_{\text{free}} = \frac{1}{6}\,, \tag{5.31}$$

where we have taken $T(\text{fund.}) = 1$ (cf. footnote 5). From this it follows that $\rho = 1/12$ in eq. (5.28).

In fig. 5.4, we report our results in terms of lower bounds on κ. We have considered the five different values $N = 2, 6, 10, 14, 18$ and report the lower bounds on κ for the case where no assumption on the spectrum is made (the lines starting from $d = 1$) and the case where the lowest-lying scalar operator in the singlet channel is assumed to have dimension $\Delta_S \geq 4$ (the other lines). The former bounds agree with previous results (e.g. compare with fig. 18 of ref. [50]). Although it is not clearly visible from the figure, we have checked that all the bounds consistently tend to the free-theory value for $d \to 1$. The latter bounds start from a given $d_{\text{cr}} > 1$ that depends on N. This is of course expected, given the known results for the upper bound on the dimension of the lowest-lying scalar singlet operator at a given d: CFTs at $d < d_{\text{cr}}$ are excluded under the assumption of a gap in the scalar singlet sector. The values of d_{cr} that we find agree with the values given in the literature (compare e.g. with the dimensions d for which $\Delta_0 = 4$ in fig. 4 of ref. [50]). The lower bounds on κ become significantly more stringent when we impose that $\Delta_S > 4$. They also decrease less rapidly when d increases compared to the unconstrained case.

In order to show how the assumption on Δ_S affects the lower bounds on κ, in fig. 5.5, we fix $N = 10$ and consider the three cases $\Delta_S \geq 2$, $\Delta_S \geq 3$ and $\Delta_S \geq 4$. As expected, the lower bound consistently becomes more severe as we increase Δ_S. As before, the bounds start at certain dimensions d_{cr} which agree with previous results (compare e.g. with the dimensions d for which $\Delta_0 = 3, 4$ in fig. 4 of ref. [50]).

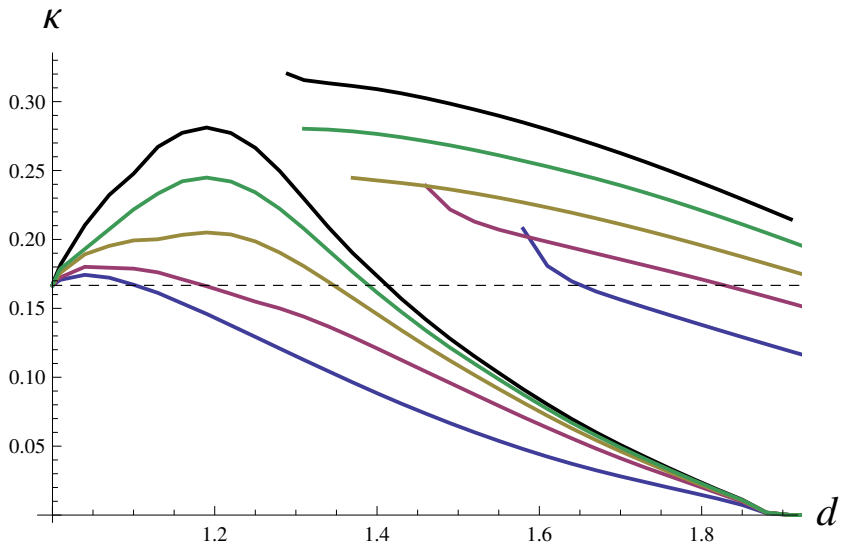

Figure 5.4: Lower bounds on the two-point function coefficient κ between two conserved SO(N) or SU($N/2$) adjoint currents as obtained from a four-point function of scalar operators in the fundamental representation with dimension d calculated at $k = 10$. From below, the lines which start at $d = 1$ correspond to $N = 2$ (blue), $N = 6$ (red), $N = 10$ (brown), $N = 14$ (green), $N = 18$ (black), with no assumption on the spectrum. In the same order and using the same color code, the lines which start at $d \simeq 1.58$, $d \simeq 1.46$, $d \simeq 1.37$, $d \simeq 1.31$ and $d \simeq 1.29$ show the bound which is obtained under the assumption that no scalar operator in the singlet channel has dimension $\Delta_S < 4$. For illustrative purposes, we show the free-theory value $\kappa_{\text{free}} = 1/6$ as a black dashed line.

5.4.2 SU(N) Global Symmetry

We consider a four-point function of complex scalars that are taken to be the components of a field in the fundamental representation of SU(N) with dimension d. The crossing symmetry relations have been derived in ref. [48]. We report them here for completeness:

$$\sum_{S^\pm} |\lambda^S_{\mathcal{O}}|^2 \begin{pmatrix} F \\ H \\ (-)^l F \\ (-)^l H \\ 0 \\ 0 \end{pmatrix} + \sum_{Ad^\pm} |\lambda^{Ad}_{\mathcal{O}}|^2 \begin{pmatrix} (1-\frac{1}{N})F \\ -(1+\frac{1}{N})H \\ (-)^{l+1}\frac{1}{N}F \\ (-)^{l+1}\frac{1}{N}H \\ (-1)^l F \\ (-)^l H \end{pmatrix} + \sum_{T^+} |\lambda^T_{\mathcal{O}}|^2 \begin{pmatrix} 0 \\ 0 \\ F \\ -H \\ F \\ -H \end{pmatrix} + \sum_{A^-} |\lambda^A_{\mathcal{O}}|^2 \begin{pmatrix} 0 \\ 0 \\ F \\ -H \\ -F \\ H \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}. \tag{5.32}$$

Here S, Ad, T and A refer respectively to the singlet, adjoint, rank-2 symmetric and rank-2 antisymmetric representations of the operators $\mathcal{O}$ which define the different conformal blocks. For the superscript $+$, even spins are included in the sum, and for $-$, odd spins are summed over. We consider here the lower bounds on κ (as defined in

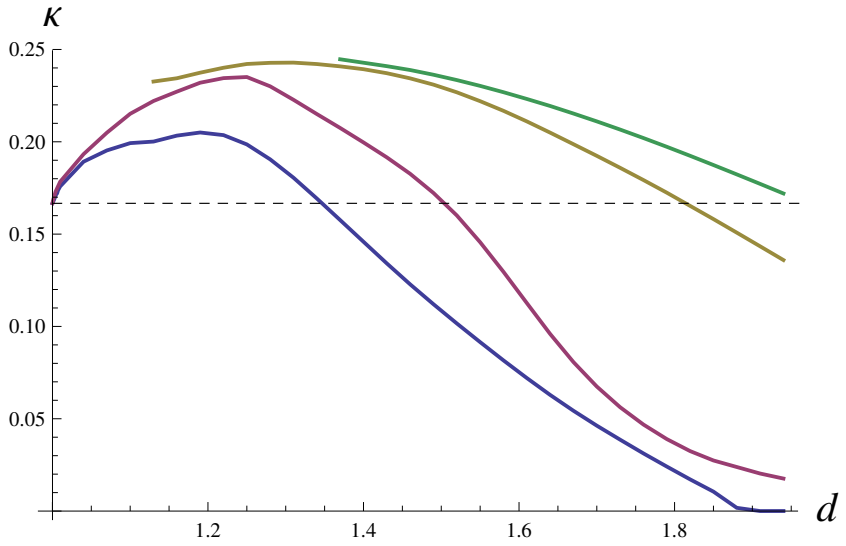

Figure 5.5: Lower bounds on the two-point function coefficient κ between two conserved $SO(10)$ or $SU(5)$ adjoint currents as obtained from a four-point function of scalar operators in the fundamental representation with dimension d calculated at $k = 10$. From below, the lines correspond to the case with no assumption on the spectrum (blue) and assuming that no scalar operator in the singlet channel has dimension $\Delta_S < 2$ (red), $\Delta_S < 3$ (brown), $\Delta_S < 4$ (green). For illustrative purposes, we show the free-theory value $\kappa_{\text{free}} = 1/6$ as a black dashed line.

eq. (5.25)) associated with the adjoint current.

As in subsection 5.4.1, we start by looking at the free theory of a complex scalar in the fundamental representation of SU(N) in order to fix the constant ρ in eq. (5.28). The free-theory values of the OPE coefficients in the four different channels read

$$\begin{aligned}
\lambda^{\text{free}}_{Ad,\ell} &= \frac{1}{\sqrt{2}}\lambda^{\text{free}}_{\ell} \qquad (\ell \text{ even } \textit{and} \text{ odd})\,, \\
\lambda^{\text{free}}_{S,\ell} &= \frac{1}{\sqrt{2N}}\lambda^{\text{free}}_{\ell} \quad (\ell \text{ even } \textit{and} \text{ odd})\ \ , \\
\lambda^{\text{free}}_{T,\ell} &= \frac{1}{\sqrt{2}}\lambda^{\text{free}}_{\ell} \qquad (\ell \text{ even})\,, \\
\lambda^{\text{free}}_{A,\ell} &= \frac{1}{\sqrt{2}}\lambda^{\text{free}}_{\ell} \qquad (\ell \text{ odd})\,,
\end{aligned} \tag{5.33}$$

where $\lambda^{\text{free}}_{\ell}$ is given in eq. (5.19). We in particular get $\lambda^{\text{free}}_{J} = \lambda^{\text{free}}_{Ad,1} = 1/\sqrt{2}$. Matching eq. (5.27) with the one-loop contribution to the β-function of a complex scalar in an SU(N) gauge theory gives

$$\kappa_{\text{free}} = \frac{1}{6}\,, \tag{5.34}$$

where we have taken $T(\text{fund.}) = 1/2$ (cf. footnote 5). From this it follows that $\rho = 1/12$ in eq. (5.28) as for SO(N).

The six crossing symmetry equations (5.32) should reduce to the three equations (6.41) when the group SU(N) is embedded in an underlying SO($2N$) group. The decomposition of the singlet, adjoint and rank-2 symmetric representations of SO($2N$) in terms of SU(N) representations reads

$$\begin{aligned} S^+_{\mathrm{SO}(2N)} &= S^+_{\mathrm{SU}(N)}\,, \\ T^+_{\mathrm{SO}(2N)} &= T^+_{\mathrm{SU}(N)} \oplus \overline{T}^+_{\mathrm{SU}(N)} \oplus \mathrm{Ad}^+_{\mathrm{SU}(N)}\,, \\ A^-_{\mathrm{SO}(2N)} &= A^-_{\mathrm{SU}(N)} \oplus \overline{A}^-_{\mathrm{SU}(N)} \oplus \mathrm{Ad}^-_{\mathrm{SU}(N)} \oplus S^-_{\mathrm{SU}(N)}\,. \end{aligned} \tag{5.35}$$

If SU(N) $\subset$ SO($2N$), for each primary operator in the A^- (T^+) representation of SU(N), there is a corresponding operator in the Ad^- and S^- (Ad^+) representation as follows from eq. (5.35). The OPE coefficients of these operators are related by the underlying SO($2N$) symmetry, $\lambda^{T^+}_{\mathrm{SO}(2N)} = \lambda^{Ad^+}_{\mathrm{SU}(N)}$, $\lambda^{A^-}_{\mathrm{SO}(2N)} = \lambda^{A^-}_{\mathrm{SU}(N)} = \sqrt{N}\lambda^{S^-}_{\mathrm{SU}(N)}$. It is straightforward to check with these identifications that eqs. (5.32) reduce to eqs. (6.41).

As we have already mentioned, the numerical results for the lower bounds on κ for SU(N) are identical to those for SO($2N$), see fig. 5.4. This suggests that, given a set of three functionals α_m that satisfy eq. (5.9) with $P = 2$, $Q = 1$ and $\eta_{F,H}$ as given by eq. (6.41), one should be able to construct a set of six functionals $\tilde{\alpha}_m$ as linear combinations of the α_m such that these functionals satisfy eq. (5.9) with $P = 3$, $Q = 3$ and $\eta_{F,H}$ as given by eq. (5.32). It would be interesting to find such a mapping and hence to understand in more analytical terms why the bounds on κ for SO($2N$) and SU(N) are equal.

5.4.3 $G_1 \times G_2$ Global Symmetries

The lower bounds on κ found in subsections 5.4.1 and 5.4.2 apply to CFTs in presence of at least one scalar field in the fundamental representation of G_1, where $G_1 = \mathrm{SO}(M)$ or SU(M). Of course, the CFT can contain additional charged fields, for example a number N of scalars in the fundamental representation of G_1, with dimensions $d_1, \ldots d_N$. In the free-theory limit of N decoupled scalars (real for SO(M), complex for SU(M)) we would simply have

$$\kappa_{\mathrm{free}} = \frac{N}{6}\,. \tag{5.36}$$

The larger N is, the more constraining (and interesting) the lower bounds are. One cannot naively rescale the results of fig. 5.4 by a factor of N in order to match the new free theory limit, however, because the interactions among the scalars will not be

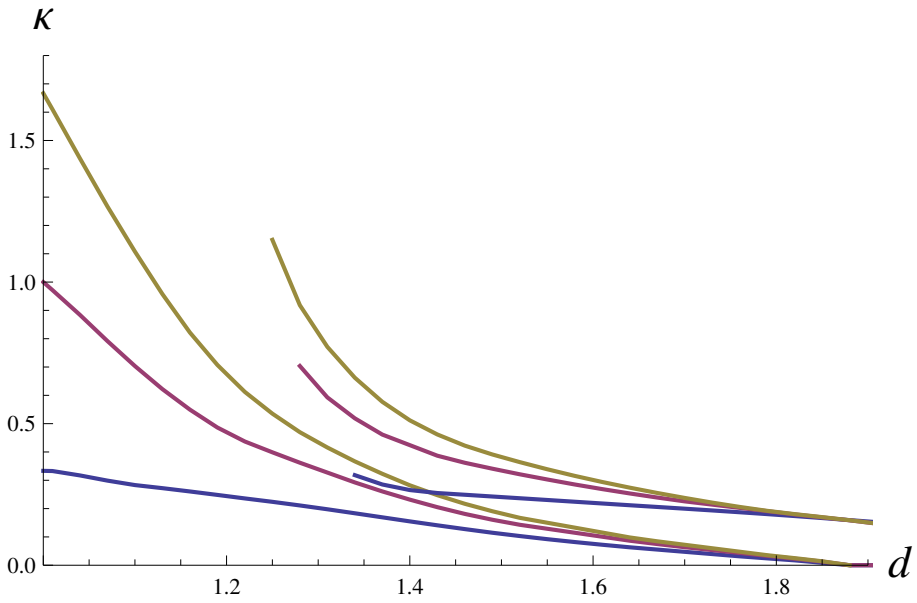

Figure 5.6: Lower bounds on the two-point function coefficient κ between two conserved SO(N) (or SU($N/2$)) adjoint currents as obtained from a four-point function of scalar operators with dimension d in the bi-fundamental representation of SO(N)×SO(M), calculated at $k = 9$. We take $N = 6$. From below, the lines which start at $d = 1$ correspond to $M = 2$ (blue), $M = 6$ (red), $M = 10$ (brown), with no assumption on the spectrum. In the same order and using the same color code, the lines which start at $d \simeq 1.34$, $d \simeq 1.28$ and $d \simeq 1.25$ show the bound which is obtained under the assumption that no scalar operator in the singlet channel has dimension $\Delta_S < 4$.

taken into account in this way. A more constraining bound could likely be obtained by studying the coupled set of four-point functions involving all N scalars. This is in general not straightforward to do, since the crossing symmetry constraints are significantly more involved in presence of fields with different scaling dimensions.[6] A simple way to mimic the presence of more fields charged under a given group, though at the cost of assuming identical scaling dimensions $d_1 = \ldots d_N = d$, is obtained by introducing a further global symmetry group G_2 and assuming that the N fields transform under some representation of G_2. This is the main motivation for us to consider global symmetries which are direct products of two simple groups: it is a way to obtain lower bounds on κ_{G_1} in presence of more than one field charged under G_1. More specifically, in the following we will consider fields in the fundamental representation of $G_2 = \mathrm{SO}(N)$.

[6] See ref. [61] for very recent progress in this direction.

$\mathbf{SO}(N)\times\mathbf{SO}(M)$

Consider a CFT with global symmetry $\mathrm{SO}(N)\times\mathrm{SO}(M)$ and one real scalar ϕ_a^i in the bi-fundamental representation of $\mathrm{SO}(N)\times\mathrm{SO}(M)$, a and i being $\mathrm{SO}(N)$ and $\mathrm{SO}(M)$ indices, respectively. In complete analogy to the $\mathrm{SO}(M)$ case discussed in ref. [48], we can impose crossing symmetry in the s- and t-channel on the four-point function $\langle\phi_a^i(x_1)\phi_b^j(x_2)\phi_c^k(x_3)\phi_d^l(x_4)\rangle$ in order to obtain the bootstrap equations. The operators appearing in the $\phi\phi$ OPE transform under $\mathrm{SO}(N)\times\mathrm{SO}(M)$ according to the decomposition of $(\mathbf{N},\mathbf{M})\otimes(\mathbf{N},\mathbf{M})$, where **N** and **M** denote the fundamental representations of respectively $\mathrm{SO}(N)$ and $\mathrm{SO}(M)$. This gives 9 different representations, consisting of pairs (ij), where $i,j=S,T,A$ refer to the singlet (S), symmetric (T) and antisymmetric (A) representations of respectively $\mathrm{SO}(N)$ and $\mathrm{SO}(M)$. Correspondingly, we get a total of $3\times3=9$ equations. We report them in eq. (D.2) in appendix D.1.

The $\mathrm{SO}(N)$ conserved current that we analyze is in the (AS) representation and is the lowest-dimensional operator appearing in the functions F_{AS} and H_{AS} defined in eq. (D.1). In fig. 5.6, we show the lower bounds on $\kappa_{\mathrm{SO}(6)}$ for the three cases $M=2,6,10$. While for the case of $\mathrm{SO}(N)$ or $\mathrm{SU}(N/2)$ considered before the lower bound first becomes significantly more stringent with growing d and only from a certain d onwards becomes less stringent, here a slight increase in the bound arises only for d very close to 1 after which the bound decreases with d. The lines starting from $d=1$ correspond to the case where no assumption on the spectrum has been made, while for the other lines we have assumed that the lowest-lying scalar operator in the SS channel has dimension $\Delta_{SS}\geq4$. The lower bounds for the latter case are stronger than for the former, but the difference is less substantial than for the groups $\mathrm{SO}(N)$ or $\mathrm{SU}(N/2)$. The lower bound $d\geq d_{\mathrm{cr}}$ on the dimension of ϕ_a^i above which the lowest-lying operator in the SS channel can have a dimension $\Delta_{SS}\geq4$ is also weaker than what was found for $\mathrm{SO}(N)$ or $\mathrm{SU}(N/2)$. This is expected, since this bound becomes the weaker the larger the group is. The correct free-theory limit is obtained in all three cases. The shape of the lower bound on κ with no assumption on the spectrum in fig. 5.6 resembles the bound found in ref. [50] for $\mathrm{SU}(N)$ singlet currents (see e.g. their fig. 19). From the $\mathrm{SO}(M)$ point of view, the $\mathrm{SO}(N)$ current is in fact a collection of $N(N-1)/2$ singlet currents. On the other hand, for $N\gg M$, we find that the lower bound on κ for $\mathrm{SO}(N)$ currents shows the characteristic bump of single $\mathrm{SO}(N)$ or $\mathrm{SU}(N/2)$ currents, well above the free-theory value, as in fig. 5.4 (a). For illustration, we show the bound on κ obtained for $N=30$ and $M=2$ in fig. 5.7. It would be interesting to further explore these bounds and to understand the origin of their different behaviours in the regimes $N\leq M$ and $N\gg M$.

As a final application of our results, in fig. 5.8, we report the lower bounds on κ for

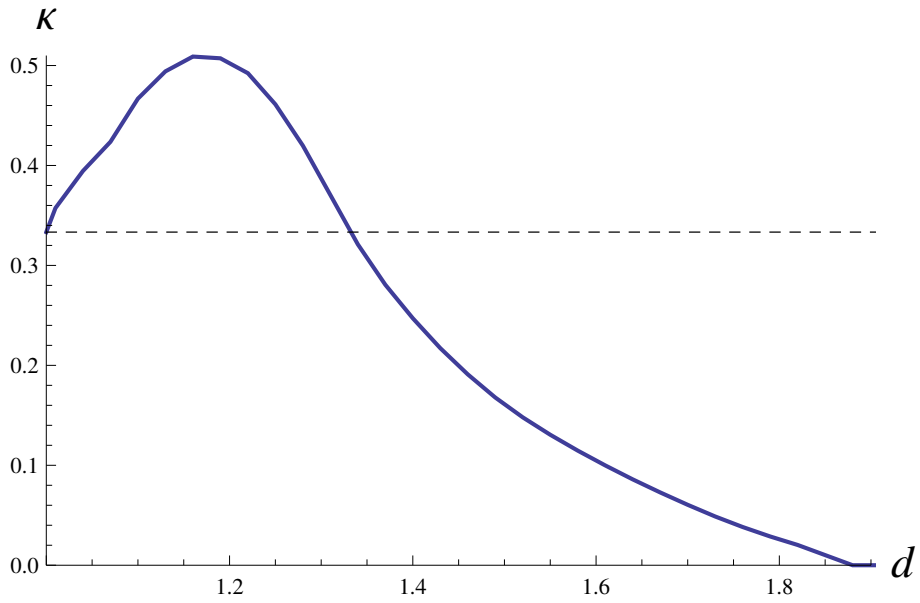

Figure 5.7: Lower bounds on the two-point function coefficient κ between two conserved SO(30) (or SU(15)) adjoint currents as obtained from a four-point function of scalar operators with dimension d in the bi-fundamental representation of SO$(30)\times$SO(2), calculated at $k = 9$. No assumption on the spectrum is made. The black dashed line corresponds to the free-theory value $\kappa_{\text{free}} = 1/3$.

the group SO$(6)\times$SO(120). We choose SO(120) because the contribution of 120 free complex scalar triplets to the SU$(3)_c \subset$ SO(6) current-current two-point function gives $\kappa = 20$. This in turn is the same value found in eq. (5.3) for the number of free fermion triplets which are needed to give mass to all the SM quarks in the SO(5)$\to$SO(4) pNGB composite Higgs model mentioned in section 5.1. We consider SO(6)$\times$SO(120) and not SU$(3)\times$SO(120) because the latter case is computationally very demanding (incidentally, in one of the models presented in ref. [100], SU$(3)_c$ was actually embedded in an underlying SO(6) flavour global symmetry). Anyhow, given the equivalence between the SO$(2N)$ and SU(N) lower bounds on κ, we believe that these results would also hold for the SU(3)$\times$SO(120) case.

As we see in fig. 5.8, assuming the absence of a relevant scalar singlet operator in the CFT does not significantly change the bounds. Furthermore, the most dangerous region regarding Landau poles, which is the region close to $d = 1$, is not consistent with the assumption of absence of relevant deformations. If we demand that no sub-Planckian Landau pole arises, then we need $d \gtrsim 1.2$, while for $d \gtrsim 1.25$, α_c remains asymptotically free.

SO$(N)\times$**SU**(M)

Consider a CFT with global symmetry SO$(N)\times$SU(M) and one complex scalar ϕ^i_a in the bi-fundamental representation of SO$(N)\times$SU(M), a and i being SO(N) and SU(M) indices, respectively. We impose crossing symmetry in the s- and t-channel on the four-point function $\langle\phi^i_a(x_1)\bar{\phi^j_b}^\dagger(x_2)\phi^k_c(x_3)\bar{\phi^l_d}^\dagger(x_4)\rangle$ and the four-point function with $x_3 \leftrightarrow x_4$. The operators appearing in the $\phi\phi$ OPE transform under SO$(N)\times$SU(M) in representations (ij), where $i = S, T, A$ refer to the singlet (S), symmetric (T) and antisymmetric (A) representations of SO(N) and $j = A, T$ refer to the symmetric (T) and antisymmetric (A) representations of SU(M). This gives 6 different representations, with even and/or odd spin operators, depending on the representation. The operators appearing in the $\phi\phi^\dagger$ OPE transform in representations (ij), with $i = S, T, A$ as before, whereas $j = S, Ad$ refer to the singlet (S) and adjoint (Ad) representations of SU(M). Taking into account that SU(M) singlet and adjoint operators appear with both even and odd spins, we get 12 different conformal blocks, for a total of 18 bootstrap equations. We report them in eqs. (D.5) and (D.6) in appendix D.2. The SO(N) conserved current that we are interested in transforms under the AS representation and is the lowest-dimensional operator appearing in the functions F^-_{AS} and H^-_{AS} defined in eq. (D.4).

As we have already mentioned, we have not numerically analyzed lower bounds on κ in this case, because the large number of bootstrap equations makes the numerical analysis computationally too demanding for our computing resources.

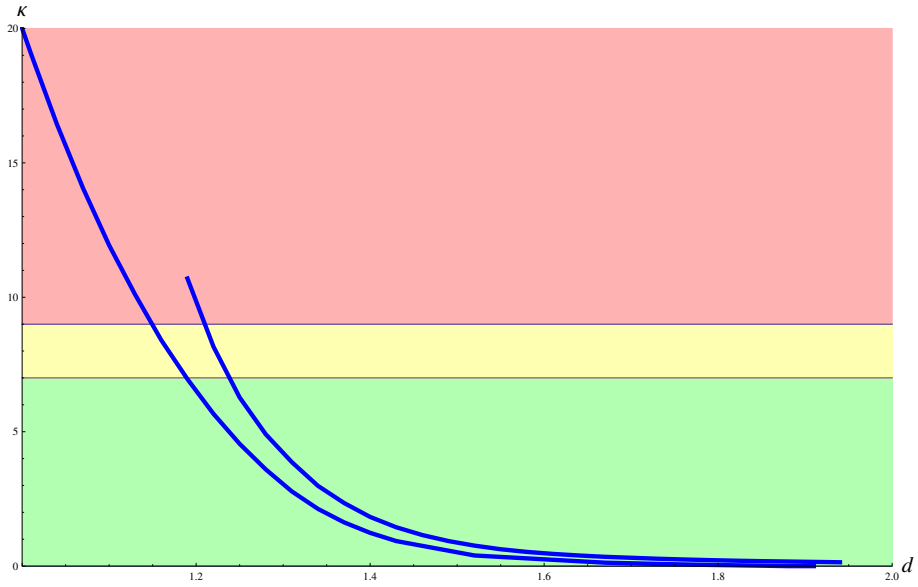

Figure 5.8: Lower bounds on the two-point function coefficient κ between two conserved $\mathrm{SO}(6) \supset \mathrm{SU}(3)_c$ adjoint currents as obtained from a four-point function of scalar operators with dimension d in the bi-fundamental representation of $\mathrm{SO}(6)\times\mathrm{SO}(120)$, calculated at $k = 9$. The line which starts at $d = 1$ corresponds to the case where no assumption is made on the spectrum, whereas for the line which starts at $d \simeq 1.19$, the CFT is assumed to have no scalar operator in the singlet channel with dimension $\Delta_S < 4$. In the green region α_c remains asymptotically free, while in the orange and red regions α_c develops trans-Planckian and sub-Planckian Landau poles, respectively.

Chapter 6
6 Multipoint Bootstrap

In this last chapter we will analyze an alternative numerical method in order to obtain constraints in general CFTs by means of the bootstrap equation 2.48. All numerical bootstrap studies up to date have used the constraints imposed by crossing symmetry on 4-point correlators evaluated at a specific value of the conformal cross-ratios, $u = v = 1/4$, or equivalently in z-coordinates at $z = \bar{z} = 1/2$ [13]. This is the point of best convergence for the combined conformal block expansions in the s and t channels. Taking higher and higher derivatives of the bootstrap equations evaluated at this point has proven to be very effective and successful in obtaining increasingly better bounds, an example of this has been shown in the previous chapter. We will denote this method in the following as the "derivative method". A drawback of the derivative method – both in its linear [9, 103, 106] or semi-definite [50, 107] programming incarnations – is the need to include a large number of operators in the bootstrap equations. This makes any, even limited, analytical understanding of the obtained results quite difficult.

A possible approximation scheme is in fact available: ref. [22] has determined the rate of convergence of the Operator Product Expansion (OPE), on which the bootstrap equations are based. This allows us to extract the maximal error from neglecting operators with dimensions larger than some cutoff Δ_* in the bootstrap equations and thus to consistently truncate them. These truncated bootstrap equations can then be evaluated at different points in the z-plane. This method, which we denote as the "multipoint method", has been previously advocated by Hogervorst and Rychkov in ref. [25] but has not yet been numerically implemented. The aim of this chapter is to provide such an implementation and study the resulting bounds. It is important to emphasize that the method of ref. [25] combines what are in principle two independent ideas: i) multipoint bootstrap and ii) truncation of the bootstrap equations. One could study i) without ii), or try to analyze ii) without i). We will not consider these other possibilities here.

We begin section 6.1 with a brief review of the results of refs. [22, 25, 41] on the convergence of the OPE. We use generalized free theories as a toy laboratory to test some of the results obtained in ref. [22]. In particular, we emphasize that in the kinematical limit $z, \bar{z} \to 1$, the behaviour of any euclidean CFT in $d > 2$ dimensions approaches that of a generalized free theory, very much like what happens in the Lorentzian case

in the limit $z \to 0$ with $\bar{z}$ fixed [23, 24]. We then generalize the results of ref. [22] for CFTs with an $O(n)$ global symmetry. For concreteness, we study bounds on operator dimensions and the central charge in 3D and 4D CFTs, with and without an $O(n)$ global symmetry (with no supersymmetry). For these bounds, extensive results are already available in the literature (see e.g. refs. [46–48, 50, 54, 58, 61, 101, 103, 105, 108, 115, 116]). In particular, we focus our attention on the regions where the 3D Ising and $O(n)$ vector models have been identified. We show how the results depend on the number N of points in the z-plane at which we evaluate the bootstrap equations and the cut-off Δ_* on the dimension of operators in the bootstrap equations. Using values for the dimension of the operator ϕ in $O(n)$ vector models available in the literature and a fit extrapolation procedure, we then determine the dimensions of the second-lowest $O(n)$ singlet and symmetric-traceless operators S' and T' for $n = 2, 3, 4$. To our knowledge, these have not been obtained before using bootstrap techniques. Our results are consistent with those from analytical calculations using the ϵ-expansion [117, 118] with a mild tension with the result of ref. [118] for the dimension of T' in the $O(2)$ model. We notice from our results that the "kink" in the bound on the dimension of the lowest scalar (singlet) operator in 3D Ising and $O(n)$ vector models is already visible for relatively small Δ_*, while the minimum in the central-charge bound is very sensitive to Δ_*. For our numerical implementation, we discretize the spectrum and formulate the bootstrap equations as a linear program which we solve using the optimizer `CPLEX`[1] by `IBM`. Since we focus on the truncated bootstrap equations with relatively low cutoff Δ_*, double precision as used by `CPLEX` is sufficient for our purposes. More refined implementations with higher numerical precision, possibly adapting the method and optimizer of refs. [103, 106], are certainly possible. More details on the numerical implementation are given in section 6.4.

6.1 Convergence of the OPE

We begin with a brief review of the results of refs. [22, 41] (see also ref. [25]) about the convergence of the OPE in a euclidean CFT in any number of dimensions.[2] For more details see the original references. Consider the 4-point function of a scalar primary

[1] `http://www-01.ibm.com/software/commerce/optimization/cplex-optimizer/`

[2] Bounds on the OPE convergence are obtained in an alternative way using crossing symmetry in ref. [119]. Interestingly, ref. [119] sets bounds which are also valid for finite values of Δ_* at $z = \bar{z} = 1/2$, though they are relative and not absolute bounds. It would be interesting to explore the approach followed in this paper further. We thank Slava Rychkov for having pointed out this reference to us.

operator ϕ with scaling dimension Δ_ϕ:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \;=\; \frac{G(u,v)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}}\,, \tag{6.1}$$

where

$$u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \quad \text{and} \quad v \equiv \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \tag{6.2}$$

are the conformally-invariant cross-ratios ($x_{ij} \equiv x_i - x_j$). Applying the OPE to the operator pairs $\phi(x_1)\phi(x_2)$ and $\phi(x_3)\phi(x_4)$ in the 4-point function, one can write

$$G(u,v) = 1 + \sum_{\Delta,\ell} \lambda_{\mathcal{O}}^2\, g_{\Delta,\ell}(z,\bar{z})\,, \tag{6.3}$$

where $u = z\bar{z}$, $v = (1-z)(1-\bar{z})$ and the sum runs over all primary operators $\mathcal{O}$ that appear in the $\phi \times \phi$ OPE with Δ and ℓ being respectively their dimension and spin. For each primary, the sum over all its descendants is encoded in the conformal block function $g_{\Delta,\ell}(z,\bar{z})$. In a euclidean CFT, $\bar{z} = z^*$ and the conformal blocks are regular everywhere in the complex z-plane, with the exception of a branch-cut along the real line $[1,+\infty)$.[3] If we impose unitarity (reflection positivity in the euclidean), the OPE coefficients $\lambda_{\mathcal{O}}$ are real and thus $\lambda_{\mathcal{O}}^2 > 0$.

Crucial for our considerations will be a bound on the remainder

$$\sum_{(\Delta \geq \Delta_*),\ell} \lambda_{\mathcal{O}}^2\, g_{\Delta,\ell}(z,\bar{z}) \tag{6.4}$$

of the sum in eq. (6.3) when it is truncated at some primary operator of dimension $\Delta = \Delta_*$. To determine this bound, one first uses that

$$|g_{\Delta,\ell}(z,\bar{z})| \;\leq\; g_{\Delta,\ell}(|z|,|\bar{z}|) \tag{6.5}$$

as follows e.g. from a representation of the conformal blocks in terms of Gegenbauer polynomials [25]. It is therefore sufficient to estimate the remainder for real $z = \bar{z}$. As was found in ref. [22], the most stringent bound is obtained by using the coordinate

$$\rho(z) = \frac{z}{(1+\sqrt{1-z})^2}\,. \tag{6.6}$$

The z-plane is mapped to the unit disk in ρ and the branch-cut is mapped to the boundary of the disk. The conformal blocks in ρ are then defined for $|\rho| < 1$. In the

[3] The branch-cut is best seen in Lorentzian signature, where z and $\bar{z}$ are two independent variables. At fixed $\bar{z}$ (z), $g_{\Delta,\ell}(z,\bar{z})$ is a true analytic function in z ($\bar{z}$) with a branch-cut along the line $[1,+\infty)$.

manifestly reflection positive configuration with $\bar{\rho} = \rho = r$, the function $G(u, v)$ in eq. (6.3) can be written as[4]

$$g(r) = 1 + \sum_{\Delta,\ell} \lambda_{\mathcal{O}}^2 \sum_{n=0}^{\infty} c_n(\Delta, \ell) r^{\Delta+n} \,, \tag{6.7}$$

where $c_n(\Delta, \ell)$ are positive coefficients whose explicit form is not important here and the sum over n takes into account the contributions from the descendants of each primary. It is convenient to rewrite $g(r)$ as

$$g(\beta) = \int_0^\infty d\Delta \, f(\Delta) e^{-\beta\Delta} \quad \text{with} \quad f(\Delta) = \sum_k \rho_k \, \delta(\Delta - \Delta_k) \,. \tag{6.8}$$

Here $\beta \equiv -\log r$, k runs over all operators (primaries and their descendants) which are exchanged in the OPE and $f(\Delta)$ is a spectral density with positive coefficients ρ_k. Again, their explicit form is not relevant for our considerations.

The behaviour of $g(\beta)$ in the limit $\beta \to 0$ (corresponding to the OPE limit $x_3 \to x_2$, in which case $z \to r \to 1$ and $1 - z \to \beta^2/4 \to 0$) is dominated by the exchange of the identity operator and one finds:[5]

$$g(\beta) \underset{\beta\to 0}{\sim} 2^{4\Delta_\phi} \beta^{-4\Delta_\phi} \,. \tag{6.9}$$

Here $a \sim b$ means that $a/b \to 1$ in the considered limit. The key observation of ref. [22] is that since the coefficients ρ_k are all positive, this asymptotic behaviour determines the leading, large-Δ behaviour of the integrated spectral density

$$F(\Delta) = \int_0^\Delta f(\Delta') \, d\Delta' \tag{6.10}$$

by means of the Hardy-Littlewood tauberian theorem (see e.g. [120]):[6]

$$F(\Delta) \underset{\Delta\to\infty}{\sim} \frac{(2\Delta)^{4\Delta_\phi}}{\Gamma(4\Delta_\phi + 1)} \,. \tag{6.11}$$

The remainder (6.4) can then be bounded as follows: We first note that

$$\sum_{(\Delta \geq \Delta_*),\ell} \lambda_{\mathcal{O}}^2 \, g_{\Delta,\ell}(\beta) \leq \int_{\Delta_*}^\infty f(\Delta) e^{-\beta\Delta} \, d\Delta \;\; , \tag{6.12}$$

[4] For simplicity, we use the same symbol to denote the functions $G(u, v)$ and $\tilde{g}(r) = G(u(r), v(r))$ etc. here and below.

[5] This is true in general only in $d > 2$ dimensions. In $d = 2$, one has to be careful since scalar operators can have arbitrarily small dimensions. See also the discussion after eq. (6.23).

[6] It is in fact sufficient that the coefficients are all positive for operators with dimension larger than some fixed value Δ_0.

since the r.h.s. contains contributions from all operators with dimension larger than Δ_*, whereas on the l.h.s. only primaries with dimension larger than Δ_* and their descendents contribute. Using eq. (6.11), the r.h.s. can in turn be bounded as

$$\begin{aligned}\int_{\Delta_*}^{\infty} f(\Delta) e^{-\beta\Delta}\, d\Delta &= \beta \int_{\Delta_*}^{\infty} e^{-\beta\Delta} (F(\Delta) - F(\Delta_*))\, d\Delta \,\leq\, \beta \int_{\Delta_*}^{\infty} e^{-\beta\Delta} F(\Delta) d\Delta \\ &\simeq \beta \int_{\Delta_*}^{\infty} e^{-\beta\Delta} \frac{(2\Delta)^{4\Delta_\phi}}{\Gamma(4\Delta_\phi+1)} d\Delta \,=\, \frac{\beta^{-4\Delta_\phi}\, 2^{4\Delta_\phi}}{\Gamma(4\Delta_\phi+1)}\, \Gamma(4\Delta_\phi+1, \beta\Delta_*)\end{aligned} \tag{6.13}$$

where $\Gamma(a,b)$ is the incomplete Gamma function. Clearly, this bound applies for parametrically large values of Δ_*, where eq. (6.11) holds. Using eq. (6.5), we finally get the bound on the remainder

$$\Big| \sum_{(\Delta\geq\Delta_*),\ell} \lambda_{\mathcal{O}}^2\, g_{\Delta,\ell}(z,\bar{z}) \Big| \,\leq\, \frac{(-\log|\rho(z)|)^{-4\Delta_\phi} 2^{4\Delta_\phi}}{\Gamma(4\Delta_\phi+1)} \Gamma(4\Delta_\phi+1, -\Delta_* \log|\rho(z)|)\,. \tag{6.14}$$

This is valid in any number $d>2$ of dimensions for 4-point functions with identical scalars.

It was pointed out in ref. [41] that the conditions for the applicability of the Hardy-Littlewood tauberian theorem in both 3 and 4 dimensions are also fulfilled for the rescaled conformal blocks

$$\tilde{g}_{\Delta,\ell}(r) \equiv (1-r^2)^\gamma g_{\Delta,\ell}(r) \tag{6.15}$$

with $\gamma = 1$. Repeating the derivation reviewed above for a remainder involving the rescaled conformal blocks, it is straightforward to get the alternative bound

$$\Big| \sum_{(\Delta\geq\Delta_*),\ell} \lambda_{\mathcal{O}}^2\, g_{\Delta,\ell}(z,\bar{z}) \Big| \,\leq\, \mathcal{R}(z,\bar{z},\Delta_*,\Delta_\phi,\gamma) \tag{6.16}$$

with

$$\mathcal{R}(z,\bar{z},\Delta_*,\Delta_\phi,\gamma) \,\equiv\, \frac{(-\log|\rho(z)|)^{-4\Delta_\phi+\gamma}\, 2^{4\Delta_\phi+\gamma}}{\Gamma(4\Delta_\phi+1-\gamma)} \frac{\Gamma(4\Delta_\phi+1-\gamma, -\Delta_*\log|\rho(z)|)}{(1-|\rho(z)|^2)^\gamma}\,. \tag{6.17}$$

For $-\Delta_* \log|\rho(z)| \gg 1$, eq. (6.17) can be approximated as

$$\mathcal{R}(z,\bar{z},\Delta_*,\Delta_\phi,\gamma) \,\approx\, \frac{2^{4\Delta_\phi+\gamma}\, \Delta_*^{4\Delta_\phi-\gamma}}{\Gamma(4\Delta_\phi+1-\gamma)} \frac{|\rho(z)|^{\Delta_*}}{(1-|\rho(z)|^2)^\gamma}\,. \tag{6.18}$$

We see that for $|\rho(z)|$ not too close to 1 and $\Delta_* \gtrsim 8\Delta_\phi$, the bound is more stringent for $\gamma=1$ than for $\gamma=0$. It was furthermore shown in ref. [41] that in $d=3$ dimensions, $\gamma=1$ is the maximal allowed value such that the Hardy-Littlewood tauberian theorem

remains applicable, whereas it was conjectured without proof that the maximal allowed value in $d = 4$ dimensions is $\gamma = 3/2$. Correspondingly we use eq. (6.17) with $\gamma = 1$ for the remainder both in 3 and 4 dimensions in our numerical implementation.[7]

The above derivations were based on the existence of a configuration for which the function $G(u, v)$ turns into a positive definite function of a single variable. The remainder is then estimated using the Hardy-Littlewood tauberian theorem. One cannot naively apply these arguments to arbitrary derivatives of $G(u, v)$ w.r.t. u and v, unless the resulting functions remain positive definite and derivatives can be brought inside the absolute value in the l.h.s. of eq. (6.16). See the appendix of ref. [123] for a recent discussion on how to estimate the remainder on derivatives of $G(u, v)$. It would be interesting to verify if this allows us to also study truncated bootstrap equations with the derivative method.

6.1.1 Comparison with Generalized Free Theories and Asymptotics for $z \to 1$

The results reviewed in the previous subsection are based on eq. (6.11) which holds in the limit $\Delta_* \to \infty$. Of course, for any practical use, we need to know the value of Δ_* beyond which we can trust eq. (6.11) and thus the bound eq. (6.16). It is difficult to determine this value for a generic CFT. But we can get useful insights by considering exactly calculable CFTs, like generalized free theories (sometimes called mean field theories) for which the CFT data are known and the function $G(u, v)$ in eq. (6.1) in any number of dimensions reads

$$G(u, v) \;=\; 1 + u^{\Delta_\phi} + \left(\frac{u}{v}\right)^{\Delta_\phi} = 1 + |z|^{2\Delta_\phi} + \left(\frac{|z|}{|1-z|}\right)^{2\Delta_\phi}. \tag{6.19}$$

For values of Δ_* such that eq. (6.11) is no good approximation, the r.h.s. of eq. (6.16) can clearly still overestimate the actual remainder, leading to no inconsistency. On the other hand, if it underestimates the actual remainder, eq. (6.16) is simply wrong. We

[7] The fact that eq. (6.16) with $\gamma = 0$ is not optimal can be traced to using the inequality (6.12) in the derivation. In order to make the bound more stringent, one could then alternatively use the series representation in ref. [25] which includes contributions from primary operators and their descendants separately. Using this series truncated at contributions corresponding to dimension Δ_* instead of the full conformal blocks $g_{\Delta,\ell}$ would make the r.h.s. of the inequality (6.12) the actual remainder to be bounded. This would thus make eq. (6.16) with $\gamma = 0$ more stringent. Here, however, we choose not to follow this approach. The reason is that the representations for the full conformal blocks $g_{\Delta,\ell}$ can be considerably faster calculated than (our implementation of) the truncated series representation of ref. [25].

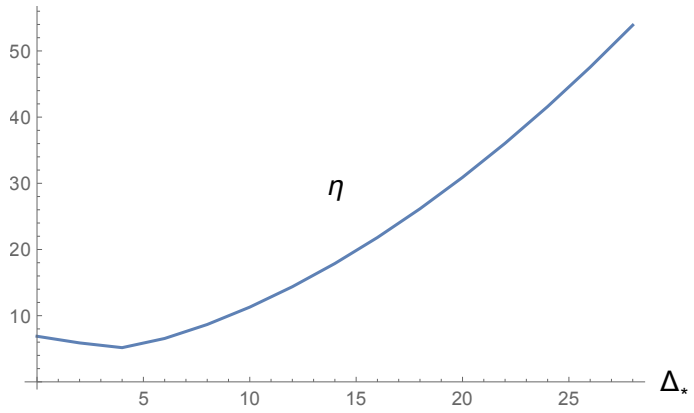

Figure 6.1: η defined in eq. (6.20) as a function of Δ_* in a generalized free theory in $d = 4$ dimensions evaluated at the symmetric point $z = \bar{z} = 1/2$. We have taken $\Delta_\phi = 1.5$ and $\gamma = 1$.

define

$$\eta \equiv \frac{\mathcal{R}(z, \bar{z}, \Delta_*, \Delta_\phi, \gamma)}{\left| \sum_{(\Delta \geq \Delta_*), \ell} \lambda^2_{\mathcal{O}}\, g_{\Delta,\ell}(z, \bar{z}) \right|} \tag{6.20}$$

and check if and when η is smaller than 1, in which case eq. (6.16) is violated. The denominator in eq. (6.20) is computed as

$$\sum_{(\Delta \geq \Delta_*), \ell} \lambda^2_{\mathcal{O}}\, g_{\Delta,\ell}(z, \bar{z}) = G(u, v) - 1 - \sum_{(\Delta < \Delta_*), \ell} \lambda^2_{\mathcal{O}}\, g_{\Delta,\ell}(z, \bar{z})\,. \tag{6.21}$$

In fig. 6.1, we show η as a function of Δ_* evaluated at the symmetric point $z = \bar{z} = 1/2$. Notice that at the point of best convergence the actual remainder is always significantly smaller than $\mathcal{R}$, and that the ratio gets bigger and bigger as Δ_* increases for large Δ_*. In particular, η is greater than 1 for *any value* of Δ_*. We have performed comparisons with GFTs in $d = 3$ dimensions with $\gamma = 0, 1$ and $d = 4$ dimensions with $\gamma = 0, 3/2$ for different values of z and Δ_ϕ within the unitary bounds, finding analogous qualitative results. Somehow unexpectedly, we find that the bound (6.16) is never violated in GFTs, for any value of Δ_*.

When $z \to 1$, both the numerator and the denominator of η in eq. (6.20) blow up, since the OPE is not convergent at $z = \bar{z} = 1$. Operators with high scaling dimension are no longer suppressed and the remainder completely dominates the OPE.[8] More precisely, we have

$$\mathcal{R}(z, \bar{z}, \Delta_*, \Delta_\phi, \gamma) \underset{z, \bar{z} \to 1^-}{\sim} 2^{4\Delta_\phi} (-\log |\rho(z)|)^{-4\Delta_\phi}\,, \tag{6.22}$$

[8] In this limit, the name remainder should actually be used for the finite sum of operators up to Δ_*.

independently of γ. On the other hand, from eq. (6.19) we have

$$G(u,v) \underset{z,\bar{z}\to 1^-}{\sim} \frac{1}{|1-z|^{2\Delta_\phi}} \underset{z,\bar{z}\to 1^-}{\sim} 2^{4\Delta_\phi}(-\log|\rho(z)|)^{-4\Delta_\phi}\,, \tag{6.23}$$

where in the last equality we have used that $|1-z| \to (\log|\rho(z)|)^2/4$ in the limit. Interestingly enough, GFTs saturate the remainder $\mathcal{R}$ in this limit, for any choice of γ and number of dimensions!

This finding has close analogies to the study of the light-cone limit of Lorentzian CFTs performed in refs. [23, 24]. It was found there that the spectrum of any CFT resembles that of a GFT for parametrically large spin operators. In particular, in ref. [23], this has been established by analyzing crossing symmetry in the limit $z \to 0$ and $\bar{z}$ fixed for $d > 2$, where large twist operators are suppressed. In the euclidean, in contrast, the twist does not play a role, and operators of any twist should be considered. The two-dimensional case is more subtle, because there is no longer a gap between the identity (which has the minimum twist zero) and the other operators. Indeed, the results of refs. [23, 24] and those of ref. [22] in the euclidean do not straightforwardly apply for $d = 2$. We see here that for large scaling dimensions any euclidean CFT seems to approach (or include) a GFT for $d > 2$. Expanded around the GFT CFT data, the inequality (6.16) might be interpreted as an integral equation for the asymptotic anomalous dimensions and OPE coefficients for parametrically large operator dimensions. It might be interesting to see if some general properties of euclidean CFTs can be extracted in this way.

6.1.2 Remainder for CFTs with $O(n)$ Symmetry

The generalization of the OPE convergence estimate to CFTs with $O(n)$ global symmetry is straightforward. For concreteness, let us consider scalars ϕ_i in the fundamental representation of $O(n)$. The only non-trivial point is to identify a proper linear combination of 4-point functions

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle \tag{6.24}$$

that leads to a positive definite series expansion, otherwise the Hardy-Littlewood tauberian theorem does not apply. A possible choice is

$$A_\eta \equiv \langle\phi_1\phi_1\phi_1\phi_1\rangle + |\eta|^2\langle\phi_2\phi_2\phi_2\phi_2\rangle + \eta\langle\phi_1\phi_1\phi_2\phi_2\rangle + \eta^*\langle\phi_2\phi_2\phi_1\phi_1\rangle = \frac{a_\eta(u,v)}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}}\,, \tag{6.25}$$

where for simplicity we have omitted the x-dependence of the fields. The parameter η can in general take an arbitrary complex value, but it is enough for our purposes to consider $\eta = \pm 1$. For $\bar{\rho} = \rho = r$ and any η, this correlator is manifestly positive definite, because it corresponds to the norm of the state

$$\phi_1|\phi_1\rangle + \eta\phi_2|\phi_2\rangle\,. \tag{6.26}$$

The leading term in $a_\eta(u,v)$ for $x_2 \to x_3$ is given by the exchange of the identity operator in the first two correlators and hence is independent of η. On the other hand, expanding in conformal blocks in the (12)-(34) channel, we have [48]

$$A_\eta = \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}}\Big(2(1+\eta)\Big(1+\sum_{S^+}\lambda_S^2\, g_{\Delta,\ell}(u,v)\Big) + 4\Big(1-\frac{1+\eta}{n}\Big)\sum_{T^+}\lambda_T^2\, g_{\Delta,\ell}(u,v)\Big)\,, \tag{6.27}$$

where S and T denote operators in the singlet and rank-two symmetric representations of $O(n)$, respectively. Both sums run over even spins. We can now repeat essentially verbatim the derivation below eq. (6.6). For $\eta = -1$, this gives rise to the bound

$$\Big|\sum_{(\Delta\geq\Delta_*),\ell}\lambda_T^2\, g_{\Delta,\ell}(z,\bar{z})\Big| \leq \frac{1}{2}\mathcal{R}(z,\bar{z},\Delta_*,\Delta_\phi,\gamma)\,, \tag{6.28}$$

where $\mathcal{R}$ is given in eq. (6.17). The factor $1/2$ with respect to the non-symmetric case arises because the identity operator is exchanged in two correlators but a factor 4 is present in the second term in the r.h.s. of eq. (6.27). For $\eta = 1$ we similarly get

$$\Big|\sum_{(\Delta\geq\Delta_*),\ell}\Big(\lambda_S^2\, g_{\Delta,\ell}(z,\bar{z}) + \Big(1-\frac{2}{n}\Big)\lambda_T^2\, g_{\Delta,\ell}(z,\bar{z})\Big)\Big| \leq \frac{1}{2}\mathcal{R}(z,\bar{z},\Delta_*,\Delta_\phi,\gamma)\,. \tag{6.29}$$

Another positive definite linear combination of correlators is

$$B_\eta \equiv \langle\phi_2\phi_1\phi_1\phi_2\rangle + |\eta|^2\langle\phi_1\phi_2\phi_2\phi_1\rangle + \eta\langle\phi_2\phi_1\phi_2\phi_1\rangle + \eta^*\langle\phi_1\phi_2\phi_1\phi_2\rangle = \frac{b_\eta(u,v)}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}}\,, \tag{6.30}$$

corresponding to the norm of the state

$$\phi_1|\phi_2\rangle + \eta\phi_2|\phi_1\rangle\,. \tag{6.31}$$

Again, we consider $\eta = \pm 1$. In the (12)-(34) channel the correlator B_η can be written

as[9]

$$B_\eta = \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \Big(2(1+\eta) \sum_{T^+} \lambda_T^2 \, g_{\Delta,\ell}(u,v) + 2(1-\eta) \sum_{A^-} \lambda_A^2 \, g_{\Delta,\ell}(u,v) \Big) \,, \tag{6.32}$$

where A stands for operators in the rank-two antisymmetric representation of $O(n)$. The first sum runs over even spins, whereas for the second one they are odd. As before, the leading term in $b_\eta(u,v)$ for $x_2 \to x_3$ is given by the exchange of the identity operator in the first two correlators and is independent of η. For $\eta = 1$, eq. (6.32) gives rise to the same bound given in eq. (6.28), while for $\eta = -1$ we have

$$\Big| \sum_{(\Delta \geq \Delta_*),\ell} \lambda_A^2 \, g_{\Delta,\ell}(z,\bar{z}) \Big| \leq \frac{1}{2} \mathcal{R}(z,\bar{z},\Delta_*,\Delta_\phi,\gamma) \,. \tag{6.33}$$

It is straightforward to see that the bounds (6.28), (6.29) and (6.33) are the best that can be obtained. Indeed, in the free-theory limit one has $\lambda_S^2 = \lambda^2/n$, $\lambda_T^2 = \lambda_A^2 = \lambda^2/2$ with λ^2 being the OPE coefficients for a single free field (see e.g. eq. 5.30). The above three bounds then reduce to eq. (6.16) which is known to give the best bound on the r.h.s. of eq. (6.12) (see however footnote 7) [22]. Any potentially better bound for $O(n)$ theories should in particular apply to the free theory, but would then be in contradiction with the results of ref. [22].

The above bounds will be used in the next section to bound the remainder of the bootstrap equations in CFTs with an $O(n)$ global symmetry.

6.2 Bootstrapping with Multiple Points

The bootstrap equation for a 4-point function with identical scalars ϕ with scaling dimension Δ_ϕ in any number of dimensions is given by the sum rule (see refs. [124, 125] for pedagogical reviews)

$$\sum_{\Delta,\ell} \lambda_{\mathcal{O}}^2 \, \mathcal{F}_{\Delta_\phi,\Delta,\ell}(z,\bar{z}) \, = \, u^{\Delta_\phi} - v^{\Delta_\phi} \,, \qquad \mathcal{F}_{\Delta_\phi,\Delta,\ell}(z,\bar{z}) \, \equiv \, v^{\Delta_\phi} g_{\Delta,\ell}(u,v) - u^{\Delta_\phi} g_{\Delta,\ell}(v,u) \,. \tag{6.34}$$

[9] In our normalization conventions for the conformal blocks, the squared OPE coefficients $\lambda_{S,T,A}^2$ are all positive.

Splitting the sum into two parts, for dimensions smaller and larger than a cutoff Δ_*, we can write

$$\sum_{(\Delta<\Delta_*),\ell} \lambda_{\mathcal{O}}^2 \, \mathcal{F}_{\Delta_\phi,\Delta,\ell}(z,\bar{z}) \;=\; u^{\Delta_\phi} - v^{\Delta_\phi} \;+\; \mathcal{E}(z,\bar{z},\Delta_*,\Delta_\phi)\,. \tag{6.35}$$

Using eq. (6.16), the remainder $\mathcal{E}$ of the sum rule is bounded by

$$|\mathcal{E}(z,\bar{z})| \;\leq\; \mathcal{E}_{\max}(z,\bar{z}) \;\equiv\; v^{\Delta_\phi}\,\mathcal{R}(z,\bar{z}) \;+\; u^{\Delta_\phi}\,\mathcal{R}(1-z,1-\bar{z})\,, \tag{6.36}$$

where we have omitted the dependence on Δ_*, Δ_ϕ and γ. The truncated sum rule (6.35) still involves a generally unknown spectrum of operators up to dimension Δ_*. In order to make it amenable to numerical analysis, we discretize the spectrum and make the ansatz[10]

$$\left\{(0,\frac{d-2}{2})\,,(0,\frac{d-2}{2}{+}\Delta_{\text{step}})\,,\ldots\,,(0,\Delta_*)\,,(2,d)\,,(2,d{+}\Delta_{\text{step}})\,,(2,\Delta_*)\,,\ldots\,,(\ell_{\max},\Delta_*)\right\} \tag{6.37}$$

for the quantum numbers (spin,dimension) of the operators that can appear in the truncated sum rule. For each spin ℓ, the dimension runs in steps of size Δ_{step} from the unitarity bound $\Delta_{\min}^{d,\ell} \equiv \ell + (d-2)/(1+\delta_{\ell 0})$ to the cutoff Δ_* (or a value close to that, depending on Δ_{step}). Accordingly, $\ell_{\max}$ is the largest spin for which the unitarity bound is still below the cutoff, $\Delta_{\min}^{d,\ell_{\max}} < \Delta_*$. In practice, we vary the step size Δ_{step} somewhat depending on the spin and dimension. This is discussed in more detail in sec. 6.4. We find that the bounds converge when going to smaller Δ_{step}, meaning that the discretization does not introduce any artifacts into our calculation.

We similarly choose a finite number of points z_i in the z-plane where the sum rule is evaluated. The details of our choice for this distribution of points are discussed in sec. 6.2.1. Together with the discretization of operator dimensions, this turns eq. (6.35) into the matrix equation

$$\mathcal{M}\cdot\vec{\rho} \;=\; \vec{\sigma} \;+\; \vec{\epsilon}\,. \tag{6.38}$$

The elements of the matrix $\mathcal{M}$ are the functions $\mathcal{F}_{\Delta_\phi,\Delta,\ell}(z,\bar{z})$ evaluated for the different quantum numbers in eq. (6.37) along the rows and for the different points z_i along the columns. Furthermore, the vector $\vec{\rho}$ consists of the squared OPE coefficients $\lambda_{\mathcal{O}}^2$ of the operators corresponding to the quantum numbers in eq. (6.37) and

$$\vec{\sigma} \;\equiv\; \begin{pmatrix} |z_1|^{2\Delta_\phi} \;-\; |1-z_1|^{2\Delta_\phi} \\ |z_2|^{2\Delta_\phi} \;-\; |1-z_2|^{2\Delta_\phi} \\ \vdots \end{pmatrix} \quad \text{and} \quad \vec{\epsilon} \equiv \begin{pmatrix} \mathcal{E}(z_1,\bar{z}_1,\Delta_*,\Delta_\phi) \\ \mathcal{E}(z_2,\bar{z}_2,\Delta_*,\Delta_\phi) \\ \vdots \end{pmatrix}. \tag{6.39}$$

[10] Alternatively, one could adapt the approach of ref. [103] to the multipoint method.

Using the bound (6.36), we then obtain the matrix inequality

$$\begin{pmatrix} \mathcal{M} \\ -\mathcal{M} \end{pmatrix} \vec{\rho} \geq \begin{pmatrix} \vec{\sigma} - \vec{\epsilon}_{\max} \\ -\vec{\sigma} - \vec{\epsilon}_{\max} \end{pmatrix} , \tag{6.40}$$

where $\vec{\epsilon}_{\max}$ is defined as $\vec{\epsilon}$ but with $\mathcal{E}$ replaced by $\mathcal{E}_{\max}$. This is the starting point for our numerical calculations. In order to determine bounds on OPE coefficients, we search for vectors $\vec{\rho}$ which satisfy eq. (6.40) and extremize the entry corresponding to that OPE coefficient. For bounds on the dimension of the lowest-lying scalar operator, on the other hand, we make an assumption on this dimension and drop all scalar operators with smaller dimension from our ansatz (6.37). This gap then allows for a consistent CFT only if there exists a vector $\vec{\rho}$ which satisfies eq. (6.40) with the reduced ansatz. By trying different assumptions, we can determine the maximal allowed gap. Both problems are linear programs which can be solved using fast numerical routines. An advantage of solving eq. (6.40) is that the vector $\vec{\rho}$ gives us the spectrum of operators and their OPE coefficients of a potential CFT living at the boundary of the allowed region. This has been used before in ref. [103].[11]

We also consider CFTs with an $O(n)$ global symmetry. For an external scalar operator in the fundamental representation of $O(n)$, the sum rule reads [48]

$$\sum_{S^+} \lambda_S^2 \begin{pmatrix} 0 \\ \mathcal{F} \\ \mathcal{H} \end{pmatrix} + \sum_{T^+} \lambda_T^2 \begin{pmatrix} \mathcal{F} \\ (1-\frac{2}{n})\mathcal{F} \\ -(1+\frac{2}{n})\mathcal{H} \end{pmatrix} + \sum_{A^-} \lambda_A^2 \begin{pmatrix} -\mathcal{F} \\ \mathcal{F} \\ -\mathcal{H} \end{pmatrix} = \begin{pmatrix} 0 \\ u^{\Delta_\phi} - v^{\Delta_\phi} \\ -u^{\Delta_\phi} - v^{\Delta_\phi} \end{pmatrix} , \tag{6.41}$$

where $\mathcal{H}_{\Delta_\phi,\Delta,l}(z,\bar{z}) \equiv v^{\Delta_\phi} g_{\Delta,l}(u,v) + u^{\Delta_\phi} g_{\Delta,l}(v,u)$ and we have suppressed the arguments of the functions $\mathcal{F}$ and $\mathcal{H}$. Splitting the sums in eq. (6.41) into two parts, for dimensions smaller and larger than a cutoff Δ_*, we can write

$$\sum_{\substack{S^+ \\ \Delta<\Delta_*}} \lambda_S^2 \begin{pmatrix} 0 \\ \mathcal{F} \\ \mathcal{H} \end{pmatrix} + \sum_{\substack{T^+ \\ \Delta<\Delta_*}} \lambda_T^2 \begin{pmatrix} \mathcal{F} \\ (1-\frac{2}{n})\mathcal{F} \\ -(1+\frac{2}{n})\mathcal{H} \end{pmatrix} + \sum_{\substack{A^- \\ \Delta<\Delta_*}} \lambda_A^2 \begin{pmatrix} -\mathcal{F} \\ \mathcal{F} \\ -\mathcal{H} \end{pmatrix} = \begin{pmatrix} \mathcal{E}_1 \\ u^{\Delta_\phi} - v^{\Delta_\phi} + \mathcal{E}_2 \\ -u^{\Delta_\phi} - v^{\Delta_\phi} + \mathcal{E}_3 \end{pmatrix} . \tag{6.42}$$

Using eqs. (6.28), (6.29) and (6.33), we obtain the bounds on the remainders

$$|\mathcal{E}_{1,2}(z,\bar{z})| \leq \mathcal{E}_{\max}(z,\bar{z}) , \qquad |\mathcal{E}_3(z,\bar{z})| \leq 2\,\mathcal{E}_{\max}(z,\bar{z}) , \tag{6.43}$$

with $\mathcal{E}_{\max}$ defined as in eq. (6.36). Discretizing the space of operator dimensions as in eq. (6.37) and evaluating the sum rule at a finite set of points z_i, we again obtain

[11] The data of CFTs at the boundary of the allowed region can also be obtained from the 'dual' method originally developed in ref. [9] by using the extremal functional method of ref. [109].

a matrix inequality of the form (6.40). This is the starting point for our numerical calculations for CFTs with $O(n)$ global symmetry.

6.2.1 Choice of Points

An important choice for the multipoint method is the distribution of points in the z-plane at which the bootstrap equations are evaluated. Using the symmetries $z \leftrightarrow \bar{z}$ and $z \leftrightarrow (1-z)$, $\bar{z} \leftrightarrow 1-\bar{z}$ of the bootstrap equations, we can restrict these points to the region $\mathrm{Re}(z) \geq 1/2$ and $\mathrm{Im}(z) \geq 0$ of the z-plane. The remainder of the truncated sum rule is controlled by $|\rho(z)|$ and $|\rho(1-z)|$ (cf. eqs. (6.18) and (6.36)). Guided by this, we introduce the measure

$$\lambda(z) \equiv |\rho(z)| + |\rho(1-z)|\,, \tag{6.44}$$

and consider points with $\lambda(z) \leq \lambda_c$ for some constant λ_c. It is desirable to choose λ_c and the distribution of points within that region in such a way that the obtained bounds are as stringent as possible. We have performed extensive scans over different values for λ_c and distributions with different density profiles and have found that a flat profile leads to as good or better bounds than more complicated profiles. We therefore choose the former and put points on a grid centered at $z = 1/2$. The grid spacing is chosen such that the desired number of points is within the region $\lambda(z) \leq \lambda_c$, $\mathrm{Re}(z) \geq 1/2$ and $\mathrm{Im}(z) \geq 0$. We have then found that

$$\lambda_c = 0.6 \tag{6.45}$$

gives the best bounds for all cases that we have studied.[12] In fig. 6.2, we show the corresponding region in the z-plane and a sample distribution of 100 points.

In order to test the influence of the choice of measure on the bounds, we have performed further scans with $\lambda(z) \equiv \max(|\rho(z)|, |\rho(1-z)|)$ proposed in ref. [25] and $\lambda(z) \equiv |z - 1/2|$ (for the latter we have removed points at or close to the branch-cuts). We have found that, once the optimal λ_c is chosen, the bounds obtained with these measures are indistinguishable from those obtained with eq. (6.44). This indicates that the precise form of the region within which points are sampled has only a marginal effect on the quality of the bounds.

[12] In more detail, we have considered bounds on the central charge and the dimension of the lowest-dimensional scalar operator, in 3D and 4D, with $O(n)$ and without symmetry, and with different choices for the number of points N and the cutoff Δ_*. It is remarkable that $\lambda_c = 0.6$ (within ± 0.02, the resolution of our scan) comes out as the optimal choice for such a variety of cases.

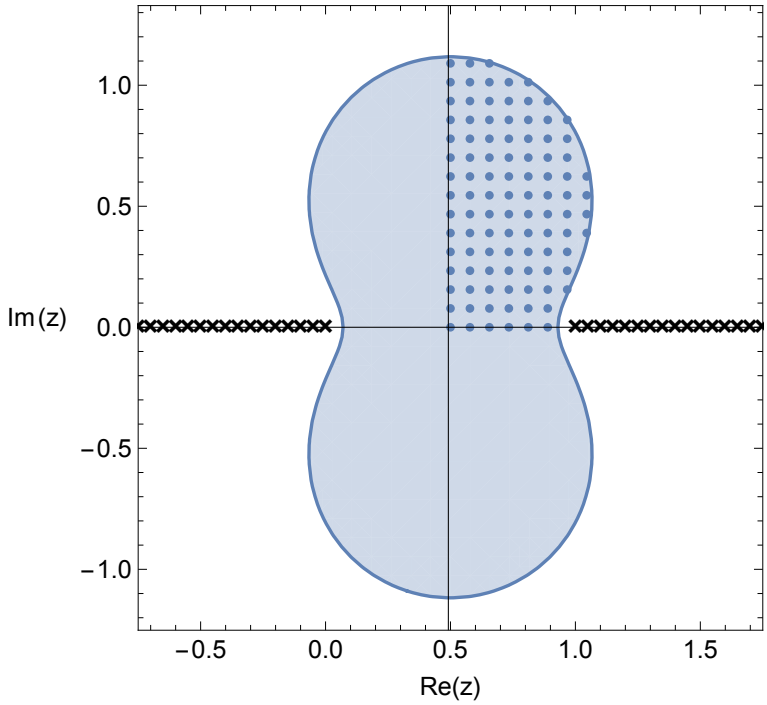

Figure 6.2: The region in the z-plane with $\lambda(z) \leq 0.6$ and a sample of 100 points in a fundamental domain of that region. The crossed lines are the two branch-cuts where the bootstrap equations do not converge.

6.3 Results

We now present the results of our numerical analysis. In subsection 6.3.1, we study bounds on the dimension of the lowest-dimensional scalar operator in the OPE and bounds on the central charge in 3D CFTs, focusing in particular on the regions where the 3D Ising and $O(n)$ models have been identified. In subsection 6.3.2 we then study the same bounds for generic 4D CFTs. We analyze in particular how our results depend on the number N of points chosen in the z-plane, and on the cutoff Δ_*. In subsection 6.3.3 we give a closer look at the spectrum of the 3D $O(n)$ models and determine the operator dimensions of the first two scalar operators in the singlet and rank-two symmetric representation of $O(n)$.

Before presenting our results, it is important to emphasize an important difference between the multipoint and the derivative bootstrap methods. As mentioned in the introduction, in the latter we do not have a reliable way of truncating the OPE series defining the bootstrap equations at some intermediate dimension Δ_*, because we do not have a reliable estimate of the resulting error. We are therefore forced to have Δ_* as large as possible to minimize this error and can only check a posteriori if the chosen Δ_* was sufficient.[13] More than Δ_* (or its analogue), the key parameter that controls

[13]We are a bit sloppy here in order to keep the discussion simple and get to the point. For instance,

the accuracy of the method is given by the total number of derivatives N_D that are applied to the bootstrap equations. Of course, the larger N_D is, the better are the bounds. The accuracy is then limited by the largest N_D that allows the calculation to be performed within an acceptable amount of time with the available computing resources.

In the multipoint method, on the other hand, we can reliably vary Δ_* due to the bound on the remainder of the truncation discussed in sec. 6.1. In addition, we can also vary the number N of points in the z-plane which is the analogue of N_D in the derivative method. The parameter region for the multipoint method corresponding to the typical bootstrap analysis with the derivative method is then very large Δ_* and N as large as possible given the available computing resources. In this paper, on the other hand, we are mostly interested in the regime where Δ_* is *not* very large, with values $\mathcal{O}(10)$-$\mathcal{O}(20)$. We find that for this range of Δ_*, the results converge for $N \sim \mathcal{O}(100)$ and do not improve further if N is increased. This corresponds to the fact that the rank of the matrix $\mathcal{M}$ in the discretized bootstrap equation (6.38) is then $\mathcal{O}(100)$. Note that since `CPLEX` is limited to double precision, we also cannot take Δ_* arbitrarily large. Due to the excellent speed of `CPLEX`, on the other hand, we have found that taking N large enough so that the bounds converge is no limiting factor.

6.3.1 3D Ising and $O(n)$ Models

The most remarkable numerical results from the conformal bootstrap have been obtained in 3D CFTs. One interesting bound to study is on the dimension of the lowest-dimensional scalar operator appearing in the OPE. We denote this operator by ϵ and the operator that is used to derive the bootstrap equations by σ. It was noted in ref. [101] that the 3D Ising model sits at a special point, a kink, at the boundary of the allowed region of Δ_ϵ as a function of Δ_σ. The Ising model is similarly special with respect to the bound on the central charge c as a function of Δ_σ, sitting again at the boundary of the excluded region, at the point where c is minimized [101, 103]. Note, however, that the theory minimizing c does not actually correspond to the 3D Ising model, but rather to some exotic theory with $\Delta_\epsilon < 1$. Most likely this theory is unphysical (though we are not aware of a solid argument to dismiss it). In practice this theory is removed by assuming a gap in the operator spectrum such that $\Delta_\epsilon > 1$. Independently of the nature of this theory, the condition $\Delta_\epsilon > 1$ is satisfied by the Ising model and can be legitimately imposed if we are interested in this particular 3D CFT.

in numerical methods based on semi-definite programming one is able to include all operator dimensions continuously up to infinity. The rough analogue of our Δ_* in that case is the maximum spin of the primary operators entering the OPE which are taken into account for the numerical implementation.

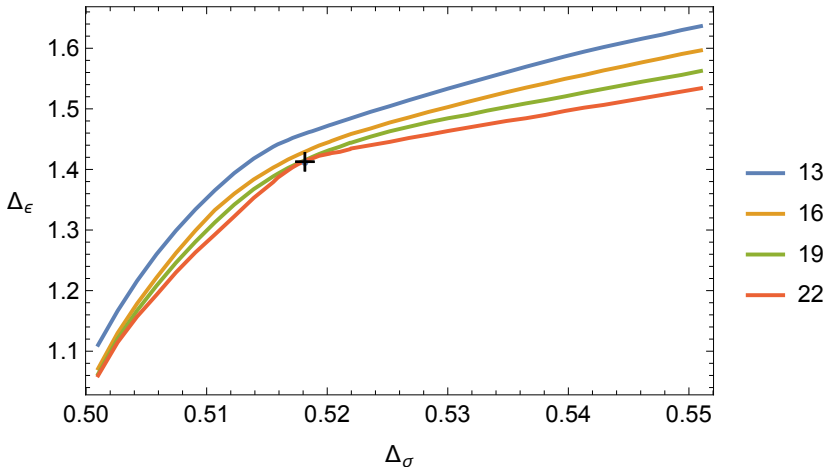

Figure 6.3: Bounds on Δ_ϵ as a function of Δ_σ for $N = 100$ points and different values of Δ_*. The regions above the lines are excluded. The black cross marks the precise values of Δ_σ and Δ_ϵ for the 3D Ising model as determined in ref. [103]. The curves and the labels in the legend have the same order from top to bottom.

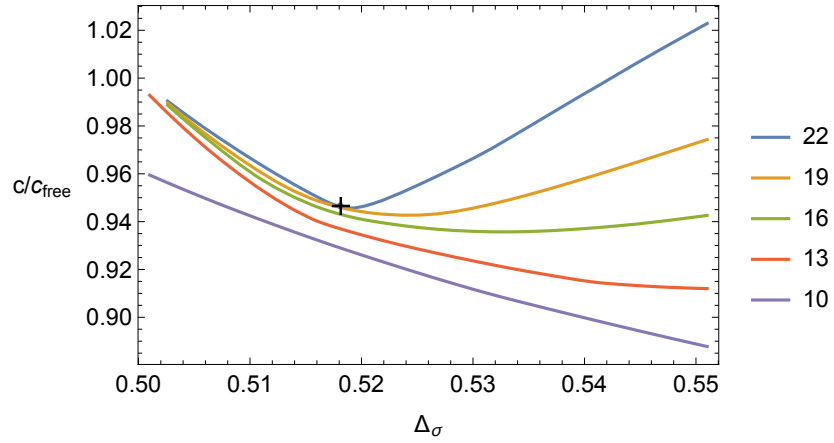

Figure 6.4: Bounds on the central charge c as a function of Δ_σ for $N = 100$ points and different values of Δ_*. A gap $\Delta_\epsilon > 1.1$ has been assumed. The regions below the lines are excluded. The black cross marks the precise values of Δ_σ and c for the 3D Ising model as determined in ref. [103]. The curves and the labels in the legend have the same order from top to bottom.

In fig. 6.3, we show the bound on Δ_ϵ as a function of Δ_σ for $N = 100$ points and different values of Δ_*. Notice how the kink shows up already for $\Delta_* = 13$ and converges quite quickly as Δ_* increases. In fig. 6.4, we show the bound on the central charge c (normalized to the central charge c_{free} of a free scalar theory) as a function of Δ_σ for $N = 100$ points and different values of Δ_*. The gap $\Delta_\epsilon > 1.1$ is assumed in the operator spectrum. A lower bound on c is obtained even for $\Delta_* = 10$, but the convergence when going to larger Δ_* is now much slower than for the bound on Δ_ϵ. A minimum is visible starting from $\Delta_* = 16$ but even at $\Delta_* = 22$ it is a bit shifted to

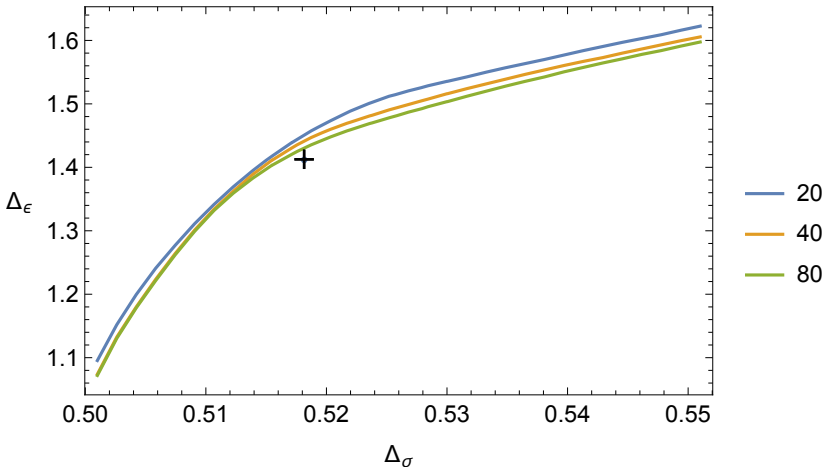

Figure 6.5: Bounds on Δ_ϵ as a function of Δ_σ for fixed $\Delta_* = 16$ and different values of N. The regions above the lines are excluded. The black cross marks the precise values of Δ_σ and Δ_ϵ for the 3D Ising model as determined in ref. [103]. The curves and the labels in the legend have the same order from top to bottom.

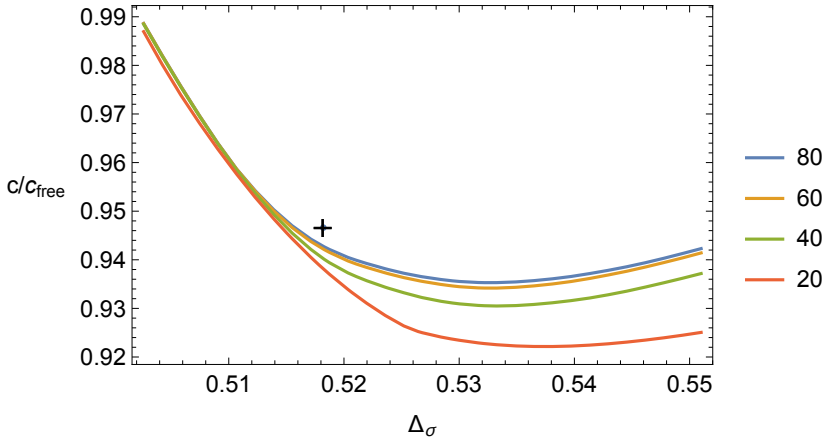

Figure 6.6: Bounds on the central charge c as a function of Δ_σ for fixed $\Delta_* = 16$ and different values of N. The gap $\Delta_\epsilon > 1.1$ is assumed. The regions below the lines are excluded. The black cross marks the precise values of Δ_σ and c for the 3D Ising model as determined in ref. [103]. The curves and the labels in the legend have the same order from top to bottom.

the right with respect to its actual value. We have still not reached the asymptotic value for Δ_*. Unfortunately, we cannot get reliable results for much higher Δ_* because the numerical accuracy of `CPLEX` is limited to double precision. Nevertheless, it is clear from comparing figs. 6.3 and 6.4 that the lower bound on c is more "UV sensitive" than the bound on Δ_ϵ. In both figures, the crosses mark the location of the 3D Ising model, as determined in ref. [103].

In order to quantify the dependence of our results on the number N of points, we show in figs. 6.5 and 6.6 the bounds on respectively Δ_ϵ and c as a function of Δ_σ for

different values of N at fixed $\Delta_* = 16$. We see that in both cases the convergence in N is quite fast, with $N = 40$ for Δ_ϵ and $N = 60$ for c being already an excellent approximation. Notice that for increasing N, the bound on Δ_ϵ converges faster than the bound on c, similar to the dependence on Δ_*. We have studied the dependence on N also for different values of Δ_* and have found as expected that the value N_* beyond which no significant improvement in the bounds is observed increases with Δ_*. The dependence is however very mild for the central charge c and barely observable for Δ_ϵ. This is still a reflection of the different "UV sensitivities" of the two quantities. In all cases, $N_* \lesssim \mathcal{O}(100)$ up to $\Delta_* = 24$.

Let us now turn to 3D CFTs with $O(n)$ global symmetry. We consider a primary operator ϕ in the fundamental representation and denote the lowest-dimensional scalar singlet operator in the $\phi \times \phi$ OPE by S. It was found in refs. [54, 108] that these CFTs have kinks in the bound on Δ_S as a function of Δ_ϕ similar to that found for the Ising model. Moreover, the kinks coincide, for all values of n that have been studied, with the values of Δ_ϕ and Δ_S associated with the 3D $O(n)$ models. On the other hand, a minimum in c no longer occurs for generic $O(n)$ models and the lower bound on c instead monotonically decreases for $n > 3$ (see ref. [54] for details).

In figs. 6.7 and 6.8, we show respectively the bound on Δ_S and c (the latter normalized to the central charge nc_{free} of n free scalars) as a function of Δ_ϕ for different $O(n)$ symmetries, at fixed $N = 80$ and $\Delta_* = 16$. For the central charge, gaps $\Delta_S > 1$ and $\Delta_T > 1$ in the spectrum of respectively singlet operators S and rank-two symmetric-traceless operators T are assumed as in ref. [54]. This assumption is satisfied for the $O(n)$ models and leads to more stringent bounds. The dashed line corresponds to the leading large-n prediction. All the qualitative behaviours found in ref. [54] are reproduced, though with milder bounds, as expected.[14] In particular, the kinks in the (Δ_ϕ-Δ_S) plane are not well visible at $\Delta_* = 16$. In figs. 6.9 and 6.10, we show the same bounds on Δ_S and c as a function of Δ_ϕ at fixed N and n, for different values of Δ_*. We see the same qualitative behaviours regarding the "UV sensitivities" found for 3D CFTs with no global symmetry (the Ising model). In particular, in fig. 6.9 we see how the kink in the bound becomes well visible at $\Delta_* = 19$ and its location is in very good agreement with that found in ref. [54]. On the other hand, the central-charge bound in fig. 6.10 is still monotonically decreasing for $\Delta_* = 19$. There are no signs of convergence comparing the bounds at $\Delta_* = 16$ and 19, indicating the need to go to larger Δ_* to approach the optimal bound.

[14] Note however that no assumption on the spectrum was made for the bounds on Δ_S presented in fig. 6.7, in contrast to fig. 2 of ref. [54] where $\Delta_T > 1$ was assumed.

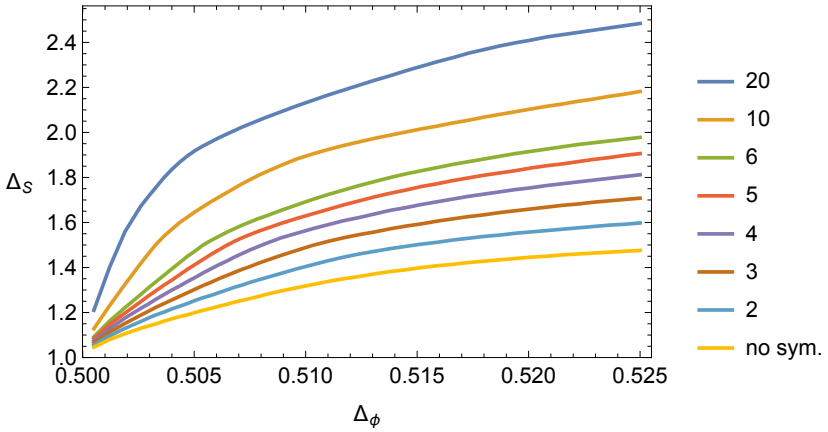

Figure 6.7: Bounds on Δ_S as a function of Δ_ϕ for 3D CFTs with different $O(n)$ symmetries, with ϕ in the fundamental representation of $O(n)$. The regions above the lines are excluded. All the bounds have been determined using $N = 80$ points and $\Delta_* = 16$. The curves and the labels in the legend have the same order from top to bottom.

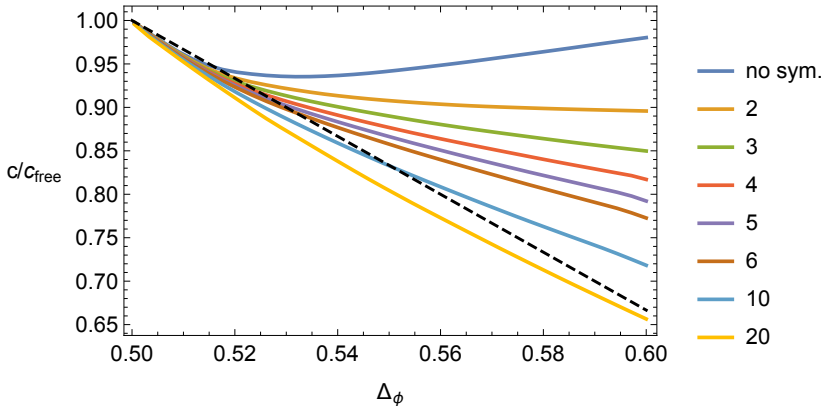

Figure 6.8: Bounds on the central charge c as a function of Δ_ϕ for 3D CFTs with different $O(n)$ symmetries, with ϕ in the fundamental representation of $O(n)$. The regions below the lines are excluded. All the bounds have been determined using $N = 80$ points and $\Delta_* = 16$ with gaps $\Delta_S > 1$ and $\Delta_T > 1$ assumed. The dashed line is the leading large-n prediction. The curves and the labels in the legend have the same order from top to bottom.

6.3.2 4D CFTs

All the above considerations can be repeated for 4D CFTs. There are no known non-super-symmetric CFTs at benchmarks points but it is still interesting to study general bounds on operator dimensions and OPE coefficients. See e.g. refs. [9, 45–48, 50, 58, 115, 116], where bounds of this kind (and others) have been determined with the derivative method using both linear and semi-definite programming.

In figs. 6.11 and 6.12, we show bounds respectively on the dimension Δ_{ϕ^2} of the

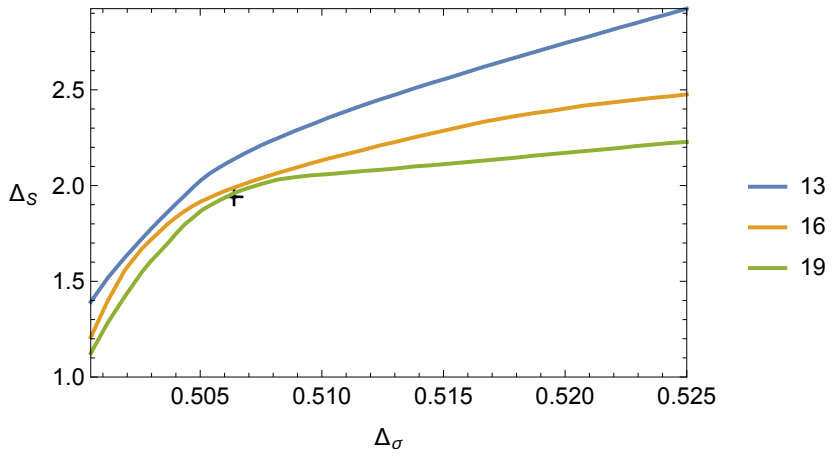

Figure 6.9: Bounds on Δ_S as a function of Δ_ϕ for $N = 100$ points and different values of Δ_* for 3D CFTs with $O(20)$ symmetry, with ϕ in the fundamental representation of $O(20)$. The regions above the lines are excluded. The black cross marks the values of Δ_ϕ and Δ_S for the $O(20)$ vector model as given in ref. [54]. The curves and the labels in the legend have the same order from top to bottom.

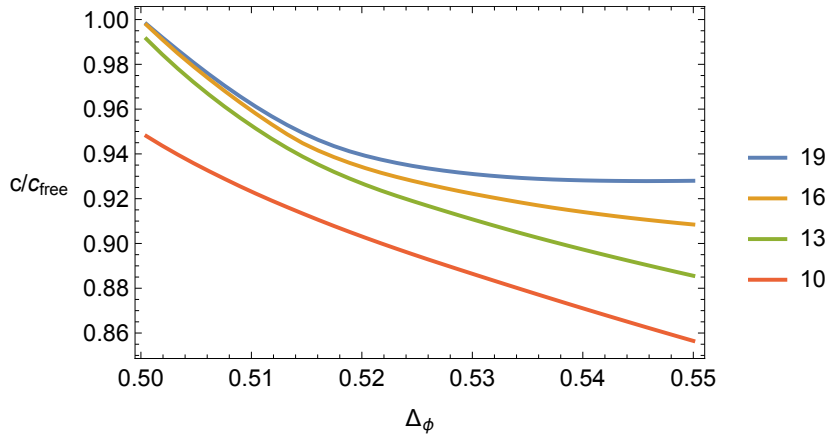

Figure 6.10: Bounds on the central charge c as a function of Δ_ϕ for $N = 100$ points and different values of Δ_* for 3D CFTs with $O(2)$ symmetry, with ϕ in the fundamental representation of $O(2)$. Gaps $\Delta_S > 1$ and $\Delta_T > 1$ are assumed. The regions below the lines are excluded. The curves and the labels in the legend have the same order from top to bottom.

lowest-dimensional scalar operator in the $\phi \times \phi$ OPE and on the central charge c as a function of Δ_ϕ for different values of Δ_*, at fixed N. The conclusions are the same as for the 3D CFTs: the bounds on the operator dimension converge faster than those on the central charge. The point of convergence of the bounds in N at fixed Δ_* is again $N_* \sim \mathcal{O}(100)$ and thus also very similar to that in 3D CFTs.

The analysis of 4D CFTs with $O(n)$ global symmetry also closely resembles its 3D counterpart. We again take the external field ϕ to transform in the fundamental

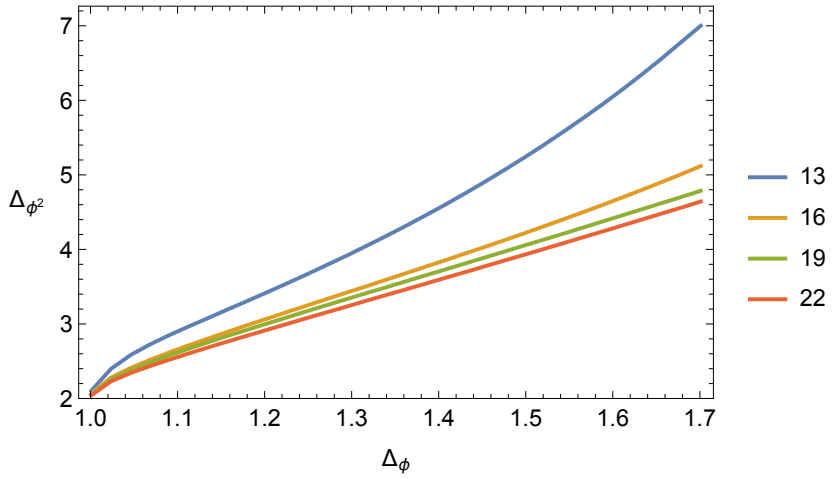

Figure 6.11: Bounds on Δ_{ϕ^2} as a function of Δ_ϕ for $N = 100$ points and different values of Δ_* for 4D CFTs with no global symmetry. The regions above the curves are excluded. The curves and the labels in the legend have the same order from top to bottom.

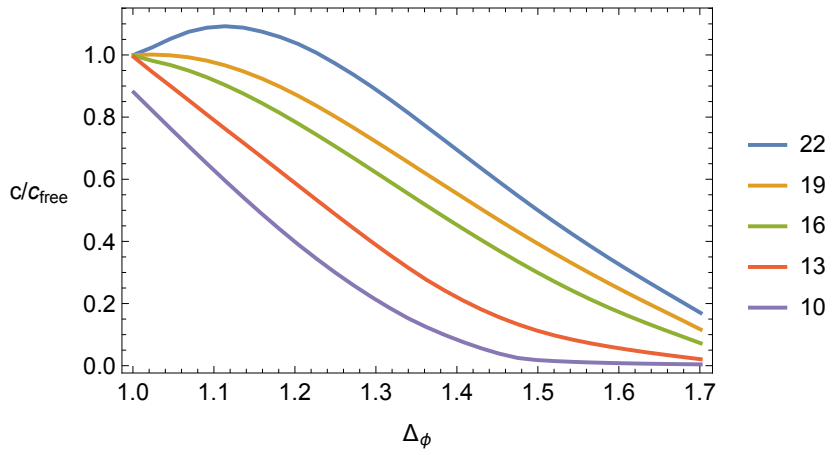

Figure 6.12: Bounds on the central charge c as a function of Δ_ϕ for $N = 100$ points and different values of Δ_* for 4D CFTs with no global symmetry. The regions below the curves are excluded. The curves and the labels in the legend have the same order from top to bottom.

representation of $O(n)$ and denote by S the lowest-dimensional singlet scalar operator that appears in the $\phi \times \phi$ OPE. For illustration, we report in fig. 6.13 the bound on Δ_S as a function of Δ_ϕ for CFTs with $O(4)$ symmetry, at fixed N and for different values of Δ_*. By comparing figs.6.11 and 6.13 we notice that the convergence in Δ_* of the operator-dimension bound in 4D CFTs with $O(4)$ symmetry is slower than its analogue with no global symmetry.

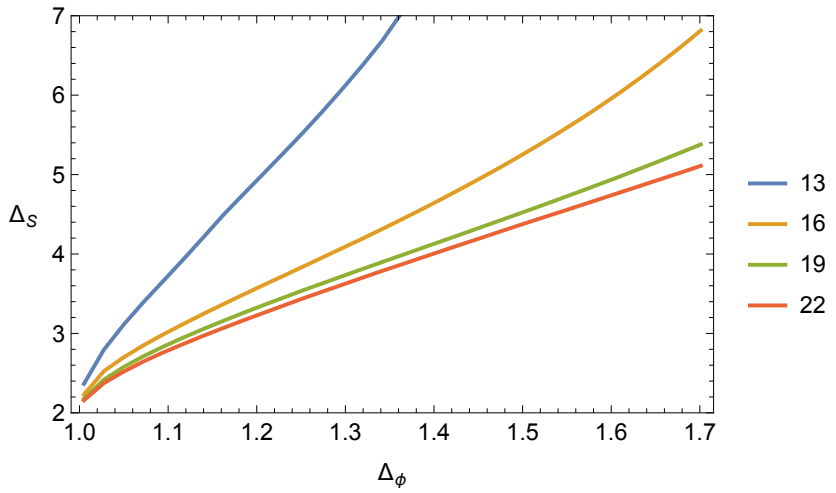

Figure 6.13: Bounds on Δ_S as a function of Δ_ϕ for $N = 100$ points and different values of Δ_* for 4D CFTs with $O(4)$ symmetry, with ϕ in the fundamental representation of $O(4)$. The regions above the curves are excluded. The curves and the labels in the legend have the same order from top to bottom.

6.3.3 A Closer Look at the Spectrum of 3D $O(n)$ Models

In the last subsections, we have shown how previously determined bounds are reproduced using the multipoint method. Here we present some new results for the spectrum of $O(n)$ models. To this end we assume, as previous analyses indicate, that the 3D $O(n)$ models sit precisely at the kink on the boundary of the excluded region in the (Δ_ϕ-Δ_S) plane (Δ_S-maximization). The vector $\vec{\rho}$ that we obtain from solving the linear program (6.40) then gives us the spectrum and OPE coefficients of the operators that are exchanged in the $\langle\phi\phi\phi\phi\rangle$ correlator of the $O(n)$ models. Here we report the scaling dimensions of the first two operators in respectively the singlet and rank-two representation of $O(n)$, S, S' and T, T', for $n = 2, 3, 4$. Scalar operators with larger scaling dimensions are physically uninteresting, whereas S' and T' are important in determining the stability of the fixed points of the $O(n)$ models (being marginal operators in the underlying UV 4D Landau-Ginzburg theory) [118].[15] Actually, one additional operator should be considered, denoted as $\mathcal{P}_{4,4}$ in ref. [118], but it transforms in the rank-four representation of $O(n)$ and hence cannot appear in the OPE of two scalar operators ϕ in the fundamental representation. Its dimension might be bounded (or computed) by considering a correlator involving, e.g., four T's. As far as we know, the scaling dimensions of S' and T' have not been previously determined using the conformal bootstrap. The best determinations of these parameters have been made

[15] See ref. [113] for a bootstrap approach to the study of the stability of fixed points in 3D $O(n)\times O(m)$ models.

n	Δ_ϕ	Δ_S	$\Delta_{S'}$	Δ_T	$\Delta_{T'}$
2	0.51905(10) [126]	$1.5118^{+0.0012}_{-0.0022}$ [54]	3.802(18) [117]	$1.23613^{+0.00058}_{-0.00158}$ [54]	3.624(10) [118]
3	0.51875(25) [127]	$1.5942^{+0.0037}_{-0.0047}$ [54]	3.794(18) [117]	$1.2089^{+0.0013}_{-0.0023}$ [54]	3.550(14) [118]
4	0.51825(40) [128]	$1.6674^{+0.0077}_{-0.0087}$ [54]	3.795(30) [117]	$1.1864^{+0.0024}_{-0.0034}$ [54]	3.493(14) [118]

Table 6.1: Scaling dimensions of the first two scalar operators in the singlet (S, S') and rank-two symmetric (T, T') representations of $O(n)$ for $n = 2, 3, 4$ determined in the literature.

n	Δ_ϕ	Δ_S	$\Delta_{S'}$	Δ_T	$\Delta_{T'}$
2	0.51905(10) [126]	1.5124(10)	3.811(10)	1.2365(16)	3.659(7)
3	0.51875(25) [127]	1.5947(35)	3.791(22)	1.2092(22)	3.571(12)
4	0.51825(40) [128]	1.668(6)	3.817(30)	1.1868(24)	3.502(16)

Table 6.2: Scaling dimensions of the first two scalar operators in the singlet (S, S') and rank-two symmetric (T, T') representations of $O(n)$ for $n = 2, 3, 4$ determined in this paper using Δ_S-maximization, the values of Δ_ϕ previously determined in the literature (first column) and the fit procedure explained in the main text. The quoted error corresponds to 1σ (68% confidence level).

using a five-loop computation in the ϵ-expansion in refs. [117] and [118].[16]

In table 6.1, we report the values of Δ_ϕ, Δ_S, $\Delta_{S'}$, Δ_T, $\Delta_{T'}$ determined in the literature, for $n = 2, 3, 4$. They should be compared with the values in table 6.2 which have been determined in this paper as follows: We take the values of Δ_ϕ for $O(n)$ models with $n = 2, 3, 4$ calculated in refs. [126–128] as input and determine the scaling dimensions Δ_S, $\Delta_{S'}$, Δ_T and $\Delta_{T'}$ using Δ_S-maximization. We repeat this procedure for the lower, central and upper value of Δ_ϕ given in these references and for different values of the cutoff $\Delta_* \in [18, 23]$ and the number of points $N \in [60, 120]$.[17] At fixed N and Δ_*, we then take the average over the scaling dimensions obtained with the different input values of Δ_ϕ. Sometimes the same operator appears twice in the spectrum, at two different but close values of the scaling dimension. In this case we take the average of these values, weighted by the size of the corresponding OPE coefficient. Let us denote the resulting scaling dimensions by $\Delta_{\mathcal{O}}(N, \Delta_*)$ for $\mathcal{O} = S, S', T, T'$. Each of these values is associated with an error, resulting from the averaging. The stepsize

[16] More precisely, $\Delta_{S'}$ has been determined also by other means, such as fixed-dimension expansion and Monte Carlo simulations. On the other hand, since $\Delta_{T'}$ has been determined only using the ϵ-expansion, we have decided to omit the other results for $\Delta_{S'}$. The interested reader can find them, e.g., in table I of ref. [118], where the coefficients $y_{4,0}$ and $y_{4,2}$ give $\Delta_{S'} = 3 - y_{4,0}$ and $\Delta_{T'} = 3 - y_{4,2}$. For completeness, we also report the relations defining Δ_S and Δ_T in the notation of ref. [118]: $\Delta_S = 3 - 1/\nu$, $\Delta_T = 3 - y_{2,2}$.

[17] Our numerical precision does not allow us to take higher values of Δ_* and N without having issues with numerical stability.

n	Δ_ϕ	Δ_S	$\Delta_{S'}$	Δ_T	$\Delta_{T'}$
2	0.51905(10) [126]	≤ 1.5145	≤ 3.852	≤ 1.2408	≤ 3.678
3	0.51875(25) [127]	≤ 1.6004	≤ 3.856	≤ 1.2116	≤ 3.588
4	0.51825(40) [128]	≤ 1.677	≤ 3.908	≤ 1.191	≤ 3.528

Table 6.3: Upper bounds on the scaling dimensions of the first two scalar operators in the singlet (S, S') and rank-two symmetric (T, T') representations of $O(n)$ for $n = 2, 3, 4$ determined in this paper using Δ_S-maximization and the values of Δ_ϕ previously determined in the literature (first column).

Δ_{step} of our discretization has been set to 10^{-4} in the region where the operators were expected to be found (the resulting uncertainty in the scaling dimensions is typically negligible compared to the other errors).

At fixed N, the results for different values of Δ_* are fitted by a function of the form $a_{\mathcal{O}}(N) + b_{\mathcal{O}}(N) \exp(-c_{\mathcal{O}}(N)\Delta_*)$, where $a_{\mathcal{O}}(N)$, $b_{\mathcal{O}}(N)$ and $c_{\mathcal{O}}(N)$ are the fit parameters. Such a dependence is roughly expected given the exponential convergence of the OPE. Somewhat surprisingly, this simplified function fits the results extremely well, see fig. 6.14 for an example of the extrapolation fit in $1/\Delta_*$. Using this fit, we have extrapolated the scaling dimensions for the different operators and values of N to $\Delta_* = \infty$. We denote the resulting scaling dimensions as $\Delta_{\mathcal{O}}(N) \equiv \Delta_{\mathcal{O}}(N, \infty) = a_{\mathcal{O}}(N)$.

We have then extrapolated to $N = \infty$ using a linear fit in $1/N$ which seems to well describe the behaviour of $\Delta_{\mathcal{O}}(N)$ as a function of $1/N$. An example of this extrapolation fit is shown in fig.6.15. We denote the resulting scaling dimensions as $\Delta_{\mathcal{O}} \equiv \Delta_{\mathcal{O}}(\infty)$.[18] We do not have an analytic understanding of why the results should scale as $1/N$ for parametrically large Δ_*. We simply take it as a working hypothesis. We expect that possible deviations from the linear behaviour should be contained within the errors of our determination (cf. fig.6.15). Note that having N as large as possible is clearly important for high precision. However, at fixed Δ_* the bounds saturate for sufficiently high N and there is no gain in taking N larger.

We have noticed that, at least for $n = 2, 3, 4$, $\Delta_{\mathcal{O}}(N, \Delta_*)$ decreases as N and/or Δ_* increase (this is obvious for S, but not for the other operators). If we assume that this is true for any N and Δ_*, we may then set rigorous upper bounds without using any fit extrapolation. These bounds are reported in table 6.3. Comparing them with the results in table 6.2 gives an idea of the impact of the fit extrapolation on the final results. As can be seen, all the scaling dimensions that we have determined are compatible with previous results in the literature. The only exception is $\Delta_{T'}$ for the $O(2)$ model for

[18] A similar linear dependence in $1/N$ has already been noticed with great accuracy in ref. [63] for the central-charge bound in 6D $\mathcal{N} = (2, 0)$ SCFTs (see their fig. 1).

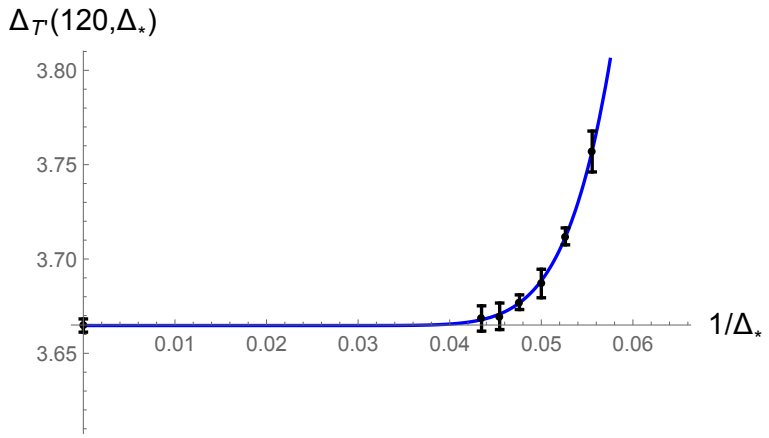

Figure 6.14: Extrapolation fit to determine the scaling dimension of the operator T' in the $O(2)$ model with $N = 120$ points at $\Delta_* = \infty$ from the results for that scaling dimension for different values of Δ_*. The vertical error bar associated with the extrapolated point on the left corresponds to 1σ (68% confidence level).

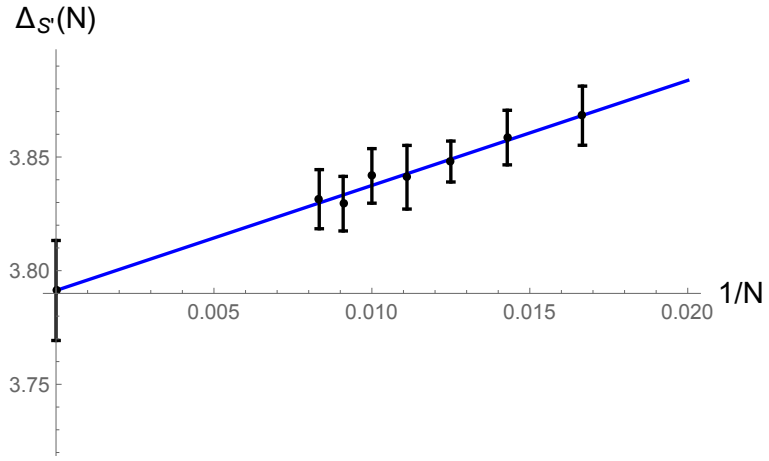

Figure 6.15: Extrapolation fit to determine the scaling dimension of the operator S' in the $O(3)$ model at $N = \infty$ from the results for that scaling dimension for different values of $1/N$. Each point corresponds to the value of $\Delta_{S'}(N)$ extracted from a fit in $1/\Delta_*$. The vertical error bar associated with each point corresponds to 1σ (68% confidence level).

which our result has an approximate 3σ tension with that of ref. [118]. Our accuracy in the determinations of Δ_S and Δ_T is comparable with that achieved in ref. [54], though it should be emphasized that the results there do not rely on extrapolations. Furthermore, our accuracy in the determinations of $\Delta_{S'}$ and $\Delta_{T'}$ is comparable with that achieved using the five-loop ϵ-expansion. This is an indication that a slightly more refined bootstrap analysis will be able to improve the determinations of these scaling dimensions.

As we mentioned at the beginning of this subsection, Δ_S-maximization also allows us to

determine the OPE coefficients $\lambda_{\phi\phi\mathcal{O}}$. We have not performed a detailed analysis with fit extrapolations as above to determine the asymptotic values of $\lambda_{\phi\phi\mathcal{O}}$ as $\Delta_*, N \to \infty$. Instead we just report $\lambda_{\phi\phi S}$ as determined with the highest values $\Delta_* = 22, 23$ and $N = 110, 120$ used in this paper:

$$\begin{aligned} O(2): &\quad \lambda_{\phi\phi S} \approx 0.686\,, \\ O(3): &\quad \lambda_{\phi\phi S} \approx 0.524\,, \\ O(4): &\quad \lambda_{\phi\phi S} \approx 0.428\,. \end{aligned} \tag{6.46}$$

We have not determined the error associated with these results and have instead rounded them to the last shown digit. The results for $O(2)$ and $O(3)$ are in agreement with the recent determination in ref. [105], whereas the result for $O(4)$ is new as far as we know.

6.4 Details of the Implementation

For the conformal blocks in $d = 4$ dimensions, we use the closed-form expression from ref. [13], normalized as in ref. [48]. For $d = 3$ dimensions, on the other hand, we use the recursion relation for the conformal blocks found in ref. [54].[19] To this end, we iterate the recursion relation up to some cutoff Δ_{rec}. We choose this cutoff large enough such that the resulting error in the conformal blocks is smaller than the error from neglecting contributions of operators with dimensions larger than the truncation cutoff Δ_*. In practice, we find that $\Delta_{\text{rec}} = \Delta_* + \text{few}$ is sufficient to ensure this.

For the ansatz (6.37) of discretized operator dimensions, we closely follow ref. [101]. We generate the discrete spectra T1 to T4 (the latter only for sufficiently large Δ_*) in their table 2, where we rescale the stepsizes δ by the factor $\Delta_{\text{step}}/(2 \cdot 10^{-5})$. We then remove duplicates from the combined spectrum and restrict to operator dimensions less than or equal to Δ_*. We have performed extensive scans using different stepsizes Δ_{step} and have found that the bounds converge for sufficiently small Δ_{step}. This is in particular satisfied for $\Delta_{\text{step}} = 2 \cdot 10^{-3}$ which we choose for all the plots in this paper. For the determination of the spectra in sec. 6.3.3 we add additional operators with stepsize $\Delta_{\text{step}} = 10^{-4}$ around the previously determined scaling dimensions for the operators S, S', T, T' in the $O(n)$ models. Furthermore, for bounds on operator dimensions for which the plots extend to bounds $\Delta_{\phi^2} > 3$ (the largest dimension of T1

[19] Alternatively, we can use the recursion relation also in $d = 4$ dimensions by setting $d = 4 + \epsilon$ (to avoid double poles that appear at $d = 4$). However, `Mathematica` evaluates the closed-form expression faster than (our implementation of) the recursion relation and we therefore choose the former.

of ref. [101]), we have included additional operators in the scalar sector so that the smallest stepsize Δ_{step} is used up to the largest bound on Δ_{ϕ^2} shown in that plot. We have also performed scans using different parametrizations for the ansatz (6.37) and have found that the bounds become indistinguishable from the bounds obtained with the ansatz discussed above for sufficiently small Δ_{step}. This gives us confidence that the discretization does not introduce any artifacts into our calculations.

We use `Mathematica` to evaluate the conformal blocks for the different operators that appear in the ansatz (6.37) and for the set of points in the z-plane. The linear progam (6.40) is then set up by a program written in Python and is subsequently solved with the optimizer `CPLEX` by `IBM` using the primal simplex algorithm. Since this optimizer is limited to double precision, it is important to reduce the spread in size of the numerical values in the problem. To this end, note that we can rescale each row of the inequality (6.40) separately by a positive number. Denoting a given row by $\mathcal{R}$, we rescale its elements by

$$\mathcal{R}_i^{\text{resc}} = \frac{\mathcal{R}_i}{\sqrt{\min_i|\mathcal{R}_i| \cdot \max_i|\mathcal{R}_i|}}\,. \tag{6.47}$$

Similarly, we can rescale each column of the matrix $\mathcal{M}$ separately by a positive number if we redefine the corresponding (squared) OPE coefficient in the vector $\vec{\rho}$. We again choose

$$\mathcal{M}_{ij}^{\text{resc}} = \frac{\mathcal{M}_{ij}}{\sqrt{\min_i|\mathcal{M}_{ij}| \cdot \max_i|\mathcal{M}_{ij}|}} \tag{6.48}$$

and correspondingly for $\vec{\rho}$. This procedure is iterated three times in our Python code, using precision arithmetic with 120 digits to ensure that no significant rounding errors are introduced in the process (the conformal blocks have been calculated with the same precision). Since we perform our own rescaling, we switch off this option in `CPLEX`.

We find that the above rescaling typically reduces the orders of magnitude in the ratio between the largest and smallest numerical value in eq. (6.40) by about half. Nevertheless, precision is a limiting factor and does not allow us to go to cutoffs Δ_* much larger than 20. The fact that double precision is sufficent for smaller cutoffs, on the other hand, makes our calculations (combined with the excellent speed of `CPLEX`) very fast.

Conclusions

In this thesis we have explored different aspects of CFTs in 4D. The first part of the thesis has focused on the mathematical properties of 4D conformal blocks for general representations of the Lorentz group $(\ell, \bar{\ell})$. The computation of such blocks opens a new window into the properties of 4D CFTs, allowing a rigorous investigation from the point of view of the Conformal Bootstrap.

In Chapter 3 we introduced a set of differential operators, eqs.(3.15), (3.16) (3.17) and (3.24), that enable us to relate different three-point functions in 4D CFTs. The 6D embedding formalism in twistor space with an index free notation, as introduced in ref. [21], and the recent classification of three-point functions in 4D CFTs [31] have allowed us to perform this task in a general setting. In particular, three-point tensor correlators with different tensor structures can always be related to a three-point function with a single tensor structure. We explicitly showed how to deconstruct any four point tensor correlators of external traceless symmetric operators in terms of a few CPW seeds. We argued that the simplest CPW seeds are those associated to the four point functions of two scalars, one $\mathcal{O}^{2\delta,0}$ and one $\mathcal{O}^{0,2\delta}$ field, that have only $2\delta+1$ independent tensor structures. The power of this method does not end here for it allows us to deconstruct any possible four point tensor correlator in terms of the same type of seeds including half integer values of δ (or as we called it in Chapter 4, for odd or even values of p).

We continued in Chapter 4 where the computation of the general set of seed conformal blocks is actually performed. The result of this chapter is a totally general expression for $G_e^{(p)}$ for any e, p, Δ, ℓ and external scaling dimensions, obtained by solving the Casimir set of differential equations, that can be written in the compact form (4.23). The shadow formalism has been of fundamental assistance to deduce it and also as a useful cross check for the validity of the results. The CBs are expressed in terms of coefficients $c^e_{m,n}$, that can be determined recursively, e.g. by means of eq.(4.100). For each CB, the coefficients $c^e_{m,n}$ span a 2D octagon-shape lattice in the (m,n) plane, with sizes that depend on p and e and increase as p increases. We have reported in Appendix C the explicit form of $c^e_{m,n}$ for the simplest case $p=1$. We have not reported the $c^e_{m,n}$ for higher values of p, since their number and complexity grows with p.

These CBs open the door to the bootstrap of universal correlators in 4D CFTs. Some obvious applications pertain the bootstrap of four-point functions of spin 1 currents or even four stress tensors, that might lead to universal bounds on symmetry representations (and central charges) and general bounds on any local CFTs.

The somewhat surprisingly simple form of the Casimir system (4.23), where at most three blocks at a time can enter in a sort of local interaction, and the geometric interpretation of the coefficients $c^e_{m,n}$ in terms of octagons, are perhaps an indication of a more fundamental symmetry principle. This should hopefully allow us to gain a better understanding of 4D CFTs or at least, less ambitiously, more compact expressions for the CBs $G^{(p)}_e$.

The last part of the thesis has dealt with numerical studies of the bootstrap equation. We focused scalar four point functions with and without global symmetries.

In Chapter 5 we have numerically studied bounds on various OPE coefficients in 4D CFTs by using standard methods based on semidefinite programming [50]. We have first studied bounds on OPE coefficients of symmetric traceless tensor operators with spins $l = 2$ and $l = 4$ as a function of their scaling dimension. Furthermore, we have analyzed how an assumption on the dimension of the lowest-lying scalar operator affects such bounds.

We have then considered 4D CFTs with a global symmetry G. When this group, or a subgroup of it, is gauged by weakly coupling external gauge fields to the CFT, the coefficient κ which enters in the two-point function of the associated conserved vector currents governs the leading CFT contribution to the one-loop β-function of the corresponding gauge coupling. In particular, if this contribution is too large, it gives rise to unwanted sub-Planckian Landau poles. Motivated by physics beyond the Standard Model, where $G_{\rm SM} \subseteq G$, we have numerically studied the lower bounds on the coefficient κ. Possible hierarchy problems are avoided by demanding that all scalar operators in the spectrum which are singlets under the global symmetry have dimensions $\Delta_S \geq 4$. More specifically, we have considered lower bounds on κ extracted from four-point functions of scalar operators ϕ_i in the fundamental representation of SO(N), or the bi-fundamental representation of SO(N)×SO(M).

The motivation behind this study was to gain some understanding on the application of this setting in the context of composite Higgs models with partial compositeness, where the CFT is assumed to have a global symmetry G and a set of fermion operators with different dimensions. The next logical step would be to repeat this exercise in the context of fermion correlators given that the conformal blocks are already known in 4D.

Chapter 6 has explored a new numerical method for the bootstrap analysis. Based on considerations given in ref. [25] we studied the bootstrap equation at different points in

the z-$\bar{z}$ plane. We wrote a linear system based on this principle and numerically solved for different assumptions of the CFTs under consideration. Using this method, we have qualitatively reproduced various results[20] that have been determined in the bootstrap literature using the more common method of taking derivatives at the symmetric point. In particular, this method can be useful at a preliminary stage when one wants to qualitatively bound or approximately compute some quantities using the bootstrap. By choosing a sufficiently low cutoff Δ_*, one can get qualitatively good results within seconds of CPU time with a standard laptop.

We have shown how, using Δ-maximization, relatively precise results can be obtained for the scaling dimensions of operators (though we relied on an extrapolation procedure). In particular, for $O(n)$ models with $n = 2, 3, 4$ we have determined the scaling dimensions of the second-lowest-dimensional operators S' and T' in the singlet and symmetric-traceless representation, respectively.

We believe that it should not be difficult to go to arbitrary precision and get rid of the discretization of the spectrum (and the extrapolation procedure) by, for instance, adapting the algorithm developed in refs. [103, 106] to multipoints. We do not exclude that bootstrapping with multipoints might then turn out to be comparable to (or better than) the derivative method for high-precision computations. From a conceptual point of view, the multipoint method is more rigorous, since the crossing equations are not truncated but bounded by an error.[21]

We have also discussed how the multipoint method is useful in understanding to which extent a given numerical result depends sensitively on the high-dimensional operators. In particular, we have noticed that bounds on operator dimensions are less sensitive in this respect than bounds on the central charge.

Ideally, one might want to push the multipoint method to the extreme "IR limit", by choosing a cutoff Δ_* so low that an analytic approach may become possible. This is certainly a very interesting direction that should be explored. Among other things, it requires to improve on the estimate of the OPE convergence given in ref. [22] that applies in the opposite regime, for parametrically large Δ_*. Perhaps the results of ref. [119] might be useful in this respect.[22]

An important line of development in the numerical bootstrap is the analysis of mixed correlators which so far are numerically accessible only using semi-definite programming

[20] In D=3 and D=4 and with or without global symmetries.

[21] Strictly speaking, this is true only when we are guaranteed to be in the regime where the Hardy-Littlewood tauberian theorem applies. But all the evidence so far indicates that this is always the case for $\Delta_* \gtrsim O(10)$.

[22] It should be mentioned that another method based on determinants has been proposed by Gliozzi [53] to study severely truncated bootstrap equations. The method sometimes works surprisingly well, but unfortunately its current implementation is not systematic or rigorous enough.

[61]. It would be very interesting to implement mixed correlators in the multipoint bootstrap, possibly by implementing Quadratic Programming solvers or variations of the semi-definite programming techniques.

A Appendix A

Relations between Four-Point Function Invariants

In this appendix we report a partial list of relations between $SU(2,2)$ invariants entering four-point functions that have been used in subsection 3.5.

The first relation is linear in the invariants and reads

$$J_{i,jl} = n_{ijkl} J_{i,kl} + n_{lijk} J_{i,jk} \,, \tag{A.1}$$

where we have defined

$$n_{ijkl} \equiv \frac{X_{ij} X_{kl}}{X_{ik} X_{jl}}. \tag{A.2}$$

The 7 relations below allow to eliminate completely products of the form $K_{i,jk}\overline{K}_{l,mn}$

$$K_{i,jk}\overline{K}_{i,jk} = \frac{1}{2} J_{j,ik} J_{k,ij} - 2 I_{jk} I_{kj} \,, \tag{A.3}$$

$$K_{i,jk}\overline{K}_{l,jk} = \sqrt{n_{ijkl}} \Big(n_{iljk} I_{jk} J_{kj,li} - \frac{1}{2}\, n_{ikjl} J_{j,ik} J_{k,jl} - 2\, I_{jk} I_{kj} \Big) \,, \tag{A.4}$$

$$K_{i,jk}\overline{K}_{j,ik} = I_{ij} J_{k,ij} + 2 I_{ik} I_{kj} \,, \tag{A.5}$$

$$K_{i,jk}\overline{K}_{j,lk} = \sqrt{n_{ijkl}} \Big(I_{kj} J_{lk,ji} + I_{lj} J_{k,ij} \Big) \,, \tag{A.6}$$

$$K_{i,jk}\overline{K}_{l,ij} = -\sqrt{n_{ilkj}} \Big(I_{ij} J_{jk,li} + I_{ik} J_{j,il} \Big) \,, \tag{A.7}$$

$$K_{i,jk}\overline{K}_{j,li} = \sqrt{n_{ilkj}} \Big(I_{ij} J_{lk,ji} - 2 I_{ik} I_{lj} \Big) \,, \tag{A.8}$$

$$K_{i,jk}\overline{K}_{i,jl} = -\sqrt{n_{ilkj}} \Big(I_{lj} J_{jk,li} + \frac{1}{2} J_{j,il} J_{lk,ji} \Big) \,. \tag{A.9}$$

Another relation is

$$J_{ji,kl} J_{lk,ij} = 4 \Big(I_{li} I_{jk} - n_{ikjl} I_{li} I_{jk} + n_{iljk} I_{ji} I_{lk} \Big) + 2 n_{iljk} \Big(I_{li} J_{jk,li} - I_{jk} J_{li,kj} \Big). \tag{A.10}$$

B Appendix B
Properties of the $\mathcal{F}$ Functions

In this Appendix we provide all the properties of the functions $\mathcal{F}^{(a,b;c)}_{\rho_1,\,\rho_2}$ needed for the system of Casimir equations and more specifically to derive eqs.(4.89)-(4.91). We will not consider the functions $\mathcal{F}^{\pm\,(a,b;c)}_{\rho_1,\,\rho_2}$ here, since their properties can trivially be deduced from the ones below by demanding both sides to be symmetric/anti-symmetric under the exchange $z \leftrightarrow \bar{z}$.

The fundamental identities to be considered can be divided in two sets, depending on whether the values (a, b, c) of the functions $\mathcal{F}$ are left invariant or not. The former identities read

$$\left(\frac{1}{z}-\frac{1}{2}\right)\mathcal{F}^{(a,b;c)}_{\rho_1,\rho_2} = \mathcal{F}^{(a,b;c)}_{\rho_1-1,\rho_2} - D^{(a,b,c)}_{\rho_1}\mathcal{F}^{(a,b;c)}_{\rho_1,\rho_2} + B^{(a,b,c)}_{\rho_1}\mathcal{F}^{(a,b;c)}_{\rho_1+1,\rho_2} \tag{B.1}$$

$$\left(\frac{1}{\bar{z}}-\frac{1}{2}\right)\mathcal{F}^{(a,b;c)}_{\rho_1,\rho_2} = \mathcal{F}^{(a,b;c)}_{\rho_1,\rho_2-1} - D^{(a,b,c)}_{\rho_2}\mathcal{F}^{(a,b;c)}_{\rho_1,\rho_2} + B^{(a,b,c)}_{\rho_2}\mathcal{F}^{(a,b;c)}_{\rho_1,\rho_2+1} \tag{B.2}$$

$$\begin{aligned} L_0\mathcal{F}^{(a,b;c)}_{\rho_1,\rho_2} = {} & \rho_2\mathcal{F}^{(a,b;c)}_{\rho_1,\rho_2-1} - \rho_1\mathcal{F}^{(a,b;c)}_{\rho_1-1,\rho_2} - (\rho_2+c-1)B^{(a,b,c)}_{\rho_2}\mathcal{F}^{(a,b;c)}_{\rho_1,\rho_2+1} + \\ & (\rho_1+c-1)B^{(a,b,c)}_{\rho_1}\mathcal{F}^{(a,b;c)}_{\rho_1+1,\rho_2} + \frac{1}{2}(2-c)(D^{(a,b,c)}_{\rho_1} - D^{(a,b,c)}_{\rho_2})\mathcal{F}^{(a,b;c)}_{\rho_1,\rho_2}, \end{aligned} \tag{B.3}$$

where $L_0 = \Big((1-\bar{z})\partial_{\bar{z}} - (1-z)\partial_z\Big)$ and we have defined

$$C^{(a,b,c)}_{\rho} = \frac{(a+\rho)(b-c-\rho)}{(c+2\rho)(c+2\rho-1)}\,, \tag{B.4}$$

$$B^{(a,b,c)}_{\rho} = C^{(a,b,c)}_{\rho}\,C^{(b-1,a,c-1)}_{\rho+1} = \frac{(\rho+a)(\rho+b)(\rho+c-b)(\rho+c-a)}{(2\rho+c)^2(c+2\rho+1)(c+2\rho-1)}\,,$$

$$D^{(a,b,c)}_{\rho} = \frac{(2a-c)(2b-c)}{2(c+2\rho)(c+2\rho-2)}\,. \tag{B.5}$$

The latter identities read

$$\mathcal{F}^{(a,b;c)}_{\rho_1,\rho_2} = \mathcal{F}^{(a,b-1;c-1)}_{\rho_1,\rho_2} - C^{(a,b,c)}_{\rho_1}\mathcal{F}^{(a,b-1;c-1)}_{\rho_1+1,\rho_2} - C^{(a,b,c)}_{\rho_2}\mathcal{F}^{(a,b-1;c-1)}_{\rho_1,\rho_2+1} + C^{(a,b,c)}_{\rho_1}C^{(a,b,c)}_{\rho_2}\mathcal{F}^{(a,b-1;c-1)}_{\rho_1+1,\rho_2+1}\,, \tag{B.6}$$

$$\mathcal{F}^{(a,b;c)}_{\rho_1,\rho_2} = \mathcal{F}^{(a-1,b;c-1)}_{\rho_1,\rho_2} - C^{(b,a,c)}_{\rho_1}\mathcal{F}^{(a-1,b;c-1)}_{\rho_1+1,\rho_2} - C^{(b,a,c)}_{\rho_2}\mathcal{F}^{(a-1,b;c-1)}_{\rho_1,\rho_2+1} + C^{(b,a,c)}_{\rho_1}C^{(b,a,c)}_{\rho_2}\mathcal{F}^{(a-1,b;c-1)}_{\rho_1+1,\rho_2+1}\,, \tag{B.7}$$

$$\frac{1}{z\bar{z}}\mathcal{F}^{(a,b;c)}_{\rho_1,\rho_2} = \mathcal{F}^{(a+1,b+1;c+2)}_{\rho_1-1,\rho_2-1}\,, \tag{B.8}$$

$$(z-\bar{z})L(a)\mathcal{F}^{(a,b;c)}_{\rho_1,\rho_2} = (\rho_2-\rho_1)\mathcal{F}^{(a,b-1;c-1)}_{\rho_1,\rho_2} - (\rho_1+\rho_2+c-1)C^{(a,b,c)}_{\rho_1}\mathcal{F}^{(a,b-1;c-1)}_{\rho_1+1,\rho_2} + (\rho_1+\rho_2+c-1)C^{(a,b,c)}_{\rho_2}\mathcal{F}^{(a,b-1;c-1)}_{\rho_1,\rho_2+1} - (\rho_2-\rho_1)C^{(a,b,c)}_{\rho_1}C^{(a,b,c)}_{\rho_2}\mathcal{F}^{(a,b-1;c-1)}_{\rho_1+1,\rho_2+1}\,, \tag{B.9}$$

$$\frac{z-\bar{z}}{z\bar{z}}L(b)\mathcal{F}^{(a,b;c)}_{\rho_1,\rho_2} = (\rho_2-\rho_1)\mathcal{F}^{(a,b+1;c+1)}_{\rho_1-1,\rho_2-1} - (\rho_1+\rho_2+c-1)C^{(b,a,c)}_{\rho_1}\mathcal{F}^{(a,b+1;c+1)}_{\rho_1,\rho_2-1} + (\rho_1+\rho_2+c-1)C^{(b,a,c)}_{\rho_2}\mathcal{F}^{(a,b+1;c+1)}_{\rho_1-1,\rho_2} - (\rho_2-\rho_1)C^{(b,a,c)}_{\rho_1}C^{(b,a,c)}_{\rho_2}\mathcal{F}^{(a,b+1;c+1)}_{\rho_1,\rho_2}\,. \tag{B.10}$$

The relations (B.1)-(B.3) were first derived in ref. [14] (see also ref. [15]), while the relations (B.9) and (B.10) are novel to this paper. It is straightforward to see that eqs.(4.89)-(4.91) can be derived using proper combinations of eqs.(B.1)-(B.10). For instance, the action of the first term appearing in the r.h.s. of eq.(4.88) is reproduced (modulo a trivial constant factor) by taking the combined action given by ((B.2)−(B.1))× (B.9)× (B.6). All other terms in eqs.(4.86)-(4.88) are similarly deconstructed.

Appendix C

The Conformal Blocks for $p = 1$

We report in this appendix the full explicit solution for the two conformal blocks $G_0^{(1)}$ and $G_1^{(1)}$ associated to the exchange of fermion operators of the kind $\mathcal{O}^{\ell,\ell+1}$ for the specific values

$$a = \frac{1}{2}\,, \qquad b = -\frac{1}{2}\,. \tag{C.1}$$

We choose as undetermined coefficient $c^1_{0,-1}$ and report below the values of the coefficients normalized to $c^1_{0,-1}$. We have

$$c^0_{-2,0} = \frac{(2+\ell)}{2\,(1+\ell)}, \quad c^0_{-1,-1} = -\frac{\ell}{2\,(1+\ell)}, \quad c^1_{-1,0} = -\frac{(3+\ell)}{1+\ell}. \tag{C.2}$$

$$\begin{aligned}
c^0_{-1,0} &= \frac{(3+\ell)(-1+2\Delta)(-1+2\ell+2\Delta)}{8(1+\ell)(-3+2\Delta)(1+2\ell+2\Delta)},\\
c^0_{-1,1} &= -\frac{(2+\ell)(5+2\ell-2\Delta)^2(-7+2\Delta)}{32(1+\ell)(3+2\ell-2\Delta)(7+2\ell-2\Delta)(-3+2\Delta)},\\
c^0_{0,-1} &= -\frac{(-1+2\Delta)(-1+2\ell+2\Delta)}{8(-3+2\Delta)(1+2\ell+2\Delta)},\\
c^0_{0,0} &= \frac{\ell(-7+2\Delta)(-1+2\ell+2\Delta)^2}{32(1+\ell)(-3+2\Delta)(-3+2\ell+2\Delta)(1+2\ell+2\Delta)},\\
c^0_{0,1} &= -\frac{(3+\ell)(5+2\ell-2\Delta)^2(-5+2\Delta)(-1+2\ell+2\Delta)}{128(1+\ell)(3+2\ell-2\Delta)(7+2\ell-2\Delta)(-3+2\Delta)(1+2\ell+2\Delta)},\\
c^0_{1,0} &= \frac{(-5+2\Delta)(-1+2\ell+2\Delta)(3+2\ell+2\Delta)^2}{128(-3+2\Delta)(1+2\ell+2\Delta)^2(5+2\ell+2\Delta)},\\
c^1_{-1,1} &= -\frac{(2+\ell)(5+2\ell-2\Delta)(-1+2\Delta)}{4(1+\ell)(7+2\ell-2\Delta)(-3+2\Delta)},\\
c^1_{0,2} &= \frac{(2+\ell)(1+2\ell-2\Delta)(5+2\ell-2\Delta)^2(-5+2\Delta)}{64(1+\ell)(3+2\ell-2\Delta)^2(7+2\ell-2\Delta)(-3+2\Delta)},\\
c^1_{1,0} &= -\frac{(-7+2\Delta)(-1+2\ell+2\Delta)(3+2\ell+2\Delta)}{16(-3+2\Delta)(1+2\ell+2\Delta)^2},\\
c^1_{1,1} &= -\frac{\ell(5+2\ell-2\Delta)(-5+2\Delta)(-1+2\ell+2\Delta)(3+2\ell+2\Delta)}{64(1+\ell)(7+2\ell-2\Delta)(-3+2\Delta)(1+2\ell+2\Delta)^2}
\end{aligned}$$

$$
\begin{aligned}
c^1_{0,0} &= \frac{1}{4(1+\ell)(11+2\ell-2\Delta)(-3+2\Delta)(-3+2\ell+2\Delta)(1+2\ell+2\Delta)} \times \\
&\Big(576-384\Delta+\ell\Big(627-2\ell(-29+2\ell(7+2\ell))-472\Delta+4\ell(-47+4\ell(3+\ell))\Delta \\
&+8(-9+\ell(19+2\ell))\Delta^2-16(-6+\ell)\Delta^3-16\Delta^4\Big)\Big), \\
c^1_{0,1} &= \frac{(5+2\ell-2\Delta)}{16(1+\ell)(3+2\ell-2\Delta)(7+2\ell-2\Delta)(-3+2\Delta)(-3+2\ell+2\Delta)(1+2\ell+2\Delta)} \times \\
&\Big(\ell(643-14\ell(-3+2\ell(9+2\ell)))+4\ell(-232+\ell(-115+4\ell(1+\ell)))\Delta+8(3+\ell) \\
&(-24+\ell(17+2\ell))\Delta^2-16(-7+\ell)(3+\ell)\Delta^3-16(3+\ell)\Delta^4+27(9+4\Delta)\Big).
\end{aligned}
$$

The asymptotic behaviour of the CBs for $z, \bar{z} \to 0$ ($z \to 0$ first) is dominated by the coefficients with $n = -1$ and the lowest value of m, i.e. $c^0_{-1,-1}$ and $c^1_{0,-1}$. For $\ell = 0$, the asymptotic behaviour of $G^{(1)}_0$ is given by the next term $c^0_{0,-1}$, since $c^0_{-1,-1}$ in eq.(C.2) vanishes. This in agreement with the asymptotic behaviour of the CBs found in subsection 4.4.1. Notice how the complexity of the $c^e_{m,n}$ varies from coefficient to coefficient. In general the most complicated ones are those in the "interior" of the octagons (hexagons only for $p = 1$).

Appendix D

D Crossing Relations for SO(N)×SO(M) and SO(N)×SU(M)

We report here the crossing symmetry constraints coming from four-point functions of scalar operators with scaling dimensions d in the bi-fundamental representation of SO(N)×SO(M) and SU(N)×SO(M).

D.1 SO(N)×SO(M)

Let ϕ_a^i be the scalar operator in the bi-fundamental representation of SO(N)×SO(M), with a and i being SO(N) and SO(M) indices, respectively. As usual, we define conformal blocks that contain the contributions of the operators appearing in the OPE of $\phi_a^i \phi_b^j$ in a given representation of the global symmetry. We have nine different conformal blocks G_{ij}, where $i, j = S, T, A$ with S, T and A corresponding to the singlet, symmetric and antisymmetric representations of SO(N) and SO(M). The first index refers to SO(N), the second one to SO(M). The spin of the operators entering in G_{ij} is even if zero or two antisymmetric representations appear and odd otherwise. In order to have reasonably compact formulas, we define the functions

$$F_{ij} \equiv \sum_{\mathcal{O}\in(i,j)-\text{sector}} |\lambda_{\mathcal{O}}^{ij}|^2 \, F_{d,\Delta,l}(z,\bar{z})\,, \qquad H_{ij} \equiv \sum_{\mathcal{O}\in(i,j)-\text{sector}} |\lambda_{\mathcal{O}}^{ij}|^2 \, H_{d,\Delta,l}(z,\bar{z})\,. \tag{D.1}$$

In terms of these, the crossing relations read

$$
\begin{aligned}
&F_{SS}-\frac{2}{M}F_{ST}-\frac{2}{N}F_{TS}+\Big(1+\frac{4}{MN}\Big)F_{TT}+F_{AT}+F_{TA}+F_{AA}=1\,,\\
&H_{SS}-\frac{2}{M}H_{ST}-\frac{2}{N}H_{TS}-\Big(1-\frac{4}{MN}\Big)H_{TT}-H_{AT}-H_{TA}-H_{AA}=-1\,,\\
&\Big(1-\frac{2}{M}\Big)F_{TT}+F_{TS}-F_{AS}+F_{TA}-\Big(1-\frac{2}{M}\Big)F_{AT}-F_{AA}=0\,,\\
&\Big(1+\frac{2}{M}\Big)H_{TT}-H_{TS}+H_{AS}+H_{TA}-\Big(1+\frac{2}{M}\Big)H_{AT}-H_{AA}=0\,,\\
&\Big(1-\frac{2}{N}\Big)F_{TT}+F_{ST}-F_{SA}+F_{AT}-\Big(1-\frac{2}{N}\Big)F_{TA}-F_{AA}=0\,,\\
&\Big(1+\frac{2}{N}\Big)H_{TT}-H_{ST}+H_{SA}+H_{AT}-\Big(1+\frac{2}{N}\Big)H_{TA}-H_{AA}=0\,,\\
&\Big(\frac{2}{M}+\frac{2}{N}\Big)F_{TT}+\frac{2}{N}F_{TA}+\frac{2}{M}F_{AT}-F_{TS}-F_{ST}-F_{SA}-F_{AS}=0\,,\\
&\Big(\frac{2}{M}-\frac{2}{N}\Big)H_{TT}-\frac{2}{N}H_{TA}+\frac{2}{M}H_{AT}-H_{TS}+H_{ST}+H_{SA}-H_{AS}=0\,,\\
&F_{TT}-F_{AT}-F_{TA}+F_{AA}=0\,.
\end{aligned}
\tag{D.2}
$$

We have verified that reflection positivity is satisfied in the appropriate channels. The values of the OPE coefficients in the free-theory limit $d\to 1$ read

$$
\begin{aligned}
\lambda_l^{TT}&=\lambda_l^{AA}=\lambda_l^{AT}=\lambda_l^{TA}=\frac{1}{2}\lambda_l^{free}\,,\\
\lambda_l^{TS}&=\lambda_l^{AS}=\frac{1}{\sqrt{2M}}\lambda_l^{free}\,,\\
\lambda_l^{ST}&=\lambda_l^{SA}=\frac{1}{\sqrt{2N}}\lambda_l^{free}\,,\\
\lambda_l^{SS}&=\frac{1}{2\sqrt{MN}}\lambda_l^{free}\,,
\end{aligned}
\tag{D.3}
$$

where λ_l^{free} is given in eq. (5.19) and l is even or odd depending on the representation. Consistency with the free-theory limit provides a further check on various signs appearing in eq. (D.2).

D.2 SO(N)×SU(M)

Let ϕ_a^i and $\phi_a^{\bar{i},\dagger}$ be a scalar operator and its complex conjugate in the bi-fundamental representation of SO(N)×SU(M), with a and i being SO(N) and SU(M) indices, respectively. As usual, we define conformal blocks that contain the contributions of the operators appearing in the OPE of $\phi_a^i\phi_b^j$ in a given representation of the global symmetry. Since operators in the singlet and adjoint representations of SU(M) can

have both even and odd spin, we define

$$F_{ij}^{+/-} \equiv \sum_{\substack{\mathcal{O}\in(i,j)-\text{sector}\\ l \text{ even/odd}}} |\lambda_{\mathcal{O}}^{ij_{+/-}}|^2 \; F_{d,\Delta,l}(z,\bar{z})\,, \qquad H_{ij}^{+/-} \equiv \sum_{\substack{\mathcal{O}\in(i,j)-\text{sector}\\ l \text{ even/odd}}} |\lambda_{\mathcal{O}}^{ij_{+/-}}|^2 \; H_{d,\Delta,l}(z,\bar{z})\,,$$
$$F_{ij} \equiv F_{ij}^{+} + F_{ij}^{-}\,, \quad \hat{F}_{ij} \equiv F_{ij}^{+} - F_{ij}^{-}\,, \qquad H_{ij} \equiv H_{ij}^{+} + H_{ij}^{-}\,, \quad \hat{H}_{ij} \equiv H_{ij}^{+} - H_{ij}^{-}\,. \tag{D.4}$$

Here i runs over the representations S, T, A of SO(N), while j runs over the singlet (S), adjoint (Ad), symmetric (T) and antisymmetric (A) representations of SU(M). Distinguishing between even and odd spins, we have a total of 18 conformal blocks and, correspondingly, a system of 18 crossing symmetry constraints. Six of these constraints arise by imposing crossing symmetry in the s- and t-channel on the four-point function $\langle \phi_a^i \phi_b^{\bar{j},\dagger} \phi_c^k \phi_d^{\bar{l},\dagger} \rangle$. They read

$$\begin{aligned}
&F_{SS} - \frac{2}{N}F_{TS} - \frac{1}{M}F_{SAd} + \Big(1 + \frac{2}{MN}\Big)F_{TAd} + F_{AAd} = 1\,,\\
&H_{SS} - \frac{2}{N}H_{TS} - \frac{1}{M}H_{SAd} - \Big(1 - \frac{2}{MN}\Big)H_{TAd} - H_{AAd} = -1\,,\\
&F_{TS} - F_{AS} + \Big(1 - \frac{1}{M}\Big)F_{TAd} - \Big(1 - \frac{1}{M}\Big)F_{AAd} = 0\,,\\
&H_{TS} - H_{AS} - \Big(1 + \frac{1}{M}\Big)H_{TAd} + \Big(1 + \frac{1}{M}\Big)H_{AAd} = 0\,,\\
&F_{TS} + F_{AS} - \Big(\frac{1}{M} + \frac{2}{N}\Big)F_{TAd} + F_{SAd} - \frac{1}{M}F_{AAd} = 0\,,\\
&H_{TS} + H_{AS} + \Big(-\frac{1}{M} + \frac{2}{N}\Big)H_{TAd} - H_{SAd} - \frac{1}{M}H_{AAd} = 0\,.
\end{aligned} \tag{D.5}$$

The remaining twelve constraints arise by imposing crossing symmetry in the s- and t-channel on the four-point function $\langle\phi_a^i\phi_b^{\bar{j},\dagger}\phi_c^{\bar{k},\dagger}\phi_d^l\rangle$. They read

$$\begin{aligned}
&\hat{F}_{SS}-\frac{2}{N}\hat{F}_{TS}-\frac{1}{M}\hat{F}_{SAd}+\frac{2}{MN}\hat{F}_{TAd}+F^+_{TT}+F^-_{AT}+F^-_{TA}+F^+_{AA}=1\,,\\
&\hat{H}_{SS}-\frac{2}{N}\hat{H}_{TS}-\frac{1}{M}\hat{H}_{SAd}+\frac{2}{MN}\hat{H}_{TAd}-H^+_{TT}-H^-_{AT}-H^-_{TA}-H^+_{AA}=-1\,,\\
&\hat{F}_{TS}-\frac{1}{M}\hat{F}_{TAd}-\frac{1}{M}\hat{F}_{AAd}+\hat{F}_{AS}+F^+_{TT}-F^-_{AT}+F^-_{TA}-F^+_{AA}=0\,,\\
&\hat{H}_{TS}-\frac{1}{M}\hat{H}_{TAd}-\frac{1}{M}\hat{H}_{AAd}+\hat{H}_{AS}-H^+_{TT}+H^-_{AT}-H^-_{TA}+H^+_{AA}=0\,,\\
&\hat{F}_{TS}-\frac{1}{M}\hat{F}_{TAd}+\frac{1}{M}\hat{F}_{AAd}-\hat{F}_{AS}-\frac{2}{N}F^+_{TT}+F^+_{ST}+F^-_{SA}-\frac{2}{N}F^-_{TA}=0\,,\\
&\hat{H}_{TS}-\frac{1}{M}\hat{H}_{TAd}+\frac{1}{M}\hat{H}_{AAd}-\hat{H}_{AS}+\frac{2}{N}H^+_{TT}-H^+_{ST}-H^-_{SA}+\frac{2}{N}H^-_{TA}=0\,,\\
&\hat{F}_{SAd}-\frac{2}{N}\hat{F}_{TAd}+F^+_{TT}-F^-_{TA}+F^-_{AT}-F^+_{AA}=0\,,\\
&\hat{H}_{SAd}-\frac{2}{N}\hat{H}_{TAd}-H^+_{TT}+H^-_{TA}-H^-_{AT}+H^+_{AA}=0\,,\\
&\hat{F}_{TAd}+\hat{F}_{AAd}+F^+_{TT}-F^-_{TA}-F^-_{AT}+F^+_{AA}=0\,,\\
&\hat{H}_{TAd}+\hat{H}_{AAd}-H^+_{TT}+H^-_{TA}+H^-_{AT}-H^+_{AA}=0\,,\\
&\hat{F}_{TAd}-\hat{F}_{AAd}-\frac{2}{N}F^+_{TT}+F^+_{ST}-F^-_{SA}+\frac{2}{N}F^-_{TA}=0\,,\\
&\hat{H}_{TAd}-\hat{H}_{AAd}+\frac{2}{N}H^+_{TT}-H^+_{ST}+H^-_{SA}-\frac{2}{N}H^-_{TA}=0\,.
\end{aligned}\tag{D.6}$$

Reflection positivity fixes the signs in both the s- and t-channel for $\langle\phi_a^i\phi_b^{\bar{j},\dagger}\phi_c^k\phi_d^{\bar{l},\dagger}\rangle$. By interchanging the coordinates of the last two fields in the former four-point function and $c\leftrightarrow d$, this then also fixes the signs in the s-channel (the channel for which the $\phi\phi^\dagger$ OPE is used) for $\langle\phi_a^i\phi_b^{\bar{j},\dagger}\phi_c^{\bar{k},\dagger}\phi_d^l\rangle$. The signs in the t-channel for the latter four-point function are in turn fixed by reflection positivity. The values of the OPE coefficients in the free-theory limit $d\to 1$ read

$$\begin{aligned}
&\lambda_l^{TAd_+}=\lambda_l^{TAd_-}=\lambda_l^{AAd_+}=\lambda_l^{AAd_-}=\lambda_l^{TT_+}=\lambda_l^{TA_-}=\lambda_l^{AT_-}=\lambda_l^{AA_+}=\frac{1}{2}\lambda_l^{\text{free}}\,,\\
&\lambda_l^{SAd_+}=\lambda_l^{SAd_-}=\lambda_l^{ST_+}=\lambda_l^{SA_-}=\frac{1}{\sqrt{2N}}\lambda_l^{\text{free}}\,,\\
&\lambda_l^{TS_+}=\lambda_l^{TS_-}=\lambda_l^{AS_+}=\lambda_l^{AS_-}=\frac{1}{2\sqrt{M}}\lambda_l^{\text{free}}\,,\\
&\lambda_l^{SS_+}=\lambda_l^{SS_-}=\frac{1}{\sqrt{2MN}}\lambda_l^{\text{free}}\,,
\end{aligned}\tag{D.7}$$

where $\lambda_l^{\rm free}$ is given in eq. (5.19) and l is even or odd depending on the representation. Consistency with the free-theory limit provides a further check on various signs appearing in eqs. (D.5) and (D.6). As a further consistency check, we have verified that eqs. (D.5) and (D.6) reduce to eqs. (D.2) when $\mathsf{SO}(N) \times \mathsf{SU}(M) \subset \mathsf{SO}(N) \times \mathsf{SO}(2M)$.

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